Rigidity of Marginally Outer Trapped Surfaces in Reissner-Nordström Spacetime

by

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Abstract

Our primary focus in this thesis is to investigate the stability vs rigidity of marginally outer trapped surfaces (MOTS) in four-dimensional Reissner-Nordström (RN) spacetime. This is connected to studying the first-order derivative of the stability operator (and hence the second derivative of the outgoing null expansion). Stability means that the principal eigenvalue is non-negative, and rigidity means that we cannot deform MOTS. The question we have addressed in this thesis is distinguishing between stability and rigidity. We study the special case of the inner horizon of Reissner-Nordström spacetimes for specific values of charge and mass is horizons can be unstable, and we ask questions whether they unstable is still rigid. To approach this question we use a technique to reduce an infinite-dimensional second variation calculation to a finite-dimensional one. We start with a brief introduction to general relativity and review some fundamental aspects of black holes. We then define the stability of MOTS in terms of the principal eigenvalue. Since the stability operator has a zero eigenvalue in our case, the MOTS admits infinitesimal deformations. In the rest of the work we use Lyapunov-Schmidt reduction to investigate whether these infinitesimal deformations can be made finite. We give evidence that suggests that the inner horizon is stable.

Keywords 1. *MOTS, RIGIDITY AND STABILITY, LYAPUNOV-SCHMIDT REDUCTION*

Lay Summary

In 1915, general relativity was first introduced by Albert Einstein. The study of black holes is the vital part of general relativity. A black hole is a region of spacetime from where nothing can escape, not even light. In this thesis we will deal with two interesting questions that arise in the analysis of black holes. Marginally outer trapped surfaces (MOTS) are black hole horizon proxies. We start our problem by studying unstable MOTS in four dimensional Reissner-Nordström spacetime. Then we address questions about rigidity vs stability of MOTS and under which conditions is it rigid or not rigid.

The geometric properties of marginally outer trapped surface are very closely related to minimal surfaces in Riemannian geometry (trace of extrinsic curvature is zero). For a more precise discussion, we can define the stability and rigidity of MOTS in more general way. A MOTS is not rigid if it can be deformed to a nearby surface which is still a MOTS. We know there are examples of stable MOTS that are rigid (outer horizon). We also know of examples of unstable MOTS that are not-rigid (the cosmological horizon in pure de Sitter spacetimes). In this thesis we examine unstable MOTS (inner RN horizons) whose rigidity is not yet known. We examine the rigidity via a Lyapunov-Schmidt reduction of the stability operator.

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$(-,+,+,+)$	metric signature
(M,g)	General spacetime
Σ	3-dimensional spacelike hypersurface of M
S	2-dimensional closed, smooth hypersurface embedded in Σ
$g_{\alpha\beta}$	Metric tensor in M
ds^2	corresponding Riemannian metric on Σ
q_{ab}	induced metric on Σ
$\tilde{q}_{AB}$	induced metric on S
∇_α	Covariant derivative on M
D_a	Covariant derivative on Σ
d_A	covariant derivative on S
Λ	Cosmological Constant
K_{ab}	Extrinsic curvature on Σ
k_{AB}	Extrinsic curvature on S
k	trace of extrinsic curvature on S
$R_{\alpha\beta}$	Ricci tensor in 4-dimensional spacetime
R	Scalar curvature for (M,g)
$T_{\alpha\beta}$	Stress energy tensor
$\mathscr{I}^\pm$	Future/past null infinity
$i^\pm$	Future/past timelike infinity
i^o	corresponding spacelike infinity
θ_ℓ	outward null expansion
θ_n	inward null expansion
r^a	outward pointing spacelike unit normal to S
u^a	timelike unit normal to Σ
L	stability operator
∇^2	Laplace operator
ω_a	connection of normal bundle on S

List of abbreviations

MOTS	Marginally Outer Trapped Surfaces
EH	Event Horizon
AH	Apparent Horizon
IH	Inner Horizon
OH	Outer Horizon
CH	Cosmological Horizon
RN	Reissner-Nordström
RNDS	Reissner-Nordström-de Sitter

Chapter 1

Introduction

1.1 Overview of General Relativity, Black holes and MOTS

The four dimensional geometry of spacetime is determined by the metric $g_{\alpha\beta}$, which defines the distance between two infinitesimally close events in spacetime and can be written as

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta. \tag{1.1}$$

The simplest example is flat spacetime where geometry is determined by the Minkowski metric which has components $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$ in Cartesian coordinates $x^0 = t, x^1 = x, x^2 = y, x^3 = z$.

Einstein's theory of gravitation states that gravity is the result of curvature in spacetime. In 1915, Einstein published his equations relating the curvature of spacetime to the stress-energy-momentum tensor of matter. We use [1–3] as references.

These field equations can be written as

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}. \tag{1.2}$$

Here

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = G_{\alpha\beta} \tag{1.3}$$

is the symmetric Einstein tensor, $R_{\alpha\beta} = R^{\gamma}_{\alpha\gamma\beta}$ is Ricci curvature and $R = R^{\gamma}_{\gamma}$ is the Ricci scalar. Ricci curvature and scalar both are contractions of the Riemann tensor $R^{\alpha}_{\beta\gamma\delta} = \Gamma^{\alpha}_{\beta\gamma,\delta} - \Gamma^{\alpha}_{\beta\delta,\gamma} + \Gamma^{\alpha}_{\mu\gamma}\Gamma^{\mu}_{\beta\delta} - \Gamma^{\alpha}_{\mu\delta}\Gamma^{\mu}_{\beta\gamma}$ and $\Gamma^{\alpha}_{\beta\gamma}$ represents the Christoffel symbols, $T_{\alpha\beta}$ is the stress-energy tensor describing all forms of matter and energy and Λ is the cosmological constant. The conservation of energy and momentum is written as

$$\nabla_{\beta} T^{\alpha\beta} = 0. \tag{1.4}$$

and for the pure vacuum spacetime

$$T_{\alpha\beta} = 0 \tag{1.5}$$

Black holes are solutions to the Einstein equations that are characterized by a region from which even light cannot escape. The black hole spacetime is divided into two regions, named the interior and the exterior region. The boundary between the interior and exterior regions is called the event horizon. Nothing can escape from the interior of a black hole.

The simplest black hole is the Schwarzschild (1916) solution to the vacuum Einstein equations. This black hole solution was also the first non-trivial solution that was found for the Einstein equations. The line element can be written in standard static spherical co-ordinates (t, r, θ, ϕ) as

$$ds^2 = -F(r)dt^2 + \frac{1}{F(r)}dr^2 + r^2\Big(d\theta^2 + \sin^2\theta d\phi^2\Big). \tag{1.6}$$

Here $F(r) = 1 - \frac{2M}{r}$, where M has the interpretation as the total mass contained in the spacetime. If r grows large then the Schwarzschild metric approaches $ds^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$, which is flat metric; the Schwarzschild solution is **asymptotically flat**. This coordinate form of the metric becomes singular at both $r = 0$ and $r = 2M$, which are a true singularity and event horizon, respectively. As is well known, the event horizon singularity can be removed by an appropriate coordinate transformation [2].

A more general black hole is the Reissner-Nordström-de Sitter black hole, which describes an asymptotically de Sitter, static, spherically symmetric black hole of mass M carrying an electric charge Q on background with cosmological constant Λ [4]. In spherical cordinates (t, r, θ, ϕ) the metric can again be written as

$$ds^2 = -F(r)dt^2 + \frac{1}{F(r)}dr^2 + r^2\Big(d\theta^2 + \sin^2\theta d\phi^2\Big), \tag{1.7}$$

here $F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2$. In the case of cosmological constant $\Lambda = 0$, the Reissner-Nordström-de Sitter spacetime reduces to the Reissner-Nordström spacetime and roots of $F(r)$ are $r = r_0, r_1$, where

$$r_0 = M - \sqrt{M^2 - Q^2} \quad \text{and} \quad r_1 = M + \sqrt{M^2 - Q^2}. \tag{1.8}$$

Now there are two horizons: r_1 is the outer horizon and r_0 is the inner horizon. The outer horizon is the location of the black hole event horizon and the inner horizon is referred to as the Cauchy horizon [1, 5, 6].

1.2 Geodesics and Minimal surfaces

In general relativity marginally outer trapped surfaces (MOTS) are a natural candidate for quasi-local black hole boundaries. We are going to be talking about MOTS but these are closely related to minimal surfaces. In particular lots of analysis involves the stability operator, which is related to the variation of area. Before talking about minimal surfaces we will talk a little bit about geodesics. Geodesics are critical points of the length functional and minimal surfaces are critical points of the area functional. The analogue of the acceleration of a curve is exactly the mean curvature of a surface. If the curve is a critical point of the length then the acceleration vanishes, i.e. it is a geodesic, and if the surface is a critical point of the area then its mean curvature vanishes, i.e it is a minimal surface.

1.2.1 Geodesics

A geodesic is sometimes thought of as the curve of shortest distance between two fixed points on a surface. But this is not a satisfactory definition. The most common examples, the geodesics on a plane, are straight lines whereas the geodesics between two points on a sphere are arcs of a great circle. There are two arcs of a great circle between two such points and only one provides the shortest path between those two points. There may exist more than one geodesic between two points on a surface. Now mathematically define geodesics in a following way.

Consider a continuously differentiable curve $\gamma : I \to M$ on Riemannian manifold M with metric tensor g, where $I = [a, b]$ is some closed interval of $\mathbb{R}$. The length $\mathcal{L}$ of γ is defined by

$$\mathcal{L}(\gamma) = \int_a^b \sqrt{g_{\mu\nu}\dot{\gamma}^\mu \dot{\gamma}^\nu} dt. \tag{1.9}$$

Here any two points $\gamma(a) = p$ and $\gamma(b) = q$ are the starting point and endpoint of the curve, respectively.

Now given a curve $\gamma : I \to M$, V a variation field and Γ a proper variation of γ, then we can write the derivative of length functional as

$$\frac{d}{ds}\bigg|_{s=0} \mathcal{L}(\Gamma_s) = -\int_a^b g_{\mu\nu} V^\mu \ddot{\gamma}^\nu dt. \tag{1.10}$$

So if γ is the critical point of the length functional then this is the geodesic. Here $\ddot{\gamma}$ is the acceleration of a curve and it is defined by

$$\ddot{\gamma} = \frac{D}{dt}\Big(\frac{d\gamma}{dt}\Big). \tag{1.11}$$

Here $\frac{D}{dt}$ is the associated covariant derivative along γ [7]. In local coordinates this can be written as

$$\ddot{\gamma}^\mu = \frac{d^2\gamma^\mu}{dt} + \Gamma^\mu_{\alpha\beta}\frac{d\gamma^\alpha}{dt}\frac{d\gamma^\beta}{dt}. \tag{1.12}$$

If γ is a critical point of $\mathcal{L}$ then the variation of $\mathcal{L}$ vanishes i.e.

$$\int_a^b g_{\mu\nu} V^\mu \ddot{\gamma}^\nu dt = 0 \tag{1.13}$$

with respect to any vector field V. This is possible only if the acceleration of the curve is zero, hence

$$\ddot{\gamma} = \frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0. \tag{1.14}$$

This is the geodesic equation and in coordinates form it can be written as

$$\frac{d^2\gamma^\mu}{dt} + \Gamma^\mu_{\alpha\beta}\frac{d\gamma^\alpha}{dt}\frac{d\gamma^\beta}{dt} = 0. \tag{1.15}$$

Here

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\nu\mu}\Big(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\Big) \tag{1.16}$$

are the Christoffel symbols associated with the metric tensor. The critical points of the first variation of length are specifically the geodesics and the second variation along a geodesic γ is said to be a Jacobi field if it satisfies Jacobi equation:

$$\frac{D^2 J^\alpha}{dt^2} + R^\alpha_{\beta\delta\eta}\dot{\gamma}^\beta J^\delta \dot{\gamma}^\eta = 0. \tag{1.17}$$

Here J^α is a vector and $R^\alpha_{\beta\delta\eta}$ is the Riemann curvature tensor. The Jacobi equation is essentially the linearized version of the deviation equation describing how nearby geodesics behave. The points along a geodesic where the Jacobi field vanishes are called **conjugate points**. A geodesic between two points is of minimal length if there is no conjugate point between the end points. Conversely, a geodesic is not of minimal length if there is a conjugate point between the end points. In other words, conjugate points tell us whether or not geodesic is the minimal length [8].

1.2.2 Minimal Surfaces

Intuitively a minimal surface is a surface that locally extremizes its area. This is exactly the same concept as a geodesic just in one higher dimension. This is equivalent to having vanishing mean curvature. One simple example is a flat surface in $\mathbb{R}^3$ and the other easy example is the catenoid which is simplest non-trivial minimal surface in 3-dimensional Euclidean space. The catenoid has parametric equations:

$$
\begin{aligned}
x &= c\cosh\frac{v}{c}\cos u \\
y &= c\cosh\frac{v}{c}\sin u \\
z &= v
\end{aligned}
$$

here the equations are parameterized by $u \in (-\pi, \pi)$ and $v \in (-\infty, \infty)$, and c is a non-zero real constant.

Minimal surfaces are the critical points of the area functional. Consider a closed surface S sitting inside Σ, where Σ is a 3-dimensional Riemannian manifold and we are calculating the area of S. One way of writing the area is

$$\text{Area}(S) = \int_S 1dA. \tag{1.18}$$

Now take the surface S and deform it in the outward normal direction by amount $t\psi$, where ψ is a smooth function on S (which is exactly the calculation we do for variation of expansion in section 2.5). Now take the derivative of the area of the deformed surface, called S_t, at $t = 0$ then we can prove

$$\frac{d}{dt}\text{Area}(S_t)\bigg|_{t=0} = \int_S k\psi dA. \tag{1.19}$$

Here k is the mean curvature of S (trace of the extrinsic curvature of S). The amount by which the area changes is some number that will be positive, negative or zero, depending on ψ. If S is a critical point of this area function then the first derivative of area is 0. That is the derivative of area in any direction is 0, i.e

$$\int_S k\psi dA = 0, \tag{1.20}$$

for any function ψ. That means k itself is 0. By definition if $k = 0$ then S is a minimal surface. So this variation of area expression shows the connection between critical points of the area functional and surfaces of zero mean curvature.

