Graceful Labellings of New Families of Windmill and Snake Graphs

by

© Ahmad H. Alkasasbeh

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Abstract

A function f is a *graceful labelling* of a graph $G = (V, E)$ with m edges if f is an injection $f : V(G) \to \{0, 1, 2, \ldots, m\}$ such that each edge $uv \in E$ is assigned the label $|f(u) - f(v)| \in \{1, 2, \ldots, m\}$, and no two edge labels are the same. If a graph G has a graceful labelling, we say that G itself is graceful. A variant is a *near graceful labelling*, which is similar, except the co-domain of f is $\{0, 1, 2, \ldots, m + 1\}$ and the set of edge labels are either $\{1, 2, \ldots, m - 1, m\}$ or $\{1, 2, \ldots, m - 1, m + 1\}$.

In this thesis, we prove any Dutch windmill with three pendant triangles is (near) graceful, which settles Rosa's conjecture for a new family of triangular cacti. Further, we introduce graceful and near graceful labellings of several families of windmills. In particular, we use Skolem-type sequences to prove (near) graceful labellings exist for windmills with C_3 and C_4 vanes, and infinite families of $3, 5$-windmills and $3, 6$-windmills. Furthermore, we offer a new solution showing that the graph obtained from the union of t 5-cycles with one vertex in common (C_5^t) is graceful if and only if $t \equiv 0, 3 \pmod 4$ and near graceful when $t \equiv 1, 2 \pmod 4$.

Also, we present a new sufficiency condition to obtain a graceful labelling for every kC_{4n} snake and use this condition to label every such snake for $n = 1, 2, \ldots, 6$. Then, we extend this result to cyclic snakes where the cycles lengths vary. Also, we obtain new results on the (near) graceful labelling of cyclic snakes based on cycles of lengths $n = 6, 10, 14$, completely solving the case $n = 6$.

General Summary

A graph is a collection of vertices (nodes) and edges (links), where each edge consists of two vertices. Graph theory is a relatively new area of mathematics, and it has numerous applications. The study of graphs has become important in several fields, for instance, in the networks and communications field. For example, Facebook can be represented by a graph, where the vertices represent the people, while the edges list all the friendships connecting them.

A graceful labelling of a graph with m edges is a labelling of its vertices with a subset of distinct non-negative integers from 0 to m, such that each vertex label is used at most once, and the edges are labelled with the absolute difference between their vertex labels, such that the edge labels are $\{1, 2, 3, \ldots, m\}$. A graph that admits a graceful labelling is called a graceful graph. A near graceful labelling is similar to a graceful labelling, except that the vertex labels are from 0 to $m + 1$, and the edge labels are $\{1, 2, 3, \ldots, m - 1, m\}$ or $\{1, 2, 3, \ldots, m - 1, m + 1\}$.

In this thesis, we prove that graceful and near graceful labellings exist for several types of windmill and snake graphs. In particular, we show that Dutch windmills of any order with three pendant triangles and infinite families of variable windmills and cyclic snakes are graceful. This work is important because we introduce more evidence that Rosa's conjecture on triangular cacti is true. Moreover, we find new results and extend the existing results to a more general class of triangular cacti and snakes. Further, we find a new sufficient condition to prove graceful labellings exist for any cyclic snake with k blocks with each cycle of length $4n$.

Besides the problem of determining which graphs are graceful, graceful labellings are useful in finding graph decompositions. Graceful labellings also have interesting applications in radio astronomy.

To my parents Hamad and Jamileh
In memory of my sister, Huda

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Statement of contribution

All the work in this thesis was done in collaboration and under the supervisor of Professor Danny Dyer.

Chapters two, three and four of this thesis consist of the following papers. Research on these papers was shared jointly, with all authors contributing equally.

Chapter 2: A. Alkasasbeh, D. Dyer, and N. Shalaby, Applying Skolem Sequences to Gracefully Label New Families of Triangular Windmills, submitted, *Australasian Journal of Combinatorics*.
Chapter 3: A. Alkasasbeh, D. Dyer, and J. Howell, Graceful Labellings of Variable Windmills Using Skolem Sequences.
Chapter 4: A. Alkasasbeh and D. Dyer, Graceful Labellings of Various Cyclic Snakes, submitted, *Discrete Mathematics journal*.

Table of contents

List of tables

List of figures

Chapter 1

Introduction and Overview

1.1 History and Elementary Results

Graph theory is a relatively new area of mathematics, first studied by Leonhard Euler in 1735. A *graph* G with n vertices and m edges consists of a vertex set $V(G) = \{v_1, \ldots, v_n\}$ and an edge set $E(G) = \{e_1, \ldots, e_m\}$, where each edge consists of two vertices called its endpoints. In this chapter, we discuss some history and terminology of graph theory, graph labellings, and Skolem-type sequences, as well as some applications. For any undefined terms we follow [29].

A *path* P_n is a sequence of n distinct vertices $v_1, v_2, \ldots, v_n$ such that every two consecutive vertices are adjacent. A *cycle* of length n, C_n, is the graph on n vertices $v_1, v_2, \ldots, v_n$ with n edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1$. A graph G is *connected* if there is at least one path between every pair of vertices. A *tree* is a connected graph with no cycles. The graph in Figure 1.1 has a path v_4, v_5, v_1, v_2 and a cycle v_1, v_2, v_3, v_5, v_1. A complete graph K_n on n vertices is a graph in which an edge connects every pair of vertices.

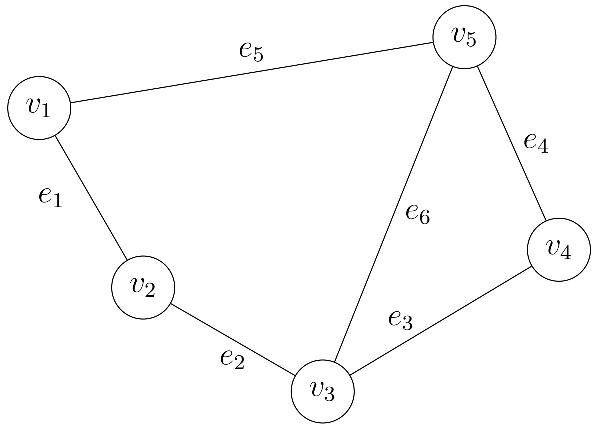


Figure 1.1: A connected graph.

A *decomposition* of a graph G is a partition of $E(G)$ into pairwise edge disjoint subgraphs. A partitioning of K_3 into P_3 and P_2 is an example of a decomposition.

In 1963, Ringel [23] posed the following conjecture.

Conjecture 1.1.1. *[23] If T is an arbitrary tree with m edges, then K_{2m+1} can be decomposed into $2m+1$ copies of T.*

Figure 1.2 presents a solution to Conjecture 1.1.1 when $m = 3$, for the tree specified in black.

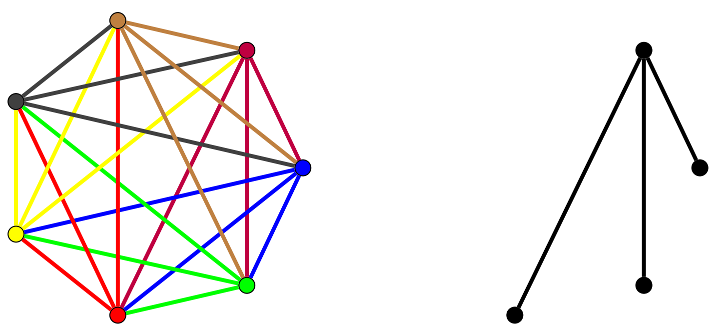


Figure 1.2: Decomposition of K_7 into seven copies of a tree.

A *cyclic decomposition* of an n-vertex graph G is a decomposition of G into n subgraphs in which each subgraph is obtained from the others by applying a (canonical) permutation. Figure 1.2 is a cyclic decomposition of K_7 into seven copies of a tree, where each distinct color corresponds to a distinct tree.

Later, in [24], Rosa introduced the Ringel-Kotzig conjecture.

Conjecture 1.1.2. *[24] If T is an arbitrary tree with m edges, then K_{2m+1} can be cyclically decomposed into $2m+1$ copies of T.*

Rosa, in 1967, introduced the concept of graph labelling, which is one of the most interesting topics in graph theory. A *graph labelling* is an assignment of labels to the vertices or the edges (or both) of a graph G under certain conditions. Rosa, in his famous paper *"On Certain Valuations of the Vertices of a Graph"* [24], introduced four different vertex labellings to aid attempts to prove the Ringel-Kotzig conjecture. One of these labellings is a *β-valuation.* In 1972, Golomb renamed a β-valuation as a *graceful labelling* [15].

In this thesis, we adopt the convention that 0 is a natural number. Further, when we write $[i, j]$ with $i, j \in \mathbb{N}$ and $i < j$, we are indicating the set $\{k \in \mathbb{N} | i \leq k \leq j\}$. Let $G = (V, E)$ be a graph with m edges. Let $f : V(G) \to [0, m]$ be a labelling of $V(G)$ and let $g : E(G) \to [1, m]$ be the induced edge labelling defined by $g(uv) = |f(u) - f(v)|$, for all $uv \in E$. The labelling f is said to be a *graceful labelling* if and only if f is an injective mapping and g is a bijection. If a graph G has a graceful labelling then we say G is *graceful.*

We will gracefully label the graph G on five edges in Figure 1.3 as follows. Let $f : V(G) \to [0, 5]$ be a vertex labelling of G and let $g : E(G) \to [1, 5]$ be the induced edge labelling $g(v_i v_j) = |f(v_i) - f(v_j)|$, for all $v_i v_j \in E$. Now we assign vertex v_1 the label 5 ($f(v_1) = 5$), and vertex v_2 the label 0 ($f(v_2) = 0$). This induces an edge label of $e_1 = 5$ ($g(e_1) = 5$). Similarly, let $f(v_3) = 4$, $f(v_4) = 3$, $f(v_5) = 2$ and $f(v_6) = 1$. These induce an edge labelling of $g(e_2) = 4$, $g(e_3) = 1$, $g(e_4) = 2$, and $g(e_5) = 3$, respectively. Thus the tree G has a graceful labelling and G is graceful.

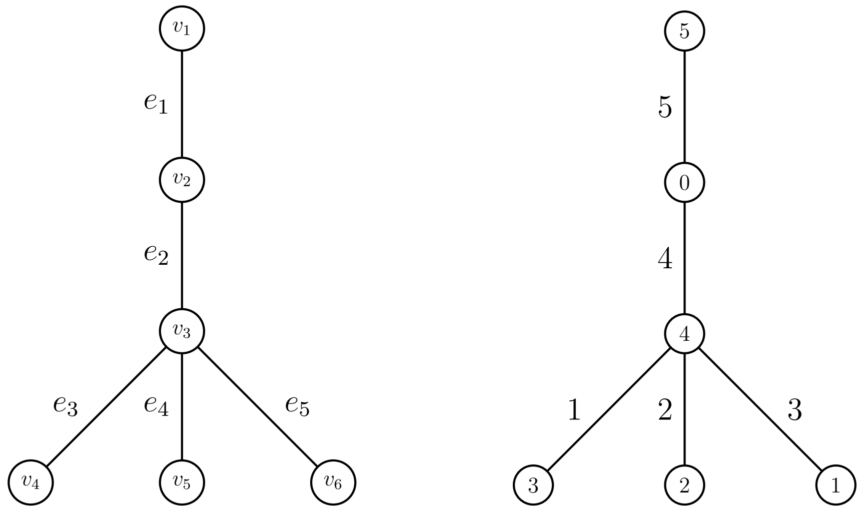


Figure 1.3: A tree G and its graceful labelling.

Rosa used the concept of graceful labelling to then show the following theorem. The proof of Theorem 1.1.3 follows [29].

Theorem 1.1.3. *[24] If a graph G with m edges has a graceful labelling, then K_{2m+1} has a (cyclic) decomposition into $2m+1$ copies of G.*

Proof. Let G be a graceful graph with m edges. Then there is graceful labelling of G that uses vertex labels from the set $[0, m]$, and the edge labels $[1, m]$. View the vertices of K_{2m+1} as the congruence classes modulo $2m+1$. We define the difference between two distinct congruence classes a and b to be $\min\{a - b \pmod{2m+1}, b - a \pmod{2m+1}\}$. Thus the possible differences are the integers $1, 2, 3, \ldots, m$.

From a graceful labelling of G, we define copies of G, denoted G_k, in K_{2m+1} for $0 \leq k \leq 2m$. We first define G_0 as a subgraph of K_{2m+1}. Let the vertices of G_0 be the set of all numbers $x \in [0, m]$ such that x is a vertex label in G. Two vertices $x, y \in V(G_0)$ are adjacent if and only if the vertices with labels x and y are adjacent in G. We define G_k as follows: the vertices of G_k are all $x + k \pmod{2m+1}$ for $x \in V(G_0)$, with $i + k$ adjacent to $j + k$ in G_k if and only if i is adjacent to j in G_0.

Let uv with $u < v$ be an edge in K_{2m+1} such that $w = v - u$. Certainly $1 \le w \le m$. Since G is graceful, there exists an edge with label w, and let x and y be the vertex labels of that edge with $x < y$. Therefore in G_0, xy is an edge with $y - x = w$. Therefore $uv \in E(G_{u-x})$, where $u - x$ is considered modulo $2m+1$. Therefore every edge in K_{2m+1} is in some G_k where $0 \le k \le 2m$.

Consider the number of edges in K_{2m+1}. Then

$$m(2m+1) = |E(K_{2m+1})| \le \left| \bigcup_{k=0}^{2m} E(G_k) \right| \le \sum_{k=0}^{2m} |E(G_k)| = m(2m+1).$$

Since the number of edges in K_{2m+1} is equal to the total number of edges of $2m+1$ copies of G, there is no edge that will be used twice in K_{2m+1}. Furthermore, the sets of edges in each copy of G are disjoint, and hence they cover all the edges in K_{2m+1} and these $2m+1$ copies of G decompose K_{2m+1}. By definition, $\pi(G_0) = G_1$, where π is the canonical permutation $(0\ 1\ 2\ \cdots 2m)$. Thus the $2m+1$ copies of G cyclically decompose K_{2m+1}. $\square$

For example, seven rotations of the star with center 3 and leaves $0, 1, 2$ decompose K_7, as in Figure 1.4.

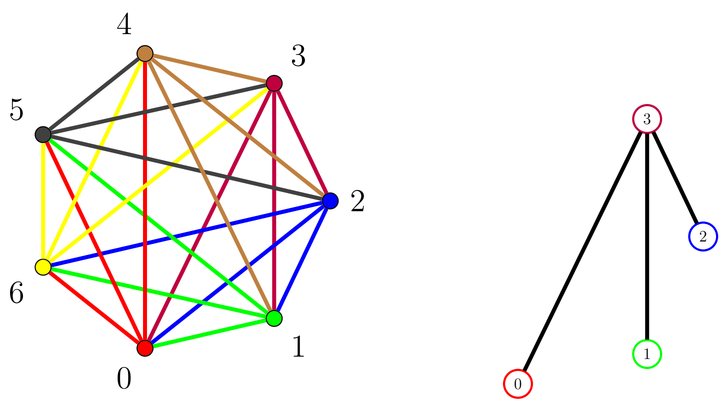


Figure 1.4: Decomposition of K_7 into seven copies of a gracefully labelled tree.

An α-labelling is another of the four graph labellings introduced by Rosa in [24]. An *α-labelling* of a graph G is a graceful labelling with the extra condition that there exists an integer w such that for any edge $uv \in E(G)$, either $f(u) \leq w < f(v)$ or $f(v) \leq w < f(u)$. Figure 1.4 includes an α-labelling of $K_{1,3}$ with $w = 2$ on the right. It is clear that a graph with an α-labelling is necessarily bipartite with one part of vertices u whose labels are less than or equal to w and the other part of vertices v whose labels are greater than w. Moreover, Rosa in [24] presented the following theorem.

Theorem 1.1.4. *[24] If G is a complete bipartite graph, then there exists an α-labelling of G.*

Every graceful labelling is not necessarily an α-labelling. If we consider the graph K_3 and label the vertices as follows $f(v_1) = 0$, $f(v_2) = 1$ and $f(v_3) = 3$ we obtain a gracefully labelled K_3 but it has no α-labelling since it has an odd cycle.

Rosa's work turned attention from proving Ringel's Conjecture to proving that all trees are graceful. This generated interest in solving the following conjecture, known as Graceful Tree Conjecture (GTC).

Conjecture 1.1.5. *(GTC [24]) All trees are graceful.*

In [17], Kotzig called the effort to prove the GTC a "disease". This famous conjecture, known also as the Ringel-Rosa-Kotzig conjecture, is still open. Recently, three mathematicians claimed a proof of Ringel's conjecture in [20], though their work proves the conjecture only for quite large n via producing cyclically invariant decompositions while applying probabilistic arguments and other involved techniques. Further, a paper in the arXiv [14] claims to be a proof of the GTC using a composition lemma. This article already has a long revision history and remains unpublished at this writing.

Many classes of trees have been shown to be graceful.

Theorem 1.1.6. *[24] All paths are graceful.*

A *caterpillar* is a tree having a path that contains at least one vertex of every edge.

Theorem 1.1.7. *[24] All caterpillars are graceful.*

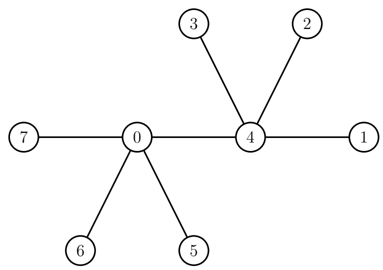


Figure 1.5: A gracefully labelled caterpillar.

A tree is *symmetrical* if it is a rooted tree in which every descendant at the same distance from the root has the same degree.

Theorem 1.1.8. *[5] All symmetrical trees are graceful.*

The graceful tree conjecture may be wrong. If then, the most straightforward proof would be to discover a counterexample tree that cannot be gracefully labelled. But, searches have only found billions of graceful trees. These currently cover every tree with up to 35 vertices. Previously it has been proven in [1] that all trees on at most 27 vertices are graceful. More recent results are in an arXiv manuscript [11], for all trees on at most 35 vertices.

Theorem 1.1.9. *[1] All trees with* $n \leq 27$ *vertices are graceful.*

Theorem 1.1.10. *[11] All trees with* $n \leq 35$ *vertices are graceful.*

Besides trees, we may also consider which graphs are graceful. Some classes of graphs have been determined to be either graceful or not graceful. Rosa in [24] gave a necessary condition for an Eulerian graph to be graceful.

Theorem 1.1.11. *[24] If G is a graceful Eulerian graph with m edges, then $m \equiv 0{,}3 \pmod 4$.*

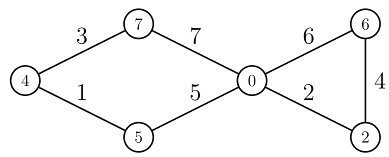


Figure 1.6: Gracefully labelled fish graph.

Further Rosa proved in the following theorem that for cycles, the necessary condition is sufficient.

Theorem 1.1.12. *[24] The cycle C_n is graceful if and only if $n \equiv 0,\ 3 \pmod 4$.*

Theorem 1.1.13. *[15, 24] All the complete bipartite graphs $K_{m,n}$ are graceful.*

Golomb in [15] proved the same result of Theorem 1.1.13, and the following theorem.

Theorem 1.1.14. *[15] The complete graph K_n $(n \geq 2)$ is graceful if and only if $n \leq 4$.*

A *wheel* graph is a graph obtained by connecting all vertices of a cycle to a single common vertex.

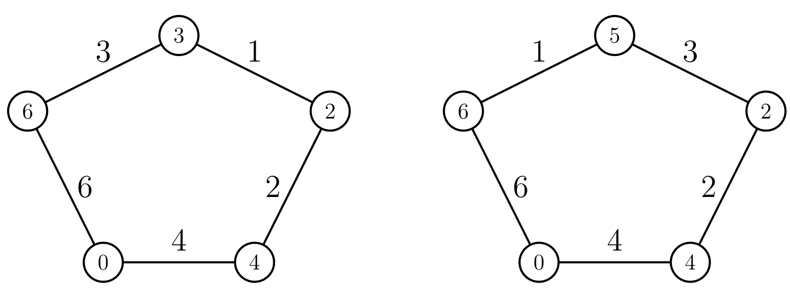
Theorem 1.1.15. *[12, 16] All wheels are graceful.*

Not all graphs are graceful. For instance, C_5 is an Eulerian graph with 5 edges and therefore it is not graceful. It is also straightforward to prove from first principles.

Suppose that C_5 is graceful, then it has 5 vertices that must be labelled from the set $[0,5]$ and 5 edges must be labelled with the set $[1,5]$; more importantly, two edges have even labels, and three have odd labels. We have two cases, either there are 3 odd vertices and 2 even vertices or 3 even vertices and 2 odd vertices. Assume we use 3 even vertices and 2 odd vertices. If the two odd vertices are adjacent, then we obtain two odd edges in C_5. If the two odd vertices are not adjacent, then we obtain four odd edges in C_5. A similar argument shows that a graceful labelling of C_5 cannot have 2 even and 3 odd vertex labels. Thus, C_5 is not graceful.

A near graceful labelling is a labelling similar to graceful labelling but with a property weaker than graceful. Let $G = (V, E)$ be a graph with m edges. Let $f : V(G) \to [0, m+1]$ be a labelling of $V(G)$ and let $g : E(G) \to A$, where A is $\{1, 2, \ldots, m-1, m\}$ or $\{1, 2, \ldots, m-1, m+1\}$, be the induced edge labelling defined by $g(uv) = |f(u) - f(v)|$, for all $uv \in E$. The labelling f is said to be a *near graceful labelling* if and only if f is an injective mapping and g is a bijection. If a graph G has a near graceful labelling then we say G is *near graceful*. Barrientos, in [4], presented the following theorem.

Theorem 1.1.16. *[4]If $n \equiv 1, 2 \pmod 4$, then the cycle C_n is near graceful.*



(a) (b)

Figure 1.7: Near graceful labellings of C_5.

We also introduce an almost graceful labelling as introduced in [21]. Let $G =$

(V, E) be a graph with m edges. Let $f : V(G) \to \{0, 1, \dots, m-1, x\}$ be a labelling of $V(G)$ where x is either m or $m+1$ and let $g : E(G) \to \{1, 2, \dots, m-1\} \cup \{x\}$ be the induced edge labelling defined by $g(uv) = |f(u) - f(v)|$, for all $uv \in E$. The labelling f is said to be an *almost graceful labelling* if and only if f is an injective mapping and g is a bijection. If a graph G has an almost graceful labelling then we say G is *almost graceful*.

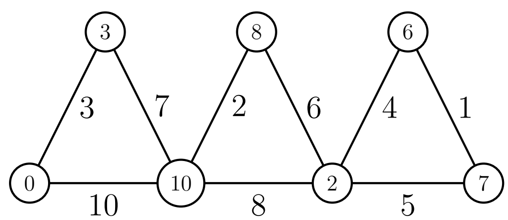


Figure 1.8: Almost graceful labelling of $3C_3$.

Based on the previous definitions we can conclude that every graceful graph is almost graceful, and every almost graceful graph is near graceful. Figure 1.7(b) is an example of a near graceful labelling that is neither graceful nor almost graceful. Figure 1.8 is an example of an almost graceful graph that we know is not graceful because it is an Eulerian graph with 9 edges. It is not clear if a graph exists that is near graceful but not almost graceful. Every near graceful graph we have considered has been shown to be almost graceful.

There are graphs that are neither graceful nor near graceful. For example, consider the complete graph K_6. We know from Theorem 1.1.14 that K_6 is not graceful. Suppose that K_6 is near graceful, then two of the six vertices are labelled 0 and 16 or 0 and 15 or 1 and 16. Then every labelling that can be obtained continuing from this point is formed in one of $3\binom{14}{4}$ ways. Through exhaustive checking, we see that none of these is near graceful.

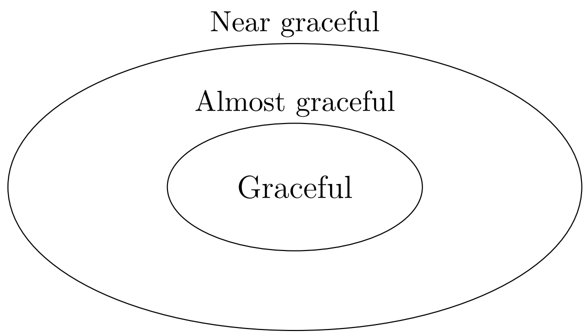


Figure 1.9: The relationship between the labellings considered in this thesis.

A *triangular cactus* is a connected graph whose blocks (or 2-connected components) are all C_3. See Figure 1.8 for an example. In [25], Rosa stated the following conjecture.

Conjecture 1.1.17. *[25] All triangular cacti with* $n \equiv 0, 1 \pmod 4$ *are graceful, and all triangular cacti with* $n \equiv 2, 3 \pmod 4$ *are near graceful.*

Let C_n be a cycle of length $n \geq 3$, and C_n^t be the graph obtained from the union of t n-cycles with one vertex in common that we will call the *central vertex*. A *Dutch windmill* C_3^t is a triangular cactus that has the property of all its blocks having a common vertex, and the blocks will be called *vanes*. The graceful labelling of the Dutch windmill was discussed in [6–8]. A *pendant triangle* is defined as a block that is added to any triangular cactus by identifying a vertex of the new block with one vertex of the cactus. Another family of triangular cacti is the class of *triangular snakes*, a type of triangular cacti whose block cutpoint graphs are paths. Rosa's conjecture was proved for triangular snakes by Moulton [21].

Theorem 1.1.18. *[21] All triangular snakes with* $n \equiv 0, 1 \pmod 4$ *are graceful, and all triangular snakes with* $n \equiv 2, 3 \pmod 4$ *are almost graceful.*

Cyclic Steiner triple systems, a class of combinatorial objects of interest to design theorists, can be thought of as triangular cacti. The concept of Skolem sequences was used to derive cyclic Steiner triple systems. Therefore, in [25], Rosa suggested using Skolem-type sequences to label various families of triangular cacti.

At this point it is important to introduce Skolem-type sequences. A *Skolem sequence* of order n is a sequence $S = (s_1, s_2, \ldots, s_{2n})$ of $2n$ integers satisfying these conditions:

1. for every $k \in \{1, 2, \ldots, n\}$, there exist exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = k$;
2. if $s_i = s_j = k$, with $i < j$, then $j - i = k$.

For example, $(5, 2, 4, 2, 3, 5, 4, 3, 1, 1)$ is a Skolem sequence of order 5.

A *hooked Skolem sequence* of order n is a sequence $S = (s_1, s_2, \ldots, s_{2n+1})$ of $2n+1$ integers satisfying these conditions:

1. for every $k \in \{1, 2, \ldots, n\}$, there exist exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = k$;
2. if $s_i = s_j = k$, with $i < j$, then $j - i = k$;
3. $s_{2n} = 0$.

For example, $(3, 1, 1, 3, 2, 0, 2)$ is a hooked Skolem sequence of order 3.

Theorem 1.1.19. *[28] A Skolem sequence of order n exists if and only if $n \equiv 0, 1 \pmod 4$.*

Theorem 1.1.20. *[22] A hooked Skolem sequence of order n exists if and only if $n \equiv 2, 3 \pmod 4$.*

Related sequences have additionally been considered and applied in the construction of combinatorial designs. In this thesis we use the definition of a Skolem-type sequence. A *Skolem-type sequence* of order n is a sequence $K = (s_1, s_2, \ldots, s_{2n})$ of $2n$ integers satisfying the conditions:

1. for some set H of n distinct positive integers, $\forall\ k \in H$, $\exists\ s_i, s_j \in K$ such that $s_i = s_j = k$;
2. if $s_i = s_j = k$, with $i < j$, then $j - i = k$.

For example, $(9, 6, 4, 1, 1, 3, 4, 6, 3, 9)$ is a Skolem-type sequence of order 5 with $H = \{1, 3, 4, 6, 9\}$. A *hooked Skolem-type sequence* of order n is a sequence $hK = (s_1, s_2, \ldots, s_{2n+1})$ of $2n+1$ integers satisfying the conditions of a Skolem-type sequence with the added condition that $s_{2n} = 0$. For example, $(4, 1, 1, 3, 4, 0, 3)$ is a hooked Skolem-type sequence of order 3 with $H = \{1, 3, 4\}$. A *(hooked) Langford sequence* with defect d and order l is a (hooked) Skolem-type sequence with $H = [d, d + l - 1]$. The necessary and sufficient conditions for the existence of Langford sequences are given in the following theorem.

Theorem 1.1.21. *[27]*

1. *A Langford sequence of order l and defect d exists if and only if*
   (a) $l \geq 2d - 1$,
   (b) $l \equiv 0, 1 \pmod 4$ *and d is odd, or*
   (c) $l \equiv 0, 3 \pmod 4$ *and d is even.*
2. *A hooked Langford sequence of order l and defect d exists if and only if*
   (a) $l(l - 2d + 1) + 2 \geq 0$,
   (b) $l \equiv 2, 3 \pmod 4$ *and d is odd, or*
   (c) $l \equiv 1, 2 \pmod 4$ *and d is even.*

In [26], Shalaby introduced the existence of (hooked) near Skolem sequences. Let m and n be positive integers, with $m \leq n$. A *near-Skolem sequence* of order n and defect m is a sequence $NS_m = (s_1, s_2, \ldots, s_{2n-2})$ of integers $s_i \in \{1,\ 2,\ \ldots,\ m-1,\ m+1,\ \ldots,\ n\}$ that satisfies the following conditions:

1. for every $k \in \{1, 2, \ldots, m-1, m+1, \ldots, n\}$, there are exactly two elements $s_i, s_j \in NS_m$ such that $s_i = s_j = k$;

2. if $s_i = s_j = k$, with $i < j$, then $j - i = k$.

For example, $(1, 1, 6, 3, 7, 5, 3, 2, 6, 2, 5, 7)$ is a 4-near Skolem sequence of order 7.

Theorem 1.1.22. *[26] An m-near Skolem sequence of order n exists if and only if $n \equiv 0, 1 \pmod 4$ and m is odd or $n \equiv 2, 3 \pmod 4$ and m is even.*

A *hooked m-near-Skolem sequence* of order n and defect m is a sequence $nS = (s_1, s_2, \ldots, s_{2n-1})$ of integers $s_i \in \{1, 2, \ldots, m-1, m+1, \ldots, n\}$ that satisfies conditions (1) and (2) for near-Skolem sequences and $s_{2n-2} = 0$.

Theorem 1.1.23. *[26] A hooked m-near Skolem sequence of order n exists if and only if $n \equiv 0, 1 \pmod 4$ and m is even or $n \equiv 2, 3 \pmod 4$ and m is odd.*

Another generalization of Skolem sequences that was considered in [3] to construct some combinatorial designs was an m-fold Skolem sequence. An *m-fold Skolem sequence* of order n is a sequence $mS = (s_1, s_2, \ldots, s_{2mn})$ of $2mn$ integers satisfying the following conditions:

1. every $k \in \{1, 2, \ldots, n\}$ occurs $2m$ times;

2. these occurrences can be partitioned into m disjoint pairs, (s_i, s_j), such that $s_i = s_j = k$ and $j - i = k$.

As an example, $(3, 1, 1, 3, 3, 1, 1, 3, 2, 2, 2, 2)$ is a 2-fold Skolem sequence of order 3.

Theorem 1.1.24. *[3] An m-fold Skolem sequence of order n exists if and only if $n \equiv 0, 1 \pmod 4$ for any m or $n \equiv 2, 3 \pmod 4$ and m is even.*

We also extend the definition of Skolem-type sequences to sequences where every pair of elements in H occurs exactly twice. A *two-fold Skolem-type sequence* of order n is a sequence $K_n^2 = (s_1, s_2, \ldots, s_{4n})$ of $4n$ positive integers such that the following conditions hold:

1. for a set H of n distinct positive integers, every $p \in H$ occurs 4 times;
2. these occurrences can be partitioned into 2 disjoint pairs, (s_i, s_j), such that $s_i = s_j = k$ and $j - i = k$.

For example, $(8, 8, 4, 4, 1, 1, 4, 4, 8, 8, 1, 1)$ is a two-fold Skolem-type sequence of order 3 with $H = \{1, 4, 8\}$.

By using Skolem-type sequences Dyer et. al. [10], proved Conjecture 1.1.17 for Dutch windmills with at most two pendant triangles.

Theorem 1.1.25. *[10] Every Dutch windmill with at most two pendant triangles is graceful or near graceful.*

In [7, 18, 19, 30–34], graceful labellings were shown for C_n^t with $n = 3, 4, 5, 6, 7, 9, 11, 13$, respectively, as well as for C_{4k}^t in [19]. In this thesis we use Skolem-type sequences to gracefully (and near gracefully) label new classes of graphs related to windmills. Figure 1.10 shows two examples of these classes.

Having previously discussed triangular snakes, we now turn to a generalization of these graphs. A *cyclic snake* (kC_n) is a connected graph with k blocks whose block-cut-point graph is a path and each of the k blocks is isomorphic to C_n.

