Global Dynamics of Two Population Models with Spatial Heterogeneity

by

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Abstract

Mathematical models provide powerful tools to explain and predict population dynamics. A central problem is to study the long-term behavior of modeling systems. The patch models and reaction-diffusion models are widely applied to describe spatial heterogeneity and habitat connectivity.

Basic reproduction number $\mathcal{R}_0$ plays an important role in mathematical biology. In epidemiology, $\mathcal{R}_0$ stands for the expected number of secondary cases produced in a completely susceptible population by a typical infective individual. The value of $\mathcal{R}_0$ can determines the persistence or extinction of population. Nowadays, characterizing the basic reproduction number due to the effects of parameters becomes very significant for predicting and controlling disease transmission.

This thesis consists of three chapters. In Chapter 1, we investigate the effect of spatial heterogeneity on the basic reproduction number for an SIS epidemic patch model, and compute $\mathcal{R}_0$ numerically to show the influence of the spatial heterogeneity and movement. Chapter 2 is devoted to the study of the global dynamics of a reaction-diffusion model arising from the dynamics of a kind of mosquitos named A. aegypti in Brazil. We first prove the global existence and boundedness of the solutions. Secondly, we establish the threshold type dynamics in terms of the basic reproduction ratio $\mathcal{R}_0$. In Chapter 3, we briefly summarize the main results and present some future works.

To my family

Lay summary

The emergence of the infectious diseases of humans or animals has resulted in serious consequences for public health, and is receiving more and more attention in mathematical epidemiology. This thesis is focused on the global dynamics of two types of population models with spatial structure.

To understand the effects of the spatial heterogeneity, we first consider an ODE SIS patch model and use the variational formula of the basic infection number R_0 to study its monotonicity with respect to some parameters. It turns out that some measures should be taken to make $R_0 < 1$ so that the disease transmission is under control. Then we investigate the global dynamics of a reaction-diffusion mosquito population model. Our analytical result indicates that the basic reproduction number R_0 is a threshold value to determine whether the population goes extinct or persists uniformly.

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Chapter 1

An SIS epidemic patch model

1.1 Introduction

In mathmatical biology, the reproduction number (ratio) $\mathcal{R}_0$ is a very important factor that stands for the expected number of secondary cases produced, in a completely susceptible population, by a typical infective individual. There are many research on characteristicing the basic reproduction ratio $\mathcal{R}_0$. Diekmann, Heesterbeek and Metz [9] introduced the next generation operators approach to $\mathcal{R}_0$ for models of infectious diseases in heterogeneous populations. And van den Driessche and Watmough [31] developed the theory of $\mathcal{R}_0$ for autonomous ordinary differential equations (ODE) models with compartmental structure. In recent years, there are numerous works about the basic reproduction ratio for various autonomous, periodic, and almost periodic evolution equation models with or without time-delay (see, e.g., [3, 17, 18, 25, 30, 31, 33, 35, 36, 38]).

In this chapter, we are interested in the following SIS patch model:

$$\begin{cases} \frac{dS_i}{dt} = d_S \sum_{j=1}^{n}(l_{ij}S_j) - \frac{\beta_i S_i I_i}{S_i+I_i} + \gamma_i I_i, & i = 1, \cdots, n,\ t \geq 0, \\ \frac{dI_i}{dt} = d_I \sum_{j=1}^{n}(l_{ij}I_j) + \frac{\beta_i S_i I_i}{S_i+I_i} - \gamma_i I_i, & i = 1, \cdots, n,\ t \geq 0, \end{cases} \tag{1.1}$$

where $n \geq 2$ is the number of patches; $S_i(t)$ and $I_i(t)$ denote the number of susceptible and infected individuals in patch i at time $t \geq 0$; $d_S > 0$ and $d_I > 0$ are the dispersal coefficients for the susceptible and infected subpopulations, respectively; l_{ij} represents the degree of movement from patch j into patch i and $-l_{jj} = \sum_{i=1, i\neq j}^{n} l_{ij}$ is the degree of movement from patch j to all other patches; $\beta_j > 0$ and $\gamma_j > 0$ are the constants that express the rate of disease transmission and recovery in patch j, respectively. Without loss of generality, throughout the whole chapter, we assume that $d_I = 1$.

The following assumptions on the initial condition and the connectivity matrix $L = (l_{ij})_{n\times n}$ are made:

(A1) $S_i(0) \geq 0$ and $I_i(0) \geq 0$ for $i = 1, \cdots, n$ and $\sum_{i=1}^n [S_i(0) + I_i(0)] = N > 0$;

(A2) L is symmetric, cooperative and irreducible.

It is easy to see that L is column zero matrix, that is $\sum_{j=1}^n l_{jk} = 0$ for all $k = 1, \cdots, n$. According to [4, Theorem 6.4.16], L has rank $n-1$, and hence, the system of linear equations

$$\begin{cases} \sum_{j=1}^n l_{ij} S_j^0 = 0, \ i = 1, \cdots, n, \\ \sum_{i=1}^n S_i^0 = N, \end{cases}$$

has a unique positive solution, denoted by $\boldsymbol{S}^0 = (S_1^0, \cdots, S_n^0)$. Then the model (1.1) admits a unique disease-free equilibrium (DFE) $E_0 = (\boldsymbol{S}^0, \mathbf{0})$. Linearizing the model system (1.1) at the DFE gives the new infection and transition matrices

$$\tilde{F} = \text{diag}\{\beta_1, \cdots, \beta_n\} \quad \text{and} \quad \tilde{V} = \text{diag}\{\gamma_1, \cdots, \gamma_n\} - L,$$

respectively. Let

$$F = \text{diag}\{\beta_1, \cdots, \beta_n\} \quad \text{and} \quad V = \text{diag}\{\gamma_1, \cdots, \gamma_n\}.$$

Following the recipe of van den Driessche and Watmough [31], the basic reproduction ratio for model (1.1) is defined by

$$\mathcal{R}_0 = r(\tilde{V}^{-1}\tilde{F}) = r(-(L-V)^{-1}F). \tag{1.2}$$

For this patch model, there are many research works about the effect of dispersal coefficients. Allen, Bolker, Lou and Nevai [2] showed that the basic reproduction number $\mathcal{R}_0$ is strictly decreasing and convex in the dispersal coefficients and studied the asymptotic behavior of $\mathcal{R}_0$ as the dispersal coefficient goes to zero and infinity when the connectivity matrix L is symmetric. Without assuming the symmetry of L, Gao and Dong [12, 13] and Chen, Shi, Shuai and Wu [5, 6] extended such results to the case where the connectivity matrix L is asymmetric.

There are also some related research for reaction-diffusion equations. Allen, Bolker, Lou and Nevai [1] introduced the basic reproduction number $\mathcal{R}_0$ for a scalar reaction-diffusion equation model by a variational formula. Wang and Zhao [36] established the theory of $\mathcal{R}_0$ for compartmental epidemic models of reaction-diffusion equation and characterized $\mathcal{R}_0$ in terms of the principal eigenvalue of an elliptic eigenvalue problem. Recently, Yang, Qarariyah and Yang [37] investigated the influence of spatial-dependent variables on the basic reproduction ratio ($\mathcal{R}_0$) for a scalar reaction-diffusion equation model using the tools established in [1, 36]. It is natural to ask how the spatial heterogeneity affects the basic reproduction ratio for a patch model. For this patch model, we prove that $\mathcal{R}_0$ could be first non-increasing and then non-decreasing with the increasing spatial heterogeneity.

The remaining of this chapter is organized as follows. In the next section, we discuss the relation between the basic reproduction ratio and coefficients representing spatial heterogeneity. In section 1.3, we compute $\mathcal{R}_0$ numerically to investigate the these influences.

1.2 Theoretical analysis

In this section, we first present basic results on the global dynamics of system (1.1), then we study the effect of spatial heterogeneity on the basic reproduction ratio for a patch model theoretically.

Let

$$X := \{(S_1, \cdots, S_n, I_1, \cdots, I_n) \in \mathbb{R}^{2n}_+ : \sum_{i=1}^{n}(S_i + I_i) = N\},$$

and

$$X_0 := \{(S_1, \cdots, S_n, I_1, \cdots, I_n) \in X : I_i > 0, \forall i = 1, 2, \cdots, n\}.$$

Then we have the following result.

Theorem 1.2.1. *If $\mathcal{R}_0 < 1$, the disease-free equilibrium E_0 is globally asymptotically stable in X, and if $\mathcal{R}_0 > 1$, there exists a unique endemic equilibrium and system* (1.1) *is uniformly persistent in X_0.*

Proof. The global stability of the disease-free equilibrium and the existence and uniqueness of the endemic equilibrium follow from [2, Lemmas 2.3 and 3.6]. It remains to show that system (1.1) is uniformly persistent. Let $\partial X_0 := X \backslash X_0$. It is easy to see that both X and X_0 are positively invariant. Set

$$M_\partial := \{(S(0), I(0)) : (S(t), I(t)) \text{ satisfies (1.1) and (S(t),I(t))} \in \partial X_0,\ \forall t \geq 0\}.$$

We claim that

$$M_\partial = \{(S, 0) : S \geq 0\}.$$

Clearly, $\{(S,0) : S \geq 0\} \subset M_\partial$. It suffices to show that $M_\partial \subset \{(S,0) : S \geq 0\}$. Let $(S(0), I(0)) \in M_\partial$ be given. Now we prove that $I(t) = 0,\ \forall t \geq 0$. Suppose not. Then there exist an $i_0,\ 1 \geq i_0 \geq n$, and a $t_0 \geq 0$ such that $I_{i_0}(t_0) > 0$. We define two sets Q_1 and Q_2 of integers such that $Q_1 \cap Q_2 = \emptyset$, $Q_1 \cup Q_2 = \{1, 2, \cdots, n\}$, $I_i(t_0) = 0$, $\forall i \in Q_1$, and $I_i(t_0) > 0$, $\forall i \in Q_2$. It then follows that Q_1 is non-empty due to the definition of M_∂, and Q_2 is non-empty since $I_{i_0}(t_0) > 0$. For any $j \in Q_1$, we have $I_j'(t_0) \geq l_{ji_0} I_{i_0}(t_0) > 0$. Thus, there is an $\epsilon_0 > 0$ such that $I_j(t) > 0$ for all $t_0 < t < t_0 + \epsilon_0$ and all $j \in Q_1$. Since

$$I_i'(t) \geq l_{ii} I_i(t) - \gamma_i I_i(t) = (l_{ii} - \gamma_i) I_i(t), \quad \forall t \geq 0,\, 1 \leq i \leq n,$$

we easily see that $I_i(t) > 0$ for all $t \geq t_0$ and all $i \in Q_2$. It then follows that $(S(t), I(t)) \notin \partial X_0$ for all $t_0 < t < t_0 + \epsilon_0$, which contradicts the assumption that $(S(0), I(0)) \in M_\partial$. Thus, $I(t) = 0,\ \forall t \geq 0$, and hence, $(S(0), I(0) \in \{(S,0) : S \geq 0\}$. This implies that $M_\partial \subset \{(S, 0) : S \geq 0\}$.

Now it is easy to see that E_0 is globally asymptotically stable for system (1.1) in M_∂. By the arguments similar to those in [34, Theorem 2.3], we can further show that E_0 is an isolated invariant set in X and $W^S(E_0) \cap X_0 = \emptyset$. By [29, Theorem 4.6], we conclude that system (1.1) is uniformly persistent in X_0. □

For a square matrix A, we define $s(A) := \max\{Re\lambda : \lambda \text{ is an eigenvalue of } A\}$ and $r(A) := \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$.