We mentioned earlier that minimal surface can be defined as critical points (minima or saddles) of the area functional. The critical points of a function are distinguished by the second derivative test. Similarly, minimal surfaces can also be classified by the second variation of the area functional.

Now take the second derivative or second variation of area

$$\frac{d^2}{dt^2}\Big(\mathrm{Area}(S_t)\Big)\bigg|_{t=0} = \frac{d}{dt}\int_{S_t} k\psi dA = \int_S \psi L\psi dA. \tag{1.21}$$

Here the stability operator $L = -d^2\psi - \frac{1}{2}\psi\Big(R - {}^{(2)}R + k_{AB}k^{AB} + k^2\Big)$ where d^2 is the Laplacian on S and R is Ricci scalar on Σ. This is a special case of the MOTS stability operator which is derived in section 2.5. From calculus we know that if $\frac{d^2}{dt^2}\Big(\mathrm{Area}(S_t)\Big)\Big|_{t=0} > 0$ then S is a local minimum of the area function. Hence we say that the minimal surface S is **strictly stable** if the second variation of area $\frac{d^2}{dt^2}\Big(\mathrm{Area}(S_t)\Big)\Big|_{t=0} > 0$ for all possible variation families S_t, **stable** if $\frac{d^2}{dt^2}\Big(\mathrm{Area}(S_t)\Big)\Big|_{t=0} \geq 0$ and unstable otherwise.

1.3 Outline

In this research work we will present the proof of certain cases of the rigidity of marginally outer trapped surfaces in static spherically symmetric spacetimes. In particular, we will distinguish between stability and rigidity. This section is organized with a synopsis of subsequent chapters.

In Chapter 2 we will discuss the geometric background and notations which will be required for the rest of the thesis. The chapter starts with establishing the notation and sign conventions. Then we will describe the stability and rigidity of MOTS in terms of the principal eigenvalue.

In Chapter 3, we will deal with unstable MOTS in RNdS spacetime. The description in this chapter will start with MOTS in Painleve-Gullstrand coordinates for RNdS spacetime. We review various horizons such as the outer black hole horizon and cosmological horizon (pure de-Sitter case) which are strictly stable. We will describe an infinite number of Reissner-Nordström spacetimes with different parameter values for which the inner black hole horizon is unstable. We will apply the formalism to unstable MOTS to examine whether or not it can be deformed.

We present our original results in Chapter 4. The new results presented in this chapter use the Lyapunov-Schmidt reduction for solving nonlinear equations. We showed that if 0 is an eigenvalue of the stability operator then rigidity can be checked by solving a system of polynomial equations. We solved them in some cases.

In Chapter 5 we summarize our results and suggest future works.

Chapter 2

Background

The existence of black holes is one of the most fundamental predictions in General Relativity. A spacetime said to contain a *black hole* if it has regions from which no null curve reaches to future null infinity. The idea of a horizon can be used to characterize black holes in a spacetime. It forms the boundary between two causally disconnected regions of a spacetime. The boundary of the region of spacetime is called an event horizon as the black hole cannot interact with the outside universe. The event horizon is a 3-dimensional hypersurface in spacetime traced by null geodesics that are neither ingoing nor outgoing [3].

The boundary separating the regions from which outward oriented light rays actually move outwards towards the asymptotically flat region instead of being forced by gravity to move inwards is a surface called the apparent horizon. More properly, it is defined as the boundary of the union of all trapped regions in an asymptotically flat Cauchy surface. There the outgoing null expansion vanishes [1,2]. In general it is distinct from the event horizon. These two horizons do coincide for Reissner-Nordström or other stationary black holes but do not coincide for non-stationary black holes such as the Vaidya spacetime. For a more complete discussion about black hole horizons see [5,9].

In this chapter we will highlight the most important aspects of black holes on which the rest of the thesis will play. We begin with the various geometric quantities that we will use in rest of the thesis.

2.1 Geometric background

In this section we will define a variety of geometric quantities that will be used extensively in the rest of the thesis. For general references on the geometry see [10–12].

Here we use Greek letters $\{\alpha, \beta, \gamma, \dots\}$ as abstract indices on the 4-dimensional spacetime M but when working with a coordinate chart switch to $\{\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \dots\}$. Similarly we use lower-case latin letters $\{a, b, c, \dots\}$ as abstract indices for tensors in 3-dimensional surfaces Σ while using $\{\hat{a}, \hat{b}, \hat{c}, \dots\}$ for coordinates and tensor components relative to coordinates. Finally upper-case latin letters $\{A, B, \dots\}$ are used as abstract indices on two dimensional surfaces S.

Let (Σ, q_{ab}, D_a) be a spacelike 3-dimensional hypersurface embedded in a (3+1) dimensional time orientable spacetime $(M, g_{\alpha\beta}, \nabla_\alpha)$ with signature $(-,+,+,+)$. Here ∇_α is the covariant derivative in $(3+1)$ dimensional spacetime and D_a is the covariant derivative in 3-dimensional spacetime.

Let $(S, \tilde{q}_{AB}, d_A)$ be a closed two-surface in the 3-dimensional slice Σ. Here $\tilde{q}_{AB}$ is the induced metric on S and d_A is the covariant derivative on S. Since M is time oriented, at each point $p \in S$ we can assemble a pair of future pointing null vectors normal to S. Consider such a pair of future pointing null vector fields ℓ^α and n^α which point outward and inward, respectively.

The pull-back operator between surfaces will be written as an e with indices indicating which spaces it operates between. If we wish to switch to coordinate charts in a particular hypersurface, which has parametric equations of the form,

$$x^{\hat{\alpha}} = x^{\hat{\alpha}}(y^{\hat{a}}), \tag{2.1}$$

where $y^{\hat{a}}$ $(\hat{a} = 1, 2, 3)$ are coordinates intrinsic to the hypersurface, then $e^{\hat{\alpha}}_{\hat{a}} = \frac{\partial x^{\hat{\alpha}}}{\partial y^{\hat{a}}}$ is the pullback/pushforward operator for one forms/vectors.

The induced metric on Σ as embedded in a spacetime M is

$$q_{ab} = e^\alpha_a e^\beta_b g_{\alpha\beta}, \tag{2.2}$$

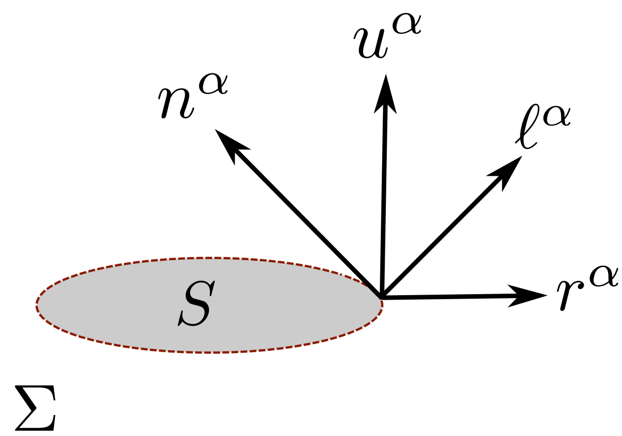

Figure 2.1: The figure shows a smooth two dimensional hypersurface S in a three dimensional slice Σ embedded in a four dimensional spacetime M. The unit outward normal to S is r^a, and u^a is a timelike normal to Σ. The outgoing and ingoing null vectors are ℓ^{α}, n^{α} (Figure taken from [13]).

and the induced metric on S is

$$\tilde{q}_{AB} = e^{\alpha}_{A} e^{\beta}_{B} g_{\alpha\beta}. \tag{2.3}$$

The direction of the two null vectors is fixed and they can be normalized as

$$\ell \cdot n = -1, \tag{2.4}$$

which leaves a single degree of rescaling freedom by an arbitrary non-zero function f:

$$\ell \to \ell f \quad \text{and} \quad n \to n/f. \tag{2.5}$$

However, independent of that particular choice of scaling we have

$$\tilde{q}^{\alpha\beta} = e^{\alpha}_{A} e^{\beta}_{B} \tilde{q}^{AB} = g^{\alpha\beta} + \ell^{\alpha} n^{\beta} + \ell^{\beta} n^{\alpha}. \tag{2.6}$$

Now we can define the outward pointing spacelike unit normal to S lying in Σ as r^a and let u^{α} be the future pointing timelike unit normal to Σ for each point on S. So for a particular choice of f now we can define the inward and outward null vectors to

S as

$$n^\alpha = \frac{1}{2}(u^\alpha - r^\beta), \tag{2.7}$$

$$\ell^\beta = (u^\alpha + r^\beta). \tag{2.8}$$

By construction we have $\ell^\alpha \ell_\alpha = n^\alpha n_\alpha = 0$ and $\tilde{q}_{\alpha\beta}\ell^\beta = \tilde{q}_{\alpha\beta}n^\beta = 0$.

Now the extrinsic curvature on Σ is defined by

$$K_{ab} = e_a^\alpha e_b^\beta \nabla_\alpha u_\beta, \tag{2.9}$$

where u^β is the future-oriented unit normal to Σ which satisfies $u^\beta u_\beta = -1$. The extrinsic curvature on S is defined by the rate of change of null normal vectors (ℓ^α, n^α) along the surface:

$$\begin{aligned} k_{AB}^{(\ell)} &= e_A^\alpha e_B^\beta \nabla_\alpha \ell_\beta \\ k_{AB}^{(n)} &= e_A^\alpha e_B^\beta \nabla_\alpha n_\beta. \end{aligned} \tag{2.10}$$

These extrinsic curvature are symmetric in A and B since ℓ^α and n^α are, by definition, surface forming.

Also, these extrinsic curvatures can be decomposed into their trace and trace free parts respectively so that

$$\begin{aligned} k_{AB}^{(\ell)} &= \frac{1}{2}\theta_{(\ell)}\tilde{q}_{AB} + \sigma_{AB}^{(\ell)}, \\ k_{AB}^{(n)} &= \frac{1}{2}\theta_{(n)}\tilde{q}_{AB} + \sigma_{AB}^{(n)}. \end{aligned} \tag{2.11}$$

Here $\theta_{(\ell)}$ and $\theta_{(n)}$ are respectively the outward and inward expansion of null geodesics:

$$\begin{aligned} \theta_{(\ell)} &= \tilde{q}^{AB}k_{AB}^{(\ell)}, \\ \theta_{(n)} &= \tilde{q}^{AB}k_{AB}^{(n)}, \end{aligned} \tag{2.12}$$

while

$$\begin{aligned} \sigma_{AB}^{(\ell)} &= (\tilde{q}_A^\alpha \tilde{q}_B^\beta - \frac{1}{2}\tilde{q}_{AB}\tilde{q}^{\alpha\beta})\nabla_\alpha \ell_\beta, \\ \sigma_{AB}^{(n)} &= (\tilde{q}_A^\alpha \tilde{q}_B^\beta - \frac{1}{2}\tilde{q}_{AB}\tilde{q}^{\alpha\beta})\nabla_\alpha n_\beta, \end{aligned} \tag{2.13}$$

are shears (trace free parts).

2.2 Gauss-Codazzi equations

The Gauss-Codazzi equations are the fundamental equations in the theory of embedded hypersurfaces of Riemannian manifolds. By [5] the 3-dimensional Riemann tensor for Σ is

$$R^d_{abc} = \Gamma^d_{ac,b} - \Gamma^d_{ab,c} + \Gamma^d_{eb}\Gamma^e_{ac} - \Gamma^d_{ec}\Gamma^e_{ab}. \tag{2.14}$$

By the Gauss equation it is determined in terms of the extrinsic curvature and the 4-dimensional Riemann tensor of M

$$R_{\alpha\beta\gamma\delta}e^\alpha_a e^\beta_b e^\gamma_c e^\delta_d = R_{abcd} - \Big(K_{ad}K_{bc} - K_{ac}K_{bd}\Big). \tag{2.15}$$

Next the Codazzi equation tells us that

$$R_{\mu\alpha\beta\gamma}u^\mu e^\alpha_a e^\beta_b e^\gamma_c = D_c K_{ab} - D_b K_{ac}. \tag{2.16}$$

Here u^μ is the future oriented unit normal to Σ. From these equations we can obtain the Hamiltonian constraint [5]:

$$G_{\alpha\beta}u^\alpha u^\beta = \frac{1}{2}\Big(R + K^2 - K_{ab}K^{ab}\Big), \tag{2.17}$$

and the diffeomorphism constraint:

$$G_{\alpha\beta}e^\alpha_a u^\beta = D_b K^b_a - D_a K, \tag{2.18}$$

here R is the 3-dimensional scalar curvature of Σ and $K = h_{ab}K^{ab}$. Applying the Einstein equations ($G_{\alpha\beta} = 8\pi T_{\alpha\beta}$) in Eqn.(2.17) and Eqn.(2.18) constrains the intrinsic and extrinsic curvatures of Σ by the stress energy content of the spacetime.

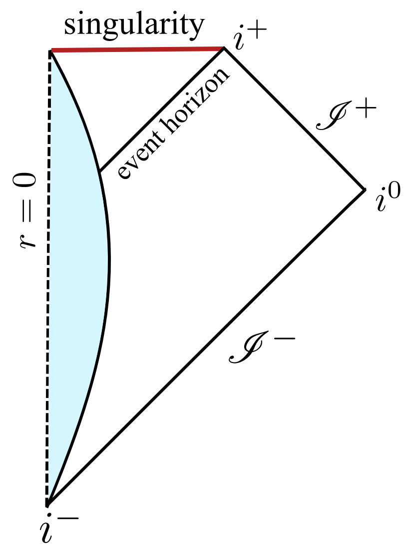

Figure 2.2: A Penrose-Carter diagram showing a spacetime of a spherically symmetric star collapsing into a black hole. Null curves are depicted with slopes ± 1, $i^{\pm}$ is future/past timelike infinity, i^0 is corresponding spacelike infinity and $\mathscr{I}^{\pm}$ is future/past null infinity (Figure based on [9, 14]).