A kC_n is an Eulerian graph, thus graceful only if $kn \equiv 0, 3 \pmod 4$. That fact motivated Barrientos to study kC_n in [4]. He proved that kC_4 is graceful for all k,

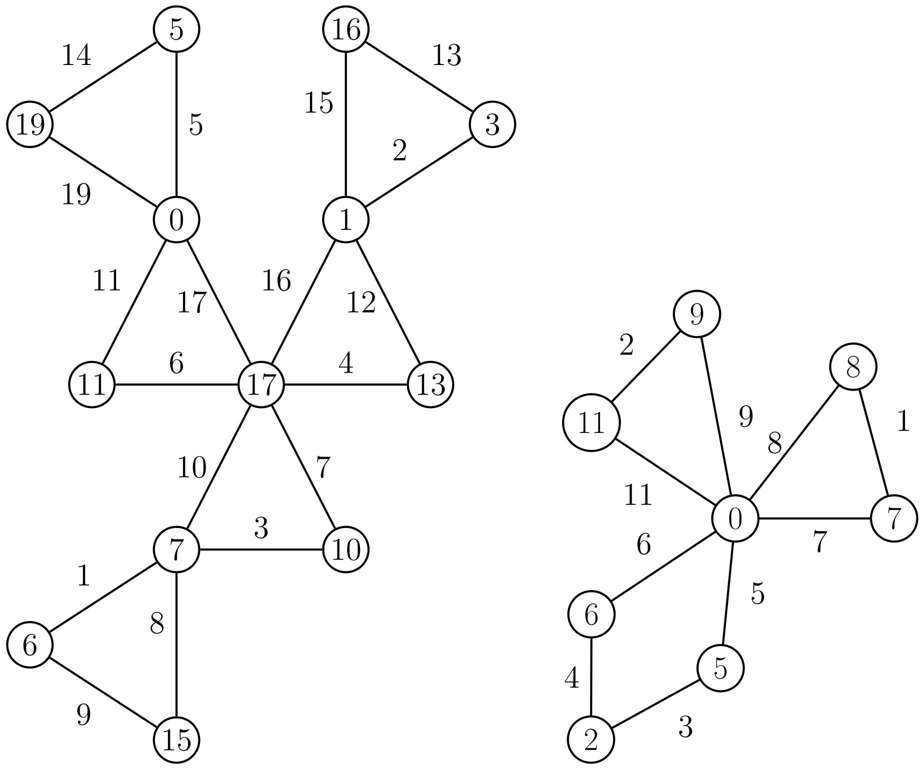


Figure 1.10: Near gracefully labelled Dutch windmill with 3 pendant triangles and near gracefully labelled variable windmill.

and that the family of kC_6 are graceful if k is even and near graceful if k is odd. He also discussed when the snakes kC_8, kC_{12} and kC_{4n} are graceful. Figure 1.11 is an example of a near gracefully labelled snake. In this thesis we prove new results related to the graceful labelling for different kC_n.

We know that graceful labellings are useful in finding graph decompositions (as we discussed before). Graceful labelling also has an interesting application in radio astronomy.

An example of a real life problem is the organization of a group of antennas to optimize the number of distinct integer distances between them [9]. The frequency is determined by the distance between two antennas; a similar spacing does not produce different frequencies. The goal becomes to use a minimum number of antennas to

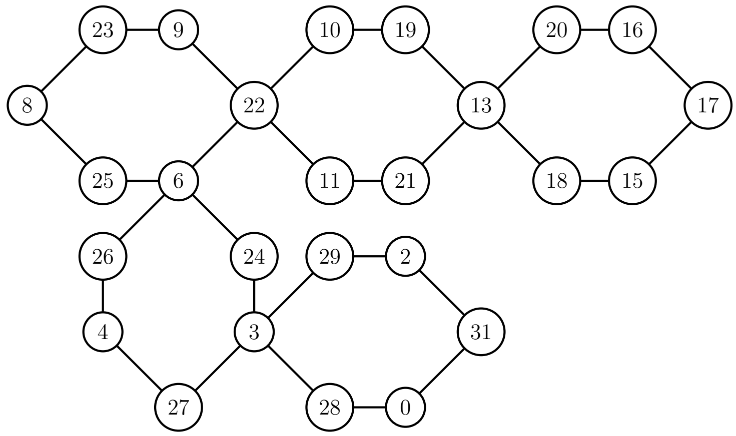


Figure 1.11: A near gracefully labelled $5C_6$.

obtain a large number of distinct frequencies. The linear array of m antennas provides $\frac{m(m-1)}{2}$ spacings between antennas. The optimal arrangement would be such that each of these spacings has different lengths. Graceful labelling of antennas will give an optimal solution for this problem if we represent the spacings as edges that are labelled by the values from $[1, \frac{m(m-1)}{2}]$. For instance, if we have 4 antennas, the maximum number of spacings is 6, so we have to find an arrangement to get the lengths from 1 to 6. In Figure 1.12 we can see the optimum arrangement for 4 antennas which is similar to a gracefully labelled K_4 graph. For more details see [7, 8].

As we mentioned earlier in Theorem 1.1.14 the complete graph K_n $(n \geq 2)$ is graceful if and only if $n \leq 4$. So the natural question now is that, "Can we find a minimum number of antennas to obtain a large number of distinct frequencies for $n > 4$?" That is, for k distinct frequencies what is the smallest number of vertex labels (or antennas) needed such that the induced edge labelling gives k distinct positive integers?

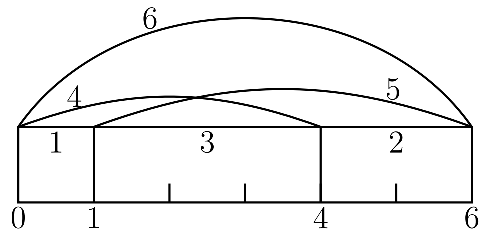


Figure 1.12: Linear configuration of 4 antennas.

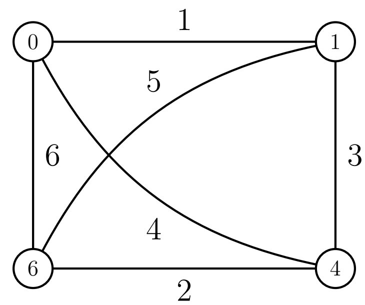


Figure 1.13: A gracefully labelled K_4.

1.2 Thesis Organization and Significance

This thesis contributes to the area of graph labelling. The purpose of this thesis is to prove that graceful and near graceful labellings exist for several types of windmill and snake graphs.

This thesis is presented in a manuscript (research paper) format. It contains five chapters. In the first chapter, we discussed the concept of graceful labelling and we give a general literature review for graceful labelling. The structure of the rest of this thesis is as follows.

In chapter two, we verify Rosa's conjecture for a new family of triangular cacti: Dutch windmills of any order with three pendant triangles, by using Skolem-type sequences. This work is more evidence that Rosa's conjecture is true and extends the result of [10] to a more general class of triangular cacti.

This chapter has four sections. In the first two sections, we discuss the history of graceful labelling of Dutch windmills and Rosa's conjecture as well as some necessary definitions and preliminary results. In the third section, we prove Rosa's conjecture for a Dutch windmill of any order with three pendant triangles. In the last section, we discuss the results of the chapter and propose some open questions.

In chapter three we have the most significant result of this thesis: proving that (near) graceful labellings exist for variable windmills by using Skolem-type sequences. In particular, we use Skolem-type sequences to prove (near) graceful labellings exist for infinite families of $C_3^tC_4^s$, $C_3^tC_5^p$ and $C_3^tC_6^h$. This is the first work on graceful labellings for windmills without constant vane length. As well, we introduce the use of Skolem-type sequences to prove that (near) graceful labellings exist for non-triangular windmills. In particular, we offer a new complete solution showing that C_5^t is (near) graceful.

There are seven sections in this chapter. In the first section, we present the concept of variable windmills, and we discuss the existing results for the graceful labelling of windmills. In the second section, we introduce Skolem-type sequences, and then present many definitions and the necessary and sufficient conditions for the existence of various Skolem-type sequences. In the third section, we discuss how to use Skolem-type sequences to label variable windmills by introducing several constructions. In the fourth, fifth and sixth sections, we prove graceful labellings exist for infinite families of $3,4$-vane windmills, $3,5$-vane windmills and $3,6$-vane windmills. Finally, in the last section, we state some open questions related to Skolem-type sequences and graceful labellings of variable windmills.

In the fourth chapter, we present a new condition showing graceful labellings of every kC_{4n} exist when satisfied. Moreover, we extend this result to snakes of varying cycle sizes. Furthermore, we expand the result of [4] on the (near) graceful labelling of cyclic snakes kC_n where $n = 6, 8, 12, 16, 20, 24$ and $k > 1$. We complete all open cases, and completely solve the problem for these snakes. Also, we prove (near) graceful labellings exist for a kC_n with $n = 10, 14$ for particular strings.

In chapter four, we have four sections. In the first section, we discuss definitions and existing results related to the graceful labelling of kC_n. In the second and third sections, we introduce new results on the graceful labelling of kC_n, and we extend published results on gracefully labelled kC_n. In section four, we discuss the results and introduce open problems linked to the graceful labelling of kC_n.

In chapter five we present a brief summary and some ideas for future work.

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Chapter 2

Applying Skolem Sequences to Gracefully Label New Families of Triangular Windmills

2.1 Introduction

In 1963, Ringel [10] posed the following problem, given an arbitrary tree T with m edges: Can K_{2m+1} be decomposed into $2m+1$ copies of T?

In 1967, Rosa introduced a new type of graph labelling called a β-*valuation*, and used the concept of β-valuations to aid attempts to prove Ringel's conjecture. Later, in 1972, Golomb renamed a β-valuation as a *graceful labelling*, as it is still known today.

In [1], Bermond studied the graceful labelling of Dutch windmills. Dutch windmills are a subtype of triangular cacti, connected graphs whose blocks are all triangles.

In [12], Rosa stated the following conjecture:

1. all triangular cacti with $n \equiv 0,1 \pmod 4$ are graceful,

2. all triangular cacti with $n \equiv 2,3 \pmod 4$ are nearly graceful.

In 1989, Moulton [8] proved Rosa's conjecture for a triangular snake, a type of triangular cactus whose block cutpoint graph is a path. Rosa, in [12], recommended using Skolem-type sequences to label various families of triangular cacti. In 2012, Dyer, Payne, Shalaby, and Wicks [3] verified Rosa's conjecture for a new class of triangular cacti: Dutch windmills with at most two pendant triangles by using Skolem-type sequences as Rosa suggested. Gallian, in his survey (A Dynamic Survey of Graph Labeling) [4], mentioned that a proof for all triangular cacti seems hopelessly difficult.

In this chapter, we develop a system in which we add pendant triangles (in all possible configurations) to a family of gracefully labelled graphs to get new gracefully labelled families. We verify Rosa's conjecture for Dutch windmills of any order with three pendant triangles, which will offer an extension of the results in [3]. We can conclude this work with the following theorem.

Theorem 2.1.1. *Every Dutch windmill with at most three pendant triangles is graceful or near graceful.*

2.2 Definitions and Preliminaries

Essential definitions are introduced in this section, as well as previously found results required to prove further results.

2.2.1 Skolem and Langford Sequences

The following definitions come from the Handbook of Combinatorial Designs [2].

A *Skolem sequence* of order n is a sequence $S = (s_1, s_2, \ldots, s_{2n})$ of $2n$ integers satisfying these conditions:

1. for every $k \in \{1, 2, \ldots, n\}$, there exist exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = k$;
2. if $s_i = s_j = k$, with $i < j$, then $j - i = k$.

For example, $S_4 = (1, 1, 4, 2, 3, 2, 4, 3)$ or equivalently $\{(1,2), (4,6), (5,8), (3,7)\}$ is a Skolem sequence of order 4. If we have a Skolem sequence $\{(a_i, b_i)\}_{i=1}^{n}$, then i is called a *pivot* of a Skolem sequence if $b_i + i \leq 2n$.

Theorem 2.2.1. *[14] A Skolem sequence of order n exists if and only if $n \equiv 0, 1 \pmod 4$.*

A *hooked Skolem sequence* of order n is a sequence $hS = (s_1, s_2, \ldots, s_{2n+1})$ of $2n+1$ integers satisfying these conditions:

1. for every $k \in \{1, 2, \ldots, n\}$, there exist exactly two elements $s_i, s_j \in hS$ such that $s_i = s_j = k$;
2. if $s_i = s_j = k$, with $i < j$, then $j - i = k$;
3. $s_{2n} = 0$.

For example, $hS_2 = (1, 1, 2, 0, 2)$ is a hooked Skolem sequence of order 2. If we have a hooked Skolem sequence $\{(a_i, b_i)\}_{i=1}^{n}$, then i is called a *pivot* of a hooked Skolem sequence if $2n \neq b_i + i \leq 2n + 1$.

Theorem 2.2.2. *[9] A hooked Skolem sequence of order n exists if and only if $n \equiv 2, 3 \pmod 4$.*

In 1897, Heffter stated two difference problems [5]. Heffter's first difference problem is: can a set $\{1, \ldots, 3n\}$ be partitioned into n ordered triples (a_i, b_i, c_i), with $1 \leq i \leq n$, such that $a_i + b_i = c_i$ or $a_i + b_i + c_i \equiv 0 \pmod{6n+1}$? If such a partition is possible, then $\{\{0, a_i+n, b_i+n\}|1 \le i \le n\}$ will be the base blocks of a cyclic Steiner triple system of order $6n+1$, $CSTS(6n+1)$. Construction 2.2.3 gives a solution to Heffter's first difference problem.

Construction 2.2.3. *[14] Consider the (hooked) Skolem sequence with pairs (a_i, b_i). The set of all triples (i, a_i+n, b_i+n), for $1 \le i \le n$, is a solution to the Heffter first difference problem. These triples yield the base blocks for a $CSTS(6n+1)$: $\{0, a_i+n, b_i+n\}$, $1 \le i \le n$. Also, $\{0, i, b_i+n\}$, $1 \le i \le n$ is another set of base blocks of a $CSTS(6n+1)$.*

Let $S_4 = (1,1,4,2,3,2,4,3)$ be a Skolem sequence of order 4, yielding the pairs $(1,2)$, $(4,6)$, $(5,8)$, $(3,7)$. These pairs yield in turn the triples $(1,5,6)$, $(2,8,10)$, $(3,9,12)$, $(4,7,11)$, forming a solution to the first Heffter problem. These triples yield the base blocks for two $CSTS(25)s$:

1. $\{0,5,6\}, \{0,8,10\}, \{0,9,12\}$, and $\{0,7,11\}$ (mod 25);

2. $\{0,1,6\}, \{0,2,10\}, \{0,3,12\}$, and $\{0,4,11\}$ (mod 25).

A *Langford sequence* of defect d and order l is a sequence $L = (l_1, l_2, \cdots, l_{2l})$ which satisfies these conditions:

1. for every $k \in \{d, d+1, \ldots, d+l-1\}$, there exist exactly two elements $l_i, l_j \in S$ such that $l_i = l_j = k$;

2. if $l_i = l_j = k$, with $i < j$, then $j - i = k$.

A *hooked Langford sequence* of defect d and order l is a sequence $L = (l_1, l_2, \cdots, l_{2l+1}$ which satisfies these conditions:

1. for every $k \in \{d, d+1, \ldots, d+l-1\}$, there exist exactly two elements $l_i, l_j \in S$ such that $l_i = l_j = k$;

2. if $l_i = l_j = k$, with $i < j$, then $j - i = k$;

3. $l_{2m} = 0$.

For example, $(4, 2, 3, 2, 4, 3)$ is a Langford sequence with $d = 2$ and $l = 3$ and $(8, 4, 7, 3, 6, 4, 3, 5, 8, 7, 6, 0, 5)$ is a hooked Langford sequence with $d = 3$ and $l = 6$.

The necessary and sufficient conditions for the existence of (hooked) Langford sequences are given in Theorem 2.2.4.

Theorem 2.2.4. *[13]*

1. *A Langford sequence of order m and defect d exists if and only if*

   (a) $m \geq 2d - 1$

   (b) $m \equiv 0, 1 \pmod 4$ *and d is odd, or*

   (c) $m \equiv 0, 3 \pmod 4$ *and d is even.*

2. *A hooked Langford sequence of order m and defect d exists if and only if*

   (a) $m(m - 2d + 1) + 2 \geq 0$

   (b) $m \equiv 2, 3 \pmod 4$ *and d is odd, or*

   (c) $m \equiv 1, 2 \pmod 4$ *and d is even.*

Though they can be thought of as a natural generalization of Skolem sequences, Langford sequences have also been classically used to build new Skolem sequences by concatenating sequences of appropriate order, possibly interlacing hooks, if required. This classic technique is a much used method to construct Skolem sequences from Langford sequences.

Lemma 2.2.5. *If a (hooked) Skolem sequence of order $d - 1$ exists, and a (hooked) Langford sequence of order l and defect d, then a (hooked) Skolem sequence of order*

$N = l + d - 1$ exists. In particular, a new Skolem sequence of order N is obtained by concatenating a Skolem sequence with a Langford sequence or by interlacing a hooked Skolem sequence and hooked Langford sequence. A new hooked Skolem sequence of order N is obtained by concatenating a Skolem sequence with a hooked Langford sequence or a hooked Skolem sequence and a Langford sequence.

Let $hS_2 = (1, 1, 2, 0, 2)$ be a hooked Skolem sequence and $hL_3^6 = (8, 4, 7, 3, 6, 4, 3, 5, 8, 7, 6, 0, 5)$ be a hooked Langford sequence. Now, if we take the reverse of the hooked Skolem sequence and by interlacing hL_3^6 and the reverse of hS_2, then we obtain a new Skolem sequence $S_8 = (8, 4, 7, 3, 6, 4, 3, 5, 8, 7, 6, 2, 5, 2, 1, 1)$ of order 8.

2.2.2 Graceful Labellings and Triangular Cacti

Let $G = (V, E)$ be a graph with m edges. Let f be a labelling defined from $V(G)$ to $\{0, 1, 2, \ldots, m\}$ and let g be the induced edge labelling defined from $E(G)$ to $\{1, 2, \ldots, m\}$ by $g(uv) = |f(u) - f(v)|$, for all $uv \in E$. The labelling f is said to be *graceful*, if f is an injective mapping and g is a bijection. If a graph G has a graceful labelling, then it is graceful.

Let $G = (V, E)$ be a graph with m edges. Let f be a labelling defined from $V(G)$ to $\{0, 1, 2, \ldots, m + 1\}$ and let g be the induced edge labelling defined from $E(G)$ to $\{1, 2, \ldots, m - 1, m\}$ or $\{1, 2, \ldots, m - 1, m + 1\}$ by $g(uv) = |f(u) - f(v)|$, for all $uv \in E$. The labelling f is said to be *near graceful*, if f is an injective mapping and g is a bijection. If a graph G has a near graceful labelling, then it is near graceful. In this chapter, we will gracefully label some new families of triangular cacti and introduce some definitions and results related to triangular cacti.

A *triangular cactus* is a connected graph whose blocks are all triangles (K_3). A triangular cactus that has the property of all its blocks having a common vertex is said to be a *Dutch windmill*, and the blocks will be called *vanes*. A *pendant triangle*

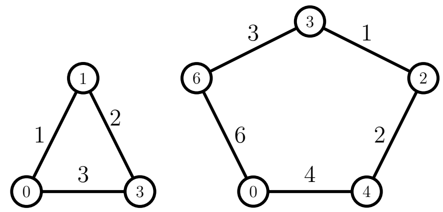


Figure 2.1: Graceful labelling of K_3 and near graceful labelling of C_5.

is defined as a block that is added to any triangular cacti. Sometimes in this chapter we call a Dutch windmill has k vanes by k-vane Dutch windmill.

The necessary conditions for gracefulness and near gracefulness of general triangular cacti are as follows:

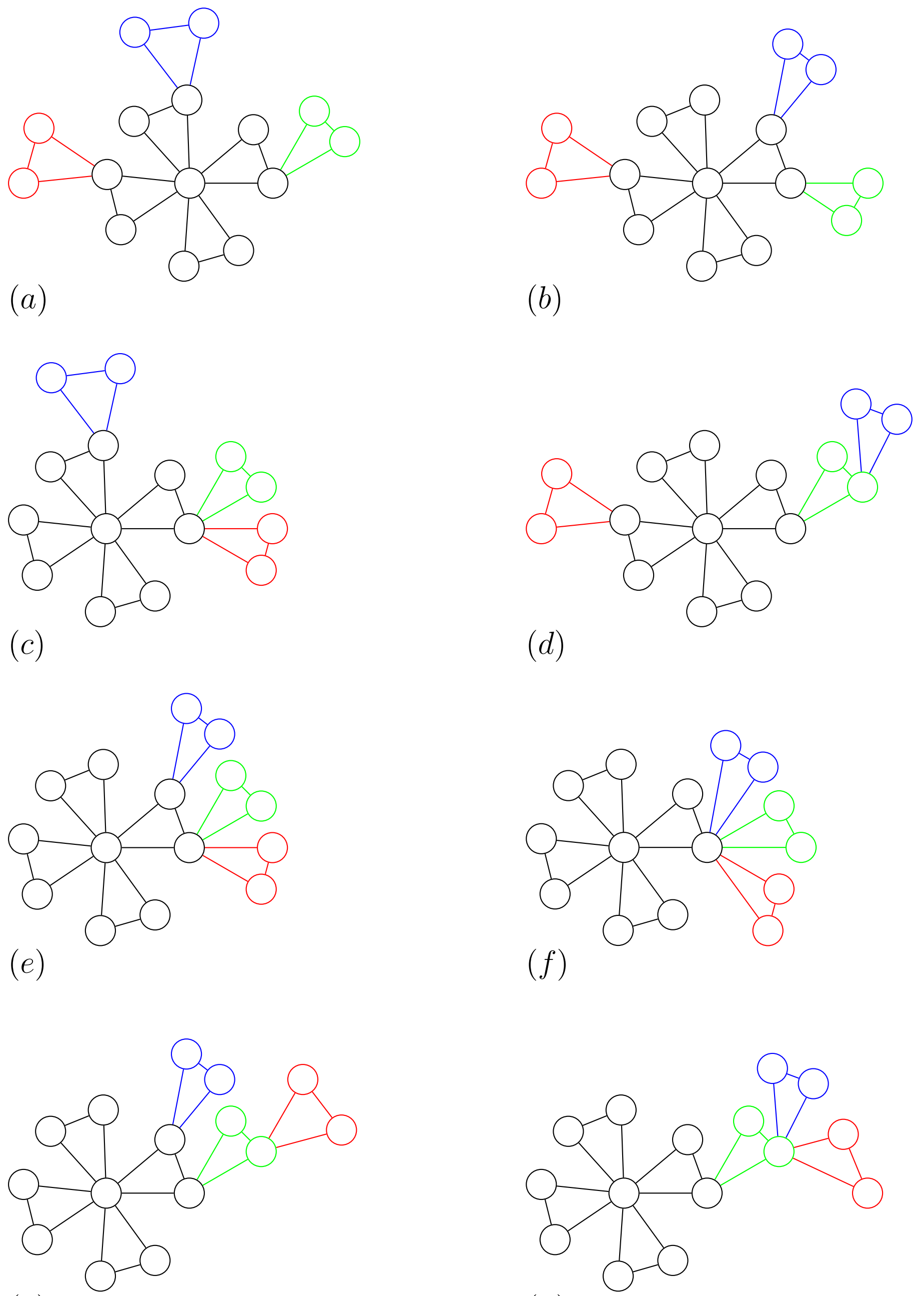
Theorem 2.2.6. *[3] Let G be a triangular cactus with n blocks. Then,*

1. *if G is graceful, then $n \equiv 0, 1 \pmod 4$, and,*

2. *if G is near graceful, then $n \equiv 2, 3 \pmod 4$.*

2.3 Dutch Windmills with Three Pendant Triangles

In order to verify Rosa's conjecture for a new family of triangular cacti, namely Dutch windmills of any order with three pendant triangles, we will use Langford sequences to obtain Skolem and hooked Skolem sequences of considerable sizes. This technique was introduced in [3]. We categorize all such cacti into one of eleven types, called Type (a) through Type (k), and then gracefully label each type.



(a) (b) (c) (d) (e) (f) (g) (h)

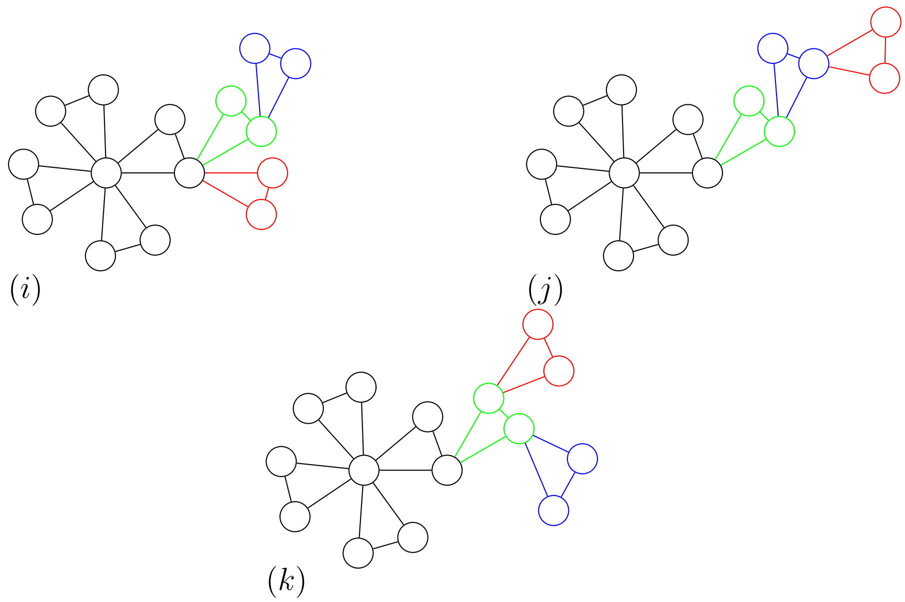


Figure 2.2: All types of Dutch windmills of order 7 with three pendant triangles.

Theorem 2.3.1. *[1] Let G be a Dutch windmill with n blocks. If there exists a Skolem sequence of order n, then G is graceful.*

Proof. Let G be a Dutch windmill with n blocks. Let S_n be a Skolem sequence of order n of the form (a_i, b_i) with $a_i < b_i$, for $i = 1, 2, \ldots, n$. These pairs give n base blocks which are $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$.

1. We can obtain three types of differences from $\{0, a_i + n, b_i + n\}$ as follows:

   (a) $A = \{(b_i + n) - (a_i + n)\} = \{1, 2, \ldots, n\}$;

   (b) $B = \{(a_i + n) - 0\}$;

   (c) $C = \{(b_i + n) - 0\}$.

   Then, $A \cup B \cup C = \{1, 2, \ldots, 3n\}$.

2. The base blocks $\{0, a_i + n, b_i + n\}_{i=1}^{n}$ give the vertex labels $\{0, 1, 2, \ldots, 3n\}$, where 0 is repeated n times, but it is a common vertex. The base blocks formed by Skolem sequences present a graceful labelling. □

Theorem 2.3.2. *[1] Let G be a Dutch windmill with n blocks. If there exists a hooked Skolem sequence of order n, then G is near graceful.*

Proof. Let G be a Dutch windmill with n blocks. Let hS_n be a hooked Skolem sequence of order n of the form (a_i, b_i) with $a_i < b_i$, for $i = 1, 2, \ldots, n$. These pairs give n base blocks which are $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$.

1. We can obtain three types of differences from $\{0, a_i + n, b_i + n\}$ as follows:
   (a) $A = \{(b_i + n) - (a_i + n)\} = \{1, 2, \ldots, n\}$;
   (b) $B = \{(a_i + n) - 0\}$;
   (c) $C = \{(b_i + n) - 0\}$.

   Then, $A \cup B \cup C = \{1, 2, \ldots, 3n - 1, 3n + 1\}$.

2. The base blocks $\{0, a_i + n, b_i + n\}_{i=1}^{n}$ give the vertex labels $\{0, 1, 2, \ldots, 3n - 1, 3n + 1\}$, where 0 is repeated n times, but it is a common vertex. The base blocks formed by a hooked Skolem sequences present a near graceful labelling. □

In the (near) graceful labelling by (hooked) Skolem sequences, we can use the base blocks of the form $\{0, a_i + n, b_i + n\}_{i=1}^{n}$ or $\{0, i, b_i + n\}_{i=1}^{n}$. They give two different vertex labels, but the same edge labels. For instance, taking the base blocks from the example in Section 2.2.1 will give us the gracefully labelled Dutch windmill of order 4 shown in Figures 2.3(a), (b), and (c). The gracefully labelled Dutch windmills in Figure 2.3(a) are formed by base blocks of the form $\{0, a_i + n, b_i + n\}$; the base blocks of the form $\{0, i, b_i + n\}$ in Figure 2.3(b); and finally by a mixed set of base blocks that come from both forms in Figure 2.3(c). We will use this technique in Section 2 when creating graceful labellings for Type (k) Dutch windmills.

Since $x - y = (x + c) - (y + c)$ for any x, y, the following lemma is straightforward.

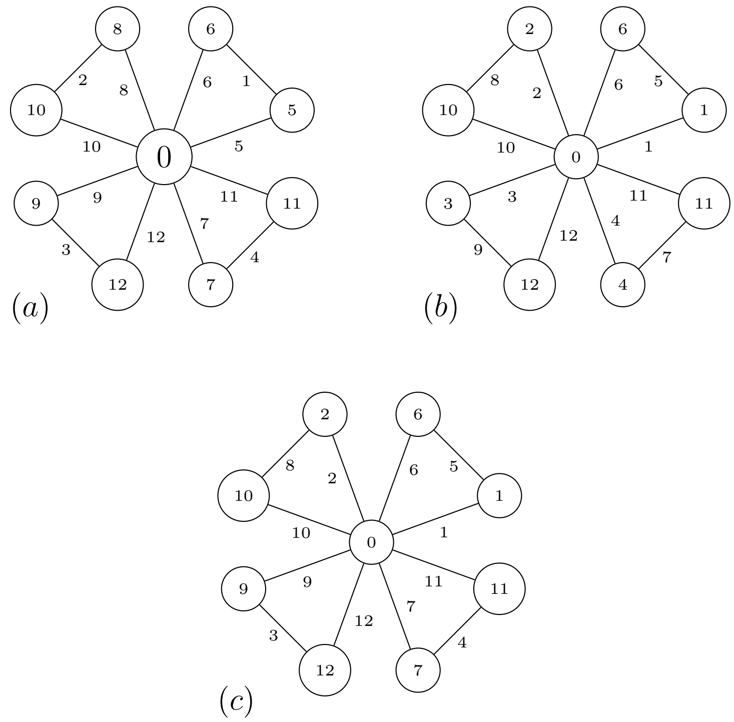


Figure 2.3: Three different gracefully labelled Dutch windmills of order 4.

Lemma 2.3.3. *If we add a constant c to each element of any triple $\{x, y, z\}$, which results in $\{x+c, y+c, z+c\}$, then the differences between the elements of $\{x+c, y+c, z+c\}$ will be the same.*

In the special case of having a triple of the form $(0, a_i+n, b_i+n)$, where a_i is a pivot, and replacing that triple with the new triple (i, a_i+i+n, b_i+i+n), we say that we are *pivoting*.

Consider the triples as given in Figure 2.3 (a), then add 2 to each element of the triple $\{0, 8, 10\}$ to obtain $\{2, 10, 12\}$. This gives the blocks $\{0, 5, 6\}$, $\{2, 10, 12\}$, $\{0, 9, 12\}$, and $\{0, 7, 11\}$. Consequently, we have obtained a graceful labelling for a Dutch windmill of order three with one pendant triangle.

Here we will present an example of how to label Figure 2.2(b) by using a hooked Skolem sequence. Let $hS_7 = (7, 4, 6, 3, 5, 4, 3, 7, 6, 5, 1, 1, 2, 0, 2)$ be a hooked Skolem sequence of order 7. This yields the pairs $\{(11, 12), (13, 15), (4, 7), (2, 6), (5, 10), (3, 9), (1, 8)\}$. This sequence gives the base blocks of the form $\{0, a_i+n, b_i+n\}$ as follows:

$A = \{(0,18,19),(0,20,22),(0,11,14),(0,9,13),(0,12,17),(0,10,16),(0,8,15)\}$. The above sequence hS_7 has six pivots, but we consider only three in particular, which are $1,4$, and 7. We obtain a new set of base blocks by pivoting as follows: $A' = \{(1,19,20),\ (0,20,22),\ (0,11,14),4,13,17),0,12,17),0,10,16),7,15,22)\}$.

We will use the set A' to label our graph as follows: label the central vertex with 0. Then label all the vanes by these base blocks containing 0. Finally, label the pendant triangles with the triples corresponding to the pivots. The blocks $\{1,19,20\}$ and $\{7,15,22\}$ each intersect $\{0,20,22\}$ at a single distinct element, namely 20 for the first block and 22 for the second block. The block $\{4,13,17\}$ intersects $\{0,12,17\}$ at a single element, namely 17.

To label Dutch windmills with n blocks that have three pendant triangles, we will use Skolem sequences of order n with at least three pivots. Since our Skolem sequence S has three distinct pivots $r,s,t,\ 1 \leq r,s,t \leq n$ we will obtain a new set of base blocks A'. Then A' contains the new triples $\{r, a_r+n+r, b_r+n+r\}$, $\{s, a_s+n+s, b_s+n+s\}$, $\{t, a_t+n+t, b_t+n+t\}$. Exactly one of r, a_r+n+r, or b_r+n+r must appear in one of the other triples and this is the label where we attach the pendant triangle; the elements of the triples in the set A' must be labelled from $\{0,1,2,\ldots,3n\}$, and these values cannot be more than $3n$, and a similar argument holds for s and t. Then, by Theorem 2.3.1, the Skolem sequence gives the vertex labels of the gracefully labelled Dutch windmill. Likewise, the same idea works for hooked Skolem sequences, and, by Theorem 2.3.2, the hooked Skolem sequence gives the vertex labels of the near gracefully labelled Dutch windmill.

In the following lemma, λ represents a letter between a and k.

Lemma 2.3.4. *If a (hooked) Skolem sequence of order n exists that gives a (near) graceful labelling of a Type (λ) Dutch windmill with n blocks then for $m \geq 3n+1$, there exists a (near) graceful labelling of a Type (λ) Dutch windmill with m blocks.*

Proof. Let G be a Type (λ) Dutch windmill with n blocks (near) gracefully labelled by S_n, a (hooked) Skolem sequence of order n. By Theorem 2.2.4, a (hooked) Langford sequence of order m exists with defect $d = n + 1$. Then by Lemma 2.2.5, we obtain a new (hooked) Skolem sequence of order m, S_m where $m \geq 3n + 1 = N(n)$. Then, since S_n had the pivot structure needed to (near) gracefully label a Type (λ) Dutch windmill with n blocks, S_m will yield the structure needed to (near) gracefully label a Type (λ) Dutch windmill with m blocks. □

2.3.1 Type (a)

For Type (a), when $n \leq 5$, there are not enough triangles to form a Type (a) Dutch windmill. When $n = 6$, see the near graceful labelling in Figure 2.4.

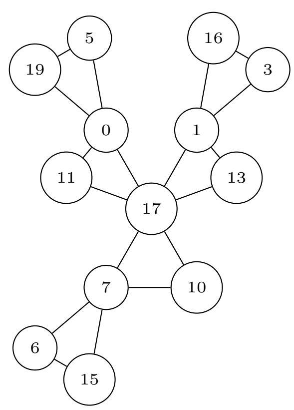


Figure 2.4: A near gracefully labelled Type (a) Dutch windmill with 6 blocks.