Proposition 1.2.1. *If $\mathcal{R}_0 > 0$, then $\lambda = \mathcal{R}_0$ is the unique solution of $s(L - V + \frac{1}{\lambda} F) = 0$.*

Proof. This can be derived by [35]. For reader's convenience, we provide a simple proof. Define

$$\hat{\mathcal{R}}_0(\lambda) := r\left(-(L-V)^{-1}\frac{F}{\lambda}\right),\ \forall \lambda > 0.$$

Clearly, $\hat{\mathcal{R}}_0(\lambda) = \frac{1}{\lambda}\mathcal{R}_0$, $\forall \lambda > 0$. It follows from [31, Theorem 2] that $s\left(L - V + \frac{F}{\lambda}\right)$ has the same sign as $\hat{\mathcal{R}}_0(\lambda) - 1 = \dfrac{1}{\lambda}(\mathcal{R}_0 - \lambda)$. This yields the desired conclusion. □

Let $\mathbf{0} = (0, \cdots, 0)^T$ be an n-dimensional vector. By the Perron-Frobenius theorem (see, e.g., [27, Theorem 4.3.1]), there is a unique column vector $\boldsymbol{\phi} = (\phi_1^*, \cdots, \phi_n^*)^T \gg 0$ with $\boldsymbol{\phi}^{*T}\boldsymbol{\phi}^* = 1$ such that

$$-(L-V)^{-1}F\boldsymbol{\phi}^* = r(-(L-V)^{-1}F)\boldsymbol{\phi}^* = \mathcal{R}_0\boldsymbol{\phi}^*,$$

which implies

$$(L - V + \frac{F}{\mathcal{R}_0})\boldsymbol{\phi}^* = 0. \tag{1.3}$$

Moreover, we have the following observation.

Lemma 1.2.1. *The following variational formula holds true:*

$$\begin{aligned}\mathcal{R}_0 &= \sup_{\phi\neq 0} \frac{\sum_{i=1}^n \beta_i\phi_i^2}{\frac{1}{2}\sum_{1\le i,j\le n} l_{ij}(\phi_i-\phi_j)^2 + \sum_{i=1}^n \gamma_i\phi_i^2} = \sup_{\phi\neq 0}\frac{\boldsymbol{\phi}^T F\boldsymbol{\phi}}{-\boldsymbol{\phi}^T L\boldsymbol{\phi} + \boldsymbol{\phi}^T V\boldsymbol{\phi}} \\ &= \frac{\sum_{i=1}^n \beta_i(\phi_i^*)^2}{\frac{1}{2}\sum_{1\le i,j\le n} l_{ij}(\phi_i^*-\phi_j^*)^2 + \sum_{i=1}^n \gamma_i(\phi_i^*)^2} = \frac{(\boldsymbol{\phi}^*)^T F\boldsymbol{\phi}^*}{-(\boldsymbol{\phi}^*)^T L\boldsymbol{\phi}^* + (\boldsymbol{\phi}^*)^T V\boldsymbol{\phi}^*}.\end{aligned} \tag{1.4}$$

Proof. We first verify the first and the second euality. The third and forth equality of (1.4) can be derived by the simple computation. Thus, it suffices to show the first and the second equality. Let $D = \mathrm{diag}\{\beta_1^{-\frac{1}{2}}, \cdots, \beta_n^{-\frac{1}{2}}\}$, $\tilde{L} = DLD = (\tilde{l}_{ij})_{n\times n}$ and $\tilde{V} = DVD = \mathrm{diag}\{\tilde{v}_1, \cdots, \tilde{v}_n\}$. It is easy to see that $\tilde{l}_{ij} = \beta_i^{-\frac{1}{2}}\beta_j^{-\frac{1}{2}} l_{ij}$ and $\tilde{v}_i = \gamma_i\beta_i^{-1}$, $\forall i, j = 1, \cdots, n$. Clearly, $\tilde{L}$ is still a symmetric, cooperative and irreducible matrix. Letting $\boldsymbol{\varphi}^* = D^{-1}\boldsymbol{\phi}^*$, we have

$$D(L - V + \frac{F}{\mathcal{R}_0})D\boldsymbol{\varphi}^* = \mathbf{0},$$

that is,

$$(-\tilde{L} + \tilde{V})\boldsymbol{\varphi}^* = \frac{1}{\mathcal{R}_0}\boldsymbol{\varphi}^*. \tag{1.5}$$

Let $\tilde{\lambda}_1$ be the smallest eigenvalue of $-\tilde{L} + \tilde{V}$. In view of [2, Appendix A], we obtain

$$\tilde{\lambda}_1 = \inf_{\varphi \neq 0} \frac{-\varphi^T \tilde{L} \varphi + \varphi^T \tilde{V} \varphi}{\varphi^T \varphi}.$$

By the Perron-Frobenius theorem, together with $F = D^{-1}D^{-1}$ and (1.5), we then obtain

$$\begin{aligned}
\mathcal{R}_0 = \tilde{\lambda}_1^{-1} &= \sup_{\varphi \neq 0} \frac{\boldsymbol{\varphi}^T \boldsymbol{\varphi}}{-\boldsymbol{\varphi}^T \tilde{L} \boldsymbol{\varphi} + \boldsymbol{\varphi}^T \tilde{V} \boldsymbol{\varphi}} \\
&= \sup_{\varphi \neq 0} \frac{\boldsymbol{\varphi}^T \boldsymbol{\varphi}}{-\boldsymbol{\varphi}^T DLD \boldsymbol{\varphi} + \boldsymbol{\varphi}^T DVD \boldsymbol{\varphi}} \\
&= \sup_{\phi \neq 0} \frac{\boldsymbol{\phi}^T F \boldsymbol{\phi}}{-\boldsymbol{\phi}^T L \boldsymbol{\phi} + \boldsymbol{\phi}^T V \boldsymbol{\phi}} \\
&= \sup_{\phi \neq 0} \frac{\sum_{i=1}^n \beta_i \phi_i^2}{\frac{1}{2} \sum_{i,j=1}^n l_{ij} (\phi_i - \phi_j)^2 + \sum_{i=1}^n \gamma_i \phi_i},
\end{aligned}$$

which is the desired conclusion. □

Lemma 1.2.2. *$\mathcal{R}_0 \geq \frac{\sum_{i=1}^n \beta_i}{\sum_{i=1}^n \gamma_i}$, and the equality holds if and only if $\phi_i^* = \phi_j^*$ for all $1 \leq i, j \leq n$.*

Proof. Since L is irreducible, $\phi_i^* > 0$ for all $i = 1, \cdots, n$. We divide the i-th equation by ϕ_i^* to obtain

$$\sum_{j=1}^n l_{ij} \frac{\phi_j^*}{\phi_i^*} - \gamma_i = -\frac{\beta_i}{\mathcal{R}_0}, \ \forall i = 1, \cdots, n.$$

Summing all above equations together, we have

$$\sum_{i=1}^n \sum_{j=1}^n l_{ij} \frac{\phi_j^*}{\phi_i^*} = \sum_{i=1}^n \gamma_i - \sum_{i=1}^n \frac{\beta_i}{\mathcal{R}_0}. \tag{1.6}$$

Since L is symmetric, so for any $i, j = 1, \cdots, n$ with $i \neq j$, we have

$$l_{ij} \frac{\phi_j^*}{\phi_i^*} + l_{ji} \frac{\phi_i^*}{\phi_j^*} = l_{ij} \left(\frac{\phi_j^*}{\phi_i^*} + \frac{\phi_i^*}{\phi_j^*} \right) \geq 2 l_{ij} \sqrt{\frac{\phi_j^*}{\phi_i^*} \frac{\phi_i^*}{\phi_j^*}} = 2 l_{ij}.$$

The equality holds if and only if $\phi_i^* = \phi_j^*$ for $1 \leq i, j \leq n$. Notice that L is column sum zero and symmetric, it is easy to see that

$$\sum_{i,j=1}^n l_{ij} \frac{\phi_i^*}{\phi_j^*} \geq 2 \sum_{1 \leq i < j \leq n} l_{ij} + \sum_{i=1}^n l_{ii} = \sum_{i,j=1}^n l_{ij} = 0.$$

In view of (1.6), we obtain

$$\sum_{i=1}^{n} \gamma_i - \sum_{i=1}^{n} \frac{\beta_i}{\mathcal{R}_0} \geq 0,$$

and hence,

$$\mathcal{R}_0 \geq \frac{\sum_{i=1}^{n} \beta_i}{\sum_{i=1}^{n} \gamma_i},$$

which leads to the desired conclusion. $\square$

To study the effect of the spatial heterogeneity on $\mathcal{R}_0$, we let

$$F(a) := \bar{F} + a\hat{F}, \text{ and } V(b) := \bar{V} + b\hat{V}. \tag{1.7}$$

Here $\bar{F} = \text{diag}\{\bar{\beta}_1, \cdots, \bar{\beta}_n\}$, $\bar{V} = \text{diag}\{\bar{\gamma}_1, \cdots, \bar{\gamma}_n\}$ with $\bar{\beta}_i = \bar{\beta}$ and $\bar{\gamma}_i = \bar{\gamma}$, $\forall i = 1, \cdots, n$, for some $\bar{\beta} > 0$ and $\bar{\gamma} > 0$, and non-vanishing matrix $\hat{F} = \text{diag}\{\hat{\beta}_1, \cdots, \hat{\beta}_n\}$ and non-vanishing matrix $\hat{V} = \text{diag}\{\hat{\gamma}_1, \cdots, \hat{\gamma}_n\}$ with $\sum_{i=1}^{n} \hat{\beta}_i = 0$ and $\sum_{i=1}^{n} \hat{\gamma}_i = 0$. Define

$$\hat{a} := \sup\{a > 0 : \bar{\beta} + a\hat{\beta}_i > 0, \ \forall i = 1, \cdots, n\},$$

and

$$\hat{b} := \sup\{b > 0 : \bar{\gamma} + b\hat{\gamma}_i > 0, \ \forall i = 1, \cdots, n\}.$$

For each $(a, b) \in [0, \hat{a}) \times [0, \hat{b})$, define $\mathcal{R}_0(a, b)$ by (1.2) with F and V replaced by $F(a) = F + a\hat{F}$ and $V(b) = V + b\hat{V}$, respectively. In the case of $a = 0$ and $b = 0$, $F(0) = \bar{F}$ and $V(0) = \bar{V}$, which corresponds to the spatial homogeneity. Thus, $F(a)$ and $V(b)$ be regarded as the heterogeneous perturbation of $\bar{F}$ and $\bar{V}$. For convenience, we write

$$\beta_i(a) := \bar{\beta} + a\hat{\beta}_i, \ \forall i = 1, \cdots, n, \ a \in (0, \hat{a}),$$

and

$$\gamma_i(b) := \bar{\gamma} + b\hat{\gamma}_i, \ \forall i = 1, \cdots, n, \ b \in (0, \hat{b}).$$

Theorem 1.2.2. *Let $a_0 \in [0, \hat{a})$ and $b_0 \in [0, \hat{b})$ be two given numbers. If $\mathcal{R}_0(a, b_0) \geq \mathcal{R}_0(a_0, b_0)$, $\forall a \in [a_0, \hat{a})$, then $\mathcal{R}_0(a, b_0)$ is convex and non-decreasing with respect to $a \in [a_0, \hat{a})$. If, in addition, $\mathcal{R}_0(a, b_0) > \mathcal{R}_0(a_0, b_0)$ for any $a \in (a_0, \hat{a})$, then $\mathcal{R}_0(a, b_0)$ is increasing with respect to $a \in [a_0, \hat{a})$.*