2.3 Event Horizon

In General Relativity the conventional approach to horizons which came from the study of static or stationary black holes is that of the **event horizon**. The event horizon is a null hypersurface in spacetime and its definition is restricted to the asymptotically flat or anti-de Sitter spacetimes. The reason why it is a null hypersurface is that it is generated by null geodesics. By [5, 9] since the event horizon is null and so coincides with a congruence of null geodesics, the Raychaudhuri equation determines its expansion:

$$\frac{d\theta_{(\ell)}}{d\lambda} = -\frac{1}{2}\theta_{(\ell)}^2 - \sigma_{AB}^{(\ell)}\sigma_{(\ell)}^{AB} - R_{\alpha\beta}\ell^\alpha\ell^\beta. \tag{2.19}$$

Here λ is an affine parameter of these geodesics, $\theta_{(\ell)}$ is the outgoing null expansion which was defined earlier and $\sigma_{AB}^{(\ell)}$ is the symmetric tracefree part as defined in section 2.1. Also $R_{\alpha\beta}$ is the Ricci tensor in 4-dimensional spacetime. An interesting thing we can see from the Raychaudhuri equation is that if the null energy condition holds, then the last term on the right hand side of Eqn.(2.19) will always be non-positive and this gives rise to the second law of black hole mechanics: the area of an event horizon is non decreasing [1].

Figure 2.2, which is based on [9], shows 3-dimensional future/past null infinity, labeled by $\mathscr{I}^{\pm}$, which contains the future/past endpoints of all outgoing/ingoing null geodesics. Where future null infinity ($\mathscr{I}^{+}$) and past null infinity ($\mathscr{I}^{-}$) meet is called two dimensional spacelike infinity and labelled by i^0, which contains the endpoints of all spacelike geodesics. In the same way, the two dimensional future/past timelike infinity denoted by $i^{\pm}$ contains future/past endpoints of timelike geodesics.

If the complement of the causal past of future null infinity ($\mathscr{I}^{+}$) is non-empty then the spacetime contains a black hole and the **event horizon** is the boundary of that region. An example is shown in Fig.2.2

The event horizon forms a causal boundary between the interior and exterior spacetime so that no geodesic can escape from the interior to exterior spacetime. As we mentioned earlier, the region inside the event horizon is a **trapped region** [1, 9] for stationary spacetimes.

2.4 Trapped Surfaces

In section 2.1 we considered two-dimensional, spacelike and closed surfaces S embedded in 4-dimensional spacetime M. We also considered that at a point p on a spacelike 2-surface S there are exactly two distinct future null directions ℓ^α (outward), n^α (inward) normal to the surface.

A closed two dimensional surface S is called **future trapped** if both null expansions (inward and outward) are strictly negative everywhere. That is for S:

$$\begin{aligned} \theta_{(n)} &< 0 \\ \theta_{(\ell)} &< 0. \end{aligned} \tag{2.20}$$

$\theta_{(n)}$ and $\theta_{(\ell)}$ are defined in Eqn.(2.12). This leads to a considerably different behaviour of the 2-surface when compared with a 2-surface embedded in Minkowski space. For a sphere (in Minkowski) an ingoing set of null rays would decrease ($\theta_{(n)} < 0$) in area while the outgoing set would increase ($\theta_{(\ell)} > 0$)

From the Raychaudhuri equation in (2.19) as well as some extra consideration

$[1, 2, 15]$, if the null energy condition holds then the existence of a trapped surface in a spacetime implies the existence of singularity somewhere in its causal future. A closed 2-surface is called a **marginally trapped surface** if the inward null expansion is negative and outward null expansion vanishes i.e.

$$\theta_{(n)} < 0 \quad \text{and} \quad \theta_{(\ell)} = 0, \tag{2.21}$$

and the surface is called a **marginally outer trapped surface** (MOTS) if

$$\theta_{(\ell)} = 0. \tag{2.22}$$

In general relativity MOTS provide the most intuitive characterization for quasi-local black hole boundaries and for those MOTS the expansion $\theta_{(n)}$ of inward null normal n^α is strictly negative. Also the surface is called **outer trapped** or **outer untrapped** respectively if

$$\theta_{(\ell)} < 0 \quad \text{or} \quad \theta_{(\ell)} > 0. \tag{2.23}$$

i.e. the outward null expansion is strictly negative or positive, respectively. For details of trapped surfaces see the references $[10, 16]$.

2.5 Deriving the variation of expansion

In this section we now consider how the expansion changes if we deform the surface S. Let the extrinsic curvature of S be defined by null normals ℓ^α and n^α, as in Eqn.(2.11).

Let r^a be the spacelike unit normal vector to the hypersurface and the trace of extrinsic curvature of S with respect to r^a is

$$k^{(r)} = \tilde{q}^{AB} k^{(r)}_{AB}, \tag{2.24}$$

and the trace of the extrinsic curvature of S with respect to the timelike unit normal u^a is

$$k^{(u)} = \tilde{q}^{ab} K^{(u)}_{ab}. \tag{2.25}$$

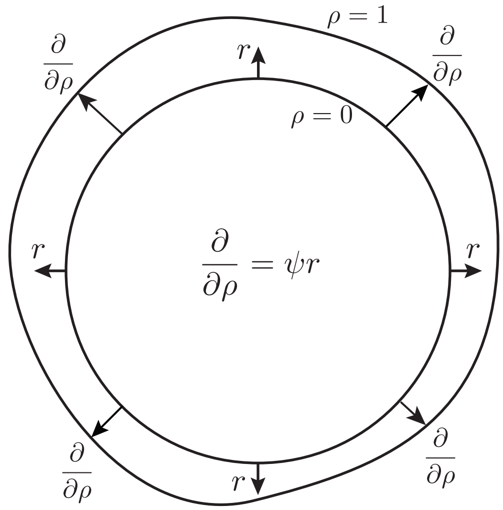

Figure 2.3: Deformation of a surface generated by a covariant vector field $\frac{\partial}{\partial \rho}$

K_{ab} is defined in Eqn.(2.9). In order to make these equations more readable we will drop the $\hat{}$ from $\hat{u}^a$ for timelike unit normal and $\hat{r}^a$ for spacelike unit normal in this section. Now to find the variation of the trace of extrinsic curvature $k^{(r)}_{AB}$, consider how $k^{(r)}$ varies if S is deformed. The simplest way to do this is imagine a deformation generated by a covariant vector field $\frac{\partial}{\partial \rho}$. Imagine S as the $\rho = 0$ surface of a family

$$y^{\hat{a}} = Y^{\hat{a}}(\theta^{\hat{A}}, \rho), \tag{2.26}$$

here $\hat{A}$ label coordinates on S. Then under a deformation $S \to S'$

$$Y^{\hat{a}}(\theta^{\hat{A}}, 0) \to Y^{\hat{a}}(\theta^{\hat{A}}, \Delta\rho) \approx Y^{\hat{a}} + \Delta\rho \frac{\partial y^{\hat{a}}}{\partial \rho}, \tag{2.27}$$

this is Taylor approximation where every component is evaluated at $\rho = 0$.

In particular we can always write $\frac{\partial}{\partial \rho} = \psi r$ for some ψ, since r_b is always perpendicular to the surfaces of constant ρ. Such a deformation is depicted in figure 2.3. Then because r is always perpendicular:

$$e^a_A \mathcal{L}_{\psi r} r_a = 0. \tag{2.28}$$

Here $\mathcal{L}_{\psi r} r_a$ is the Lie derivative of r_a along ψr. Then we can write

$$\begin{aligned} e^a_A\Big(\psi r^c D_c r_a + r_c D_a[\psi r^c]\Big) &= 0 \\ \implies \psi e^a_A r^c D_c r_a + d_A \psi &= 0. \end{aligned}$$

Hence,

$$r^c D_c r_a = -\frac{1}{\psi} d_a \psi, \tag{2.29}$$

since $r^a r^c D_c r_a = 0$.

Now consider the variation of the trace of extrinsic curvature $k^{(r)}$ i.e.

$$\begin{aligned} \frac{\partial}{\partial \rho}(k^{(r)}) &= \psi r^c D_c(\tilde{q}^{ab} D_a r_b) \\ &= \psi r^c D_c\Big([q^{ab} - r^a r^b] D_a r_b\Big) \\ &= \psi q^{ab} r^c D_c D_a r_b \\ &= \psi q^{ab} r^c \Big(R_{cabd} r^d + D_a D_c r_b\Big), \end{aligned}$$

where in the last line we used the fact that for the 3-dimensional Riemann tensor

$$R_{cabd} r^d = D_c D_a r_b - D_a D_c r_b. \tag{2.30}$$

Then

$$\frac{\partial}{\partial \rho}(k^{(r)}) = \psi q^{ab} r^c \Big(R_{cabd} r^d + D_a D_c r_b\Big). \tag{2.31}$$

Now from (2.31) we can write

$$\psi q^{ab} r^c R_{cabd} r^d = -\psi q^{ac} r^b r^d R_{abcd}. \tag{2.32}$$

We know that the three-dimensional Ricci scalar is

$$R = q^{ac} q^{bd} R_{abcd}. \tag{2.33}$$

Then it follows that:

$$
\begin{aligned}
R &= \Big(\tilde{q}^{ac} + r^a r^c\Big)\Big(\tilde{q}^{bd} + r^b r^d\Big) R_{abcd} \\
&= \tilde{q}^{ac}\tilde{q}^{bd} R_{abcd} + 2\tilde{q}^{ac} r^b r^d R_{abcd} \\
&= \tilde{q}^{ac}\tilde{q}^{bd} R_{abcd} + 2q^{ac} r^b r^d R_{abcd}.
\end{aligned}
$$

Hence

$$
q^{ac} r^b r^d R_{abcd} = \frac{1}{2}\Big(R - \tilde{q}^{ac}\tilde{q}^{bd} R_{abcd}\Big), \tag{2.34}
$$

and from section 2.2 we can apply the Gauss equation to get

$$
q^{ac} r^b r^d R_{abcd} = \frac{1}{2}\Big(R - {}^{(2)}R - k^{(r)}_{ab} k^{ab}_{(r)} + k^2\Big). \tag{2.35}
$$

Hence from Eqn. (2.32) we get,

$$
\psi q^{ab} r^c R_{cabd} r^d = -\frac{1}{2}\psi\Big(R - {}^{(2)}R - k^{(r)}_{ab} k^{ab}_{(r)} + k^2\Big). \tag{2.36}
$$

Now again from Eqn.(2.31)

$$
\begin{aligned}
\psi q^{ab} r^c D_a D_c r_b &= \psi\tilde{q}^{ab} r^c D_a D_c r_b + \psi r^a r^b r^c D_a D_c r_b \\
&= \psi\tilde{q}^{ab} D_a\Big(r^c D_c r_b\Big) - \psi\tilde{q}^{ab}\Big(D_a r^c\Big)\Big(D_c r_b\Big) + \psi r^a r^c D_a\Big(r^b D_c r_b\Big) \\
&- \psi r^a r^c\Big(D_a r^b\Big)\Big(D_c r_b\Big) \\
&= \psi\tilde{q}^{ab} d_a\Big(-\frac{1}{\psi} d_a\psi\Big) - \psi k^{(r)}_{ab} k^{ab}_{(r)} - \psi\Big(r^a D_a r^b\Big)\Big(r^c D_c r_b\Big) \\
&= -d^2\psi + \frac{1}{\psi}\Big(d_A\psi\Big)\Big(d^A\psi\Big) - \psi k^{(r)}_{AB} k^{AB}_{(r)} - \frac{1}{\psi}\Big(d_A\psi\Big)\Big(d^A\psi\Big).
\end{aligned}
$$

Hence

$$
\psi q^{ab} r^c D_a D_c r_b = -d^2\psi - \psi k^{(r)}_{AB} k^{AB}_{(r)}. \tag{2.37}
$$

From Eqn.(2.31) combining Eqn.(2.36) and Eqn.(2.37) we get

$$
\frac{\partial}{\partial\rho}(k^{(r)}) = -d^2\psi - \psi k^{(r)}_{AB} k^{AB}_{(r)} - \frac{1}{2}\psi\Big(R - {}^{(2)}R - k^{(r)}_{AB} k^{AB}_{(r)} + k^2_{(r)}\Big), \tag{2.38}
$$

i.e.

$$\frac{\partial}{\partial\rho}(k^{(r)}) = -d^2\psi - \frac{1}{2}\psi\Big(R - {}^{(2)}R + k^{(r)}_{AB}k^{AB}_{(r)} + k^2_{(r)}\Big). \tag{2.39}$$

Next the variation of the extrinsic curvature of S with respect to timelike unit normal u is

$$\frac{\partial}{\partial\rho}(k^{(u)}) = \psi r^c D_c\Big(\tilde{q}^{ab}K_{ab}\Big). \tag{2.40}$$

This implies that

$$\begin{aligned}\frac{\partial}{\partial\rho}(k^{(u)}) &= \psi r^c D_c\Big(K - K_{ab}r^a r^b\Big)\\ &= \psi\Big(r^c D_c K - r^a r^b r^c D_c K_{ab} - K_{ab}r^a r^c D_c r^b - K_{ab}r^b r^c D_c r^a\Big)\\ &= \psi\Big(r^c D_c K - r^a\Big[q^{bc} - \tilde{q}^{bc}\Big]D_c K_{ab} - 2\psi K_{ab}r^a\Big[r^c D_c r^b\Big]\Big)\end{aligned}$$

i.e.

$$\frac{\partial}{\partial\rho}(k^{(u)}) = \psi r^c\Big(D_c K - D_b K^b_c\Big) + \psi r^a\tilde{q}^{bc}D_c K_{ab} - 2\psi K_{ab}r^a\Big(-\frac{1}{\psi}d^b\psi\Big). \tag{2.41}$$

