Some aspects of rings whose elements satisfy a special property

by

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A thesis submitted to the
School of Graduate Studies
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Department of Mathematics and Statistics
Memorial University of Newfoundland

August, 2021

St. John's, Newfoundland and Labrador, Canada

Abstract

Throughout, rings are associative with identity $1 \neq 0$. The main focus of this thesis is on the study of rings whose elements are sums of certain special elements and, in this case, these elements are from the sets of nilpotents, units, idempotents, involutions, and tripotents.

In Chapter 1, we provide some basic definitions and results in ring theory which are needed for this thesis.

In Chapter 2, we study rings whose elements are sums of a nilpotent and an idempotent (i.e., nil-clean rings). The motivation is the open question, raised by Breaz et al., whether the ring of linear transformations of a countable dimensional vector space over $\mathbb{F}_2$ is a nil-clean ring. In Section 2.1, we first prove that for a semisimple module M over a ring R with $R/J(R)$ Boolean, every endomorphism of M is a sum of an idempotent endomorphism and a locally nilpotent endomorphism. As a consequence, it is proved that, for a vector space V over a division ring D, every linear transformation of V is a sum of an idempotent linear transformation and a locally nilpotent linear transformation if and only if $D \cong \mathbb{F}_2$. In Section 2.2, we study nil $*$-clean rings with emphasis on matrix rings. We show that a $*$-ring R is a nil $*$-clean ring if and only if $J(R)$ is nil and $R/J(R)$ is a nil $*$-clean ring. Particularly, it is shown that for a 2-primal $*$-ring R, with involution $*$ given by $(a_{ij})^* = (a_{ij}^*)^T$, $\mathbb{M}_n(R)$ is a nil $*$-clean ring if and only if $J(R)$ is nil, $R/J(R)$ is Boolean, $a^* - a \in J(R)$ for all $a \in R$ and $\mathbb{M}_n(\mathbb{F}_2)$ is nil $*$-clean. Notice that the structure of nil-clean rings is unknown. We give a description of nil-clean rings with nilpotency index at most 2 in Section 2.3.

In Chapter 3 we first give a review of known results regarding rings whose elements are sums of nilpotents, idempotents, or tripotents. In Section 3.2 we determine the structure for a more general class of rings i.e., rings in which every element is a sum of a nilpotent and three tripotents that commute with one another and, in the last section, we discuss when a group ring has such property.

In Chapter 4, we focus on decomposing a matrix as a sum of certain special matrices. In Section 4.1 we give the necessary and sufficient conditions for an $n \times n$ matrix over an integral domain to be a sum of involutions and, respectively, a sum of tripotents. We completely determine the integral domains over which every $n \times n$ matrix is a sum of involutions and, respectively, a sum of tripotents. We also show that every $n \times n$ matrix over an integral domain R is a sum of two tripotents if and only if $R \cong \mathbb{F}_p$ where $p = 2$, 3, or 5.

The last chapter of the thesis is about rings whose elements are left annihilator-stable. An element a in a ring R is left annihilator-stable (or left AS) if, whenever $Ra + \mathbf{l}(b) = R$ with $b \in R$, then $a - u \in \mathbf{l}(b)$ for some unit u in R, and the ring R is a left AS ring if each of its elements is left AS. In this chapter, we show that the left AS elements in a ring form a multiplicatively closed set, giving an affirmative answer to a question of Nicholson [66]. This result is further used to obtain a necessary and sufficient condition for a formal triangular matrix ring to be left AS. As an application, we provide examples of left AS rings R over which the triangular matrix rings $\mathbb{T}_n(R)$ are not left AS for all $n \geq 2$. These examples give a negative answer to another question of Nicholson [66] whether $R/J(R)$ being left AS implies that R is left AS.

General Summary

Rings whose elements are sums of certain special elements have been widely studied in ring theory. Some of the most interesting cases are elements from the sets of units, nilpotents, idempotents, involutions, and tripotents. In this thesis we present our study on:

(1) Rings whose elements are sums of a nilpotent and an idempotent.

(2) Rings whose elements are a sum of a nilpotent and three tripotents that commute with one another.

(3) Decompositions of matrices as sums of idempotents, involutions, and tripotents.

(4) Rings whose elements are uniquely generated.

Statement of Contributions

This dissertation is my joint work under the supervision of Professor Yiqiang Zhou. It combines the following six papers:

i. G. Tang, G. Xia, Y. Zhou, When is every linear transformation a sum of an idempotent one and a locally nilpotent one? *Linear Algebra Appl.* **543** (2018), 226-233.

ii. J. Cui, G. Xia, Y. Zhou, Nil-clean rings with involution, *Algebra Colloquium*, **28** (2021), 367-378.

iii. Y. Li, X. Quan, G. Xia, Nil-clean rings of nilpotency index at most two with application to involution-clean rings, *Commun. Korean Math. Soc.* **33**(3) (2018), 751-758.

iv. J. Cui, G. Xia, Rings in which every element is a sum of a nilpotent and three tripotents, *Bull. Korean Math. Soc.* **58** (2021), 47-58.

v. G. Xia, G. Tang, Y. Zhou, When is a matrix a sum of involutions or tripotents? *Comm. Algebra*, **49** (2021), 1717-1724.

vi. G. Xia, Y. Zhou, Annihilator-stability and two questions of Nicholson, *Glasgow Math. J.*, **63** (2021), 258-265.

Acknowledgements

First and foremost, I would like to thank my supervisor, Professor Yiqiang Zhou for his invaluable guidance, for his financial support, encouragement and patience throughout my Ph.D. study. Without his insightful guidance and constant feedback, my work would not have been achievable.

I also would like to thank Dr. Cox, Dr. Xiao, and Dr. Martinez-Pedroza for teaching me graduate courses. I sincerely thank all the members of the Examination Committee for taking time to review this thesis.

The findings in this dissertation have been combined efforts with myself, Gaohua Tang, Jian Cui, Yu Li, Xiaoshan Quan and Yiqiang Zhou. I thank them for allowing me to include our results into my dissertation.

I gratefully acknowledge the financial support provided by the School of Graduate Studies and Department of Mathematics and Statistics in the form of Graduate Fellowships and Teaching Assistantships.

In addition, I would like to thank my family for all their unconditional support. Finally, I would like to thank Joshua for his understanding, support, and sympathetic ear throughout my Ph.D. study.

List of Symbols

M_R	right R-module M over a ring R
$\mathrm{End}(M_R)$	ring of endomorphisms of a right R-module M
$\mathbb{M}_n(R)$	$n \times n$ matrix ring over a ring R
$\mathbb{T}_n(R)$	$n \times n$ upper triangular matrix ring over a ring R
rad(M)	radical of a module M
$J(R)$	Jacobson radical of a ring R
$R[x]$	polynomial ring over a ring R with indeterminate x
$R[[x]]$	ring of formal power series over a ring R with indeterminate x
$\mathbb{Z}$	ring of integers
$\mathbb{Z}_n$	ring of integers modulo n
$\widehat{\mathbb{Z}}_p$	ring of p-adic integers
$U(R)$	unit group of a ring R
$\mathrm{Nil}(R)$	set of nilpotent elements of a ring R
ch(R)	characteristic of a ring R
$tr(A)$	trace of a matrix A
rank(A)	rank of a matrix A
$\mathbf{l}(a)$	left annihilator of an element a in a ring
$\mathbf{r}(a)$	right annihilator of an element a in a ring
E_{ij}	matrix with (i,j)-entry 1 and 0 elsewhere
$\mathbb{F}_p$	field of order p

Contents

Introduction

Let R be an associative ring with $1 \neq 0$. An element $r \in R$ is called **nil-clean** if there is an idempotent $e \in R$ and a nilpotent $b \in R$ such that $r = e+b$; the element r is further called **strongly nil-clean** if the idempotent and the nilpotent can be chosen such that $be = eb$. A ring is called **nil-clean** (**strongly nil-clean**) if each of its elements is nil-clean (strongly nil-clean). The notion of nil-clean rings was first introduced by Diesl in [31] and has been studied by many researchers for the last decade. Before we get to know more about nil-clean rings, it is worthwhile to mention that the study of nil-clean rings was motivated by the study of clean rings. Introduced by Nicholson in [64] and [65], an element a of R is defined to be **clean** if $a = e + u$, where $e \in R$ is an idempotent and $u \in U(R)$; moreover, a is called **strongly clean** if the idempotent and the unit can be chosen such that $eu = ue$. A ring is defined to be (**strongly**) **clean** if each of its elements is (strongly) clean. The class of clean rings contains semiperfect rings, unit-regular rings, strongly π-regular rings, etc. (See [17], [18], and [15]). Clean rings have been extensively investigated in the past forty years with studies related to topology, functional analysis, and the Köthe conjecture (see [3], [5], [16], [19], [37], [45], [46], [54], [72], etc). Clean rings were initially studied in [64] as a class of rings satisfying the exchange property which was introduced by Crawley and J*ó*nsson in 1964 in [26]. A module M is said to have the **exchange property** if for any module G and any two decompositions $G = M' \oplus N = \sum_{i\in I} A_i$ where $M' \cong M$, there are submodules $A_i' \subseteq A_i$ such that $G = M' \oplus \sum A_i'$; the module M is said to satisfy the finite exchange property if this holds whenever the index set I is finite. In [79], Warfield introduced exchange rings. A ring R is called an **exchange ring** if the regular module R_R has the exchange property. It is well-known that the definition of an exchange ring is right-left symmetric and the module M_R has the finite exchange property if and only if the endomorphism ring $\mathrm{End}(M_R)$ is an exchange ring [79]. In [64], Nicholson proved that clean rings are exchange rings and, conversely, exchange rings with central idempotents are clean rings. In [38], Bergman constructed an exchange ring which is not clean. Therefore, clean rings form a proper subclass of exchange rings. An **involution** of a ring R is an operation $* : R \to R$ such that $(x+y)^* = x^* + y^*$, $(xy)^* = y^* x^*$ and $(x^*)^* = x$ for all x, y in R. Clearly, the identity mapping $\mathbf{id}_R$ is an involution of R if and only if R is commutative. A ring R with involution $*$ is called a **$*$-ring**. An element e in a $*$-ring R is called a **projection** if $e^2 = e = e^*$. Following L.Vaš [77], a $*$-ring is called $*$-clean if each of its elements is a sum of a unit and a projection, and a $*$-ring is called a strongly $*$-clean ring if each of its elements is a sum of a unit and a projection that commute. Various characterizations of $*$-clean rings and strongly $*$-clean rings have been obtained (see [77] and [55]).

While nil-clean rings form a proper subclass of clean rings, some fundamental properties have been proved in [32] and many interesting results have been published on this class of rings in the past decade (for example, see [14], [49], [60], [32], [50], [61]). In [32] Diesl proved that a ring R is strongly nil-clean if and only if R is strongly π-regular and $U(R) = 1 + \mathrm{Nil}(R)$. Notice that the structure theorem for nil-clean rings is till unknown. In [50], Kosan, Wang, and Zhou completely characterized strongly nil-clean rings and they proved that a ring R is strongly nil-clean if and only if $R/J(R)$ is Boolean and $J(R)$ is nil. In [61] the researchers proved that for a commutative ring R and an abelian group G, the group ring RG is nil-clean if and only if R is nil-clean and G is a 2-group (see [61, Theorem 2.6]). Notice that every abelian group is a nilpotent group. In [69], Sahinkaya, Tang and Zhou discussed when the group ring of a nilpotent group over an arbitrary ring is nil-clean. They showed that for a ring R and a nilpotent group G, RG is nil-clean if and only if R is a nil-clean ring and G is a 2-group (see [69, Theorem 2.7]); they also proved that RS_3 is a nil-clean ring if and only if both R and $\mathbb{M}_2(R)$ are nil-clean rings (see [69, Proposition2.9]). This shows that a group G being a 2-group is not a necessary condition for the group ring RG

to be nil-clean and the question is reduced to determining when a matrix ring is nil-clean. Let A be a square matrix over a field. Then $A = E + W$ where E is similar to a diagonal matrix, W is nilpotent, and $EW = WE$. Such a decomposition is called the Jordan-Chevalley decomposition in Lie theory (see [43]). A strongly nil-clean decomposition of a square matrix over a field implies the well-known Jordan-Chevalley decomposition in Lie theory. In [32], Diesl showed that the matrix ring $\mathbb{M}_n(R)$ is nil-clean if and only if $\mathbb{M}_n(R/J(R))$ is nil-clean and $\mathbb{M}_n(J(R))$ is nil, and he raised the following question:

- Does R nil-clean imply that the matrix ring $\mathbb{M}_n(R)$ is nil-clean?

In [14], Breaz et al. proved that for a field F, the matrix ring $\mathbb{M}_n(F)$ is nil-clean if and only if $F \cong \mathbb{F}_2$. In [73], Šter strengthened this result by showing that every $n \times n$ matrix A over $\mathbb{F}_2$ can be decomposed as $A = E + Q$ where E is an idempotent and Q is a nilpotent with $Q^4 = 0$. The authors in [14] also raised a question: whether their result holds for a division ring? This question was affirmatively answered by Kosan, Lee, and Zhou in [49] and they proved that, for a division ring D and an integer $n \geq 1$, $\mathbb{M}_n(D)$ is nil-clean if and only if $D \cong \mathbb{F}_2$ (see [49, Theorem 3]). The **prime radical** $\text{Nil}_*(R)$ of a ring R is defined to be the intersection of all the prime ideals in R. A **2-primal ring** is a ring such that $\text{Nil}_*(R) = \text{Nil}(R)$. In [50], Kosan, Wang and Zhou proved that for a 2-primal ring R and an integer $n \geq 1$, $\mathbb{M}_n(R)$ is nil-clean if and only if $J(R)$ is nil and $R/J(R)$ is Boolean (see [50, Theorem 6.1]). Köthe's conjecture, formulated in 1930, asks whether the sum of two nil left ideals is nil in any ring, or equivalently if the matrix ring over any nil ring is again nil. It is known that Köthe's conjecture has a positive solution if and only if it has positive solution for algebras over fields (see [52, Theorem 6]). In [60], Matczuk proved that the matrix ring over any nil-clean ring is nil-clean if and only if Köthe's conjecture has a positive answer for the class of $\mathbb{F}_2$-algebras. Since the conjecture is still unsettled, this suggests the difficulty of Diesl's question on nil-clean matrix rings as well as the difficulty of determining the nil-cleanness of a group ring in general.

As mentioned before, in [49], Kosan et al. proved that for a division ring D, the matrix ring $\mathbb{M}_n(D)$ is nil-clean if and only if $D \cong \mathbb{F}_2$ ([14, Theorem 3]). For the property of being clean, it is known that every linear transformation of a countably infinite dimensional vector space over a division ring is clean. The authors in [14] raised a similar question for nil-clean rings:

- Is the ring of linear transformations of a countably infinite dimensional vector space over $\mathbb{F}_2$ a nil-clean ring? (Breaz et al)

An endomorphism f of a module M is called **locally nilpotent** if for any $x \in M$, $f^n(x) = 0$ for some $n > 0$. It is easily seen that, for a finitely generated module M, an endomorphism of M is locally nilpotent if and only if it is nilpotent. Thus, the result of Kosan et al. in [49] can be phrased as follows: for a **finite dimensional** vector space V over a division ring D, every linear transformation of V is a sum of an idempotent linear transformation and a locally nilpotent linear transformation if and only if $D \cong \mathbb{F}_2$. Also, the question of Breaz et al. in [14] has the following "local" version:

- For a vector space V of arbitrary dimension over a division ring D, is it true that every linear transformation of V is a sum of an idempotent linear transformation and a locally nilpotent linear transformation if and only if $D \cong \mathbb{F}_2$? (Question 1)

It is clear that the ring $\mathbb{M}_n(\mathbb{F}_2)$ is nil-clean. Observe that, aside for just being nil-clean, every matrix in $\mathbb{M}_2(\mathbb{F}_2)$ is a sum of a symmetric idempotent matrix and a nilpotent matrix. For a $*$-ring R, there is an involution of $\mathbb{M}_n(R)$ given by $(a_{ij})^* = (a_{ij}^*)^T$ where A^T denotes the transpose of matrix A. Particularly, for a commutative ring R, $\mathbb{M}_n(R)$ is a $*$-ring where $A^* = A^T$ for any $A \in \mathbb{M}_n(R)$. Thus, the above example indicates that, with $*$ being the transpose, every element in $\mathbb{M}_2(\mathbb{F}_2)$ is a sum of a projection and a nilpotent.

In other words, $\mathbb{M}_2(\mathbb{F}_2)$ is a nil $*$-clean ring, i.e., every element is a sum of a projection and a nilpotent. Thus, the question follows:

- For a $*$-ring R, when is the matrix ring $\mathbb{M}_n(R)$ a nil $*$-clean ring where $(a_{ij})^* = (a_{ij}^*)^T$? (Question 2)

In [41], Hirano and Tominaga considered a class of rings in which each of its elements is the sum of two idempotents. For convenience, a ring is called a HT-**ring** if each of its elements is a sum of two idempotents. The structure of HT-rings is unknown so far. An element a in a ring is called **tripotent** if $a^3 = a$. Clearly, if $a^2 = a$ then both a and $-a$ are tripotents. In [82], the authors determined rings in which every element is a sum of an idempotent and a tripotent that commute, and rings in which every element is a sum of two commuting tripotents, which extended Hirano and Tominaga's work [41]. In 2017, Chen and Sheibani [24] proved that every element of a ring is a sum of a nilpotent and a tripotent that commute if and only if each of its elements is a sum of a nilpotent and two idempotents that commute with each other. Furthermore, in [83] the author studied rings for which every element is a sum of a nilpotent and two tripotents that commute with each other, and many equivalent conditions for such rings are given. Motivated by the above, we consider the rings for which every element is a sum of a nilpotent and three tripotents that commute with one another and we aim to determine the structure of such rings.

In [41] Hirano and Tominaga proved that for a ring R and an integer $n > 1$, the matrix ring $\mathbb{M}_n(R)$ is not a HT-ring, i.e., some element cannot be written as a sum of two idempotent matrices (see [41, Lemma 3]). On the other hand, in [30] the author proved that for a field F with $\text{char}(F) \leq 3$ and an integer $n \geq 1$, every matrix of $\mathbb{M}_n(F)$ is a sum of three idempotents (see [30, Proposition 9]). Thus, $k = 3$ is the smallest positive integer such that, for some ring R, every matrix over R is a sum of k idempotents. In [75], Tang, Zhou, and Su characterized the commutative reduced rings R in which every matrix in $\mathbb{M}_n(R)$ is a sum of three idempotents. Particularly, for a field F, they proved that every matrix over F is a sum of three idempotents if and only if $F \cong \mathbb{F}_2$ or $F \cong \mathbb{F}_3$ (see [75, Theorem 2.6]). For a ring R with $2 \in U(R)$ and an integer $k \geq 1$, the researchers in [75] also pointed out that every element of R is a sum of k idempotents if and only if every element of R is a sum of k involutions (see [75, Lemma 4.1]); further they proved that for a ring R, every element of R is a sum of two involutions if and only if every element of R is a sum of three commuting involutions if and only if R is a subdirect product of $\mathbb{F}_3$'s (see [75, Theorem 4.2]). From this result we can see that for any ring R and integer $n \geq 2$, there exists a matrix in $\mathbb{M}_n(R)$ which is not a sum of two involutive matrices. On the other hand, every matrix in $\mathbb{M}_n(\mathbb{F}_3)$ is a sum of three idempotent matrices. Therefore, every matrix in $\mathbb{M}_n(\mathbb{F}_3)$ is a sum of three involutive matrices (by [75, Lemma 4.1]). Hence, $k = 3$ is the smallest positive integer such that, for some ring R, every matrix over R is a sum of k involutive matrices. In [75], the authors also determined commutative rings over which every matrix is a sum of three involutive matrices. They showed that for a commutative ring R, every matrix over R is a sum of three involutive matrices if and only if R is a subdirect product of $\mathbb{F}_3$'s (see [75, Theorem 4.3]). In [40] the authors showed that, an $n \times n$ matrix A over a field F of characteristic 0 is a sum of idempotents if and only if $tr(A) = k \cdot 1_F$, where $k \in \mathbb{Z}$ and $k \geq \text{rank(A)}$. Motivated by the work of Merino, Paras, and Pelejo in [63], where the authors presented the necessary and sufficient conditions for an $n \times n$ matrix over K (for $K = \mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Z}_k$) to be a sum of involutions, we have the following question:

- When is an $n \times n$ matrix over a commutative ring a sum of tripotents or involutions? (Question 3)

Let R be a field (or a commutative ring). As a more general case than being a sum of three idempotents or three involutions, we have the following question:

- When is every matrix in $\mathbb{M}_n(R)$ a sum of three tripotents? (Question 4)

Responding to the questions above, we organize the first four chapters of the thesis as follows.

In Chapter 1, we give preliminaries for later use.

In Chapter 2, we first answer Question 1 affirmatively in Section 2.1. We prove that for a semisimple right R-module M in a ring R with $R/J(R)$ Boolean, every endomorphism of M is a sum of an idempotent endomorphism and a locally nilpotent endomorphism. Further, we show that for a vector space V with arbitrary dimension over a division ring D, every linear transformation of V is a sum of an idempotent linear transformation and a locally nilpotent linear transformation if and only if $D \cong \mathbb{F}_2$. In Section 2.2, regarding to Question 2, for a large class of rings, we reduce the question to the nil $*$-cleanness of the matrix ring over $\mathbb{F}_2$. We show that, for a 2-primal $*$-ring R, with involution $*$ given by $(a_{ij})^* = (a_{ij}^*)^T$, $\mathbb{M}_n(R)$ is a nil $*$-clean ring if and only if $J(R)$ is nil, $R/J(R)$ is Boolean, $a^* - a \in J(R)$ for all $a \in R$ and $\mathbb{M}_n(\mathbb{F}_2)$ is nil $*$-clean. For a 2-primal $*$-ring R, we verify: (1) $\mathbb{M}_2(R)$ is a nil $*$-clean ring if and only if $J(R)$ is nil, $R/J(R)$ is Boolean, $a^* - a \in J(R)$ for all $a \in R$; (2) $\mathbb{M}_n(R)$ is not a nil $*$-clean ring for $n = 3, 4$. We have also observed that for a commutative ring R and $n \in \{2, 3, 4\}$, every symmetric matrix in $\mathbb{M}_n(R)$ is a sum of symmetric idempotent matrix and a nilpotent matrix if and only if $J(R)$ is nil and $R/J(R)$ is Boolean. The structure of nil-clean rings is still unknown, in Section 2.3 we give a description of nil-clean rings of nilpotency index at most 2, and use it to describe involution-clean rings introduced in [29].

In Chapter 3, we prove the structure theorem for rings whose elements are sums of a nilpotent and three tripotents that commute with one another, and we discuss when a group ring has this property.

In Chapter 4 we give the necessary and sufficient conditions for an $n \times n$ matrix over an integral domain to be a sum of involutions and, respectively, a sum of tripotents. We also completely determine the integral domains over which every $n \times n$ matrix is a sum of involutions and, respectively, a sum of tripotents. In Section 4.2 we prove that for an integral domain R and an integer $n \geq 1$, every matrix over R is a sum of two tripotents if and only if $R \cong \mathbb{F}_p$ where $p = 2$, 3, or 5. In Section 4.3 we present some partial results related to Question 4.

An element a in a ring R is left **uniquely generated (or left** UG**)** if, for any $b \in R$, $Ra = Rb$ implies $a = ub$ for some unit u in R, and the ring R is a left UG **ring** if each of its elements is left UG. In [48], a left UG element is called an element with the left unique generator property. It is unknown whether a left UG ring is right UG. The study of left UG rings was initiated in 1949 by Kaplansky [47] in his work on matrices admitting diagonal reduction, and has been continued by a number of authors (see, for instance, [2, 1, 6, 20, 39, 48, 59, 66, 70]). Kaplansky [47] gave some first known examples of left UG rings: any ring whose zero-divisors are contained in its Jacobson radical (see [47, Lemma 2.1], for example, domains and local rings) and the matrix rings over a left Hermite domain [47, Theorem 3.8]. Commutative UG rings, under the name of associate rings or strongly associate rings, have been extensively discussed in [2, 1, 6, 34, 70]. For example, every commutative ring can be embedded into a commutative UG ring by [70, Theorem 14]; every commutative p.p. ring is UG by [2, Theorem 11]. It is proved that polynomial rings over commutative UG rings need not be UG (see [2, Example 19]). However, it is open whether R being commutative UG implies that the formal power series ring $R[[t]]$ is UG [2, Question 21]. Khurana and Lam [48, Theorem 6.2] showed that every (von Neumann) regular element in a ring is unit-regular if and only if every regular element in the ring is left UG and, independently, Marks [59, Theorem] proved that a regular ring is unit-regular if and only if it is left UG. An earlier result of Hartwig and Luh [39, Theorem 2B], generalizing the two results, states that a regular element a in a ring R is unit-regular if and only if, whenever $Ra = Rb$ with b

unit-regular, $a = ub$ for a unit u in R. In 1964, Bass introduced the concept of stable range in his study of the stability of the general liner group (see [9]). An element $a \in R$ is said to have left **stable range 1** if $Ra + Rb = R$ and $b \in R$ implies $a - u \in Rb$ for a unit u in R. A ring R is said to have stable range 1 (denoted by SR1) if every element of R has stable range 1. This condition is left-right symmetric by [78, Theorem 2]. The class of SR1 rings contains strongly π-regular rings, unit-regular rings, semiperfect rings, etc. For an exchange ring R, it is proved R is right UG if and only if R has SR1 (see [48, Theorem 6.5]). In 1995, Canfell [20, Corollary 4.4] obtained the following characterizations of left UG rings:
Canfell's Theorem. For any ring R, the following are equivalent:

- If $Ra + \mathbf{l}(b) = R$, a, $b \in R$, then $a - u \in \mathbf{l}(b)$ for some unit $u \in R$.
- R is left UG.
- If $Ra = Rb$, a, $b \in R$, then $a = vb$ for some unit $v \in R$.

From Canfell's Theorem we can see that SR1 rings are left (and right) UG rings but the converse fails as the ring of integer $\mathbb{Z}$ is a UG ring but not a SR1 ring. In [66], Nicholson was interested in the first condition of Canfell's Theorem and introduced left annihilator-stable (or left AS) elements and rings as natural generalizations of elements and rings with left stable range 1. Following [66], an element a in a ring R is called **left annihilator-stable (or left** AS) if $Ra + \mathbf{l}(b) = R$, a, $b \in R$, implies that $a - u \in \mathbf{l}(b)$ for some unit $u \in R$. A ring R is called a **left annihilator-stable ring (or left** AS **ring)** if every element of R is left AS. Various characterizations of left AS rings were obtained in [20, Corollary 4.4] and [66, Theorem 5]. Particularly, Canfell's Theorem shows that a ring is left AS if and only if it is left UG. Thus, Marks' Theorem [59, Theorem] can be restated as follows: A regular ring is unit-regular if and only if it is left AS. The element-wise version of this result is obtained by Nicholson [66, Lemma 24]: An element in a ring is unit-regular if and only if it is regular and left AS. It is known that the product of two SR1 elements is again SR1 (see [23, Lemma 17]). The following questions are raised by Nicholson in [66]:

- Is the product of two left AS elements again left AS? ([66, Question 1])
- If $R/J(R)$ is left AS does it follow that R is left AS? What if R is exchange? Clean? ([66, Question 3])

As a continuation of the study of left AS elements and left AS rings, Chapter 5 is organized as follows.

Section 5.1 is mainly about properties of left AS elements and left UG elements. Although left AS and left UG are not equivalent element-wise , a UG element does represent a sort of annihilator-stability: An element $b \in R$ is left UG if and only if whenever $Ra + \mathbf{l}(b) = R$ with $a \in R$, $a - u \in \mathbf{l}(b)$ for a unit u. In this section, we also show that the left AS elements in a ring form a multiplicatively closed set, which gives an affirmative answer to the question of Nicholson above. Using this result, we establish a necessary and sufficient condition for a formal triangular matrix ring to be left AS in Section 5.2, and further produce examples of left AS rings R over which the triangular matrix rings $\mathbb{T}_n(R)$ are not left AS for all $n \geq 2$. These examples give a negative answer to the second question of Nicholson above.

Chapter 1

Preliminaries

This chapter is divided into two sections: (1) Semisimplicity and (2) Subdirect products. Some basic definitions and results in ring theory, needed in the thesis, are stated in these sections. In particular, the notation of subdirect product is used throughout.

1.1 Semisimplicity

In this section, we state several basic definitions and results in ring theory which are needed throughout the dissertation.

Definition 1.1.1 *Let R be a ring, and M be a left R-module.*

(1) *M is called a simple module if $M \neq 0$ and M has no proper submodule.*

(2) *M is called a semisimple module if every submodule of M is a direct summand of M, or equivalently, M is a direct sum of simple submodles.*

For example, a module M over $\mathbb{Z}$ is simple if and only if $M \cong (\mathbb{Z}/p\mathbb{Z})_{\mathbb{Z}}$ for a prime p. Therefore, the semisimple $\mathbb{Z}$-modules are just direct sums of copies of $\mathbb{Z}_p$'s. A ring R is called a **simple ring** if $R \neq 0$, and R has only trivial ideals. A ring R is said to be a **left semisimple ring** if the left regular R-module ${}_RR$ is semisimple.

Definition 1.1.2 *Let R be a ring, and M a left R-module.*

(1) *M is called artinian if M has the descending chain condition* (DCC) *on submodules, i.e., for any descending chain submodules $M_1 \supseteq M_2 \supseteq \cdots$, there exists an integer $n \geq 1$ such that $M_n = M_{n+1} = \cdots$.*

2) *R is called left artinain if the module ${}_RR$ is artinian.*

A submodule $N \subseteq M$ is said to be **maximal** if $N \neq M$ and $N \subseteq X \subseteq M$ with $N \neq X$ implies $X = M$. The **radical** of M is defined to be the intersection of all maximal submodules of M, and is denoted by $\mathrm{rad}(M)$. We set $\mathrm{rad}(M) = M$ if M does not have maximal submodules. For a ring R, $\mathrm{rad}({}_RR) = \mathrm{rad}(R_R)$, which is called the Jacobson radical of R, and is denoted by $J(R)$. So $J(R)$ equals the intersection of all maximal left ideals of R.

Theorem 1.1.1 *Let R be a ring. Then*

(1) *An element $y \in J(R)$ if and only if $1 - xy \in U(R)$ for every $x \in R$.*

(2) *$J(R/J(R)) = 0$.*

Definition 1.1.3 *Let R be a ring and I an ideal of R.*

(1) *I is said to be a prime ideal of R if $I \neq R$ and, for ideals I_1, $I_2 \subseteq R$, $I_1 \cdot I_2 \subseteq I$ implies that $I_1 \subseteq I$ or $I_2 \subseteq I$.*

(2) *I is said to be a semiprime ideal of R if, for any ideal I' of R, $I'^2 \subseteq I$ implies that $I' \subseteq I$.*

A ring R is called a **prime** (respectively, **semiprime**) ring if (0) is a prime (respectively, semiprime) ideal. The prime radical $\mathrm{Nil}_*(R)$ of a ring R is defined to be the intersection of prime ideals of R.

We list the following well-known results in ring theory.

Theorem 1.1.2 *(Wedderburn-Artin) The following are equivalent for a ring R:*

(1) *R is left semisimple.*

(2) *R is semiprime left artinian.*

(3) *R is a finite direct product of matrix rings over division rings.*

Theorem 1.1.3 *(Wedderburn) The following are equivalent for a ring R:*

(1) *R is simple left artinian.*

(2) *R is prime left artinian.*

(3) *R is the matrix ring over a division ring.*

Proposition 1.1.4 *Let R be a left artinian ring. Then $J(R)$ is nilpotent and $R/J(R)$ is left semisimple.*

1.2 Subdirect products and regularity

Let $\prod_{\alpha} R_\alpha$ be the direct product of a family of rings $\{R_\alpha : \alpha \in \Lambda\}$. A ring R is called a **subdirect product** of $\{R_\alpha\}$ if there exists a one to one homomorphism $\sigma : R \to \prod_{\alpha} R_\alpha$ such that $\pi_\alpha \circ \sigma : R \to R_\alpha$ is onto for each $\alpha \in \Lambda$, where π_α is a projection from $\prod_{\alpha} R_\alpha$ to R_α.

Example 1.2.1 (1) *The direct product $\prod_\alpha R_\alpha$ is a subdirect product of $\{R_\alpha\}$ by taking σ as the identity map.*

(2) *$\mathbb{Z}$ is a subdirect product of $\{\mathbb{Z}_{p^i} | i \geq 1\}$ where p is a prime, $\mathbb{Z}_{p^i} \cong \mathbb{Z}/p^i\mathbb{Z}$ and $\bigcap_{i} p^i\mathbb{Z} = 0$. Also, $\mathbb{Z}$ is a subdirect product of $\{\mathbb{Z}_p | p \text{ is a prime}\}$, where $\bigcap_p p\mathbb{Z} = 0$.*

Theorem 1.2.2 *Let R be a ring. Then R is a subdirect product of a family of rings $\{R_\alpha | \alpha \in \Lambda\}$ if and only if for each $\alpha \in \Lambda$, there exists an onto homomorphism $f_\alpha : R \to R_\alpha$ such that $\bigcap_{\alpha\in\Lambda} Ker(f_\alpha) = 0$.*

Proof. Suppose that R is a subdirect product of a family of rings $\{R_\alpha | \alpha \in \Lambda\}$. Then there exists a one to one homomorphism $\sigma : R \to \prod_{\alpha} R_\alpha$ such that $\pi_\alpha\sigma(R) = R_\alpha$ for all α. Let $f_\alpha := \pi_\alpha\sigma$. Then $f_\alpha : R \to R_\alpha$ is onto. Given any element $x \in \bigcap_{\alpha\in\Lambda} Ker(f_\alpha)$, write $\sigma(x) = (x_\alpha) \in \Pi_\alpha R_\alpha$. Then $0 = f_\alpha(x) = \pi_\alpha(\sigma(x)) = \pi_\alpha(x_\alpha) = x_\alpha$ for all α. It follows that $\sigma(x) = 0$. Furthermore, we have $x = 0$ since σ is one to one.

