Symmetry analysis of hydrodynamic-type systems

by

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A thesis submitted to the School of Graduate Studies

in partial fulfillment of the requirements

for the degree of Doctor of Philosophy

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Memorial University of Newfoundland

February 2021

St. John's Newfoundland

Abstract

Using advantages of nonstandard computational techniques based on the light-cone variables, we explicitly find the algebra of generalized symmetries of the (1+1)-dimensional Klein–Gordon equation. This allows us to describe this algebra in terms of the universal enveloping algebra of the essential Lie invariance algebra of the Klein–Gordon equation. Then we single out variational symmetries of the corresponding Lagrangian and compute the space of local conservation laws of this equation, which turns out to be generated, up to the action of generalized symmetries, by a single first-order conservation law.

We study the hydrodynamic-type system of differential equations modeling isothermal no-slip drift flux. Using the facts that the system is partially coupled and its essential subsystem reduces to the (1+1)-dimensional Klein–Gordon equation, we exhaustively describe generalized symmetries, cosymmetries and local conservation laws of this system. A generating set of local conservation laws under the action of generalized symmetries is proved to consist of two zeroth-order conservation laws. The subspace of translation-invariant conservation laws is singled out from the entire space of local conservation laws. The essential subsystem possesses three first-order hydrodynamic-type Hamiltonian operators, two of which are prolonged nonlocally to the entire system.

The (1+2)-dimensional hydrodynamic-type system governing the shallow water model is studied from the symmetry-analysis point of view. Its complete point symmetry group is found with the help of the automorphism-based algebraic method. Lie reductions of both codimensions one and two are classified. We exhaustively describe the algebra of differential invariants of the point symmetry group of the system using the method of moving frames.

We construct for the first time classes of differential equations with nontrivial generalized equivalence groups, i.e. whose equivalence-transformation components corresponding to independent and dependent variables locally depend on nonconstant arbitrary elements of the class. We rigourously construct extended generalized equivalence groups of several classes of differential equations as well. The new notion of effective generalized equivalence group is introduced.

General summary

Physical phenomena are governed by systems of differential equations, which are seldom completely integrable. In view of this it is necessary either to find their particular solutions or to simplify models via physically reasonable assumptions. One of the most common ways to find particular solutions of differential equations is employing their symmetries to carry out Lie reductions. The idea of Lie symmetries is naturally generalized to the notion of higher symmetries. They are important because, for example, the existence of an infinite hierarchy thereof may testify to a complete integrability of the system.

Using the machinery of symmetry analysis we study the Klein–Gordon equation, which is a fundamental equation of quantum mechanics. Our interest in this equation lies in the fact that an isothermal no-slip drift flux model, which is a submodel of the two-phase flow model, reduces to the Klein–Gordon equation. Thus, any result on the Klein–Gordon equation can be prolonged to a result for the drift flux model. An interesting mathematical twist here is that not every *local* result on the Klein–Gordon equation has a local counterpart. For example, some local generalized symmetries of the Klein–Gordon equation have nonlocal counterparts. Nonlocal symmetry analysis is a recent field, which draws more and more attention in both the mathematical and physical communities.

Averaging nonlinear differential equations used in numerical simulations may result in a loss of some of their internal properties. This is why it may be necessary to use a parameterization scheme, i.e. to replace processes that are too small-scale or complex to be mathematically represented in the model by simplified processes. We study the shallow water model with the aim to find invariant and conservative parameterizations schemes preserving symmetries and conservation laws of the model by describing its conservation laws and the algebra of differential invariants of its symmetry group.

When a system of differential equations governing a physical phenomenon involves some parameters, the problem of group classification arises. To distinguish equivalent systems, some notion of equivalence is necessary. In the thesis we give the first nontrivial examples of generalized equivalence groups, which induce the above equivalence, and we develop the theory behind for these groups.

Acknowledgments

I would like to express my gratitude to my supervisor Prof. Dr. Alexander Bihlo for his support along the way, Prof. Dr. Roman Popovych for his continuous impact on my research, Prof. Dr. Artur Sergyeyev for introducing me to the subject of Hamiltonian structures of differential equations, and all the academic and administrative staff of the Department of Mathematics and Statistics of Memorial University for creating an inspiring environment for studying and research. The research was supported by NSERC's Alexander Graham Bell Canada Graduate Scholarship – Doctoral Fellowship, the Canada Research Chairs Program, the NSERC Discovery grant program and the School of Graduate Studies. I also acknowledge the hospitality of the University of Vienna, where a part of this thesis was completed.

Co-authorship statement

Some results of this thesis were published in [110, 111, 113, 114, 115], the results of Chapter 4 have not been published before.

Chapters 1 and 2, resp. papers [115] and [113], are a natural continuation of my MSc thesis. The arguments to prove Theorem 2.4 with the help of the Shapovalov–Shirokov theorem and to prove Theorem 2.8 with the help of Martínez Alonso lemma belong to Prof. Dr. Popovych. The computation of generalized and variational symmetries and conservation laws in [115] is my contribution.

Prof. Dr. Popovych supervised the computation of generalized symmetries, cosymmetries and conservation laws in Sections 3.4, 3.5 and 3.6. Other sections of Chapter 2 are entirely my contribution. The results of Sections 3.7 and 3.8 are unpublished yet. Prof. Dr. Bihlo and Prof. Dr. Sergyeyev doublechecked the proofs and took part in the manuscript [113] preparation.

The initiation of the studying of the shallow water model via the moving frames method in Chapter 4 is due to Prof. Dr. Bihlo, while computations and the extensions results to Lie reductions are my contribution.

The examples of nontrivial generalized and extended generalized equivalence groups, which are the results included in the thesis, were my contributions to [110, 111, 114], although the guidance of and discussions with Prof. Dr. Popovych, especially concerning the idea of the notion of effective generalized equivalence groups, are highly appreciated. Prof. Dr. Bihlo and Prof. Dr. Boyko doublechecked the proofs and took part in the above manuscripts preparation.

Additional work

Only the examples of generalized equivalence from [110, 111, 114] are included in this thesis. More precisely, the above papers are dedicated to the exhaustive group classification problems of the classes of general Burgers–Korteweg–de Vries and variable-coefficient Burgers equations and of a class of reaction–diffusion equations, respectively. These results constitute the core of my PhD thesis [109] defended at the Institute of Mathematics of National Academy of Sciences of Ukraine.

The paper [108] would be a good contribution to Section 5 of the thesis but is not included due to the present thesis being already quite substantial in volume.

Results of my work as a graduate student at Memorial University of Newfoundland are published as the following papers.

[99] Opanasenko S., Equivalence groupoid of a class of general Burgers–Korteweg–de Vries equations with space-dependent coefficients, in *Collection of Works of Institute of Mathematics*, vol. 16, no. 1, Institute of Mathematics, Kyiv, pp. 131–154, 2019, `arXiv:1909.00036`.

[100] Opanasenko S., Bihlo A. and Popovych R.O., Group analysis of general Burgers–Korteweg–de Vries equations, *J. Math. Phys.* **58** (2017), 081511, `arXiv:1703.06932`.

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[103] Opanasenko S., Bihlo A., Popovych R.O. and Sergyeyev A., Generalized symmetries, conservation laws and Hamiltonian operators of isothermal no-slip drift flux model, *Physica D* **411** (2020), 132546, `arXiv:1908.00034`.

[104] Opanasenko S., Boyko V. and Popovych R.O., Enhanced group classification of reaction–diffusion equations with gradient-dependent diffusion, *J. Math. Anal. Appl.* **484** (2020), 123739, `arXiv:1804.08776`.

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Glossary

$\mathcal{L}$	a system or a class of systems of differential equations
κ	arbitrary-elements tuple
$\mathcal{L}_\theta$	a system of differential equations in the class $\mathcal{L}$
$\mathcal{G}^\sim$	equivalence groupoid of a class
$G^\sim$	usual equivalence group of a class
$\bar{G}^\sim$	generalized equivalence group of a class
$\hat{G}^\sim$	extended generalized equivalence group of a class
$\breve{G}^\sim$	effective generalized equivalence group
$\mathrm{J}^r = \mathrm{J}^r(x\|u)$	jet space of order r
	in independent variables x and dependent variables u
v, X	vector field
$\mathrm{pr}^{(r)}\mathrm{v}$, $X^{(r)}$	rth prolongation of a vector field
$\langle \dots \rangle$	a span
Σ	the algebra of generalized vector fields
Σ^{triv}	the algebra of trivial generalized vector fields
Σ^{q}	the quotient space $\Sigma/\Sigma^{\mathrm{triv}}$
$\hat{\Sigma}^{\mathrm{q}}$	the algebra of generalized symmetries reduced in view of the equation
$\hat{\Sigma}^n$	the algebra of generalized symmetries in reduced form of order up to n
$\tilde{\Sigma}^n$	the algebra of generalized symmetries of order up to n
$\hat{\Sigma}^{[n]}$	the algebra of generalized symmetries in reduced form of order n
D_x	the total derivative operator with respect to x
$\mathcal{D}_x$	the total derivative operator with respect to x reduced in view of $\mathcal{L}$

Contents

Introduction

In the 1870s Sophus Lie started developing a theory for integrating ordinary differential equations to equal and even surpass his compatriot Abel's theory of solvability of algebraic equations. At the heart of his theory lies the notion of symmetries of a differential equation, that is, continuous transformations of independent and dependent variables under which the equation is invariant. Since to find such symmetries one needs to solve a system of nonlinear differential equations, it is more convenient to work with their infinitesimal counterparts which are solutions of a linear system of equations. Lie's ideas are fundamental for several fields of mathematics, including Lie groups, Lie algebras and what is commonly known today as symmetry analysis of differential equations.

Initiated in my Master thesis was a study of an isothermal no-slip drift flux model within the framework of symmetry analysis. In particular, we computed generalized symmetries and conservation laws of order not greater than one and first-order local Hamiltonian structures of hydrodynamic type. More importantly, it was noted that the system $\mathcal{S}$ governing the model is partially decoupled, while its essential subsystem $\mathcal{S}_0$ reduces to the (1+1)-dimensional Klein–Gordon equation via the rank-two hodograph transformation. This equation is linear and therefore is easier to study than the quasilinear system $\mathcal{S}$. Furthermore, the Klein–Gordon equation is a basic equation in quantum mechanics and is of interest per se. Thus, Chapter 2 of this thesis is devoted to the Klein–Gordon equation.

The system $\mathcal{S}$ is degenerate in two ways. Besides being partially decoupled, it is not a genuinely nonlinear hydrodynamic-type system. This double degeneracy allows us to partition every problem concerning the system $\mathcal{S}$ into two stages. The first stage is to solve the counterpart-problem for the Klein–Gordon equation and transfer the result to the system $\mathcal{S}$. This step is not always straightforward. Thus, any local conservation law of the Klein–Gordon equation has a local counterpart for the system $\mathcal{S}$. At the same time, not all generalized symmetries have them. This way the prolongation problem arises. Similarly to generalized symmetries, not all local Hamiltonian structures for the system $\mathcal{S}_0$ have local counterparts for the system $\mathcal{S}$. Thereby, we enter the territory of nonlocal symmetry analysis of differential equations. The second stage is to deal with the equation complementary to the system $\mathcal{S}_0$ in $\mathcal{S}$. This step is much easier and, in fact, it was somewhat considered in my Master thesis. The system $\mathcal{S}$ is studied in Chapter 3.

In Chapter 4 we go to a multidimensional case and consider a shallow water model which is governed by a (1+2)-dimensional hydrodynamic-type system. This model is used in weather prediction, which despite all the progress is still insufficiently accurate. One way to improve it is to use better parameterization schemes for the model. It is known that in numerical simulations one often uses averaging of differential equations, which may lead to a loss of crucial data. It is possible to circumvent the problem by choosing a closure scheme and by parameterizing unresolved terms. Physicists usually do not pay special attention to parameterizations preserving geometric properties of an initial model, such as e.g. symmetries, conservation laws or Hamiltonian structures. Our aim is to change the priority: one should choose a parameterization scheme from the set of "geometry-preserving" parameterization schemes. For this end, we study the question of conservation laws, symmetries and invariants for the shallow water model.

Chapter 5 concerns equivalence groups of classes of differential equations. Such groups arise in group classification problems, i.e. problems of classifying Lie symmetries of parameterized equations, and give rise to the equivalence therein. For years researchers used usual equivalence groups in this regard, often assuming that there are no nontrivial examples of also known generalized equivalence groups. Such examples are found in this thesis. We also consider rigorous construction of extended generalized equivalence groups and we delve into their theory.

Chapter 1

Symmetry-like objects for differential equations

In this introductory chapter we get a reader acquainted with the geometric interpretation of differential equations and give definitions for basic objects of symmetry analysis, which are quite loosely called “symmetry-like objects” in the thesis. For more details see one of the classic textbooks on symmetry analysis of differential equations [22, 23, 24, 26, 78, 103, 104, 116] or classic reference papers [2, 3, 4, 130, 161, 166]. We primarily use the textbook [103] as a reference source, while indicating other sources when needed.

To begin with, we need to introduce a space on which (systems of) differential equations live. It should accommodate not only values of a function — a solution of a differential equation — but also the values of all its derivatives. Given a smooth real-valued function $f(x) = f(x^1, \dots, x^n)$ of n independent variables, there are $n_k = \binom{n+k-1}{k}$ different kth order partial derivatives of f. Hereafter $J = (j_1, \dots, j_k)$ denotes an unordered k-tuple of integers and $\partial_J = \dfrac{\partial^k}{\partial x^{j_1} \cdots \partial x^{j_k}}$ is the corresponding derivative of order $\#J = k$. For a given smooth function $f\colon X \to U$ with $X \equiv \mathbb{R}^n$, $U \equiv \mathbb{R}^m$, there exist mn_k different kth order partial derivatives $u^\alpha_J = \partial_J f^\alpha(x)$ of components of f at a given point x. The total number of partial derivatives of all orders from 0 to r is then $m^{(r)} :== m\binom{r+n}{r}$. Thus, one can define $U^{(r)}$ to be a Euclidean space of dimension $m^{(r)}$, with its coordinates being all possible partial derivatives of u of order from 0 up to r.

Definition 1.1. The r-jet space $J^r(x|u) = X \times U^{(r)}$ of the underlying space $X \times U$ is a Euclidean space of dimension $n + m^{(r)}$, whose coordinates represent the independent variables, the dependent variables and the derivatives of the dependent variables up to order r. We call the inverse limit $J^\infty(x|u)$ of $J^r(x|u)$ the space of infinite jets.

Remark 1.2. Though some differential equations are defined only on some open subsets of the underlying space $X \times U$, we will avoid this technical remark when possible.

Now we adapt the notion of a solution of the system to jet spaces. This is done via the prolongation of the function to the space $U^{(r)}$. Given a smooth function $u = f(x)$, such that $f\colon X \to U$, we define its rth *prolongation* $u^{(r)} = \mathrm{pr}^{(r)} f(x)\colon X \to U^{(r)}$ as $u^\alpha_J = \partial_J f^\alpha(x)$. Thus $\mathrm{pr}^{(r)} f(x)$ is a vector-function whose coordinates represent the values of f and all its derivatives up to order r at the given point x.

To formulate this geometrically, given a function $u = f(x)$ whose graph lies in $X \times U$, its rth prolongation $\mathrm{pr}^{(r)} f(x)$ is a function whose graph lies in a jet space $J^r(x|u)$. Alternatively [104], the rth prolongation of a function f can be determined as a section $F(x)$ of $J^r(x|u)$ such that the pullbacks of ω^a_J by the function F vanish, $F^*\omega^a_J = 0$ for any $a = 1, \ldots, m$ and any multiindex J, $0 \leqslant \#j < n$. Here the differential forms $\omega^a_J = \mathrm{d}u^a_J - \sum\limits_{i=1}^{n} u_{J,i}\mathrm{d}x^i$ are called the *contact forms*. A space spanned by these forms is called the *contact structure* of the jet space $J^\infty(x|u)$.

Finally, we can determine a geometric interpretation of differential equations. Let here and in what follows the system $\mathcal{L}$ of differential equations consist of l equations of the form

$$L^\mu(x, u_{(r)}) = 0, \quad \mu = 1, \ldots, l,$$

where the symbol $u_{(r)}$ denotes all derivatives of the functions u with respect to x of order not greater than r, including u's as derivatives of order zero. An alternative geometric definition of a system $\mathcal{L}$ is the subvariety

$$\big\{\big(x, u^{(r)}\big) \mid L^\mu(x, u_{(r)}) = 0 \text{ for all } \mu\big\} \subset J^r(x|u), \tag{1.1}$$

of a jet space $J^r(x|u)$, that is, the subset of the r-jet space, where the maps L^μ vanish.

When considering higher-order symmetry structures we need to consider a system with all its differential consequences. This new system is considered to be a subvariety $\mathcal{L}^{(\infty)}$ of the jet space $J^\infty(x|u)$. Abusing notation, we denote the above subvariety again by $\mathcal{L}$. Similarly, a function $u = f(x)$ is called a solution of the system $\mathcal{L}$ if the graph of its prolongation $\mathrm{pr}^{(r)}f(x)$ lies within the subvariety $\mathcal{L}$.

A smooth function f depending on x and a finite number of derivatives of u (i.e., a smooth function on an open set of $\mathrm{J}^\infty(x|u)$ with finite number of arguments and with values in the underlying field) is called a *differential function* of u, and it is denoted by $f = f[u]$. The order $\mathrm{ord}\, f$ of a differential function f is the highest order of derivatives involved in f, and, if f does not depend on derivatives of u, $\mathrm{ord}\, f := -\infty$.

1.1 Lie symmetries

Now we want to apply a notion of symmetry to systems of differential equations. Similarly to algebraic equations as known in Galois theory, a symmetry of a system of differential equations is a certain transformation mapping its solutions into solutions of the same system. Let us now define rigorously these transformations.

Given a local group[1] of transformations G acting on the space $X \times U$ of independent and dependent variables, that is, a group of local diffeomorphisms of the space, one can define the rth prolongation of G denoted by $\mathrm{pr}^{(r)}G$, which is in fact the induced local action of G on the r-jet space $U^{(r)}$ transforming the derivatives of functions $u = f(x)$ into the corresponding derivatives of the transformed function $\tilde{u} = \tilde{f}(\tilde{x})$. The action of this group is defined via

$$\mathrm{pr}^{(r)}g \cdot (x_0, u_0^{(r)}) = \left(\tilde{x}_0, \mathrm{pr}^{(r)}(g \cdot f)(\tilde{x}_0)\right),$$

whenever $(\tilde{x}_0, \tilde{u}_0) = g \cdot (x_0, u_0)$, $u_0^{(r)} = \mathrm{pr}^{(r)}f(x_0)$ and $g \in G$. Taking into account the local action of the group of transformations, we can restrict ourselves to groups acting on local

[1] In case when a "symmetry" group is infinite-dimensional, e.g., it is parameterized by an arbitrary smooth function of its arguments or by a solution of a system of PDEs, it is more appropriate to say a "pseudogroup" or a "Lie pseudogroup", but we prefer to keep language simple.

subsets of the space $X \times U$. So we can determine the symmetry group of the system of differential equations as follows.

Proposition 1.3. *Let M be an open subset of $X \times U$ and $\mathcal{L}$ an rth order system of differential equations defined over M, with the corresponding subvariety defined by* (1.1)*. Let a local group of transformations G act on M so that its prolongation leaves the subvariety invariant. Then G is a symmetry group of the system $\mathcal{L}$ of differential equations.*

In practice, it is much easier to work with infinitesimal generators of symmetry transformations. It is possible to determine infinitesimal generators of the prolonged group action via the corresponding infinitesimal generators of the underlying group.

Definition 1.4. Let M be an open subset of the space $X \times U$ of independent and dependent variables and v a vector field on M with corresponding one-parameter group $\exp(\varepsilon \mathrm{v})$. The rth prolongation $\mathrm{pr}^{(r)}\mathrm{v}$ of a vector field v is a vector field on the jet space $J^r(x|u)$, defined as the infinitesimal generator of the corresponding prolonged one-parameter group $\mathrm{pr}^{(r)}[\exp(\varepsilon \mathrm{v})]$,

$$\mathrm{pr}^{(r)}\mathrm{v}|_{(x,u_{(r)})} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \mathrm{pr}^{(r)}[\exp(\varepsilon \mathrm{v})](x, u_{(r)}) \quad \text{for any } (x, u_{(r)}) \in J^r(x|u).$$

Having at our disposal all these tools, we can derive the infinitesimal condition for a group G to be a symmetry group of a given system of differential equations. Nonetheless, there are technical conditions on systems of differential equations that make all the constructions work. Systems satisfying these conditions are called *totally nondegenerate* [103] or *normal* [26]. We notice only that systems of evolution equations, systems of *Cauchy–Kovalevskaya form*,[2] and systems of *extended Kovalevskaya form*[3] are normal. Without further ado, we give the invariance criterion to determine the symmetry group of a normal system.

[2] A system is called of Kovalevskaya form it can be rewritten as $\frac{\partial^r u^\mu}{\partial t^r} = L^\mu(t, x, u_{(r)})$, $\mu = 1, \ldots, l$, where the functions L^μ's are analytic functions of their arguments and the derivatives $\partial^r u^\mu / \partial t^r$ do not arise on the right hand side.

[3] A system of partial differential equation is called of extended Kovalevskaya form if its equations can be written as $\frac{\partial^{r^a} u^a}{\partial (x^n)^{r_a}} = H^a(x, \widetilde{u_{(r)}})$, $a = 1, \ldots, m$, where $0 \leqslant r_a \leqslant r$ and $\widetilde{u_{(r)}}$ denotes all deratives of the functions u with respect to x up to order r, where each u^b is differentiated with respect to x_n at most $r_b - 1$ times, $b = 1, \ldots, m$.

Theorem 1.5. *Let $\mathcal{L}$ be a normal system of differential equations over $M \subset X \times U$. If G is a local group of transformations acting on M, and $\mathrm{pr}^{(r)}\mathrm{v}[L^\mu] = 0$ for an appropriate point in the subvariety $\mathcal{L}$ and every infinitesimal generator* v *of elements G, then G is the symmetry group of the system.*

In view of this theorem the only task remaining for us is to find an explicit formula for the prolongation of a vector field. In spite of the complexity of the prolonged group action, the calculation of prolonged vector fields is straightforward. The cornerstone of most of the computations is the notion of total derivative operators.

Definition 1.6. Let $P[x]$ be a differential function. Its total derivative with respect to x^i is the differential function $\mathrm{D}_i P[x]$ such that

$$\mathrm{D}_i P(x, \mathrm{pr}^{(r+1)} f(x)) = \partial_i \Big(P(x, \mathrm{pr}^{(r)} f(x)) \Big) \quad \text{for any smooth function } f.$$

Using the straightforward chain rule argument one defines the general formula to determine the action of the total derivative D_i,

$$\mathrm{D}_i P = \partial_i P + \sum_{\alpha=1}^{m} \sum_J u^\alpha_{J,i} \frac{\partial P}{\partial u^\alpha_J}, \quad \text{where } J, i \text{ is the multi-index } (j_1, \ldots, j_k, i).$$

Theorem 1.7. *Let $\mathrm{v} = \sum\limits_{i=1}^{n} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum\limits_{\alpha=1}^{m} \eta_\alpha(x,u) \frac{\partial}{\partial u^\alpha}$ be a vector field defined on an open subset $M \subset X \times U$. Its rth prolongation $\mathrm{pr}^{(r)}\mathrm{v}$ is the vector field*

$$\mathrm{pr}^{(r)}\mathrm{v} = \mathrm{v} + \sum_{\alpha=1}^{m} \sum_J \eta^J_\alpha(x, u_{(r)}) \frac{\partial}{\partial u^\alpha_J}$$

defined in the jet space $J^r(x|u)$, the multi-indices $J = (j_i, \ldots, j_k)$ run through all possible indices with $1 \leqslant j_k \leqslant n$ and $1 \leqslant k \leqslant r$. The components η^J_α of $\mathrm{pr}^{(r)}\mathrm{v}$ are determined as

$$\eta^J_\alpha(x, u_{(r)}) = \mathrm{D}_J \left(\eta_\alpha - \sum_{i=1}^{n} \xi^i u^\alpha_i \right) + \sum_{i=1}^{n} \xi^i u^\alpha_{J,i}. \tag{1.2}$$

Similarly to infinitesimal generators of the symmetry group of a system of differential equations, their prolongations also form a Lie algebra.

1.2 Generalized symmetries

A vector field $\mathrm{v} = \sum\limits_{i=1}^{n} \xi_i(x,u)\frac{\partial}{\partial x^i} + \sum\limits_{\alpha=1}^{m} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha}$ defined on some open subset M of the space of independent and dependent variables $X\times U$ has a geometric sense, generating a one-parameter transformation acting pointwise on $X\times U$. Letting vector-field components depend on derivatives of dependent variables, this sense is evidently being lost. Nonetheless, this idea has another important interpretation. It provides a connection with conservation laws, which are of significant importance in both physics and mathematics. We call such vector fields generalized and discuss them in the remainder of this section.

Definition 1.8. A generalized vector field is a formal expression of the form

$$\mathrm{v} = \sum_{i=1}^{n} \xi_i[u]\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{m} \eta_\alpha[u]\frac{\partial}{\partial u^\alpha} \tag{1.3}$$

where ξ_i's and ϕ_α's are smooth differential functions.

Just as for ordinary geometric vector fields, we can define the prolongation of a generalized vector field $\mathrm{pr}^{(r)}\mathrm{v} = \mathrm{v} + \sum\limits_{\alpha=1}^{m}\sum\limits_{J} \eta_\alpha^J[u]\frac{\partial}{\partial u_J^\alpha}$ whose coefficients are determined by the prolongation formula (1.2). Similarly to Lie symmetries, there is the invariance criterion generalized symmetries.

Definition 1.9. A generalized vector field v is a generalized infinitesimal symmetry of a system $\mathcal{L}$ of differential equations if and only if $\left(\mathrm{pr}^{(r)}\,\mathrm{v}\right) L^\mu = 0$ for any $\mu = 1,\ldots,l$ and any solution $u = f(x)$ of the system $\mathcal{L}$.

Another name for generalized symmetries is higher symmetries [26]. Among all the generalized vector fields defined by (1.3), those for which the coefficients $\xi^i[u]$ vanish play a distinguished role.

Definition 1.10. An m-tuple $Q[u] = (\chi_1[u],\ldots,\chi_m[u])$ is called the characteristic of the evolutionary generalized vector field $\mathrm{v} = \sum\limits_{\alpha=1}^{m} \chi_\alpha[u]\frac{\partial}{\partial u^\alpha}$.

The characteristic of a generalized vector filed is also known as its *generating function* [26].

Note that the rth prolongation of an evolutionary vector field is an evolutionary vector field of the form

$$\mathrm{pr}^{(r)}\,\mathrm{v}_\chi = \sum_{\alpha,J} \mathrm{D}_J \chi_\alpha[u] \frac{\partial}{\partial u_J^\alpha}.$$

Any vector field v, geometric or generalized, has the associated evolutionary representative v_χ with the characteristic χ defined by

$$\chi_\alpha = \eta_\alpha - \sum_{i=1}^{n} \xi^i u_i^\alpha, \quad \alpha = 1, \ldots, m. \tag{1.4}$$

Thus, every geometric vector field has the evolutionary representative with characteristic depending on at most first-order derivatives. At the same time, not every first-order evolutionary vector field has a geometric counterpart. This is the case only when its characteristic is of the specific form (1.4), with ξ^i and η^α not depending on derivatives of u.

Theorem 1.11. *A generalized vector field* v *is a symmetry of a system of differential equations if and only if its evolutionary representative* v_χ *is.*

This property makes evolutionary vector fields distinguished. The generalized vector field is called *trivial* if its characteristic vanishes on solutions of the system $\mathcal{L}$. Two generalized symmetries are called *equivalent* if they differ by a trivial one. This gives rise to an equivalence relation on the space of generalized symmetries of the system. In particular, the geometric symmetry and its evolutionary counterpart are equivalent.

Similarly to Lie symmetries, there are determining equations for generalized symmetries of a system of differential equations. To state it, we need to introduce an additional object. Let $\mathcal{A}$ be the algebra of differential functions on the jet space $J^\infty(x|u)$, and $\mathcal{A}^l$ be the algebra of their l-tuples.

Definition 1.12. The *Fréchet derivative* of a differential function $P[u] \in \mathcal{A}^l$ is called the differential operator $\mathbf{D}_P \colon \mathcal{A}^m \to \mathcal{A}^l$ defined for any $Q \in \mathcal{A}^m$ so that

$$\mathbf{D}_P(Q) = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} P\big[u + \varepsilon Q[u]\big].$$

The Fréchet derivative of an l-tuple $P = (P_1, \dots, P_l)$ is represented by the $m \times l$-matrix differential operator with entries $(\mathbf{D}_P)_{\mu\nu} = \sum\limits_J \frac{\partial P_\mu}{\partial u^\nu_J} \mathrm{D}_J$, where $\mu = 1, \dots, l$, $\nu = 1, \dots, m$ and the sum is running over all possible unordered multi-indices J. There is an alternative name for the Fréchet derivative of a differential function P in the literature – the *linearization operator* ℓ_P of the differential function P, cf. [26].

It turns out that the infinitesimal invariance criterion of systems of differential equations can be reformulated in terms of the Fréchet derivative, which is based on the following proposition.

Proposition 1.13. *If $L \in \mathcal{A}^l$ and $\chi \in \mathcal{A}^m$, then $\mathbf{D}_L(\chi) = \mathrm{pr}\, \mathrm{v}_\chi(L)$.*

Corollary 1.14. *A generalized vector field* v *with characteristic χ is a generalized infinitesimal symmetry of a system $\mathcal{L}$ of differential equations if and only if $\mathbf{D}_L(\chi) = 0$, where $L = (L_1, \dots, L^l)$ on solutions of the system $\mathcal{L}$.*

There is a way to obtain new generalized symmetries of a system of differential equations from known ones. This operation is realized by so-called recursion operators and will be used in the thesis.

Definition 1.15. A recursion operator for a system $\mathcal{L}$ of differential equations is a linear differential operator $\mathfrak{R}\colon \mathcal{A}^m \to \mathcal{A}^m$ such that the image of any generalized symmetry v_χ of the system $\mathcal{L}$ is the generalized symmetry $\mathrm{v}_{\tilde\chi}$ of the same system, where $\tilde\chi = \mathfrak{R}\chi$, and χ and $\tilde\chi$ are characteristics of the corresponding evolutionary vector fields.

1.3 Conservation laws

Definition 1.16. A *conserved current* of the system $\mathcal{L}$ is an n-tuple of differential functions $F = (F^1[u], \dots, F^n[u])$ the total divergence of which vanishes on the solutions of $\mathcal{L}$,

$$(\mathrm{Div}\, F)\big|_{\mathcal{L}} = 0. \tag{1.5}$$

Hereafter, the total divergence operator is defined by $\mathrm{Div}\, F = \mathrm{D}_i F^i$, and $\mathrm{D}_i = \mathrm{D}_{x_i}$ denotes the operator of total differentiation with respect to the variable x_i.

The validity of (1.5) on the solution set of $\mathcal{L}$ is significant for relating the conserved current F to $\mathcal{L}$. A conserved current F is *trivial* if it is represented as $F = \hat{F} + \check{F}$, where $\hat{F}$ and $\check{F}$ are n-tuples of differential functions such that the components of $\hat{F}$ vanish on the solutions of $\mathcal{L}$ and $\check{F}$ is a null divergence. By null divergence it is meant that $\operatorname{Div} \check{F} = 0$ holds unrestricted of the system $\mathcal{L}$.

Two conserved currents F and F' are called *equivalent* if their difference $F - F'$ is a trivial conserved current. It is obvious that for any system $\mathcal{L}$ its set of conserved currents, denoted by $\mathrm{CC}(\mathcal{L})$, is a linear space. Likewise, the subset of trivial conserved currents, denoted by $\mathrm{CC}_0(\mathcal{L})$, is a linear subspace of $\mathrm{CC}(\mathcal{L})$. The set of equivalence classes of $\mathrm{CC}(\mathcal{L})$ with respect to the above equivalence relation on conserved currents is the quotient space $\mathrm{CC}(\mathcal{L})/\mathrm{CC}_0(\mathcal{L})$, which is denoted by $\mathrm{CL}(\mathcal{L})$.

Definition 1.17. The linear space $\mathrm{CL}(\mathcal{L})$ is called the *space of (local) conservation laws* of the system $\mathcal{L}$. Its elements are called *(local) conservation laws* of the system $\mathcal{L}$.

If the system $\mathcal{L}$ is totally nondegenerate, then it is possible to use the Hadamard lemma and 'integration by parts' to represent the definition of conserved current (1.5) in the form

$$\operatorname{Div} F = \lambda^1 L^1 + \cdots + \lambda^l L^l. \tag{1.6}$$

Definition 1.18. The l-tuple of differential functions $\lambda = (\lambda^1, \ldots, \lambda^l)$ and the equation (1.6) are called the *characteristic* and the *characteristic form* of the conservation law corresponding to the conserved current F, respectively.

The *Euler operator* $\mathsf{E} = (\mathsf{E}^1, \ldots, \mathsf{E}^m)$ is the m-tuple of differential operators defined by

$$\mathsf{E}^a = (-\mathrm{D})^\alpha \partial_{u^a_\alpha}, \quad a = 1, \ldots, m, \quad \text{where} \quad (-\mathrm{D})^\alpha = (-\mathrm{D}_1)^{\alpha_1} \cdots (-\mathrm{D}_n)^{\alpha_n}.$$

A differential function f is a total divergence, meaning that $f = \operatorname{Div} F$ for some n-tuple of differential functions F, if and only if it is annihilated by the Euler operator, $\mathsf{E}^a f = 0$. Using this property of the Euler operator and applying it to the characteristic form of conservation laws (1.6), one obtains $\mathsf{E}^a(\lambda^1 L^1 + \cdots + \lambda^l L^l) = 0$, which is a necessary and sufficient condition for the tuple λ to be a conservation-law characteristic of the system $\mathcal{L}$.

The notion of triviality extends to conservation-law characteristics as well. A characteristic λ is called trivial if it vanishes for all solutions of $\mathcal{L}$. The existence of trivial characteristics makes it necessary to introduce equivalent characteristics. If the difference $\lambda - \tilde{\lambda}$ of characteristics λ and $\tilde{\lambda}$ is a trivial characteristic, then the characteristics λ and $\tilde{\lambda}$ are called *equivalent*. Similarly to conserved currents, the set of characteristics of $\mathcal{L}$, denoted by $\mathrm{Ch}(\mathcal{L})$, is a linear space with the subset $\mathrm{Ch}_0(\mathcal{L})$ of trivial characteristics being a linear subspace thereof.

In the literature, characteristics of conservation laws are also called their multipliers [22, 23] and generating functions [26].

Finally, it is necessary to state the fundamental Noether theorem relating symmetries of a system of differential equations with its conservation laws. Let a system $\mathcal{L}$ be Euler–Lagrange equations with the Lagrangian L, that is, $\mathrm{E}(\mathrm{L}) = 0$. A generalized vector field X is called a variational symmetry for L if $X(\mathrm{L}) = 0$ on solutions of $\mathcal{L}$.

Theorem 1.19. *Suppose that* $\mathcal{L} = \{\mathrm{E}(\mathrm{L}) = 0\}$ *is an Euler–Lagrange system for the Lagrangian* L. *Then an evolutionary vector field* $\chi\partial_u$ *is a variational symmetry for the Lagrangian* L *if and only if* χ *is the characteristic of a conservation law of the system* $\mathcal{L}$.

1.4 Cosymmetries

In the study of conservation laws of systems of differential equations one needs to consider formally adjoint operators to the Fréchet derivatives of differential functions.

Definition 1.20. Given a differential operator $\mathcal{D} = \sum\limits_J P_J[u]\mathrm{D}_J$, its formal adjoint is the differential operator $\mathcal{D}^*$ such that

$$\int_\Omega P \cdot \mathcal{D}Q \;\mathrm{d}x = \int_\Omega Q \cdot \mathcal{D}^* P \;\mathrm{d}x$$

for every pair of differential functions P and Q in $\mathcal{A}$ which vanish when $u = 0$, every domain $\Omega \in \mathbb{R}^n$ and every function $u = f(x)$ of compact support in Ω.

Given a differential operator $\mathcal{D}$ as in the above definition, its formal adjoint is determined by the action on a differential function $Q \in \mathcal{A}$ as follows $\mathcal{D}^*Q = \sum\limits_J (-D)_J(P_J Q)$. Similarly, a matrix differential operator $\mathcal{D}\colon \mathcal{A}^p \to \mathcal{A}^q$ with entries $\mathcal{D}_{\mu\nu}$ has as the formal adjoint the operator $\mathcal{D}^*\colon \mathcal{A}^q \to \mathcal{A}^p$ with entries $\mathcal{D}^*_{\mu\nu} = (\mathcal{D}_{\nu\mu})^*$.

Definition 1.21. An operator $\mathcal{D}$ is formally self-adjoint if $\mathcal{D}^* = \mathcal{D}$, it is formally skew-adjoint if $\mathcal{D}^* = -\mathcal{D}$.

Finally, the formally adjoint operator $\mathbf{D}^*_P\colon \mathcal{A}^r \to \mathcal{A}^m$ of the Fréchet derivative of the differential function $P \in \mathcal{A}^r$ has entries $(\mathbf{D}^*_P)_{\nu\mu} = \sum\limits_J (-\mathrm{D})_J \cdot \frac{\partial P_\mu}{\partial u^\nu_J}$, where $\mu = 1, \dots, r$ and $\nu = 1, \dots, m$.

Definition 1.22. A tuple of differential functions $\chi = (\chi_1, \dots, \chi_l)$ is called a cosymmetry of the system $\mathcal{L}$, if it satisfies the condition $\mathbf{D}^*_{L^\mu}(\chi) = 0$ on solutions of the system $\mathcal{L}$.

For example, characteristics of conserved currents of $\mathcal{L}$ are cosymmetries thereof. Similarly to higher symmetries and conservation laws, one can define trivial cosymmetries of $\mathcal{L}$ and an equivalence relation among them. Cosymmetries are also called adjoint-symmetries in the literature [3, 4]. Recently, it was shown [9] that cosymmetries of $\mathcal{L}$ can be geometrically viewed as certain vertical 1-forms on $\mathcal{L}^{(\infty)}$.

1.5 Hamiltonian systems of evolution equations

Consider the algebra $\mathcal{A}$ of differential functions over $M = X \times U$. Each differential function $P \in \mathcal{A}$ determines the functional $\int P\, \mathrm{d}x$. We define the space $\mathcal{F}$ of functionals as the set of equivalence classes on the algebra $\mathcal{A}$ under the equivalence relation $\tilde{P} \sim P$ if and only if $\tilde{P} = P + \mathrm{Div}\ Q$ for some $Q \in \mathcal{A}^n$.

Definition 1.23. A Poisson bracket of functionals on a smooth manifold M is an operation that assigns a functional $\{\mathcal{P}, \mathcal{Q}\}$ on M to each pair $\mathcal{P}, \mathcal{Q} \in \mathcal{F}$, with the basic properties (a) Bilinearity: $\{a\mathcal{P} + \mathcal{Q}, \mathcal{R}\} = a\{\mathcal{P}, \mathcal{R}\} + \{\mathcal{Q}, \mathcal{R}\}$, $\{\mathcal{P}, a\mathcal{Q} + \mathcal{R}\} = a\{\mathcal{P}, \mathcal{Q}\} + \{\mathcal{P}, \mathcal{R}\}$; (b) Skew-symmetry: $\{\mathcal{P}, \mathcal{Q}\} = -\{\mathcal{Q}, \mathcal{P}\}$; (c) Jacobi identity: $\{\{\mathcal{P}, \mathcal{Q}\}, \mathcal{R}\} + \{\{\mathcal{Q}, \mathcal{R}\}, \mathcal{P}\} + \{\{\mathcal{R}, \mathcal{P}\}, \mathcal{Q}\} = 0$ for any $a \in \mathbb{R}$ and $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in \mathcal{F}$.

Consider a linear differential operator $\mathfrak{D}\colon \mathcal{A}^m \to \mathcal{A}^m$ on the space of m-tuples of differential functions and associate to it the bracket $\{\mathcal{P}, \mathcal{Q}\} := \int \delta\mathcal{P} \cdot \mathfrak{D}\delta\mathcal{Q}\mathrm{d}x$, where $\cdot$ stands for the inner product in $\mathbb{R}^m$.

Definition 1.24. A linear differential operator $\mathfrak{D}$ is called Hamiltonian if its associated bracket is Poisson.

The equilibrium solutions of the equations of nondissipative continuum mechanics are usually found by minimizing an appropriate variational integral. Therefore, smooth solutions satisfy the Euler–Lagrange equations for the relevant functional and thus one works in the Lagrangian framework discussed above. Nevertheless, for the full dynamical problem described by a system of evolution equations Lagrangian formalism may not be applicable and then the Hamiltonian formulation thereof comes into the scene.

Having the definition of the Poisson bracket of functionals we can introduce the Hamiltonian formalism of systems of evolution equations of the form $u_t = K[u]$, where K is a differential function depending on u and its spatial derivatives. We call the system Hamiltonian if it can be written as $u_t = \mathfrak{D}\delta\mathcal{H}$ for some $\mathcal{H} \in \mathcal{F}$ called the Hamiltonian of the system. Thus to verify that a differential operator is Hamiltonian, one must check that operator is formally skew-adjoint and it satisfies the Jacobi identity.

From the symmetry analysis point of view, Hamiltonian operators are important since they relate cosymmetries of a system of differential equations with its generalized symmetries.

Chapter 2

Generalized symmetries and conservation laws of (1+1)-dimensional Klein–Gordon equation

2.1 Introduction

Noether's idea of generalizing the notion of Lie symmetries of systems of differential equations was to allow components of vector fields to depend on derivatives of unknown functions, which led to the notion of generalized (or higher) symmetries [26, 103]. This way, symmetries lose their geometric charm but become a powerful tool, e.g., for finding, with Noether's theorem, conservation laws of systems that are systems of Euler–Lagrange equations for some Lagrangians. Although the general procedure of finding generalized symmetries is similar to its counterpart for Lie symmetries, computational difficulty increases rapidly as the order of symmetries to be found increases. Even low-order generalized symmetries may be hard to compute, in spite of the possibility of using specialized computer algebra packages [13, 36] in such computations. The situation with (local) conservation laws is alike, see for instance remarks in [37] on computational complexity of the problem on conservation laws of the Euler and the Navier–Stokes equations of order less than or equal to two. Besides, given a system of differential equations, a computer cannot handle the construction of all generalized symmetries or conservation laws of this system unless there exist upper bounds on their orders, and these bounds are quite low and are found independently. In view of this, the complete descriptions of generalized symmetries and/or of conservation laws are known for not so many systems of differential equations important for real-world applications as may be expected, taking into account the intensive research activity in the related field.

The above approach with computing the upper bound of orders of generalized symmetries, cosymmetries or conservation laws was applied for a number of systems of differential equations for which such bounds exist. This includes conservation laws of the BBM equations [44, 100], of the k-ε turbulence model [75], of (1+1)-dimensional even-order linear evolution equations [132, Corollary 6] and of the equation $u_t = u_{xxx} + xu$ [132, Example 6], the classification of conservation laws of second-order evolution equations [131] up to contact equivalence, generalized symmetries of the Bakirov system [136] as well as generalized symmetries and conservation laws of the Navier–Stokes equations [67], of the (1+3)-dimensional, (1+2)-dimensional and axisymmetric Khokhlov–Zabolotskaya equations [141], of non-integrable compacton $K(m,m)$-equations [163] and of generalized Kawahara equations [160]. There exist no more or less general results on such upper bounds, except the well-known upper bound for orders of conservation laws of even-order (1+1)-dimensional evolution equations and the extension of this bound in [72] to a wider class of systems of differential equations.

For (integrable) systems admitting (co)symmetries of arbitrary high order, it may be possible to find recursion operators [78, 102, 103, 138] for symmetries and/or for cosymmetries with subsequent determining which cosymmetries are associated with conservation laws. At the same time, recursion operators are not guaranteed to yield all (co)symmetries and so there remains a problem of proving nonexistence of other (co)symmetries. Another point is that recursion operators do not always generate local objects, with generalized symmetries of the Korteweg–de Vries equation and the Lenard recursion operator [63] as an example here, so it is necessary to pick the local ones post factum or prove that the generated hierarchy is local [137]. Amongst known examples of complete descriptions of infinite hierarchies of generalized symmetries and conservation laws are those for the Korteweg–de Vries equation [71, 79, 84, 152], for its linear counterpart $u_t = u_{xxx}$ [132, Example 5], of the vacuum Einstein equations in the four-dimensional spacetime [11], for free Maxwell's equations in (3+1)-dimensional Minkowski space [5, 7], for massless free fields of spin $s \geqslant 1/2$ [6, 122] and for an isothermal no-slip drift flux model [113]. All the generalized symmetries of the Yang–Mills equations on Minkowski space with a semi-simple structure group were computed in [121]. Symmetry operators of the one-dimensional Schrödinger equation were studied in [62, 95]. See also [43, 150] for a general theory of hydrodynamic systems, where infinite hierarchies of conservation laws and symmetries, though often nonlocal, are common, and [92, 138, 139, 162] for some related examples.

In the present chapter, we exhaustively describe generalized symmetries and local conservation laws of the (1+1)-dimensional (real) Klein–Gordon equation, which takes, in natural units, the form $\Box u + m^2 u = 0$, where u is the real-valued unknown function of the real independent variables x_0 and x_1, $\Box$ is the d'Alembert operator in (1+1) dimensions, $\Box = \partial^2/\partial x_0^2 - \partial^2/\partial x_1^2$, and m denotes the nonzero mass parameter.[1] Without loss of generality, the mass parameter can be set to be equal one by simultaneous scaling of the independent variables. We work with this equation in the characteristic, or light-cone, variables $x = (x_0 + x_1)/2$ and $y = (-x_0 + x_1)/2$,

$$\mathcal{K}\colon \quad u_{xy} = u.$$

In what follows we use the same notation $\mathcal{K}$ for the solution set of the equation $\mathcal{K}$ as well as for the set defined by $\mathcal{K}$ and its differential consequences in the corresponding infinite-order jet space.

Our specific interest to the equation $\mathcal{K}$ originated from the study of the hydrodynamic-type system $\mathcal{S}$ of differential equations modeling an isothermal no-slip drift flux, see Chap-

[1]The zero value of m, which corresponds to the wave equation, is singular in all properties related to symmetry analysis of differential equations, including Lie, contact and generalized symmetries and conservation laws; cf. [70, Section 18.4] and [127].

ter 3. It turned out that the (nonlinear) system $\mathcal{S}$ is reduced to the (linear) equation $\mathcal{K}$ by the composition of a simple point transformation and a rank-two hodograph transformation. The family of regular solutions of $\mathcal{S}$ is parameterized by an arbitrary solution of $\mathcal{K}$ and by an arbitrary function of a single argument. Moreover, finding generalized symmetries and local conservation laws of the system $\mathcal{S}$ reduces to the analogous problems for the equation $\mathcal{K}$. At the same time, we did not find exhaustive and trusted solutions of the latter problems in the literature, which motivated our study of the Klein–Gordon equation.

The Lie invariance algebra $\mathfrak{g}$ of the equation $\mathcal{K}$ was computed by Sophus Lie himself in the course of the group classification of second-order linear equations with two independent variables [82, Section 9]. The equation $\mathcal{K}$ appeared there as the simplest particular member of a parameterized family of inequivalent equations that admit three-dimensional Lie-symmetry extensions in comparison with the general case.[2] The algebra $\mathfrak{g}$ is spanned by the vector fields

$$\partial_x,\ \partial_y,\ x\partial_x - y\partial_y,\ u\partial_u,\ f(x,y)\partial_u,$$

where the function $f = f(x,y)$ runs through the solution set of $\mathcal{K}$. This algebra is represented as the semidirect sum, $\mathfrak{g} = \mathfrak{g}^{\rm ess} \ltimes \mathfrak{g}^{\infty}$, of the so-called (finite-dimensional) essential Lie invariance subalgebra $\mathfrak{g}^{\rm ess} := \langle \partial_x,\, \partial_y,\, x\partial_x - y\partial_y,\, u\partial_u\rangle$ and the (infinite-dimensional) Abelian ideal $\mathfrak{g}^{\infty} := \langle f(x,y)\partial_u,\ f \in \mathcal{K}\rangle$ related to the linear superposition of solutions of $\mathcal{K}$. Note that Sophus Lie carried out the group classification over the complex field under supposing all objects, like equation coefficients and components of vector fields, to be analytic. This is why his results are directly extended to hyperbolic equations over the real field.

Since the equation $\mathcal{K}$ is the Euler–Lagrange equation of the Lagrangian

$$\mathrm{K} = -\frac{1}{2}(u_xu_y + u^2),$$

[2]The same classification case was represented in [116, Section 9.6] by another family, which is similar to the family singled out by Lie with respect to a point transformation but is more cumbersome. Under this representation, the relation of the Klein–Gordon equation to Lie-symmetry extensions within the class of second-order linear equations with two independent variables is not so obvious as in Lie's paper [82].

its local conservation laws can be constructed using Noether's theorem. Conservation laws associated with essential variational Lie symmetries of the Lagrangian K are well known and admit an obvious physical interpretation. These are the conservations of energy-momentum and of relativistic angular momentum, which are respectively related, via Noether's theorem, to spacetime translations and to Lorentz transformations; see [148] for a good pedagogical presentation.

In the course of a general discussion of quadratic conserved quantities in free-field theories in [76], it was shown that the (1+3)-dimensional Klein–Gordon equation possesses an infinite-dimensional space of conservation laws with conserved currents whose components are quadratic expressions in derivatives of the dependent variable with constant coefficients; in fact, the specific dimension (1+3) is not essential in this result. Tsujishita [151] proved that for the $(1+n)$-dimensional Klein–Gordon equation with $n \geqslant 2$, this space coincides with the space of conservation laws containing the conserved currents whose components are differential polynomials with constant coefficients; see also [152] and references therein. At the same time, the Klein–Gordon equation obviously possesses other conservation laws. There are such conservation laws even among conservation laws associated with Lie variational symmetries of the corresponding Lagrangian, e.g., the conservations of relativistic angular momentum.

Having generalized the notion of Killing vector, in [94] Nikitin introduced the notions of generalized Killing tensors and generalized conformal Killing tensors of arbitrary rank and arbitrary order in the $(p+q)$-dimensional pseudo-Euclidean space $\mathbb{R}^{p,q}$ of signature (p,q) with arbitrary $p,q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $p+q > 1$. The explicit form of these tensors was found therein and then used for the study of linear symmetry operators of the Klein–Gordon–Fock equation in $\mathbb{R}^{p,q}$. See also [96] for a more detailed exposition of the above results and [62], where a number of results on linear symmetry operators of linear systems of differential equations arising as models in quantum mechanics are collected.

Shapovalov and Shirokov stated in [140] that for any $r \in \mathbb{N}_0$, an arbitrary linear second-order partial differential equation with nondegenerate symbol and more than two independent variables possesses only a finite number of linearly independent linear symmetry operators up to order r and admits no nonlinear generalized symmetries, that is, symmetries equivalence classes of which do not have elements with characteristics not affine in derivatives of a dependent variable. Therein, they also described the algebra of generalized symmetries of the Laplace–Beltrami equation in the space $\mathbb{R}^{p,q}$ in terms of the universal enveloping algebra of the essential Lie invariance algebra of this equation; see [45] for a further deeper study of the algebra of generalized symmetries of the Laplace equation.

Note that the algebra of generalized symmetries and the spaces of local conservation laws and variational symmetries of the associated Lagrangian of the allied (1+1)-dimensional wave equation $u_{xy} = 0$ are known, see [70, Section 18.4] and [127], and they essentially differ from the corresponding objects for the equation $\mathcal{K}$. Nonlinear wave equations of the form $u_{xy} = f(u)$ admitting generalized symmetries whose characteristics do not depend on the independent variables were singled out in [169]; see also [70, Section 21.2]. The complete classification of local conservation laws of equations in this class was initiated and partially carried out in [60].

The results of the present chapter are published in [115] and its structure is as follows. In Section 2.2 we explicitly describe the quotient algebra Σ^{q} of generalized symmetries of the (1+1)-dimensional Klein–Gordon equation $\mathcal{K}$ with respect to the standard equivalence of generalized symmetries by presenting a naturally isomorphic space of representatives for equivalence classes of generalized symmetries. This leads to the description of the algebra Σ^{q} in terms of the universal enveloping algebra of the essential Lie invariance algebra of $\mathcal{K}$. The related computations are essentially simplified by using advantages of the characteristic independent variables for the equation $\mathcal{K}$, which are specific for the (1+1)-dimensional case. As another optimization, we avoid the direct integration of the system of determining equations for generalized symmetries of $\mathcal{K}$. Instead of this integration, which is realizable but quite cumbersome, we estimate the number of independent linear symmetries of an arbitrary fixed order, apply the Shapovalov–Shirokov theorem [140] and explicitly present the same number of appropriate linear symmetries. In Section 2.3 we recall the variational interpretation of the equation $\mathcal{K}$ and accurately single out the space of variational symmetries of the Lagrangian K from the entire space of generalized symmetries of $\mathcal{K}$. Finally, in Section 2.4 we find the space of local conservation laws of $\mathcal{K}$ using Noether's theorem for constructing a space of conserved currents that is naturally isomorphic to the space of local conservation laws. In the course of this construction, we select conserved currents of minimal order among the equivalent ones, which immediately specifies the spaces of conservation laws of each fixed order. We also show that, up to the action of generalized symmetries, the entire space of conservation laws of the equation under study is generated by a single conservation law. In Section 2.5 we underscore all the techniques and ideas, especially specific to the present chapter, which we use in the course of the study.

2.2 Generalized symmetries

Here we revisit the construction of the algebra Σ of generalized symmetries of the (1+1)-dimensional Klein–Gordon equation with some enhancements. Computing generalized symmetries, without loss of generality we can consider only evolutionary generalized vector fields and evolutionary representatives of generalized symmetries [103, p. 291] and thus assume that the algebra Σ is constituted by such representatives for the above equation,

$$\Sigma = \big\{X = \eta[u]\partial_u \mid \mathrm{D}_x\mathrm{D}_y\eta[u] = \eta[u] \text{ on } \mathcal{K}\big\},$$

where $\eta[u]$ denotes a differential function of u, and D_x and D_y are the operators of total derivatives in x and y, respectively; see [103, Definition 2.34]. We denote by $\Sigma^{\rm triv}$ the algebra of trivial generalized symmetries of the equation $\mathcal{K}$, which is an ideal of Σ. It consists of all generalized vector fields in the evolutionary form (with the independent variables (x, y) and the dependent variable u) whose characteristics vanish on solutions of $\mathcal{K}$. The quotient algebra $\Sigma^{\rm q} = \Sigma/\Sigma^{\rm triv}$ is naturally isomorphic[3] to the algebra of

[3]There are two similar kinds of *natural (*or *canonical) isomorphisms* in this chapter—those related to quotient linear spaces and those related to quotient Lie algebras. Given a linear space V and its subspaces U and W such that $V = U \dotplus W$, where "$\dotplus$" denotes the direct sum of subspaces, the natural isomorphism between V/U and W is established in the way that each coset of U corresponds to the unique element of W belonging to this coset. In a similar way, natural isomorphisms are established

canonical representatives in the reduced evolutionary form,

$$\hat{\Sigma}^{\mathrm{q}} = \big\{X = \eta[u]\partial_u \in \Sigma \mid \eta[u] = \eta(x,y,u_{-n},\dots,u_n) \text{ for some } n \in \mathbb{N}_0\big\}.$$

Here x, y, $u_0 := u$, $u_k := \partial_x^k u$ and $u_{-k} := \partial_y^k u$, $k \in \mathbb{N}$, constitute the standard coordinates on the manifold defined by the equation $\mathcal{K}$ and its differential consequences in the infinite-order jet space $\mathrm{J}^\infty(x,y|u)$ with the independent variables (x,y) and the dependent variable u. Negative indices were used in view of the equality $u_{xy} = u$ on $\mathcal{K}$. The Lie bracket on $\hat{\Sigma}^{\mathrm{q}}$ is defined as the reduced Lie bracket of generalized vector fields, where all arising mixed derivatives of u are substituted in view of the equation $\mathcal{K}$ and its differential consequences,

$$[\eta^1\partial_u, \eta^2\partial_u] = \sum_{k=0}^{\infty}(\eta^2_{u_k}\mathrm{D}_x^k\eta^1 - \eta^1_{u_k}\mathrm{D}_x^k\eta^2)\partial_u + \sum_{k=1}^{\infty}(\eta^2_{u_{-k}}\mathrm{D}_y^k\eta^1 - \eta^1_{u_{-k}}\mathrm{D}_y^k\eta^2)\partial_u,$$

where D_x and D_y are the reduced operators of total derivatives with respect to x and y,

$$\mathrm{D}_x := \partial_x + \sum_{k=-\infty}^{+\infty} u_{k+1}\partial_{u_k}, \quad \mathrm{D}_y := \partial_y + \sum_{k=-\infty}^{+\infty} u_{k-1}\partial_{u_k}.$$

The subspace $\Sigma^n = \big\{[X] \in \Sigma^{\mathrm{q}} \mid \exists\, \eta[u]\partial_u \in [X]\colon \operatorname{ord}\eta[u] \leqslant n\big\}$, $n \in \mathbb{N}_0 \cup \{-\infty\}$, of Σ^{q} is the space of generalized symmetries of order less than or equal to n.[4] It is naturally isomorphic to the subspace of canonical representatives in the reduced evolutionary form with characteristics of order less than or equal to n,

$$\hat{\Sigma}^n = \big\{\eta[u]\partial_u \in \hat{\Sigma}^{\mathrm{q}} \mid \operatorname{ord}\eta[u] \leqslant n\big\}, \quad n \in \mathbb{N}_0 \cup \{-\infty\}.$$

Note that the subspace $\hat{\Sigma}^{-\infty}$ can be identified with the subalgebra of Lie symmetries of $\mathcal{K}$ associated with the linear superposition of solutions of $\mathcal{K}$, $\hat{\Sigma}^{-\infty} = \{f(x,y)\partial_u \mid f \in \mathcal{K}\}$, i.e., with f running through the solution set of $\mathcal{K}$. The subspace family $\{\Sigma^n \mid n \in \mathbb{N}_0 \cup \{-\infty\}\}$

between $\mathfrak{a}/\mathfrak{i}$ and $\mathfrak{b}$, where $\mathfrak{a}$ is a Lie algebra, and $\mathfrak{b}$ and $\mathfrak{i}$ are its subalgebra and its ideal, respectively, such that $\mathfrak{a} = \mathfrak{b} \Subset \mathfrak{i}$.

[4]The order $\operatorname{ord} F[u]$ of the differential function $F[u]$ is the highest order of derivatives of u involved in $F[u]$ if there are such derivatives, and $\operatorname{ord} F[u] = -\infty$ otherwise. If $X = \eta[u]\partial_u$, then $\operatorname{ord} X := \operatorname{ord}\eta[u]$. For $[X] \in \Sigma^{\mathrm{q}}$, $\operatorname{ord}[X] = \min\big\{\operatorname{ord}\eta[u] \mid \eta[u]\partial_u \in [X]\big\}$.

filters the algebra $\Sigma^{\rm q}$. Consider the quotient spaces $\Sigma^{[n]}=\Sigma^n/\Sigma^{n-1}$ for $n\in\mathbb N$ and $\Sigma^{[0]}=\Sigma^0/\Sigma^{-\infty}$ and denote $\Sigma^{[-\infty]}:=\Sigma^{-\infty}$. The space $\Sigma^{[n]}$ can be assumed as the space of nth order generalized symmetries of $\mathcal K$, $n\in\mathbb N_0\cup\{-\infty\}$.

An algebra of linear generalized symmetries of the equation $\mathcal K$ is

$$\Lambda=\left\{\eta[u]\partial_u\in\Sigma\;\middle|\;\eta=\mathfrak Du\ \text{ for some }\ \mathfrak D=\sum_{|\alpha|\leqslant n}\zeta^\alpha(x,y)\mathrm D_x^{\alpha_1}\mathrm D_y^{\alpha_2},\ n\in\mathbb N_0\right\}.$$

Recall that $\alpha=(\alpha_1,\alpha_2)\in\mathbb N_0^{\,2}$ is a multiindex, and $|\alpha|=\alpha_1+\alpha_2$. The subalgebra $\Lambda^{\rm triv}$ of trivial linear generalized symmetries coincides with $\Lambda\cap\Sigma^{\rm triv}$. The quotient algebra $\Lambda^{\rm q}=\Lambda/\Lambda^{\rm triv}$ can be embedded into $\Sigma^{\rm q}$ as the subalgebra of cosets of $\Sigma^{\rm triv}$ that contain linear generalized symmetries. The subspace $\Lambda^n=\Lambda^{\rm q}\cap\Sigma^n$ with $n\in\mathbb N_0$ is naturally isomorphic to the space $\hat\Lambda^n$ of evolutionary generalized symmetries whose characteristics are of the reduced form, where the mixed derivatives of u are excluded in view of $\mathcal K$,

$$\eta[u]=\sum_{k=-n}^{n}\eta^k(x,y)u_k. \tag{2.1}$$

Elements of $\hat\Lambda^n$ are canonical representatives of cosets of $\Sigma^{\rm triv}$ constituting the space Λ^n. The quotient spaces $\Lambda^{[n]}=\Lambda^n/\Lambda^{n-1}$, $n\in\mathbb N$, and the subspace $\Lambda^{[0]}=\Lambda^0$ are naturally embedded into the respective spaces $\Sigma^{[n]}$'s, $n\in\mathbb N_0$. We interpret the space $\Lambda^{[n]}$ as the space of nth order linear generalized symmetries of $\mathcal K$, $n\in\mathbb N_0$. This space is isomorphic to the space of the pairs (η^n,η^{-n}) such that the differential function $\eta[u]$ defined by (2.1) with some values of the other coefficients η's is the characteristic of an element of $\hat\Lambda^n$.

Lemma 2.1. *$\dim\Lambda^{[n]}=2n+1$, $n\in\mathbb N_0$.*

Proof. For generalized symmetries with characteristics of the form (2.1), the invariance criterion for $\mathcal K$, $\mathfrak D_x\mathfrak D_y\eta=\eta$, implies the following system of determining equations:

$$\Delta_k\colon\ \eta^k_{xy}+\eta^{k-1}_y+\eta^{k+1}_x=0,\quad k=-n-1,-n,\dots,n,n+1,$$

where we assume η^{-n-2}, η^{-n-1}, η^{n+1} and η^{n+2} to vanish. These symmetries are of (essential) order n if and only if at least one of the coefficients η^{-n} and η^n does not vanish.

Suppose the coefficient η^{-n} does not vanish. We integrate the equation Δ_{-n-1}: $\eta^{-n}_x{=}0$, which gives $\eta^{-n} = \theta(y)$ for some smooth function θ of y. After substituting the obtained value of η^{-n} into Δ_{-n} and Δ_{-n+1}, we consider the set $\Delta_{[-n,n-1]}$ of the equations Δ_k with $k = -n, -n+1, \dots, n-1$ as a system of inhomogeneous linear differential equations with respect to the other η's. The equation Δ_{-n} takes the form $\eta^{-n+1}_x = 0$, and it is convenient to represent the equations Δ_k with $k = -n+1, -n+2, \dots, n-1$ as $\eta^{k+1}_x = -\eta^k_{xy} - \eta^{k-1}_y$. To find a particular solution of the system $\Delta_{[-n,n-1]}$, we successively integrate its equations with respect to x, taking the antiderivatives 0 and $x^{n+1}/(n+1)$ for 0 and x^n, respectively. We can neglect the solutions of the homogeneous counterpart of $\Delta_{[-n,n-1]}$ since they correspond to the zero value of η^{-n}. After the integration, we derive an expression for η^n,

$$\eta^n = \frac{(-1)^n}{n!}\frac{\mathrm{d}^n\theta}{\mathrm{d}y^n}x^n + R,$$

where R is a polynomial in x with $\deg_x R < n$, whose coefficients depend linearly and homogeneously on derivatives of θ of order greater than n. Substituting this expression into the equation Δ_n: $\eta^n_y = 0$ and splitting with respect to x, we obtain the equation $\mathrm{d}^{n+1}\theta/\mathrm{d}y^{n+1} = 0$. Since the derivative η^{n-1}_y is of the same structure as R, the equation Δ_{n-1}: $\eta^{n-1}_y = 0$ is identically satisfied in view of the equation for θ. As a result, we have $n+1$ linearly independent values of the coefficient η^{-n}, say, 1, y, $\dots$, y^n, and, therefore, $n+1$ linearly independent generalized symmetries with characteristics of the form (2.1) with nonvanishing coefficient η^{-n}. Moreover, only one of these symmetries, with $\eta^{-n} = y^n$, has a nonvanishing value of the coefficient η^n.

Since the problem is symmetric with respect to x and y, after supposing that the coefficient η^n does not vanish, we turn the above procedure around by permuting x and y and by changing the direction of the successive integration. This leads to $n+1$ linearly independent generalized symmetries with characteristics of the form (2.1) with nonvanishing coefficient η^n, where similarly to the above case, only one of these symmetries has a nonvanishing value of the coefficient η^{-n}.

Therefore, in total there exist precisely $2n+1$ linearly independent nth order generalized symmetries with characteristics of the form (2.1). $\square$

Corollary 2.2. $\dim \Lambda^n = \sum\limits_{k=0}^{n} \dim \Lambda^{[k]} = (n+1)^2 < +\infty$, $n \in \mathbb{N}_0$.

Lemma 2.3. *The space $\Sigma^{[n]}$ with $n \in \mathbb{N}_0$ is naturally isomorphic to the subspace*

$$\tilde\Sigma^{[n]} = \big\langle (\mathrm{J}^n u)\partial_u,\ (\mathrm{J}^k \mathrm{D}_x^{n-k} u)\partial_u,\ (\mathrm{J}^k \mathrm{D}_y^{n-k} u)\partial_u,\ k = 0, \dots, n-1 \big\rangle$$

of Λ, where $\mathrm{J} := x\mathrm{D}_x - y\mathrm{D}_y$. Here each element X of $\tilde\Sigma^{[n]}$ corresponds to the element of $\Sigma^{[n]}$ that, as a coset of Σ^{n-1} in Σ^n, contains an element of Σ^n that, as a coset of Σ^{triv} in Σ, contains X.

Proof. In view of the Shapovalov–Shirokov theorem [140, Theorem 4.1], Lemma 2.1 implies that $\Sigma^{[n]} = \Lambda^{[n]}$ for $n \in \mathbb{N}_0$.

The differential functions $\mathrm{D}_x u = u_x$, $\mathrm{D}_y u = u_y$ and $\mathrm{J}u = xu_x - yu_y$ are the characteristics of the Lie symmetries $-\partial_x$, $-\partial_y$ and $y\partial_y - x\partial_x$ of $\mathcal{K}$, respectively, and hence the operators D_x, D_y and J are its recursion operators. Therefore, any operator $\mathfrak{D}$ in the universal enveloping algebra generated by these operators is a symmetry operator of $\mathcal{K}$, that is, a generalized vector field $(\mathfrak{D}u)\partial_u$ is a generalized symmetry of $\mathcal{K}$. Thus, $\tilde\Sigma^{[n]} \subset \Lambda \subset \Sigma$.

The space $\tilde\Sigma^{[n]}$ contains no nonzero trivial generalized symmetries of $\mathcal{K}$. Indeed, suppose that an element $X \in \tilde\Sigma^{[n]}$ with characteristic

$$X[u] = a\mathrm{J}^n u + \sum_{k=0}^{n-1} \big(b_k \mathrm{J}^k \mathrm{D}_x^{n-k} u + c_k \mathrm{J}^k \mathrm{D}_y^{n-k} u\big)$$

is a trivial symmetry, that is, $X[u]$ vanishes on solutions of $\mathcal{K}$. Here a, b's and c's are constants. Consider the solution $u^\lambda = \mathrm{e}^{\lambda x + \lambda^{-1} y}$ of the equation $\mathcal{K}$, which is parameterized by $\lambda \in \mathbb{R}/\{0\}$. The expression $\mathrm{e}^{-\lambda x - \lambda^{-1} y} X[u^\lambda]$ is a polynomial in $\lambda x - \lambda^{-1} y$, $\lambda x + \lambda^{-1} y$, λ and λ^{-1}, whose collection of terms of maximal total degree, which equals n, coincides with $a\big(\lambda x - \lambda^{-1} y\big)^n + \sum_{k=0}^{n-1} \big(\lambda x - \lambda^{-1} y\big)^k \big(b_k \lambda^{n-k} + c_k \lambda^{k-n}\big)$. Then the condition $X[u^\lambda] = 0$ implies that $a = 0$ and $b_k = c_k = 0$, $k = 0, \dots, n-1$.