Lemma 2.3.5. *Any Type (a) Dutch windmill with at least 6 blocks is graceful or near graceful.*

Proof. We begin by constructing a (hooked) Skolem sequence which we will use to label a Type (a) Dutch windmill of order 7. Consider the (hooked) Skolem sequence, $hS_7 = (3, 4, 7, 3, 2, 4, 2, 5, 6, 7, 1, 1, 5, 0, 6)$. This sequence gives triples of the form

$(0, a_i + n, b_i + n)$ as follows: $(0, 18, 19)$, $(0, 12, 14)$, $(0, 8, 11)$, $(0, 9, 13)$, $(0, 15, 20)$, $(0, 16, 22)$, $(0, 10, 17)$. The above sequence hS_7 has three pivots, which are $1, 3$, and 4. We convert these triples by pivoting as follows: $(1, 19, 20)$, $(0, 12, 14)$, $(3, 11, 14)$, $(4, 13, 17)$, $(0, 15, 20)$, $(0, 16, 22)$, $(0, 10, 17)$. These triples near gracefully label the Dutch windmill with 7 blocks.

We have an order 7 hooked Skolem sequence. Then by Theorem 2.2.4, a hooked Langford sequence of order m with $d = 8$ exists for $m \geq 15$, with m congruent to 1 and 2 (mod 4). By Lemma 2.2.5, we obtain an associated Skolem sequence of order n, and by Lemma 2.3.4, we obtain a gracefully labelled Type (a) Dutch windmill with n blocks for $n \geq 22$. Likewise, By Theorem 2.2.4, a Langford sequence of order m with $d = 8$ exists for $m \geq 15$ with m congruent to 0 and 3 (mod 4). By Lemma 2.2.5, we obtain an associated hooked Skolem sequence of order n, and by Lemma 2.3.4, we obtain a near gracefully labelled Type (a) Dutch windmill with n blocks for $n \geq 22$. Now, we can see from the above steps and Theorem 2.2.6 that a Type (a) Dutch windmill of order $n \geq 22$ with three pendant triangles is graceful or near graceful.

Skolem and hooked Skolem sequences with three pivots of order $8 \leq n \leq 21$ are given in Table 2.1. By using the same construction as hS_7 with the given pivots, then we have the labellings for Type (a) Dutch windmill with n blocks for $8 \leq n \leq 21$. In this table and all subsequent tables, we will represent 10 by A, 11 by B, 12 by C, and so on.

Therefore, for $n \geq 6$, a Type (a) Dutch windmill with n vanes is graceful or near graceful, as required.

n	Skolem or hooked Skolem sequence	Pivots
8	4857411568723263	$1, 2$ and 4
9	759242574869311368	$2, 4$ and 5
10	$A853113598A7426249706$	$5, 8$ and A
11	$B68527265A8B7941134A309$	$2, 7$ and B
12	$A8531135C8A6B9742624C79B$	$3, 6$ and A
13	$B97D4262479B6CA8D5311358AC$	$3, 7$ and B
14	$CA75311357EACDB864292468EBD09$	$4, 6$ and C
15	$FDB9753EC3579BDF6A84CE64118A202$	$4, 9$ and F
16	$9FDBG864292468BDFECAG75311357ACE$	$3, 4$ and F
17	$FDB9H64282469BDF8GECAH75311357ACEG$	$4, 5$ and 9
18	$GEC9753113579ICEGHFDA8642B2468AIDFH0B$	$4, 6$ and 9
19	$IGEC9753113579JCEGIHFDA8642B2468AJDFH0B$	$7, 9$ and C
20	$JHFDB9753CK3579BDFHJICEGA86411K468A2E2IG$	$1, 6$ and 9
21	$KIGEC9753113579LCEGIKAJHFDB8642A2468LBDFHJ$	$5, A$ and K

Table 2.1: Skolem and hooked Skolem sequences with three pivots for Type (a).

□

2.3.2 Type (b)

For $n \leq 4$, there are not enough triangles to form a Type (b) Dutch windmill.

Lemma 2.3.6. *Any Type (b) Dutch windmill with at least 5 blocks is graceful or near graceful.*

Proof. For Type (b), consider $S_5 = (4, 1, 1, 5, 4, 2, 3, 2, 5, 3)$. This sequence has three pivots, which are $1, 2$, and 4. This gives us a graceful labelling of a Type (b) Dutch windmill with 5 blocks. The rest of the proof is analogous to the steps taken to prove Lemma 2.3.5, by using Lemma 2.3.4 with S_5 to obtain a (near) graceful labelling of a Type (b) Dutch windmill with n blocks for $n \geq 16$. Table 2.2 provides Skolem and hooked Skolem sequences with three pivots of order $6 \leq n \leq 15$, each of which gives a (near) graceful labelling of a Type (b) Dutch windmill with n blocks.

n	Skolem or hooked Skolem sequence	Pivots
6	5611453643202	$1, 3$ and 5
7	746354376511202	$1, 4$ and 7
8	3723258476541186	$1, 3$ and 5
9	759242574869311368	$1, 4$ and 5
10	2529115784A694738630A	$2, 4$ and 5
11	B68527265A8B7941134A309	$5, 7$ and B
12	A8531135C8A6B9742624C79B	$5, 6$ and A
13	B97D4262479B6CA8D5311358AC	$1, 6$ and 7
14	CA75311357$EACDB$864292468EBD09	$3, 4$ and C
15	FDB9753EC3579BDF6A84CE64118A202	$6, 9$ and F

Table 2.2: Skolem and hooked Skolem sequences with three pivots for Type (b).

□

2.3.3 Type (c)

For $n \leq 4$, there are not enough triangles to form a Type (c) Dutch windmills.

Lemma 2.3.7. *Any Type* (c) *Dutch windmill with at least* 5 *blocks is graceful or near graceful.*

Proof. For Type (c), consider $S_5 = (3, 5, 2, 3, 2, 4, 5, 1, 1, 4)$. This sequence has three pivots, which are $1, 2$, and 3. This gives us a graceful labelling of a Type (c) Dutch windmill with 5 blocks. Following the method of Lemma 2.3.5, we use S_5 to obtain a (near) graceful labelling of a Type (c) Dutch windmill with n blocks for $n \geq 16$. Table 2.3 provides Skolem and hooked Skolem sequences with three pivots of order

$6 \leq n \leq 15$, each of which gives a (near) graceful labelling of a Type (c) Dutch windmill with n blocks.

n	Skolem or hooked Skolem sequence	Pivots
6	6451146523203	$1, 5$ and 6
7	746354376511202	$4, 6$ and 7
8	3723258476541186	$1, 4$ and 5
9	372329687115649854	$2, 3$ and 7
10	$36232A768119574A85409$	$1, 2$ and 3
11	$35232B549A841167B98A607$	$1, 2$ and 3
12	$3A232C78119AB7685C49654B$	$1, 2$ and 3
13	$CA8531135D8AC6B9742624D79B$	$2, 4$ and 6
14	$3B932A211DE9B6CA485647D5E8C07$	$4, 6$ and A
15	$ECA8642D2468ACEFB953D735119B70F$	$4, 5$ and E

Table 2.3: Skolem and hooked Skolem sequences with three pivots for Type (c).

□

2.3.4 Type (d)

For $n \leq 4$, there are not enough triangles to form a Type (d) Dutch windmill. For $n = 5$, a graceful labelling of the 5-vane Dutch windmill appears in [12]. When $n = 6$, see the near graceful labelling in Figure 2.5.

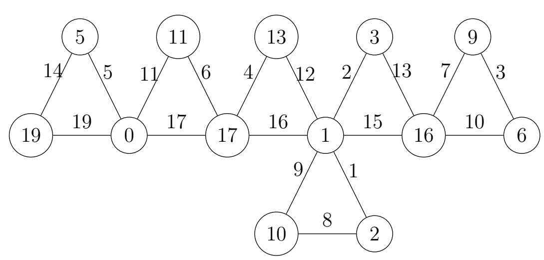


Figure 2.5: A near gracefully labelled Type (d) Dutch windmill with 6 blocks.

Lemma 2.3.8. *Any Type* (d) *Dutch windmill with at least* 5 *blocks is graceful or near graceful.*

Proof. For Type (d), consider $hS_7 = (3,4,7,3,2,4,2,5,6,7,1,1,5,0,6)$. This sequence has three pivots, which are $1,2$, and 3. This gives us a near graceful labelling of a Type (d) Dutch windmill with 7 blocks. The rest of the proof is analogous to the steps taken to prove Lemma 2.3.5, by using Lemma 2.3.4 with hS_7 to obtain a (near) graceful labelling of Type (d) Dutch windmill with n blocks for $n \geq 22$. Table 2.4 provides Skolem and hooked Skolem sequences with three pivots of order $8 \leq n \leq 21$, each of which gives a (near) graceful labelling of a Type (d) Dutch windmill with n blocks.

n	Skolem or hooked Skolem sequence	Pivots
8	4857411568723263	$1,4$ and 5
9	759242574869311368	$1,5$ and 7
10	$A853113598A7426249706$	$3,8$ and A
11	$B68527265A8B7941134A309$	$2,5$ and 7
12	$A85311135C8A6B9742624C79B$	$6,8$ and A
13	$B97D4262479B6CA8D5311358AC$	$4,6$ and 7
14	$CA75311357EACDB864292468EBD09$	$1,3$ and A
15	$FDB9753EC3579BDF6A84CE64118A202$	$5,B$ and F
16	$9FDBG864292468BDFECAG75311357ACE$	$4,B$ and D
17	$FDB9H64282469BDF8GECAH75311357ACEG$	$2,6$ and 9
18	$GEC9753113579ICEGHFDA8642B2468AIDFH0B$	$3,6$ and 9
19	$IGEC9753113579JCEGIHFDA8642B2468AJDFH0B$	$1,5$ and E
20	$JHFDB9753CK3579BDFHJICEGA86411K468A2E2IG$	$6,D$ and F
21	$KIGEC9753113579LCEGIKAJHFDB8642A2468LBDFHJ$	$6,E$ and G

Table 2.4: Skolem and hooked Skolem sequences with three pivots for Type (d).

□

2.3.5 Type (e)

For $n \leq 3$, there are not enough triangles to form Type (e) Dutch windmills. For $n = 4$, a graceful labelling of the 4-vane Dutch windmill appears in [12].

Lemma 2.3.9. *Any Type (e) Dutch windmill with at least 4 blocks is graceful or near graceful.*

Proof. For Type (e), consider $S_5 = (2, 4, 2, 3, 5, 4, 3, 1, 1, 5)$. This sequence has three pivots, which are $1, 2$, and 3. This gives us a graceful labelling of a Type (e) Dutch windmill with 5 blocks. The rest of the proof is analogous to the steps taken to prove Lemma 2.3.5, by using Lemma 2.3.4 with S_5 to obtain a (near) graceful labelling of a Type (e) Dutch windmill with n blocks for $n \geq 16$. Table 2.5 provides Skolem and hooked Skolem sequences with three pivots of order $6 \leq n \leq 15$, each of which gives a (near) graceful labelling of a Type (e) Dutch windmill with n blocks.

n	Skolem or hooked Skolem sequence	Pivots
6	5611453643202	$3, 4$ and 5
7	746354376511202	$1, 6$ and 7
8	3723258476541186	$1, 2$ and 3
9	572825967348364911	$2, 3$ and 7
10	$36232A768119574A85409$	$2, 3$ and 6
11	$B68527265A8B7941134A309$	$4, 6$ and B
12	$A85311 35C8A6B9742624C79B$	$4, 5$ and 6
13	$39B32D258A9C5B6784DA647C11$	$4, 8$ and B
14	$DB964E1146C9BDA8537E35C8A7202$	$3, 6$ and B
15	$CE3693BF262DC97EAB1187F4D5A4805$	$1, 7$ and B

Table 2.5: Skolem and hooked Skolem sequences with three pivots for Type (e).

□

2.3.6 Type (f)

For $n \le 3$, there are not enough triangles to form Type (f) Dutch windmills. For $n = 4$, a graceful labelling of the 4-vane Dutch windmill appears in [12].

Lemma 2.3.10. *Any Type (f) Dutch windmill with at least 4 blocks is graceful or near graceful.*

Proof. For Type (f), consider $S_5 = (2,4,2,3,5,4,3,1,1,5)$. This sequence has three pivots, which are $1, 3$, and 4. This gives us a graceful labelling of a Type (f) Dutch windmill with 5 blocks. The rest of the proof is analogous to the steps taken to prove Lemma 2.3.5, by using Lemma 2.3.4 with S_5 to obtain a (near) graceful labelling of a Type (f) Dutch windmill with n blocks for $n \ge 16$. Table 2.6 provides Skolem and hooked Skolem sequences with three pivots of order $6 \le n \le 15$, each of which gives a (near) graceful labelling of a Type (f) Dutch windmill with n blocks.

n	Skolem or hooked Skolem sequence	Pivots
6	6451146523203	$2, 5$ and 6
7	746354376511202	$5, 6$ and 7
8	3723258476541186	$4, 5$ and 7
9	746394376825291158	$2, 6$ and 7
10	$A869117468A4973523205$	$2, 3$ and A
11	$378392A2768B5946A54110B$	$4, 5$ and 9
12	$2529115B86AC947684B3A73C$	$4, 6$ and 9
13	$9BD3753AC957B82D2A46C84116$	$2, 7$ and 9
14	$E7D6C5B47654A8EDCB2928A311309$	$5, 6$ and 7
15	$3C9382B2E1198CDFAB5647E546AD70F$	$4, 5$ and B

Table 2.6: Skolem and hooked Skolem sequences with three pivots for Type (f).

□

2.3.7 Type (g)

For $n \leq 3$, there are not enough triangles to form Type (g) Dutch windmills. For $n = 4$, a graceful labelling of the 4-vane Dutch windmill appears in [12].

Lemma 2.3.11. *Any Type (g) Dutch windmill with at least 4 blocks is graceful or near graceful.*

Proof. For Type (g), consider $S_5 = (3, 4, 5, 3, 2, 4, 2, 5, 1, 1)$. This sequence has three pivots, which are $2, 3$, and 4. This gives us a graceful labelling of a Type (g) Dutch windmill with 5 blocks. The rest of the proof is analogous to the steps taken to prove Lemma 2.3.5, by using Lemma 2.3.4 with S_5 to obtain a (near) graceful labelling of a Type (g) Dutch windmill with n blocks for $n \geq 16$. Table 2.7 provides Skolem and hooked Skolem sequences with three pivots of order $6 \leq n \leq 15$, each of which gives a (near) graceful labelling of a Type (g) Dutch windmill with n blocks.

n	Skolem or hooked Skolem sequence	Pivots
6	6451146523203	$1, 2$ and 4
7	746354376511202	$1, 4$ and 5
8	4857411568723263	$1, 2$ and 5
9	759242574869311368	$1, 4$ and 7
10	$2529115784A694738630A$	$1, 4$ and 5
11	$B68527265A8B7941134A309$	$2, 5$ and B
12	$A85311 35C8A6B9742624C79B$	$5, 6$ and 8
13	$B97D4262479B6CA8D5311358AC$	$1, 4$ and 6
14	$CA75311357EACDB864292468EBD09$	$1, 3$ and C
15	$FDB9753EC3579BDF6A84CE64118A202$	$5, 6$ and F

Table 2.7: Skolem and hooked Skolem sequences with three pivots for Type (g).

□

2.3.8 Type (h)

For $n \leq 3$, there are not enough triangles to form Type (h) Dutch windmills. For $n = 4, 5$ a graceful labelling of the 4-vane and 5-vane Dutch windmill appears in [12].

Lemma 2.3.12. *Any Type (h) Dutch windmill with at least 4 blocks is graceful or near graceful.*

Proof. For Type (h), consider $hS_6 = (4, 5, 3, 6, 4, 3, 5, 1, 1, 6, 2, 0, 2)$. This sequence has three pivots, which are $1, 3$, and 4. This gives us a near graceful labelling of a Type (h) Dutch windmill with 6 blocks. The rest of the proof is analogous to the steps taken to prove Lemma 2.3.5, again by using Lemma 2.3.4 with hS_6 to obtain a (near) graceful labelling of a Type (h) Dutch windmill with n blocks for $n \geq 19$. Table 2.8 provides Skolem and hooked Skolem sequences with three pivots of order $7 \leq n \leq 18$, each of which gives a (near) graceful labelling of a Type (h) Dutch windmill with n blocks.

n	Skolem or hooked Skolem sequence	Pivots
7	746354376511202	$3, 4$ and 5
8	1157468543763282	$3, 4$ and 5
9	736931176845924258	$1, 3$ and 7
10	$5262854A674981137A309$	$1, 4$ and 6
11	$635A37659B117A842924B08$	$5, 6$ and 7
12	$52426549B86CA711983B73AC$	$4, 5$ and 6
13	$86272C568B75D9A11C34B394AD$	$1, 5$ and 8
14	$2726AD1176C4EBA458D935C3B8E09$	$4, 6$ and 7
15	$8C3473D48117ACEFB96D52A2659BE0F$	$2, 5$ and C
16	$637A3E6FC7D52A2G58BEC9FD48114B9G$	$2, 5$ and 7
17	$962D2G5649754EFHD78CAGB1138E3FACHB$	$4, 5$ and D
18	$5D9485E4F1198BDG6HIAEC6FB7232A3G7CH0I$	$8, 9$ and E

Table 2.8: Skolem and hooked Skolem sequences with three pivots for Type (h).

$\square$

2.3.9 Type (i)

For $n \leq 3$, there are not enough triangles to form Type (i) Dutch windmills. For $n = 4, 5$ a graceful labelling of the 4-vane and 5-vane Dutch windmill appears in [12]. When $n = 6$, see the near graceful labelling in Figure 2.6.

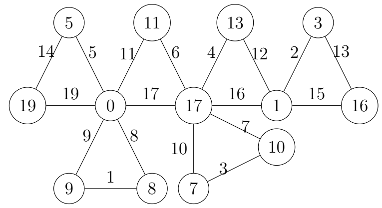


Figure 2.6: A near gracefully labelled Type (i) Dutch windmill with 6 blocks.

Lemma 2.3.13. *Any Type (i) Dutch windmill with at least 4 blocks is graceful or near graceful.*

Proof. For Type (i), consider $hS_7 = (7, 4, 6, 3, 5, 4, 3, 7, 6, 5, 1, 1, 2, 0, 2)$. This sequence has three pivots, which are $4, 5$, and 6. This gives us a near graceful labelling of a Type (i) Dutch windmill with 7 blocks. The rest of the proof is analogous to the steps taken to prove Lemma 2.3.5, by using Lemma 2.3.4 with hS_7 to obtain a (near) graceful labelling of a Type (i) Dutch windmill with n blocks for $n \geq 22$. Table 2.9 provides Skolem and hooked Skolem sequences with three pivots of order $8 \leq n \leq 21$, each of which gives a (near) graceful labelling of a Type (i) Dutch windmill with n blocks.

n	Skolem or hooked Skolem sequence	Pivots
8	7536835726248114	$1, 6$ and 7
9	975386357946824211	$2, 6$ and 7
10	$A853113598A7426249706$	$2, 3$ and 8
11	$B68527265A8B7941134A309$	$4, 8$ and B
12	$A8531135C8A6B9742624C79B$	$4, 6$ and 8
13	$CA8531135D8AC6B9742624D79B$	$4, 6$ and A
14	$DB964E1146C9BDA8537E35C8A7202$	$3, 9$ and B
15	$ECA8642D2468ACEFB953D735119B70F$	$3, A$ and C
16	$9FDBG864292468BDFECAG75311357ACE$	$2, 6$ and 9
17	$FDB9H64282469BDF8GECAH75311357ACEG$	$3, D$ and F
18	$IGECA8642H2468ACEGIFD9753BH357911DF0B$	$6, 7$ and I
19	$IGEC9753113579JCEGIHFDA8642B2468AJDFH0B$	$2, 5$ and E
20	$JHFDB9753CK3579BDFHJICEGA86411K468A2E2IG$	$6, F$ and J
21	$KIGEC9753113579LCEGIKAJHFDB8642A2468LBDFHJ$	$6, G$ and K

Table 2.9: Skolem and hooked Skolem sequences with three pivots for Type (i)

□

2.3.10 Type (j)

For $n \leq 3$, there are not enough triangles to form Type (j) Dutch windmills. For $n = 4$ a graceful labelling of the 4-vane Dutch windmill appears in [12].

Lemma 2.3.14. *Any Type (j) Dutch windmill with at least 4 blocks is graceful or near graceful.*

Proof. For Type (j), consider $S_5 = (2, 3, 2, 5, 3, 4, 1, 1, 5, 4)$. This sequence has three pivots, which are $1, 2$, and 3. This gives us a graceful labelling of a Type (j) Dutch windmill with 5 blocks. The rest of the proof is analogous to the steps taken to prove Lemma 2.3.5, by using Lemma 2.3.4 with S_5 to obtain a (near) graceful labelling of a

Type (j) Dutch windmill with n blocks for $n \geq 16$. Table 2.10 provides Skolem and hooked Skolem sequences with three pivots of order $6 \leq n \leq 15$, each of which gives a (near) graceful labelling of a Type (j) Dutch windmill with n blocks.

n	Skolem or hooked Skolem sequence	Pivots
6	2326351146504	$1, 2$ and 3
7	232437546115706	$2, 3$ and 4
8	3753811576428246	$1, 3$ and 5
9	759242574869311368	$1, 2$ and 7
10	$A853113598A7426249706$	$1, 3$ and 8
11	$35232B549A841167B98A607$	$2, 4$ and 5
12	$A8531135C8A6B9742624C79B$	$3, 6$ and 8
13	$A8D536C358A6B97D42C2479B11$	$3, 4$ and A
14	$7A8C53E7358ADB9C6411E469BD202$	$3, 4$ and A
15	$DB964F1146E9BD7CA853F735E8AC202$	$4, 5$ and B

Table 2.10: Skolem and hooked Skolem sequences with three pivots for Type (j).

□

2.3.11 Type (k)

Figure 2.7 represents a portion of a type (k) Dutch windmill. For this type we can label all the triangles by triples of the form $(0, a_i + n, b_i + n)$, where $1 \leq i \leq n$, using the method of the previous types with the exception of the three pendent triangles $2, 3$ and 4. Let triangle 1 be labelled by $(0, a_j + n, b_j + n)$; we proceed under the assumption that the remaining triangles can be labelled by the pivoting technique. By pivoting we label triangle 2 by $(k, a_k + n + k, b_k + n + k)$, where $b_j + n = a_k + n + k$ or $a_j + n = a_k + n + k$. We will consider $b_j + n = a_k + n + k$ at vertex c in Figure 2.7. Again by pivoting we label triangle 3 by $(l, a_l + n + l, b_l + n + l)$. We will consider

$b_k + n + k = a_l + n + l$, at vertex a, and hence vertex b is labelled k. Note that $1 \leq j, k, l \leq n$, and all are distinct. If we pivot triangle 4 we label it by the triple $(s, a_s + n + s, b_s + n + s)$. However $k \neq s$ and $\min(a_s + n + s, b_s + n + s) > n \geq k$. Since this is impossible we must abandon the pivoting method. Instead we will add some constant c to obtain the triple $(c, a_s + c + n, b_s + c + n)$, and we will use triples of the form: $(0, i, b_i + n)$ to avoid any conflicts created between the vertex labels. By Lemma 2.3.3, adding any constant gives the same differences. (This is a generalization of the idea of pivoting.) The same approach works for hooked Skolem sequences.

For this type, we will introduce the sequences and the corresponding triples to indicate forms of the triples we use.

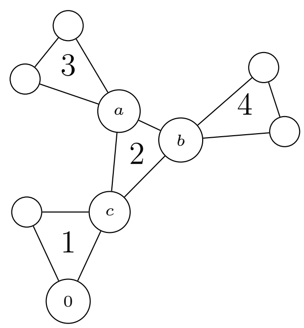


Figure 2.7: Illustration for Type (k) Dutch windmill labelling.

For $n \leq 3$, there are not enough triangles to form Type (k) Dutch windmills. For $n = 4$, a graceful labelling of the Dutch windmill with 4 vanes appears in [12].

Lemma 2.3.15. *Any Type* (k) *Dutch windmill with at least* 5 *blocks is graceful or near graceful.*

Proof. For Type (k), consider $S_5 = (2, 3, 2, 5, 3, 4, 1, 1, 5, 4)$. This sequence gives triples of the form $(0, a_i + n, b_i + n)$ as follows: $(0, 12, 13)$, $(0, 6, 8)$, $(0, 7, 10)$, $(0, 11, 15)$, $(0, 9, 14)$. The above sequence S_5 has three pivots, but it does not work for this case and so we will use mixed forms of the triples to gracefully label a Type (k) Dutch windmill with 5 blocks and the triples are: $(1, 13, 14)$, $(1, 7, 9)$, $(3, 10, 13)$, $(0, 11, 15)$, $(0, 5, 14)$. The first and the third triples are formed by pivoting the elements, the second triple is formed by adding 1 to each element, the fourth triple is formed by the base block of the form $(0, a_i + n, b_i + n)$ and the fifth triple is formed by the base block of the form $(0, i, b_i + n)$. This gives us a graceful labelling of a Type (k) Dutch windmill with 5 blocks.

We have an order 5 Skolem sequence S_5. By Theorem 2.2.4, a hooked Langford sequence of order m with $d = 6$ exists for $m \geq 11$. By Lemma 2.2.5, we obtain an associated Skolem sequence of order n, and by Lemma 2.3.4, we obtain a gracefully labelled Type (k) Dutch windmill with n blocks for $n \geq 16$. Likewise, By Theorem 2.2.4, a Langford sequence of order m with $d = 6$ exists for $m \geq 11$. By Lemma 2.2.5, we obtain an associated hooked Skolem sequence of order n, and by Lemma 2.3.4, we obtain a near graceful labelled Type (k) Dutch windmill with n blocks for $n \geq 16$. Now, we can see from the above steps and Theorem 2.2.6 that a Type (k) Dutch windmill of order $n \geq 16$ with three pendant triangles is graceful or near graceful.

Skolem and hooked Skolem sequences with corresponding triples of order $6 \leq n \leq 15$ are given in Table 2.11, where the triples in bold are formed in an unusual way (i.e formed by one of the following forms: $(i, a_i + i + n, b_i + i + n)$, $(c,\ a_i + c + n,$ $b_i + c + n$, for some c, or $(0, i, b_i + n))$ and the rest of the triples are formed by the typical $(0, a_i + n, b_i + n)$ construction. By using a similar structure as S_5 with the given triples, then we have the labellings for Type (k) Dutch windmills with n blocks for $6 \leq n \leq 15$.

Therefore, for $n \geq 5$, a Type (k) Dutch windmill with n vanes is graceful or near graceful, as required.

n	Skolem or hooked Skolem sequence	Triples
6	2326351146504	$(\mathbf{1{,}14{,}15}),(\mathbf{1{,}8{,}10}),(\mathbf{3{,}11{,}14}),(0,15,19),(0,12,17),(\mathbf{0{,}6{,}16})$.
7	746354376511202	$(\mathbf{0{,}1{,}19}),(\mathbf{0{,}2{,}22}),(\mathbf{4{,}15{,}18}),(\mathbf{7{,}11{,}20}),(0,12,17),(0,10,16),(7,15,22)$.
8	3753811576428246	$(\mathbf{1{,}15{,}16}),(0,20,22),(\mathbf{2{,}5{,}14}),(0,19,23),(\mathbf{5{,}16{,}21}),(0,18,24),(0,10,17),(0,13,21)$.
9	736931176845924258	$(\mathbf{7{,}8{,}23}),(\mathbf{0{,}2{,}25}),(\mathbf{3{,}14{,}17}),(0,20,24),(0,21,26),(0,12,18),(\mathbf{7{,}17{,}24}),(0,19,27),(0,13,22)$.
10	$5262854A674981137A309$	$(\mathbf{1{,}25{,}26}),(0,12,14),(0,26,29),(\mathbf{1{,}5{,}22}),(0,11,16),(\mathbf{6{,}19{,}25}),(0,20,27),(0,15,23),(\mathbf{0{,}9{,}31}),(0,18,28)$.
11	$635A37659B117A842924B08$	$(0,22,23),(0,28,30),(0,13,16),(0,27,31),(\mathbf{7{,}12{,}26}),(\mathbf{6{,}18{,}24}),(\mathbf{7{,}24{,}31}),(\mathbf{0{,}8{,}34}),(0,20,29),(0,15,25),(0,21,32)$.
12	$A8531135C8A6B9742624C79B$	$(0,17,18),(0,29,31),(0,16,19),(0,28,32),(\mathbf{6{,}11{,}26}),(\mathbf{6{,}30{,}36}),(0,27,34),(\mathbf{8{,}22{,}30}),(\mathbf{0{,}9{,}35}),(0,13,23),(0,25,36),(0,21,33)$.
13	$3113692D286C7B9A5847D54CBA$	$(\mathbf{5{,}6{,}21}),(\mathbf{2{,}22{,}24}),(0,14,17),(0,32,36),(0,30,35),(\mathbf{6{,}24{,}30}),(0,26,33),(0,23,31),(0,19,28),(0,29,39),(0,27,38),(0,25,37),(\mathbf{0{,}13{,}34})$.
14	$B36A349C647BDAE9578C25211D80E$	$(0,38,39),(\mathbf{2{,}37{,}39}),(\mathbf{2{,}5{,}21}),(0,20,24),(0,31,36),(0,17,23),(0,25,32),(0,33,41),(\mathbf{0{,}9{,}30}),(0,18,28),(\mathbf{11{,}26{,}37}),(0,22,34),(0,27,40),(0,29,43)$.
15	$11FB479242DE76B9CF86A35D3E85C0A$	$(\mathbf{2{,}3{,}19}),(0,23,25),(\mathbf{3{,}40{,}43}),(0,20,24),(0,38,43),(0,29,35),(0,21,28),(0,34,42),(\mathbf{9{,}31{,}40}),(0,36,46),(\mathbf{0{,}11{,}30}),(0,32,44),(0,26,39),(0,27,41),(0,18,33)$.

Table 2.11: Skolem and hooked Skolem sequences with the triples for Type (k).

□

2.4 Some Remarks

In this chapter, we proved Rosa's conjecture for a new family of triangular cacti: Dutch windmills of any order with three pendant triangles. This result, combined with those of Dyer, et al [3], gives the following theorem.

Theorem 2.4.1. *Every Dutch windmill with at most three pendant triangles is graceful or near graceful.*

Question: Can we gracefully label a Dutch windmill consisting of m triangles and n 4-cycles, all joined at a common point?

Langford sequences have been classically used to build new Skolem sequences. In this chapter, we use this technique to gracefully label Dutch windmills with three pendant triangles. However, in our construction, the Langford sequence implicitly contains a pivot, which when pivoted can gracefully label Dutch windmills with four pendant triangles. Thus we can in many cases label the Dutch windmills with four pendant triangles, for Dutch windmills of a large order.

Theorem 2.4.2. *There exists M such that for every Dutch windmill of order $m > M$ with exactly four pendant triangles, where one pendant is attached to a vane containing no other pendants, is graceful or near graceful.*

For example, Let $S_8 = (4, 8, 5, 7, 4, 1, 1, 5, 6, 8, 7, 2, 3, 2, 6, 3)$ be a Skolem sequence and $L_9^{17} = (24, 17, 21, 22, 18, 14, 11, 19, 25, 23, 10, 20, 9, 16, 13, 15, 12, 11, 17, 14, 10, 9, 18, 21, 24, 22, 19, 13, 12, 16, 15, 20, 23, 25)$ be a Langford sequence. Now, if we take the Skolem sequence S_8 and the Langford sequence L_9^{17}, then we obtain a new Skolem sequence $S_{25} = (4, 8, 5, 7, 4, 1, 1, 5, 6, 8, 7, 2, 3, 2, 6, 3, 24, 17, 21, 22, 18, 14, 11, 19, 25, 23, 10, 20, 9, 16, 13, 15, 12, 11, 17, 14, 10, 9, 18, 21, 24, 22, 19, 13, 12, 16, 15, 20, 23, 25)$ of order 25. If we take that triples obtained from S_{25} and pivot the triples $1, 2, 4$ and 11 we can gracefully label the Dutch windmill of order 25 with four pendant triangles.

Furthermore, using the Langford sequences technique to gracefully label triangular cacti, we obtain the following theorem.

Theorem 2.4.3. *Let G be a graph on m edges that can be (near) gracefully labelled. Let x be a vertex of G that obtains the label zero under some (near) graceful labelling.*

1. *If G is near gracefully labelled we can obtain a new graceful labelled graph G^* of size $m + 3(l + 1)$ by adding an $l + 1$ triangular vanes at the vertex x.*

2. *If G is gracefully labelled we can obtain a new graceful labelled graph G^* of size $m+3l$ edges by adding an l triangular vanes at the vertex x.*

3. *If G is gracefully labelled we can obtain a new near graceful labelled graph G^* of size $m+3l$ by adding an l triangular vanes at the vertex x.*

4. *If G is near gracefully labelled we can obtain a new near graceful labelled graph G^* of size $m+3(l+1)$ by adding an $l+1$ triangular vanes at the vertex x.*

In [6] Linek and Jiang studied p-extended Langford sequences. Here we will present the definition of a p-extended Langford sequence because we will use it in the proof of Theorem 2.4.3.

A *p-extended Langford sequence* of defect d and m differences is a sequence $S = (s_1, s_2, \ldots, s_{2m+1})$ which satisfies these conditions:

1. for every $k \in \{d, d+1, \ldots, d+m-1\}$, there exist exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = k$;

2. if $s_i = s_j = k$, with $i < j$, then $j - i = k$;

3. $s_p = 0$, for some $1 \leq p \leq 2m+1$.

Now we will present the proof of Theorem 2.4.3.

Proof. We present a proof for the first case. The proof of the other three cases will be similar.

Let G be a near gracefully labelled graph whose vertex labels are a subset of $\{0, 1, 2, \ldots, m+1\}$ and whose edge labels are exactly $\{1, 2, \ldots, m-1, m+1\}$.