Proof. We first show that $\mathcal{R}_0(a, b_0)$ is non-decreasing with respect to $a \in [a_0, \hat{a})$. For any given $a_1 > a_2 > a_0$, there exists $\tau \in (0, 1)$ such that $a_2 = \tau a_1 + (1-\tau)a_0$. It is easy to see that $\beta_i(a_2) = \tau\beta_i(a_1) + (1-\tau)\beta_i(a_0)$, $\forall i = 1, \cdots, n$. Let $\hat{\boldsymbol{\phi}}^* = (\hat{\phi}_1^*, \cdots, \hat{\phi}_n^*)$ be the unique positive eigenvector of (1.3) with F and V replaced by $F(a_2)$ and $V(b_0)$, which corresponds the eigenvalue $\mathcal{R}_0(a_2, b_0)$. By Lemma 1.2.1, we have

$$
\begin{aligned}
\mathcal{R}_0(a_2,b_0) =& \frac{\sum_{i=1}^n \beta_i(a_2)(\hat{\phi}_i^*)^2}{\frac{1}{2}\sum_{i,j=1}^n l_{ij}(\hat{\phi}_i^* - \hat{\phi}_j^*)^2 + \sum_{i=1}^n \gamma_i(b_0)(\hat{\phi}_i^*)^2} \\
=& \frac{\sum_{i=1}^n [\tau\beta_i(a_1) + (1-\tau)\beta_i(a_0)](\hat{\phi}_i^*)^2}{\frac{1}{2}\sum_{i,j=1}^n l_{ij}(\hat{\phi}_i^* - \hat{\phi}_j^*)^2 + \sum_{i=1}^n \gamma_i(b_0)(\hat{\phi}_i^*)^2} \\
=& \frac{\tau\sum_{i=1}^n \beta_i(a_1)(\hat{\phi}_i^*)^2}{\frac{1}{2}\sum_{i,j=1}^n l_{ij}(\hat{\phi}_i^* - \hat{\phi}_j^*)^2 + \sum_{i=1}^n \gamma_i(b_0)(\hat{\phi}_i^*)^2} \\
& + \frac{(1-\tau)\sum_{i=1}^n \beta_i(a_0)(\hat{\phi}_i^*)^2}{\frac{1}{2}\sum_{i,j=1}^n l_{ij}(\hat{\phi}_i^* - \hat{\phi}_j^*)^2 + \sum_{i=1}^n \gamma_i(b_0)(\hat{\phi}_i^*)^2} \\
\le& \tau \sup_{\phi\neq 0} \frac{\sum_{i=1}^n \beta_i(a_1)\phi_i^2}{\frac{1}{2}\sum_{i,j=1}^n l_{ij}(\phi_i - \phi_j)^2 + \sum_{i=1}^n \gamma_i(b_0)\phi^2} \\
& + (1-\tau)\sup_{\phi\neq 0} \frac{\sum_{i=1}^n \beta_i(a_0)\phi_i^2}{\frac{1}{2}\sum_{i,j=1}^n l_{ij}(\phi_i - \phi_j)^2 + \sum_{i=1}^n \gamma_i(b_0)\phi_i^2} \\
=& \tau\mathcal{R}_0(a_1,b_0) + (1-\tau)\mathcal{R}_0(a_0,b_0).
\end{aligned}
$$

Since $\mathcal{R}_0(a,b_0) \ge \mathcal{R}_0(a_0,b_0)(\mathcal{R}_0(a,b_0) > \mathcal{R}_0(a_0,b_0))$, $\forall a \in [a_0,\hat{a})$, it follows that $\mathcal{R}_0(a_2,b_0) \le \mathcal{R}_0(a_1,b_0)(\mathcal{R}_0(a_2,b_0) < \mathcal{R}_0(a_1,b_0))$ for any $a \in [a_0,\hat{a})$. The convexity can be derived by repeating the above arguments. □

Remark 1.2.1. *Let $a_0 \in [0,\hat{a})$ and $b_0 \in [0,\hat{b})$ be two given numbers. If $\mathcal{R}_0(a,b_0) \ge \mathcal{R}_0(a_0,b_0)$, $\forall a \in [0,a_0]$, then $\mathcal{R}_0(a,b_0)$ is convex and non-increasing with respect to $a \in [0,a_0]$. If, in addition, $\mathcal{R}_0(a,b_0) > \mathcal{R}_0(a_0,b_0)$ for any $a \in [0,a_0]$, then $\mathcal{R}_0(a,b_0)$ is decreasing with respect to $a \in [0,a_0]$.*

Theorem 1.2.3. *Let $a_0 \in [0,\hat{a})$ and $b_0 \in [0,\hat{b})$ be two given numbers. If $\mathcal{R}_0(a_0,b) \ge \mathcal{R}_0(a_0,b_0)$, $\forall b \in [b_0,\hat{b})$, then $\mathcal{R}_0(a_0,b)$ is non-decreasing with respect to $b \in [b_0,\hat{b})$.*

Proof. For any given $b_1 > b_2 > b_0$, there exists $\tau \in (0,1)$ such that $b_2 = \tau b_1$. It's easy to see that $V(b_2) = \tau V(b_1) + (1-\tau)V(b_0)$. Let $\hat{\boldsymbol{\phi}}^* = (\hat{\phi}_1^*, \cdots, \hat{\phi}_n^*)$ be the unique positive eigenvector of (1.3) with F and V replaced by $F(a_0)$ and $V(b_2)$, which corresponds to the eigenvalue $\mathcal{R}_0(a_0, b_2)$. By Lemma 1.2.1, we have

$$
\begin{aligned}
&\mathcal{R}_0(a_0, b_2)\\
&= \frac{(\boldsymbol{\phi}^*)^T F(a_0)\boldsymbol{\phi}^*}{-(\boldsymbol{\phi}^*)^T L\boldsymbol{\phi}^* + (\boldsymbol{\phi}^*)^T V(b_2)\boldsymbol{\phi}^*}\\
&= \frac{(\boldsymbol{\phi}^*)^T F(a_0)\boldsymbol{\phi}^*}{-(\boldsymbol{\phi}^*)^T L\boldsymbol{\phi}^* + (\boldsymbol{\phi}^*)^T [\tau V(b_1) + (1-\tau)V(b_0)]\boldsymbol{\phi}^*}\\
&= \frac{(\boldsymbol{\phi}^*)^T F(a_0)\boldsymbol{\phi}^*}{\tau[-(\boldsymbol{\phi}^*)^T L\boldsymbol{\phi}^* + (\boldsymbol{\phi}^*)^T V(b_1)\boldsymbol{\phi}^*] + (1-\tau)[-(\boldsymbol{\phi}^*)^T L\boldsymbol{\phi}^* + (\boldsymbol{\phi}^*)^T V(b_0)\boldsymbol{\phi}^*]}\\
&\le \max\left\{\frac{(\boldsymbol{\phi}^*)^T F(a_0)\boldsymbol{\phi}^*}{-(\boldsymbol{\phi}^*)^T L\boldsymbol{\phi}^* + (\boldsymbol{\phi}^*)^T V(b_1)\boldsymbol{\phi}^*}, \frac{(\boldsymbol{\phi}^*)^T F(a_0)\boldsymbol{\phi}^*}{-(\boldsymbol{\phi}^*)^T L\boldsymbol{\phi}^* + (\boldsymbol{\phi}^*)^T V(b_0)\boldsymbol{\phi}^*}\right\}\\
&\le \max\left\{\sup_{\boldsymbol{\phi}\neq 0}\frac{\boldsymbol{\phi}^T F(a_0)\boldsymbol{\phi}}{-\boldsymbol{\phi}^T L\boldsymbol{\phi} + \boldsymbol{\phi}^T V(b_1)\boldsymbol{\phi}}, \sup_{\boldsymbol{\phi}\neq 0}\frac{\boldsymbol{\phi}^T F(a_0)\boldsymbol{\phi}}{-\boldsymbol{\phi}^T L\boldsymbol{\phi} + \boldsymbol{\phi}^T V(b_0)\boldsymbol{\phi}}\right\}\\
&\le \max\{\mathcal{R}_0(a_0, b_1), \mathcal{R}_0(a_0, b_0)\}.
\end{aligned}
$$

In view of $\mathcal{R}_0(a_0, b_1) \ge \mathcal{R}_0(a_0, b_0)$, $\forall b \in [b_0, \hat{b})$, we get $\mathcal{R}_0(a_0, b_2) \le \mathcal{R}_0(a_0, b_1)$. □

Remark 1.2.2. *Let $a_0 \in [0, \hat{a})$ and $b_0 \in [0, \hat{b})$ be two given numbers. If $\mathcal{R}_0(a_0, b) \ge \mathcal{R}_0(a_0, b_0)$, $\forall b \in [0, b_0]$, then $\mathcal{R}_0(a_0, b)$ is non-increasing with respect to $b \in [0, b_0]$.*

Lemma 1.2.3. *Let $b_0 \in [0, \hat{b})$ be a given number and $G = \mathrm{diag}\{g_1, \cdots, g_n\}$ be a given diagonal matrix with $\sum_{i=1}^n g_i = 0$. If $\hat{F} = \hat{V} = G$, then $\mathcal{R}_0(b_0\bar{\beta}\bar{\gamma}^{-1}, b_0) = \bar{\beta}\bar{\gamma}^{-1}$.*

Proof. Let $a_0 = b_0\bar{\beta}\bar{\gamma}^{-1}$ and $\mathbf{1} = (1, \cdots, 1)^T$ be an n-dimensional vector. Since $F(a_0) = \bar{F} + a_0 G = \bar{F} + b_0\bar{\beta}\bar{\gamma}^{-1}G = \bar{\beta}\bar{\gamma}^{-1}(\bar{V} + b_0 G)$, it is easy to see that $\bar{\beta} + a_0\hat{g}_i > 0$, $\forall i = 1, \cdots, n$. By Lemma 1.2.2, it suffices to show that $\boldsymbol{\phi} = \mathbf{1}$ is an eigenvector of (1.3) corresponding to the eigenvalue $\bar{\beta}\bar{\gamma}^{-1}$. Notice that L is symmetric and column sum zero, we have $L\mathbf{1} = \mathbf{0}$. Then we have

$$(L - V + \frac{F}{\bar{\beta}\bar{\gamma}^{-1}})\mathbf{1} = (-V + \frac{F}{\bar{\beta}\bar{\gamma}^{-1}})\mathbf{1}.$$

For each $i = 1, \cdots, n$, an easy computation yields that

$$
\begin{aligned}
-\gamma_i + \frac{\beta_i}{\bar{\beta}\bar{\gamma}^{-1}} &= -(\bar{\gamma} + b_0 g_i) + (\frac{\bar{\beta} + b_0\bar{\beta}\bar{\gamma}^{-1}g_i}{\bar{\beta}\bar{\gamma}^{-1}})\\
&= -(\bar{\gamma} + b_0 g_i) + (\bar{\gamma} + b_0 g_i)\\
&= 0.
\end{aligned}
$$

We obtain that

$$(L - V + \frac{F}{\bar{\beta}\bar{\gamma}^{-1}})\mathbf{1} = \mathbf{0},$$

which implies the desired conclusion. $\square$

As a consequence of Theorem 1.2.2, Remark 1.2.1 and Lemma 1.2.3, we have the following theorem.

Theorem 1.2.4. *Let $b_0 \in [0, \hat{b})$ be a given number and $G = \mathrm{diag}\{g_1, \cdots, g_n\}$ be a given non-zero matrix with $\sum_{i=1}^{n} g_i = 0$. If $\hat{F} = \hat{V} = G$, then $\mathcal{R}_0(a, b_0)$ is decreasing with respect to $a \in [0, b_0\bar{\beta}\bar{\gamma}^{-1}]$ and increasing with respect to $a \in [b_0\bar{\beta}\bar{\gamma}^{-1}, \hat{a})$.*

The following result follows from Theorem 1.2.3, Remark 1.2.2 and Lemma 1.2.3.