In other words, different elements of $\tilde\Sigma^{[n]}$ belong to different cosets of Σ^{triv} in Σ, which are elements of Σ^{q}. Moreover, the order of each of these cosets is n, and $\dim \tilde\Sigma^{[n]} = 2n+1$. In view of Lemma 2.1, the space $\tilde\Sigma^{[n]}$ is canonically isomorphic to the space $\Lambda^{[n]} = \Sigma^{[n]}$. □

It follows from Lemma 2.3 that $\Sigma^{\mathrm{q}} = \Lambda^{\mathrm{q}} \Subset \Sigma^{-\infty} \simeq \tilde\Sigma^{\mathrm{q}} = \tilde\Lambda^{\mathrm{q}} \Subset \tilde\Sigma^{-\infty}$, where

$$\Lambda^{\mathrm{q}} \simeq \tilde\Lambda^{\mathrm{q}} := \big\langle (\mathrm{J}^k u)\partial_u,\ (\mathrm{J}^k\mathrm{D}_x^l u)\partial_u,\ (\mathrm{J}^k\mathrm{D}_y^l u)\partial_u,\ k\in\mathbb{N}_0,\ l\in\mathbb{N}\big\rangle,$$

$$\Sigma^{-\infty} \simeq \tilde\Sigma^{-\infty} := \hat\Sigma^{-\infty} = \big\{f(x,y)\partial_u \mid f\in\mathcal{K}\big\},$$

and all the above isomorphisms are natural as related to quotient spaces. They become natural isomorphisms related to quotient Lie algebras if we define the Lie bracket on the space $\tilde\Sigma^{\mathrm{q}}$ as the Lie bracket of generalized vector fields, where mixed derivatives arising due to the action of the operators D_x and D_y not involved in J should be substituted in view of the equation $\mathcal{K}$ and its differential consequences.

The essential Lie invariance algebra $\mathfrak{g}^{\mathrm{ess}}$ of the equation $\mathcal{K}$ is spanned by the vector fields ∂_x, ∂_y, $x\partial_x - y\partial_y$ and $u\partial_u$, cf. [62]. It can be identified with the quotient $\mathfrak{g}/\tilde\Sigma^{-\infty}$ of the Lie invariance algebra $\mathfrak{g}$ of with respect to the abelian ideal $\tilde\Sigma^{-\infty}$ corresponding to the linear superposition of solutions of $\mathcal{K}$. Thus, the algebra $\mathfrak{g}^{\mathrm{ess}}$ is isomorphic to the direct sum of the pseudo-Euclidean algebra $\mathfrak{e}(1,1)$ (the Poincaré algebra $\mathfrak{p}(1,1)$ in another terminology or the algebra $\mathrm{g}_{3.4}^{-1}$ in Mubarakzyanov's classification of low-dimensional Lie algebras [93]) and the one-dimensional (abelian) algebra $\mathfrak{a}_1$, $\mathfrak{g}^{\mathrm{ess}} \simeq \mathfrak{e}(1,1)\oplus\mathfrak{a}_1$. Note also that $\mathfrak{g}^{\mathrm{ess}} \simeq \Lambda^1 \simeq \Sigma^1/\Sigma^{-\infty}$. Let

$$\phi\colon \mathfrak{g}^{\mathrm{ess}} \to \mathfrak{e}(1,1)\oplus\mathfrak{a}_1$$

be the isomorphism with $\phi(u\partial_u) = e_0$, $\phi(\partial_x) = e_1$, $\phi(\partial_y) = e_2$ and $\phi(x\partial_x - y\partial_y) = e_3$, where $\langle e_0\rangle = \mathfrak{a}_1$ and the basis (e_1,e_2,e_3) of $\mathfrak{e}(1,1)$ is related to the standard basis $(\tilde e_1,\tilde e_2,\tilde e_3)$ by $\tilde e_1 = e_1+e_2$, $\tilde e_2 = e_1-e_2$, $\tilde e_3 = e_3$. The canonical commutation relations of $\mathfrak{e}(1,1)$ are $[\tilde e_1,\tilde e_2]=0$, $[\tilde e_1,\tilde e_3]=\tilde e_2$ and $[\tilde e_2,\tilde e_3]=\tilde e_1$, which in the basis (e_1,e_2,e_3) take the form $[e_1,e_2]=0$, $[e_1,e_3]=e_1$ and $[e_2,e_3]=-e_2$. Thus, the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}^{\mathrm{ess}})$ of the algebra $\mathfrak{g}^{\mathrm{ess}}$ is isomorphic to the quotient of the tensor algebra $\mathrm{T}(\mathfrak{e}(1,1)\oplus\mathfrak{a}_1)$ by the two-sided ideal I generated by $e_1\otimes e_2 - e_2\otimes e_1$, $e_1\otimes e_3 - e_3\otimes e_1 - e_1$, $e_2\otimes e_3 - e_3\otimes e_2 + e_2$, $e_0\otimes e_i - e_i\otimes e_0$, $i=1,2,3$.

Theorem 2.4. *The quotient algebra Σ^{q} of generalized symmetries of the Klein–Gordon equation $\mathcal{K}$ is naturally isomorphic to the algebra $\tilde{\Sigma}^{\mathrm{q}}$, which is the semidirect sum of the algebra*

$$\tilde{\Lambda}^{\mathrm{q}} = \big\langle (\mathrm{J}^k u)\partial_u,\ (\mathrm{J}^k\mathrm{D}_x^l u)\partial_u,\ (\mathrm{J}^k\mathrm{D}_y^l u)\partial_u,\ k\in\mathbb{N}_0,\ l\in\mathbb{N}\big\rangle \simeq \mathfrak{U}\big(\mathfrak{e}(1,1)\oplus\mathfrak{a}_1\big)/\mathfrak{I}$$

with the abelian algebra $\tilde{\Sigma}^{-\infty} = \{f(x,y)\partial_u \mid f\in\mathcal{K}\}$. Here $\mathfrak{I}$ is the two-sided ideal of the universal enveloping algebra $\mathfrak{U}(\mathfrak{e}(1,1)\oplus\mathfrak{a}_1)$ that is generated by the cosets $e_1\otimes e_2 - e_0 + I$ and $e_0\otimes e_j - e_j + I$, $j=0,1,2,3$.

Moreover, for each $X\in\tilde{\Lambda}^{\mathrm{q}}$ we denote by $\mathfrak{X}$ the linear operator in total derivatives with coefficients depending on x and y that is associated with X, $X[u]=\mathfrak{X}u$. In this terminology the operators 1, D_x, D_y and J are associated with the evolutionary forms of the Lie symmetries $u\partial_u$, $-\partial_x$, $-\partial_y$ and $y\partial_y - x\partial_x$ of the Klein–Gordon equation $\mathcal{K}$, respectively. Note that $\mathfrak{L}\mathfrak{X}=\mathfrak{X}\mathfrak{L}$ for any $X\in\tilde{\Lambda}^{\mathrm{q}}$.

Corollary 2.5. $\hat{\Sigma}^{\mathrm{q}} = \hat{\Lambda}^{\mathrm{q}} \Subset \hat{\Sigma}^{-\infty}$, *where*

$$\hat{\Lambda}^{\mathrm{q}} := \big\langle (\mathfrak{J}^k u)\partial_u,\ (\mathfrak{J}^k\mathfrak{D}_x^l u)\partial_u,\ (\mathfrak{J}^k\mathfrak{D}_y^l u)\partial_u,\ k\in\mathbb{N}_0,\ l\in\mathbb{N}\big\rangle \simeq \tilde{\Lambda}^{\mathrm{q}},$$

$$\hat{\Sigma}^{-\infty} = \tilde{\Sigma}^{-\infty} := \big\{f(x,y)\partial_u \mid f\in\mathcal{K}\big\} \quad \text{and} \quad \mathfrak{J} := x\mathfrak{D}_x - y\mathfrak{D}_y.$$

2.3 Variational symmetries

The (1+1)-dimensional Klein–Gordon equation $\mathcal{K}$ is the Euler–Lagrange equation for the Lagrangian $\mathrm{K} = -(u_xu_y+u^2)/2$. Therefore, the spaces Σ, Σ^{triv} and Σ^{q} respectively coincide with their counterparts for cosymmetries. Moreover, in view of Noether's theorem [103, Theorem 5.58] a differential function is a conservation-law characteristic of $\mathcal{K}$ if and only if it is the characteristic of a (generalized) variational symmetry of K.

Since a generalized vector field is a variational symmetry of a Lagrangian if and only if its evolutionary representative is [103, Proposition 5.32], we work only with evolutionary representatives of variational symmetries. Denote by Υ, Υ^{triv} and Υ^{q} the algebra (of evolutionary representatives) of variational symmetries of the Lagrangian K, its subalgebra of trivial variational symmetries and the quotient algebra of variational symmetries of this Lagrangian, i.e., $\Upsilon\subset\Sigma$, $\Upsilon^{\rm triv}:=\Upsilon\cap\Sigma^{\rm triv}$ and $\Upsilon^{\rm q}:=\Upsilon/\Upsilon^{\rm triv}$. In contrast to $\Sigma^{\rm triv}$, the algebra $\Upsilon^{\rm triv}$ does not consist of all generalized vector fields in the evolutionary form whose characteristics vanish on solutions of $\mathcal K$. This is why one should carefully use reductions of generalized symmetries by excluding derivatives in view of $\mathcal K$ when working with variational symmetries, the space of which may not be closed with respect to such a reduction. We also define the subspace of variational symmetries of order less than or equal to n,

$$\Upsilon^n=\big\{[X]\in\Upsilon^{\rm q}\mid\exists\,\eta[u]\partial_u\in[X]\colon\ \operatorname{ord}\eta[u]\leqslant n\big\},\quad n\in\mathbb N_0\cup\{-\infty\},$$

and denote $\Upsilon^{[n]}=\Upsilon^n/\Upsilon^{n-1}$ for $n\in\mathbb N$, $\Upsilon^{[0]}=\Upsilon^0/\Upsilon^{-\infty}$ and $\Upsilon^{[-\infty]}:=\Upsilon^{-\infty}$. The space $\Upsilon^{[n]}$ can be interpreted as the space of nth order variational symmetries of K, $n\in\mathbb N_0\cup\{-\infty\}$.

Lemma 2.6. *If a linear generalized symmetry $X\in\Lambda$ of the Klein–Gordon equation is a variational symmetry of the Lagrangian* K*, then* $\operatorname{ord}X\in2\mathbb N_0+1$.

Proof. In order for a generalized vector field X in Λ to be a variational symmetry of K, its characteristic $\mathfrak Xu$ has to satisfy the criterion [103, Proposition 5.49]

$$\mathsf D^\dagger_{\mathfrak Xu}(\mathfrak Ku)+\mathsf D^\dagger_{\mathfrak Ku}(\mathfrak Xu)=(\mathfrak X^\dagger\mathfrak K+\mathfrak K^\dagger\mathfrak X)u=0$$

on the entire infinite-order jet space $\mathrm J^\infty(x,y|u)$. Here the operator in total derivatives $\mathfrak X$ corresponds to X, $\mathfrak K$ is the operator in total derivatives that is associated with the equation $\mathcal K$, $\mathfrak K=\mathrm D_x\mathrm D_y-1$, a constant summand in a differential operator denotes the multiplication operator by this constant, $\mathsf D_F$ denotes the Fréchet derivative of a differential function F, and $\mathfrak B^\dagger$ denotes the formal adjoint to a differential operator $\mathfrak B$. Hence we have the operator equality $\mathfrak X^\dagger\mathfrak K+\mathfrak K^\dagger\mathfrak X=0$. Since the equation $\mathcal K$ is the Euler–Lagrange equation of a Lagrangian, the operator $\mathfrak K$ is formally self-adjoint, $\mathfrak K^\dagger=\mathfrak K$. If $\operatorname{ord}X$ were even, then the principal symbol of the left-hand side of the operator equality $\mathfrak X^\dagger\mathfrak K+\mathfrak K^\dagger\mathfrak X=0$ would be equal to the product of the principal symbols of $\mathfrak X$ and $\mathfrak K$ multiplied by two, and hence this left-hand side could not be equal to zero. Therefore, $\operatorname{ord}X$ is odd. □

Corollary 2.7. *A linear generalized symmetry $X \in \tilde{\Lambda}^{\rm q}$ of the Klein–Gordon equation $\mathcal{K}$ is a variational symmetry of the Lagrangian* K *if and only if the corresponding operator $\mathfrak{X}$ is formally skew-adjoint, $\mathfrak{X}^\dagger = -\mathfrak{X}$.*

Proof. For $X \in \tilde{\Lambda}^{\rm q}$, the operators $\mathfrak{K}$ and $\mathfrak{X}$ commute, $\mathfrak{L}\mathfrak{X} = \mathfrak{X}\mathfrak{L}$. This implies

$$0 = \mathfrak{X}^\dagger\mathfrak{K} + \mathfrak{K}^\dagger\mathfrak{X} = \mathfrak{X}^\dagger\mathfrak{K} + \mathfrak{K}\mathfrak{X} = \mathfrak{X}^\dagger\mathfrak{K} + \mathfrak{X}\mathfrak{K} = (\mathfrak{X}^\dagger + \mathfrak{X})\mathfrak{K},$$

and, therefore, $\mathfrak{X}^\dagger + \mathfrak{X} = 0$. Turning all implications around completes the proof. $\square$

Remark. A thorough inspection of the proof of Lemma 2.6 shows that the same assertion holds for linear variational symmetries of any Lagrangian of one dependent variable whose Euler–Lagrange equation is linear. The assertion analogous to Corollary 2.7 additionally needs commuting differential operators associated with these symmetries and with the Euler–Lagrange equation.

We change the basis of the algebra $\tilde{\Lambda}^{\rm q}$ to $\big((\mathfrak{X}_{kl}u)\partial_u, k, l \in \mathbb{N}_0, (\bar{\mathfrak{X}}_{kl}u)\partial_u, k \in \mathbb{N}_0, l \in \mathbb{N}\big)$, where basis' elements are respectively associated with the operators

$$\mathfrak{X}_{kl} = \left(\mathrm{J} + \frac{l}{2}\right)^k \mathrm{D}_x^l, \quad k, l \in \mathbb{N}_0, \qquad \bar{\mathfrak{X}}_{kl} = \left(\mathrm{J} - \frac{l}{2}\right)^k \mathrm{D}_y^l, \quad k \in \mathbb{N}_0, \quad l \in \mathbb{N}. \tag{2.2}$$

The algebra $\tilde{\Lambda}^{\rm q}$ is decomposed into the direct sum of two subspaces, $\tilde{\Lambda}^{\rm q} = \tilde{\Lambda}^{\rm q}_- \dotplus \tilde{\Lambda}^{\rm q}_+$, where $\tilde{\Lambda}^{\rm q}_-$ (resp. $\tilde{\Lambda}^{\rm q}_+$) is the subspace of elements in $\tilde{\Lambda}^{\rm q}$ associated with formally skew-adjoint (resp. self-adjoint) operators. Since

$$\mathrm{D}_x^\dagger = -\mathrm{D}_x, \quad \mathrm{D}_y^\dagger = -\mathrm{D}_y, \quad \mathrm{J}^\dagger = -\mathrm{J}, \quad \mathrm{D}_x\mathrm{J} = (\mathrm{J}+1)\mathrm{D}_x, \quad \mathrm{D}_y\mathrm{J} = (\mathrm{J}-1)\mathrm{D}_y,$$

we have $\mathfrak{X}_{kl}^\dagger = (-\mathrm{D}_x)^l\left(-\mathrm{J} + \frac{l}{2}\right)^k = (-1)^{k+l}\mathrm{D}_x^l\left(\mathrm{J} - \frac{l}{2}\right)^k = (-1)^{k+l}\mathfrak{X}_{kl}$ and similarly $\bar{\mathfrak{X}}_{kl}^\dagger = (-1)^{k+l}\bar{\mathfrak{X}}_{kl}$. Therefore, the generalized vector fields corresponding to the operators (2.2) with odd (resp. even) values of $k+l$ constitute a basis of the space $\tilde{\Lambda}^{\rm q}_-$ (resp. $\tilde{\Lambda}^{\rm q}_+$),

$$\tilde{\Lambda}^{\rm q}_- = \langle\, (\mathfrak{X}_{k'0}u)\partial_u,\ k' \in 2\mathbb{N}_0{+}1,\ (\mathfrak{X}_{kl}u)\partial_u,\ (\bar{\mathfrak{X}}_{kl}u)\partial_u,\ k \in \mathbb{N}_0,\ l \in \mathbb{N},\ k+l \in 2\mathbb{N}_0{+}1\rangle,$$

$$\tilde{\Lambda}^{\rm q}_+ = \langle\, (\mathfrak{X}_{k'0}u)\partial_u,\ k' \in 2\mathbb{N}_0,\ (\mathfrak{X}_{kl}u)\partial_u,\ (\bar{\mathfrak{X}}_{kl}u)\partial_u,\ k \in \mathbb{N}_0,\ l \in \mathbb{N},\ k+l \in 2\mathbb{N}_0\rangle.$$

Theorem 2.8. *The quotient algebra $\Upsilon^{\rm q}$ of variational symmetries of the Lagrangian* K *is naturally isomorphic to the algebra* $\tilde\Upsilon^{\rm q}=\tilde\Lambda^{\rm q}_-\Subset\tilde\Sigma^{-\infty}$.

Proof. We revert to the coordinates (x_0,x_1,u) and solve the equation $\mathcal K$ with respect to the derivative $\partial^2u/\partial x_0^2$, $\partial^2u/\partial x_0^2=\partial^2u/\partial x_1^2-u$. This gives a representation of $\mathcal K$ in the (extended) Kovalevskaya form. Lemma 3 in [86] (which was given in [103] as Lemma 4.28) reformulated for Euler–Lagrange equations in terms of variational symmetries of corresponding Lagrangians implies that for an arbitrary generalized vector field X in Υ, the corresponding element $[X]_{\rm var}$ of $\Upsilon^{\rm q}$ contains, as the coset $X+\Upsilon^{\rm triv}$ in Υ, a generalized vector field $\breve X$ in the reduced form that is obtained by excluding all derivatives of u with more than one differentiation with respect to x_0 in view of $\mathcal K$. Moreover, $\breve X$ is the only generalized vector field in the above reduced form that belongs to the coset $X+\Upsilon^{\rm triv}$ in Υ. It is also the only generalized vector field in the above reduced form that belongs to the coset $X+\Sigma^{\rm triv}$ in Σ. The coset $X+\Sigma^{\rm triv}$ necessarily contains exactly one element of $\tilde\Sigma^{\rm q}=\tilde\Lambda^{\rm q}\Subset\tilde\Sigma^{-\infty}$, which we denote by $\tilde X$. Note that the used coordinate change preserves the linearity of elements of Λ. Therefore, $\breve X$ is the reduced form of $\tilde X$, and hence $\breve X\in\Lambda\Subset\tilde\Sigma^{-\infty}$. Now we can revert to the coordinates (x,y,u).

For any linear system of differential equations, characteristics of its Lie symmetries associated with the linear superposition of solutions are conservation-law characteristics of this system. Therefore, $\tilde\Sigma^{-\infty}\subset\Upsilon$. Since different elements in $\tilde\Sigma^{-\infty}$ belong to different elements in the quotient space $\Upsilon^{\rm q}$ as cosets of $\Upsilon^{\rm triv}$ in Υ, and $\mathrm{ord}[X]=-\infty$ for each $X\in\tilde\Sigma^{-\infty}$, the algebra $\tilde\Sigma^{-\infty}$ is naturally isomorphic to $\Upsilon^{[-\infty]}$.

By $\tilde\Lambda^{[n]}_-$ we denote the subspace of $\tilde\Lambda^{\rm q}_-$ that is spanned by basis elements of $\tilde\Lambda^{\rm q}_-$ of order n. We have $\tilde\Lambda^{[n]}_-=\{0\}$ for even n, and if n is odd, then

$$\tilde\Lambda^{[n]}_-=\big\langle(\mathfrak X_{n0}u)\partial_u,\,(\mathfrak X_{k,n-k}u)\partial_u,\,(\bar{\mathfrak X}_{k,n-k}u)\partial_u,\,k=0,\dots,n-1\big\rangle.$$

Lemma 2.6 implies that if $X\in\Lambda\cap\Upsilon$, then $\mathrm{ord}\,X$ is odd. Therefore, $\dim\Upsilon^{[n]}=0=\dim\tilde\Lambda^{[n]}_-$ for even n. For odd n, $\dim\Upsilon^{[n]}\leqslant\dim\Sigma^{[n]}=\dim\tilde\Sigma^{[n]}=\dim\tilde\Lambda^{[n]}_-<+\infty$. On the other hand, $\tilde\Lambda^{[n]}_-\subset\Upsilon$, and $\mathrm{ord}[X]=n$ for each nonzero $X\in\tilde\Lambda^{[n]}_-$. Hence different elements in $\tilde{\Lambda}_-^{[n]}$ belong to cosets of $\Upsilon^{\rm triv}$ in Υ that are elements of Υ^n and belong to different cosets of Υ^{n-1} in Υ^n. Recall that the latter cosets are considered as elements of the twice quotient space $\Upsilon^{[n]}$. This implies that $\dim\tilde{\Lambda}_-^{[n]}\leqslant\dim\Upsilon^{[n]}$. In total, for odd n this gives that $\dim\tilde{\Lambda}_-^{[n]}=\dim\Upsilon^{[n]}$, and the subspace $\tilde{\Lambda}_-^{[n]}$ of Υ is naturally isomorphic to the space $\Upsilon^{[n]}$ via taking quotients twice. Therefore, the subspace Υ^n of $\Upsilon^{\rm q}$ is naturally isomorphic to the subspace $\tilde{\Sigma}^{-\infty}\dotplus\tilde{\Lambda}_-^{[0]}\dotplus\dots\dotplus\tilde{\Lambda}_-^{[n]}$ of Υ. Then the algebra $\Upsilon^{\rm q}$ is naturally isomorphic to the algebra $\tilde{\Upsilon}^{\rm q}=\tilde{\Lambda}_-^{\rm q}\ni\tilde{\Sigma}^{-\infty}$. Here the Lie bracket on $\tilde{\Upsilon}^{\rm q}$ is defined similarly to the Lie bracket on $\tilde{\Sigma}^{\rm q}$, i.e., as the Lie bracket of generalized vector fields, where mixed derivatives arising due to the action of D_x and D_y not involved in J should be substituted in view of the equation $\mathcal{K}$ and its differential consequences. $\square$

Remark 2.9. Cosets of $\Upsilon^{\rm triv}$ in Υ do not necessarily intersect the algebra $\hat{\Sigma}^{\rm q}$, i.e., they do not have canonical representatives in the evolutionary form reduced on solutions of the equation $\mathcal{K}$. For example, the reduced counterpart $(\mathscr{J}^3u)\partial_u$ of the variational symmetry $(\mathfrak{X}_{30}u)\partial_u=(\mathrm{J}^3u)\partial_u$ of K is not a variational symmetry of K since the difference

$$(\mathscr{J}^3u)\partial_u-(\mathrm{J}^3u)\partial_u=3xy\mathrm{J}(u_{xy}-u)\partial_u$$

is not. Recall that $\mathscr{J}:=x\mathscr{D}_x-y\mathscr{D}_y$. In other words, the reduced evolutionary form of generalized symmetries of the Klein–Gordon equation $\mathcal{K}$ is not appropriate in the course of the study of variational symmetries of K.

2.4 Conservation laws

For each element in a set spanning the space $\tilde{\Upsilon}^{\rm q}$, we construct a conserved current of the corresponding conservation law. Moreover, these conserved currents are of the simplest form and of minimal order among equivalent conserved currents, that is, their orders coincide with the orders of conservation laws containing them. In the course of this construction, we multiply the differential function $\mathfrak{K}u$ by the characteristic of a variational symmetry of K and rewrite, "integrating by parts", this expression in the form of a total divergence of a tuple of differential functions, which is nothing else but a conserved current of $\mathcal{K}$.

Thus, for any element $f(x,y)\partial_u$ of $\tilde\Sigma^{-\infty}$, the function $f=f(x,y)$ is a solution of $\mathcal K$, and we have $f\mathfrak K u=\mathrm D_x(fu_y)+\mathrm D_y(-f_xu)=\mathrm D_x(-f_yu)+\mathrm D_y(fu_x)$, which yields the equivalent first-order conserved currents

$$\mathrm C^0_f=(fu_y,-f_xu)\quad\text{and}\quad\bar{\mathrm C}^0_f=(-f_yu,fu_x).$$

Using a similar trick we derive a conserved current of $\mathcal K$ for any $X=(\mathfrak X u)\partial_u\in\tilde\Lambda^{\mathrm q}$,

$$\begin{aligned}\mathrm D_x(-u\mathrm D_y\mathfrak Xu)+\mathrm D_y(u_x\mathfrak Xu)&=u_{xy}\mathfrak Xu-u\mathrm D_x\mathrm D_y\mathfrak Xu=(\mathfrak Xu)\mathfrak Ku-u\mathfrak K\mathfrak Xu\\&=(\mathfrak Xu)\mathfrak Ku-u\mathfrak X\mathfrak Ku=(\mathfrak Xu-\mathfrak X^\dagger u)\mathfrak Ku+(\mathfrak X^\dagger u)\mathfrak Ku-u\mathfrak X\mathfrak Ku.\end{aligned}$$

Here we take into account that $\mathfrak K\mathfrak X=\mathfrak X\mathfrak K$ for $X\in\tilde\Lambda^{\mathrm q}$. The Lagrange identity (also called generalized Green's formula [168, Section 12]) implies that the differential function $(\mathfrak X^\dagger u)\mathfrak Ku-u\mathfrak X\mathfrak Ku$ is the total divergence of a pair of differential functions bilinearly depending on the tuples of total derivatives of u and $\mathfrak Ku$; cf. [168, Proposition A.4], i.e., it is the total divergence of a trivial conserved current of the equation $\mathcal K$. Therefore, $(\mathfrak X-\mathfrak X^\dagger)u$ is a characteristic of the conservation law of $\mathcal K$ that contains the conserved current $\tilde{\mathrm C}_{\mathfrak X}=(-u\mathrm D_y\mathfrak Xu,\,u_x\mathfrak Xu)$. For any $X\in\tilde\Lambda^{\mathrm q}_+$, we have $\mathfrak X^\dagger=\mathfrak X$, i.e., the corresponding conservation law is zero. For any $X\in\tilde\Lambda^{\mathrm q}_-$, we have $\mathfrak X^\dagger=-\mathfrak X$ and thus obtain the characteristic $2\mathfrak Xu$ of a nonzero conservation law of . Running X through the basis of $\tilde\Lambda^{\mathrm q}_-$ gives conservation laws that are linearly independent since their characteristics are. In view of Theorem 2.8, these conservation laws jointly with those containing conserved currents $\mathrm C^0_f$, $f\in\mathcal K$, span the entire space of conservation laws of $\mathcal K$.

Proposition 2.10. *The space of conservation laws of the* $(1+1)$*-dimensional Klein–Gordon equation* $\mathcal K$ *is naturally isomorphic to the space spanned by the conserved currents* $\mathrm C^0_f$ *and* $\tilde{\mathrm C}_{\mathfrak X}$*, where the parameter function* $f=f(x,y)$ *runs through the solution set of* $\mathcal K$*, and the operator* $\mathfrak X$ *runs through the basis of* $\tilde\Lambda^{\mathrm q}_-$*,*

$$(\,\mathfrak X_{k'0},\ k'\in2\mathbb N_0+1,\ \mathfrak X_{kl},\ \bar{\mathfrak X}_{kl},\ k\in\mathbb N_0,\ l\in\mathbb N,\ k+l\in2\mathbb N_0+1\,).$$

Corollary 2.11. *Under the action of generalized symmetries of the* $(1+1)$*-dimensional Klein–Gordon equation* $\mathcal{K}$ *on the space of conservation laws of this equation, a generating set of conservation laws of* $\mathcal{K}$ *is constituted by the single conservation law containing the conserved current* $(-u^2, u_x^2)$*.*

Proof. The actions of the generalized symmetries $\frac{1}{2}f_y\partial_u$ and $\frac{1}{2}(\mathrm{D}_y\mathfrak{X}u)\partial_u$ on the conserved current $(-u^2, u_x^2)$ of the equation $\mathcal{K}$ give the conserved currents

$$\bar{\mathrm{C}}^0_f = (-f_y u, f u_x) \quad \text{and} \quad (-u\mathrm{D}_y\mathfrak{X}u, u_x\mathrm{D}_x\mathrm{D}_y\mathfrak{X}u),$$

which are equivalent to C^0_f and $\tilde{\mathrm{C}}_{\mathfrak{X}}$, respectively. □

The order of the conserved current $\tilde{\mathrm{C}}_{\mathfrak{X}}$ is greater than the order of the corresponding conservation law. This is why we compute a conserved current of minimal order with characteristic $\mathfrak{X}u$, where the generalized vector field $(\mathfrak{X}u)\partial_u$ runs through the chosen basis elements $(\mathfrak{X}_{kl}u)\partial_u$ of $\tilde{\Lambda}^{\mathrm{q}}_-$, for each of which $k+l$ is odd. We consider two cases, when k is odd and when k is even.

In the first case, we denote $k'=(k-1)/2$ and $l'=l/2$. Note that $\mathrm{J}=\mathrm{D}_x\circ x-\mathrm{D}_y\circ y$. Hence $\mathfrak{X}_{kl}=\mathrm{D}_x^{l'}\mathrm{J}^k\mathrm{D}_x^{l'}$ and

$$\begin{aligned}(\mathfrak{X}_{kl}u)\mathfrak{K}u = {}& \mathrm{D}_x\sum_{l''=0}^{l'-1}(-1)^{l''}\left(\mathrm{D}_x^{l'-l''-1}\mathrm{J}^k\mathrm{D}_x^{l'}u\right)\mathrm{D}_x^{l''}\mathfrak{K}u \\ &+\mathrm{J}\sum_{k''=0}^{k'-1}(-1)^{l'+k''}\left(\mathrm{J}^{2k'-k''}\mathrm{D}_x^{l'}u\right)\mathrm{J}^{k''}\mathrm{D}_x^{l'}\mathfrak{K}u \\ &+\frac{(-1)^{l'+k'}}{2}\left(x\mathrm{D}_y(\mathrm{D}_x\mathrm{J}^{k'}\mathrm{D}_x^{l'}u)^2-y\mathrm{D}_x(\mathrm{D}_y\mathrm{J}^{k'}\mathrm{D}_x^{l'}u)^2-\mathrm{J}(\mathrm{J}^{k'}\mathrm{D}_x^{l'}u)^2\right),\end{aligned}$$

which gives, up to the equivalence of conserved currents of $\mathcal{K}$ and their rescaling, the conserved current

$$\mathrm{C}^1_{k'l'}=\left(-y(\mathrm{D}_y\mathrm{J}^{k'}\mathrm{D}_x^{l'}u)^2-x(\mathrm{J}^{k'}\mathrm{D}_x^{l'}u)^2,\ x(\mathrm{D}_x\mathrm{J}^{k'}\mathrm{D}_x^{l'}u)^2+y(\mathrm{J}^{k'}\mathrm{D}_x^{l'}u)^2\right)$$

of order $k'+l'+1=(k+l+1)/2$, which is minimal for the conserved currents related to the characteristic $\mathfrak{X}_{kl}u$.

If k is even, then l is odd and we denote $k'=k/2$ and $l'=(l-1)/2$. Hence

$$
\begin{aligned}
\mathfrak{X}_{kl} &= \mathrm{D}_x^{l'}(\mathrm{J}+1/2)^{k'}\mathrm{D}_x(\mathrm{J}-1/2)^{k'}\mathrm{D}_x^{l'} = \mathrm{D}_x^{l'+1}(\mathrm{J}-1/2)^{k}\mathrm{D}_x^{l'} = \mathrm{D}_x^{l'}(\mathrm{J}+1/2)^{k}\mathrm{D}_x^{l'+1},\\
(\mathfrak{X}_{kl}u)\mathfrak{K}u &= \mathrm{D}_x\sum_{l''=0}^{l'-1}(-1)^{l''}\left(\mathrm{D}_x^{l'-l''}\left(\mathrm{J}-\frac{1}{2}\right)^{k}\mathrm{D}_x^{l'}u\right)\mathrm{D}_x^{l''}\mathfrak{K}u\\
&\quad+\mathrm{J}\sum_{k''=0}^{k'-1}(-1)^{l'+k''}\left(\left(\mathrm{J}+\frac{1}{2}\right)^{k-k''-1}\mathrm{D}_x^{l'+1}u\right)\left(\mathrm{J}-\frac{1}{2}\right)^{k''}\mathrm{D}_x^{l'}\mathfrak{K}u\\
&\quad+\frac{(-1)^{l'+k'}}{2}\left(\mathrm{D}_y\left(\mathrm{D}_x\left(\mathrm{J}-\frac{1}{2}\right)^{k'}\mathrm{D}_x^{l'}u\right)^2-\mathrm{D}_x\left(\left(\mathrm{J}-\frac{1}{2}\right)^{k'}\mathrm{D}_x^{l'}u\right)^2\right).
\end{aligned}
$$

Up to the equivalence of conserved currents of $\mathcal{K}$ and multiplying them by constants, this leads to the conserved current

$$
\mathrm{C}^2_{k'l'}=\left(-\left(\left(\mathrm{J}-\frac{1}{2}\right)^{k'}\mathrm{D}_x^{l'}u\right)^2,\ \left(\mathrm{D}_x\left(\mathrm{J}-\frac{1}{2}\right)^{k'}\mathrm{D}_x^{l'}u\right)^2\right)
$$

of order $k'+l'+1=(k+l+1)/2$, which is again minimal for the conserved currents related to the characteristic $\mathfrak{X}_{kl}u$. Since the permutation of x and y is a discrete point symmetry transformation of $\mathcal{K}$, a conserved current associated with the vector field $(\bar{\mathfrak{X}}_{kl}u)\partial_u$, for which $k+l$ is odd, can be constructed by this permutation either from the conserved current $\mathrm{C}^1_{k'l'}$ if k is odd or from the conserved current $\mathrm{C}^2_{k'l'}$ if k is even, where again k' and l' denote the integer parts of $k/2$ and $l/2$, respectively. We obtain

$$
\begin{aligned}
\bar{\mathrm{C}}^1_{k'l'} &= \left(y(\mathrm{D}_y\mathrm{J}^{k'}\mathrm{D}_y^{l'}u)^2+x(\mathrm{J}^{k'}\mathrm{D}_y^{l'}u)^2,\ -x(\mathrm{D}_x\mathrm{J}^{k'}\mathrm{D}_y^{l'}u)^2-y(\mathrm{J}^{k'}\mathrm{D}_y^{l'}u)^2\right),\\
\bar{\mathrm{C}}^2_{k'l'} &= \left(\left(\mathrm{D}_y\left(\mathrm{J}+\frac{1}{2}\right)^{k'}\mathrm{D}_y^{l'}u\right)^2,\ -\left(\left(\mathrm{J}+\frac{1}{2}\right)^{k'}\mathrm{D}_y^{l'}u\right)^2\right).
\end{aligned}
$$

Theorem 2.12. *The space of conservation laws of the* $(1+1)$*-dimensional Klein–Gordon equation* $\mathcal{K}$ *is naturally isomorphic to the space spanned by the conserved currents*

$$
\mathrm{C}^1_{k'l'},\ k'\in\mathbb{N}_0,\ l'\in\mathbb{N},\quad \bar{\mathrm{C}}^1_{k'l'},\ \mathrm{C}^2_{k'l'},\ \bar{\mathrm{C}}^2_{k'l'},\ k',l'\in\mathbb{N}_0,\quad \mathrm{C}^0_f,
$$

where the parameter function $f=f(x,y)$ *runs through the solution set of* $\mathcal{K}$*. The order of conserved currents* $\mathrm{C}_{k'l'}$*'s is equal to* $k'+l'+1$*, and* $\operatorname{ord}\mathrm{C}^0_f=1$*.*

In other words, the conserved currents $\mathrm{C}^1_{k'l'}$, $k' \in \mathbb{N}_0$, $l' \in \mathbb{N}$, $\bar{\mathrm{C}}^1_{k'l'}$, $\mathrm{C}^2_{k'l'}$, $\bar{\mathrm{C}}^2_{k'l'}$, $k', l' \in \mathbb{N}_0$, with $k' + l' = n - 1$ represent a complete (up to adding lower-order conservation laws) set of linearly independent nth order conservation laws of $\mathcal{K}$ if $n \geqslant 2$. The space of first-order conservation laws is spanned by those with conserved currents $\bar{\mathrm{C}}^1_{00}$, C^2_{00}, $\bar{\mathrm{C}}^2_{00}$ and C^0_f, where the parameter function $f = f(x, y)$ runs through the solution set of $\mathcal{K}$.

Corollary 2.13. *Up to adding low-order conservation laws, the Klein–Gordon equation $\mathcal{K}$ possesses $4n-1$ linearly independent conservation laws of order n if $n \geqslant 2$, and an infinite number of linearly independent first-order conservation laws.*

Remark. Replacing the operators D_x, D_y and J by $\mathcal{D}_x$, $\mathcal{D}_y$ and $\mathcal{J}$, respectively, in constructed conserved currents, we obtain equivalent conserved currents that are reduced in view of the solution set of $\mathcal{K}$.

2.5 Conclusion

The consideration in the present chapter has several interesting aspects, which are worth recalling. Its main specific feature is that it is essentially based on the representation $\mathcal{K}$: $u_{xy} = u$ of the (1+1)-dimensional Klein–Gordon equation in the light-cone variables, which cannot be adapted, in contrast to the representation in the standard spacetime variables, as an (extended) Kovalevskaya form of this equation.[5]

There are only a few papers in the literature, where the entire spaces of generalized symmetries and, especially, conservation laws were computed for (systems of) differential equations that are inconvenient for representing in the extended Kovalevskaya form [44, 141] or lack such a representation at all [5, 6, 7, 11, 121, 122]. Moreover, in [44, 141] the least upper bounds for orders of reduced cosymmetries were low, 2 and $-\infty$, respectively, each equivalence class of cosymmetries contained a conservation-law characteristic, and the sufficient number of linearly independent conservation laws had been known [44]

[5]See [125] for the definition of the extended Kovalevskaya form of systems of differential equations and a discussion of significance of this form in the theory of conservation laws. Systems of a bit more restrictive form are called normal systems [86] or Cauchy–Kowalevsky systems in a weak sense (resp., pseudo CK systems in short) [152].

or could be easily derived directly [141]. This is why employing the equation representations different from the extended Kovalevskaya form created no obstacles for selecting conservation-law characteristics among cosymmetries in these papers although, in general, such a selection may be a nontrivial problem. Thus, the present chapter provides one of a few examples of studying conservation laws of a system of differential equations that is not in the extended Kovalevskaya form and possesses conservation laws of arbitrarily high order as well as cosymmetries of arbitrarily high order that are not equivalent to conservation-law characteristics, cf. [5, 6, 7, 8].

To get around the complication in the course of selecting variational symmetries among generalized ones for the representation of the Klein–Gordon equation $\mathcal{K}$ in the light-cone variables x and y, we have temporarily switched to the standard form of the Klein–Gordon equation for applying the Martínez Alonso lemma [86, Lemma 3]. That the transitions between the standard spacetime and the light-cone variables preserve the linearity of characteristics of generalized symmetries allowed us to prove that each nonnegative-order coset of variational symmetries contains a linear symmetry. All the other computations were carried out in the light-cone variables.

Despite the above complication, the representation of the Klein–Gordon equation $\mathcal{K}$ in the light-cone variables x and y is preferable to the standard one. The choice of it is paid off by virtue of the facts that it is more compact and the differentiations with respect to x and y are inverse to each other, $\mathrm{D}_x\mathrm{D}_y = 1$, on solutions of $\mathcal{K}$. The latter enables us to choose the jet coordinates $(t, x, u_k, k \in \mathbb{Z})$ on $\mathcal{K}^{(\infty)}$, which are numerated by a single integer. This simplifies the entire consideration, including the reduced operators of total derivatives $\mathcal{D}_x$ and $\mathcal{D}_y$, the determining equations for generalized symmetries of $\mathcal{K}$ and the process of solving thereof.

In contrast to the standard spacetime coordinates, the use of light-cone variables in the course of confining to the solution set of the Klein–Gordon equation also allows us to preserve the equality of independent variables, which is intrinsic to this equation. As a result, both the constructed spaces of canonical representatives for equivalence classes of generalized symmetries of $\mathcal{K}$ admit bases that are symmetrical with respect to x and y.

The procedure of finding generalized symmetries of $\mathcal{K}$ includes the standard techniques of computing the dimension of the space of reduced generalized symmetries of each finite order and of generating the necessary amount of linearly independent symmetries by recursion operators. In fact, for the latter it suffices to use only the recursion operators, corresponding to the Lie symmetries ∂_x, ∂_y and $x\partial_x - y\partial_y$ of $\mathcal{K}$. To show that the generation produces no trivial symmetries, we have evaluated the constructed generalized symmetries on a family of solutions of $\mathcal{K}$ parameterized by a nonzero real constant, see the proof of Lemma 2.3. From this perspective, the entire algebra $\tilde{\Sigma}^{\rm q}$ (resp. $\hat{\Sigma}^{\rm q}$) of canonical representatives for equivalence classes of generalized symmetries of $\mathcal{K}$ is spanned by the generalized vector fields that are related to the linear superposition of solutions of $\mathcal{K}$ or generated from the single Lie symmetry $u\partial_u$ of $\mathcal{K}$ by means of the recursion operators D_x, D_y and J (resp. $\mathscr{D}_x$, $\mathscr{D}_y$ and $\mathscr{J}$). The algebra $\hat{\Sigma}^{\rm q}$ is the collection of generalized symmetries of $\mathcal{K}$ reduced on the solution set of $\mathcal{K}$, thus being a standard object. Moreover, the elements of $\hat{\Sigma}^{\rm q}$ are represented in a compact form, in particular, due to the obtained compact representation of the reduced operators of total derivatives $\mathscr{D}_x$ and $\mathscr{D}_y$. Nevertheless, the algebra $\hat{\Sigma}^{\rm q}$ is inappropriate for use in the description of variational symmetries of the equation $\mathcal{K}$, see Remark 2.9. This is why we have paid a more attention to another collection of canonical representatives for equivalence classes of generalized symmetries of $\mathcal{K}$, the algebra $\tilde{\Sigma}^{\rm q}$, which does not have the above disadvantage of the algebra $\hat{\Sigma}^{\rm q}$. In order to efficiently single out variational symmetries among elements of the algebra $\tilde{\Sigma}^{\rm q}$, we have made a basis change in this algebra, so that the subspace of skew-adjoint operators, which are naturally associated with variational symmetries, is evident in the new basis.

The space of conservation laws of $\mathcal{K}$ is expectedly computed using Noether's theorem. It is convenient to represent this space as the direct sum of two infinite-dimensional subspaces. The first subspace is of the kind that is common for linear systems of differential equations. It consists of the (first-order) linear conservation laws of $\mathcal{K}$. Such conservation laws are necessarily of order one, and their (reduced) characteristics are of order $-\infty$. For $\mathcal{K}$ as the Euler–Lagrange equation of the Lagrangian K, these characteristics are characteristics of generalized symmetries of order $-\infty$ of $\mathcal{K}$, which constitute the algebra $\tilde{\Sigma}^{-\infty}$

and are associated with the linear superposition of solutions of $\mathcal{K}$. The second subspace is specific and is exhausted by the quadratic conservation laws of $\mathcal{K}$. They admit linear characteristics being characteristics of linear variational symmetries from the algebra $\tilde{\Lambda}^{\rm q}_{-}$. We have derived canonical representatives of two kinds for conserved currents contained in quadratic conservation laws. The first kind of representatives is uniform for all quadratic conservation laws and is convenient in the course of the study how generalized symmetries of the equation $\mathcal{K}$ act on its conservation laws. It was an unexpected result for us that the so huge space of conservation laws of diverse structures is generated, under the action of generalized symmetries, by a single first-order quadratic conservation law. We have also computed a conserved current of minimal order for each basis quadratic conservation law. For computational and presentation reasons, in the course of this computation we partition the chosen basis of variational symmetries of nonnegative order into four families, which leads to the associated partition for quadratic conservation laws. We have constructed conserved currents of minimal order for two of these four families of conservation laws and then used the permutation of x and y, which is a discrete point symmetry transformation of $\mathcal{K}$, to obtain conserved currents of minimal order for the other two families from the constructed ones.

An additional advantage of using the operators D_x and D_y over their rivals $\mathcal{D}_x$ and $\mathcal{D}_y$ is a more clear insight into generalizing results of the present chapter to the multi-dimensional Klein–Gordon equation. In view of the greater number of independent variables, it possesses more translations and Lorentz transformations (usual and hyperbolic rotations) than the equation $\mathcal{K}$ does but the principal structure of the algebra of generalized symmetries should be similar to that for $\mathcal{K}$, cf. [45, 94, 96, 140]. The techniques applied in the present chapter for singling out variational symmetries and computing associated conserved currents of minimal order may still be employed for constructing the entire space of conservation laws of the multi-dimensional Klein–Gordon equation, including the translation-noninvariant ones, which were not considered in [76, 151].

Chapter 3

Extended symmetry analysis of an isothermal no-slip drift flux model

3.1 Introduction

The drift flux model introduced in [170] is a simplified model of a well-known two-phase flow phenomenon [73, 167]. The former system takes the form

$$
\begin{aligned}
&\partial_t(a_1\rho^1) + \partial_x(a_1u_1\rho^1) = 0,\\
&\partial_t(a_2\rho^2) + \partial_x(a_2u_2\rho^2) = 0,\\
&\partial_t(a_1u_1\rho^1 + a_2u_2\rho^2) + \partial_x(a_1u_1^2\rho^1 + a_2u_2^2\rho^2 + p) = Q,
\end{aligned}
$$

where $a_i(t,x)$ are the volume fractions, $u_i(t,x)$ are the velocities and $\rho_i(t,x)$ are the densities of phases, $Q(t,x)$ is a source term, with $a_1 + a_2 = 1$. It was thoroughly studied in [46, 47, 48, 49], where several submodels easier to tackle but still real-world applicable were suggested. In particular, the simplifying *slip condition* was considered, $u_1 - u_2 = \Phi(u_1, u_2, p)$. In [12] a further simplification was made, assuming the equality of the volume fractions, $a_1 = a_2 = a$, a vanishing slip function $\Phi = 0$, an absence of the source term $Q = 0$ and an isothermal equation of state $p = a(\rho_1+\rho_2)$. The resulting isothermal no-slip drift flux model is governed by the system

$$
\begin{aligned}
&\rho^1_t + u\rho^1_x + u_x\rho^1 = 0,\\
&\rho^2_t + u\rho^2_x + u_x\rho^2 = 0,\\
&(\rho^1+\rho^2)(u_t+uu_x) + a^2(\rho^1_x+\rho^2_x) = 0,
\end{aligned}
$$

which we denote by $\mathcal{S}$. This model describes the mixing motion of liquids (or gases) rather than their individual phases. Here $u = u(t,x)$ is the common velocity, $\rho^1 = \rho^1(t,x)$ and $\rho^2 = \rho^2(t,x)$ are the densities of the liquids, and the constant parameter a can be set to 1 by scaling (x,u) with a. Any constraint meaning that ρ^1 and ρ^2 are proportional, e.g., $\rho^2 = \rho^1$ or $\rho^2 = 0$, reduces $\mathcal{S}$ to the system $\tilde{\mathcal{S}}_0$ describing one-dimensional isentropic gas flows with constant sound speed, cf. the system (3)–(4) with $\nu = 0$ in [133, Section 2.2.7]. The system $\mathcal{S}$ is a diagonalizable hydrodynamic-type system since it admits an equivalent form

$$\mathfrak{r}^1_t + (\mathfrak{r}^1+\mathfrak{r}^2+1)\mathfrak{r}^1_x = 0, \tag{3.1a}$$
$$\mathfrak{r}^2_t + (\mathfrak{r}^1+\mathfrak{r}^2-1)\mathfrak{r}^2_x = 0, \tag{3.1b}$$
$$\mathfrak{r}^3_t + (\mathfrak{r}^1+\mathfrak{r}^2)\mathfrak{r}^3_x = 0 \tag{3.1c}$$

by changing the dependent variables (u,ρ^1,ρ^2) to the Riemann invariants[1] $(\mathfrak{r}^1,\mathfrak{r}^2,\mathfrak{r}^3)$ via $\mathfrak{r}^1 = \frac{u+\ln(\rho^1+\rho^2)}{2}$, $\quad \mathfrak{r}^2 = \frac{u-\ln(\rho^1+\rho^2)}{2}$, $\quad \mathfrak{r}^3 = \frac{\rho^2}{\rho^1}$.

The corresponding characteristic velocities

$$V^1 = \mathfrak{r}^1+\mathfrak{r}^2+1,\quad V^2 = \mathfrak{r}^1+\mathfrak{r}^2-1,\quad V^3 = \mathfrak{r}^1+\mathfrak{r}^2 \tag{3.2}$$

are distinct, meaning that the system $\mathcal{S}$ is strictly hyperbolic. Besides, the characteristic velocities satisfy the system

$$\partial_{\mathfrak{r}^i}\frac{V^k_{\mathfrak{r}^j}}{V^j-V^k} = \partial_{\mathfrak{r}^j}\frac{V^k_{\mathfrak{r}^i}}{V^i-V^k} \qquad \text{for all}\quad i,j,k\in\{1,2,3\}\quad \text{with}\quad i,j\neq k.$$

[1] Riemann invariants are dependent variables, in which a hydrodynamic-type systems takes a diagonalized form.

Thus, the system $\mathcal{S}$ is semi-Hamiltonian and, since $V^3_{\mathfrak{r}^3}=0$, it is not genuinely nonlinear with respect to $\mathfrak{r}^3$; see [150] for related definitions. The system $\mathcal{S}$ is also partially coupled. The essential subsystem $\mathcal{S}_0$ consisting of the equations (3.1a)–(3.1b) coincides with the diagonalized form of the system $\tilde{\mathcal{S}}_0$ [133, Section 2.2.7, Eq. (16)].

Hydrodynamic-type systems are extensively studied in the literature in view of their various physical applications in fluid mechanics, acoustics and gas and shock dynamics [133, 164] and rich differential geometry [41, 43, 149, 150]. See [21, 30, 51, 54, 64, 66, 119, 138, 139] and references therein for an assortment of examples.

In view of the above properties, the system $\mathcal{S}$ can be integrated in an implicit form. In my MSc thesis, results of which were published in [112], for this system we expressed the general solution in terms of the general solution of the (1+1)-dimensional Klein–Gordon equation using the generalized hodograph transformation [149] and described the entire set of local solutions via the linearization of the subsystem $\mathcal{S}_0$ to the same equation. Since the practical use of the derived representations for solutions of $\mathcal{S}$ is limited because of their implicit form and complicated structure, in [112] we also began the extended classical symmetry analysis of the system $\mathcal{S}$. In particular, for this system we constructed the maximal Lie invariance algebra $\mathfrak{g}$, the algebra of generalized symmetries of order not greater than one, the complete point symmetry group and group-invariant solutions. Thus, the algebra $\mathfrak{g}$ is spanned by the vector fields

$$
\begin{aligned}
&\hat{\mathcal{D}}=t\partial_t+x\partial_x,\quad \hat{\mathcal{G}}_1=t\partial_x+\partial_{\mathfrak{r}^1},\quad \hat{\mathcal{G}}_2=\partial_{\mathfrak{r}^1}-\partial_{\mathfrak{r}^2},\\
&\hat{\mathcal{P}}^t=\partial_t,\quad \hat{\mathcal{P}}^x=\partial_x,\quad \hat{\mathcal{W}}(\Omega)=\Omega(\mathfrak{r}^3)\partial_{\mathfrak{r}^3},
\end{aligned}
$$

where Ω runs through the set of smooth functions of $\mathfrak{r}^3$. The maximal Lie invariance algebra $\mathfrak{g}_0$ of the essential subsystem $\mathcal{S}_0$ is wider than the projection of the algebra $\mathfrak{g}$ to the space with the coordinates $(t,x,\mathfrak{r}^1,\mathfrak{r}^2)$ and is spanned by the vector fields

$$
\begin{aligned}
&\breve{\mathcal{D}}=t\partial_t+x\partial_x,\ \breve{\mathcal{G}}_1=t\partial_x+\partial_{\mathfrak{r}^1},\ \breve{\mathcal{G}}_2=\partial_{\mathfrak{r}^1}-\partial_{\mathfrak{r}^2},\ \breve{\mathcal{P}}(\tau^0,\xi^0)=\tau(\mathfrak{r}^1,\mathfrak{r}^2)\partial_t+\xi(\mathfrak{r}^1,\mathfrak{r}^2)\partial_x,\\
&\breve{\mathcal{J}}=\left(\frac{1}{2}x-t(\mathfrak{r}^1+\mathfrak{r}^2)\right)\partial_t+t\left(\mathfrak{r}^1-\mathfrak{r}^2-\frac{1}{2}(\mathfrak{r}^1+\mathfrak{r}^2)^2+\frac{1}{2}\right)\partial_x+\mathfrak{r}^1\partial_{\mathfrak{r}^1}-\mathfrak{r}^2\partial_{\mathfrak{r}^2},
\end{aligned}
$$

where (τ, ξ) is a tuple of smooth functions of $(\mathfrak{r}^1, \mathfrak{r}^2)$, running through the solution set of the system $\xi_{\mathfrak{r}^1} = V^2\tau_{\mathfrak{r}^1}$, $\xi_{\mathfrak{r}^2} = V^1\tau_{\mathfrak{r}^2}$. In [112], for the system $\mathcal{S}$ we also found the zeroth-order local conservation laws using the direct method and, following [40], constructed the entire space of first-order conservation laws with (t, x)-translation-invariant densities of and a subspace of (t, x)-translation-invariant conservation laws of arbitrarily high order. Building on the description of the algebra of generalized symmetries of order not greater than one, we obtained an infinite-dimensional subspace of generalized symmetries of arbitrarily high order for $\mathcal{S}$. (In the present section we show that this subspace is an ideal in the entire algebra of generalized symmetries of the system $\mathcal{S}$.)

At the same time, the system $\mathcal{S}$ possesses two properties that allow us to exhaustively describe the entire spaces of generalized symmetries, cosymmetries and local conservation laws (see [78] for definitions). Firstly, the system is partially coupled with the essential subsystem $\mathcal{S}_0$ being linearizable through the rank-two hodograph transformation to the (1+1)-dimensional Klein–Gordon equation, which was thoroughly studied in Chapter 2, published as [115], from the point of view of generalized and variational symmetries and local conservation laws. Secondly, in addition to being not genuinely nonlinear with respect to $\mathfrak{r}^3$, the system $\mathcal{S}$ is decoupled with respect to $\mathfrak{r}^3$, and the third equation of $\mathcal{S}$ is linear in $\mathfrak{r}^3$. Thus, speaking of the degeneracy of the system $\mathcal{S}$, we mean both its linear degeneracy and decoupling with respect to $\mathfrak{r}^3$. Due to the dual nature of this degeneracy, the system $\mathcal{S}$ admits not only an infinite number of linearly independent conservation laws of arbitrarily high order, that are related to the degeneracy, cf. [40, 142], but also similar generalized symmetries.

Substantially generalizing results of [112], in the present section we comprehensively study generalized symmetries, cosymmetries and local conservation laws of the system $\mathcal{S}$. This includes both a description of the corresponding spaces and their interrelations, which are described in terms of recursion operators and Noether and Hamiltonian operators. Our *modus operandi* to study the system $\mathcal{S}$ is to select appropriate symmetry-like objects of the Klein–Gordon equation (generalized symmetries, cosymmetries and conservation laws), to find their counterparts for the system $\mathcal{S}$ and to complement these counterparts with the objects of the same kind that are related to the degeneracy of the system. Then we prove that the constructed objects span the entire spaces of objects of the corresponding kinds for the system $\mathcal{S}$. As a result, we obtain one more example, in addition to a few ones existing in the literature, where generalized symmetries and local conservation laws are exhaustively described for a model arising in real-world applications and possessing symmetry-like objects of arbitrarily high order.

All results of this section except for original Sections 3.7 and 3.8 were published in [113]. The structure of this section is as follows. In Section 3.2 we reduce the system $\mathcal{S}$ to the (1+1)-dimensional Klein–Gordon equation and show that any regular solution of the former is expressed in terms of solutions of the latter. In Section 3.3 we lay out notations and auxiliary results to be used throughout the remainder of the section. It is proved in Section 3.4 that the algebra of reduced generalized symmetries of the system $\mathcal{S}$ is a (non-direct) sum of an ideal related to the degeneracy of $\mathcal{S}$ and consisting of generalized vector fields with zero $\mathfrak{r}^1$- and $\mathfrak{r}^2$-components and of a subalgebra stemming from generalized symmetries of the Klein–Gordon equation. At the same time, not all generalized symmetries of the Klein–Gordon equation have counterparts among those of the system $\mathcal{S}$, and we solve the problem on selecting appropriate elements of the algebra of generalized symmetries of the Klein–Gordon equation. This differs from cosymmetries and conservation laws of $\mathcal{S}$, for which there are injections from the corresponding spaces for the Klein–Gordon equation to those for the system $\mathcal{S}$, see Sections 3.5 and 3.6, respectively. The space of conservation laws of $\mathcal{S}$ is proved to be generated, under the action of generalized symmetries of $\mathcal{S}$, by two zeroth-order conservation laws. We also find the space of conservation-law characteristics of $\mathcal{S}$. The knowledge of them helps us to single out the conservation laws of orders zero and one as well as the (t,x)-translation-invariant ones. Using the simplest conservation laws of the system $\mathcal{S}$ we construct a covering thereof and study its symmetries in an attempt to prolong all the symmetries of the Klein–Gordon equation to the system $\mathcal{S}$ in Section 3.7. In Section 3.8 we construct nonlocal Hamiltonian operators for the system $\mathcal{S}$ as prolongation on $\mathfrak{r}^3$ local hydrodynamic-type Hamiltonian operators of the subsystem $\mathcal{S}_0$. For each of local Hamiltonian operators found in [112] we find the space of its distinguished (Casimir) functionals and the associated algebra of Hamiltonian symmetries. Section 3.9 is left for the conclusions, where we underline the nontrivial features encountered in the course of the study of the system $\mathcal{S}$ in the present section and discuss further problems to be considered for this system within the framework of symmetry analysis of differential equations.

3.2 Solution through linearization of the essential subsystem

Using the facts that the system $\mathcal{S}$ is partially coupled and the subsystem $\mathcal{S}_0$ can be linearized, we construct an implicit representation of the general solution for the diagonalized form (3.1) of the system $\mathcal{S}$ in terms of the general solution of the (1+1)-dimensional Klein–Gordon equation; cf. [112, Section 8]. At first, we reduce the system (3.1) by a point transformation to a system containing the (1+1)-dimensional Klein–Gordon equation. It is convenient to derive this transformation as a chain of simpler point transformations. We begin with the rank-two hodograph transformation[2], where

$y = \mathfrak{r}^1/2, \quad z = -\mathfrak{r}^2/2$ are the new independent variables and

$p = t, \quad \hat{q} = x, \quad s = \mathfrak{r}^3$ are the new dependent variables.

This transformation maps the system (3.1) to the system

$$\hat{q}_z - (2y - 2z + 1)p_z = 0, \tag{3.3a}$$

$$\hat{q}_y - (2y - 2z - 1)p_y = 0, \tag{3.3b}$$

$$s_y p_z + s_z p_y = 0. \tag{3.3c}$$

After representing the equation (3.3a) in the form $\big(\hat{q} - (2y - 2z + 1)p\big)_z - 2p = 0$, it becomes natural to make the change $\check{q} = \hat{q} - (2y - 2z + 1)p$ of $\hat{q}$. Then the equations (3.3a)

[2]Recall that every (1+1)-dimensional hydrodynamic-type system with two dependent variables is linearizable via the hodograph transformation.

and (3.3b) take the form $p = \check{q}_z/2$ and $\check{q}_y + 2p_y + 2p = 0$, respectively. Excluding p from the second equation in view of the first one, we obtain the second-order linear partial differential equation $\check{q}_{yz} + \check{q}_y + \check{q}_z = 0$ in $\check{q}$, which reduces by the change $q = \mathrm{e}^{y+z}\check{q}$ of $\check{q}$ to the (1+1)-dimensional Klein–Gordon equation for q in light-cone variables, $q_{yz} = q$. Carrying out this chain of two transformations in the whole system (3.3), we obtain the system $\mathcal{K}$, which reads

$$q_{yz} = q, \tag{3.4a}$$

$$K^1 s_y = K^2 s_z, \quad \text{where} \quad K^1 := q_{zz} - 2q_z + q, \quad K^2 := q_y + q_z - 2q. \tag{3.4b}$$

We have $K^1 = (\mathrm{D}_z - 1)^2 q$ and, on solutions of (3.4a), $K^2 = -(\mathrm{D}_y - 1)(\mathrm{D}_z - 1)q$, $\mathrm{D}_y K^1 = K^2$ and $\mathrm{D}_z K^2 = K^1$. Here D_y and D_z are the total derivative operators with respect to y and z, respectively. We exclude p from the system (3.4) in view of the equation

$$p = \frac{1}{2}\mathrm{e}^{-y-z}(q_z - q) \tag{3.5}$$

as well as we neglect this equation itself. The composition of the above three transformations is the transformation

$$\mathcal{T}\colon \quad y = \frac{\mathfrak{r}^1}{2}, \quad z = -\frac{\mathfrak{r}^2}{2}, \quad p = t, \quad q = \mathrm{e}^{(\mathfrak{r}^1 - \mathfrak{r}^2)/2}\big(x - (\mathfrak{r}^1 + \mathfrak{r}^2 + 1)t\big), \ s = \mathfrak{r}^3. \tag{3.6}$$

Therefore, to make the inverse transition from the system (3.4) to the system (3.1), we should attach the equation (3.5) to the system (3.4), thus extending the tuple of dependent variables (q, s) by p, and carry out the inverse to the transformation (3.6),

$$\hat{\mathcal{T}}\colon \quad t = p, \quad x = \mathrm{e}^{-y-z}q + (2y - 2z + 1)p, \quad \mathfrak{r}^1 = 2y, \quad \mathfrak{r}^2 = -2z, \quad \mathfrak{r}^3 = s. \tag{3.7}$$

It is convenient to collect the expressions for low-order derivatives of p and q and for their combinations in terms of the old variables in view of the system (3.1), which will be needed below:

$$
\begin{gathered}
p_y = -\frac{1}{\mathfrak{r}^1_x}, \quad p_z = -\frac{1}{\mathfrak{r}^2_x}, \quad K^1 = -\frac{2}{\mathfrak{r}^2_x}\mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}, \quad K^2 = \frac{2}{\mathfrak{r}^1_x}\mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}, \\
\frac{s_y}{K^2} = \mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\frac{\mathfrak{r}^3_x}{2}, \quad q_y = \mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\left(\frac{2}{\mathfrak{r}^1_x} + x - V^1t - 2t\right), \quad q_z = \mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}(x - V^2t), \\
q_{zz} = \mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\left(-\frac{2}{\mathfrak{r}^2_x} + x - V^2t + 2t\right).
\end{gathered}
$$

Following the procedure analogous to that in [112], we find the complete set of local solutions of the system (3.1) via the linearization of the subsystem (3.1a)–(3.1b).

We are allowed to make the point transformation (3.6) if and only if the nondegeneracy condition $\mathfrak{r}^1_t\mathfrak{r}^2_x - \mathfrak{r}^1_x\mathfrak{r}^2_t \neq 0$ holds, which is equivalent, on solutions of (3.1), to the inequality $\mathfrak{r}^1_x\mathfrak{r}^2_x \neq 0$. Therefore, $\mathfrak{r}^1_t\mathfrak{r}^2_t \neq 0$ as well, and thus both Riemann invariants $\mathfrak{r}^1$ and $\mathfrak{r}^2$ are not constants. In this case, we introduce the "pseudopotential" Ψ defined by the potential system $\Psi_y = q - \Psi$, $\Psi_z = q_z - \Psi$ for the equation (3.4a). In fact, this "pseudopotential" is a modification, $\Psi = \mathrm{e}^{-y-z}\tilde\Psi$, of the standard potential $\tilde\Psi$ for the equation (3.4a) associated with the conserved current $(\mathrm{e}^{y+z}q_z, -\mathrm{e}^{y+z}q)$ of this equation via the potential system $\tilde\Psi_y = \mathrm{e}^{y+z}q$, $\tilde\Psi_z = \mathrm{e}^{y+z}q_z$. It is easily seen that the function Ψ satisfies the Klein–Gordon equation $\Psi_{yz} = \Psi$. Moreover, solutions of the equations (3.4a), (3.4b) and (3.5) are locally expressed in terms of Ψ,

$$
q = \Psi_y + \Psi, \quad p = \frac{1}{2}\mathrm{e}^{-y-z}(\Psi_z - \Psi_y), \quad s = W\left(\mathrm{e}^{y+z}(\Psi_y + \Psi_z - 2\Psi)\right).
$$

Here and in what follows W is an arbitrary smooth function of its argument. Returning to the old coordinates, we obtain the regular family of solutions of the system (3.1), which is expressed in terms of the general solution of the Klein–Gordon equation. Note that the nondegeneracy condition for this inverse transformation is $K^1K^2 \neq 0$, where, in terms of Ψ,

$$
K^1 = \Psi_{zz} - \Psi_z + \Psi_y - \Psi, \quad K^2 = \Psi_{yy} - \Psi_y + \Psi_z - \Psi.
$$

In view of the Klein–Gordon equation $\Psi_{yz} = \Psi$, the inequalities $K^1 \neq 0$ and $K^2 \neq 0$ are equivalent to each other as well as to the condition $\Psi \notin \langle \mathrm{e}^{-y-z}, \mathrm{e}^{y+z}, (y-z)\mathrm{e}^{y+z}\rangle$.

If the nondegeneracy condition $\mathfrak{r}^1_t\mathfrak{r}^2_x - \mathfrak{r}^1_x\mathfrak{r}^2_t \neq 0$ does not hold, then at least one of the Riemann invariants $\mathfrak{r}^1$ and $\mathfrak{r}^2$ is a constant. If only one Riemann invariant is a constant, we derive the singular family of solutions of (3.1). Let $\mathfrak{r}^1$ be a constant, $\mathfrak{r}^1 = c$. Then the equation (3.1a) is trivially satisfied, and we make the rank-one hodograph transformation $\bar t = t$, $\bar z = \mathfrak{r}^2$, $\bar q = x$, $\bar s = \mathfrak{r}^3$ in the two remaining equations (3.1b) and (3.1c), exchanging the roles of x and $\mathfrak{r}^2$, that is, $\bar t$ and $\bar z$ are the new independent variables, $\bar q$ and $\bar s$ are the new dependent variables. This yields the system $\bar q_{\bar t} = \bar z + c - 1$, $\bar s_{\bar z} + \bar q_{\bar z}\bar s_{\bar t} = 0$. Integrating the first equation to $\bar q = (\bar z + c - 1)\bar t + \mathrm{e}^{\bar z}\Theta^2_{\bar z}$, where Θ^2 is an arbitrary function of $\bar z$. It is chosen with a help of a hindsight to represent the general solution of the second equation in the form $\bar s = W(\mathrm{e}^{-\bar z}\bar t - \Theta^2_{\bar z} - \Theta^2)$. The consideration when $\mathfrak{r}^2$ being a constant is similar. When the both $\mathfrak{r}^1$ and $\mathfrak{r}^2$ are constants, we obtain an ultra-singular family of solutions.

Theorem 3.1. *Any solution of the system* (3.1) *(locally) belongs to one of the following families; below W is an arbitrary function of its argument.*

1. The regular family, where both the Riemann invariants $\mathfrak{r}^1$ and $\mathfrak{r}^2$ are not constants (the general solution):

$$
\begin{aligned}
&t = -\mathrm{e}^{(\mathfrak{r}^2-\mathfrak{r}^1)/2}(\Psi_{\mathfrak{r}^1} + \Psi_{\mathfrak{r}^2}), \quad x = \mathrm{e}^{(\mathfrak{r}^2-\mathfrak{r}^1)/2}\big((2\Psi_{\mathfrak{r}^1} + \Psi) - (\mathfrak{r}^1 + \mathfrak{r}^2 + 1)(\Psi_{\mathfrak{r}^1} + \Psi_{\mathfrak{r}^2})\big),\\
&\mathfrak{r}^3 = W\big(\mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}(\Psi_{\mathfrak{r}^1} - \Psi_{\mathfrak{r}^2} - \Psi)\big).
\end{aligned}
$$

Here the function $\Psi = \Psi(\mathfrak{r}^1, \mathfrak{r}^2)$ runs through the set of solutions of the Klein–Gordon equation $\Psi_{\mathfrak{r}^1\mathfrak{r}^2} = -\Psi/4$ with $\Psi \notin \langle\, \mathrm{e}^{(\mathfrak{r}^2-\mathfrak{r}^1)/2},\ \mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2},\ (\mathfrak{r}^1 + \mathfrak{r}^2)\mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\,\rangle$.