Then from Eqn.(2.41) we get

$$\begin{aligned}\frac{\partial}{\partial\rho}(k^{(u)}) &= -\psi G_{\alpha\beta}u^\alpha r^\beta + \psi\tilde{q}^{bc}D_c\Big(K_{ab}r^a\Big) - \psi\tilde{q}^{bc}K_{ab}D_c\Big(r^a\Big) + 2\Big[\tilde{q}^b_a K_{bc}u^c\Big]d^a\psi\\ &= -\psi G_{\alpha\beta}u^\alpha r^\beta + \psi\tilde{q}^{bc}D_c\Big(\tilde{q}^d_b K_{da}r^a + r_b\Big[K_{da}r^d r^a\Big]\Big)\\ &- \psi\tilde{q}^{bc}K_{ba}\Big[\tilde{q}^a_d + r_d r^a\Big]D_c r^d + 2\tilde{\omega}^A d_A\psi.\end{aligned}$$

This implies

$$\frac{\partial}{\partial\rho}(k^{(u)}) = -\psi G_{\alpha\beta}u^\alpha r^\beta + \psi d_C\tilde{\omega}^C + \psi\Big(K_{ab}r^a r^b\Big)k^{(r)} - \psi k^{(u)}_{AB}k^{AB}_{(r)} + 2\tilde{\omega}^A d_A\psi, \tag{2.42}$$

where $\tilde{\omega}^A$ is the connection of normal bundle defined by

$$\begin{aligned}\tilde{\omega}_A &= e^\beta_A N_\alpha \nabla_\beta \ell^\alpha \\ &= -\frac{1}{2} e^\beta_A \Big(u_\alpha - r_\alpha\Big) \nabla_\beta \Big(u_\alpha + r_\alpha\Big) \\ &= e^\beta_A r^\alpha \nabla_\alpha u_\beta,\end{aligned}$$

and so in terms of the extrinsic curvature,

$$\tilde{\omega}_A = e^\beta_A K_{\beta\alpha} r^\alpha. \tag{2.43}$$

Now going back to Eqn.(2.39), viewing Σ as a hypersurface and applying the Hamiltonian constraint we have

$$\frac{1}{2} R = G_{\alpha\beta} u^\alpha u^\beta - \frac{1}{2}\Big(K^2 - K_{ab}K^{ab}\Big). \tag{2.44}$$

Also,

$$\begin{aligned}K_{ab} &= k^{(u)}_{ab} + \tilde{\omega}_a r_b + \tilde{\omega}_b r_a + K_{rr} r_a r_b, \\ \implies K &= k^{(u)} + K_{rr}\end{aligned}$$

and

$$K_{ab}K^{ab} = k^{(u)}_{ab} k^{ab}_{(u)} + 2\tilde{\omega}^A \tilde{\omega}_A + K^2_{rr}.$$

Combining K and $K_{ab}K^{ab}$ we can write

$$K^2 - K_{ab}K^{ab} = k^2_{(u)} + 2K_{rr}k^{(u)} - k^{(u)}_{AB} k^{AB}_{(u)} - 2\|\tilde{\omega}\|^2, \tag{2.45}$$

where $\|\tilde{\omega}\|^2 = \tilde{\omega}^A \tilde{\omega}_A$. By using Eqn.(2.45) in Eqn.(2.44) we get

$$\frac{1}{2} R = G_{\alpha\beta} u^\alpha u^\beta - \frac{1}{2}\Big(k^2_{(u)} + 2K_{rr}k^{(u)} - k^{(u)}_{AB} k^{AB}_{(u)} - 2\|\tilde{\omega}\|^2\Big). \tag{2.46}$$

Therefore looking back to Eqn.(2.39)

$$\frac{\partial}{\partial\rho}\Big(k^{(r)}\Big) = -d^2\psi + \frac{\psi}{2}\Big({}^{(2)}R - k_{(r)}^2 - k^{(r)}_{AB}k^{AB}_{(r)}\Big) - \psi\Big(G_{\alpha\beta}u^\alpha u^\beta + \frac{1}{2}\Big[k^{AB}_{(u)}k^{(u)}_{AB} - k^2_{(u)}\Big] - K_{rr}k^{(u)} + \|\tilde{\omega}\|^2\Big) \tag{2.47}$$

we know that outward null vector is $\ell = u + r$ and outward null expansion

$$\theta_{(\ell)} = k_{(u)} + k_{(r)}, \tag{2.48}$$

we defined $\theta_{(\ell)}$ in Eqn.(2.12). Then the variation of outward null expansion is

$$\frac{\partial}{\partial\rho}\Big(\theta_{(\ell)}\Big) = \frac{\partial}{\partial\rho}\Big(k_{(u)}\Big) + \frac{\partial}{\partial\rho}\Big(k_{(r)}\Big). \tag{2.49}$$

By adding Eqn.(2.42) and Eqn.(2.47) then Eqn.(2.48) becomes

$$\begin{aligned}\frac{\partial}{\partial\rho}\Big(\theta_{(\ell)}\Big) = {} & -\psi G_{\alpha\beta}u^\alpha r^\beta + \psi d_C\tilde{\omega}^C + \psi\Big(K_{ab}r^a r^b\Big)k^{(r)} - \psi k^{(u)}_{AB}k^{AB}_{(r)} + 2\tilde{\omega}^A d_A\psi - d^2\psi \\ & + \frac{\psi}{2}\Big({}^{(2)}R - k^2_{(r)} - k^{(r)}_{AB}k^{AB}_{(r)}\Big) - \psi\Big(G_{\alpha\beta}u^\alpha u^\beta + \frac{1}{2}\Big[k^{AB}_{(u)}k^{(u)}_{AB} - k^2_{(u)}\Big] \\ & - K_{rr}k^{(u)} + \|\tilde{\omega}\|^2\Big),\end{aligned} \tag{2.50}$$

and this becomes

$$\begin{aligned}\frac{\partial}{\partial\rho}\Big(\theta_{(\ell)}\Big) = {} & -d^2\psi + 2\tilde{\omega}^A d_A\psi + \psi\Big(\frac{1}{2}\,{}^{(2)}R - \|\tilde{\omega}\|^2 + d_A\tilde{\omega}^A - G_{\alpha\beta}u^\alpha\ell^\beta\Big) \\ & + \psi K_{rr}\Big(k_{(u)} + k_{(r)}\Big) + \psi\Big(-k^{(u)}_{AB}k^{AB}_{(r)} - \frac{1}{2}k^2_{(r)} - \frac{1}{2}k^{(r)}_{AB}k^{AB}_{(r)} \\ & + \frac{1}{2}k^2_u - \frac{1}{2}k^{AB}_{(u)}k^{(u)}_{AB}\Big),\end{aligned} \tag{2.51}$$

but $\theta_{(\ell)} = k_{(u)} + k_{(r)}$ so we can write Eqn.(2.51) as

$$\frac{\partial}{\partial \rho}\Big(\theta_{(\ell)}\Big) = -d^2\psi + 2\tilde{\omega}^A d_A\psi + \psi\Big(\frac{1}{2}\,{}^{(2)}R - \|\tilde{\omega}\|^2 + d_A\tilde{\omega}^A - G_{\alpha\beta}u^\alpha\ell^\beta\Big) + \psi K_{rr}\theta_{(\ell)}$$
$$+\psi\Big(\frac{1}{2}\Big(k^2_{(u)} - k^2_{(r)}\Big) - \Big[k^{(u)}_{AB} + k^{(r)}_{AB}\Big]\Big[k^{AB}_{(u)} + k^{AB}_{(r)}\Big]\Big), \tag{2.52}$$

here $k^2_{(u)} - k^2_{(r)} = \theta_{(\ell)}\theta_{(r)}$ and $\Big[k^{(u)}_{AB} + k^{(r)}_{AB}\Big]\Big[k^{AB}_{(u)} + k^{AB}_{(r)}\Big] = k^{(\ell)}_{AB}k^{AB}_{(\ell)} = \frac{1}{2}\theta^2_{(\ell)} - \sigma^{AB}_{(\ell)}\sigma^{(\ell)}_{AB}$. Hence Eqn.(2.52) becomes

$$\frac{\partial}{\partial \rho}\Big(\theta_{(\ell)}\Big) = -d^2\psi + 2\tilde{\omega}^A d_A\psi + \psi\Big(\frac{1}{2}\,{}^{(2)}R - \|\tilde{\omega}\|^2 + d_A\tilde{\omega}^A - G_{\alpha\beta}u^\alpha\ell^\beta\Big) + \psi K_{rr}\theta_{(\ell)}$$
$$+\psi\Big(\theta_{(\ell)}\theta_{(r)} - \frac{1}{2}\theta^2_{(\ell)} - \sigma^{AB}_{(\ell)}\sigma^{(\ell)}_{AB}\Big). \tag{2.53}$$

Now, $\theta_{(\ell)} = 0$ if S is a marginally outer trapped surface (MOTS). The details of MOTS are explained in section 2.4 . Hence finally the variation of outward null expansion of a MOTS can be written as

$$\frac{\partial}{\partial \rho}\Big(\theta_{(\ell)}\Big)\Big|_{\rho=0} = -d^2\psi + 2\tilde{\omega}^A d_A\psi + \psi\Big(\frac{1}{2}\,{}^{(2)}R - \|\tilde{\omega}\|^2 + d_A\tilde{\omega}^A - \sigma^{AB}_{(\ell)}\sigma^{(\ell)}_{AB} - G_{\alpha\beta}u^\alpha\ell^\beta\Big), \tag{2.54}$$

and we will use this to define the stability operator. We want to know whether a MOTS can be deformed or not and this differential operator tells us how $\theta_{(\ell)}$ changes as S is deformed. This is the topic of the next section.

2.6 Principal eigenvalues and the Stability Operator

The derivative $\frac{\partial}{\partial\rho}\theta_{(\ell)}$ is a second order elliptic operator for ψ on S. We define the stability operator as:

$$L\psi = \frac{\partial}{\partial \rho}\Big(\theta_{(\ell)}\Big)\Big|_{\rho=0} \tag{2.55}$$

Note that if a MOTS can be continuously deformed then under the deformation

$$L\psi = \frac{\partial}{\partial \rho}\Big(\theta_{(\ell)}\Big) = 0. \tag{2.56}$$

Hence the existence of a ψ for which $L\psi = 0$ is a necessary condition for such a deformation to exist. Equivalently L must have a vanishing eigenvalue for such a deformation to exist [10, 17–19]. Hence let us consider the operator in more detail. From 2.54.

$$L\psi = -d^2\psi + 2\tilde{\omega}^A d_A\psi + \psi\Big(\frac{1}{2}\,{}^{(2)}R - \|\tilde{\omega}^2\| + d_A\tilde{\omega}^A - \sigma^{AB}_{(\ell)}\sigma^{(\ell)}_{AB} + G_{\alpha\beta}\ell^\alpha n^\beta\Big). \tag{2.57}$$

As a result of the presence of the first-order term in (2.57), L is not self-adjoint in general. Seeing that L is linear and elliptic and S is compact, the corresponding eigenfunction are regular with discrete eigenvalues, however in general they are complex [18]. The following definition arises from the above discussion:

Definition 1. *The eigenvalue of L with smallest real part is called the **principal eigenvalue**.*

Recall minimal surfaces in Euclidean geometry. A minimal surface is one for which the mean curvature vanishes. Equivalently, since this trace is rate of change of the area if the surface is deformed in the normal direction, this is a critical point of the area functional. Then the derivative of the trace of the extrinsic curvature is the second derivative of the area. Then a stable surface has non-negative eigenvalues (it is a minimum of the area) while an unstable one has one or more negative eigenvalues.

There is a closely related notion of stability for MOTS. In this case the direction of vanishing expansion ℓ is different from that of deformation ($\hat{r}$) but otherwise things are very similar. We proceed as follows starting with the eigenvalues of L [16]. Hence we are very interested in the eigenvalues of L.

The following lemma holds for second order elliptic operators of the form of L.

Lemma 1 ([20])**.** *The principal eigenvalue λ of L is real and the corresponding eigenfunction (the principal eigenfunction) ψ is either everywhere positive or everywhere negative.*

The following definition given by [20] determines if the variation of expansion can be everywhere non-negative or somewhere positive.

Definition 2. *A marginally outer trapped surface S is called* ***stably outermost*** *if there exists a function $\psi \geq 0, \psi \neq 0$ on S such that $L\psi \geq 0$. S is called* ***strictly stably outermost*** *if $L\psi \neq 0$ somewhere on S.*

Note that this is equivalent to saying that there is an (infinitesimal) variation generated by $\psi\hat{r}$ for which S becomes outer untrapped and another, generated by $-\psi\hat{r}$ for which it becomes outer trapped. Now to describe the relation between stability and the sign of principal eigenvalue the following lemma holds:

Lemma 2 ([20]). *Let S be a MOTS and λ be the principal eigenvalue of the operator L. Then S is* ***stably outermost*** *if and only if $\lambda \geq 0$, and* ***strictly stably outermost*** *if and only if $\lambda > 0$.*

Definition 3. *The marginally outer trapped surface S is* ***stable*** *if the principal eigenvalue $\lambda \geq 0$ and* ***strictly stable*** *if the principal eigenvalue is positive* $(\lambda > 0)$.

The MOTS is stably outermost if and only if it is stable and it is strictly stably outermost if and only if it is strictly stable. So that we can say the above definition 2 and definition 3 are equivalent. The following definition given in [20] is motivated by above discussion:

Definition 4. *A marginally outer trapped surface S is called* ***locally outermost*** *in Σ iff there exists a two sided neighbourhood U of S such that the outer part of U does not contain a weakly outer trapped surface .*

The proposition below from [20] describes the association between definition 3 and definition 4.

Proposition 1.

1. *If the principal eigenvalue $\lambda > 0$ then the surface S is locally outermost.*
2. *A* ***locally outermost*** *surface S is stably outermost (which is $\lambda \geq 0$).*

2.7 Stability Vs Rigidity

We now define rigidity. This is essentially the finite version of invertibility of L.