Theorem 1.2.5. *Let $a_0 \in [0, \hat{a})$ be a given number and $G = \mathrm{diag}\{g_1, \cdots, g_n\}$ be a given non-zero matrix with $\sum_{i=1}^{n} g_i = 0$. If $\hat{F} = \hat{V} = G$, then $\mathcal{R}_0(a_0, b)$ is decreasing with respect to $b \in [0, a_0\bar{\beta}^{-1}\bar{\gamma}]$ and increasing with respect to $b \in [a_0\bar{\beta}^{-1}\bar{\gamma}, \hat{b})$.*

1.3 Numerical simulations

In this section, we investigate the effect of spatial heterogeneity on the basic reproduction ratio numerically. We choose $n = 3$, $\bar{F} = \mathrm{diag}\{1.1, 1.1, 1.1\}$, $\bar{V} = \mathrm{diag}\{0.9, 0.9, 0.9\}$ and $\hat{F} = \hat{V} = \mathrm{diag}\{1, -2, 1\}$. We first consider the effect of a and b on $\mathcal{R}_0(a, b)$.

Example 1.3.1. *Let*

$$L = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

We show $\mathcal{R}_0(a, b)$ versus the parameters a and b in Figure 1.1. (a) and (b) indicate that $\mathcal{R}_0(a, b)$ can be increasing or non-monotone with respect to a when $b = 0$ and $b = 1$, respectively. In (c) and (d), $\mathcal{R}_0(a, b)$ can also be increasing or non-monotone with respect to b when $a = 0$ and $a = 1$, respectively.

In the previous section, we assume that L is symmetric. It is natural to ask whether the similar results hold for the asymmetric connectivity matrix L. The following example gives a confirmative answer numerically.

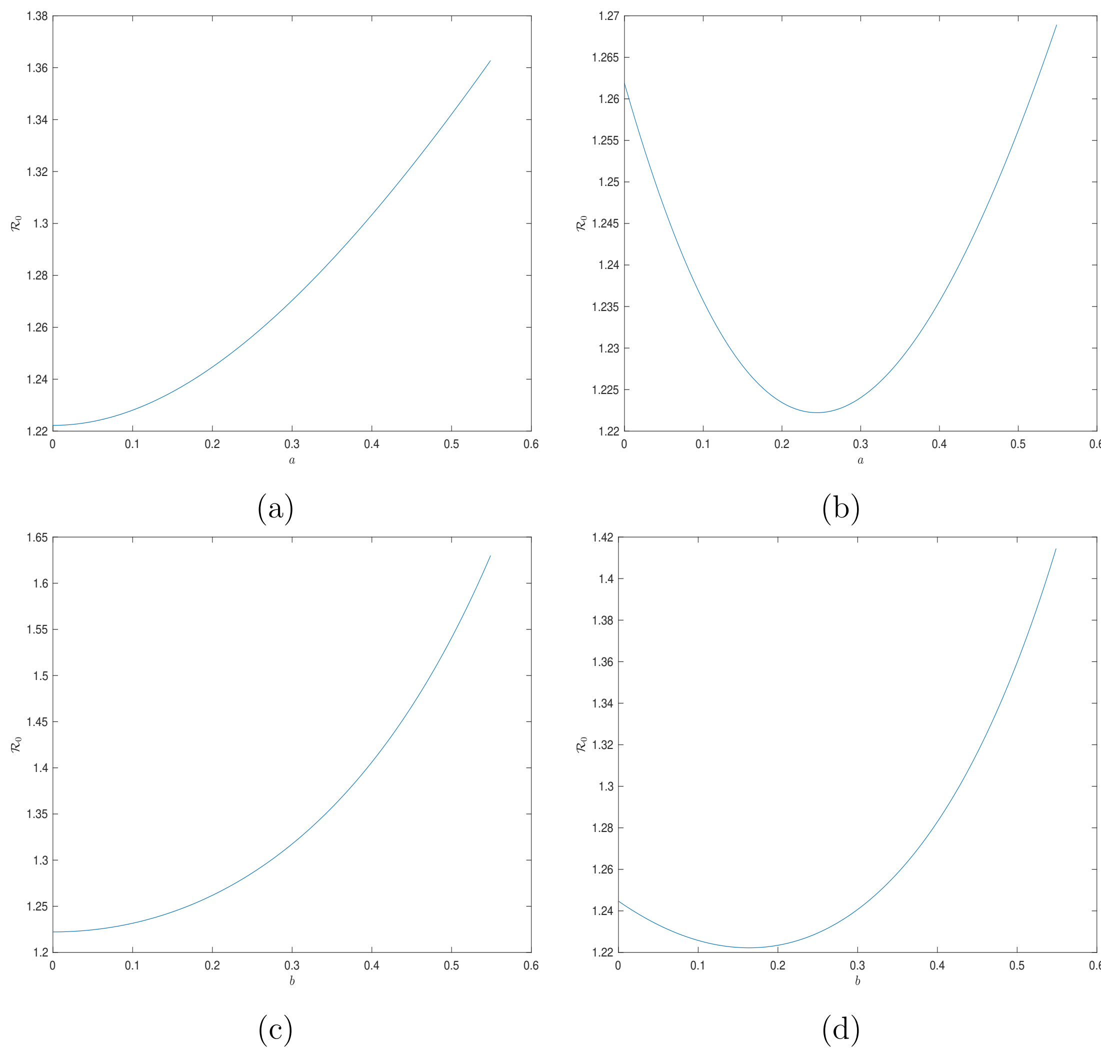


Figure 1.1: Relationship between $\mathcal{R}_0(a, b)$ and the coefficient a and b under four different scenarios: (a) $b = 0$; (b) $b = 1$; (c) $a = 0$; (d) $a = 1$.

Example 1.3.2. *Choose L having the following form*

$$L_1 = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \ L_2 = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \ \textit{and } L_3 = \begin{pmatrix} -2 & 0 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Figure 1.2(a)(b) and (c) suggest that the influence of changes in coefficients a and b could be determined no matter whether L is symmetric or not. Figure 1.2(d) shows that L_1 and L_2 share the same minimum $\mathcal{R}_0$ at the same a.

Example 1.3.3. *Using the formula (1.2), we compute the basic reproduction $\mathcal{R}_0$ numerically under the assumption that*

$$L = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, F = \mathrm{diag}\{0.4, 0.5, 0.6\}, V = \mathrm{diag}\{1, 1.5, 2\},$$

we obtain $\mathcal{R}_0 = 0.3364 < 1$. In this case, the disease will die out eventually(see Figure1.3(a)(b)). If we choose $F = \mathrm{diag}\{1.5, 1.5, 1.6\}$, $V = \mathrm{diag}\{0.3, 0.4, 0.5\}$, then $\mathcal{R}_0 = 3.8581 > 1$. The long-term behaviors of the suspicious and the infectious populations are shown in Figure1.3(c)(d), which implies that the disease will persist eventually. These simulations are consistent with our theoretical results.

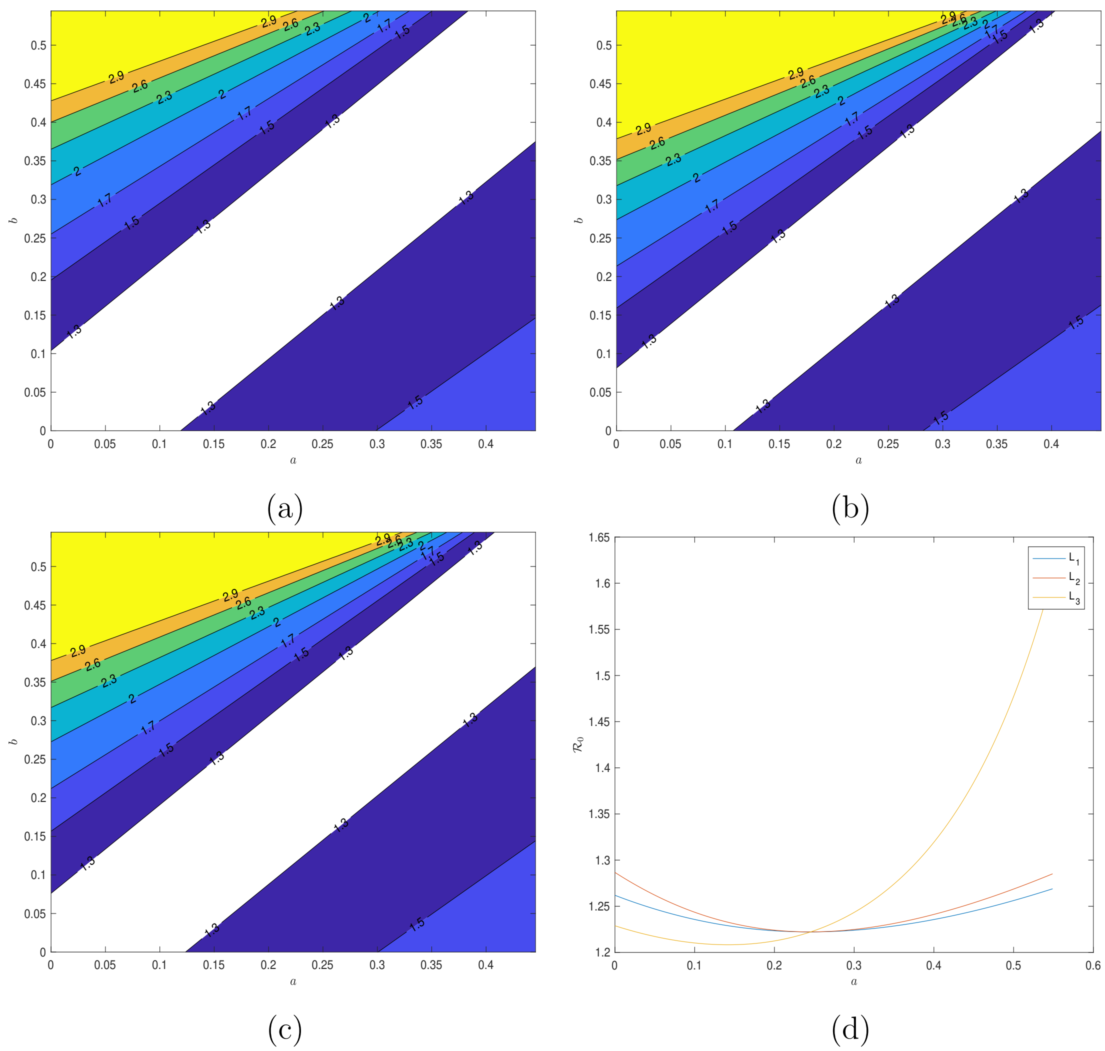


Figure 1.2: The contour plot of the reproduction number $\mathcal{R}_0$ under three different scenarios: (a) $L = L_1$; (b) $L = L_2$; (c) $L = L_3$. The figure (d) shows, for a given $b_0 = 0.2$, the relation between $\mathcal{R}_0$ and a under three different L which is either symmetric or asymmetric.