Conversely, define $\sigma : R \to \prod_{\alpha} R_\alpha$ such that $\sigma(x) = (f_\alpha(x))$. Then $\pi_\alpha \sigma = f_\alpha$ is onto. If $\sigma(x) = 0$, then $f_\alpha(x) = 0$ for all α. Therefore, $x \in Ker(f_\alpha)$ for all $\alpha \in \Lambda$. It follows that $x \in \bigcap_{\alpha \in \Lambda} Ker(f_\alpha) = 0$. Hence $x = 0$, and so σ is one to one. □

Corollary 1.2.3 *Let R be a ring with a family of ideals $\{I_\alpha : \alpha \in \Lambda\}$. Let $I := \bigcap_{\alpha} I_\alpha$. Then R/I is a subdirect product of $\{R/I_\alpha : \alpha \in \Lambda\}$.*

Proof. Define $f_\alpha : R/I \to R/I_\alpha$ by $f_\alpha(x+I) = x + I_\alpha$ for every $x \in R$. Then f_α is an onto homomorphism. If $x + I \in \bigcap_{\alpha} \mathrm{Ker}(f_\alpha)$, then $x \in I_\alpha$ for each α. So $x \in \bigcap_{\alpha} I_\alpha = I$ and $x + I = 0$. □

Theorem 1.2.4 *(Jacobson) If R is a ring and, for any $a \in R$, there exists some integer $n(a) > 1$ such that $a^{n(a)} = a$. Then R is commutative.*

Proposition 1.2.5 *[53, Ex. 12.11] Let p be a fixed prime. Define a nonzero ring R to be a p-ring if $a^p = a$ and $pa = 0$ for all $a \in R$. A ring R is a p-ring if and only if it is a subdirect product of $\mathbb{F}_p$'s.*

Proof. First observe that p-rings are closed with respect to the direct products and subrings. Secondly, $\mathbb{F}_p$ is a p-ring. In fact, it's clear that $a^p = a$ for $a = 0$. The nonzero elements of $\mathbb{F}_p$ form a group of order $p-1$ under multiplication. By the fact that $a^{|G|} = 1_G$ for any element a in a finite group G, we have that all $0 \neq a \in \mathbb{F}_p$ satisfy $a^{p-1} = 1$, i.e., $a^p = a$. From these, it follows that any subdirect product of $\mathbb{F}_p$'s is a p-ring. Conversely, let R be a p-ring. Then R is commutative by Jacobson's Theorem. Let $\{m_i : i \in \Lambda\}$ be the family of maximal ideals of R. We have $\bigcap_{i} m_i = 0$. Therefore, by Corollary 1.2.3, R is a subdirect product of fields $F_i = R/m_i$, which clearly remains p-rings. But in any field the equation $x^p = x$ has at most p solutions, so $F_i \cong \mathbb{F}_p$. □

Definition 1.2.1 *A ring R is called subdirectly irreducible if R has a smallest nonzero ideal (or equivalently, the intersection of all nonzero ideals of R*

is nonzero).

Example 1.2.6 (1) *Any simple ring R is subdirectly irreducible.(Its minimal ideal is given by R itself.)*

(2) $\mathbb{Z}/p^i\mathbb{Z}$ *($i \geq 1$ and p a prime) are subdirectly irreducible. (Their minimal ideals are $p^{i-1}\mathbb{Z}/p^i\mathbb{Z}$.)*

The role of subdirectly irreducible rings is seen from the following result.

Theorem 1.2.7 *(Birkhoff) Any nonzero ring R is a subdirect product of subdirectly irreducible rings.*

Proof. Given a nonzero element $x \in R$, consider the set

$$S = \{I | I \text{ is an ideal of } R \text{ with } x \notin I\}.$$

Then $(S, \subseteq)$ is a partially ordered set. For any chain $\{I_\alpha\}$ in S, $I_0 := \bigcup_\alpha I_\alpha$ is an upper bound of the chain $\{I_\alpha\}$. By Zorn's Lemma, S has a maximal member, say I_x. Notice that $x \notin I_x$, we have $\bigcap_{0\neq x\in R} I_x = 0$. In fact, suppose the intersection is not zero, let $0 \neq y \in R$ and $y \in \bigcap_{0\neq x} I_x$. Then $y \in I_y$, a contradiction. Then by Corollary 1.2.3, R is a subdirect product of $\{R/I_x\}$. It remains to show that each R/I_x is subdirectly irreducible. Let A be a nonzero ideal of R/I_x and write $A = K/I_x$ where K is an ideal of R with $I_x \subsetneqq K$. By the choice of I_x, we have $x \in K$, so $\bar{0} \neq x + I_x \in A$. Therefore, $\bigcap\{A | A \text{ is a nonzero ideal of } R/I_x\} \neq 0$ and R/I_x is subdirectly irreducible. □

An element a in a ring R is called **nilpotent** if $a^n = 0$ for some integer n. A ring R is called **reduced** if R has no nonzero nilpotent (or equivalently, for any element $a \in R$, $a^2 = 0$ implies $a = 0$.)

Proposition 1.2.8 *Let R be a reduced ring. Then the intersection of all prime ideals of R is zero, i.e.,* $\mathrm{Nil}_*(R) = 0$.

Proof. Suppose $\mathrm{Nil}_*(R) \neq 0$. Then there exists an element $0 \neq a \in \mathrm{Nil}_*(R)$. Since R is reduced, $a^n \neq 0$ for all $n \in \mathbb{N}$. Let $A = \{a^n : n \in \mathbb{N}\}$ and $X = \{I | I \text{ is an ideal of } R \text{ and } I \bigcap A = \emptyset\}$. Note that $0 \in X$, $X \neq \emptyset$. By Zorn's Lemma, $(X, \subseteq)$ has a maximal member, say P. To get a contradiction, we only need to show P is a prime ideal. Suppose I_1, I_2 are ideals of R such that $I_1 I_2 \subseteq P$. If $I_1 \nsubseteq P$ and $I_2 \nsubseteq P$, then $a^n \in I_1$ and $a^m \in I_2$ for some n, $m \in \mathbb{N}$. It follows that $a^{n+m} \in I_1 I_2 \subseteq P$, a contradiction. Therefore, we have $I_1 \subseteq P$ or $I_2 \subseteq P$ and P is a prime ideal. Hence, $a \in P$, contradicting to $P \in X$. □

Theorem 1.2.9 *(Andrunakievich-Ryabukhin) A nonzero ring R is reduced if and only if R is a subdirect product of domains.*

Proof. First assume that R is a subdirect product of a family of domains $\{R_i\}$. If $a \in R$ is nilpotent, then a maps to zero in each R_i, so $a = 0$.

Conversely, assumed that R is reduced. Let $\mathfrak{p}$ be a minimal prime ideal of R. Next we show that $R/\mathfrak{p}$ is a domain. Let $S = R \setminus \mathfrak{p}$. Then it is easily verified that $R/\mathfrak{p}$ is domain if and only if S is multiplicatively closed. Let S' be the set of all elements of R which are a finite product of some elements of S. Then clearly, S' is multiplicatively closed, $S \subseteq S'$ and S is multiplicatively closed if and only if $S = S'$. First we show that $0 \notin S'$. For otherwise we have $s_1 s_2 \cdots s_n = 0$ with all $s_i \in S$ and with n minimal. Clearly $n \geq 2$. Since R is reduced and $(s_n R s_1 \cdots s_{n-1})^2 = 0$, we have $s_n R s_1 \cdots s_{n-1} = 0$. Now since $\mathfrak{p}$ is prime ideal, $s_n R s_1$ cannot be a subset of $\mathfrak{p}$ otherwise we'd have $s_n \in \mathfrak{p}$ or $s_1 \in \mathfrak{p}$. Thus, $s_n R s_1 \bigcap S \neq \varnothing$. Let $s \in s_n R s_1 \bigcap S$. Then $s s_2 \cdots s_{n-1} \in s_n R s_1 s_2 \cdots s_n = 0$, which contradicts the minimality of n. So we have shown $0 \notin S'$. Second, we show that $S = S'$. Let $\Gamma = \{A | A \text{ is an ideal of } R \text{ and } A \bigcap S' = \emptyset\}$. Since $0 \in \Gamma$, we have $\Gamma \neq \emptyset$. Therefore, by Zorn's Lemma, $(\Gamma, \subseteq)$ has a maximal member T and T is a prime ideal of R. Since $T \bigcap S' = \emptyset$, $T \bigcap S = \emptyset$, thus $T \subseteq \mathfrak{p}$. It follows that $T = \mathfrak{p}$ since $\mathfrak{p}$ is a minimal prime ideal. So $S' \bigcap \mathfrak{p} = \emptyset$, which means $S' \subseteq S$.

Hence, $S = S'$. So far we have shown that $R/\mathfrak{p}$ is a domain for any minimal prime ideal $\mathfrak{p}$. To continue on the proof, let $\{\mathfrak{p}_i\}$ be the family of minimal prime ideals in R. Since every prime ideal contains a minimal prime ideal, we have

$$\bigcap_i \mathfrak{p}_i = \mathrm{Nil}_*(R) = 0.$$

Therefore, R is a subdirect product of $\{R/\mathfrak{p}_i\}$ where each $R/\mathfrak{p}_i$ is a domain. □

A ring R is (von Neumann) **regular** if for every element $x \in R$, there exists $y \in R$ such that $x = xyx$.

Lemma 1.2.10 *[36, Theorem 1.1] For a ring R, the following are equivalent:*

(1) *R is regular.*

(2) *Every principal right (left) ideal of R is generated by an idempotent.*

(3) *Every finitely generated right (left) ideal of R is generated by an idempotent.*

Proof. (1) ⇒ (2) Let xR be a principal right ideal of R. Then $x = xyx$ for some $y \in R$ and $e := xy$ is an idempotent. It follows that $xR = xyxR \subseteq xyR \subset xR$, i.e., $xR = eR$.

(2) ⇒ (3) It suffices to show that $xR + yR$ is principal for any $x, y \in R$. Now $xR = eR$ for some idempotent $e \in R$, and since $y - ey \in xR + yR$ we see that $xR + yR = eR + (y - ey)R$. There is an idempotent $f \in R$ such that $fR = (y - ey)R$, and we note that $ef = 0$. Consequently, $g := f - fe$ is an idempotent orthogonal to e. Observing that $fg = g$ and $gf = f$, we see that $gR = fgR \subseteq fR = gfR \subseteq gR$, i.e., $gR = fR$, whence $xR + yR = eR + gR$. Since $eg = ge = 0$, we conclude that $xR + yR = (e + g)R$.

(3) ⇒ (1) Given $x \in R$, there exists an idempotent $e \in R$ such that $eR = xR$. Then $e = xy$ for some $y \in R$. As $x = er$ for some $r \in R$, $(1 - e)x = 0$, i.e., $x = ex = xyx$. □

According to the lemma above, every finitely generated one-sided ideal of a regular ring R is a direct summand of R.

An element $a \in R$ is called **strongly regular** if $a^2x = a = ya^2$ for some $x, y \in R$ (or equivalently, $a \in Ra^2 \bigcap a^2R$). An element $a \in R$ is called right π-regular if it satisfies the following equivalent conditions:

(1) $a^n \in a^{n+1}R$ for some integer $n \geq 1$.

(2) $a^nR = a^{n+1}R$ for some integer $n \geq 1$.

(3) The chain $aR \supseteq a^2R \supseteq \cdots$ terminates.

The **left π-regular** elements are defined analogously. An element $a \in R$ is defined to be **strongly π-regular** if it is both left and right π-regular. A ring R is called a strongly π-regular ring if every element is strongly π-regular.

Let $a \in R$ be a strongly π-regular element. Then $a^nR = a^{n+1}R$ and $Ra^m = Ra^{m+1}$ for some integers $n, m \geq 1$. Write $a^n = a^{n+1}x$ and $a^m = ya^{m+1}$ for some $x, y \in R$. Then we have the following cases:

- If $m \leq n$, $ya^{n+1} = ya^{m+1}a^{n-m} = a^m a^{n-m} = a^n$
- If $m > n$, $a^n = a^{n+1}x$ implies that

$$a^n = a^{n+1}x = a^{n+2}x^2 = \cdots = a^m x^{m-n}.$$

  Thus $ya^{n+1} = yaa^n = yaa^m x^{m-n} = a^m x^{m-n} = a^n$.

Therefore, $a^{n+1}x = a^n = ya^{n+1}$. This shows that an element a is strongly π-regular if and only if $a^n \in Ra^{n+1} \bigcap a^{n+1}R$ for some integer $n \geq 1$. Since $a^n = a^{n+1}x$ implies that $a^n = a^{n+1}x = a^{n+2}x^2 = \cdots = a^{2n}x^n$. Similarly, by $a^n = ya^{n+1}$, we have $a^n = y^n a^{2n}$. Therefore, a^n is strongly regular.

Lemma 1.2.11 *[7, Lemma 1] If $a \in R$ is a strongly π-regular element and $a^nR = a^{n+1}R$ for some integer $n \geq 1$, then a^n is strongly regular and there exists $b \in R$ such that $ab = ba$ and $a^n = a^{n+1}b$.*

Proof. It is clear now that a^n is strongly regular. Write $a^n = a^{2n}x$ and $a^n = ya^{2n}$ for some elements $x, y \in R$. Then $a^nx = ya^{2n}x = ya^n$, so that

$a^n x^2 = ya^n x = y^2 a^n$. We have $a^n x a^n = ya^{2n} = a^n$ and $a^n y a^n = a^{2n} x = a^n$. Now put $b = a^n x^2$. Then

$$a^n b = (a^{2n} x)x = (a^n y a^n)x = a^n x = ya^n = y(a^n x a^n) = a^n x^2 a_n = ba^n.$$

Further, we have

$$\begin{aligned} a^{2n} b &= a^n b a^n = a^{2n} x = a^n \\ a^n b^2 &= ya^n b = y(a^{2n} x)x = (ya^n)x = b \end{aligned}$$

Now let c be any element such that $ac = ca$. Then

$$ba^n c = bca^n = bca^{2n} b = ba^{2n} cb = a^n cb = ca^n b, \quad \text{i.e., } c \text{ commutes with } a^n b = ba^n.$$

It follows that $bc = a^n b^2 c = ba^n bc = bca^n b = ba^n cb = cba^n b = ca^n b^2 = cb$. This shows that $ab = ba$ and b commutes with any element that commutes with a. The last assertion follows as $a^n = a^{2n} b = a^{n+1} \cdot a^{n-1} b$, denoting $a^{n-1} b$ again by b, then we have $ab = ba$ and $a^n = a^{n+1} b$. □

This leads to the following characterization of strongly π-regular elements.

Lemma 1.2.12 *[65, Proposition 1] Let R be a ring and let $a \in R$. Then a is strongly π-regular if and only if there exists an integer $n \geq 1$ such that $a^n = eu = ue$ where $e^2 = e$, $u \in U(R)$ and a, e, and u all commute with each other.*

Proof. If a is strongly π-regular, Lemma 1.2.11 provides $b \in R$ and an integer $n \geq 1$ such that $ab = ba$ and $a^n = a^{n+1} b$. Thus $a^n = a^n ab = a^{n+1} b(ab) = a^{n+2} b^2$. By iterating this, we get $a^n = a^{n+2} b^2 = \cdots = a^{2n} b^n = a^n b^n a^n$. So $e = a^n b^n = b^n a^n$ is an idempotent, $ae = ea$ and $a^n e = a^n$. Write $c = b^n e$. Then $c \in eRe$ and $a^n c = ca^n = e$. Hence $u = a^n + (1 - e)$ is a unit with $u^{-1} = c + (1 - e)$ and $eu = ue = a^n$.

Conversely, by the given condition, we have $a^n u^{-1} a^n = ea^n = a^n$ and $au^{-1} = u^{-1}a$. It follows that a^n is strongly regular. Therefore, a is strongly π-regular. □

Based on the characterization of strongly π-regular endomorphisms in terms of direct sum decomposition in [11], the author in [32] gave an element-wise decomposition of a strongly π-regular element as follows. We are going to provide a direct proof for the following lemma.

Lemma 1.2.13 *[32, Proposition 2.5] Let R be a ring. An element $a \in R$ is strongly π-regular if and only if there is an idempotent $e \in R$ and a unit $u \in R$ such that $a = e + u$, $ae = ea$ and eae is nilpotent.*

Proof. Suppose a is strongly π-regular. Then it follows from Lemma 1.2.12 that there exists an integer $n \geq 1$ such that $a^n = eu = ue$ where $e^2 = e$, $u \in U(R)$ and a, e, and u all commute. Let

$$\begin{aligned} w &= a - (1-e) \\ v &= a^{n-1}u^{-1}e - (1 + a + a^2 + \cdots + a^{n-1})(1-e) \end{aligned}$$

Then $wv = vw$. It follows from $w = ae - (1-a)(1-e)$ that

$$\begin{aligned} wv &= [ae - (1-a)(1-e)][a^{n-1}u^{-1}e - (1 + a + a^2 + \cdots + a^{n-1})(1-e)] \\ &= e + (1-e)(1-a^n) \\ &= 1 \quad (as \quad ea^n = a^n) \end{aligned}$$

Thus, $a = (1-e) + w$, where $(1-e)w = w(1-e)$ and $(a(1-e))^n = a^n - a^n e = 0$.

Conversely, let a, e, and u as be given. Then a, e, u and u^{-1} all commute. Since ea is nilpotent, $(ea)^n = ea^n = 0$ for some integer n. Then we have $a^n u^{-1}(a-u) = a^n u^{-1} e = ea^n u^{-1} = 0$. So $a^n = a^n u^{-1} a$. It follows that

$$a^n = (a^n u^{-1} a)(u^{-1} a) = a^n u^{-2} a^2 = \cdots = a^n u^{-n} a^n.$$

Let $f = a^n u^{-n}$. Then $a^n = fu^n$ where $f^2 = f$ and a, f, u^n all commute. Hence, by Lemma 1.2.12, a is strongly π-regular. $\square$

Chapter 2

Rings whose elements are sums of a nilpotent and an idempotent

In this chapter we focus on rings in which every element is a sum of a nilpotent and an idempotent (i.e., nil-clean rings). In Section 2.1, we prove that for a vector space V of arbitrary dimension over a division ring D, every linear transformation of V is a sum of an idempotent linear transformation and a locally nilpotent linear transformation if and only if $D \cong \mathbb{F}_2$. In Section 2.2, we discuss when $\mathbb{M}_n(R)$ is a nil $*$-clean ring where $(a_{ij})^* = (a_{ij}^*)^T$ for a $*$-ring R. We show that for a 2-primal $*$-ring R, with involution $*$ given by $(a_{ij})^* = (a_{ij}^*)^T$, $\mathbb{M}_n(R)$ is a nil $*$-clean ring if and only if $J(R)$ is nil, $R/J(R)$ is Boolean, $a^* - a \in J(R)$ for all $a \in R$ and $\mathbb{M}_n(\mathbb{F}_2)$ is nil $*$-clean. In Section 2.3, we give a description of nil-clean rings with nilpotency index at most 2.

2.1 Ring of linear transformations

In [14], Breaz et al. proved that for a field F, the matrix ring $\mathbb{M}_n(F)$ is nil-clean if and only if $F \cong \mathbb{F}_2$. Toward extending this result, the authors in [14] raised two questions: (i) Does the result hold for a division ring? (ii) Is the ring of linear transformations of a countably infinite dimensional vector space over $\mathbb{F}_2$ a nil-clean ring? While the first question was affirmatively answered in [49], the second one remains open.

Recall that an endomorphism f of a module M is **locally nilpotent** if for any $x \in M$, $f^n(x) = 0$ for some $n > 0$. It is clear that a nilpotent endomorphism is clearly locally nilpotent, but the converse is false. For example, let V be a countably infinite dimensional right vector space over a division ring D with a basis $\{v_1, v_2, \ldots\}$. Let $f \in \mathrm{End}(V_D)$ be given by $f(v_1) = 0$ and $f(v_{i+1}) = v_i$ for all $i \geq 1$. Then f is locally nilpotent but not nilpotent. If M_R is a finitely generated module, then $f \in \mathrm{End}(M_R)$ is locally nilpotent if and only if it is nilpotent. Thus, the aforementioned results can be phrased as that, for a finite dimensional vector space V over a division ring D, every linear transformation of V is a sum of an idempotent linear transformation and a locally nilpotent linear transformation if and only if $D \cong \mathbb{F}_2$. Also, the question of Breaz et al. [14] has the following "local" version: For a vector space V with arbitrary dimension over a division ring D, is it true that every linear transformation of V is a sum of an idempotent linear transformation and a locally nilpotent linear transformation if and only if $D \cong \mathbb{F}_2$? In this section, we answer this question in the affirmative. Indeed, we are able to show that, for any semisimple module over a ring R with $R/J(R)$ Boolean, every endomorphism of M is a sum of an idempotent endomorphism of M and a locally nilpotent endomorphism of M.

Lemma 2.1.1 *[62, Lemma 2.2] Let M be a module and let η be any locally nilpotent endomorphism of M. Then $1 + \eta$ is an automorphism of M.*

Proof. It is clear that $1+\eta$ is also an endomorphism of M. To show that $1+\eta$ is bijective let $m \neq 0$ be an arbitrary element of M. Since η is locally nilpotent, there exists an integer $n \geq 1$ such that $\eta^n(m) = 0$ and $\eta^{n-1}(m) \neq 0$ where $\eta^0 = 1$.

Since $\eta^{n-1}(\eta+1)(m) = \eta^{n-1}(m) \neq 0$, it follows that $\mathrm{Ker}(\eta+1) = 0$. Also, $(\eta+1)(1-\eta+\eta^2-\eta^3+\cdots+\eta^{n-1})(m) = m$ which shows that $1+\eta$ is surjective. □

When $M_R = R_R$, the following lemma recaptures [32, Proposition 3.14] that the element 2 in a nil-clean ring R is nilpotent and is contained in $J(R)$.

Lemma 2.1.2 *Let M be a right R-module. If $2 \in \mathrm{End}(M_R)$ is a sum of an idempotent endomorphism and a locally nilpotent endomorphism, then 2 is locally nilpotent in $\mathrm{End}(M_R)$ and $2 \in J(\mathrm{End}(M_R))$.*

Proof. Write $2 = g+h$ where $g^2 = g \in \mathrm{End}(M_R)$ and $h \in \mathrm{End}(M_R)$ is locally nilpotent. Then $2-h = (2-h)^2 = 4-4h+h^2$, showing that $2 = (3-h)h$ is locally nilpotent in $\mathrm{End}(M_R)$, which is central. Thus, for any $f \in \mathrm{End}(M_R)$, $2f$ is locally nilpotent in $\mathrm{End}(M_R)$, so $1+2f$ is an automorphism of M by Lemma 2.1.1. This shows that $2 \in J(\mathrm{End}(M_R))$. □

The following lemma is the main result in [49].

Lemma 2.1.3 *[49, Theorem 3] Let D be a division ring and let $n \geq 1$. Then $\mathbb{M}_n(D)$ is a nil-clean ring if and only if $D \cong \mathbb{F}_2$.*

Notice that the **shift operator** as defined below is proved to be clean in [19]. In the following lemma, we show the "shift operator" is a sum of an idempotent and a locally nilpotent endomorphism.

Lemma 2.1.4 *Let M_R be an R-module and $f \in \mathrm{End}(M_R)$. If $2^m = 0$ in R and f is a shift operator on M_R, i.e., $M = \bigoplus_{n\geq 0} f^n(x)R$ for some $x \in M$,* *such that* $\mathbf{r}(x) = \mathbf{r}(f^i(x))$ *for all* i*, then* $f = \beta + \gamma$*, where* $\beta^2 = \beta \in \mathrm{End}(M_R)$ *and* $\gamma^{3m} = 0$ *in* $\mathrm{End}(M_R)$

Proof. Define

$$\begin{aligned} \beta : M &\to M \\ f^{3n}(x) &\mapsto f^{3n}(x) \\ f^{3n+1}(x) &\mapsto f^{3n}(x) + f^{3n+1}(x) - f^{3n+2}(x) \\ f^{3n+2}(x) &\mapsto f^{3n}(x) \end{aligned}$$

and

$$\begin{aligned} \gamma : M &\to M \\ f^{3n}(x) &\mapsto -f^{3n}(x) + f^{3n+1}(x) \\ f^{3n+1}(x) &\mapsto -f^{3n}(x) - f^{3n+1}(x) + 2f^{3n+2}(x) \\ f^{3n+2}(x) &\mapsto -f^{3n}(x) + f^{3n+3}(x) \end{aligned}$$

for $n = 0, 1, \ldots$. Because of $\mathbf{r}(x) = \mathbf{r}(f^i(x))$ for all $i > 0$, one easily sees that β, γ are well-defined and

$$f = \beta + \gamma \text{ and } \beta^2 = \beta.$$

We next verify that $\gamma^{3m} = 0$. We have

$$\begin{aligned} \gamma^2(f^{3n}(x)) &= 2\big(- f^{3n+1}(x) + f^{3n+2}(x)\big) \\ \gamma^2(f^{3n+1}(x)) &= 2\big(- f^{3n+2}(x) + f^{3n+3}(x)\big) \\ \gamma^3(f^{3n+2}(x)) &= 2\big(f^{3n+1}(x) - f^{3n+2}(x) - f^{3n+4}(x) + f^{3n+5}(x)\big) \end{aligned}$$

for $n \geq 0$. This implies that, for any $v \in M$, $\gamma^3(v) = 2w$ for some $w \in M$, and hence $\gamma^{3m}(v) = 2^m z$ for some $z \in M$. As $2^m = 0$ in R, $\gamma^{3m}(v) = 0$ for every $v \in M$. Hence $\gamma^{3m} = 0$. $\square$

Theorem 2.1.5 *Let R be a ring such that $R/J(R)$ is Boolean, and let M be a semisimple right R-module. Then every endomorphism of M is a sum of an idempotent endomorphism and a locally nilpotent endomorphism.*

Proof. Let $f \in \mathrm{End}(M_R)$. Let $\mathcal{S}$ be the set of all ordered pairs (W, g) where W is an f-invariant submodule of M, $g^2 = g \in \mathrm{End}(W_R)$ and $f|_W - g$ is locally nilpotent in $\mathrm{End}(W_R)$. Clearly, $((0), 0) \in \mathcal{S}$. Define a partial ordering on $\mathcal{S}$ by setting $(W', g') \leq (W, g)$ whenever both are in $\mathcal{S}$, $W' \subseteq W$ and $g' = g|_{W'}$. Suppose that $\{(W_\alpha, g_\alpha) : \alpha \in \Lambda\}$ is a totally ordered subset of $\mathcal{S}$. Define $g \in \mathrm{End}((\cup W_\alpha)_R)$ by setting $g(x) = g_\alpha(x)$ $(\alpha \in \Lambda, x \in W_\alpha)$, and it is easy to see that $(\cup W_\alpha, g) \in \mathcal{S}$ and $(W_\alpha, g_\alpha) \leq (\cup W_\alpha, g)$ for all $\alpha \in \Lambda$. It follows from Zorn's Lemma that there exists a maximal element (W, g) in $\mathcal{S}$; we finish the proof by showing that $W = M$. Assume to the contrary that $W \neq M$.

Claim. $f(x) \notin W$ for any $0 \neq x \in M$ with $W \cap xR = 0$.

Proof of Claim. If $f(x) \in W$ for some $0 \neq x \in M$ with $W \cap xR = 0$, then we can extend g to $g' \in \mathrm{End}((W \oplus xR)_R)$ by setting $g'(x) = 0$ and $g'|_W = g$. So g' is an idempotent of $\mathrm{End}((W \oplus xR)_R)$. For $w + xa \in W \oplus xR$ with $w \in W$ and $a \in R$, $(f - g')(w + xa) = (f - g)(w) + f(x)a \in W$. As $f|_W - g \in \mathrm{End}(W_R)$ is locally nilpotent, there exists $n > 0$ such that $(f - g)^n\big((f - g')(w + xa)\big) = 0$. That is, $(f - g')^{n+1}(w + xa) = 0$. Thus, $f - g' \in \mathrm{End}((W \oplus xR)_R)$ is locally nilpotent. So $(W \oplus xR, g') \in \mathcal{S}$, contradicting the maximality of (W, g). This proves the Claim.

Choose a simple submodule xR of M with $W \cap xR = 0$. We proceed with two cases.

Case 1: $xc_0 + f(x)c_1 + \cdots + f^l(x)c_l \in W$ for some $l \geq 0$, where $c_i \in R$ for each i and at least one of the terms $f^i(x)c_i$ is nonzero. We can assume that l is the smallest non-negative integer such that $xc_0 + f(x)c_1 + \cdots + f^l(x)c_l \in W$ for some $c_0, \ldots, c_l \in R$ with not all $f^i(x)c_i$ equal to 0. Then $l > 0$ by hypothesis, and $f^l(x)c_l \neq 0$ by the choice of l. Let $V := xR + f(x)R + \cdots + f^{l-1}(x)R$. By the choice of l, $V = xR \oplus f(x)R \oplus \cdots \oplus f^{l-1}(x)R$ is a direct sum and

$W \cap V = 0$. Note that

$$\begin{aligned} 0 &\neq f^l(x)c_l \\ &= \big(xc_0 + f(x)c_1 + \cdots + f^l(x)c_l\big) - \big(xc_0 + f(x)c_1 + \cdots + f^{l-1}(x)c_{l-1}\big) \in W \oplus V. \end{aligned}$$

As xR is simple, $f^l(x)c_lR = f^l(xc_lR) = f^l(xR) = f^l(x)R \subseteq W \oplus V$, so $W \oplus V$ is f-invariant. Write

$$f^l(x) = w_0 + xa_0 + f(x)a_1 + \cdots + f^{l-1}(x)a_{l-1},$$

where $w_0 \in W$ and $a_i \in R$ for $i = 0, \ldots, l-1$. Let $\alpha \in \mathrm{End}(V_R)$ be given by

$$\begin{aligned} \alpha(x) &= f(x), \\ \alpha(f(x)) &= f^2(x), \\ &\vdots \\ \alpha(f^{l-2}(x)) &= f^{l-1}(x), \\ \alpha(f^{l-1}(x)) &= f^{l-1}(x)a_{l-1} + \cdots + f(x)a_1 + xa_0. \end{aligned}$$

One sees that α is well-defined. As xR is simple,

$$xR \cong f(x)R \cong \cdots \cong f^{l-1}(x)R,$$

so $\mathrm{End}(V_R) \cong \mathrm{End}((xR)^l_R) \cong \mathbb{M}_l(D)$ where $D = \mathrm{End}((xR)_R)$. Moreover, $I := \mathbf{r}(x)$ is a maximal right ideal of R, so $J(R) \subseteq I$. As $R/J(R)$ is Boolean, every right ideal of $R/J(R)$ is an ideal. Hence, $I/J(R)$ is an ideal of $R/J(R)$, so I is an ideal of R. Hence $R/I \cong \mathbb{F}_2$. Thus,

$$D = \mathrm{End}((xR)_R) \cong \mathrm{End}((R/I)_R) \cong \mathrm{End}((R/I)_{R/I}) \cong R/I \cong \mathbb{F}_2.$$

By Lemma 2.1.3, we see that $\mathrm{End}(V_R)$ is nil-clean. Write $\alpha = \beta + \gamma$ where $\beta^2 = \beta \in \mathrm{End}(V_R)$ and $\gamma^k = 0$ in $\mathrm{End}(V_R)$ for some $k \geq 1$. Extend g to $g' \in \mathrm{End}((W \oplus V)_R)$ by setting $g'|_W = g$ and $g'|_V = \beta$. Then it follows that

$g'^2 = g' \in \mathrm{End}((W \oplus V)_R)$. Now we verify that $f|_{W\oplus V} - g'$ is locally nilpotent in $\mathrm{End}((W \oplus V)_R)$.

Note that $(f-\alpha)V \subseteq w_0R$. For $w+v \in W+V$ where $w \in W$ and $v \in V$, write $v = xb_0 + f(x)b_1 + \cdots + f^{l-1}(x)b_{l-1}$. Then

$$\begin{aligned}(f-g')(w+v) &= (f-g')w + (f-g')v \\ &= (f-g)w + (f-\alpha)v + (\alpha-\beta)v \\ &= w_1 + v_1,\end{aligned}$$

where $w_1 = (f-g)w + (f-\alpha)v \in W$ and $v_1 = (\alpha-\beta)v \in V$. Similarly,

$$(f-g')^2(w+v) = (f-g')(w_1+v_1) = w_2 + v_2$$

where $w_2 \in W$ and $v_2 = (\alpha-\beta)^2 v \in V$. In general, for $n \geq 1$, we have

$$(f-g')^n(w+v) = w_n + v_n$$

where $w_n \in W$ and $v_n = (\alpha-\beta)^n v \in V$. In particular, as $(\alpha-\beta)^k = \gamma^k = 0$, we have $(f-g')^k(u+w) = w_k + (\alpha-\beta)^k v = w_k \in W$. Since $f-g$ is locally nilpotent in $\mathrm{End}(W_R)$, there exists $m > 0$ such that $(f-g)^m(w_k) = 0$. Therefore,

$$\begin{aligned}(f-g')^{m+k}(w+v) &= (f-g')^m\big((f-g')^k(w+v)\big) \\ &= (f-g')^m(w_k) = (f-g)^m(w_k) = 0.\end{aligned}$$

This verifies that $f-g'$ is locally nilpotent in $\mathrm{End}((W \oplus V)_R)$. Thus, we have $(W \oplus V, g') \in \mathcal{S}$, contradicting the maximality of (W, g).

Case 2: No linear combination $xa_0 + f(x)a_1 + \cdots + f^l(x)a_l$ $(l \geq 0, a_i \in R)$ is contained in W unless $f^i(x)a_i = 0$ for all i. Thus,

$$\begin{aligned}V :&= xR + f(x)R + \cdots + f^i(x)R + \cdots \\ &= xR \oplus f(x)R \oplus \cdots \oplus f^i(x)R \oplus \cdots .\end{aligned}$$

We see that V is f-invariant, f acts as the shift operator on V and $W \cap V = 0$. Because of the Claim, a simple induction shows that $f^i(x) \neq 0$ for all i. As xR is simple, each $f^i(x)R$ is simple. So $\mathbf{r}(x)$ and $\mathbf{r}(f^i(x))$ are maximal right ideals of R and $\mathbf{r}(x) \subseteq \mathbf{r}(f^i(x))$, and it follows that $I := \mathbf{r}(x) = \mathbf{r}(f^i(x))$ (for all i). As $R/J(R)$ is Boolean, I is a two-sided ideal of R. So $VI = 0$, and V is a right R/I-module. Thus, an endomorphism of V_R can be regarded as an endomorphism of $V_{R/I}$. Especially, f is the shift operator on V as a semisimple R/I-module. As $2 = 0$ in R/I, by Lemma 2.1.4,

$$f|_V = \beta + \gamma, \ \ \beta^2 = \beta \ \text{ and } \ \gamma^3 = 0, \tag{2.1}$$

where $f|_V, \beta$ and γ are endomorphisms of the module $V_{R/I}$. Certainly, (2.1) holds when $f|_V, \beta$ and γ are regarded as endomorphisms of the module V_R. Extend g to $g' \in \text{End}((W \oplus V)_R)$ by setting $g'|_W = g$ and $g'|_V = \beta$. Then $g'^2 = g' \in \text{End}((W \oplus V)_R)$. To see that $f - g'$ is locally nilpotent in the endomorphism ring $\text{End}((W \oplus V)_R)$, let $w + v \in W \oplus V$ with $w \in W$ and $v \in V$. As $f - g$ is locally nilpotent in $\text{End}(W_R)$, there exists $m \geq 3$ such that $(f - g)^m w = 0$. Thus, $(f|_V - \beta)^m = \gamma^m = 0$, and so

$$\begin{aligned}(f - g')^m(w + v) &= (f - g')^m w + (f - g')^m v = (f - g)^m w + (f - \beta)^m v \\ &= 0 + 0 = 0,\end{aligned}$$

so $f - g'$ is locally nilpotent in $\text{End}((W \oplus V)_R)$. Hence $(W \oplus V, g') \in \mathcal{S}$, contradicting the maximality of (W, g). Therefore, $W = M$ and the proof is complete. □

A ring is called **right quasi-duo** if every maximal right ideal is a two-sided ideal. For instance, commutative rings and local rings are right quasi-duo rings.