2. The two singular families, where exactly one of the Riemann invariants $\mathfrak{r}^1$ and $\mathfrak{r}^2$ is a constant:

$$
\begin{aligned}
&\mathfrak{r}^1 = c, \quad x = (\mathfrak{r}^2 + c - 1)t + \mathrm{e}^{\mathfrak{r}^2}\Theta^2_{\mathfrak{r}^2}, \quad \mathfrak{r}^3 = W(\mathrm{e}^{-\mathfrak{r}^2}t - \Theta^2_{\mathfrak{r}^2} - \Theta^2);\\
&\mathfrak{r}^2 = c, \quad x = (\mathfrak{r}^1 + c + 1)t + \mathrm{e}^{-\mathfrak{r}^1}\Theta^1_{\mathfrak{r}^1}, \quad \mathfrak{r}^3 = W(\mathrm{e}^{\mathfrak{r}^1}t + \Theta^1_{\mathfrak{r}^1} - \Theta^1).
\end{aligned}
$$

Here c is an arbitrary constant and $\Theta^1 = \Theta^1(\mathfrak{r}^1)$ and $\Theta^2 = \Theta^2(\mathfrak{r}^2)$ are arbitrary functions of their arguments.

3. The ultra-singular family, with arbitrary constants $\mathfrak{r}^1$ and $\mathfrak{r}^2$ and $\mathfrak{r}^3 = W(x - (\mathfrak{r}^1 + \mathfrak{r}^2)t)$.

The regular, singular and ultra-singular families of solutions of the system $\mathcal{S}$ are associated with solutions of the subsystem $\mathcal{S}_0$ of rank 2, 1 and 0, respectively; cf. [65].

Alternatively, to get the subfamily of regular solutions with nonconstant parameter function W, one can employ the generalized hodograph transformation [149], see details in [112, Section 9].

3.3 Preliminaries

Given a system $\mathcal{L}$ of differential equations, we denote by $\mathcal{L}^{(\infty)}$ the manifold defined by the system $\mathcal{L}$ and its differential consequences in the associated jet space. A local object associated with $\mathcal{L}$ within the framework of symmetry analysis of differential equations, like a generalized symmetry, a conserved current of a local conservation law, a conservation-law characteristic or a cosymmetry, is called trivial if it vanishes on solutions of $\mathcal{L}$ or, equivalently, on $\mathcal{L}^{(\infty)}$. Two such local objects of the same kind are naturally assumed equivalent if their difference is trivial, and thus such local objects of the same kind in total are considered up to this equivalence relation.

The system $\mathcal{S}$ given by (3.1) is of the evolution form. The jet variables t, x and $\mathfrak{r}^i_\kappa = \partial^\kappa\mathfrak{r}^i/\partial x^\kappa$, $i=1,2,3$, $\kappa\in\mathbb{N}_0$, constitute the standard coordinates on the manifold $\mathcal{S}^{(\infty)}$. Therefore, up to the above equivalence relation on solutions of $\mathcal{S}$, for the coset of each of local symmetry-like objects associated with $\mathcal{S}$ we can consider a representative whose components do not depend on the derivatives of $\mathfrak{r}$ involving differentiation with respect to t.[3] A symbol with $[\mathfrak{r}]$, like $f[\mathfrak{r}]$, denotes a differential function of $\mathfrak{r}$ that depends at most on t, x and a finite number of derivatives of $\mathfrak{r}$ with respect to x, $f=f(t,x,\mathfrak{r}_0,\dots,\mathfrak{r}_\kappa)$, $\kappa\in\mathbb{N}_0$. Below we consider only such differential functions and assume that the components of any local symmetry-like objects associated with $\mathcal{S}$ are such differential functions. For $i\in\{1,2,3\}$, the order $\operatorname{ord}_{\mathfrak{r}^i} f[\mathfrak{r}]$ of a differential function $f[\mathfrak{r}]$ with respect to $\mathfrak{r}^i$ is defined to be equal $\max\{\kappa\in\mathbb{N}_0 \mid f_{\mathfrak{r}^i_\kappa}\neq 0\}$ unless this set is empty and $-\infty$ otherwise.

We restrict the total derivative operators D_x and D_t with respect to x and t to the set of above differential functions of $\mathfrak{r}$, and additionally exclude the derivatives of $\mathfrak{r}$ that

[3]Here, for conservation-law characteristics we need to use Lemma 3 in [86], see also [103, Lemma 4.28].

involve differentiation with respect to t from D_t in view of the system $\mathcal{S}$, respectively obtaining the (commuting) operators

$$\mathcal{D}_x := \partial_x + \sum_{\kappa=0}^{\infty}\sum_{i=1}^{3} \mathfrak{r}^i_{\kappa+1}\partial_{\mathfrak{r}^i_\kappa}, \quad \mathcal{D}_t := \partial_t - \sum_{\kappa=0}^{\infty}\sum_{i=1}^{3} \mathcal{D}_x^\kappa(V^i\mathfrak{r}^i_1)\partial_{\mathfrak{r}^i_\kappa}.$$

We define the commuting operators $\mathcal{A} := \mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathcal{D}_x$ and $\mathcal{B} := \mathcal{D}_t + (\mathfrak{r}^1+\mathfrak{r}^2)\mathcal{D}_x$, $\mathcal{AB} = \mathcal{BA}$.

It is convenient to introduce the modified coordinates t, x, $r^j_\kappa = \mathfrak{r}^j_\kappa$ and $\omega^\kappa := \mathcal{A}^\kappa\mathfrak{r}^3$ for $\kappa \in \mathbb{N}_0$ and $j = 1, 2$ on the manifold $\mathcal{S}^{(\infty)}$ instead of the standard ones.[4] In this notation, we have

$$\begin{aligned}
&\mathcal{A}\omega^\kappa = \omega^{\kappa+1}, \quad \mathcal{B}\omega^\kappa = 0, \quad \kappa\in\mathbb{N}_0, \quad \mathcal{B}r^1 = -r^1_1, \quad \mathcal{B}r^2 = r^2_1,\\
&\mathcal{D}_x = \partial_x + \sum_{\kappa=0}^{\infty}\big(r^1_{\kappa+1}\partial_{r^1_\kappa} + r^2_{\kappa+1}\partial_{r^2_\kappa} + \mathrm{e}^{r^1-r^2}\omega^{\kappa+1}\partial_{\omega^\kappa}\big),\\
&\mathcal{D}_t = \partial_t - \sum_{\kappa=0}^{\infty}\big(\mathcal{D}_x^\kappa(V^1r^1_1)\partial_{r^1_\kappa} + \mathcal{D}_x^\kappa(V^2r^2_1)\partial_{r^2_\kappa} + (r^1+r^2)\mathrm{e}^{r^1-r^2}\omega^{\kappa+1}\partial_{\omega^\kappa}\big).
\end{aligned}$$

We define the orders $\operatorname{ord}_{r^j} f$, $j = 1, 2$, and $\operatorname{ord}_\omega f$ of a differential function $f = f[\mathfrak{r}]$ with respect to r^j and "ω" to be equal $\max\{\kappa \mid f_{r^j_\kappa} \neq 0\}$ and $\max\{\kappa \mid f_{\omega^\kappa} \neq 0\}$, respectively, unless the corresponding set is empty and $-\infty$ otherwise. Note that $\operatorname{ord}_\omega f = \operatorname{ord}_{\mathfrak{r}^3} f$. The notation like $f[r^1, r^2]$, or equivalently $f[\mathfrak{r}^1, \mathfrak{r}^2]$, denotes a differential function f of $(r^1, r^2) = (\mathfrak{r}^1, \mathfrak{r}^2)$.

Lemma 3.2. *A differential function $f = f[\mathfrak{r}]$ satisfies the equation $\mathcal{B}f = 0$ if and only if it is a smooth function of a finite number of ω's, $f = f(\omega^0, \dots, \omega^\kappa)$ with $\kappa \in \mathbb{N}_0$.*

Proof. Provided f being a smooth function of a finite number of ω's, it satisfies the equation $\mathcal{B}f = 0$ because of $\mathcal{B}\omega^\kappa = 0$ for all $\kappa \in \mathbb{N}_0$.

Conversely, using the modified coordinates on $\mathcal{S}^{(\infty)}$ we denote $\kappa_j = \operatorname{ord}_{r^j} f$, $j = 1, 2$. Suppose that $\kappa_j \geqslant 0$ for some j. Then collecting coefficients of $r^j_{\kappa_j+1}$ in the equation $\mathcal{B}f = 0$ yields $\partial f/\partial r^j_{\kappa_j} = 0$, which gives a contradiction. Hence the function f does not depend on r^j_κ, $\kappa \in \mathbb{N}_0$. The equation $\mathcal{B}f = 0$ takes the form $f_t + (r^1 + r^2)f_x = 0$, splitting with respect to (r^1, r^2) to $f_t = f_x = 0$. □

[4]The operator $\mathcal{A}$ and the modified coordinates are related to the degeneration of V^3 meaning, that $V^3_{\mathfrak{r}^3} = 0$; cf. [40, Theorem 5.2].

As the standard coordinates on the manifold $\mathcal{K}^{(\infty)}$ associated with the system (3.4), we can take the jet variables y, z, $q_\iota = \partial^\iota q/\partial y^\iota$ if $\iota \geqslant 0$ and $q_\iota = \partial^{-\iota} q/\partial z^{-\iota}$ if $\iota < 0$, $\iota \in \mathbb{Z}$, $s_\kappa = \partial^\kappa s/\partial y^\kappa$, $\kappa \in \mathbb{N}_0$. In these coordinates, the restrictions of the total derivative operators with respect to y and z respectively take the form

$$\mathcal{D}_y = \partial_y + \sum_{\iota=-\infty}^{+\infty} q_{\iota+1}\partial_{q_\iota} + \sum_{\kappa=0}^{+\infty} s_{\kappa+1}\partial_{s_\kappa}, \quad \mathcal{D}_z = \partial_z + \sum_{\iota=-\infty}^{+\infty} q_{\iota-1}\partial_{q_\iota} + \sum_{\kappa=0}^{+\infty} \mathcal{D}_y^\kappa\left(\frac{K^1}{K^2}s_1\right)\partial_{s_\kappa},$$

where $K^1 := q_{-2} - 2q_{-1} + q_0$, $K^2 := q_1 + q_{-1} - 2q_0$. The infinite prolongation of the transformation (3.6) induces pushing forward of the operators $\mathcal{D}_t$, $\mathcal{D}_x$, $\mathcal{A}$ and $\mathcal{B}$ to the operators

$$\begin{gathered}\hat{\mathcal{D}}_t = -\frac{\mathrm{e}^{y+z}}{K^2}(2y-2z+1)\mathcal{D}_y - \frac{\mathrm{e}^{y+z}}{K^1}(2y-2z-1)\mathcal{D}_z, \quad \hat{\mathcal{D}}_x = \frac{\mathrm{e}^{y+z}}{K^2}\mathcal{D}_y + \frac{\mathrm{e}^{y+z}}{K^1}\mathcal{D}_z,\\ \hat{\mathcal{A}} = \frac{\mathrm{e}^{-y-z}}{K^2}\mathcal{D}_y + \frac{\mathrm{e}^{-y-z}}{K^1}\mathcal{D}_z, \quad \hat{\mathcal{B}} = -\frac{\mathrm{e}^{y+z}}{K^2}\mathcal{D}_y + \frac{\mathrm{e}^{y+z}}{K^1}\mathcal{D}_z, \quad \hat{\mathcal{A}}\hat{\mathcal{B}} = \hat{\mathcal{B}}\hat{\mathcal{A}}\end{gathered}$$

A symbol with $[q,s]$, like $f[q,s]$, denotes a differential function of (q,s) that depends at most on y, z and a finite, but unspecified number of q_ι, $\iota \in \mathbb{Z}$, and s_κ, $\kappa \in \mathbb{N}_0$. The order $\operatorname{ord}_s f$ of a differential function $f = f[q,s]$ with respect to s is defined to be equal $\max\{\kappa \in \mathbb{N}_0 \mid f_{s_\kappa} \neq 0\}$ unless this set is empty and $-\infty$ otherwise. Analogously, a symbol with $[q]$, like $f[q]$, denotes a differential function of q that depends at most on y, z and a finite, but unspecified number of q_ι, $\iota \in \mathbb{Z}$. We also use the modified coordinates y, z, $\hat{q}_\iota = q_\iota$, $\iota \in \mathbb{Z}$ and $\hat{\omega}^\kappa = \hat{\mathcal{A}}^\kappa s$, $\kappa \in \mathbb{N}_0$, on the manifold $\mathcal{K}^{(\infty)}$.

Corollary 3.3. *A differential function $f = f[q,s]$ satisfies the equation $\hat{\mathcal{B}}f = 0$, i.e., $K^1\mathcal{D}_y f = K^2\mathcal{D}_z f$, if and only if it is a smooth function of a finite number of $\hat{\omega}$'s, $f = f(\hat{\omega}^0, \dots, \hat{\omega}^\kappa)$ with $\kappa \in \mathbb{N}_0$.*

The infinite prolongation of the transformation (3.7) induces pushing forward of the operators $\mathcal{D}_y$ and $\mathcal{D}_z$ to the (commuting) operators

$$\tilde{\mathcal{D}}_y := -\frac{1}{\mathfrak{r}^1_x}\big(\mathcal{D}_t + (\mathfrak{r}^1 + \mathfrak{r}^2 - 1)\mathcal{D}_x\big), \quad \tilde{\mathcal{D}}_z := -\frac{1}{\mathfrak{r}^2_x}\big(\mathcal{D}_t + (\mathfrak{r}^1 + \mathfrak{r}^2 + 1)\mathcal{D}_x\big).$$

3.4 Generalized symmetries

The following two facts allow us to exhaustively describe generalized symmetries of the system (3.1). Firstly, the equation (3.1c) is partially coupled with the equations (3.1a) and (3.1b). Secondly, the subsystem (3.1a)–(3.1b) is linearized by the hodograph transformation, and the associated linear system reduces to the (1+1)-dimensional Klein–Gordon equation.

We denote by Σ the algebra of generalized symmetries of the system (3.1), and by $\Sigma^{\rm triv}$ the algebra of its trivial generalized symmetries, whose characteristics vanish on solutions of (3.1). The quotient algebra $\Sigma^{\rm q}=\Sigma/\Sigma^{\rm triv}$ can be identified, e.g., with the subalgebra of canonical representatives in the reduced evolutionary form, $\hat\Sigma^{\rm q}=\big\{\sum_{i=1}^3\eta^i[\mathfrak r]\partial_{\mathfrak r^i}\in\Sigma\big\}$. The criterion of invariance of the system (3.1) with respect to the generalized vector field $\sum_{i=1}^3\eta^i[\mathfrak r]\partial_{\mathfrak r^i}$ results in the system of three determining equations for the components η^i,

$$\mathrm D_t\eta^1+(\mathfrak r^1+\mathfrak r^2+1)\mathrm D_x\eta^1+\mathfrak r^1_x(\eta^1+\eta^2)=0, \tag{3.8a}$$

$$\mathrm D_t\eta^2+(\mathfrak r^1+\mathfrak r^2-1)\mathrm D_x\eta^2+\mathfrak r^2_x(\eta^1+\eta^2)=0, \tag{3.8b}$$

$$\mathrm D_t\eta^3+(\mathfrak r^1+\mathfrak r^2)\mathrm D_x\eta^3+\mathfrak r^3_x(\eta^1+\eta^2)=0. \tag{3.8c}$$

Lemma 3.4. *For any generalized vector field $\sum_{i=1}^3\eta^i[\mathfrak r]\partial_{\mathfrak r^i}$ from $\hat\Sigma^{\rm q}$, its components η^1 and η^2 do not depend on derivatives of $\mathfrak r^3$, i.e., $\eta^1=\eta^1[\mathfrak r^1,\mathfrak r^2]$ and $\eta^2=\eta^2[\mathfrak r^1,\mathfrak r^2]$.*

Proof. Suppose that $\kappa_j:=\operatorname{ord}_{\mathfrak r^3}\eta^j\geqslant0$ for some $j\in\{1,2\}$. Collecting the coefficients of the jet variable $\mathfrak r^3_{\kappa_j+1}$ in the jth equation of (3.8) yields the equation $\partial\eta^j/\partial\mathfrak r^3_{\kappa_j}=0$, which contradicts the assumption. Hence $\kappa_j=-\infty$ for any $j=1,2$. □

Lemma 3.4 is the manifestation of partial coupling of the system (3.1). In view of this lemma, the subalgebra $\hat\Sigma^{\rm q}_3$ of $\hat\Sigma^{\rm q}$ constituted by elements with vanishing η^1 and η^2 is an ideal of $\hat\Sigma^{\rm q}$, and the quotient algebra $\Sigma^{\rm q}_{12}:=\hat\Sigma^{\rm q}/\hat\Sigma^{\rm q}_3$ is isomorphic to the subalgebra of reduced generalized symmetries of the subsystem (3.1a)–(3.1b) that admit local prolongations to $\mathfrak r^3$. The ideal $\hat\Sigma^{\rm q}_3$ is described by the following corollary of Lemma 3.2.

Corollary 3.5. *A generalized vector field $\eta^3\partial_{\mathfrak{r}^3}$ belongs to $\hat\Sigma^{\rm q}$ if and only if the coefficient η^3 is a smooth function of a finite number of ω's.*

Proof. The invariance of the system (3.1) with respect to the generalized vector field $\eta^3\partial_{\mathfrak{r}^3}$ leads to the single determining equation $\mathcal{B}\eta^3=0$. Further we use Lemma 3.2. □

Therefore, the infinite prolongation of an element $f\partial_{\mathfrak{r}^3}$ of $\hat\Sigma^{\rm q}$ is equal to $\sum_{\iota=0}^\infty(\hat{\mathcal{A}}^\iota f)\partial_{\omega^\iota}$, and thus the commutator of elements $f^1\partial_{\mathfrak{r}^3}$ and $f^2\partial_{\mathfrak{r}^3}$ of $\hat\Sigma^{\rm q}$ is

$$\sum_{\iota=0}^\infty\big((\hat{\mathcal{A}}^\iota f^1)f^2_{\omega^\iota}-(\hat{\mathcal{A}}^\iota f^2)f^1_{\omega^\iota}\big)\partial_{\mathfrak{r}^3},\ \text{where}\ \hat{\mathcal{A}}=\sum_{\kappa=0}^\infty\omega^{\kappa+1}\partial_{\omega^\kappa}.$$

We specify the form of canonical representatives of cosets of $\hat\Sigma^{\rm q}_3$.

Lemma 3.6. *Each coset of $\hat\Sigma^{\rm q}_3$ contains a generalized vector field of the form*

$$\eta^1[\mathfrak{r}^1,\mathfrak{r}^2]\partial_{\mathfrak{r}^1}+\eta^2[\mathfrak{r}^1,\mathfrak{r}^2]\partial_{\mathfrak{r}^2}+{\rm e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathfrak{r}^3_x\hat\eta^3[\mathfrak{r}^1,\mathfrak{r}^2]\partial_{\mathfrak{r}^3},\tag{3.9}$$

where the coefficients η^1, η^2 and $\hat\eta^3$ satisfy the system of equations (3.8a), (3.8b) *and*

$$\mathcal{D}_t\hat\eta^3+(\mathfrak{r}^1+\mathfrak{r}^2)\mathcal{D}_x\hat\eta^3+{\rm e}^{\mathfrak{r}^1-\mathfrak{r}^2}(\eta^1+\eta^2)=0.\tag{3.10}$$

Proof. In view of Lemma 3.4 and Corollary 3.5, it suffices to show that the third components of canonical representatives for elements from the quotient algebra $\Sigma^{\rm q}_{12}$ can be chosen to be of the form $\eta^3={\rm e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathfrak{r}^3_x\hat\eta^3[\mathfrak{r}^1,\mathfrak{r}^2]$. After substituting the representation $\eta^3={\rm e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathfrak{r}^3_x\hat\eta^3[\mathfrak{r}]$ into the equation (3.8c), we derive the equation (3.10). We use the modified coordinates on the manifold $\mathcal{S}^{(\infty)}$. If the coefficient $\hat\eta^3$ depends on ω^κ for some $\kappa\in\mathbb{N}_0$, then a differential function of $(\mathfrak{r}^1,\mathfrak{r}^2)$ obtained from $\hat\eta^3$ by fixing values of all involved ω^κ's in the domain of $\hat\eta^3$ is also a solution of (3.10) for the same value of (η^1,η^2). □

The elements of the form (3.9) from the algebra $\hat\Sigma^{\rm q}$ constitute a subalgebra of this algebra, which we denote by $\bar\Sigma^{\rm q}_{12}$. Unfortunately, the algebras $\Sigma^{\rm q}_{12}$ and $\bar\Sigma^{\rm q}_{12}$ are not isomorphic. Although $\hat\Sigma^{\rm q}=\bar\Sigma^{\rm q}_{12}+\hat\Sigma^{\rm q}_3$, this sum is not direct since $\bar\Sigma^{\rm q}_{12}\cap\hat\Sigma^{\rm q}_3=\langle{\rm e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathfrak{r}^3_x\partial_{\mathfrak{r}^3}\rangle$. The algebra $\Sigma^{\rm q}_{12}$ is naturally isomorphic to the quotient algebra $\bar\Sigma^{\rm q}_{12}/\langle{\rm e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathfrak{r}^3_x\partial_{\mathfrak{r}^3}\rangle$.

Deriving the exhaustive description of the algebra Σ^{q}_{12} is quite complicated. For this purpose, we reduce the system (3.1) to a system (3.4) containing the (1+1)-dimensional Klein–Gordon equation. Similarly to the system (3.1), we denote by $\mathfrak{S}$ the algebra of generalized symmetries of the system (3.4), and by $\mathfrak{S}^{\mathrm{triv}}$ the algebra of its trivial generalized symmetries, whose characteristics vanish on solutions of (3.4). The quotient algebra $\mathfrak{S}^{\mathrm{q}} = \mathfrak{S}/\mathfrak{S}^{\mathrm{triv}}$ can be identified, e.g., with the subalgebra of canonical representatives in the evolutionary form, $\hat{\mathfrak{S}}^{\mathrm{q}} = \{\chi[q,s]\partial_q + \theta[q,s]\partial_s \in \mathfrak{S}\}$. The Lie bracket on $\hat{\mathfrak{S}}^{\mathrm{q}}$ is defined as the modified Lie bracket of generalized vector fields in the jet space with the independent variables (y,z) and the dependent variables (q,s), where all arising mixed derivatives of q and all arising derivatives of s that involve differentiation with respect to y are substituted in view of the system (3.4) and its differential consequences. The system of determining equations for components of elements of $\hat{\mathfrak{S}}^{\mathrm{q}}$ is

$$\mathcal{D}_y\mathcal{D}_z\chi = \chi, \tag{3.11a}$$

$$s_1(\mathcal{D}_z - 1)^2\chi + K_1\mathcal{D}_y\theta = \frac{K^1}{K^2}s_1(\mathcal{D}_y + \mathcal{D}_z - 2)\chi + K^2\mathcal{D}_z\theta. \tag{3.11b}$$

The algebra $\hat{\mathfrak{S}}^{\mathrm{q}}$ is isomorphic to the algebra $\hat{\Sigma}^{\mathrm{q}}$. This isomorphism is induced by the pushforward of Σ onto $\mathfrak{S}$ that is generated by the point transformation (3.6), excluding the derivatives of p (including p itself) in view of the equation (3.5) and its differential consequences and the successive projection of the obtained generalized vector fields to the jet space with the independent variables (y,z) and the dependent variables (q,s). To map $\mathfrak{S}$ into Σ, we need to prolong the elements of $\mathfrak{S}$ to p according the equation (3.5) and make the pushforward by the point transformation (3.7).

Lemma 3.7. *The q-component of every element of $\hat{\mathfrak{S}}^{\mathrm{q}}$ does not depend on s and its derivatives.*

Proof. Suppose that $X = \chi\partial_q + \theta\partial_s \in \hat{\mathfrak{S}}^{\mathrm{q}}$, and $\kappa := \mathrm{ord}_s\,\chi \geqslant 0$. Then invariance criterion for the equation $q_{yz} = q$ and the generalized vector field X implies, after collecting coefficients of $s_{\kappa+2}$, the equation $\chi_{s_\kappa} = 0$, which contradicts the assumption. This is why $\mathrm{ord}_s\,\chi = -\infty$. □

Remark 3.8. The only essential feature for Lemma 3.7 is that K^1 and K^2 do not vanish simultaneously, not the specific form thereof.

Lemma 3.7 is the counterpart of Lemma 3.4 for the system (3.4) and is the manifestation of partial coupling of this system. In view of Lemma 3.7, the subalgebra $\hat{\mathfrak{S}}^{\mathrm{q}}_s$ of $\hat{\mathfrak{S}}^{\mathrm{q}}$ constituted by elements with vanishing q-components is an ideal of $\hat{\mathfrak{S}}^{\mathrm{q}}$. In view of Corollary 3.3 (or Corollary 3.5), this ideal consists of generalized vector fields of the form $\theta\partial_s$, where θ is a smooth function of a finite, but unspecified number of $\hat{\omega}$'s. Since the ideal $\hat{\mathfrak{S}}^{\mathrm{q}}_s$ of $\hat{\mathfrak{S}}^{\mathrm{q}}$ corresponds to and is isomorphic to the ideal $\hat{\Sigma}^{\mathrm{q}}_3$ of $\hat{\Sigma}^{\mathrm{q}}$, for our purpose it suffices to describe the quotient algebra $\mathfrak{S}^{\mathrm{q}}_q := \hat{\mathfrak{S}}^{\mathrm{q}}/\hat{\mathfrak{S}}^{\mathrm{q}}_s$.

Denote by $\hat{\mathfrak{K}}^{\mathrm{q}}$ the algebra of reduced generalized symmetries of the (1+1)-dimensional Klein–Gordon equation (3.4a), $\hat{\mathfrak{K}}^{\mathrm{q}} = \{\chi[q]\partial_q \mid \mathcal{D}_y\mathcal{D}_z\chi = \chi\}$. The quotient algebra $\mathfrak{S}^{\mathrm{q}}_q$ is naturally isomorphic to the subalgebra $\mathfrak{A}$ of $\hat{\mathfrak{K}}^{\mathrm{q}}$ that consists of elements of $\hat{\mathfrak{K}}^{\mathrm{q}}$ admitting local prolongations to s. It was proved in Section 2.2 that the algebra $\hat{\mathfrak{K}}^{\mathrm{q}}$ is the semi-direct sum of its subalgebra $\hat{\Lambda}^{\mathrm{q}}$ and its ideal $\hat{\mathfrak{K}}^{-\infty}$, $\hat{\mathfrak{K}}^{\mathrm{q}} = \hat{\Lambda}^{\mathrm{q}} \in\!\!\!\!+ \hat{\mathfrak{K}}^{-\infty}$, where

$$
\begin{aligned}
&\hat{\Lambda}^{\mathrm{q}} := \langle\, (\mathcal{J}^\kappa q)\partial_q,\ (\mathcal{D}_y^\iota\mathcal{J}^\kappa q)\partial_q,\ (\mathcal{D}_z^\iota\mathcal{J}^\kappa q)\partial_q,\ \kappa\in\mathbb{N}_0,\ \iota\in\mathbb{N}\,\rangle,\\
&\hat{\mathfrak{K}}^{-\infty} := \{f(y,z)\partial_q \mid f\in\mathrm{KG}\},
\end{aligned}
$$

$\mathcal{J} := y\mathcal{D}_y - z\mathcal{D}_z$, and KG denotes the solution set of the (1+1)-dimensional Klein–Gordon equation (3.4a), i.e., $f\in\mathrm{KG}$ means that $f_{yz} = f$.

Lemma 3.9. $\mathfrak{A} = \big\{X^{\zeta,c} := \big((\mathcal{D}_y+1)\zeta + cq\big)\partial_q \mid \zeta = \zeta[q]\colon \mathcal{D}_y\mathcal{D}_z\zeta = \zeta,\ c\in\mathbb{R}\big\}$, *and an appropriate prolongation of the generalized vector field $X^{\zeta,c}$ to s is given by*

$$
\theta = \frac{s_1}{K^2}(\mathcal{D}_y + \mathcal{D}_z - 2)\zeta. \tag{3.12}
$$

Proof. Denote $\tilde{\mathfrak{A}} = \big\{X^{\zeta,c} := \big((\mathcal{D}_y+1)\zeta + cq\big)\partial_q \mid \zeta = \zeta[q]\colon \mathcal{D}_y\mathcal{D}_z\zeta = \zeta,\ c\in\mathbb{R}\big\}$. Note that here the form of ζ is defined up to summands proportional to e^{-y-z}.

For any solution ζ of the equation $\mathcal{D}_y\mathcal{D}_z\zeta = \zeta$, the differential functions $\chi = (\mathcal{D}_y+1)\zeta$ and θ defined by (3.12) satisfy the system (3.11). The tuple $(\chi,\theta) = (q,0)$ is a solution of (3.11) as well. Hence $\mathfrak{A} \supseteq \tilde{\mathfrak{A}}$.

Suppose that a generalized vector field $\chi[q]\partial_q$ belongs to $\mathfrak{A}$. This means that there exists $\theta = \theta[q,s]$ such that $\chi\partial_q + \theta\partial_s \in \hat{\mathfrak{G}}^{\mathrm{q}}$. Then the tuple (χ,θ) satisfies the system (3.11). By the substitution $\theta = s_1(K^2)^{-1}\tilde\theta$, the equation (3.11b) is reduced to

$$K^1(\mathcal{D}_y+1)\tilde\theta - K^2(\mathcal{D}_z+1)\tilde\theta = K^1(\mathcal{D}_y+\mathcal{D}_z-2)\chi - K^2(\mathcal{D}_z-1)^2\chi. \tag{3.13}$$

We use the modified coordinates on the manifold $\mathcal{K}^{(\infty)}$. If the function $\tilde\theta$ depends on ω^κ for some $\kappa\in\mathbb{N}_0$, then a differential function of q obtained from $\tilde\theta$ by fixing values of all involved $\hat\omega^\kappa$'s in the domain of $\tilde\theta$ is also a solution of (3.13) for the same value of χ. Therefore, without loss of generality we can assume that $\tilde\theta = \tilde\theta[q]$. Then the equation (3.13) rewritten in the form

$$K^1\big((\mathcal{D}_y+1)\tilde\theta - (\mathcal{D}_y+\mathcal{D}_z-2)\chi\big) = K^2\big((\mathcal{D}_z+1)\tilde\theta - (\mathcal{D}_z-1)^2\chi\big)$$

implies that there exists a differential function $\mu = \mu[q]$ such that

$$(\mathcal{D}_y+1)\tilde\theta - (\mathcal{D}_y+\mathcal{D}_z-2)\chi = \mu K^2, \quad (\mathcal{D}_z+1)\tilde\theta - (\mathcal{D}_z-1)^2\chi = \mu K^1. \tag{3.14}$$

We exclude $\tilde\theta$ from these equations by acting the operators $\mathcal{D}_z+1$ and $\mathcal{D}_y+1$ on the first and the second equations, respectively, and subtracting the first obtained equation from the second one, which gives the equation on μ alone, $K^1\mathcal{D}_y\mu = K^2\mathcal{D}_z\mu$. In view of Corollary 3.3, μ is a constant, and hence equations (3.14) can be rewritten as

$$(\mathcal{D}_y+1)\tilde\theta = (\mathcal{D}_y+\mathcal{D}_z-2)(\chi+\mu q), \quad (\mathcal{D}_z+1)\tilde\theta = (\mathcal{D}_z-1)^2(\chi+\mu q). \tag{3.15}$$

We subtract the second equation from the result of acting the operator $\mathcal{D}_z$ on the first equation and thus derive the equation $(\mathcal{D}_y\mathcal{D}_z-1)\tilde\theta = 0$. Then the differential function $\zeta = \zeta[q]$ that is defined by $\zeta := -\frac14\big(\tilde\theta - (\mathcal{D}_z+1)(\chi+\mu q)\big)$ satisfies the same equation, $(\mathcal{D}_y\mathcal{D}_z-1)\zeta = 0$. We express $\tilde\theta$ from the equality defining ζ, $\tilde\theta = -4\zeta + (\mathcal{D}_z+1)(\chi+\mu q)$, and substitute the obtained expression into (3.15), deriving the equations $-4(\mathcal{D}_y+1)\zeta = -4(\chi+\mu q)$ and $-4(\mathcal{D}_z+1)\zeta = -4\mathcal{D}_z(\chi+\mu q)$. The first of these equations gives the required representation for χ, $\chi = (\mathcal{D}_y + 1)\zeta - \mu q$. The second equation is identically satisfied in view of the above representation for χ and the equation $\mathcal{D}_y\mathcal{D}_z\zeta = \zeta$. We also get $\tilde\theta = -4\zeta + (\mathcal{D}_z + 1)(\mathcal{D}_y + 1)\zeta = (\mathcal{D}_y + \mathcal{D}_z - 2)\zeta$. Therefore, $\mathfrak{A} \subseteq \tilde{\mathfrak{A}}$, i.e., $\mathfrak{A} = \tilde{\mathfrak{A}}$, and the equality (3.12) defines an appropriate prolongation of $X^{\zeta,c} \in \mathfrak{A}$ to s. □

In other words, Lemma 3.9 implies that an element of $\hat\Lambda^{\mathrm{q}}$ can be mapped to a generalized symmetry of the system (3.1) if and only if the associated operator belongs to the subspace

$$\langle\, 1,\ (\mathcal{D}_y + 1)\mathcal{D}_y^\iota\mathfrak{J}^\kappa,\ (\mathcal{D}_z + 1)\mathcal{D}_z^\iota\mathfrak{J}^\kappa,\ \kappa, \iota \in \mathbb{N}_0 \,\rangle.$$

In particular, this subspace contains all polynomials of $\mathcal{D}_y$ and all polynomials of $\mathcal{D}_z$. A complement subspace to it in the entire space of operators associated with elements of $\hat\Lambda^{\mathrm{q}}$ is $\langle\, \mathfrak{J}^\kappa,\ \kappa \in \mathbb{N}\,\rangle$. Elements of $\hat\Lambda^{\mathrm{q}}$ associated with operators from the complement subspace are mapped to *nonlocal* symmetries of the system (3.1). Such nonlocal symmetries are generalized symmetries of certain potential systems for the system (3.1) that are related to potential systems for the (1+1)-dimensional Klein–Gordon equation (3.4a), see Section 3.7 for more details.

Completing the above consideration, we prove the following theorem.

Theorem 3.10. *The quotient algebra Σ^{q} of generalized symmetries of the system* (3.1) *is naturally isomorphic to the algebra $\hat\Sigma^{\mathrm{q}}$ spanned by the generalized vector fields*

$$\begin{aligned}
&\check{\mathcal{W}}(\Omega) = \Omega\partial_{\mathfrak{r}^3}, \quad \check{\mathcal{P}}(\Phi) = \mathrm{e}^{(\mathfrak{r}^2-\mathfrak{r}^1)/2}\left((\Phi + 2\Phi_{\mathfrak{r}^1})\mathfrak{r}^1_x\partial_{\mathfrak{r}^1} + (\Phi - 2\Phi_{\mathfrak{r}^2})\mathfrak{r}^2_x\partial_{\mathfrak{r}^2} + 2\Phi\mathfrak{r}^3_x\partial_{\mathfrak{r}^3}\right),\\
&\check{\mathcal{D}} = \big(x - (\mathfrak{r}^1 + \mathfrak{r}^2 + 1)t\big)\mathfrak{r}^1_x\partial_{\mathfrak{r}^1} + \big(x - (\mathfrak{r}^1 + \mathfrak{r}^2 - 1)t\big)\mathfrak{r}^2_x\partial_{\mathfrak{r}^2} + \big(x - (\mathfrak{r}^1 + \mathfrak{r}^2)t\big)\mathfrak{r}^3_x\partial_{\mathfrak{r}^3},\\
&\check{\mathcal{R}}(\Gamma) = \mathrm{e}^{(\mathfrak{r}^2-\mathfrak{r}^1)/2}\Big((\tilde{\mathcal{D}}_y\Gamma + \Gamma)\mathfrak{r}^1_x\partial_{\mathfrak{r}^1} + (\tilde{\mathcal{D}}_z\Gamma + \Gamma)\mathfrak{r}^2_x\partial_{\mathfrak{r}^2} + 2\Gamma\mathfrak{r}^3_x\partial_{\mathfrak{r}^3}\Big),
\end{aligned}$$

where Γ runs through the set $\{\tilde{\mathfrak{J}}^\kappa\tilde q,\ \tilde{\mathcal{D}}_y^\iota\tilde{\mathfrak{J}}^\kappa\tilde q,\ \tilde{\mathcal{D}}_z^\iota\tilde{\mathfrak{J}}^\kappa\tilde q,\ \kappa \in \mathbb{N}_0,\ \iota \in \mathbb{N}\}$ with

$$\begin{aligned}
&\tilde{\mathcal{D}}_y := -\frac{1}{\mathfrak{r}^1_x}\big(\mathcal{D}_t + (\mathfrak{r}^1 + \mathfrak{r}^2 - 1)\mathcal{D}_x\big), \quad \tilde{\mathcal{D}}_z := -\frac{1}{\mathfrak{r}^2_x}\big(\mathcal{D}_t + (\mathfrak{r}^1 + \mathfrak{r}^2 + 1)\mathcal{D}_x\big),\\
&\tilde{\mathfrak{J}} := \frac{\mathfrak{r}^1}{2}\tilde{\mathcal{D}}_y + \frac{\mathfrak{r}^2}{2}\tilde{\mathcal{D}}_z, \quad \tilde q := \mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\big(x - (\mathfrak{r}^1 + \mathfrak{r}^2 + 1)t\big),
\end{aligned}$$

the parameter function $\Phi = \Phi(\mathfrak{r}^1, \mathfrak{r}^2)$ *runs through the solution set of the Klein–Gordon equation* $\Phi_{\mathfrak{r}^1\mathfrak{r}^2} = -\Phi/4$, *and the parameter function* Ω *runs through the set of smooth functions of a finite, but unspecified number of* $\omega^\kappa := (\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathcal{D}_x)^\kappa\mathfrak{r}^3$, $\kappa \in \mathbb{N}_0$.

Proof. For computing the counterpart of an element $X = \chi\partial_q + \theta\partial_s \in \hat{\mathfrak{G}}^{\mathrm{q}}$ in $\hat{\Sigma}^{\mathrm{q}}$, one should make the following steps:

- prolong the generalized vector field X to p in view of (3.5),
- push forward the prolonged vector field by an appropriate prolongation of the transformation (3.7),
- convert the obtained image to the evolutionary form and
- substitute for all derivatives of $\mathfrak{r}$ with differentiation with respect to t in view of the system (3.1) and its differential consequences.

This procedure gives the generalized vector field

$$\tilde{X} = -\mathrm{e}^{(\mathfrak{r}^2-\mathfrak{r}^1)/2}\tilde{\chi}\mathfrak{r}^1_x\partial_{\mathfrak{r}^1} - \mathrm{e}^{(\mathfrak{r}^2-\mathfrak{r}^1)/2}(\tilde{\mathcal{D}}_z\tilde{\chi})\mathfrak{r}^2_x\partial_{\mathfrak{r}^2} + \big(\theta - \tfrac{1}{2}\mathrm{e}^{(\mathfrak{r}^2-\mathfrak{r}^1)/2}(\tilde{\mathcal{D}}_z\tilde{\chi} + \tilde{\chi})\mathfrak{r}^3_x\big)\partial_{\mathfrak{r}^3}.$$

Here and in what follows tildes mark the counterparts of involved operators and differential functions that are computed according to the procedure.

The ideal $\hat{\mathfrak{G}}^{\mathrm{q}}_s$ of $\hat{\mathfrak{G}}^{\mathrm{q}}$ corresponds to and is isomorphic to the ideal $\hat{\Sigma}^{\mathrm{q}}_3$ of $\hat{\Sigma}^{\mathrm{q}}$, and the form of elements of $\hat{\Sigma}^{\mathrm{q}}_3$, $\check{\mathcal{W}}(\Omega)$, is already known. The generalized vector field $q\partial_q$ is mapped to $-\check{\mathcal{D}}$. We also prolong each generalized vector field of the form $X^{\zeta,0} := (\mathcal{D}_y + 1)\zeta\partial_q$ from $\mathfrak{A}$ to s according to (3.12) and then employ the above procedure, getting the generalized vector field

$$\tilde{X}^{\zeta,0} = -\mathrm{e}^{(\mathfrak{r}^2-\mathfrak{r}^1)/2}\left(\mathfrak{r}^1_x(\tilde{\mathcal{D}}_y + 1)\tilde{\zeta}\partial_{\mathfrak{r}^1} + \mathfrak{r}^2_x(\tilde{\mathcal{D}}_z + 1)\tilde{\zeta}\partial_{\mathfrak{r}^2} + 2\mathfrak{r}^3_x\tilde{\zeta}\partial_{\mathfrak{r}^3}\right),$$

where $\zeta = \zeta[q]$ runs through the characteristics of generalized vector fields in $\hat{\mathfrak{K}}^{\mathrm{q}}$ and is defined up to summands proportional to e^{-y-z}, and $\tilde{\zeta}$ denotes the pullback of ζ by the infinite prolongation of the transformation (3.6). According to the splitting $\hat{\mathfrak{K}}^{\mathrm{q}} = \hat{\Lambda}^{\mathrm{q}} \Subset \hat{\mathfrak{K}}^{-\infty}$, for $\zeta\partial_q \in \hat{\Lambda}^{\mathrm{q}}$ and $\zeta\partial_q \in \hat{\mathfrak{K}}^{-\infty}$ we obtain generalized vector fields of the forms

$-\check{\mathcal{R}}(\Gamma)$ and $-\check{\mathcal{P}}(\Phi)$, respectively, where $\Gamma\partial_q$ can be assumed to run through the chosen basis of $\hat{\Lambda}^{\mathrm{q}}$, and the parameter function $\Phi=\Phi(\mathfrak{r}^1,\mathfrak{r}^2)$ runs through the solution set of the Klein–Gordon equation $\Phi_{\mathfrak{r}^1\mathfrak{r}^2}=-\Phi/4$ and is defined up to summands proportional to $\mathrm{e}^{(\mathfrak{r}^2-\mathfrak{r}^1)/2}$. □

Remark 3.11. The subspaces $\mathcal{I}^1$ and $\mathcal{I}^2$ that consist of all generalized vector fields of the forms $\check{\mathcal{P}}(\Phi)$ and $\check{\mathcal{W}}(\Omega)$ from the algebra $\hat{\Sigma}^{\mathrm{q}}$, respectively, are (infinite-dimensional) ideals of $\hat{\Sigma}^{\mathrm{q}}$. Moreover, the ideal $\mathcal{I}^1$ is commutative. Since $\check{\mathcal{P}}(\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1})=\check{\mathcal{W}}(\omega^1)=\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathfrak{r}^3_x\partial_{\mathfrak{r}^3}$, these ideals are not disjoint, $\mathcal{I}^1\cap\mathcal{I}^2=\langle\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathfrak{r}^3_x\partial_{\mathfrak{r}^3}\rangle$, which displays the above indeterminacy of Φ.

Remark 3.12. The algebra of first-order reduced generalized symmetries of the system (3.1) can be identified with the subspace of $\hat{\Sigma}^{\mathrm{q}}$ spanned by $\check{\mathcal{D}}$, $\check{\mathcal{R}}(\tilde{q})$, $\check{\mathcal{R}}(\tilde{\mathrm{D}}_z\tilde{q})$, $\check{\mathcal{P}}(\Phi)$, $\check{\mathcal{W}}(\Omega)$, where the parameter function $\Phi=\Phi(\mathfrak{r}^1,\mathfrak{r}^2)$ runs through the solution set of the Klein–Gordon equation $\Phi_{\mathfrak{r}^1\mathfrak{r}^2}=-\Phi/4$, and the parameter function Ω runs through the set of smooth functions of $\omega^0=\mathfrak{r}^3$ and $\omega^1=\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathfrak{r}^3_x$. As was noted in [112, Remark 19], this subspace is a Lie algebra since it is closed with respect to the Lie bracket of generalized vector fields. The indicated property is shared by all strictly hyperbolic diagonalizable hydrodynamic-type systems. In the notation of [112, Theorem 18],

$$\check{\mathcal{R}}(\tilde{q})=2(\check{\mathcal{D}}-\check{\mathcal{G}}_1),\quad \check{\mathcal{R}}(\tilde{\mathrm{D}}_z\tilde{q})=2(\check{\mathcal{D}}+\check{\mathcal{G}}_1+\check{\mathcal{G}}_2),$$

where $\check{\mathcal{G}}_1=(t\mathfrak{r}^1_x-1)\partial_{\mathfrak{r}^1}+t\mathfrak{r}^2_x\partial_{\mathfrak{r}^2}+t\mathfrak{r}^3_x\partial_{\mathfrak{r}^3}$ and $\check{\mathcal{G}}_2=\partial_{\mathfrak{r}^1}-\partial_{\mathfrak{r}^2}$. Moreover, the generalized vector fields

$$\check{\mathcal{D}},\quad \check{\mathcal{G}}_1,\quad \check{\mathcal{G}}_2,\quad \check{\mathcal{P}}\big((\mathfrak{r}^1+\mathfrak{r}^2)\mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\big),\quad \check{\mathcal{P}}\big(\mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\big),\quad \check{\mathcal{W}}(\Omega) \tag{3.16}$$

with an arbitrary Ω depending on $\mathfrak{r}^3$ only are the evolutionary forms of Lie-symmetry vector fields $-\hat{\mathcal{D}}$, $-\hat{\mathcal{G}}_1$, $\hat{\mathcal{G}}_2$, $2\hat{\mathcal{P}}^t$, $-2\hat{\mathcal{P}}^x$ and $\hat{\mathcal{W}}(\Omega)$ of the system (3.1), respectively, which span the entire Lie invariance algebra of this system. Therefore, any element of $\hat{\Sigma}^{\mathrm{q}}$ that does not belong to the span of (3.16) is a genuinely generalized symmetry of the system (3.1).

3.5 Cosymmetries

The space Υ of cosymmetries of the system (3.1) can be computed in a way that is similar to the computation of generalized symmetries and involves the partial coupling of this system and the linearizability of the subsystem (3.1a)–(3.1b) by the hodograph transformation. Let $\Upsilon^{\rm triv}\subset\Upsilon$ denote the space of trivial cosymmetries of the system (3.1), which vanish on solutions thereof. The quotient space $\Upsilon^{\rm q}=\Upsilon/\Upsilon^{\rm triv}$ can be identified, e.g., with the subspace that consists of canonical representatives of cosymmetries, $\hat\Upsilon^{\rm q}=\big\{(\lambda^i[\mathfrak r],i=1,2,3)\in\Upsilon\big\}$.

Theorem 3.13. *The space $\hat\Upsilon^{\rm q}$ of canonical representatives of cosymmetries is spanned by cosymmetries from three families,*

1. $\mathrm e^{\mathfrak r^1-\mathfrak r^2}\big(\Omega,-\Omega,(\hat{\mathcal A}\Omega)/\omega^1\big)$ *with the operator* $\hat{\mathcal A}=\sum_{\kappa=0}^\infty\omega^{\kappa+1}\partial_{\omega^\kappa}$ *and with Ω running through the space of smooth functions of a finite, but unspecified number of* $\omega^\kappa=(\mathrm e^{\mathfrak r^2-\mathfrak r^1}\mathrm D_x)^\kappa\mathfrak r^3$, $\kappa\in\mathbb N_0$.

2. $\mathrm e^{(\mathfrak r^1-\mathfrak r^2)/2}(-2\Phi_{\mathfrak r^1},\,\Phi,0)$, *where the parameter function $\Phi=\Phi(\mathfrak r^1,\mathfrak r^2)$ runs through the solution space of the Klein–Gordon equation* $\Phi_{\mathfrak r^1\mathfrak r^2}=-\Phi/4$.

3. $\mathrm e^{(\mathfrak r^1-\mathfrak r^2)/2}\big(-\tilde{\mathrm D}_y\tilde{\mathfrak X}\tilde q,\,\tilde{\mathfrak X}\tilde q,\,0\big)$, *where the operator $\tilde{\mathfrak X}$ runs through the set*

$$\big\{\tilde{\mathfrak J}^\kappa,\,\tilde{\mathfrak J}^\kappa\tilde{\mathrm D}_y^\iota,\,\tilde{\mathfrak J}^\kappa\tilde{\mathrm D}_z^\iota,\,\kappa\in\mathbb N_0,\,\iota\in\mathbb N\big\},$$

and

$$\tilde{\mathrm D}_y:=-\frac{1}{\mathfrak r^1_x}\big(\mathrm D_t+(\mathfrak r^1+\mathfrak r^2-1)\mathrm D_x\big),\quad \tilde{\mathrm D}_z:=-\frac{1}{\mathfrak r^2_x}\big(\mathrm D_t+(\mathfrak r^1+\mathfrak r^2+1)\mathrm D_x\big),$$
$$\tilde{\mathfrak J}:=\frac{\mathfrak r^1}{2}\tilde{\mathrm D}_y+\frac{\mathfrak r^2}{2}\tilde{\mathrm D}_z,\quad \tilde q:=\mathrm e^{(\mathfrak r^1-\mathfrak r^2)/2}\big(x-(\mathfrak r^1+\mathfrak r^2+1)t\big).$$

Proof. The space $\hat\Upsilon^{\rm q}$ coincides with the solution space of the system

$$\mathrm D_t\lambda^1+(\mathfrak r^1+\mathfrak r^2+1)\mathrm D_x\lambda^1=\mathfrak r^2_x(\lambda^2-\lambda^1)+\mathfrak r^3_x\lambda^3,\tag{3.17a}$$
$$\mathrm D_t\lambda^2+(\mathfrak r^1+\mathfrak r^2-1)\mathrm D_x\lambda^2=\mathfrak r^1_x(\lambda^1-\lambda^2)+\mathfrak r^3_x\lambda^3,\tag{3.17b}$$
$$\mathrm D_t\lambda^3+(\mathfrak r^1+\mathfrak r^2)\mathrm D_x\lambda^3+(\mathfrak r^1_x+\mathfrak r^2_x)\lambda^3=0,\tag{3.17c}$$

which is formal adjoint to the system (3.8) for generalized symmetries of (3.1). The substitution $(\lambda^1,\lambda^2,\lambda^3)=\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}(\tilde\lambda^1,\tilde\lambda^2,\tilde\lambda^3)$ reduces the system (3.17) to

$$\mathrm{D}_t\tilde\lambda^1+(\mathfrak{r}^1+\mathfrak{r}^2+1)\mathrm{D}_x\tilde\lambda^1=\mathfrak{r}^2_x(\tilde\lambda^1+\tilde\lambda^2)+\mathfrak{r}^3_x\tilde\lambda^3, \tag{3.18a}$$

$$\mathrm{D}_t\tilde\lambda^2+(\mathfrak{r}^1+\mathfrak{r}^2-1)\mathrm{D}_x\tilde\lambda^2=\mathfrak{r}^1_x(\tilde\lambda^1+\tilde\lambda^2)+\mathfrak{r}^3_x\tilde\lambda^3, \tag{3.18b}$$

$$\mathrm{D}_t\tilde\lambda^3+(\mathfrak{r}^1+\mathfrak{r}^2)\mathrm{D}_x\tilde\lambda^3=0. \tag{3.18c}$$

We again use the modified coordinates on $\mathcal{S}^{(\infty)}$. We will show below that the general solution of the system (3.18) can be represented in the form

$$\tilde\lambda^1=\tilde\lambda^{1\mathrm{h}}+\Omega,\quad \tilde\lambda^2=\tilde\lambda^{2\mathrm{h}}-\Omega,\quad \tilde\lambda^3=\frac{\hat{\mathcal{A}}\Omega}{\omega^1}, \tag{3.19}$$

where $\hat{\mathcal{A}}=\sum_{\kappa=0}^{\infty}\omega^{\kappa+1}\partial_{\omega^\kappa}$, Ω runs through the space of smooth functions of a finite, but unspecified number of ω's, and $(\tilde\lambda^{1\mathrm{h}},\tilde\lambda^{2\mathrm{h}})$ with $\tilde\lambda^{j\mathrm{h}}=\tilde\lambda^{j\mathrm{h}}[\mathfrak{r}^1,\mathfrak{r}^2]$, $j=1,2$, is the general solution of the subsystem (3.18a)–(3.18b) with $\tilde\lambda^3=0$,

$$\mathrm{D}_t\tilde\lambda^{1\mathrm{h}}+(\mathfrak{r}^1+\mathfrak{r}^2+1)\mathrm{D}_x\tilde\lambda^{1\mathrm{h}}=\mathfrak{r}^2_x(\tilde\lambda^{1\mathrm{h}}+\tilde\lambda^{2\mathrm{h}}),$$
$$\mathrm{D}_t\tilde\lambda^{2\mathrm{h}}+(\mathfrak{r}^1+\mathfrak{r}^2-1)\mathrm{D}_x\tilde\lambda^{2\mathrm{h}}=\mathfrak{r}^1_x(\tilde\lambda^{1\mathrm{h}}+\tilde\lambda^{2\mathrm{h}}).$$

The counterpart $(\lambda^{1\mathrm{h}},\lambda^{2\mathrm{h}})=\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}(\tilde\lambda^{1\mathrm{h}},\tilde\lambda^{2\mathrm{h}})$ of $(\tilde\lambda^{1\mathrm{h}},\tilde\lambda^{2\mathrm{h}})$ satisfies the subsystem (3.17a)–(3.17b) with $\lambda^3=0$,

$$\mathrm{D}_t\lambda^{1\mathrm{h}}+(\mathfrak{r}^1+\mathfrak{r}^2+1)\mathrm{D}_x\lambda^{1\mathrm{h}}=\mathfrak{r}^2_x(\lambda^{2\mathrm{h}}-\lambda^{1\mathrm{h}}), \tag{3.20a}$$

$$\mathrm{D}_t\lambda^{2\mathrm{h}}+(\mathfrak{r}^1+\mathfrak{r}^2-1)\mathrm{D}_x\lambda^{2\mathrm{h}}=\mathfrak{r}^1_x(\lambda^{1\mathrm{h}}-\lambda^{2\mathrm{h}}). \tag{3.20b}$$

Therefore, the triple $\lambda=(\lambda^1,\lambda^2,\lambda^3)$ belongs to $\hat\Upsilon^{\mathrm{q}}$ if and only if it can be represented, in the above notation, in the form

$$\lambda=\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\big(\Omega,-\Omega,(\hat{\mathcal{A}}\Omega)/\omega^1\big)+(\lambda^{1\mathrm{h}},\lambda^{2\mathrm{h}},0). \tag{3.21}$$

The substitution $(\lambda^{1\mathrm{h}},\lambda^{2\mathrm{h}})=\mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}(\hat\lambda^1,\hat\lambda^2)$ reduces the system (3.20) to the system

$$\mathcal{D}_t\hat{\lambda}^1 + (\mathfrak{r}^1 + \mathfrak{r}^2 + 1)\mathcal{D}_x\hat{\lambda}^1 = \mathfrak{r}^2_x\hat{\lambda}^2,$$
$$\mathcal{D}_t\hat{\lambda}^2 + (\mathfrak{r}^1 + \mathfrak{r}^2 - 1)\mathcal{D}_x\hat{\lambda}^2 = \mathfrak{r}^1_x\hat{\lambda}^1,$$

which can be rewritten in terms of the operators $\tilde{\mathcal{D}}_y$ and $\tilde{\mathcal{D}}_z$ as $\tilde{\mathcal{D}}_z\hat{\lambda}^1 = -\hat{\lambda}^2$, $\tilde{\mathcal{D}}_y\hat{\lambda}^2 = -\hat{\lambda}^1$. Therefore, both the components $\hat{\lambda}^1$ and $\hat{\lambda}^2$ satisfy the image of the equation (3.11a) under the transformation (3.7) and thus are the reduced forms of the pullbacks of characteristics of generalized vector fields from $\hat{\mathfrak{K}}^{\mathrm{q}}$ by this transformation. As a result, we obtain the families of cosymmetries of the system (3.1) that are presented in the theorem. The first and second summands in (3.21) correspond to the first family and the span of the second and the third families, respectively.

Now we prove the representation (3.19) by induction on the order $\mathrm{ord}_\omega(\tilde{\lambda}^1 - \tilde{\lambda}^2) \in \{-\infty\} \cup \mathbb{N}_0$. In view of Lemma 3.2, any solution of the equation (3.18c), which can be shortly rewritten as $\mathcal{B}\tilde{\lambda}^3 = 0$, is a smooth function of a finite number of ω's. We take the sum and the difference of the equations (3.18a) and (3.18b), additionally writing them, after multiplying by $\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}$, in terms of the operator $\mathcal{A}$ and $\mathcal{B}$,

$$\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathcal{B}(\tilde{\lambda}^1 + \tilde{\lambda}^2) + \mathcal{A}(\tilde{\lambda}^1 - \tilde{\lambda}^2) = (\tilde{\lambda}^1 + \tilde{\lambda}^2)\mathcal{A}(\mathfrak{r}^1 + \mathfrak{r}^2) + 2\omega^1\tilde{\lambda}^3, \tag{3.22a}$$
$$\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathcal{B}(\tilde{\lambda}^1 - \tilde{\lambda}^2) + \mathcal{A}(\tilde{\lambda}^1 + \tilde{\lambda}^2) = (\tilde{\lambda}^1 + \tilde{\lambda}^2)\mathcal{A}(\mathfrak{r}^2 - \mathfrak{r}^1). \tag{3.22b}$$

Base case. Let $\mathrm{ord}_\omega(\tilde{\lambda}^1 - \tilde{\lambda}^2) = -\infty$. The equation (3.22b) implies $\mathrm{ord}_\omega(\tilde{\lambda}^1 + \tilde{\lambda}^2) = -\infty$ as well, i.e., both $\tilde{\lambda}^1$ and $\tilde{\lambda}^2$ do not depend on ω's. Then we obtain from the equation (3.22a) that the summand $2\omega^1\tilde{\lambda}^3$ does not depend on ω's as well. Recalling that $\tilde{\lambda}^3$ depends at most on a finite number of ω's, we educe that $c := \omega^1\tilde{\lambda}^3$ is a constant, i.e., $\tilde{\lambda}^3 = c/\omega^1 = c\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}/\mathfrak{r}^3_x$. We substitute $(\lambda^1, \lambda^2) = \mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}(\hat{\lambda}^1, \hat{\lambda}^2)$ into the equations (3.18a) and (3.18b) and rewrite them in the notation of Section 3.2 as

$$\tilde{\mathcal{D}}_z\hat{\lambda}^1 = -\hat{\lambda}^2 + \frac{c}{2}\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}K^1, \quad \tilde{\mathcal{D}}_y\hat{\lambda}^2 = -\hat{\lambda}^1 - \frac{c}{2}\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}K^2.$$

We carry out the transformation (3.6) restricted to the spaces with the coordinates $(t, x, \mathfrak{r}^1, \mathfrak{r}^2)$ and (y, z, p, q) and then exclude derivatives of p in view of the equation (3.5) and its differential consequences. As a result, we derive the system

$$\mathrm{D}_z\breve\lambda^1=-\breve\lambda^2+\frac{c}{2}\mathrm{e}^{2y+2z}K^1,\quad \mathrm{D}_y\breve\lambda^2=-\breve\lambda^1-\frac{c}{2}\mathrm{e}^{2y+2z}K^2,\tag{3.23}$$

where the differential function $\breve\lambda^i=\breve\lambda^i[q]$ is the image of $\hat\lambda^i$ under the above transformation, $i=1,2$, see the notation in Section 3.3. We solve the first equation of (3.5) with respect to $\breve\lambda^2$ and substitute the obtained expression $\breve\lambda^2=-\mathrm{D}_z\breve\lambda^1+\frac12 c\mathrm{e}^{2y+2z}K^1$ into the second equation, deriving

$$\mathrm{D}_y\mathrm{D}_z\beta[q]=\beta[q]+c\mathrm{e}^{2y+2z}(q_{zz}+q_y-q_z-q).\tag{3.24}$$

with respect to $\beta:=\breve\lambda^1$. Therefore, the system (3.18) with $\tilde\lambda^3=c/\omega^1$ has a solution if and only if the equation (3.24) has. In this way, we reduce the proof in the base case to studying the existence of solutions of the equation (3.24).

Given a differential function $\alpha=\alpha[q]$ that is affine in totality of involved derivatives of q, any solution $\beta=\beta[q]$ of the equation $\mathrm{D}_y\mathrm{D}_z\beta=\beta+\alpha$ has the same property. Indeed, we fix an arbitrary solution β of (3.24) and substitute $q=q^0+\varepsilon_1q^1+\varepsilon_2q^2$ into it. Here ε_1 and ε_2 are constant parameters, and q^0, q^1 and q^2 are arbitrary solutions of the equation (3.4a), which is the (1+1)-dimensional Klein–Gordon equation for q in light-cone variables, $q^i_{yz}=q^i$, $i=0,1,2$. We take the mixed derivative of the equation for β with substituted q with respect to $(\varepsilon_1,\varepsilon_2)$ at $(\varepsilon_1,\varepsilon_2)=(0,0)$ to derive the equation

$$\sum_{\iota,\iota'=-\operatorname{ord}\beta}^{\operatorname{ord}\beta}\big(\mathrm{D}_y\mathrm{D}_z(\beta_{q_\iota q_{\iota'}}[q^0]q^1_\iota q^2_{\iota'})-\beta_{q_\iota q_{\iota'}}[q^0]q^1_\iota q^2_{\iota'}\big)=0,\tag{3.25}$$

which can be split with respect to $\{q_\iota,q_{\iota'},\iota,\iota'=-\operatorname{ord}\beta-1,\dots,\operatorname{ord}\beta+1\}$. Suppose that $\beta_{q_\iota q_{\iota'}}\neq0$ for some (ι,ι'). Let $\iota_0=\max\big\{\iota\mid\exists\iota'\colon\beta_{q_\iota q_{\iota'}}\neq0\big\}$ and $\iota_0'=\min\big\{\iota'\mid\beta_{q_{\iota_0}q_{\iota'}}\neq0\big\}$. Collecting the coefficients of $q^1_{\iota_0+1}q^2_{\iota_0'-1}$ in the equation (3.25) gives $\beta_{q_{\iota_0}q_{\iota_0'}}=0$ contradicting the inequality $\beta_{q_{\iota_0}q_{\iota_0'}}\neq0$. Therefore, $\beta_{q_\iota q_{\iota'}}=0$ for any $(\iota,\iota')\in\mathbb{Z}^2$.

In view of the claim proved in the previous paragraph, we can represent each fixed solution of the equation (3.24) in the form

$$\beta=\sum_{\iota=-n}^{n}\beta^\iota(y,z)q_\iota+\beta^{00}(y,z),$$

where $n := \operatorname{ord}\beta$, and the coefficients β^ι, $\iota = -n, \dots, n$, and β^{00} are smooth functions of (y, z). Without loss of generality, we can assume that $n > 2$ and $\beta^{00} = 0$. The equation (3.24) splits into the following system for the coefficients of β:

$$
\begin{aligned}
&\Delta_{-n-1}\colon\ \beta^{-n}_y = 0, \quad \Delta_{-n}\colon\ \beta^{-n+1}_y = 0,\\
&\Delta_{\kappa-n}\colon\ \beta^{\kappa-n-1}_z + \beta^{\kappa-n}_{yz} + \beta^{\kappa-n+1}_y = \alpha^{\kappa-n}, \quad \kappa = 1, \dots, 2n-1,\\
&\Delta_n\colon\ \beta^{n-1}_z = 0, \quad \Delta_{n+1}\colon\ \beta^n_z = 0,
\end{aligned}
$$

where $\alpha^{-2} = -\alpha^{-1} = -\alpha^0 = \alpha^1 = c\mathrm{e}^{2y+2z}$ and the other α^ι are zero. The equation Δ_ι is constituted by the coefficients of q_ι. For each $\kappa \in \{1, \dots, 2n-1\}$, we solve the equation $\Delta_{\kappa-n}$ with respect to $\beta^{\kappa-n+1}_y$, differentiate the result κ times with respect to y,

$$
\frac{\partial^{\kappa+1}\beta^{\kappa-n+1}}{\partial y^{\kappa+1}} = \frac{\partial^\kappa\alpha^{\kappa-n}}{\partial y^\kappa} - \frac{\partial^{\kappa+1}\beta^{\kappa-n-1}}{\partial y^\kappa\partial z} - \frac{\partial^{\kappa+2}\beta^{\kappa-n}}{\partial y^{\kappa+1}\partial z}
$$

and substitute for the last two derivatives in view of differential consequences of the previous equations. As a result, we obtain

$$
\frac{\partial^{\kappa+1}\beta^{\kappa-n+1}}{\partial y^{\kappa+1}} = c_\kappa\mathrm{e}^{2y+2z},
$$

where $c_\kappa = 0$, $\kappa = 1, \dots, n-3$, $c_{n-2} = 2^{n-2}c$, $c_{n-1} = -3 \cdot 2^{n-1}c$, $c_n = 2^{n+2}c$, and $c_{n+1} = -2^{n+3}c$.

We can prove by induction on κ that $c_\kappa = (-1)^{\kappa-n}2^{\kappa+2}c$, $\kappa = n+1, \dots, 2n-1$. The base case $\kappa = n$ is given by the above equality $c_{n+1} = -2^{n+3}c$, and the induction step follows from the equality $c_{\kappa+1} = -4(c_\kappa + c_{\kappa-1})$ for $\kappa > n+1$. Therefore, the equation Δ_{n+1}: $\beta^n_z = 0$ implies that $c = 0$, and we obtain the representation (3.19) with $\Omega = 0$.

Induction step. Suppose that the representation (3.19) holds if $\operatorname{ord}_\omega(\tilde\lambda^1 - \tilde\lambda^2) < \kappa \in \mathbb{N}_0$ and prove this representation for $\operatorname{ord}_\omega(\tilde\lambda^1 - \tilde\lambda^2) = \kappa$. In view of the equation (3.22b), under the last condition we have $\operatorname{ord}_\omega(\tilde\lambda^1 + \tilde\lambda^2) < \kappa$. Then the equation (3.22a) implies that $\operatorname{ord}_\omega(\omega^1\tilde\lambda^3) = \kappa + 1$. Differentiating the equation (3.22a) with respect to $\omega^{\kappa+1}$, we derive $(\tilde\lambda^1 - \tilde\lambda^2)_{\omega^\kappa} = 2(\omega^1\tilde\lambda^3)_{\omega^{\kappa+1}}$, and thus both the left and the right hand sides of the last equality depends at most on $\omega^0, \ldots, \omega^\kappa$, i.e., there exists a smooth function $\Upsilon = \Upsilon(\omega^0, \ldots, \omega^\kappa)$ such that

$$(\tilde\lambda^1 - \tilde\lambda^2)_{\omega^\kappa} = 2(\omega^1\tilde\lambda^3)_{\omega^{\kappa+1}} = 2\Upsilon.$$

Let $\breve\Upsilon = \breve\Upsilon(\omega^0, \ldots, \omega^\kappa)$ be a fixed antiderivative of Υ with respect to ω^κ, $\breve\Upsilon_{\omega^\kappa} = \Upsilon$. Define

$$\breve\lambda^1 := \tilde\lambda^1 - \breve\Upsilon, \quad \breve\lambda^2 := \tilde\lambda^2 + \breve\Upsilon, \quad \breve\lambda^3 := \tilde\lambda^3 - \frac{\hat{\mathscr{A}}\breve\Upsilon}{\omega^1}.$$

The tuple $(\breve\lambda^1, \breve\lambda^2, \breve\lambda^3)$ satisfies the system (3.18), and

$$(\breve\lambda^1 - \breve\lambda^2)_{\omega^\kappa} = (\tilde\lambda^1 - \tilde\lambda^2)_{\omega^\kappa} - 2\Upsilon = 0,$$

i.e., $\breve\kappa := \mathrm{ord}_\omega(\breve\lambda^1 - \breve\lambda^2) < \kappa$. By the induction hypothesis, this tuple can be represented in the form (3.19) with some smooth function $\breve\Omega = \breve\Omega(\omega^0, \ldots, \omega^{\breve\kappa})$. Setting $\Omega = \breve\Omega + \breve\Upsilon$, we derive the representation (3.19) for $(\tilde\lambda^1, \tilde\lambda^2, \tilde\lambda^3)$ with the same $\lambda^{1\mathrm{h}}$ and $\lambda^{2\mathrm{h}}$. □

Remark 3.14. The first and the second families from Theorem 3.13, which are linear spaces, are not disjoint in the sense of linear spaces. Their intersection is one-dimensional and is spanned by the cosymmetry $\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}(1, -1, 0)$ corresponding to $\Omega = 1$ and $\Phi = -\mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}$. The span of these two families has the zero intersection with the span of the third family.