Definition 5. *A MOTS S is **rigid** if there exists an $\epsilon > 0$ such that the only MOTS of the form $S + \psi$, with $\|\psi\|_{C^{2,\alpha}} < \epsilon$, is S itself.*

In the above definition $\|\psi\|_{C^{2,\alpha}}$ is α^{th}-Hölder norm and the space function $C^{2,\alpha}$ consists of those functions ψ that are second derivative continuous and whose second partial derivatives are bounded and Hölder continuous with exponent α [21]. Here $S + \psi$ represents the surface deformed in the normal direction by the function ψ. In section 2.6 we defined the stability of a MOTS in terms of the principal eigenvalue. Also Eqn(2.56) says that, if there is a solution to $L\psi = 0$, then L is not invertible, so S may or may not be rigid. But an unstable MOTS could be rigid or not rigid.

We can write the eigenvalue equation with corresponding eigenfunction ψ i.e

$$L\psi = \lambda\psi. \tag{2.58}$$

From the eigenvalue equation, $L\psi = 0$ has a solution precisely if any of the eigenvalues are zero.

We can prove something about rigidity: when 0 is not an eigenvalue of the stability operator L then the marginally outer trapped surface (MOTS) is **rigid** and hence strict stability implies rigidity. See theorem 1 in chapter 4 for a more precise statement. Also the MOTS is strictly stable if the principal eigenvalue λ is positive. In particular positive means that all of the eigenvalues have positive real parts and hence are non zero.

Chapter 3

Marginally Outer Trapped Surfaces in RNdS spacetime

As we discussed earlier in section 2.6 and section 2.7, the stability of a marginally outer trapped surface is defined in terms of its principal eigenvalue. In this chapter first we begin with reviewing some equations for unstable marginally outer trapped surface (MOTS) in RNdS spacetime which can be infinitesimally deformed and develop the formalism to understand whether or not these deformations can be made finite. We use the formalism based on the results of [10, 22].

3.1 MOTS in Painlevé-Gullstrand coordinates for RNdS spacetime

In this section we consider the stability of marginally outer trapped surfaces in RNdS spacetime. We look at the horizons in RNdS spacetime and discover that most are stable (outer black hole horizon, cosmological horizon $M \neq 0$ case). However, there are also an infinite number of spacetimes with inner horizons that are unstable MOTS. We will start with introducing RNdS spacetime in Painlevé-Gullstrand coordinates.

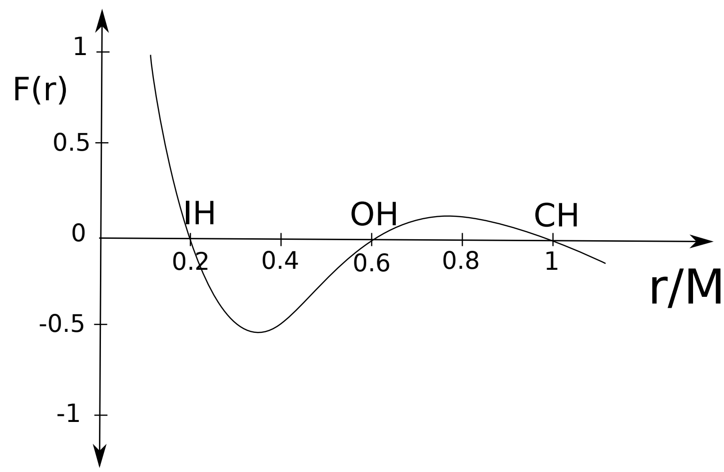

Figure 3.1: $F(r)$ vs $\frac{r}{M}$ for typical cosmological black hole solution. IH, OH, CH denote the inner horizon, outer horizon and cosmological horizon of black hole respectively. **Figure taken from [10]**

3.1.1 Reissner-Nordström-de Sitter spacetime and coordinates

The 4-dimensional RNdS spacetime including cosmological constant Λ in terms of Painlevé-Gullstrand coordinates (T, r, θ, ϕ) is described by the metric [10, 23]

$$ds^2 = -F(r)dT^2 + 2\sqrt{1 - F(r)}dTdr + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.1}$$

where

$$F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2, \tag{3.2}$$

with $M > 0$, electric charge $Q > 0$ and cosmological constant $\Lambda > 0$. This metric represents a spherically symmetric charged black hole. One of the most important characteristics of PG coordinates is that the spatial slices $T=$ constant are intrinsically flat [23] in these coordinates. To see this note that a surface at constant T has $dT = 0$. Then Eqn.(3.1) becomes

$$ds_3^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.3}$$

which is the three dimensional Euclidean metric in spherical coordinates (r, θ, ϕ).

3.1.2 MOTS in PG coordinates

We will look for MOTS in surfaces of constant T. For that it will be useful to have the extrinsic curvature of S with respect to $\hat{u}$ and $\hat{r}$.

First consider that the future-oriented unit timelike normal to Σ is

$$\hat{u}^a \partial_a = \partial_T - \sqrt{1 - F(r)}\partial_r. \tag{3.4}$$

The extrinsic curvature of Σ is then

$$K_{ab} = \begin{bmatrix} \frac{F'(r)}{2\sqrt{1-F(r)}} & 0 & 0 \\ 0 & -r\sqrt{1-F(r)} & 0 \\ 0 & 0 & -r\sin^2\theta\sqrt{1-F(r)} \end{bmatrix}, \tag{3.5}$$

where $F'(r)$ represents the derivative of F with respect to r.

Next consider a surface in Σ that is defined by:

$$r = r_0 + \rho\psi \tag{3.6}$$

for some function ψ. Here r_0 is inner horizon and $\rho\psi$ is the finite perturbation with ρ parameterized in magnitude. Hence

$$r_0 = r - \rho\psi. \tag{3.7}$$

and a spacelike normal to S in Σ is

$$d(r - \rho\psi) = dr - \rho\psi_\theta d\theta - \rho\psi_\phi d\phi, \tag{3.8}$$

which can be normalized as:

$$\hat{r}_a = \frac{r\sin\theta}{\sqrt{\rho^2\psi_\theta^2\sin^2\theta + \rho^2\psi_\phi^2 + r^2\sin^2\theta}}\Big[1, \rho\psi_\theta, -\rho\psi_\phi\Big]. \tag{3.9}$$

Now by $\tilde{q}_{ab} = h_{ab} - \hat{r}_a \hat{r}_b$ the induced metric on S is

$$\tilde{q}_{\hat{a}\hat{b}} = \frac{1}{\frac{\rho^2\psi_\theta^2}{r^2} + \frac{\rho^2\psi_\phi^2}{r^2\sin^2\theta} + 1} \begin{bmatrix} \frac{\rho^2\psi_\theta^2}{r^2} + \frac{\rho^2\psi_\phi^2}{r^2\sin^2\theta} & \rho\psi_\theta & \rho\psi_\phi \\ \rho\psi_\theta & r^2 + \frac{\rho^2\psi_\phi^2}{r^2\sin^2\theta} & -\rho^2\psi_\theta\psi_\phi \\ \rho\psi_\phi & -\rho^2\psi_\theta\psi_\phi & \sin^2\theta(\rho^2\psi_\theta^2 + r^2) \end{bmatrix} \tag{3.10}$$

and note that by substituting $\rho = 0$ we get the induced two metric in the expected form,

$$\tilde{q}_{\hat{a}\hat{b}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}. \tag{3.11}$$

Meanwhile, the trace of the extrinsic curvature of S in Σ with respect to $\hat{r}$ is

$$k_{(\hat{r})} \equiv \tilde{q}^{ab} D_a \hat{r}_b. \tag{3.12}$$

It follows that

$$\begin{aligned} k_{(\hat{r})} := \frac{1}{r\Big(\rho^2\Big(\sin^2\theta\psi_\theta^2 + \psi_\phi^2\Big) + r^2\sin^2\theta\Big)^{\frac{3}{2}}} \Big[&- \rho\sin^2\theta\cos\theta\psi_\theta\Big(\rho^2 + r^2\Big) \\ &-\rho r^2\sin\theta\Big(\psi_{\phi\phi} + \sin^2\theta\psi_{\theta\theta}\Big) + \rho^3\sin\theta\Big(2\psi_\theta\psi_\phi\psi_{\phi\theta} - \psi_\phi^2\psi_{\theta\theta} \\ &-\psi_\theta^2\psi_{\phi\phi}\Big) + 2r^3\sin^3\theta + 3r\rho^2\sin\theta\Big(\sin\theta\psi_\theta^2 + \psi_\phi^2\Big)\Big]. \end{aligned} \tag{3.13}$$

Again checking the $\rho = 0$ case, we find the expected

$$k_{(\hat{r})} = \frac{2}{r}. \tag{3.14}$$

The trace of extrinsic curvature of S with respect to $\hat{u}$ is

$$k_{(\hat{u})} \equiv \tilde{q}^{ab} D_a u_b = \tilde{q}^{ab} K_{ab}, \tag{3.15}$$

and computing $k_{(\hat{u})}$, we find that

$$k_{(\hat{u})} := \frac{r\rho^2 F'(r)\Big[\psi_\theta^2 \sin^2\theta + \psi_\phi^2\Big] + 2\Big(1 - F(r)\Big)\Big[-2r^2\sin^2\theta - \rho^2\Big(\psi_\phi^2 + \sin^2\theta\psi_\theta^2\Big)\Big]}{2r\sqrt{1-F(r)}\Big(\rho^2\psi_\theta^2\sin^2\theta + \rho^2\psi_\phi^2 + r^2\sin^2\theta\Big)}. \tag{3.16}$$

For $\rho = 0$ this reduces to

$$k_{(\hat{u})} = -\frac{2\sqrt{1-F(r)}}{r}. \tag{3.17}$$

The outward null expansion is $\theta_{(\ell)} = k_{(\hat{r})} + k_{(\hat{u})}$ and we get

$$\begin{aligned}
\theta_{(\ell)} = {} & \frac{r\rho^2 F_1(r)\Big(\psi_\theta^2\sin^2\theta + \psi_\phi^2\Big) - 2\Big(1-F(r)\Big)\Big[2r^2\sin^2\theta + \rho^2\Big(\psi_\theta^2 + \psi_\phi^2\Big)\Big]}{2r\sqrt{1-F(r)}\Big[\rho^2\Big(\psi_\theta^2\sin^2\theta + \psi_\phi^2\Big) + r^2\sin^2\theta\Big]} \\
& + \frac{1}{\Big[r\Big(\rho^2\Big(\psi_\theta^2\sin^2\theta + \psi_\phi^2\Big) + r^2\sin^2\theta\Big)^{\frac{3}{2}}\Big]}\Big[\rho\sin\theta\Big(-\rho^2\psi_\phi^2 - r^2\sin^2\theta\Big)\psi_{\theta\theta} \\
& + 2\rho^3\psi_\phi\psi_{\phi\theta}\psi_\theta\sin\theta - \rho\psi_{\phi\phi}\sin\theta\Big(\rho^2\psi_\theta^2 + r^2\Big) - \rho^3\psi_\theta^3\cos\theta\sin^2\theta \\
& + 3r\rho^2\psi_\theta^2\sin^3\theta + \rho\cos\theta\Big(-r^2\sin^2\theta - 2\rho^2\psi_\phi^2\Big)\psi_\theta \\
& - 2r\sin\theta\Big(-r^2\sin^2\theta - \frac{3}{2}\rho^2\psi_\phi^2\Big)\Big]
\end{aligned} \tag{3.18}$$

By substituting $r = r_0$, $\rho = 0$ in Eqn.(3.18) the outward null expansion $\theta_{(\ell)}$ becomes

$$\theta_{(\ell)}\Big|_{r=r_0} = \frac{-2\sqrt{1-F(r_0)} + 2}{r_0}. \tag{3.19}$$

Hence we have a MOTS when $F(r_0) = 0$.

In this case we can also directly find the stability operator by differentiating Eqn.(3.18) with respect to ρ as

$$L\psi = \frac{\partial}{\partial\rho}\Big(\theta_{(\ell)}\Big)\Big|_{r=r_0} = \frac{F'(r_0)\psi}{r_0} - \frac{\psi_{\theta\theta}}{r_0^2} - \frac{\psi_\theta\cos\theta}{r_0^2\sin\theta} - \frac{\psi_{\phi\phi}}{r_0^2\sin^2\theta}. \tag{3.20}$$

The last three terms of Eqn.(3.20) can be combined to give

$$\frac{\partial}{\partial\rho}\Big(\theta_{(\ell)}\Big)\Big|_{r=r_0} = \frac{F'(r_0)}{r_0}\psi - \frac{1}{r_0^2}\Delta\psi, \tag{3.21}$$

where $\Delta\psi$ is the Laplacian on the unit sphere and Eqn.(3.21) is a general formula for L, whether or not $\frac{\partial}{\partial\rho}\Big(\theta_{(\ell)}\Big) = 0$.

Later on we will also find it useful to know the second variation of outgoing null expansion. So by differentiating Eqn.(3.18) two times we get the second variation of outgoing null expansion which takes the form:

$$\begin{aligned}\frac{\partial^2}{\partial^2\rho}\Big(\theta_{(\ell)}\Big)\Big|_{\rho=0} = \frac{2\psi_\theta^2}{{r_0}^3} + \frac{4}{{r_0}^3}\Delta\psi + \frac{F'(r_0)\psi_\theta^2}{{r_0}^2} + \frac{F'(r_0)^2\psi^2}{2r_0} + \frac{\psi^2F''(r_0)}{r_0} - \frac{2\psi^2F'(r_0)}{{r_0}^2}\\ +\frac{4\psi_\theta\psi\cos\theta}{{r_0}^3\sin\theta} + \frac{1}{\sin^2\theta}\left(\frac{2\psi_\phi^2}{{r_0}^3} + \frac{\psi_\phi^2F'(r_0)}{{r_0}^2} + \frac{4}{{r_0}^3}\Delta\psi\right).\end{aligned} \tag{3.22}$$

Now (3.22) can be simplified assuming $\frac{\partial}{\partial\rho}\Big(\theta_{(\ell)}\Big)\Big|_{\rho=0} = 0$:

$$\begin{aligned}\frac{\partial^2}{\partial^2\rho}\Big(\theta_{(\ell)}\Big)\Big|_{\rho=0} = \left(\frac{F'(r_0)^2}{2r_0} + \frac{2F'(r_0)}{{r_0}^2} + \frac{F''(r_0)}{r_0}\right)\psi^2 + \Big(\frac{2}{{r_0}^3} + \frac{F'(r_0)}{{r_0}^2}\Big)\psi_\theta^2\\ +\frac{1}{\sin^2\theta}\left(\frac{2}{{r_0}^3} + \frac{F'(r_0)}{r_0^2}\right)\psi_\phi^2.\end{aligned} \tag{3.23}$$

All of these calculations were obtained using Maple.