By Theorem 2.1.5, we have the following corollary:

Corollary 2.1.6 *Let R be a right quasi-duo ring. The following are equiva-*

lent:

(1) $R/J(R)$ *is Boolean.*

(2) *For any semisimple right* R*-module* M*, every endomorphism of* M *is a sum of an idempotent endomorphism and a locally nilpotent endomorphism.*

Proof. (1) $\Rightarrow$ (2). This follows from Theorem 2.1.5.

(2) $\Rightarrow$ (1). For any maximal right ideal I of R, $(R/I)^2$ is a 2-generated semisimple right R-module, so $\mathrm{End}((R/I)^2_R)$ is nil-clean by (2). On the other hand, since $\mathrm{End}((R/I)^2_R) \cong \mathbb{M}_2(\mathrm{End}((R/I)_R))$, $\mathbb{M}_2(\mathrm{End}((R/I)_R))$ is nil-clean, and so $\mathrm{End}((R/I)_R) \cong \mathbb{F}_2$ by Lemma 2.1.3. As R is right quasi-duo, I is an ideal. Thus $\mathrm{End}((R/I)_R) \cong \mathrm{End}((R/I)_{R/I}) \cong R/I$, so $R/I \cong \mathbb{F}_2$. Because $J(R)$ is the intersection of all maximal (right) ideals of R, we deduce that $R/J(R)$ is a subdirect product of the $\mathbb{F}_2$'s. It follows that $R/J(R)$ is Boolean. □

The following lemma is implicit in the proof of [49, Theorem 3].

Lemma 2.1.7 *[49] Let* D *be a division ring of characteristic* 2*. If an element* $a \in D\backslash\{0,1\}$*, then* $aE_{11} \in \mathbb{M}_n(D)$ *is not nil-clean for any* $n \geq 1$*, where* E_{11} *is the matrix with* $(1,1)$*-entry* 1 *and all others* 0*.*

Lemma 2.1.3 (or [49, Theorem 3]) is a special case of the following theorem where $V_D = (D^n)_D$.

Theorem 2.1.8 *Let* V *be a non-trivial vector space over a division ring* D*. Then every linear transformation of* V *is a sum of an idempotent linear transformation and a locally nilpotent linear transformation if and only if* $D \cong \mathbb{F}_2$*.*

Proof. ($\Longleftarrow$). This is by Theorem 2.1.5.

($\Longrightarrow$). By Lemma 2.1.2, D has characteristic 2. Assume to the contrary that $D \ncong \mathbb{F}_2$. Choose $a \in D\backslash\{0,1\}$. Take a nonzero element $v \in V$ and write $V = vD \oplus U$. Let $f \in \mathrm{End}(V_D)$ be such that $f(vd + u) = vad$ for all $d \in D$ and $u \in U$. Then, by hypothesis, $f = g + h$ where $g^2 = g \in \mathrm{End}(V_D)$

and $h \in \mathrm{End}(V_D)$ is locally nilpotent. So, $h^n(v) = 0$ for some $n > 0$. Then $W := \mathrm{span}\{v, h(v), \ldots, h^{n-1}(v)\}$ is h-invariant. As $\mathrm{im}(f) = vD \subseteq W$, W is f-invariant, so it is also g-invariant. Therefore, $f|_W = g|_W + h|_W$ where $g|_W$ is an idempotent of $\mathrm{End}(W_D)$ and $h|_W$ is nilpotent in $\mathrm{End}(W_D)$ (as $\dim(W) < \infty$). That is, $f|_W$ is nil-clean in $\mathrm{End}(W_D)$. As $vD \subseteq W$, $W = vD \oplus W'$ with $W' = W \cap U$. Fix a basis $B = \{v = v_1, v_2, \ldots, v_k\}$ of W where $\{v_2, \ldots, v_k\}$ is a basis of W'. With respect to the basis B, we identify $\mathrm{End}(W_D)$ with the matrix ring $\mathbb{M}_k(D)$ and $f|_W$ with the matrix aE_{11}. Thus, by what we showed above, $aE_{11} \in \mathbb{M}_k(D)$ is nil-clean, contradicting to the Lemma 2.1.7. □

We give a non nil-clean endomorphism that is a sum of an idempotent endomorphism and a locally nilpotent endomorphism.

Example 2.1.9 *Let $M := \mathbb{Z}_{p^\infty}$ be the Prüfer abelian group (p is a prime). Then $p \in \mathrm{End}(M_{\mathbb{Z}})$ is locally nilpotent. But it is not nil-clean in* $\mathrm{End}(M_{\mathbb{Z}})$. *In fact, note that $\mathrm{End}(M_{\mathbb{Z}}) \cong \widehat{\mathbb{Z}}_p$, the ring of p-adic integers, which is a local domain. So the trivial idempotents are the only nil-clean elements of* $\mathrm{End}(M_{\mathbb{Z}})$. *Hence, $p \in \mathrm{End}(M_{\mathbb{Z}})$ is not nil-clean.*

Remark 2.1.10 *The question, left in [14], whether the ring of linear transformations of a countably infinite dimensional vector space over $\mathbb{F}_2$ is nil-clean remains open.*

Statement 2.1.11 *The material in this section is taken from [74]. My contribution in this joint work is to come up with the decomposition of a shift operator of an endomorphism ring of vector space over a field, which motivates Lemma 2.1.4.*

2.2 Nil-clean rings with involution

Recall that an element of a ring is called **nil-clean** if it is a sum of an idempotent and a nilpotent, and the ring is called **nil-clean** if each of its elements is nil-clean. One result on nil-clean rings states that the ring $\mathbb{M}_n(\mathbb{F}_2)$ is nil-clean (see [32, Example 4.5]), or more generally, for a 2-primal ring R, $\mathbb{M}_n(R)$ is nil-clean if and only if $J(R)$ is nil and $R/J(R)$ is Boolean (see [50, Theorem 6.1]). The motivation here is the observation that, not just being nil-clean, every matrix in $\mathbb{M}_2(\mathbb{F}_2)$ is indeed a sum of a symmetric idempotent matrix and a nilpotent matrix. To explain what this might mean, we consider the notion of a $*$-ring. An element e in a $*$-ring R is called a projection if $e^2 = e = e^*$. For a $*$-ring R, there is an involution of $\mathbb{M}_n(R)$ given by $(a_{ij})^* = (a_{ij}^*)^T$ where A^T denotes the transpose of matrix A. Particularly, for a commutative ring R, $\mathbb{M}_n(R)$ is a $*$-ring where $A^* = A^T$ for any $A \in \mathbb{M}_n(R)$. Thus, the above example indicates that, with $*$ being the transpose, every element in $\mathbb{M}_2(\mathbb{F}_2)$ is a sum of a projection and a nilpotent. In other words, $\mathbb{M}_2(\mathbb{F}_2)$ is a nil $*$-clean ring, where a $*$-ring is called nil $*$-clean if every element is a sum of a projection and a nilpotent. One therefore is led to the question: For a $*$-ring R, when is $\mathbb{M}_n(R)$ a nil $*$-clean ring where $(a_{ij})^* = (a_{ij}^*)^T$?

In this section, we first show that a $*$-ring R is a nil $*$-clean ring if and only if $J(R)$ is nil and $R/J(R)$ is a nil $*$-clean ring. Regarding the question above, for a large class of rings, we reduce the question to the nil $*$-cleanness of the matrix ring over $\mathbb{F}_2$. In fact, it is shown that, for a 2-primal $*$-ring R, with involution $*$ given by $(a_{ij})^* = (a_{ij}^*)^T$, then $\mathbb{M}_n(R)$ is a nil $*$-clean ring if and only if $J(R)$ is nil, $R/J(R)$ is Boolean, $a^* - a \in J(R)$ for all $a \in R$ and $\mathbb{M}_n(\mathbb{F}_2)$ is nil $*$-clean. We verify that $\mathbb{M}_2(\mathbb{F}_2)$ is nil $*$-clean and $\mathbb{M}_n(\mathbb{F}_2)$ is not nil $*$-clean for $n = 3, 4$, but have been unable to prove that $\mathbb{M}_n(\mathbb{F}_2)$ is not nil $*$-clean for all $n > 2$. Thus, for a 2-primal $*$-ring R, (1) $\mathbb{M}_2(R)$ is a nil $*$-clean ring if and only if $J(R)$ is nil, $R/J(R)$ is Boolean, $a^* - a \in J(R)$ for all $a \in R$; (2) $\mathbb{M}_n(R)$ is not a nil $*$-clean ring for $n = 3, 4$. It is also observed that, for a commutative ring R and $n \in \{2,3,4\}$, every symmetric matrix in $\mathbb{M}_n(R)$ is a sum of symmetric idempotent matrix and a nilpotent matrix if and only if $J(R)$ is nil and $R/J(R)$ is Boolean.

We start with the following observations.

Lemma 2.2.1 *Let R be a $*$-ring and $e^2 = e \in R$.*

(1) If $1+(e-e^*)^*(e-e^*) \in U(R)$, then there exists $p^2 = p^* = p \in R$ such that $eR = pR$ and $e-p \in \mathrm{Nil}(R)$.

(2) If $e-e^* \in J(R)$, then there exists $p^2 = p^* = p \in R$ such that $eR = pR$ and $e-p \in J(R) \cap \mathrm{Nil}(R)$.

(3) If $e-e^* \in \mathrm{Nil}(R)$, then there exists $p^2 = p^* = p \in R$ such that $eR = pR$ and $e-p \in \mathrm{Nil}(R)$.

Proof. (1) Write $x := 1+(e-e^*)^*(e-e^*)$. Then $xe = ee^*e = ex$ and $x^* = x$, so $xe^* = e^*x$ and $e = x^{-1}ee^*e$. Thus $x^{-1}ee^* = x^{-1}ee^*x^{-1}ee^*$. Let $p = x^{-1}ee^*$. Then $p = p^2 = p^*$. Since $(1-e)p = (1-e)x^{-1}ee^* = 0$, we have $p = ep$. Since $pe = (x^{-1}ee^*)e = x^{-1}(ee^*e) = x^{-1}xe = e$, we have $eR = pR$ and $(e-p)^2 = e+p-ep-pe = 0$. Therefore, $e-p \in \mathrm{Nil}(R)$.

(2) Since $e-e^* \in J(R)$, $x = 1+(e-e^*)^*(e-e^*) \in U(R)$. By (1), there exists $p^2 = p^* = p \in R$ such that $eR = pR$ and $e-p \in \mathrm{Nil}(R)$. So $e = pe$ and $p = ep = p* = pe^*$, which implies $e-p = pe-pe^* = p(e-e^*) \in J(R)$. Thus, $e-p \in J(R) \cap \mathrm{Nil}(R)$.

(3) Note that $(e-e^*)^*(e-e^*) = (e-e^*)(e-e^*)^*$. So it is clear that $x = 1+(e-e^*)^*(e-e^*) \in U(R)$ as $e-e^* \in \mathrm{Nil}(R)$. Thus, the claim follows from (1). □

Recall that a ring R is **abelian** if every idempotent is central. The authors in [55] gave a sufficient condition for a $*$-ring to be abelian as follows

Lemma 2.2.2 *[55, Lemma 2.1] A $*$-ring R is abelian if every idempotent of R is a projection.*

Proof. Let $e^2 = e \in R$ and $r \in R$. Then $e+(er-ere)$ is an idempotent, and hence is a projection by hypothesis. So it follows that

$$e + (er - ere) = [e + (er - ere)]^* = e + r^*e - er^*e,$$

which gives

$$er - ere = r^*e - er^*e, \qquad (1)$$

Multiplying (1) by $1 - e$ from the right and from the left, respectively, yields $er = ere$. Similarly, we have $re = ere$ by taking idempotent $(1-e)+(1-e)re$. This shows that e is central. □

In an abelian $*$-ring, an idempotent need not be a projection. In fact, if $R = S \times S$, where S is a commutative ring, with involution $*$ given by $(a, b)^* = (b, a)$, then R is a $*$-ring that is abelian, but the idempotent $(1, 0)$ is not a projection. In the next lemma we explains when an abelian $*$-ring has the property that every idempotent is a projection.

Lemma 2.2.3 *The following are equivalent for a $*$-ring R :*

(1) *For each $e^2 = e \in R$, $e - e^* \in J(R)$ and R is abelian.*

(2) *For each $e^2 = e \in R$, $e - e^* \in \mathrm{Nil}(R)$ and R is abelian.*

(3) *For each $e^2 = e \in R$, $e - e^* \in J(R)$ and $ee^* = e^*e$.*

(4) *For each $e^2 = e \in R$, $e - e^* \in \mathrm{Nil}(R)$ and $ee^* = e^*e$.*

(5) *Every idempotent of R is a projection.*

Proof. (5) ⇒ (i) (for $i = 1, 2, 3, 4$). They are clear by Lemma 2.2.2.

(1) ⇒ (5) and (2) ⇒ (5). By Lemma 2.2.1, there exists a projection $p \in R$ such that $eR = pR$. Since R is abelian, we obtain $e = pe = ep = p$, as required.

(3) ⇒ (5). Assume that (3) holds. Write $j := e - e^*$. Then it follows that $e(1 - e^*) = j(1 - e^*) \in J(R)$ and $e^*(1 - e) = -j(1 - e) \in J(R)$. Since $ee^* = e^*e$, $e(1 - e^*)$ and $e^*(1 - e)$ are idempotents. Therefore, $e(1 - e^*) = 0$ and $e^*(1 - e) = 0$. Thus $e = ee^* = e^*e = e^*$. So every idempotent of R is a projection.

(4) $\Rightarrow$ (5). It is similar to the proof of "(3) $\Rightarrow$ (5)". $\square$

An ideal I of a $*$-ring R is called a $*$-ideal if $I^* \subseteq I$. For a $*$-ideal I of a $*$-ring R, R/I is a $*$-ring where $* : R/I \to R/I$ is given by $\overline{a}^* = \overline{a^*}$. In particular, if R is a $*$-ring, then $R/J(R)$ is a $*$-ring.

Let R be a ring and I be an ideal of R. We say that idempotents can be lifted modulo I if, whenever $a^2 - a \in I$, there exists $e^2 = e \in R$ such that $a - e \in I$.

Lemma 2.2.4 *Let R be a $*$-ring. If idempotents lift modulo $J(R)$, then every projection of $R/J(R)$ is lifted to a projection of R.*

Proof. Let $\overline{e} = \overline{e}^* = \overline{e}^2 \in R/J(R)$. Since idempotents lift modulo $J(R)$, we may assume that $e^2 = e \in R$. Since $\overline{e} = \overline{e}^* = \overline{e^*}$, $e - e^* \in J(R)$. By Lemma 2.2.1, there exists $p^2 = p^* = p \in R$ such that $e - p \in J(R)$, as desired. $\square$

Comparing with a result of Diesl [32] that a ring R is nil clean if and only if $J(R)$ is nil and $R/J(R)$ is nil clean, we have the next theorem.

Theorem 2.2.5 *Let R be a $*$-ring. Then R is nil $*$-clean if and only if $J(R)$ is nil and $R/J(R)$ is nil $*$-clean.*

Proof. ($\Rightarrow$) If R is nil $*$-clean, then $R/J(R)$ is clearly nil $*$-clean. As R is nil clean, $J(R)$ is nil by [32].

($\Leftarrow$) Let $a \in R$. Since $R/J(R)$ is nil $*$-clean, $\overline{a} = \overline{e} + \overline{b}$, where $\overline{b} \in R/J(R)$ is nilpotent and $\overline{e}$ is a projection in $R/J(R)$. Since $J(R)$ is nil, $b \in R$ is nilpotent and every projection of $R/J(R)$ is lifted to a projection of R by Lemma 2.2.4. Thus, we can assume that $e \in R$ is a projection. Hence $a = e + (b + j)$ for some $j \in J(R)$. It is clear that $b + j$ is nilpotent. $\square$

Next we discuss the matrix rings. The following example is easily verified.

Example 2.2.6 *For each $A \in \mathbb{M}_2(\mathbb{F}_2)$, $A = B + E$, where $B^2 = 0$ and $E^2 = E = E^T$.*

Note that, for a ring R and $n \geq 2$, the transpose on $\mathbb{M}_n(R)$ is an involution if and only if R is commutative. Thus, the following question is motivated by Example 2.2.6: Let R be a commutative ring and $n \geq 2$. The involution $\mathbb{M}_n(R) \to \mathbb{M}_n(R)$ given by $(a_{ij}) \mapsto (a_{ij})^T$ is still denoted by $*$. When is $\mathbb{M}_n(R)$ a nil $*$-clean ring?

Before addressing this question, we point out some needed facts: If matrix $B \in \mathbb{M}_n(F)$ is a nilpotent matrix where F is a field, then $B^n = 0$ and $tr(B) = 0$. Indeed, it is well-known that B is similar to a nilpotent Jordan matrix C. As C is strictly upper triangular, $C^n = 0$, so $B^n = 0$. Moreover, the trace of a matrix is similarity-invariant. Thus, $tr(B) = tr(C) = 0$.

Lemma 2.2.7 *Let R be a commutative ring and $n \geq 2$. With $*$ being the transpose, the following are equivalent:*

(1) $\mathbb{M}_n(R)$ *is a nil $*$-clean ring.*

(2) $J(R)$ *is nil and* $\mathbb{M}_n(R/J(R))$ *is nil $*$-clean.*

(3) $R/J(R)$ *is Boolean with* $J(R)$ *nil and* $\mathbb{M}_n(\mathbb{F}_2)$ *is nil $*$-clean.*

Proof. (1) $\Rightarrow$ (3) Assume that $\mathbb{M}_n(R)$ is a nil $*$-clean ring. Then, by [50, Theorem 6.1], $J(R)$ is nil and $R/J(R)$ is Boolean. So, there exists an ideal I of R such that $R/I \cong \mathbb{F}_2$. Let $\varphi : \mathbb{M}_n(R) \to \mathbb{M}_n(R/I)$ be the natural homomorphism given by $(a_{ij}) \mapsto (a_{ij} + I)$. If $E^2 = E = E^*$ in $\mathbb{M}_n(R)$, then $\varphi(E)^* = \varphi(E^*) = \varphi(E) = \varphi(E)^2$. It follows that $\mathbb{M}_n(R/I)$ is nil $*$-clean.

(3) $\Rightarrow$ (2) For any matrix A in $\mathbb{M}_n(R/J(R))$, to show that A is nil $*$-clean in $\mathbb{M}_n(R/J(R))$ it suffices to show that A is nil $*$-clean in $\mathbb{M}_n(S)$, where S is the subring of $R/J(R)$ generated by the entries of A and the identity of $R/J(R)$. Here, S is a direct product of finitely many copies of $\mathbb{F}_2$. Since $\mathbb{M}_n(\mathbb{F}_2)$ is nil $*$-clean, $\mathbb{M}_n(S)$ is nil $*$-clean, so A is nil $*$-clean in $\mathbb{M}_n(S)$.

(2) $\Rightarrow$ (1) Since $J(R)$ is nil and R is commutative, $J(\mathbb{M}_n(R)) = \mathbb{M}_n(J(R))$ is nil. So, the implication holds by Theorem 2.2.5. □

Corollary 2.2.8 *Let R be a commutative ring. Then, with $*$ being the transpose,*

(1) $\mathbb{M}_2(R)$ *is a nil $*$-clean ring if and only if* $J(R)$ *is nil and* $R/J(R)$ *is a Boolean ring.*

(2) $\mathbb{M}_n(R)$ *is not a nil $*$-clean ring for* $n = 3, 4$.

Proof. (1) By Lemma 2.2.7, it suffices to show that $\mathbb{M}_2(\mathbb{F}_2)$ is nil $*$-clean. For $(a_{ij}) \in \mathbb{M}_2(\mathbb{F}_2)$, let $x = a_{11} + a_{12}a_{21}$ and $y = a_{22} + a_{12}a_{21}$. Then

$$(a_{ij}) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} + \begin{pmatrix} a_{12}a_{21} & a_{12} \\ a_{21} & a_{12}a_{21} \end{pmatrix}$$

with

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}^2 = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}^T, \text{ and } \begin{pmatrix} a_{12}a_{21} & a_{12} \\ a_{21} & a_{12}a_{21} \end{pmatrix}^2 = 0.$$

So $\mathbb{M}_2(\mathbb{F}_2)$ is nil $*$-clean.

(2) By Lemma 2.2.7, it suffices to show that $\mathbb{M}_n(\mathbb{F}_2)$ is not nil $*$-clean.

It is an easy exercise that $E^2 = E = E^T$ in $\mathbb{M}_3(\mathbb{F}_2)$ if and only if matrix $E = \begin{pmatrix} a & x & x \\ x & b & x \\ x & x & c \end{pmatrix}$, where $ax = bx = cx$. We show that $A := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is not nil $*$-clean in $\mathbb{M}_3(\mathbb{F}_2)$. Indeed, if $A = E + B$ where $E^2 = E = E^T$ and B is nilpotent, then $E = \begin{pmatrix} a & x & x \\ x & b & x \\ x & x & c \end{pmatrix}$ with $ax = bx = cx$ and $B = A - E$. That is, $B = \begin{pmatrix} a & x & 1+x \\ 1+x & b & x \\ x & 1+x & c \end{pmatrix}$. Since B is nilpotent, $B^3 = 0$. So, $0 = \det(B^3) = \det(B)^3 = \det(B)$. But $\det(B) = 1 + abc$, so $a = b = c = 1$. On the other hand, $0 = tr(B) = a + b + c = 1 + 1 + 1 = 1$, a contradiction.

Let $A = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & A_1 \end{pmatrix}$ where $A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. We next show that matrix

$A \in \mathbb{M}_4(\mathbb{F}_2)$ is not nil $*$-clean. Assume to the contrary that $A \in \mathbb{M}_4(\mathbb{F}_2)$ is nil $*$-clean. Then there exists $E^2 = E = E^T \in \mathbb{M}_4(\mathbb{F}_2)$ such that $A + E$ is nilpotent (indeed, $(A+E)^4 = 0$) and $tr(E) = tr(A+E) = 0$. Write $E = \begin{pmatrix} e & \alpha \\ \alpha^T & E_1 \end{pmatrix}$, where $e \in \mathbb{F}_2, \alpha = (x_1, x_2, x_3)$, and $E_1 \in \mathbb{M}_3(\mathbb{F}_2)$. From $E^2 = E = E^T$, we see

$$\begin{cases} \alpha\alpha^T = 0, \ \ E_1^T = E_1, \\ \alpha = e\alpha + \alpha E_1, \\ E_1 = \alpha^T\alpha + E_1^2. \end{cases} \tag{2.2}$$

So, we have

$$\begin{aligned} 0 &= (A+E)^4 \\ &= \begin{pmatrix} e + \alpha(A_1 + I_3)^2\alpha^T & e\alpha(A_1 + I_3) + \alpha(A_1 + I_3)(\alpha^T\alpha + (A_1 + E_1)^2) \\ * & * \end{pmatrix} \end{aligned} \tag{2.3}$$

We next show that $e \neq 0$. In fact, if $e = 0$, then, as $\alpha\alpha^T = 0$ and also $\alpha(A_1 + I_3)^2\alpha^T = 0$, we have

$$x_1 + x_2 + x_3 = 0 \ \text{ and } \ x_1 + x_2 + x_3 + x_1x_2 + x_1x_3 + x_2x_3 = 0.$$

If $x_1 = 1$, then $x_2 + x_3 = 1$ and $x_2x_3 = 1$, which is impossible in $\mathbb{F}_2$. If $x_1 = 0$, then $x_2 = x_3$ and $x_2x_3 = 0$; so $x_2 = x_3 = 0$. Thus $\alpha = 0$, so $E_1^2 = E_1 = E_1^T$ and $A_1 + E_1$ is nilpotent. This is a contradiction, because A_1 is not nil $*$-clean in $\mathbb{M}_3(\mathbb{F}_2)$. Hence, we have proved that $e = 1$.

By (2.2), $\alpha^T\alpha = E_1 - E_1^2$. Thus, by (2.3), we have

$$\begin{aligned}
0 &= \alpha(A_1 + I_3)(I_3 + \alpha^T\alpha + A_1^2 + A_1E_1 + E_1A_1 + E_1^2) \\
&= \alpha(A_1 + I_3)(I_3 + E_1 + A_1^2 + A_1E_1 + E_1A_1) \\
&= \alpha \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} + E_1 + A_1E_1 + E_1A_1 \right).
\end{aligned} \tag{2.4}$$

Since $\alpha\alpha^T = 0$ and $1 + \alpha(A_1 + I_3)^2\alpha^T = 0$, we have $x_1 + x_2 + x_3 = 0$ and $x_1 + x_2 + x_3 + x_1x_2 + x_1x_3 + x_2x_3 = 1$. Thus $x_1 + x_2x_3 = 1$. So either $x_1 = 0$ and $x_2 = x_3 = 1$, or $x_1 = x_2 = 1$ and $x_3 = 0$, or $x_1 = x_3 = 1$ and $x_2 = 0$. We proceed with three cases.

Case 1: $x_1 = 0$ and $x_2 = x_3 = 1$. Since $E^2 = E = E^T$ and $tr(E) = 0$, it must be that

$$E = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & a & a \\ 1 & a & 1+a & 1+a \\ 1 & a & 1+a & 1+a \end{pmatrix} \quad \text{with} \;\; a \in \{0, 1\}.$$

Thus, by (2.4),

$$0 = \alpha \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} + E_1 + A_1E_1 + E_1A_1 \right) = (a, 1, 1 + a),$$

which is a contradiction.

Case 2: $x_1 = x_3 = 1$ and $x_2 = 0$. Since $E^2 = E = E^T$ and $tr(E) = 0$, it must be that

$$E = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1+a & a & 1+a \\ 0 & a & 1 & a \\ 1 & 1+a & a & 1+a \end{pmatrix} \text{ with } a \in \{0,1\}.$$

Thus, by (2.4),

$$0 = \alpha \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \Big(\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} + E_1 + A_1E_1 + E_1A_1 \Big) = (1+a, a, 1),$$

which is a contradiction.

Case 3: $x_1 = x_2 = 1$ and $x_3 = 0$. Since $E^2 = E = E^T$ and $tr(E) = 0$, it must be that

$$E = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & a & a & 1+a \\ 1 & a & a & 1+a \\ 0 & 1+a & 1+a & 1 \end{pmatrix} \text{ with } a \in \{0,1\}.$$

Thus, by (2.4),

$$0 = \alpha \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \Big(\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} + E_1 + A_1E_1 + E_1A_1 \Big) = (1, a, 1+a),$$

which is a contradiction. □

Next we consider a more general setting. Let R be a $*$-ring. There is an involution $\mathbb{M}_n(R) \to \mathbb{M}_n(R)$, still denoted by $*$, given by $(a_{ij}) \mapsto (a_{ij}^*)^T$. When is $\mathbb{M}_n(R)$ a nil $*$-clean ring?

Recall a ring R is called subdirectly irreducible if R has a smallest nonzero ideal (or equivalently, the intersection of all nonzero ideals of R is nonzero). A $*$-ring is called **$*$-subdirectly irreducible** if the intersection of all nonzero $*$-ideals of R is nonzero.

A routine application of Birkhoff's theorem gives that every $*$-ring R is a subdirect product of $*$-subdirectly irreducible rings. An element a in a $*$-ring R is called **symmetric** if $a^* = a$. For a $*$-subdirectly irreducible ring that is Boolean we have the following:

Lemma 2.2.9 *If R is a $*$-subdirectly irreducible ring that is Boolean, then either $R = \mathbb{F}_2$ or $R = \{0, 1, a, 1 + a\}$ with $a^* = 1 + a$.*

Proof. The hypothesis immediately implies that 0 and 1 are the only symmetric elements of R. We can assume that $R \neq \{0, 1\}$. For any $x \in R \backslash \{0, 1\}$, $x^* + x \neq 0$ and so $x^* + x = 1$ as $x^* + x$ is a symmetric element. Hence, $x^* = 1 + x$ for all $x \in R \backslash \{0, 1\}$. Take $a \in R \backslash \{0, 1\}$. Then $1 + a \in R \backslash \{0, 1\}$. We claim that $R = \{0, 1, a, 1 + a\}$. Assume to the contrary that $R \neq \{0, 1, a, 1 + a\}$. Take $y \in R \backslash \{0, 1, a, 1 + a\}$. Then $1 + y \in R \backslash \{0, 1, a, 1 + a\}$. Note that $ay \neq 0$. In fact, $ay = 0$ would imply that

$$0 = (ay)^* = y^* a^* = (1 + y)(1 + a) = 1 + a + y + ay = 1 + a + y,$$

i.e., $y = 1 + a$, a contradiction. Moreover, $ay \neq 1$. Hence $1 + ay = (ay)^* = y^* a^* = (1 + y)(1 + a) = 1 + a + y + ay$, i.e., $y = a$, a contradiction. □

Lemma 2.2.10 *Let R be a Boolean ring of four elements $\{0, 1, a, 1 + a\}$, with involution $*$ given by $0^* = 0, 1^* = 1, a^* = 1 + a$ and $(1 + a)^* = a$. Then, with involution given by $(a_{ij})^* = (a_{ij}^*)^T$, $\mathbb{M}_n(R)$ is not nil $*$-clean.*

Proof. Let $aE_{11} \in \mathbb{M}_n(R)$, where E_{11} is the matrix with $(1, 1)$-entry 1 and $0's$ elsewhere and write $A = aE_{11}$. Assume to the contrary that $A = E + B$ where $E^2 = E = E^*$ and $B = A + E$ is nilpotent. Write $E = (e_{ij})$, where $e_{ii}^* = e_{ii} \in \{0, 1\}$ for each i. As $R \cong \mathbb{F}_2 \times \mathbb{F}_2$, we deduce that $tr(B) = 0$ in view of the remark prior to Lemma 2.2.7. Thus, $0 = tr(B) = tr(A) + tr(E) = a + (e_{11} + \cdots + e_{nn})$, so $a = e_{11} + \cdots + e_{nn} \in \{0, 1\}$, a contradiction. $\square$

We need the following description of nil-clean matrix ring over a 2-primal ring.

Lemma 2.2.11 *[50, Theorem 6.1] Let R be a 2-primal ring and $n \geq 1$. Then $\mathbb{M}_n(R)$ is nil-clean if and only if $J(R)$ is nil and $R/J(R)$ is Boolean.*

For a 2-primal $*$-ring, we have the following lemma.

Lemma 2.2.12 *Let R be a 2-primal $*$-ring. If $\mathbb{M}_n(R)$ is a nil $*$-clean ring with involution given by $(r_{ij})^* = (r_{ij}^*)^T$, then $\mathbb{M}_n(\mathbb{F}_2)$ is a nil $*$-clean ring.*

Proof. Suppose that $\mathbb{M}_n(R)$ is nil $*$-clean. Then $\mathbb{M}_n(R)$ is nil clean, so $R/J(R)$ is Boolean by Lemma 2.2.11. Let $\varphi : \mathbb{M}_n(R) \to \mathbb{M}_n(R/J(R))$ be the natural homomorphism given by $(r_{ij}) \mapsto (r_{ij} + J(R))$. If $E^2 = E = E^*$ in $\mathbb{M}_n(R)$, then $\varphi(E)^* = \varphi(E^*) = \varphi(E) = \varphi(E)^2$. It follows that $\mathbb{M}_n(R/J(R))$ is nil $*$-clean. Thus, we can assume that R is Boolean.

As R is a subdirect product of $*$-subdirectly irreducible rings, there exists a $*$-ideal I such that R/I is $*$-subdirectly irreducible. In particular, R/I is a $*$-ring where $(x + I)^* = x^* + I$. Let $\Phi : \mathbb{M}_n(R) \to \mathbb{M}_n(R/I)$ be the natural homomorphism given by $(r_{ij}) \mapsto (r_{ij} + I)$. If $E^2 = E = E^*$ in $\mathbb{M}_n(R)$, then $\Phi(E)^* = \Phi(E^*) = \Phi(E) = \Phi(E)^2$. It follows that $M_n(R/I)$ is nil $*$-clean. Since $S := R/I$ is a $*$-subdirectly irreducible that is Boolean, by Lemma 2.2.9 we see that $S = \mathbb{F}_2$ or $S = \{0, 1, a, 1 + a\}$ is a Boolean $*$-ring with $a^* = 1 + a$. Thus, $S = \mathbb{F}_2$ by Lemma 2.2.10. $\square$

Next we give equivalent conditions for a matrix ring over a 2-primal $*$-ring to be nil $*$-clean.

Theorem 2.2.13 *Let R be a 2-primal $*$-ring. With involution given by $(a_{ij})^* = (a_{ij}^*)^T$, the following are equivalent:*

(1) *$\mathbb{M}_n(R)$ is a nil $*$-clean ring.*

(2) *$J(R)$ is nil and $\mathbb{M}_n(R/J(R))$ is a nil $*$-clean ring.*

(3) *$J(R)$ is nil, $R/J(R)$ is a Boolean ring, $a^* - a \in J(R)$ for all $a \in R$, and $\mathbb{M}_n(\mathbb{F}_2)$ is nil $*$-clean.*

Proof. (1) $\Rightarrow$ (3). Suppose that $\mathbb{M}_n(R)$ is nil $*$-clean. Then $J(R)$ is nil and $R/J(R)$ is Boolean by Lemma 2.2.11. Moreover, by Lemma 2.2.12, $\mathbb{M}_n(\mathbb{F}_2)$ is nil $*$-clean.

Assume to the contrary that $a^* - a \notin J(R)$. Then $\bar{a}^* \neq \bar{a}$ in $R/J(R)$. Let $\varphi : \mathbb{M}_n(R) \to \mathbb{M}_n(R/J(R))$ be the natural homomorphism given by $(a_{ij}) \mapsto (a_{ij} + J(R))$. If $E^2 = E = E^*$ in $\mathbb{M}_n(R)$, then it is easily to verfy that $\varphi(E)^* = \varphi(E^*) = \varphi(E) = \varphi(E)^2$. It follows that $\mathbb{M}_n(R/J(R))$ is nil $*$-clean. Thus, to get a contradiction, we can assume that R is Boolean and $a^* \neq a$ for some $a \in R$.