3.6 Conservation laws

Theorem 3.15. *The space of conservation laws of the system* (3.1) *is naturally isomorphic to the space spanned by the following conserved currents of this system:*

1. $\big(\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\Omega,\ (\mathfrak{r}^1+\mathfrak{r}^2)\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\Omega\big)$, *where the parameter function* Ω *runs through the space of smooth functions of a finite, but unspecified number of* $\omega^\kappa = (\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathrm{D}_x)^\kappa\mathfrak{r}^3$, $\kappa \in \mathbb{N}_0$, *and such two functions should be assumed equivalent if their difference belongs to the image of the operator* $\hat{\mathscr{A}} = \sum_{\kappa=0}^{\infty}\omega^{\kappa+1}\partial_{\omega^\kappa}$.

2. $\big(\mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}(2\Phi_{\mathfrak{r}^1}+\Phi),\ \mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}(2(\mathfrak{r}^1+\mathfrak{r}^2+1)\Phi_{\mathfrak{r}^1}+(\mathfrak{r}^1+\mathfrak{r}^2-1)\Phi)\big)$, *where the parameter function* $\Phi=\Phi(\mathfrak{r}^1,\mathfrak{r}^2)$ *runs through the solution space of the Klein–Gordon equation* $\Phi_{\mathfrak{r}^1\mathfrak{r}^2}=-\Phi/4$.

3. $\big(\mathfrak{r}^2_x\tilde\rho+\mathfrak{r}^1_x\tilde\sigma,\ (\mathfrak{r}^1+\mathfrak{r}^2-1)\mathfrak{r}^2_x\tilde\rho+(\mathfrak{r}^1+\mathfrak{r}^2+1)\mathfrak{r}^1_x\tilde\sigma\big)$ *with* $\tilde\rho=-\tilde q\tilde{\mathcal{D}}_z\tilde{\mathfrak{X}}\tilde q$, $\tilde\sigma=(\tilde{\mathcal{D}}_y\tilde q)\tilde{\mathfrak{X}}\tilde q$, *where the operator* $\tilde{\mathfrak{X}}$ *runs through the set*

$$\big\{\,\tilde{\mathcal{J}}^{\kappa'},\ \kappa'\in2\mathbb{N}_0+1,\ (\tilde{\mathcal{J}}+\iota/2)^{\kappa}\tilde{\mathcal{D}}_y^{\iota},\ (\tilde{\mathcal{J}}-\iota/2)^{\kappa}\tilde{\mathcal{D}}_z^{\iota},\ \kappa\in\mathbb{N}_0,\ \iota\in\mathbb{N},\ \kappa+\iota\in2\mathbb{N}_0+1\,\big\},$$

and

$$\tilde{\mathcal{D}}_y:=-\frac{1}{\mathfrak{r}^1_x}\big(\mathcal{D}_t+(\mathfrak{r}^1+\mathfrak{r}^2-1)\mathcal{D}_x\big),\quad \tilde{\mathcal{D}}_z:=-\frac{1}{\mathfrak{r}^2_x}\big(\mathcal{D}_t+(\mathfrak{r}^1+\mathfrak{r}^2+1)\mathcal{D}_x\big),$$
$$\tilde{\mathcal{J}}:=\frac{\mathfrak{r}^1}{2}\tilde{\mathcal{D}}_y+\frac{\mathfrak{r}^2}{2}\tilde{\mathcal{D}}_z,\quad \tilde q:=\mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\big(x-(\mathfrak{r}^1+\mathfrak{r}^2+1)t\big).$$

Proof. We compute the space of local conservation laws of the system (3.1) combining the direct method of finding conservation laws [128, 166], which is based on the definition of conserved currents, with using the linearization of the essential subsystem (3.1a)–(3.1b) to the (1+1)-dimensional Klein–Gordon equation. Up to the equivalence of conserved currents, meaning that they coincide on the solution set of the corresponding system of differential equations, it suffices to consider only reduced conserved currents of the system (3.1), which are of the form (ρ,σ), where $\rho=\rho[\mathfrak{r}]$ and $\sigma=\sigma[\mathfrak{r}]$. A tuple $(\rho[\mathfrak{r}],\sigma[\mathfrak{r}])$ is a conserved current of the system (3.1) if and only if $\mathcal{D}_t\rho+\mathcal{D}_x\sigma=0$. We should also take into account the equivalence of conserved currents up to adding null divergences, which means that conserved currents $(\rho[\mathfrak{r}],\sigma[\mathfrak{r}])$ and $(\rho'[\mathfrak{r}],\sigma'[\mathfrak{r}])$ belong to the same conservation law if and only if there exists a differential function $f=f[\mathfrak{r}]$ such that $\rho'=\rho+\mathcal{D}_xf$ and $\sigma'=\sigma-\mathcal{D}_tf$.

We associate an arbitrary reduced conserved current $(\rho[\mathfrak{r}],\sigma[\mathfrak{r}])$ of the system (3.1) with the modified density $\breve\rho:=\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\rho$ and the modified flux $\breve\sigma=\sigma-(\mathfrak{r}^1+\mathfrak{r}^2)\rho$, i.e.,

$$(\rho,\sigma)=\big(\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\breve\rho,(\mathfrak{r}^1+\mathfrak{r}^2)\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\breve\rho+\breve\sigma\big),$$

and $\mathcal{D}_t\rho+\mathcal{D}_x\sigma=\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}(\mathcal{B}\breve\rho+\mathcal{A}\breve\sigma)$. Therefore, the equality $\mathcal{D}_t\rho+\mathcal{D}_x\sigma=0$ for conserved currents is equivalent to the equality $\mathcal{B}\breve\rho+\mathcal{A}\breve\sigma=0$ for modified conserved currents, and the equivalence of conserved currents up to adding a null divergence is modified to $\breve\rho'=\breve\rho+\mathcal{A}f$ and $\breve\sigma'=\breve\sigma-\mathcal{B}f$.

Fixing a reduced conserved current $(\rho[\mathfrak{r}],\sigma[\mathfrak{r}])$ and using the modified coordinates on $\mathcal{S}^{(\infty)}$, we define $\kappa:=\max(\operatorname{ord}_\omega\breve\rho,\operatorname{ord}_\omega\breve\sigma)$ and prove by mathematical induction with respect to $\kappa\in\{-\infty\}\cup\mathbb{N}_0$ that up to adding a modified null divergence we have the representation $\breve\rho=\breve\rho^1[r^1,r^2]+\breve\rho^0(\omega^0,\dots,\omega^\kappa)$ for some differential functions $\breve\rho^0=\breve\rho^0(\omega^0,\dots,\omega^\kappa)$ and $\breve\rho^1=\breve\rho^1[r^1,r^2]$, and $\breve\sigma=\breve\sigma[r^1,r^2]$.

The base case $\kappa=-\infty$ is obvious.

For the inductive step, we fix $\kappa\in\mathbb{N}_0$, suppose that the above claim is true for all $\kappa'<\kappa$ and prove it for κ. Collecting coefficients of $\omega^{\kappa+1}$ in the equality $\mathcal{B}\breve\rho+\mathcal{A}\breve\sigma=0$, we derive $\breve\sigma_{\omega^\kappa}=0$, i.e., in fact $\operatorname{ord}_\omega\breve\sigma<\kappa$. Then we differentiate the same equality twice with respect to ω^κ, which leads to $\mathcal{B}\breve\rho_{\omega^\kappa\omega^\kappa}=0$. In view of Lemma 3.2, this means that the $\breve\rho_{\omega^\kappa\omega^\kappa}$ can depend at most on $(\omega^0,\dots,\omega^\kappa)$. Therefore, there exist differential functions $\breve\rho^{10}=\breve\rho^{10}(\omega^0,\dots,\omega^\kappa)$, $\breve\rho^{11}=\breve\rho^{11}[\mathfrak{r}]$ and $\breve\rho^{12}=\breve\rho^{12}[\mathfrak{r}]$ such that $\operatorname{ord}_\omega\breve\rho^{11}<\kappa$, $\operatorname{ord}_\omega\breve\rho^{12}<\kappa$ and $\breve\rho=\breve\rho^{12}[\mathfrak{r}]\omega^\kappa+\breve\rho^{11}[\mathfrak{r}]+\breve\rho^{10}(\omega^0,\dots,\omega^\kappa)$. Since $\mathcal{B}\breve\rho^{10}=0$, the tuple $(\breve\rho^{10},0)$ is a modified conserved current of the system (3.1). Hence the tuple $(\breve\rho^{12}\omega^\kappa+\breve\rho^{11},\breve\sigma)$ is a modified conserved current of this system as well. Adding the modified null divergence $(-\mathcal{A}\int\breve\rho^{12}\,\mathrm{d}\omega^{\kappa-1},\mathcal{B}\int\breve\rho^{12}\,\mathrm{d}\omega^{\kappa-1})$ to the latter modified conserved current, we obtain an equivalent modified conserved current $(\breve\rho',\breve\sigma')$ with $\max(\operatorname{ord}_\omega\breve\rho',\operatorname{ord}_\omega\breve\sigma')<\kappa$. The induction hypothesis implies that up to adding a modified null divergence, the component $\breve\rho'$ admits the representation $\breve\rho'=\breve\rho^{21}[r^1,r^2]+\breve\rho^{20}(\omega^0,\dots,\omega^\kappa)$ for some differential functions $\breve\rho^{20}=\breve\rho^{20}(\omega^0,\dots,\omega^\kappa)$ and $\breve\rho^{21}=\breve\rho^{21}[r^1,r^2]$, and $\breve\sigma'=\breve\sigma'[r^1,r^2]$. Setting $\breve\rho^0=\breve\rho^{10}+\breve\rho^{20}$, $\breve\rho^1=\breve\rho^{21}$ and $\breve\sigma=\breve\sigma'$, we complete the inductive step.

In other words, we have proved that up to adding a null divergence, any conserved current of the system (3.1) can be represented as the sum of a conserved current from the first theorem's family and of a conserved current of the form $(\rho[r^1,r^2],\sigma[r^1,r^2])$. The subspace of conserved currents of the latter forms is the pullback of the space of reduced conserved currents of the essential subsystem (3.1a)–(3.1b) by the projection $(t,x,\mathfrak{r})\to$

$(t,x,\mathfrak{r}^1,\mathfrak{r}^2)$; cf. [80, Proposition 3]. The latter space is naturally isomorphic to the space of conservation laws of the essential subsystem (3.1a)–(3.1b), which is the pullback of the space of conservation laws of the Klein–Gordon equation (3.4a) with respect to the composition of the restriction of the transformation (3.6) to the space with coordinates $(t,x,\mathfrak{r}^1,\mathfrak{r}^2)$ (i.e., the s-component of this transformation should be neglected) with the projection $(y,z,q,p)\to(y,z,q)$. We take the space of conservation laws of the (1+1)-dimensional Klein–Gordon equation, which was constructed in Section 2.4, and perform the above pullbacks,

$$\rho=-\frac{1}{2}(\mathfrak{r}^2_x\tilde\rho_{\rm KG}+\mathfrak{r}^1_x\tilde\sigma_{\rm KG}),\quad \sigma=-\frac{1}{2}(V^2\mathfrak{r}^2_x\tilde\rho_{\rm KG}+V^1\mathfrak{r}^1_x\tilde\sigma_{\rm KG}),$$

where $\tilde\rho_{\rm KG}$ and $\tilde\sigma_{\rm KG}$ are, as differential functions, the pullbacks of the density $\rho_{\rm KG}$ and the flux $\sigma_{\rm KG}$ of a conserved current of (3.4a), respectively; see [128, Section III] or [130, Proposition 1]. As a result, we obtain, up to the equivalence on solutions of the system (3.1) and up to rescaling of conserved currents, the other families of the conserved currents of this system that are presented in the theorem.

More specifically, the equation (3.4a) is the Euler–Lagrange equation for the Lagrangian $\mathrm{K}=-(q_yq_z+q^2)/2$. Hence characteristics of generalized symmetries of this equation are also its cosymmetries, and vice versa. The quotient algebra $\mathfrak{K}^{\rm q}=\mathfrak{K}/\mathfrak{K}^{\rm triv}$ of generalized symmetries of (3.4a), where $\mathfrak{K}$ and $\mathfrak{K}^{\rm triv}$ are the algebra (of evolutionary representatives) of generalized symmetries of the Lagrangian (3.4a) and its ideal of trivial generalized symmetries, is naturally isomorphic to the algebra $\tilde{\mathfrak{K}}^{\rm q}=\tilde\Lambda^{\rm q}\Subset\tilde{\mathfrak{K}}^{-\infty}$, where

$$\tilde\Lambda^{\rm q}:=\big\langle\,(\mathrm{J}^\kappa q)\partial_q,\,(\mathrm{J}^\kappa\mathrm{D}_y^\iota q)\partial_q,\,(\mathrm{J}^\kappa\mathrm{D}_z^\iota q)\partial_q,\,\kappa\in\mathbb{N}_0,\,\iota\in\mathbb{N}\big\rangle$$

is a subalgebra and $\tilde{\mathfrak{K}}^{-\infty}:=\{f(y,z)\partial_q\mid f\in\mathrm{KG}\}$ is an abelian ideal, see Theorem 2.4. Here D_y and D_z are the operators of total derivatives in y and z, respectively, and $\mathrm{J}:=y\mathrm{D}_y-z\mathrm{D}_z$. Denote by Υ, $\Upsilon^{\rm triv}$ and $\Upsilon^{\rm q}$ the algebra (of evolutionary representatives) of variational symmetries of the Lagrangian K, its ideal of trivial variational symmetries and the quotient algebra of variational symmetries of this Lagrangian, i.e., $\Upsilon\subset\mathfrak{K}$, $\Upsilon^{\rm triv}:=$

$\Upsilon\cap\mathfrak{K}^{\rm triv}$ and $\Upsilon^{\rm q}:=\Upsilon/\Upsilon^{\rm triv}$. The quotient algebra $\Upsilon^{\rm q}$ is naturally isomorphic to the algebra $\tilde\Upsilon^{\rm q}=\tilde\Lambda^{\rm q}_-\Subset\tilde\Sigma^{-\infty}$, where

$$\tilde\Lambda^{\rm q}_-:=\big\langle\,(\mathfrak{X}_{\kappa'0}q)\partial_q,\ \kappa'\in2\mathbb{N}_0{+}1,\ \ (\mathfrak{X}_{\kappa\iota}q)\partial_q,\ (\bar{\mathfrak{X}}_{\kappa\iota}q)\partial_q,\ \kappa\in\mathbb{N}_0,\ \iota\in\mathbb{N},\ \kappa{+}\iota\in2\mathbb{N}_0{+}1\,\big\rangle$$

with

$$\mathfrak{X}_{\kappa\iota}=\Big({\rm J}+\frac{\iota}{2}\Big)^\kappa{\rm D}_y^\iota,\quad \kappa,\iota\in\mathbb{N}_0,\qquad \bar{\mathfrak{X}}_{\kappa\iota}=\Big({\rm J}-\frac{\iota}{2}\Big)^\kappa{\rm D}_z^\iota,\quad \kappa\in\mathbb{N}_0,\quad \iota\in\mathbb{N},$$

is the subspace of $\tilde\Lambda^{\rm q}$ that is associated with the space of formally skew-adjoint differential operators generated by ${\rm D}_y$, ${\rm D}_z$ and J. Note that in the context of Noether's theorem, we need to consider the algebra $\tilde{\mathfrak{K}}^{\rm q}$ instead of the algebra $\hat{\mathfrak{K}}^{\rm q}$ of reduced generalized symmetries of (3.4a), which is mentioned in Section 3.4, since cosets of $\Upsilon^{\rm triv}$ in Υ do not necessarily intersect the algebra $\hat{\mathfrak{K}}^{\rm q}$, see Remark 2.9. The space of conservation laws of (3.4a) is naturally isomorphic to the space spanned by the conserved currents

$$\bar{\rm C}^0_f=(-f_zq,fq_y),\quad {\rm C}_{\mathfrak{X}}=(-q{\rm D}_z\mathfrak{X}q,\,q_y\mathfrak{X}q),$$

where the parameter function $f=f(y,z)$ runs through the solution set of (3.4a), and the operator $\mathfrak{X}$ runs through the basis of $\tilde\Lambda^{\rm q}_-$, see Proposition 2.10. The conserved current $\bar{\rm C}^0_f$ is equivalent to the conserved current ${\rm C}^0_f=(fq_z,-f_yq)$.

We map conserved currents of the form ${\rm C}_{\mathfrak{X}}$, where $\mathfrak{X}q\partial_q$ runs through the basis of $\tilde\Lambda^{\rm q}_-$, to conserved currents of the system (3.1), which leads to the third family of the theorem. Possible modifications of the form of these conserved currents up to recombining them and adding null divergences are discussed in Remark 3.23 below.

At the same time, it is convenient to modify conserved currents of the form $\bar{\rm C}^0_f$ before their mapping in order to directly obtain hydrodynamic conservation laws.[5] We reparameterize these conserved currents, representing the parameter function f in the form $f=\bar f_y+\bar f_z+2\bar f$, where the function $\bar f=\bar f(y,z)$ also runs through the solution set of the (1+1)-dimensional Klein–Gordon equation (3.4a). Then $f_z=\bar f_{zz}+2\bar f_z+\bar f$. Adding the

[5]Recall that a conservation law is called *hydrodynamic* if its density ρ is a function of dependent variables only.

null divergence $(\mathrm{D}_z R, -\mathrm{D}_y R)$ with $R := \bar f q_z - \bar f_z q - 2\bar f q$ to $-\bar{\mathrm{C}}^0_f$, we obtain the equivalent conserved current $(\bar f K_1, -\bar f_y K_2)$, which is mapped to the conserved current from the second family with $\Phi = \bar f(\mathfrak{r}^1/2, -\mathfrak{r}^2/2)$.

Note that the first and second theorem's families are in fact subspaces in the space of conserved currents of the system (3.1). Analyzing the equivalence of modified conserved currents, we see that conserved currents from the first theorem's family are equivalent if and only if the difference of corresponding Ω's belongs to the image of the operator $\hat{\mathcal{A}} = \sum_{\kappa=0}^{\infty} \omega^{\kappa+1}\partial_{\omega^\kappa}$. The intersection of the first and the second families is one-dimensional and spanned by the conserved current $\left(\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}, (\mathfrak{r}^1+\mathfrak{r}^2)\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\right)$. The sum of these two families does not intersect the span of the third family. The equivalence of conserved currents within the span of all the three families is generated by the equivalence of conserved currents within the first family. □

Remark 3.16. The kernel $\ker \mathsf{E}$ of the operator $\mathsf{E} = \sum_{\kappa=1}^{\infty}\sum_{\kappa'=0}^{\kappa-1} \omega^{\kappa-\kappa'}(-\hat{\mathcal{A}})^{\kappa'}\partial_{\omega^\kappa} - 1$ is contained in the kernel $\ker \mathsf{E}'$ of the operator $\mathsf{E}' = \sum_{\kappa=0}^{\infty}(-\hat{\mathcal{A}})^{\kappa}\partial_{\omega^\kappa}$, $\ker \mathsf{E} \subset \ker \mathsf{E}'$, since the operator identity $\hat{\mathcal{A}}\mathsf{E} = -\omega^1\mathsf{E}'$ holds. In view of [103, Theorem 4.26], Theorem 3.18 below implies that (locally) the image of the operator $\hat{\mathcal{A}}$ coincides with $\ker \mathsf{E} \cap \ker \mathsf{E}' = \ker \mathsf{E}$. The kernel $\ker \mathsf{E}'$ of E' is spanned by the constant function 1 and the image of $\hat{\mathcal{A}}$. Hence $\operatorname{im} \hat{\mathcal{A}} = \ker \mathsf{E} \subsetneq \ker \mathsf{E}'$.

Remark 3.17. The conserved currents from Theorem 3.15 that are associated with

$$\begin{aligned}
&\Omega = \frac{\mathfrak{r}^3}{\mathfrak{r}^3+1}, \quad \Omega = \frac{1}{\mathfrak{r}^3+1}, \quad \Omega = 1,\\
&\Phi = \mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}(\mathfrak{r}^1+\mathfrak{r}^2-1), \quad \Phi = \frac{1}{8}\mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\left((\mathfrak{r}^1+\mathfrak{r}^2)^2 - 4\mathfrak{r}^2\right)
\end{aligned}$$

correspond to the conservation of masses of the both individual phases and of mixture mass as well as the conservation of mixture momentum and of energy in the drift flux model, respectively, cf. [73, Chapter 13]. The related equations in conserved form are

$$\begin{aligned}
&\rho^1_t + (\rho^1 u)_x = 0, \quad \rho^2_t + (\rho^2 u)_x = 0, \quad (\rho^1+\rho^2)_t + \big((\rho^1+\rho^2)u\big)_x = 0,\\
&\big((\rho^1+\rho^2)u\big)_t + \big((\rho^1+\rho^2)(u^2+1)\big)_x = 0,
\end{aligned}$$

$$\left((\rho^1+\rho^2)\left(\frac{u^2}{2}+\ln(\rho^1+\rho^2)\right)\right)_t+\left((\rho^1+\rho^2)\left(\frac{u^2}{2}+\ln(\rho^1+\rho^2)+1\right)u\right)_x=0.$$

In particular, the magnitude $\ln(\rho^1+\rho^2)$ can be interpreted as (proportional to) the internal mixture energy. The first, second and fourth equations constitute the conserved form of the system $\mathcal{S}$ in the original variables (ρ^1,ρ^2,u).

Theorem 3.18. *In the notation of Theorem 3.15, the associated reduced conservation-law characteristics of the system* (3.1) *are respectively*

1. $\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\left(\Omega-\sum\limits_{\kappa=1}^{\infty}\sum\limits_{\kappa'=0}^{\kappa-1}\omega^{\kappa-\kappa'}(-\hat{\mathcal{A}})^{\kappa'}\Omega_{\omega^\kappa},\ \sum\limits_{\kappa=1}^{\infty}\sum\limits_{\kappa'=0}^{\kappa-1}\omega^{\kappa-\kappa'}(-\hat{\mathcal{A}})^{\kappa'}\Omega_{\omega^\kappa}-\Omega,\ \sum\limits_{\kappa=0}^{\infty}(-\hat{\mathcal{A}})^{\kappa}\Omega_{\omega^\kappa}\right).$
2. $\mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\big(2\Phi_{\mathfrak{r}^1\mathfrak{r}^1}+2\Phi_{\mathfrak{r}^1}+\tfrac12\Phi,\ \Phi_{\mathfrak{r}^2}-\Phi_{\mathfrak{r}^1}-\Phi,\ 0\big).$
3. $\mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\big(-\tilde{\mathfrak{D}}_y\tilde{\mathfrak{X}}\tilde{q},\ \tilde{\mathfrak{X}}\tilde{q},\ 0\big).$[6]

The space spanned by these characteristics is naturally isomorphic to the quotient space of conservation-law characteristics of the system (3.1).

Proof. Since the system (3.1) is a system of evolution equations, its conservation-law characteristics can be found from reduced densities of the associated conservation laws by acting the Euler operator,

$$\mathsf{E}=\left(\sum_{\kappa=0}^{\infty}(-\mathrm{D}_x)^\kappa\partial_{\mathfrak{r}^i_\kappa},\ i=1,2,3\right),$$

see e.g. [152, Proposition 7.41]. This perfectly works for characteristics related to the second family of conserved currents presented in Theorem 3.15 but since characteristics related to the first and third families are not in $\mathfrak{r}$'s coordinates, while the Euler operator is, it is better to use different methods.

Characteristics related to the third family can be obtained from conservation-law characteristics of the (1+1)-dimensional Klein–Gordon equation (3.4a). A characteristic of the conservation law of (3.4a) containing the conserved current $\mathrm{C}_{\mathfrak{X}}$ is $\lambda=(\mathfrak{X}-\mathfrak{X}^\dagger)q=2\mathfrak{X}q$ for $(\mathfrak{X}q)\partial_q\in\tilde{\Lambda}^{\mathrm{q}}_{-}$. It is trivially prolonged to the conservation-law characteristic $(\lambda,0)$ of the

[6]Here we omitted the multiplier -2, which is needed for the direct correspondence between these conservation-law characteristics and conserved currents from the third family of Theorem 3.15.

system (3.4a), (3.5). Denote by R^1, R^2, L^1 and L^2 the differential functions associated with the equations (3.1a), (3.1b), (3.4a) and (3.5), respectively,

$$
\begin{aligned}
&R^1 := \mathfrak{r}^1_t + (\mathfrak{r}^1 + \mathfrak{r}^2 + 1)\mathfrak{r}^1_x, \quad R^2 := \mathfrak{r}^2_t + (\mathfrak{r}^1 + \mathfrak{r}^2 - 1)\mathfrak{r}^2_x,\\
&L^1 := q_{yz} - q, \quad L^2 := p - \frac{1}{2}\mathrm{e}^{-y-z}(q_z - q).
\end{aligned}
$$

These differential functions are related via the transformation $\mathcal{T}$, namely $\hat{\mathcal{T}}^*(R^1, R^2)^\mathsf{T} = \mathfrak{M}(L^1, L^2)^\mathsf{T}$ with

$$
\mathfrak{M} = \begin{pmatrix} 0 & -\dfrac{4}{\Delta} \\ \dfrac{2}{\Delta}\mathrm{e}^{-y-z} & \dfrac{4}{\Delta}(\mathrm{D}_y + 1) \end{pmatrix}, \quad \text{and} \quad \mathfrak{M}^\dagger = \begin{pmatrix} 0 & \dfrac{2}{\Delta}\mathrm{e}^{-y-z} \\ -\dfrac{4}{\Delta} & -(\mathrm{D}_y - 1)\circ\dfrac{4}{\Delta} \end{pmatrix},
$$

where $\Delta = (\mathrm{D}_y\hat{\mathcal{T}}^t)(\mathrm{D}_z\hat{\mathcal{T}}^x) - (\mathrm{D}_z\hat{\mathcal{T}}^t)(\mathrm{D}_y\hat{\mathcal{T}}^x)$, $\mathcal{T}^*\Delta = -4(\mathfrak{r}^1_t\mathfrak{r}^2_x - \mathfrak{r}^1_x\mathfrak{r}^2_t)$. The conservation-law characteristic (λ^1, λ^2) of the system (3.1a), (3.1b) that is associated with the conservation-law characteristic $(\lambda, 0)$ of the system (3.4a), (3.5) is defined by $\mathfrak{M}^\dagger(\Delta\hat{\mathcal{T}}^*\lambda^1, \Delta\hat{\mathcal{T}}^*\lambda^2)^\mathsf{T} = (\lambda, 0)^\mathsf{T}$. Therefore, the conservation-law characteristic λ of (3.4a) is mapped to the conservation-law characteristic $\frac{1}{2}\mathrm{e}^{y+z}(-\mathcal{D}_y\lambda, \lambda, 0)$ of the system $\mathcal{S}$, where all values should be expressed in terms of the variables $(t, x, \mathfrak{r})$. This gives a conservation-law characteristic from the third family of the theorem.

Characteristics related to the first family are found following the procedure of defining them via the formal integration by parts, cf. [103, p. 266]. We denote by A and B the counterparts of the operators $\mathcal{A}$ and $\mathcal{B}$, respectively, in the complete total derivative operators with respect to t and x, $\mathrm{A} := \mathrm{e}^{\mathfrak{r}^2 - \mathfrak{r}^1}\mathrm{D}_x$, $\mathrm{B} := \mathrm{D}_t + (\mathfrak{r}^1 + \mathfrak{r}^2)\mathrm{D}_x$. Then

$$
\mathrm{D}_t(\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\Omega) + \mathrm{D}_x\big((\mathfrak{r}^1+\mathfrak{r}^2)\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\Omega\big) = \mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\Omega\mathcal{E}^1 - \mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\Omega\mathcal{E}^2 + \sum_{\kappa=0}^{\infty}\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\Omega_{\omega^\kappa}\mathrm{B}\omega^\kappa. \quad (3.26)
$$

Here $\mathcal{E}^k$ denotes the left-hand side of the kth equation of the system (3.1), $\mathcal{E}^k = \mathfrak{r}^k_t + V^k\mathfrak{r}^k_x$, $k = 1, 2, 3$. Note that $\mathcal{E}^3 = \mathrm{B}\mathfrak{r}^3$. Since Ω depends on a finite number of ω's, there is no issue with convergence.

We derive using the mathematical induction with respect to ι that

$$\mathrm{B}\omega^{\kappa}=\mathrm{A}^{\kappa}\mathcal{E}^3+\sum_{\kappa'=0}^{\kappa-1}\mathrm{A}^{\kappa'}\big(\omega^{\kappa-\kappa'}(\mathcal{E}^2-\mathcal{E}^1)\big). \tag{3.27}$$

Indeed, for the base case $\kappa=0$, we have $\mathrm{B}\omega^0=\mathrm{B}\mathfrak{r}^3=\mathcal{E}^3$. The induction step follows from the equality $\mathrm{B}\omega^{\kappa+1}=\mathrm{BA}\omega^{\kappa}=\mathrm{AB}\omega^{\kappa}+\omega^{\kappa+1}(\mathcal{E}^2-\mathcal{E}^1)$.

Using again the mathematical induction with respect to κ, we prove the counterpart of the Lagrange identity in terms of the operator A,

$$\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}F\mathrm{A}^{\kappa}G=\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\big((-\mathrm{A})^{\kappa}F\big)G+\mathrm{D}_x\sum_{\kappa'=0}^{\kappa-1}\big((-\mathrm{A})^{\kappa'}F\big)\mathrm{A}^{\kappa-\kappa'-1}G,\quad \kappa\in\mathbb{N}_0,$$

for any differential functions F and G of $\mathfrak{r}$. We apply this identity to each summand of the expression $\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\Omega_{\omega^\kappa}\mathrm{B}\omega^{\kappa}$ expanded in view of (3.6), which gives

$$\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\Omega_{\omega^\kappa}\mathrm{B}\omega^{\kappa}=\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\big((-\mathrm{A})^{\kappa}\Omega_{\omega^\kappa}\big)\mathcal{E}^3+\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\sum_{\kappa'=0}^{\kappa-1}\big((-\mathrm{A})^{\kappa'}\Omega_{\omega^\kappa}\big)\omega^{\kappa-\kappa'}(\mathcal{E}^2-\mathcal{E}^1)+\mathrm{D}_xH,$$

where H is a differential function of $\mathfrak{r}$ that vanishes on the manifold $\mathcal{S}^{(\infty)}$ and whose precise form is not essential. When acting on functions of ω's, the operator A can be replaced by the operator $\hat{\mathcal{A}}=\sum_{\kappa=0}^{\infty}\omega^{\kappa+1}\partial_{\omega^\kappa}$. Substituting the derived expression for $\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\Omega_{\omega^\kappa}\mathrm{B}\omega^{\kappa}$ into (3.26) and collecting coefficients of $\mathcal{E}^1$, $\mathcal{E}^2$ and $\mathcal{E}^3$, we obtain a characteristic from the first family of the theorem. □

Remark 3.19. Since the common element $\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}(1,-1,0)$ of cosymmetry families, which is mentioned in Remark 3.14, is a conservation-law characteristic of the system $\mathcal{S}$, it was expected that the families of conserved currents and of conservation-law characteristics from Theorems 3.15 and 3.18 have the same properties as the properties of cosymmetry families indicated in Remark 3.14. Thus, the above conservation-law characteristic, which spans the intersection of the first and the second families from Theorem 3.18, corresponds to the conserved current $\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}(1,\mathfrak{r}^1+\mathfrak{r}^2)$ spanning the intersection of the respective families from Theorem 3.15, cf. the end of the proof of this theorem.

Remark 3.20. The second family of cosymmetries from Theorem 3.13 coincides with the second family of conservation-law characteristics from Theorem 3.18 up to reparameterization. In other words, each cosymmetry in this family is a conservation-law characteristic. This is not the case for the first[7] and third families of cosymmetries from Theorem 3.13, which properly contain the first and third families of conservation-law characteristics from Theorem 3.18, respectively.

Theorem 3.21. *Under the action of generalized symmetries of the system* (3.1) *on its space of conservation laws, a generating set of conservation laws of this system is constituted by the two zeroth-order conservation laws respectively containing the conserved currents*

$$\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\big(\,\mathfrak{r}^3,\ (\mathfrak{r}^1+\mathfrak{r}^2)\mathfrak{r}^3\,\big), \tag{3.28a}$$

$$\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\big(x-V^3t, V^3(x-V^3t)-t\big)\quad \textit{with}\quad V^3:=\mathfrak{r}^1+\mathfrak{r}^2. \tag{3.28b}$$

Proof. The action of the generalized symmetry $\Omega\partial_{\mathfrak{r}^3}$ on the conserved current (3.28a) gives the conserved current $\big(\,\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\Omega,\ (\mathfrak{r}^1+\mathfrak{r}^2)\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\Omega\,\big)$. Varying the parameter function Ω through the space of smooth functions of a finite, but unspecified number of $\omega^\kappa=(\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathcal{D}_x)^\kappa\mathfrak{r}^3$, $\kappa\in\mathbb{N}_0$, we obtain the first family of conserved currents from Theorem 3.15.

Conserved currents from the other two families are constructed by mapping conserved currents of the (1+1)-dimensional Klein–Gordon equation (3.4a) in the way described in the proof of Theorem 3.15. In view of Corollary 2.11, a generating set of conservation laws of (3.4a) is constituted, under the action of generalized symmetries of (3.4a) on conservation laws thereof, by the single conservation law containing the conserved current $(q_z^2,-q^2)$. The counterpart of this conserved current for the system (3.1) is the conserved current

[7] In the notation of Remark 3.16, upon formally interpreting ω^0 as a single dependent variable of a single independent variable, say ς, and ω^1, ω^2, … as the successive derivatives of ω^0, the operators $\partial_\varsigma+\hat{\mathcal{A}}$ and E' become the total derivative operator with respect to ς and the Euler operator with respect to ω^0, respectively. Suppose that a smooth function Ω of a finite number of ω's belongs to $\operatorname{im}\mathsf{E}$. Then $(\hat{\mathcal{A}}\Omega)/\omega^1\in\operatorname{im}\mathsf{E}'$ and thus the Fréchet derivative of $(\hat{\mathcal{A}}\Omega)/\omega^1$ with respect to ω^0 is a formally self-adjoint operator. This is not the case for any Ω of even positive order. Therefore, any cosymmetry from the first family of Theorem 3.13 with Ω of even positive order is not a conservation-law characteristic of the system (3.1).

$$\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\Big(\mathfrak{r}^2_x(x-V^2t)^2-\mathfrak{r}^1_x(x-V^1t)^2,\ V^2\mathfrak{r}^2_x(x-V^2t)^2-V^1\mathfrak{r}^1_x(x-V^1t)^2\Big),$$

which is equivalent to the conserved current (3.28b) multiplied by 2. It follows from Lemma 3.9 that not all generalized symmetries of (3.4a) can be naturally mapped to those of the system (3.1). This is why we need to carefully analyze the result on generating conservation laws of (3.4a) before adopting it for the system (3.1).

The conserved current $\mathrm{C}^0_f=(fq_z,-f_yq)$ of the equation (3.4a) can be obtained by acting the generalized symmetry $\frac12 f_y\partial_q\in\hat{\mathfrak{K}}^{-\infty}$ of this equation on the chosen conserved current $(q_z^2,-q^2)$. Here the parameter function $f=f(y,z)$ runs through the solution set of (3.4a). Each conserved current from the second family of Theorem 3.15 is the image of a conserved current of the form C^0_f, and each Lie symmetry vector field $f\partial_q$ of (3.4a) is mapped to an element of the ideal $\mathcal{I}^1$ of the algebra $\hat{\Sigma}^{\mathrm{q}}$. Therefore, the second family of conserved currents from Theorem 3.15 is generated by acting the elements of $\mathcal{I}^1$ on the conserved current (3.28b).

The action of the generalized symmetry $\frac12(\mathrm{D}_y\mathfrak{X}q)\partial_q$, where $(\mathfrak{X}q)\partial_q\in\tilde{\Lambda}^{\mathrm{q}}$, on the conserved current $(q_z^2,-q^2)$ gives the conserved current $(q_z\mathrm{D}_y\mathrm{D}_z\mathfrak{X}q,-q\mathrm{D}_y\mathfrak{X}q)$, which is equivalent to the conserved currents $(q_z\mathfrak{X}q,-q\mathrm{D}_y\mathfrak{X}q)$ and, therefore, to $\mathrm{C}_{\mathfrak{X}}=(-q\mathrm{D}_z\mathfrak{X}q,\ q_y\mathfrak{X}q)$. The conservation law containing the obtained conserved currents has the characteristic $(\mathfrak{X}-\mathfrak{X}^\dagger)q$.

We denote by $\mathfrak{V}$ the subalgebra of $\tilde{\Lambda}^{\mathrm{q}}$ constituted by the elements of $\tilde{\Lambda}^{\mathrm{q}}$ that have counterparts among generalized symmetries of the system (3.1), and $\mathfrak{J}:=\langle(\mathrm{J}^\kappa q)\partial_q,\ \kappa\in\mathbb{N}\rangle$. We also introduce the corresponding spaces $\mathfrak{V}_-$ and $\mathfrak{J}_-$ of linear generalized symmetries associated with formally skew-adjoint counterparts $\frac12(\mathfrak{X}-\mathfrak{X}^\dagger)$ of operators $\mathfrak{X}$ from $\mathfrak{V}$ and $\mathfrak{J}$, respectively. Note that $\mathfrak{V}_-\supsetneq\mathfrak{V}\cap\tilde{\Lambda}^{\mathrm{q}}_-$ and $\mathfrak{J}_-=\mathfrak{J}\cap\tilde{\Lambda}^{\mathrm{q}}_-$. In view of Lemma 3.9, $(\mathfrak{X}q)\partial_q\in\mathfrak{V}$ if and only if the operator $\mathfrak{X}$ is represented in the form $\mathfrak{X}=(\mathrm{D}_y+1)\mathfrak{X}_1+(\mathrm{D}_z+1)\mathfrak{X}_2+c$ for some $\mathfrak{X}_1\in\langle\,\mathrm{D}_y^\iota\mathrm{J}^\kappa,\ \kappa,\iota\in\mathbb{N}_0\rangle$, some $\mathfrak{X}_2\in\langle\,\mathrm{D}_z^\iota\mathrm{J}^\kappa,\ \kappa,\iota\in\mathbb{N}_0\rangle$ and some $c\in\mathbb{R}$. Hence $\tilde{\Lambda}^{\mathrm{q}}$ is the direct sum of $\mathfrak{V}$ and $\mathfrak{J}$ as vector spaces, $\tilde{\Lambda}^{\mathrm{q}}=\mathfrak{V}\dotplus\mathfrak{J}$, and thus $\tilde{\Lambda}^{\mathrm{q}}_-=\mathfrak{V}_-+\mathfrak{J}_-$, where the sum is not direct by now. We are going to show that $\mathfrak{V}_-\supset\mathfrak{J}_-$, which implies that $\tilde{\Lambda}^{\mathrm{q}}_{-} = \mathfrak{V}_{-}$. Indeed, for any $\mathfrak{X} := (\mathrm{D}_y + 1)(\mathrm{J} - 1/2)^\kappa$ with $\kappa \in 2\mathbb{N}_0 + 1$ we have

$$\mathfrak{X} - \mathfrak{X}^\dagger = (\mathrm{D}_y + 1)(\mathrm{J} - 1/2)^\kappa - (\mathrm{J} + 1/2)^\kappa(\mathrm{D}_y - 1) = (\mathrm{J} - 1/2)^\kappa + (\mathrm{J} + 1/2)^\kappa,$$

i.e., $\big((\mathrm{J} - 1/2)^\kappa q + (\mathrm{J} + 1/2)^\kappa q\big)\partial_q \in \mathfrak{V}_{-}$ since $(\mathfrak{X} q)\partial_q \in \mathfrak{V}$. Therefore,

$$\mathfrak{J}_{-} = \big\langle (\mathrm{J}^\kappa q)\partial_q,\ \kappa \in 2\mathbb{N}_0 + 1\big\rangle = \big\langle \big((\mathrm{J} - 1/2)^\kappa q + (\mathrm{J} + 1/2)^\kappa q\big)\partial_q,\ \kappa \in 2\mathbb{N}_0 + 1\big\rangle \subset \mathfrak{V}_{-}.$$

As a result, for any $(\mathfrak{X} q)\partial_q \in \tilde{\Lambda}^{\mathrm{q}}$ the conserved current $\mathrm{C}_{\mathfrak{X}}$ is equivalent to a conserved current of (3.4a) that is obtained by the action of a generalized symmetry from $\mathfrak{V}$ on the chosen conserved current $(q_z^2, -q^2)$. For the system (3.1), this means that the third family of conserved currents from Theorem 3.15 is generated by acting the generalized symmetries of the form $\check{\mathcal{R}}(\Gamma)$ on the conserved current (3.28b). □

Remark 3.22. The conserved currents from the second family of Theorem 3.15 can be represented in a more symmetrical form. Reparameterizing them in terms of the potential $\bar{\Phi}$ defined via Φ by the system $\bar{\Phi}_{\mathfrak{r}^1} + \frac{1}{2}\bar{\Phi} = 2\Phi_{\mathfrak{r}^1}$, $-\bar{\Phi}_{\mathfrak{r}^2} + \frac{1}{2}\bar{\Phi} = \Phi$, cf. Section 3.2, we obtain another representation for these conserved currents,

$$\mathrm{e}^{(\mathfrak{r}^1 - \mathfrak{r}^2)/2}\big(\,\bar{\Phi}_{\mathfrak{r}^1} - \bar{\Phi}_{\mathfrak{r}^2} + \bar{\Phi},\ (\mathfrak{r}^1 + \mathfrak{r}^2 + 1)\bar{\Phi}_{\mathfrak{r}^1} - (\mathfrak{r}^1 + \mathfrak{r}^2 - 1)\bar{\Phi}_{\mathfrak{r}^2} + (\mathfrak{r}^1 + \mathfrak{r}^2)\bar{\Phi}\,\big),$$

where the parameter function $\bar{\Phi} = \bar{\Phi}(\mathfrak{r}^1, \mathfrak{r}^2)$ runs through the solution space of the Klein–Gordon equation $\bar{\Phi}_{\mathfrak{r}^1\mathfrak{r}^2} = -\bar{\Phi}/4$ as well. The successive point transformation $\tilde{\Phi} = \mathrm{e}^{(\mathfrak{r}^1 - \mathfrak{r}^2)/2}\bar{\Phi}$ reduces the above representation to $\big(\,\tilde{\Phi}_{\mathfrak{r}^1} - \tilde{\Phi}_{\mathfrak{r}^2},\ (\mathfrak{r}^1 + \mathfrak{r}^2 + 1)\tilde{\Phi}_{\mathfrak{r}^1} - (\mathfrak{r}^1 + \mathfrak{r}^2 - 1)\tilde{\Phi}_{\mathfrak{r}^2}\,\big)$, where the parameter function $\tilde{\Phi} = \tilde{\Phi}(\mathfrak{r}^1, \mathfrak{r}^2)$ runs through the solution space of the equation $2\tilde{\Phi}_{\mathfrak{r}^1\mathfrak{r}^2} = \tilde{\Phi}_{\mathfrak{r}^2} - \tilde{\Phi}_{\mathfrak{r}^1}$. It is the last representation that was employed in [112, Theorem 22]. In terms of $\tilde{\Phi}$, the associated characteristics take the form $(\tilde{\Phi}_{\mathfrak{r}^1\mathfrak{r}^1} - \tilde{\Phi}_{\mathfrak{r}^1\mathfrak{r}^2}, \tilde{\Phi}_{\mathfrak{r}^1\mathfrak{r}^2} - \tilde{\Phi}_{\mathfrak{r}^2\mathfrak{r}^2}, 0)$.

Remark 3.23. The advantage of using conserved currents of the form $\mathrm{C}_{\mathfrak{X}}$ for mapping to conserved currents of the system $\mathcal{S}$ is that we obtain a uniform representation for elements of the third family of Theorem 3.15. At the same time, it is not obvious how to find equivalent conserved currents of minimal order for elements of this family or how to single out conserved currents in this family that are equivalent to ones not depending on (t,x) explicitly. The former problem can be solved by replacing conserved currents of the form $\mathrm{C}_{\mathfrak{X}}$ in the mapping by equivalent conserved currents $\mathrm{C}^1_{\kappa\iota}$, $\kappa\in\mathbb{N}_0$, $\iota\in\mathbb{N}$, $\bar{\mathrm{C}}^1_{\kappa\iota}$, $\mathrm{C}^2_{\kappa\iota}$, $\bar{\mathrm{C}}^2_{\kappa\iota}$, $\kappa,\iota\in\mathbb{N}_0$, presented in Section 2.4 although an additional "integration by parts" may still be needed for lowest values of (κ,ι) after the mapping, cf. the proof of Theorem 3.21. For solving the latter problem, we use an analog of the trick used to prove Theorem 3.15 for deriving the second family of conserved currents, which leads to Theorem 3.26 below.

Corollary 3.24. *(i) The space of hydrodynamic conservation laws of the system* (3.1) *is infinite-dimensional and is naturally isomorphic to the space spanned by the conserved currents from the second family of Theorem 3.15 and from the first family with Ω running through the space of smooth functions of $\omega^0:=\mathfrak{r}^3$.*

(ii) The space of zeroth-order conservation laws of the system (3.1) *is naturally isomorphic to the space spanned by its hydrodynamic conserved currents and the conserved current* (3.28b).

Proof. This assertion was proved in [112, Theorem 22] by the direct computation. At the same time, it is a simple corollary of Theorems 3.15 and 3.18. Indeed, when linearly combining conserved currents from different families of Theorem 3.15, the maximum of their orders is preserved. The selection of zeroth-order conserved currents from the first and the second families is obvious. Theorem 3.18 implies that the space of zeroth-order characteristics related to the third family is one-dimensional and spanned by the characteristic $\mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\big(\,\tilde q,\ -\tilde{\mathfrak{D}}_z\tilde q,\ 0\,\big)$ of the conservation law containing the conserved current (3.28b). $\square$

Corollary 3.25. *The space of zeroth- and first-order conservation laws of the system* (3.1) *is naturally isomorphic to the space spanned by the conserved currents from the second family of Theorem 3.15 and from the first family, where the parameter function Ω runs through the space of smooth functions of $(\omega^0,\omega^1):=(\mathfrak{r}^3,\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathfrak{r}^3_x)$ and such two functions should be assumed equivalent if their difference is of the form $f(\omega^0)\omega^1$, as well as the conserved currents from the third family, where the operator $\tilde{\mathfrak{X}}$ runs through the set $\big\{\tilde{\mathfrak{D}}_z,\ \tilde{\mathfrak{D}}_y,\ \tilde{\mathfrak{J}},\ \tilde{\mathfrak{D}}_z^3,\ (\tilde{\mathfrak{J}}-1)\tilde{\mathfrak{D}}_z^2\big\}$.*

Proof. In the same spirit as in the proof of Corollary 3.24, we select the zeroth- and first-order conserved currents equivalent to those listed in Theorem 3.15 using Theorem 3.18 for estimating the orders of the associated conservation laws. Thus, the selection of the conserved currents from the second family is again obvious since all of then are of order zero. The order of a conservation law related to the first family coincides with the minimal order of the associated Ω's. In general, for zeroth- and first-order conservation laws of the system (3.1), the order of corresponding reduced characteristics is not greater than two. This is why a conservation law related to the span of the third family is of order not greater than one if and only if it contains a conserved current corresponding to $\tilde{\mathfrak{X}}\in\big\langle\tilde{\mathfrak{D}}_z,\,\tilde{\mathfrak{D}}_y,\,\tilde{\mathfrak{J}},\,\tilde{\mathfrak{D}}_z^3,\,(\tilde{\mathfrak{J}}-1)\tilde{\mathfrak{D}}_z^2\big\rangle$. □

Theorem 3.26. *The space of (t,x)-translation-invariant conservation laws of the system* (3.1) *is naturally isomorphic to the space spanned by the conserved currents from the first and second families of Theorem 3.15 as well as the conserved currents from the span of the third family that have the form $\tilde{\mathrm{C}}_{\tilde{\mathfrak{X}}}$ of elements of this family,*

$$
\begin{aligned}
&\big(\mathfrak{r}^2_x\tilde\rho+\mathfrak{r}^1_x\tilde\sigma,\ (\mathfrak{r}^1+\mathfrak{r}^2-1)\mathfrak{r}^2_x\tilde\rho+(\mathfrak{r}^1+\mathfrak{r}^2+1)\mathfrak{r}^1_x\tilde\sigma\big)\\
&\qquad\quad \textit{with }\tilde\rho=-\tilde q\tilde{\mathfrak{D}}_z\tilde{\mathfrak{X}}\tilde q,\ \ \tilde\sigma=(\tilde{\mathfrak{D}}_y\tilde q)\tilde{\mathfrak{X}}\tilde q,
\end{aligned}
\tag{3.29}
$$

where the operator $\tilde{\mathfrak{X}}$ runs through the set $\mathfrak{T}$ constituted by the operators

$$
\begin{aligned}
&\tilde{\mathfrak{Z}}_{\kappa\iota}:=(\tilde{\mathfrak{D}}_z+1)^2(\tilde{\mathfrak{J}}-\iota/2)^\kappa\tilde{\mathfrak{D}}_z^\iota(\tilde{\mathfrak{D}}_z-1)^2,\ \tilde{\mathfrak{Y}}_{\kappa,\iota+4}:=(\tilde{\mathfrak{D}}_y+1)^2(\tilde{\mathfrak{J}}+\iota/2)^\kappa\tilde{\mathfrak{D}}_y^\iota(\tilde{\mathfrak{D}}_y-1)^2,\\
&\qquad\kappa,\iota\in\mathbb{N}_0\ \textit{with}\ \kappa+\iota\in2\mathbb{N}_0+1,\\
&\tilde{\mathfrak{Y}}_{\kappa1}:=(\tilde{\mathfrak{J}}+1/2)^\kappa(\tilde{\mathfrak{D}}_y+\tilde{\mathfrak{D}}_z-2)+(\tilde{\mathfrak{D}}_z+2)(\tilde{\mathfrak{J}}-1/2)^\kappa(\tilde{\mathfrak{D}}_z-1)^2,\quad\kappa\in2\mathbb{N}_0,\\
&\tilde{\mathfrak{Y}}_{\kappa2}:=2\tilde{\mathfrak{J}}^\kappa(\tilde{\mathfrak{D}}_y+\tilde{\mathfrak{D}}_z-2)+(\tilde{\mathfrak{J}}+1)^\kappa(\tilde{\mathfrak{D}}_y-1)^2+(\tilde{\mathfrak{J}}-1)^\kappa(\tilde{\mathfrak{D}}_z-1)^2,\quad\kappa\in2\mathbb{N}_0+1,\\
&\tilde{\mathfrak{Y}}_{\kappa3}:=(\tilde{\mathfrak{J}}-1/2)^\kappa(\tilde{\mathfrak{D}}_y+\tilde{\mathfrak{D}}_z-2)+(\tilde{\mathfrak{D}}_y+2)(\tilde{\mathfrak{J}}+1/2)^\kappa(\tilde{\mathfrak{D}}_y-1)^2,\quad\kappa\in2\mathbb{N}_0.
\end{aligned}
$$

Proof. Denote by $\bar{\mathfrak{T}}$ a complementary subspace of the span of $\mathfrak{T}$ in the span of the set run by $\tilde{\mathfrak{X}}$ in the third family of Theorem 3.15. Since conserved currents from the first and second families of Theorem 3.15 are (t,x)-translation-invariant, it suffices to prove that conserved currents of the form (3.29) with $\tilde{\mathfrak{X}}\in\mathfrak{T}$ (resp. with nonzero $\tilde{\mathfrak{X}}\in\bar{\mathfrak{T}}$) are equivalent (resp. not equivalent) to (t,x)-translation-invariant ones.

For each $\tilde{\mathfrak{X}}\in\mathfrak{T}$ we explicitly construct a related (t,x)-translation-invariant conserved current by considering the associated operator $\mathfrak{X}$ in $\tilde\Lambda^{\rm q}_-$, choosing an appropriate conserved current of the Klein–Gordon equation (3.4a) among those equivalent to $\mathrm{C}_{\mathfrak{X}}$ and mapping it to a conserved current of (3.1). Each operator $\mathfrak{X}\in\tilde\Lambda^{\rm q}_-$ associated with some $\tilde{\mathfrak{X}}\in\mathfrak{T}$ is equivalent to an operator of the form $(\mathrm{D}_z{+}1)^2\mathfrak{P}(\mathrm{D}_z-1)^2$ with $(\mathfrak{P}q)\partial_q\in\tilde\Lambda^{\rm q}_-$, where the operator $\mathfrak{P}$ coincides with $(\mathrm{J}-\iota/2)^\kappa\mathrm{D}_z^\iota$, $(\mathrm{J}+\iota/2+2)^\kappa\mathrm{D}_z^{\iota+4}$, $(\mathrm{J}+1/2)^\kappa\mathrm{D}_z$, $(\mathrm{J}+1)^\kappa\mathrm{D}_z^2$, $(\mathrm{J}+3/2)^\kappa\mathrm{D}_z^3$ for $\tilde{\mathfrak{Z}}_{\kappa\iota}$, $\tilde{\mathfrak{Y}}_{\kappa,\iota+4}$, $\tilde{\mathfrak{Y}}_{\kappa1}$, $\tilde{\mathfrak{Y}}_{\kappa2}$ and $\tilde{\mathfrak{Y}}_{\kappa3}$, respectively. For such $\mathfrak{X}$ we obtain

$$
\begin{aligned}
&\mathrm{C}_{\mathfrak{X}}\sim\big(-K^1\mathfrak{D}_z\mathfrak{P}K^1,K^2\mathfrak{P}K^1\big)\\
&\quad\mapsto\ 2\mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\left((\tilde{\mathfrak{D}}_z+1)\tilde{\mathfrak{P}}\frac{\mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}}{\mathfrak{r}^2_x},\ (V^2\tilde{\mathfrak{D}}_z+V^1)\tilde{\mathfrak{P}}\frac{\mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}}{\mathfrak{r}^2_x}\right),
\end{aligned}
$$

which is obviously a (t,x)-translation-invariant conserved current of the system (3.1).

As a subspace complementary to the span of $\mathfrak{T}$, we can choose

$$
\bar{\mathfrak{T}}=\big\langle\,\mathrm{J}^{2\kappa+1},\ (\mathrm{J}+1)^{2\kappa+1}\mathrm{D}_z^2,\ (\mathrm{J}+1/2)^{2\kappa}\mathrm{D}_z,\ (\mathrm{J}+3/2)^{2\kappa}\mathrm{D}_z^3,\ \kappa\in\mathbb{N}_0\,\big\rangle.
$$

We prove by contradiction that for any nonzero $\tilde{\mathfrak{X}}\in\bar{\mathfrak{T}}$, i.e.,

$$
\tilde{\mathfrak{X}}=\sum_{\kappa=0}^{N}\Big(c_{0\kappa}\mathrm{J}^{2\kappa+1}+c_{2\kappa}(\mathrm{J}+1)^{2\kappa+1}\mathrm{D}_z^2+c_{1\kappa}(\mathrm{J}+1/2)^{2\kappa}\mathrm{D}_z+c_{3\kappa}(\mathrm{J}+3/2)^{2\kappa}\mathrm{D}_z^3\Big)
$$

for some $N\in\mathbb{N}_0$ and some constants c's with $(c_{0N},c_{1N},c_{2N},c_{3N})\neq(0,0,0,0)$, the corresponding conserved current of the form (3.29) is not equivalent to a (t,x)-translation-invariant one. Suppose that this is not the case. If a conservation law of the system (3.1) is (t,x)-translation-invariant, then its characteristic is also (t,x)-translation-invariant. The conservation-law characteristic associated with $\tilde{\mathfrak{X}}$ (see Theorem 3.18) does not depend on x and t if and only if $(\tilde{\mathfrak{X}}\tilde q)_x=\tilde{\mathfrak{X}}\mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}=0$ and $(\tilde{\mathfrak{X}}\tilde q)_t=-\tilde{\mathfrak{X}}\big((\mathfrak{r}^1+\mathfrak{r}^2+1)\mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\big)=0$. In the coordinates (3.6), these conditions, after re-combining, take the form $\mathfrak{X}\mathrm{e}^{y+z}=0$, $\mathfrak{X}\big((y-z)\mathrm{e}^{y+z}\big)=\mathfrak{X}\mathrm{J}\mathrm{e}^{y+z}=0$, or, equivalently,

$$R^1 := \sum_{\kappa=0}^{N}\Big(c_{0\kappa}\mathrm{J}^{2\kappa+1}+c_{2\kappa}(\mathrm{J}+1)^{2\kappa+1}+c_{1\kappa}(\mathrm{J}+1/2)^{2\kappa}+c_{3\kappa}(\mathrm{J}+3/2)^{2\kappa}\Big)\mathrm{e}^{y+z}=0,$$
$$R^2 := \sum_{\kappa=0}^{N}\Big(c_{0\kappa}\mathrm{J}^{2\kappa+2}+c_{2\kappa}(\mathrm{J}+1)^{2\kappa+1}(\mathrm{J}-2)+c_{1\kappa}(\mathrm{J}+1/2)^{2\kappa}(\mathrm{J}-1)$$
$$+c_{3\kappa}(\mathrm{J}+3/2)^{2\kappa}(\mathrm{J}-3)\Big)\mathrm{e}^{y+z}=0.$$

The left-hand sides of these equations, R^1 and R^2, are polynomials of $y-z$ and $y+z$ multiplied by e^{y+z}, and the highest degrees of $y-z$ correspond to the highest degrees of J. Recombining these equations to

$$R^2-\mathrm{J}R^1=-\sum_{\kappa=0}^{N}\Big(2c_{2\kappa}(\mathrm{J}+1)^{2\kappa+1}+c_{1\kappa}(\mathrm{J}+1/2)^{2\kappa}+3c_{3\kappa}(\mathrm{J}+3/2)^{2\kappa}\Big)\mathrm{e}^{y+z}=0,$$
$$R^2-(\mathrm{J}-2)R^1=\sum_{\kappa=0}^{N}\Big(2c_{0\kappa}\mathrm{J}^{2\kappa+1}+c_{1\kappa}(\mathrm{J}+1/2)^{2\kappa}-c_{3\kappa}(\mathrm{J}+3/2)^{2\kappa}\Big)\mathrm{e}^{y+z}=0,$$

we easily see that $c_{0N}=c_{2N}=0$ and thus also $c_{1N}=c_{2N}=0$, which contradicts the supposition $(c_{0N},c_{1N},c_{2N},c_{3N})\neq(0,0,0,0)$. $\square$

In order to construct a lowest-order (t,x)-translation-invariant conserved current for conservation laws associated with operators from $\mathfrak{T}$, for the respective operator $\mathfrak{P}$ we should take the respective (up to a constant multiplier) conserved current among $\mathrm{C}^1_{\kappa\iota}$, $\kappa\in\mathbb{N}_0$, $\iota\in\mathbb{N}$, $\bar{\mathrm{C}}^1_{\kappa\iota}$, $\mathrm{C}^2_{\kappa\iota}$, $\bar{\mathrm{C}}^2_{\kappa\iota}$, $\kappa,\iota\in\mathbb{N}_0$, presented in Section 2.4, formally replace (x,y,u) by (y,z,K^1) and map the obtained conserved current. In particular, linearly independent (t,x)-translation-invariant inequivalent conserved currents up to order two from the span of the third family of Theorem 3.15 are exhausted by the following:

$$\tilde{\mathfrak{X}}=\tilde{\mathfrak{Y}}_{01}=\tilde{\mathfrak{D}}_z^3-2\tilde{\mathfrak{D}}_z+\tilde{\mathfrak{D}}_y\colon\quad \mathfrak{P}=\mathfrak{D}_y,\quad \mathrm{C}_{\tilde{\mathfrak{X}}}\sim\big(-(K^1)^2,\,(K^2)^2\big)$$
$$\mapsto\ 2\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\left(\frac{1}{\mathfrak{r}^2_x}-\frac{1}{\mathfrak{r}^1_x},\ \frac{V^2}{\mathfrak{r}^2_x}-\frac{V^1}{\mathfrak{r}^1_x}\right),$$
$$\tilde{\mathfrak{X}}=\tilde{\mathfrak{Y}}_{03}=\tilde{\mathfrak{D}}_y^3-2\tilde{\mathfrak{D}}_y+\tilde{\mathfrak{D}}_z\colon\quad \mathfrak{P}=\mathfrak{D}_y^3,\quad \mathrm{C}_{\tilde{\mathfrak{X}}}\sim\big((K^2)^2,\,-(\mathfrak{D}_yK^2)^2\big)$$
$$\mapsto\ \frac{2}{(\mathfrak{r}^1_x)^5}\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\Big((2\mathfrak{r}^1_{xx}+\mathfrak{r}^1_x\mathfrak{r}^2_x)^2-\mathfrak{r}^2_x(\mathfrak{r}^1_x)^3,\ V^1(2\mathfrak{r}^1_{xx}+\mathfrak{r}^1_x\mathfrak{r}^2_x)^2-V^2\mathfrak{r}^2_x(\mathfrak{r}^1_x)^3\Big),$$
$$\tilde{\mathfrak{X}}=\tilde{\mathfrak{Z}}_{01}=\tilde{\mathfrak{D}}_z^5-2\tilde{\mathfrak{D}}_z^3+\tilde{\mathfrak{D}}_z\colon\quad \mathfrak{P}=\mathfrak{D}_z,\quad \mathrm{C}_{\tilde{\mathfrak{X}}}\sim\big((\mathfrak{D}_zK^1)^2,\,-(K^1)^2\big)$$
$$\mapsto\ \frac{-2}{(\mathfrak{r}^2_x)^5}\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\Big((2\mathfrak{r}^2_{xx}-\mathfrak{r}^1_x\mathfrak{r}^2_x)^2-\mathfrak{r}^1_x(\mathfrak{r}^2_x)^3,\ V^2(2\mathfrak{r}^2_{xx}-\mathfrak{r}^1_x\mathfrak{r}^2_x)^2-V^1\mathfrak{r}^1_x(\mathfrak{r}^2_x)^3\Big),$$

$$\tilde{\mathfrak{X}} = \tilde{\mathfrak{Z}}_{10} := (\tilde{\mathfrak{D}}_z+1)^2\tilde{\mathfrak{J}}(\tilde{\mathfrak{D}}_z-1)^2: \quad \mathfrak{P} = \mathfrak{J},$$

$$\mathrm{C}_{\mathfrak{X}} \sim \big(-y(K^1)^2-z(\mathfrak{D}_zK^1)^2,\, y(K^2)^2+z(K^1)^2\big) \mapsto -\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}(\mathfrak{z}^1+\mathfrak{z}^2,\, V^1\mathfrak{z}^1+V^2\mathfrak{z}^2),$$

$$\mathfrak{z}^1 := \frac{\mathfrak{r}^1}{\mathfrak{r}^1_x} - \frac{\mathfrak{r}^2\mathfrak{r}^1_x}{(\mathfrak{r}^2_x)^2}, \quad \mathfrak{z}^2 := \frac{\mathfrak{r}^2}{(\mathfrak{r}^2_x)^5}(2\mathfrak{r}^2_{xx}-\mathfrak{r}^1_x\mathfrak{r}^2_x)^2 - \frac{\mathfrak{r}^1}{\mathfrak{r}^2_x}.$$

3.7 Simplest potential symmetries

To begin with, we introduce the homogeneous notation in this section, which differs from that of the present chapter. Namely, for a system $\mathcal{L}$ we denote $\operatorname{sym}\mathcal{L}$ the algebra of canonical representatives of its generalized symmetries in the reduced form. Being in the reduced form means that the functions belonging to the characteristic-tuple of an evolutionary vector field depend only on coordinates of the manifold $\mathcal{L}^{(\infty)}$ defined by the system in a corresponding infinite-dimensional jet space. The chosen coordinates are to be indicated explicitly for every system encountered below.

Using the conserved current $\Big(\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}, \mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}(\mathfrak{r}^1+\mathfrak{r}^2)\Big)$ of the system $\mathcal{S}$, we introduce the potential ϕ to obtain the potential system $\tilde{\mathcal{S}}$ (or *covering*) for the system $\mathcal{S}$,

$$\phi_t = -\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}(\mathfrak{r}^1+\mathfrak{r}^2), \quad \phi_x = \mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2},$$

$$\mathfrak{r}^1_t + (\mathfrak{r}^1+\mathfrak{r}^2+1)\mathfrak{r}^1_x = 0, \quad \mathfrak{r}^2_t + (\mathfrak{r}^1+\mathfrak{r}^2-1)\mathfrak{r}^2_x = 0, \quad \mathfrak{r}^3_t + (\mathfrak{r}^1+\mathfrak{r}^2)\mathfrak{r}^3_x = 0.$$

Recalling the operators $\mathcal{A}$ and $\mathcal{B}$, we see that the potential ϕ satisfies the system $\mathcal{B}\phi = 0$, $\mathcal{A}\phi = 1$. We can equivalently rewrite the system $\tilde{\mathcal{S}}$ as follows

$$2\phi_t\phi_x\phi_{xx} + 2\phi_t\phi_x\phi_{tx} - \phi_t^2\phi_{xx} - \phi_x^2\phi_{xx} - \phi_{tt}\phi_x = 0, \quad \phi_t\mathfrak{r}^3_x - \phi_x\mathfrak{r}^3_t = 0$$

$$\mathfrak{r}^1 = \frac{\phi_x\ln\phi_x - \phi_t}{2\phi_x}, \quad \mathfrak{r}^2 = -\frac{\phi_x\ln\phi_x + \phi_t}{2\phi_x}.$$

As for coordinates on the manifold defined by the system $\tilde{\mathcal{S}}$, then it is convenient to choose ϕ, $\mathfrak{D}_x^\iota\mathfrak{r}^1$, $\mathfrak{D}_x^\iota\mathfrak{r}^2$ and ω^ι, $\iota \in \mathbb{N}_0$, in order to efficiently single out nonlocal symmetries of $\mathcal{S}$ associated with the covering $\tilde{\mathcal{S}}$. In a sense, the associated potential symmetries of the system $\mathcal{S}$ are the simplest in their class as the potential ϕ corresponds to the conserved current belonging to the intersection of first two families of conserved currents presented in Theorem 3.15. Upon transition from the system $\mathcal{S}$ to the system $\mathcal{X}$ of the equations (3.4)–(3.5) by the point transformation (3.6), the above conserved current is mapped to the conserved current $(\mathrm{e}^{y+z}q_z, -\mathrm{e}^{y+z}q)$ of $\mathcal{X}$ and the potential ϕ to the corresponding potential ψ, satisfying the system

$$\psi_y = q - \psi, \quad \psi_z = q_z - \psi,$$

see the transformation $\mathcal{T}$ below. Recall that the potential ψ satisfies the Klein–Gordon equation $\psi_{yz} = \psi$, while the coefficients K^1 and K^2 in the equation (3.4b) and the dependent variable p can be represented as differential functions of ψ, see Section 3.2,

$$K^1[\psi] = \psi_{zz} + \psi_y - \psi_z - \psi, \quad K^2[\psi] = \psi_{yy} - \psi_y + \psi_z - \psi, \quad p = -\frac{1}{2}\mathrm{e}^{-y-z}(\psi_y - \psi_z).$$

Thus, we have constructed the covering $\tilde{\mathcal{X}}$ for the system $\mathcal{X}$, which can be rewritten in the equivalent form

$$\psi_{yz} = \psi, \quad q = \psi_y + \psi, \quad p = -\frac{1}{2}\mathrm{e}^{-y-z}(\psi_y - \psi_z), \quad K^1[\psi]s_y = K^2[\psi]s_z.$$

We also prolong the point transformation (3.6) to the potential ϕ obtaining the point transformation $\mathcal{T}$ mapping the system $\tilde{\mathcal{S}}$ to the system $\tilde{\mathcal{X}}$,

$$\mathcal{T}\colon \quad \begin{aligned} & y = \mathfrak{r}^1/2, \quad z = -\mathfrak{r}^2/2, \quad p = t, \quad q = \mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}(x - (\mathfrak{r}^1 + \mathfrak{r}^2 + 1)t), \quad s = \mathfrak{r}^3, \\ & \psi = -\frac{1}{2}\left(\mathrm{e}^{(\mathfrak{r}^2-\mathfrak{r}^1)/2}\phi + \mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}(x - (\mathfrak{r}^1 + \mathfrak{r}^2)t)\right). \end{aligned}$$

Denote by $\hat{\mathcal{T}}$ the inverse to the transformation $\mathcal{T}$.