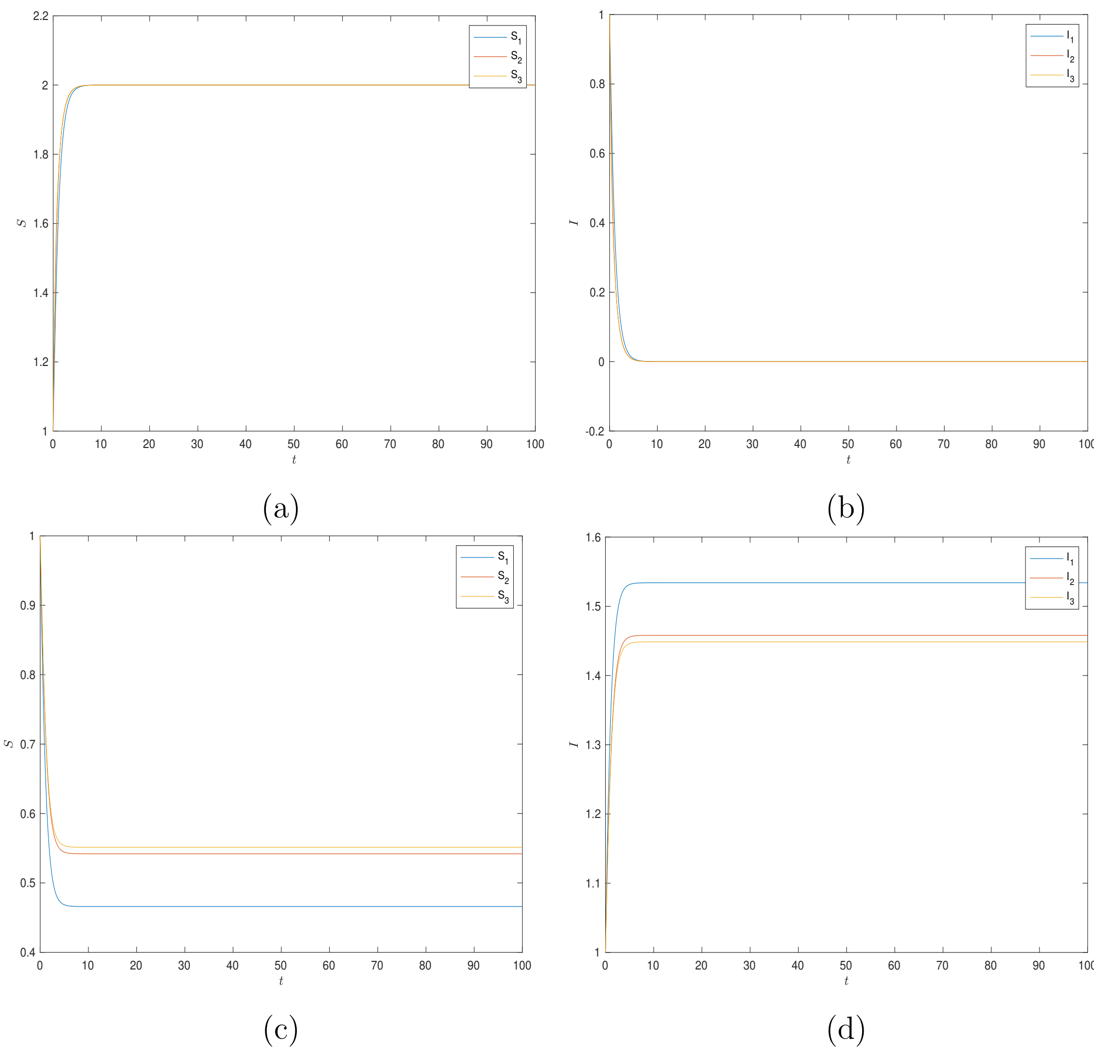


Figure 1.3: Long term behaviors of the suspicious and the infectious populations in system (1.1) when $\mathcal{R}_0 = 0.3364$ and $\mathcal{R}_0 = 3.8581$.

Chapter 2

A reation-diffusion population model

2.1 Introduction

Dengue is a mosquito-borne disease which is transmitted by arthropods of the species Aedes aegypti, a mosquito found in the places where a hot and humid climate is predominant [32]. The mosquito A. aegypti inhabits mainly human houses and bites at any time during the day, which makes it a very efficient vector. Infectious individuals, either humans or mosquitoes, can start a dengue epidemic in human populations very quickly when placed in a previously A. aegypti infested region. Dengue is a particularly serious public health problem in Brazil due to favourable climate and environmental conditions for A. aegypti population expansion. A. aegypti was first detected in Brazil in 1923. Since then, the disease subsequently spread into many states in Brazil (see e.g. [7, 10]).

To describe the biological vital dynamics of the A. aegypti, we consider two subpopulations: the winged form (mature female mosquitoes) and an aquatic population, which includes eggs, larvae and pupae. Winged female A. aegypti in search of human blood or places for oviposition are the main reason for local population dispersal and the slow advance of a mosquito infestation. On the other hand, wind currents may also result in an advection movement of large masses of mosquitoes and consequently cause a quick advance of infestation.

The spatial density of the winged A. aegypti at point x and time t is denoted by $M(x,t)$. The aquatic form is denoted likewise by $A(x,t)$. The mathematical model that describes the spatial dynamics of the A. aegypti is then governed by the following reaction-diffusion system:

$$\begin{cases} \frac{\partial}{\partial t}M = \frac{\partial^2}{\partial x^2}M - v\frac{\partial}{\partial x}M + \frac{\gamma(x)}{k(x)}A(1-M) - \mu_1(x)M, & x \in (0,L), t > 0, \\ \frac{\partial}{\partial t}A = k(x)(1-A)M - (\mu_2(x) + \gamma(x))A, & x \in (0,L), t > 0, \end{cases} \tag{2.1}$$

subject to the boundary conditions:

$$\frac{\partial}{\partial x}M(0,t) = \frac{\partial}{\partial x}M(L,t) = 0, \quad t > 0,$$

and the initial conditions:

$$M(x,0) = M_0(x) \geq 0, A(x,0) = A_0(x) \geq 0, \quad x \in [0,L],$$

where v is the wind advection rate, $\mu_1(x)$ and $\mu_2(x)$ are mortality rates of the winged and aquatic population, respectively, $\gamma(x)$ denotes the rate of maturation of the aquatic form into winged form, $k(x)$ is the rate of oviposition by the winged mosquitoes. We assume all parameters are positive and continous with respect to x. And the aquatic form is in a sessile state, that is, no diffusion term for the aquatic form. This model was established in [28] and the propagation dynamics was also studied there.

The purpose of this chapter is to study the threshold type dynamics in terms of the basic reproduction ratio $\mathcal{R}_0$. The main difficulty is the lack of compactness of solution maps of the system (2.1) due to the loss of diffusion term in the second eqution of (2.1). To handle this problem, we first prove that the solution map associated with a linearized system around the trivial solution is κ-contraction, where κ is the Kuratowski measure of noncompactness (see, e.g., [8]). Moreover, we also prove that the solution map of system (2.1) is asymptotically compact on any bounded set by using the ideas in [14, 15]. Thus, in the case where $\mathcal{R}_0 \leq 1$, the trivial solution is stable due to comparison arguments; in the case where when $\mathcal{R}_0 > 1$, by a generalized Krein-Rutman Theorem, the principal eigenvalue of the associated eigenvalue problems exists, and hence, the global dynamics can be derived by the theory of monotone dynamical systems developed in [27, 39].

The rest of this chapter is organized as follows. In the next section, we address the well-posedness of the model. In section 2.3, we establish the threshold dynamics of the system in terms of $\mathcal{R}_0$.

2.2 The well-posedness

In this section, we study the well-posedness of the initial-boundary-value problem (2.1). Let $X = C([0, L], \mathbb{R}^2)$ be the Banach space with the norm

$$\|\psi\| = \max_{1\leq i\leq 2} \max_{x\in[0,L]} \psi_i(x), \ \forall \psi = (\psi_1, \psi_2)^T \in X,$$

and $X_+ = C([0, L], \mathbb{R}^2_+)$. It is easy to see that (X, X_+) is an ordered Banach space. Let $\Lambda = [0, 1] \times [0, 1]$ and

$$X_\Lambda := \{(\psi_1, \psi_2)^T \in X : 0 \leq \psi_1(x) \leq 1, 0 \leq \psi_2(x) \leq 1, x \in [0, L]\}.$$

Definition 2.2.1. *A family of bounded linear operators $\{S(t)\}_{t\geq 0}$ on a Banach space $\mathcal{X}$ is called a C_0-semigroup provided that*

(i) $S(0) = I$.

(ii) $S(t + s) = S(t)S(s)$, $\forall t, s \geqslant 0$.

(iii) *For each $x \in \mathcal{X}$, $S(t)x$ is continuous in $t \geqslant 0$.*

Lemma 2.2.1. *For any $\psi \in X_\Lambda$, system (2.1) has a unique global solution $u(\cdot, t, \psi)$ on $[0, +\infty)$ with $u(\cdot, 0, \psi) = \psi$. Moreover, $u(\cdot, t, \psi) \in X_\Lambda$ for all $t \geq 0$.*

Proof. Let $T_1(t)$ be the C_0 semigroup (see, e.g., [24, Chapters 1 and 7]) on $C([0, L], \mathbb{R})$ of

$$\frac{\partial}{\partial t}\psi_1 = \frac{\partial^2}{\partial x^2}\psi_1 - v\frac{\partial}{\partial x}\psi_1 - \mu_1(\cdot)\psi_1,$$

subject to the Neumann boundary condition. Let $T_2(t)$ be a family of bounded linear operators on $C([0, L], \mathbb{R})$ defined by $T_2(t)\psi_2 = e^{-(\mu_2(\cdot)+\gamma(\cdot))t}\psi_2$, $\forall t \geq 0$, $\psi_2 \in C([0, L], \mathbb{R})$. It is easy to see $T_2(t)$ is a semigroup on $C([0, L], \mathbb{R})$. Define $J : X \to X$

by

$$[J\psi](x) := \begin{pmatrix} \frac{\gamma(x)}{k(x)}\psi_2(x)[1-\psi_1(x)] \\ k(x)\psi_1(x)[1-\psi_2(x)] \end{pmatrix}, \ \forall x \in [0, L], \ \psi = (\psi_1, \psi_2)^T \in X.$$

Then system (2.1) can be written as the following integral equation:

$$u(\cdot, t, \psi) = T(t)\psi + \int_0^t T(t-s)J(u(\cdot, s, \psi))\mathrm{d}s.$$

where $T(t) = \mathrm{diag}(T_1(t), T_2(t))$, $u(x, t, \psi) = (M(x, t), A(x, t))^T$ and $\psi = (M_0(x), A_0(x))^T$. For any given $x \in [0, L]$, $h \geq 0$ and $\psi \in X_\Lambda$, we have

$$\psi(x) + hJ(\psi)(x) = \begin{pmatrix} \psi_1(x) + \frac{\gamma(x)}{k(x)}h\psi_2(x)[1-\psi_1(x)] \\ \psi_2(x) + k(x)h[1-\psi_2(x)]\psi_1(x) \end{pmatrix}.$$

In view of $0 \leq \psi_1(x), \psi_2(x) \leq 1$, it is easy to see that

$$0 \leq \psi_1(x) + \frac{\gamma(x)h}{k(x)}\psi_2(x)(1-\psi_1(x)) \leq 1,$$

and

$$0 \leq \psi_2(x) + k(x)h(1-\psi_2(x))\psi_1(x) \leq 1.$$

In the case where $\psi_1(x) = 1$,

$$\psi_1(x) + \frac{\gamma(x)h}{k(x)}\psi_2(x)(1-\psi_1(x)) = 1.$$

In the case where $0 \leq \psi_1(x) < 1$, there is a small number $\hat{h}_1 = \hat{h}_1(\psi) > 0$ such that

$$0 \leq \psi_1(x) + \frac{\gamma(x)h}{k(x)}\psi_2(x)(1-\psi_1(x)) \leq 1, \ \forall h \in [0, \hat{h}_1].$$