Since R is a subdirect product of $*$-subdirectly irreducible rings, there exist $*$-ideals $\{I_\alpha\}$ such that $\cap I_\alpha = 0$. Since $a^* - a \neq 0$, $a^* - a \notin I_\alpha$ for some α. Since I_α is a $*$-ideal, R/I_α is a $*$-ring where $(x + I_\alpha)^* = x^* + I_\alpha$. Define a natural homomorphism $\Phi : \mathbb{M}_n(R) \to \mathbb{M}_n(R/I_\alpha)$ which is given by $(a_{ij}) \mapsto (a_{ij} + I_\alpha)$. If $E^2 = E = E^*$ in $\mathbb{M}_n(R)$, then we have that $\Phi(E)^* = \Phi(E^*) = \Phi(E) = \Phi(E)^2$. It follows that $M_n(R/I_\alpha)$ is nil $*$-clean. Since R/I_α is a $*$-subdirectly irreducible that is Boolean, by Lemma 2.2.9 we have $R/I_\alpha \cong \mathbb{F}_2$ or $R/I_\alpha = \{0, 1, y, 1+y\}$ with $y^* = 1+y$. But, as $a^* - a \notin I_\alpha$, $(a + I_\alpha)^* \neq a + I_\alpha$. So $R/I_\alpha \ncong \mathbb{F}_2$, and thus $R/I_\alpha = \{0, 1, y, 1 + y\}$ with $y^* = 1 + y$. By Lemma 2.2.10, $\mathbb{M}_n(R/I_\alpha)$ is not nil $*$-clean, a contradiction.

(3) $\Rightarrow$ (2). Since $\bar{a}^* = \bar{a}$ for all $a \in R$, $\mathbb{M}_n(R/J(R))$ is a nil $*$-clean ring by Lemma 2.2.7, where the involution $*$ on $\mathbb{M}_n(R/J(R))$ is given by $\left(\overline{a_{ij}}\right)^* = \left(\overline{a_{ij}}\right)^T$.

(2) $\Rightarrow$ (1). Since $J(R)$ is nil and R is 2-primal, $J(R) = \mathrm{Nil}_*(R)$, the lower nilradical. So, $J\big(\mathbb{M}_n(R)\big) = \mathbb{M}_n(J(R)) = \mathbb{M}_n(\mathrm{Nil}_*(R)) = \mathrm{Nil}_*(\mathbb{M}_n(R))$ is nil by [53, p.172]. By Theorem 2.2.5, $\mathbb{M}_n(R)$ is nil $*$-clean. □

Corollary 2.2.14 *Let R be a 2-primal $*$-ring. Then, with involution given by $(a_{ij})^* = (a_{ij}^*)^T$,*

(1) $\mathbb{M}_2(R)$ *is a nil* $*$*-clean ring if and only if* $J(R)$ *is nil and* $R/J(R)$ *is a Boolean ring, and* $a^* - a \in J(R)$ *for all* $a \in R$.

(2) $\mathbb{M}_n(R)$ *is not a nil* $*$*-clean ring for* $n = 3, 4$.

Proof. By Corollary 2.2.8, $\mathbb{M}_2(\mathbb{F}_2)$ is nil $*$-clean and $\mathbb{M}_n(\mathbb{F}_2)$ is not nil $*$-clean for $n = 3, 4$. Thus the claims follow from Theorem 2.2.13. □

Next we give some examples of nil $*$-clean matrix rings with involution given by $(a_{ij})^* = (a_{ij}^*)^T$.

Example 2.2.15 *Let* R *be a Boolean ring and* $n \geq 3$. *Let*

$$S = \left\{ \begin{pmatrix} a_1 & \cdots & a_{n-2} & x & y \\ & a_1 & \cdots & a_{n-2} & z \\ & & a_1 & \cdots & a_{n-2} \\ & & & \ddots & \vdots \\ & & & & a_1 \end{pmatrix} : a_1, \ldots, a_{n-2}, x, y, z \in R \right\}.$$

Then $J(S)$ *is nil and* $S/J(S)$ *is Boolean. The mapping* $* : S \to S$ *given by*

$$\begin{pmatrix} a_1 & \cdots & a_{n-2} & x & y \\ & a_1 & \cdots & a_{n-2} & z \\ & & a_1 & \cdots & a_{n-2} \\ & & & \ddots & \vdots \\ & & & & a_1 \end{pmatrix} \mapsto \begin{pmatrix} a_1 & \cdots & a_{n-2} & z & y \\ & a_1 & \cdots & a_{n-2} & x \\ & & a_1 & \cdots & a_{n-2} \\ & & & \ddots & \vdots \\ & & & & a_1 \end{pmatrix}$$

is an involution on S *and* $* \neq \mathbf{id}_S$. *As* $s^* - s \in J(S)$ *for all* $s \in S$, *by Corollary 2.2.14* $\mathbb{M}_2(S)$ *is nil* $*$*-clean where* $\big(s_{ij}\big)^* = \big(s_{ij}^*\big)^T$.

Example 2.2.16 *Let* R *be a Boolean ring and* $S = \mathbb{T}_2(R)$, *the* 2×2 *upper triangular matrix ring over* R. *Then* $J(S)$ *is nil and* $S/J(S)$ *is Boolean. The* *mapping* $*: S \to S$ *given by*

$$\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \mapsto \begin{pmatrix} b & x+a+b \\ 0 & a \end{pmatrix}$$

is an involution on S *and and* $* \neq \mathbf{id}_S$. *As* $s^* - s \in J(S)$ *for all* $s \in S$, *by Corollary 2.2.14* $\mathbb{M}_2(S)$ *is nil* $*$*-clean where* $(s_{ij})^* = (s_{ij}^*)^T$.

Example 2.2.17 *Let* R *be a* $*$*-ring that is Boolean. We write the set* $S = \{a \in R : a^* = a\}$. *Then* S *is a subring of* R *and* R *is an* (S, S)*-bimodule. Let* $T = S \propto R := \left\{ \begin{pmatrix} s & r \\ 0 & s \end{pmatrix} : s \in S, r \in R \right\}$ *be the trivial extension of* S *by* R. *Then* $J(T)$ *is nil and* $T/J(T)$ *is Boolean. The mapping* $* : T \to T$ *given by*

$$\begin{pmatrix} s & r \\ 0 & s \end{pmatrix} \mapsto \begin{pmatrix} s & r^* \\ 0 & s \end{pmatrix}$$

is an involution on T. *As* $t^* - t \in J(T)$ *for all* $t \in T$, *by Corollary 2.2.14* $\mathbb{M}_2(T)$ *is nil* $*$*-clean where* $(t_{ij})^* = (t_{ij}^*)^T$.

Although not every matrix in $\mathbb{M}_n(R)$ is nil $*$-clean by Corollary 2.2.8, we next show that symmetric matrices in $\mathbb{M}_n(R)$ (where $R/J(R)$ is Boolean with $J(R)$ nil) are nil $*$-clean.

For convenience, we call a $*$-ring R **partially nil $*$-clean** if every symmetric element is nil $*$-clean.

Lemma 2.2.18 *Let* R *be a commutative ring and* $n \geq 2$. *With* $*$ *being the transpose, the following are equivalent:*

(1) $\mathbb{M}_n(R)$ *is partially nil* $*$*-clean.*

(2) $J(R)$ *is nil and* $\mathbb{M}_n(R/J(R))$ *is partially nil* $*$*-clean.*

(3) $J(R)$ *is nil,* $R/J(R)$ *is Boolean and* $\mathbb{M}_n(\mathbb{F}_2)$ *is partially nil* $*$*-clean.*

Proof. (1) $\Rightarrow$ (3). Let $a \in R$. Then $aI_n \in \mathbb{M}_n(R)$ is nil clean, and hence strongly nil-clean as aI_n is central in $\mathbb{M}_n(R)$. So, by [69, Lemma 2.4], we have $aI_n - (aI_n)^2$ is nilpotent and hence $a - a^2$ is nilpotent. Thus, $a \in R$ is strongly nil-clean by [69, Lemma 2.4]. So, by Lemma 2.2.11, $J(R)$ is nil and $R/J(R)$ is Boolean. Thus, there exists an ideal I of R with $J(R) \subseteq I$ such that $R/I \cong \mathbb{F}_2$. Let $\big(\alpha_{ij}\big) \in \mathbb{M}_n(R/I)$ be symmetric. Since R is commutative, there exists a symmetric matrix (r_{ij}) in $\mathbb{M}_n(R)$ such that $\big(\alpha_{ij}\big) = \big(\overline{r_{ij}}\big)$. As $\mathbb{M}_n(R)$ is partially nil $*$-clean, (r_{ij}) is nil $*$-clean in $\mathbb{M}_n(R)$, it follows that $\big(\alpha_{ij}\big) \in \mathbb{M}_n(R/I)$ is nil $*$-clean.

(3) $\Rightarrow$ (2). For any symmetric matrix A in $\mathbb{M}_n(R/J(R))$, to show that A is nil $*$-clean in $\mathbb{M}_n(R/J(R))$ it suffices to show that A is nil $*$-clean in $\mathbb{M}_n(S)$, where S is the subring of $R/J(R)$ generated by the entries of A and the identity of $R/J(R)$. Thus, S is a direct product of finitely many copies of $\mathbb{F}_2$. Since $\mathbb{M}_n(\mathbb{F}_2)$ is partially nil $*$-clean, $\mathbb{M}_n(S)$ is partially nil $*$-clean, so A is nil $*$-clean in $\mathbb{M}_n(S)$.

(2) $\Rightarrow$ (1). Let $A \in \mathbb{M}_n(R)$ be symmetric. Then $\overline{A} \in \mathbb{M}_n(R/J(R))$ is symmetric, so $\overline{A} = \overline{E} + \overline{B}$ where $\overline{E}^2 = \overline{E} = \overline{E}^T$ and $\overline{B}$ is nilpotent. As $J(R)$ is nil and R is commutative, $J(\mathbb{M}_n(R))$ is nil, so idempotents lift modulo $J(\mathbb{M}_n(R))$. So, by Lemma 2.2.4, projections of $\mathbb{M}_n(R)/J(\mathbb{M}_n(R))$ can be lifted to projections of $\mathbb{M}_n(R)$. Thus we can assume that $E^2 = E = E^T$, and B is nilpotent. So $A = E + B + X$ where $X \in J(\mathbb{M}_n(R))$. As $B + X$ is nilpotent, A is nil $*$-clean. □

A square matrix P is called **orthogonal** if $P^{-1} = P^T$. A **permutation matrix** is a square matrix that has precisely one entry of 1 in each row and each column and 0's elsewhere. It is known that every permutation matrix is orthogonal and that every orthogonal matrix over $\mathbb{F}_2$ is a permutation matrix. Thus, for any diagonal matrix A in $\mathbb{M}_n(\mathbb{F}_2)$ and any orthogonal matrix P in $\mathbb{M}_n(\mathbb{F}_2)$, P^TAP is again a diagonal matrix, and the next lemma follows.

Lemma 2.2.19 *Let P be an orthogonal matrix in $\mathbb{M}_n(\mathbb{F}_2)$ and $A \in \mathbb{M}_n(\mathbb{F}_2)$. Then A is a sum of a diagonal matrix and a nilpotent matrix if and only if P^TAP is as well.*

Theorem 2.2.20 *Let R be a commutative ring. Then, with $*$ being the transpose, $\mathbb{M}_n(R)$ is a partially nil $*$-clean ring if and only if $J(R)$ is nil and $R/J(R)$ is Boolean for $n = 2, 3, 4$.*

Proof. The necessity follows from Lemma 2.2.18. For the sufficiency, it suffices to show that $\mathbb{M}_n(\mathbb{F}_2)$ is partially nil $*$-clean by Lemma 2.2.18. So we can assume that $R = \mathbb{F}_2$.

(1) For any symmetric matrix $A = \begin{pmatrix} a & x \\ x & b \end{pmatrix}$ in $\mathbb{M}_2(R)$, write

$$A = \begin{pmatrix} a+x & 0 \\ 0 & b+x \end{pmatrix} + \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

Then A is a sum of a diagonal idempotent and a nilpotent, so A is nil $*$-clean.

(2) Consider a symmetric matrix $A = \begin{pmatrix} a & x & y \\ x & b & z \\ y & z & c \end{pmatrix}$ in $\mathbb{M}_3(R)$. If the sum $x + y + z = 0$, then $A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} + \begin{pmatrix} 0 & x & y \\ x & 0 & z \\ y & z & 0 \end{pmatrix}$ is a sum of a diagonal idempotent and a nilpotent. So we can assume that $x + y + z = 1$. Thus either $x = y = z = 1$ or $xy = 0$ and $x + y + z = 1$.

If $x = y = z = 1$, then $A = \begin{pmatrix} 1+a & 0 & 0 \\ 0 & 1+b & 0 \\ 0 & 0 & c \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ is a sum of a diagonal idempotent and a nilpotent. If $x + y + z = 1$ and $xy = 0$, then

$$A = \begin{pmatrix} a+x+y & 0 & 0 \\ 0 & 1+b+y & 0 \\ 0 & 0 & 1+c+x \end{pmatrix} + \begin{pmatrix} x+y & x & y \\ x & 1+y & z \\ y & z & 1+x \end{pmatrix}$$

is a sum of a diagonal idempotent and a nilpotent. It is worthy of noting that A is always a sum of a diagonal matrix and a nilpotent matrix.

(3) For a symmetric matrix $A = \begin{pmatrix} a & x & y & z \\ x & b & u & v \\ y & u & c & w \\ z & v & w & d \end{pmatrix}$ in $\mathbb{M}_4(R)$, let matrix $B = \begin{pmatrix} 0 & x & y & z \\ x & 0 & u & v \\ y & u & 0 & w \\ z & v & w & 0 \end{pmatrix}$ in $\mathbb{M}_4(R)$. To show that A is a sum of a symmetric idempotent and a nilpotent, it suffices to show that B is a sum of a diagonal matrix and a nilpotent. Next we proceed with the following cases.

Case 1: $x = y = z = 0$. The claim follows by the proof of (2).

Case 2: One of x, y, z is 1 and the others are 0. By Lemma 2.2.19, without loss of generality, we can assume that $x = 1$ and $y = z = 0$. That is, $B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & u & v \\ 0 & u & 0 & w \\ 0 & v & w & 0 \end{pmatrix}$.

If $u = v = 0$, then B is a sum of a diagonal matrix and a nilpotent matrix by Lemma 2.2.19 and Case 1. If $u = v = 1$, then for $E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w \end{pmatrix}$, $(B+E)^4 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}^2 = 0$, and so $B = E + (B+E)$ is a sum of a diagonal matrix and a nilpotent matrix.

If one of u and v is 1 and the other is 0, then for $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$,

$P^TBP = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & v & u \\ 0 & v & 0 & w \\ 0 & u & w & 0 \end{pmatrix}$, so we can also assume that $u = 1$ and $v = 0$ by Lemma 2.2.19. Moreover, we can further assume that $w = 1$ because of Case 1. Thus, $B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Now let $E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Then we have $(B+E)^4 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}^2 = 0$, and so $B = E + (B+E)$ is a sum of a diagonal matrix and a nilpotent matrix.

Case 3: Two of x, y, z are 1 and the other is 0. By Lemma 2.2.19, without loss of generality, we can assume that $x = y = 1$ and $z = 0$. That is, $B = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & u & v \\ 1 & u & 0 & w \\ 0 & v & w & 0 \end{pmatrix}$. Let $P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ and then we have $P^TBP = \begin{pmatrix} 0 & v & w & 0 \\ v & 0 & u & 1 \\ w & u & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$, so we can assume that $v = w = 1$ because of Cases 1 and 2, and hence $B = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & u & 1 \\ 1 & u & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$. Let $E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and then

$(B+E)^2 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & u & u & 1 \\ 1 & u & u & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}^2 = 0$, and so $B = E + (B+E)$ is a sum of a diagonal matrix and a nilpotent matrix.

Case 4: $x = y = z = 1$. That is, $B = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & u & v \\ 1 & u & 0 & w \\ 1 & v & w & 0 \end{pmatrix}$. For matrix $P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $P^T BP = \begin{pmatrix} 0 & 1 & u & v \\ 1 & 0 & 1 & 1 \\ u & 1 & 0 & w \\ v & 1 & w & 0 \end{pmatrix}$, so we can also assume $u = v = 1$ because of Cases 2 and 3. Thus, for $P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, we have $P^T BP = \begin{pmatrix} 0 & 1 & 1 & w \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ w & 1 & 1 & 0 \end{pmatrix}$, so we can also assume $w = 1$ because of Case 3. That is, $B = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$. Then $(B + I_4)^2 = 0$, so $B = I_4 + (B + I_4)$ is a sum of a diagonal matrix and a nilpotent matrix. □

Statement 2.2.21 *The material in this section is taken from [28]. My contribution in this joint work is to provide the decompositions of the matrices in the proofs of Corollary 2.2.8 and Theorem 2.2.20.*

2.3 Nil-clean rings of nilpotentcy index at most 2 with application to involution-clean rings

The structure of strongly nil-clean rings is already obtained as follows (see [50, Theorem 2.7]): A ring R is strongly nil-clean if and only if $R/J(R)$ is Boolean and $J(R)$ is nil. It's also proved that a ring R is nil-clean if and only if $J(R)$ is nil and $R/J(R)$ is nil-clean (see [32, Corollary 3.17]). However, the structure of nil-clean rings is still unknown. As a special case, in this section we give a description of nil-clean rings of nilpotency index at most 2, and further apply this result to describing involution-clean rings. A ring R is said to be of **nilpotency index at most** n if $a^n = 0$ for all $a \in \mathrm{Nil}(R)$. We start with the following lemma.

Lemma 2.3.1 *The following are equivalent for a ring R:*

(1) *Every element of R is a sum of an idempotent and a square-zero element.*

(2) *R is nil-clean of nilpotency index ≤ 2.*

(3) *$R/J(R)$ is nil-clean of nilpotency index ≤ 2, and $a^2 = 0$ for all $a \in J(R) \cup \mathrm{Nil}(R)$.*

Proof. (1) $\Rightarrow$ (2) It suffices to show that $a^{n+1} = 0$ whenever $a^{n+2} = 0$ in R for $n \geq 1$. Write $1 + a = b + e$ where $b^2 = 0$ and $e^2 = e$. Let $f = 1 - e$. Then $f + a = b$, so

$$0 = (f + a)^2 = f + fa + af + a^2.$$

Thus $0 = (f + fa + af + a^2)a^{n+1} = fa^{n+1} + afa^{n+1} = (1 + a)fa^{n+1}$, so $fa^{n+1} = 0$ (as $1 + a$ is a unit). Hence,

$$0 = (f + fa + af + a^2)a^n = fa^n + afa^n = (1 + a)fa^n,$$

so $fa^n = 0$. Thus,

$$0 = (f + fa + af + a^2)a^{n-1} = fa^{n-1} + afa^{n-1} + a^{n+1} = (1 + a)fa^{n-1} + a^{n+1},$$

so $0 = a[(1 + a)fa^{n-1} + a^{n+1}] = (1 + a)afa^{n-1}$, and hence $afa^{n-1} = 0$. It follows that $fa^{n-1} + a^{n+1} = 0$. So

$$0 = f(fa^{n-1} + a^{n+1}) = fa^{n-1} + fa^{n+1} = fa^{n-1}.$$

It follows that $a^{n+1} = 0$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (1) The implications are clear in view of [32, Corollary 3.17]. $\square$

A ring R is of index n if n is the minimal integer such that $a^n = 0$ for any $a \in \mathrm{Nil}(R)$. It is clear that if R is a ring of nilpotency index at most 2 then R is a ring of index 2.

Lemma 2.3.2 *[4, Theorem 1] Let R be a semiprime ring of index n. Then R is a subdirect product of prime rings of nilpotency index $\leq n$.*

Let $(r_\alpha) \in \prod\{R_\alpha : \alpha \in \Gamma\}$. The support of (r_α) is the subset $\Lambda = \{\alpha \in \Gamma : r_\alpha \neq 0\}$.

Now we give a description of a nil-clean ring of nilpotency index ≤ 2.

Theorem 2.3.3 *The following are equivalent for a ring R:*

(1) R is a nil-clean ring of nilpotency index ≤ 2.

(2) $a^2 = 0$ for all $a \in \mathrm{J}(R) \cup \mathrm{Nil}(R)$ and $R/J(R)$ is a subdirect product of rings $\{R_\alpha : \alpha \in \Gamma\}$, where $R_\alpha = \mathbb{F}_2$ or $\mathbb{M}_2(\mathbb{F}_2)$, such that whenever $(x_\alpha)_\Lambda \in R/J(R)$ with $x_\alpha^3 = 1$ and $x_\alpha \neq 1$ for all $\alpha \in \Lambda$, there exists $(y_\alpha)_\Lambda \in R/J(R)$ with $y_\alpha \neq 0$ and $y_\alpha^2 = 0$ for all $\alpha \in \Lambda$.

Proof. (1) $\Rightarrow$ (2) By Lemma 2.3.1, $a^2 = 0$ for all $a \in J(R) \cup \mathrm{Nil}(R)$. Moreover, $R/J(R)$ is nil-clean of nilpotency index ≤ 2. As $R/J(R)$ is a semiprime ring of index 2, by Lemma 2.3.2, $R/J(R)$ is a subdirect product of prime rings

$\{R_\alpha : \alpha \in \Gamma\}$ of nilpotency index ≤ 2. Hence, [10, Corollary 6], for each α, $R_\alpha \cong \mathbb{M}_n(D)$ where D is a division ring and $n \leq 2$. As $\mathbb{M}_n(D)$ is still nil-clean, $D \cong \mathbb{F}_2$ by [49, Theorem 3]. So $R_\alpha \cong \mathbb{F}_2$ or $R_\alpha \cong \mathbb{M}_2(\mathbb{F}_2)$. Identify $R/J(R)$ as a subring of $\prod_\Gamma R_\alpha$.

If $R/J(R)$ contains an element $x := (x_\alpha)_\Lambda$ where $1 \neq x_\alpha \in R_\alpha$ with $x_\alpha^3 = 1$ for all $\alpha \in \Lambda$, then, as x is nil-clean in $R/J(R)$, there exits a nilpotent $y \in R/J(R)$ such that $x+y$ is an idempotent. Write $y = (y_\alpha)$ where $y_\alpha \in R_\alpha$. It must be that $y_\alpha = 0$ for $\alpha \in \Gamma \setminus \Lambda$ and $y_\alpha \neq 0$ for $\alpha \in \Lambda$. So $y = (y_\alpha)_\Lambda$.

(2) $\Rightarrow$ (1) We only need to show that R is nil-clean. As $J(R)$ is nil, it suffices to show that $R/J(R)$ is nil- clean by [32, Corollary 3.17]. Regard $R/J(R)$ as a subring of $\prod_\Gamma R_\alpha$.

Let $x \in R/J(R)$. Write $x = (x_\alpha)$ where $x_\alpha \in R_\alpha$. In R_α, there are four types of elements b: $b^2 = 0$; $b^2 = b$; $b^2 = 1$ with $b \neq 1$; $b^3 = 1$ with $b \neq 1$. Thus, we can write Γ as a disjoint union of Λ_1, Λ_2, Λ_3 and Λ_4 such that $x_\alpha^2 = 0$ if and only if $\alpha \in \Lambda_1$; $x_\alpha^2 = x_\alpha$ if and only if $\alpha \in \Lambda_2$; $x_\alpha^2 = 1$ with $x_\alpha \neq 1$ if and only if $\alpha \in \Lambda_3$; $x_\alpha^3 = 1$ with $x_\alpha \neq 1$ if and only if $\alpha \in \Lambda_4$. Without loss of generality, we can denote $x = (x_\alpha) = ((x_\alpha)_{\Lambda_1}, (x_\alpha)_{\Lambda_2}, (x_\alpha)_{\Lambda_3}, (x_\alpha)_{\Lambda_4})$. We have

$$x + x^7 = ((x_\alpha)_{\Lambda_1}, \mathbf{0}, \mathbf{0}, \mathbf{0}),$$

$$x^2 + x^5 = (\mathbf{0}, \mathbf{0}, \mathbf{1} + (x_\alpha)_{\Lambda_3}, \mathbf{0}),$$

$$(x^2 + x^5 + x^6 + x^7)^2 = (\mathbf{0}, \mathbf{0}, \mathbf{0}, (x_\alpha)_{\Lambda_4}).$$

So $(x_\alpha)_{\Lambda_4} \in R/J(R)$. By our assumption, there exists $(y_\alpha)_{\Lambda_4} \in R/J(R)$ with $y_\alpha \neq 0$ and $y_\alpha^2 = 0$ for all $\alpha \in \Lambda_4$. Note that $(x_\alpha)_{\Lambda_4} + (y_\alpha)_{\Lambda_4} \in R/J(R)$ is an idempotent. We see that

$$\begin{aligned} y :&= ((x_\alpha)_{\Lambda_1}, \mathbf{0}, \mathbf{1} + (x_\alpha)_{\Lambda_3}, (y_\alpha)_{\Lambda_4}) \\ &= ((x_\alpha)_{\Lambda_1}, \mathbf{0}, \mathbf{0}, \mathbf{0}) + (\mathbf{0}, \mathbf{0}, \mathbf{1} + (x_\alpha)_{\Lambda_3}, \mathbf{0}) + (\mathbf{0}, \mathbf{0}, \mathbf{0}, (y_\alpha)_{\Lambda_4}) \in R/J(R) \end{aligned} \tag{2.5}$$

is nilpotent, and

$$(\mathbf{0}, (x_\alpha)_{\Lambda_2}), \mathbf{1}, (x_\alpha)_{\Lambda_4}) + (y_\alpha)_{\Lambda_4}) = x + y \in R/J(R)$$

is an idempotent. Therefore, $x = y + (x + y)$ is nil-clean in $R/J(R)$. So $R/J(R)$ is nil-clean. □

Corollary 2.3.4 *If $R/J(R) \cong S \oplus (\prod \mathbb{M}_2(\mathbb{F}_2))$ for a Boolean ring S and if $J(R)$ is nil such that $a^2 = 0$ for all $a \in \mathrm{Nil}(R)$, then R a is nil-clean ring of nilpotency index at most 2.*

A subdirect product of a Boolean ring and a family of copies of $\mathbb{M}_2(\mathbb{F}_2)$ need not be a nil-clean ring.

Example 2.3.5 *Let $T = \prod_{n=1}^{\infty} R_i$ where $R_i = \mathbb{M}_n(\mathbb{F}_2)$ for all $i \geq 1$. Let element $z = (z_i) \in T$ where $z_i = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in \mathbb{M}_2(\mathbb{F}_2)$. Let S be the subring of T generated by z, i.e., $S = \{0, 1, z, 1 + z\}$ where $z^2 = 1 + z$. Let $R = (\oplus_{n=1}^{\infty} R_i) + S$. Then R is a subdirect product of $\{R_i\}$, so $J(R) = 0$ and R has nilpotency index 2. However, although R contains z, R does not contain a nilpotent (y_i) with $y_i \neq 0$ for all $i \geq 1$. So R is not nil-clean by Theorem 2.3.3.*

We can see that a nil-clean ring of nilpotency index at most 2 also pass to its corners.

Corollary 2.3.6 *If R is a nil-clean ring of nilpotency index at most 2, then so is eRe for all $e^2 = e \in R$.*

Proof. Let $S = eRe$. Then $J(S) = eJ(R)e \subseteq J(R)$ and $\mathrm{Nil}(S) \subseteq \mathrm{Nil}(R)$. Since R is a nil-clean ring of nilpotency index at most 2, $a^2 = 0$ for all $a \in J(R) \cup \mathrm{Nil}(R)$ by Theorem 2.3.3, so $a^2 = 0$ for all $a \in J(S) \cup Nil(S)$. Moreover, $\overline{R} := R/J(R)$ is a subdirect product of $\{R_\alpha : \alpha \in \Gamma\}$ where either $R_\alpha \cong \mathbb{F}_2$ or $R_\alpha \cong \mathbb{M}_2(\mathbb{F}_2)$. That is, $\overline{R}$ is a subring of $\prod R_\alpha$ such that

$\pi_\alpha(\overline{R}) = R_\alpha$ where $\pi_\alpha : \prod R_\alpha \to R_\alpha$ is the natural projection for all $\alpha \in \Gamma$. Let $\bar{e} = e + J(R) \in \overline{R}$. Write $\bar{e} = (e_\alpha)$ where $e_\alpha \in R_\alpha$ is an idempotent. It is easily seen that $\bar{e}\overline{R}\bar{e}$ is a subring of $\prod e_\alpha R_\alpha e_\alpha$ with $\pi_\alpha(\bar{e}\overline{R}\bar{e}) = e_\alpha R_\alpha e_\alpha$ for all α. That is, $\bar{e}\overline{R}\bar{e}$ is a subdirect product of $\{e_\alpha R_\alpha e_\alpha\}$. We notice that, if $R_\alpha \cong \mathbb{F}_2$, then $e_\alpha R_\alpha e_\alpha = 0$ or $e_\alpha R_\alpha e_\alpha \cong \mathbb{F}_2$, and that, if $R_\alpha \cong \mathbb{M}_2(\mathbb{F}_2)$, then $e_\alpha R_\alpha e_\alpha = 0$, or $e_\alpha R_\alpha e_\alpha \cong \mathbb{F}_2$, or $e_\alpha R_\alpha e_\alpha \cong \mathbb{M}_2(\mathbb{F}_2)$(this only occurs when e_α is the identity of R_α). Suppose that $x = (x_\alpha)_\Lambda \in \bar{e}\overline{R}\bar{e}$ where $e_\alpha \neq x_\alpha \in e_\alpha R_\alpha e_\alpha$ with $x_\alpha^3 = e_\alpha$ for all $\alpha \in \Lambda$. It must be that, for each $\alpha \in \Lambda$, $R_\alpha \cong \mathbb{M}_2(\mathbb{F}_2)$ and $e_\alpha = 1_{R_\alpha}$. Then by Theorem 2.3.3, there exists $y = (y_\alpha)_\Lambda \in \overline{R}$ such that $y_\alpha \neq 0$ and $y_\alpha^2 = 0$. But $y = \bar{e}\bar{y}\bar{e} \in \bar{e}\overline{R}\bar{e}$. Note that $S/J(S) = eRe/eJ(R)e = eRe/(eRe \cap J(R)) \cong (eRe + J(R))/J(R) = \bar{e}\overline{R}\bar{e}$. Hence, by Theorem 2.3.3, S is a nil-clean ring of nilpotency index at most 2. □

Recall that an element of a ring R is strongly π-regular if $a^n \in a^{n+1}R \bigcap Ra^{n+1}$ for some $n > 0$. A ring is strongly π-regular if each of its elements is strongly π-regular. It is proved that nil-clean rings need not to be strongly π-regular (see [73, Example 3.1]). However, we have the following corollary for a nil-clean ring of nilpotency index ≤ 2:

Corollary 2.3.7 *If R is nil-clean of nilpotency index ≤ 2, then R is strongly π-regular.*

Proof. If $a \in J(R)$, then $a^2 = 0$. Suppose that $a \notin J(R)$. Then we write $x = \bar{a} \in R/J(R)$. As in the proof of Theorem 2.3.3, we have

$$x = (x_\alpha) = ((x_\alpha)_{\Lambda_1}, (x_\alpha)_{\Lambda_2}, (x_\alpha)_{\Lambda_3}, (x_\alpha)_{\Lambda_4}).$$

Moreover, $x + x^7 = ((x_\alpha)_{\Lambda_1}, \mathbf{0}, \mathbf{0}, \mathbf{0})$, so $(x + x^7)^2 = \bar{0}$, i.e., $(a + a^7)^2 \in J(R)$. Hence, $a^4(1+a^6)^4 = (a+a^7)^4 = ((a+a^7)^2)^2 = 0$, showing that $a^4 \in a^5R \cap Ra^5$. So R is strongly π-regular. □

Following in [29], a ring is an **involution-clean** ring if every element is a sum of an idempotent and an involution. The following result is proved in

[29].

Lemma 2.3.8 *[29] Let R be a ring. Then R is an involution-clean ring if and only if $R = A \times B$, where A is a nil-clean ring with $a^2 + 2a = 0$ for all $a \in \mathrm{Nil}(A)$ and B is a zero or a subdirect product of $\mathbb{F}_3$'s.*

Lemma 2.3.9 *[32, Proposition 3.14] Let R be a nil-clean ring. Then the element 2 is a (central) nilpotent and $2 \in J(R)$.*

Proof. Since R is nil-clean, write $2 = e + b$ as a sum of an idempotent and a nilpotent, then $1 - e = b - 1$ is both an idempotent and a unit. Therefore, $b - 1 = 1$ and $b = 2$. □

Lemma 2.3.10 *[32, Corollary 3.17] A ring R is a nil-clean ring if and only if $R/J(R)$ is nil-clean and $J(R)$ nil.*

We give a further description of the subring A in Lemma 2.3.8.

Lemma 2.3.11 *A ring R is nil-clean with $a^2 + 2a = 0$ for all $a \in \mathrm{Nil}(R)$ if and only if $R/J(R)$ is nil-clean of nilpotency index ≤ 2, $J(R)$ nil and $a^2 + 2a = 0$ for all $a \in \mathrm{Nil}(R)$.*

Proof. Since R is nil-clean, by Lemma 2.3.9, 2 is nilpotent and $2 \in J(R)$. By Lemma 2.3.10, we see that R is nil-clean with $a^2 + 2a = 0$ for all $a \in \mathrm{Nil}(R)$ $\Leftrightarrow$ $R/J(R)$ is nil-clean with $J(R)$ nil and with $a^2 + 2a = 0$ for all $a \in \mathrm{Nil}(R)$ $\Leftrightarrow$ $R/J(R)$ is nil-clean of nilpotency index ≤ 2, $J(R)$ nil and $a^2 + 2a = 0$ for all $a \in \mathrm{Nil}(R)$. □

Now we have the following description for involution-clean rings.

Theorem 2.3.12 *Let R be a ring. Then R is an involution-clean ring if and only if $R \cong A \times B$, where*

(1) *B is zero or a subdirect product of $\mathbb{F}_3$'s.*

(2) *$J(A)$ is nil, $a^2 + 2a = 0$ for all $a \in \mathrm{Nil}(A)$, and $A/J(A)$ is a subdirect product of rings $\{A_\alpha : \alpha \in \Gamma\}$, where $A_\alpha = \mathbb{F}_2$ or $\mathbb{M}_2(\mathbb{F}_2)$, such that whenever*

$(x_\alpha)_\Lambda \in A/J(A)$ with $x_\alpha^3 = 1$ and $x_\alpha \neq 1$ for all $\alpha \in \Lambda$, there exists an element $(y_\alpha)_\Lambda \in A/J(A)$ with $y_\alpha \neq 0$ and $y_\alpha^2 = 0$ for all $\alpha \in \Lambda$.

Proof. This is by Lemma 2.3.8, 2.3.11 and Theorem 2.3.3. □

Corollary 2.3.13 *If $R/J(R) \cong S \oplus (\prod \mathbb{M}_2(\mathbb{F}_2))$ for a Boolean ring S with $J(R)$ nil such that $a^2 + 2a = 0$ for all $a \in \mathrm{Nil}(R)$, then R is an involution-clean ring.*

As seen in Example 2.3.5, a subdirect product of a Boolean ring and a family of copies of $\mathbb{M}_2(\mathbb{F}_2))$ need not be an involution-clean ring.