We choose the following coordinates on the manifold $\tilde{\mathcal{X}}^{(\infty)}$: y, z, $\psi_\kappa = \partial^\kappa\psi/\partial y^\kappa$, $\psi_{-\kappa} = \partial^\kappa\psi/\partial z^\kappa$ and $s_\kappa = \partial^\kappa s/\partial y^\kappa$, $\kappa \in \mathbb{N}_0$. The elements of the algebra $\operatorname{sym}\tilde{\mathcal{X}}$ are of the form $X = \zeta\partial_\psi + \chi\partial_q + \rho\partial_p + \theta\partial_s$, where ζ, ψ, ρ and θ are smooth functions of the above coordinates. The point transformation $\mathcal{T}$ induces an isomorphism of the algebras of generalized symmetries of the systems $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{X}}$, $\operatorname{sym}\tilde{\mathcal{X}} \simeq \operatorname{sym}\tilde{\mathcal{S}}$.

In view of partially coupling of the system $\tilde{\mathcal{X}}$, the components of any generalized vector field $X \in \operatorname{sym}\tilde{\mathcal{X}}$ do not depend on p, q and their derivatives. Therefore, there is a well-defined projection from $\operatorname{sym}\tilde{\mathcal{X}}$ onto the algebra $\operatorname{sym}\mathcal{P}$ of the subsystem $\mathcal{P}$ thereof, consisting of the equations $\psi_{yz} = \psi$, $K^1[\psi]s_y = K^2[\psi]s_z$. Vice versa, given a generalized symmetry $X \in \operatorname{sym}\mathcal{P}$, we can locally and uniquely prolong it to the generalized symmetry $\tilde{X} \in \operatorname{sym}\tilde{\mathcal{X}}$ as variables p and q are defined as differential functions of ψ on $\tilde{\mathcal{X}}^{(\infty)}$. Thus, there is a natural one-to-one correspondence between the algebras $\operatorname{sym}\mathcal{P}$ and $\operatorname{sym}\tilde{\mathcal{X}}$. More specifically, $\operatorname{sym}\tilde{\mathcal{X}} \simeq \operatorname{sym}\mathcal{P}$. Hence, it is more convenient to study generalized symmetries of the system $\mathcal{P}$.

Remark 3.8 implies that for any generalized vector field $X \in \operatorname{sym}\mathcal{P}$, its ψ-component does not depend on s and its derivatives. Furthermore, the elements in the algebra $\operatorname{sym}\mathcal{P}$ with the vanishing ψ-component form an ideal $\operatorname{sym}_s\mathcal{P}$ of $\operatorname{sym}\mathcal{P}$, $\operatorname{sym}_s\mathcal{P} = \{\theta[s,\psi]\partial_s\}$. Similarly to Section 3.4, in order to describe $\operatorname{sym}_s\mathcal{P}$ we introduce the modified coordinates on the manifold $\mathcal{P}^{(\infty)}$. With this aim we prolong differential operators $\mathcal{D}_y$ and $\mathcal{D}_z$ to ψ and define the differential operator $\hat{\mathcal{A}}$ and the modified coordinates $\hat{\omega}$'s on the manifold $\mathcal{P}^{(\infty)}$.

$$\mathcal{D}_y = \partial_y + \sum_{\iota=-\infty}^{+\infty} q_{\iota+1}\partial_{q_\iota} + \sum_{\iota=0}^{+\infty} s_{\iota+1}\partial_{s_\iota} + \sum_{\iota=-\infty}^{+\infty} \psi_{\iota+1}\partial_{\psi_\iota},$$

$$\mathcal{D}_z = \partial_z + \sum_{\iota=-\infty}^{+\infty} q_{\iota-1}\partial_{q_\iota} + \sum_{\iota=0}^{+\infty} \mathcal{D}_y^\iota\left(\frac{K^1}{K^2}s_1\right)\partial_{s_\iota} + \sum_{\iota=-\infty}^{+\infty} \psi_{\iota-1}\partial_{\psi_\iota},$$

$$\hat{\mathcal{A}} = \frac{\mathrm{e}^{-y-z}}{K^2[\psi]}\mathcal{D}_y + \frac{\mathrm{e}^{-y-z}}{K^1[\psi]}\mathcal{D}_z, \quad \hat{\omega}^\iota = \hat{\mathcal{A}}^\kappa s, \ \kappa \in \mathbb{N}_0.$$

As before $\hat{\omega}$'s belong to the kernel of the differential operator $\hat{\mathcal{B}}$, which in the new coordinates is defined to be

$$\hat{\mathcal{B}} = -\frac{\mathrm{e}^{y+z}}{K^2[\psi]}\mathcal{D}_y + \frac{\mathrm{e}^{y+z}}{K^1[\psi]}\mathcal{D}_z.$$

Note that here the operators $\mathcal{D}_y$ and $\mathcal{D}_z$ do not coincide with those above, and are used in this section only. For the future use we also redefine $\mathcal{J} = y\mathcal{D}_y - z\mathcal{D}_z$. Following the logic of Corollary 3.5, we prove the proposition.

Lemma 3.27. *The algebra* $\operatorname{sym}_s \mathcal{P}$ *is spanned by generalized vector fields of the form* $\Omega\partial_s$*, where* Ω *runs through the set of smooth functions of* $\beta = \mathrm{e}^{y+z}(\mathcal{D}_y + \mathcal{D}_z - 2)\psi$ *and a finite, but unspecified number of* $\hat{\omega}^\kappa$*,* $\kappa \in \mathbb{N}_0$.

The only difference with Corollary 3.5 is a presence of the differential function β in the kernel of the operator $\hat{\mathcal{B}}$, which can be explained by the fact that the potential ϕ is in the kernel of the operator $\mathcal{B}$, which manifests itself after mapping to the system $\tilde{\mathcal{X}}$.

The quotient algebra $\operatorname{sym}\mathcal{P}/\operatorname{sym}_s\mathcal{P}$ can be identified with the algebra $\operatorname{sym}_\psi\mathcal{P}$ of generalized symmetries of the Klein–Gordon equation $\psi_{yz} = \psi$ that are locally prolonged to an element of $\operatorname{sym}\mathcal{P}$. Elements of both the algebras $\operatorname{sym}_s\mathcal{P}$ and $\operatorname{sym}_\psi\mathcal{P}$ are easy to prolong to the full system $\operatorname{sym}\tilde{\mathcal{X}}$. In particular, $\operatorname{sym}_s\tilde{\mathcal{X}} = i_*\operatorname{sym}_s\mathcal{P}$, where the map $i\colon \mathcal{P}^{(\infty)} \hookrightarrow \tilde{\mathcal{X}}^{(\infty)}$ is the inclusion, while a prolongation of an element of $\operatorname{sym}_\psi\mathcal{P}$ requires a simple inspection of the invariance criterion only.

Proposition 3.28. *The algebra* $\operatorname{sym}_\psi\tilde{\mathcal{X}}$ *of generalized vector fields that are prolongations of elements of* $\operatorname{sym}_\psi\mathcal{P}$ *to* p*,* q *and* s *is spanned by the generalized vector fields*

$$\zeta\partial_\psi - \frac{1}{2}\mathrm{e}^{-y-z}(\mathcal{D}_y - \mathcal{D}_z)\zeta\partial_p + (\mathcal{D}_y + 1)\zeta\partial_q + \frac{s_1}{K^2[\psi]}(\mathcal{D}_y + \mathcal{D}_z - 2)\zeta\partial_s,$$

where ζ *runs through the set*

$$\{\mathcal{J}^\kappa\psi,\ \mathcal{D}_y^\iota\mathcal{J}^\kappa\psi,\ \mathcal{D}_z^\iota\mathcal{J}^\kappa\psi,\ \kappa \in \mathbb{N}_0,\ \iota \in \mathbb{N}\} \bigcup \{f = f(y,z)\colon f_{yz} = f\}.$$

Proof. Let $\zeta\partial_\psi$ be a generalized symmetry of the Klein–Gordon equation $\psi_{yz} = \psi$. Then by definition, the algebra $\operatorname{sym}_\psi\tilde{\mathcal{X}}$ consists of generalized symmetries of the system $\tilde{\mathcal{X}}$, which are of the form $\zeta\partial_\psi + \rho\partial_p + \chi\partial_q + \theta\partial_s$. The invariance criterion then reads

$$\begin{gathered}\mathcal{D}_y\mathcal{D}_z\zeta = \zeta, \quad \chi = (\mathcal{D}_y + 1)\zeta, \quad \rho = -\frac{1}{2}\mathrm{e}^{-y-z}(\mathcal{D}_y - \mathcal{D}_z)\zeta, \\ (\mathcal{D}_z - \mathcal{D}_y)(\mathcal{D}_z - 1)\zeta s_1 + K^1[\psi]\mathcal{D}_y\theta = K^2[\psi]\mathcal{D}_z\theta + (\mathcal{D}_y - \mathcal{D}_z)(\mathcal{D}_y - 1)\zeta\frac{K^1[\psi]}{K^2[\psi]}s_1.\end{gathered}$$

The latter equation is first-order inhomogeneous equation on θ with a particular solution $\theta = s_1(\mathcal{D}_y + \mathcal{D}_z - 2)\psi/K^2[\psi]$. $\square$

Remark 3.29. Note also that given $\zeta = \psi$, the last determining equation degenerates and becomes homogeneous, and hence we can also take $\theta = 0$ as a prolongation of the generalized symmetry $\psi\partial_\psi$ of the Klein–Gordon equation. It is perfectly fine from the point of view of closedness under the Lie bracket of vector fields, but it causes complications for the vector space structure of the set of generalized symmetries. Therefore, it is better to choose prolongation of the generalized vector field $\psi\partial_\psi$ as in the theorem.

Following the proof of Theorem 3.10, we can deduce the algebra of generalized symmetries of the system $\tilde{\mathcal{S}}$.

Theorem 3.30. *The algebra* $\operatorname{sym}\tilde{\mathcal{S}}$ *of generalized symmetries of the system* $\tilde{\mathcal{S}}$ *is spanned by the generalized vector fields of the form*

$$
\begin{aligned}
&\mathrm{e}^{(\mathfrak{r}^2-\mathfrak{r}^1)/2}\left(\mathfrak{r}^1_x(\tilde{\mathcal{D}}_y+1)\Gamma\partial_{\mathfrak{r}^1}+\mathfrak{r}^2_x(\tilde{\mathcal{D}}_z+1)\Gamma\partial_{\mathfrak{r}^2}+2\mathfrak{r}^3_x\Gamma\partial_{\mathfrak{r}^3}\right)+2\mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\Gamma\partial_\phi,\\
&\mathrm{e}^{(\mathfrak{r}^2-\mathfrak{r}^1)/2}\left((\Phi+2\Phi_{\mathfrak{r}^1})\mathfrak{r}^1_x\partial_{\mathfrak{r}^1}+(\Phi-2\Phi_{\mathfrak{r}^2})\mathfrak{r}^2_x\partial_{\mathfrak{r}^2}+2\Phi\mathfrak{r}^3_x\partial_{\mathfrak{r}^3}\right)+2\mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\Phi\partial_\phi,\quad \Omega\partial_{\mathfrak{r}^3},
\end{aligned}
$$

where Φ *runs through the set of smooth solutions of the Klein–Gordon equation* $\Phi_{\mathfrak{r}^1\mathfrak{r}^2} = -\Phi/4$, Ω *runs through the set of smooth functions of a finite, but unspecified number of* ω^κ

$$
\omega^\kappa = \mathcal{T}^*\hat{\omega}^\kappa = (\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathcal{D}_x)^\kappa\mathfrak{r}^3,\ \kappa\in\mathbb{N}_0
$$

and Γ *runs through the set*

$$
\begin{aligned}
&\{\tilde{\mathfrak{J}}^\kappa\tilde{\psi},\ \tilde{\mathcal{D}}_y^\iota\tilde{\mathfrak{J}}^\kappa\tilde{\psi},\ \tilde{\mathcal{D}}_z^\iota\tilde{\mathfrak{J}}^\kappa\tilde{\psi},\kappa\in\mathbb{N}_0,\ \iota\in\mathbb{N}\},\ \textit{where}\\
&\tilde{\mathcal{D}}_y=\hat{\mathcal{T}}_*\mathcal{D}_y=-\frac{1}{\mathfrak{r}^1_x}\big(\mathcal{D}_t+(\mathfrak{r}^1+\mathfrak{r}^2-1)\mathcal{D}_x\big),\quad \tilde{\mathcal{D}}_z=\hat{\mathcal{T}}_*\mathcal{D}_z=-\frac{1}{\mathfrak{r}^2_x}\big(\mathcal{D}_t+(\mathfrak{r}^1+\mathfrak{r}^2+1)\mathcal{D}_x\big),\\
&\tilde{\mathfrak{J}}=\frac{\mathfrak{r}^1}{2}\tilde{\mathcal{D}}_y+\frac{\mathfrak{r}^2}{2}\tilde{\mathcal{D}}_z,\quad \tilde{\psi}=\mathcal{T}^*\psi=-\frac{1}{2}\left(\mathrm{e}^{(\mathfrak{r}^2-\mathfrak{r}^1)/2}\phi-\mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}(x-(\mathfrak{r}^1+\mathfrak{r}^2)t)\right).
\end{aligned}
$$

Remark 3.31. It is easily seen that the list of potential symmetries in Theorem 3.30 contains all generalized symmetries listed in Theorem 3.10 but is not exhausted by them. Thanks to the choice of coordinates on the manifold $\tilde{\mathcal{S}}^{(\infty)}$ we can single out nonlocal symmetries quite easily. Indeed, a potential symmetry is nonlocal if and only if it characteristic-tuple depends nontrivially on ϕ.

Just like with generalized symmetries of the system $\mathcal{S}$, a generalized symmetry of the Klein–Gordon equation can be mapped to a generalized symmetry of the system $\tilde{\mathcal{S}}$ if and only if the associated operator belongs to the subspace

$$\langle\, 1,\, (\mathfrak{D}_y+1)\mathfrak{D}_y^{\iota}\mathfrak{J}^{\kappa},\, (\mathfrak{D}_z+1)\mathfrak{D}_z^{\iota}\mathfrak{J}^{\kappa},\, \kappa,\iota\in\mathbb{N}_0\,\rangle.$$

Thus, to prolong generalized symmetries of the form $\mathfrak{J}^{\kappa}$, $\kappa\in\mathbb{N}$, one needs to construct another covering of the system $\mathcal{S}$.

3.8 Hamiltonian structures of hydrodynamic type

Local Hamiltonian structures of the system $\mathcal{S}$ were found in my MSc thesis. Recall that Hamiltonian operators of the form $\mathfrak{H}^{ij}_k = g^{ij}_k \mathrm{D}_x + b^{ij}_{kl}\mathfrak{r}^l_x$ are common for (1+1)-dimensional hydrodynamic-type systems. The Hamiltonian properties impose strong conditions on the coefficients of these operators. In particular, g should be a flat (pseudo)-Riemannian metric, see e.g. [42].

Theorem 3.32. *The system* (3.1) *admits an infinite family of compatible local Hamiltonian structures $\mathfrak{H}_\Theta$ parameterized by a smooth function Θ of $\mathfrak{r}^3$,*

$$\hat{\mathfrak{H}}^1_\Theta = \mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\,\mathrm{diag}\big(-1,1,\Theta(\mathfrak{r}^3)\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\big)\mathrm{D}_x - \frac{1}{2}\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\begin{pmatrix} \mathfrak{r}^2_x-\mathfrak{r}^1_x & \mathfrak{r}^1_x-\mathfrak{r}^2_x & -2\mathfrak{r}^3_x \\ \mathfrak{r}^2_x-\mathfrak{r}^1_x & \mathfrak{r}^1_x-\mathfrak{r}^2_x & -2\mathfrak{r}^3_x \\ 2\mathfrak{r}^3_x & 2\mathfrak{r}^3_x & -2f^{33} \end{pmatrix}$$

with the corresponding family of Hamiltonians $\hat{\mathcal{H}}^1_{\Xi,c_2}=\int \hat{H}^1_{\Xi,c_2}\mathrm{d}x$ defined by densities

$$\hat{H}^1_{\Xi,c_2} = \left(\frac{1}{4}(\mathfrak{r}^1+\mathfrak{r}^2)^2+\frac{1}{2}\big(\mathfrak{r}^1-\mathfrak{r}^2\big)+\Xi(\mathfrak{r}^3)\right)\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}+c_2(\mathfrak{r}^1+\mathfrak{r}^2).$$

Here $f^{33}:=\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\big((\mathfrak{r}^2_x-\mathfrak{r}^1_x)\Theta+\frac{1}{2}\mathfrak{r}^3_x\Theta_{\mathfrak{r}^3}\big)$, the function Ξ of $\mathfrak{r}^3$ and an arbitrary constant c_2 satisfy the differential equation $\Xi_{\mathfrak{r}^3\mathfrak{r}^3}\Theta+\frac{1}{2}\Theta_{\mathfrak{r}^3}\Xi_{\mathfrak{r}^3}=2c_2$.

Two Hamiltonian operators are called compatible if any of their linear combination is a Hamiltonian operator as well. Two nondegenerate hydrodynamic-type Hamiltonian operators for a hydrodynamic-type system is compatible if the Nijenhuis tensor $\mathcal{N}$ of the tensor (s^i_j) defined by $s^i_j = \tilde{g}^{il} g_{lj}$ vanishes,

$$\mathcal{N}^i_{jk} := s^l_j \partial_{u^l} s^i_k - s^l_k \partial_{u^l} s^i_j - s^i_l(\partial_{u^j} s^l_k - \partial_{u^k} s^l_j) = 0,$$

see [55, 89]. Here g and $\tilde{g}$ are the metrics corresponding to the Hamiltonian operators. In terms of g and $\tilde{g}$, the condition of vanishing the Nijenhuis tensor $\mathcal{N}$ takes the form

$$\nabla^i \nabla^j \tilde{g}^{kl} + \nabla^k \nabla^l \tilde{g}^{ij} - \nabla^i \nabla^k \tilde{g}^{jl} - \nabla^j \nabla^l \tilde{g}^{ik} = 0. \tag{3.30}$$

The covariant differentiation in (3.30) corresponds to the metric g. The conditions (3.30) are preserved by the permutation of g and $\tilde{g}$, so that they are indeed the compatibility conditions of the two metrics.

When the tensor g degenerates at some point, the associated hydrodynamic-type system loses its geometric charm and one needs to proceed otherwise. To show that the bracket of a skew-symmetric Noether operator $\mathfrak{N}$ for $\mathcal{E}$ satisfies the Jacobi identity, one may equivalently check that the variational Schouten bracket $[\![\mathfrak{N}, \mathfrak{N}]\!]$ vanishes. To show the compatibility of two hydrodynamic-type Hamiltonian operators $\mathfrak{H}_1$ and $\mathfrak{H}_2$, $\mathfrak{H}^{ij}_k = g^{ij}_k \mathrm{D}_x + b^{ij}_{kl} \mathfrak{r}^l_x$, $k = 1, 2$, one may check that $[\![\mathfrak{H}_1, \mathfrak{H}_2]\!] = 0$, cf. [78, Section 10.1]. Since $\mathcal{E}$ is a system of evolution equations, one may consider the cotangent covering $T^*\mathcal{E}$ of $\mathcal{E}$ (i.e., the joint system $F = 0$, $\ell^\dagger_F(\lambda) = 0$) and substitute the latter condition by the equivalent one

$$\mathsf{E} \sum_{j=1}^{n} \Big((\mathsf{E}_{u^j} \mathrm{F}_{\mathfrak{H}_1})(\mathsf{E}_{\lambda^j} \mathrm{F}_{\mathfrak{H}_2}) + (\mathsf{E}_{\lambda^j} \mathrm{F}_{\mathfrak{H}_1})(\mathsf{E}_{u^j} \mathrm{F}_{\mathfrak{H}_2}) \Big) = 0,$$

where $\mathrm{F}_{\mathfrak{H}_k} = \sum_{i,j} \big(g^{ij}_k (\mathrm{D}_x \lambda^i) \lambda^j + b^{ij}_{kl} \mathfrak{r}^l_x \lambda^i \lambda^j \big)$, $k = 1, 2$, and $\mathsf{E} = (\mathsf{E}_{u^1}, \dots, \mathsf{E}_{u^n}, \mathsf{E}_{\lambda^1}, \dots, \mathsf{E}_{\lambda^n})$ is the Euler operator on $T^*\mathcal{E}$.

For the system (3.1) the tensor (s^i_j) takes the simple form, $(s^i_j) = \mathrm{diag}(1, 1, \tilde{\Theta}/\Theta)$, where Θ and $\tilde{\Theta}$ are functions of $\mathfrak{r}^3$ parameterizing the metrics g and $\tilde{g}$. It is trivial to verify that its Nijenhuis tensor vanishes. Since eigenvalues of (s^i_j) are not distinct, we need also to verify the conditions (3.30), and they also hold.

If Θ is a somewhere vanishing function, then the geometric reasoning for Hamiltonian operators is no longer available, and we should proceed by establishing that the corresponding variational Schouten brackets vanish, which is done symbolically.

Below we consider only canonical representatives of symmetry-type objects, where derivatives involving differentiations with respect to t are replaced by their expressions in view of the system $\mathcal{S}$, which is necessary for relating different kinds of such objects via Hamiltonian structures.

For any Hamiltonian operator $\mathfrak{H}_\Theta$ from Theorem 3.32, we can endow the space $\hat\Upsilon^{\rm q}$ of canonical representatives for cosymmetries of $\mathcal{S}$ with a Lie-algebra structure, cf. [61] and [20, Section 3.1], where the corresponding Lie bracket is defined by

$$[\gamma^1,\gamma^2]_{\mathfrak{H}_\Theta}=\ell_{\gamma^2}\mathfrak{H}_\Theta\gamma^1+\ell^\dagger_{\mathfrak{H}_\Theta\gamma^1}\gamma^2+(\ell_{\gamma^1}-\ell^\dagger_{\gamma^1})\mathfrak{H}_\Theta\gamma^2$$

for any $\gamma^1,\gamma^2\in\hat\Upsilon^{\rm q}$. Here ℓ_γ and $\ell^\dagger_\gamma$ are the universal linearization operator of $\gamma\in\hat\Upsilon^{\rm q}$ and its formal adjoint, respectively. Denote the Lie algebra with the underlying space $\hat\Upsilon^{\rm q}$ and the Lie bracket $[\cdot,\cdot]_{\mathfrak{H}_\Theta}$ by $\hat\Upsilon^{\rm q}_\Theta$. The operator $\mathfrak{H}_\Theta$ establishes a homomorphism from the Lie algebra $\hat\Upsilon^{\rm q}_\Theta$ to the Lie algebra $\hat\Sigma^{\rm q}$. The image $\mathfrak{H}_\Theta\hat\Upsilon^{\rm q}_\Theta$ of this homomorphism is a proper subalgebra of $\hat\Sigma^{\rm q}$ of canonical representatives for generalized symmetries of the system $\mathcal{S}$. More specifically, the image $\mathfrak{H}_\Theta\hat\Upsilon^{\rm q}_\Theta$ is spanned by generalized symmetries from three families that are the images of the respective families from Theorem 3.13 and whose elements are, in the notation of Theorems 3.10 and 3.13, of the following form:

1. $\check{\mathcal{W}}(\bar\Omega^\Theta)$, where $\bar\Omega^\Theta=\hat{\mathcal{A}}\big((\hat{\mathcal{A}}\Omega)\Theta/\omega^1\big)$,
2. $\check{\mathcal{P}}(\bar\Phi)$, where $\bar\Phi=\Phi_{\mathfrak{r}^1}-\frac12\Phi$, and thus the parameter function $\bar\Phi=\bar\Phi(\mathfrak{r}^1,\mathfrak{r}^2)$ runs through the solution space of the Klein–Gordon equation $\bar\Phi_{\mathfrak{r}^1\mathfrak{r}^2}=-\bar\Phi/4$ as well,
3. $\check{\mathcal{R}}(\bar\Gamma)$, where $\bar\Gamma=\frac12(\tilde{\mathfrak{D}}_y-1)\tilde{\mathfrak{X}}\tilde q$.

For the nonvanishing function Θ, the kernel of the above homomorphism is two-dimensional and spanned by the cosymmetries ${\rm e}^{\mathfrak{r}^1-\mathfrak{r}^2}(1,-1,0)$ and ${\rm e}^{\mathfrak{r}^1-\mathfrak{r}^2}(\bar\Theta,-\bar\Theta,\bar\Theta_{\mathfrak{r}^3})$ with an antiderivative $\bar\Theta$ of $1/\Theta$, $\bar\Theta_{\mathfrak r^3}=1/\Theta$. The former cosymmetry is special due to being a single (up to linear independence) common element of the first and the second families from Theorem 3.13, see Remark 3.14. Both the cosymmetries are conservation-law characteristics and are associated with the conserved currents $\mathrm e^{\mathfrak r^1-\mathfrak r^2}(1,\mathfrak r^1+\mathfrak r^2)$ and $\mathrm e^{\mathfrak r^1-\mathfrak r^2}\big(\bar\Theta,(\mathfrak r^1+\mathfrak r^2)\bar\Theta\big)$, which belong to the first family of Theorem 3.15. As a result, the space of distinguished (Casimir) functionals of the Hamiltonian operator $\mathfrak H_\Theta$ is spanned by two functionals,

$$\mathcal C_1:=\int\mathrm e^{\mathfrak r^1-\mathfrak r^2}\,\mathrm dx,\quad \mathcal C_2^\Theta:=\int\mathrm e^{\mathfrak r^1-\mathfrak r^2}\bar\Theta(\mathfrak r^3)\,\mathrm dx.$$

In the degenerate case with $\Theta\equiv0$, the kernel of the above homomorphism is infinite-dimensional and coincides with the first family of Theorem 3.13. Elements of this family are conservation-law characteristics if and only if they belong to the first family of Theorem 3.18 and are thus associated with conserved currents from the first family of Theorem 3.15. This means that the space of distinguished (Casimir) functionals of the Hamiltonian operator $\mathfrak H_0$ consists of the functionals

$$\int\mathrm e^{\mathfrak r^1-\mathfrak r^2}\Omega(\omega^0,\omega^1,\dots)\,\mathrm dx,$$

where the parameter function Ω runs through the space of smooth functions of a finite, but unspecified number of $\omega^\kappa=(\mathrm e^{\mathfrak r^2-\mathfrak r^1}\mathcal D_x)^\kappa\mathfrak r^3$, $\kappa\in\mathbb N_0$.

Consider the constraints that single out the space of canonical representatives conservation-law characteristics of $\mathcal S$, which is described in Theorem 3.18, from the space $\hat\Upsilon^{\mathrm q}$ of canonical representatives of cosymmetries of $\mathcal S$. Imposing these constraints on Ω and $\tilde{\mathfrak X}$ that parameterize families spanning $\mathfrak H_\Theta\hat\Upsilon^{\mathrm q}$, we single out the algebra of Hamiltonian symmetries of $\mathcal S$ associated with the Hamiltonian operator $\mathfrak H_\Theta$.

Theorem 3.33. *Given a smooth function Θ of $\omega^0:=\mathfrak r^3$, the algebra of Hamiltonian symmetries of the system* (3.1) *for the Hamiltonian operator $\mathfrak H_\Theta$ is spanned by the generalized vector fields*

$$\check{\mathcal W}(\bar\Omega^\Theta)=\bar\Omega^\Theta\partial_{\mathfrak r^3},\ \check{\mathcal P}(\Phi)=\mathrm e^{(\mathfrak r^2-\mathfrak r^1)/2}\left((\Phi+2\Phi_{\mathfrak r^1})\mathfrak r^1_x\partial_{\mathfrak r^1}+(\Phi-2\Phi_{\mathfrak r^2})\mathfrak r^2_x\partial_{\mathfrak r^2}+2\Phi\mathfrak r^3_x\partial_{\mathfrak r^3}\right),$$

$$\check{\mathcal{R}}(\bar{\Gamma}) = \mathrm{e}^{(\mathfrak{r}^2-\mathfrak{r}^1)/2}\Big((\tilde{\mathfrak{D}}_y\bar{\Gamma}+\bar{\Gamma})\mathfrak{r}^1_x\partial_{\mathfrak{r}^1} + (\tilde{\mathfrak{D}}_z\bar{\Gamma}+\bar{\Gamma})\mathfrak{r}^2_x\partial_{\mathfrak{r}^2} + 2\bar{\Gamma}\mathfrak{r}^3_x\partial_{\mathfrak{r}^3}\Big),$$

where $\bar{\Omega}^\Theta = \hat{\mathcal{A}}\big(\Theta\sum_{\kappa=0}^\infty(-\hat{\mathcal{A}})^\kappa\Omega_{\omega^\kappa}\big)$ *with the operator* $\hat{\mathcal{A}} = \sum_{\kappa=0}^\infty \omega^{\kappa+1}\partial_{\omega^\kappa}$ *and with* Ω *running through the space of smooth functions of a finite, but unspecified number of* $\omega^\kappa = (\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathrm{D}_x)^\kappa\mathfrak{r}^3$*,* $\kappa\in\mathbb{N}_0$*, the parameter function* $\Phi=\Phi(\mathfrak{r}^1,\mathfrak{r}^2)$ *runs through the solution space of the Klein–Gordon equation* $\Phi_{\mathfrak{r}^1\mathfrak{r}^2} = -\Phi/4$*, and* $\bar{\Gamma} = \frac12(\tilde{\mathfrak{D}}_y - 1)\tilde{\mathfrak{X}}\tilde{q}$ *with the operator* $\tilde{\mathfrak{X}}$ *running through the set*

$$\big\{\tilde{\mathfrak{J}}^{\kappa'},\ \kappa'\in 2\mathbb{N}_0+1,\ (\tilde{\mathfrak{J}}+\iota/2)^\kappa\tilde{\mathfrak{D}}_y^\iota,\ (\tilde{\mathfrak{J}}-\iota/2)^\kappa\tilde{\mathfrak{D}}_z^\iota,\ \kappa\in\mathbb{N}_0,\ \iota\in\mathbb{N},\ \kappa+\iota\in 2\mathbb{N}_0+1\big\}.$$

$$\text{where}\quad \tilde{\mathfrak{D}}_y := -\frac{1}{\mathfrak{r}^1_x}\big(\mathfrak{D}_t + (\mathfrak{r}^1+\mathfrak{r}^2-1)\mathfrak{D}_x\big),\quad \tilde{\mathfrak{D}}_z := -\frac{1}{\mathfrak{r}^2_x}\big(\mathfrak{D}_t + (\mathfrak{r}^1+\mathfrak{r}^2+1)\mathfrak{D}_x\big),$$
$$\tilde{\mathfrak{J}} := \frac{\mathfrak{r}^1}{2}\tilde{\mathfrak{D}}_y + \frac{\mathfrak{r}^2}{2}\tilde{\mathfrak{D}}_z,\quad \tilde{q} := \mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\big(x-(\mathfrak{r}^1+\mathfrak{r}^2+1)t\big).$$

The system $\mathcal{S}_0$ describes one-dimensional isentropic gas flows with constant sound speed and is known to possess three compatible Hamiltonian structures of Dubrovin–Novikov type [97]. In Riemann invariants those are

$$\mathfrak{H}^1 = \mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\left(\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\mathrm{D}_x - \frac12\begin{pmatrix}\mathfrak{r}^2_x-\mathfrak{r}^1_x & \mathfrak{r}^1_x-\mathfrak{r}^2_x\\ \mathfrak{r}^2_x-\mathfrak{r}^1_x & \mathfrak{r}^1_x-\mathfrak{r}^2_x\end{pmatrix}\right),$$
$$\mathfrak{H}^2 = \mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\mathrm{D}_x + \frac12\begin{pmatrix}\mathfrak{r}^2_x-\mathfrak{r}^1_x & -\mathfrak{r}^1_x-\mathfrak{r}^2_x\\ \mathfrak{r}^1_x+\mathfrak{r}^2_x & \mathfrak{r}^2_x-\mathfrak{r}^1_x\end{pmatrix}\right),$$
$$\mathfrak{H}^3 = \mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\left(\begin{pmatrix}\mathfrak{r}^1 & 0\\ 0 & \mathfrak{r}^2\end{pmatrix}\mathrm{D}_x + \frac12\begin{pmatrix}(1-\mathfrak{r}^1)\mathfrak{r}^1_x+\mathfrak{r}^1\mathfrak{r}^2_x & -\mathfrak{r}^2\mathfrak{r}^1_x-\mathfrak{r}^1\mathfrak{r}^2_x\\ \mathfrak{r}^2\mathfrak{r}^1_x+\mathfrak{r}^1\mathfrak{r}^2_x & -\mathfrak{r}^2\mathfrak{r}^1_x+(1+\mathfrak{r}^2)\mathfrak{r}^2_x\end{pmatrix}\right),$$

with the associated families of Hamiltonians

$$\mathcal{H}^1_{c_1,c_2} = \left(\frac14(\mathfrak{r}^1+\mathfrak{r}^2)^2 + \frac12(\mathfrak{r}^1-\mathfrak{r}^2)+c_1\right)\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2} + c_2(\mathfrak{r}^1+\mathfrak{r}^2),$$
$$\mathcal{H}^2_{c_1,c_2} = -(\mathfrak{r}^1+\mathfrak{r}^2)\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2} + \mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\left(c_1\sin\frac{\mathfrak{r}^1+\mathfrak{r}^2}{2} + c_2\cos\frac{\mathfrak{r}^1+\mathfrak{r}^2}{2}\right),$$
$$\mathcal{H}^3_{c_1,c_2} = -2\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2} + c_1\,\mathrm{erf}\left(\frac{\sqrt{\mathfrak{r}^2}+\sqrt{-\mathfrak{r}^1}}{\sqrt2}\right) + c_2\,\mathrm{erf}\left(\frac{\sqrt{\mathfrak{r}^2}-\sqrt{-\mathfrak{r}^1}}{\sqrt2}\right).$$

The multipliers of parameterizing constants c_1 and c_2 are densities of the Casimir functionals of the corresponding Hamiltonian operators, and therefore, by and large, there is only one essential Hamiltonian for every family of the above Hamiltonian operators.

The first family of Hamiltonian structures are locally prolonged to that of the system $\mathcal{S}$, cf. Theorem 3.32. Let us investigate what happens with the Hamiltonian operators $\mathfrak{H}^2$ and $\mathfrak{H}^3$ upon a prolongation to the third equation. For this aim, we consider nonlocal Noether operators of the hydrodynamic type

$$\mathcal{N}^{ij} = g^{ij}\mathrm{D}_x - g^{is}\Gamma^j_{sk}\mathfrak{r}^k_x + \sum_{\alpha=1}^{3} \epsilon_\alpha w^i_{\alpha k}\mathfrak{r}^k_x\ \mathrm{D}_x^{-1} \circ w^j_{\alpha l}\mathfrak{r}^l_x,$$

where the metric components g^{ij}, the affinors components $w^i_{\alpha k}$ and the Christoffel symbols Γ^j_{sk} are smooth functions of $(\mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{r}^3)$, and $\epsilon_\alpha \in \{-1, 1\}$, cf. [52, 54, 91]. Of course, we want the restriction of the local part of this operator to coincide with the operator $\mathfrak{H}^2$. Plugging $\mathcal{N}$ into the determining equations $\eta = \mathcal{N}\lambda$, where η is the symmetry-tuple (a row) and λ is the cosymmetry-tuple (a column) of the system $\mathcal{S}$, we see that along the way there arise two types of nonlocalities we need to deal with, $\mathrm{D}_x^{-1}(A^i_\alpha\lambda^\alpha)$ and $\mathrm{D}_x^{-1}\mathrm{D}_t(A^i_\alpha\lambda^\alpha)$, where $A^i_\alpha = w^i_{\alpha l}\mathfrak{r}^l_x$. To get rid of them, we must ensure that either coefficients sum up to 0, or they are local magnitudes to begin with. The first type above is no doubt nonlocal and equating their coefficients to 0 we find the following constraints on the affinors w_α,

$$w_\alpha = \mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathrm{diag}\left(\Psi^\alpha_{\mathfrak{r}^1}, -\Psi^\alpha_{\mathfrak{r}^2}, \Phi^\alpha + \Psi^\alpha\right),$$

where Φ's are arbitrary smooth functions of $\mathfrak{r}^3$ and $\Psi = \Psi(\mathfrak{r}^1, \mathfrak{r}^2)$'s are arbitrary solutions of the Klein–Gordon equation $\Psi^\alpha_{\mathfrak{r}^2} - \Psi^\alpha_{\mathfrak{r}^1} = 2\Psi^\alpha_{\mathfrak{r}^1\mathfrak{r}^2}$. In fact, the very same conditions ensure the nonlocalities of the second type be local and we may consider the determining equations for $\mathfrak{N}$ without further ado.

The consideration until this point was valid for both $\mathfrak{H}^2$ and $\mathfrak{H}^3$. Now we prolong the operator $\mathfrak{H}^2$, while the consideration for $\mathfrak{H}^3$ is very similar. Solving the determining equations above one finds the general form of the Noether operator,

$$\mathfrak{N}^2 = \mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathrm{diag}\left(1,1,\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\Theta\right)\mathrm{D}_x + \sum_{\alpha=1}^{3}\epsilon_\alpha w^i_{\alpha k}\mathfrak{r}^k_x\,\mathrm{D}_x^{-1}\circ w^j_{\alpha l}\mathfrak{r}^l_x$$
$$+\frac{1}{2}\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\begin{pmatrix} \mathfrak{r}^2_x-\mathfrak{r}^1_x & -\mathfrak{r}^1_x-\mathfrak{r}^2_x & -2\mathfrak{r}^3_x \\ \mathfrak{r}^1_x+\mathfrak{r}^2_x & \mathfrak{r}^2_x-\mathfrak{r}^1_x & 2\mathfrak{r}^3_x \\ 2\mathfrak{r}^3_x & -2\mathfrak{r}^3_x & 2\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}(\mathfrak{r}^2_x-\mathfrak{r}^1_x)\Theta+\bar{\Theta} \end{pmatrix}$$

for some smooth functions Θ of $\mathfrak{r}^3$ and $\bar{\Theta}$ of $(\mathfrak{r}^3, \mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathfrak{r}^3_x)$. Only hydrodynamic-type Noether operator of this form are formally skew-adjoint. Indeed, $\mathfrak{N}^2$ is formally skew-adjoint if and only if $\bar{\Theta} = \mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathfrak{r}^3_x\Theta_{\mathfrak{r}^3}$. Moreover, there are additional constraints on the arbitrary functions Φ's and Ψ's,

$$\sum_{\alpha=1}^{3}\epsilon_\alpha(\Phi^\alpha+\Psi^\alpha)\Psi^\alpha_{\mathfrak{r}^1} = -\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}, \tag{3.31a}$$
$$\sum_{\alpha=1}^{3}\epsilon_\alpha(\Phi^\alpha+\Psi^\alpha)\Psi^\alpha_{\mathfrak{r}^2} = \mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}, \tag{3.31b}$$
$$\sum_{\alpha=1}^{3}\epsilon_\alpha\Psi^\alpha_{\mathfrak{r}^1}\Psi^\alpha_{\mathfrak{r}^2} = 0. \tag{3.31c}$$

Those are the very conditions to ensure that ${R^{ij}}_{kl} = \sum_\alpha(w^i_{\alpha k}w^j_{\alpha l} - w^j_{\alpha k}w^i_{\alpha l})$, which are required by the Jacobi identity. The other such conditions [52], commutativity of the affinors ω_α, $j-k$ symmetry of $\nabla_k w^i_{\alpha j}$ and $i-j$ symmetry of $g_{ik}w^k_{\alpha j}$ are also satisfied.

The first two equations integrate to $\sum_{\alpha=1}^{3}\epsilon_\alpha\left(\Phi^\alpha+\frac{1}{2}\Psi^\alpha\right)\Psi^\alpha = \Omega(\mathfrak{r}^3) - \mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}$ for some function Ω of $\mathfrak{r}^3$, while the third one ensures that $\sum_{\alpha=1}^{3}\epsilon_\alpha\left(\Psi^\alpha\right)^2$ satisfies the Klein–Gordon equation as well.

Theorem 3.34. *The system $\mathcal{S}$ admits two families of nonlocal first-order Hamiltonian operators of hydrodynamic type,*

$$\hat{\mathfrak{H}}^2_{\Theta,\alpha} = \mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\left(\mathrm{diag}\left(1,1,\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\Theta\right)\mathrm{D}_x + \frac{1}{2}\begin{pmatrix} \mathfrak{r}^2_x-\mathfrak{r}^1_x & -\mathfrak{r}^1_x-\mathfrak{r}^2_x & -2\mathfrak{r}^3_x \\ \mathfrak{r}^1_x+\mathfrak{r}^2_x & \mathfrak{r}^2_x-\mathfrak{r}^1_x & 2\mathfrak{r}^3_x \\ 2\mathfrak{r}^3_x & -2\mathfrak{r}^3_x & 2f^{33} \end{pmatrix}\right)$$
$$+\sum_{\alpha=1}^{3}\epsilon_\alpha w^i_{\alpha k}\mathfrak{r}^k_x\,\mathrm{D}_x^{-1}\circ w^j_{\alpha l}\mathfrak{r}^l_x,$$

$$
\begin{aligned}
\hat{\mathfrak{H}}^3_{\Theta,\alpha} &= \mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathrm{diag}\left(\mathfrak{r}^1,\mathfrak{r}^2,\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\Theta\right)\mathrm{D}_x+\sum_{\alpha=1}^{3}\epsilon_\alpha w^i_{\alpha k}\mathfrak{r}^k_x\ \mathrm{D}_x^{-1}\circ w^j_{\alpha l}\mathfrak{r}^l_x\\
&+\frac{1}{2}\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\begin{pmatrix}\mathfrak{r}^1_x+\mathfrak{r}^1(\mathfrak{r}^2_x-\mathfrak{r}^1_x) & -\mathfrak{r}^2\mathfrak{r}^1_x-\mathfrak{r}^1\mathfrak{r}^2_x & -2\mathfrak{r}^1\mathfrak{r}^3_x\\ \mathfrak{r}^2\mathfrak{r}^1_x+\mathfrak{r}^1\mathfrak{r}^2_x & \mathfrak{r}^2_x+\mathfrak{r}^2(\mathfrak{r}^2_x-\mathfrak{r}^1_x) & 2\mathfrak{r}^2\mathfrak{r}^3_x\\ 2\mathfrak{r}^1\mathfrak{r}^3_x & -2\mathfrak{r}^2\mathfrak{r}^3_x & 2f^{33}\end{pmatrix}.
\end{aligned}
$$

where $f^{33}=\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}((\mathfrak{r}^2_x-\mathfrak{r}^1_x)\Theta+\frac{1}{2}\Theta_{\mathfrak{r}^3}\mathfrak{r}^3_x)$, $w_\alpha=\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathrm{diag}\left(\Psi^\alpha_{\mathfrak{r}^1},-\Psi^\alpha_{\mathfrak{r}^2},\Phi^\alpha+\Psi^\alpha\right)$, Φ^α*'s are arbitrary smooth functions of* $\mathfrak{r}^3$ *and* Ψ^α*'s run through the solution set of the Klein–Gordon equation* $\Psi^\alpha_{\mathfrak{r}^2}-\Psi^\alpha_{\mathfrak{r}^1}=2\Psi^\alpha_{\mathfrak{r}^1\mathfrak{r}^2}$ *and additionally satisfy the joint condition* $\sum_{\alpha=1}^3\epsilon_\alpha\Psi^\alpha_{\mathfrak{r}^1}\Psi^\alpha_{\mathfrak{r}^2}=0$, *and two individual ones for each case,*

$$
\begin{aligned}
\hat{\mathfrak{H}}^2_{\Theta,\alpha}\colon\ &\sum_{\alpha=1}^{3}\epsilon_\alpha\left(\Phi^\alpha+\frac{1}{2}\Psi^\alpha\right)\Psi^\alpha=\Omega(\mathfrak{r}^3)-\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2},\\
\hat{\mathfrak{H}}^3_{\Theta,\alpha}\colon\ &\sum_{\alpha=1}^{3}\epsilon_\alpha\left(\Phi^\alpha+\frac{1}{2}\Psi^\alpha\right)\Psi^\alpha=\Omega(\mathfrak{r}^3)-\frac{1}{2}(\mathfrak{r}^1+\mathfrak{r}^2)\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}
\end{aligned}
$$

for an arbitrary function Ω *of its argument and* $\epsilon_\alpha\in\{-1,1\}$.

Let us study the above system on parameterizing functions (Φ^i,Ψ^i) in more detail. Firstly, the system admits the discrete symmetry transformation $(\Phi^i,\Psi^i)\mapsto(\Phi^j,\Psi^j)$ and the gauge transformation $(\Phi^i,\Psi^i)\mapsto(\Phi^i-c,\Psi^i+c)$.

Assume first that not all Φ^α's are constants. Up to the above discrete symmetry, we can assume without loss of generality that $\Phi^3_{\mathfrak{r}^3}\neq 0$. Differentiating the equations (3.31a)–(3.31b) twice with respect to $\mathfrak{r}^3$, we can get the system

$$
\Psi^1_{\mathfrak{r}^1}\phi^1+\Psi^2_{\mathfrak{r}^1}\phi^2=0,\quad \Psi^1_{\mathfrak{r}^2}\phi^1+\Psi^2_{\mathfrak{r}^2}\phi^2=0,
$$

where $\phi^\alpha:=\epsilon_\alpha(\Phi^\alpha_{\mathfrak{r}^3}\Phi^3_{\mathfrak{r}^3\mathfrak{r}^3}-\Phi^3_{\mathfrak{r}^3}\Phi^\alpha_{\mathfrak{r}^3\mathfrak{r}^3})$. Assume first that $\operatorname{rk}J=0$, where $J:=(\Psi^i_{\mathfrak{r}^j})^2_{i,j=1}$. Then all Φ^α's are constants in view of (3.31c), which contradicts (3.31b).

When $\operatorname{rk}J=1$, then there exists a nonzero constant c_1 such that $\Psi^\alpha=\Psi^\alpha(\omega)$, where $\omega=\mathfrak{r}^1+c_1\mathfrak{r}^2$, $\alpha=1,2$. One of these Ψ^α's must be nonconstant, and without loss of generality we assume $\Psi^1_\omega\neq 0$. Then $\phi^1=-\Psi^2_\omega\phi^2/\Psi^1_\omega$, which is possible only when there exists a constant c_2 such that $\Psi^2_\omega=c_2\Psi^1_\omega$. Thus $\Psi^2=c_2\Psi^1+c_3$ and $\Phi^1=-\epsilon_1\epsilon_2c_2\Phi^2+c_4\Phi^3$

up to the gauge symmetry for some constants c_i, $i=3,4,5$. Splitting (3.31a) with respect to Φ^3 one finds that $\Psi^3=-\epsilon_1\epsilon_3c_4\Psi^1+c_5$, which reduces (3.31c) to $\epsilon_1+\epsilon_2c_2^2+\epsilon_3c_4^2=0$. The system (3.31a)— (3.31b) integrates to

$$\sum_{\alpha=1}^{3}\epsilon_\alpha\left(\Phi^\alpha+\frac{1}{2}\Psi^\alpha\right)\Psi^\alpha=\Omega(\mathfrak{r}^3)-\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}$$

for an arbitrary function Ω of $\mathfrak{r}^3$. Plugging all the above expression in this equation and separating $\mathfrak{r}^3$-part from it, one finds $\epsilon_2c_3\Phi^2+\epsilon_3c_5\Phi^3=\Omega-c_6$ for some constant c_6 and

$$\Phi^1=-\epsilon_1\epsilon_2c_2\Phi^2+c_4\Phi^3,\quad \Psi^1(\mathfrak{r}^1,\mathfrak{r}^2)=c_6+\frac{\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}}{\epsilon_2c_2c_3-\epsilon_1c_4c_6},\quad \Psi^2=c_2\Psi^1+c_3,$$
$$\Psi^3=-\epsilon_1\epsilon_3c_4\Psi^1+c_5,$$

where $\epsilon_1+\epsilon_2c_2^2+\epsilon_3c_4^2=0$, Φ^3 runs through the set of arbitrary functions of $\mathfrak{r}^3$ and expression in the denominator of the equality for Ψ^1 is nonvanishing. In particular, ϵ's can not be of the same sign, and $c_1=-1$.

If J is nondegenerate, then immediately $\Phi^1=c_1\Phi^3$, $\Phi^2=c_2\Phi^3$ up to the gauge transformation and the splitting of (3.31a) with respect to Φ^3 gives $\Psi^3=c_3-\epsilon_1\epsilon_3c_1\Psi^1-\epsilon_2\epsilon_3c_2\Psi^2$. Integrating (3.31a)–(3.31b) gives a quadratic polynomial in Ψ^1 and Ψ^2, which does not violate the nondegeneracy condition for J if and only if its a polynomial in a single variable. Without loss of generality, let Ψ^1 be this variable. Then $\epsilon_2=-\epsilon_3$, $c_2=\pm1$, $c_1=c_3=0$ and $\Psi^1(\mathfrak{r}^1,\mathfrak{r}^2)=\pm\sqrt{c_4-2\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}}$, which is not a solution to the Klein–Gordon equation for any c_4.

Examples of admissible tuples of functions $\{\Phi^\alpha,\Psi^\alpha\}$ for $-\epsilon_1=-\epsilon_2=\epsilon_3=1$ are

$$\begin{aligned}\hat{\mathfrak{H}}^2_{\Theta,\Omega}\colon\ &\Phi^1=-\frac{\Omega}{2},\quad \Phi^2=\frac{1}{2},\quad \Phi^3=0,\quad \Psi^1=\frac{1}{2},\\ &\Psi^2(\mathfrak{r}^1,\mathfrak{r}^2)=2\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}=-\Psi^3(\mathfrak{r}^1,\mathfrak{r}^2);\\ \hat{\mathfrak{H}}^3_{\Theta,\Omega}\colon\ &\Phi^1=-\frac{\Omega}{2},\quad \Phi^2=1,\quad \Phi^3=0,\quad \Psi^1=\frac{1}{2},\\ &\Psi^2(\mathfrak{r}^1,\mathfrak{r}^2)=(\mathfrak{r}^1+\mathfrak{r}^2)\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2},\ \Psi^3(\mathfrak{r}^1,\mathfrak{r}^2)=(\mathfrak{r}^1+\mathfrak{r}^2)\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}+\frac{1}{2}.\end{aligned}$$

When all Ψ^α's are constants, they can be assumed to be equal to zero up to the gauge symmetry. Thus, the solutions Ψ^α's of the Klein–Gordon equation enjoy the system

$$\sum_{\alpha=1}^{3} \epsilon_\alpha (\Psi^\alpha)^2 = C - 2\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}, \quad \sum_{\alpha=1}^{3} \epsilon_\alpha \Psi^\alpha_{\mathfrak{r}^1} \Psi^\alpha_{\mathfrak{r}^2} = 0.$$

Provided in [54, p. 11] is a list of situations in which nonlocal Hamiltonian operators naturally arise. An above prolongation procedure gives yet another such situation.

Alternatively, one can study nonlocal Hamiltonian operators of the system $\mathcal{S}$ as follows. It is known that any nonlocal Hamiltonian operator of a hydrodynamic-type system can be reduced to a local one via a reciprocal transformation [53]. Thus, a reciprocal transformation associated with a solution $\Psi(\mathfrak{r}^1, \mathfrak{r}^2) = \mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}$ of the Klein–Gordon equation[8],

$$\mathrm{d}\tilde{x} = \mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}(\mathrm{d}x - (\mathfrak{r}^1 + \mathfrak{r}^2)\mathrm{d}t), \quad \mathrm{d}\tilde{t} = \mathrm{d}t$$

maps the system $\mathcal{S}$ to the hydrodynamic-type system $\tilde{\mathcal{S}}$

$$\mathfrak{r}^1_{\tilde{t}} = -\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\mathfrak{r}^1_{\tilde{x}}, \quad \mathfrak{r}^2_{\tilde{t}} = \mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\mathfrak{r}^2_{\tilde{x}}, \quad \mathfrak{r}^3_{\tilde{t}} = 0,$$

which possesses the three families of local Hamiltonian operators,

$$\tilde{\mathfrak{H}}^1_{\tilde{\Theta}} = \mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\left(\mathrm{diag}(-1, 1, \mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\tilde{\Theta})\mathrm{D}_{\tilde{x}} + \frac{1}{2}\begin{pmatrix} \mathfrak{r}^1_{\tilde{x}} - \mathfrak{r}^2_{\tilde{x}} & \mathfrak{r}^2_{\tilde{x}} - \mathfrak{r}^1_{\tilde{x}} & 0 \\ \mathfrak{r}^1_{\tilde{x}} - \mathfrak{r}^2_{\tilde{x}} & \mathfrak{r}^2_{\tilde{x}} - \mathfrak{r}^1_{\tilde{x}} & 0 \\ 0 & 0 & \mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathfrak{r}^3_{\tilde{x}}\tilde{\Theta}_{\mathfrak{r}^3} \end{pmatrix}\right),$$

$$\tilde{\mathfrak{H}}^2_{\tilde{\Theta}} = \mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\left(\mathrm{diag}(1, 1, \mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\tilde{\Theta})\mathrm{D}_{\tilde{x}} + \frac{1}{2}\begin{pmatrix} \mathfrak{r}^1_{\tilde{x}} - \mathfrak{r}^2_{\tilde{x}} & \mathfrak{r}^1_{\tilde{x}} + \mathfrak{r}^2_{\tilde{x}} & 0 \\ -\mathfrak{r}^1_{\tilde{x}} - \mathfrak{r}^2_{\tilde{x}} & \mathfrak{r}^1_{\tilde{x}} - \mathfrak{r}^2_{\tilde{x}} & 0 \\ 0 & 0 & \mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathfrak{r}^3_{\tilde{x}}\tilde{\Theta}_{\mathfrak{r}^3} \end{pmatrix}\right),$$

$$\tilde{\mathfrak{H}}^3_{\tilde{\Theta}} = \mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\mathrm{diag}(\mathfrak{r}^1, \mathfrak{r}^2, \mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\tilde{\Theta})\mathrm{D}_{\tilde{x}}$$

[8] More precisely, it is associated with the conservation law $\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}(1, \mathfrak{r}^1 + \mathfrak{r}^2)$ of the system $\mathcal{S}$. This connection implies that the 1-form $\mathrm{d}\tilde{x}$ is closed.

$$
+\frac{1}{2}\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}\begin{pmatrix} (1+\mathfrak{r}^1)\mathfrak{r}^1_{\tilde{x}}-\mathfrak{r}^1\mathfrak{r}^2_{\tilde{x}} & \mathfrak{r}^2\mathfrak{r}^1_{\tilde{x}}+\mathfrak{r}^1\mathfrak{r}^2_{\tilde{x}} & 0 \\ -\mathfrak{r}^2\mathfrak{r}^1_{\tilde{x}}-\mathfrak{r}^1\mathfrak{r}^2_{\tilde{x}} & \mathfrak{r}^2\mathfrak{r}^1_{\tilde{x}}+(1-\mathfrak{r}^2)\mathfrak{r}^2_{\tilde{x}} & 0 \\ 0 & 0 & \mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathfrak{r}^3_{\tilde{x}}\tilde{\Theta}_{\mathfrak{r}^3} \end{pmatrix}
$$

parameterized by an arbitrary function $\tilde{\Theta}$ of $\mathfrak{r}^3$, with associated families of Hamiltonians

$$
\begin{aligned}
\mathcal{H}^1_{c_1,c_2,\tilde{\Xi}}(\mathfrak{r}^1,\mathfrak{r}^2,\mathfrak{r}^3) &= \frac{1}{4}(\mathfrak{r}^1+\mathfrak{r}^2)^2-\mathfrak{r}^1+c_1\mathrm{e}^{\mathfrak{r}^1-\mathfrak{r}^2}+c_2(\mathfrak{r}^1+\mathfrak{r}^2)+\tilde{\Xi}(\mathfrak{r}^3),\\
\mathcal{H}^2_{c_1,c_2,\tilde{\Xi}}(\mathfrak{r}^1,\mathfrak{r}^2,\mathfrak{r}^3) &= \mathrm{e}^{(\mathfrak{r}^2-\mathfrak{r}^1)/2}\left(c_1\sin\frac{\mathfrak{r}^1+\mathfrak{r}^2}{2}+c_2\cos\frac{\mathfrak{r}^1+\mathfrak{r}^2}{2}\right)-(\mathfrak{r}^1+\mathfrak{r}^2)+\tilde{\Xi}(\mathfrak{r}^3),\\
\mathcal{H}^3_{c_1,c_2,\tilde{\Xi}}(\tilde{\mathfrak{r}}^1,\tilde{\mathfrak{r}}^2,\mathfrak{r}^3) &= \sum_{i=1}^{2}\int\left(2\sqrt{\pi}\,\mathrm{erf}(\tilde{\mathfrak{r}}^i)+c_i\right)\mathrm{e}^{(\tilde{\mathfrak{r}}^i)^2}\mathrm{d}\tilde{\mathfrak{r}}^i+\tilde{\Xi}(\mathfrak{r}^3)
\end{aligned}
$$

parameterized by an arbitrary smooth function $\tilde{\Xi}$ of $\mathfrak{r}^3$ which additionally satisfies

$$
2\tilde{\Theta}\tilde{\Xi}_{\mathfrak{r}^3\mathfrak{r}^3}+\tilde{\Theta}_{\mathfrak{r}^3}\tilde{\Xi}_{\mathfrak{r}^3}=0.
$$

Here $\tilde{\mathfrak{r}}^1=(\sqrt{\mathfrak{r}^2}+\sqrt{-\mathfrak{r}^1})/\sqrt{2}$ and $\tilde{\mathfrak{r}}^2=(\sqrt{\mathfrak{r}^2}-\sqrt{-\mathfrak{r}^1})/\sqrt{2}$.

3.9 Conclusion

To study the diagonalized form (3.1) of the system $\mathcal{S}$, we heavily rely on its two primary features. The first feature is the degeneracy of $\mathcal{S}$ in the sense that this system is not genuinely nonlinear with respect to $\mathfrak{r}^3$ and, moreover, it is partially decoupled since the first two equations of $\mathcal{S}$ do not involve $\mathfrak{r}^3$. To take into account the degeneracy efficiently, we introduce the modified coordinates on $\mathcal{S}^{(\infty)}$, where derivatives of $\mathfrak{r}^3$ are replaced by ω's constituting a functional basis of the kernel of the operator $\mathfrak{B}$. This operator is nothing else but the differential operator in the total derivatives that is associated with the equation on $\mathfrak{r}^3$. From another perspective, the infinite tuple of ω's, $\omega^0:=\mathfrak{r}^3$, $\omega^{\kappa+1}:=\mathcal{A}\omega^\kappa$, $\kappa\in\mathbb{N}_0$, can be seen to be generated by the differential operator $\mathcal{A}:=\mathrm{e}^{\mathfrak{r}^2-\mathfrak{r}^1}\mathfrak{D}_x$, commuting with $\mathfrak{B}$, $[\mathcal{A},\mathfrak{B}]=0$, cf. [40]. The introduction of the modified coordinates essentially simplifies computations of all kinds of symmetry-like objects for the system $\mathcal{S}$. Due to the partial decoupling of the system $\mathcal{S}$, we recognize its essential subsystem $\mathcal{S}_0$ constituted by the equations (3.1a), (3.1b). The second primary feature of $\mathcal{S}$ is the linearization of $\mathcal{S}_0$ to the (1+1)-dimensional Klein–Gordon equation, which was thoroughly studied from the point of view of generalized symmetries and conservation laws in [112].

In turn, these features allow us to describe symmetry-like objects for the system $\mathcal{S}$ by working within the following general approach. For a given kind of symmetry-like objects for $\mathcal{S}$, we show that the chosen space U of canonical representatives of equivalence classes of such objects is the sum of three subspaces, $U = U_1 + U_2 + U_3$. One of them, say, U_1, stems from the degeneracy of $\mathcal{S}$, and thus its elements are parameterized by an arbitrary function of a finite but unspecified number of ω's. The other two subspaces, U_2 and U_3, are related to the linearization of $\mathcal{S}_0$ to the (1+1)-dimensional Klein–Gordon equation (3.4a). Singling out these two subspaces is induced by decomposing the objects of the same kind for the Klein–Gordon equation as sums of those underlaid by linear superposition of solutions of (3.4a) and those associated with linear generalized symmetries of (3.4a). This is why the elements of the subspaces U_2 and U_3 are parameterized by an arbitrary solution of the (1+1)-dimensional Klein–Gordon equation and by characteristics of reduced linear generalized symmetries of this equation, respectively. Although $(U_1+U_2)\cap U_3 = \{0\}$, the sum $U_1 + U_2 + U_3$ is not direct since the subspaces U_1 and U_2 are not disjoint, and their intersection is one-dimensional.

The first kind of objects we exhaustively describe for the system $\mathcal{S}$ is given by generalized symmetries. Not all generalized symmetries of the Klein–Gordon equation (3.4a) have counterparts among generalized symmetries of the system $\mathcal{S}$, which was also noted in [112] for first-order generalized symmetries. The most difficult problem here, which is solved in Lemma 3.9, is to single out the subalgebra $\mathfrak{A}$ of canonical representatives of generalized symmetries of the Klein–Gordon equation (3.4a) that have such counterparts. A complementary subalgebra to $\mathfrak{A}$ is $\bar{\mathfrak{A}} = \langle\, (\mathcal{J}^\kappa q)\partial_q,\ \kappa \in \mathbb{N}\,\rangle$. We conjecture that elements of $\bar{\mathfrak{A}}$ have counterparts among nonlocal, or specifically potential, symmetries of the system $\mathcal{S}$. In fact, we consider the simplest potential system for the system $\mathcal{S}$ in Section 3.7, but to no avail. In future research, we plan to study certain Abelian coverings and potential symmetries of the system $\mathcal{S}$ and of the Klein–Gordon equation (3.4a). We expect that the main role in this consideration will be played by the conservation laws of the Klein–Gordon equation (3.4a) with characteristics of the form $\mathcal{J}^\kappa e^{y+z}$, $\kappa \in \mathbb{N}_0$, and by their counterparts for the system $\mathcal{S}$.

Considering cosymmetries and local conservation laws, we do not need to make the selection among those for the Klein–Gordon equation (3.4a) since all of them have counterparts for the system $\mathcal{S}$. For conservation laws, this follows directly from the general assertion proved in [80, Theorem 1]. Amongst cosymmetries, local conservation laws and their characteristics, the complete description of the space of cosymmetries for the system $\mathcal{S}$ is the most complicated since it requires utilizing a couple of nontrivial tricks within the framework of our general approach.

To construct the space of local conservation laws of $\mathcal{S}$, we have to make use of the direct method [128, 166] whose essence is the direct construction of conserved currents canonically representing conservation laws using the definitions of conserved currents and of their equivalence. The standard approach [26] based on singling out conservation-law characteristics among cosymmetries is not effective for the system $\mathcal{S}$ since its application to $\mathcal{S}$ leads to too cumbersome computations. At the same time, we still need to know conservation-law characteristics for the system $\mathcal{S}$, in particular, to look for special-feature conservation laws, like low-order and translation-invariant ones. The known formula [152, Proposition 7.41] relating characteristics of conservation laws of systems in the extended Kovalevskaya form [125, Definition 4] to densities of these conservation laws gives suitable expressions only for characteristics of conservation laws from the second family of Theorem 3.15, which are of order zero. The other two families should be tackled differently. For the first family, we in fact derive an analogue of the above formula in terms of the operator $\mathcal{A}$ using the formal integration by parts. Characteristics of conservation laws from the third family are constructed from their counterparts being variational symmetries of the Klein–Gordon equation (3.4a). We also prove that under the action of generalized symmetries of the system $\mathcal{S}$ on its space of conservation laws, a generating set of conservation laws of this system is constituted by two zeroth-order conservation laws. One of them belongs to and generates the first subspace of conservation laws, which is related to the degeneracy of $\mathcal{S}$. The other is the counterpart of a single generating conservation law of the Klein–Gordon equation (3.4a). It belongs to the third subspace of conservation laws of $\mathcal{S}$ but generates the second subspace as well. The claim on generation of the entire third subspace is unexpected since only a proper part of linear generalized symmetries of the Klein–Gordon equation (3.4a) are naturally mapped to generalized symmetries of $\mathcal{S}$ but the amount of the images still suffices for generating all required conservation laws.

Interrelating generalized symmetries and cosymmetries, constructed in my MSc thesis was a family of compatible Hamiltonian operators for the system $\mathcal{S}$ parameterized by an arbitrary function of $\mathfrak{r}^3$, and a Hamiltonian operator from this family is degenerate if the corresponding value of the parameter function vanishes at some point. This fundamentally differs from the case of genuinely nonlinear hydrodynamic-type systems, for which the number of local Hamiltonian operators of hydrodynamic type is known not to exceed $n+1$, where n is the number of dependent variables, see [57]. In this thesis, we find even more Hamiltonian operators although they all are nonlocal, see Section 3.8. Their existence stems from the observation that the subsystem $\mathcal{S}_0$ possesses three hydrodynamic-type Hamiltonian structures. Each of them can be prolonged to the entire system likewise the procedure for both the generalized symmetries and conservation laws. It turns out that only one prolongation is local and leads to the above family of Hamiltonian structures, while another two are nonlocal. Thus, such a prolongation gives another natural construction of nonlocal Hamiltonian operators, cf. [54, p. 11]. The system $\mathcal{S}_0$ possesses a third-order Hamiltonian operator $\mathfrak{H}_3$ but it is not of hydrodynamic-type and neither its prolongation will be. While third-order nonlocal hydrodynamic-type Hamiltonian operators are well-studied, cf. [32], the operator $\mathfrak{H}_3$ may be specific to the subsystem $\mathcal{S}_0$ and may not have even a nonlocal prolongation, cf. Section 4.3.2.

We should like to emphasize that the local description of the solution set of the system $\mathcal{S}$ in Theorem 3.1 is implicit and involves the general solution of the (1+1)-dimensional Klein–Gordon equation. This is why it is difficult to further use this description, and thus it is still worthwhile to comprehensively study the system $\mathcal{S}$ within the framework of symmetry analysis of differential equations.

As the essential subsystem $\mathcal{S}_0$ coincides with the diagonalized form of the system describing one-dimensional isentropic gas flows with constant sound speed [133, Section 2.2.7, Eq. (16)], symmetry-like objects of $\mathcal{S}_0$ deserve a separate consideration but in fact they are implicitly described in the present section. In contrast to the system $\mathcal{S}$, all the quotient spaces of symmetry-like objects of the subsystem $\mathcal{S}_0$ are isomorphic to their counterparts for the system (3.4a), (3.5) and thus to their counterparts for the Klein–Gordon equation (3.4a). Therefore, to construct an algebra of canonical representatives of generalized symmetries for the subsystem $\mathcal{S}_0$, we take the respective algebra for the equation (3.4a) and follow the procedure given in the first paragraph of the proof of Theorem 3.10, just ignoring the $\mathfrak{r}^3$-components in the point transformation (3.7) and in the vector field $\tilde{X}$. As a result, we obtain that the quotient algebra of generalized symmetries of the subsystem $\mathcal{S}_0$ is naturally isomorphic to the algebra spanned by the generalized vector fields

$$\begin{aligned}
&\big(x-(\mathfrak{r}^1+\mathfrak{r}^2+1)t\big)\mathfrak{r}^1_x\partial_{\mathfrak{r}^1}+\big(x-(\mathfrak{r}^1+\mathfrak{r}^2-1)t\big)\mathfrak{r}^2_x\partial_{\mathfrak{r}^2},\ \mathrm{e}^{(\mathfrak{r}^2-\mathfrak{r}^1)/2}\big(\Gamma\mathfrak{r}^1_x\partial_{\mathfrak{r}^1}+\tilde{\mathcal{D}}_z\Gamma\mathfrak{r}^2_x\partial_{\mathfrak{r}^2}\big),\\
&\mathrm{e}^{(\mathfrak{r}^2-\mathfrak{r}^1)/2}\left((\Phi+2\Phi_{\mathfrak{r}^1})\mathfrak{r}^1_x\partial_{\mathfrak{r}^1}+(\Phi-2\Phi_{\mathfrak{r}^2})\mathfrak{r}^2_x\partial_{\mathfrak{r}^2}\right),
\end{aligned}$$

where the parameter function $\Phi=\Phi(\mathfrak{r}^1,\mathfrak{r}^2)$ runs through the solution set of the Klein–Gordon equation $\Phi_{\mathfrak{r}^1\mathfrak{r}^2}=-\Phi/4$, Γ runs through the set $\{\tilde{\mathfrak{J}}^\kappa\tilde{q},\tilde{\mathcal{D}}_y^\iota\tilde{\mathfrak{J}}^\kappa\tilde{q},\tilde{\mathcal{D}}_z^\iota\tilde{\mathfrak{J}}^\kappa\tilde{q},\kappa\in\mathbb{N}_0,\iota\in\mathbb{N}\}$ with

$$\begin{aligned}
&\tilde{\mathcal{D}}_y:=-\frac{1}{\mathfrak{r}^1_x}\big(\mathcal{D}_t+(\mathfrak{r}^1+\mathfrak{r}^2-1)\mathcal{D}_x\big),\quad \tilde{\mathcal{D}}_z:=-\frac{1}{\mathfrak{r}^2_x}\big(\mathcal{D}_t+(\mathfrak{r}^1+\mathfrak{r}^2+1)\mathcal{D}_x\big),\\
&\tilde{\mathfrak{J}}:=\frac{\mathfrak{r}^1}{2}\tilde{\mathcal{D}}_y+\frac{\mathfrak{r}^2}{2}\tilde{\mathcal{D}}_z,\quad \tilde{q}:=\mathrm{e}^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\big(x-(\mathfrak{r}^1+\mathfrak{r}^2+1)t\big),
\end{aligned}$$

and one should use the restrictions to $(\mathfrak{r}^1,\mathfrak{r}^2)$, $\mathcal{D}_x:=\partial_x+\sum_{\kappa=0}^\infty\sum_{i=1}^2\mathfrak{r}^i_{\kappa+1}\partial_{\mathfrak{r}^i_\kappa}$, $\mathcal{D}_t:=\partial_t-\sum_{\kappa=0}^\infty\sum_{i=1}^2\mathcal{D}_x^\kappa(V^i\mathfrak{r}^i_1)\partial_{\mathfrak{r}^i_\kappa}$. of the complete operators $\mathcal{D}_t$ and $\mathcal{D}_x$ defined in Section 3.3. The descriptions of cosymmetries and conservation laws of $\mathcal{S}_0$ are derived from those for the system $\mathcal{S}$ by excluding the first families of cosymmetries and conservation laws, which are related to the degeneracy of $\mathcal{S}$, in Theorems 3.13 and 3.15.