In the case where $\psi_2(x) = 1$,

$$\psi_2(x) + k(x)h(1-\psi_2(x))\psi_1(x) = 1.$$

In the case where $0 \le \psi_2(x) < 1$, there is a small number $\hat{h}_2 = \hat{h}_2(\psi) > 0$ such that

$$0 \le \psi_2(x) + k(x)h(1 - \psi_2(x))\psi_1(x) \le 1, \ \forall h \in [0, \hat{h}_2].$$

Thus, $\psi(x) + hJ(\psi)(x) \in \Lambda$ for sufficiently small $h > 0$. It then follows that

$$\lim_{h \to 0^+} \frac{1}{h}\text{dist}(\psi(x) + hJ(\psi)(x), \Lambda) = 0, \ \forall \psi \in X_\Lambda \text{ and } x \in [0, L].$$

By [21, Corollary 4] with $\tau = 0$, system (2.1) has a unique global solution $u(\cdot, t, \psi)$ on $[0, +\infty)$ with $u(\cdot, 0, \psi) = \psi$, and $u(\cdot, t, \psi) \in X_\Lambda$, $\forall t \ge 0$. $\square$

Lemma 2.2.2. *For any $\psi \in X_+$, system (2.1) has a unique solution $u(\cdot, t, \psi)$ on $[0, \infty)$ with $u(\cdot, 0, \psi) = \psi$, and there exists some $T_0 > 0$ such that $u(\cdot, t, \psi) \in X_\Lambda$ for all $t \ge T_0$.*

Proof. According to [21, Corollary 4] with $\tau = 0$, system (2.1) admits a unique non-continuable solution $u(\cdot, t, \psi)$ on $[0, t_\psi)$ for some $0 < t_\psi \le \infty$ with $u(\cdot, 0, \psi) = \psi$, and $u(\cdot, t, \psi) \in X_+$, $\forall t \in [0, t_\psi)$. Define

$$\underline{\mu}_1 := \min_{x \in [0,L]} \mu_1(x), \ \underline{\mu}_2 := \min_{x \in [0,L]} \mu_2(x), \ \underline{\gamma} := \min_{x \in [0,L]} \gamma(x), \ \underline{k} := \min_{x \in [0,L]} k(x),$$

$$\overline{k} := \max_{x \in [0,L]} k(x), \ \text{and } \overline{\gamma} := \max_{x \in [0,L]} \gamma(x),$$

Let

$$f_1(\overline{M}, \overline{A}) = \begin{cases} \frac{\overline{\gamma}}{\underline{k}}\overline{A}(1 - \overline{M}), & \overline{M} \le 1, \\ 0, & \overline{M} > 1, \end{cases}$$

and

$$f_2(\overline{M}, \overline{A}) = \begin{cases} \overline{k}(1 - \overline{A})\overline{M}, & \overline{A} \le 1, \\ 0, & \overline{A} > 1. \end{cases}$$

Let $\overline{u}(x, t, \psi)$ be the solution of the following cooperative system:

$$\begin{cases} \frac{\partial}{\partial t}\overline{M} = \frac{\partial^2}{\partial x^2}\overline{M} - v\frac{\partial}{\partial x}\overline{M} + f_1(\overline{M}, \overline{A}) - \underline{\mu}\overline{M}, & x \in (0, L), \ t > 0, \\ \frac{\partial}{\partial t}\overline{A} = f_2(\overline{M}, \overline{A}) - (\underline{\mu}_2 + \underline{\gamma})\overline{A}, & x \in (0, L), \ t > 0, \end{cases} \tag{2.2}$$

subject to the Neumann boundary condition $\frac{\partial}{\partial x}M(0, t) = \frac{\partial}{\partial x}M(L, t) = 0, \quad t > 0$ with initial data ψ. Denote $\overline{\psi} = (\overline{\psi}_1, \overline{\psi}_2)^T$ with $\overline{\psi}_1 = \max_{x \in [0,L]} \psi_1(x)$ and $\overline{\psi}_2 =$

$\max_{x\in[0,L]}\psi_2(x)$. Clearly, $\overline{u}(x,t,\overline{\psi})$ is the solution of the following ODE system

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\overline{M} = f_1(\overline{M},\overline{A}) - \underline{\mu}_1\overline{M}, & t>0, \\ \frac{\mathrm{d}}{\mathrm{d}t}\overline{A} = f_2(\overline{M},\overline{A}) - (\underline{\mu}_2+\underline{\gamma})\overline{A}, & t>0. \end{cases} \tag{2.3}$$

We first prove that there exists T_0 such that $\bar{M}(t)\le 1$, $\forall\ t>T_0$. If $\bar{M}_0>1$,then

$$\frac{d}{dt}\bar{M} = -\mu_1\bar{M},\ \forall\bar{M}>1.$$

It is easy to see that $\bar{M}(t)=\bar{M}_0e^{-\mu_1 t},\ \ \forall\bar{M}>1$, and hence, there exists $t_0>0$ such that $\bar{M}(t_0)\le 1$. It suffices to proof that if $\bar{M}(t_1)\le 1$, then $\bar{M}(t)\le 1$, $\forall\ t>t_1$. Assume by contradiction, there exists a $t_2>t_1$ such that $\bar{M}(t_2)>1$. Since $\bar{M}$ is continous with respect to t, there exists a $t_3<t_2$ such that $\bar{M}(t_3)=1$ and $\bar{M}(t)\ge 1$, $\forall t\in[t_3,t_2]$. By the Lagrange mean value formula, we have

$$-\mu_1\bar{M}(\xi) = \bar{M}'(\xi) = \frac{\bar{M}(t_2)-\bar{M}(t_3)}{t_2-t_3} > 0,$$

where $\xi\in[t_3,t_2]$ and $\mu_1>0$. This leads to a contradiction. Similarly, there exists T_1 such that $\bar{A}(t)\le 1$, $\forall\ t>T_1$.

In conclusion, there exists $T_2=\max\{T_0,T_1\}>0$ such that $\overline{u}(\cdot,t,\overline{\psi})\in X_\Lambda$ for all $t\ge T_2$. Notice that $u(x,t,\psi)$ is the subsolution of (2.2), by the comparison principle, we obtain

$$\overline{u}(x,t,\overline{\psi}) \ge \overline{u}(x,t,\psi) \ge u(x,t,\psi),\ \forall x\in[0,L],\ \ t\ge 0.$$

This shows that $u(\cdot,t,\psi)\in X_\Lambda$ for all $t\ge T_0$. □

2.3 The threshold dynamics

In this section, we study the global dynamics of system (2.1). We start with some basic concepts.

Definition 2.3.1. *A closed operator $\mathcal{B}$ in $\mathcal{X}$ is called resolvent-positive if the resolvent set of $\mathcal{B}$, $\rho(\mathcal{B})$, contains a ray (ω,∞) and $(\lambda-\mathcal{B})^{-1}$ is a positive operator for all $\lambda>\omega$.*

Definition 2.3.2. *Let $\sigma(\mathcal{L})$ denote the spectrum of $\mathcal{L}$, i.e., for a bounded linear operator the set of eigenvalues of $\mathcal{L}$. The spectral radius $r(\mathcal{L})$ of a bounded linear* *operator $\mathcal{L}$ is defined as the supremum of the magnitudes of elements in the spectrum:*

$$r(\mathcal{L}) = \sup\{|\lambda| : \ \lambda \in \sigma(\mathcal{L})\}.$$

The spectral bound $s(\mathcal{L})$ of a bounded linear operator $\mathcal{L}$ is defined as the supremum real part of magnitudes of elements:

$$s(\mathcal{L}) = \sup\{\Re\lambda : \ \lambda \in \sigma(\mathcal{L})\}.$$

Definition 2.3.3. *Let $\mathcal{X}$ be a Banach space, $\mathcal{K} \subset \mathcal{X}$ be a cone with $\mathcal{K} \neq \{0\}$, and $\mathcal{B}$ be a resolvent-positive and closed operator. $s(\mathcal{B})$ is called the principal eigenvalue if there exists some $x \in \mathcal{K}$ such that $\mathcal{B}x = s(\mathcal{B})x$.*

Since the second equation of system (2.1) has no diffusion term, the solution maps of that lose the compactness. To overcome this difficulty, we introduce the Kuratowski measure of noncompactness κ on X:

$$\kappa(\mathcal{D}) := \inf\{r : \mathcal{D} \text{ has a finite cover of diameter } r\}$$

for any bounded subset $\mathcal{D}$ of X. We set $\kappa(\mathcal{D}) = +\infty$ whenever $\mathcal{D}$ is unbounded. It is easy to see that $\mathcal{D}$ is precompact if and only if $\kappa(\mathcal{D}) = 0$.

Linearizing system (2.1) at $E_0 = (0, 0)$, we get the following linear system:

$$\begin{cases} \frac{\partial}{\partial t}M = \frac{\partial^2}{\partial x^2}M - v\frac{\partial}{\partial x}M + \frac{\gamma(x)}{k(x)}A - \mu_1(x)M, & x \in (0, L), \ t > 0, \\ \frac{\partial}{\partial t}A = k(x)M - (\mu_2(x) + \gamma(x))A, & x \in (0, L), \ t > 0, \\ \frac{\partial}{\partial x}M(0, t) = \frac{\partial}{\partial x}M(L, t) = 0, & t > 0. \end{cases} \tag{2.4}$$

We substitute $M(x, t) = e^{\lambda t}\phi_1(x)$ and $A(x, t) = e^{\lambda t}\phi_2(x)$ into (2.4) to obtain the following eigenvalue problem:

$$\begin{cases} \lambda\phi_1(x) = \phi_1''(x) - v\phi_1'(x) + \frac{\gamma(x)}{k(x)}\phi_2(x) - \mu_1(x)\phi_1(x), & x \in (0, L), \\ \lambda\phi_2(x) = k(x)\phi_1(x) - (\mu_2(x) + \gamma(x))\phi_2(x), & x \in (0, L), \\ \frac{\partial}{\partial x}\phi_1(0) = \frac{\partial}{\partial x}\phi_1(L) = \frac{\partial}{\partial x}\phi_2(0) = \frac{\partial}{\partial x}\phi_2(L) = 0. \end{cases} \tag{2.5}$$

Let $\Phi(t)$ be a C_0 semigroup on X of (2.4), and its generator B can be written as

$$B\psi := \begin{pmatrix} \frac{\partial^2}{\partial x^2}\psi_1 - v(\cdot)\frac{\partial}{\partial x}\psi_1 - \mu_1(\cdot)\psi_1 + \frac{\gamma(\cdot)}{k(\cdot)}\psi_2, \\ k(\cdot)\psi_1 - [\mu_2(\cdot) + \gamma(\cdot)]\psi_2 \end{pmatrix}, \ \forall \psi = (\psi_1, \psi_2)^T \in X.$$

It then follows that B is a closed and resolvent-positive operator (see [30, Theorem 3.12]). Let $s(B)$ be the spectral bound of B and $r(\Phi(t))$ be spectral radius of the linear operator $\Phi(t)$. Then we have the following two results.