Next, we determine when a (formal or triangular) matrix ring is involution-clean.

Proposition 2.3.14 *Let S, T be rings and M a non-trivial (S,T)-bimodule. Then the formal matrix ring $\begin{pmatrix} S & M \\ 0 & T \end{pmatrix}$ is an involution-clean ring if and only if S, T are involution-clean rings and* $\mathrm{Nil}(S)M = M\mathrm{Nil}(T) = 2M = 0$.

Proof. (⇒) If $x \in M$, then $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}^2 + 2\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = 0$, and this implies $2x = 0$. Hence $2M = 0$. Let $a \in \mathrm{Nil}(S)$ and $x \in M$. Then $\begin{pmatrix} a & x \\ 0 & 0 \end{pmatrix}^2 + 2\begin{pmatrix} a & x \\ 0 & 0 \end{pmatrix} = 0$, and this shows that $ax = -2x = 0$. So $\mathrm{Nil}(S)M = 0$. Similarly $M\mathrm{Nil}(T) = 0$. As images of $\begin{pmatrix} S & M \\ 0 & T \end{pmatrix}$, S and T are clearly involution-clean rings.

(⇐) Write $S = A \oplus A'$ and $T = B \oplus B'$ where $8 = 0$ in A and in B, $A' \oplus B'$ is zero or a subdirect product of $\mathbb{F}_3$'s. Write $1_S = 1_A + 1_{A'}$ and $1_T = 1_B + 1_{B'}$. From $2M = 0$, one deduces that $1_{A'}M = 0$, $M1_{B'} = 0$, and $1_A x = x1_B = x$ for all $x \in M$. Therefore,

$$\begin{pmatrix} S & M \\ 0 & T \end{pmatrix} \cong \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \times A' \times B'.$$

Thus, we only need to show that $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ is an involution-clean ring. Let $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. Write $a = e + v$ and $b = f + w$ where $e^2 = e$, $v^2 = 1$, $f^2 = f$ and $w^2 = 1$. Then $(1+v)^2 = 2(1+v) \in J(A)$, so $(1+v)x = 0$. Similarly, $x(1 + w) = 0$. Thus $vx + xw = (1 + v)x + x(1 + w) - 2x = 0$, therefore, $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} + \begin{pmatrix} v & x \\ 0 & w \end{pmatrix}$ is a sum of an idempotent and an involution. □

Recall a result that for a 2-primal ring R and $n \geq 1$, $\mathbb{M}_n(R)$ is nil-clean if and only if $J(R)$ is nil and $R/J(R)$ is Boolean (see [50, Theorem 6.1]). Clearly this result also implies the following corollary.

Corollary 2.3.15 *[50, Corollary 6.3] Let R be a reduced ring. Then $\mathbb{M}_n(R)$ is nil-clean if and only if R is Boolean.*

Next we characterize when a formal triangular matrix ring and, more generally, a matrix ring is involution-clean.

Theorem 2.3.16 *Let R be a ring and $n \geq 2$. The following are equivalent:*

(1) $\mathbb{T}_n(\mathbb{R})$ *is an involution-clean ring.*

(2) $\mathbb{T}_n(\mathbb{R})$ *is a nil-clean ring of nilpotency index ≤ 2.*

(3) $n = 2$ *and R is Boolean.*

(4) $\mathbb{M}_n(R)$ *is a nil-clean ring of nilpotency index ≤ 2.*

(5) $\mathbb{M}_n(R)$ *is an involution-clean ring.*

Proof. (1) ⇒ (3) Write $\mathbb{T}_n(R) = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$ where $S = \mathbb{T}_{n-1}(R)$ and $M = \mathbb{M}_{(n-1)\times 1}(R)$. By Proposition 2.3.14, $2 = 0$ in R and $\mathrm{Nil}(S)M = 0$, from which we deduce that $n = 2$ and R is a reduced ring. As an image of $\mathbb{T}_2(R)$, R is involution-clean. Thus, R is a subdirect product of involution-clean domains in which 2 is zero. One easily sees that each of the domains is isomorphic to $\mathbb{F}_2$, so R is Boolean.

(3) $\Rightarrow$ (2) Let $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \mathbb{T}_2(R)$. Then $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ is a sum of an idempotent and a square-zero element.

(2) $\Rightarrow$ (1) As $2 \in \mathrm{Nil}(R)$, $2E_{11} + E_{12}$ is nilpotent, so $0 = (2E_{11} + E_{12})^2 = 4E_{11} + 2E_{12}$. This shows that $2 = 0$ in R. For $A \in \mathbb{T}_n(R)$, write $A = E + B$ where $E^2 = E$ and $B^2 = 0$. Then $A = (1 + E) + (1 + B)$ is a sum of an idempotent and an involution.

(5) $\Rightarrow$ (4) By Lemma 2.3.8, $\mathbb{M}_n(R) \cong A \times B$, where $8 = 0$ in A and B is zero or a subdirect product of $\mathbb{F}_3$'s. Thus, there exists a central idempotent e of R such that $A \cong \mathbb{M}_n(eR)$ and $B \cong \mathbb{M}_n((1-e)R)$. As $n \geq 2$, it follows from Lemma 2.3.8 that $e = 1$, so $8 = 0$ in $\mathbb{M}_n(R)$. As $E_{12} \in \mathbb{M}_n(R)$ is nilpotent, $(E_{12})^2 + 2E_{12} = 0$, showing that $2 = 0$ in R. For $A \in \mathbb{M}_n(R)$, write $A = E + V$ where $E^2 = E$ and $V^2 = I$. Then $A = (I + E) + (I + V)$ is a sum of an idempotent and a square-zero element.

(4) $\Rightarrow$ (3) If $x^2 = 0$ in R, $xE_{11} + E_{12} \in \mathbb{M}_n(R)$ is nilpotent; so $xE_{12} = (xE_{11} + E_{12})^2 = 0$, showing $x = 0$. Hence, R is a reduced ring. As $\mathbb{M}_n(R)$ is nil-clean, R is Boolean by [50, Corollary 6.3]. Assume that $n > 2$. Then, as $E_{12} + E_{23} \in \mathbb{M}_n(R)$ is nilpotent, $E_{23} = (E_{12} + E_{23})^2 = 0$. This contradiction shows that $n = 2$.

(3) $\Rightarrow$ (5) As Boolean rings are reduced, by Lemma 2.3.15, R is nil-clean. By Lemma 2.3.8, it suffices to show that $A^2 = 0$ for any nilpotent matrix A in $\mathbb{M}_2(R)$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be nilpotent in $\mathbb{M}_n(R)$. Then the determinant

of A must be zero, so $ad = bc$. We have $A^2 = \begin{pmatrix} a+bc & ab+bd \\ ac+cd & bc+d \end{pmatrix}$, and

$$\begin{aligned} A^3 &= \begin{pmatrix} a+bc\cdot d & ab+b\cdot ad+bc+bd \\ ac+bc+c\cdot ad+cd & a\cdot bc+d \end{pmatrix} \\ &= \begin{pmatrix} a+ad & ab+bc+bcbd \\ ac+bc+bc+cd & ad+d \end{pmatrix} \\ &= \begin{pmatrix} a+bc & ab+bd \\ ac+cd & bc+d \end{pmatrix} = A^2. \end{aligned}$$

It follows that $A^2 = 0$.

Example 2.3.17 *$\mathbb{Z}_8$ is an involution-clean ring, but 2 is not a sum of an idempotent and a square-zero element. The trivial extension $\mathbb{Z}_4 \propto \mathbb{Z}_4$ is not an involution-clean, but a nil-clean ring with nilpotency index ≤ 2.*

Statement 2.3.18 *The material in this section is taken from [56]. The authors contributed equally to the joint work in [56].*

Chapter 3

Rings whose elements are sums of nilpotents, idempotents, and tripotents

In this chapter, we study rings whose elements are sums of nilpotents, idempotents, and tripotents. In [41], the authors first studied rings whose elements are sums of two idempotents that commute. Instead of two idempotents, the authors in [82] considered rings in which every element is a sum of an idempotent and a tripotent that commute and respectively, rings in which every element is a sum of two tripotents that commute. In [24], the authors studied rings whose elements are sums of a nilpotent and two idempotents that commute with one another. To go one step further, the author in [83] considered rings whose elements are sums a nilpotent and two tripotents that commute with one another. In this chapter, we come up with a new type of rings which generalizes the cases mentioned above.

3.1 A review of the known results

The results reviewed in this section are direct motivation of our work in this chapter.

Sum of two idempotents

In [41], Hirano and Tominaga considered a class of rings in which every element is the sum of two idempotents in R. For convenience, we call a ring R a HT-ring if every element of R is a sum of two idempotents. The structure of HT-rings is still unknown, but the authors proved the following results in [41]:

Theorem 3.1.1 *[41, Theorem 1] The following are equivalent for a ring R:*

(1) *R is a commutative* HT-*ring.*

(2) *Every element of R is a sum of two commutating idempotents.*

(3) *R has the identity $x^3 = x$.*

The authors in [82] obtained the following reduction result:

Lemma 3.1.2 *[82] A ring R is a* HT-*ring if and only if $R = A \times B$, where A is a* HT-*ring of characteristic* 2 *and B is a subdirect product of $\mathbb{F}_3$'s.*

Sum of an idempotent and a tripotent

In [82], the authors considered the rings in which every element is a sum of an idempotent and a tripotent that commute and obtained the following structure theorem.

Theorem 3.1.3 *[82, Theorem 3.10] The following are equivalent for a ring R:*

(1) *Every element of R is a sum of an idempotent and a tripotent that commute.*

(2) *R has the identity $x^6 = x^4$.*

(3) *R is one of the following types:*

(a) $R/J(R)$ is Boolean and $U(R)$ is a group of exponent 2.

(b) R is a subdirect product of $\mathbb{F}_2$'s.

(c) $R \cong A \times B$, where $A/J(A)$ is Boolean with $U(A)$ a group of exponent 2, and B is a subdirect product of $\mathbb{F}_3$'s.

Sum of two tripotents

In [82], the authors also considered the rings in which every element is a sum of two tripotents that commute and obtained the following structure theorem.

Theorem 3.1.4 *[82, Theorem 5.2] The following are equivalent for a ring R:*

(1) *Every element of R is a sum of two commuting tripotents.*

(2) *$R \cong A \times B \times C$, where $A/J(A)$ is Boolean with $U(A)$ a group of exponent 2, B is a subdirect product of $\mathbb{F}_3$'s, and C is a subdirect product of $\mathbb{F}_5$'s.*

Sum of a nilpotent and two idempotents

In [24], the authors studied rings whose elements are sums of a nilpotent and two idempotents that commute with one another and they proved the following results.

Theorem 3.1.5 *[24] Let R be a ring. Then the following are equivalent:*

(1) *Every element of R is a sum of a nilpotent and two idempotents that commute with one another.*

(2) *Every element of R is a sum of a nilpotent and a tripotent that commute.*

(3) $a - a^3$ *is nilpotent for all* $a \in R$.

(4) $J(R)$ *is nil and* $R/J(R)$ *has identity* $x^3 = x$.

(5) $R \cong R_1 \times R_2$, *where* R_1 *is zero or* $R_1/J(R_1)$ *is Boolean with* $J(R_1)$ *nil,* R_2 *is zero or* $R_2/J(R_2)$ *is a subdirect product of* $\mathbb{F}_3$*'s.*

Sum of a nilpotent and two tripotents

Furthermore, the author in [83] studied rings for which every element is a sum of a nilpotent, an idempotent and a tripotent that commute with one another, and rings for which every element is a sum of a nilpotent and two tripotents that commute with one another. The structures of these rings are completely determined.

Theorem 3.1.6 *[83, Theorem 2.11] The following are equivalent for a ring R:*

(1) *Every element of R is a sum of a nilpotent and two tripotents that commute with one another.*

(2) $R \cong A \times B \times C$, *where A is zero or* $A/J(A)$ *is Boolean with* $J(A)$ *nil, B is zero or* $B/J(B)$ *is a subdirect product of* $\mathbb{F}_3$*'s with* $J(B)$ *nil, and C is zero or* $C/J(C)$ *is a subdirect product of* $\mathbb{F}_5$*'s with* $J(C)$ *nil.*

(3) $J(R)$ *is nil and* $R/J(R)$ *has the identity* $x^5 = x$.

(4) $a^5 - a$ *is nilpotent for all* $a \in R$.

Theorem 3.1.7 *[83, Theorem 2.12] The following are equivalent for a ring R:*

(1) *Every element of R is a sum of a nilpotent, an idempotent and a tripotent that commute with one another.*

(2) *Every element of R is a sum of a nilpotent, and two idempotents that commute with one another.*

(3) *Every element of R is a sum of a nilpotent and tripotent that commute.*

(4) $R \cong A \times B$, *where A is zero or $A/J(A)$ is Boolean with $J(A)$ nil, and B is zero or $B/J(B)$ is a subdirect product of $\mathbb{F}_3$'s with $J(B)$ nil.*

(5) $J(R)$ *is nil and* $R/J(R)$ *has the identity* $x^3 = x$.

(6) $a^3 - a$ *is nilpotent for all* $a \in R$.

3.2 Rings in which every element is a sum of a nilpotent and three tripotents

To continue on the work in [41], [82], [24], and [83], in this section our main objective is to determine rings in which every element is a sum of a nilpotent and three tripotents that commute with one another. For convenience, we begin with the following definition.

Definition 3.2.1 *An element $a \in R$ is said to have the $\mathcal{P}$* **property** *if a is a sum of a nilpotent and three tripotents that commute with one another. A ring R is said to have the $\mathcal{P}$ property if every element of R has the $\mathcal{P}$ property.*

One can easily check that the class of rings that have $\mathcal{P}$ property is closed under finite direct products and homomorphic images.

Lemma 3.2.1 *Let R be a ring. Then R has the $\mathcal{P}$ property if and only if $R = R_1 \times R_2 \times R_3 \times R_4$ where R_1, R_2, R_3, R_4 have the $\mathcal{P}$ property with $2 \in \mathrm{Nil}(R_1)$, $3 \in \mathrm{Nil}(R_2)$, $5 \in \mathrm{Nil}(R_3)$, and $7 \in \mathrm{Nil}(R_4)$.*

Proof. It suffices to show the necessity. We first show that $2 \cdot 3 \cdot 5 \cdot 7$ is nilpotent in R. Write $4 = b + e + f + g$, where $b \in \mathrm{Nil}(R)$, $e^3 = e$, $f^3 = f$, $g^3 = g$ and b, e, f, g commute with one another. Note that $(4-b)^3-(4-b) = (15 - 8b + b^2)(4 - b)$ and $4 - b = e + f + g$. Let $t := 15 - 8b + b^2$. Then we obtain

$$\begin{aligned} t(e + f + g) &= (e + f + g)^3 - (e + f + g) \\ &= 3e^2f + 3e^2g + 3ef^2 + 3gf^2 + 3eg^2 + 3fg^2 + 6efg \end{aligned} \tag{3.1}$$

Multiplying $e^2f^2g^2$ on both sides of Eq. (3.1) gives

$$6efg = (t - 6)efg(ef + eg + fg).$$

Therefore,

$$\begin{aligned}12efg &= (t-6)efg(2ef+2eg+2fg)\\ &= (t-6)efg[(4-b)^2-e^2-f^2-g^2]\\ &= (t-6)efg(4-b)^2-(t-6)efg(e^2+f^2+g^2)\\ &= (t-6)efg(4-b)^2-3(t-6)efg,\end{aligned}$$

which implies that $[(t-6)(4-b)^2-3(t-6)-12]efg=0$. As b is nilpotent, we deduce that $(9\cdot 4^2-27-12)efg = 105efg$ is nilpotent.

Now multiplying e^2, f^2, g^2 on both sides of Eq. (3.1) respectively, we have the following equations:

$$(3.2)\ \ te+(t-3)e^2f+(t-3)e^2g = 3ef^2+3eg^2+3e^2f^2g+3e^2fg^2+6efg;$$

$$(3.3)\ \ tf+(t-3)ef^2+(t-3)gf^2 = 3e^2f+3g^2f+3e^2f^2g+3ef^2g^2+6efg;$$

$$(3.4)\ \ tg+(t-3)eg^2+(t-3)fg^2 = 3e^2g+3gf^2+3e^2fg^2+3ef^2g^2+6efg.$$

Then (3.2) + (3.3) + (3.4) gives

$$(t-3)(e^2f+e^2g+ef^2+gf^2+eg^2+fg^2) = 6efg(2+ef+eg+fg).$$

Since $105efg$ is nilpotent, it follows that

$$35(t-3)(e^2f+e^2g+ef^2+gf^2+eg^2+fg^2) = 105\cdot 2efg(2+ef+eg+fg)\in \mathrm{Nil}(R).$$

In view of Eq. (3.1), we have

$$\begin{aligned}35(t-3)t(4-b) &= 35(t-3)(3e^2f+3e^2g+3ef^2+3gf^2+3eg^2+3fg^2+6efg)\\ &= 35(t-3)(3e^2f+3e^2g+3ef^2+3gf^2+3eg^2+3fg^2)\\ &\quad +105efg\cdot 2(t-3)\end{aligned}$$

is nilpotent. Since b is nilpotent, one has $2^4\cdot 3^2\cdot 5^2\cdot 7\in \mathrm{Nil}(R)$, and thus, $2\cdot 3\cdot 5\cdot 7\in \mathrm{Nil}(R)$. Hence there exists an integer $n\geq 1$ such that

$$2^nR\cap 3^nR\cap 5^nR\cap 7^nR=0.$$

Therefore, by the Chinese Reminder Theorem, $R=R_1\times R_2\times R_3\times R_4$ where $R_1\cong R/2^nR$, $R_2\cong R/3^nR$, $R_3\cong R/5^nR$, $R_4\cong R/7^nR$ and R_1, R_2, R_3, R_4 have the $\mathcal{P}$ property. □

Example 3.2.2 *Let R be a ring. Suppose R is a subdirect product of $\mathbb{F}_7's$. Then every element of R is a sum of three tripotents.*

Proof. Let R be a subdirect product of $\{R_\alpha:\alpha\in\Lambda\}$ where $R_\alpha=\mathbb{F}_7$ for all $\alpha\in\Lambda$. Then R is a subring of ΠR_α. Let $x=(x_\alpha)\in R$ and let Λ be a disjoint union of Λ_0, Λ_1, Λ_2, Λ_3, Λ_4, Λ_5, Λ_6 such that

$$x_\alpha=i\Leftrightarrow \alpha\in\Lambda_i \ \ for \ \ i=0,1,2,3,4,5,6.$$

Without lose of generality, let $x=(0_{\Lambda_0},1_{\Lambda_1},2_{\Lambda_2},3_{\Lambda_3},4_{\Lambda_4},5_{\Lambda_5},6_{\Lambda_6})$. Next set $e_i=(0_{\Lambda_0},0_{\Lambda_1},...,1_{\Lambda_i},...,0_{\Lambda_6})$ for $i=1,...,6$. Write

$$\begin{aligned}e &:= e_1+e_2+e_3+6e_4+6e_5+6e_6,\\ f &:= e_2+e_3+6e_4+6e_5,\\ g &:= e_3+6e_4.\end{aligned}$$

Then $e^3=e, f^3=f$, $g^3=g$ and $x=e+f+g$. □

By the proof of Lemma 3.2.1, we can see that if a ring has the $\mathcal{P}$ property, then $2\cdot 3\cdot 5\cdot 7 \in \mathrm{Nil}(R)$. Further, we discuss the $\mathcal{P}$ property of rings for which $2 \in \mathrm{Nil}(R)$, $3 \in \mathrm{Nil}(R)$, $5 \in \mathrm{Nil}(R)$, and $7 \in \mathrm{Nil}(R)$, respectively. First, we recall a characterization of strongly nil-clean elements (a sum of a nilpotent and an idempotent that commute).

Lemma 3.2.3 *[82] An element $a \in R$ is strongly nil-clean if and only if $a - a^2$ is nilpotent.*

Proof. If $a = e + b$, where $e^2 = e \in R$, $b \in \mathrm{Nil}(R)$, and $be = eb$, then $a^2 = e + 2eb + b^2$, so $a - a^2 = b - 2eb - b^2 = (1 - 2e - b)b$ is nilpotent.

Conversely, assume $a - a^2$ is nilpotent. Write $b = 1 - a$. We have $ab = ba = a - a^2$ is nilpotent. So $(ab)^m = 0$ for some integer $m > 1$. By the Binomial Theorem,

$$\begin{aligned} 1 &= (a+b)^{2m} \\ &= a^{2m} + r_1 a^{2m-1} b + \cdots + r_m a^m b^m + r_{m+1} a^{m-1} b^{m+1} + \cdots + b^{2m}, \end{aligned}$$

where the r_i's are integers. Let

$$e = a^{2m} + r_1 a^{2m-1} b + \cdots + r_m a^m b^m$$

$$f = r_{m+1} a^{m-1} b^{m+1} + \cdots + b^{2m}$$

Since $a^m b^m = b^m a^m = 0$, we have $ef = 0$, and so $e = e(e+f) = e^2$. It is clear that $e = \theta(a)$ for a monic polynomial $\theta(t)$ over $\mathbb{Z}$. Since $ab = ba$ is nilpotent, $e - a^{2m} = r_1 a^{m-1} b + \cdots + r_m a^m b^m$ is nilpotent. It follows that

$$\begin{aligned} a - e &= (a - a^{2m}) - (e - a^{2m}) \\ &= (a - a^2)(1 + a + \cdots + a^{2m-2}) - (e - a^{2m}) \end{aligned}$$

is nilpotent. So $a = (a - e) + e$ is a strongly nil-clean element. $\square$

Proposition 3.2.4 *Let R be a ring with $2 \in \mathrm{Nil}(R)$. The following are equivalent:*

(1) *R has the $\mathcal{P}$ property.*

(2) *Every element of R is a sum of a nilpotent and an idempotent that commute.*

(3) *$a - a^2$ is nilpotent for all $a \in R$.*

Proof. (2) $\Leftrightarrow$ (3). By Lemma 3.2.3, an element a in a ring R is strongly nil-clean if and only if $a - a^2 \in \mathrm{Nil}(R)$.

(2) $\Rightarrow$ (1). The implication is clear.

(1) $\Rightarrow$ (3). Let $a \in R$. $a = b + e + f + g$, where $b \in \mathrm{Nil}(R)$, $e^3 = e$, $f^3 = f$, $g^3 = g$ and b, e, f, g commute with one another. Notice that if x is a tripotent of R, then $x - x^2 \in \mathrm{Nil}(R)$. Indeed, $(x - x^2)^2 = 2(x^2 - x) \in \mathrm{Nil}(R)$ as $2 \in \mathrm{Nil}(R)$, and so $x - x^2 \in \mathrm{Nil}(R)$. As e, f, g are tripotents, it follows that $a - a^2 = b(1 - b - 2e - 2f - 2g) - 2(ef + eg + fg) + (e - e^2) + (f - f^2) + (g - g^2)$. So $a - a^2 \in \mathrm{Nil}(R)$. □

Lemma 3.2.5 *[83, Lemma 2.6] If $2 \in U(R)$ and $a^3 - a$ is nilpotent, then there exists a polynomial $\theta(t) \in \mathbb{Z}[t]$ such that $\theta(a)^3 = \theta(a)$ and $a - \theta(a)$ is nilpotent.*

Proof. As $2 \in U(R)$, we have $a = b - c$, where $b := \frac{1}{2}(a^2 + a)$ and $c := \frac{1}{2}(a^2 - a)$. We calculate that

$$b^2 - b = \frac{1}{4}(a^4 + 2a^3 - a^2 - 2a) = \frac{1}{4}(a + 2)(a^3 - a), \quad \textit{and}$$

$$c^2 - c = \frac{1}{4}(a^4 - 2a^3 - a^2 + 2a) = \frac{1}{4}(a - 2)(a^3 - a).$$

Since $a^3 - a$ is nilpotent, both $b^2 - b$ and $c^2 - c$ are nilpotent. Therefore, from the proof of Lemma 3.2.3, there exist polynomials $\vartheta(t)$, $\xi(t)$ in $\mathbb{Z}[t]$ such that $\vartheta(b)$, $\xi(c)$ are idempotents of R, and $b - \vartheta(b)$, $c - \xi(c)$ are nilpotent. As a difference of two commuting idempotents, $\theta(a) := \vartheta(b) - \xi(c)$ is a tripotent.

Finally, we check that

$$a - \theta(a) = (b - c) - (\vartheta(b) - \xi(c)) = (b - \vartheta(b)) - (c - \xi(c))$$

is a difference of two commuting nilpotents. So $a - \theta(a)$ is nilpotent. □

Proposition 3.2.6 *Let R be a ring with $3 \in \mathrm{Nil}(R)$. The following are equivalent:*

(1) *R has the $\mathcal{P}$ property.*

(2) *Every element of R is a sum of a nilpotent and a tripotent that commute.*

(3) *$a - a^3$ is nilpotent for all $a \in R$.*

Proof. (2) ⇒ (3). Given an element $a \in R$, write $a = b + f$ where $b \in \mathrm{Nil}(R)$ and $f^3 = f$. Then $a^3 = (b + f)^3 = b^3 + 3b^2 f + 3bf^2 + f$, and it follows that $a^3 - a = b(b^2 + 3bf + 3f^2 - 1) \in \mathrm{Nil}(R)$.

(3) ⇒ (2). It follows from Lemma 3.2.5.

(2) ⇒ (1). As idempotents are tripotents, the implication follows.

(1) ⇒ (3). Let $a \in R$. Then $a = b + e + f + g$ where $b \in \mathrm{Nil}(R)$, $e^3 = e$, $f^3 = f$, $g^3 = g$ and b, e, f, g commute with one another. As $3 \in \mathrm{Nil}(R)$, it follows that

$$\begin{aligned} a - a^3 = b(&1 - b^2 - 3be - 3bf - 3bg - 3e^2 - 3f^2 - 3g^2 - 6ef - 6eg - 6fg) \\ &- 3(e^2 f + e^2 g + ef^2 + eg^2 + f^2 g + fg^2 + 2efg). \end{aligned}$$

So $a - a^3 \in \mathrm{Nil}(R)$. □

Lemma 3.2.7 *[53, Proposition 21.20] Let e, f be idempotents in a ring R. Then $eR \cong fR$ as right R-modules if and only if there exist a, $b \in R$ such that $e = ab$ and $f = ba$.*

Proof. (⇒) Let $\theta : eR \to fR$ be an R-isomorphism. Then $\theta(e) = b = fx$ for some b, $x \in R$. Further, for $\theta^{-1} : fR \to eR$, $\theta^{-1}(f) = a = ey$ for some a,

$y \in R$. Then, we have $e = \theta^{-1}\theta(e) = \theta^{-1}(b) = \theta^{-1}(fx)$. It follows that $ab = \theta^{-1}(f)b = \theta^{-1}(fb) = \theta^{-1}(fx) = e$. Similarly, $f = \theta\theta^{-1}(f) = \theta(a) = \theta(ey)$, so $ba = \theta(e)a = \theta(ea) = \theta(ey) = f$.

($\Leftarrow$) Given such a, b in R, we have $be = b(ab) = (ba)b = fb \in fR$ and $af = a(ba) = (ab)a = ea \in eR$. Define $\theta : eR \to fR$ and $\theta' : fR \to eR$ by $\theta(x) = bx \in fR$ and $\theta'(y) = ay \in eR$. Then $\theta'\theta(e) = \theta'(be) = abe = e^2 = e$, $\theta\theta'(f) = \theta(af) = baf = f^2 = f$. Hence, $\theta'\theta = 1$ and $\theta\theta' = 1$, as desired.

Lemma 3.2.8 *[58, Theorem 2.1] Let R be a ring with $J(R) = 0$ such that every nonzero right ideal contains a nonzero idempotent. If $a^n = 0$ but $a^{n-1} \neq 0$, then there exists an idempotent $f^2 = f \in RaR$ such that $fRf \cong \mathbb{M}_n(T)$ for some non-trivial ring T.*

Proof. By the assumption, the nonzero right ideal $a^{n-1}R$ contains a nonzero idempotent e. Set $e = a^{n-1}b_0$, so $e = e^2 = a^{n-1}b_0e = a^{n-1}(b_0e)$. Let $b = b_0e$. Then $be = b_0e^2 = b_0e = b$. Thus, we have $0 \neq e = a^{n-1}b$ and $b = be$. We set $e_i = a^{n-i}ba^{i-1} = a^{n-i}bea^{i-1}$, $i = 1, \cdots, n$. Then $e_i^2 = e_i$ and $e_1 = e$. Since $a^{i-1}e_ia^{n-i}b = e \neq 0$, we have $e_i \neq 0$. If $i > j$, then $a^{n+i-j-1} = 0$ and $e_ie_j = a^{n-i}ba^{i-1}a^{n-j}ba^{j-1} = a^{n-i}ba^{n+i-j-1}ba^{j-1} = 0$. For two elements x, $y \in R$, we denote by $x \circ y = x + y - xy$. We set $f := e_1 \circ e_2 \circ \cdots \circ e_n$. Then it is readily verified by induction on n that f is an idempotent and $e_if = fe_i = e_i$ for $i = 1, \cdots, n$. Set $B := fRf$. Then B is a ring with identity element f and B contains all the idempotents e_i. Since $e_ie_j = 0$ for $i > j$, we prove by induction that $B_B = \oplus_{i=1}^{n} e_iB$. For $k = 2$, write $f' = e_1 \circ e_2$. By the above proof, we have $f'^2 = f'$, $f'e_1 = e_1 \in B$, and $f'e_2 = e_2 \in B$. Then $e_1B = f'e_1B \subseteq f'B$, $e_2B = f'e_2B \subseteq f'B$. As $f' = e_1(e_1 - e_2) + e_2$, $f'B = e_1B + e_2B$. If $x = e_1b_1 = e_2b_2$ for some elements b_1, $b_2 \in B$, then $x = e_2^2b_2 = e_2e_1b_1 = 0$. Hence, $f'B = e_1B \oplus e_2B$. Suppose the claim holds for $k = n-1$, i.e., with $f'' = e_1 \circ \cdots \circ e_{n-1}$, $f''B = e_1B \oplus \cdots \oplus e_{n-1}B$. Now for $k = n$, $f = f'' \circ e_n$. As $ff'' = (f'' \circ e_n)f'' = f''$ and $fe_n = e_n$, it follows that

$f''B \subseteq fB$ and $e_nB \subseteq fB$. Therefore, by the case for $k = 2$, $B_B = fB = f''B \oplus e_nB = \oplus_{i=1}^n e_iB$. In addition, $(a^{n-i}b)a^{i-1} = e_i$ and $a^{i-1}(a^{n-i}b) = e_1$. By Lemma 3.2.7, $e_iB \cong e_1B$ for all i. That is, $B_B \cong (e_1B)^n$. As $B \cong \mathrm{End(B_B)}$, it follows that

$$B \cong \mathrm{End((e_1B)^n)} \cong \mathbb{M}_n(\mathrm{End(e_1B)}) \cong \mathbb{M}_n(e_1Be_1).$$

Therefore, we have $fRf \cong \mathbb{M}_n(T)$, where $T = e_1Be_1$. □

Notice that for a ring R with $5 \in \mathrm{Nil}(R)$, the following related results have been obtained in [83].

Lemma 3.2.9 *[83, Proposition 2.10] Let R be a ring with $5 \in \mathrm{Nil}(R)$. The following are equivalent:*

(1) *Every element of R is a sum of a nilpotent and two tripotents that commute with one another.*

(2) $a^5 - a \in \mathrm{Nil}(R)$ *for all $a \in R$.*

(3) $J(R)$ *is nil and $R/J(R)$ is a subdirect product of $\mathbb{F}_5' s$.*

Proof. (1) $\Rightarrow$ (2) Let $a \in R$ and write $a = b + e + f$, where $b \in \mathrm{Nil(R)}$, $e^3 = e \in R$, $f^3 = f \in R$ and b, e, f commute with one another. Then $a^5 = (b + e + f)^5 \equiv b^5 + e^5 + f^5 = b^5 + e + f (\mathrm{mod 5R})$. Thus, it follows that $a^5 - a \equiv (b^5 + e + f) - (b + e + f) = b^5 - b (\mathrm{mod 5R})$. As $b^5 - b$ and 5 are nilpotent, we deduce that $a^5 - a$ is nilpotent.

(2) $\Rightarrow$ (1) Let $a \in R$. Write $x := 3a + a^2 + a^4$ and $y := 3a - a^2 - a^4$. Regarding 5 as 0 and a^5 as a, we calculate

$$x^2 \equiv a + a^2 + a^3 + 2a^4, \quad x^3 \equiv x,$$
$$y^2 \equiv -a + a^2 - a^3 + 2a^4, \quad y^3 \equiv y.$$

As $a^5 - a$ and 5 are nilpotent, it follows that $x^3 - x$, $y^3 - y$ are nilpotent. As $2 \in R$ is a unit, by Lemma 3.2.5 there exist $\theta_1(t)$, $\theta_2(t) \in \mathbb{Z}[t]$ such that

$\theta_1(a)^3 = \theta_1(a)$, $\theta_2(a)^3 = \theta_2(a)$, $x - \theta_1(a)$ and $y - \theta_2(a)$ are nilpotent. Take $\theta[t] \in \mathbb{Z}[t]$ such that $\theta(a) = (x - \theta_1(a)) + (y - \theta_2(a)) - 5a$. Then $\theta(a)$ is nilpotent and $a = \theta(a) + \theta_1(a) + \theta_2(a)$.

(3) $\Rightarrow$ (2) The implication is clear.

(2) $\Rightarrow$ (3) Let $a \in J(R)$. As $a(1 - a^4) = a - a^5$ is nilpotent and $1 - a^4 \in U(R)$, it follows that a is nilpotent. So $J(R)$ is nil. For any $a \in R$, $(a - a^5)^n = 0$ for some $n \geq 1$. Then $a^n(1 - a^4)^n = 0$, and it follows that $a^n \in a^{n+1}R \cap Ra^{n+1}$. So R, and hence $R/J(R)$ are strongly π-regular. As $x^5 - x$ is nilpotent for all $x \in R/J(R)$, we can assume that $J(R) = 0$. We next show that R is reduced. Assume that $a^2 = 0$ for some $0 \neq a \in R$. Then, by Lemma 3.2.8, there exists $0 \neq g^2 = g \in RaR$ such that $gRg \cong \mathbb{M}_2(S)$ where S is a nontrivial ring. Let $y = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \in \mathbb{M}_2(S)$. We have $y^5 = \begin{pmatrix} 10 & 4 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 4 & 2 \end{pmatrix}$, so $y^5 - y = \begin{pmatrix} 3 & 3 \\ 3 & 2 \end{pmatrix}$. We calculate that $(y^5 - y)^2 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, which is not nilpotent. So $y^5 - y$ is not nilpotent. This contradiction shows that R is reduced. So $a^5 = a$ for all $a \in R$, and hence R is a subdirect product of $\mathbb{F}_5$'s by Proposition 1.2.5. □

Proposition 3.2.10 *Let R be a ring with $5 \in \mathrm{Nil}(R)$. The following are equivalent:*

(1) *R has the $\mathcal{P}$ property.*

(2) *$J(R)$ is nil and $R/J(R)$ is a subdirect product of $\mathbb{F}_5's$.*

(3) *$a - a^5$ is nilpotent for all $a \in R$.*

(4) *Every element of R is a sum of a nilpotent and two tripotents that commute.*

Proof. In view of Lemma 3.2.9, the equivalence (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) follows.