Chapter 4

Symmetry analysis of shallow water equations

4.1 Introduction

The shallow water equations are a submodel of the Euler equations for an ideal fluid. The principal simplifications are (i) density is constant, (ii) the hydro-static approximation is valid and (iii) motions along the vertical are of scales much smaller than motions in the horizontal directions. These assumptions allow us to derive the shallow water equations, which in nondimensional form read [120]

$$
\begin{aligned}
&u_t + uu_x + vu_y + h_x = 0,\\
&v_t + uv_x + vv_y + h_y = 0,\\
&h_t + uh_x + vh_y + h(u_x + v_y) = 0.
\end{aligned}
\tag{4.1}
$$

In this system, u, v are the velocity components in x- and y-directions, and h is the height of the fluid column over a fixed reference level. As we assume for now that there is no bottom topography, the reference level can be taken as the lower boundary of the fluid, in which case h denotes the total fluid height.

The system (4.1) is a (1+2)-dimensional hydrodynamic-type system. Hydrodynamic-type systems attracted enormous interest [56, 90, 113, 139, 150] in the integrability community since the seminal paper [41] on their geometric interpretation. Among the points of interest are[1] Hamiltonian structures [33, 134, 135, 143], exact solutions [31, 81, 145] and integrability in general, Lie symmetries [17, 19, 34, 35, 81], conservation laws [18, 33] and the underlying geometry [33].

The primary applications of the system under study are tsunami propagation models [29, 146, 147] and a test case for numerical approaches for more advanced weather and climate models, see [1, 28, 29, 38, 39, 59, 135] and references therein.

The problem of parameterization lies in the necessity of incorporating unresolved processes in terms of resolved ones. To be more precise, after averaging nonlinear differential equations they become unclosed and there is a need to design effective closure schemes. This way unresolved terms appear and they must be parameterized by resolved averaged quantities. Such parameterizations as it is noted in [144] should retain geometric characteristics of the initial unaveraged equations. Oberlack [98, 99] was first to incorporate Lie symmetries for the turbulence closure scheme for the Navier–Stokes equations, having postulated the so-called *invariant parameterization problem*. Recently, there was a string of works that not only follow this procedure and but also extend the theoretical results, see [15, 16, 124]. Another possible direction is a *conservative parameterization problem* [14, 125], where instead of Lie symmetries the conservation laws are retained in a closure scheme. Both the methodologies may also be combined.

The principal aim of the this chapter is to make a preliminary mathematical step towards the geometric parameterization of the shallow water system. In other words, to describe in detail the algebra of differential invariants of the point symmetry group and conservation laws of the system (4.1). Conservation laws up to order one are well-known and it is hypothesized that they are the only conservation laws of the system (4.1), see Section 4.5. We give a new geometric proof of the result [18] on a generating set of conservation laws. For a description of the algebra of differential invariants, including a generating set of differential invariants and a set of the lowest-order syzygies, that is, functional relations among differential invariants, we utilize the method of moving frames [50].

[1] Cited are papers on the shallow-water system only.

The structure of the chapter is as follows. In Section 4.2 we recall the maximal algebra of Lie invariance for the system (4.1) and compute its complete point symmetry group using the automorphism-based algebraic method. The algebra of differential invariants for the above group is described in Section 4.4. Section 4.5 collects the results on the conservation laws and the Hamiltonian structure. Section 4.6 concerns the question of parameterization problem and may be viewed as plans for future research. In Section 4.3 we classify one- and two-dimensional reductions of the system (4.1) and find some of its group-invariant solutions. In particular, ∂_y-reduction is considered in Section 4.3.2. The reduced system is a (1+1)-dimensional non-genuinely nonlinear hydrodynamic-type system and investigating it is very similar to the study of the hydrodynamic-type system $\mathcal{S}$ in Chapter 3. We do not study the reduced system exhaustively, but we show that although the system (4.1) has very few symmetries and conservation laws it possesses a plethora of their hidden counterparts. Also we repeat the same trick with the Hamiltonian operators as we did in Section 3.8, namely we locally and nonlocally prolong Hamiltonian structures of the essential subsystem of the reduced system to the third equation.

4.2 Symmetries of the shallow water equations

The maximal Lie invariance algebra $\mathfrak{g}$ of the shallow water equations (4.1) is generated by the vector fields $\mathcal{P}^t = \partial_t$, $\mathcal{D}^1 = 2t\partial_t + x\partial_x + y\partial_y - u\partial_u - v\partial_v - 2h\partial_h$, $\mathcal{K} = t^2\partial_t + tx\partial_x + ty\partial_y + (x - tu)\partial_u + (y - tv)\partial_v - 2th\partial_h$, $\mathcal{D}^2 = x\partial_x + y\partial_y + u\partial_u + v\partial_v + 2h\partial_h$, $\mathcal{J} = -y\partial_x + x\partial_y - v\partial_u + u\partial_v$, $\mathcal{P}^x = \partial_x$, $\mathcal{P}^y = \partial_y$, $\mathcal{G}^x = t\partial_x + \partial_u$, $\mathcal{G}^y = t\partial_y + \partial_v$, see e.g. [34, 118]. The corresponding Lie group G^0 of continuous symmetries of (4.1) is constituted by the point transformations of the form

$$\tilde{t} = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \frac{\sigma\varepsilon}{\gamma t + \delta} O \begin{pmatrix} x \\ y \end{pmatrix} + \frac{\alpha t + \beta}{\gamma t + \delta} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix},$$

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \frac{\varepsilon(\gamma t + \delta)}{\sigma} O \begin{pmatrix} u \\ v \end{pmatrix} - \frac{\varepsilon\gamma}{\sigma} O \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \tilde{h} = \frac{\varepsilon^2(\gamma t + \delta)^2}{\sigma^2} h,$$

where α, β, γ and δ are arbitrary constants such that $\alpha\delta-\beta\gamma>0$ and their tuple is defined up to a positive multiplier, $\sigma:=\sqrt{\alpha\delta-\beta\gamma}$, $O\in\mathrm{SO}(2)$, $\varepsilon>0$ and μ's and ν's are arbitrary constants.

The algebra $\mathfrak g$ has the structure $\mathfrak g=\mathfrak f\ni\mathfrak r$, where $\mathfrak r=\langle\mathcal D^2,\mathcal J,\mathcal P^x,\mathcal P^y,\mathcal G^x,\mathcal G^y\rangle$ is the radical of $\mathfrak g$ and $\mathfrak f=\langle\mathcal P^t,\mathcal D^1,\mathcal K\rangle\simeq\mathfrak{sl}_2(\mathbb R)$ is its Levi factor. According to the Malcev–Harish-Chandra theorem, a Levi factor of a Lie algebra is defined up to inner automorphisms generated by elements of the nilradical of the algebra. This fact is of use in the sequel and finding the nilradical of the algebra $\mathfrak g$ is our first priority. It is straightforward to verify that $\mathfrak n=\langle\mathcal P^x,\mathcal P^y,\mathcal G^x,\mathcal G^y\rangle$ is a nilpotent ideal of $\mathfrak g$. Both $\mathcal J$ and $\mathcal D^2$ are not ad-nilpotent, so Engel's theorem together with the fact that the nilradical of the algebra is contained in the radical thereof yield that $\mathfrak n$ is the nilradical of the algebra $\mathfrak g$.

Having at our disposal the structure of the algebra $\mathfrak g$ we are ready to find the complete point symmetry group of the system (4.1). Since the algebra $\mathfrak g$ does not possess a sufficient number of fully characteristic ideals [68, 126] (in fact, $\mathfrak r$, $\mathfrak r'$ and $\mathfrak n$ are the only ones), it is reasonable to apply the automorphism-based version of algebraic method [69, 77] to compute that group. It is based on the fact that any symmetry transformation $\mathcal T$ of a system of differential equations induces the automorphism on the maximal Lie invariance algebra $\mathfrak h$ thereof via the pushforward of vector fields, $\mathcal T_*\mathfrak h\in\mathfrak h$.

Recall that discrete symmetries of a system of differential equations are elements of the quotient group H/H^0, where H and H^0 are the complete point symmetry group and the group of continuous symmetries thereof, and hence are cosets of H^0 in H. In particular, discrete symmetries are defined up to combining with continuous symmetries and the coset $\mathrm{Id}\,H^0$ of the identity transformation $\mathrm{Id}\in H$ is also a discrete symmetry. Also, composing representatives of two different cosets we obtain a discrete symmetry which is not in a sense essential. What we need to compute is *independent* discrete symmetries, which are different up to combining with discrete and continuous symmetries. Given a group of canonical representatives of H/H^0 (it always exists if H^0 is a normal subgroup of H, and it exists for the system (4.1)), the discrete symmetries are generators thereof.

Proposition 4.1. *The system* (4.1) *admits only two independent discrete symmetries,*

$$(t, x, y, u, v, h) \mapsto (-t, -x, -y, u, v, h) \text{ and } (t, x, y, u, v, h) \mapsto (t, x, -y, u, -v, h).$$

In particular, the discrete symmetry group of the system (4.1) *is isomorphic to the Klein Vierergruppe* $\mathbb{Z}_2^2$.

Proof. As we are interested in discrete symmetries of the system (4.1) only, we factor the inner automorphisms out from the automorphism group of the Lie algebra $\mathfrak{g}$ as they are generated by the transformations in G^0. In particular, according to Malcev–Harish-Chandra theorem, we can determine the Levi factor $\mathfrak{f}$ of $\mathfrak{g}$ up to outer automorphisms of $\mathfrak{f}$ by inner automorphisms generated by elements of $\mathfrak{n}$. As $\mathfrak{f} \simeq \mathfrak{sl}_2(\mathbb{R})$, its outer automorphism group is $\operatorname{diag}(\varepsilon, 1, \varepsilon)$, where $\varepsilon = \pm 1$, (a basis of $\mathfrak{f}$ here is $(\mathcal{P}^t, \mathcal{D}^1, \mathcal{K})$), cf. [58]. Hence one needs to find those automorphisms of the algebra $\mathfrak{g}$, which are of the form $A = \operatorname{diag}(\varepsilon, 1, \varepsilon) \oplus \tilde{A}$, where $\tilde{A}$ is a nondegenerate 6×6 matrix. This problem is easily solved symbolically. Thus, in a basis $(\mathcal{P}^t, \mathcal{D}^1, \mathcal{K}, \mathcal{D}^2, \mathcal{J}, \mathcal{P}^x, \mathcal{P}^y, \mathcal{G}^x, \mathcal{G}^y)$ of $\mathfrak{g}$,

$$A = \operatorname{diag}(\varepsilon, 1, \varepsilon, 1, \varepsilon') \oplus \varepsilon \begin{pmatrix} \varepsilon' a & b \\ -\varepsilon' b & a \end{pmatrix} \oplus \begin{pmatrix} \varepsilon' a & b \\ -\varepsilon' b & a \end{pmatrix},$$

where $\varepsilon, \varepsilon' = \pm 1$, $a^2 + b^2 \neq 0$. Besides, b can be set to 0 by the inner automorphism of $\mathfrak{g}$, generated by the element $\mathcal{J}$ ($\mathcal{J} \notin \mathfrak{n}$). Therefore, the final form of automorphisms to be considered is $A = \operatorname{diag}(\varepsilon, 1, \varepsilon, 1, \varepsilon', \varepsilon\varepsilon' a, \varepsilon a, \varepsilon' a, a)$. Symmetry transformations $\mathcal{T} \in G$, $(t, x, y, u, v, h) \to (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}, \tilde{h})$, which define such automorphisms are found from the system of the linear equations $\mathcal{T}_* \mathfrak{g} \subset \mathfrak{g}$,

$$\mathcal{T}_* \mathcal{P}^t = \varepsilon \tilde{\mathcal{P}}^t, \quad \mathcal{T}_* \mathcal{D}^1 = \tilde{\mathcal{D}}^1, \quad \mathcal{T}_* \mathcal{K} = \varepsilon \tilde{\mathcal{K}}, \quad \mathcal{T}_* \mathcal{D}^2 = \tilde{\mathcal{D}}^2, \quad \mathcal{T}_* \mathcal{J} = \varepsilon' \tilde{\mathcal{J}},$$
$$\mathcal{T}_* \mathcal{P}^x = \varepsilon\varepsilon' a \tilde{\mathcal{P}}^x, \quad \mathcal{T}_* \mathcal{P}^y = \varepsilon a \tilde{\mathcal{P}}^y, \quad \mathcal{T}_* \mathcal{G}^x = \varepsilon' a \tilde{\mathcal{G}}^x, \quad \mathcal{T}_* \mathcal{G}^y = a \tilde{\mathcal{G}}^y,$$

solving which for the transformation components yields $\tilde{t} = \varepsilon t$, $\tilde{x} = \varepsilon\varepsilon' a x$, $\tilde{y} = \varepsilon a y$, $\tilde{u} = \varepsilon' a u$, $\tilde{v} = a v$, $\tilde{h} = c h$, where $c \neq 0$. Since not all automorphisms of $\mathfrak{g}$ are realized as the pushforwards of Lie symmetry vector fields, see [112, Remark 12], we additionally need to single out genuine point symmetry transformations of (4.1) from transformation of the above form, which is realized by the direct method. This gives a constraint $c = a^2$ and using the symmetry transformation corresponding to $\mathcal{D}^2$ (not belonging to $\mathfrak{n}$ as well) we can set a to be equal to $a := \varepsilon'' = \pm 1$. Finally, taking into account that the simultaneous reflection in the planes (y, v) and (x, u), which corresponds to the transformation with $\varepsilon'\varepsilon'' = -1$, can be factored out because it belongs to G^0, one shows that there exist only two independent discrete symmetries which are written out above. $\square$

Corollary 4.2. *The complete point symmetry group G of the system* (4.1) *is constituted by the point transformations of the form*

$$\begin{gathered}\tilde t = \frac{\alpha t+\beta}{\gamma t+\delta}, \quad \begin{pmatrix}\tilde x\\ \tilde y\end{pmatrix} = \frac{\alpha\delta-\beta\gamma}{\gamma t+\delta}O\begin{pmatrix}x\\ y\end{pmatrix} + \frac{\alpha t+\beta}{\gamma t+\delta}\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix} + \begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix},\\ \begin{pmatrix}\tilde u\\ \tilde v\end{pmatrix} = (\gamma t+\delta)O\begin{pmatrix}u\\ v\end{pmatrix} - \gamma O\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}, \quad \tilde h = (\gamma t+\delta)^2 h,\end{gathered} \tag{4.2}$$

where α, β, γ, δ, κ, μ's and ν's are arbitrary constants, with $\alpha\delta - \beta\gamma \neq 0$, and $O \in \mathrm{O}(2, \mathbb{R})$.

The above parameterization is not completely correct, because there is no one-to-one correspondence between transformations and the values of parameters. Thus, both the values of the parameter-tuples

$$\begin{gathered}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \quad O = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \quad \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0, \text{ and}\\ \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad O = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0,\end{gathered}$$

correspond to the identity transformation. Therefore, the transformation corresponding to the former parameter-tuple should be factored out from the above group. Finally, a Levi factor of the algebra $\mathfrak{g}$ corresponds to a subgroup $\mathrm{PSL}^{\pm}(2, \mathbb{R})$ of the group G.

4.3 Lie reductions

Lie reduction is one of the most reliable methods of finding particular solutions of a differential equations. Particular solutions even if they are quite trivial are valuable since they can be used to verify an accuracy of a numerical scheme. By providing the first systematic study of Lie symmetries reductions of the system (4.1) we aim to expand a list of its known particular solutions given for instance in [165].

To carry out Lie reductions of the system (4.1) we first need to classify Lie subalgebras of $\mathfrak{g}$. Since the latter is not solvable and of dimensional nine it is difficult to use brute force, and instead we build upon the well known list of subalgebras of $\mathfrak{f}$, which is isomorphic to $\mathrm{sl}_2(\mathbb{R})$, see e.g. [117], and incorporate the elements of the radical $\mathfrak{r}$ of $\mathfrak{g}$.

Ansatzes associated with one-dimensional subalgebras of $\mathfrak{g}$ reduce the system (4.1) to systems of three partial differential equations in the dependent variables (w^1, w^2, w^3) and the independent variables (z_1, z_2), while those associated with two-dimensional subalgebras reduce the system (4.1) to a system of ODEs with the dependent variables $(\varphi^1, \varphi^2, \varphi^3)$ and the independent variable ω. Below for each equivalence class of the one- and two-dimensional subalgebras, we present an ansatz constructed for (u, v, h) and the corresponding reduced system. For each codimension one reduction we present the maximal Lie invariance algebra $\mathfrak{a}$ of the reduced system in attempt to find hidden symmetries of (4.1), i.e. symmetries of the reduced systems which are not induced by symmetries of the initial system. In general, the criterion to determine their existence is to check that $\dim \mathfrak{a} > \dim \mathrm{N}_\mathfrak{g}(\mathfrak{g}^{j.i}) - j$, where $\mathfrak{g}^{j.i}$ is an j-dimensional Lie algebra an ansatz was constructed with and $\mathrm{N}_\mathfrak{g}(\mathfrak{g}^{j.i})$ is its normalizer in $\mathfrak{g}$. It turns out that only one reduced system has hidden symmetries, namely the one associated with the subalgebra $\mathcal{P}^y$.

The last four codimension one reduced systems are of hydrodynamic-type, so it makes sense to discuss reductions of the Hamiltonian structure of (4.1) and hidden Hamiltonian structures thereof. Note that the hydrodynamic-type Hamiltonian structures for inhomogeneous hydrodynamic-type systems were introduced in [42]. When performing Lie reductions with respect to a given vector field v, one makes a change of variables so that the vector field is straightened, $\tilde{\mathrm{v}} = \partial_z$, and then carries the reduction out by assuming that new dependent variables do not depend on the new independent variable z. This is the philosophy we follow when carrying out reductions of the Hamiltonian structure. Additionally, all new obtained coordinate charts turn out to be physically relevant.

4.3.1 Codimension one reductions

Denote the vector fields used to describe maximal Lie invariance algebras of reduced system as

$$\begin{aligned}
&\tilde{\mathcal{P}}^1 := \partial_{z_1}, \quad \tilde{\mathcal{P}}^2 := \partial_{z_2}, \quad \tilde{\mathcal{J}} := -z_2\partial_{z_1} + z_1\partial_{z_2} - w^2\partial_{w^1} + w^1\partial_{w^2}, \quad \tilde{\mathcal{D}}^1 := z_1\partial_{z_1},\\
&\tilde{\mathcal{D}}^2 = z_2\partial_{z_2}, \quad \tilde{\mathcal{D}}^3 := w^1\partial_{w^1} + w^2\partial_{w^2} + 2w^3\partial_{w^3}, \quad \tilde{\mathcal{J}}(f) = f\partial_{z_2} + f_{z_1}\partial_{w^1} - 2f\partial_{w^2},\\
&\tilde{\mathcal{K}}_\kappa := z_1^2\partial_{z_1} - \frac{\kappa}{\kappa^2+1}z_1\partial_{z_2} - \left(2z_1w^1 - \frac{1}{\kappa^2+1}\right)\partial_w^1 - \left(\frac{\kappa}{\kappa^2+1} + 2z_1w^2\right)\partial_w^2 - 4z_1w^3\partial_w^3.
\end{aligned}$$

Since only Reduction 1.12 gives hidden symmetries for the system (4.1), first we write down all the maximal Lie invariance algebras and the normalizers in $\mathfrak{g}$ of the algebras $\langle \mathrm{v}\rangle$, where v is the vector field with respect to which Lie reduction is taken (below f is running through the set of smooth function of w^2),

$$\begin{aligned}
&\mathfrak{a}^{1.1}_{\nu\kappa} = \langle \tilde{\mathcal{D}}^1 + \tilde{\mathcal{D}}^2 + \tilde{\mathcal{D}}^3, \tilde{\mathcal{J}}\rangle \text{ if } (\nu,\kappa) \neq (0,0), \quad \mathfrak{a}^{1.1}_{00} = \langle \tilde{\mathcal{P}}^1, \tilde{\mathcal{P}}^2, \tilde{\mathcal{D}}^1 + \tilde{\mathcal{D}}^2, \tilde{\mathcal{D}}^3, \tilde{\mathcal{J}}\rangle,\\
&\mathfrak{a}^{1.2} = \langle \tilde{\mathcal{P}}^1, \tilde{\mathcal{P}}^2, 2\tilde{\mathcal{D}}^1 + 2\tilde{\mathcal{D}}^2 + \tilde{\mathcal{D}}^3\rangle, \quad \mathfrak{a}^{1.3}_{\nu\kappa} = \langle \tilde{\mathcal{D}}^1 + \tilde{\mathcal{D}}^2 + \tilde{\mathcal{D}}^3, \tilde{\mathcal{J}}\rangle, \quad \mathfrak{a}^{1.4} = \langle \tilde{\mathcal{P}}^1, \tilde{\mathcal{P}}^2\rangle,\\
&\mathfrak{a}^{1.5}_{\nu\kappa} = \langle \tilde{\mathcal{D}}^1 + \tilde{\mathcal{D}}^2 + \tilde{\mathcal{D}}^3, \tilde{\mathcal{J}}\rangle, \quad \mathfrak{a}^{1.6} = \langle \tilde{\mathcal{P}}^1, \tilde{\mathcal{P}}^2\rangle, \quad \mathfrak{a}^{1.7}_{\kappa} = \langle \tilde{\mathcal{P}}^1, \tilde{\mathcal{P}}^2, \tilde{\mathcal{D}}^1 - \tilde{\mathcal{D}}^3, \tilde{\mathcal{K}}_\kappa\rangle,\\
&\mathfrak{a}^{1.8} = \langle \tilde{\mathcal{P}}^1, \tilde{\mathcal{P}}^2, \tilde{\mathcal{D}}^1 - \tilde{\mathcal{D}}^3, \tilde{\mathcal{K}}_0 + z_1\partial_{z_2}\rangle, \ \mathfrak{a}^{1.9} = \langle \tilde{\mathcal{P}}^1, \tilde{\mathcal{P}}^2, \tilde{\mathcal{D}}^2 + \tilde{\mathcal{D}}^3, \tilde{\mathcal{J}}(\sin 2z_1), \tilde{\mathcal{J}}(\cos 2z_1)\rangle,\\
&\mathfrak{a}^{1.10} = \langle \tilde{\mathcal{P}}^1, \tilde{\mathcal{P}}^2, z_1\partial_{z_2} + \partial_{w^1}, \tilde{\mathcal{D}}^1 - \tilde{\mathcal{D}}^3, \tilde{\mathcal{D}}^2 + \tilde{\mathcal{D}}^3, f(w^2)\partial_{w^2}\rangle;\\
&\mathrm{N}_{\mathfrak{g}}(\mathfrak{g}^{1.1}_{\nu\kappa}) = \langle \mathcal{D}^1, \mathcal{D}^2, \mathcal{J}\rangle \text{ if } (\nu,\kappa) \neq (0,0), \quad \mathrm{N}_{\mathfrak{g}}(\mathfrak{g}^{1.1}_{00}) = \langle \mathcal{P}^t, \mathcal{D}^1, \mathcal{D}^2, \mathcal{J}, \mathcal{P}^x, \mathcal{P}^y\rangle,\\
&\mathrm{N}_{\mathfrak{g}}(\mathfrak{g}^{1.2}) = \langle \mathcal{P}^t + \mathcal{G}^y, \mathcal{D}^1 + 3\mathcal{D}^2, \mathcal{P}^x, \mathcal{P}^y\rangle, \quad \mathrm{N}_{\mathfrak{g}}(\mathfrak{g}^{1.3}_{\nu\kappa}) = \langle \mathcal{D}^1, \mathcal{D}^2, \mathcal{J}\rangle,\\
&\mathrm{N}_{\mathfrak{g}}(\mathfrak{g}^{1.4}) = \langle \mathcal{D}^1 - \mathcal{D}^2, \mathcal{P}^x, \mathcal{P}^y\rangle, \quad \mathrm{N}_{\mathfrak{g}}(\mathfrak{g}^{1.5}_{\nu\kappa}) = \langle \mathcal{P}^t + \mathcal{K}, \mathcal{D}^2, \mathcal{J}\rangle,\\
&\mathrm{N}_{\mathfrak{g}}(\mathfrak{g}^{1.6}) = \langle \mathcal{P}^t + \mathcal{K} + \mathcal{J}, \mathcal{P}^x + \mathcal{G}^y, \mathcal{P}^y - \mathcal{G}^x\rangle, \quad \mathrm{N}_{\mathfrak{g}}(\mathfrak{g}^{1.7}_{\kappa}) = \langle \mathcal{P}^t, \mathcal{D}^1, \mathcal{K}, \mathcal{D}^2, \mathcal{J}\rangle,\\
&\mathrm{N}_{\mathfrak{g}}(\mathfrak{g}^{1.8}) = \langle \mathcal{P}^t, \mathcal{D}^1, \mathcal{K}, \mathcal{D}^2, \mathcal{J}\rangle, \quad \mathrm{N}_{\mathfrak{g}}(\mathfrak{g}^{1.9}) = \langle \mathcal{P}^t + \mathcal{K} + \mathcal{J}, \mathcal{D}^2, \mathcal{P}^x, \mathcal{P}^y, \mathcal{G}^x, \mathcal{G}^y\rangle,\\
&\mathrm{N}_{\mathfrak{g}}(\mathfrak{g}^{1.10}) = \langle \mathcal{P}^t, \mathcal{D}^1, \mathcal{D}^2, \mathcal{P}^x, \mathcal{P}^y, \mathcal{G}^x, \mathcal{G}^y\rangle.
\end{aligned}$$

Ansatzes constructed with these subalgebras and the corresponding reduced systems have the following forms:

1.1. $\mathfrak{g}^{1.1}_{\nu\kappa}=\langle\mathcal{P}^t+\nu\mathcal{D}^2+\kappa\mathcal{J}\rangle_{(\nu=1,\kappa\geqslant0)\vee(\nu=0,\kappa\in\{0,1\})}$:

$$u=\mathrm{e}^{\nu t}(w^1\cos\kappa t-w^2\sin\kappa t)+\nu x-\kappa y,\ v=\mathrm{e}^{\nu t}(w^1\sin\kappa t+w^2\cos\kappa t)+\kappa x+\nu y,\ h=\mathrm{e}^{2\nu t}w^3,$$

where $z_1=\mathrm{e}^{-\nu t}(x\cos\kappa t+y\sin\kappa t)$, $z_2=\mathrm{e}^{-\nu t}(-x\sin\kappa t+y\cos\kappa t)$;

$$\begin{aligned}
&w^1w^1_1+w^2w^1_2+w^3_1+2\nu w^1-2\kappa w^2+(\nu^2-\kappa^2)z_1-2\nu\kappa z_2=0,\\
&w^1w^2_1+w^2w^2_2+w^3_2+2\nu w^2+2\kappa w^1+(\nu^2-\kappa^2)z_2+2\nu\kappa z_1=0,\\
&w^1w^3_1+w^2w^3_2+w^3w^1_1+w^3w^2_2+4\nu w^3=0.
\end{aligned}$$

1.2. $\mathfrak{g}^{1.2}=\langle\mathcal{P}^t+\mathcal{G}^y\rangle$: $u=w^1$, $v=w^2+t$, $h=w^3$ with $z_1=x$, $z_2=y-\dfrac{t^2}{2}$;

$$\begin{aligned}
&w^1w^1_1+w^2w^1_2+w^3_1=0,\\
&w^1w^2_1+w^2w^2_2+w^3_2+1=0,\\
&w^1w^3_1+w^2w^3_2+w^3w^1_1+w^3w^2_2=0.
\end{aligned}$$

1.3. $\mathfrak{g}^{1.3}_{\nu\kappa}=\langle\mathcal{D}^1+2\nu\mathcal{D}^2+2\kappa\mathcal{J}\rangle_{\kappa\geqslant0}$:

$$\begin{aligned}
&u=t^{\nu-1/2}(w^1\cos\tau-w^2\sin\tau)+(\nu+\tfrac12)t^{-1}x-\kappa t^{-1}y,\\
&v=t^{\nu-1/2}(w^1\sin\tau+w^2\cos\tau)+\kappa t^{-1}x+(\nu+\tfrac12)t^{-1}y,\quad h=t^{2\nu-1}w^3,
\end{aligned}$$

where $z_1=t^{-\nu-1/2}(x\cos\tau+y\sin\tau)$, $z_2=t^{-\nu-1/2}(-x\sin\tau+y\cos\tau)$, $\tau:=\kappa\ln|t|$;

$$\begin{aligned}
&w^1w^1_1+w^2w^1_2+w^3_1+2\nu w^1-2\kappa w^2+(\nu^2-\kappa^2-\tfrac14)z_1-2\nu\kappa z_2=0,\\
&w^1w^2_1+w^2w^2_2+w^3_2+2\nu w^2+2\kappa w^1+(\nu^2-\kappa^2-\tfrac14)z_2+2\nu\kappa z_1=0,\\
&w^1w^3_1+w^2w^3_2+w^3w^1_1+w^3w^2_2+4\nu w^3=0.
\end{aligned}$$

1.4. $\mathfrak{g}^{1.4} = \langle \mathcal{D}^1 - \mathcal{D}^2 + 2\mathcal{P}^y \rangle$: $u = \dfrac{w^1}{t}$, $v = \dfrac{w^2+1}{t}$, $h = \dfrac{w^3}{t^2}$ with $z_1 = x$, $z_2 = y - \ln|t|$;

$$
\begin{aligned}
&w^1 w^1_1 + w^2 w^1_2 + w^3_1 - w^1 = 0,\\
&w^1 w^2_1 + w^2 w^2_2 + w^3_2 - w^2 - 1 = 0,\\
&w^1 w^3_1 + w^2 w^3_2 + w^3 w^1_1 + w^3 w^2_2 - 2w^3 = 0.
\end{aligned}
$$

1.5. $\mathfrak{g}^{1.5}_{\nu,\kappa} = \langle \mathcal{P}^t + \mathcal{K} + \nu\mathcal{D}^2 + \kappa\mathcal{J} \rangle$:

$$
\begin{aligned}
u &= \frac{\mathrm{e}^{\nu\tau}}{\sqrt{t^2+1}}(w^1\cos\kappa\tau - w^2\sin\kappa\tau) + \frac{\nu x - \kappa y + tx}{t^2+1},\\
v &= \frac{\mathrm{e}^{\nu\tau}}{\sqrt{t^2+1}}(w^1\sin\kappa\tau + w^2\cos\kappa\tau) + \frac{\nu y + \kappa x + ty}{t^2+1}, \quad h = \frac{\mathrm{e}^{2\nu\tau}w^3}{t^2+1},
\end{aligned}
$$

where $z_1 = \dfrac{\mathrm{e}^{-\nu\tau}}{\sqrt{t^2+1}}(x\cos\kappa\tau + y\sin\kappa\tau)$, $z_2 = \dfrac{\mathrm{e}^{-\nu\tau}}{\sqrt{t^2+1}}(y\cos\kappa\tau - x\sin\kappa\tau)$, $\tau := \arctan t$;

$$
\begin{aligned}
&w^1 w^1_1 + w^2 w^1_2 + w^3_1 + 2\nu w^1 - 2\kappa w^2 + (\nu^2 - \kappa^2 + 1)z_1 - 2\nu\kappa z_2 = 0,\\
&w^1 w^2_1 + w^2 w^2_2 + w^3_2 + 2\nu w^2 + 2\kappa w^1 + (\nu^2 - \kappa^2 + 1)z_2 + 2\nu\kappa z_1 = 0,\\
&w^1 w^3_1 + w^2 w^3_2 + w^3 w^1_1 + w^3 w^2_2 + 4\nu w^3 = 0.
\end{aligned}
$$

1.6. $\mathfrak{g}^{1.6} = \langle \mathcal{P}^t + \mathcal{K} + \mathcal{J} + \mathcal{G}^x - \mathcal{P}^y \rangle$:

$$
u = \frac{tw^1 + w^2}{t^2+1} + \frac{t(x+1) - y}{t^2+1}, \quad v = \frac{-w^1 + tw^2}{t^2+1} + \frac{ty + x - 1}{t^2+1}, \quad h = \frac{w^3}{t^2+1},
$$

where $z_1 = \dfrac{tx - y}{t^2+1} - \arctan t$, $z_2 = \dfrac{x + ty}{t^2+1}$;

$$
\begin{aligned}
&w^1 w^1_1 + w^2 w^1_2 + w^3_1 - 2w^2 = 0,\\
&w^1 w^2_1 + w^2 w^2_2 + w^3_2 + 2w^1 + 2 = 0,\\
&w^1 w^3_1 + w^2 w^3_2 + w^3 w^1_1 + w^3 w^2_2 = 0.
\end{aligned}
$$

1.7. $\mathfrak{g}^{1.7}_\kappa = \langle \mathcal{D}^2 + \kappa\mathcal{J}\rangle_{\kappa\geqslant 0}$:

$$u = (x-\kappa y)w^1-(\kappa x+y)w^2, \quad v = (\kappa x+y)w^1+(x-\kappa y)w^2, \quad h = (\kappa^2+1)(x^2+y^2)w^3,$$

where $z_1 = t,\ z_2 = \dfrac{1}{\kappa^2+1}\Big(\arctan\dfrac{y}{x} - \dfrac{\kappa}{2}\ln(x^2+y^2)\Big)$;

$$\begin{aligned}
&w^1_1 + w^2w^1_2 + (w^1)^2 - (w^2)^2 - 2\kappa w^1w^2 + 2w^3 = 0,\\
&w^2_1 + w^2w^2_2 + w^3_2 + \kappa(w^1)^2 - \kappa(w^2)^2 + 2w^1w^2 - 2\kappa w^3 = 0,\\
&w^3_1 + w^2w^3_2 + w^3w^2_2 + 4(w^1 - \kappa w^2)w^3 = 0.
\end{aligned}$$

The system (4.1) in (modified) "polar" coordinates (z_1, z_2, z_3) reads

$$\begin{aligned}
&w^1_1 + w^3_3 + w^1w^1_3 + w^2w^1_2 + (w^1)^2 - (w^2)^2 - 2\kappa w^1w^2 + 2w^3 = 0,\\
&w^2_1 + w^2w^2_2 + w^1w^2_3 + w^3_2 + \kappa(w^1)^2 - \kappa(w^2)^2 + 2w^1w^2 - 2\kappa w^3 = 0,\\
&w^3_1 + w^1w^3_3 + w^2w^3_2 + w^3w^1_3 + w^3w^2_2 + 4(w^1 - \kappa w^2)w^3 = 0
\end{aligned}$$

and is Hamiltonian with the Hamiltonian operator $\mathfrak{H}$,

$$\mathfrak{H} = \mathrm{e}^{6\kappa z_2 - 6z_3}\begin{pmatrix} 0 & q & -\mathrm{D}_{z_3}+4\\ -q & 0 & -\mathrm{D}_{z_2}-4\kappa\\ -\mathrm{D}_{z_3}+2 & -\mathrm{D}_{z_2}-2\kappa & 0\end{pmatrix},$$

where $q = (w^2_3 - w^1_2 + 2\kappa w^1 + 2w^2)/w^3$. The reduced system under consideration is also the reduced system for the system above with respect to ∂_{z_3}, but since $\mathfrak{H}$ explicitly depends on z_3 it is impossible to get the Hamiltonian operator for the reduced system by the simple reduction. Here $z_3 = \dfrac{1}{\kappa^2+1}\Big(\kappa\arctan\dfrac{y}{x} + \dfrac{1}{2}\ln(x^2+y^2)\Big)$.

1.8. $\mathfrak{g}^{1.8} = \langle\mathcal{J}\rangle$:

$$u = xw^1 - yw^2, \quad v = yw^1 + xw^2, \quad h = (x^2+y^2)w^3$$

with $z_1 = t$, $z_2 = \dfrac{1}{2}\ln(x^2{+}y^2)$;

$$
\begin{aligned}
&w^1_1 + w^1 w^1_2 + w^3_2 + (w^1)^2 - (w^2)^2 + 2w^3 = 0,\\
&w^2_1 + w^1 w^2_2 + 2w^1 w^2 = 0,\\
&w^3_1 + w^1 w^3_2 + w^3 w^1_2 + 4w^1 w^3 = 0.
\end{aligned}
$$

The system (4.1) in (modified) polar coordinates z_1, z_2 and $z_3 = \arctan\dfrac{y}{x}$ takes the form

$$
\begin{aligned}
&w^1_1 + w^1 w^1_2 + w^2 w^1_3 + (w^1)^2 - (w^2)^2 + w^3_2 + 2w^3 = 0,\\
&w^2_1 + w^1 w^2_2 + w^2 w^2_3 + 2w^1 w^2 + w^3_3 = 0,\\
&w^3_1 + (w^1 w^3)_2 + (w^2 w^3)_3 + 4w^1 w^3 = 0.
\end{aligned}
$$

It is Hamiltonian with the Hamiltonian operator $\mathfrak{H}$, and the corresponding reduced system is also the reduced system for the system above with respect to ∂_{z_3}. A Hamiltonian structure $\bar{\mathfrak{H}}$ of the reduced system is therefore inherited from that of (4.1),

$$
\mathfrak{H} = \mathrm{e}^{-6z_2}\begin{pmatrix} 0 & \dfrac{w^2_2 - w^1_3 + 2w^2}{w^3} & -\mathrm{D}_{z_2} + 4 \\ -\dfrac{w^2_2 - w^1_3 + 2w^2}{w^3} & 0 & -\mathrm{D}_{z_3} \\ -\mathrm{D}_{z_2} + 2 & -\mathrm{D}_{z_3} & 0 \end{pmatrix},
$$

$$
\bar{\mathfrak{H}} = \mathrm{e}^{-6z_2}\begin{pmatrix} 0 & \dfrac{w^2_2 + 2w^2}{w^3} & -\mathrm{D}_{z_2} + 4 \\ -\dfrac{w^2_2 + 2w^2}{w^3} & 0 & 0 \\ -\mathrm{D}_{z_2} + 2 & 0 & 0 \end{pmatrix}.
$$

There are no other (hidden) Hamiltonian structures of the reduced system.

1.9. $\mathfrak{g}^{1.9} = \langle \mathcal{G}^x - \mathcal{P}^y \rangle$:

$$
u = \frac{w^1 - tw^2 + tx - y}{t^2+1}, \quad v = \frac{tw^1 + w^2 + x + ty}{t^2+1}, \quad h = \frac{w^3}{t^2+1},
$$

where $z_1=\arctan t,\ \ z_2=\dfrac{x+ty}{t^2+1}$;

$$
\begin{aligned}
&w^1_1+w^1w^1_2+w^3_2-2w^2=0,\\
&w^2_1+w^1w^2_2+2w^1=0,\\
&w^3_1+w^1w^3_2+w^3w^1_2=0.
\end{aligned}
$$

Recall [34] that the system (4.1) is equivalent to the system describing rotating shallow water model with constant Coriolis force f,

$$
\begin{aligned}
&w^1_1+w^1w^1_2+w^2w^1_3-fw^2+w^3_2=0,\\
&w^2_1+w^1w^2_2+w^2w^2_3+fw^1+w^3_3=0,\\
&w^3_1+(w^1w^3)_2+(w^2w^3)_3=0.
\end{aligned}
$$

The latter system is also known to be Hamiltonian [143] with the Hamiltonian operator $\mathfrak{H}_f$, but the reduced system under question is a reduction of the rotating shallow water system with $f=2$ with respect to ∂_{z_3}. The Hamiltonian operator $\mathfrak{H}_2$ reduces to $\bar{\mathfrak{H}}_2$ accordingly,

$$
\mathfrak{H}_f=\begin{pmatrix}0 & \frac{f+w^2_2-w^1_3}{w^3} & -\mathrm{D}_{z_2}\\ -\frac{f+w^2_2-w^1_3}{w^3} & 0 & -\mathrm{D}_{z_3}\\ -\mathrm{D}_{z_2} & -\mathrm{D}_{z_3} & 0\end{pmatrix},\quad \bar{\mathfrak{H}}_2=\begin{pmatrix}0 & \frac{w^2_2+2}{w^3} & -\mathrm{D}_{z_2}\\ -\frac{w^2_2+2}{w^3} & 0 & 0\\ -\mathrm{D}_{z_2} & 0 & 0\end{pmatrix}.
$$

Direct computation shows that there are no (hidden) Hamiltonian structures of the reduced system.

1.10. $\mathfrak{g}^{1.10}=\langle\mathcal{P}^y\rangle$: $\ u=w^1,\ \ v=w^2,\ \ h=w^3$ with $\ z_1=t,\ \ z_2=x$;

$$
\begin{aligned}
&w^1_1+w^1w^1_2+w^3_2=0,\\
&w^2_1+w^1w^2_2=0,\\
&w^3_1+w^1w^3_2+w^3w^1_2=0.
\end{aligned}
$$

Hidden symmetries: $f\partial_{w^2}$, where f runs through the set of smooth functions of w^2.

4.3.2 $\mathcal{P}^y$-reduction

The obtained reduced hydrodynamic-type system

$$w^1_1 + w^1 w^1_2 + w^3_2 = 0, \quad w^2_1 + w^1 w^2_2 = 0, \quad w^3_1 + w^1 w^3_2 + w^3 w^1_2 = 0,$$

can be diagonalized via the change of variables $w^1 = 2(r^1 + r^2),\ w^2 = r^3,\ w^3 = (r^1 - r^2)^2$ to the system $\mathcal{S}$,

$$r^1_t + (3r^1 + r^2)r^1_x = 0, \quad r^2_t + (r^1 + 3r^2)r^2_x = 0, \quad r^3_t + 2(r^1 + r^2)r^3_x = 0.$$

Thus, the r's are the Riemann invariants for $\mathcal{S}$, while $V^1 = 3r^1 + r^2$, $V^2 = r^1 + 3r^2$ and $V^3 = 2(r^1 + r^2)$ are its characteristic velocities. The system $\mathcal{S}$ is partially coupled and is not genuinely nonlinear as $V^3_3 = 0$. Here and in what follows the index i denotes the differentiation with respect to the Riemann invariant r^i, $i = 1, 2, 3$. Moreover, $\mathcal{S}$ is semi-Hamiltonian and thus can be solved via the generalized hodograph transformation [150], that is, its solutions satisfying $r^i_x \neq 0$ can be implicitly presented as $x - V^i(r)t = W^i(r)$, where $r = (r^1, r^2, r^3)$ and W's satisfy the system $W^i_j/(W^j - W^i) = V^i_j/(V^j - V^i)$ for $i \neq j$,

$$\begin{aligned}
&\frac{2W^1_2}{W^2 - W^1} = \frac{1}{r^2 - r^1}, \quad W^2_1 = W^1_2, \quad W^1_3 = W^2_3 = 0,\\
&\frac{W^3_1}{W^1 - W^3} = \frac{2}{r^1 - r^2}, \quad \frac{W^3_2}{W^2 - W^3} = \frac{2}{r^2 - r^1}.
\end{aligned}$$

Introducing the potential $\Lambda(r^1, r^2)$ via $W^1 = \Lambda_1$ and $W^2 = \Lambda_2$, one can derive from the first three equations that it satisfies the Euler–Poisson–Darboux equation $2(r^2 - r^1)\Lambda_{12} = \Lambda_2 - \Lambda_1$. The general solution of the overdetermined system of the last two equations on W^3 is $W^3(r^1, r^2, r^3) = F(r^3)/(r^1 - r^2)^2 + \Phi(r^1, r^2)$, where F runs through the set of smooth functions of r^3 and Φ is a particular solution of the system $(r^1 - r^2)\Phi_1 = 2(\Lambda_1 - \Phi),\ (r^2 - r^1)\Phi_2 = 2(\Lambda_2 - \Phi)$. It can be seen that Φ satisfies the Euler–Poisson–Darboux equation $2(r^2 - r^1)\Phi_{12} = 3(\Phi_2 - \Phi_1)$.

Let us now consider solutions which are not caught by the generalized hodograph transformation, that is, when $r^i_x = 0$ for some i('s). First of all, the solutions with $r^3_x = 0$

are naturally embedded in the above family, cf. Theorem 3.1. Let $r^1_x = 0$ but $r^2_x \neq 0$. Then $r^1 = c^1$ is a constant and we obtain a hydrodynamic-type system on (r^2, r^3) which is linearized via the rank-1 hodograph transformation, $\tilde{t} = t$, $\tilde{x} = r^2$, $\tilde{r}^2 = x$ and $\tilde{r}^3 = r^3$ with $(\tilde{t}, \tilde{x})$ being the new independent variables to get $\tilde{r}^2_{\tilde{t}} - 3\tilde{x} - c_2 = 0$, $\tilde{r}^2_{\tilde{x}}\tilde{r}^3_{\tilde{t}} = (\tilde{x} - c_2)\tilde{r}^3_{\tilde{x}}$, which is readily solved. An approach when r^2 is a constant, while r^1 is not, is very similar. When both r^1 and r^2 are constants one has a transport equation on r^3.

Theorem 4.3. *Any solution of the system $\mathcal{S}$ (locally) belongs to one of the following families; below W is an arbitrary function of its argument.*

1. The regular family, where both the Riemann invariants r^1 and r^2 are not constants (the general solution):

$$
\begin{aligned}
x - (3r^1 + r^2)t &= \frac{1}{2}(r^1 - r^2)\Phi_1 + \Phi, \\
x - (r^1 + 3r^2)t &= \frac{1}{2}(r^2 - r^1)\Phi_2 + \Phi, \\
x - 2(r^1 + r^2)t &= \frac{F}{(r^1 - r^2)^2} + \Phi,
\end{aligned}
$$

where Φ is a smooth function of (r^1, r^2) which runs through the set of solutions of the equation $2(r^2 - r^1)\Phi_{12} = 3(\Phi_2 - \Phi_1)$ and the function F runs through the set of smooth functions of r^3.

2. The two singular families, where exactly one of the Riemann invariants r^1 and r^2 is a constant:

$$
\begin{aligned}
&r^1 = c, \quad x = (3r^2 + c)t + \Theta^2_{r^2}/(c - r^2), \quad r^3 = W\left((c - r^2)^3 t - 2\Theta^2 - \Theta^2_{r^2}(c - r^2)\right); \\
&r^2 = c, \quad x = (3r^1 + c)t + \Theta^1_{r^1}/(c - r^1), \quad r^3 = W\left((c - r^1)^3 t - 2\Theta^1 - \Theta^1_{r^1}(c - r^1)\right).
\end{aligned}
$$

Here c is an arbitrary constant and $\Theta^1 = \Theta^1(r^1)$ and $\Theta^2 = \Theta^2(r^2)$ are arbitrary functions of their arguments.

3. The ultra-singular family, where the Riemann invariants r^1 and r^2 are arbitrary constants and $r^3 = W(x - 2(r^1 + r^2)t)$.

The regular, singular and ultra-singular families of solutions of the system $\mathcal{S}$ are associated with solutions of the subsystem $\mathcal{S}_0$ of rank 2, 1 and 0, respectively; cf. [65]. Perhaps, the more instructive but less standard way to solve the system $\mathcal{S}$ is via linearizing its subsystem $\mathcal{S}_0$ of the first two equations,

$$r^1_t + (3r^1 + r^2)r^1_x = 0, \quad r^2_t + (r^1 + 3r^2)r^2_x = 0$$

using the rank-2 hodograph transformation $y = r^1$, $z = r^2$, $p = t$, $q = x$ to the equation

$$3(p_y - p_z) = 2(y - z)p_{yz}. \tag{4.3}$$

Due to the fact that the system $\mathcal{S}$ is not genuinely nonlinear one can introduce special coordinates $\omega_i = \left((r^1 - r^2)^{-2}\mathrm{D}_x\right)^i r^3$, $i \in \mathbb{N}_0$ to show existence of an infinite hierarchy of conservation laws, cf. [40] and Section 3.6, and (by virtue of partial coupling of $\mathcal{S}$) of higher symmetries, cf. [112, 113] and Section 3.4.

Indeed, the generalized vector fields of the form $\Omega(\omega^0, \omega^1, \ldots, \omega^n)\partial_{r^3}$, where Ω runs through the set of smooth functions of any finite number of ω's, form an ideal Σ_3 in the algebra Σ of nontrivial higher symmetries of $\mathcal{S}$. The subalgebra Σ/Σ_3 is isomorphic to the algebra that consists of the generalized symmetries of the essential subsystem $\mathcal{S}_0$ which can be locally prolonged to the third equation. As an example, the subalgebra thereof, that is constituted by higher symmetries of genuine order one (no Lie symmetries) is as follows,

$$\left\langle \left(\frac{1}{2}(r^1 - r^2)\Phi_1 + \Phi\right) r^1_x\partial_{r^1} + \left(\frac{1}{2}(r^2 - r^1)\Phi_2 + \Phi\right) r^2_x\partial_{r^2} + \left(\frac{F}{(r^1 - r^2)^2} + \Phi\right) r^3_x\partial_{r^3} \right\rangle,$$

where $\Phi = \Phi(r^1, r^2)$ runs through the set of solutions of the equation $2(r^2 - r^1)\Phi_{12} = 3(\Phi_2 - \Phi_1)$ and the function F runs through the set of smooth functions of r^3.

It was shown in [40] that not genuinely nonlinear hydrodynamic-type systems admit nontrivial conservation laws of arbitrary high order, parameterized by a smooth function of finitely many ω's. Besides, the system $\mathcal{S}$ inherits conservation laws from the subsystem $\mathcal{S}_0$, or, equivalently, from the equation (4.3). Let us describe the zeroth-order conservation laws first. Using the standard techniques the characteristics thereof are easily found,

$$(r^1-r^2)\left(2\Omega-\gamma(x-V^1t)+\frac{\Phi_1}{r^1-r^2},-2\Omega+\gamma(x-V^2t)+\frac{\Phi_2}{r^1-r^2},\Omega_3(r^1-r^2)\right),$$

where γ is an arbitrary constant, Φ and Ω run through the set of smooth functions of (r^1,r^2) and r^3, respectively, with Φ satisfying the equation of the form (4.3). The associated conserved currents are then recovered,

$$\begin{aligned}&(r^1-r^2)^2\left(\Omega,V^3\Omega\right),\quad \left(\Phi_1+\Phi_2,V^1\Phi_1+V^2\Phi_2-4\Phi\right),\\ &(r^1-r^2)^2\left(2(x+V^3t),\left((V^1)^2+(V^2)^2-(r^1-r^2)^2\right)t-2V^3x\right).\end{aligned}$$

Furthermore, following [40, Theorem 5.1] we may construct all first-order conserved currents of $\mathcal{S}$, whose densities are (t,x)-independent. The space thereof is spanned by $\dfrac{r^1-r^2}{r^1_xr^2_x}\,(r^2_x-r^1_x,V^1r^2_x-V^2r^1_x)$, $(r^1-r^2)^2\,(\Omega,V^3\Omega)$, where Ω runs through the space of smooth functions of $(\omega^0,\omega^1)=(r^3,r^3_x/(r^1-r^2)^2)$.

It was our conjecture that the system (4.1) admits only first-order conservation laws of specific type and no nontrivial higher symmetries of the higher order at all. Nevertheless, it turns out to possess a plethora of hidden higher symmetries and conservation laws of arbitrary order.

Let us look how the system $\mathcal{S}$ (which we will write in (t,x,u,v,h)-coordinates in the remainder of the subsection),

$$u_t+uu_x+h_x=0,\quad h_t+uh_x+hu_x=0,\quad v_t+uv_x=0,$$

inherits the Hamiltonian structure of (1+1)-dimensional system $\mathcal{S}_0$ of equations of gas dynamics with $\gamma=2$, i.e., the decoupled subsystem on (u,h). Recall that the latter system is known to be quadri-Hamiltonian, that is, it admits four different Hamiltonian structures. Three corresponding Hamiltonian operators are of order one [97] and one of order three [101],

$$\mathfrak{H}^1_{\mathcal{S}_0} = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathrm{D}_x, \quad \mathfrak{H}^2_{\mathcal{S}_0} = \begin{pmatrix} 2 & u \\ u & 2h \end{pmatrix} \mathrm{D}_x + \begin{pmatrix} 0 & u_x \\ 0 & h_x \end{pmatrix},$$

$$\mathfrak{H}^3_{\mathcal{S}_0} = \begin{pmatrix} 2u & u^2/2+2h \\ u^2/2+2h & 2uh \end{pmatrix} \mathrm{D}_x + \begin{pmatrix} u_x & uu_x+h_x \\ h_x & uh_x+hu_x \end{pmatrix},$$

$$\mathfrak{H}^4_{\mathcal{S}_0} = \mathrm{D}_x \circ U \circ \mathrm{D}_x \circ U \circ \sigma^1 \circ \mathrm{D}_x, \quad \text{where}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U = \frac{1}{\delta}\begin{pmatrix} u_x & -h_x/h \\ -h_x & u_x \end{pmatrix}, \quad \delta = u_x^2 - \frac{h_x^2}{h}.$$

Moreover, it is easy to check that the system $\mathcal{S}_0$ also admits the zeroth-order Noether operator

$$\mathfrak{N}^4_{\mathcal{S}_0} = \begin{pmatrix} h_x/h & u_x \\ u_x & h_x \end{pmatrix}.$$

Below we may use the alternative notation (u^1, u^2, u^3) for the dependent variables (u, h, v) in summation formulae. To construct a Hamiltonian structure of the system $\mathcal{S}$, we first find all Noether operators $\mathfrak{N}$ thereof. These are matrix-operators mapping cosymmetries of $\mathcal{S}$ into its symmetries. Recall that cosymmetries are solutions to the system adjoint to that used for finding generalized symmetries of the same system. Since $\mathcal{S}$ is a hydrodynamic-type system we consider only (local) hydrodynamic operators [41, 43, 150] of the form $\mathfrak{N} = g^{ij}\mathrm{D}_x + g^{is}\Gamma^j_{sk}u^k$. A direct computation shows that they are of the form

$$\mathfrak{N}_{\theta,\zeta} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \dfrac{\theta}{h^2} \end{pmatrix} \mathrm{D}_x + \begin{pmatrix} 0 & 0 & \dfrac{v_x}{h} \\ 0 & 0 & 0 \\ \dfrac{-v_x}{h} & 0 & -\dfrac{\theta h_x}{h^3} + \dfrac{\zeta}{h} \end{pmatrix},$$

where θ and ζ run through the set of smooth functions of $(v, v_x/h)$. In order to qualify as a Hamiltonian operator, $\mathfrak{N}_{\theta,\zeta}$ must be skew-adjoint and satisfy the Jacobi identity. The first requirements gives $\theta = \theta(v)$ and $\zeta(v, v_x/h) = v_x\theta_v/(2h)$, while the other is identically satisfied in view of the fact that the metric $(g_{ij}) = (g^{ij})^{-1}$ is flat, cf. [41]. The net result is the family of Hamiltonian operators parameterized by a function of a single argument,

$$\mathfrak{H}^1_{\mathcal{S};\theta} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \dfrac{\theta}{h^2} \end{pmatrix} \mathrm{D}_x + \begin{pmatrix} 0 & 0 & \dfrac{v_x}{h} \\ 0 & 0 & 0 \\ \dfrac{-v_x}{h} & 0 & -\dfrac{\theta h_x}{h^3} + \dfrac{v_x \theta_v}{2h^2} \end{pmatrix},$$

which is obviously a prolongation of the Hamiltonian operator $\mathfrak{H}^1_{\mathcal{S}_0}$ to the entire system $\mathcal{S}$. At the same time, when $\theta = 0$ it is a reduction of the Hamiltonian operator of (4.1). For all other θ's the associated Hamiltonian operators are hidden Hamiltonian operators for (4.1). The associated Hamiltonians are parameterized by a smooth function of r^3 and two constants,

$$\int \left(\frac{h^2}{2} + h \left(\frac{u^2}{2} + \Psi(v) \right) + c_1 u + c_2 \right) \mathrm{d}x,$$

where Ψ and c_1 additionally satisfy the equation $2\theta \Psi_{vv} + \theta_v \Psi_v - 2c_1 = 0$. Of course, constants c_1 and c_2 are associated with the Casimir functionals of the Hamiltonian operator $\mathfrak{H}^1_{\mathcal{S};\theta}$.

But the underlying $(1+1)$-dimensional gas-dynamics system admits three hydrodynamic-type Hamiltonian structures. Let us investigate what happens with the other two upon a prolongation to the third equation. For this aim we consider nonlocal Noether operators of the form

$$\mathcal{N}^{ij} = g^{ij} \mathrm{D}_x - g^{is} \Gamma^j_{sk} u^k_x + \sum_{\alpha=1}^{3} \epsilon_\alpha w^i_{\alpha k} u^k_x \mathrm{D}_x^{-1} \circ w^j_{\alpha l} u^l_x,$$

where the functions g^{ij}, $w^i_{\alpha k}$ and Γ^j_{sk} are smooth functions of (u, v, h), see [52, 54, 91] and Section 3.8. The Einstein summation convention is utilized. The functions $w^i_{\alpha k}$ in the geometric interpretations of hydrodynamic-type systems play the role of affinors, i.e. 1-contravariant, 1-covariant tensors. Following the procedure in the aforementioned section we find that $w_\alpha = w_{\Phi^\alpha}$, where

$$w_\Phi = \begin{pmatrix} \Phi_{uu} & \Phi_{uh} & 0 \\ h\Phi_{uh} & \Phi_{uu} & 0 \\ 0 & 0 & \Phi_h \end{pmatrix}.$$

The functions Φ^1, Φ^2 and Φ^3 of (u,h,v) satisfy the differential constraints $\Phi^\alpha_{uuv} = \Phi^\alpha_{uhv} = 0$ and the differential equation $\Phi^i_{uu} = h\Phi^i_{hh} + \Phi^i_h$, which is the form of the Euler–Poisson–Darboux equation (4.3) in (u,h)-variables. The Φ^α's can be presented explicitly as

$$\Phi^\alpha = a^\alpha(v)\ln|h| + b^\alpha(v)u + c^\alpha(v) + d^\alpha(u,h), \quad \alpha = 1,2,3,$$

for smooth functions a^α, b^α, c^α, d^α of their arguments, with d^α satisfying the equations $d^\alpha_{uu} = hd^\alpha_{hh} + d^\alpha_h$. Overall, the Noether operators take the form

$$\mathfrak{N}_{\Phi,\bar\theta,\bar\zeta} = \begin{pmatrix} 2 & u & 0 \\ u & 2h & 0 \\ 0 & 0 & \dfrac{\bar\theta}{h^2} \end{pmatrix} \mathrm{D}_x + \begin{pmatrix} 0 & u_x & -\dfrac{uv_x}{h} \\ 0 & h_x & -2v_x \\ \dfrac{uv_x}{h} & 2v_x & -\dfrac{\bar\theta h_x}{h^3} + \dfrac{\bar\zeta}{h} \end{pmatrix} + \sum_{\alpha=1}^{3} \epsilon_\alpha w^i_{\alpha k} u^k_x \mathrm{D}_x^{-1} \circ w^j_{\alpha l} u^l_x$$

for some smooth functions $\bar\theta$ and $\bar\zeta$ of $(v, v_x/h)$. Moreover, there are three more constraints on the functions Φ's,

$$\Psi_u = 0, \quad h^2\Psi_h + 2h\Psi + 2 = 0, \quad \sum \epsilon_\alpha (\Phi^\alpha_{uu})^2 = h\sum \epsilon_\alpha (\Phi^\alpha_{uh})^2, \quad \text{where } \Psi := \sum \epsilon_\alpha (\Phi^\alpha_h)^2.$$

It is straightforward that $\Psi = Ch^{-2} - 2h^{-1}$, where C is a constant.

Let us now single out the values of parameters which make $\mathfrak{N}_{\Phi,\bar\theta,\bar\zeta}$ Hamiltonian. The skew-symmetry of $\mathfrak{N}_{\Phi,\bar\theta,\bar\zeta}$ is equivalent to g_{ij} being a metric tensor and Γ^j_{sk} its Levi-Civita connection, which is ensured by the conditions $\bar\theta(v, v_x/h) = \theta(v)$, $\bar\zeta(v, v_x/h) = \zeta(v)v_x/h$ and $\zeta = \theta_v/2$. The operator $\mathfrak{N}_{\Phi,\theta}$ satisfies the Jacobi identity if and only if the set of affinors[2] is commutative, $[w_\alpha, w_\beta] = 0$, the raised Riemann tensor is $R^{ij}{}_{kl} = \sum_\alpha (w^i_{\alpha k} w^j_{\alpha l} - w^j_{\alpha k} w^i_{\alpha l})$, $\nabla_k w^i_{\alpha j} = \nabla_j w^i_{\alpha k}$, $g_{ik} w^k_{\alpha j} = g_{jk} w^k_{\alpha i}$, see [52]. It turns out though that all these conditions are automatically satisfied.

[2] An affinor is a $\binom{1}{1}$-tensor.

The similar approach is taken to compute a nonlocal prolongation of the operator $\mathfrak{H}^3_{\mathcal{S}}$.

Theorem 4.4. *The system $\mathcal{S}$ admits three families of first-order Hamiltonian operators of hydrodynamic type,*

$$\mathfrak{H}^1_{\mathcal{S};\theta} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \dfrac{\theta}{h^2} \end{pmatrix} \mathrm{D}_x + \begin{pmatrix} 0 & 0 & \dfrac{v_x}{h} \\ 0 & 0 & 0 \\ \dfrac{-v_x}{h} & 0 & -\dfrac{\theta h_x}{h^3} + \dfrac{\theta_v v_x}{2h^2} \end{pmatrix},$$

$$\mathfrak{H}^2_{\mathcal{S};\Phi,\theta} = \begin{pmatrix} 2 & u & 0 \\ u & 2h & 0 \\ 0 & 0 & \dfrac{\theta}{h^2} \end{pmatrix} \mathrm{D}_x + \begin{pmatrix} 0 & u_x & -\dfrac{uv_x}{h} \\ 0 & h_x & -2v_x \\ \dfrac{uv_x}{h} & 2v_x & -\dfrac{\theta h_x}{h^3} + \dfrac{\theta_v v_x}{2h^2} \end{pmatrix} + \sum_{\alpha=1}^{3} w^i_{\alpha k} u^k_x \mathrm{D}_x^{-1} \circ w^j_{\alpha l} u^l_x,$$

$$\mathfrak{H}^3_{\mathcal{S};\Phi,\theta} = \begin{pmatrix} 2u & \dfrac{u^2}{2}+2h & 0 \\ \dfrac{u^2}{2}+2h & 2uh & 0 \\ 0 & 0 & \dfrac{\theta}{h^2} \end{pmatrix} \mathrm{D}_x + \begin{pmatrix} u_x & uu_x + h_x & -\dfrac{(u^2+4h)v_x}{2h} \\ h_x & uh_x + hu_x & -2uv_x \\ \dfrac{(u^2+4h)v_x}{2h} & 2uv_x & -\dfrac{\theta h_x}{h^3} + \dfrac{\theta_v v_x}{2h^2} \end{pmatrix}$$
$$+ \sum_{\alpha=1}^{3} w^i_{\alpha k} u^k_x \mathrm{D}_x^{-1} \circ w^j_{\alpha l} u^l_x,$$

which can be seen as prolongations of the corresponding Hamiltonian operators of $\mathcal{S}_0$. Here $(u^1, u^2, u^3) = (u, h, v)$, θ runs through the set of smooth functions of v,

$$(w^i_{\alpha k}) = \begin{pmatrix} \Phi^\alpha_{uu} & \Phi^\alpha_{uh} & 0 \\ h\Phi^\alpha_{uh} & \Phi^\alpha_{uu} & 0 \\ 0 & 0 & \Phi^\alpha_h \end{pmatrix}, \qquad \Phi^\alpha = a^\alpha(v) \ln |h| + b^\alpha(v) u + c^\alpha(v) + d^\alpha(u, h),$$

for the smooth functions a^α, b^α, c^α, d^α of their arguments, with $d^\alpha(u,h)$ satisfying the equations $d^\alpha_{uu} = hd^\alpha_{hh} + hd^\alpha_h$. The functions Φ^α additionally satisfy the system of PDEs

$$\sum \epsilon_\alpha (\Phi^\alpha_{uu})^2 = h \sum \epsilon_\alpha (\Phi^\alpha_{uh})^2, \quad \sum \epsilon_\alpha (\Phi^\alpha_h)^2 = \Psi,$$

where $\Psi = Ch^{-2} - 2h^{-1}$ for $\mathfrak{H}^2_{\mathcal{S};\Phi,\theta}$ and $\Psi = Ch^{-2} - 2uh^{-1}$ for $\mathfrak{H}^3_{\mathcal{S};\Phi,\theta}$, C is an arbitrary constant.

It worth noticing that one function Φ^α is not enough to construct a Hamiltonian structure, as the associated system thereon would be inconsistent.