Form a p-extended Langford sequence (ELS) with defect $d = m+2$ and the differences $d, d+1, \ldots, d+l-1$ and the extended position at the place $p = s_{2l+1} - (m-1)$. Following the constructions in [7], form a p-extended Langford sequence with $d = m+2$. Form a new sequence ELS^* with all of the elements from ELS along with the element m at the position k and s_{2l+2}. Then from ELS^* we can construct triples of the form $D = \{0, a_i+d+l-1, b_i+d+l-1\}, 1 \le i \le d+l-1$. These base blocks give the vertex labels for $l+1$ triangles, where 0 is repeated $l+1$ times, as a common vertex. We obtain the vertex labels from the set $\{0, d+l, d+l+1, \ldots, d+3l-1\}$, where 0 is a common vertex and edge labels exactly are $\{d+l, d+l+1, \ldots, d+3l-1\} \cup \{m\}$.

The base blocks formed by the ELS^* sequence with the original labelling of the graph G give a graceful labelling for a new graph G^*, formed by attaching $l+1$ triangles at 0 on the graph G.

For the second statement, we use the same technique as in the first with L as a Langford sequence with defect $d = m+1$ and order l. For the third statement, we use the same technique as in the first with hL as a hooked Langford sequence with defect $d = m+1$ and order l. For the fourth statement, we use the same technique as in the first with W as a hooked extended Langford sequence with defect $d = m+2$ and order l and use W^*. □

In fact, by using Theorem 2.4.3, we can (near) gracefully label new graphs. For instance, let $G = C_5$ be a nearly gracefully labelled graph where the edge labels are exactly $\{1,2,3,4,6\}$; see Figure 2.1. Let $ELS_7^{13} = (11, 14, 15, 16, 17, 18, 19, 7, 8, 9, 10, 11, 12, 13, 7, 14, 8, 15, 9, 16, 10, 17, 0, 18, 12, 19, 13)$ be an extended Langford sequence and $ELS_7^{*13} = (11, 14, 15, 16, 17, 18, 19, 7, 8, 9, 10, 11, 12, 13, 7, 14, 8, 15, 9, 16, 10, 17, \mathbf{5}, 18, 12, 19, 13, \mathbf{5})$ be a modified extended Langford sequence.

This sequence, ELS_7^{*13}, gives triples of the form $(0, a_i+19, b_i+19)$ as follows: $(0,42,47)$, $(0,27,34)$, $(0,28,36)$, $(0,29,38)$, $(0,30,40)$, $(0,20,31)$, $(0,32,44)$, $(0,33,46)$, $(0,21,35)$, $(0,22,37)$, $(0,23,39)$, $(0,24,41)$, $(0,25,43)$, $(0,26,45)$. The base blocks formed by the ELS_7^{*13} sequence with the original labelling of the graph $G = C_5$ give a graceful labelling for a new graph G^*, formed by attaching 14 triangles at 0 on the graph G.

In 1989, Moulton [8] proved Rosa's conjecture for a triangular snake, a type of triangular cactus whose block cutpoint graph is a path. Finally we pose the following question.

Question: Can we use Skolem type sequences to gracefully label triangular snakes? Can we use them to gracefully label triangular snakes with pendant triangles?

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Chapter 3

Graceful Labellings of Variable Windmills Using Skolem Sequences

3.1 Introduction

In [13], Rosa introduced a new type of graph labelling known as a β-labelling, or graceful labelling, as it was renamed later. Let $G = (V, E)$ be a graph with m edges. Let $f : V(G) \to \{0, 1, 2, \ldots, m\}$ be a labelling of V of G and let $g : E(G) \to \{1, 2, \ldots, m\}$ be the induced edge labelling defined by $g(uv) = |f(u) - f(v)|$, for all $uv \in E$. The labelling f is said to be a *graceful labelling* if and only if f is an injective mapping and g is a bijection. If a graph G has a graceful labelling then we say G is graceful.

A near graceful labelling of a graph $G = (V, E)$ with m edges is defined in a similar way. Let $f : V(G) \to \{0, 1, 2, \ldots, m + 1\}$ be a labelling of V of G and let $g : E(G) \to A$ be the induced edge labelling defined by $g(uv) = |f(u) - f(v)|$, for all $uv \in E$, where A is $\{1, 2, \ldots, m - 1, m\}$ or $\{1, 2, \ldots, m - 1, m + 1\}$. The labelling f is said to be a *near graceful labelling* if and only if f is an injective mapping and g is a bijection. If a graph G has a near graceful labelling then we say G is near graceful. In this chapter, all near graceful labellings constructed will omit the vertex label m and the edge label m.

In this chapter, we adopt the convention that 0 is a natural number. So, when we write $[a, b]$ with $a, b \in \mathbb{N}$ and $a < b$, we are indicating the set $\{x \in \mathbb{N} | a \leq x \leq b\}$.

Bermond, in [4], proved that Dutch windmills (the graphs consisting of t copies of K_3 with one vertex in common) are graceful. Let C_n be a cycle of length $n \geq 3$, and C_n^t be the graph obtained from the union of t n-cycles with one vertex in common that we will call the *central vertex*. In [10], the authors stated the following conjecture.

Conjecture 3.1.1. *[10] C_n^t is graceful if and only if $nt \equiv 0, 3$ $(mod\ 4)$.*

This conjecture has been shown to hold for $n = 3, n = 4, n = 5$, and $n = 6$ and $n = 4k$ (k any positive integer) in [5], [9], [19], and [10], respectively. Also, in [17, 18, 20, 21], the authors show that the graceful labellings exist for the C_n^t with $n = 7, 9, 11, 13$, respectively. In 2012, Dyer, et al. [6], use Skolem sequences to prove that all Dutch windmills with zero, one or two pendant triangles are (near) graceful. A comprehensive survey of graceful labelling can be found in [7].

We define an m, n-windmill to be a graph $G = C_m^s C_n^t$ obtained from identifying the central vertices of C_m^s and C_n^t, where $m \neq n$. In other words, the graph $C_m^s C_n^t$ is a windmill with s m-cycle vanes and t n-cycle vanes. More generally, we call any windmill made up of two or more cycle lengths a *variable windmill*.

In this chapter we use Skolem-type sequences to show (near) graceful labellings exist for the $G = C_n^t C_m^s$ graphs where $n = 3$ and $m = 4, 5, 6$. An example of a graceful labelling of $C_3^4 C_4^3$ is given in Figure 3.1.

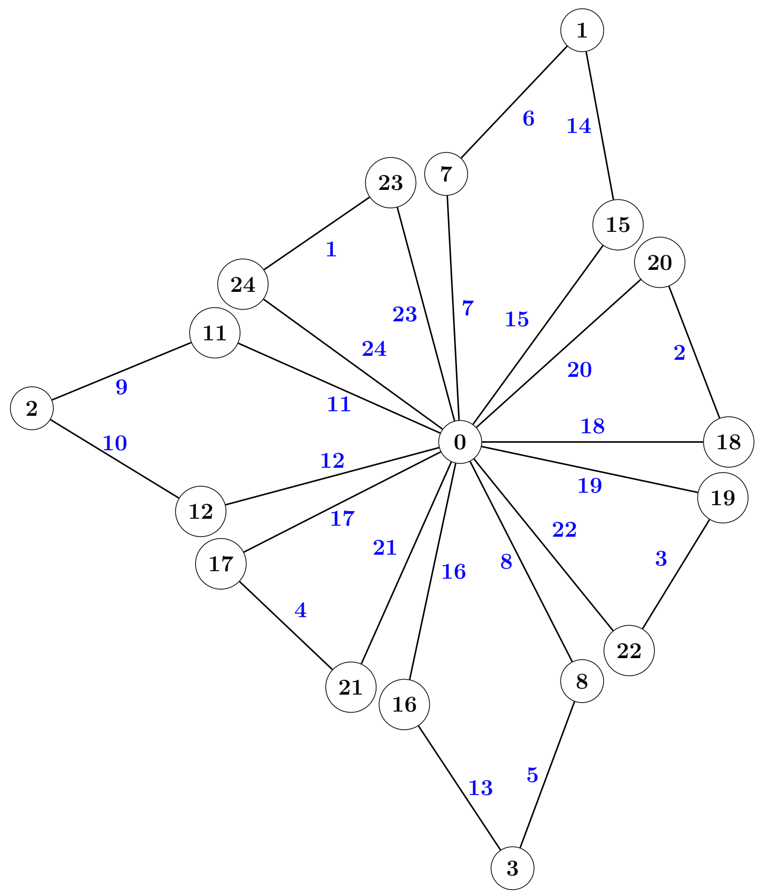

Figure 3.1: Graceful labelling of $C_3^4 C_4^3$.

3.2 Skolem-type Sequences

In this section we begin by defining a Skolem-type sequence, and then present several definitions and known results for Skolem and Langford sequences.

A *Skolem-type sequence* of order n is a sequence $K = (s_1, s_2, \ldots, s_{2n})$ of $2n$ integers satisfying the conditions:

1. for some set H of n distinct positive integers, $\forall\, k \in H$, $\exists\, s_i, s_j \in K$ such that $s_i = s_j = k$;
2. if $s_i = s_j = k$, with $i < j$, then $j - i = k$.

For example, $(6,4,1,1,3,4,6,3)$ is a Skolem-type sequence of order 4 with $H = \{1,3,4,6\}$. We can also write Skolem-type sequences by specifying the ordered pairs where identical elements of H occur, as follows $\{(a_i, b_i) : i \in H, b_i - a_i = i\}$. So, we could equivalently write the sequence $\{(3,4), (5,8), (2,6), (1,7)\}$. In the Skolem-type sequences if $b_i > a_i$ we call b_i *right endpoint*. The definition of the Skolem-type sequence introduced in [2].

A *hooked Skolem-type sequence* of order n is a sequence $hK = (s_1, s_2, \ldots, s_{2n+1})$ of $2n+1$ integers satisfying the conditions of a Skolem-type sequence with the added condition that $s_{2n} = 0$. For example, $(5,3,1,1,3,5,2,0,2)$ or equivalently $\{(3,4), (7,9), (2,5), (1,6)\}$ is a hooked Skolem-type sequence of order 4 with $H = \{1,2,3,5\}$.

A *(hooked) Langford sequence* with defect d and order l is a (hooked) Skolem-type sequence with $H = [d, d+l-1]$. Necessary and sufficient conditions for the existence of Langford sequences are given in [15].

Note that in this chapter, in the constructions, a_i and b_i or c_j and d_j or e_j and f_j represent the two positions in the Skolem-type sequence of the element i and j, with $a_i < b_i$, $c_j < d_j$ and $e_j < f_j$ with $1 \leq i, j \leq n$. When written in this form, we say that a_i, c_j, and e_j are left endpoints, and b_i, d_j, and f_j are right endpoints.

Construction 3.2.1. *From Table 3.1, we can construct a Langford sequence with defect $d \geq 1$ and order $2d-1$, (omitting row 2 when $d = 1$). We define L_d^{2d-1} to be exactly this sequence for $d \geq 1$.*

	i	a_i	b_i	
1	$d+2r$	$d-r$	$2d+r$	$0 \leq r \leq d-1$
2	$d+2r+1$	$2d-1-r$	$3d+r$	$0 \leq r \leq d-2$

Table 3.1: Langford sequence, L_d^{2d-1}.

Construction 3.2.1 is new and it is straightforward to check that Construction 3.2.1 gives a Langford sequence.

A *(hooked) Skolem sequence* of order n is a (hooked) Skolem-type sequence of order n with $H = [1, n]$. Necessary and sufficient conditions for the existence of Skolem sequences are given in [16].

Let m and n be positive integers, with $m \leq n$. A *(hooked) near-Skolem sequence* of order n and defect m is a sequence $nSm = (s_1, s_2, \ldots, s_{2n-2})$ is a (hooked) Skolem-type sequence of order $n-1$ with $H = [1, m-1] \cup [m+1, n]$. For example, $(1, 1, 6, 3, 7, 5, 3, 2, 6, 2, 5, 7)$ is a 4-near-Skolem sequence of order 7 and $(2, 5, 2, 4, 6, 7, 5, 4, 1, 1, 6, 0, 7)$ is a hooked 3-near-Skolem sequence of order 7. Necessary and sufficient conditions for the existence of near-Skolem sequences are given in [14].

Notice that there is a different use of defect in Langford and near-Skolem sequences. In Langford sequences the defect (d) is the smallest integer in the sequence, but in near-Skolem sequences the defect (m) is the integer omitted from the sequence.

Construction 3.2.2. *From Table 3.2, we can construct a hooked near-Skolem sequence of order $n = 4m + 1$ with $m \geq 3$ and defect $n - 1$, (omitting row 5 when $m = 3$).*

	i	a_i	b_i	
1	$2r+1$	$2m+1-r$	$2m+2+r$	$1 \leq r \leq 2m$
2	$4m-4$	$2m+2$	$6m-2$	–
3	$4m-2$	$2m+1$	$6m-1$	–
4	$2m+2r$	$5m-r$	$7m+r$	$0 \leq r \leq m-3$
5	$2r+4$	$6m-r-3$	$6m+r+1$	$0 \leq r \leq m-4$
6	$2m-2$	$6m$	$8m-2$	–
7	1	$7m-2$	$7m-1$	–
8	2	$8m-1$	$8m+1$	–

Table 3.2: Hooked near-Skolem sequence from Construction 3.2.2.

Construction 3.2.3. *From Table 3.3, we can construct a near-Skolem sequence of order $n = 4m+3$ with $m \geq 2$ and defect $n-1$, (omitting row 4, 5 when $m = 2$).*

	i	a_i	b_i	
1	$2r+1$	$2m+2-r$	$2m+3+r$	$1 \leq r \leq 2m+1$
2	$4m$	$2m+2$	$6m+2$	$-$
3	$4m-2$	$2m+3$	$6m+1$	$-$
4	$2m+2+2r$	$5m+2-r$	$7m+4+r$	$0 \leq r \leq m-3$
5	$2r+4$	$6m-r$	$6m+4+r$	$0 \leq r \leq m-3$
6	1	$7m+2$	$7m+3$	$-$
7	$2m$	$6m+3$	$8m+3$	$-$
8	2	$8m+2$	$8m+4$	$-$

Table 3.3: Near-Skolem sequence from Construction 3.2.3.

It is straightforward to check that Constructions 3.2.2 and 3.2.3 both produce the desired sequences. These constructions establish Lemma 3.2.4, which will be useful in Section 3.3 and 3.5 when labelling C_5^p and $C_3^t C_5^p$.

Lemma 3.2.4. *If $n = 2k+1$, with $k \geq 5$, then there exists a (hooked) near-Skolem sequence of order n, with defect $n-1$, which has no right endpoints in the positions $[1, k+2]$.*

We now extend our definition of Skolem-type sequences to sequences where every pair of elements in H occurs exactly twice. A *two-fold Skolem-type sequence* of order n is a sequence $(s_1, s_2, \ldots, s_{4n})$ of $4n$ positive integers such that the following conditions hold:

1. for a set H of n distinct positive integers, for every $p \in H$, there exist exactly 2 disjoint pairs (s_i, s_j) and $(s_{i'}, s_{j'})$ where $s_i, s_j, s_{i'}, s_{j'} \in K_n^2$ such that $s_i = s_j = s_{i'} = s_{j'} = p$,
2. if $s_i = s_j = p$ and $s_{i'} = s_{j'} = p$ with $i < j$ and $i' < j'$, then $j - i = p$ and $j' - i' = p$.

For example, $(2, 3, 2, 3, 3, 2, 3, 2)$ is a two-fold Skolem-type sequence of order 2 with

$H = \{2, 3\}$.

Construction 3.2.5. *From Table 3.4, we can construct a two-fold Skolem-type sequence construction of order $n \geq 1$, where $H = \{1\} \cup \{4i | 1 \leq i \leq n-1\}$.*

	j	(c_j, d_j)	(e_j, f_j)	
1	1	$(2n-1, 2n)$	$(4n-1, 4n)$	–
2	$4r$	$(2n-2r-1, 2n+2r-1)$	$(2n-2r, 2n+2r)$	$1 \leq r \leq n-1$

Table 3.4: Two-fold Skolem-type sequence from Construction 3.2.5.

For example, $(8, 8, 4, 4, 1, 1, 4, 4, 8, 8, 1, 1)$ is a two-fold Skolem-type sequence of order 3 with $H = \{1, 4, 8\}$, created using Construction 3.2.5.

A $(2n+1, 2n+2)$-*extended two-fold Skolem-type sequence* of order n is a sequence $(\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_{4n+2})$ of $4n+2$ positive integers that satisfies the above conditions as well as $\hat{s}_{2n+1} = \hat{s}_{2n+2} = 0$.

Proposition 3.2.6. *The following sequences are two-fold Skolem-type sequences:*

$C^0 = \emptyset$, *(the sequence of length 0),*

$C^1 = (2, 2, 2, 2)$, *where* $H = \{2\}$,

$C^2 = (2, 3, 2, 3, 3, 2, 3, 2)$, *where* $H = \{2, 3\}$,

$C^3 = (2, 2, 2, 2, 5, 3, 5, 3, 3, 5, 3, 5)$, *where* $H = \{2, 3, 5\}$, *and*

$C^4 = (6, 6, 2, 2, 2, 2, 6, 6, 5, 3, 5, 3, 3, 5, 3, 5)$, *where* $H = \{2, 3, 5, 6\}$,.

By placing restrictions on the nature of H, we can define two-fold Skolem and Langford sequences, similar to the way we defined the Skolem and Langford sequences.

A *double Skolem-type sequence* of order l and defect d is a sequence obtained by concatenating two existing Skolem-type sequences with the same defect and order, or by interlacing a hooked Skolem-type sequence with its reverse. For example, $(4, 2, 3, 2, 4, 3)$ is a Langford sequence with $d = 2$ and $l = 3$ and (4, 2, 3, 2, 4, 3, 4, 2, 3, 2, 4, 3) is a double Langford sequence with $d = 2$ and $l = 3$. Also, (8, 4, 7, 3, 6, 4, 3, 5, 8, 7, 6, 0, 5) is a hooked Langford sequence with $d = 3$ and $l = 6$, and (8, 4, 7, 3, 6, 4, 3, 5, 8, 7, 6, 5, 5, 6, 7, 8, 5, 3, 4, 6, 3, 7, 4, 8) is a double Langford sequence with $d = 3$ and $l = 6$.

A *two-fold Skolem sequence* of order n is a two-fold Skolem-type sequence $S_n^2 = (s_1, s_2, \ldots, s_{4n})$ with $H = [1, n]$. The sequence $(3, 1, 1, 3, 2, 2, 2, 2, 3, 1, 1, 3)$ is an example of a two-fold Skolem sequence of order 3. It is straightforward to show that two-fold Skolem sequences exist for any order n by concatenating Skolem sequences (possibly interlacing their hooks). However, in Tables 3.5 and 3.6 we introduce a different construction for two-fold Skolem sequences which will be useful later.

	j	(c_j, d_j)	(e_j, f_j)	
1	$2r+1$	$\left(\frac{n+1}{2} - r, \frac{n+3}{2} + r\right)$	$\left(\frac{3n+3}{2} - r, \frac{3n+5}{2} + r\right)$	$0 \le r \le \frac{n-1}{2}$
2	$n-1$	$(2n+3, 3n+2)$	$\left(\frac{5n+5}{2}, \frac{7n+3}{2}\right)$	–
3	$2r+2$	$\left(\frac{5n+3}{2} - r, \frac{5n+3}{2} + r + 2\right)$	$\left(\frac{7n+1}{2} - r, \frac{7n+1}{2} + r + 2\right)$	$0 \le r \le \frac{n-5}{2}$

Table 3.5: Two-fold Skolem sequence construction of odd order n.

	j	(c_j, d_j)	(e_j, f_j)	
1	$2r+1$	$\left(\frac{n}{2} - r, \frac{n+2}{2} + r\right)$	$\left(\frac{3n}{2} - r, \frac{3n+2}{2} + r\right)$	$0 \le r \le \frac{n-2}{2}$
2	n	$(2n+1, 3n+1)$	$\left(\frac{5n+2}{2}, \frac{7n+2}{2}\right)$	–
3	$2r+2$	$\left(\frac{5n}{2} - r, \frac{5n}{2} + r + 2\right)$	$\left(\frac{7n}{2} - r, \frac{7n}{2} + r + 2\right)$	$0 \le r \le \frac{n-4}{2}$

Table 3.6: Two-fold Skolem sequence construction of even order n.

A *two-fold Langford sequence* of order l and defect d is a two-fold Skolem-type sequence of order l, $(l_1, l_2, \ldots, l_{4l})$ with $H = [d, d + l - 1]$. The sequence $(7, 5, 7, 5, 6, 6, 5, 7, 5, 7, 6, 6)$ is an example of a two-fold Langford sequence with defect 5 and order

3. Again, it is straightforward to show that a two-fold Langford sequence can be obtained by concatenating (hooked) Langford sequences. In Construction 3.2.7, we give a two-fold Langford sequence that is not constructed by concatenation that will be useful later.

Construction 3.2.7. *From Table 3.7, we can construct a two-fold Langford sequence with defect* $6k-1$ *and order* $4k-1$*, where* $k \geq 1$*.*

	j	(c_j, d_j)	(e_j, f_j)	
1	$10k-2-2r$	$(r, 10k-2-r)$	$(2k-1+r, 12k-3-r)$	$1 \leq r \leq 2k-1$
2	$10k-1-2r$	$(4k-2+r, 14k-3-r)$	$(6k-2+r, 16k-3-r)$	$1 \leq r \leq 2k$

Table 3.7: Two-fold Langford sequence with defect $6k-1$ and order $4k-1$, where $k \geq 1$.

We summarize necessary and sufficient conditions for the existence of various useful Skolem-type sequences in Table 3.8 as an aid for the reader.

3.3 Labellings from Skolem-type sequences

In this section, we use Skolem-type sequences to label variable windmills of different orders.

3.3.1 Constructions using Skolem-type sequences

To begin, we discuss how to label C_3^t and C_5^p by using Skolem-type sequences. In [19], Yang et al. proved that the C_5^p is graceful when $p \equiv 0, 3 \pmod 4$. Bermond in [4] proved that the C_3^t is graceful when $t \equiv 0, 1 \pmod 4$.

Construction 3.3.1. *From a Skolem-type sequence or a hooked Skolem-type sequence of order* t*, construct the pairs* (a_i, b_i) *such that* $b_i - a_i = i$ *for* $1 \leq i \leq t$*. From these*

Sequence	Necessary and Sufficient conditions	Reference
Skolem sequence of order n	$n \equiv 0, 1 \pmod 4$	[16]
hooked Skolem sequence of order n	$n \equiv 2, 3 \pmod 4$	[12]
Langford sequence of order l and defect d	$l \geq 2d-1$, $l \equiv 0, 1 \pmod 4$ and d is odd, or $l \equiv 0, 3 \pmod 4$ and d is even	[15]
hooked Langford sequence of order l and defect d	$l\,(l-2d+1)+2 \geq 0$, $l \equiv 2, 3 \pmod 4$ and d is odd, or $l \equiv 1, 2 \pmod 4$ and d is even	[15]
m-near-Skolem sequence of order n	$n \equiv 0, 1 \pmod 4$ and m is odd, or $n \equiv 2, 3 \pmod 4$ and m is even	[14]
hooked m-near-Skolem sequence of order n	$n \equiv 0, 1 \pmod 4$ and m is even, or $n \equiv 2, 3 \pmod 4$ and m is odd	[14]
m-fold Skolem sequence of order n	$n \equiv 0, 1 \pmod 4$ and any m, or $n \equiv 2, 3 \pmod 4$ and m is even	[3]
hooked m-fold Skolem sequence of order n	$n \equiv 2, 3 \pmod 4$ and m is odd	[3]

Table 3.8: Summary of necessary and sufficient conditions for the existence of various Skolem-type sequences.

pairs, along with an arbitrary positive integer c, form the triples $(0, a_i + c, b_i + c)$, $1 \leq i \leq t$. The triples give a labelling of a C_3^t, with the central vertex labelled 0.

Lemma 3.3.2. *Let $c \geq t$ be an arbitrary positive integer.*

1. *Using any Skolem sequence of order t, with Construction 3.3.1 gives the edge labels $[1, t] \cup [c+1, c+2t]$, and the vertex labels from $\{0\} \cup [c+1, c+2t]$, such that each nonzero label occurs exactly once, and*
2. *using any hooked Skolem sequence of order t, with Construction 3.3.1 gives the edge labels $[1, t] \cup [c+1, c+2t-1] \cup \{c+2t+1\}$, and the vertex labels $\{0\} \cup [c+1, c+2t-1] \cup \{c+2t+1\}$, such that each nonzero label occurring exactly once.*

Proof. Start with a Skolem sequence S_t of order t, and construct the triples $(0, a_i + c, b_i + c)_{i=1}^{t}$ by using Construction 3.3.1. Since $1 \leq a_i \leq 2t-1$ and $2 \leq b_i \leq 2t$, and all a_i and b_i are distinct, the vertex labels are $\{0\} \cup [c+1, c+2t]$, where no vertex label other than 0 is repeated. Considering the edge labels, we see that they are $b_i - a_i = i$, $a_i + c$, and $b_i + c$. Since by construction, $1 \leq i \leq t$ and all the a_i and b_i are distinct, we obtain edge labels $[1, t] \cup [c+1, c+2t]$, a union of disjoint sets since $c \geq t$, all of which are distinct.

A similar argument holds for hooked Skolem sequences. □

Construction 3.3.3. *From a Skolem-type sequence or a hooked Skolem-type sequence of order t, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq t$. From these pairs, along with an arbitrary positive integer c, form the triples $(0, i, b_i + c)$, $1 \leq i \leq t$. The triples give a labelling of a C_3^t, with the central vertex labelled 0.*

In the same fashion as Lemma 3.3.2, we obtain Lemma 3.3.4.

Lemma 3.3.4.

1. *Using any Skolem sequence with Construction 3.3.3 and $c \geq t$ gives the edge labels $[1, t] \cup [c+1, c+2t]$ and the vertex labels from $\{0\} \cup [1, t] \cup [c+2, c+2t]$, each nonzero label occurring exactly once, and*

2. *using any hooked Skolem sequence with Construction 3.3.3 and $c \geq t$ gives the edge labels $[1, t] \cup [c+1, c+2t-1] \cup \{c+2t+1\}$ and the vertex labels from $\{0\} \cup [1, t] \cup [c+2, c+2t-1,] \cup \{c+2t+1\}$, each nonzero label occurring exactly once.*

To illustrate Construction 3.3.1 and 3.3.3, consider for example, (3, 1, 1, 3, 2, 0, 2), a hooked Skolem sequence of order 3 which yields the pairs $(2, 3)$, $(5, 7)$, $(1, 4)$. Letting $c = 3$, these pairs yield the triples to near gracefully label a C_3^3:

1. $(0, 5, 6)$, $(0, 8, 10)$, and $(0, 4, 7)$ by Construction 3.3.1,

2. $(0, 1, 6)$, $(0, 2, 10)$, and $(0, 3, 7)$ by Construction 3.3.3.

Taking $c = t$ and considering only Skolem and hooked Skolem sequences in Constructions 3.3.1 and 3.3.3, we can use the resulting triples to (near) gracefully label Dutch windmills (C_3^t) for any $t \geq 1$, as well as some related graphs. See [6] and [1].

We make note of one more special case of a Skolem-type sequence used with Construction 3.3.1.

Lemma 3.3.5. *Using a Langford sequence with Construction 3.3.1 and* $c \geq d + l - 1$ *gives edge labels* $[d, d + l - 1] \cup [c + 1, c + 2l]$ *and vertex labels from* $\{0\} \cup [c + 1, c + 2l]$*, each nonzero label occurring exactly once.*

Koh, Rogers, Lee, and Toh [10] conjectured that C_n^t is graceful if and only if $nt \equiv 0, 3 \pmod 4$. In 2005, Yang et al. in [19] have shown the conjecture true for $n = 5$. In Theorem 3.3.7, we prove that $G = C_5^p$ is near graceful when $p \equiv 1, 2 \pmod 4$, and verify that G is graceful when $p \equiv 0, 3 \pmod 4$ through the use of (hooked) Skolem and (hooked) near-Skolem sequences. Then, in Section 5, we use this construction to prove (near) graceful labellings exist for families of $C_3^t C_5^p$. In Construction 3.3.6 we will use (hooked) Skolem and (hooked) near-Skolem sequences together to form the 5-tuples to label a C_5^p.

Construction 3.3.6. *Let* $p > 0$*. Given a (hooked) Skolem sequence* S_1 *of order* $p = 4m + k$ *where* $0 \leq k \leq 3$*, construct the pairs* (a_i, b_i) *such that* $b_i - a_i = i$ *for* $1 \leq i \leq p$*. Select Skolem sequence* S_2 *and construct the pairs* (c_j, d_j) *as follows.*

1. *For* $k = 0$*, select the Skolem sequence of order* $2p$ *using Table 3.12, in Appendix A.*

2. *For* $k = 1$*, select the hooked Skolem sequence of order* $2p$ *using Table 3.14, in Appendix A.*

3. *For* $k = 2$*, select the hooked near-Skolem sequence of order* $2p+1$ *using Table 3.2.*

4. *For* $k = 3$*, select the near-Skolem sequence of order* $2p + 1$ *using Table 3.3.*

Then form the 5*-tuple* $(0, d_{b_i} + p, b_i, a_i, d_{a_i} + p)$ *for each* $1 \leq i \leq p$*.*

Note that by construction, for $k = 0$ and $k = 1$, no right endpoints occur in positions 1 to p of S_2. Similarly, S_2 has no right endpoints in positions $1, 2, \ldots, p-1$, nor in $p+1$ when $k = 2$ and $k = 3$.

Theorem 3.3.7. *If $G = C_5^p$, then G is near graceful when $p \equiv 1, 2 \pmod 4$ and G is graceful when $p \equiv 0, 3 \pmod 4$.*

Proof.

Case 1: Let $p \equiv 1 \pmod 4$. Form the 5-tuples $(0, d_{b_i} + p, b_i, a_i, d_{a_i} + p)$ for each $1 \le i \le p$ as indicated in Construction 3.3.6. We begin by considering the vertex labels used by the 5-tuples.

Note that from the Skolem sequence S_1 the entries a_i and b_i in the third and fourth entries of the 5-tuples give the distinct numbers $[1, 2p]$. As we know, there are no right endpoints (d_j) in the first p positions in the hooked Skolem sequence. The first right endpoint is in position $p+1$, so the set of possible positions for right endpoints are $[p+1, 4p-1] \cup \{4p+1\}$. Thus, the second and fifth entries of the 5-tuples are all elements of $[2p+1, 5p-1] \cup \{5p+1\}$. In Skolem sequences, a_i and b_i are distinct, so in the hooked Skolem sequence, d_{a_i} and d_{b_i} are distinct. Further, we know that the entries a_i and b_i on the third and fourth entries of the 5-tuples give the distinct numbers $[1, 2p]$, and $d_{a_i}, d_{b_i} \ge p+1$ so the minimum value of the second and fifth entries of the 5-tuples is $2p+1$. Therefore, all the nonzero entries of the 5-tuples are distinct.

From the above discussion, it is clear that the only vertex label repeated is 0 (p times), and that all vertices are distinct and come from the set $[0, 5p-1] \cup \{5p+1\}$.

We now consider the edge labels defined by the difference between subsequent entries (taken cyclically) in the 5-tuple $(0, d_{b_i} + p, b_i, a_i, d_{a_i} + p)$.

Since $b_i - a_i = i$, these differences are all distinct and comprise the set $[1, p]$. Based on our previous discussion, the differences between $d_{b_i}+p$ and 0 and the differences between $d_{a_i}+p$ and 0 give distinct numbers from the set $[2p+1, 5p-1]\cup\{5p+1\}$. Considering the remaining differences, we see that $(d_{b_i}+p)-b_i = (d_{b_i}-b_i)+p = c_{b_i}+p$ and $(d_{a_i}+p)-a_i = (d_{a_i}-a_i)+p = c_{a_i}+p$. These differences are all distinct numbers in the set $[p+1, 5p-1]$. Since $\bigcup_{i=1}^{p}\{a_i, b_i\} = [1, p]$, then $\bigcup_{i=1}^{p}\{c_{a_i}, c_{b_i}, d_{a_i}, d_{b_i}\} = [1, 4p-1]\cup\{4p+1\}$, and hence $\bigcup_{i=1}^{p}\{c_{a_i}+p, c_{b_i}+p, d_{a_i}+p, d_{b_i}+p\}$ is exactly $[p+1, 5p-1]\cup\{5p+1\}$, since all the c_j and d_j are distinct.

From the above discussion, it is clear that all edges are distinct and are exactly the set $[1, 5p-1]\cup\{5p+1\}$. We can conclude that since the vertex labels are a subset of $[0, 5p-1]\cup\{5p+1\}$, each nonzero label occurring exactly once, and the edge labels are exactly $[1, 5p-1]\cup\{5p+1\}$, C_5^p can be near gracefully labelled when $p \equiv 1 \pmod 4$.

Case 2: The case $p \equiv 0 \pmod 4$, is proved similarly to Case 1. Use a Skolem sequence to construct the pairs a_i and b_i. Instead of a hooked Skolem sequence, use a Skolem sequence to construct the pairs d_{a_i} and d_{b_i}. The resulting C_5^p can be gracefully labelled when $p \equiv 0 \pmod 4$.

Case 3: If $p \equiv 2 \pmod 4$ and $p > 2$, the statement is proved similarly to Case 1. Instead of a Skolem sequence, use a hooked Skolem sequence to construct the pairs a_i and b_i. Instead of a hooked Skolem sequence, use a hooked near-Skolem sequence to construct the pairs d_{a_i} and d_{b_i}. We know such a sequence exists by Lemma 3.2.4. For $p = 2$, use the 5-tuples $(0, 11, 2, 9, 1)$ and $(0, 6, 3, 7, 5)$. Therefore, C_5^p can be near gracefully labelled when $p \equiv 2 \pmod 4$.

Case 4: If $p \equiv 3 \pmod 4$ and $p > 3$, the statement is proved similarly to Case 1. Instead of a Skolem sequence, use a hooked Skolem sequence to construct the pairs a_i and b_i. Instead of a hooked Skolem sequence, use a near-Skolem sequence to construct the pairs d_{a_i} and d_{b_i}. We know such a sequence exists by Lemma 3.2.4. For $p = 3$, use the 5-tuples $(0, 15, 1, 14, 12)$, $(0, 5, 6, 3, 10)$ and $(0, 9, 13, 2, 8)$. Therefore, C_5^p can be gracefully labelled when $p \equiv 3 \pmod 4$. □

3.3.2 Constructions using two-fold Skolem-type sequences

In this section we discuss how to label C_4^s using two-fold Skolem-type sequences.