Lemma 2.3.1. *$r(\Phi(t)) = e^{s(B)t}$ for any $t \geq 0$.*

Proof. The proof is essentially the same as that of [23, Lemma 3.1]. For completeness, here we provide the details. Let $\overline{\omega}(\Phi)$ be the exponential growth bound of the semigroup $\Phi(t)$. It follows from [30, Formula (3.4)] that $\overline{\omega}(\Phi) = \frac{\ln r(\Phi(t_0))}{t_0}$ for any $t_0 > 0$. Moreover, [30, Theorem 3.14(iv)] implies that $\overline{\omega}(\Phi) = s(B)$. Thus, we have $r(\Phi(t_0)) = e^{s(B)t_0}$ for any $t_0 > 0$. □

Lemma 2.3.2. *If $s(B) \geq 0$, then $\lambda^*{:=}s(B)$ is the principal eigenvalue of system (2.5) with a positive eigenfunction.*

Proof. Let $\tilde{T}_1(t)$ be the analytic semigroup on $C([0, L], \mathbb{R})$ generated by

$$\frac{\partial}{\partial t}M = \frac{\partial^2}{\partial x^2}M - v\frac{\partial}{\partial x}M - \mu_1(x)M, \ x \in (0, L), \ t > 0,$$

subject to the Neumann boundary condition $\frac{\partial}{\partial x}M(0, t) = \frac{\partial}{\partial x}M(L, t) = 0, \quad t > 0,$. Let $\tilde{T}_2(t)$ be a family of bounded linear operators on $C([0, L], \mathbb{R})$ define by $\tilde{T}_2(t)\varphi_2 = e^{-(\mu_2(\cdot)+\gamma(\cdot))t}\varphi_2$. For any given $t \geq 0$, define a linear operator $P(t)$ and a linear $Q(t)$ on X by

$$P(t)\varphi := (0, \tilde{T}_2(t)\varphi_2)^T, \ \forall \varphi = (\varphi_1, \varphi_2)^T \in X,$$

and

$$Q(t)\varphi := \left(M(\cdot, t, \varphi), k(\cdot) \int_0^t e^{-(\mu_2(\cdot)+\gamma)(t-s)} M(\cdot, s, \varphi)\, ds \right)^T, \ \forall \varphi = (\varphi_1, \varphi_2)^T \in X,$$

where

$$M(\cdot, t, \varphi) = \tilde{T}_1(t)\varphi_1 + \int_0^t \tilde{T}_1(t-s) \left[\frac{\gamma(\cdot)}{k(\cdot)} A(\cdot, s, \varphi) \right] \mathrm{ds}.$$

It is easy to see that

$$\Phi(t)\varphi = P(t)\varphi + Q(t)\varphi, \ \forall \varphi \in X, \ t \geq 0.$$

Since

$$\|P(t)\varphi\| \leq \left\|\tilde{T}_2(t)\varphi_2\right\| \leq e^{-(\mu_2(\cdot)+\gamma(\cdot))t}\varphi_2 \leq e^{-(\mu_2(\cdot)+\gamma(\cdot))t}\varphi,$$

we have $\|P(t)\| \leq e^{-(\mu_2(\cdot)+\gamma(\cdot))t}$. By the boundedness of $\Phi(t)$ and the compactness of $\tilde{T}_1(t)$ for $t > 0$, it follows that $Q(t) : X \to X$ is compact for each $t > 0$. For any bounded set B_0 in X, there holds $\kappa(Q(t)B_0) = 0$ since $Q(t)B_0$ is precompact, and consequently,

$$\kappa(\Phi(t)B_0) \leq \kappa(P(t)B_0) + \kappa(Q(t)B_0) \leq \|P(t)\| \, \kappa(B_0) \leq e^{-(\mu_2(\cdot)+\gamma(\cdot))t}\kappa(B_0), \ \forall t > 0.$$

That is, $\Phi(t)$ is κ-contraction on X for each $t > 0$. Note that

$$r_e(\Phi(t)) \leq e^{-(\mu_2(\cdot)+\gamma(\cdot))t} < 1 \leq e^{s(B)t} = r(\Phi(t)), \ \forall t > 0,$$

where $r_e(\Phi(t))$ and $r(\Phi(t))$ represent the essential spectral radius and the spectral radius of the operator $\Phi(t)$, respectively. This inequality shows that the essential spectral radius is strictly less than the spectral radius. By a generalized Krein-Rutman theorem (see, e.g., [22, Corollary 2.2] or [16, Appendix A]), $s(B)$ is the principal eigenvalue of system (2.5). □

In the following, we adopt the idea of next geneation operators to define the basic reproduction ratio. Let $S(t) : X \to X$ be the C_0 semigroup generated by the following system:

$$\begin{cases} \frac{\partial}{\partial t}M = \frac{\partial^2}{\partial x^2}M - v\frac{\partial}{\partial x}M - \mu_1(x)M, & x \in (0, L), \ t > 0, \\ \frac{\partial}{\partial t}A = -(\mu_2(x) + \gamma(x))A, & x \in (0, L), \ t > 0, \end{cases} \tag{2.6}$$

subject to the Neumann boundary condition $\frac{\partial}{\partial x}M(0, t) = \frac{\partial}{\partial x}M(L, t) = 0, \forall t > 0$.
Define $F(x) = \begin{pmatrix} 0 & \dfrac{\gamma(x)}{k(x)} \\ k(x) & 0 \end{pmatrix}$ and $S(t)\varphi$ is the distribution of population under the influence of mobility and mortality.

Let $L := X \to X$ be defined by

$$L(\varphi)(x) = \int_0^{+\infty} F(x)[S(t)\varphi](x)\, dt.$$

It then follows that $L(\varphi)(x)$ represents the distribution of new total population generated by initial Aedes aegypti. Following [36], we define the spectral radius of L as the basic reproduction ratio for system (2.1), that is,

$$\mathcal{R}_0 := r(L).$$

By [36, Theorem 3.1], we have the following observation.

Lemma 2.3.3. *$\mathcal{R}_0 - 1$ has the same sign as λ^*.*

Define the solution semiflow $\Psi(t) : X \to X,\ t \geq 0$ by:

$$\Psi(t)\psi = u(\cdot, t, \psi),\ \forall t \geq 0,$$

where $u(\cdot, t, \psi)$ is the unique solution of system (2.1) with $u(\cdot, 0, \psi) = \psi \in X$. By Lemma 2.2.1, we can easily see that X_Λ is a positively invariant set for $\Psi(t)$, that is, for any $\psi \in X_\Lambda$, there holds $\Psi(t)\psi \in X_\Lambda,\ \forall t \geq 0$.

Lemma 2.3.4. *The solution semiflow $\Psi(t)$ is κ-contracting in the sense that*

$$\lim_{t\to\infty} \kappa(\Psi(t)\mathcal{D}) = 0$$

for any bounded set $\mathcal{D} \subset X_\Lambda$.

Proof. Let $g = (g_1, g_2)$ be defined as

$$\begin{cases} g_1(x, A, M) = \frac{\gamma(x)}{k(x)} A(1 - M) - \mu_1(x)M,\ \forall x \in [0, L] \\ g_2(x, A, M) = k(x)(1 - A)M - (\mu_2(x) + \gamma(x))A,\ \forall x \in [0, L]. \end{cases} \tag{2.7}$$

It is easy to see that there exists a constant $\hat{r} = \min(\mu_2(x) + \gamma(x)) > 0,\ x \in [0, L]$ such that

$$\frac{\partial g_2(x, A, M)}{\partial A} = -k(x)M - (\mu_2(x) + \gamma(x)) \leq -(\mu_2(x) + \gamma(x)) \leq -\hat{r},\ \forall x \in [0, L]. \tag{2.8}$$

By [14, Lemma 4.1], it follows that $\Psi(t)$ is asymptotically compact on $\mathcal{D}$ in the sense that for any sequences $\varphi_n \in \mathcal{D}$ and $t_n \to \infty$, there exist subsequences φ_{n_i} and $t_{n_i} \to \infty$ such that $\Psi(t_{n_i})\varphi_{n_i}$ converges in X as $i \to \infty$. By [26, Lemma 23.1(2)], it follows that the omega-limit set $\omega(\mathcal{D})$ of $\mathcal{D}$ is nonempty, compact and invariant in X_Λ, and $\omega(\mathcal{D})$ attracts $\mathcal{D}$. In view of [20, Lemma 2.1(b)], we have

$$\kappa(\Psi(t)\mathcal{D}) \leq \kappa(\omega(\mathcal{D})) + \delta(\Psi(t)\mathcal{D}, \omega(\mathcal{D})) \to 0 \text{ as } t \to \infty,$$

where

$$\delta(\Psi(t)\mathcal{D}, \omega(\mathcal{D})) := \sup_{y \in \Psi(t)(\mathcal{D})} d(y, \omega(\mathcal{D})), \text{ and } d(y, \omega(\mathcal{D})) := \inf_{z \in \omega(\mathcal{D})} d(y, z).$$

This completes the proof. □

Recall that a square matrix is said to be cooperative if its off-diagonal elements are nonnegative, and irreducible if it is not similar, via a permutation, to a block upper triangular matrix.
Now we are in a position to prove the main result of this chapter.

Theorem 2.3.1. *The following statements are valid:*

(i) *If $\mathcal{R}_0 \leq 1$, then $E_0 := (0, 0)$ is globally asymptotically stable for system (2.1) in X_+.*

(ii) *If $\mathcal{R}_0 > 1$, then system (2.1) has a unique positive steady state $E_* := (M_*, A_*)$ and E_* is globally asymptotically stable for system (2.1) in $X_+ \backslash \{0\}$.*

Proof. Without loss of generality, we assume that $\psi \in X_\Lambda$ due to Lemma 2.2.2. Let $g = (g_1, g_2)$ be defined as in (2.7), and let

$$\tilde{X}_\Lambda := \{(\psi_1, \psi_2)^T \in X : 0 < \psi_1(x) < 1, 0 < \psi_2(x) < 1, \forall x \in [0, L]\}.$$

Then the Jacobi matrix of $g(x, \psi_1, \psi_2)$ with respect to (ψ_1, ψ_2) is cooperative and irreducible at any point in $\tilde{X}_\Lambda$. Define an operator G on X_Λ by

$$[G(\psi)](x) := g(x, \psi_1(x), \psi_2(x)), \ \forall x \in [0, L], \ \psi = (\psi_1, \psi_2)^T \in X_\Lambda.$$

It follows that G is strictly subhomogeneous on X_Λ in the sense that $G(s\psi) > sG(\psi)$ for any $0 < s < 1$ and $\psi \in X_\Lambda$ with $\psi \gg 0$. By the arguments similiar to those in the proof of [27, Theorem 7.4.1] and [11, Theorem 2.2] respectively, we see that solution map $\Psi(t)$ is strongly monotone and strictly subhomogenous on X_Λ for each $t > 0$.

In the case where $\mathcal{R}_0 < 1$, Lemma 2.3.3 implies that $s(B) < 0$. By [30, Theorem 3.14], we have $\overline{\omega}(\Phi) = s(B) < 0$. In view of the definition of $\overline{\omega}(\Phi)$, it follows that $\lim_{t\to\infty} \|\Phi(t)\| = 0$. By the comparison principle, we have

$$\Psi(t)\psi \leq \Phi(t)\psi, \ \forall t \geq 0, \ \psi \in X_\Lambda.$$

This implies that E_0 is globally attractive in X_+. By the continous-time version of [39, Lemma 2.2.1], E_0 is Liapunov stable, and hence, E_0 is globally asymptotically stable for system (2.1) in X_+.

In the case where $\mathcal{R}_0 \geq 1$, Lemma 2.3.3 implies that $s(B) \geq 0$. For any given $t_0 > 0$, $\Psi(t_0)$ is strongly monotone and strictly subhomogenous on X_Λ, $\Psi(t_0)E_0 = E_0$ and the Fréchet derivative $D[\Psi(t_0)](E_0) = \Phi(t_0)$. In view of Lemma 2.3.2, we easily see that $r(\Phi(t_0))$ is a positive eigenvalue of $\Phi(t_0)$ and $r(\Phi(t_0)) = e^{s(B)t_0} \geq 1$.