4.3.3 Codimension two reductions

A list of G-inequivalent two-dimensional subalgebras of the algebra $\mathfrak{g}$ is as follows,

$$\mathfrak{g}^{2.1}_{\nu\kappa} = \langle \mathcal{P}^t, \mathcal{D}^1 + \nu\mathcal{D}^2 + \kappa\mathcal{J}\rangle_{\kappa\geqslant 0}, \quad \mathfrak{g}^{2.2}_{\kappa_1\kappa_2} = \langle \mathcal{P}^t + \kappa_1\mathcal{J}, \mathcal{D}^2 + \kappa_2\mathcal{J}\rangle_{\kappa_1\in\{0,1\},\kappa_2\geqslant 0},$$
$$\mathfrak{g}^{2.3}_{\nu} = \langle \mathcal{P}^t + \nu\mathcal{D}^2, \mathcal{J}\rangle_{\nu\in\{0,1\}}, \quad \mathfrak{g}^{2.4}_{\kappa_1\kappa_2} = \langle \mathcal{D}^1 + \kappa_1\mathcal{J}, \mathcal{D}^2 + \kappa_2\mathcal{J}\rangle_{\kappa_1\geqslant 0},$$
$$\mathfrak{g}^{2.5}_{\nu} = \langle \mathcal{D}^1 + \nu\mathcal{D}^2, \mathcal{J}\rangle_{\nu\geqslant 0}, \quad \mathfrak{g}^{2.6}_{\kappa_1\kappa_2} = \langle \mathcal{P}^t + \mathcal{K} + \kappa_1\mathcal{J}, \mathcal{D}^2 + \kappa_2\mathcal{J}\rangle_{\kappa_1,\kappa_2\geqslant 0},$$
$$\mathfrak{g}^{2.7}_{\nu} = \langle \mathcal{P}^t + \mathcal{K} + \nu\mathcal{D}^2, \mathcal{J}\rangle_{\nu\geqslant 0}, \quad \mathfrak{g}^{2.8}_{\nu} = \langle \mathcal{P}^t + \mathcal{K} + \nu\mathcal{D}^2 + \mathcal{J}, \mathcal{G}^x - \mathcal{P}^y\rangle_{\nu>0},$$
$$\mathfrak{g}^{2.9}_{\mu} = \langle \mathcal{P}^t + \mathcal{K} + \mathcal{J} + \mu(\mathcal{P}^x + \mathcal{G}^y), \mathcal{G}^x - \mathcal{P}^y\rangle_{\mu\geqslant 0}, \quad \mathfrak{g}^{2.10} = \langle \mathcal{D}^2, \mathcal{J}\rangle,$$
$$\mathfrak{g}^{2.11} = \langle \mathcal{D}^2, \mathcal{G}^x - \mathcal{P}^y\rangle, \quad \mathfrak{g}^{2.12}_{\mu} = \langle \mathcal{P}^x + \mathcal{G}^x, \mathcal{P}^y + \mu\mathcal{G}^x\rangle_{\mu>0}, \quad \mathfrak{g}^{2.13} = \langle \mathcal{P}^y, \mathcal{P}^t + \mathcal{D}^2\rangle,$$
$$\mathfrak{g}^{2.14}_{\mu} = \langle \mathcal{P}^y, \mathcal{P}^t + \mu\mathcal{G}^x + \nu\mathcal{G}^y\rangle_{\mu,\nu\geqslant 0,\ \mu^2+\nu^2\in\{0,1\}}, \quad \mathfrak{g}^{2.15}_{a} = \langle \mathcal{P}^y, \mathcal{D}^1 + a\mathcal{D}^2\rangle_{a\geqslant 0},$$
$$\mathfrak{g}^{2.16}_{\mu} = \langle \mathcal{P}^y, \mathcal{D}^1 + \mathcal{D}^2 + \mu\mathcal{G}^x + \nu\mathcal{G}^y\rangle_{\mu,\nu\geqslant 0,\ \mu^2+\nu^2=1}, \quad \mathfrak{g}^{2.17}_{a} = \langle \mathcal{P}^y, \mathcal{D}^1 - \mathcal{D}^2 + a\mathcal{P}^x\rangle_{a>0},$$
$$\mathfrak{g}^{2.18} = \langle \mathcal{P}^y, \mathcal{D}^2\rangle, \quad \mathfrak{g}^{2.19} = \langle \mathcal{P}^y, \mathcal{P}^x + \mathcal{G}^y\rangle, \quad \mathfrak{g}^{2.20} = \langle \mathcal{P}^y, \mathcal{P}^x\rangle,$$
$$\mathfrak{g}^{2.21}_{\mu} = \langle \mathcal{P}^y, \mathcal{G}^x + \mu\mathcal{G}^y\rangle_{\mu\geqslant 0}, \quad \mathfrak{g}^{2.22} = \langle \mathcal{P}^y, \mathcal{G}^y\rangle.$$

We will not consider Lie reductions of codimension two constructed with the help of the above algebras which have $\mathcal{P}^y$ as their basis element, because the reduced system 1.10 was completely integrated. One may find particular solutions to the Euler–Poisson–Darboux equation and prolong them to the solution of the reduced system 1.10. All the reduced systems are systems of first-order ODEs, and therefore have infinite-dimensional maximal Lie invariance algebras, but they are not systematically constructable. We try to give as many solutions to the reduced systems below as possible. Usually, a simple set of solutions can be found by considering $\varphi^1 = 0$. Below c's are constants.

2.1. $\mathfrak{g}^{2.1}_{\nu\kappa} = \langle \mathcal{P}^t, \mathcal{D}^1 + \nu\mathcal{D}^2 + \kappa\mathcal{J}\rangle_{\nu\neq -1,\ \kappa\geqslant 0}$:

$$u = (x^2+y^2)^{-\frac{1}{\nu+1}}\left(((\nu+1)x - \kappa y)\varphi^1 - (\kappa x + (\nu+1)y)\varphi^2\right),$$

$$v=(x^2+y^2)^{-\frac{1}{\nu+1}}\left((\kappa x+(\nu+1)y)\varphi^1+((\nu+1)x-\kappa y)\varphi^2\right),$$
$$h=\frac{(x^2+y^2)^{\frac{\nu-1}{\nu+1}}\varphi^3}{\kappa^2+(\nu+1)^2},$$

where $\omega=\dfrac{1}{\kappa^2+(\nu+1)^2}\left(\dfrac{\kappa}{2}\ln(x^2+y^2)-(\nu+1)\arctan\dfrac{y}{x}\right)$;

$$\varphi^2\varphi^1_\omega+\frac{\kappa\nu}{\nu+1}\left((\varphi^1)^2-(\varphi^2)^2\right)+2(\nu-1)\varphi^1\varphi^2-\frac{\nu-1}{(\kappa^2+(\nu+1)^2)^2}\varphi^3=0,$$
$$\varphi^2\varphi^2_\omega-\frac{2\kappa(\nu-1)\varphi^3+(\nu+1)\varphi^3_\omega}{(\nu+1)(\kappa^2+(\nu+1)^2)^2}-(\nu-1)\left((\varphi^1)^1-(\varphi^2)^2\right)+\frac{2\kappa\nu}{\nu+1}\varphi^1\varphi^2=0,$$
$$\varphi^2\varphi^3_\omega+\varphi^3\varphi^2_\omega-2(2\nu-1)\left(\varphi^1-\frac{\kappa}{\nu+1}\varphi^2\right)\varphi^3=0.$$

$\mathfrak{g}^{2.1}_{-1,\kappa}=\langle\mathcal{P}^t,\mathcal{D}^1-\mathcal{D}^2+\kappa\mathcal{J}\rangle_{\kappa>0}$:

$$u=\mathrm{e}^{-2\alpha/\kappa}\frac{x\varphi^1-y\varphi^2}{x^2+y^2},\quad v=\mathrm{e}^{-2\alpha/\kappa}\frac{y\varphi^1+x\varphi^2}{x^2+y^2},\quad h=\mathrm{e}^{-4\alpha/\kappa}\frac{\varphi^3}{x^2+y^2},$$

where $\omega=\dfrac{\kappa}{2}\ln(x^2+y^2)$ and $\alpha=\arctan\dfrac{y}{x}$;

$$\varphi^1\varphi^1_\omega+\varphi^3_\omega-\kappa\left((\varphi^1)^2+(\varphi^2)^2\right)-2\varphi^1\varphi^2-2\kappa\varphi^3=0,$$
$$\varphi^1\varphi^2_\omega-2(\varphi^2)^2-4\varphi^3=0,$$
$$\varphi^1\varphi^3_\omega+\varphi^3\varphi^1_\omega-2(\kappa\varphi^1+3\varphi^2)\varphi^3=0.$$

One can express $\varphi^3=(\varphi^1\varphi^2_\omega-2(\varphi^2)^2)/4$ from the second equation and eliminate φ^1_ω from the first and the third equations, which results in the cubic equation on φ^1,

$$4\left(2\kappa\varphi^2_\omega+\varphi^2_{\omega\omega}\right)(\varphi^1)^3-\left(\varphi^2_\omega\left(\varphi^2_{\omega\omega}-4\kappa\varphi^2_\omega+8\right)+8\kappa(\varphi^2)^2\right)(\varphi^1)^2$$
$$-\left(2(\varphi^2)^2(2\kappa\varphi^2_\omega-\varphi^2_{\omega\omega})-2(\varphi^2_\omega)^2(2\varphi^2-1)+16(\varphi^2)^2(\varphi^2-1)\right)\varphi^1-4(\varphi^2)^2\varphi^2_\omega(\varphi^2-1)=0.$$

At the same time, the solution to this equation and the resulting ODE on φ^2 are too cumbersome to be presented here.

For the algebra $\mathfrak{g}^{2.1}_{-1,0}$ the local transversality condition does not hold and therefore one can not carry out a classical Lie reduction. Note that it is still possible to consider Lie reductions for some algebras with this property [10] but not for $\mathfrak{g}^{2.1}_{-1,0}$.

2.2. $\mathfrak{g}^{2.2}_{\kappa_1\kappa_2} = \langle \mathcal{P}^t + \kappa_1\mathcal{J}, \mathcal{D}^2 + \kappa_2\mathcal{J}\rangle_{\kappa_1\in\{0,1\},\kappa_2\geqslant 0}$:

$$u = (x - \kappa_2 y)\varphi^1 - (\kappa_2 x + y)\varphi^2, \quad v = (\kappa_2 x + y)\varphi^1 + (x - \kappa_2 y)\varphi^2,$$
$$h = (\kappa_2^2 + 1)(x^2 + y^2)\varphi^3,$$

where $\omega = \dfrac{1}{\kappa_2^2+1}\Big(\kappa_1 t - \arctan\dfrac{y}{x} + \dfrac{\kappa_2}{2}\ln(x^2+y^2)\Big)$;

$$\begin{aligned}
&\left(\frac{\kappa_1}{\kappa_2^2+1} - \varphi^2\right)\varphi^1_\omega + (\varphi^1)^2 - (\varphi^2)^2 - 2\kappa_2\varphi^1\varphi^2 + 2\varphi^3 = 0,\\
&\left(\frac{\kappa_1}{\kappa_2^2+1} - \varphi^2\right)\varphi^2_\omega - \varphi^3_\omega + \kappa_2(\varphi^1)^2 - \kappa_2(\varphi^2)^2 + 2\varphi^1\varphi^2 - 2\kappa_2\varphi^3 = 0,\\
&\left(\frac{\kappa_1}{\kappa_2^2+1} - \varphi^2\right)\varphi^3_\omega - \varphi^3\varphi^2_\omega + 4(\varphi^1 - \kappa_2\varphi^2)\varphi^3 = 0.
\end{aligned}$$

2.3. $\mathfrak{g}^{2.3}_{\nu} = \langle \mathcal{P}^t + \nu\mathcal{D}^2, \mathcal{J}\rangle_{\nu\in\{0,1\}}$:

$$u = x\varphi^1 - y\varphi^2, \quad v = y\varphi^1 + x\varphi^2, \quad h = (x^2+y^2)\varphi^3,$$

where $\omega = \nu t - \dfrac{1}{2}\ln(x^2+y^2)$;

$$\begin{aligned}
&(\varphi^1 - \nu)\varphi^1_\omega + \varphi^3_\omega - (\varphi^1)^2 + (\varphi^2)^2 - 2\varphi^3 = 0,\\
&(\varphi^1 - \nu)\varphi^2_\omega - 2\varphi^1\varphi^2 = 0,\\
&(\varphi^1 - \nu)\varphi^3_\omega + (\varphi^1_\omega - 4\varphi^1)\varphi^3 = 0.
\end{aligned}$$

When $\nu = 0$, the system is completely integrable.

Thus, $(\varphi^1, \varphi^2, \varphi^3) = (0, \pm\sqrt{2f - f_\omega}, f)$ is a solution for any smooth function f of ω, for which $2f \geqslant f_\omega$. An alternative representation of this solution is $(\varphi^1, \varphi^2, \varphi^3) = (0, f, c_3\mathrm{e}^{2\omega} + g)$, where c_3 is an arbitrary constant, f is an arbitrary function of ω and g is a particular solution of the equation $g_\omega - 2g + f^2 = 0$.

If $\varphi^1 \neq 0$, then $\varphi^2 = c_2 \mathrm{e}^{2\omega}$, $\varphi^3 = c_3 \mathrm{e}^{4\omega}/\varphi^1$ and φ^1 satisfies the ODE.

$$\left((\varphi^1)^3 - c_3 \mathrm{e}^{4\omega}\right) \varphi^1_\omega = (\varphi^1)^4 - c_2 \mathrm{e}^{4\omega} (\varphi^1)^2 - 2 c_3 \mathrm{e}^{4\omega} \varphi^1,$$

It has the first integral

$$(c_2^2 + 2c_3(\varphi^1)^{-1})\mathrm{e}^{2\omega} + \mathrm{e}^{-2\omega}(\varphi^1)^2.$$

Denote a constant value of this first integral on a solution of the equation by $-c_1$. In other words, the function φ^1 satisfies the cubic equation

$$(\varphi^1)^3 + (c_2^2 \mathrm{e}^{4\omega} + c_1 \mathrm{e}^{2\omega})\varphi^1 + 2c_3 \mathrm{e}^{4\omega} = 0.$$

This equation may have one, two or three distinct real-valued solution depending on the sign of $\Delta(\omega) = 27 c_3^2 \mathrm{e}^{8\omega} + (c_2^2 \mathrm{e}^{4\omega} + c_1 \mathrm{e}^{2\omega})^3$. As its sign may change as ω varies, with parameters c's fixed, on some intervals a real-valued solution may degenerate into a complex-valued one, and a complex-valued solution may regularize into a real-valued one. Three solutions of the cubic equation are

$$\begin{aligned}
&\varphi^1_1(\omega) = \varphi_-, \quad \varphi^1_2(\omega) = \frac{1}{2}(i\sqrt{3}\varphi_+ - \varphi_-); \quad \varphi^1_3(\omega) = -\frac{1}{2}(i\sqrt{3}\varphi_+ + \varphi_-);\\
&\text{where} \quad \varphi_\pm = \frac{\psi(\omega)}{3} \pm \frac{c_2^2 \mathrm{e}^{4\omega} + 2c_1 \mathrm{e}^{2\omega}}{\psi(\omega)}, \quad \psi(\omega) = \left(3\sqrt{3\Delta(\omega)} - 27 c_3 \mathrm{e}^{4\omega}\right)^{\frac{1}{3}}.
\end{aligned}$$

Recall that real-valued solutions of a cubic equation not always can be written as a function of real arguments, and therefore $\varphi^1_2(\omega)$ and $\varphi^1_3(\omega)$ may still be real-valued. Analogous solutions exist for several other reduced systems below.

If $\nu = 1$, then introducing the function ϕ of ω such that $\varphi^1 = \phi/\phi_\omega + 1$ one yields $\varphi^2(\omega) = c_2 \mathrm{e}^{2\omega}(\phi(\omega))^2$ and $\varphi^3(\omega) = c_3 \mathrm{e}^{4\omega}(\phi(\omega))^3 \phi_\omega(\omega)$, and the first equation reduces to

$$\frac{\phi}{\phi_\omega}\left(\frac{\phi}{\phi_\omega}\right)_\omega - \left(\frac{\phi}{\phi_\omega} + 1\right)^2 + c_3(\mathrm{e}^{4\omega}\phi^3\phi_\omega)_\omega - 2c_3 \mathrm{e}^{4\omega}\phi^3\phi_\omega + c_2^2 \mathrm{e}^{4\omega}\phi^4 = 0.$$

Alternatively, introducing the function ϕ of ω satisfying $\varphi^1 = \phi_\omega/(\phi_\omega - 2\phi)$ one obtains

$\varphi^2 = c_2\phi$, $\varphi^3 = c_3\phi(\phi_\omega - 2\phi)$, and the first equation reduces to

$$\frac{2\phi}{\phi_\omega - 2\phi}\left(\frac{\phi_\omega}{\phi_\omega - 2\phi}\right)_\omega + c_3\left(\phi(\phi_\omega - 2\phi)\right)_\omega - \left(\frac{\phi_\omega}{\phi_\omega - 2\phi}\right)^2 + c_2^2\phi^2 - 2c_3\phi(\phi_\omega - 2\phi) = 0.$$

One can reduce the order of this autonomous equation by choosing $\theta(\phi) = \phi_\omega$ to be a new dependent variable,

$$\begin{aligned}&-(c_3\theta^3 - 6c_3\phi\theta^2 + 12c_3\phi^2\theta - 8c_3\phi^3 - 4\phi)\phi\theta\theta_\phi = c_3\theta^5 - 12c_3\phi\theta^4\\ &+\left((c_2^2{+}52c_3)\phi^2 - 1\right)\theta^3 - \left((6c_2^2{+}104c_3)\phi^2 - 6\right)\phi\theta^2 + 12(c_2^2{+}8c_3)\phi^4\theta - 8(c_2^2{+}4c_3)\phi^5.\end{aligned}$$

2.4. $\mathfrak{g}^{2.4}_{\kappa_1\kappa_2} = \langle \mathcal{D}^1 + \kappa_1\mathcal{J}, \mathcal{D}^2 + \kappa_2\mathcal{J}\rangle_{\kappa_1\geqslant 0}$:

$$\begin{aligned}u &= \frac{(x-\kappa_2 y)\varphi^1 - (\kappa_2 x + y)\varphi^2}{t}, \quad v = \frac{(\kappa_2 x + y)\varphi^1 + (x - \kappa_2 y)\varphi^2}{t},\\ h &= \frac{(\kappa_2^2+1)(x^2+y^2)\varphi^3}{t^2},\end{aligned}$$

where $\omega = \dfrac{1}{(\kappa_2^2+1)}\left(\dfrac{\kappa_1-\kappa_2}{2}\ln|t| - \arctan\dfrac{y}{x} + \dfrac{\kappa_2}{2}\ln(x^2+y^2)\right)$;

$$\begin{aligned}&\left(\frac{\kappa_1-\kappa_2}{2(\kappa_2^2+1)} - \varphi^2\right)\varphi^1_\omega + (\varphi^1)^2 - (\varphi^2)^2 - 2\kappa_2\varphi^1\varphi^2 - \varphi^1 + 2\varphi^3 = 0,\\ &\left(\frac{\kappa_1-\kappa_2}{2(\kappa_2^2+1)} - \varphi^2\right)\varphi^2_\omega - \varphi^3_\omega + \kappa_2(\varphi^1)^2 - \kappa_2(\varphi^2)^2 + 2\varphi^1\varphi^2 - \varphi^2 - 2\kappa_2\varphi^3 = 0,\\ &\left(\frac{\kappa_1-\kappa_2}{2(\kappa_2^2+1)} - \varphi^2\right)\varphi^3_\omega - \varphi^3\varphi^2_\omega + 4(\varphi^1 - \kappa_2\varphi^2)\varphi^3 - 2\varphi^3 = 0.\end{aligned}$$

2.5. $\mathfrak{g}^{2.5}_{\nu} = \langle \mathcal{D}^1 - (1+2\nu)\mathcal{D}^2, \mathcal{J}\rangle_{\nu\leqslant -1/2}$:

$$u = \frac{x\varphi^1 - y\varphi^2}{t}, \quad v = \frac{y\varphi^1 + x\varphi^2}{t}, \quad h = \frac{(x^2+y^2)\varphi^3}{t^2},$$

where $\omega = \ln|t| + \dfrac{1}{2\nu}\ln(x^2+y^2)$;

$$(\varphi^1+\nu)\varphi^1_\omega + \varphi^3_\omega + \nu\left((\varphi^1)^2 - (\varphi^2)^2 - \varphi^1 + 2\varphi^3\right) = 0,$$

$$
\begin{aligned}
&(\varphi^1+\nu)\varphi^2_\omega+\nu\left(2\varphi^1-1\right)\varphi^2=0,\\
&(\varphi^1+\nu)\varphi^3_\omega+(\varphi^1_\omega+4\nu\varphi^1-2\nu)\varphi^3=0.
\end{aligned}
$$

If $\nu=-1/2$, then the system is completely integrable.

If $\varphi^1=1/2$, then the system reduces to the equation $2\varphi^3_\omega-2\varphi^3+(\varphi^2)^2+\frac14=0$, which can be easily solved with respect to either φ^2 or φ^3.

If $\varphi^1\neq 1/2$, then the second and the third equations of the system give $\varphi^2(\omega)=c_2\mathrm{e}^\omega$ and $\varphi^3(\omega)=c_3\mathrm{e}^{2\omega}/(\varphi^1(\omega)-1/2)$, respectively, and the first equation then becomes an ODE with respect to φ^1, which possesses the first integral

$$
\mathrm{e}^{-\omega}\left(\varphi^1-(\varphi^1)^2-\frac{4c_3\mathrm{e}^{2\omega}}{2\varphi^1-1}\right)-c_2^2\mathrm{e}^\omega.
$$

In other words, φ^1 satisfies the cubic equation

$$
2(\varphi^1)^3-3(\varphi^1)^2+(2c_2^2\mathrm{e}^{2\omega}+2c_1\mathrm{e}^\omega+1)\varphi^1+(4c_3-c_2^2)\mathrm{e}^{2\omega}-c_1\mathrm{e}^\omega=0.
$$

Three solutions of this cubic equation are

$$
\begin{aligned}
&\varphi^1_1(\omega)=\varphi_-+\frac12,\quad \varphi^1_2(\omega)=\frac12(i\sqrt3\varphi_+-\varphi_-+1);\quad \varphi^1_3(\omega)=\frac12(-i\sqrt3\varphi_+-\varphi_-+1);\\
&\text{where}\quad \varphi_\pm=\frac{\psi(\omega)}{6}\pm\frac{4c_2^2\mathrm{e}^{2\omega}+2c_1\mathrm{e}^\omega-1}{2\psi(\omega)},\\
&\psi(\omega)=\left(3\sqrt{3(4c_2^2\mathrm{e}^{2\omega}+8c_1\mathrm{e}^\omega-1)^3+5184c_3^2\mathrm{e}^{4\omega}}+27c_1\mathrm{e}^\omega-216c_3\mathrm{e}^{2\omega}\right)^{1/3}.
\end{aligned}
$$

If $\nu\neq 1/2$, we can introduce the function ϕ of ω such that $\varphi^1=\nu(\phi_\omega-\phi)/(2\nu\phi+\phi_\omega)$. Then the functions φ^2 and φ^3 can be expressed as $\varphi^2=c_2\phi$ and $\varphi^3=c_3(2\nu\phi+\phi_\omega)\phi$. The first equation reduces to the autonomous ODE

$$
\begin{aligned}
&\left(c_3\phi{\phi_\omega}^3+6c_3\nu\phi^2{\phi_\omega}^2+12c_3\nu^2\phi^3\phi_\omega+8c_3\nu^3\phi^4-(2\nu+1)^2\nu^2\phi^2\right)\phi_{\omega\omega}\\
&\qquad+c_3{\varphi_\omega}^5+12c_3\nu\phi{\phi_\omega}^4+\left((52c_3\nu-c_2^2)\nu\phi^2+\nu^3\right){\phi_\omega}^3+\left((104c_3\nu-6c_2^2)\phi^2+6\nu^2\right)\nu^2\phi{\phi_\omega}^2\\
&\qquad+12(8c_3\nu-c_2^2)\nu^3\phi^4\phi_\omega+2\nu^4\left(4(4c_3\nu-c_2^2)\phi^2-1\right)\phi^3=0,
\end{aligned}
$$

which can be further reduced to a first-order ODE with the help of standard methods.

2.6. $\mathfrak{g}^{2.6}_{\kappa_1\kappa_2} = \langle \mathcal{P}^t + \mathcal{K} + \kappa_1\mathcal{J}, \mathcal{D}^2 + \kappa_2\mathcal{J}\rangle_{\kappa_1,\kappa_2\geqslant 0}$:

$$
\begin{aligned}
&u = \frac{(x-\kappa_2 y)\varphi^1 - (\kappa_2 x + y)\varphi^2}{\sqrt{\kappa_2^2+1}(t^2+1)} + \frac{tx}{t^2+1}, \;\; v = \frac{(\kappa_2 x + y)\varphi^1 + (x-\kappa_2 y)\varphi^2}{\sqrt{\kappa_2^2+1}(t^2+1)} + \frac{ty}{t^2+1},\\
&h = \frac{(x^2+y^2)\varphi^3}{(t^2+1)^2},
\end{aligned}
$$

where $\omega = \dfrac{\arctan\dfrac{y}{x} - \dfrac{\kappa_2}{2}\ln(x^2+y^2) - \kappa_1\arctan t + \dfrac{\kappa_2}{2}\ln(t^2+1)}{\kappa_2^2+1}$;

$$
\begin{aligned}
&\left(\varphi^2 - \kappa_1/\sqrt{\kappa_2^2+1}\right)\varphi^1_\omega + (\varphi^1)^2 - (\varphi^2)^2 - 2\kappa_2\varphi^1\varphi^2 + 2\varphi^3 + 1 = 0,\\
&\left(\varphi^2 - \kappa_1/\sqrt{\kappa_2^2+1}\right)\varphi^2_\omega + \varphi^3_\omega + \kappa_2\left((\varphi^1)^2 - (\varphi^2)^2\right) + 2\varphi^1\varphi^2 - \kappa_2(2\varphi^3+1) = 0,\\
&\left(\varphi^2 - \kappa_1/\sqrt{\kappa_2^2+1}\right)\varphi^3_\omega + \varphi^3\varphi^2_\omega - 4\left(\kappa_2\varphi^2 - \varphi^1\right)\varphi^3 = 0.
\end{aligned}
$$

2.7. $\mathfrak{g}^{2.7}_{\nu} = \langle \mathcal{P}^t + \mathcal{K} + \nu\mathcal{D}^2, \mathcal{J}\rangle_{\nu\geqslant 0}$:

$$
u = \frac{x(\varphi^1+t) - y\varphi^2}{t^2+1}, \quad v = \frac{y(\varphi^1+t) + x\varphi^2}{t^2+1}, \quad h = \frac{(x^2+y^2)\varphi^3}{(t^2+1)^2},
$$

where $\omega = \nu\arctan t - \dfrac{1}{2}\ln\dfrac{x^2+y^2}{t^2+1}$;

$$
\begin{aligned}
&(\varphi^1-\nu)\varphi^1_\omega + \varphi^3_\omega - (\varphi^1)^2 + (\varphi^2)^2 - 2\varphi^3 - 1 = 0,\\
&(\varphi^1-\nu)\varphi^2_\omega - 2\varphi^1\varphi^2 = 0,\\
&(\varphi^1-\nu)\varphi^3_\omega + (\varphi^1_\omega - 4\varphi^1)\varphi^3 = 0.
\end{aligned}
$$

The system is completely integrable for $\nu = 0$.

If $\varphi^1 = 0$, then the system reduces to the equation $\varphi^3_\omega + (\varphi^2)^2 - 2\varphi^3 - 1 = 0$, which can be easily solved with respect to either φ^2 or φ^3.

If $\varphi^1 \neq 0$, then one yields from the second and the third equations that $\varphi^2(\omega) = c_2\mathrm{e}^{2\omega}$ and $\varphi^3(\omega) = c_3\mathrm{e}^{4\omega}(\varphi^1(\omega))^{-1}$, respectively, and the first equation becomes an ODE with the first integral

$$(c_2^2+2c_3(\varphi^1)^{-1})\mathrm{e}^{2\omega}+((\varphi^1)^2+1)\mathrm{e}^{-2\omega}.$$

In other words, φ^1 satisfies the cubic equation

$$(\varphi^1)^3+(c_2^2\mathrm{e}^{4\omega}+c_1\mathrm{e}^{2\omega}+1)\varphi^1+2c_3\mathrm{e}^{4\omega}=0,$$

whose solutions are

$$\begin{aligned}
&\varphi^1_1(\omega)=\varphi_-,\quad \varphi^1_2(\omega)=\frac{1}{2}(i\sqrt{3}\varphi_+-\varphi_-);\quad \varphi^1_3(\omega)=-\frac{1}{2}(i\sqrt{3}\varphi_++\varphi_-);\\
&\text{where}\quad \varphi_\pm=\frac{\psi(\omega)}{3}\pm\frac{c_2^2\mathrm{e}^{4\omega}+c_1\mathrm{e}^{2\omega}+1}{\psi(\omega)},\\
&\psi(\omega)=\left(3\sqrt{3(c_2^2\mathrm{e}^{4\omega}+c_1\mathrm{e}^{2\omega}+1)^3+81c_3^2\mathrm{e}^{8\omega}}-27c_3\mathrm{e}^{4\omega}\right)^{1/3}.
\end{aligned}$$

If $\nu\neq 0$, the system can be reduced to a first-order ODE by introducing the function ϕ of ω such that $\varphi^1=\nu\phi_\omega/(\phi_\omega-2\phi)$. Then immediately $\varphi^2=c_2\phi$, $\varphi^3=c_3(\phi_\omega-2\phi)\phi$ and

$$\begin{aligned}
&(c_3\phi{\phi_\omega}^3-6c_3\phi^2{\phi_\omega}^2+12c_3\phi^3\phi_\omega-8c_3\phi^4-4\nu^2\phi^2)\phi_{\omega\omega}+c_3{\phi_\omega}^5-12c_3\phi{\phi_\omega}^4\\
&\quad+\big((c_2^2+52c_3)\phi^2-\nu^2-1\big)\,{\phi_\omega}^3-\big((6c_2^2+104c_3)\phi^2-6(\nu^2+1)\big)\,\phi{\phi_\omega}^2\\
&\quad+12\,\big((c_2^2+8c_3)\phi^2-1\big)\,\phi^2\phi_\omega-8\,\big((c_2^2+4c_3)\phi^2-1\big)\,\phi^3=0.
\end{aligned}$$

This equation can be further reduced with the help of the standard substitution $\theta(\phi)=\phi_\omega$.

2.8. $\mathfrak{g}^{2.8}_\nu=\langle\mathcal{P}^t+\mathcal{K}+\nu\mathcal{D}^2+\mathcal{J},\mathcal{G}^x-\mathcal{P}^y\rangle_{\nu>0}$:

$$\begin{aligned}
&u=\frac{(x+ty)(\varphi^1-t\varphi^2)+(tx-y)(t^2+1)}{(t^2+1)^2},\quad v=\frac{(x+ty)(t\varphi^1+\varphi^2+t^2+1)}{(t^2+1)^2},\\
&h=\frac{(x+ty)^2\varphi^3}{(t^2+1)^3},
\end{aligned}$$

where $\omega = \ln \dfrac{|x+ty|}{t^2+1} - \nu \arctan t$;

$$(\varphi^1-\nu)\varphi^1_\omega + \varphi^3_\omega + (\varphi^1)^2 - 2\varphi^2 + 2\varphi^3 = 0,$$
$$(\varphi^1-\nu)\varphi^2_\omega + \varphi^1\varphi^2 + 2\varphi^1 = 0,$$
$$(\varphi^1-\nu)\varphi^3_\omega + (\varphi^1_\omega + 3\varphi^1)\varphi^3 = 0.$$

2.9. $\mathfrak{g}^{2.9}_\mu = \langle \mathcal{P}^t + \mathcal{K} + \mathcal{J} + \mu(\mathcal{P}^x + \mathcal{G}^y), \mathcal{G}^x - \mathcal{P}^y \rangle_{\mu\geqslant 0}$:

$$u = \frac{\varphi^1 - t\varphi^2 + tx - y + \mu}{t^2+1}, \quad v = \frac{t\varphi^1 + \varphi^2 + x + ty + \mu t}{t^2+1}, \quad h = \frac{\varphi^3}{t^2+1},$$

where $\omega = \mu \arctan t - \dfrac{x+ty}{t^2+1}$;

$$\varphi^1\varphi^1_\omega + \varphi^3_\omega + 2\varphi^2 = 0,$$
$$\varphi^1\varphi^2_\omega - 2\varphi^1 - 2\mu = 0,$$
$$\varphi^1\varphi^3_\omega + \varphi^3\varphi^1_\omega = 0.$$

First, if $\varphi^1 = 0$, then $\mu = 0$ and $\varphi^2 = -\varphi^3_\omega/2$, that is, $(0, -f_\omega/2, f)$ is a solution of the reduced system for any sufficiently smooth function f of ω.

Consider two cases when $\varphi^1 \neq 0$: $\mu = 0$ and $\mu \neq 0$. In the former case we have immediately $\varphi^3 = c_3/\varphi^1$ and $\varphi^2 = 2\omega + c_2/2$. The first equation gives

$$(\varphi^1)^3 + (\omega^2 + c_2\omega + c_1)\varphi^1 + c_3 = 0.$$

This cubic equation has the solutions

$$\varphi^1_1(\omega) = \varphi_-, \quad \varphi^1_2(\omega) = \frac{1}{2}(i\sqrt{3}\varphi_+ - \varphi_-); \quad \varphi^1_3(\omega) = -\frac{1}{2}(i\sqrt{3}\varphi_+ + \varphi_-);$$
$$\text{where} \quad \varphi_\pm = \frac{\psi(\omega)}{6} \pm \frac{2(\omega^2 + c_2\omega + c_1)}{\psi(\omega)},$$
$$\psi(\omega) = \left(12\sqrt{12(\omega^2 + c_2\omega + c_1)^3 + 81c_3^2} - 108c_3\right)^{1/3}.$$

In the latter case, we have $\varphi^3 = c_3/\varphi^1$, $\varphi^2 = -(\varphi^1\varphi^1_\omega + \varphi^3_\omega)/2$ and the second equation results in $(\phi_\omega/\phi^3)_\omega - c_3\phi_{\omega\omega} = 4(1+\mu\phi)$ for $\phi = 1/\varphi^1$. Taking $\theta(\phi) = \phi_\omega$ as a new unknown function of ϕ, the equation integrates to

$$\theta^2(\phi) = \frac{-4\phi^4}{(1-c_3\phi^3)^2}\left(c_3\mu\phi^4 + 2c_3\phi^3 + c_4\phi^2 + 2\mu\phi + 1\right).$$

Alternatively, one can express φ^1 via φ^2, $\varphi^1 = 2\mu/(\varphi^2_\omega - 2)$, then immediately obtain $\varphi^3 = c_3(\varphi^2_\omega - 2)$ and plug in these expressions for φ^1 and φ^3 into the first equation,

$$(c_3(\varphi^2_\omega)^3 - 6c_3(\varphi^2_\omega)^2 + 12c_3\varphi^2_\omega - 4\mu^2 - 8c_3)\varphi^2_{\omega\omega} + (2(\varphi^2_\omega)^3 - 12(\varphi^2_\omega)^2 + 24\varphi^2_\omega - 16)\varphi^2 = 0.$$

This equation has a first integral, which allows us to reduce the above equation to

$$c_3(\varphi^2_\omega)^2 + 2c_1 + 2(\varphi^2)^2 + \frac{8\mu^2(\varphi^2_\omega - 1)}{(\varphi^2_\omega - 2)^2} = 0.$$

Unlike the equation given by the representation above for $\theta(\phi)$, this equation can be easily solved with respect to the independent variable,

$$(\varphi^2)^2 = -\frac{1}{2}c_3(\varphi^2_\omega)^2 - c_1 - \frac{4\mu^2}{\varphi^2_\omega - 2} - \frac{4\mu^2}{(\varphi^2_\omega - 2)^2},$$

and present a solution in an implicit form. Indeed, on the interval (t_1, t_2), where the function $g(z) = \pm\sqrt{-\frac{1}{2}c_3z^2 - c_1 - \frac{4\mu^2}{z-2} - \frac{4\mu^2}{(z-2)^2}}$ is strictly monotonous, the solution to the above equation may be written as

$$\omega = c_2 + \int_{\eta_0}^{\varphi^2_\omega} \frac{g_z(z)}{z}\mathrm{d}z, \quad \varphi^2 = g(z).$$

2.10. $\mathfrak{g}^{2.10} = \langle \mathcal{D}^2, \mathcal{J}\rangle$: $u = x\varphi^1 - y\varphi^2$, $v = y\varphi^1 + x\varphi^2$, $h = (x^2+y^2)\varphi^3$ with $\omega = t$;

$$\begin{aligned}
&\varphi^1_\omega + (\varphi^1)^2 - (\varphi^2)^2 + 2\varphi^3 = 0,\\
&\varphi^2_\omega + 2\varphi^1\varphi^2 = 0,\\
&\varphi^3_\omega + 4\varphi^1\varphi^3 = 0.
\end{aligned}$$

Making the Ricatti substitution $\varphi^1 = \phi_\omega/\phi$ we immediately find $\varphi^2 = c_1\phi^{-2}$ and $\varphi^3 = c_2\phi^{-4}$ for arbitrary constants c's, while the first equation results in $\phi_{\omega\omega}+(2c_2-c_1^2)\phi^{-3} = 0$, which integrates directly first to ${\phi_\omega}^2 = (2c_2 - c_1^2)\phi^{-2} + c_3$ and then a second time to

$$\phi(\omega) = \begin{cases} \sqrt[4]{8c_3 - 4c_2^2}\sqrt{|\omega|} & \text{if} \quad c_4 = 0, \\ \sqrt[4]{\frac{c_2^2-2c_3}{c_4}}\sqrt{|\omega^2 + c_5|} & \text{otherwise} \end{cases}$$

Here c_4 is an arbitrary constant; $2c_3-c_2^2 > 0$ for the first solution, and $\operatorname{sgn}(c_2^2-2c_3) = \operatorname{sgn} c_4$ for the second one.

2.11. $\mathfrak{g}^{2.11} = \langle \mathcal{D}^2, \mathcal{G}^x - \mathcal{P}^y \rangle$:

$$u = \frac{z_2(\varphi^1 - t\varphi^2) + tx - y}{t^2+1}, \quad v = \frac{z_2(t\varphi^1 + \varphi^2) + x + ty}{t^2+1}, \quad h = \frac{z_2^2\varphi^3}{t^2+1},$$

where $\omega = \arctan t, \;\; z_2 = \dfrac{x+ty}{t^2+1}$;

$$\begin{aligned} &\varphi^1_\omega + (\varphi^1)^2 - 2\varphi^2 + 2\varphi^3 = 0, \\ &\varphi^2_\omega + \varphi^1\varphi^2 + 2\varphi^1 = 0, \\ &\varphi^3_\omega + 3\varphi^1\varphi^3 = 0. \end{aligned}$$

Making the Ricatti substitution $\varphi^1 = \phi_\omega/\phi$ we immediately find $\varphi^2 = c_2\phi^{-1} - 2$ and $\varphi^3 = c_3\phi^{-3}$ for arbitrary constants c's, and the function ϕ satisfies $\phi_{\omega\omega}-2(c_2-2\phi)+2c_3\phi^{-2} = 0$, which is integrated for ϕ_ω to ${\phi_\omega}^2 = 4(c_2\phi - \phi^2 + c_3\phi^{-1} + c_1)$. It implies that

$$\omega + c_4 = \int \frac{\mathrm{d}\phi}{2\sqrt{c_2\phi - \phi^2 + c_3\phi^{-1} + c_1}}.$$

Making the substitution $\phi = \sqrt{\psi/(a - b\psi)}$, where a and b are constants to be specified, the integral transforms to

$$\int \frac{2b^{3/2}\psi^2\mathrm{d}\psi}{(a\psi^2+1)\sqrt{A\psi^6 + B\psi^4 + C\psi^2 + c_3}},$$

where A, B, C depend on some of the constants c_1, c_2, c_3, a, b. Choosing appropriate a and b and assuming $c_3 \neq 0$, which is a natural assumption as otherwise $h = 0$, one can show that the above integral can be represented as a linear combination of incomplete elliptic integrals of the first and the third kinds, although their arguments may be complex and so are their values. We list below some of the real-valued cases.

Taking $c_1 = c_2 = 0$, one obtains $\phi(\omega) = c_3^{1/3}\,(\sin(3\omega + c_4))^{2/3}$ and thus

$$\varphi^1(\omega) = 2\cot(3\omega), \quad \varphi^2(\omega) = -2, \quad \varphi^3(\omega) = \sin^{-2}(3\omega) \pmod{G}.$$

Setting $c_1 = (c_5^2 - c_2^2)/3$ and $c_3 = (c_2^3 - 3c_2c_5^2 + 2c_5^3)/27$ for a new constant c_5 such that $c_2 > c_5 > 0$, and denoting $\tilde{c}_2 = (c_2 + 2c_5)/6$, $\tilde{c}_5 = (c_2 - c_5)/3$, one obtains an implicit expression for the corresponding value of the function ϕ,

$$\omega{+}c_4 = \arcsin\frac{\tilde{c}_2{-}\phi}{\tilde{c}_2} + \sqrt{\frac{\tilde{c}_5}{2\tilde{c}_2{-}\tilde{c}_5}}\,\ln\left|\frac{(\tilde{c}_5{-}\tilde{c}_2)\phi - \sqrt{\tilde{c}_5(2\tilde{c}_2{-}\tilde{c}_5)}\sqrt{\phi(2\tilde{c}_2{-}\phi)} - \tilde{c}_2\tilde{c}_5}{\tilde{c}_5 - \phi}\right|.$$

The case $\tilde{c}_5 = 0$ corresponds to the trivial values $c_1 = c_3 = 0$ and thus $h = 0$, and in this case the function ϕ can be explicitly expressed, $\phi(\omega) = \tilde{c}_2 - \sin(\tilde{c}_2\omega) \pmod{G}$.

2.12. $\mathfrak{g}^{2.12}_{\mu} = \langle \mathcal{G}^x - \mathcal{P}^y, \mathcal{G}^y + \mu\mathcal{P}^x \rangle_{\mu>0}$:

$$u = \frac{t\varphi^1 + \mu\varphi^2 + tx - \mu y}{t^2 + \mu}, \quad v = \frac{\varphi^1 + t\varphi^2 + x + ty}{t^2 + \mu}, \quad h = \frac{\varphi^3}{t^2 + \mu}, \quad \text{where} \ \ \omega = t;$$

$$\varphi^1_\omega = 0, \quad \varphi^2_\omega = 0, \quad \varphi^3_\omega = 0.$$

4.4 Differential invariants for the shallow water equations

In order to set up a moving frame, we have to define a coordinate cross-section that allows us to solve for the group parameters, see details on the moving frame method in [50, 106]. As the shallow water equations admit the nine-dimensional maximal point symmetry group G, nine normalization conditions are to be chosen. The group action must be smooth to define a bona fide moving frame, so we consider in what follows the action of the identity component G^0 of the maximal point symmetry group G, consisting of the transformations

$$T = \mathrm{e}^{\varepsilon_6-\varepsilon_7}\left(\frac{t}{1-\varepsilon_9 t}+\varepsilon_1\right), \quad \begin{pmatrix} X \\ Y \end{pmatrix} = \frac{\mathrm{e}^{\varepsilon_6}}{1-\varepsilon_9 t}\left(O\begin{pmatrix} x \\ y \end{pmatrix}+\begin{pmatrix} \varepsilon_4 \\ \varepsilon_5 \end{pmatrix}t+\begin{pmatrix} \varepsilon_2 \\ \varepsilon_3 \end{pmatrix}\right),$$

$$\begin{pmatrix} U \\ V \end{pmatrix} = \mathrm{e}^{\varepsilon_7}\left((1-\varepsilon_9 t)O\begin{pmatrix} u \\ v \end{pmatrix}+\varepsilon_9 O\begin{pmatrix} x \\ y \end{pmatrix}+\begin{pmatrix} \varepsilon_4+\varepsilon_2\varepsilon_9 \\ \varepsilon_5+\varepsilon_3\varepsilon_9 \end{pmatrix}\right), \quad H = \mathrm{e}^{2\varepsilon_7}(1-\varepsilon_9 t)^2 h,$$

where $O := \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$, $c := \cos\varepsilon_8$, $s := \sin\varepsilon_8$ and ε's are arbitrary constants with $\varepsilon_9 \neq 1/t$ for any value of t. In this section we use Cartan's notational convention using capital letters instead of tildes to denote the target coordinates. In general, the existence of a moving frame is linked to the freeness property of a Lie group. It is clear that G^0 cannot act freely on the jet space $\mathrm{J}^0(t,x,y|u,v,h)$ for dimensional reasons. Therefore, it is necessary to prolong the action at least to $\mathrm{J}^1(t,x,y|u,v,h)$. To be more precise, we consider an action of the group G^0 on the open subset $\{h>0,\ h_x h_y>0,\ 2h+t(h_t+uh_x+vh_y)>0\}$ thereof.

Let $\mathrm{D}_t = \partial_t + \sum_{\alpha,\kappa} w^\kappa_{\alpha+\delta_1}\partial_{w^\kappa_\alpha}$, $\mathrm{D}_x = \partial_x + \sum_{\alpha,\kappa} w^\kappa_{\alpha+\delta_2}\partial_{w^\kappa_\alpha}$ and $\mathrm{D}_y = \partial_y + \sum_{\alpha,\kappa} w^\kappa_{\alpha+\delta_3}\partial_{w^\kappa_\alpha}$ be the usual operators of total differentiation. Here and in what follows we denote by α the tuple $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in \mathbb{N}_0^3$, $|\alpha| = \alpha_1+\alpha_2+\alpha_3$ and set $\delta_1 = (1,0,0)$, $\delta_2 = (0,1,0)$ and $\delta_3 = (0,0,1)$, $w^\kappa_\alpha = w^\kappa_{\alpha_1\alpha_2\alpha_3}$ are the jet coordinates with $w^\kappa_\alpha := \partial^{|\alpha|}w^\kappa/\partial t^{\alpha_1}\partial x^{\alpha_2}\partial y^{\alpha_3}$, $w^\kappa_{000} := w^\kappa$, $w^1 := u$, $w^2 := v$ and $w^3 := h$.

In order to show the above prolongation explicitly, it is necessary to determine the dual total differentiation operators D_T, D_X and D_Y. They are defined via $\mathrm{D}_{W^i} = \sum_{j=1}^3 (\mathrm{J}^{ij})^{-1}\mathrm{D}_{w^j}$, where J is the total Jacobian matrix, which in case of projectable group actions is the usual Jacobian matrix $\mathrm{J} = \frac{\partial(T,X,Y)}{\partial(t,x,y)}$. The notation W^i for the transformed variables are in accordance with Cartan's convention. Thus,

$$\mathrm{D}_X = (1-\varepsilon_9 t)\mathrm{e}^{-\varepsilon_6}(c\mathrm{D}_x - s\mathrm{D}_y), \quad \mathrm{D}_Y = (1-\varepsilon_9 t)\mathrm{e}^{-\varepsilon_6}(s\mathrm{D}_x + c\mathrm{D}_y),$$
$$\mathrm{D}_T = \mathrm{e}^{\varepsilon_7}\left(\frac{1-\varepsilon_9 t}{\mathrm{e}^{\varepsilon_6}}\Big[(1-\varepsilon_9 t)\mathrm{D}_t - \varepsilon_9(x\mathrm{D}_x+y\mathrm{D}_y)\Big] - (\varepsilon_4+\varepsilon_2\varepsilon_9)\mathrm{D}_X - (\varepsilon_5+\varepsilon_3\varepsilon_9)\mathrm{D}_Y\right).$$

These operators can be used to compute the transformed derivatives W^κ_α. To determine a well-defined moving frame $\rho : M \to G^0$, we only need the explicit expressions of three first-order derivatives, and we choose H_T, H_X and H_Y,

$$
\begin{aligned}
H_T &= \mathrm{e}^{3\varepsilon_7-\varepsilon_6}(1-\varepsilon_9 t)^3\Big[(1-\varepsilon_9 t)h_t - 2\varepsilon_9 h - \Big(\varepsilon_9 x + c(\varepsilon_4+\varepsilon_2\varepsilon_9) + s(\varepsilon_5+\varepsilon_3\varepsilon_9)\Big)h_x \\
&\qquad - \Big(\varepsilon_9 y + c(\varepsilon_5+\varepsilon_3\varepsilon_9) - s(\varepsilon_4+\varepsilon_2\varepsilon_9)\Big)h_y\Big], \\
H_X &= \mathrm{e}^{2\varepsilon_7-\varepsilon_6}(1-\varepsilon_9 t)^3(ch_x - sh_y), \quad H_Y = \mathrm{e}^{2\varepsilon_7-\varepsilon_6}(1-\varepsilon_9 t)^3(sh_x + ch_y),
\end{aligned}
$$

and a coordinate cross-section,

$$
(T, X, Y, U, V, H, H_T, H_X, H_Y) = (0, 0, 0, 0, 0, 1, 0, 0, 1). \tag{4.4}
$$

Solving for the group parameters leads to

$$
\begin{aligned}
&\varepsilon_1 = -\frac{tS}{2h}, \quad \varepsilon_2 = \frac{(y-tv)h_x - (x-tu)h_y}{\sqrt{h_x^2+h_y^2}}, \quad \varepsilon_3 = \frac{(tu-x)h_x + (tv-y)h_y}{\sqrt{h_x^2+h_y^2}}, \\
&\varepsilon_4 = \frac{vh_x - uh_y}{\sqrt{h_x^2+h_y^2}}, \quad \varepsilon_5 = -\frac{uh_x + vh_y}{\sqrt{h_x^2+h_y^2}}, \quad \varepsilon_6 = \frac{1}{2}\ln\frac{4(h_x^2+h_y^2)}{S^2}, \\
&\varepsilon_7 = \frac{1}{2}\ln\frac{S^2}{4h^3}, \quad \varepsilon_8 = \arctan\frac{h_x}{h_y}, \quad \varepsilon_9 = \frac{h_t + uh_x + vh_y}{S},
\end{aligned} \tag{4.5}
$$

where $S = 2h + t(h_t + uh_x + vh_y)$. With the aid of this moving frame, it is possible to derive a functionally independent list of differential invariants upon normalizing those coordinate functions of the jet space that have not been involved in setting up the moving frame, $I^{w^\kappa}_\alpha := \iota(w^\kappa_\alpha)$, where ι is the invariantization map. According to the theorem on bases of the algebra of differential invariants [105, Theorem 7.1], a (not necessarily minimal) generating set of all differential invariants is constituted by the following *edge* differential invariants,

$$
\begin{aligned}
I^u_{100} &= \frac{h_y(u_t + uu_x + vu_y) - h_x(v_t + uv_x + vv_y)}{h_x^2+h_y^2}, \\
I^u_{010} &= \frac{(h_x^2+h_y^2)(h_t + uh_x + vh_y) + 2h(u_x h_y^2 + v_y h_x^2) - 2hh_xh_y(v_x+u_y)}{2\sqrt{h}(h_x^2+h_y^2)^{3/2}}, \\
I^u_{001} &= \frac{\sqrt{h}(h_xh_y(u_x-v_y) + u_y h_y^2 - v_x h_x^2)}{(h_x^2+h_y^2)^{3/2}},
\end{aligned}
$$

$$
\begin{aligned}
I^v_{100} &= \frac{h_x(u_t + uu_x + vu_y) + h_y(v_t + uv_x + vv_y)}{h_x^2 + h_y^2}, \\
I^v_{010} &= \frac{\sqrt{h}(h_x h_y(u_x - v_y) - u_y h_x^2 + v_x h_y^2)}{(h_x^2 + h_y^2)^{3/2}}, \\
I^v_{001} &= \frac{(h_x^2 + h_y^2)(h_t + uh_x + vh_y) + 2h(u_x h_x^2 + v_y h_y^2) + 2hh_x h_y(v_x + u_y)}{2\sqrt{h}(h_x^2 + h_y^2)^{3/2}}, \\
I^h_{200} &= \frac{-3(uh_x + vh_y + h_t)^2 + 2h(2uvh_{xy} + v^2 h_{yy} + u^2 h_{xx} + 2vh_{ty} + 2uh_{tx} + h_{tt})}{2h(h_x^2 + h_y^2)} \\
I^h_{110} &= \frac{(-h_x(uh_{xy} + vh_{yy} + h_{ty}) + h_y(vh_{xy} + uh_{xx} + h_{tx}))\sqrt{h}}{(h_x^2 + h_y^2)^{3/2}} \\
I^h_{101} &= \frac{2h(h_x(vh_{xy}{+}uh_{xx}{+}h_{tx}) + h_y(uh_{xy}{+}vh_{yy}{+}h_{ty})) - 3(h_x^2 + h_y^2)(h_t{+}uh_x{+}vh_y)}{2\sqrt{h}(h_x^2 + h_y^2)^{3/2}}, \\
I^h_{020} &= \frac{h}{(h_x^2 + h_y^2)^2}(h_x^2 h_{yy} + h_y^2 h_{xx} - 2h_x h_y h_{xy}), \\
I^h_{011} &= \frac{h}{(h_x^2 + h_y^2)^2}(h_{xy}(h_y^2 - h_x^2) + h_x h_y(h_{xx} - h_{yy})), \\
I^h_{002} &= \frac{h}{(h_x^2 + h_y^2)^2}(h_x^2 h_{xx} + h_y^2 h_{yy} + 2h_x h_y h_{xy}).
\end{aligned}
$$

All the other differential invariants can be derived upon functional recombination of the basis elements and by acting on them with the operators of invariant differentiation. These operators are the invariantization of the three operators of total differentiation using the normalization (4.4) and are

$$
\begin{aligned}
&\mathrm{D}^{\mathrm{i}}_t = \sqrt{\frac{h}{h_x^2 + h_y^2}}(\mathrm{D}_t + u\mathrm{D}_x + v\mathrm{D}_y), \quad \mathrm{D}^{\mathrm{i}}_x = \frac{h}{h_x^2 + h_y^2}(h_y\mathrm{D}_x - h_x\mathrm{D}_y), \\
&\mathrm{D}^{\mathrm{i}}_y = \frac{h}{h_x^2 + h_y^2}(h_x\mathrm{D}_x + h_y\mathrm{D}_y).
\end{aligned}
\tag{4.6}
$$

Remark 4.5. The normalized differential invariants can be used to derive the formulation of the shallow water equations in terms of fundamental differential invariants. This is done upon replacing each term in the system (4.1) by its invariantized counterpart, which is called the Replacement Theorem [50]. In view of the normalization (4.4) this invariantized representation reads $\iota(u_t) = 0$, $\iota(v_t) + 1 = 0$, $\iota(u_x) + \iota(v_y) = 0$. Explicitly, this gives

$$
\begin{aligned}
&\frac{1}{h_x^2+h_y^2}(h_y(u_t+uu_x+vu_y+h_x)-h_x(v_t+uv_x+vv_y+h_y))=0,\\
&\frac{1}{h_x^2+h_y^2}(h_x(u_t+uu_x+vu_y+h_x)+h_y(v_t+uv_x+vv_y+h_y))=0,\\
&\frac{1}{\sqrt{h(h_x^2+h_y^2)}}(h_t+uh_x+vh_y+h(u_x+v_y))=0.
\end{aligned}
\tag{4.7}
$$

As of now we have a generating set of the algebra of differential invariants for the system (4.1), but differential invariants are not necessarily functionally independent. In what follows, we aim to find all such dependencies. The systematic way of doing it is finding all so-called syzygies. The first kind of syzygies is associated with the commutator formula $[\mathrm{D}^{\mathrm{i}}_j,\mathrm{D}^{\mathrm{i}}_k]=\sum_{l=1}^3 Y^l_{jk}\mathrm{D}^{\mathrm{i}}_l$, where Y's are certain differential functions. To avoid the direct cumbersome computation we use the fact that these Y arise also in $\mathrm{d_h}\omega^l=-\sum_{j<k}Y^l_{jk}\omega^j\wedge\omega^k$, where $\omega^1=\iota(\mathrm{d}t)$, $\omega^2=\iota(\mathrm{d}x)$ and $\omega^3=\iota(\mathrm{d}y)$ are the invariantized Maurer–Cartan forms. To find the left hand side of these identities we will need the horizontal part of the universal recurrence formula,

$$
\mathrm{d}\iota(\Omega)=\iota[\mathrm{d}\Omega+\mathfrak{L}_{\mathbf{v}^{(n)}}(\Omega)],
\tag{4.8}
$$

for Ω running through the set of functions used in choosing the cross-section. Here $\mathbf{v}^{(n)}$ is the nth prolongation of the general infinitesimal generator $\mathbf{v}$ of the group G^0 and $\mathfrak{L}_{\mathbf{v}^{(n)}}$ is the Lie derivative with respect to $\mathbf{v}^{(n)}$, see [106, 107] for more details. For the computations we need $n=2$. The left hand side of (4.8) is identically zero since $\iota(\Omega)$ are all constants. The collection of all such so-called *phantom recurrence relations* forms a linear system of algebraic equations that can be solved for the *invariantized Maurer–Cartan forms.* Plugging them in the remaining recurrence relations then yields a complete and closed description of the relations between all invariantly differentiated differential invariants and the normalized differential invariants.

In order to evaluate the general recurrence formula (4.8) for differential functions Ω, one also needs the prolongations of the infinitesimal generators $\mathbf{v}=\tau\partial_t+\xi^x\partial_x+\xi^y\partial_y+\phi^u\partial_u+\phi^v\partial_v+\phi^h\partial_h$ that generate the maximal Lie invariance algebra of the shallow water equations. The coefficients of the second prolongation of the vector field $\mathbf{v}$,

$$\mathbf{v}^{(2)} = \mathbf{v} + \sum_{\kappa} \Big(\phi^{\kappa,100} \partial_{w^\kappa_{100}} + \phi^{\kappa,010} \partial_{w^\kappa_{010}} + \phi^{\kappa,001} \partial_{w^\kappa_{001}} +$$
$$\phi^{\kappa,200} \partial_{w^\kappa_{200}} + \phi^{\kappa,110} \partial_{w^\kappa_{110}} + \phi^{\kappa,101} \partial_{w^\kappa_{101}} + \phi^{\kappa,020} \partial_{w^\kappa_{020}} + \phi^{\kappa,011} \partial_{w^\kappa_{011}} + \phi^{\kappa,002} \partial_{w^\kappa_{002}} \Big),$$

are given by the general prolongation formula [103]

$$\phi^{\kappa,\alpha} = \mathrm{D}_t^{\alpha_1} \mathrm{D}_x^{\alpha_2} \mathrm{D}_y^{\alpha_3} (\phi^\kappa - \tau w^\kappa_{\delta_1} - \xi^x w^\kappa_{\delta_2} - \xi^y w^\kappa_{\delta_3}) + \tau w^\kappa_{\alpha+\delta_1} + \xi^x w^\kappa_{\alpha+\delta_2} + \xi^y w^\kappa_{\alpha+\delta_3}.$$

We already know from the basis theorem that the invariants I^u_{100}, I^u_{010}, I^u_{001}, I^v_{100}, I^v_{010}, I^u_{001}, I^h_{020}, I^h_{011} I^h_{002}, I^h_{200}, I^h_{110} and I^h_{101} form a generating set (possibly not minimal) of differential invariants for the symmetry group G^0 of the shallow water equations. The purpose of evaluating the recurrence relations for the low-order differential invariants is primary to eventually find a minimal set of generating differential invariants. This is why we only have to evaluate the recurrence relations for those differential invariants that belong to the above basis. In order to do this, we only need the prolongations of vector field coefficients $\phi^{\kappa,100}$, $\phi^{\kappa,010}$, $\phi^{\kappa,001}$, $\phi^{3,200}$, $\phi^{3,110}$, $\phi^{3,101}$, $\phi^{3,020}$, $\phi^{3,011}$ and $\phi^{3,002}$. It turns out the invariantizations $\hat{\phi}^{\kappa,\alpha} := \iota(\phi^{\kappa,\alpha})$ are conveniently expressed via

$$\hat{\tau} = \iota(\tau), \quad \hat{\xi}^x = \iota(\xi^x), \quad \hat{\xi}^y = \iota(\xi^y), \quad \hat{\tau}_t = \iota(\tau_t), \quad \hat{\tau}_{tt} = \iota(\tau_{tt}),$$
$$\hat{\xi}^x_t = \iota(\xi^x_t), \quad \hat{\xi}^x_x = \iota(\xi^x_x), \quad \hat{\xi}^x_y = \iota(\xi^x_y), \quad \hat{\xi}^y_t = \iota(\xi^y_t),$$

and are

$$\hat{\phi}^{1,100} = (\hat{\xi}^x_x - 2\hat{\tau}_t) I^u_{100} + \hat{\xi}^x_y I^v_{100} - \hat{\xi}^x_t I^u_{010} - \hat{\xi}^y_t I^u_{001},$$
$$\hat{\phi}^{1,010} = \frac{1}{2}\hat{\tau}_{tt} - \hat{\tau}_t I^u_{010} + \hat{\xi}^x_y (I^u_{001} + I^v_{010}), \quad \hat{\phi}^{1,001} = -\hat{\tau}_t I^u_{001} + \hat{\xi}^x_y (I^v_{001} - I^u_{010}),$$
$$\hat{\phi}^{2,100} = -\hat{\xi}^x_y I^u_{100} + (\hat{\xi}^x_x - 2\hat{\tau}_t) I^v_{100} - \hat{\xi}^x_t I^v_{010} - \hat{\xi}^y_t I^v_{001},$$
$$\hat{\phi}^{2,010} = -\hat{\xi}^x_y I^u_{010} - \hat{\tau}_t I^v_{010} + \hat{\xi}^x_y I^v_{001}, \quad \hat{\phi}^{2,001} = \frac{1}{2}\hat{\tau}_{tt} - \hat{\xi}^x_y (I^u_{001} + I^v_{010}) - \hat{\tau}_t I^v_{001},$$
$$\hat{\phi}^{3,100} = -\hat{\tau}_{tt} - \hat{\xi}^y_t, \quad \hat{\phi}^{3,010} = \hat{\xi}^x_y, \quad \hat{\phi}^{3,001} = \hat{\xi}^x_x - 2\hat{\tau}_t,$$

$$
\begin{aligned}
\hat{\phi}^{3,200} &= 2(\hat{\xi}^x_x - 2\hat{\tau}_t)I^h_{200} - 2\hat{\xi}^x_t I^h_{110} - 2\hat{\xi}^y_t I^h_{101},\\
\hat{\phi}^{3,110} &= -(3\hat{\tau}_t - \hat{\xi}^x_x)I^h_{110} + \hat{\xi}^x_y I^h_{101} - \hat{\xi}^x_t I^h_{020} - \hat{\xi}^y_t I^h_{011},\\
\hat{\phi}^{3,101} &= -(3\hat{\tau}_t - \hat{\xi}^x_x)I^h_{101} - \hat{\xi}^x_y I^h_{110} - \hat{\xi}^x_t I^h_{011} - \hat{\xi}^y_t I^h_{002}, \quad \hat{\phi}^{3,020} = -2\hat{\tau}_t I^h_{020} + 2\hat{\xi}^x_y I^h_{011},\\
\hat{\phi}^{3,011} &= -2\hat{\tau}_t I^h_{011} + \hat{\xi}^x_y (I^h_{002} - I^h_{020}), \quad \hat{\phi}^{3,002} = -2\hat{\tau}_t I^h_{002} - 2\hat{\xi}^x_y I^h_{011}.
\end{aligned}
$$

We have now prepared all ingredients to evaluate the phantom recurrence relations,

$$
\begin{aligned}
&0 = \mathrm{d_h}\iota(t) = \omega^1 + \hat{\tau}, \quad 0 = \mathrm{d_h}\iota(x) = \omega^2 + \hat{\xi}^x, \quad 0 = \mathrm{d_h}\iota(y) = \omega^3 + \hat{\xi}^y,\\
&0 = \mathrm{d_h}\iota(u) = I^u_{100}\omega^1 + I^u_{010}\omega^2 + I^u_{001}\omega^3 + \hat{\xi}^x_t,\\
&0 = \mathrm{d_h}\iota(v) = I^v_{100}\omega^1 + I^v_{010}\omega^2 + I^v_{001}\omega^3 + \hat{\xi}^y_t, \quad 0 = \mathrm{d_h}\iota(h) = \omega^3 - 2(\hat{\tau}_t - \hat{\xi}^x_x),\\
&0 = \mathrm{d_h}\iota(h_t) = I^h_{200}\omega^1 + I^h_{110}\omega^2 + I^h_{101}\omega^3 - \hat{\tau}_{tt} - \hat{\xi}^y_t,\\
&0 = \mathrm{d_h}\iota(h_x) = I^h_{110}\omega^1 + I^h_{020}\omega^2 + I^h_{011}\omega^3 + \hat{\xi}^x_y,\\
&0 = \mathrm{d_h}\iota(h_y) = I^h_{101}\omega^1 + I^h_{011}\omega^2 + I^h_{002}\omega^3 - 2\hat{\tau}_t + \hat{\xi}^x_x,
\end{aligned}
\tag{4.9}
$$

where we used the fact that the group G^0 acts projectably and thus the forms ω's are horizontal. These phantom recurrence relations allow us to solve for the invariantized Maurer–Cartan forms, which are

$$
\begin{aligned}
&\hat{\tau} = -\omega^1, \quad \hat{\xi}^x = -\omega^2, \quad \hat{\xi}^y = -\omega^3, \quad \hat{\tau}_t = I^h_{101}\omega^1 + I^h_{011}\omega^2 + \left(I^h_{002} - \frac{1}{2}\right)\omega^3,\\
&\hat{\xi}^x_t = -(I^u_{100}\omega^1 + I^u_{010}\omega^2 + I^u_{001}\omega^3), \quad \hat{\xi}^x_x = I^h_{101}\omega^1 + I^h_{011}\omega^2 + (I^h_{002} - 1)\omega^3,\\
&\hat{\xi}^x_y = -(I^h_{110}\omega^1 + I^h_{020}\omega^2 + I^h_{011}\omega^3), \quad \hat{\xi}^y_t = -(I^v_{100}\omega^1 + I^v_{010}\omega^2 + I^v_{001}\omega^3),\\
&\hat{\tau}_{tt} = (I^v_{100} + I^h_{200})\omega^1 + (I^v_{010} + I^h_{110})\omega^2 + (I^v_{001} + I^h_{101})\omega^3.
\end{aligned}
\tag{4.10}
$$

Before we present the lowest non-phantom recurrence relations, we determine the commutation relations between the operators of invariant differentiation (4.6). This is done in the following way. Specifying the universal recurrence relation (4.8) for the basis horizontal forms $\Omega \in \{\mathrm{d}t, \mathrm{d}x, \mathrm{d}y\}$ and only evaluating the horizontal components of this relation (denoted by the $\equiv$ sign), we derive

$$
\begin{aligned}
\mathrm{d_h}\omega^1 &\equiv \mathrm{d}\iota(\mathrm{d}t) = \iota(\mathrm{d}\tau) = \iota(\tau_t)\wedge\omega^1 = -I^h_{011}\omega^1\wedge\omega^2 - \left(I^h_{002}-\frac{1}{2}\right)\omega^1\wedge\omega^3,\\
\mathrm{d_h}\omega^2 &\equiv \iota(\mathrm{d}\xi^x) = (I^u_{010}+I^h_{101})\omega^1\wedge\omega^2 + (I^u_{001}-I^h_{110})\omega^1\wedge\omega^3 - (I^h_{020}+I^h_{002}-1)\omega^2\wedge\omega^3,\\
\mathrm{d_h}\omega^3 &\equiv \iota(\mathrm{d}\xi^y) = (I^v_{010}+I^h_{110})\omega^1\wedge\omega^2 + (I^v_{001}+I^h_{101})\omega^1\wedge\omega^3,
\end{aligned}
$$

where we have used the expressions for the invariantized Maurer–Cartan forms (4.10) that we derived from the phantom recurrence relations (4.9).