3.2 Locating Unstable MOTS in RNdS spacetime

We now consider the stability of MOTS in RNdS. In general $F(r)$ from Eqn.(3.2) has three roots, as shown in figure.(3.1). There are the inner black/white hole (r_{IH}), outer black/white hole (r_{OH}) and cosmological future/past (r_{CH}) horizon shown in increasing order. From Eqn.(3.20) the stability operator evaluated at constant r_0 is

$$\frac{\partial}{\partial\rho}\Big(\theta_{(\ell)}\Big)\Big|_{\rho=0} = L\psi = -\frac{\Delta\psi}{r_0^2} + \frac{F'(r_0)}{r_0}\psi. \tag{3.24}$$

The eigenfunctions and eigenvalues of this operator can be found directly. The eigenfunctions here are the usual spherical harmonics which have real and imaginary parts. But for the deformation of MOTS we will consider only real valued eigenfunctions. Hence instead of standard spherical harmonics (Y_l^m) we will use associated Legendre polynomial (P_l^m) multiplied by trigonometric function of ϕ. For any non-negative value l and one for each integer m with $-l \leq m \leq l$, then the general form for eigenfunctions can be written as

$$\psi = P_l^m(\cos\theta)(A_{lm}\cos(m\phi) + B_{lm}\sin(m\phi)) \tag{3.25}$$

where A_{lm}, B_{lm} are constants. Later on in our calculations we will choose a particular basis of eigenfunctions which is defined by

$$S_l^m(\theta,\phi) = \begin{cases} \cos(m\phi)P_l^m(\cos\theta) & m \geq 0 \\ \sin(m\phi)P_l^m(\cos\theta) & m < 0 \end{cases} \tag{3.26}$$

Now the corresponding eigenvalues of the stability operator L are $\frac{l(l+1)}{r^2} + \frac{F'(r)}{r}$ for $l = 0, 1, 2, \ldots$. Then if $F'(r) > 0$ all eigenvalues are positive. Negative eigenvalues can only occur if $F'(r) < 0$. Refering to Fig.3.1 this means that the outer horizon is always stable while the inner horizon and cosmological horizon could be unstable depending on the magnitude of $F'(r)$.

The family of (l, m) eigenfunctions have vanishing eigenvalues if

$$l(l+1) + rF'(r) = 0 \tag{3.27}$$

for some $l = 0, 1, 2, \ldots$. The possible values of $-rF'(r)$ are shown for all possible RNdS spacetimes are shown in figure.3.2. Then it is again clear that for r_{OH} all eigenvalues are positive and so all outer horizons are strictly stable.

The cosmological horizon is more interesting. In that case we can see that all $l > 1$ eigenvalues are positive. However for $l = 1$ we can have vanishing eigenvalue if $M = 0$, $Q = 0$ (the top of the blue sheet). In $l = 1$, the only case is $-rF'(r) = 2$ along with $M = 0$, this corresponds to pure-de Sitter case where the cosmological horizon is neither stable nor rigid: this cosmological horizon may be "translated" anywhere in this homogeneous spacetime.

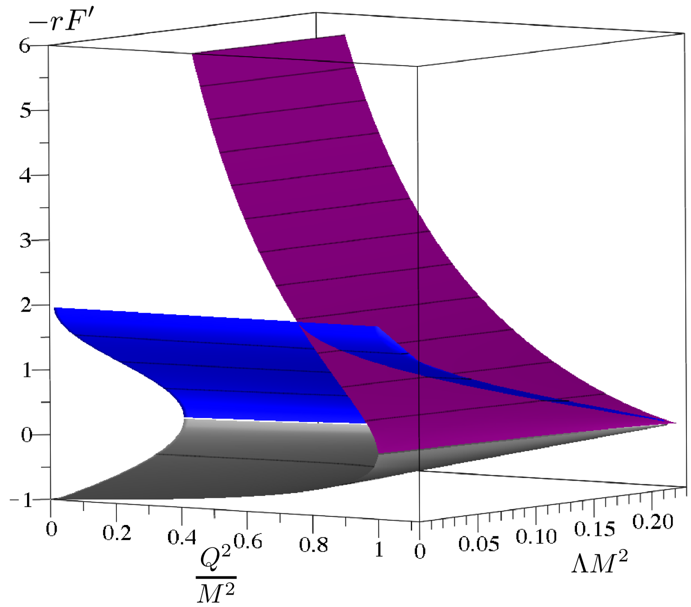

Figure 3.2: The values of $-rF'(r)$ for outer horizon (grey), cosmological horizon (blue), inner horizon (purple). For $-rF'(r) = l(l+1), l$ positive integer hold when potential instabilities exist.

Figure taken from [10]

In earlier section 3.1.2 we have calculated the extrinsic curvature K_{ab} at $T =$ constant surfaces in Painlevé-Gullstrand coordinates (pure-de Sitter case). For that spacetime

$$F(r) = 1 - \frac{\Lambda}{3}r^2. \tag{3.28}$$

Now from Eqn.(3.17) we get the trace of extrinsic curvature is

$$k_{(\hat{u})} = \tilde{q}^{ab}K_{ab} = -2\sqrt{\frac{\Lambda}{3}} \tag{3.29}$$

which is constant for any surface with same $k_{(\hat{r})}$ and it is MOTS i.e $\theta_{(\ell)} = 0$. Also in $T =$ constant surface, any sphere of radius $r = \sqrt{\frac{3}{\Lambda}}$ has

$$k_{(\hat{r})} = 2\sqrt{\frac{\Lambda}{3}}. \tag{3.30}$$

Then any such sphere has $\theta_{(\ell)} = k_{(\hat{u})} + k_{(\hat{r})} = 0$. Hence none of these are rigid: the non rigidity shows up as the ability to move the spheres around while leaving $\theta_{(\ell)} = 0$.

Even more interesting is r_{IH}. There since $r_{IH}F'(r_{IH})$ can be arbitrarily large we can make any eigenvalue vanish with a careful choice of physical parameter. In particular for

$$-r_{IH}F'(r_{IH}) = l(l+1) \tag{3.31}$$

0 is an eigenvalue of L of multiplicity $2l+1$, with eigenfunctions of the form Eqn.(3.25) for $-l \leq m \leq l$. Hence we now have concrete examples of horizons that are stable and rigid (outer horizons) and as well as unstable and not rigid (cosmological in pure de-Sitter). However we also have a family of MOTS (the inner horizons) that are unstable but we don't know whether or not they are rigid. It turns out that this family continues to exist for Reissner-Nordström spacetime with vanishing Λ, so for simplicity we reduce to that case. We will investigate these cases in the next chapter.

Chapter 4

Lyapunov-Schmidt Reduction and MOTS

The previous chapter found a class of RN spacetimes for which the stability operator on the inner horizon has a 0 eigenvalue. This is essentially the first derivative test with 0 eigenvalue, i.e., the derivative of expansion is zero. However, in this chapter, we are interested in the second derivative of the expansion. We will introduce the Lyapunov-Schmidt reduction to study the differential operator to a finite system of polynomial equations. Lyapunov-Schmidt reduction is used for solving non-linear equations, i.e $\theta_{(\ell)} = 0$, when the implicit function theorem does not work.

Let $S \subset \Sigma$ be a MOTS so that $\theta_{(\ell)} = 0$. For any function ψ on S from (2.55) we have

$$L\psi = \frac{\partial}{\partial \rho}\Big(\theta_{(\ell)}\Big)\Big|_{\rho=0}, \tag{4.1}$$

which defines the stability operator L. If L is invertible, i.e does not have 0 as an eigenvalue, then S will be rigid in Σ. Thus one can prove the following theorem [20]

Theorem 1. *Let S be a MOTS and assume that the stability operator L is invertible then there exists $\epsilon > 0$ such that the only MOTS of the form $S + \psi$, with $\|\psi\|_{C^{2,\alpha}} < \epsilon$, is S itself.*

The theorem does not apply if L has a zero eigenvalue.

4.1 The Lyapunov-Schmidt Reduction for MOTS

To examine the rigidity we will introduce Lyapunov-Schmidt reduction beyond looking at the stability operator. The Lyapunov-Schmidt reduction is applied to solve non linear equations when the inverse function theorem does not work.

Now consider the case that L is formally self adjoint (does not contain first derivative term); this will be the case when the spacelike hypersurface M is time symmetric. Later we view L as a bounded linear operator such that $L : C^{2,\alpha}(S) \to C^{\alpha}(S)$. Assume $\ker L$ is non trivial so that for some smooth functions $\{\psi_i\}$

$$\ker L = \mathrm{span}\{\psi_1, \psi_2, \dots, \psi_N\}. \tag{4.2}$$

Suppose $X \subset C^{2,\alpha}(S)$ denotes the L^2- orthogonal complement of $\ker L$. Since all functions in $\ker L$ are smooth, we get the decomposition $C^{2,\alpha}(S) = X \oplus \ker L$ [17]. Also let $Y \subset C^{\alpha}(S)$ is the range of L, and P be the orthogonal projection operator onto Y then

$$X = \Big\{\phi \in C^{2,\alpha}(S) : \int_S \phi\psi_i = 0 \text{ for all i}\Big\}, \tag{4.3}$$

By construction $PL|_X : X \to Y$ is invertible. The implicit function theorem for Banach spaces guarantees that there exist neighbourhoods $0 \in U \subset \ker L$ and $0 \in V \subset X$ and a map $g : U \to V$ such that

$$P\theta\Big(\psi + g(\psi)\Big) = 0 \tag{4.4}$$

for all $\psi \in U$. Note that $P\theta$ is a nonlinear map from $C^{2,\alpha}(S) \to C^{\alpha}(S)$, and its linearization is precisely PL. Moreover the inverse function theorem guarantees that every solution to $P\theta(\phi) = 0$ near 0 (i.e in the neighbourhood $U \times V$) must be of the form $\phi = \psi + g(\psi)$.

We can see that $\theta(\phi) = 0$ if and only if $P\theta(\phi) = 0$ and $(I - P)\theta(\phi) = 0$. Therefore there will be a small solution to $\theta(\phi) = 0$ if and only if there exists $\psi \in U \subset \ker L$ for which

$$(I - P)\,\theta\,(\psi + g(\psi)) = 0. \tag{4.5}$$

Since $\ker L$ is N dimensional and the projection $(I-P)$ maps onto $(\operatorname{ran} L)^{\perp} = \ker L^{*} = \ker L$, (4.5) can be written as $N \times N$ system of equations.

4.2 The Reduced equation

We now study the reduced system of equations (4.5) and identify $\ker L$ with $\mathbb{R}^N$ by mapping

$$\big(t_1, t_2, \ldots, t_N\big) \mapsto t_1\psi_1 + \cdots + t_N\psi_N. \tag{4.6}$$

Therefore we can write $g(\psi) = g(t)$, where $t = \big(t_1, \ldots, t_N\big)$ hence (4.5) becomes

$$(I-P)\theta\big(t_1\psi_1 + \cdots + t_N\psi_N + g(t)\big) = 0. \tag{4.7}$$

Our goal is to prove the rigidity of MOTS and for that it is sufficient to show (4.7) has no non-trivial solutions in a neighbourhood of $t = 0 \in \mathbb{R}^N$.
Now for further study define a map $\alpha : \mathbb{R}^N \to \mathbb{R}^N$ by

$$(t_1, \ldots, t_N) \mapsto$$
$$\left(\int_S \psi_1\theta\Big(t_1\psi_1 + \cdots + t_N\psi_N + g(t)\Big), \ldots, \int_S \psi_N\theta\Big(t_1\psi_1 + \cdots + t_N\psi_N + g(t)\Big)\right) \tag{4.8}$$

observe that $\alpha(0) = 0$ and from the above construction we have the following result:

Theorem 2. *The MOTS S is* ***rigid*** *if and only if $t = (t_1, \ldots, t_N) = 0$ is an isolated solution of the equation $\alpha(t) = 0$ (i.e in a neighbourhood of the origin there is no solution of $\alpha(t) = 0$ other than $t = 0$).*

The equation $\alpha(t) = 0$ (finite dimensional but nonlinear) gives a necessary and sufficient condition for MOTS S to be rigid. Now for any j we can compute the first derivate

$$\left.\frac{\partial \alpha}{\partial t_j}\right|_{t=0} = \left(\int_S \psi_1 L\psi_j, \ldots, \int_S \psi_N L\psi_j\right) = 0 \in \mathbb{R}^N \tag{4.9}$$

as $L\psi_j = 0$ and the t derivative of $g(t)$ vanishes at $t = 0$. For the second derivative we find

$$\left.\frac{\partial^2 \alpha}{\partial t_j \partial t_k}\right|_{t=0} = \left(\int_S \psi_1 Q(\psi_j, \psi_k), \ldots, \int_S \psi_N Q(\psi_j, \psi_k) \right), \tag{4.10}$$

here Q denotes the second variation of expansion θ and it can be written as

$$Q(\psi, \psi) = \left.\frac{d^2}{d\rho^2}\right|_{\rho=0} \theta(\rho\psi) \tag{4.11}$$

and $Q(\psi, \phi)$ can be obtained using the polarization identity as

$$Q(\psi, \phi) = \frac{Q(\psi + \phi, \psi + \phi) - Q(\psi - \phi, \psi - \phi)}{4}. \tag{4.12}$$

In particular for the ith component of α Eqn.(4.10) written as:

$$\left.\frac{\partial^2 \alpha_i}{\partial t_j \partial t_k}\right|_{t=0} = \int_S \psi_i Q(\psi_j, \psi_k). \tag{4.13}$$

We can view this as a family of $N \times N$ matrices, one for each i.
Hence the MOTS S will be rigid if there do not exist numbers $(t_1, \ldots, t_N)$ not identically zero, that *simultaneously* satisfy the system of N quadratic equations

$$\sum_{j,k=1}^{N} t_j t_k \int\limits_S \psi_i Q(\psi_j, \psi_k) = 0, \quad 1 \leq i \leq N. \tag{4.14}$$

Now if we are looking back to Eqn.(3.27) the first order condition $-rF'(r) = l(l+1)$, we can say that in spherical harmonic form for a given value of l there are $2l + 1$ independent solutions of the eigenvalue equations.