(4) $\Rightarrow$ (1). The implication is clear.

(1) $\Rightarrow$ (3). Let $a \in R$. $a = b + e + f + g$, where $b \in \mathrm{Nil}(R)$, $e^3 = e$, $f^3 = f$, $g^3 = g$ and b, e, f, g commute with one another. Then there

exist polynomials $\theta(t_1,t_2,t_3,t_4)$ and $\eta(t_1,t_2,t_3,t_4)$ in $\mathbb{Z}[t_1,t_2,t_3,t_4]$ such that $a^5 - a = 5\cdot\theta(b,e,f,g) + b\cdot\eta(b,e,f,g)$. So $a^5 - a \in \mathrm{Nil}(R)$. $\square$

Proposition 3.2.11 *Let R be a ring with $7 \in \mathrm{Nil}(R)$. The following are equivalent:*

(1) *R has the $\mathcal{P}$ property.*

(2) *$a - a^7$ is nilpotent for all $a \in R$.*

(3) *$J(R)$ is nil and $R/J(R)$ is a subdirect product of $\mathbb{F}_7'$s.*

Proof. (1) $\Rightarrow$ (2). Let $a \in R$. Then $a = b + e + f + g$, where $b \in \mathrm{Nil}(R)$, $e^3 = e$, $f^3 = f$, $g^3 = g$ and b, e, f, g commute with one another. Then there exist polynomials $\theta(t_1,t_2,t_3,t_4)$ and $\eta(t_1,t_2,t_3,t_4)$ in $\mathbb{Z}[t_1,t_2,t_3,t_4]$ such that $a^7 - a = 7\cdot\theta(b,e,f,g) + b\cdot\eta(b,e,f,g)$. So, $a^7 - a \in \mathrm{Nil}(R)$.

(2) $\Rightarrow$ (3). Let $j \in J(R)$. Then $j - j^7 = j(1-j^6)$ is nilpotent. As $1 - j^6$ is a unit in R, j is nilpotent. Hence $J(R)$ is nil. For any $a \in R$, $a - a^7$ is nilpotent, then there exists an integer $n \geq 1$ such that $a^n \in a^{n+1}R \cap Ra^{n+1}$. So R, and further $\overline{R} := R/J(R)$ are strongly π-regular rings. Now suppose $\overline{a}^2 = \overline{0}$ for some $0 \neq \bar{a} \in \bar{R}$. By Lemma 3.2.8, there exists $\bar{w}^2 = \bar{w} \in \bar{R}\bar{a}\bar{R}$ where $\bar{w} \neq 0$ such that $\overline{w}\overline{R}\overline{w} \cong \mathbb{M}_2(T)$ where T is a nontrivial ring. Let $x = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $x \in \mathbb{M}_2(T)$. As $7 \in J(R)$, 7=0 in $\overline{R}$ and further, it implies that $x^7 - x = \begin{pmatrix} -1 & -2 \\ -2 & 1 \end{pmatrix}$. Therefore, it follows that $(x^7 - x)^2 = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ $\in U(\overline{R})$, a contradiction. This shows that $\overline{R}$ is reduced. It follows that $\overline{R}$ has the identity $x^7 = x$ for all $x \in \overline{R}$. Hence $\overline{R}$ is a subdirect product of $\mathbb{F}_7'$s.

(3) $\Rightarrow$ (1). Let $a \in R$. By Example 3.2.2, $\overline{a} = \overline{e_0} + \overline{f_0} + \overline{g_0}$, where $\overline{e_0}^3 = \overline{e_0}$, $\overline{f_0}^3 = \overline{f_0}$, $\overline{g_0}^3 = \overline{g_0}$ and $\overline{e_0}$, $\overline{f_0}$, $\overline{g_0}$ commute with one another. As $J(R)$ is nil, $e_0 - e_0^3$, $f_0 - f_0^3$, $g_0 - g_0^3$ are nilpotent. Notice that $2 \in U(R)$. By Lemma 3.2.5, there exist tripotents $e^3 = e$, $f^3 = f$, $g^3 = g \in \mathbb{Z}[a]$ such that $e_0 = b_1 + e$, $f_0 = b_2 + f$, $g_0 = b_3 + g$ where b_1, b_2, b_3 are commuting nilpotents. As $a - e_0 - f_0 - g_0 \in J(R)$, there exists $j \in J(R)$ such that $a = j + e_0 + f_0 + g_0$.

Write $b := j + b_1 + b_2 + b_3$. Then $a = b + e + f + g$ where $b \in \text{Nil}(R)$, $e^3 = e$, $f^3 = f$, $g^3 = g$ and b, e, f, g commute with one another. □

Based on the results above, we have our structure theorem for the class of rings that has the $\mathcal{P}$ property.

Theorem 3.2.12 *The following are equivalent for a ring R:*

(1) *R has the $\mathcal{P}$ property.*

(2) *$R = R_1 \times R_2 \times R_3 \times R_4$, where R_i is zero or $R_i/J(R_i)$ is a subdirect product of C_i's with $J(R_i)$ nil for $i = 1, 2, 3, 4$, with $C_1 = \mathbb{F}_2$, $C_2 = \mathbb{F}_3$, $C_3 = \mathbb{F}_5$, and $C_4 = \mathbb{F}_7$.*

(3) *$R = A \times B$, where $a - a^5 \in \text{Nil}(A)$ for all $a \in A$ and $b - b^7 \in \text{Nil}(B)$ for all $b \in B$ with $7 \in \text{Nil}(B)$.*

Proof. (1) $\Leftrightarrow$ (2). The equivalence follows from Lemma 3.2.1, Proposition 3.2.4, Proposition 3.2.6, Proposition 3.2.10, and Proposition 3.2.11.

(2) $\Rightarrow$ (3). Suppose that (2) holds. Let $A = R_1 \times R_2 \times R_3$ and $B = R_4$. Then, $a - a^5 = (1 + a + a^2 + a^3)(a - a^2) = (1 + a^2)(a - a^3) \in \text{Nil}(A)$ for all $a \in A$. By Proposition 3.2.11, $b - b^7 \in \text{Nil}(B)$ for all $b \in B$ with $7 \in \text{Nil}(B)$.

(3) $\Rightarrow$ (1). In view of Theorem 3.1.6, every element of A is a sum of a nilpotent and two tripotents that commute, so A has the $\mathcal{P}$ property. By Proposition 3.2.11, B has the $\mathcal{P}$ property. Hence R has the $\mathcal{P}$ property. □

The following proposition can be used to determine if a ring has the $\mathcal{P}$ property.

Proposition 3.2.13 *A ring R has the $\mathcal{P}$ property if and only if*

(1) *$13 \in U(R)$;*

(2) *$a - a^{13}$ is nilpotent for all $a \in R$;*

(3) *$1 + a + a^2 \in U(R)$ for all $a \in R$ whenever $2 \in \text{Nil}(R)$.*

Proof. ($\Rightarrow$). In view of the proof of Lemma 3.2.1, $2 \cdot 3 \cdot 5 \cdot 7 = 210 \in \text{Nil}(R)$, and so $1 + 210 \cdot 6 = 13 \cdot 97 \in U(R)$, which implies that $13 \in U(R)$, and (1) follows. By Theorem 3.2.12, R has the decomposition as stated in Theorem 3.2.12 (3). If $a \in A$, then $a - a^{13} = (a - a^5)(1 + a^4 + a^8)$ is nilpotent. If $a \in B$, then $a - a^{13} = (a - a^7)(1 + a^6)$ is nilpotent. Thus, (2) holds. Now suppose that $2 \in \mathrm{Nil}(R)$. By Proposition 3.2.4, $a - a^2$ is nilpotent for all $a \in R$. So $1 + a + a^2 = 1 + [(a - a^2) + 2a^2] \in U(R)$, and therefore, (3) follows.

($\Leftarrow$). As $2^{13} - 2 = 2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \in \mathrm{Nil}(R)$ and $13 \in U(R)$, we have $2 \cdot 3 \cdot 5 \cdot 7$ is nilpotent. By the Chinese Reminder Theorem, $R = R_1 \times R_2 \times R_3 \times R_4$, where $R_1 \cong R/2^nR$, $R_2 \cong R/3^nR$, $R_3 \cong R/5^nR$, $R_4 \cong R/7^nR$ and $J(R_i)$ is nil for $i = 1, 2, 3, 4$. For $a \in R_1$, $(a - a^4)^4 = a^3(a - a^{13}) + 2(a^{16} - 2a^{13} + 3a^{10} - 2a^7)$. As $2 \in \mathrm{Nil}(R_1)$, $(a - a^4)^4 \in \mathrm{Nil}(R_1)$, so $a - a^4 = (a - a^2)(1 + a + a^2)$ is nilpotent. It follows that $a - a^2$ is nilpotent since $1 + a + a^2 \in U(R_1)$. By Proposition 3.2.4, R_1 has the $\mathcal{P}$ property. For $a \in R_2$, $(a - a^5)^3 = a^2(a - a^{13}) - 3(a^7 - a^{11})$ belongs to $\mathrm{Nil}(R_2)$, by [83, Theorem 2.11], every element of R_2 is a sum of a nilpotent and two tripotents that commute. Hence, R_2 has the $\mathcal{P}$ property. For $a \in R_3$, $\overline{a} - \overline{a}^{13}$ is nilpotent in $\overline{R_3} := R_3/J(R_3)$. As $\overline{R_3}$ is semiprimitive, $\overline{R_3}$ is a subdirect product of a family of right primitive rings $\{R_\alpha : \alpha \in \Lambda\}$. For each R_α, suppose that R_α is not simple Artinian. Then for any integer $n \geq 1$ there exists a subring $S_\alpha \subseteq R_\alpha$ such that $\mathbb{M}_n(D)$ is a factor ring of S_α, where D is a division ring. It follows that $y - y^{13} \in \mathrm{Nil}(\mathbb{M}_n(D))$ for any $y \in \mathbb{M}_n(D)$. Take $y = \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}$. Noting that $5 = 0$ in R_α, we have $y - y^{13} = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$, and further $(y - y^{13}) \cdot \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \in U(\mathbb{M}_n(D))$, so $y - y^{13}$ is not a nilpotent, a contradiction. This shows that R_α is simple Artinian for each $\alpha \in \Lambda$, so $R_\alpha \cong \mathbb{M}_n(D)$ for some $n \geq 1$. As argued above, $n = 1$, i.e., $R_\alpha \cong D$ is a division ring for each α. Therefore R_α has the identity $\overline{x} = \overline{x}^{13}$. Further, let $\overline{b} \in R_\alpha$. Then $(\overline{b} - \overline{b}^5)^5 = \overline{b}^5 - \overline{b}^{25} = \overline{b}^5 - \overline{b}^{13} = \overline{b}^5 - \overline{b} \in R_\alpha$. If $\overline{b} - \overline{b}^5 \neq \overline{0}$, then $(\overline{b} - \overline{b}^5)^4 = -1$, and $(\overline{b} - \overline{b}^5)^{12} = -1$, so $(\overline{b} - \overline{b}^5)^{13} = -(\overline{b} - \overline{b}^5)$. Note that R_α has the identity $\overline{x} = \overline{x}^{13}$. So $2(\overline{b} - \overline{b}^5) = 0$, and then $\overline{b} - \overline{b}^5 = 0$, a contradiction. Hence $\overline{b} = \overline{b}^5$ in each R_α. It follows that $\overline{a} - \overline{a}^5 = \overline{0}$ for all $a \in R_3$. So, $a - a^5 \in J(R_3)$ is a nilpotent. By Proposition 3.2.10, R_3 has the

$\mathcal{P}$ property. Similarly, since $\overline{R_4} := R_4/J(R_4)$ is a subdirect product of right primitive rings $\{R_\beta : \beta \in \Gamma\}$ where each R_β has the property that $z - z^{13}$ is nilpotent. Take $z = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Then $z - z^{13} = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \in U(\mathbb{M}_n(D))$, where D is a division ring. As proved above, $R_\beta \cong D$, where D is a division ring. So R_β has the identity $\overline{x} = \overline{x}^{13}$. It follows that $(\overline{x} - \overline{x}^7)(\overline{1} + \overline{x}^6) = \overline{0}$ in each R_β. Next we show that $\overline{x} - \overline{x}^7 = \overline{0}$ in R_β. Suppose that there exists $\overline{x} \in R_\beta$ such that $\overline{x} - \overline{x}^7 \neq \overline{0}$. Then $\overline{1} + \overline{x}^6 = \overline{0}$. So, $\overline{x}^7 = -\overline{x}$ and $(\overline{1} + \overline{x})^7 = \overline{1} + \overline{x}^7 = \overline{1} - \overline{x}$. Since $(\overline{1} + \overline{x}) - (\overline{1} + \overline{x})^7 = \overline{x} - \overline{x}^7 \neq \overline{0}$ and $[(\overline{1} + \overline{x}) - (\overline{1} + \overline{x})^7][1 + (\overline{1} + \overline{x})^6] = \overline{0}$, we have $\overline{1} + (\overline{1} + \overline{x})^6 = \overline{0}$, i.e., $(\overline{1}+\overline{x})^7 = -(\overline{1}+\overline{x})$. Hence, $\overline{1}-\overline{x} = -\overline{1}-\overline{x}$, i.e. $\overline{2} = \overline{0}$ in R_β, a contradiction. Therefore, $a - a^7 \in J(R_4)$ is nilpotent. By Proposition 3.2.11, R_4 has the $\mathcal{P}$ property. Hence R has $\mathcal{P}$ property. □

Corollary 3.2.14 *If a ring R has the $\mathcal{P}$ property, then so does its center $C(R)$.*

Remark 3.2.15 (1) *The condition $13 \in U(R)$ in Proposition 3.2.13 is not superfluous. Let $R = \mathbb{F}_{13}$. Then $2 \in U(R)$ and $a - a^{13} = 0$ for all $a \in R$, but $13 \notin U(R)$. It is easy to see that all tripotents of R are $0, 1$ and 12. So R does not have the $\mathcal{P}$ property.*

(2) *Let $R = \mathbb{M}_2(\mathbb{F}_2)$. Then $13 \in U(R)$ and it is easy to calculate that $a - a^{13}$ is nilpotent for all $a \in R$. Note that $2 \in \mathrm{Nil}(R)$). If take $a = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in R$ then $1 + a + a^2 = 0$. In view of Proposition 3.2.4, R does not have the $\mathcal{P}$ property since R is not strongly nil clean.*

Next we consider the $\mathcal{P}$ property for some extensions of rings. By Theorem 3.2.12, if R has the $\mathcal{P}$ property, then $R/J(R)$ is commutative. Thus any matrix ring of size greater then 1 does not have the $\mathcal{P}$ property.

Proposition 3.2.16 *Let R be a ring. If R has the $\mathcal{P}$ property then so does eRe for any $e^2 = e \in R$.*

Proof. By Theorem 3.2.12, $R = A \times B$, where $a - a^5 \in \text{Nil}(A)$ for all $a \in A$ and $b - b^7 \in \text{Nil}(B)$ for all $b \in B$ with $7 \in \text{Nil}(B)$. Let $e^2 = e \in R$. Then $e = (e_1, e_2)$ for some ${e_1}^2 = e_1 \in A$ and ${e_2}^2 = e_2 \in B$. It follows that $eRe = e_1Ae_1 \times e_2Be_2$. Since e_1Ae_1 is a subring of A and e_2Be_2 is a subring of B, $x - x^5 \in \text{Nil}(e_1Ae_1)$ for all $x \in e_1Ae_1$ and $y - y^7 \in \text{Nil}(e_2Be_2)$ for all $y \in e_2Be_2$. As 7 is central in B, $7e_2 \in \text{Nil}(e_2Be_2)$. Hence eRe has the $\mathcal{P}$ property. □

Proposition 3.2.17 *Suppose I is a nil ideal of a ring R. Then R has the $\mathcal{P}$ property if and only if R/I has the $\mathcal{P}$ property.*

Proof. The necessity is obvious; the sufficiency is a quick consequence of Proposition 3.2.13. □

Corollary 3.2.18 *Let R be a ring. Then R has the $\mathcal{P}$ property if and only if $\mathbb{T}_n(R)$ has the $\mathcal{P}$ property.*

Proof. Let I be an ideal consisting of all strictly upper triangular matrices of $\mathbb{T}_n(R)$. Then I is nilpotent and $\mathbb{T}_n(R)/I \cong R^n$. So the result follows from Proposition 3.2.17. □

Corollary 3.2.19 *A ring R has the $\mathcal{P}$ property if and only if $R/J(R)$ has the $\mathcal{P}$ property and $J(R)$ is nil.*

3.3 Group rings whose every element is a sum of a nilpotent and three tripotents

In this section, we determine when a group ring of an abelian group has $\mathcal{P}$ property. The center of a group G is denoted by $\mathcal{Z}(G)$. A group G is called **locally finite** if every finitely generated subgroup of G is finite. Let p be a prime number. A group G is called a p-group if the order of each element of G is a power of p. The cyclic group of order n is denoted by C_n.

If R is a ring and G is a group, RG denotes the group ring of the group G over R. The ring homomorphism $\omega : RG \to R$, $\Sigma r_g g \mapsto \Sigma r_g$ is called the augmentation map, and $\ker(\omega)$ is called the augmentation ideal of the group ring RG and is denoted by $\triangle(RG)$. Note that if the group ring RG has property $\mathcal{P}$, so does R.

Lemma 3.3.1 *[25, Theorem 9] $\Delta(RG)$ is nilpotent if and only if G is a finite p-group and $p \in \mathrm{Nil}(R)$.*

Lemma 3.3.2 *Let R be a ring and G be a group. Suppose that RG has the $\mathcal{P}$ property.*

(1) *If $2 \in J(R)$, then $\mathcal{Z}(G)$ is a 2-group.*

(2) *If $3 \in J(R)$, then $\mathcal{Z}(G)$ is a direct product of a group of exponent 2 and a 3-group.*

(3) *If $5 \in J(R)$, then $\mathcal{Z}(G) = H \times K$, where $h^4 = 1$ for all $h \in H$ and K is a 5-group.*

(4) *If $7 \in J(R)$, then $\mathcal{Z}(G)$ is a direct product of a group of exponent 2, a group of exponent 3 and a 7-group.*

Proof. Let $p \in \{2, 3, 5, 7\}$. If $p \in J(R)$, then $(R/J(R))G$ has the $\mathcal{P}$ property and $p = 0$ in $R/J(R)$. So, without loss of generality, we can assume $J(R) = 0$. Then $x - x^p$ is nilpotent for all $x \in RG$ by Propositions 3.2.4, 3.2.6, 3.2.10 and 3.2.11. For $g \in \mathcal{Z}(G)$, $g - g^p$ is nilpotent, so $1 - g^{p-1}$ is nilpotent. Thus, there exists $n > 0$ such that $(1 - g^{p-1})^{p^n} = 0$, i.e., $g^{(p-1)\cdot p^n} = 1$ as $p = 0$ in R.

If $p = 2$, then for each $g \in \mathcal{Z}(G)$, $g^{2^n} = 1$ for some $n \geq 1$; so $\mathcal{Z}(G)$ is a 2-group.

If $p = 3$, then for each $g \in \mathcal{Z}(G)$, $g^{2\cdot 3^n} = 1$ for some $n \geq 1$; so $\mathcal{Z}(G)$ is a direct product of its 2-component and 3-component. Write $\mathcal{Z}(G) = H \times K$ where H is a 2-group and K is a 3-group. For each 2- element $h \in H$, we have $(h^2)^{3^n} = 1$. Since h^2 is a 2-element, we obtain that $h^2 = 1$. Hence H is a group of exponent 2.

If $p = 5$, then for each $g \in \mathcal{Z}(G)$, $g^{4\cdot 5^n} = 1$ for some $n \geq 1$; so $\mathcal{Z}(G)$ is a direct product of its 2-component and 5-component. Write $\mathcal{Z}(G) = H \times K$ where H is a 2-group and K is a 5-group. As argued above, $h^4 = 1$ for all $h \in H$.

If $p = 7$, then for each $g \in \mathcal{Z}(G)$, $g^{2\cdot 3\cdot 7^n} = 1$ for some $n \geq 1$; so $\mathcal{Z}(G)$ is a direct product of its 2-component, 3-component and 7-component. Write $\mathcal{Z}(G) = H \times K \times J$ where H is a 2-group, K is a 3-group and J is a 7-group. As argued above, for $h \in H$ and $k \in K$, $h^6 = 1$ and $k^6 = 1$. As H is a 2-group and K is a 3-group, it follows that $h^2 = 1$ and $k^3 = 1$. So H is a group of exponent 2 and K is a group of exponent 3. □

Lemma 3.3.3 *Let $p \in \{2, 3, 5, 7\}$. If R has the $\mathcal{P}$ property with $p \in J(R)$ and G is a locally finite p-group, then RG has the $\mathcal{P}$ property.*

Proof. As G is locally finite, we can assume that G is a finite p-group. By Theorem 3.2.12, $J(R)$ is nil, so $p \in J(R)$ is nilpotent. By Lemma 3.3.1, $\triangle(RG)$ is nilpotent. Since $(RG)/\triangle(RG) \cong R$, it follows from Proposition 3.2.17 that RG has the $\mathcal{P}$ property. □

Theorem 3.3.4 *Let R be a ring and G be an abelian group. Then RG has the $\mathcal{P}$ property if and only if one of the following cases holds:*

(1) *$R \cong A$ and G is a 2-group,*

(2) $R \cong B$ *and* G *is a direct product of a group of exponent* 2 *and a* 3*-group.*

(3) $R \cong C$ *and* $G = H \times K$, *where* $h^4 = 1$ *for all* $h \in H$ *and* K *is a* 5*-group.*

(4) $R \cong D$ *and* $G = H \times K \times J$, *where* H *is a group of exponent* 2, K *is a group of exponent* 3 *and* J *is a* 7*-group.*

(5) $R \cong A \times C$, *and* $g^4 = 1$ *for all* $g \in G$.

(6) $R \in \{A \times B, A \times D, B \times C, B \times D, C \times D, A \times B \times C, A \times B \times D, \quad B \times C \times D, A \times B \times C \times D\}$, *and* G *is a group of exponent* 2,

where $A/J(A)$ *is Boolean with* $J(A)$ *nil,* $B/J(B)$ *is a subdirect product of* $\mathbb{F}_3$*'s with* $J(B)$ *nil,* $C/J(C)$ *is a subdirect product of* $\mathbb{F}_5$*'s with* $J(C)$ *nil and* $D/J(D)$ *is a subdirect product of* $\mathbb{F}_7$*'s with* $J(D)$ *nil.*

Proof. ($\Rightarrow$). Suppose that RG has the $\mathcal{P}$ property. Then R has the $\mathcal{P}$ property, so, by Theorem 3.2.12, $R = A \times B \times C \times D$ where A is zero or $A/J(A)$ is Boolean with $J(A)$ nil, B is zero or $B/J(B)$ is a subdirect product of $\mathbb{F}_3$'s with $J(B)$ nil, C is zero or $C/J(C)$ is a subdirect product of $\mathbb{F}_5$'s with $J(C)$ nil, and D is zero or $D/J(D)$ is a subdirect product of $\mathbb{F}_7$'s. Then one of the following cases occurs, in view of Lemma 3.3.2.

Case 1: $R = A$. Then G is 2-group.

Case 2: $R = B$. Then G is a direct product of a group of exponent 2 and a 3-group.

Case 3: $R = C$. Then $G = H \times K$, where $h^4 = 1$ and K is a 5-group.

Case 4: $R = D$. Then G is a direct product of a group of exponent 2, a group of exponent 3 and a 7-group.

Case 5: $A \neq 0$, and $B \neq 0$ or $D \neq 0$. Then G satisfies the conditions in Cases 1,2 and 4, so G is a group of exponent 2.

Case 6: $R = A \times C$. Then G satisfies the conditions in Cases 1 and 3, so $g^4 = 1$ for all $g \in G$.

($\Leftarrow$). Firstly, by Lemma 3.3.3, (1) implies that RG has the $\mathcal{P}$ property.

We next show that, if G is a group of exponent 2, then $(A \times B \times C \times D)G$

has the $\mathcal{P}$ property. Indeed, $(A\times B\times C\times D)G \cong AG\times(B\times C\times D)G$. As AG has the $\mathcal{P}$ property by (1), we only need to show that $(B\times C\times D)G$ has the $\mathcal{P}$ property, and we can assume that G is a finite group. So G is a ditrct product of finite copies of C_2, and hence, as 2 is a unit in $B \times C \times D$, $(B \times C \times D)G$ is a direct sum of finite copies of $B \times C \times D$. Hence, $(B \times C \times D)G$ has the $\mathcal{P}$ property. Thus, (6) implies RG has the $\mathcal{P}$ property.

Suppose (2) holds. Write $G = H \times K$ where H is a group of exponent 2 and K is a 3-group. Then $RH \cong BH$ has the $\mathcal{P}$ property by (6), so $RG \cong (BH)K$ has the $\mathcal{P}$ property by Lemma 3.3.3.

Suppose that (3) holds. Write $G = H \times K$, where $h^4 = 1$ for all $h \in H$ and K is a 5-group. Then $RG \cong (CH)K$, thus to show RG has the $\mathcal{P}$ property it suffices to show that CH has the $\mathcal{P}$ property by Lemma 3.3.3. We can assume that H is a finite group. Thus, H is a direct product of finite copies of C_2 and finite copies of C_4. Therefore, we only need to show that CC_2 and CC_4 have the $\mathcal{P}$ property. Note that CC_2 has the $\mathcal{P}$ property by (6). Since $J(C)$ is nil, $J(C)C_4$ is nil. As $(CC_4)/(J(C)C_4) \cong (C/J(C))C_4$, to show CC_4 has the $\mathcal{P}$ property it suffices to show that $(C/J(C))C_4$ has the $\mathcal{P}$ property by Proposition 3.2.17. As $C/J(C)$ is commutative with $x^5 = x$ for all $x \in C/J(C)$ and $g^5 = g$ for all $g \in C_4$, one quickly sees that $y^5 = y$ for all $y \in (C/J(C))C_4$, so $(C/J(C))C_4$ has the $\mathcal{P}$ property by Proposition 3.2.10. Hence (3) implies RG has the $\mathcal{P}$ property.

Suppose (4) holds. Write $G = H \times K \times J$, where H is a group of exponent 2, K is a group of exponent 3 and J is a 7-group. Then it follows that $RG \cong DG \cong ((DH)K)J$. Thus to show RG has the $\mathcal{P}$ property it suffices to show that $(DH)K$ has the $\mathcal{P}$ property by Lemma 3.3.3. By (6), DH has the $\mathcal{P}$ property. Since $J(DH)$ is nil, $J(DH)K$ is nil. Note that we have $(DH)K/J(DH)K \cong ((DH)/J(DH))K$, to show that $(DH)K$ has the $\mathcal{P}$ property it suffices to show that $((DH)/J(DH))K$ has the $\mathcal{P}$ property by Proposition 3.2.17. As $(DH)/J(DH)$ is commutative with $x^7 = x$ for all $x \in (DH)/J(DH)$ and $g^7 = g$ for all $g \in K$, one quickly sees that $y^7 = y$

for all $y \in ((DH)/J(DH))K$, so $((DH)/J(DH))K$ has the $\mathcal{P}$ property by Proposition 3.2.11. Hence (4) implies RG has the $\mathcal{P}$ property.

Finally suppose (5) holds. Then $RG \cong AG \times CG$. By (1), AG has the $\mathcal{P}$ property. By (3), CG has the $\mathcal{P}$ property. So RG has the $\mathcal{P}$ property. $\square$

Statement 3.3.5 *The materials of Section 3.2 and Section 3.3 are taken from [27]. My contribution in this joint work is to provide the characterizations of rings that have the $\mathcal{P}$ property as in Section* 3.2*. Section* 3.3 *is the outcome of the discussions between the authors in [27].*

Chapter 4

Decompositions of matrices as sums of idempotents, involutions, and tripotents

In this chapter, we discuss decomposing matrices into a sum of involutive matrices or a sum of tripotent matrices. Decomposing matrices has been an active topic (see [12], [13], [30], [40]). For instance, Hartwig and Putcha [40] showed that, an $n \times n$ matrix A over a field F of characteristic 0 is a sum of idempotents if and only if $\mathrm{tr(A)} = k \cdot 1_F$, where $k \in \mathbb{Z}$ and $k \geq \mathrm{rank(A)}$, and de Seguins Pazzis [30] obtained that an $n \times n$ matrix A over a field F of characteristic $p > 0$ is a sum of idempotents if and only if $\mathrm{tr(A)} \in \mathbb{F}_p$, the prime subfield of F. In particular, in this chapter we determine the integral domains over which every matrix can be decomposed as a sum of involutions or a sum of tripotents, respectively.

4.1 Matrices over rings as sums of involutions or tripotents

In this section we discuss decomposing matrices over integral domains as sums of involutions or tripotents. Motivated by the work of Merino, Paras, and Pelejo in [63], where the authors presented the necessary and sufficient conditions for an $n \times n$ matrix over K (for $K = \mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Z}_k$) to be a sum of involutions, as an extension of the work in [63], we give the necessary and sufficient conditions for an $n \times n$ matrix over an integral domain to be a sum of involutions and, respectively, a sum of tripotents. We also completely determine the integral domains over which every $n \times n$ matrix is a sum of involutions and a sum of tripotents, respectively.

We start with some useful observations on matrices over a general ring. The first three statements in the following lemma are from [63].

Lemma 4.1.1 *Let R be a ring and $a \in R$ and $n \geq 2$. The following hold in $\mathbb{M}_n(R)$:*

(1) *If $i \neq j$, then $aE_{ij} = (I + aE_{ij} - 2E_{jj}) + (2E_{jj} - I)$, a sum of two involutions.*

(2) *If k is even, then $kE_{11} = \frac{k}{2}I + \frac{k}{2}(2E_{11} - I)$ where I and $2E_{11} - I$ are involutions.*

(3) *If n is odd, then*

$$E_{11} = \begin{pmatrix} 1 & & & & & \\ & 0 & 1 & & & \\ & 1 & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & & & & & \\ & 1 & -1 & & & \\ & 0 & -1 & & & \\ & & & \ddots & & \\ & & & & 1 & -1 \\ & & & & 0 & -1 \end{pmatrix} + \begin{pmatrix} -1 & & & & & \\ & -1 & 0 & & & \\ & -1 & 1 & & & \\ & & & \ddots & & \\ & & & & -1 & 0 \\ & & & & -1 & 1 \end{pmatrix}$$

is a sum of three involutions.

(4) *The matrix* $A = \begin{pmatrix} a & 0 & & & \\ 0 & -a & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}$ *is a sum of involutions.*

Proof. The statement (1)-(3) are directly verified.

(4) Write $X = \begin{pmatrix} a-1 & 2a-a^2 & & & \\ 1 & 1-a & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{pmatrix}$. Then we have

$A = (I - 2E_{22}) + X - E_{21} + a(a-2)E_{12}$, where $(I - 2E_{22})$ and X are involutions and $-E_{21} + a(a-2)E_{12}$ is a sum of involutions by (1). □

Proposition 4.1.2 *Let R be a ring and $A \in \mathbb{M}_n(R)$ with $tr(A) = k \cdot 1_R$ for some $k \in \mathbb{Z}$.*

(1) *If k is even, then A is a sum of involutions.*

(2) *If n is odd, then A is a sum of involutions.*

Proof. We can assume that $n \geq 2$. Write $A = (a_{ij})$, so $k \cdot 1_R = a_{11} + \cdots + a_{nn}$. Then

$$A = kE_{11} + \sum_{i=2}^{n}(-a_{ii}E_{11} + a_{ii}E_{ii}) + \sum_{i \neq j} a_{ij}E_{ij}$$

For each $i \geq 2$, $-a_{ii}E_{11} + a_{ii}E_{ii}$ is similar to $-a_{ii}E_{11} + a_{ii}E_{22}$, which is a sum of involutions by Lemma 4.1.1(4). Hence $\sum_{i=2}^{n}(-a_{ii}E_{11} + a_{ii}E_{ii})$ is a sum of involutions. Moreover, $\sum_{i \neq j} a_{ij}E_{ij}$ is a sum of involutions by Lemma 4.1.1(1). Therefore, to show that A is a sum of involutions, it suffices to show that kE_{11} is a sum of involutions.

(1) If k is even, then kE_{11} is a sum of involutions by Lemma 4.1.1(2); so (1) holds.

(2) The claim is certainly true for even k. If k is odd, then it follows that $kE_{11} = E_{11} + (k-1)E_{11}$. As $k-1$ is even, $(k-1)E_{11}$ is a sum of involutions by (1). On the other hand, as n is odd, E_{11} is a sum of involutions by Lemma 4.1.1(3). Hence, (2) holds. □

The proof of the Proposition 4.1.2 implies the following.

Corollary 4.1.3 *Let R be a ring and $n \geq 1$. The following are equivalent:*

(1) *$E_{11} \in \mathbb{M}_n(R)$ is a sum of involutions;*

(2) *Every matrix $A \in \mathbb{M}_n(R)$ with $tr(A) \in \mathbb{Z} \cdot 1_R$ is a sum of involutions.*

Proof. (2) ⇒ (1) This is clear.

(1) ⇒ (2) Let $A \in \mathbb{M}_n(R)$ with $tr(A) = k \cdot 1_R$. By the proof of Proposition 4.1.2, it suffices to show that kE_{11} is a sum of involutions. Noting that $kE_{11} = E_{11} + (k-1)E_{11}$ and $E_{11} \in \mathbb{M}_n(R)$ is a sum of involutions, kE_{11} is a sum of involutions for any $k \in \mathbb{Z}$. □

We next focus on matrices over integral domains.