Construction 3.3.8. *From a two-fold Skolem-type sequence of order s, construct the pairs of the form (c_j, d_j) and (e_j, f_j) where c_j, d_j, e_j, and f_j are the entries where j occurs in the two-fold Skolem-type sequence, $c_j < d_j, e_j < f_j$, with $j \in H$ and $d_j - c_j = f_j - e_j = j$. From these pairs we can obtain s quadruples of the form $(0, d_j + c, j, f_j + c)$, where c is a fixed positive integer. These quadruples admit a labelling of C_4^s.*

In this chapter, we will use Construction 3.3.8 with a variety of two-fold Skolem-type sequences which will produce edge and vertex labels as given in Table 3.9. The symbols t and c are constants, s and l are the order of the sequences, and d is the defect of the Langford sequence.

Lemma 3.3.9. *The results in Table 3.9 are correct, with each nonzero label occurring exactly once.*

Proof. We will prove that the result in the first row of Table 3.9 is correct. The other rows follow similarly.

From a two-fold Skolem sequence S_s^2 made up of pairs (c_j, d_j) and (e_j, f_j), construct the quadruples $(0, d_j + c, j, f_j + c)_{j=1}^{s}$ as given by Construction 3.3.8 with $c \geq s - 1$. As we know $2 \leq d_j \leq 4s - 1$ and $4 \leq f_j \leq 4s$ and $1 \leq j \leq s$, all the vertex labels will be distinct and from the set $[1, s] \cup [c + 2, c + 4s]$, except 0, which will be repeated s times. Considering the differences of these quadruples, we see that

Sequence	Edge labels are:	Vertex labels from:
Two-fold Skolem sequences of order $s \leq c+1$	$[c+1, c+4s]$	$[0, s] \cup [c+2, c+4s]$
Two-fold Skolem sequences from Table 3.5 of order $s \leq 2c+1$ and s is odd	$[c+1, c+4s]$	$[0, s] \cup \left[c+\frac{s+3}{2}, c+4s\right]$
Two-fold Skolem sequences from Table 3.6 of order $s \leq 2c$ and s is even	$[c+1, c+4s]$	$[0, s] \cup \left[c+\frac{s+2}{2}, c+4s\right]$
Double Langford sequence from Table 3.1 with defect $d \leq c+1$ and order l	$[c+1, c+4l]$	$\{0\} \cup [d, 3d-2] \cup [c+2d, c+4l]$
Double Skolem sequences of order $s \leq 2c+2$ from Table 3.12, 3.14	$[c+1, c+4s]$	$[0, s] \cup \left[c+\frac{s+4}{2}, c+4s\right]$
Double Skolem sequences of order $s \leq 2c+1$ from Table 3.13, 3.15	$[c+1, c+4s]$	$[0, s] \cup \left[c+\frac{s+3}{2}, c+4s\right]$
Two-fold Skolem-type sequence of order $s \leq {}^{(c+4)}/_{2}$ from Table 3.4	$[c+1, c+4s]$	$\{0\} \cup [c+2s+1, c+4s] \cup H$
Two-fold Skolem-type sequence C^1, with $c \geq 0$	$[c+1, c+4]$	$\{0, 2\} \cup [c+3, c+4]$
Two-fold Skolem-type sequence C^2, with $c \geq 1$	$[c+1, c+8]$	$\{0, 2, 3\} \cup \{c+3, c+5, c+7, c+8\}$
Two-fold Skolem-type sequence C^3, with $c \geq 3$	$[c+1, c+12]$	$\{0, 2, 3, 5\} \cup [c+3, c+4] \cup [c+9, c+12]$
Two-fold Skolem-type sequence C^4, with $c \geq 2$	$[c+1, c+16]$	$\{0, 2, 3, 5, 6\} \cup [c+5, c+8] \cup [c+13, c+16]$
Two-fold Langford sequences from Table 3.7 with $d = 6k-1$ and $l = 4k-1$, and $c \geq 2k-1$	$[c+1, c+16k-4]$	$\{0\} \cup [6k-1, 10k-3] \cup [c+8k-1, c+16k-4]$

Table 3.9: Summary of results of Construction 3.3.8 with a variety of two-fold Skolem-type sequences.

$(d_j+c)-j = c_j+c, (f_j+c)-j = e_j+c, d_j+c-0 = d_j+c$, and $f_j+c-0 = f_j+c$. Since, by construction, c_j, d_j, e_j, f_j are all distinct, we obtain $[c+1, c+4s]$ as the set of distinct edge labels. □

Note that for the result in the first row of Table 3.9 there are no restrictions on the sequence so the set of vertex labels is large. If we restrict the sequence, as in the second and third rows of Table 3.9, we will refine the set of vertex labels.

In Constructions 3.3.10–3.3.13, we will give labellings of $C_3^t C_4^s$ when $2t < s \leq {}^{(13t+37)}/_{2}$, with the central vertex labelled 0.

Let P_x be the two-fold Skolem-type sequence of order x given by Construction 3.2.5. Let P_x' be the the sequence obtained from P_x by removing the pair $(1,1)$ from the end of the sequence. Define P_0' to be the sequence $(1, 1)$.

Construction 3.3.10. *From a double Langford sequence with defect* $d = t + 1$ *and order* $l = 2t + 1$*, construct the quadruples of the form* $(0, d_j + c, j, f_j + c)$ *as indicated in Construction 3.3.8 with* $c = t$*. From a Skolem sequence of order* t*, construct the triples* $(0, a_i + c, b_i + c)$ *as indicated in Construction 3.3.1 with* $c = 4l + t$*. These triples and quadraples give a labelling for a* $C_3^t C_4^s$ *where* $s = 2t + 1$*, with the central vertex labelled* 0*.*

Construction 3.3.11. *By concatenating a double Langford sequence with defect* $d = t + 1$ *and order* $l = 2t + 1$*, with a two-fold Skolem sequence of order* k $(k \leq t)$*, construct the quadruples of the form* $(0, d_j + c, j, f_j + c)$ *with* $c = t$ *as indicated in Construction 3.3.8. From a Skolem sequence of order* t*, construct the triples* $(0, a_i + c, b_i + c)$ *as indicated in Construction 3.3.1 with* $c = 4k + 4l + t$*. These triples and quadruples give a labelling for a* $C_3^t C_4^s$ *where* $2t + 2 \leq s \leq 3t + 1$*, with the central vertex labelled* 0*.*

Construction 3.3.12. *By concatenating a two-fold Skolem-type sequence* P_{x-1}' *of order* $x - 1$ *and* $1 \leq x \leq (t+3)/2$*, with a double Langford sequence with defect* $d = t + 4x - 1$ *and order* $l = 2t + 8x - 3$*, with the sequence* P_0'*, and a two-fold Skolem-type sequence from Proposition 3.2.6 of order* y*, for* $0 \leq y \leq 4$*, construct the quadruples of the form* $(0, d_j + c, j, f_j + c)$ *with* $c = t$ *as indicated in Construction 3.3.8. From a Skolem sequence of order* t *construct the triples* $(0, a_i + c, b_i + c)$ *as indicated in Construction 3.3.1 with* $c = 4l + t + 4x + 4y + 4$*. These triples and quadruples give a labelling for a* $C_3^t C_4^s$ *where* $s = 2t + 9x + y - 3$ *and* $3t + 2 \leq s \leq (13t+37)/2$*, with the central vertex labelled* 0*.*

Construction 3.3.13. *By concatenating a two-fold Skolem-type sequence* P_x *of order* x *with* $1 \leq x \leq (t+3)/2$*, with a double Langford sequence with defect* $d = t + 4x + 1$ *and* *order* $l = 2t + 8x + 1$*, with a two-fold Skolem-type sequence from Proposition 3.2.6 of order* y*, with* $0 \le y \le 4$*, construct the quadruples of the form* $(0, d_j + c, j, f_j + c)$ *with* $c = t$ *as indicated in Construction 3.3.8. From a Skolem sequence of order* t *construct the triples* $(0, a_i + c, b_i + c)$ *as indicated in Construction 3.3.1 with* $c = 4l+t+4x+4y$*. These triples and quadruples give a labelling for a* $C_3^t C_4^s$ *where* $s = 2t + 9x + y + 1$ *and* $3t + 2 \le s \le {}^{(13t+37)}/_2$*, with the central vertex labelled* 0*.*

Note that Constructions 3.3.12 and 3.3.13 will cover all the values of $s \in [3t + 2, {}^{(13t+37)}/_2]$.

The bound $s \le {}^{(13t+37)}/_2$ comes from Constructions 3.3.12 and 3.3.13 as we know $s = 2t + 9x + z$ or $x = {}^{(s-2t-z)}/_9$. Since $t \ge 2x - 3$ then $s \le {}^{(13t+2z+27)}/_2$ and since $z \in [-3, 5]$ we obtain that $s \le {}^{(13t+37)}/_2$.

Lemma 3.3.14. *In the labellings of* $C_3^t C_4^s$ *given by Constructions 3.3.10 - 3.3.13 for* $4 \le t \le s \le {}^{(13t+37)}/_2$*, and* $t \ge 2x - 3$ *when* $s \ge 3t + 2$*,*

1. *if* $t \equiv 0, 1 \pmod 4$*, then the edge labels used are* $[1, 4s + 3t]$ *and the vertex labels used are from* $\{0\} \cup [1, 4s + 3t]$*, where each nonzero label occurs exactly once,*
2. *if* $t \equiv 2, 3 \pmod 4$ *then the edge labels used are* $[1, 4s + 3t - 1] \cup \{4s + 3t + 1\}$ *and the vertex labels used are from* $\{0\} \cup [1, 4s + 3t - 1] \cup \{4s + 3t + 1\}$*, where each nonzero label occurs exactly once.*

Proof. We will prove these results for Construction 3.3.13. The proof for Construction 3.3.12 follows in the same fashion as Construction 3.3.13. For Constructions 3.3.10 and 3.3.11 the proofs also follow in the same fashion as Construction 3.3.13, but with no Skolem-type sequences.

Let $t \equiv 0, 1 \pmod 4$. We begin by considering the edge labels. Consider the quadruples formed by the concatenated sequence in Construction 3.3.13, with $c = t$. Those quadruples corresponding to P_x yield edge labels $[t + 1, t + 4x]$, by Table 3.9

(row 7). For the quadruples corresponding to the double Langford sequence, we obtain the edge labels $[t+4x+1, 4l+t+4x]$, by Table 3.9 (row 4), by considering $c = t+4x$ (the length of P_x). For the quadruples corresponding to the two-fold Skolem-type sequence from Proposition 3.2.6, we obtain the edge labels $[4l+t+4x+1, 4l+t+4x +4y]$, by Table 3.9 (rows 9-12), by considering $c = 4l+t+4x$ (the length of P_x and the double Langford sequence). Note that if $y = 0$, then we are considering C^0, the empty sequence, and hence produce no edge labels.

Consider the triples formed by the Skolem sequence of order t with $c = 4l + t + 4x + 4y$ (the length of P_x, the double Langford sequence, and the two-fold Skolem-type sequence). By Lemma 3.3.2(1), this construction yields edge labels $[1, t] \cup [4l+t+4x+4y+1, 4l+3t+4x+4y]$.

From the above discussion, it is clear that all edges are distinct and are exactly the set $[1, 4s+3t]$, where $s = l+x+y$.

We now consider the vertex labels. Consider the quadruples formed by the concatenated sequence in Construction 3.3.13, with $c = t$. Those quadruples corresponding to P_x yield vertex labels that are a subset of $\{0, 1\} \cup [t+2x, t+4x] \cup \{4i|1 \leq i \leq x-1\}$, by Table 3.9 (row 7) and since $t \geq 2x-3$ there are no vertices repeated. For the quadruples corresponding to the double Langford sequence, we obtain vertex labels from $\{0\} \cup [t+4x+1, 9t+36x+4]$, by Table 3.9 (row 4), by considering $c = t+4x$ (the length of P_x). For the quadruples corresponding to the two-fold Skolem-type sequence from Proposition 3.2.6, we obtain vertex labels from $\{0, 2, 3, 5, 6\} \cup [9t+36x+7, 9t+36x+4y+4]$, by Table 3.9 (rows 9-12), by considering $c = 4l+t+4x$ (the length of P_x and double Langford sequence). The labels $\{0, 2, 3, 5, 6\}$ are only used once so they do not conflict with any other vertex labels, since $t+2x \geq 6$ and none of these labels is a multiple of four. Note that if $y = 0$, then we are considering C^0, the empty sequence, and hence produce no vertex labels.

Consider the triples formed by the Skolem sequence of order t with $c = 4l + t + 4x + 4y$ (the length of P_x, double Langford sequence, and the two-fold Skolem-type sequence). By Lemma 3.3.2(1), this construction yields vertex labels from $\{0\} \cup [9t + 36x + 4y + 5, 11t + 36x + 4y + 4]$.

From the above discussion, it is clear that all vertex labels are distinct and are from the set $[0, 4s + 3t]$ where $s = l + x + y$.

If $t \equiv 2, 3 \pmod 4$, we proceed similarly to the proof for the case $t \equiv 0, 1 \pmod 4$, but use a hooked Skolem sequence instead of Skolem sequence with Construction 3.3.1 and Lemma 3.3.2(2). □

3.4 $C_3^t C_4^s$

In this section, we prove (near) graceful labellings exist for $C_3^t C_4^s$.

Theorem 3.4.1. *If $G = C_3^t C_4^s$, where $t \geq s \geq 1$, then G is graceful when $t \equiv 0, 1 \pmod 4$ and near graceful when $t \equiv 2, 3 \pmod 4$.*

Proof. **Case 1**: $t \equiv 0, 1 \pmod 4$.

Use a two-fold Skolem sequence of order s, with Construction 3.3.8 and with $c = t$, to get $(0, d_j + c, j, f_j + c)$, $1 \leq j \leq s$. Using a Skolem sequence of order t in Construction 3.3.1 with $c = 4s + t$ and $t \geq s$, gives $(0, a_i + c, b_i + c)$, $1 \leq i \leq t$. These sequences are known to exist; see Table 3.8.

These two constructions give the vertex labels and the induced edge labels of G. By Table 3.9 (row 1) the quadruples use vertex labels in $[0, s] \cup [t + 2, 4s + t]$ and edge labels $[t + 1, 4s + t]$ and by Lemma 3.3.2(1) the triples use vertex labels in $\{0\} \cup [4s + t + 1, 4s + 3t]$ and edge labels $[1, t] \cup [4s + t + 1, 4s + 3t]$. Thus, this is a graceful labelling.

Case 2: $t \equiv 2, 3 \pmod 4$.

This is similar to Case 1 using a hooked Skolem sequence and Lemma 3.3.2(2) instead of a Skolem sequence and Lemma 3.3.2(1). □

As an example, by using a two-fold Skolem sequence of order 3 with a Skolem sequence of order 4 we can label $G = C_3^4 C_4^3$. Consider the two-fold Skolem sequence $(3, 1, 1, 3, 2, 2, 2, 2, 3, 1, 1, 3)$. This sequence gives quadruples $(0, 7, 1, 15)$, $(0, 11, 2, 12)$, $(0, 8, 3, 16)$. Consider the Skolem sequence $(4, 2, 3, 2, 4, 3, 1, 1)$. This sequence gives triples $(0, 23, 24)$, $(0, 18, 20)$, $(0, 19, 22)$, $(0, 17, 21)$. These quadruples and triples together gracefully label $G = C_3^4 C_4^3$ (see Figure 3.1).

Theorem 3.4.2. *If $G = C_3^t C_4^s$, where $4 \leq t \leq s \leq {}^{(13t+37)}/_2$, then G is graceful when $t \equiv 0, 1 \pmod 4$ and is near graceful when $t \equiv 2, 3 \pmod 4$.*

Proof. Case 1: $t \equiv 0, 1 \pmod 4$.

For $4 \leq t < s \leq 2t$ and s odd, use a two-fold Skolem sequence of order s as given in Table 3.5, with Construction 3.3.8 and with $c = t$, to get $(0, d_j + c, j, f_j + c)$, $1 \leq j \leq s$. Using a Skolem sequence of order t in Construction 3.3.1 with $c = 4s + t$ and $t < s$, gives $(0, a_i + c, b_i + c)$, $1 \leq i \leq t$. These sequences exist, as detailed in Table 3.8.

These two constructions give the vertex labels and the induced edge labels of G. By Table 3.9 (row 2) the quadruples use vertex labels in $[0, s] \cup [{}^{(s+3)}/_2 + t, 4s + t]$ and edge labels $[t + 1, 4s + t]$ and by Lemma 3.3.2(1) the triples use vertex labels in $\{0\} \cup [4s + t + 1, 4s + 3t]$ and produce edge labels $[1, t] \cup [4s + t + 1, 4s + 3t]$. Thus, this is a graceful labelling.

For $4 \leq t < s \leq 2t$ and s even, use a two-fold Skolem sequence of order s as given in Table 3.6, with Construction 3.3.8 and $c = t$, to get $(0, d_j + c, j, f_j + c)$, $1 \leq j \leq s$. Using a Skolem sequence of order t in Construction 3.3.1 with $c = 4s + t$ and $t < s$, gives $(0, a_i + c, b_i + c)$, $1 \leq i \leq t$. These sequences exist, as detailed in Table 3.8.

These two constructions give the vertex labels and the induced edge labels of G. By Table 3.9 (row 2) the quadruples use vertex labels in $[0, s] \cup [(s+2)/2 + t, 4s + t]$ and edge labels $[t + 1, 4s + t]$ and by Lemma 3.3.2(1) the triples use vertex labels in $\{0\} \cup [4s + t + 1, 4s + 3t]$ and edge labels $[1, t] \cup [4s + t + 1, 4s + 3t]$. Thus, this is a graceful labelling.

For $4 \leq t < s = 2t + 1$ use Construction 3.3.10, and for $2t + 2 \leq s \leq 3t + 1$ use Construction 3.3.11. These constructions give the vertex labels and the induced edge labels of G. By Lemma 3.3.14(1) the quadruples and triples use vertex labels in $[0, 4s + 3t]$ and edge labels $[1, 4s + 3t]$. Thus, this is a graceful labelling.

We now consider the case $3t+2 \leq s \leq (13t+37)/2$. Define $I_x = [2t + 9x - 3, 2t + 9x + 1]$, and $J_x = [2t + 9x + 1, 2t + 9x + 5]$ for $x \geq 1$ and fixed t. Define $K_x = [2t + 9x - 3, 2t + 9x + 5] = I_x \cup J_x$. Note that $K_x \cap K_{x+1} = \emptyset$, but for any interval K_x the largest element is $2t + 9x + 5$ and the smallest element in K_{x+1} is $2t + 9x + 6$. Note $\bigcup_{x \geq 1} K_x = [2t + 6, \infty)$, and so for all integers $s \in [2t + 6, \infty)$, there exists x such that $s \in K_x$, and hence $s \in I_x$ or $s \in J_x$. That is, s can be written either in the form $s = 2t + 9x + y - 3$ or the form $s = 2t + 9x + y + 1$, where $0 \leq y \leq 4$, for some x.

For $s = 2t + 9x + y - 3$, and $3t + 2 \leq s \leq (13t+37)/2$ use Construction 3.3.12, and for $s = 2t + 9x + y + 1$, and $3t + 2 \leq s \leq (13t+37)/2$ use Construction 3.3.13. These constructions give the vertex labels and the induced edge labels of G. By Lemma 3.3.14(1) the quadruples and triples use vertex labels in $[0, 4s + 3t]$ and edge labels $[1, 4s + 3t]$. Thus, this is a graceful labelling.

Case 2: $t \equiv 2, 3 \pmod 4$.

Similar to Case 1, but use a hooked Skolem sequence, Lemma 3.3.2(2), and Lemma 3.3.14(2) instead of a Skolem sequence, Lemma 3.3.2(1) and Lemma 3.3.14(1). Then, this is a near graceful labelling. □

In Theorem 3.4.1 and Theorem 3.4.2 we proved (near) graceful labellings exist for $C_3^t C_4^s$ if $1 \leq s \leq t$ or $4 \leq t \leq s \leq {}^{(13t+37)}/_{2}$, omitting the cases for $t = 1, 2, 3$. So, in the following lemmas we consider those cases.

Lemma 3.4.3. *For $k \geq 2$, if ${}^{(k-2)}/_{2} \leq s \leq {}^{(6k-5)}/_{4}$ and a graceful labelling of $C_3^1 C_4^s$ exists, then a graceful labelling of $C_3^1 C_4^{s+4k-1}$ exists.*

Proof. Suppose that $k \geq 2$, with ${}^{(k-2)}/_{2} \leq s \leq {}^{(6k-5)}/_{4}$ such that a graceful labelling of $C_3^1 C_4^s$ exists. As we know the labelling of $C_3^1 C_4^s$ uses edge labels $[1, 4s+3]$ and vertex labels from $[0, 4s+3]$.

By using a two-fold Langford sequence with defect $d = 6k-1$ and order $l = 4k-1$ from Table 3.7 with Construction 3.3.8, $c = 4s+3$, we can label a C_4^{4k-1} with edge labels $[4s+4, 16k+4s-1]$ and vertex labels from $\{0\} \cup [6k-1, 10k-3] \cup [8k+4s+2, 16k+4s-1]$.

Note that to make sure any vertex label is only used at most once, $6k-1$ must be greater than $4s+3$ and since ${}^{(6k-5)}/_{4} \geq s$, we have no vertex labels used twice. Also, to avoid a conflict in vertex labelling, $8k+4s+2 > 10k-3$ and since $s \geq {}^{(k-2)}/_{2}$, we have no vertex labels used twice.

By identifying the vertices with label zero in $C_3^1 C_4^s$ and C_4^{4k-1}, the labelling we obtain is a graceful labelling of $C_3^1 C_4^{s+4k-1}$ with edge labels $[1, 16k+4s-1]$ and vertex labels from $[0, 16k+4s-1]$. □

Lemma 3.4.4. *For $k \geq 3$, if ${}^{(2k-5)}/_{4} \leq s \leq {}^{(3k-3)}/_{2}$ and a near graceful labelling of $C_3^2 C_4^s$ exists which contains a triangle $(0, 4s+5, 4s+7)$, then a near graceful labelling of $C_3^2 C_4^{s+4k-1}$ exists which contains a triangle $(0, 16k+4s+1, 16k+4s+3)$.*

Proof. Suppose that $k \geq 2$, with ${}^{(2k-5)}/_{4} \leq s \leq {}^{(3k-3)}/_{2}$ such that a graceful labelling of $C_3^2 C_4^s$ exists with the specified triangle. As we know the labelling of $C_3^2 C_4^s$ uses edge labels $[1, 4s+5] \cup \{4s+7\}$ and vertex labels from $[0, 4s+5] \cup \{4s+7\}$.

By using a two-fold Langford sequence with defect $d = 6k-1$ and order $l = 4k-1$ from Table 3.7 with Construction 3.3.8, $c = 4s+4$, we can label a C_4^{4k-1} with edge labels $[4s+5, 16k+4s]$ and vertex labels from $\{0\} \cup [6k-1, 10k-3] \cup [8k+4s+3, 16k+4s]$.

Note that to make sure any vertex label only used once, $6k-1$ must be greater than $4s+4$ and since $(3k-3)/2 \geq s$, we have no vertices are used twice. Also, to avoid a conflict in vertex labelling, $8k+4s+3 > 10k-3$ and since $s \geq (2k-5)/4$, we have no vertices are used twice.

If we replace the triangle containing edge length 2, $(0, 4s+5, 4s+7)$, by $(0, 16k+4s+1, 16k+4s+3)$ and identify the vertices with label zero in the $C_3^2C_4^s$ and the C_4^{4k-1} labelling, then we obtain a near graceful labelled $C_3^2C_4^{s+4k-1}$ with edge labels $[1, \ldots, 16k+4s+1] \cup \{16k+4s+3\}$ and vertex labels from $[0, 16k+4s+1] \cup \{16k+4s+3\}$. □

Lemma 3.4.5. *For $k \geq 4$, if $(k-4)/2 \leq s \leq (6k-7)/4$ and a near graceful labelling of $C_3^3C_4^s$ exists which contains triangles $(0, 4s+7, 4s+8)$ and $(0, 4s+6, 4s+10)$, then a near graceful labelling of $C_3^3C_4^{s+4k-1}$ exists which contain triangles $(0, 16k+4s+3, 16k+4s+4)$ and $(0, 16k+4s+2, 16k+4s+6)$.*

Proof. Suppose that $k \geq 2$, with $(k-4)/2 \leq s \leq (6k-7)/4$ such that a graceful labelling of $C_3^3C_4^s$ exists with the specified triangles. As we know the labelling of $C_3^3C_4^s$ uses edge labels $[1, 4s+8] \cup \{4s+10\}$ and vertex labels from $[0, 4s+8] \cup \{4s+10\}$.

By using a two-fold Langford sequence with defect $d = 6k-1$ and order $l = 4k-1$ from Table 3.7 with Construction 3.3.8, $c = 4s+5$, we can label a C_4^{4k-1} with edge labels $[4s+6, 16k+4s+1]$ and vertex labels from $\{0\} \cup [6k-1, 10k-3] \cup [8k+4s+4, 16k+4s+1]$.

Note that to make sure any vertex label only used once, $6k-1$ must be greater than $4s+5$ and since ${}^{(6k-7)}/_{4} \geq s$, we have no vertices are used twice. Also, to avoid a conflict in vertex labelling, $8k+4s+4 > 10k-3$ and since $s \geq {}^{(k-4)}/_{2}$, then we have no vertices are used twice.

Replace the triangles containing edge lengths 1 and 4, $(0, 4s+7, 4s+8)$ and $(0, 4s+6, 4s+10)$, by $(0, 16k+4s+3, 16k+4s+4)$ and $(0, 16k+4s+2, 16k+4s+6)$. If we then identify the vertices with label zero in the $C_3^3C_4^s$ and the C_4^{4k-1} labelling, we obtain a near graceful labelled $C_3^3C_4^{s+4k-1}$ with edge labels $[1, \ldots, 16k+4s+4] \cup \{16k+4s+6\}$ and vertex labels from $[0, 16k+4s+4] \cup \{16k+4s+6\}$. □

Theorem 3.4.6. *If $s \in \mathbb{N}$, and $1 \leq t \leq 3$ then $C_3^tC_4^s$ can be gracefully labelled if $t = 1$ and near gracefully labelled if $t = 2, 3$.*

Proof. Case 1: $t = 1$.

For $C_3^1C_4^s$ with $s = 1, 2, \ldots, 20$, explicit labellings are given in Appendix B. Define I_k to be the real interval $\left[{}^{(9k-4)}/_{2}, {}^{(22k-9)}/_{4}\right]$ for fixed k. If $k \geq {}^{19}/_{4}$ then ${}^{(9(k+1)-4)}/_{2} \leq {}^{(22k-9)}/_{4}$. That is, $I_k \cap I_{k+1} \neq \emptyset$. This implies $\bigcup_{k\geq 5} I_k = \left[{}^{41}/_{2}, \infty\right)$, and so for all $s \in \left[{}^{41}/_{2}, \infty\right)$, there exists k such that $s \in I_k$.

We proceed by induction on s. Let s be an integer and $s \geq 21$. There exists some k such that $s \in I_k$. Therefore, letting $s = 4k + s' - 1$, then ${}^{(k-2)}/_{2} \leq s' \leq {}^{(6k-5)}/_{4}$. By induction, a graceful labelling of $C_3^1C_4^{s'}$ exists, and by Lemma 3.4.3 a graceful labelling of $C_3^1C_4^s$ exists.

Case 2: $t = 2$.

This case is proved similarly to Case 1. For $s = 1, 2, \ldots, 20$, explicit labellings are given in Appendix C. And, define $I_k = \left[{}^{(18k-9)}/_{4}, {}^{(11k-5)}/_{2}\right]$ with $s \in \left[{}^{41}/_{2}, \infty\right)$, there exists k such that $s \in I_k$.

Case 3: $t = 3$.

This case is proved similarly to Case 1. For $s = 1, 2, \ldots, 19$, explicit labellings are given in Appendix D. And, define $I_k = [{(9k-6)}/{2}, {(22k-11)}/{4}]$ with $s \in [{39}/{2}, \infty)$, there exists k such that $s \in I_k$. $\square$

3.5 $C_3^t C_5^p$

Using Theorem 3.3.7 with Langford sequences, we can obtain (near) graceful labellings of $C_3^t C_5^p$.

In Constructions 3.5.1 and 3.5.3, we will give labellings of $C_3^t C_5^p$, with the central vertex labelled 0.

Construction 3.5.1. *Given a C_5^p gracefully labelled by the 5-tuples $(0, d_{b_i} + p, b_i, a_i, d_{a_i} + p)$ for each $1 \leq i \leq p$ formed by one of either Construction 3.3.6(1) or Construction 3.3.6(4) and $t \geq 2p+1$, replace the 5-tuples with $(0, d_{bi} + p + 3t, b_i, a_i, d_{ai} + p + 3t)$ for each $1 \leq i \leq p$. From a Langford sequence with defect $p+1$ and order t, form triples via Construction 3.3.1 with $c = p + t$. These 5-tuples and triples give a labelling of a $C_3^t C_5^p$, with the central vertex labelled 0.*

Recall the result of Theorem 3.3.7: if $G = C_5^p$, then G is near graceful when $p \equiv 1, 2 \pmod 4$ and G is graceful when $p \equiv 0, 3 \pmod 4$. In order to apply Construction 3.5.1 we have two cases: $p \equiv 0 \pmod 4$ and $p \equiv 3 \pmod 4$.

Lemma 3.5.2. *The labelling of $C_3^t C_5^p$ given by Construction 3.5.1 uses edge labels $[1, 5p + 3t]$ exactly once and vertex labels from $[0, 5p + 3t]$, each nonzero label occurring exactly once.*

Proof. We have two cases: $p \equiv 0 \pmod 4$ and $p \equiv 3 \pmod 4$ because we are using gracefully labelled C_5^p.

Case 1: If $p \equiv 0 \pmod 4$, form the 5-tuples $(0, d_{b_i} + p + 3t, b_i, a_i, d_{a_i} + p + 3t)$ for each $1 \leq i \leq p$. We begin by considering vertex labels.

In a Skolem-type sequence, the a_i, b_i, c_i and d_i are all unique. From the Skolem sequence S_1 of order p indicated in Construction 3.3.6, we notice that the entries a_i and b_i in the third and fourth entries of the 5-tuples give the distinct numbers $[1, 2p]$. Recall that the Skolem sequence S_2 of order $2p$ used in Construction 3.3.6 contains no right endpoints in the first p positions, with the first right endpoint occurring in position $p+1$. Note $\bigcup_{i=1}^{p} \{d_{a_i}, d_{b_i}\}$ is the set of all right endpoints, and so a subset of $[p+1, 4p]$. From the 5-tuples, we use the right endpoints as indicated in the second and fifth entries. We add $p + 3t$ to each element based on our construction so we obtain a subset of $[2p + 3t + 1, 5p + 3t]$.

Consider the triples of the form $(0, a_i + p + t, b_i + p + t)$ with $1 \leq i \leq t$ given by using Construction 3.3.1 with $c = p + t$. These triples give the vertex labels for t triangles C_3^t, where 0 is repeated t times as a common vertex. Thus by Lemma 3.3.5 these triples use vertex labels from $\{0\} \cup [p + t + 1, p + 3t]$.

Thus, in this labelling, the common vertex 0 is repeated $p+t$ times. The remaining vertices are all distinct and from the union of disjoint sets $[1, 2p] \cup [p + t + 1, p + 3t] \cup [2p + 3t + 1, 5p + 3t]$.

We now examine the edge labels from this construction, by considering the differences between subsequent entries (taken cyclically) in $(0, d_{b_i}+p+3t, b_i, a_i, d_{a_i}+p+3t)$. The differences between b_i and a_i produce the distinct numbers $[1, p]$.

The differences $(d_{b_i} + p + 3t) - 0$ and the differences $(d_{a_i} + p + 3t) - 0$ produce distinct numbers. Call this set of distinct numbers A and observe that $A \subseteq [2p + 3t + 1, 5p + 3t]$. Also, from $(d_{b_i} + p + 3t) - b_i = (d_{b_i} - b_i) + p + 3t = c_{b_i} + p + 3t$ and $(d_{a_i} + p + 3t) - a_i = (d_{a_i} - a_i) + p + 3t = c_{a_i} + p + 3t$, we get distinct numbers. Call this set of distinct numbers B and observe that $B \subseteq [p + 3t + 1, 5p + 3t - 1]$. We know in a Skolem sequence, the c_i and d_i are all unique so A and B are disjoint. Note that $|A \cup B| = 4p$. Now, we can conclude that all the previous differences give exactly the edge labels $[p+3t+1, 5p+3t]$ exactly once.

Consider the differences from the triples of the form $(0, a_i+p+t, b_i+p+t)$ with $1 \leq i \leq t$ given by using Construction 3.3.1 with $c = p+t$. By Lemma 3.3.5 these triples use edge labels $[p+1, p+3t]$. If we take the union of the sets of edge labels, we obtain $[1, 5p+3t]$.

Case 2: The case $p \equiv 3 \pmod 4$, is proved similarly to Case 1. Instead of a Skolem sequence for S_1, use a hooked Skolem sequence of order p to construct the entries a_i and b_i. Instead of a Skolem sequence for S_2, use a near-Skolem sequence of order $2p+1$ to construct the entries d_{a_i} and d_{b_i}. The result follows in a similar fashion. □

Construction 3.5.3. *Given a near gracefully labelled* C_5^p *by the 5-tuples* $(0, d_{b_i}+p, b_i, a_i, d_{a_i}+p)$ *for each* $1 \leq i \leq p$ *formed by either of Construction 3.3.6(2) or Construction 3.3.6(3) and* $t \geq 2p+1$*, replace the 5-tuples with* $(0, d_{bi}+p+3t, b_i, a_i, d_{ai}+p+3t)$ *for each* $1 \leq i \leq p$*. Form a Langford sequence with defect* $p+1$ *and order* t*, then use this sequence to form triples via Construction 3.3.1 with* $c = p+t$*. These 5-tuples and triples give a labelling of a* $C_3^tC_5^p$*, with the central vertex labelled* 0*.*

In order to apply Construction 3.5.3 we will consider two cases: $p \equiv 1 \pmod 4$ and $p \equiv 2 \pmod 4$.