4.3 Higher derivatives of the expansion evaluated on RN spacetime

Now in this section we evaluate Eqn.(3.23) at the inner horizon in Reissner-Nordström spacetime. Writing the metric function $F(r)$ as

$$F(r) = \frac{(r - r_0)(r - r_1)}{r^2}, \tag{4.15}$$

where in terms of physical quantities

$$r_0 = M - \sqrt{M^2 - Q^2} \quad \text{and} \quad r_1 = M + \sqrt{M^2 - Q^2}. \tag{4.16}$$

Here r_0 is the inner horizon and r_1 is the outer horizon. We have

$$F'(r_0) = \frac{r_0 - r_1}{r_0^2} \tag{4.17}$$

and

$$F''(r_0) = \frac{2}{{r_0}^2} - \frac{4(r_0 - r_1)}{{r_0}^3}, \tag{4.18}$$

so that

$$r_0 F'(r_0) = \frac{-2\sqrt{M^2 - Q^2}}{\left(M - \sqrt{M^2 - Q^2}\right)}. \tag{4.19}$$

Now we can write Eqn.(3.27) explicitly in terms of M and Q as

$$l(l+1) = -\frac{-2\sqrt{M^2 - Q^2}}{\left(M - \sqrt{M^2 - Q^2}\right)}. \tag{4.20}$$

Now L has 0 eigenvalue exactly when M and Q satsify this equation for some integer l.

So that Eqn.(3.23) becomes

$$\begin{aligned}\frac{\partial^2}{\partial^2\rho}\Big(\theta_{(\ell)}\Big)\Big|_{\rho=0} &= \left(\frac{(r_0-r_1)^2}{2r_0^5}+\frac{2(r_0-r_1)}{r_0^4}+\frac{2}{r_0^3}-\frac{4(r_0-r_1)}{r_0^4}\right)\psi^2 \\ &+\left(\frac{2}{r_0^3}+\frac{(r_0-r_1)}{r_0^4}\right)\psi_\theta^2+\frac{1}{\sin^2\theta}\left(\frac{2}{r_0^3}+\frac{(r_0-r_1)}{r_0^4}\right)\psi_\phi^2.\end{aligned} \tag{4.21}$$

This equation is the second variation of outgoing null expansion. Now from equation (4.21) we have the second variation of expansion $Q\Big(\psi,\psi\Big)$ is

$$\begin{aligned}Q\big(\psi,\psi\big) &= \Big(\frac{r_0-r_1}{r_0^3}\Big)\left(\frac{r_0-r_1}{2r_0^2}+\frac{2}{r_0}+\frac{2}{r_0-r_1}-\frac{4}{r_0}\right)\psi^2 \\ &+\left(\frac{2}{r_0^3}+\frac{r_0-r_1}{r_0^4}\right)\psi_\theta^2+\frac{1}{\sin^2\theta}\left(\frac{2}{r_0^3}+\frac{r_0-r_1}{r_0^4}\right)\psi_\phi^2.\end{aligned} \tag{4.22}$$

Hence from this equation we can write the second variation of expansion as

$$Q\big(\psi_j,\psi_k\big) = \frac{(r_1+r_0)^2}{2r_0^5}\psi_j\psi_k+\frac{3r_0-r_1}{r_0^4}\nabla\psi_j\cdot\nabla\psi_k \tag{4.23}$$

Here $\nabla\psi_j\cdot\nabla\psi_k$ is an inner product with respect to the metric on a unit sphere. Now from Eqn.(3.21) we can write,

$$-\Delta\psi+\frac{F'(r)}{r}\psi=0 \tag{4.24}$$

By factoring the metric function $F(r)$, from Eqn.(4.15) we have

$$F(r)=\frac{(r-r_0)(r-r_1)}{r^2} \tag{4.25}$$

For further progress now we combine Eqn.(3.27) and Eqn.(4.25) which gives us

$$\frac{r_1-r_0}{r_0}=l(l+1) \tag{4.26}$$

this implies

$$r_1=(l^2+l+1)r_0. \tag{4.27}$$

Using the relation between r_0 and r_1 in Eqn.(4.23) we can write the second variation of expansion as

$$Q\big(\psi_j, \psi_k\big) = \frac{(l^2+l+2)^2}{2r_0^3}\psi_j\psi_k - \frac{(l^2+l-2)}{r_0^3}\nabla\psi_j \cdot \nabla\psi_k. \tag{4.28}$$

Therfore,

$$Q(\psi_1, \psi_2) = \frac{-(l^2+l-2)}{r_0^3}\left(\nabla\psi_1 \cdot \nabla\psi_2 - \frac{(l^2+l+2)^2}{2(l^2+l-2)}\psi_1\psi_2\right). \tag{4.29}$$

4.3.1 Identities for Integrals involving three Eigenfunctions

Lemma 3. *Let S be a compact manifold without boundary, and suppose ψ^i is a set of eigenfunctions for the same eigenvalue $\lambda > 0, i.e. -\Delta\psi^i = \lambda\psi^i$ for each i. Then*

$$I = \int_S \psi^i\Big(\nabla\psi^j \cdot \nabla\psi^k\Big) = \frac{\lambda}{2}\int_S \psi^i\psi^j\psi^k \tag{4.30}$$

for all i, j, k

Proof. Integrating by parts in different ways, we get

$$I = -\int_S \psi^j\Big(\nabla\psi^i \cdot \nabla\psi^k\Big) + \lambda\int_S \psi^i\psi^j\psi^k \tag{4.31}$$

and

$$I = -\int_S \psi^k\Big(\nabla\psi^i \cdot \nabla\psi^j\Big) + \lambda\int_S \psi^i\psi^j\psi^k \tag{4.32}$$

Comparing Eqn.(4.31) and Eqn.(4.32) we see that the left-hand side of Eqn.(4.30) is symmetric in the indices i, j, k and hence

$$\int_S \psi^i\Big(\nabla\psi^j \cdot \nabla\psi^k\Big) = -\int_S \psi^i\Big(\nabla\psi^j \cdot \nabla\psi^k\Big) + \lambda\int_S \psi^i\psi^j\psi^k \tag{4.33}$$

which proves the result. □

Now applying this to $S_l^{m_i}(\theta,\phi)$ we can see that

$$\int_{S^2} S_l^{m_i}(\theta,\phi)\Big[\nabla S_l^{m_i}(\theta,\phi)\cdot\nabla S_l^{m_k}(\theta,\phi)\Big]d\Omega = \frac{l(l+1)}{2}\int_{S^2} S_l^{m_i}(\theta,\phi)S_l^{m_j}(\theta,\phi) S_l^{m_k}(\theta,\phi)d\Omega, \tag{4.34}$$

here $d\Omega = \sin\theta d\theta d\phi$. Now compute the integrals

$$M^i_{jk} = \int_{S^2} S^i_l Q(S^j_l S^k_l)d\Omega, \tag{4.35}$$

here

$$Q(\psi_1,\psi_2) = \frac{-(l^2+l-2)}{r_0^3}\left(\nabla\psi_1\cdot\nabla\psi_2 - \frac{(l^2+l+2)^2}{2(l^2+l-2)}\psi_1\psi_2\right), \tag{4.36}$$

which we get from Eqn.(4.29). Now from lemma 3 we get

$$\int_{S^2} S^i_l Q(S^j_l S^k_l)d\Omega = \frac{-(l^2+l-2)}{r_0^3}\left[\int_{S^2} S^i_l(\nabla S^j_l\cdot\nabla S^k_l)d\Omega - \frac{(l^2+l+2)^2}{2(l^2+l-2)}\int_{S^2} S^i_l S^j_l S^k_l d\Omega\right].$$

This implies

$$M^i_{jk} = \frac{-(l^2+l-2)}{r_0^3}\left[\frac{l(l+1)}{2} - \frac{(l^2+l+2)^2}{2(l^2+l-2)}\right]\int_{S^2} S^i_l S^j_l S^k_l d\Omega \tag{4.37}$$

For real value of l the term in bracket never vanishes so that we can write Eqn.(4.37) as

$$M^i_{jk} = c(l)\int_{S^2} S^i_l S^j_l S^k_l d\Omega \tag{4.38}$$

here $c(l)$ is a nonzero constant that depends only on l. Next by solving Eqn.(4.38) we will get system of polynomial equations and we will solve the polynomial equations to check the rigidity of inner horizon.

4.3.2 The $l = 0$ case

Consider ker$L$= span $\{\psi\}$ is one dimensional and in Eqn.(4.8) we defined the map $\alpha : \mathbb{R} \to \mathbb{R}$,

$$\alpha(t) = \int_S \psi\theta(t\psi + g(t))d\Omega \tag{4.39}$$

with $\alpha(0) = \alpha'(0) = 0$ and

$$\alpha''(0) = \int_S \psi Q(\psi, \psi)d\Omega. \tag{4.40}$$

Now by using Eqn.(4.38) for $l = 0$ in the case of real eigenfunction S_l^m, the MOTS will be rigid if M_{11}^1 is non-zero and we get by direct calculations

$$\begin{aligned} M_{11}^1 &= c(l)\int_{S^2} S_0^0 S_0^0 S_0^0 d\Omega & (4.41)\\ &= c(l)(S_0^0)^3 \cdot 4\pi & (4.42)\\ &\neq 0 & (4.43)\end{aligned}$$

since the function $S_0^0(\theta, \phi)$ is a non-zero constant. Therefore the inner horizon is rigid in this case.

4.3.3 The $l = 1$ case

From Eqn.(4.38) now we are checking for $\ell = 1$ hence we get

$$M^1 = M^2 = M^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.44}$$

The above solutions obtained by using Maple. From this solution and checking for different values of l we can find that by using symmetry argument (in symmetric level $-l$ to $+l$ is always zero) for all odd cases of l we always have $M_{jk}^i = 0$, $\forall i, j, k$. So that the solution is trivial and method is inconclusive for all l odd cases.

Now for all even values of l we will get non-trivial system of polynomial equations. In the next sections we will check for l even cases.

4.3.4 The $l = 2$ case

In the case $l = 2$ we will get $2l + 1 = 5$ independent eigenfunctions. Hence from Eqn.(4.38) we can write directly for the rigidity of MOTS, the $N \times N$ matrix for the single value of i becomes,

$$M^i_{jk} = c(l) \int_{S^2} S^i_l S^j_l S^k_l d\Omega \tag{4.45}$$

Now by direct computation using Maple, solving Eqn. (4.45) we can get 5×5 matrix. Here $S^m_l(\theta, \phi)$ associated with Legendre polynomial for the real valued eigenfunctions which are defined earlier in Eqn.(3.26)

Then Eqn.(4.45) becomes

$$M^1 = \begin{pmatrix} 0 & 0 & \frac{\pi}{42} & 0 & 0 \\ 0 & 0 & 0 & \frac{\pi}{14} & 0 \\ \frac{\pi}{42} & 0 & 0 & 0 & 0 \\ 0 & \frac{\pi}{14} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.46}$$

Similarly we can find

$$M^2 = \begin{pmatrix} 0 & 0 & 0 & \frac{\pi}{14} & 0 \\ 0 & 0 & -\frac{\pi}{21} & 0 & \frac{2\pi}{7} \\ 0 & -\frac{\pi}{21} & 0 & 0 & 0 \\ \frac{\pi}{14} & 0 & 0 & 0 & 0 \\ 0 & \frac{2\pi}{7} & 0 & 0 & 0 \end{pmatrix}, \tag{4.47}$$

$$M^3 = \begin{pmatrix} \frac{\pi}{42} & 0 & 0 & 0 & 0 \\ 0 & -\frac{\pi}{21} & 0 & 0 & 0 \\ 0 & 0 & -\frac{8\pi}{7} & 0 & 0 \\ 0 & 0 & 0 & -\frac{12\pi}{7} & 0 \\ 0 & 0 & 0 & 0 & \frac{96\pi}{7} \end{pmatrix}, \tag{4.48}$$

$$M^4 = \begin{pmatrix} 0 & \frac{\pi}{14} & 0 & 0 & 0 \\ \frac{\pi}{14} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{12\pi}{7} & 0 \\ 0 & 0 & -\frac{12\pi}{7} & 0 & -\frac{72\pi}{7} \\ 0 & 0 & 0 & -\frac{72\pi}{7} & 0 \end{pmatrix}, \tag{4.49}$$

and

$$M^5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2\pi}{7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{96\pi}{7} \\ 0 & 0 & 0 & -\frac{72\pi}{7} & 0 \\ 0 & 0 & \frac{96\pi}{7} & 0 & 0 \end{pmatrix}. \tag{4.50}$$

To find the solution for Eqn.(4.45) now consider

$$t = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix} \tag{4.51}$$

This yields the system of quadratic equations

$$t^T M^1 t = \frac{\pi}{7}\Big(\frac{1}{3}t_3t_1 + t_4t_2\Big) = 0 \tag{4.52}$$

$$t^T M^2 t = \frac{\pi}{7}\Big(t_4t_1 + \Big(-\frac{2}{3}t_3 + 4t_5\Big)t_2\Big) = 0 \tag{4.53}$$

$$t^T M^3 t = \frac{\pi}{7}\Big(\frac{1}{6}t_1^2 - \frac{1}{3}t_2^2 - 8t_3^2 - 12t_4^2 + 96t_5^2\Big) = 0 \tag{4.54}$$

$$t^T M^4 t = \frac{\pi}{7}\Big(t_2t_1 - 24t_4\Big(t_3 + 6t_5\Big)\Big) = 0 \tag{4.55}$$

$$t^T M^5 t = \frac{2\pi}{7}\Big(t_2^2 + 96t_5t_3 - 36t_4^2\Big) = 0 \tag{4.56}$$

The above five equations are polynomial equations with five unknowns. Now to examine the rigidity of MOTS, we will solve those five system of equations for unknown t_1, t_2, t_3, t_4, t_5.