If $\mathcal{R}_0 = 1$, then $r(\Phi(t_0)) = 1$, and hence, [39, Theorem 2.3.4(a) and Remark 2.1.4] imply that every forward orbit for the map $\Psi(t_0)$ in X_Λ converges to E_0.

If $\mathcal{R}_0 > 1$, then $r(\Phi(t_0)) > 1$. By [39, Theorem 2.3.4(b) and Remark 2.1.4], it follows that there exists a unique fixed point $E_* \gg 0$ of $\Psi(t_0)$ in X_Λ such that every forward orbit for the map $\Psi(t_0)$ in $X_\Lambda \backslash \{\mathbf{0}\}$ converges to E_*. For any $t \geq 0$, $\Psi(t)E_* = \Psi(t)\Psi(t_0)E_* = \Psi(t_0)\Psi(t)E_*$, and hence, $\Psi(t)E_*$ is also a fixed point of $\Psi(t_0)$. By the uniqueness of the positive fixed point for $\Psi(t_0)$, we have $\Psi(t)E_* = E_*$ for all $t \geq 0$. Thus, E_* is a steady state of system (2.1) in X_Λ and

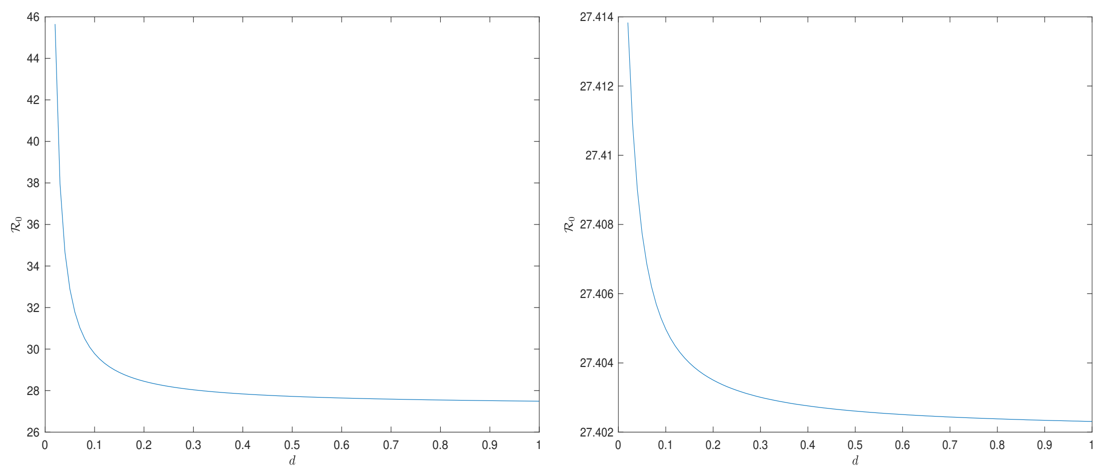
$$\lim_{t\to\infty} \|\Psi(t)\psi - E_*\| = \lim_{t\to\infty} \|\Psi(t)\psi - \Psi(t)E_*\| = 0, \ \forall \psi \in X_\Lambda \backslash \{\mathbf{0}\}.$$

Again by the continous-time version of [39, Lemma 2.2.1], we see that E_* is globally asymptotically stable for system (2.1) in $X_+ \backslash \{0\}$. $\square$

2.4 Numerical simulations

In this section, we study the relation between the basic reproduction number ratio $\mathcal{R}_0$ and the coefficient of the diffusion term d via numerical simulations.

Example 2.4.1. *Let $L = 1$, we numerically approximate $\mathcal{R}_0$ for (2.4). Most parameters are taken from [28]: ν=0.08164, γ=0.25, k=0.00666, μ_2=0.000333. We first consider $\mu_1 = 0.00133[1 - \cos(\pi x)]$ where $x \in [0, 1]$ and $\mu_2 = 0.000333$. Then we choose $\mu_1 = 0.00133$ and $\mu_2 = 0.000333(1 - \cos(\pi x))$. Different trends of $\mathcal{R}_0$ in 2D are observed in Figure 2.1. It shows that $\mathcal{R}_0$ is a decreasing function with respect to d.*



(a) (b)

Figure 2.1: Relationship between $\mathcal{R}_0(a, b)$ and the coefficient d under 2 different scenarios: (a) $\mu_1 = 0.00133[1 - \cos(\pi x)]$ where $x \in [0, 1]$ and $\mu_2 = 0.000333$; (b) $\mu_1 = 0.00133$ and $\mu_2 = 0.000333(1 - \cos(\pi x))$.

Example 2.4.2. *In this example, we present long-term behavior numerical analyses based on the parameter values from [28]. Firstly, let us choose the following parameter values and the others stay the same as above: $\nu = 0.08164$, $\gamma = 0.25$, $k = 0.00666$, $\mu_1 = 0.00133$, $\mu_2 = 0.000333$. We numerically compute the basic reproduction ratios and obtain $\mathcal{R}_0 = 27.4022 > 1$. Setting the initial values as $A_0(x) = M_0(x) = 1$. The long-term behaviors of M and A are shown in Figure 2.2(a)(b). Then we choose $\mu_1 = 0.133$, $\mu_2 = 0.0333$ and other parameters are the same as above, we obtain*

$\mathcal{R}_0 = 0.5678 < 1$. *The long-term behaviors of M and A are shown in Figure2.2(c)(d), which is coincident with Theorem 2.3.*

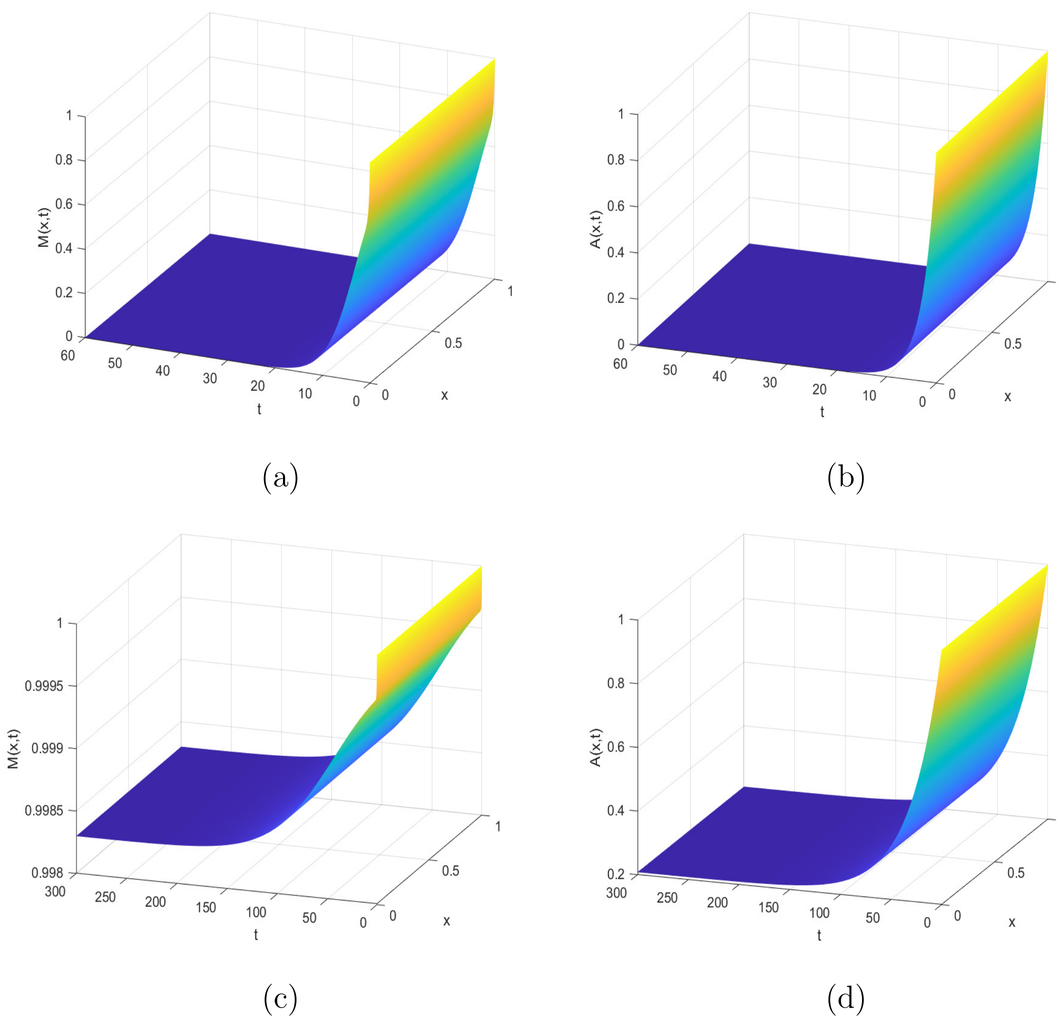


Figure 2.2: Long term behaviors of of two sub-populations in system (2.1) when $\mathcal{R}_0 = 0.3364$ and $\mathcal{R}_0 = 3.8581$

Chapter 3

Summary and future works

In this chapter, we first briefly summarize the main results in the thesis, and then present some possible future works.

3.1 Summary

In this thesis, we have studied the R_0 properties and the global dynamics for two population models. In Chapter 1, we discussed the relationship between the basic reproduction ratio and the coefficients representing spatial heterogeneity. We used the variational formula of $\mathcal{R}_0$ to explore the effect of the spatial heterogeneity on $\mathcal{R}_0$. From Theorems 1.2.4 and 1.2.5, we see that $\mathcal{R}_0$ can be monotone or non-monotone with the increasing spatial heterogeneity. We also computed $\mathcal{R}_0$ numerically to investigate these influences. The long term behavior of solutions are simulated for two cases $R_0 < 1$ and $R_0 > 1$, respectively. Biologically, we should take some measures to make $R_0 < 1$ so that the disease transmission can be controlled.

In Chapter 2, we established the threshold type result on the global stability for system (2.1) in terms of $\mathcal{R}_0$. To analyse the long term behavior of two sub-populations, we adopted the idea of next generation operators to define the basic reproduction ratio. Although the solution maps are not compact, we are able to prove that the solution semiflow is asymptotically compact. From Theorem 2.3.1, we see that if $\mathcal{R}_0 \leq 1$, $E_0 := (0, 0)$ is globally asymptotically stable; while $\mathcal{R}_0 > 1$, system (2.1) has an unique positive steady state $E_* := (M_*, A_*)$ and E_* is globally asymptotically stable for system (2.1). This indicates that R_0 is a threshold value to determine whether the population goes extinct or persists uniformly.

3.2 Future works

Related to the projects in this thesis, there are some interesting but challenging problems for future investigation.

In Chapter 1, we assumed that L is symmetric. However, it seems that the same result still holds true when L is asymmetric (see Figure 1.2). This motivates us to further prove it analytically. We may also consider the global dynamics and the monotonicity of $\mathcal{R}_0$ for the model system in an environment with seasonality. In such a case, the coefficients in system (1.1) depend on both x and t and are ω-periodic in t for some real number $\omega > 0$. This gives rise to a time-periodic version of system (1.1).

We may study travelling waves and spreading properties of solutions for system (2.1) in the case where the spatial variable x changes in the real line $\mathbb{R}$ by appealing to the theory developed in [19]. In addition, a possible extension of this spatial model is to replace the Laplacian term with a non-local dispersal one. As such, we should first study an associated principal eigenvalue problem with nonlocal dispersal and then discuss the global dynamics in terms of R_0.

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