Lemma 3.5.4. *The labelling of* $C_3^tC_5^p$ *given by Construction 3.5.3 uses edge labels* $[1, 5p+3t-1] \cup \{5p+3t+1\}$ *exactly once and vertex labels from* $[0, 5p+3t-1] \cup \{5p+3t+1\}$*, each nonzero label occurring exactly once.*

Proof. The proof is similar to the proof of Lemma 3.5.2, except the type of sequence used is as follows.

If $p \equiv 1 \pmod 4$, use a hooked Skolem sequence of order p for S_1 to obtain the entries a_i and b_i and use a hooked Skolem sequence of order $2p$ for S_2 to obtain the entries d_{a_i} and d_{b_i}.

If $p \equiv 2 \pmod 4$, use a hooked Skolem sequence of order p for S_1 to obtain the entries a_i and b_i and use a hooked near-Skolem sequence of order $2p+1$ for S_2 to obtain the entries d_{a_i} and d_{b_i}. □

We give an example of a labelling of $G = C_3^9 C_5^4$. Consider the graceful labelling of C_5^4 obtained using Construction 3.3.6 with the vertices of the vanes labelled by the 5-tuples $(0,16,7,8,13)$, $(0,11,4,2,10)$, $(0,17,3,6,12)$, $(0,19,1,5,20)$. Use $t=9$ with Construction 3.5.1 to obtain $(0,43,7,8,40)$, $(0,38,4,2,37)$, $(0,44,3,6,39)$, $(0,46,1,5,47)$ as the vertex labels of C_5 vanes in $C_3^9 C_5^4$. Apply Construction 3.3.1 with the Langford sequence, $L_5^9 = (13,11,9,7,5,12,10,8,6,5,7,9,11,13,6,8,10,12)$, and $c=13$ to construct the triples $(0,18,23)$, $(0,22,28)$, $(0,17,24)$, $(0,21,29)$, $(0,16,25)$, $(0,20,30)$, $(0,15,26)$, $(0,19,31)$, $(0,14,27)$. Use these triples to label the vertices of the C_3 vanes in $C_3^9 C_5^4$. Together, the labellings of C_5 and C_3 vanes give a graceful labelling the graph G.

Theorem 3.5.5. *If $G = C_3^t C_5^p$ and $t \geq 2p+1$ then*

1. *G is graceful when $p \equiv 0 \pmod 4$ and $t \equiv 0,1 \pmod 4$,*
2. *G is graceful when $p \equiv 3 \pmod 4$ and $t \equiv 0,3 \pmod 4$,*
3. *G is near graceful when $p \equiv 1 \pmod 4$ and $t \equiv 0,3 \pmod 4$,*
4. *G is near graceful when $p \equiv 2 \pmod 4$ and $t \equiv 0,1 \pmod 4$.*

Proof. The (near) graceful C_5^p exists by Theorem 3.3.7. Construction 3.5.1 gives the edge and vertex labels of G when $p \equiv 0,3 \pmod 4$ and in these cases G is graceful by Lemma 3.5.2. Also Construction 3.5.3 gives the edge and vertex labels of G when $p \equiv 1,2 \pmod 4$ and in these cases G is near graceful by Lemma 3.5.4 □

Theorem 3.5.5 only contains results for half of the possible combinations of p and t and only for large t. That is not to say the other cases are not (near) graceful. For example, the case $p = t = 1$ does not lend itself to our construction. However the labelling of $C_3^1C_5^1$ with the vertices of the vanes labelled by $(0, 5, 7)$ and $(0, 8, 4, 3, 6)$ is graceful.

3.6 $C_3^tC_6^h$

In this section, we extend the technique of [4] to obtain labellings for $C_3^tC_6^h$.

Lemma 3.6.1. *Suppose there exists a labelling of a windmill with two triangles labelled $(0, i, b_i + n)$ and $(0, j, b_j + n)$, where (a_i, b_i) and (a_j, b_j) are the positions of i and j in a Skolem sequence. If these two triangles are removed and replaced by a C_6 with vertex labels $(0, b_i + n, i, i + j, j, b_j + n)$, for $1 \leq i, j \leq n$ and $i \neq j$, then the edge labels are preserved.*

Proof. The edge labels induced by the triples $(0, i, b_i + n)$ and $(0, j, b_j + n)$ are $\{i, j, a_j + n, b_j + n, a_i + n, b_i + n\}$. The edge labels induced by $(0,\ b_i + n,\ i,\ i + j,\ j,\ b_j + n)$ are the same. □

Construction 3.6.2. *From a (near) gracefully labelled C_3^n labelled by $(0, i, b_i + n)$ with $1 \leq i \leq n$ formed by one of the (hooked) Skolem sequences of order n in Table 3.12 - 3.15, in Appendix A as in Construction 3.3.3, replace $2h$ triples ($h \leq \lfloor (2n+1)/5 \rfloor$) with h 6-tuples by replacing the pair of triples $(0, i, b_i + n)$ and $(0, j, b_j + n)$ with $(0, b_i + n, i, i + j, j, b_j + n)$, with pairs indicated as in Table 3.10.*

Table 3.10 gives a family of possible pairs (i, j) corresponding to triples $(0, i, b_i + n)$ and $(0, j, b_j + n)$ with $1 \leq i, j \leq n$ that may be paired to form hexagons of the form $(0, b_i + n, i, i + j, j, b_j + n)$ using the vertex label $i + j$. If the value of $i + j$ does not conflict with any other vertex labels, we obtain a (near) graceful labelling of $C_3^t C_6^h$, where $n = t + 2h$ and $1 \leq t, h \leq n$.

We notice from Table 3.10, when $n \in \{5k, 5k+1\}$ with $k \geq 1$, we can obtain up to $2k$ distinct values of $i + j$. By way of contradiction for $n = 5k$ suppose there are $2k+m$ pairs where $1 \leq m \leq \left\lfloor \frac{k}{2} \right\rfloor$. We use $2k+m$ pairs to provide a bound. If we have $2k + m$ pairs then we can use $4k + 2m$ elements. The maximum sum of the $4k + 2m$ elements used in these pairs is

$$\sum_{i=k-2m+1}^{5k} i = 12k^2 + 2km + 2k - 2m^2 + m. \tag{3.1}$$

Now we compute the minimum sum of the sums of the pairs. In Construction 3.6.2, before any 6-tuples have been constructed, we have used the vertex labels $[1, 5k]$. Thus, the smallest possible sum of the $2k + m$ pairs is

$$\sum_{i=5k+1}^{7k+m} i = 12k^2 + 7km + k + \tfrac{m^2}{2} + \tfrac{m}{2}. \tag{3.2}$$

Now subtracting Equation 3.1 from Equation 3.2 we obtain

$$5km - k + \tfrac{5m^2}{2} - \tfrac{m}{2} = k\left(5m - 1\right) + m\left(\tfrac{5m-1}{2}\right) > 0. \tag{3.3}$$

From Equation 3.3 notice that the minimum sum of the $2k+m$ pairs is greater than the maximum sum of the $4k + 2m$ elements used in pairs; hence we have a contradiction. So for $n = 5k$, the maximum number of pairs of triples of this form that can be replaced by hexagons is $2k$. In a similar way, for $n = 5k + 1$ we can obtain up to $2k$ distinct values of $i + j$ and for $n \in \{5k + 2, 5k + 3, 5k + 4\}$, it can be shown that $2k + 1$ is the maximum number of pairs that can be achieved. Thus, the construction in Table 3.10 is best possible using this method of generating hexagons.

The bound $h \leq \lfloor {}^{(2n+1)}/_{5} \rfloor$ comes from the maximum number of possible pairs of triples formed by using the method of Table 3.10. Combining this with the fact $n = t+2h$, it is straightforward to show these restrictions are equivalent to $h \leq 2t+1$.

n	(i,j)	
$5k$	$(k+z, 4k+z+1)$, $(2k+z'+1, 3k+z'+1)$	$0 \leq z, z' \leq k-1$
$5k+1$	$(k+z+1, 4k+z+1)$, $(2k+z'+2, 3k+z'+1)$	$0 \leq z \leq k$, $0 \leq z' \leq k-2$
$5k+2$	$(k+z+1, 4k+z+2)$, $(2k+z'+2, 3k+z'+2)$	$0 \leq z \leq k$, $0 \leq z' \leq k-1$
$5k+3$	$(k+z+1, 4k+z+3)$, $(2k+z'+2, 3k+z'+3)$	$0 \leq z \leq k$, $0 \leq z' \leq k-1$
$5k+4$	$(k+z+1, 4k+z+4)$, $(2k+z'+3, 3k+z'+3)$	$0 \leq z \leq k$, $0 \leq z' \leq k-1$

Table 3.10: Possible pairs of triples corresponding to i and j that may be paired to form hexagons with $i+j$.

Recall the following example from Section 3.3. From the hooked Skolem sequence $(3,1,1,3,2,0,2)$ consider the pairs $(2,3),(5,7),(1,4)$. These pairs yield the triples $(0,1,6)$, $(0,2,10)$, and $(0,3,7)$, to near gracefully label C_3^3, by Construction 3.3.3. Pair the triples $(0,1,6)$ and $(0,3,7)$ corresponding to $i=1$ and $j=3$ to form the 6-tuple $(0,6,1,4,3,7)$ using the label $i+j=4$. The label 4 has not already been used, so we obtain a near graceful labelling of $C_3^1 C_6^1$. We cannot pair $(0,1,6)$ with $(0,2,10)$ because $1+2=3$ which duplicates another vertex label.

In the following example we will present how we can use Construction 3.6.2 with Table 3.10 to get a family of possible pairs of triples corresponding to i and j that may be paired to form hexagons with $i+j$.

Start with $(8,6,4,2,7,2,4,6,8,3,5,7,3,1,1,5)$ the Skolem sequence of order $n=8$ constructed from Table 3.12, in Appendix A. Take the triples of the form $(0,i,b_i+8)$, where $1 \leq i \leq 8$ as follows: $(0,1,23)$, $(0,2,14)$, $(0,3,21)$, $(0,4,15)$, $(0,5,24)$, $(0,6,16)$, $(0,7,20)$, $(0,8,17)$. Since $n=8=5\,(1)+3$, from Table 3.10, we can get up to three pairs $(2,7)$, $(3,8)$, and $(4,6)$ which give three different gracefully labelled graphs. We have three possible replacements: replace $(0,2,14)$ and $(0,7,20)$ by $(0,14,2,9,7,20)$; replace $(0,3,21)$ and $(0,8,17)$ by $(0,21,3,11,8,17)$; and replace $(0,4,15)$ and $(0,6,16)$ by $(0,15,4,10,6,16)$. We can gracefully label $C_3^6C_6^1$ by using any one of these replacements, $C_3^4C_6^2$ by using any two, and $C_3^2C_6^3$ by simultaneously using all three.

Note that the first right endpoint in the Skolem sequences of order n described in Tables 3.12–3.15, in Appendix A, is always at $\lceil (n+3)/2 \rceil$. This fact will be useful later in the proof of Theorem 3.6.3.

In Theorem 3.6.3, we use Construction 3.6.2 with the appropriate Skolem sequence of order n which will give (near) graceful labellings for $C_3^tC_6^h$ and $h \leq 2t+1$ as given in Table 3.11.

$n = t + 2h$	Graceful	Near graceful
$5k$	$k \equiv 0,1 \pmod 4$	$k \equiv 2,3 \pmod 4$
$5k+1$	$k \equiv 0,3 \pmod 4$	$k \equiv 1,2 \pmod 4$
$5k+2$	$k \equiv 2,3 \pmod 4$	$k \equiv 0,1 \pmod 4$
$5k+3$	$k \equiv 1,2 \pmod 4$	$k \equiv 0,3 \pmod 4$
$5k+4$	$k \equiv 0,1 \pmod 4$	$k \equiv 2,3 \pmod 4$

Table 3.11: Summary of results of (near) graceful labelling $C_3^tC_6^h$ and $h \leq 2t+1$.

Theorem 3.6.3. *If $G = C_3^tC_6^h$ with $h \leq 2t+1$, then G is graceful or near graceful.*

Proof. Let $n = 2h + t$. We begin by considering $n = 5k$. Proofs for other n follow in the same fashion.

Start with a (near) gracefully labelled C_3^n with $n = 5k$ with $k \equiv 0,1 \pmod 4$ labelled by $(0, i, b_i + n)$ with $1 \leq i \leq n$ formed by a Skolem sequence as indicated in Construction 3.6.2, where $2h + t = n$ and $h \leq 2t+1$. By Construction 3.6.2, we replace $2h$ triples with h 6-tuples by $(0, b_i + n, i, i + j, j, b_j + n)$. By Lemma 3.6.1 the edge labels will not be changed. Therefore, in this new labelling we use the edge labels

$[1, 3n]$. The new labelling uses the same vertex labels, as well as some new labels of the form $i + j$.

We constructed $(0, i, b_i + n)$ with $1 \leq i \leq n$ by using Table 3.12 or Table 3.13, in Appendix A. These triples use the elements i in the interval $[1, n]$ and use the elements $b_i + n$ in the interval $[\lceil (3n+3)/2 \rceil, 3n]$. Thus any subset of these triples do not use any vertex labels in the interval $[n + 1, \lceil (3n+3)/2 \rceil - 1]$.

By construction, the sums $i + j$ are all distinct and in the interval $[n + 1, \lceil (3n+3)/2 \rceil - 1]$. The new vertex labels do not duplicate any previously used labels; this new vertex labelling is injective and uses labels from the set $[1, 3n]$.

Finally, we can conclude that since the vertex labels are a subset of $[0, 3n]$ and the edge labels are exactly $[1, 3n]$, that $G = C_3^t C_6^h$ with $2h + t = n$ and $h \leq 2t + 1$ can be gracefully labelled when $k \equiv 0, 1 \pmod 4$.

The case $n = 5k$ with $k \equiv 2, 3 \pmod 4$, works in the same way except the construction of $(0, i, b_i + n)$ with $1 \leq i \leq n$ is by using Table 3.14 or Table 3.15, in Appendix A. This construction then uses the edge labels $[1, 3n - 1] \cup \{3n + 1\}$ and vertex labels from $[0, 3n - 1] \cup \{3n + 1\}$. We conclude that $G = C_3^t C_6^h$ with $2h+t = 5k$ and $h \leq 2t + 1$ can be near gracefully labelled when $k \equiv 2, 3 \pmod 4$. □

Theorem 3.6.3 only contains results for $h \leq 2t + 1$. We cannot (near) gracefully label $G = C_3^t C_6^h$ with $h > 2t + 1$ by our construction, so these problems remain open.

3.7 Discussion

In Section 3.4, we proved (near) graceful labellings exist for the $C_3^t C_4^s$ for positive integers s and t with $t \geq s$ by Theorem 3.4.1; for $t = 1, 2, 3$ and any s by Theorem 3.4.6; and for any positive integer t with $4 \leq t \leq s \leq (13t+37)/2$, by Theorem 3.4.2. But for $t \geq 4$ and $s > (13t+37)/2$ the problem is still open.

Also in Section 3.3, we found a new solution showing that C_5^t is near graceful when $t \equiv 1, 2 \pmod 4$, which let us prove in Section 3.5 that (near) graceful labellings exist for $C_3^t C_5^p$ for $t \geq 2p+1$, by using Skolem, Langford and near-Skolem sequences except when $p \equiv 0 \pmod 4$ and $t \equiv 2, 3 \pmod 4$, when $p \equiv 3 \pmod 4$ and $t \equiv 1, 2 \pmod 4$, when $p \equiv 1 \pmod 4$ and $t \equiv 1, 2 \pmod 4$, and when $p \equiv 2 \pmod 4$ and $t \equiv 2, 3 \pmod 4$. By Theorem 4.3(2) in [1], we could prove that graceful labellings exist for the $C_3^t C_5^p$ with t sufficiently larger than p, but our work in this chapter gives a nice lower bound for half of the cases. Finally, in Section 3.6, we proved (near) graceful labellings exist for the $C_3^t C_6^h$ for $h \leq 2t + 1$, by using Skolem sequences.

Besides the cases indicated above, finding graceful labellings for $C_3^t C_5^p$ windmills when $t \leq 2p + 1$ and for $C_3^t C_6^h$ when $2t + 1 \leq h$ remains open.

Many of the techniques of this chapter can be combined. From the example in Section 3.5 we gracefully labelled $C_3^9 C_5^4$. If we replace the triples $(0, 16, 25), (0, 20, 30), (0, 19, 31)$ by $(0, 9, 25), (0, 10, 30), (0, 12, 31)$, we notice that the edge labels are the same and the new vertex labels do not appear elsewhere in the labelling of $C_3^9 C_5^4$. Thus we have obtained another graceful labelling of $C_3^9 C_5^4$. Now by the same technique we used in Section 3.6, we can obtain a graceful labelling for $C_3^7 C_5^4 C_6^1$ by replacing $(0, 9, 25), (0, 10, 30)$ with $(0, 25, 9, 19, 10, 30)$.

Question: What windmills of cycles using one, two, or three different lengths are graceful or near graceful?

In [17, 18, 20, 21], it was proved that graceful labellings exist for C_7^e, C_9^e, C_{11}^e, and C_{13}^e respectively. By Theorem 4.3(2) in [1], we can prove that graceful labellings exist for the $C_3^t C_7^e, C_3^t C_9^e, C_3^t C_{11}^e$, and $C_3^t C_{13}^e$, respectively with t sufficiently larger than e.

In this chapter, we were using several types of Skolem-type sequences to gracefully label variable windmills. Frequently, our constructions relied on the positions of the right endpoints of these sequences, and particularly where the first right endpoint occurs, such as in the proof of Theorem 3.3.7. For example, let $(4,1,1,2,4,2,4,1,1,2,4,2)$ be a 2-fold Skolem-type sequence of order 3 with $H = \{1,2,4\}$. By Construction 3.3.8 we can obtain 3 quadruples $(0,3,1,9)$, $(0,6,2,12)$, and $(0,5,4,11)$. These quadruples gracefully label a C_4^3 windmill. In this example the set H is exactly the same as of the set of left endpoints in the sequence that is used to gracefully label the C_4^3 windmill. That leads us to the following questions.

Question: Can we use Skolem-type sequences to gracefully label all C_4 windmills?

Question: Can we find a family of m-fold Skolem-type sequences with first right endpoint as large as possible?

In the process of finding some of the constructions in this chapter, the authors became interested in m-fold Langford sequences. While it is straightforward to concatenate Langford sequences, it is actually possible to find sequences of much smaller order than would be possible by concatenation. For example, a Langford sequence with defect 5 must have order at least 9, so a 2-fold Langford sequence formed by concatenation would obviously have the same restrictions. However, we were capable of finding 2-fold Langford sequences of defect 5 and order 3, for example: $(7,5,7,5,6,6,5,7,5,7,6,6)$. Thus, we pose the following question.

Question: What are necessary and sufficient conditions for the existence of m-fold Langford sequences with $m \geq 2$?

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Appendix A: Skolem Constructions

For $n \equiv 0 \pmod 4$, we recall the following construction which is the reverse of Skolem's solution [16]. For $n = 4m$ and $m \geq 1$, follow the construction given in Table 3.12.

	i	a_i	b_i	
1	$2r+2$	$2m-r$	$2m+2+r$	$0 \leq r \leq 2m-1$
2	1	$7m$	$7m+1$	–
3	$4m-1$	$2m+1$	$6m$	–
4	$2m+2r+1$	$5m+1-r$	$7m+r+2$	$0 \leq r \leq m-2$
5	$2m-1$	$4m+2$	$6m+1$	–
6	$2m-3-2r$	$5m+2+r$	$7m-1-r$	$0 \leq r \leq m-3$

Table 3.12: Skolem sequence construction of order $n = 4m$.

For $n \equiv 1 \pmod 4$, we recall the following construction which was introduced by Marsh [11]. For $n = 4m+1$ and $m \geq 2$, follow the construction given in Table 3.13.

	i	a_i	b_i	
1	$2r$	$2m+1-r$	$2m+1+r$	$1 \leq r \leq 2m$
2	$4m+1$	$2m+1$	$6m+2$	–
3	$2m-1+2r$	$5m+2-r$	$7m+1+r$	$1 \leq r \leq m$
4	$2m-1$	$6m+3$	$8m+2$	–
5	1	$5m+2$	$5m+3$	–
6	$2r+1$	$6m+2-r$	$6m+3+r$	$1 \leq r \leq m-2$

Table 3.13: Skolem sequence construction of order $n = 4m+1$.

For $n \equiv 2 \pmod 4$, we recall the following construction which was introduced by Hilton [8]. For $n = 4m+2$ and $m \geq 2$, follow the construction given in Table 3.14.

	i	a_i	b_i	
1	$2r$	$2m+2-r$	$2m+2+r$	$1 \leq r \leq 2m+1$
2	1	$7m+4$	$7m+5$	–
3	$1+2r$	$6m+2-r$	$6m+3+r$	$1 \leq r \leq m$
4	$2m+3$	$6m+2$	$8m+5$	–
5	$2m+3+2r$	$5m+2-r$	$7m+5+r$	$1 \leq r \leq m-2$
6	$4m+1$	$2m+2$	$6m+3$	–

Table 3.14: Hooked Skolem sequence construction of order $n = 4m+2$.

For $n \equiv 3 \pmod 4$, we recall the following construction which was introduced by Hilton [8]. For $n = 4m+3$ and $m \geq 1$, follow the construction given in Table 3.15.

	i	a_i	b_i	
1	$2r$	$2m+2-r$	$2m+2+r$	$1 \leq r \leq 2m+1$
2	1	$5m+4$	$5m+5$	–
3	$1+2r$	$6m+5-r$	$6m+6+r$	$1 \leq r \leq m-1$
4	$2m+1$	$6m+6$	$8m+7$	–
5	$2m+1+2r$	$5m+4-r$	$7m+5+r$	$1 \leq r \leq m$
6	$4m+3$	$2m+2$	$6m+5$	–

Table 3.15: Hooked Skolem sequence construction of order $n = 4m+3$.

Appendix B

We include here graceful labellings of $C_3^1 C_4^s$, were where $1 \leq s \leq 20$.

For $s=1$, use $(0,3,2,6)$, $(0,5,7)$.

For $s=2$, use $(0,9,1,11)$, $(0,3,2,6)$, $(0,5,7)$.

For $s=3$, use $(0,11,1,15)$, $(0,12,4,13)$, $(0,3,2,6)$, $(0,5,7)$.

For $s=4$, use $(0,13,1,19)$, $(0,14,4,15)$, $(0,16,8,17)$, $(0,3,2,6)$, $(0,5,7)$.

For $s=5$, use $(0,15,1,19)$, $(0,16,4,17)$, $(0,20,11,22)$, $(0,21,13,23)$, $(0,3,2,6)$, $(0,5,7)$.

For $s=6$, use $(0,17,1,21)$, $(0,18,4,19)$, $(0,22,12,25)$, $(0,23,14,26)$, $(0,24,16,27)$, $(0,3,2,6)$, $(0,5,7)$.

For $s=7$, use $(0,19,1,21)$, $(0,22,10,27)$, $(0,23,12,28)$, $(0,24,14,29)$, $(0,25,16,30)$, $(0,26,18,31)$, $(0,3,2,6)$, $(0,5,7)$.

For $s=8$, use $(0,24,16,27)$, $(0,23,14,26)$, $(0,22,12,25)$, $(0,31,17,35)$, $(0,30,15,34)$, $(0,29,13,33)$, $(0,28,11,32)$, $(0,3,2,6)$, $(0,5,7)$.

For $s = 9$, use $(0, 23, 1, 39)$, $(0, 24, 10, 31)$, $(0, 25, 12, 32)$, $(0, 26, 14, 33)$, $(0, 27, 16, 34)$, $(0, 28, 18, 35)$, $(0, 29, 20, 36)$, $(0, 30, 22, 37)$, $(0, 3, 2, 6)$, $(0, 5, 7)$.

For $s = 10$, use $(0, 23, 1, 41)$, $(0, 42, 4, 43)$, $(0, 24, 10, 31)$, $(0, 25, 12, 32)$, $(0, 26, 14, 33)$, $(0, 27, 16, 34)$, $(0, 28, 18, 35)$, $(0, 29, 20, 36)$, $(0, 30, 22, 37)$, $(0, 3, 2, 6)$. $(0, 5, 7)$.

For $s = 11$, use $(0, 28, 13, 36)$, $(0, 29, 15, 37)$, $(0, 30, 17, 38)$, $(0, 44, 18, 45)$, $(0, 31, 19, 39)$, $(0, 32, 21, 40)$, $(0, 46, 22, 47)$, $(0, 33, 23, 41)$, $(0, 34, 25, 42)$, $(0, 35, 27, 43)$, $(0, 3, 2, 6)$, $(0, 5, 7)$.

For $s = 12$, use $(0, 34, 26, 39)$, $(0, 33, 24, 38)$, $(0, 32, 22, 37)$, $(0, 31, 20, 36)$, $(0, 30, 18, 35)$, $(0, 45, 27, 51)$, $(0, 44, 25, 50)$, $(0, 43, 23, 49)$, $(0, 42, 21, 48)$, $(0, 41, 19, 47)$, $(0, 40, 17, 46)$, $(0, 3, 2, 6)$, $(0, 5, 7)$.

For $s = 13$, use $(0, 38, 26, 43)$, $(0, 37, 24, 42)$, $(0, 36, 22, 41)$, $(0, 35, 20, 40)$, $(0, 34, 18, 39)$, $(0, 49, 27, 55)$, $(0, 48, 25, 54)$, $(0, 47, 23, 53)$, $(0, 46, 21, 52)$, $(0, 45, 19, 51)$, $(0, 44, 17, 50)$, $(0, 3, 2, 6)$, $(0, 5, 7)$.

For $s = 14$, use $(0, 42, 26, 47)$, $(0, 41, 24, 46)$, $(0, 40, 22, 45)$, $(0, 39, 20, 44)$, $(0, 38, 18, 43)$, $(0, 53, 27, 59)$, $(0, 52, 25, 58)$, $(0, 51, 23, 57)$, $(0, 50, 21, 56)$, $(0, 49, 19, 55)$, $(0, 48, 17, 54)$, $(0, 3, 2, 6)$, $(0, 5, 7)$.

For $s = 15$, use $(0, 55, 1, 59)$, $(0, 56, 4, 57)$, $(0, 60, 10, 61)$, $(0, 28, 13, 36)$, $(0, 62, 14, 63)$, $(0, 29, 15, 37)$, $(0, 30, 17, 38)$, $(0, 44, 18, 45)$, $(0, 31, 19, 39)$, $(0, 32, 21, 40)$, $(0, 46, 22, 47)$, $(0, 33, 23, 41)$, $(0, 34, 25, 42)$, $(0, 35, 27, 43)$, $(0, 3, 2, 6)$, $(0, 5, 7)$.

For $s = 16$, use $(0, 44, 36, 51)$, $(0, 43, 34, 50)$, $(0, 42, 32, 49)$, $(0, 41, 30, 48)$, $(0, 40, 28, 47)$, $(0, 39, 26, 46)$, $(0, 38, 24, 45)$, $(0, 59, 37, 67)$, $(0, 58, 35, 66)$, $(0, 57, 33, 65)$, $(0, 56, 31, 64)$, $(0, 55, 29, 63)$, $(0, 54, 27, 62)$, $(0, 53, 25, 61)$, $(0, 52, 23, 60)$, $(0, 3, 2, 6)$, $(0, 5, 7)$.

For $s = 17$, use $(0, 48, 36, 55)$, $(0, 47, 34, 54)$, $(0, 46, 32, 53)$, $(0, 45, 30, 52)$, $(0, 44, 28, 51)$, $(0, 43, 26, 50)$, $(0, 42, 24, 49)$, $(0, 63, 37, 71)$, $(0, 62, 35, 70)$, $(0, 61, 33, 69)$, $(0, 60, 31, 68)$, $(0, 59, 29, 67)$, $(0, 58, 27, 66)$, $(0, 57, 25, 65)$, $(0, 56, 23, 64)$, $(0, 3, 2, 6)$, $(0, 5, 7)$.

For $s = 18$, use $(0, 52, 36, 59)$, $(0, 51, 34, 58)$, $(0, 50, 32, 57)$, $(0, 49, 30, 56)$, $(0, 48, 28, 55)$, $(0, 47, 26, 54)$, $(0, 46, 24, 53)$, $(0, 67, 37, 75)$, $(0, 66, 35, 74)$, $(0, 65, 33, 73)$, $(0, 64, 31, 72)$, $(0, 63, 29, 71)$, $(0, 62, 27, 70)$, $(0, 61, 25, 69)$, $(0, 60, 23, 68)$, $(0, 3, 2, 6)$, $(0, 5, 7)$.

For $s = 19$, use $(0, 56, 36, 63)$, $(0, 55, 34, 62)$, $(0, 54, 32, 61)$, $(0, 53, 30, 60)$, $(0, 52, 28, 59)$, $(0, 51, 26, 58)$, $(0, 50, 24, 57)$, $(0, 71, 37, 79)$, $(0, 70, 35, 78)$, $(0, 69, 33, 77)$, $(0, 68, 31, 76)$, $(0, 67, 29, 75)$, $(0, 66, 27, 74)$, $(0, 65, 25, 73)$, $(0, 64, 23, 72)$, $(0, 3, 2, 6)$, $(0, 5, 7)$.

For $s = 20$, use $(0, 61, 1, 80)$, $(0, 81, 4, 82)$, $(0, 73, 9, 83)$, $(0, 65, 10, 69)$, $(0, 66, 12, 70)$, $(0, 75, 13, 76)$, $(0, 67, 14, 71)$, $(0, 68, 16, 72)$, $(0, 40, 17, 46)$, $(0, 30, 18, 35)$, $(0, 41, 19, 47)$, $(0, 31, 20, 36)$, $(0, 42, 21, 48)$, $(0, 32, 22, 37)$, $(0, 43, 23, 49)$, $(0, 33, 24, 38)$, $(0, 44, 25, 50)$, $(0, 34, 26, 39)$, $(0, 45, 27, 51)$, $(0, 3, 2, 6)$, $(0, 5, 7)$.

Appendix C

We include here near graceful labellings of $C_3^2C_4^s$, were where $1 \leq s \leq 20$.

For $s = 1$, use $(0, 5, 2, 6)$, $(0, 7, 8)$, $(0, 9, 11)$.

For $s = 2$, use $(0, 10, 1, 12)$, $(0, 5, 2, 6)$, $(0, 7, 8)$, $(0, 15, 13)$.

For $s = 3$, use $(0, 12, 1, 16)$, $(0, 13, 4, 14)$, $(0, 5, 2, 6)$,$(0, 7, 8)$, $(0, 19, 17)$.

For $s = 4$, use $(0, 14, 1, 20)$, $(0, 15, 4, 16)$, $(0, 17, 8, 18)$, $(0, 5, 2, 6)$, $(0, 7, 8)$, $(0, 23, 21)$.

For $s = 5$, use $(0, 16, 1, 20)$, $(0, 17, 4, 18)$, $(0, 21, 11, 23)$, $(0, 22, 13, 24)$, $(0, 5, 2, 6)$, $(0, 7, 8)$, $(0, 27, 25)$.

For $s = 6$, use $(0, 18, 1, 22)$, $(0, 19, 4, 20)$, $(0, 23, 12, 26)$, $(0, 24, 14, 27)$, $(0, 25, 16, 28)$, $(0, 5, 2, 6)$, $(0, 7, 8)$, $(0, 31, 29)$.

For $s = 7$, use $(0, 20, 1, 22)$, $(0, 23, 10, 28)$, $(0, 24, 12, 29)$, $(0, 25, 14, 30)$, $(0, 26, 16, 31)$, $(0, 27, 18, 32)$, $(0, 5, 2, 6)$, $(0, 7, 8)$, $(0, 33, 35)$.

For $s = 8$, use $(0, 25, 16, 28)$, $(0, 24, 14, 27)$, $(0, 23, 12, 26)$, $(0, 32, 17, 36)$, $(0, 31, 15, 35)$, $(0, 30, 13, 34)$, $(0, 29, 11, 33)$, $(0, 5, 2, 6)$, $(0, 7, 8)$, $(0, 39, 37)$.

For $s = 9$, use $(0, 24, 1, 40)$, $(0, 25, 10, 32)$, $(0, 26, 12, 33)$, $(0, 27, 14, 34)$, $(0, 28, 16, 35)$, $(0, 29, 18, 36)$, $(0, 30, 20, 37)$, $(0, 31, 22, 38)$, $(0, 5, 2, 6)$, $(0, 7, 8)$, $(0, 43, 41)$.

For $s = 10$, use $(0, 24, 1, 42)$, $(0, 43, 4, 44)$, $(0, 25, 10, 32)$, $(0, 26, 12, 33)$, $(0, 27, 14, 34)$, $(0, 28, 16, 35)$, $(0, 29, 18, 36)$, $(0, 30, 20, 37)$, $(0, 31, 22, 38)$, $(0, 5, 2, 6)$, $(0, 7, 8)$,
$(0, 47, 45)$.

For $s = 11$, use $(0, 29, 13, 37)$, $(0, 30, 15, 38)$, $(0, 31, 17, 39)$, $(0, 45, 18, 46)$, $(0, 32, 19, 40)$, $(0, 33, 21, 41)$, $(0, 47, 22, 48)$, $(0, 34, 23, 42)$, $(0, 35, 25, 43)$, $(0, 36, 27, 44)$, $(0, 5, 2, 6)$, $(0, 7, 8)$, $(0, 51, 49)$.

For $s = 12$, use $(0, 35, 26, 40)$, $(0, 34, 24, 39)$, $(0, 33, 22, 38)$, $(0, 32, 20, 37)$, $(0, 31, 18, 36)$, $(0, 46, 27, 52)$, $(0, 45, 25, 51)$, $(0, 44, 23, 50)$, $(0, 43, 21, 49)$, $(0, 42, 19, 48)$, $(0, 41, 17, 47)$, $(0, 5, 2, 6)$, $(0, 7, 8)$, $(0, 55, 53)$.

For $s = 13$, use $(0, 39, 26, 44)$, $(0, 38, 24, 43)$, $(0, 37, 22, 42)$, $(0, 36, 20, 41)$, $(0, 35, 18, 40)$, $(0, 50, 27, 56)$, $(0, 49, 25, 55)$, $(0, 48, 23, 54)$, $(0, 47, 21, 53)$,

$(0,46,19,52)$, $(0,45,17,51)$, $(0,10,1,12)$, $(0,5,2,6)$, $(0,7,8)$, $(0,59,57)$.