We start with considering the case $t_3 = 0$. If $t_3 = 0$ then from Eqn.(4.52) we get either $t_2 = 0$ or $t_4 = 0$. Now consider the case for $t_2 = 0, t_3 = 0$ then from Eqn.(4.56) we get $t_4 = 0$. Similarly if $t_4 = 0, t_3 = 0$ then Eqn.(4.56) shows $t_2 = 0$. Now by substituting $t_2 = t_3 = t_4 = 0$ in Eqn.(4.54) we get

$$\frac{1}{6}t_1^2 + 96t_5^2 = 0. \tag{4.57}$$

So the only real solution to that is $t_1 = t_5 = 0$. Hence in case of $t_3 = 0$ we find that everything vanishes.

Next we consider $t_3 \neq 0$, then from Eqn.(4.52) and Eqn.(4.56) respectively we get

$$t_1 = \frac{-3t_4t_2}{t_3} \tag{4.58}$$

$$t_5 = \frac{-t_2^2 + 36t_4^2}{96t_3} \tag{4.59}$$

Now substitute t_1 and t_5 in Eqn.(4.53) we get

$$t_4\left(\frac{-3t_4t_2}{t_3}\right) - \frac{2}{3}t_3t_2 + 4\left(\frac{-t_2^2 + 36t_4^2}{96t_3}\right)t_2 = 0, \tag{4.60}$$

this implies

$$t_2\left(t_2^2 + 16t_3^2 + 36t_4^2\right) = 0 \tag{4.61}$$

which shows either

$$t_2 = 0 \quad \text{or} \quad t_2^2 + 16t_3^2 + 36t_4^2 = 0. \tag{4.62}$$

Similarly by substituting t_1, t_5 in Eqn.(4.55) we get either $t_4 = 0$ or same as Eqn.(4.62). For Eqn.(4.62) we can say that it has complex roots for t_2, t_3, t_4 but the only real roots are $t_2 = t_3 = t_4 = 0$.

In our case we are looking for real solution as we cannot deform MOTS in complex direction and the only real solution is $t_1 = t_2 = t_3 = t_4 = t_5 = 0$. Here we can conclude by saying that there is no real solution apart from the trivial one and this is enough to tell us that we cannot deform this unstable MOTS.

4.3.5 The $l = 3$ case

Now in case of $l = 3$, we will get 7×7 matrix and all the entries of the matrix become zero which is true for all odd cases of l.

4.3.6 The $l = 4$ case

Now check for $l = 4$ we will get 9×9 matrix and it becomes

$$M^1 = \frac{\pi}{286} \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{2520} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{252} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{2520} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{252} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{14} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.63}$$

$$M^2 = \frac{\pi}{143}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{504} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{420} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{126} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{7} & 0 & 4 \\ 0 & -\frac{1}{420} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{504} & 0 & -\frac{1}{126} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{7} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.64}$$

$$M^3 = \frac{\pi}{143}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{28} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{126} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{143}{8190} & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & -\frac{2}{7} & 0 & 2 & 0 \\ 0 & 0 & -\frac{143}{8190} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{126} & 0 & -\frac{2}{7} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{28} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.65}$$

$$M^4 = \frac{\pi}{143}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{7} & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & -\frac{2}{7} & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & \frac{9}{35} & 0 & -\frac{36}{7} & 0 & 0 \\ 0 & 0 & 0 & \frac{9}{35} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{7} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{7} & 0 & -\frac{36}{7} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.66}$$

$$M^5 = \frac{9\pi}{143}\begin{pmatrix} \frac{1}{45360} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3780} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{143}{73710} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{35} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{16}{7} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{80}{7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{22880}{91} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6720 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 35840 \end{pmatrix}. \tag{4.67}$$

$$M^6 = \frac{\pi}{143}\begin{pmatrix} 0 & \frac{1}{504} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{504} & 0 & -\frac{1}{126} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{126} & 0 & -\frac{2}{7} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{7} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{720}{7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{720}{7} & 0 & \frac{14400}{7} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{14400}{7} & 0 & 14400 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 14400 & 0 & -403200 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -403200 & 0 \end{pmatrix}. \tag{4.68}$$

$$M^7 = \frac{\pi}{143}\begin{pmatrix} 0 & 0 & \frac{1}{28} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{7} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{28} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{7} & 0 & -\frac{36}{7} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{205920}{91} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{14400}{7} & 0 & 14400 & 0 \\ 0 & 0 & 0 & 0 & -\frac{205920}{91} & 0 & 0 & 518400 & 0 \\ 0 & 0 & 0 & 0 & 0 & 14400 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 518400 & 0 & 0 \end{pmatrix}. \tag{4.69}$$

$$M^8 = \frac{\pi}{143}\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -60480 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 14400 & 0 & -403200 \\ 0 & 0 & 0 & 0 & 0 & 14400 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -60480 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -403200 & 0 & 0 & 0 \end{pmatrix}. \quad (4.70)$$

$$M^9 = \frac{4\pi}{143}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 80640 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -100800 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 129600 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -100800 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 80640 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.71)$$

Now the system of quadratic equations in that case are:

$$t^T M^1 t = \frac{1}{180}t_5 t_1 + \frac{1}{18}t_6 t_2 + t_7 t_3 + 14 t_8 t_4 = 0 \tag{4.72}$$

$$t^T M^2 t = \frac{1}{36}t_1 t_6 - \frac{1}{30}t_5 t_2 - \frac{1}{9}t_3 t_6 + 2 t_4 t_7 + 56 t_4 t_9 = 0 \tag{4.73}$$

$$t^T M^3 t = \frac{1}{2}t_1 t_2 - \frac{1}{9}t_6 t_2 - \frac{143}{585}t_5 t_3 - 56 t_3 t_9 - 4 t_4 t_6 + 28 t_4 t_8 = 0 \tag{4.74}$$

$$t^T M^4 t = 7 t_1 t_8 + 2 t_7 t_2 + 56 t_9 t_2 - 4 t_6 t_3 + 28 t_3 t_8 + \frac{18}{5}t_5 t_4 - 72 t_4 t_7 = 0 \tag{4.75}$$

$$\begin{aligned} t^T M^5 t = {} & \frac{1}{720}t_1^2 - \frac{1}{60}t_2^2 - \frac{143}{1170}t_3^2 + \frac{9}{5}t_4^2 + 144 t_5^2 + 720 t_6^2 - 15840 t_7^2 \\ & - 423360 t_8^2 + 2257920 t_9^2 = 0 \end{aligned} \tag{4.76}$$

$$\begin{aligned} t^T M^6 t = {} & \frac{1}{36}t_1 t_2 - \frac{1}{9}t_3 t_2 - 4 t_4 t_3 + 1440 t_5 t_6 + 28800 t_6 t_7 + 201600 t_7 t_8 \\ & - 5644800 t_8 t_9 = 0 \end{aligned} \tag{4.77}$$

$$\begin{aligned} t^T M^7 t = {} & \frac{1}{14}t_1 t_3 + \frac{1}{7}t_2 t_4 - \frac{36}{7}t_4^2 - \frac{411840}{91}t_5 t_7 + \frac{14400}{7}t_6^2 + 28800 t_6 t_8 \\ & + 1036800 t_7 t_9 = 0 \end{aligned} \tag{4.78}$$

$$t^T M^8 t = t_1 t_4 + 4 t_3 t_4 - 120960 t_5 t_8 + 28800 t_6 t_7 - 806400 t_6 t_9 = 0 \tag{4.79}$$

$$t^T M^9 t = 8 t_2 t_4 - 4 t_3^2 + 645120 t_9 t_5 - 806400 t_6 t_8 + 518400 t_7^2 = 0 \tag{4.80}$$

So in $l = 4$ we can find the polynomials (from the matrices above), but so far we have not been able to solve these equations (i.e can not prove $l = 4$ case, there are no non-zero solutions) Hence for $l = 0, 2$ cases we analyze explicitly and proved there are no solutions, so that MOTS is rigid but not stable.

Chapter 5

Conclusion

In this chapter, we summarize our results. This thesis examines the rigidity of marginally outer trapped surfaces. We study this in the case of the Reissner-Nordström spacetime. Our analysis indicated that in the case of the outer black hole horizon, all eigenvalues of the stability operator are positive so the horizon is strictly stable and hence rigid. The case of the cosmological horizon is more subtle. This has vanishing eigenvalues for the $l = 1$ mode and in the pure de-Sitter case this horizon is neither stable nor rigid. The more interesting case here is the inner horizon which is unstable and we have looked here for the rigidity of the unstable inner horizon. To study this, we introduced Lyapunov-Schmidt reduction. This method works when the inverse function theorem doesn't. We have conclusively proved that spacetimes for which the $l = 0$ or $l = 2$ eigenvalues vanish are unstable but still rigid. For odd values the method does not work at all and for even values (> 2) it works in principle but we did not do it yet algebrically. Given these matrices M_1 to M_9 we can write down the polynomial equations analogous to the previous chapter, but it is too complicated to prove. It is still an open problem to resolve this for bigger even values of l and all odd values of l. However we conjecture that all of these unstable cases are still rigid. Note that, in axisymmetry many of these cases were dealt with ad-hoc methods in [10].

Bibliography

[1] S. Hawking and G. Ellis, *The Large Scale Structure of Space-Time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2, 2011.

[2] R. M. Wald, *General relativity*. University of Chicago press, 2010.

[3] J. Pullin, "Numerical relativity: Solving einstein's equations on the computer," 2011.

[4] C. M. Chambers, "The Cauchy horizon in black hole de sitter space-times," *Annals Israel Phys. Soc.* **13** (1997) 33, `arXiv:gr-qc/9709025`.

[5] E. Poisson, *A relativist's toolkit: the mathematics of black-hole mechanics*. Cambridge university press, 2004.

[6] J. B. Hartle, "Gravity: an introduction to Einstein's general relativity/james hartle," 2003.

[7] M. P. do Carmo, *Riemannian geometry / Manfredo do Carmo ; translated by Francis Flaherty*. Boston : Birkhäuser, 1992. 1992.

[8] J. J. Jurgen Jost, *Riemannian Geometry and Geometric Analysis*. Springer, 2005, illustrated ed., 2005.

[9] I. Booth, "Black hole boundaries," *Can. J. Phys.* **83** (2005) 1073–1099, `arXiv:gr-qc/0508107`.

[10] I. Booth, H. K. Kunduri, and A. O'Grady, "Unstable marginally outer trapped surfaces in static spherically symmetric spacetimes," *Physical Review D* **96** no. 2, (Jul, 2017) . `http://dx.doi.org/10.1103/PhysRevD.96.024059`.

[11] W. Kavanagh and I. Booth, "Spacetimes containing slowly evolving horizons," *Phys. Rev. D* **74** (2006) 044027, `arXiv:gr-qc/0603074`.

[12] I. Booth, "Spacetime near isolated and dynamical trapping horizons," *Phys. Rev. D* **87** no. 2, (2013) 024008, `arXiv:1207.6955 [gr-qc]`.

[13] O. Dreyer, B. Krishnan, D. Shoemaker, and E. Schnetter, "Introduction to isolated horizons in numerical relativity," *Phys. Rev. D* **67** (Jan, 2003) 024018. `https://link.aps.org/doi/10.1103/PhysRevD.67.024018`.

[14] R. Bousso and N. Engelhardt, "New area law in general relativity.," *Physical review letters* **115 8** (2015) 081301.

[15] R. Penrose, "Gravitational collapse and space-time singularities," *Phys. Rev. Lett.* **14** (1965) 57–59.

[16] L. Andersson and J. Metzger, "The Area of horizons and the trapped region," *Commun. Math. Phys.* **290** (2009) 941–972, `arXiv:0708.4252 [gr-qc]`.

[17] L. Andersson, M. Mars, J. Metzger, and W. Simon, "The Time evolution of marginally trapped surfaces," *Class. Quant. Grav.* **26** (2009) 085018, `arXiv:0811.4721 [gr-qc]`.

[18] L. Andersson, M. Mars, and W. Simon, "Stability of marginally outer trapped surfaces and existence of marginally outer trapped tubes," *Adv. Theor. Math. Phys.* **12** no. 4, (2008) 853–888, `arXiv:0704.2889 [gr-qc]`.

[19] D. Pook-Kolb, O. Birnholtz, B. Krishnan, and E. Schnetter, "The existence and stability of marginally trapped surfaces," *Physical Review D* **99** (2019) .

[20] L. Andersson, M. Mars, and W. Simon, "Local existence of dynamical and trapping horizons," *Phys. Rev. Lett.* **95** (2005) 111102, `arXiv:gr-qc/0506013 [gr-qc]`.

[21] L. C. Evans, *Partial differential equations.* American Mathematical Society, Providence, R.I., 2010.

[22] I. Booth and S. Fairhurst, "Isolated, slowly evolving, and dynamical trapping horizons: Geometry and mechanics from surface deformations," *Phys. Rev. D* **75** (2007) 084019, `arXiv:gr-qc/0610032`.

[23] K. Martel and E. Poisson, "Regular coordinate systems for Schwarzschild and other spherical space-times," *Am. J. Phys.* **69** (2001) 476–480, `arXiv:gr-qc/0001069`.

Appendix A

Horizons

Name	$\theta_{(\ell)}$	$\theta_{(n)}$
Trapped	< 0	< 0
Marginally Trapped	0	< 0
Outer trapped	< 0	anything
Marginally outer trapped	0	anything
Weakly trapped	≤ 0	≤ 0

Table A.1: Classification of different types of surfaces in terms of null expansion (outward and inward)