Lemma 4.1.4 *Let R be an integral domain. If $A^2 = I$ in $\mathbb{M}_n(R)$, then $tr(A) = k \cdot 1_R$, where $k \in \mathbb{Z}$ with $n \equiv k \pmod 2$.*

Proof. Let Q be the field of quotients of R. Then by [63, Proposition 4], $tr(A) = k \cdot 1_Q$ where $k \in \mathbb{Z}$ with $n \equiv k \pmod 2$. As $1_R = 1_Q$, $tr(A) = k \cdot 1_R$. □

Theorem 4.1.5 *Let R be an integral domain and $A \in \mathbb{M}_n(R)$.*

(1) *If the order n is even, then A is a sum of involutions if and only if $tr(A) \in 2\mathbb{Z} \cdot 1_R$.*

(2) *If the order n is odd, then A is a sum of involution if and only if $tr(A) \in \mathbb{Z} \cdot 1_R$.*

Proof. The necessity follows from Lemma 4.1.4, and the sufficiency follows from Proposition 4.1.2. □

Corollary 4.1.6 *Let R be an integral domain and $n \geq 1$.*

(1) *If n is even, then every matrix in $\mathbb{M}_n(R)$ is a sum of involutions if and only if $R \cong \mathbb{F}_p$ where $p > 2$ is a prime.*

(2) *If n is odd, then every matrix in $\mathbb{M}_n(R)$ is a sum of involutions if and only if $R \cong \mathbb{Z}$ or $R \cong \mathbb{F}_p$ where $p \geq 2$ is a prime.*

Proof. (1) By Theorem 4.1.5, every matrix in $\mathbb{M}_n(R)$ is a sum of involutions if and only if $R = 2\mathbb{Z} \cdot 1_R$, if and only if 2 is a unit of R and R is an image of $\mathbb{Z}$, if and only if $R \cong \mathbb{F}_p$ where $p > 2$ is a prime.

(2) The proof is similar to (1). □

Lemma 4.1.7 *[44, p.192] Let K be a field and $n \geq 2$. Then every matrix $A \in \mathbb{M}_n(K)$ is similar to its rational canonical form $B = \begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_s \end{pmatrix}$, where $s \geq 1$, B_i is a companion matrix of size n_i and $1 \leq n_1 \leq n_2 \leq \cdots \leq n_s$.*

Lemma 4.1.8 *Let R be an integral domain. If $A^3 = A$ in $\mathbb{M}_n(R)$, then $tr(A) \in \mathbb{Z} \cdot 1_R$.*

Proof. We embed the integral domain R into its field of fractions Q, and further into the algebraic closure $\overline{Q}$. From equality $A^3 = A$ we deduce that each eigenvalue $\lambda \in \overline{Q}$ of A satisfies $\lambda^3 = \lambda$, hence it must be $\lambda = 0, 1$ or -1. Since $\mathrm{tr}(A)$ is the sum of all eigenvalues of A in $\overline{Q}$ counted with multiplicities, it follows that $\mathrm{tr}(A) = k \cdot 1_{\overline{Q}} = k \cdot 1_R$ for some $k \in \mathbb{Z}$. □

Note that tripotents are a common generalization of idempotents and involutions.

Theorem 4.1.9 *Let R be an integral domain and $n \geq 1$. Then $A \in \mathbb{M}_n(R)$ is a sum of tripotents if and only if $tr(A) \in \mathbb{Z} \cdot 1_R$.*

Proof. The necessity is by Lemma 4.1.8. For the sufficiency, let $A \in \mathbb{M}_n(R)$ with $tr(A) = k \cdot 1_R$ for some $k \in \mathbb{Z}$. If k is even, then A is a sum of involutions by Proposition 4.1.2. If k is odd, then $tr(A - E_{11}) = (k-1) \cdot 1_R$ with $k-1$ even, so $A - E_{11}$ is a sum of involutions, and hence, $A = E_{11} + (A - E_{11})$ is a sum of idempotent and involutions. $\square$

Corollary 4.1.10 *Let R be an integral domain and $n \geq 1$. Then every matrix in $\mathbb{M}_n(R)$ is a sum of tripotents if and only if $R \cong \mathbb{Z}$ or $R \cong \mathbb{F}_p$ where $p \geq 2$ is a prime.*

Corollary 4.1.11 *Let $n \geq 1$ and $k \geq 2$. Then every matrix in $\mathbb{M}_n(\mathbb{Z}_k)$ is a sum of tripotents.*

Proof. Every matrix in $\mathbb{M}_n(\mathbb{Z})$ is a sum of tripotents by Corollary 4.1.10 and, furthermore, $\mathbb{M}_n(\mathbb{Z}_k)$ is an homomorphic image of $\mathbb{M}_n(\mathbb{Z})$. So every matrix in $\mathbb{M}_n(\mathbb{Z}_k)$ is a sum of tripotents. $\square$

4.2 Matrices over rings as sums of two tripotents

The following fact is implicit in the proof of [82, Theorem 5.2].

Lemma 4.2.1 *[82] If* 3 *is a sum of two tripotents in a ring* R*, then* $2^3{\cdot}3{\cdot}5 = 0$ *in* R*.*

Proof. Write $3 = e + f$ where e, f are tripotents in R. Then

$$8(e+f) = 24 = 3^3 - 3 = (e+f)^3 - (e+f) = 3e^2f + 3ef^2.$$

Multiplying both sides by ef gives $8e^2f + 8ef^2 = 3ef^2 + 3e^2f$, that is , $5(e^2f + ef^2) = 0$. So $2^3 \cdot 3 \cdot 5 = 5 \cdot 24 = 3 \cdot 5(e^2f + ef^2) = 0$. □

Let R be a ring. The $n \times n$ companion matrix associated with a_0, a_1, $\cdots$, $a_{n-1} \in R$ is of the form

$$C(P) := \begin{pmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_{n-2} \\ 0 & 0 & \cdots & 1 & a_{n-1} \end{pmatrix}.$$

Lemma 4.2.2 *Let* R *be a ring and* $n \geq 2$*. For any idempotent* $(n-1) \times (n-1)$ *block* E *and for any* $(n-1) \times 1$ *block* X *over* R*, the* $n \times n$ *matrices*

$$\begin{pmatrix} E & X \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -E & X \\ 0 & 1 \end{pmatrix}$$

are tripotents.

Proof. The proof is a straightforward verification. □

Lemma 4.2.3 *Let R be a ring and let* C(P) *be an $n \times n$ companion matrix associated with a_0, a_1, $\cdots$, $a_{n-1} \in R$. If $a_{n-1} \in \{-2, -1, 0, 1, 2\}$, then* C(P) *is a sum of two tripotents in $\mathbb{M}_n(R)$.*

Proof. The claim is obviously true if $n = 1$, so we are assuming $n \geq 2$. In each of the cases below we will explicitly decompose $\mathrm{C(P)} = E + F$, where $E^3 = E$ is a trivial verification and $F^3 = F$ is a consequence of Lemma 4.2.2 for each of the cases.

Case 1: n is even. If $a_{n-1} = 1$,

$$C(P) = \begin{pmatrix} 1 & 0 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 1 & 0 \\ & & & 1 & 0 \end{pmatrix} + \begin{pmatrix} -1 & & & & & & a_0 \\ & 0 & 0 & & & & a_1 \\ & 1 & -1 & & & & a_2 \\ & & & \ddots & & & \vdots \\ & & & & 0 & 0 & a_{n-3} \\ & & & & 1 & -1 & a_{n-2} \\ & & & & & & 1 \end{pmatrix}.$$

If $a_{n-1} = 2$,

$$C(P) = \begin{pmatrix} 0 & 0 & & & \\ 1 & 1 & & & \\ & & \ddots & & \\ & & & 0 & 0 \\ & & & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & & & & & & a_0 \\ & -1 & 0 & & & & a_1 \\ & 1 & 0 & & & & a_2 \\ & & & \ddots & & & \vdots \\ & & & & -1 & 0 & a_{n-3} \\ & & & & 1 & 0 & a_{n-2} \\ & & & & & & 1 \end{pmatrix}.$$

If $a_{n-1} = -1$,

$$C(P) = \begin{pmatrix} -1 & 0 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & -1 & 0 \\ & & & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & & & & & & a_0 \\ & 0 & 0 & & & & a_1 \\ & 1 & 1 & & & & a_2 \\ & & & \ddots & & & \vdots \\ & & & & 0 & 0 & a_{n-3} \\ & & & & 1 & 1 & a_{n-2} \\ & & & & & & -1 \end{pmatrix}.$$

If $a_{n-1} \in \{0, -2\}$,

$$C(P) = \begin{pmatrix} 0 & 0 & & & & & \\ 1 & -1 & & & & & \\ & & \ddots & & & & \\ & & & 0 & 0 & & \\ & & & 1 & -1 & & \\ & & & & & 0 & 0 \\ & & & & & 1 & \pm 1 \end{pmatrix} + \begin{pmatrix} 0 & & & & & & a_0 \\ & 1 & 0 & & & & a_1 \\ & 1 & 0 & & & & a_2 \\ & & & \ddots & & & \vdots \\ & & & & 1 & 0 & a_{n-3} \\ & & & & 1 & 0 & a_{n-2} \\ & & & & & & -1 \end{pmatrix}.$$

Case 2: $n \geq 3$ is odd. If $a_{n-1} = 1$,

$$C(P) = \begin{pmatrix} 0 & & & & & \\ & 1 & 0 & & & \\ & 1 & 0 & & & \\ & & & \ddots & & \\ & & & & 1 & 0 \\ & & & & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & & & & a_0 \\ 1 & -1 & & & & a_1 \\ & & \ddots & & & \vdots \\ & & & 0 & 0 & a_{n-3} \\ & & & 1 & -1 & a_{n-2} \\ & & & & & 1 \end{pmatrix}.$$

If $a_{n-1} = 2$,

$$C(P) = \begin{pmatrix} 1 & & & & & & \\ & 0 & 0 & & & & \\ & 1 & 1 & & & & \\ & & & \ddots & & & \\ & & & & & 0 & 0 \\ & & & & & 1 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 & & & & a_0 \\ 1 & 0 & & & & a_1 \\ & & \ddots & & & \vdots \\ & & & -1 & 0 & a_{n-3} \\ & & & 1 & 0 & a_{n-2} \\ & & & & & 1 \end{pmatrix}.$$

If $a_{n-1} = -1$,

$$C(P) = \begin{pmatrix} 0 & & & & & & \\ & -1 & 0 & & & & \\ & 1 & 0 & & & & \\ & & & \ddots & & & \\ & & & & & -1 & 0 \\ & & & & & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & & & & a_0 \\ 1 & 1 & & & & a_1 \\ & & \ddots & & & \vdots \\ & & & 0 & 0 & a_{n-3} \\ & & & 1 & 1 & a_{n-2} \\ & & & & & -1 \end{pmatrix}.$$

If $a_{n-1} \in \{0, -2\}$,

$$C(P) = \begin{pmatrix} -1 & & & & & & & \\ & 0 & 0 & & & & & \\ & 1 & -1 & & & & & \\ & & & \ddots & & & & \\ & & & & 0 & 0 & & \\ & & & & 1 & -1 & & \\ & & & & & & 0 & 0 \\ & & & & & & 1 & \pm 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & & & & a_0 \\ 1 & 0 & & & & a_1 \\ & & \ddots & & & \vdots \\ & & & 1 & 0 & a_{n-3} \\ & & & 1 & 0 & a_{n-2} \\ & & & & & -1 \end{pmatrix}.$$

□

Theorem 4.2.4 *Let R be an integral domain and $n \geq 1$. Then every matrix* *in* $\mathbb{M}_n(R)$ *is a sum of two tripotents if and only if* $R \cong \mathbb{F}_p$ *for* $p = 2, 3,$ *or* 5.

Proof. ($\Rightarrow$) Suppose that every matrix in $\mathbb{M}_n(R)$ is a sum of two tripotents. Then by Lemma 4.2.6, $2^3 \cdot 3 \cdot 5 = 0$ in R, so $\mathrm{ch}(R) = p$ where $p = 2, 3$, or 5. Let $a \in R$. Then $aE_{11} = E + F$ where $E^3 = E$ and $F^3 = F$. By Lemma 4.1.8, $a = tr(aE_{11}) = tr(E) + tr(F) \in \mathbb{Z} \cdot 1_R$. So, $R = \mathbb{Z} \cdot 1_R \cong \mathbb{F}_p$, with $p = 2, 3$, or 5.

($\Leftarrow$) The sufficiency follows from Lemma 4.1.7 and Lemma 4.2.3. □

We next extend Theorem 4.2.4 from an integral domain to a commutative reduced ring.

Lemma 4.2.5 *[75, Lemma 3.3] Let R be a ring and $n \geq 1$. Then every element of $\mathbb{M}_n(R)$ is a sum of three idempotents if and only if every element of $\mathbb{M}_n(R/I)$ is a sum of three idempotents for all indecomposable factor rings R/I of R.*

Proof. ($\Rightarrow$) This is clear.

($\Leftarrow$) Assume to the contrary that some $(a_{ij}) \in \mathbb{M}_n(R)$ is not a sum of three idempotent matrices. Then

$$\mathcal{F} = \{I \lhd R : (\overline{a_{ij}}) \in \mathbb{M}_n(R/I) \text{ is not a sum of three idempotents}\}$$

is not empty. For a chain $\{I_\lambda\}$ of elements of $\mathcal{F}$, let $I = \cup_\lambda I_\lambda$. Then I is an ideal of R. Assume that $(\overline{a_{ij}}) \in \mathbb{M}_n(R/I)$ is a sum of three idempotents. Then there exist $(\overline{e_{ij}})$, $(\overline{f_{ij}})$, $(\overline{g_{ij}}) \in \mathbb{M}_n(R/I)$ such that

$$(\overline{a_{ij}}) = (\overline{e_{ij}}) + (\overline{f_{ij}}) + (\overline{g_{ij}}), \qquad (4.0)$$
$$(\overline{e_{ij}})(\overline{e_{ij}}) = (\overline{e_{ij}}), \quad (\overline{f_{ij}})(\overline{f_{ij}}) = (\overline{f_{ij}}), \quad (\overline{g_{ij}})(\overline{g_{ij}}) = (\overline{g_{ij}}).$$

Thus, all the following elements are in $\mathbb{M}_n(I)$:

$$(a_{ij}) - (e_{ij}) - (f_{ij}) - (g_{ij}),$$
$$(e_{ij}) - (e_{ij})(e_{ij}), \quad (f_{ij}) - (f_{ij})(f_{ij}), \quad (g_{ij}) - (g_{ij})(g_{ij}).$$

Because $\{I_\lambda\}$ is a chain, there exists some I_λ such that all these elements are in $\mathbb{M}_n(I_\lambda)$. Hence, (4.0) holds in $\mathbb{M}_n(R/I_\lambda)$. So, $(\overline{a_{ij}}) \in \mathbb{M}_n(R/I_\lambda)$ is a sum of three idempotents. This contradiction shows that I is in $\mathcal{F}$. So $\mathcal{F}$ is an inductive set. By Zorn's Lemma, $\mathcal{F}$ has a maximal element, say I. We next show that R/I is indecomposable. In fact, if R/I is decomposable, then there exist ideals I_1, I_2 of R such that $I \subsetneqq I_k \subsetneqq R (k = 1, 2)$, $R = I_1 + I_2$ and $I_1 \cap I_2 = I$. So we have the isomorphism

$$R/I \cong R/I_1 \times R/I_2 \quad \text{via } r + I \longmapsto (r + I_1, r + I_2),$$

which induces an isomorphism

$$\mathbb{M}_n(R/I) \cong \mathbb{M}_n(R/I_1) \times \mathbb{M}_n(R/I_2) \;\; \text{via } r_{ij} + I \longmapsto ((r_{ij} + I_1), (r_{ij} + I_2)).$$

By the maximality of I, $(\overline{a_{ij}}) \in \mathbb{M}_n(R/I_k)$ is a sum of three idempotents for $k = 1, 2$. It follows that $(\overline{a_{ij}}) \in \mathbb{M}_n(R/I)$ is a sum of three idempotents. This contradiction shows that R/I is indecomposable. But by the hypothesis, every matrix in $\mathbb{M}_n(R/I)$ is a sum of three idempotents, contradicting that $I \in \mathcal{F}$. □

Similarly, we have the following lemma.

Lemma 4.2.6 *Let R be a ring and $n \geq 1$. Then every element of $\mathbb{M}_n(R)$ is a sum of two tripotents if and only if every element of $\mathbb{M}_n(R/I)$ is a sum of two tripotents for all indecomposable factor rings R/I of R.*

Proof. The proof is similar to that of Lemma 4.2.5. □

Let R be a ring. Notice that for any element $x \in R$, $x^5 = x$ if and only if $R \cong A \times B \times C$, where A is zero or A is Boolean, B is zero or B is a subdirect product of $\mathbb{F}_3$'s, and C is zero or C is a subdirect product of $\mathbb{F}_5$'s (see Lemma 3.1.6).

Theorem 4.2.7 *Let R be a commutative reduced ring and $n \geq 1$. Every matrix in $\mathbb{M}_n(R)$ is a sum of two tripotents if and only if $x^5 = x$ for all $x \in R$.*

Proof. ($\Rightarrow$) Suppose that every matrix in $\mathbb{M}_n(R)$ is a sum of two tripotents. As a reduced ring, R is a subdirect product of integral domains $\{R_\alpha\}$. For each α, $\mathbb{M}_n(R_\alpha)$ is a homomorphic image of $\mathbb{M}_n(R)$, so every matrix in $\mathbb{M}_n(R_\alpha)$ is a sum of two tripotents. Thus $R_\alpha \cong \mathbb{F}_p$ with $p \in \{2, 3, 5\}$ by Theorem 4.2.4. So, $x^5 = x$ for all $x \in R_\alpha$ and hence $x^5 = x$ for all $x \in R$.

($\Leftarrow$) Suppose that $x^5 = x$ for all $x \in R$. Let S be an indecomposable factor ring of R. Then $x^5 = x$ for all $x \in S$. For any nonzero $a \in S$, a^4 is a nonzero idempotent, so $a^4 = 1$ and hence a is invertible. Thus, S is a field of at most 5 elements, and it is easily seen that S is isomorphic to either $\mathbb{F}_2$ or $\mathbb{F}_3$ or $\mathbb{F}_5$. So, by Theorem 4.2.4, every matrix in $\mathbb{M}_n(S)$ is a sum of two tripotents. Thus, by Lemma 4.2.6, every matrix in $\mathbb{M}_n(R)$ is a sum of two tripotents. □

Statement 4.2.8 *The materials of Section* 4.1 *and Section* 4.2 *are taken from [80]. My contribution in this joint work is to determine the integral domains over which every matrix can be decomposed as a sum of two tripotents as in Section* 4.2.

4.3 Some other decompositions of matrices

Recall that the following two results have been obtained in [75]:

- Let F be a field and $n \geq 1$. Then every matrix in F is a sum of three idempotents if and only if $F \cong \mathbb{F}_2$ or $F \cong \mathbb{F}_3$.
- Let R be a commutative ring and $n \geq 1$. Then every matrix in $\mathbb{M}_n(R)$ is a sum of three involutive matrices if and only if R is a subdirect product of $\mathbb{F}_3$'s.

In Section 4.2 we have shown that every matrix ring over an integral domain R is a sum of two tripotents if and only if $R \cong \mathbb{F}_p$ for $p = 2, 3$, or 5. As a more generalized case of the above result, we have the following:

Question 4.3.1 *Let R be a field (or a commutative ring). When is every matrix in $\mathbb{M}_n(R)$ a sum of three tripotents?*

In this section we discuss some progress towards this question.

Lemma 4.3.2 *If 4 is a sum of two idempotents and an involution in a ring R, then $2^4 \cdot 3^3 \cdot 5 = 0$.*

Proof. Write $4 = e + f + g$ where $e^2 = e$, $f^2 = f$ and $g^2 = 1$. We have $16 - 7e = (4 - e)^2 = (f + g)^2 = f + fg + gf + 1$. So

$$15 - 7e - f = fg + gf.$$

It follows that $15 - 7(4 - f - g) - f = fg + gf$. That is

$$-13 + 6f + 7g = fg + gf.$$

So $f(-13 + 6f + 7g) = f(fg + gf)$, i.e.,$-7f + 6fg = fgf$. Moreover,

$$(-13 + 6f + 7g)f = (fg + gf)f, \quad i.e., \quad -7f + 6gf = fgf.$$

It follows that

$$6fg = 6gf.$$

Thus $6(-7f + 6fg) = 6(fgf) = f(6gf) = 6fg$. So $30fg = 42f$, and hence $30f = 42fg$. By $30f = 42fg = 30fg + 12fg = 42f + 12fg$, we have $12fg = -12f$. Thus $2 \cdot 42f = 60fg = 5 \cdot 12fg = 5 \cdot (-12f) = -60f$. Hence $144f = 0$. Similarly, $144e = 0$.

As $4 = e + f + g$, we have

$$17 - 8g = (4 - g)^2 = (e + f)^2 = e + ef + fe + f = (4 - g) + ef + fe.$$

Hence, $13-7g = ef+fe$. So $144(13-7g) = 144(ef+fe) = 144ef+144fe = 0$. It follows that $7 \cdot 144g = 13 \cdot 144$, and

$$7 \cdot 144 \cdot 4 = 7 \cdot 144(e + f + g) = 7 \cdot 144(e + f) + 7 \cdot 144g = 13 \cdot 144.$$

Thus $2160 = 2^4 \cdot 3^3 \cdot 5 = 0$. □

Let E_n and F_n be the $n \times n$ block diagonal matrices given by

$$E_n = \begin{pmatrix} 1 & 0 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 1 & 0 \\ & & & 1 & 0 \end{pmatrix} \quad \textit{and} \quad F_n = \begin{pmatrix} 0 & 0 & & & \\ 1 & 1 & & & \\ & & \ddots & & \\ & & & 0 & 0 \\ & & & 1 & 1 \end{pmatrix}.$$

It is clear that E_n and F_n are idempotents and both always have even orders.

Set

$$V_0 = \begin{pmatrix} -1 & & & & a_0 \\ & -1 & & & a_1 \\ & & \ddots & & \vdots \\ & & & -1 & a_{n-2} \\ & & & & 1 \end{pmatrix}$$

It is easily to verify that V_0 is an involution. We denote $\mathbf{0}$ by matrix with only zero entries. We have the following lemmas.

Lemma 4.3.3 *For $n \geq 1$, every $n \times n$ companion matrix $C(P)$ over $\mathbb{F}_2$ is a sum of two idempotents and an involution.*

Proof. The claim is true for $n = 1$. Assume that $n \geq 2$. As shown below, $C(P)$ is a sum of two idempotents and an involution.

Case 1: If n is even, then

$$C(P) = E_n + \begin{pmatrix} 0 & \mathbf{0} & 0 \\ \mathbf{0} & E_{n-2} & \mathbf{0} \\ 0 & \mathbf{0} & 1 + a_{n-1} \end{pmatrix} + V_0.$$

Case 2: If n is odd, then

$$C(P) = \begin{pmatrix} E_{n-1} & \mathbf{0} \\ \mathbf{0} & 1 + a_{n-1} \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & E_{n-1} \end{pmatrix} + V_0.$$

□

Lemma 4.3.4 *For each number $n \geq 1$, every $n \times n$ companion matrix $C(P)$ over $\mathbb{F}_3$ is a sum of two idempotents and an involution.*

Proof. The claim is true if $n = 1$. Assume that $n \geq 2$. As shown below, $C(P)$ is a sum of two idempotents and an involution.

Case 1**:** n is even.

If $a_{n-1} \neq 0$, then

$$C(P) = E_n + \begin{pmatrix} 0 & \mathbf{0} & 0 \\ \mathbf{0} & E_{n-2} & \mathbf{0} \\ 0 & \mathbf{0} & a_{n-1} - 1 \end{pmatrix} + V_0.$$

If $a_{n-1} = 0$, then

$$C(P) = \begin{pmatrix} 1 & \mathbf{0} & 0 \\ \mathbf{0} & F_{n-2} & \mathbf{0} \\ 0 & \mathbf{0} & 1 \end{pmatrix} + F_n + V_0.$$

Case 2: n is odd.

If $a_{n-1} \neq 0$, then

$$C(P) = \begin{pmatrix} E_{n-1} & \mathbf{0} \\ \mathbf{0} & a_{n-1} - 1 \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & E_{n-1} \end{pmatrix} + V_0.$$

If $a_{n-1} = 0$, then

$$C(P) = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & F_{n-1} \end{pmatrix} + \begin{pmatrix} F_{n-1} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} + V_0.$$

□

Lemma 4.3.5 *For $n \geq 1$, every $n \times n$ companion matrix $C(P)$ over $\mathbb{F}_5$ is a sum of two idempotents and an involution.*

Proof. **Case** 1: n is even.

If $a_{n-1} = 0$, then

(i) for $n = 2$, $\begin{pmatrix} 0 & a_0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a_0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$

(ii) for $n = 4$,

$$C(P) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & -1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1+a_2) \\ 0 & 1 & 1 & 1+a_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$+ \begin{pmatrix} -1 & 0 & 0 & a_0 \\ 0 & -1 & 3 & 2+a_1+a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(iii) for $n > 4$

$$C(P) = \begin{pmatrix} F_{n-4} & \mathbf{0} \\ \mathbf{0} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & -1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & F_{n-4} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{pmatrix} 0 & 0 & -(1+a_{n-2}) \\ 1 & 1 & 1+a_{n-2} \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix}$$

$$+ \begin{pmatrix} -1 & & & & & & a_0 \\ & -1 & & & & & a_1 \\ & & \ddots & & & & \vdots \\ & & & -1 & & & a_{n-4} \\ & & & & -1 & 3 & 2+a_{n-2}+a_{n-3} \\ & & & & & 1 & 0 \\ & & & & & & 1 \end{pmatrix}.$$

If $a_{n-1} \in \{1, 2\}$, then

$$C(P) = E_n + \begin{pmatrix} 0 & \mathbf{0} & 0 \\ \mathbf{0} & E_{n-2} & \mathbf{0} \\ 0 & \mathbf{0} & a_{n-1} - 1 \end{pmatrix} + V_0.$$

If $a_{n-1} = 0$, then

$$C(P) = \begin{pmatrix} 1 & \mathbf{0} & 0 \\ \mathbf{0} & F_{n-2} & \mathbf{0} \\ 0 & \mathbf{0} & 1 \end{pmatrix} + F_n + V_0$$

If $a_{n-1} = -1$, then

$$C(P) = \begin{pmatrix} E_{n-2} & \mathbf{0} \\ \mathbf{0} & \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & E_{n-4} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & a_{n-2}+2 \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix}$$

$$+ \begin{pmatrix} -1 & & & & & a_0 \\ & -1 & & & & a_1 \\ & & \ddots & & & \vdots \\ & & & -1 & & a_{n-3} \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}.$$

Case 2: n is odd.
If $a_{n-1} = 0$, then
(i) for $n = 3$,

$$C(P) = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 0 & 1 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -(1+a_1) \\ 1 & 1 & 1+a_1 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 3 & 2+a_0+a_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(ii) for $n > 3$,

$$C(P) = \begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & F_{n-5} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & -1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} F_{n-3} & \mathbf{0} \\ \mathbf{0} & \begin{pmatrix} o & 0 & -(1+a_{n-2}) \\ 1 & 1 & 1+a_{n-2} \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix}$$

$$+ \begin{pmatrix} -1 & & & & & & a_0 \\ & -1 & & & & & a_1 \\ & & \ddots & & & & \vdots \\ & & & -1 & & & a_{n-4} \\ & & & & -1 & 3 & 2 + a_{n-2} + a_{n-3} \\ & & & & & 1 & 0 \\ & & & & & & 1 \end{pmatrix}.$$

If $a_{n-1} \in \{1, 2\}$, then

$$C(P) = \begin{pmatrix} E_{n-1} & \mathbf{0} \\ \mathbf{0} & a_{n-1} - 1 \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & E_{n-1} \end{pmatrix} + V_0.$$

If $a_{n-1} = 3$, then

$$C(P) = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & F_{n-1} \end{pmatrix} + \begin{pmatrix} F_{n-1} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} + V_0.$$

If $a_{n-1} = -1$, then

$$C(P) = \begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & E_{n-3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} E_{n-3} & \mathbf{0} \\ \mathbf{0} & \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & a_{n-2}+2 \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix}$$

$$+ \begin{pmatrix} -1 & & & & & a_0 \\ & -1 & & & & a_1 \\ & & \ddots & & & \vdots \\ & & & -1 & & a_{n-3} \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}.$$

□

Theorem 4.3.6 *Let R be an integral domain and $n \geq 1$. Then the following are equivalent:*

(1) *Every matrix in $\mathbb{M}_n(R)$ is a sum of two idempotents and an involution.*

(2) *Every matrix in $\mathbb{M}_n(R)$ is a sum of two tripotents.*

(3) *$R \cong \mathbb{F}_p$ where $p = 2, 3,$ or 5.*

Proof. (2) ⇔ (3) This follows from Theorem 4.2.4.

(1) ⇒ (3) If every matrix in $\mathbb{M}_n(R)$ is a sum of two idempotents and an involution, then $2^4 \cdot 3^3 \cdot 5 = 0$ in R by Lemma 4.3.2, so ch(R) $= p$, where $p = 2, 3$, or 5. Let $a \in R$. Then $aE_{11} = E_1 + E_2 + V$ where $E_1^2 = E_1$, $E_2^2 = E_2$, and $V^2 = I$. As idempotents and involutions are tripotents, by Lemma 4.1.8, $a = tr(aE_{11}) = tr(E_1) + tr(E_2) + tr(V) \in \mathbb{Z} \cdot 1_R$. So $R = \mathbb{Z} \cdot 1_R \cong \mathbb{F}_p$, with $p = 2, 3$, or 5.

(3) ⇒ (1) This follows from Lemma 4.3.3, 4.3.4, and Lemma 4.3.5. □

To discuss the particular cases of being a sum of three tripotent matrices, we have the following observations.

Lemma 4.3.7 (1) *Every matrix in* $\mathbb{M}_2(\mathbb{F}_7)$ *is a sum of an idempotent and two tripotents.*

(2) *Not every matrix in* $\mathbb{M}_2(\mathbb{F}_7)$ *is a sum of two idempotents and a tripotent.*

(3) *Not every matrix in* $\mathbb{M}_2(\mathbb{F}_{11})$ *is a sum of an idempotent and two tripotents.*

Proof. (1) Let $A \in \mathbb{M}_2(\mathbb{F}_7)$. Because of Lemma 4.1.7, to show A is a sum of an idempotent and two tripotents, we can assume that

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad or \quad A = \begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix}.$$

First, assume $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Every element in $\mathbb{F}_7$, except 4, is a sum of an idempotent and two tripotents. Thus, to show A is a sum of an idempotent and two tripotents, we can assume that at least one of a and b is 4, and, without loss of generality, that $a = 4$. As below, we can write $A = \begin{pmatrix} 4 & 0 \\ 0 & b \end{pmatrix}$ as a sum of an idempotent and two tripotents:

$$\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -2 & 1 \\ 1 & -4 \end{pmatrix} + \begin{pmatrix} -2 & -1 \\ -1 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 4 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ -1 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 4b-2 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 4b-2 \end{pmatrix} \text{ if } b = 2 \text{ or } -1,$$

$$\begin{pmatrix} 4 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 1-b & -1 \\ -1 & b \end{pmatrix} + \begin{pmatrix} b-4 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ if } b = 3 \text{ or } -2,$$

$$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 4 & -1 \\ -2 & 4 \end{pmatrix}.$$

Next, assume $A = \begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix}$. As below, we can write $A = \begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix}$ as a sum of an idempotent and two tripotents:

$$\begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \text{ if } b = 0, 1 \text{ or } -1,$$
$$\begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & b-2 \end{pmatrix} \text{ if } b = 2 \text{ or } 3,$$
$$\begin{pmatrix} 0 & a \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix},$$
$$\begin{pmatrix} 0 & a \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2+a \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}.$$

(2) If $G = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is a tripotent in $\mathbb{M}_2(\mathbb{F}_7)$, then $a, b \in \{0, \pm 1\}$; so $\mathrm{tr}(G) \in \{0, \pm 1, \pm 2\}$. If $G = \begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix}$ is a tripotent, then

$$G \in \{\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\};$$

so $\mathrm{tr}(G) \in \{0, \pm 1\}$. As every tripotent in $\mathbb{M}_2(\mathbb{F}_7)$ is similar to either $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ or $\begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix}$, we deduce that $\mathrm{tr}(G) \in \{0, \pm 1, \pm 2\}$ for every tripotent G in $\mathbb{M}_2(\mathbb{F}_7)$. Next we show that $A := \begin{pmatrix} 0 & 0 \\ 1 & 5 \end{pmatrix} \in \mathbb{M}_2(\mathbb{F}_7)$ is not a sum of two idempotents and a tripotent.

Assume $A = E + F + G$, where $E^2 = E$, $F^2 = F$ and $G^3 = G$.

If $E = 0$, then $A = F + G$, and F must be a non-trivial idempotent. So

$\mathrm{tr}(F) = 1$, which implies $\mathrm{tr}(G) = 4$, a contradiction.

If $E = I_2$, then F must be a non-trivial idempotent. So $\mathrm{tr}(F) = 1$, and hence $\mathrm{tr}(G) = 2$. Thus, $G = I_2$, which is impossible.

So, E is a non-trivial idempotent. Similarly, F is a non-trivial idempotent. Thus, $\mathrm{tr}(E) = \mathrm{tr}(F) = 1$, so $\mathrm{tr}(G) = \mathrm{tr}(A) - \mathrm{tr}(E) - \mathrm{tr}(F) = 3$, which is impossible.

(3) As seen in the proof of (2), we have that $\mathrm{tr}(G) \in \{0, \pm 1, \pm 2\}$ for every tripotent G in $\mathbb{M}_2(\mathbb{F}_{11})$. Moreover, $\mathrm{tr}(E) \in \{0, 1, 2\}$ for every idempotent E. Next we show that $A := \begin{pmatrix} 0 & 0 \\ 1 & 6 \end{pmatrix} \in \mathbb{M}_2(\mathbb{F}_{11})$ is not a sum of an idempotent and two tripotents.

Assume that $A = E + F + G$, where $E^2 = E$, $F^3 = F$ and $G^3 = G$. Then $6 = \mathrm{tr}(E) + \mathrm{tr}(F) + \mathrm{tr}(G)$, and this happens only when $\mathrm{tr}(E) = \mathrm{tr}(F) = \mathrm{tr}(G) = 2$, or equivalently, $E = F = G = I_2$. But this is clearly impossible. $\square$

We end this section by proving the following lemma.