For $s=14$, use $(0,43,26,48)$, $(0,42,24,47)$, $(0,41,22,46)$, $(0,40,20,45)$, $(0,39,18,44)$, $(0,54,27,60)$, $(0,53,25,59)$, $(0,52,23,58)$, $(0,51,21,57)$, $(0,50,19,56)$, $(0,49,17,55)$, $(0,12,1,16)$, $(0,13,4,14)$, $(0,5,2,6)$, $(0,7,8)$, $(0,63,61)$.

For $s=15$, use $(0,56,1,60)$, $(0,57,4,58)$, $(0,61,10,62)$, $(0,29,13,37)$, $(0,63,14,64)$, $(0,30,15,38)$, $(0,31,17,39)$, $(0,45,18,46)$, $(0,32,19,40)$, $(0,33,21,41)$, $(0,47,22,48)$, $(0,34,23,42)$, $(0,35,25,43)$, $(0,36,27,44)$, $(0,5,2,6)$, $(0,7,8)$, $(0,67,65)$.

For $s=16$, use $(0,45,36,52)$, $(0,44,34,51)$, $(0,43,32,50)$, $(0,42,30,49)$, $(0,41,28,48)$, $(0,40,26,47)$, $(0,39,24,46)$, $(0,60,37,68)$, $(0,59,35,67)$, $(0,58,33,66)$, $(0,57,31,65)$, $(0,56,29,64)$, $(0,55,27,63)$, $(0,54,25,62)$, $(0,53,23,61)$, $(0,5,2,6)$, $(0,7,8)$, $(0,71,69)$.

For $s=17$, use $(0,49,36,56)$, $(0,48,34,55)$, $(0,47,32,54)$, $(0,46,30,53)$, $(0,45,28,52)$, $(0,44,26,51)$, $(0,43,24,50)$, $(0,64,37,72)$, $(0,63,35,71)$, $(0,62,33,70)$, $(0,61,31,69)$, $(0,60,29,68)$, $(0,59,27,67)$, $(0,58,25,66)$, $(0,57,23,65)$, $(0,10,1,12)$, $(0,5,2,6)$, $(0,7,8)$, $(0,75,73)$.

For $s=18$, use $(0,53,36,60)$, $(0,52,34,59)$, $(0,51,32,58)$, $(0,50,30,57)$, $(0,49,28,56)$, $(0,48,26,55)$, $(0,47,24,54)$, $(0,68,37,76)$, $(0,67,35,75)$, $(0,66,33,74)$, $(0,65,31,73)$, $(0,64,29,72)$, $(0,63,27,71)$, $(0,62,25,70)$, $(0,61,23,69)$, $(0,12,1,16)$, $(0,13,4,14)$, $(0,5,2,6)$, $(0,7,8)$, $(0,79,77)$.

For $s=19$, use $(0,57,36,64)$, $(0,56,34,63)$, $(0,55,32,62)$, $(0,54,30,61)$, $(0,53,28,60)$, $(0,52,26,59)$, $(0,51,24,58)$, $(0,72,37,80)$, $(0,71,35,79)$, $(0,70,33,78)$, $(0,69,31,77)$, $(0,68,29,76)$, $(0,67,27,75)$, $(0,66,25,74)$, (0, 65, 23, 73), (0, 14, 1, 20), (0, 15, 4, 16), (0, 17, 8, 18), (0, 5, 2, 6),

(0, 7, 8), (0, 83, 81).

For $s = 20$, use (0, 62, 1, 81), (0, 82, 4, 83), (0, 74, 9, 84), (0, 66, 10, 70), (0, 67, 12, 71), (0, 76, 13, 77), (0, 68, 14, 72), (0, 69, 16, 73), (0, 41, 17, 47), (0, 31, 18, 36), (0, 42, 19, 48), (0, 32, 20, 37), (0, 43, 21, 49), (0, 33, 22, 38), (0, 44, 23, 50), (0, 34, 24, 39), (0, 45, 25, 51), (0, 35, 26, 40), (0, 46, 27, 52), (0, 5, 2, 6), (0, 7, 8), (0, 87, 85).

Appendix D

We include here near graceful labellings of $C_3^3C_4^s$, were where $1 \leq s \leq 19$.

For $s = 1$, use (0, 8, 2, 9), (0, 3, 5), (0, 11, 12), (0, 10, 14).

For $s = 2$, use (0, 11, 1, 13), (0, 8, 2, 9), (0, 3, 5), (0, 14, 18), (0, 15, 16).

For $s = 3$, use (0, 13, 1, 17), (0, 14, 4, 15), (0, 8, 2, 9), (0, 3, 5), (0, 22, 18), (0, 20, 19).

For $s = 4$, use (0, 15, 1, 19), (0, 16, 4, 17), (0, 20, 10, 21), (0, 8, 2, 9), (0, 3, 5), (0, 26, 22), (0, 24, 23).

For $s = 5$, use (0, 17, 1, 21), (0, 18, 4, 19), (0, 22, 11, 24), (0, 23, 13, 25), (0, 8, 2, 9), (0, 3, 5), (0, 30, 26), (0, 28, 27).

For $s = 6$, use (0, 19, 1, 23), (0, 20, 4, 21), (0, 24, 12, 27), (0, 25, 14, 28), (0, 26, 16, 29), (0, 8, 2, 9), (0, 3, 5), (0, 34, 30), (0, 32, 31).

For $s = 7$, use (0, 21, 1, 23), (0, 24, 10, 29), (0, 25, 12, 30),(0, 26, 14, 31), (0, 27, 16, 32), (0, 28, 18, 33), (0, 8, 2, 9), (0, 3, 5), (0, 38, 34), (0, 36, 35).

For $s = 8$, use (0, 26, 16, 29), (0, 25, 14, 28), (0, 24, 12, 27), (0, 33, 17, 37),

$(0, 32, 15, 36)$, $(0, 31, 13, 35)$, $(0, 30, 11, 34)$, $(0, 8, 2, 9)$, $(0, 3, 5)$, $(0, 42, 38)$, $(0, 40, 39)$.

For $s = 9$, use $(0, 25, 1, 41)$, $(0, 26, 10, 33)$, $(0, 27, 12, 34)$, $(0, 28, 14, 35)$, $(0, 29, 16, 36)$, $(0, 30, 18, 37)$, $(0, 31, 20, 38)$, $(0, 32, 22, 39)$, $(0, 8, 2, 9)$, $(0, 3, 5)$, $(0, 46, 42)$, $(0, 44, 43)$.

For $s = 10$, use $(0, 25, 1, 43)$, $(0, 44, 4, 45)$, $(0, 26, 10, 33)$, $(0, 27, 12, 34)$, $(0, 28, 14, 35)$, $(0, 29, 16, 36)$, $(0, 30, 18, 37)$, $(0, 31, 20, 38)$, $(0, 32, 22, 39)$, $(0, 8, 2, 9)$, $(0, 3, 5)$, $(0, 50, 46)$, $(0, 48, 47)$.

For $s = 11$, use $(0, 30, 13, 38)$, $(0, 31, 15, 39)$, $(0, 32, 17, 40)$, $(0, 46, 18, 47)$, $(0, 33, 19, 41)$, $(0, 34, 21, 42)$, $(0, 48, 22, 49)$, $(0, 35, 23, 43)$, $(0, 36, 25, 44)$, $(0, 37, 27, 45)$, $(0, 8, 2, 9)$, $(0, 3, 5)$,$(0, 54, 50)$, $(0, 52, 51)$.

For $s = 12$, use $(0, 36, 26, 41)$, $(0, 35, 24, 40)$, $(0, 34, 22, 39)$, $(0, 33, 20, 38)$, $(0, 32, 18, 37)$, $(0, 47, 27, 53)$, $(0, 46, 25, 52)$, $(0, 45, 23, 51)$, $(0, 44, 21, 50)$, $(0, 43, 19, 49)$, $(0, 42, 17, 48)$, $(0, 8, 2, 9)$, $(0, 3, 5)$, $(0, 58, 54)$, $(0, 56, 55)$.

For $s = 13$, use $(0, 40, 26, 45)$, $(0, 39, 24, 44)$, $(0, 38, 22, 43)$, $(0, 37, 20, 42)$, $(0, 36, 18, 41)$, $(0, 51, 27, 57)$, $(0, 50, 25, 56)$, $(0, 49, 23, 55)$, $(0, 48, 21, 54)$, $(0, 47, 19, 53)$, $(0, 46, 17, 52)$, $(0, 11, 1, 13)$, $(0, 8, 2, 9)$, $(0, 3, 5)$, $(0, 59, 60)$, $(0, 58, 62)$.

For $s = 14$, use $(0, 55, 1, 59)$, $(0, 56, 4, 57)$, $(0, 60, 10, 61)$, $(0, 30, 13, 38)$, $(0, 31, 15, 39)$, $(0, 32, 17, 40)$, $(0, 46, 18, 47)$, $(0, 33, 19, 41)$, $(0, 34, 21, 42)$, $(0, 48, 22, 49)$, $(0, 35, 23, 43)$, $(0, 36, 25, 44)$, $(0, 37, 27, 45)$, $(0, 8, 2, 9)$, $(0, 3, 5)$, $(0, 66, 62)$, $(0, 64, 63)$.

For $s = 15$, use $(0, 57, 1, 61)$, $(0, 58, 4, 59)$, $(0, 62, 10, 63)$, $(0, 30, 13, 38)$, $(0, 64, 14, 65)$, $(0, 31, 15, 39)$, $(0, 32, 17, 40)$, $(0, 46, 18, 47)$, $(0, 33, 19, 41)$, $(0, 34, 21, 42)$, $(0, 48, 22, 49)$, $(0, 35, 23, 43)$, $(0, 36, 25, 44)$, $(0, 37, 27, 45)$,

$(0,8,2,9)$, $(0,3,5)$, $(0,70,66)$, $(0,68,67)$.

For $s = 16$, use $(0,46,36,53)$, $(0,45,34,52)$, $(0,44,32,51)$, $(0,43,30,50)$, $(0,42,28,49)$, $(0,41,26,48)$, $(0,40,24,47)$, $(0,61,37,69)$, $(0,60,35,68)$, $(0,59,33,67)$, $(0,58,31,66)$, $(0,57,29,65)$, $(0,56,27,64)$, $(0,55,25,63)$, $(0,54,23,62)$, $(0,8,2,9)$, $(0,3,5)$, $(0,71,72)$, $(0,70,74)$.

For $s = 17$, use $(0,50,36,57)$, $(0,49,34,56)$, $(0,48,32,55)$, $(0,47,30,54)$, $(0,46,28,53)$, $(0,45,26,52)$, $(0,44,24,51)$, $(0,65,37,73)$, $(0,64,35,72)$, $(0,63,33,71)$, $(0,62,31,70)$, $(0,61,29,69)$, $(0,60,27,68)$, $(0,59,25,67)$, $(0,58,23,66)$, $(0,11,1,13)$, $(0,8,2,9)$, $(0,3,5)$, $(0,78,74)$, $(0,76,75)$.

For $s = 18$, use $(0,54,36,61)$, $(0,53,34,60)$, $(0,52,32,59)$, $(0,51,30,58)$, $(0,50,28,57)$, $(0,49,26,56)$, $(0,48,24,55)$, $(0,69,37,77)$, $(0,68,35,76)$, $(0,67,33,75)$, $(0,66,31,74)$, $(0,65,29,73)$, $(0,64,27,72)$, $(0,63,25,71)$, $(0,62,23,70)$, $(0,13,1,17)$, $(0,14,4,15)$, $(0,8,2,9)$, $(0,3,5)$, $(0,78,82)$, $(0,79,80)$.

For $s = 19$, use $(0,58,36,65)$, $(0,57,34,64)$, $(0,56,32,63)$, $(0,55,30,62)$, $(0,54,28,61)$, $(0,53,26,60)$, $(0,52,24,59)$, $(0,73,37,81)$, $(0,72,35,80)$, $(0,71,33,79)$, $(0,70,31,78)$, $(0,69,29,77)$, $(0,68,27,76)$, $(0,67,25,75)$, $(0,66,23,74)$, $(0,15,1,19)$, $(0,16,4,17)$, $(0,20,10,21)$, $(0,8,2,9)$, $(0,3,5)$, $(0,82,86)$, $(0,84,83)$.

Chapter 4

Graceful Labellings of Various Cyclic Snakes

4.1 Introduction

Let $G = (V, E)$ be a graph with m edges. Let f be a labelling defined from $V(G)$ to $\{0, 1, 2, \ldots, m\}$ and let g be the induced edge labelling defined from $E(G)$ to $\{1, 2, \ldots, m\}$ given by $g(uv) = |f(u) - f(v)|$, for all $uv \in E$. The labelling f is *graceful* if f is an injective mapping and g is a bijection. If a graph G has a graceful labelling, then it is graceful.

Alternatively, let f be defined from $V(G)$ to $\{0, 1, 2, \ldots, m + 1\}$ and let g be the induced edge labelling defined from $E(G)$ to A, where A is $\{1, 2, \ldots, m - 1, m\}$ or $\{1, 2, \ldots, m - 1, m + 1\}$ given by $g(uv) = |f(u) - f(v)|$, for all $uv \in E$. Then f is *near graceful* if f is an injective mapping and g is a bijection. If a graph G has a near graceful labelling, then it is near graceful. In this chapter, every near graceful labelling we find will omit the vertex label m and the edge label m; that is the codomain of f will be $\{1, 2, \ldots, m - 1, m + 1\}$ and the codomain of g will be $\{1, 2, \ldots, m - 1, m + 1\}$.

A *cyclic snake* is a connected graph whose block-cutpoint graph is a path and each of the blocks is isomorphic to a fixed cycle. We define kC_n to be a cyclic snake with k blocks each of which is C_n. The *string* of a kC_n is a sequence of integers $(d_1, d_2, d_3, \dots, d_{k-2})$ where d_i is the distance between the ith and $(i+1)$th cut vertex, counting cut vertices from one end of the snake to the other. Certainly, for fixed n and k, the string is uniquely determined by the snake and vice versa. Note that if $k = 1$ or $k = 2$ the snake kC_n has no string. A kC_n is *linear* if all entries in its string are $\lfloor \frac{n}{2} \rfloor$, and it is *even* if all entries in its string are even numbers.

In [5], Rosa showed that all cycles C_n with $n \equiv 0$ or $3 \pmod 4$ are graceful. Further, he introduced a necessary condition for an Eulerian graph to be graceful, namely if G is a graceful Eulerian graph with n edges, then $n \equiv 0$ or $3 \pmod 4$. A kC_n is an Eulerian graph, and hence graceful only if $kn \equiv 0$ or $3 \pmod 4$.

Moulton, in [4], proved that a graceful labelling exists for every kC_3. Barrientos, in [1], proved that the cycle C_n has a near graceful labelling if and only if $n \equiv 1$ or $2 \pmod 4$. This chapter also showed that a graceful labelling exists for every kC_4, and for particular cases of snakes for C_6, C_8, and C_{12}. A complete survey of graph labellings is presented in *A Dynamic Survey of Graph Labelling* [2].

We define a variable snake, $n_1C_{m_1}n_2C_{m_2}\dots n_iC_{m_i}$, to be a combination of different $n_jC_{m_j}$, where $n_jC_{m_j}$ is connected with $n_{j+1}C_{m_{j+1}}$ by identifying a vertex in the last cycle of the $n_jC_{m_j}$ with a vertex in the first cycle of the $n_{j+1}C_{m_{j+1}}$ (other than the cut vertex). The string for a variable snake is similar to the string for a kC_n.

We will represent all the cycle labellings in this chapter as n-tuples, with the overline elements indicating the cut vertices, when necessary.

In Figure 4.1, we have a near gracefully labelled $5C_6$ with string $(3, 1, 2)$. We can represent the labelling of the $5C_6$ in Figure 4.1 in five 6-tuples as follows: $(20, 16, 17, 15, 18, \overline{13})$, $(\overline{13}, 21, 11, \overline{22}, 10, 19)$, $(\overline{22}, \overline{6}, 25, 8, 23, 9)$, $(\overline{6}, 26, 4, 27, \overline{3}, 24)$, $(\overline{3}, 29, 2, 31, 0, 28)$.

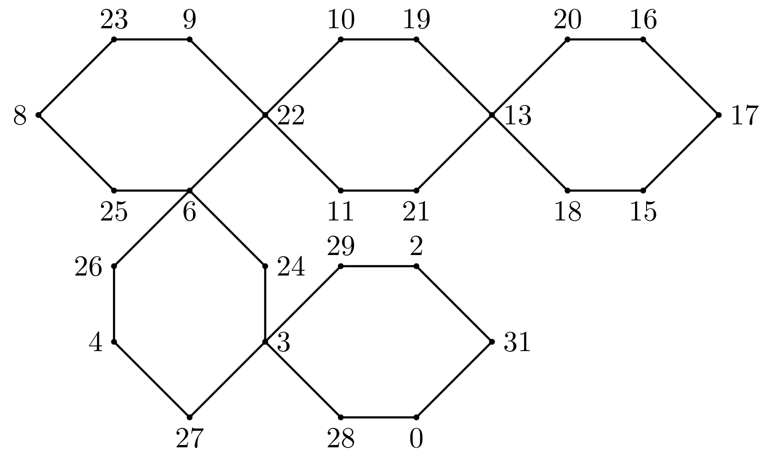

Figure 4.1: A near gracefully labelled $5C_6$.

In [5], Rosa also introduced an α-labelling. An *α-labelling* of a graph G is a graceful labelling with an extra condition which is there exists an integer w such that for any edge $uv \in G$, either $f(u) \leq w < f(v)$ or $f(v) \leq w < f(u)$. Any graph with an α-labelling is necessarily bipartite.

We would also like to introduce an analogue of near graceful labellings. An *$\hat{\alpha}$-labelling* of a graph G is a near graceful labelling with an extra condition which is there exists an integer w such that for any edge $uv \in G$, either $f(u) \leq w < f(v)$ or $f(v) \leq w < f(u)$. Thus the snake in Figure 4.1 has an $\hat{\alpha}$-labelling with $w = 16$. In fact, by the nature of our constructions, the main results of this chapter could be given in terms of α- or $\hat{\alpha}$-labellings. While noting this to be true, we choose instead to state our results in the more familiar language of graceful and near graceful labellings.

In Lemmas 4.1.1 to 4.1.3, we give some results about (not necessarily graceful) labellings that will be useful later. In this chapter, we adopt the convention that 0 is a natural number. Then, when we write $[x, y]$ with $x, y \in \mathbb{N}$ and $x < y$, we are indicating the set $\{z \in \mathbb{N} | x \leq z \leq y\}$.

If G is any graph and f is any labelling G then we can relabel G by adding a constant c, $h(v) = f(v) + c$. This technique preserves edge labels.

Lemma 4.1.1. *If c is an arbitrary integer and f is a labelling of a graph $G = (V, E)$, defined by $f : V(G) \to [0, m]$ then $h : V \to [c, c + m]$ defined by $h(v) = f(v) + c$, is a labelling that preserves edge labels.*

Proof. If $v_1v_2 \in E$ and f is a labelling then $f(v_1v_2) = |f(v_1) - f(v_2)|$. By definition, $h(v_1v_2) = |h(v_1) - h(v_2)| = |(f(v_1) + c) - (f(v_2) + c)| = |f(v_1) - f(v_2)| = f(v_1v_2)$. Therefore, h preserves edge labels. □

Lemma 4.1.2. *If c is an arbitrary integer and f is a labelling of a graph $G = (V, E)$, defined by $f : V(G) \to [0, m]$ then $h : V \to [c - m, c]$ defined by $h(v) = c - f(v)$, is a labelling that preserves edge labels. Further if f is graceful and $c = m$, then $h(v) = m - f(v)$ is graceful.*

Proof. This proof follows the same argument as Lemma 4.1.1.

If f is graceful and $c = m$, then $h(v) = m - f(v)$ is graceful. Let $v_1, v_2 \in V$ such that $h(v_1) = h(v_2)$. Then $m - f(v_1) = m - f(v_2)$ which implies $f(v_1) = f(v_2)$; since f is injective, then $v_1 = v_2$. Therefore, h is injective. Since h is also edge-preserving, it is graceful. □

In Lemma 4.1.3 we present a similar result for near graceful labelling; we omit 1 from the range of f and the domain of h, because if $f(v) = 1$ then $h(v) = m$, and this would contradict the definition of a near graceful labelling.

Lemma 4.1.3. *If f is a near graceful labelling of a graph $G = (V, E)$, $f : V(G) \to \{0, 2, 3, \ldots, m - 1, m + 1\}$ then $h : V(G) \to \{0, 2, 3, \ldots, m - 1, m + 1\}$ defined by $h(v) = (m + 1) - f(v)$ is a near graceful labelling.*

Proof. This proof follows the same argument as Lemma 4.1.2. □

We describe the technique in Lemmas 4.1.2 and 4.1.3 as taking the complement of a (near) graceful labelling.

In [5] Rosa introduced a graceful labelling for C_{4n} with $n \geq 1$.

Lemma 4.1.4. *[5] Let C_{4n} be a cycle with $4n$ edges and vertices v_i, for $1 \leq i \leq 4n$. Then the following labelling f shows that C_{4n} is graceful:*

$$f(v_i) = \begin{cases} \frac{i-1}{2} & \text{if } i \text{ is odd,} \\ 4n+1-\frac{i}{2} & \text{if } i \text{ is even, } i \leq \frac{t}{2}, \\ 4n-\frac{i}{2} & \text{if } i \text{ is even, } i > \frac{t}{2}. \end{cases} \tag{4.1}$$

Barrientos in [1] obtained the following results.

Theorem 4.1.5. *[1] The kC_4 has a graceful labelling for any string.*

Theorem 4.1.6. *[1] The linear kC_6 is near graceful if k is odd and graceful if k is even.*

Theorem 4.1.7. *[1] The even kC_8 and kC_{12} are graceful graphs.*

Theorem 4.1.8. *[1] The even kC_{4n} with string $(d_1, d_2, \ldots, d_{k-2})$, where $d_i \in \{2, 4\}$, has a graceful labelling.*

Theorem 4.1.9. *[1] The even kC_{4n}, $4 \leq n \leq 5$, with string $(d_1, d_2, \ldots, d_{k-2})$, where $d_i \in \{2, 4, 2n\}$ has a graceful labelling.*

In this chapter, we introduce a new sufficiency condition to get a graceful labelling for every kC_{4n}. Then, we extend this result to variable $n_1C_{m_1}n_2C_{m_2}\ldots n_iC_{m_i}$. Further, we extend the results in Theorem 4.1.6 to 4.1.9 on (near) gracefully labelled kC_n where $n = 6, 8, 12, 16, 20, 24$ for all possible strings. Also, we present new results on the (near) graceful labelling of kC_n where $n = 10, 14$ and $k > 1$.

4.2 Graceful Labelling of kC_m for $m \equiv 0 \pmod 4$

Since the size of a kC_{4n} is $4kn \equiv 0 \pmod 4$, we have the potential to find a graceful labelling for any kC_{4n}. In Theorem 4.2.1, we give a new sufficient condition which, when satisfied, shows there is a graceful labelling of a kC_{4n} for any string.

For fixed even t and an arbitrary fixed positive integer s, an *s, t-useful cycle with even distance* d is a t-cycle with vertices labelled from $\left[0, \frac{t}{2} - 1\right] \cup \left[st - \frac{t}{2}, st\right]$ and edge labels $[st - t + 1, st]$, with a vertex labelled 0 and a vertex labelled $\frac{t}{2} - 1$ at distance d. Similarly, an *s, t-odd useful cycle with odd distance* d is a t-cycle with vertices labelled from $\left[0, \frac{t}{2}\right] \cup \left[st - \frac{t}{2} + 1, st\right]$ and edge labels $[st - t + 1, st]$, with a vertex labelled 0 and vertex labelled $st - \frac{t}{2} + 1$ at distance d.

A *complete s, t-useful cycle set* is a set of s, t-useful cycles of even and odd distances, the union of whose distances is $\left\{1, 2, \ldots, \frac{t}{2}\right\}$. Let C_t^d be an element of a complete s, t-useful cycle set such that the distance between the vertices labelled 0 and $\frac{t}{2} - 1$ is d if d is even, and the distance between the vertices labelled 0 and $st - \frac{t}{2} + 1$ is d if d is odd.

Theorem 4.2.1. *If there is a complete $s, 4n$-useful cycle set with $s \geq 1$, then there exists a graceful labelling of any kC_{4n}.*

Proof. To prove this result, we will in fact prove a slightly more complex result: namely, that given a complete $s, 4n$-useful cycle set with $s \geq 1$ and for any $k \geq 1$, then there exists a graceful labelling of any kC_{4n} with 0 in the last cycle in any position except the cut vertex.

If 0 is in the last cycle of a kC_{4n}, then up to symmetry its position is uniquely determined by the distance from the last cut vertex. These distances, d, can only be $1, 2, \ldots, 2n$.

We proceed by induction on k. For $k = 1$ use the graceful labelling for C_t in

Lemma 4.1.4, letting $t = 4n$. The vertex labels for this graceful $1C_{4n}$ are a subset of $[0, 4n]$ and the edge labels are exactly $[1, 4n]$.

For $k = 2$, we label $2C_{4n}$ while obtaining a vertex with label 0 at even distance d from the unique cut vertex as follows. We label one cycle with the labelling used for $1C_{4n}$, with $2n - 1$ added to each vertex, so that the vertex formerly labelled 0, and now labelled $2n - 1$, is the cut vertex of $2C_{4n}$. The vertices have been labelled from the set $[2n - 1, 6n - 1]$ and, by Lemma 4.1.1, the edge labels are $[1, 4n]$. Apply the labelling C_{4n}^{d} to the second cycle where d is even, with the cut vertex receiving the label $2n - 1$. Then this labelling of $2C_{4n}$ has all vertex labels from $[0, 8n]$, and the edge labels are exactly $[1, 8n]$, with no repeated vertex or edge label. That is, it is a graceful labelling of $2C_{4n}$ with the vertex labelled 0 at even distance d from the cut vertex.

In the same way we label $2C_{4n}$ while obtaining a vertex with label 0 at odd distance d from the unique cut vertex. We label one cycle with the labelling used for $1C_{4n}$, but replace each vertex label x by $6n + 1 - x$. The vertices have been labelled from the set $[2n + 1, 6n + 1]$ and, by Lemma 4.1.1, the edge labels are $[1, 4n]$. Apply the labelling C_{4n}^{d} to the second cycle where d is odd, with the cut vertex receiving the label $6n + 1$. Then this labelling of $2C_{4n}$ has all vertex labels from $[0, 8n]$, and the edge labels are exactly $[1, 8n]$, with no repeated vertex or edge label. That is, it is a graceful labelling of $2C_{4n}$ with the vertex labelled 0 at odd distance d from the cut vertex.

Consider an arbitrary kC_{4n} with $k \geq 3$, with the last entry in its string d_{k-2}. Let G be the $(k - 1)C_{4n}$ obtained by deleting a last cycle from this kC_{4n}. By the induction hypothesis, there is a graceful labelling of G with a 0 on the vertex distance d_{k-2} from the previous cut vertex. This labelling has vertex labels that are a subset of $[0, 4nk - 4n]$ and the edge labels are exactly $[1, 4kn - 4n]$.

We label kC_{4n} obtaining a vertex with label 0 at even distance d from the unique cut vertex as follows. For the first $k-1$ cycles, use the labelling of G and add $2n-1$ to each vertex label, so that the final cut vertex receives label $2n-1$. Thus, the vertices have been labelled from the set $[2n-1, 4kn-2n-1]$ and by Lemma 4.1.1, the edge labels are $[1, 4kn-4n]$. Apply the labelling C_{4n}^d to the final cycle, with the cut vertex receiving label $2n-1$. Then this labelling of kC_{4n} has all vertices labelled from $[0, 4kn]$, and the edge labels are exactly $[1, 4kn]$, with no repeated vertex or edge label. Thus, there is a graceful labelling of kC_{4n} with the vertex labelled 0 at even distance d from the cut vertex.

We label kC_{4n} obtaining a vertex with label 0 at odd distance d from the unique cut vertex as follows. For the first $k-1$ cycles, use the labelling of G with a 0 on the vertex distance d_{k-2} from the previous cut vertex. Then subtract each vertex label from $4kn-2n+1$. Thus, the vertices have been labelled from the set $[2n+1, 4kn-2n+1]$ and by Lemma 4.1.1, the edge labels are $[1, 4kn-4n]$. Apply the labelling C_{4n}^d to the final cycle, with the cut vertex receiving label $4kn-2n+1$. Then this labelling of kC_{4n} has all vertices labelled from $[0, 4kn]$, and the edge labels are exactly $[1, 4kn]$, with no repeated vertex or edge label. Thus, there is a graceful labelling of kC_{4n} with the vertex labelled 0 at odd distance d from the cut vertex. $\square$

In Table 4.1, we give labellings for C_{4n}^{2j} where $1 \leq j \leq n$ and $n \leq 6$. For each C_{4n}^{2j}, we can use Lemma 4.1.2 with $c = t$ to obtain C_{4n}^{2j-1}. Then, $\{C_{4n}^m \,|1 \leq m \leq 2n\}$ is a complete $s, 4n$-useful cycle set. Combining these sets with Theorem 4.2.1, we obtain the following corollary.

Corollary 4.2.2. *If* $1 \leq n \leq 6$ *and* $k \geq 1$ *then every* kC_{4n} *is graceful.*

By adding $t-4n$ to all even vertices in Equation (4.1) in Lemma 4.1.4 we obtain

	Labelling	
C_4^2	$(0, t, \overline{1}, t-2)$	[1, 3]
C_8^2	$(0, t, \overline{3}, t-4, 2, t-3, 1, t-1)$	
C_8^4	$(0, t, 1, t-1, \overline{3}, t-4, 2, t-3)$	[1]
C_{12}^2	$(0, t, \overline{5}, t-6, 4, t-5, 3, t-4, 2, t-2, 1, t-1)$	[1]
C_{12}^4	$(0, t, 4, t-5, \overline{5}, t-6, 2, t-4, 3, t-2, 1, t-1)$	
C_{12}^6	$(0, t, 1, t-1, 2, t-6, \overline{5}, t-5, 4, t-2, 3, t-4)$	[1]
C_{16}^2	$(0, t, \overline{7}, t-8, 6, t-7, 5, t-6, 4, t-5, 3, t-3, 2, t-2, 1, t-1)$	
C_{16}^4	$(0, t, 1, t-1, \overline{7}, t-8, 6, t-7, 5, t-6, 4, t-5, 2, t-2, 3, t-3)$	
C_{16}^6	$(0, t, 1, t-1, 2, t-2, \overline{7}, t-8, 6, t-7, 5, t-6, 4, t-3, 3, t-5)$	
C_{16}^8	$(0, t, 1, t-1, 2, t-2, 4, t-3, \overline{7}, t-8, 6, t-7, 5, t-6, 3, t-5)$	[1]
C_{20}^2	$(0, t, \overline{9}, t-10, 8, t-9, 7, t-8, 6, t-7, 5, t-6, 4, t-4, 3, t-3, 2, t-2, 1, t-1)$	
C_{20}^4	$(0, t, 1, t-1, \overline{9}, t-10, 8, t-9, 7, t-8, 6, t-7, 5, t-6, 3, t-4, 4, t-2, 2, t-3)$	
C_{20}^6	$(0, t, 1, t-1, 2, t-10, \overline{9}, t-9, 8, t-8, 7, t-7, 6, t-3, 5, t-6, 4, t-2, 3, t-4)$	
C_{20}^8	$(0, t, 1, t-4, 3, t-6, 8, t-9, \overline{9}, t-10, 6, t-7, 4, t-8, 7, t-3, 5, t-1, 2, t-2)$	
C_{20}^{10}	$(0, t, 1, t-1, 2, t-2, 3, t-3, 5, t-4, \overline{9}, t-10, 8, t-9, 7, t-8, 6, t-6, 4, t-7)$	[1]
C_{24}^2	$(0, t, \overline{11}, t-12, 10, t-11, 9, t-10, 8, t-9, 7, t-8, 6, t-7, 5, t-5, 4, t-4, 3, t-3, 2, t-2, 1, t-1)$	
C_{24}^4	$(0, t, 1, t-4, \overline{11}, t-12, 10, t-11, 9, t-10, 8, t-9, 7, t-7, 6, t-3, 5, t-5, 2, t-1, 3, t-8, 4, t-2)$	
C_{24}^6	$(0, t, 1, t-1, 2, t-12, \overline{11}, t-11, 10, t-10, 9, t-9, 8, t-8, 7, t-5, 6, t-7, 3, t-2, 4, t-3, 5, t-4)$	
C_{24}^8	$(0, t, 1, t-4, 2, t-1, 3, t-8, \overline{11}, t-12, 10, t-11, 9, t-7, 8, t-10, 7, t-5, 5, t-9, 4, t-3, 6, t-2)$	
C_{24}^{10}	$(0, t, 1, t-4, 2, t-1, 3, t-8, 4, t-5, \overline{11}, t-12, 10, t-11, 9, t-10, 8, t-9, 6, t-7, 7, t-3, 5, t-2)$	
C_{24}^{12}	$(0, t, 1, t-1, 2, t-2, 3, t-3, 4, t-5, 6, t-4, \overline{11}, t-12, 10, t-11, 9, t-10, 8, t-9, 7, t-7, 5, t-8)$	

Table 4.1: Useful labellings C_{4n}^{2j}, where $t = 4kn$.

another labelling.

$$g(v_i) = \begin{cases} \frac{i-1}{2} & \text{if } i \text{ is odd,} \\ t + 1 - \frac{i}{2} & \text{if } i \text{ is even,} \quad i \leq 2n, \\ t - \frac{i}{2} & \text{if } i \text{ is even,} \quad i > 2n. \end{cases} \tag{4.2}$$

From Equation (4.2) we obtain C_{4n}^2 and $n \geq 1$, since the distance between the labels 0 and $2n-1$ in C_{4n} is 2. The same labelling also works as C_{4n}^3, since the distance between the labels 0 and $t-(2n-1)$ is 3. Further, applying Lemma 4.1.2 with $c = t$ to the labelling C_{4n}^2 gives a new labelling with distance 4 between the labels 0 and $2n-1$, C_{4n}^4. Then, $S = \{C_{4n}^m \,|2 \leq m \leq 4\}$ is a $s, 4n$-useful cycle set (though not complete). Combining the set S with Theorem 4.2.1, we obtain the following theorem.