Lemma 4.3.8 *If R is a ring such that $4 = e+f+g$ where e, f are idemotents and g is a tripotent, then $2^9 \cdot 3^6 \cdot 5^2 \cdot 7 = 0$ in R.*

Proof. We have $g^2 = (4 - e - f)^2 = 16 - 7e - 7f + ef + fe$, so

$$\begin{aligned} 0 = g(g^2 - 1) &= (4 - e - f)(15 - 7e - 7f + ef + fe) \\ &= 60 - 36e - 36f + 10ef + 10fe - efe - fef. \end{aligned} \tag{4.1}$$

Multiplying (4.1) on the left by e gives

$$24e - 26ef + 9efe = efef, \tag{4.2}$$

and multiplying (4.2) on the right by f gives

$$2ef = 8efef. \tag{4.3}$$

Similarly, we have

$$2fe = 8fefe. \tag{4.4}$$

We calculate

$$\begin{aligned}0 = g^2(g^2-1) &= (16-7e-7f+ef+fe)(15-7e-7f+ef+fe)\\ &= 240-168e-168f+66ef+66fe-13(efe+fef)+efef+fefe.\end{aligned}$$

In view of (4.1), (4.3) and (4.4), we have

$$\begin{aligned}0 =&240\cdot 8-168\cdot 8e-168\cdot 8f+66\cdot 8ef+66\cdot 8fe\\ &-13\cdot 8\big(60-36e-36f+10ef+10fe\big)+2ef+2fe.\end{aligned}$$

That is,

$$4320 = 2400e+2400f-510ef-510fe. \tag{4.5}$$

Multiplying (4.5) by e on the left and, respectively, on the right gives

$$\begin{aligned}1920e &= 1890ef-510efe\\ 1920e &= 1890fe-510efe.\end{aligned} \tag{4.6}$$

It follows that

$$1890ef = 1890fe.$$

So, $510\cdot 63efe = 17\cdot 1890efe = 17\cdot 1890ef = 510\cdot 63ef$, and hence, by (4.6),

$$1920\cdot 63e = 1890\cdot 63ef-510\cdot 63ef = 1380\cdot 63ef. \tag{4.7}$$

Multiplying (4.7) by f on the right gives $540\cdot 63ef = 0$. Hence, by (4.7),

$$1920\cdot 63\cdot 9e = 1380\cdot 63\cdot 9ef = 23\cdot 63\cdot 540ef = 0. \tag{4.8}$$

Similarly,

$$1920 \cdot 63 \cdot 9f = 0. \tag{4.9}$$

Thus, $4 \cdot 1920 \cdot 63 \cdot 9 = 1920 \cdot 63 \cdot 9(e + f + g) = 1920 \cdot 63 \cdot 9g$. On the other hand,

$$g = g^3 = (4 - e - f)^3 = 64 - 37e - 37f + 10ef + 10fe - efe - fef.$$

So, by (4.8) and (4.9), $4 \cdot 1920 \cdot 63 \cdot 9 = 1920 \cdot 63 \cdot 9g = 1920 \cdot 63 \cdot 9 \cdot 64$, showing that $60 \cdot 1920 \cdot 63 \cdot 9 = 0$. That is, $2^9 \cdot 3^6 \cdot 5^2 \cdot 7 = 0$. $\square$

Remark 4.3.9 *We do not have a result similar to Lemma 4.3.8 regarding to decomposing matrices into sum of three tripotents.*

Chapter 5

Rings in which every element is left annihilator-stable

In this chapter we discuss rings in which every element is left uniquely generated (or left UG rings). Canfell's theorem [20] asserts that a ring R is left UG if and only if every element of R is annihilator-stable (R is left AS ring). We will explore the properties of left annihilator-stable elements and compare them with left uniquely generated elements. We present the necessary and sufficient condition for a formal triangular matrix ring to be left AS. As consequences, two open questions of Nicholson [66] are answered.

5.1 Left AS elements and left UG elements

In this section, we compare left AS elements with left UG elements. As a main result, we show that the left AS elements in a ring form a multiplicatively closed set, answering a question of Nicholson [66] in the affirmative. We start with the following notions.

Definition 5.1.1 (1) *An element a in a ring R is left annihilator-stable (or left* AS*) if, whenever $Ra + \mathbf{l}(b) = R$ with $b \in R$, $a - u \in \mathbf{l}(b)$ for a unit u in*

R, and the ring R is a left AS *ring if each of its elements is left* AS*.*

(2) *An element a in a ring R is left uniquely generated (or left* UG*) if, whenever $Ra = Rb$, $a = ub$ for a unit u in R, and the ring R is a left* UG *ring if each of its elements is left* UG*.*

(3) *An element a in a ring R has left stable range* 1 (SR1) *(or a is called a* SR1 *element) if, whenever $Ra + Rb = R$ for some $b \in R$, $a - u \in Rb$ for a unit u in R, and the ring R has left* SR1 *if each of its elements has left* SR1*.*

Naturally, left SR1 elements are left AS; the converse is not true because $\mathbb{Z}$ is left AS but does not have SR1. Kaplansky proved in [47, Lemma 2.1] that a ring R is left UG if every zero-divisor is in $J(R)$, for instance, domains and local rings.

The following result was proved by Canfell [20, Corollary 4.4] (also see [66, Theorem 5]).

Theorem 5.1.1 *[20] A ring is left* UG *if and only if it is left* AS*.*

Proof. Suppose R is a left UG ring. If $Ra + \mathbf{l}(b) = R$, then $Rab = Rb$, so $ab = ub$ with $u \in U(R)$. Hence, $a - u \in \mathbf{l}(b)$. Conversely, suppose R is a left AS ring. If $Ra = Rb$, then $a = pb$ and $b = qa$ for some $p, q \in R$. Then $a = pqa$, so $Rq + \mathbf{l}(a) = R$. It follows that $q - v \in \mathbf{l}(a)$ for some $v \in U(R)$. Therefore, $b = va$. □

In [47] Kaplansky also gave an example of commutative ring that is not UG.

Example 5.1.2 *[47, Page 466] Let $K_5 = \{(n, \lambda) \in \mathbb{Z} \times \mathbb{Z}_5[x] \mid \lambda(\bar{0}) = \bar{n}\}$, where $\bar{k} = k + 5\mathbb{Z}$ in $\mathbb{Z}_5$. Then $(0, x)$ and $(0, \bar{2}x)$ generate the same ideal of K_5 but are not unit multiples. Hence K_5 is a commutative, noetherian, reduced ring (rings with nonzero nilpotents) that is not* UG *(and so not* AS *by Theorem 5.1.1).*

Further, Nicholson considered an extension of Kaplansky's ring in Example 5.1.2 as follows.

Example 5.1.3 *[66, Example 8] Let $p \in \mathbb{Z}$ be a prime, write $D = \mathbb{Z}_p[x]$, write $\bar{k} = k + p\mathbb{Z}$ for all $k \in \mathbb{Z}$, and define*

$$K_p = \{(n, \lambda) | n \in \mathbb{Z}, \lambda \in D, \lambda(\bar{0}) = \bar{n}\}, \quad \textit{a subring of } \mathbb{Z} \times D.$$

Then K_2 and K_3 are both AS. *But if $p \geq 5$ then K_p is not* AS *(and so not* UG*).*

Proof. Observe first that the units of K_p are $(1, \bar{1})$ and $(-1, \overline{-1})$. Indeed, if $u = (n, \lambda) \in K_p$ is a unit then $n = \pm 1$ in $\mathbb{Z}$ and $\lambda = \bar{\omega} \neq 0$ in $\mathbb{Z}_p$. But $\bar{n} = \bar{\omega}$ because $\lambda(\bar{0}) = \bar{n}$.

Let $a = (n, \lambda) \in K_p$ and $b = (m, \gamma) \in K_p$ satisfy $K_p a + \mathbf{l}(b) = K_p$. We must find a unit $\varepsilon \in K_p$ such that $a - \varepsilon \in \mathbf{l}(b)$. There exist $(d, \pi) \in K_p$ and $(k, \beta) \in K_p$ such that $(d, \pi)(n, \lambda) + (k, \beta) = (1, \bar{1})$ and $(k, \beta)(m, \gamma) = 0$. Hence

$$dn + k = 1 \text{ in } \mathbb{Z},\ \pi\lambda + \beta = \bar{1} \text{ in } D,\ km = 0 \text{ in } \mathbb{Z} \text{ and } \beta\gamma = \bar{0} \text{ in } D. \quad (*)$$

Case I: $\gamma = \bar{0}$. If $m = 0$ then $b = 0$ so $a - \varepsilon \in K_p = \mathbf{l}(b)$ for any unit $\varepsilon \in K_p$. So assume $m \neq 0$. Then $\mathbf{l}(b) = \{(0, \varphi) | \varphi(\bar{0}) = \bar{0}\}$. Also $k = 0$ by $(*)$, whence $dn = 1$ in $\mathbb{Z}$. Thus $n = \pm 1$, so $a = (\pm 1, \lambda)$. If $n = 1$ then $\lambda(\bar{0}) = \bar{1}$ so $a - (1, \bar{1}) = (0, \lambda - \bar{1}) \in \mathbf{l}(b)$. Similarly, if $n = -1$ then $a - (-1, \overline{-1}) = (0, \lambda + \bar{1}) \in \mathbf{l}(b)$. Hence a is AS in both cases.

Case II: $\gamma \neq \bar{0}$. Then $\beta = 0$ by $(*)$, so $\pi\lambda = \bar{1}$ in D. Thus λ is a unit in D, say $\lambda = \bar{k}$, $\bar{0} \neq \bar{k} \in \mathbb{Z}_p$. Hence $a = (n, \bar{k}) = (n, \bar{n})$ because $a \in K_p$. If $u \in K_p$ denotes a unit then $a - u$ becomes

$$a - (1, \bar{1}) = (n - 1, \bar{n} - 1) \text{ or } a - (-1, \overline{-1}) = (n + 1, \bar{n} + \bar{1}).$$

As $b = (m, \gamma)$ and $\gamma \neq 0$, this shows that $a - u \in \mathbf{l}(b)$ for some unit u if $n = \pm 1$. This is always the case if $p = 2$ or $p = 3$, so this shows that K_2 and K_3 are AS. But if $p \geq 5$ then $(n, \bar{n})$ is not AS if $n \in \{2, 3, \cdots, p - 2\}$, so K_p

is not AS for $p \geq 5$, therefore not UG. $\square$

Note that for a unit-regular element $a \in R$, $a = aua$ for some unit u in R. It follows that $ua = (ua)^2$. Write $v = u^{-1}$ and $e = ua$, then $a = ve$. Therefore, an element in a ring is unit-regular if and only if it is a product of an idempotent and a unit (in either order). In [36] Goodearl characterized regular rings that have SR1 as follows.

Theorem 5.1.4 *[36, Proposition 4.12] A regular ring R has* SR1 *if and only if it is unit-regular.*

Proof. First assume that R has SR1. Given any $a \in R$, there exists $x \in R$ such that $a = axa$. As $1 = xa + (1 - xa)$, $R = Ra + R(1 - xa)$; hence, there exists $y \in R$ such that $a + y(1 - xa)$ is a unit of R. Then there is a unit $u \in R$ for which $u[a + y(1 - xa)] = 1$, whence

$$a = axa = a \cdot xa = au[a + y(1 - xa)] \cdot xa = aua.$$

Therefore R is unit-regular.

Conversely, suppose R is unit-regular, and let $Ra + Rb = R$. As $a = ve$ where v is a unit and $e^2 = e$, $Re + Rb = R$. Let $1 - re \in Rb$ where $r \in R$, and define $u := 1 - (1 - e)re$. Then u is a unit, and $e - u = (e - 1) + (1 - e)re = (e - 1)(1 - re) \in Rb$. It follows that $a - w \in Rb$ where $w = vu$ is a unit. Hence, R has SR1. $\square$

We recall a theorem of Marks [59, Theorem] as follows.

Theorem 5.1.5 *[59] A regular ring is unit-regular if and only if it is left* UG

Proof. Suppose R is left UG. For any $x \in R$, choose $y \in R$ such that $x = xyx$; then $Rx = Ryx$ implies that $yx = ux$ for some unit $u \in U(R)$, whence $x = xyx = xux$. So R is unit-regular.

Conversely, assume that R is unit-regular. Let $Ra = Rb$ for some a, $b \in R$. Choose units u, $v \in U(R)$ such that $a = aua$, $b = bvb$. As $a = pb$, $b = qa$ for some p, $q \in R$, $1 - pq \in \mathbf{l}(a) = R(1 - au)$. It follows that $Rq + R(1 - au) = R$. By Theorem 5.1.4, R has SR1, hence there exists some $r \in R$ such that $w := q + r(1 - au)$ is a unit of R. As $b = qa = wa$, R is left UG. $\square$

Thus, Theorem 5.1.5 can be restated as

Corollary 5.1.6 *A regular ring is unit-regular if and only if it is left* AS.

The next result of Nicholson [66], an element-wise version of Corollary 5.1.6, shows the significance of left AS elements.

Theorem 5.1.7 *[66] An element $a \in R$ is unit-regular if and only if it is regular and left* AS.

Proof. Suppose a is regular and left AS. Let $a = axa$ where $x \in R$. We may assume that $xax = x$ too (with x replaced by $x \mapsto xax$ when necessary). It follows that $1 - xa \in \mathbf{l}(x) = \mathbf{l}(xa)$ so $Ra + \mathbf{l}(xa) = R$. As a is left AS, let $a - u \in \mathbf{l}(xa) = \mathbf{l}(x)$ for some unit u in R. Hence $ax = ux$, so $a = axa = uxa$. Thus $u^{-1}a = xa$, and so $au^{-1}a = axa = a$.

Conversely, write $a = ve$ where v is a unit in R and $e^2 = e$. Suppose $Ra + \mathbf{l}(b) = R$. Then $ra + x = rve + x = 1$ where $r \in R$ and $xb = 0$. Then $(rve + x)(1 - e) = 1 - e$ gives $x + (1 - x)e = 1$, hence $Rx + Re = R$. From the proof of Theorem 5.1.4, we can see that idempotents have SR1. Therefore, $e - u \in Rx$ for some unit u. It follows that $eb = ub$ and $ab = vub$, so a is left AS. $\square$

However, when comparing to the above result, regular left UG elements have a different consequence. A ring R is called **directly finite** if $ab = 1$ in R implies $ba = 1$.

Proposition 5.1.8 *If an element $a \in R$ regular and left* UG*, then a is unit-regular; but the converse is false.*

Proof. Write $a = aba$ with $b \in R$, and let $e = ba$. Then $e^2 = e$ and $Ra = Re$. Since a is left UG, $a = ue$ for some unit u in R. So a is unit-regular.

To see the converse fails, consider a ring R that is not directly finite. Thus, there exist $a, b \in R$ such that $ab = 1$ but $ba \neq 1$. Thus, $R = Rb$. Note that $1 \in R$ is unit-regular. We claim that 1 is not left UG. Otherwise, we would have $1 = ub$ for a unit u in R. Then $b = u^{-1}$ is a unit and, as $ab = 1$, it follows that $ba = 1$, a contradiction. □

As noticed by Nicholson [66], left UG and left AS are not equivalent for elements. In fact, in the proof of [66, Theorem 6] (note that [66, Theorem 6] has been corrected in [67]), Nicholson gave a left AS element that is not left UG in a commutative ring. In [6, Example 3.5(2)], the authors showed that in $C(\mathbb{R})$, the ring of all continuous real-valued functions on $\mathbb{R}$, there is a UG element that is not AS.

In order to detect the relation between left UG and left AS, we give the following definition.

Definition 5.1.2 *An element $b \in R$ is called left modified* AS (*or left* MAS) *if $Ra + \mathbf{l}(b) = R$, $a \in R$, implies $a - u \in \mathbf{l}(b)$ for some $u \in U(R)$, and the ring R is left* MAS *if every element in R is left* MAS.

Obviously, a ring is left AS if and only if it is left MAS. Thus, we give the next statement which can be viewed as an element-wise version of Theorem 5.1.1.

Proposition 5.1.9 *An element $b \in R$ is left* UG *if and only if b is left* MAS.

Proof. ($\Rightarrow$). Let $Ra + \mathbf{l}(b) = R$. Then $Rab = Rb$, so, by hypothesis, $b = uab$ for some $u \in U(R)$, i.e., $u^{-1}b = ab$. Thus, $a - u^{-1} \in \mathbf{l}(b)$. So b is left MAS.

($\Leftarrow$). Let $Ra = Rb$. Then $a = xb$ and $b = ya$ where $x, y \in R$. So $b = yxb$ or $(1 - yx) \in l(b)$. Thus, $Ryx + \mathbf{l}(b) = R$, and so $Rx + \mathbf{l}(b) = R$. As b is left

MAS, $x - u \in \mathbf{l}(b)$ for some $u \in U(R)$. Hence, $a = xb = ub$, i.e., $b = u^{-1}a$. So b is left UG. □

Given an element $a \in R$ and any units u, $v \in R$, by [66, Lemma 12], if a is left AS, then uav is left AS; if a has SR1, then uav has SR1. For left UG elements we have the following similar result.

Proposition 5.1.10 *Let a be an element in a ring R. If a is left* UG*, then uav is left* UG *for any units u, $v \in R$.*

Proof. Given any units u, $v \in R$, then it is clear that ua is left UG. Let $Rb + \mathbf{l}(uav) = R$ for some $b \in R$, then $Rb + \mathbf{l}(ua) = R$. Since ua is left UG, it is left MAS. So there exists an unit $w \in R$ such that $b - w \in \mathbf{l}(ua) = \mathbf{l}(uav)$. Hence, uav is left UG by Proposition 5.1.9. □

We write $Z_r(R)$ for the right singular ideal of R and ureg(R) for the set of all unit-regular elements in R. For convenience, let $\mathrm{as}_l(R)$ be the set of all left AS elements in R. Some notable properties of $\mathrm{as}_l(R)$ are proved in [66, Example 13; Lemma 35]: (1) $J(R) \cup Z_r(R) \cup \mathrm{ureg}(R) \subseteq \mathrm{as}_l(R)$; (2) $\mathrm{as}_l(R) + J(R) = \mathrm{as}_l(R)$.

Motivated by the result that the product of two SR1 elements is again SR1 [23, Lemma 17], the following question is raised by Nicholson [66, Question 1]:

Question 5.1.11 *[66] Is the product of two left* AS *elements again left* AS*?*

This question is answered in the affirmative. Noting that the product of two UG elements need not be UG (see [6, Example 3.11]), we give the following surprising contrast.

Theorem 5.1.12 *If $a, b \in R$ are left* AS*, then ab is left* AS.

Proof. Assume that $Rab + \mathbf{l}(c) = R$ with $c \in R$. Then $1 = rab + x$ where $r \in R$ and $x \in \mathbf{l}(c)$, so $c = rabc$. From $Rab + \mathbf{l}(c) = R$, it follows that

$Rb + \mathbf{l}(c) = R$. Since b is left AS, $b - u \in \mathbf{l}(c)$ for some unit $u \in R$. Thus, $bc = uc$, and so $abc = auc$ and $c = rabc = rauc$. Hence, $1 - rau \in \mathbf{l}(c)$, so $Rau + \mathbf{l}(c) = R$. Since a is left AS and u is a unit, au is left AS by [66, Lemma 12]. It follows that $au - v \in \mathbf{l}(c)$ for a unit v in R. Thus, $auc = vc$. As $auc = abc$, we obtain that $abc = vc$, i.e., $ab - v \in \mathbf{l}(c)$. Hence ab is left AS. □

As seen in the next section, Theorem 5.1.12, together with the other known properties of left AS elements, is quite useful in constructing new examples of left AS rings. But directly from Theorem 5.1.12, we see an important fact that $\mathrm{as}_l(R)$ possesses an algebraic structure.

Corollary 5.1.13 *For a ring R, $\mathrm{as}_l(R)$ is a submonoid of the multiplicative monoid $(R, \cdot)$.*

5.2 Going up and going down

Let B be a subring of A with $1_B = 1_A$. As usual, it is interesting to know if the left AS property is a going-up property or a going-down property: That is, (1) if B is left AS, does it imply that every element of B is left AS in A? and (2) if every element of B is left AS in A, does it imply that B is left AS?

By [70, Theorem 14], every commutative ring is embeddable in a commutative UG ring. So, a subring of a left UG ring need not be left UG. Hence, a subring of a left AS ring need not be left AS, and this shows that the left AS property is not a going-down property. For instance, Kapanlsky's ring K_5 is a non-AS subring of the commutative AS ring $\mathbb{Z} \times \mathbb{Z}_5[x]$.

For the going-up, we start with the following simple example:

Example 5.2.1 *Let R be a commutative ring and $a \in R$. If a is left* UG *in R then a is left* UG *in $R[[x]]$.*

Proof. Suppose $R[[x]]a = R[[x]]f(x)$ for some $f(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n + \cdots$ in $R[[x]]$. Then $f(x) = ah(x)$ and $a = g(x)f(x)$ where

$h(x) = d_0 + d_1x + d_2x^2 + \cdots \in R[[x]]$ and $g(x) = c_0 + c_1x + c_2x^2 + \cdots \in R[[x]]$. It follows that $a = (c_0 + c_1x + c_2x^2 + \cdots)(b_0 + b_1x + b_2x^2 + \cdots)$ and $b_0 + b_1x + b_2x^2 + \cdots = a(d_0 + d_1x + d_2x^2 \cdots)$. Therefore, we have $a = c_0b_0 \in Rb_0$ and $b_0 = ad_0 \in Ra$, so $Ra = Rb_0$. As a is left UG, there exists an unit $u_0 \in U(R)$ such that $a = u_0b_0$. Take a unit $u(x) := u_0 + u_1x + u_2x^2 + u_3x^3 + \cdots \in R[[x]]$ where $u_n = -u_0(u_0d_n + u_1d_{n-1} + u_2d_{n-2} + \cdots + u_{n-1}d_1)$ for $n = 1, 2, 3, \cdots$. Then it is easily to verify that $a = u(x)f(x)$. Hence a is left UG in $R[[x]]$. □

Next we consider a more restrictive situation.

Question 5.2.2 *Let B be a subring of A with $1_B = 1_A$ such that $A = B + J(A)$. If B is left* AS*, does it imply that every element of B is left* AS *in A?*

It would be convenient to give examples of left AS rings if the anwser to Question 5.2.2 was in the affirmative. Meanwhile, Question 5.2.2 is related to another question of Nicholson [66, Question 3]. A ring is clean if every element is the sum of an idempotent and a unit (see [64]).

Question 5.2.3 *[66] If $R/J(R)$ is left* AS *does it follow that R is left* AS*? What if R is exchange? Clean?*

Note that R has $SR1$ if and only if $R/J(R)$ has $SR1$. Moreover, by [66, Corollary 19], if $R/J(R)$ is unit-regular and idempotents lift modulo $J(R)$, then the matrix ring $\mathbb{M}_n(R)$ is left AS for all $n \geq 1$; in particular, R is left AS. Furthermore, every unit-regular ring is clean. So, Question 5.2.3 is well-motivated.

To answer Question 5.2.2 and Question 5.2.3, we first prove a necessary and sufficient condition for a triangular matrix ring to be left AS as an application of Theorem 5.1.12.

Theorem 5.2.4 *Let A, B be rings and M be an (A, B)-bimodule. The following are equivalent:*

(1) *The triangular matrix ring* $R := \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *is left* AS.

(2) (*a*) *Whenever* $(1 - a'a)r = 0$ *and* $(1 - a'a)x \in Ms$, $a, a', r \in A$, $s \in B$ *and* $x \in M$, *there exists a unit* $u \in U(A)$ *such that* $(1 - ua)r = 0$ *and* $(1 - ua)x \in Ms$.

(*b*) B *is left* AS.

Proof. (1) $\Rightarrow$ (2). Suppose that $(1 - a'a)r = 0$ and $(1 - a'a)x \in Ms$, where $a, a', r \in A$, $s \in B$ and $x \in M$. Let $\alpha = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} r & x \\ 0 & s \end{pmatrix}$, and write $(1 - a'a)x = x's$ with $x' \in M$. Then $\begin{pmatrix} 1 - a'a & -x' \\ 0 & 0 \end{pmatrix} \in \mathbf{l}(\beta)$, and it follows that $R\alpha + \mathbf{l}(\beta) = R$. As α is left AS, there is a unit $\gamma := \begin{pmatrix} u & y \\ 0 & v \end{pmatrix}$ in R such that $\alpha - \gamma \in \mathbf{l}(\beta)$. It follows that u is a unit in A, $(a - u)r = 0$ and $(a - u)x \in Ms$. Hence, $(1 - u^{-1}a)r = 0$ and $(1 - u^{-1}a)x \in Ms$; so (2*a*) holds.

It follows from [66, Theorem 36(1)$\Rightarrow$(2)] that B is left AS.

(2) $\Rightarrow$ (1). As $\begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \subseteq J(R)$, to show (1) it suffices to show that every $\alpha = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in R$ is left AS in R by [66, Lemma 35]. As $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$, we only need to show that both $\alpha_1 := \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ and $\alpha_2 := \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$ are left AS in R by Theorem 5.1.12.

Assume that $R\alpha_1 + \mathbf{l}(\beta) = R$ where $\beta = \begin{pmatrix} r & x \\ 0 & s \end{pmatrix}$. Then there exists

$\begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \in R$ such that

$$0 = \Big(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \alpha_1 \Big) \beta$$
$$= \begin{pmatrix} (1-a'a)r & (1-a'a)x - x's \\ 0 & -b's \end{pmatrix}.$$

That is, $(1-a'a)r = 0$ and $(1-a'a)x \in Ms$. By (2)(a), there exists $u \in U(A)$ such that $(1-ua)r = 0$ and $(1-ua)x \in Ms$. Write $(1-ua)x = ys$ with $y \in M$. Then $\gamma := \begin{pmatrix} u^{-1} & -u^{-1}y \\ 0 & 1 \end{pmatrix}$ is a unit in R and $\alpha_1 - \gamma \in \mathbf{l}(\beta)$. So α_1 is left AS in R.

Assume that $R\alpha_2 + \mathbf{l}(\beta) = R$ where $\beta = \begin{pmatrix} r & x \\ 0 & s \end{pmatrix}$. Then there exists $\begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \in R$ such that

$$0 = \Big(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \alpha_2 \Big) \beta$$
$$= \begin{pmatrix} (1-a')r & (1-a')x - x'bs \\ 0 & (1-b'b)s \end{pmatrix}.$$

Thus, $(1-a')r = 0$ and $(1-a')x \in Mbs$. By (2)(a), there exists $u \in U(A)$ such that $(1-u)r = 0$ and $(1-u)x \in Mbs$. Write $(1-u)x = ybs$ with $y \in M$. Moreover, from $(1-b'b)s = 0$, we see that $Bb + \mathbf{l}(s) = B$. Hence, by (2)(b), $b - v \in \mathbf{l}(s)$ for some $v \in U(B)$. Now $\gamma := \begin{pmatrix} u & yb \\ 0 & v \end{pmatrix}$ is a unit in R and $\alpha_2 - \gamma \in \mathbf{l}(\beta)$. So α_2 is left AS in R. □

Corollary 5.2.5 *The upper triangular matrix ring* $\mathbb{T}_n(\mathbb{Z})$ $(n \geq 2)$ *is not left* AS.

Proof. By [66, Theorem 30], the left AS condition also passes to corners. Therefore, it suffices to show that $\mathbb{T}_2(\mathbb{Z})$ is not left AS. Considering Theorem 5.2.4(2a) and considering $a' = 2, a = 3, x = 1$ and $s = 5$ with $A = B = M = \mathbb{Z}$, we have $(1 - 2 \cdot 3) \cdot 1 \in 5\mathbb{Z}$. That is,

$$(1 - a'a)x \in Ms.$$

We next see that a' cannot be replaced by a unit u in A. $\mathbb{Z}$ has two units 1 and -1. If $a' = 1$, then $-2 = (1 - a'a)x \in 5\mathbb{Z}$, a contradiction. If $a' = -1$, then $4 = (1 - a'a)x \in 5\mathbb{Z}$, again a contradiction. So, Theorem 5.2.4(2a) is not satisfied. Hence, $\mathbb{T}_2(\mathbb{Z})$ is not left AS. □

In the next example, we give a direct proof that $\mathbb{T}_2(\mathbb{Z})$ is not left AS.

Example 5.2.6 *Let* $R = \mathbb{T}_2(\mathbb{Z})$*. If* $p \in \mathbb{Z}$ *is a prime, then* $\alpha := \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ *is not left* AS *in* R*.*

Proof. We have $R\alpha = \begin{pmatrix} p\mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$. Take $1 < q \in \mathbb{Z}$ such that

$$\gcd\big(q,\, p(p^2 - 1)\big) = 1.$$

Let $\beta = \begin{pmatrix} 0 & 1 \\ 0 & q \end{pmatrix} \in R$. Then $\mathbf{l}(\beta) = \left\{ \begin{pmatrix} qn & -n \\ 0 & 0 \end{pmatrix} : n \in \mathbb{Z} \right\}$. As $\gcd(q, p) = 1$, $p\mathbb{Z} + q\mathbb{Z} = \mathbb{Z}$, so $R\alpha + \mathbf{l}(\beta) = R$.

We next show that for any unit $\gamma = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ in R, $\alpha - \gamma \notin \mathbf{l}(\beta)$. Assume that $\alpha - \gamma \in \mathbf{l}(\beta)$. Then $0 = \begin{pmatrix} p - x & -y \\ 0 & 1 - z \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & q \end{pmatrix} = \begin{pmatrix} 0 & (p - x) - qy \\ 0 & q(1 - z) \end{pmatrix}$. It follows that $p - x = qy$. As γ is a unit in R, $x = \pm 1$. But this would yield $p - 1 = qy$ or $p + 1 = qy$ in $\mathbb{Z}$, contradicting the choice of q. Hence, $\alpha - \gamma \notin \mathbf{l}(\beta)$ for any unit γ in R. So α is not left AS in R. □

Example 5.2.7 *Let* $R = \begin{pmatrix} \mathbb{Z}_2 & M \\ 0 & \mathbb{Z} \end{pmatrix}$*, where* M *is a* $(\mathbb{Z}_2, \mathbb{Z})$*-bimodule. Then* R *is left* AS.

Proof. Since $\mathbb{Z}$ is left AS, by Theorem 5.2.4, it suffices to verify that, whenever $(\bar{1} - a'a)r = 0$ and $(\bar{1} - a'a)x \in Ms$, $a, a', r \in \mathbb{Z}_2$, $s \in \mathbb{Z}$ and $x \in M$, we have $(\bar{1} - a)r = 0$ and $(\bar{1} - a)x \in Ms$. This is certainly the case if $a' = \bar{1}$. So we can assume that $a' = \bar{0}$, which implies that $r = \bar{0}$. Thus, $(\bar{1} - a)r = 0$, and $x = (\bar{1} - a'a)x \in Ms$. It follows that $(1 - a)x \in Ms$. □

We now give answers to both Question 5.2.2 and Question 5.2.3.

Theorem 5.2.8 (1) *The answer to Question 5.2.2 is in the negative.*

(2) *There exists a ring* R *such that* $R/J(R)$ *is left* AS*, but* R *is not left* AS.

(3) *If* R *is an exchange ring, then* R *is left* AS *if and only if* $R/J(R)$ *is left* AS.

Proof. (1) Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$. Then $R = S + J(R)$, where $S = \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z} \end{pmatrix}$. Here $S \cong \mathbb{Z} \times \mathbb{Z}$ is left AS. But the element $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ in S is not left AS in R by Example 5.2.6. Hence, the answer to Question 5.2.2 is in the negative.

(2) Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$. Then $J(R) = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$, so $R/J(R) \cong \mathbb{Z} \times \mathbb{Z}$ is left AS. But R is not left AS by Example 5.2.6.

(3) Since R is an exchange ring, R is left AS if and only if R has SR1 and, respectively, $R/J(R)$ is left AS if and only if $R/J(R)$ has SR1 by Theorem 5.1.1 and [48, Theorem 6.5]. Moreover, it is known that R has SR1 if and only if $R/J(R)$ has SR1, so it follows that R is left AS if and only if $R/J(R)$ is left AS. □

It is unknown whether every left AS ring is right AS (see [20, Remark 4.9], [48, p. 218], [66, Question 4]). However, we have

Theorem 5.2.9 *A left* AS *element in a ring need not be right* AS*.*

Proof. Let $R = \mathbb{T}_2(\mathbb{Z})$ and $\alpha = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \in R$, where $p \in \mathbb{Z}$ is a prime. By Example 5.2.6, α is not left AS in R. We next show that α is right AS in R.

Suppose that $\alpha R + \mathbf{r}(\beta) = R$, where $\beta = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ in R. As $\alpha R = \begin{pmatrix} p\mathbb{Z} & p\mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$, it follows from $\alpha R + \mathbf{r}(\beta) = R$ that $x = 0$, and so $\beta = \begin{pmatrix} 0 & y \\ 0 & z \end{pmatrix}$. Then $\alpha - I_2 = \begin{pmatrix} p-1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbf{r}(\beta)$. So α is right AS in R. □

It is open whether R being commutative UG implies that $R[[t]]$ is UG (see [2, Question 21]). We end this section by giving a *UG* ring R such that $R[[t]]$ is not *UG*. Note that $\mathbb{M}_n(\mathbb{Z})$ is a *UG* ring by [47, Theorem 3.8].

Example 5.2.10 *Let* $R = \mathbb{M}_n(\mathbb{Z})$ $(n \geq 2)$*. Then* $R[[t]]$ *is not left* UG*.*

Proof. As $\mathbb{M}_n(\mathbb{Z})[[t]] \cong \mathbb{M}_n(\mathbb{Z}[[t]])$, it suffices to show that $\mathbb{M}_2(\mathbb{Z}[[t]])$ (which is isomorphic to $\mathbb{M}_2(\mathbb{Z})[[t]]$) is not left AS by Theorem 5.1.1 and [66, Theorem 30]. Hence, we can assume that $n = 2$.

Let $\alpha = a_0 + a_1 t$ and $\beta = b_0 + b_1 t$, where $a_0 = \begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}$, $a_1 = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}$, $b_0 = \begin{pmatrix} 0 & 0 \\ 17 & 17 \end{pmatrix}$, and $b_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, and let $a = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$\begin{aligned}(1 - \alpha a)\beta &= [(1 - a_0 a) - a_1 a t](b_0 + b_1 t) \\ &= (1 - a_0 a)b_0 + [(1 - a_0 a)b_1 - a_1 a b_0]t - a_1 a b_1 t^2 \\ &= 0.\end{aligned}$$

So $R[[t]]a + \mathbf{l}(\beta) = R[[t]]$. We next show that for any unit $\gamma = r_0 + r_1 t + \cdots$ in $R[[t]]$, $a - \gamma \notin \mathbf{l}(\beta)$. Assume that $a - \gamma \in \mathbf{l}(\beta)$. Then it follows that $(a - r_0)b_0 = 0$ and $(a - r_0)b_1 = r_1 b_0$ with r_0 a unit in R. Write $r_0 = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$. From $(a - r_0)b_0 = 0$, it follows that $u_2 = 0$ and $u_4 = 1$. So $r_0 = \begin{pmatrix} u_1 & 0 \\ u_3 & 1 \end{pmatrix}$ and $u_1 = \pm 1$. Thus, from $(a - r_0)b_1 = r_1 b_0$, it follows that $5 - u_1 = 17k$ for some $k \in \mathbb{Z}$. But this is impossible as $u_1 = \pm 1$. Therefore, we have proved that a is not left AS in $R[[t]]$. $\square$

By Example 5.2.10, the ring $\mathbb{M}_n(\mathbb{Z})[[t]]$ $(n \geq 2)$ also satisfies Theorem 5.2.8(2).

Statement 5.2.11 *The material in this chapter is taken from [81]. My contribution in this joint work is to give an answer to one of Nicholson's questions as stated in Section* 5.1.

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