Theorem 4.2.3. *The snake kC_{4n} with string $(d_1, d_2, \ldots, d_{k-2})$ has a graceful labelling if $d_i \in \{2, 3, 4\}$ for all i.*

From the previous discussion we can gracefully label a variable snake made from any kC_{4n} with $n \leq 6$. As an example of a variable snake, consider $3C_8$, gracefully labelled via Theorem 4.2.1, with the vertices labelled from the set $[0, 24]$ and the edge labels $[1, 24]$. Then form a new labelling via Lemma 4.1.1 with $c = 5$, so that a vertex in the last cycle obtains the label 5. Then add any C_{12}^{2j} (from Table 4.1) to $3C_8$. We obtain a gracefully labelled $3C_8 1C_{12}$. More generally, if we have a complete $k, 4i$-useful cycle set for all $1 \leq i \leq n$, then, by combining these sets with Theorem 4.2.1, we obtain the following corollary.

Corollary 4.2.4. *If there is a complete $s, 4i$-useful cycle set with $s \geq 1$ for all $1 \leq i \leq n$, $j \geq 1$, and $1 \leq m_1, m_2, \ldots, m_j \leq n$, with $n_1, n_2, \ldots, n_j$ positive integers then every $n_1 C_{4m_1} n_2 C_{4m_2} \ldots n_j C_{4m_j}$ is graceful.*

Proof. We follow the same method as in the proof of Theorem 4.2.1. That is, we will proceed by induction on the number of cycles $k = n_1 + n_2 + \cdots + n_j$, and will prove the slightly more complex result that if there is a complete $s, 4i$-useful cycle set with $s \geq 1$ for all $1 \leq i \leq n$, $j \geq 1$, and $1 \leq m_1, m_2, \ldots, m_j \leq n$, with positive integers $n_1, n_2, \ldots, n_j$ then every $n_1 C_{4m_1} n_2 C_{4m_2} \ldots n_j C_{4m_j}$ is graceful with the label 0 in any position in the last cycle except the cut vertex. For $k = 1$ we have a $k, 4i$-useful cycle set from Theorem 4.2.1, so there exists a graceful labelling of any $1C_{4n}$ with 0 in any position in the last cycle.

Consider an arbitrary $n_1 C_{4m_1} n_2 C_{4m_2} \ldots n_j C_{4m_j}$ with $k > 1$ and $m = \sum_{l=1}^{j} n_l \, (4m_l)$, the total number of edges. Let G be the graph obtained by deleting a last $4m_j$

cycle from $n_1C_{4m_1}n_2C_{4m_2}\dots n_jC_{4m_j}$. By the induction hypothesis, there is a graceful labelling of G with a 0 on the vertex distance d_{k-2} from the previous cut vertex. This labelling has vertex labels that are a subset of $[0, m-4m_j]$ and the edge labels are exactly $[1, m-4m_j]$.

We label $n_1C_{4m_1}n_2C_{4m_2}\dots n_jC_{4m_j}$ obtaining a vertex with label 0 at even distance d from the last cut vertex as follows. For the first $k-1$ cycles, use the labelling of G and add $2m_j-1$ to each vertex label, so that the final cut vertex receives label $2m_j-1$. Thus, the vertices have been labelled from the set $[2m_j-1, m+2m_j-1]$ and by Lemma 4.1.1, the edge labels are $[1, m-4m_j]$.

Now the labelling of G has the label $2m_j-1$ at the last cycle. So apply the labelling of $C^d_{4m_j}$ to the last cycle of G, with the cut vertex receiving label $2m_j-1$ and 0 at position d (even distance) from the cut vertex. Then this labelling has all vertices labelled from $[0, m]$, and edge labels exactly $[1, m]$, with no repeated vertex or edge label. Thus, there is a graceful labelling of $n_1C_{4m_1}n_2C_{4m_2}\dots n_iC_{4m_j}$ with the vertex labelled 0 at even distance d from the cut vertex.

We label $n_1C_{4m_1}n_2C_{4m_2}\dots n_jC_{4m_j}$ to obtain a vertex with label 0 at odd distance d from the last cut vertex as follows. For the first $k-1$ cycles, use the labelling of G and subtract each vertex label from $m-2m_j+1$, so that the final cut vertex receives label $m-2m_j+1$. Thus, the vertices have been labelled from the set $[2m_j-1, m-2m_j+1]$ and by Lemma 4.1.1, the edge labels are $[1, m-2m_j+1]$.

Now the labelling of G has the label $m-2m_j+1$ at the last cycle. So apply the labelling of $C^d_{4m_j}$ to the last cycle of G, with the cut vertex receiving label $m-2m_j+1$ and 0 at position d (odd distance) from the cut vertex. Then this labelling has all vertices labelled from $[0, m]$, and edge labels exactly $[1, m]$, with no repeated vertex or edge label. Thus, there is a graceful labelling of $n_1C_{4m_1}n_2C_{4m_2}\dots, n_iC_{4m_j}$ with the vertex labelled 0 at odd distance d from the cut vertex. $\square$

4.3 Graceful Labelling of kC_m for $m \equiv 2 \pmod 4$

In Section 4.2, we proved that if there is a complete $s, 4n$-useful cycle set with $s \geq 1$, then there exists a graceful labelling of any kC_{4n} for $k \geq 1$. The next natural question is can we prove the same results for kC_{4n+2}?

In kC_{4n} we have the nice property that kC_{4n} is always graceful regardless of the parity of k. For kC_{4n+2} we will obtain graceful or near graceful labellings depending on the value of k, because the size of a kC_{4n+2} is $4kn + 2k$ which is congruent to 0 modulus 4 for k even and $4kn+2k$ which is congruent to 2 modulus 4 for k odd. Thus, we would essentially need to find two complete useful cycle sets for $4n+2$ because we are trying to change a graceful labelling to a near graceful one, or the reverse.

For kC_{4n} we obtained the complete cycle set by taking the complement as in Lemmas 4.1.2 to 4.1.3. Here for kC_{4n+2} we need to omit 1 from the labelling of the the near graceful cycles, because if we use 1 and take the complement we will obtain a labelling that is not near graceful. For example, if we take $(0, 7, 3, 1, 2, 5)$ as a labelling of $1C_6$, the complement would have 6 in the resulting vertex labelling and hence would not be near graceful. In this section we prove that there exists a (near) graceful labelling of any kC_6 (Theorem 4.3.1), because we found an analogue of a complete $k, 6$-useful cycle set in Table 4.2, and did not use 1 for the near graceful useful cycles. An exhaustive analysis shows that no labelling of C_{10} exists that uses the labels $[0, 5]$ and $[t-5, t+1] \setminus \{t\}$ that omits the label 1. Thus, despite their effectiveness in the $4n$-cycle case, complete cycle sets cannot help us label even kC_{10}.

In [1], Barrientos proved that the kC_4 has a graceful labelling with any string, as summarized in Theorem 4.1.5. Recall the result of Barrientos from Theorem 4.1.6: that the linear kC_6 is near graceful if k is odd and graceful if k is even. In Theorem 4.3.1 we prove (near) graceful labellings exist for any kC_6.

In Table 4.2 we see four labellings of C_6. The labellings C_6^a and C_6^b use edge labels $[6k-5, 6k+1]\backslash\{6k\}$. The labellings C_6^c and C_6^d use edge labels $[6k-6, 6k]\backslash\{6k-5\}$.

	Labelling
C_6^a	$\big(0, 6k+1, 2, 6k-1, \overline{3}, 6k-2\big)$
C_6^b	$\big(0, 6k+1, \overline{3}, 6k-2, 2, 6k-1\big)$
C_6^c	$\big(0, 6k, \overline{3}, 6k-3, 1, 6k-1\big)$
C_6^d	$\big(0, 6k, 1, 6k-1, \overline{3}, 6k-3\big)$

Table 4.2: Some useful labellings of C_6.

Theorem 4.3.1. *If $k \geq 1$ then there exists a (near) graceful labelling of any kC_6.*

Proof. As in proof of Theorem 4.2.1, we prove a slightly more complex result. Namely, we prove that if $k \geq 1$, then there exists a (near) graceful labelling of any kC_6 with 0 in the last cycle in any position except the cut vertex.

If 0 is in the last cycle of a kC_6, then up to symmetry its position is uniquely determined by the distance from the last cut vertex. These distances, d, can only be $1, 2$ or 3.

We proceed by induction on k. For $k = 1$, use C_6^a or C_6^b in Table 4.2 with $k = 1$ which will make it near graceful. For $k = 2$, use $\big(4, 7, 2, \overline{9}, 5, 6\big)$ and $\big(\overline{9}, 0, 12, 1, 11, 3\big)$ to obtain a labelling with $d = 1$; $\big(8, 5, 10, \overline{3}, 7, 6\big)$ and $\big(\overline{3}, 12, 0, 11, 1, 9\big)$ to obtain a labelling with $d = 2$; and $\big(4, 7, 2, \overline{9}, 5, 6\big)$ and $(9, 1, 11, 0, 12, 3)$ to obtain a labelling with $d = 3$.

Case 1: Consider an arbitrary kC_6 with $k \geq 4$ and k even, with the last entry in the string d_{k-2}. Let G be the $(k-1)C_6$ obtained by deleting a last cycle from this kC_6. By the induction hypothesis, there is a near graceful labelling of G with a 0 on the vertex distance d_{k-2} from the previous cut vertex. This labelling has vertex labels that are a subset of $[0, 6k-5] \setminus \{6k-6\}$ and the edge labels are exactly $[1, 6k-5] \setminus \{6k-6\}$.

We label kC_6 obtaining a vertex with label 0 at distance d from the cut vertex as follows. For the first $k-1$ cycles, use the labelling of G and add 3 to each vertex, so that the final cut vertex receives label 3. Thus, the vertices have been labelled from the set $[3, 6k-2] \setminus \{6k-3\}$ and by Lemma 4.1.1, the edge labels are $[1, 6k-5] \setminus \{6k-6\}$.

Apply the labelling C_6^c or C_6^d to the final cycle, with the cut vertex receiving label 3. Then this labelling of kC_6 has all vertices labelled from $[0, 6k]$, and the edge labels are exactly $[1, 6k]$, with no repeated vertex or edge label. Then there is a graceful labelling of kC_6 with 0 in the $d = 2$ position.

We obtain a labelling with 0 in the $d = 1$ position by using the previously discussed labelling, ending with the C_6^c-labelling in the last cycle, then applying Lemma 4.1.2. We obtain $d = 3$ by using the C_6^d-labelling in the last cycle, and applying Lemma 4.1.2.

Case 2: Consider an arbitrary kC_6 with $k \geq 3$ and k odd, with the last entry in the string d_{k-2}. We proceed in the same fashion as in Case 1, labelling all vertices except those in the final cycle, with vertices labelled from the set $[3, 6k-3]$ and edges labelled $[1, 6k-6]$.

Apply the labelling C_6^a or C_6^b to the final cycle, with the cut vertex receiving label 3. Then this labelling of kC_6 has all vertices labelled from $[0, 6k+1] \setminus \{1, 6k\}$, and the edge labels are exactly $[1, 6k+1] \setminus \{6k\}$, with no repeated vertex or edge label. Then there is a graceful labelling of kC_6 with 0 in the $d = 2$ position.

We obtain a labelling with 0 in the $d = 1$ position by using the previously discussed labelling, ending with the C_6^b-labelling in the last cycle, then applying Lemma 4.1.3. We obtain $d = 3$ by using the C_6^a-labelling in the last cycle, and applying Lemma 4.1.3.

□

In Theorem 4.3.2 we prove (near) graceful labellings exist for some kC_{10}. In Table 4.3 we see 7 labellings of C_{10}. The labellings of $C_{10}^a, C_{10}^e, C_{10}^g$, and C_{10}^h use edge labels $[10k-9, 10k+1]\backslash\{10k\}$. The labellings of $C_{10}^b, C_{10}^c, C_{10}^d$, and C_{10}^f use edge labels $[10k-10, 10k]\backslash\{10k-9\}$. Since we cannot rely on the uniformity of a complete cycle set, this theorem uses a variety of different techniques to achieve similar effects.

	Labelling
C_{10}^a	$(0, 10k+1, 4, 10k-2, 3, 10k-4, \overline{5}, 10k-3, 1, 10k-1)$
C_{10}^b	$(0, 10k, 1, 10k-7, 3, 10k-3, \overline{4}, 10k-1, 2, 10k-2)$
C_{10}^c	$(0, 10k, 2, 10k-4, \overline{4}, 10k-6, 1, 10k-2, 3, 10k-1)$
C_{10}^d	$(0, 10k+1, 3, 10k-2, 1, 10k-3, 4, 10k-4, \overline{5}, 10k-1)$
C_{10}^e	$(0, 10k, 2, 10k-2, 1, 10k-7, 3, 10k-3, \overline{4}, 10k-1)$
C_{10}^f	$(0, 10k+1, \overline{5}, 10k-4, 4, 10k-3, 3, 10k-2, 1, 10k-1)$
C_{10}^g	$(0, 10k, \overline{4}, 10k-6, 2, 10k-4, 3, 10k-2, 1, 10k-1)$

Table 4.3: Some useful labellings of C_{10}.

Theorem 4.3.2. *The kC_{10} $(k \geq 1)$ with string $(d_1, d_2, \ldots, d_{k-2})$ is graceful if k is even and near graceful if k is odd and one of the following is true:*

1. $d_i \in \{4, 5\}$ *if i is odd and* $d_i = 5$ *if i is even, where* $1 \leq i \leq k-2$,
2. $d_i \in \{3, 4\}$ *if i is odd and* $d_i = 4$ *if i is even, where* $1 \leq i \leq k-2$,
3. $d_i \in \{2, 3\}$ *if i is odd and* $d_i = 3$ *if i is even, where* $1 \leq i \leq k-2$, *or*
4. $d_i \in \{1, 2\}$ *if i is odd and* $d_i = 2$ *if i is even, where* $1 \leq i \leq k-2$.

Proof. The proof is similar to the proof of Theorem 4.3.1 with some changes to the relabelling technique we use on vertex and edge labels.

Case 1: $d_i \in \{4, 5\}$ if i is odd and $d_i = 5$ if i even.

As in the proof of Theorem 4.3.1, we prove that the kC_{10} $(k \geq 1)$ with string $(d_1, d_2, \ldots, d_{k-2})$ satisfying the condition of part one with 0 in the $d = 4$ or $d = 5$ position in the last cycle and near graceful if k is odd with $10k + 1$ in the $d = 5$

position in the last cycle.

We proceed by induction on k. For $k = 1$, use the labelling of C_{10}^a in Table 4.3 with $k = 1$ which will make it near graceful. For $k = 2$, use $\left(11, 8, 13, 9, \overline{16}, 5, 14, 6, 12, 10\right)$ and $\left(0, 20, 2, 18, 1, \overline{16}, 3, 17, 7, 19\right)$ to get $d = 5$, and take this labelling with Lemma 4.1.2 to get $d = 4$.

Case 1a: The proof is similar to the proof of Case 1 in the proof of Theorem 4.3.1. Consider kC_{10} to be an arbitrary snake with a string as indicated in the condition of Case 1 with $k \geq 4$ and k even. The labelling of $(k-1)C_{10}$ by the induction hypothesis has vertex labels that are a subset of $[0, 10k-9] \setminus \{2, 10k-10\}$ and the edge labels are exactly $[1, 10k-9] \setminus \{10k-10\}$ with $10k-9$ in the vertex distance d_{k-2} from the previous cut vertex.

We label kC_{10} as follows. For the first $k-1$ cycles, use the labelling of $(k-1)\, C_{10}$ obtained by induction and then subtract each vertex label from $10k-5$. Thus, the vertices have been labelled from the set $[4, 10k-5] \setminus \{5, 10k-7\}$ and by Lemma 4.1.2, the edge labels are $[1, 10k-9] \setminus \{10k-10\}$.

Apply the labelling C_{10}^b to the final cycle, with the cut vertex receiving label 4. Then this labelling of kC_{10} has all vertices labelled from $[0, 10k]$, and the edge labels are exactly $[1, 10k]$, with no repeated vertex or edge label. By induction, a graceful labelling of kC_{10} exists, with 0 in the $d = 4$ position. (Note that a possible conflict occurs as $(10k-5) - 2 = 10k-7$, however, in the labelling of the $(k-1)C_{10}$ no vertex is labelled 2, therefore we can use C_{10}^b without any restriction).

We obtain a labelling with 0 in the $d = 5$ position by using the previously discussed labelling, ending with the C_{10}^b-labelling in the last cycle, then applying Lemma 4.1.2.

Case 1b: Consider kC_{10} to be an arbitrary snake with a string as indicated in the condition of Case 1 with $k \geq 3$ and k odd. The proof is similar to the proof of Case 1a, but instead of subtracting each vertex label from $10k-5$, add 5 to each vertex and applying the labelling C_{10}^a to the final cycle, with the cut vertex receiving label 5. Then this labelling of kC_{10} has all vertices labelled from $[0, 10k] \setminus \{2\}$ and the edge labels are exactly $[1, 10k+1] \setminus \{10k\}$, with no repeated vertex or edge label. Then we obtain a near graceful labelling of kC_{10} with $10k+1$ in the $d=5$ position from the cut vertex.

Case 2: $d_i \in \{3, 4\}$ if i is odd and $d_i = 4$ if i is even.

As in the proof of Case 1, we prove that kC_{10} $(k \geq 1)$ with string $(d_1, d_2, \ldots, d_{k-2})$ satisfying the condition of part two with 0 in the $d=3$ or $d=4$ position in the last cycle and near graceful if k is odd with 0 in the $d=4$ position in the last cycle.

We proceed by induction on k. For $k=1$, use the labelling of C_{10}^a in Table 4.3 with $k=1$ which will make it near graceful. For $k=2$, use $\big(10, 9, 11, 5, 13, \overline{4}, 15, 8, 12, 7\big)$ and $\big(0, 20, 2, 16, \overline{4}, 14, 1, 18, 3, 19\big)$ to get $d=4$, and take this labelling with Lemma 4.1.2 to get $d=3$.

Case 2a: Consider kC_{10} to be an arbitrary snake with a string as indicated in the condition of Case 2 with $k \geq 4$ and k even. The proof is similar to the proof of Case 1a.

We label kC_{10} as follows. For the first $k-1$ cycles, use the labelling of $(k-1)\,C_{10}$ obtained by induction and then add 4 to each vertex label. Thus, the vertices have been labelled from the set $[4, 10k-5] \setminus \{6, 10k-6\}$ and by Lemma 4.1.1, the edge labels are $[1, 10k-9] \setminus \{10k-10\}$.

Apply the labelling C_{10}^c to the final cycle, with the cut vertex receiving label 4. Then there is a graceful labelling of kC_{10} and 0 in the $d=4$ position, from the cut vertex. (Note that a possible conflict occurs as $(10k-6)+4 = 10k-2$, however, in the labelling of the $(k-1)C_{10}$ no vertex is labelled $10k-2$, therefore we can use C_{10}^c without any restriction).

We obtain a labelling with 0 in the $d=3$ position by using the previously discussed labelling, ending with the C_{10}^c-labelling in the last cycle, then applying Lemma 4.1.2.

Case 2b: Consider kC_{10} to be an arbitrary snake with a string as indicated in the condition of Case 2 with $k \geq 3$ and k odd. The proof is similar to the proof of Case 1b, using the labelling of $(k-1)\,C_{10}$ and adding 5 to each vertex. Finally, apply the labelling C_{10}^a to the final cycle, with the cut vertex receiving label 5. Then we obtain a near graceful labelling of kC_{10} and 0 in the $d = 4$ position, relative to cut vertex.

Case 3: $d_i \in \{2, 3\}$ if i is odd and $d_i = 3$ if i is even.

We prove that the kC_{10} $(k \geq 1)$ with string $(d_1, d_2, \ldots, d_{k-2})$ satisfying the condition of part three with 0 in the $d = 2$ or $d = 3$ position in the last cycle and near graceful if k is odd with $10k + 1$ in the $d = 3$ position in the last cycle.

For $k = 1$, use the labelling of C_{10}^d in Table 4.3 with $k = 1$ which will make it near graceful. For $k = 2$, use $\big(12, 6, 13, 8, \overline{16}, 5, 14, 10, 11, 9\big)$ with (0, 20, 1, $\overline{16}$, 3, 17, 7, 19, 2, 18) to get $d = 3$, and take this labelling with Lemma 4.1.2 to get $d = 2$. The rest of the proof is similar to the proof of Case 1, but uses C_{10}^d instead of C_{10}^a, and C_{10}^e instead of C_{10}^b.

Case 4: $d_i \in \{1, 2\}$ if i is odd and $d_i = 2$ if i is even.

We prove that kC_{10} $(k \geq 1)$ with string $(d_1, d_2, \ldots, d_{k-2})$ satisfying the condition of part four with 0 in the $d = 1$ or $d = 2$ position in the last cycle and near graceful if k is odd with $10k + 1$ in the $d = 1$ position in the last cycle.

For $k = 1$, use the labelling of C_{10}^f in Table 4.3 with $k = 1$ which will make it near graceful. For $k = 2$, use $\big(8, 14, 7, 12, \overline{4}, 15, 6, 10, 9, 11\big)$ with (0, 20, 2, 18, 1, 13, 3, 17, $\overline{4}$, 19) to get $d = 2$, and take this labelling with Lemma 4.1.2 to get $d = 1$. The rest of the proof is similar to the proof of Case 2, but uses C_{10}^f instead of C_{10}^a, and C_{10}^g instead of C_{10}^c. □

In Theorem 4.3.3 we prove (near) graceful labellings exist for a kC_{14} for particular strings. In Table 4.4 we see two labellings of C_{14}. The labelling C_{14}^a uses edge labels $[14k-13, 14k+1] \setminus \{14k\}$. The labelling C_{14}^b uses edge labels $[14k-14, 14k] \setminus \{14k-13\}$.

	Labelling
C_{14}^a	$(0, 14k+1, 4, 14k-2, 5, 14k-4, 6, 14k-5, \overline{7}, 14k-6, 2, 14k-3, 1, 14k-1)$
C_{14}^b	$(0, 14k, 4, 14k-10, 2, 14k-3, 3, 14k-5, \overline{6}, 14k-4, 5, 14k-2, 1, 14k-1)$

Table 4.4: Some useful labellings of C_{14}.

Theorem 4.3.3. *If $k \geq 1$ and $d_i \in \{6, 7\}$ if i is odd and $d_i = 7$ if i is even, for $1 \leq i \leq k-2$, then kC_{14} with string $(d_1, d_2, \ldots, d_{k-2})$ is graceful if k is even and near graceful if k is odd.*

Proof. The proof is similar to the proof of Theorem 4.3.1.

As in the proof of Theorem 4.3.1, we prove that if $k \geq 1$ and $d_i \in \{6, 7\}$ if i is odd and $d_i = 7$ if i is even, for $1 \leq i \leq k-2$, then the kC_{14} with string $(d_1, d_2, \ldots, d_{k-2})$ is graceful if k is even with 0 in the $d = 6$ or $d = 7$ position in the last cycle and near graceful if k is odd, with $14k+1$ in the $d = 7$ position in the last cycle.

We proceed by induction on k. For $k = 1$, use the labelling of C_{14}^a in Table 4.4 which will make it near graceful. For $k = 2$ use (21, 8, 20, 10, 19, 13, 14, 12, 15, 11, 16, 9, 17, $\overline{6}$) and (0, 28, 4, 18, 2, 25, 3, 23, $\overline{6}$, 24, 5, 26, 1, 27) to get $d = 6$, and take this labelling with Lemma 4.1.2 to get $d = 7$.

Case 1: Consider kC_{14} to be an arbitrary snake with 0 in the $d = 6$ or $d = 7$ position in the last cycle with $k \geq 4$ and k even. The labelling of $(k-1)C_{14}$ by the induction hypothesis has vertex labels that are a subset of $[0, 14k-15] \setminus \{3, 14k-14\}$ and the edge labels are exactly $[1, 14k+1] \setminus \{14k\}$ with $14k-13$ in the vertex distance

d_{k-2} from the previous cut vertex.

We label kC_{14} as follows. Subtract each vertex label from $14k-7$ for the first $k-1$ cycles of the $(k-1)C_{14}$, so that the last cut vertex receives label 6. Then, apply the labelling C_{14}^b to the final cycle, with the cut vertex receiving label 6. Then this labelling of kC_{14} has all vertices labelled from $[0, 14k]$ and the edge labels are exactly $[1, 14k]$, with no repeated vertex or edge labels. Hence, we obtain a graceful labelling of kC_{14} and 0 in the $d = 6$ position.

By using the previously discussed labelling, ending with the C_{14}^b-labelling in the last cycle, then applying Lemma 4.1.2 we obtain a labelling with 0 in the $d = 7$ position.

Case 2: Consider kC_{14} to be an arbitrary snake with 0 in the $d = 6$ position in the last cycle with $k \geq 3$ and k odd. The proof is similar to the proof of Case 1. Add 7 to each vertex label instead of subtracting each vertex label from $14k-7$ and apply the labelling C_{14}^a to the final cycle instead of C_{14}^b, with the cut vertex receiving label 7 instead of 6. Then this labelling of kC_{14} has all vertices labelled from $[0, 14k+1] \setminus \{3, 14k\}$, and the edge labels are exactly $[1, 14k+1] \setminus \{14k\}$, with no repeated vertex or edge label. Thus, we obtain a near graceful labelling of kC_{14} and $14k+1$ in the $d = 7$ position. □

Recall that a kC_t is linear if all entries in its string are equal to $\lfloor \frac{t}{2} \rfloor$. So, based on the results of Theorems 4.3.2 and 4.3.3 we now state a corollary for linear kC_{10} and kC_{14}, following the style of Theorem 4.1.6.

Corollary 4.3.4. *If $k \geq 1$ then the linear kC_{10} and linear kC_{14} are graceful if k is even and nearly graceful if k is odd.*

4.4 Discussion

In this chapter we (near) gracefully labelled several type of snakes. In section 4.2, we presented a new sufficient condition which when satisfied shows there is a graceful labelling of a kC_{4n} for any string. By using a complete s,t-useful cycle set we proved that if there is a complete $s,4n$-useful cycle set with $s \geq 1$, then there exists a graceful labelling of any kC_{4n}. We used the results in [5] with our results for kC_{4n} and proved that a graceful labelling exists for particular kC_{4n} with string $(d_1, d_2, \ldots, d_{k-2})$, where $d_i \in \{2,3,4\}$. Expanding these results for any n and d is possible but hard to apply for large n. We extended our main result to the case of cyclic snakes with cycles of varying sizes. Further, we extended the results in Theorems 4.1.6 to 4.1.9 on (near) gracefully labelled kC_n where $n = 6, 8, 12, 16, 20, 24$ for all possible snakes.

As we discussed in Section 4.3, new approaches must be found to gracefully label kC_{4n+2} snakes, even for fixed n. Our collections of ad hoc methods work to give classes for fixed n, but do not seem to generalize, even for "nice" subfamilies, such as linear snakes. Thus we pose the following open question.

Question: Can we (near) gracefully label every kC_m with $k \geq 1$ and $m \equiv 2 \pmod 4$?

In fact the technique we used is more general than indicated in our theorems. Suppose we have a gracefully labelled bipartite graph $G = K_{3,4}$ as in Figure 4.2. If we add 3 to each vertex label and use the cycle $H = C_8^2$ from Table 4.1 then we obtain a new gracefully labelled graph as in Figure 4.3. Thus we can in several cases gracefully label new graphs.

Theorem 4.4.1. *If G is graceful and H is a kC_{4n} with $1 \leq n \leq 6$, then the graph GH^* obtained by identifying any vertex in G that can be labelled 0 in some graceful labelling with any vertex in the first cycle of H is graceful.*

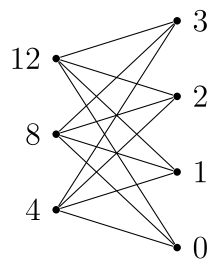

Figure 4.2: Gracefully labelled $K_{3,4}$.

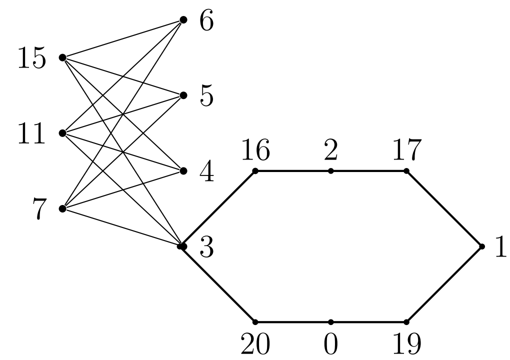

Figure 4.3: Gracefully labelled GH^*.

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Chapter 5

Conclusion and Future Works

In this chapter, we succinctly summarize the main results in this thesis and present some questions for future research goals.

In chapter two, we proved Rosa's conjecture for a new family of triangular cacti: Dutch windmills of any order with three pendant triangles, by using Skolem-type sequences. We can summarize the results in this chapter in the following theorem.

Theorem 5.1.1. *Every Dutch windmill with three pendant triangles is graceful or near graceful.*

Although Rosa's conjecture is still open after more than thirty years, there are still particular problems that are natural to consider based on our research.

Problem 1. Can we use Skolem sequences to prove that (near) graceful labellings exist for all possible windmills with four (or more) pendant triangles?

Problem 2. Can we use Skolem type sequences to prove that (near) graceful labellings exist for triangular snakes? Can we use them to gracefully label triangular snakes with pendant triangles?

In chapter three, we used many types of Skolem-type sequences to (near) gracefully label variable windmills. We used Skolem-type sequences to prove (near) graceful labellings exist for the $C_3^tC_4^s$, $C_3^tC_5^p$ and $C_3^tC_6^h$. Furthermore, we used near-Skolem sequences along with Skolem sequences to prove that C_5^t is (near) graceful. We can conclude the results in this chapter in the following theorems.

Theorem 5.1.2. *If $G = C_5^p$, then G is near graceful when $p \equiv 1, 2 \pmod 4$ and G is graceful when $p \equiv 0, 3 \pmod 4$.*

Theorem 5.1.3. *If $G = C_3^tC_4^s$, where $t \geq s \geq 1$, then G is graceful when $t \equiv 0, 1 \pmod 4$ and near graceful when $t \equiv 2, 3 \pmod 4$.*

Theorem 5.1.4. *If $s \in \mathbb{N}$, and $1 \leq t \leq 3$ then $C_3^tC_4^s$ can be gracefully labelled if $t = 1$ and near gracefully labelled if $t = 2, 3$.*

Theorem 5.1.5. *If $G = C_3^tC_4^s$, where $4 \leq t \leq s \leq {}^{(13t+37)}/_{2}$, then G is graceful when $t \equiv 0, 1 \pmod 4$ and near graceful when $t \equiv 2, 3 \pmod 4$.*

Theorem 5.1.6. *For $t \geq 2p + 1$, $G = C_3^tC_5^p$ is graceful when $(p, t) \equiv (0, 0), (0, 1), (3, 0), (3, 3) \pmod 4$ and near graceful when $(p, t) \equiv (1, 0), (1, 3), (2, 0), (2, 1) \pmod 4$.*

Theorem 5.1.7. *If $G = C_3^tC_6^h$ with $h \leq 2t + 1$ then G is graceful or near graceful.*

The following are some open problems that relate directly to the new results in chapter three.

Problem 3: Do (near) graceful labellings exist for all possible $C_3^tC_4^s$, $C_3^tC_5^p$ and $C_3^tC_6^h$ for any t, s, p, h?

Problem 4: Do (near) graceful labellings exist for $C_i^mC_j^n$ for any i, j, m, n? Can we prove (near) graceful labellings exist for $C_i^lC_j^mC_k^n$ for any i, j, k, l, m, n?

Problem 5: Can we find a family of m-fold Skolem-type sequences with the first right endpoint as large as possible? What are the necessary and sufficient conditions for the existence of m-fold Langford sequences with $m \geq 2$?

Note that in our constructions to label the windmills in chapters two and three we have the label 0 at the central vertex, so that leads us to ask the following.

Problem 6: Can we gracefully label windmills with any label other than 0 at the central vertex?

In chapter four we introduced a new condition showing graceful labelling of every kC_{4n} under certain conditions. Additionally, we extended this result to variable snakes. Furthermore, we found results on the (near) graceful labelling of cyclic snakes kC_n where $n = 6, 8, 12, 16, 20, 24$ and $k > 1$. Also, we proved (near) graceful labellings exist for a kC_n with $n = 10, 14$ for particular strings. We can summarize the results in this chapter in the following theorems.

Theorem 5.1.8. *If there is a complete $k, 4i$-useful cycle set with $k \geq 1$ for all $1 \leq i \leq n$, $j \geq 1$, and $1 \leq m_1, m_2, \ldots, m_j \leq n$, with positive integers $n_1, n_2, \ldots, n_j$ then every $n_1C_{4m_1}n_2C_{4m_2} \ldots n_jC_{4m_j}$-snake is graceful.*

Theorem 5.1.9. *If $k \geq 1$ then there exists a (near) graceful labelling of any kC_m, where $m = 4, 6, 8, 12, 16, 20, 24$.*

Theorem 5.1.10. *The kC_{10} $(k \geq 1)$ with string $(d_1, d_2, \ldots, d_{k-2})$ is graceful if k is even and near graceful if k is odd and one of the following is true:*

1. *$d_i \in \{4, 5\}$ if i is odd and $d_i = 5$ if i is even, where $1 \leq i \leq k - 2$,*
2. *$d_i \in \{3, 4\}$ if i is odd and $d_i = 4$ if i is even, where $1 \leq i \leq k - 2$,*
3. *$d_i \in \{2, 3\}$ if i is odd and $d_i = 3$ if i is even, where $1 \leq i \leq k - 2$, or*
4. *$d_i \in \{1, 2\}$ if i is odd and $d_i = 2$ if i is even, where $1 \leq i \leq k - 2$.*

Theorem 5.1.11. *If $k \geq 1$ and $d_i \in \{6, 7\}$ if i is odd and $d_i = 7$ if i is even, for $1 \leq i \leq k - 2$, then the kC_{14} with string $(d_1, d_2, \ldots, d_{k-2})$ is graceful if k is even and near graceful if k is odd.*

For kC_{10} and kC_{14} we presented new partial results on (near) graceful labellings when $k > 1$. Thus, it is a natural place to start and expand the results. Further for future research, the most obvious open problem is the following.

Problem 7: Can we prove (near) graceful labellings exist for all (or any) kC_m with $k \geq 1$, $m \geq 18$ and $m \equiv 2 \pmod 4$?