From this result, it is possible to read off the commutator formulae for the operators of invariant differentiation (4.6), which are

$$[\mathrm{D}^{\mathrm{i}}_t, \mathrm{D}^{\mathrm{i}}_x] = I^h_{011}\mathrm{D}^{\mathrm{i}}_t - (I^u_{010}+I^h_{101})\mathrm{D}^{\mathrm{i}}_x - (I^v_{010}+I^h_{110})\mathrm{D}^{\mathrm{i}}_y, \tag{4.11a}$$

$$[\mathrm{D}^{\mathrm{i}}_t, \mathrm{D}^{\mathrm{i}}_y] = \left(I^h_{002}-\frac{1}{2}\right)\mathrm{D}^{\mathrm{i}}_t - (I^u_{001}-I^h_{110})\mathrm{D}^{\mathrm{i}}_x - (I^v_{001}+I^h_{101})\mathrm{D}^{\mathrm{i}}_y, \tag{4.11b}$$

$$[\mathrm{D}^{\mathrm{i}}_x, \mathrm{D}^{\mathrm{i}}_y] = (I^h_{020}+I^h_{002}-1)\mathrm{D}^{\mathrm{i}}_x. \tag{4.11c}$$

The next step in the description of the algebra of differential invariants is the computation of the syzygies, meaning the functional dependency of certain differentiated differential invariants. They are

$$
\begin{aligned}
\mathrm{D}^{\mathrm{i}}_t I^u_{001} - \mathrm{D}^{\mathrm{i}}_y I^u_{100} =& I^h_{110}(I^u_{010}-I^v_{001}) - I^h_{101}I^u_{001} + I^h_{011}I^v_{100} + I^h_{002}I^u_{100}\\
&- I^u_{001}(I^v_{001}+I^u_{010}),
\end{aligned} \tag{4.12a}
$$

$$\mathrm{D}^{\mathrm{i}}_t I^h_{011} - \mathrm{D}^{\mathrm{i}}_y I^h_{110} = -I^h_{011}I^h_{101} + I^h_{110}\left(I^h_{020}+I^h_{002}-\frac{1}{2}\right) - I^h_{020}I^u_{001} - I^h_{011}I^v_{001}, \tag{4.12b}$$

$$
\begin{aligned}
\mathrm{D}^{\mathrm{i}}_t I^v_{001} - \mathrm{D}^{\mathrm{i}}_y I^v_{100} =& \frac{1}{2}I^h_{200} - I^h_{101}I^v_{001} - (I^v_{001})^2 + \left(I^h_{002}+\frac{1}{2}\right)I^v_{100}\\
&+ I^h_{110}(I^v_{010}+I^u_{001}) - I^v_{010}I^u_{001} - I^h_{011}I^u_{100}
\end{aligned} \tag{4.12c}
$$

$$\mathrm{D}^{\mathrm{i}}_t I^h_{002} - \mathrm{D}^{\mathrm{i}}_y I^h_{101} = I^h_{110}I^h_{011} + I^h_{101} + \left(\frac{3}{2}-I^h_{002}\right)I^v_{001} - I^h_{011}I^u_{001} \tag{4.12d}$$

$$
\begin{aligned}
\mathrm{D}^{\mathrm{i}}_t I^h_{101} - \mathrm{D}^{\mathrm{i}}_y I^h_{200} =& (I^h_{110})^2 + \left(2I^h_{002}-\frac{3}{2}\right)I^h_{200} - 2I^h_{101}I^v_{001} - 2(I^h_{101})^2\\
&+ \left(I^h_{002}-\frac{3}{2}\right)I^v_{100} - 2I^h_{110}I^u_{001} + I^h_{011}I^u_{100}
\end{aligned} \tag{4.12e}
$$

$$
\begin{aligned}
\mathrm{D}^{\mathrm{i}}_t I^u_{010} - \mathrm{D}^{\mathrm{i}}_x I^u_{100} =& \frac{1}{2} I^h_{200} - I^h_{101} I^u_{010} - I^h_{110}(I^v_{010} + I^u_{001}) + \left(I^h_{020} + \frac{1}{2} \right) I^v_{100} \\
& - (I^u_{010})^2 + I^h_{011} I^u_{100} - I^u_{001} I^v_{010}
\end{aligned}
\tag{4.12f}
$$

$$
\begin{aligned}
\mathrm{D}^{\mathrm{i}}_t I^v_{010} - \mathrm{D}^{\mathrm{i}}_x I^v_{100} =& - I^h_{101} I^v_{010} - I^h_{110} \left(I^v_{001} - I^u_{010} \right) - I^h_{020} I^u_{100} + I^h_{011} I^v_{100} \\
& - I^v_{010}(I^u_{010} + I^v_{001})
\end{aligned}
\tag{4.12g}
$$

$$
\mathrm{D}^{\mathrm{i}}_t I^h_{020} - \mathrm{D}^{\mathrm{i}}_x I^h_{110} = -I^h_{020} \left(I^h_{101} + I^u_{010} \right) - I^h_{011} I^v_{010} \tag{4.12h}
$$

$$
\mathrm{D}^{\mathrm{i}}_t I^h_{011} - \mathrm{D}^{\mathrm{i}}_x I^h_{101} = \left(I^h_{110} + I^v_{010} \right) \left(\frac{3}{2} - I^h_{002} \right) - I^u_{010} I^h_{011} \tag{4.12i}
$$

$$
\mathrm{D}^{\mathrm{i}}_t I^h_{110} - \mathrm{D}^{\mathrm{i}}_x I^h_{200} = I^h_{011}(2I^h_{200} + I^v_{100}) - (3I^h_{101} + 2I^u_{010}) I^h_{110} - 2I^v_{010} I^h_{101} + I^u_{100} I^h_{020} \tag{4.12j}
$$

$$
\mathrm{D}^{\mathrm{i}}_x I^u_{001} - \mathrm{D}^{\mathrm{i}}_y I^u_{010} = I^h_{002} I^u_{010} + I^h_{020}(I^u_{010} - I^v_{001}) + I^h_{011} I^v_{010} - \frac{1}{2}(I^u_{010} + I^v_{001}) - \frac{1}{2} I^h_{101} \tag{4.12k}
$$

$$
\mathrm{D}^{\mathrm{i}}_x I^v_{001} - \mathrm{D}^{\mathrm{i}}_y I^v_{010} = \frac{1}{2} I^h_{110} - I^h_{011} I^u_{010} + I^h_{002} I^v_{010} + I^h_{020}(I^u_{001} + I^v_{010}) \tag{4.12l}
$$

$$
\begin{aligned}
\mathrm{D}^{\mathrm{i}}_x I^h_{101} - \mathrm{D}^{\mathrm{i}}_y I^h_{110} =& I^h_{110}(I^h_{020} + 2I^h_{002} - 2) - \left(I^h_{101} + I^v_{001} - I^u_{010} \right) I^h_{011} \\
& + \left(I^h_{002} - \frac{3}{2} \right) I^v_{010} - I^h_{020} I^u_{001}
\end{aligned}
\tag{4.12m}
$$

$$
\mathrm{D}^{\mathrm{i}}_x I^h_{011} - \mathrm{D}^{\mathrm{i}}_y I^h_{020} = I^h_{020}(I^h_{020} + I^h_{002} - 1) \tag{4.12n}
$$

$$
\mathrm{D}^{\mathrm{i}}_x I^h_{002} - \mathrm{D}^{\mathrm{i}}_y I^h_{011} = I^h_{011}(I^h_{020} + I^h_{002} - 1) \tag{4.12o}
$$

The systems (4.12b), (4.12d) and (4.12h), (4.12i) are linear inhomogeneous systems on pairs (I^u_{001}, I^v_{001}) and (I^u_{010}, I^v_{010}) of differential invariants, respectively, whose solutions are combinations of differential invariants I^h_{110}, I^h_{101}, I^h_{020}, I^h_{011}, I^h_{002} and their certain invariant derivatives. Via the same differential invariants one may also express I^u_{100}, I^v_{100} and I^h_{200} from the equations (4.12m), (4.12a) and (4.12c), respectively. The differential invariants may be excluded from the above generating set by expressing them from the system obtained be applying the commutator relation (4.11c) to I^h_{110} and I^h_{101}. After substituting all the obtained values in the remaining syzygies, their orders raise by one and neither of the differential invariants I^h_{110}, I^h_{101} and I^h_{011} can be expressed therefrom. In this way, we have proven the following statement.

Theorem 4.6. *The algebra of differential invariants for the group G^0 of the shallow water equations* (4.1) *is generated by the three normalized differential invariants I^h_{110}, I^h_{101} and I^h_{011} along with the three operators of invariant differentiation* (4.6).

4.5 Conservation laws

Computed in the course of classifying conservation laws of the shallow water equations with variable bottom topography in [18] was a complete space of zeroth order conservation laws of (4.1). In the notation

$$\begin{aligned}
&\Lambda^0 := (-yh, xh, xv - yu), \quad \Lambda^1(f) := \big(-2fuh + f_t xh, -2fvh + f_t yh,\\
&\qquad -f(u^2+v^2+2h) + f_t(xu+yv) - \frac{1}{2}f_{tt}(x^2+y^2)\big),\\
&\Lambda^2(f) := (fh, 0, fu - f_t x),\ \Lambda^3(f) := (0, fh, fv - f_t y),\ \Lambda^4(f) := (0,0,f),
\end{aligned}$$

the space of characteristics of (4.1) is spanned by

$$\Lambda^0,\ \Lambda^1(1),\ \Lambda^1(t),\ \Lambda^1(t^2),\ \Lambda^2(1),\ \Lambda^2(t),\ \Lambda^3(1),\ \Lambda^3(t),\ \Lambda^4(1). \tag{4.13}$$

The associated conserved currents of (4.1) are then (up to sign when necessary)

$$\begin{aligned}
\mathrm{CL}_1 &= \left(h(vx-uy), hu(vx-uy) - \frac{1}{2}h^2 y, hv(vx-uy) + \frac{1}{2}h^2 x\right),\\
\mathrm{CL}_2 &= \big(h(h+v^2+u^2), hu(u^2+v^2+2h), hv(u^2+v^2+2h)\big),\\
\mathrm{CL}_3 &= \Big(h^2 t + hu(tu-x) + hv(tv-y),\\
&\qquad h^2\left(2ut - \frac{1}{2}x\right) + hu^2(tu-x) + huv(tv-y),\\
&\qquad h^2\left(2vt - \frac{1}{2}y\right) + hv^2(tv-y) + huv(tu-x)\Big),\\
\mathrm{CL}_4 &= \Big(ht^2(h+u^2+v^2) - 2ht(ux+vy) + h(x^2+y^2),\\
&\qquad hut^2(u^2+v^2+2h) - th(hx+2u^2x+2vuy) + uh(x^2+y^2),\\
&\qquad hvt^2(u^2+v^2+2h) - th(hy+2v^2y+2uvx) + vh(x^2+y^2)\Big),\\
\mathrm{CL}_5 &= \left(hu, \frac{1}{2}h^2 + hu^2, huv\right),\ \mathrm{CL}_6 = \left(h(x-tu), h\left(ux - \frac{1}{2}ht - tu^2\right), hv(x-tu)\right),\\
\mathrm{CL}_7 &= \left(hv, huv, hv^2 + \frac{h^2}{2}\right),\ \mathrm{CL}_8 = \left(h(y-tv), hu(y-tv), h(vy - tv^2 - \frac{1}{2}ht)\right),\\
\mathrm{CL}_9 &= (h, hu, hv)\,.
\end{aligned}$$

Conserved currents CL_1, CL_2, CL_5, CL_7 and CL_9 are associated with the conservation of angular momentum, energy, x-momentum, y-momentum and mass, respectively [153].

If one is to preserve the conservation laws of an equation upon parameterization, in other words, to use conservative parameterization schemes [14], one may use the fact that the conservation laws are preserved under the action of point symmetries of this equation [26, 130]. Thus, using an invariant parameterization scheme [124] preserving a conservation law one in fact may preserve other conservation laws for free. Therefore, to proceed effectively one needs to determine a generating set of conservation laws of the equation [74], which is a minimal set of conservation laws which generates under the action of point symmetries the entire space of conservation laws.

$\{\Lambda^0, \Lambda^1(t)\}$ is such a set for (4.1), see [18]. Here we reprove this result geometrically.

Theorem 4.7. *A generating set of zero-order conservation laws of the system* (4.1) *consists of the conserved currents* CL_1 *and* CL_2.

Proof. Recall [26, 152] that the conserved current $\mathrm{CL} = (\rho, \sigma_x, \sigma_y)$ of (4.1) is associated with the horizontal 2-form $\mathrm{CL} = -\rho \mathrm{d}x \wedge \mathrm{d}y + \sigma_x \mathrm{d}t \wedge \mathrm{d}y - \sigma_y \mathrm{d}t \wedge \mathrm{d}x$ on the jet space $J^\infty(t, x, y|u, v, h)$. The condition $\mathrm{Div}\,\mathrm{CL} = 0$ is equivalent to $\mathrm{d}(\mathrm{CL}) = 0$, where d is the exterior derivative, which means that conserved currents are closed 2-forms and the equivalence of conserved currents is analogous to the equivalence of closed forms. Thus, conservation laws are elements of the so-called $(n-1)$st horizontal cohomology group on $J^\infty(t, x, y|u, v, h)$. The action of the point symmetry group G on conservation laws of (4.1) is induced by the pullback of differential forms by transformations in G. Let us make short-hand notations for some point symmetry transformations of (4.1),

$$
\begin{aligned}
&\mathcal{G}^x_{\varepsilon_4}\colon\quad \tilde t = t, \quad \tilde x = x + \varepsilon_4 t, \quad \tilde y = y + \varepsilon_4, \quad \tilde u = u, \quad \tilde v = v, \quad \tilde h = h,\\
&\mathcal{G}^y_{\varepsilon_5}\colon\quad \tilde t = t, \quad \tilde x = x + \varepsilon_5 t, \quad \tilde y = y, \quad \tilde u = u + \varepsilon_5, \quad \tilde v = v, \quad \tilde h = h,\\
&\mathcal{I}_{\varepsilon_9}\colon\quad \tilde t = \frac{t}{1-\varepsilon_9 t}, \quad \tilde x = \frac{x}{1-\varepsilon_9 t}, \quad \tilde y = \frac{y}{1-\varepsilon_9 t},\\
&\qquad\quad \tilde u = u(1-\varepsilon_9 t) + \varepsilon_9 x, \quad \tilde v = (1-\varepsilon_9 t)v + \varepsilon_9 y, \quad \tilde h = (1-\varepsilon_9 t)^2 h,\\
&\mathcal{S}^x_{\varepsilon_2}\colon\quad \tilde t = t, \quad \tilde x = x + \varepsilon_2, \quad \tilde y = y, \quad \tilde u = u, \quad \tilde v = v, \quad \tilde h = h,\\
&\mathcal{S}^y_{\varepsilon_3}\colon\quad \tilde t = t, \quad \tilde x = x, \quad \tilde y = y + \varepsilon_3, \quad \tilde u = u, \quad \tilde v = v, \quad \tilde h = h,
\end{aligned}
$$

Then we can express some of the conservation laws as

$$
\begin{aligned}
&\mathrm{CL}_3=\mathrm{CL}_2+\frac{1}{4}\mathrm{CL}_4-\mathcal{I}^*_{1/2}\widetilde{\mathrm{CL}_2},\quad \mathrm{CL}_4=\mathcal{I}^*_1\widetilde{\mathrm{CL}_2}+2\mathrm{CL}_3-\mathrm{CL}_2,\\
&\mathrm{CL}_5=\mathrm{CL}_1-(\mathcal{S}^y_1)^*\widetilde{\mathrm{CL}_1},\quad \mathrm{CL}_8=\mathrm{CL}_1-(\mathcal{G}^x_1)^*\widetilde{\mathrm{CL}_1},\quad \mathrm{CL}_6=(\mathcal{G}^y_1)^*\widetilde{\mathrm{CL}_1}-\mathrm{CL}_1,\\
&\mathrm{CL}_9=\mathrm{CL}_1-(\mathcal{G}^x_1\circ\mathcal{S}^y_1)^*\widetilde{\mathrm{CL}_1}-\mathrm{CL}_8-\mathrm{CL}_5,\quad \mathrm{CL}_7=(\mathcal{S}^x_1)^*\widetilde{\mathrm{CL}_1}-\mathrm{CL}_1.
\end{aligned}
$$

We put the tilde over 2-forms to distinguish different systems of coordinates. Under no point symmetry $\mathcal{T}\in G$ can a conserved current CL_1 be related to CL_2 which is readily seen from the transformation of the above currents under a general point symmetry $\mathcal{T}\in G$. But in view of the resulting expression being overly cumbersome we will not present it here. □

As for the conservation laws of (4.1) of higher order, then to begin with, the system (4.1) is Hamiltonian [134], $\mathrm{w}=\mathfrak{H}\delta\mathcal{H}$, where δ stands for the variational derivative with respect to the tuple of the dependent variables $\mathrm{w}:=(u,v,h)^\top$, and

$$
\mathfrak{H}=\begin{pmatrix} 0 & q & -\mathrm{D}_x\\ -q & 0 & -\mathrm{D}_y\\ -\mathrm{D}_x & -\mathrm{D}_y & 0\end{pmatrix},\quad \mathcal{H}:=\frac{1}{2}\iint h(u^2+v^2+h)\mathrm{d}x\mathrm{d}y
$$

are the associated Hamiltonian operator and Hamiltonian functional, and $q=(v_x-u_y)/h$ is the shallow water potential vorticity. Note that $\mathfrak{H}$ is a Hamiltonian operator of hydrodynamic-type [41, 56, 90]. Elements of the kernel of $\mathfrak{H}$ are called Casimir functionals of $\mathfrak{H}$ and they are associated with conservation laws of (4.1). It was shown in [143] that Casimir functionals of (4.1) are functionals of the form $\iint hR(q)\mathrm{d}x\mathrm{d}y$, where R runs through the set of smooth function of q. The associated family $\mathcal{C}_R$ of conserved currents are $hR(q)(1,u,v)$ and they are of order zero if $R':=\mathrm{d}R/\mathrm{d}q=0$ and of order one otherwise. Additionally, their characteristics are of the form $\Lambda_R:=(\mathrm{D}_yR'(q),-\mathrm{D}_xR'(q),R(q)-qR'(q))$. Among elements of the family $\mathcal{C}_R$ are functionals associated with the conservation of mass ($R=1$), the trivial conservation of circulation ($R=q$) and the conservation of potential enstrophy ($R=q^2/2$), cf. [153].

Are there any other conservation laws? There is a strong belief in scientific circles that the answer is negative. This claim is supported by direct computer computations of low-order conserved currents. Moreover, there is also a strong belief that there are no nontrivial higher symmetries (of order greater than one) for the system (4.1), computation of which is less costly.

Conjecture. The space of higher symmetries of (4.1) is exhausted by the Lie symmetries thereof.

Assuming the conjecture is true, the space of conservation laws of the system (4.1) is spanned by its zero-order conservation laws and the first-order conservation laws from the family $\mathcal{C}_R$.

The Hamiltonian operator $\mathfrak{H}$ maps characteristics of cosymmetries of (4.1) to that of symmetries thereof. If the above conjecture is assumed to hold, the image of $\mathcal{H}$ consists of nonzero characteristics of order no greater than one and therefore the order of mapped characteristics should be of order 0 or $-\infty$. Moreover, the kernel of $\mathfrak{H}$ is already known as well as the corresponding cosymmetries' characteristics Λ_R. All of these cosymmetries are known and they do not amount to anything beyond conservation laws in the statement.

4.6 Toward geometric parameterization

Representing the dependent variables in the system (4.1) as sums of the mean (resolved or grid-scale) parts and the departure from the mean part (subgrid-scale parts), $u = \bar{u} + u'$, $v = \bar{v} + v'$, $h = \bar{h} + h'$, and applying the Reynolds averaging rule $\overline{ab} = \bar{a}\bar{b} + \overline{a'b'}$, one yields

$$
\begin{aligned}
&\bar{u}_t + \bar{u}\bar{u}_x + \bar{v}\bar{u}_y + \bar{h}_x = w_1,\\
&\bar{v}_t + \bar{u}\bar{v}_x + \bar{v}\bar{v}_y + \bar{h}_y = w_2,\\
&\bar{h}_t + \bar{u}\bar{h}_x + \bar{v}\bar{h}_y + \bar{h}(\bar{u}_x + \bar{v}_y) = w_3,
\end{aligned}
\tag{4.14}
$$

where w's do not depend on resolved only expressions and whose explicit form is of no importance here. Since there are no equations for the subgrid-scale parts of u, v and h, one should parameterize w's via the grid-scale parts $\bar{u}$, $\bar{v}$ and $\bar{h}$, i.e. introduce a parameterization scheme. There are a lot of obstacles to finding a physically reasonable parameterization scheme for the system (4.1) [85]. Therefore, we would like to move in a parallel course.

More specifically, we want a parameterization scheme to be symmetry-preserving, i.e. it should admit the same Lie symmetries the initial system (4.1) does. In order to achieve this, we can apply the moving frame from Section 4.4 to the system (4.14) (now the barred variables are "physical" variables). The obtained system will look like the system (4.5) with nonzero right sides depending on differential invariants for the system (4.1),

$$
\begin{aligned}
&\frac{1}{h_x^2+h_y^2}(h_y(u_t+uu_x+vu_y+h_x)-h_x(v_t+uv_x+vv_y+h_y))=\iota(w_1),\\
&\frac{1}{h_x^2+h_y^2}(h_x(u_t+uu_x+vu_y+h_x)+h_y(v_t+uv_x+vv_y+h_y))=\iota(w_2),\\
&\frac{1}{\sqrt{h(h_x^2+h_y^2)}}(h_t+uh_x+vh_y+h(u_x+v_y))=\iota(w_3).
\end{aligned}
$$

We can reduce it to the inhomogeneous form of the system (4.1),

$$
\begin{aligned}
&u_t+uu_x+vu_y+h_x=h_y\iota(w_1)+h_x\iota(w_2),\\
&v_t+uv_x+vv_y+h_y=h_y\iota(w_2)-h_x\iota(w_1),\\
&h_t+uh_x+vh_y+h(u_x+v_y)=\sqrt{h(h_x^2+h_y^2)}\iota(w_3).
\end{aligned}
\tag{4.15}
$$

Let us specify the form of $\iota(w)$'s. To begin with, it is physically reasonable that the right hand sides should not depend on the time derivatives in order to keep the evolutionary form of equations. Thus, $\iota(w)$'s are the functions of $(I^u_{001}, I^v_{010}, I^u_{010}-I^v_{001}, I^h_{020}, I^h_{011}, I^h_{002})$. It might be reasonable to drop the dependence of $\iota(w_3)$ on the first three arguments, while the dependence of $\iota(w_1)$ and $\iota(w_2)$ should also be extended to the second-order $\iota(u_{xx})$, $\iota(v_{yy})$ et cetera since many parameterization schemes are diffusive [15, 16], but they are too cumbersome to be presented here.

One may further try to incorporate conservation laws in the parameterization scheme, which results in a conservative-invariant parameterization scheme. To this aim, one should parameterize functions $\iota(w)$'s so that the system (4.15) admits some of characteristics (4.13) of the system (4.1).

Chapter 5

A zoo of equivalence groups

5.1 Introduction

In many applications it is natural to consider not single (systems of) differential equations, but sets thereof, parameterized by arbitrary elements that can be constants or functions which satisfy some auxiliary differential relations. These sets are called classes of differential equations, and the procedure of finding Lie symmetries of equations in the class depending on values of arbitrary elements – the group classification problem. The idea to consider such problems is twofold. First, some physical processes are governed by differential equations with parameters corresponding to independent factors like a bottom topography or a diffusion coefficient. Second, the same differential equation may govern different processes and therefore it is reasonable to study this mathematical model per se. But where there is a classification problem, there is an equivalence. This way the notion of the equivalence group of a class of differential equations appears. The most common representative thereof is a so-called usual equivalence group, that is, a group with independent-variables and dependent-variables parameters not depending on the arbitrary elements of the class. Although a generalization of such notion via relaxing the above dependence — a generalized equivalence group — was introduced [87, 88], for many years it was assumed that only trivial examples are possible and there were doubts about the necessity of such a notion at all. In this chapter we provide the first nontrivial examples of generalized and extended generalized equivalence group as well as some insight into their theory.

Let $\mathcal{L}_\theta$ denote a system of differential equations of the form $L(x, u^{(r)}, \theta(x, u^{(r)})) = 0$. Here, $x = (x_1, \dots, x_n)$ are the n independent variables, $u = (u^1, \dots, u^m)$ are the m dependent variables, and L is a tuple of differential functions in u. We use the standard short-hand notation $u^{(r)}$ to denote the tuple of derivatives of u with respect to x up to order r, which also includes the u's as the derivatives of order zero. The system $\mathcal{L}_\theta$ is parameterized by the tuple of functions $\theta = (\theta^1(x, u^{(r)}), \dots, \theta^k(x, u^{(r)}))$, called the arbitrary elements, which runs through the solution set $\mathcal{S}$ of an auxiliary system of differential equations and inequalities in θ, $S(x, u^{(r)}, \theta^{(q)}(x, u^{(r)})) = 0$ and, e.g., $\Sigma(x, u^{(r)}, \theta^{(q)}(x, u^{(r)})) \neq 0$. Here, the notation $\theta^{(q)}$ encompasses the partial derivatives of the arbitrary elements θ up to order q with respect to both x and $u^{(r)}$. Thus, the *class of (systems of) differential equations* $\mathcal{L}|_\mathcal{S}$ is the parameterized family of systems $\mathcal{L}_\theta$'s, such that θ lies in $\mathcal{S}$.

For the specific class of general Burgers–KdV equations considered below,

$$u_t + C(t,x)uu_x = \sum_{k=1}^{r} A^k(t,x)u_k, \quad u_k = \partial^k u/\partial_x^k, \tag{5.1}$$

we have $n = 2$, $m = 1$, and $x_1 = t$ and $x_2 = x$. The tuple of arbitrary elements is $\theta = (A^0, \dots, A^r, B, C)$, which runs through the solution set of the auxiliary system

$$A^k_{u_\alpha} = 0, \quad k = 0, \dots, r, \quad B_{u_\alpha} = 0, \quad C_{u_\alpha} = 0, \quad |\alpha| \leqslant r, \quad CA^r \neq 0,$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi-index, $\alpha_1, \alpha_2 \in \mathbb{N} \cup \{0\}$, $|\alpha| = \alpha_1 + \alpha_2$, and $u_\alpha = \partial^{|\alpha|}u/\partial t^{\alpha_1}\partial x^{\alpha_2}$. Satisfying the auxiliary differential equations is equivalent to the fact that the arbitrary elements do not depend on derivatives of u. The inequality $A^rC \neq 0$ ensures that equations from the class (5.1) are both nonlinear and of order r.

Group classification of differential equations is based on studying how systems from a given class are mapped to each other. This study is formalized in the notion of *admissible transformations*, which constitute the *equivalence groupoid* of the class $\mathcal{L}|_\mathcal{S}$.

Definition 5.1. An admissible transformation is a triple $(\theta, \tilde{\theta}, \varphi)$, where $\theta, \tilde{\theta} \in \mathcal{S}$ are arbitrary-element tuples associated with systems $\mathcal{L}_\theta$ and $\mathcal{L}_{\tilde{\theta}}$ from the class $\mathcal{L}_\mathcal{S}$ that are similar to each other, and φ is a point transformation in the space of (x, u) that maps $\mathcal{L}_\theta$ to $\mathcal{L}_{\tilde{\theta}}$.

A related notion of relevance in the group classification of differential equations is that of *equivalence transformations*.

Definition 5.2. Usual equivalence transformations are point transformations in the joint space of independent variables, derivatives of u up to order r and arbitrary elements that are projectable to the space of $(x, u^{(r')})$ for each $r' = 0, \dots, r$, respect the contact structure of the rth order jet space coordinatized by the r-jets $(x, u^{(r)})$ and map every system from the class $\mathcal{L}|_{\mathcal{S}}$ to a system from the same class.

The Lie (pseudo)group constituted by the equivalence transformations of $\mathcal{L}|_{\mathcal{S}}$ is called the *usual equivalence group* of this class and denoted by $G^{\sim}$. If the arbitrary elements depend at most on derivatives of u up to order $\hat{r} < r$, then one can assume that equivalence transformations act in the space of $(x, u^{(\hat{r})}, \theta)$ instead of the space of $(x, u^{(r)}, \theta)$.

The usual equivalence group $G^{\sim}$ gives rise to a subgroupoid of the equivalence groupoid $\mathcal{G}^{\sim}$ since each equivalence transformation $\mathcal{T} \in G^{\sim}$ generates a family of admissible transformations parameterized by θ,

$$G^{\sim} \ni \mathcal{T} \to \big\{(\theta, \mathcal{T}\theta, \pi_*\mathcal{T}) \mid \theta \in \mathcal{S}\big\} \subset \mathcal{G}^{\sim}.$$

Here π denotes the projection of the space of $(x, u^{(r)}, \theta)$ to the space of equation variables only, $\pi(x, u^{(r)}, \theta) = (x, u)$. The pushforward $\pi_*\mathcal{T}$ of $\mathcal{T}$ by π is then just the restriction of $\mathcal{T}$ to the space of (x, u).

In the case when the arbitrary elements θ's are functions of (x, u) only, we can assume that equivalence transformations of the class $\mathcal{L}|_{\mathcal{S}}$ are point transformations of (x, u, θ) mapping every system from the class $\mathcal{L}|_{\mathcal{S}}$ to a system from the same class. The projectability property for equivalence transformations is neglected here. Then these equivalence transformations constitute a Lie (pseudo)group $\bar{G}^{\sim}$ called the *generalized equivalence group* of the class $\mathcal{L}|_{\mathcal{S}}$. See the first discussion of this notion in [87, 88] with no relevant examples and the further development in [123, 129]. Often the generalized equivalence group coincides with the usual one; this situation is considered as trivial. Each element of $\bar{G}^{\sim}$ generates a family of admissible transformations parameterized by θ,

$$\bar G^\sim \ni \mathcal T \to \big\{(\theta', \mathcal T\theta', \pi_*(\mathcal T|_{\theta=\theta'(x,u)})) \mid \theta' \in \mathcal S\big\} \subset \mathcal G^\sim,$$

and thus the generalized equivalence group $\bar G^\sim$ also generates a subgroupoid $\bar{\mathcal H}$ of the equivalence groupoid $\mathcal G^\sim$.

Definition 5.3. We call any minimal subgroup of $\bar G^\sim$ that generates the same subgroupoid of $\mathcal G^\sim$ as the entire group $\bar G^\sim$ does an *effective generalized equivalence group* of the class $\mathcal L|_{\mathcal S}$.

The uniqueness of an effective generalized equivalence group is obvious if the entire group $\bar G^\sim$ is effective itself. At the same time, there exist classes of differential equations, where effective generalized equivalence groups are proper subgroups of the corresponding generalized equivalence groups that are not even normal. Hence each of these effective generalized equivalence groups is not unique since it differs from some of subgroups non-identically similar to it, and all of these subgroups are also effective generalized equivalence groups of the same class. See the discussion of particular examples in Remark 5.14 below.

Suppose that the class $\mathcal L|_{\mathcal S}$ possesses parameterized non-identity usual equivalence transformations and some of its arbitrary elements are constants. Then this class necessarily admits purely generalized equivalence transformations. Indeed, we can set all parameters of elements from the usual equivalence group $G^\sim$ depending on constant arbitrary elements, which gives generalized equivalence transformations. The set $\bar G^\sim_0$ of such transformations is a subgroup of the generalized equivalence group $\bar G^\sim$. If $\bar G^\sim_0 = \bar G^\sim$, the usual equivalence group $G^\sim$ is an effective generalized equivalence group of the class $\mathcal L|_{\mathcal S}$.

The property for equivalence transformations to be point transformations with respect to arbitrary elements can also be weakened. We formally extend the arbitrary-element tuple θ of the class $\mathcal L|_{\mathcal S}$ with virtual arbitrary elements that are related to initial arbitrary elements by differential equations and thus expressed via initial arbitrary elements in a nonlocal way. Denote the reparameterized class by $\hat{\mathcal L}|_{\mathcal S}$. Suppose that the usual (resp. generalized or effective generalized) equivalence group $\hat G^\sim$ of $\hat{\mathcal L}|_{\mathcal S}$ induces the maximal subgroupoid of the equivalence groupoid $\mathcal G^\sim$ among the classes obtained from $\mathcal L|_{\mathcal S}$ by similar reparameterizations, and the extension of the arbitrary-element tuple θ for $\hat{\mathcal L}|_{\mathcal S}$ is minimal among the reparameterized classes giving the same subgroupoid of $\mathcal{G}^{\sim}$ as $\hat{\mathcal{L}}|_{\mathcal{S}}$. Then we call the group $\hat{G}^{\sim}$ an *extended equivalence group* (resp. an *extended generalized equivalence group*) of the class $\mathcal{L}|_{\mathcal{S}}$.

The class of differential equations $\mathcal{L}|_{\mathcal{S}}$ is *normalized* in the usual (resp. generalized, extended, extended generalized) sense if the subgroupoid induced by its usual (resp. generalized, extended, extended generalized) equivalence group coincides with the entire equivalence groupoid $\mathcal{G}^{\sim}$ of $\mathcal{L}|_{\mathcal{S}}$. The normalization of $\mathcal{L}|_{\mathcal{S}}$ in the usual sense is equivalent to the following conditions. The transformational part φ of each admissible transformation $(\theta', \theta'', \varphi) \in \mathcal{G}^{\sim}$ does not depend on the fixed initial value θ' of the arbitrary-element tuple θ and, therefore, is appropriate for any initial value of θ. Moreover, the prolongation of φ to the space of $(x, u^{(r)})$ and the further extension to the arbitrary elements according to the relation between θ' and θ'' gives a point transformation in the joint space of $(x, u^{(r)}, \theta)$.

If the class $\mathcal{L}|_{\mathcal{S}}$ is normalized in the generalized sense, the expression for transformational parts of admissible transformations may involve arbitrary elements but only in a quite specific way. The equivalence groupoid is partitioned into families of admissible transformations parameterized by the source arbitrary-element tuple, and the transformational parts of admissible transformations from each of these families jointly give, after the extension to the arbitrary elements according to the relation between the corresponding source and target arbitrary elements, a point transformation in the joint space of (x, u, θ).

To establish the normalization properties of the class $\mathcal{L}|_{\mathcal{S}}$ one should compute its equivalence groupoid $\mathcal{G}^{\sim}$, which is realized using the direct method. Here one fixes two arbitrary systems from the class, $\mathcal{L}_\theta$: $L(x, u^{(r)}, \theta(x, u^{(r)})) = 0$ and $\mathcal{L}_{\tilde{\theta}}$: $L(\tilde{x}, \tilde{u}^{(r)}, \tilde{\theta}(\tilde{x}, \tilde{u}^{(r)})) = 0$, and aims to find the (nondegenerate) point transformations, φ: $\tilde{x}_i = X^i(x, u)$, $\tilde{u}^a = U^a(x, u)$, $i = 1, \dots, n$, $a = 1, \dots, m$, connecting them. For this, one changes the variables in the system $\mathcal{L}_{\tilde{\theta}}$ by expressing the derivatives $\tilde{u}^{(r)}$ in terms of $u^{(r)}$ and derivatives of the functions X^i and U^a as well as by substituting X^i and U^a for $\tilde{x}_i$ and $\tilde{u}^a$, respectively. The requirement that the resulting transformed system has to be satisfied identically for solutions of $\mathcal{L}_\theta$ leads to the system of determining equations for the components of the transformation φ.

In the case of a single dependent variable ($m=1$), all the above notions involving point transformations can be directly extended to contact transformations.

See more details on theory of symmetry analysis of classes of differential equations in [110, 129, 154].

5.2 Generalized equivalence groups

Our first example of generalized equivalence groups comes from discussing the group classification problem of the class of general Burgers–Korteweg–de Vries equations (5.1). It was shown that the best gauge for classification purposes is $(C,A^1)=(1,0)$, which preserves the normalization in the usual sense. On the other hand, here we are interested in another gauge, $(A^r,A^1)=(1,0)$, which provided the first example for a generalized equivalence group containing transformations whose components for equation variables depend on a nonconstant arbitrary element.

The A^r-component of equivalence transformations in the class (5.1) is $\tilde A^r=\frac{(X_x)^r}{T_t}A^r$. If $A^r=1$ and $\tilde A^r=1$, then the parameters of the admissible transformations in the subclass of (5.1) singled out by the constraint $A^r=1$ satisfy the constraint $(X_x)^r=T_t$, i.e., $X=X^1(t)x+X^0(t)$, where $(X^1)^r=T_t$.

Proposition 5.4. *The subclass of the class* (5.1) *singled out by the constraint* $A^r=1$ *is normalized in the usual sense. Its usual equivalence group is constituted by the transformations of the form*

$$
\begin{aligned}
&\tilde t=T(t),\quad \tilde x=X^1(t)x+X^0(t),\quad \tilde u=U^1(t)u+U^0(t,x),\\
&\tilde A^l=\frac{(X^1)^l}{T_t}A^l,\quad \tilde A^1=\frac{X^1}{T_t}\left(A^1+\frac{U^0}{U^1}C-\frac{X^1_tx+X^0_t}{X^1}\right),\\
&\tilde A^0=\frac{1}{T_t}\left(A^0+\frac{U^1_t}{U^1}+\frac{U^0_x}{U^1}C\right),\quad \tilde C=\frac{X^1}{T_tU^1}C,\\
&\tilde B=\frac{U^1}{T_t}B+\frac{U^0_t}{T_t}+\frac{U^0_x}{T_t}\left(\frac{U^0}{U^1}C-\frac{X^1_tx+X^0_t}{X^1}\right)-\frac{U^0_r}{(X^1)^r}-\sum_{k=0}^{r-1}\frac{U^0_k}{(X^1)^k}\tilde A^k,
\end{aligned}
$$

where $l=2,\dots,r-1$, *and* $T=T(t)$, $X^0=X^0(t)$, $U^1=U^1(t)$ *and* $U^0=U^0(t,x)$ *are arbitrary smooth functions of their arguments such that* $T_tU^1\neq0$, *as well as* $X^1=(T_t)^{1/r}$ *if* r *is odd and* $T_t>0$, $X^1=\varepsilon(T_t)^{1/r}$ *with* $\varepsilon=\pm1$ *if* r *is even.*

The gauge $A^1=0$ leads to the appearance of the arbitrary element C in the u-component of admissible transformations since then we have

$$U^0=\frac{X^1_tx+X^0_t}{X^1C}U^1.$$

Denote by θ' the arbitrary-element tuple of the subclass $\mathcal{L}_1$ obtained as a result of the double gauge $(A^r,A^1)=(1,0)$,

$$\theta'=(A^0,A^2,\dots,A^{r-1},B,C).$$

Proposition 5.5. *The equivalence groupoid of the subclass $\mathcal{A}_1$ of the class* (5.1) *singled out by the constraints $A^r=1$ and $A^1=0$ consists of the triples $(\theta',\tilde\theta',\varphi)$'s, where the point transformation φ is of the form*

$$\tilde t=T(t),\quad \tilde x=X^1(t)x+X^0(t),\quad \tilde u=U^1(t)u+U^0,\quad U^0:=\frac{X^1_tx+X^0_t}{X^1C}U^1, \tag{5.2a}$$

the arbitrary-element tuples θ' and $\tilde\theta'$ are related according to

$$\tilde A^l=\frac{(X^1)^l}{T_t}A^l,\quad \tilde A^0=\frac{1}{T_t}\left(A^0+\frac{U^1_t}{U^1}+\frac{U^0_x}{U^1}C\right),\quad \tilde C=\frac{X^1}{T_tU^1}C, \tag{5.2b}$$

$$\tilde B=\frac{U^1}{T_t}B+\frac{U^0_t}{T_t}+\frac{U^0_x}{T_t}\left(\frac{U^0}{U^1}C-\frac{X^1_tx+X^0_t}{X^1}\right)-\frac{U^0_r}{(X^1)^r}-\sum_{l=2}^{r-1}\frac{U^0_l}{(X^1)^l}\tilde A^l-U^0\tilde A^0, \tag{5.2c}$$

with $l=2,\dots,r-1$, and $T=T(t)$, $X^0=X^0(t)$ and $U^1=U^1(t)$ being arbitrary smooth functions of t such that $T_tU^1\neq0$, as well as $X^1=(T_t)^{1/r}$ if r is odd and $T_t>0$, $X^1=\varepsilon(T_t)^{1/r}$ with $\varepsilon=\pm1$ if r is even.

The subclass $\mathcal{A}_1$ is not normalized in the usual sense since the parameter U^0 appearing in the transformation-component for u depends on the arbitrary element U^0, and therefore the equivalence group above is generalized. If U^0 is C-independent, then we obtain the usual equivalence group of the class $\mathcal{A}_1$, which is constituted by the point transformations of the form (5.2) in the joint space of the variables (t,x,u) and the arbitrary elements θ', where parameters satisfy more constraints, $T_{tt}=X^0_t=0$, and thus $X^1_t=0$ and $U^0=0$.

All the components of (5.2) locally depend on C, and, moreover, the expressions for $\tilde{A}^0$ and $\tilde{B}$ involve derivatives of C with respect to t and x. This is why, to interpret (5.2) as generalized equivalence transformations, we need to formally extend the arbitrary-element tuple θ' with the derivatives of C as new arbitrary elements, $Z^0 := C_t$ and $Z^k := C_k$, $k = 1, \dots, r$, and prolong equivalence transformations to them,

$$\tilde{Z}^0 = \frac{X^1}{T_t^2 U^1} Z^0 + \left(\frac{X^1}{T_t U^1}\right)_t \frac{C}{T_t}, \quad \tilde{Z}^k = \frac{(X^1)^{1-k}}{T_t^2 U^1} Z^k, \quad k = 1, \dots, r. \tag{5.3}$$

The derivatives of U^0 in the expressions for $\tilde{A}^0$ and $\tilde{B}$ should be expanded and then derivatives of C should be replaced by the corresponding Z's.

We denote by $\bar{\mathcal{A}}_1$ the class of equations of the form (5.1) with $(A^r, A^1) = (1, 0)$ and the extended arbitrary-element tuple $\bar{\theta}' = (A^0, A^2, \dots, A^{r-1}, B, C, Z^0, \dots, Z^r)$, where the relations defining Z^0, ..., Z^r are assumed as additional auxiliary equations for arbitrary elements.

Theorem 5.6. *The class $\bar{\mathcal{A}}_1$ is normalized in the generalized sense. Its generalized equivalence group $\bar{G}^{\sim}_{\bar{\mathcal{A}}_1}$ coincides with its effective generalized equivalence group and consists of the point transformations in the joint space of the variables (t, x, u) and the arbitrary elements $\bar{\theta}'$ with components of the form (5.2), (5.3) and the same constraints for parameters as in Proposition 5.5, where partial derivatives of U^0 are replaced by the corresponding restricted total derivatives with $\bar{\mathrm{D}}_t = \partial_t + Z^0\partial_C$ and $\bar{\mathrm{D}}_x = \partial_x + Z^1\partial_C + Z^2\partial_{Z^1} + \dots + Z^r\partial_{Z^{r-1}}$.*

Proof. The point transformations of the above form constitute a group G, which generates the entire equivalence groupoid of the class $\bar{\mathcal{A}}_1$ and is minimal among point-transformation groups in the joint space of $(t, x, u, \bar{\theta}')$ that have this generation property. Therefore, G is an effective generalized equivalence group of the class $\bar{\mathcal{A}}_1$. We are going to prove that the group G coincides with $\bar{G}^{\sim}_{\bar{\mathcal{A}}_1}$. Indeed, substituting every particular value of $\bar{\theta}'$ to any element of $\bar{G}^{\sim}_{\bar{\mathcal{A}}_1}$ gives an admissible transformation of the class $\bar{\mathcal{A}}_1$. This implies that elements of $\bar{G}^{\sim}_{\bar{\mathcal{A}}_1}$ are of the form (5.2), (5.3), where the parameter functions T, X^0 and X^1 may depend on arbitrary elements, and the partial derivatives of these functions are replaced by the corresponding total derivatives prolonged to the arbitrary elements of the class $\bar{\mathcal{A}}_1$. At the same time, these parameters satisfy the condition $\mathrm{D}_xT = \mathrm{D}_xX^0 = \mathrm{D}_xX^1 = 0$ with the prolonged total derivative operator D_x. This condition implies via splitting with respect to unconstrained derivatives of arbitrary elements that the parameters T, X^0 and X^1 are functions of t only. Hence $\bar{G}^{\sim}_{\bar{\mathcal{A}}_1} = G$. □

Although $\mathcal{A}_1$ is the first known class that admits a nontrivial generalized equivalence group, the situation with its effective generalized group is trivial: it coincides with the entire generalized equivalence group.

Our second example is in a sense opposite to the first. Its generalized equivalence group is finite-dimensional, i.e. the arbitrary elements of the class under question are constants, but its effective generalized equivalence groups (plural) have absolutely exquisite properties. Also, although we emphasized that our first example was the first example of the class with nonconstant arbitrary elements and with nontrivial generalized equivalence group, we did not mean that the constant-arbitrary elements case was well-studied. Indeed, all the known "finite-dimensional generalized equivalence groups" were effective generalized equivalence groups and thus either were trivial or were just subgroups of the generalized equivalence groups. And it was not evident from their construction, what was the case. Below we provide an example of a class whose generalized equivalence group is much wider than its effective generalized equivalence groups, with neither of them containing the usual equivalence group of the class.

Consider the class $\mathcal{F}$ of nonlinear "filtration"[1] equations

$$u_t = f(u_x)u_{xx} + g, \tag{5.4}$$

where f is a nonzero arbitrary function of u_x and g is a constant. We encountered this class while classifying a class of reaction–diffusion equations [114].

Proposition 5.7. *The generalized equivalence group $\bar{G}^{\sim}_{\mathcal{F}}$ of the class $\mathcal{F}$ is constituted by the point transformations in the space with the coordinates (t, x, u, u_x, f, g), whose components are of the form*

[1] In filtration equations the arbitrary element g is equal to 0, but we can reduce the equation equations under study to filtration equations by a simple point transformation.

$$
\begin{aligned}
&\tilde t=\bar T^1t+\bar T^0,\quad \tilde x=\bar X^1x+\bar X^2u-g\bar X^2t+\bar X^0,\\
&\tilde u=\bar U^1x+\bar U^2u+(\bar T^1\bar F-g\bar U^2)t+\bar U^0,\\
&\tilde u_{\tilde x}=\frac{\bar U^1+\bar U^2u_x}{\bar X^1+\bar X^2u_x},\quad \tilde f=\frac{(\bar X^1+\bar X^2u_x)^2}{\bar T^1}f,\quad \tilde g=\bar F,
\end{aligned}
$$

where $\bar T$'s, $\bar X$'s, $\bar U$'s and $\bar F$ are arbitrary functions of g with $\bar T^1(\bar X^1\bar U^2-\bar X^2\bar U^1)\bar F_g\neq 0$.

The usual equivalence group $G^\sim_{\mathcal F}$ is a (finite-dimensional) subgroup of the generalized equivalence group $\bar G^\sim_{\mathcal F}$ that is singled out from $\bar G^\sim_{\mathcal F}$ by the following system of constraints for the group parameters:

$$
\bar T^0_g=\bar T^1_g=0,\quad \bar X^0_g=\bar X^1_g=0,\quad \bar X^2=0,\quad \bar U^0_g=\bar U^1_g=\bar U^2_g=0,\quad \bar T^1\bar F_g=\bar U^2.
$$

Denote by $\mathcal G^\sim_{\mathcal F}$ the equivalence groupoid of the class $\mathcal F$ and by $\mathcal S^\sim_{\mathcal F}$ the subgroupoid of $\mathcal G^\sim_{\mathcal F}$ generated by the generalized equivalence group $\bar G^\sim_{\mathcal F}$. The subgroupoid of $\mathcal G^\sim_{\mathcal F}$ generated by the usual equivalence group $G^\sim_{\mathcal F}$ is a proper subgroupoid of $\mathcal S^\sim_{\mathcal F}$. Hence the group $\bar G^\sim_{\mathcal F}$ is an example of a nontrivial generalized equivalence group. The dependence of group parameters on g is needless for generating admissible transformations in the class $\mathcal F$ and is merely a manifestation of the fact that the arbitrary element g is constant within the subclass $\mathcal F$. This is why we need to consider an effective generalized equivalence group of the class $\mathcal F$, which is a minimal subgroup of $\bar G^\sim_{\mathcal F}$ generating the subgroupoid $\mathcal S^\sim_{\mathcal F}$ of $\mathcal G^\sim_{\mathcal F}$. The only dependence on g that is essential for generalized equivalence is the explicit involvement of g in the t-coefficient of the x-component. At the same time, setting the group parameters $\bar T$'s, $\bar X$'s, $\bar U$'s and $\bar T^1\bar F-\bar U^2$ to be constants singles out the subset of elements from $\bar G^\sim_{\mathcal F}$ that is not a subgroup of $\bar G^\sim_{\mathcal F}$ although this subset is minimal among subsets of $\bar G^\sim_{\mathcal F}$ generating $\mathcal S^\sim_{\mathcal F}$. The construction of an effective generalized equivalence group of the class $\mathcal F$ is in fact more tricky.

Proposition 5.8. *An effective generalized equivalence group $\hat G^\sim_{\mathcal F}$ of the subclass $\mathcal F$ is constituted by the point transformations*

$$
\tilde t=T_1t+T_0,\quad \tilde x=X_1x+X_2u-X_2gt+X_0,
$$

$$
\tilde{u} = U_1 x + U_2 u + (1-U_2)gt + U_3 t + \frac{T_0}{T_1} g + U_0, \quad \tilde{u}_{\tilde{x}} = \frac{U_1 + U_2 u_x}{X_1 + X_2 u_x},
$$
$$
\tilde{f} = \frac{(X_1 + X_2 u_x)^2}{T_1} f, \quad \tilde{g} = \frac{g + U_3}{T_1},
$$

where T's, X's and U's are arbitrary constants with $T_1(X_1U_2 - X_2U_1) \neq 0$.

Proof. Consider the set H_1 of the point transformations in the space with the coordinates (t, x, u, u_x, f, g), whose components are of the form

$$
\begin{aligned}
&\tilde{t} = T_1 t + T_0, \quad \tilde{x} = X_1 x + X_2 u + (A_{11} g + A_{10})t + B_{11} g + B_{10}, \\
&\tilde{u} = U_1 x + U_2 u + (A_{21} g + A_{20})t + B_{21} g + B_{20}, \\
&\tilde{u}_{\tilde{x}} = \frac{U_1 + U_2 u_x}{X_1 + X_2 u_x}, \quad \tilde{f} = \frac{(X_1 + X_2 u_x)^2}{T_1} f, \quad \tilde{g} = \frac{C_1 g + C_0}{T_1},
\end{aligned}
\tag{5.5}
$$

where T's, X's, U's, A's, B's and C's are arbitrary constants with $T_1(X_1U_2 - X_2U_1)C_1 \neq 0$. It is obvious that this set is closed with respect to the composition of transformations and taking the inverse, i.e., it is a (local) transformation group with $\dim H_1 = 16$. Then the intersection $H_0 := H_1 \cap \bar{G}^{\sim}_{\mathcal{F}}$ of H_1 with $\bar{G}^{\sim}_{\mathcal{F}}$, which is singled out from H_1 by the constraints $A_{10} = 0$, $A_{11} = -X_2$, $A_{20} = C_0$ and $A_{21} = C_1 - U_2$, is also a group, and $\dim H_0 = 12$. The subgroup H_0 of $\bar{G}^{\sim}_{\mathcal{F}}$ generates the entire subgroupoid $\mathcal{S}^{\sim}_{\mathcal{F}}$ of $\mathcal{G}^{\sim}_{\mathcal{F}}$, which is generated by $\bar{G}^{\sim}_{\mathcal{F}}$. At the same time, for each fixed pair of the arbitrary elements (f, g), the subgroupoid $\mathcal{S}^{\sim}_{\mathcal{F}}$ contains a precisely nine-parameter family of admissible transformations with the source (f, g). This is why we should try to find three more constraints for group parameters of the group H_1 in order to construct a nine-dimensional subgroup of H_0 that still generates the entire $\mathcal{S}^{\sim}_{\mathcal{F}}$.

We analyze the composition of two arbitrary elements from the group H_0, $\hat{\mathcal{T}} = \tilde{\mathcal{T}}\mathcal{T}$ with $\tilde{\mathcal{T}}, \mathcal{T} \in H_0$. These generalized equivalence transformations have the general form (5.5), where group parameters satisfy the above constraints for the subgroup H_0. We additionally reparameterize H_0 with replacing the parameter B_{21} by $B'_{21} + T_0/T_1$ and mark the group-parameter values corresponding to $\hat{\mathcal{T}}$ and $\tilde{\mathcal{T}}$ by hats and tildes, respectively. We obtain, in particular, the following expressions for group-parameter values of the composition $\hat{\mathcal{T}}$:

$$\hat{C}_1 = \tilde{C}_1 C_1, \quad \hat{B}_{11} = \tilde{X}_1 B_{11} + \tilde{X}_2 B'_{21} + \frac{\tilde{B}_{11}}{T_1}, \quad \hat{B}'_{21} = \tilde{U}_1 B_{11} + \tilde{U}_2 B'_{21} + \frac{\tilde{B}'_{21}}{T_1},$$

which imply that the constraints $C_1 = 1$, $B_{11} = B'_{21} = 0$ singling out $\hat{G}^{\sim}_{\mathcal{F}}$ from the subgroup H_0 are preserved by the composition of transformations and taking the inverse in H_0. Therefore, $\hat{G}^{\sim}_{\mathcal{F}}$ is really a group. It generates the entire subgroupoid $\mathcal{S}^{\sim}_{\mathcal{F}}$ of $\mathcal{G}^{\sim}_{\mathcal{F}}$, and any its proper subset does not possess this property, i.e., it is a minimal subgroup of $\bar{G}^{\sim}_{\mathcal{F}}$ with this property. □

The usual equivalence group $G^{\sim}_{\mathcal{F}}$ of the subclass $\mathcal{F}$ is not contained in the effective generalized equivalence group $\hat{G}^{\sim}_{\mathcal{F}}$ constructed in Proposition 5.8. The intersection $G^{\sim}_{\mathcal{F}} \cap \hat{G}^{\sim}_{\mathcal{F}}$ is singled out from $G^{\sim}_{\mathcal{F}}$ by the constraints $T_0 = 0$ and $U_2 = 1$.

To prove an assertion generalizing the above claim, we need to consider the infinitesimal counterparts of related groups. For convenience, we introduce the following dual notation for relevant vector fields on the space with the coordinates (t, x, u, u_x, f, g):

$$\begin{aligned}
&X^1 = P^t = \partial_t, \quad X^2 = D^t = t\partial_t - f\partial_f - g\partial_g, \; X^3 = P^x = \partial_x, \\
&X^4 = D^x = x\partial_x - u_x\partial_{u_x} + 2f\partial_f, \quad X^5 = P^u = \partial_u, \; X^6 = D^u = u\partial_u + u_x\partial_{u_x} + g\partial_g, \\
&X^7 = Z^t = t\partial_u + \partial_g, \; X^8 = Z^x = x\partial_u + \partial_{u_x}, \; X^9 = R = (u - gt)\partial_x - u_x^2\partial_{u_x} + 2u_x f\partial_f.
\end{aligned}$$

Up to the anticommutativity of the Lie bracket, the nonzero commutation relations between these vector fields are exhausted by

$$\begin{aligned}
&[P^t, D^t] = P^t, \; [P^x, D^x] = P^x, \; [P^u, D^u] = P^u, \; [P^t, Z^t] = P^u, \; [P^x, Z^x] = P^u, \\
&[Z^t, D^t] = -Z^t, \; [Z^x, D^x] = -Z^x, \; [Z^t, D^u] = Z^t, \; [Z^x, D^u] = Z^x, \\
&[P^t, R] = -gP^x, \; [P^u, R] = P^x, \; [D^x, R] = -R, \; [D^u, R] = R, \\
&[Z^x, R] = D^x - D^u + gZ^t.
\end{aligned}$$

The Lie algebras $\mathfrak{g}^{\sim}_{\mathcal{F}}$, $\bar{\mathfrak{g}}^{\sim}_{\mathcal{F}}$ and $\hat{\mathfrak{g}}^{\sim}_{\mathcal{F}}$ of the groups $G^{\sim}_{\mathcal{F}}$, $\hat{G}^{\sim}_{\mathcal{F}}$ and $\bar{G}^{\sim}_{\mathcal{F}}$ are naturally called the usual equivalence algebra, the generalized equivalence algebra and an effective generalized equivalence algebra of the class $\mathcal{F}$, respectively. Each of them is merely the set of infinitesimal generators of one-parameter subgroups of the corresponding groups. In order to construct all such generators, we successively take one of the group parameters in the respective general form of group elements to depend on a continuous subgroup parameter δ and set the other parameter-functions to their values corresponding to the identity transformations, which are $T_1 = X_1 = U_2 = 1$ and $T_0 = X_0 = X_2 = U_0 = U_1 = U_3 = 0$ for the groups $G^{\sim}_{\mathcal{F}}$ and $\hat{G}^{\sim}_{\mathcal{F}}$ (the parameter X_2 is relevant only for $\hat{G}^{\sim}_{\mathcal{F}}$) and similarly $\bar{T}^1 = \bar{X}^1 = \bar{U}^2 = 1$, $\bar{T}^0 = \bar{X}^0 = \bar{X}^2 = \bar{U}^0 = \bar{U}^1 = 0$ and $\bar{F} = g$ for the group $\bar{G}^{\sim}_{\mathcal{F}}$. Then we differentiate the transformation components with respect to δ and evaluate the result at $\delta = 0$. As a result, we derive that

$$\mathfrak{g}^{\sim}_{\mathcal{F}} = \langle X^1, \dots, X^8 \rangle, \quad \bar{\mathfrak{g}}^{\sim}_{\mathcal{F}} = \left\{ \sum_{i=1}^{9} \vartheta^i(g) X^i \right\},$$

$$\hat{\mathfrak{g}}^{\sim}_{\mathcal{F}} = \langle P^t + gP^u,\ D^t,\ P^x,\ D^x,\ P^u,\ D^u - gZ^t,\ Z^t,\ Z^x,\ R \rangle,$$

where the coefficients ϑ's run through the set of smooth functions of g, i.e., the algebra $\bar{\mathfrak{g}}^{\sim}_{\mathcal{F}}$ is the module over the ring of smooth functions of g with basis $(X^1, \dots, X^9)$ equipped with the Lie bracket of vector fields.

Theorem 5.9. *Any effective generalized equivalence group of the class $\mathcal{F}$ does not contain the usual equivalence group $G^{\sim}_{\mathcal{F}}$ of this class.*

Proof. We prove the re-formulated assertion: Suppose that a subgroup of the generalized equivalence group $\bar{G}^{\sim}_{\mathcal{F}}$ of the class $\mathcal{F}$ contains the usual equivalence group $G^{\sim}_{\mathcal{F}}$ of this class and generates the same subgroupoid of the equivalence groupoid $\mathcal{G}^{\sim}_{\mathcal{F}}$ as the entire group $\bar{G}^{\sim}_{\mathcal{F}}$ does. Then this subgroup is not an effective generalized equivalence group of $\mathcal{F}$.

A complete list of discrete usual equivalence transformations of the class $\mathcal{F}$ that are independent up to combining with each other and with continuous usual equivalence transformations of this class is exhausted by the involutions I^t, I^x and I^u alternating the signs of (t, f, g), (x, u_x) and (u, u_x, g), respectively. Among generalized equivalence transformations, there is one more independent discrete transformation I^g: $(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{u}_{\tilde{x}}, \tilde{f}, \tilde{g}) = (t, x, u - 2gt, u_x, f, -g)$. Discrete equivalence transformations play an auxiliary role in the course of the proof.

It suffices to prove the infinitesimal counterpart of the above assertion, which states the following. Let a subalgebra $\mathfrak{h}$ of $\bar{\mathfrak{g}}^{\sim}_{\mathcal{F}}$ contain $\mathfrak{g}^{\sim}_{\mathcal{F}}$ and a vector field $X = \sum_{i=1}^{9} \zeta^i X^i$, where $\zeta^i = \zeta^i(g)$ are smooth functions of g with $\zeta^9 \neq 0$, be invariant with respect to discrete transformations in $G^{\sim}_{\mathcal{F}}$, $I^t_*\mathfrak{h}, I^x_*\mathfrak{h}, I^u_*\mathfrak{h} \subseteq \mathfrak{h}$, and be associated with a transformation (pseudo)group. Then this subalgebra properly contains another subalgebra $\mathfrak{s}$ among whose elements there are $K^j = \sum_{i=1}^{9} \chi^{ij} X^i$, where χ^{ij}, $i,j = 1,\dots,9$, are smooth functions of g with $\det(\chi^{ij}) \neq 0$, and which is also invariant with respect to I^t_*, I^x_* and I^u_* and is associated with a transformation (pseudo)group. Here the subscript "*" combined with the notation of a point transformation denotes pushing forward vector fields on the same manifold by this transformation.

If the algebra $\mathfrak{h}$ contains the pure vector field R, then we commute R with elements of $\mathfrak{g}^{\sim}_{\mathcal{F}}$ and successively obtain that

$$[R, P^t] = gP^x \in \mathfrak{h}, \quad [gP^x, Z^x] = gP^u \in \mathfrak{h}, \quad [Z^x, R] = D^t - D^u + gZ^t \in \mathfrak{h}.$$

Hence $gZ^t \in \mathfrak{h}$, i.e., $\mathfrak{h} \supseteq \mathfrak{g}^{\sim}_{\mathcal{F}} + \langle gP^x,\, gP^u,\, gZ^t \rangle \supsetneq \hat{\mathfrak{g}}^{\sim}_{\mathcal{F}}$. We can choose $\mathfrak{s} = \hat{\mathfrak{g}}^{\sim}_{\mathcal{F}}$. Then we also have $I^t_*\mathfrak{s} = I^x_*\mathfrak{s} = I^u_*\mathfrak{s} = I^g_*\mathfrak{s} = \mathfrak{s}$. Otherwise, we compute the commutators

$$\begin{aligned}
&[X, D^x] = \zeta^9 R - \zeta^8 Z^x + \zeta^3 P^x \in \mathfrak{h},\\
&[\zeta^9 R - \zeta^8 Z^x + \zeta^3 P^x, D^x] = \zeta^9 R + \zeta^8 Z^x + \zeta^3 P^x \in \mathfrak{h},\\
&[\zeta^9 R - \zeta^8 Z^x + \zeta^3 P^x, D^t + D^u] = -\zeta^9 R - \zeta^8 Z^x \in \mathfrak{h},
\end{aligned}$$

and thus derive that $\zeta^9 R \in \mathfrak{h}$, and $\zeta^9 \neq \mathrm{const}$. In the same way, we can show that for any element $\sum_{i=1}^{9} \vartheta^i(g) X^i \in \mathfrak{h}$, the element $\vartheta^3 P^x$ and thus the element $\vartheta^3 P^u = [\vartheta^3 P^x, Z^x]$ also belong to $\mathfrak{h}$. Taking two more commutators,

$$[Z^x, \zeta^9 R] = \zeta^9 (D^x - D^u + gZ^t) \in \mathfrak{h}, \quad [Z^x, \zeta^9 (D^x - D^u + gZ^t)] = -2\zeta^9 Z^x \in \mathfrak{h},$$

we get $\zeta^9 Z^x \in \mathfrak{h}$. Consider the span

$$\mathfrak{s} = \langle P^t,\, D^t,\, Z^t,\, D^x,\, D^u,\, \beta P^x,\, \beta P^u,\, \alpha R,\, \alpha(D^x - D^u + gZ^t),\, \alpha Z^x \mid \alpha R, \beta P^x \in \mathfrak{h}\rangle.$$

It is a subalgebra of $\mathfrak{h}$. Since the entire algebra $\mathfrak{h}$ is invariant with respect to I^t_*, I^x_* and I^u_* and is associated with a transformation (pseudo)group, the subalgebra $\mathfrak{s}$ has the same properties. In view of $R \notin \mathfrak{h}$, the parameter function α does not take constant values. Hence $Z^x \notin \mathfrak{s}$, i.e., $\mathfrak{s} \subsetneq \mathfrak{h}$. As the required elements K^j, $j = 1, \dots, 9$, we can choose P^t, D^t, Z^t, D^x, D^u, P^x, P^u, $\zeta^9 R$ and $\zeta^9 Z^x$.

Therefore, the algebra $\mathfrak{h}$ is not an effective generalized equivalence algebra of $\mathcal{F}$. □

It is worth to note that since the first example of a nontrivial generalized equivalence group, the cornucopia of new examples was found, with the paper [108] being El Dorado.

5.3 Extended generalized equivalence groups

To provide examples of extended generalized equivalence groups we return to the class of general Burgers–Korteweg–de Vries equations and it subclasses. Consider first the subclass of the class (5.1) with coefficients depending at most on t. To study its admissible and equivalence transformations it is convenient to start with a wider class, which is the subclass $\mathcal{K}_0$ of the class (5.1) singled out by the constraint $C_x = 0$ (resp. $A^r_x = 0$) implying $X_{xx} = 0$ for admissible transformations.

Proposition 5.10. *The class $\mathcal{K}_0$ is normalized in the usual sense. Its usual equivalence group is constituted by the transformations of the form*

$$\begin{aligned}
&\tilde t = T(t), \quad \tilde x = X^1(t)x + X^0(t), \quad \tilde u = U^1(t)u + U^0(t,x),\\
&\tilde A^j = \frac{(X^1)^j}{T_t}A^j, \quad \tilde A^1 = \frac{X^1}{T_t}\left(A^1 + \frac{U^0}{U^1}C - \frac{X^1_t x + X^0_t}{X^1}\right),\\
&\tilde A^0 = \frac{1}{T_t}\left(A^0 + \frac{U^1_t}{U^1} + \frac{U^0_x}{U^1}C\right),\\
&\tilde B = \frac{U^1}{T_t}B + \frac{U^0_t}{T_t} + \frac{U^0_x}{T_t}\left(\frac{U^0}{U^1}C - \frac{X^1_t x + X^0_t}{X^1}\right) - \sum_{k=0}^{r}\frac{U^0_k}{(X^1)^k}\tilde A^k, \quad \tilde C = \frac{X^1}{T_tU^1}C,
\end{aligned}$$

where $j = 2, \dots, r$, and $T = T(t)$, $X^1 = X^1(t)$, $X^0 = X^0(t)$, $U^1 = U^1(t)$ and $U^0 = U^0(t,x)$ are arbitrary smooth functions of their arguments with $T_tX^1U^1 \neq 0$.

Consider the subclass $\mathcal{K}_1$ obtained by attaching the constraints $A^0_x = 0$, $A^1_{xx} = 0$, $A^j_x = 0$, $j = 2, \dots, r$, $C_x = 0$ and $B_{xx} = 0$ to the auxiliary system for arbitrary elements. It is also normalized in the usual sense and its usual equivalence group is the subgroup of the usual equivalence group $G^\sim_{\mathcal{K}_0}$ of the class $\mathcal{K}_0$ that is associated with the constraint $U^0_{xx} = 0$, i.e., $U^0 = U^{01}(t)x + U^{00}(t)$. Note that we can reparameterize the class $\mathcal{K}_1$ by representing $B = B^1(t)x + B^0(t)$, $A^1 = A^{11}(t)x + A^{10}(t)$ and assuming the coefficients B^1, B^0, A^{11} and A^{10} as arbitrary elements instead of B and A^1. The transformation component for B simplifies to

$$\tilde{B} = \frac{U^1}{T_t}B + \frac{U^0_t}{T_t} - \frac{U^0_x}{T_t}A^1 - \frac{U^0}{T_t}\left(A^0 + \frac{U^1_t}{U^1} + \frac{U^0_x}{U^1}C\right).$$

The next intermediate subclass $\mathcal{K}_2$ is singled out by strengthening the constraint for A^1 to $A^1_x = 0$. In fact, this can be realized by gauging A^1 in the class $\mathcal{K}_0$ up to $G^\sim_{\mathcal{K}_0}$-equivalence. Since the arbitrary element C is still not gauged to one, it parameterizes the u-component of admissible transformations in $\mathcal{K}_2$, $U^{01} = X^1_tU^1/(X^1C)$, and this fact can again be interpreted in terms of generalized equivalence groups.

Theorem 5.11. *The equivalence groupoid of the subclass $\mathcal{K}_2$ of the class* (5.1) *singled out by the constraints $A^k_x = 0$, $k = 0, \dots, r$, $C_x = 0$ and $B_{xx} = 0$ consists of the triples $(\theta, \tilde{\theta}, \varphi)$'s, where the point transformation φ is of the form*

$$\tilde{t} = T, \quad \tilde{x} = X^1x + X^0, \quad \tilde{u} = U^1u + \frac{X^1_tU^1}{X^1C}x + U^{00}, \tag{5.6a}$$

the arbitrary-element tuples θ and $\tilde{\theta}$ are related according to

$$\tilde{A}^j = \frac{(X^1)^j}{T_t}A^j,\ \tilde{A}^1 = \frac{X^1}{T_t}\left(A^1 + \frac{U^{00}}{U^1}C - \frac{X^0_t}{X^1}\right),\ \tilde{A}^0 = \frac{1}{T_t}\left(A^0 + \frac{U^1_t}{U^1} + \frac{X^1_t}{X^1}\right), \tag{5.6b}$$

$$\tilde{B} = \frac{U^1}{T_t}B + \left(\frac{X^1_tU^1}{X^1C}\right)_t\frac{x}{T_t} + \frac{U^{00}_t}{T_t} - \frac{X^1_tU^1}{X^1C}\frac{A^1}{T_t} - \left(\frac{X^1_tU^1}{X^1C}x + U^{00}\right)\tilde{A}^0, \tag{5.6c}$$

$$\tilde{C} = \frac{X^1}{T_tU^1}C, \tag{5.6d}$$

with $j = 2, \dots, r$, and $T = T(t)$, $X^1 = X^1(t)$, $X^0 = X^0(t)$, $U^1 = U^1(t)$ and $U^{00} = U^{00}(t)$ are arbitrary smooth functions of t with $T_tX^1U^1 \neq 0$.

The usual equivalence group $G^{\sim}_{\mathcal{K}_2}$ of the subclass $\mathcal{K}_2$ is constituted by the transformations (5.6a)–(5.6d) additionally satisfying the constraint $X^1_t = 0$. Hence it is clear that the subclass $\mathcal{K}_2$ is not normalized in the usual sense.

The equation (5.6c) hints that the proper treatment of the related generalized equivalence group within the framework of point transformations needs considering the derivative C_t as an additional arbitrary element Z^0 and prolonging the relation (5.6d) to Z^0 as a derivative of C,

$$\tilde{Z}^0 = \frac{X^1}{T_t^2 U^1} Z^0 + \left(\frac{X^1}{T_t U^1}\right)_t \frac{C}{T_t}. \tag{5.6e}$$

We denote by $\bar{\mathcal{K}}_2$ the class $\mathcal{K}_2$ in which the tuple of arbitrary elements θ is formally extended to $\bar{\theta} = (A^0, \dots, A^r, B, C, Z^0)$ with $Z^0 := C_t$.

Corollary 5.12. *The class $\bar{\mathcal{K}}_2$ is normalized in the generalized sense. The group $\breve{G}^{\sim}_{\bar{\mathcal{K}}_2}$ constituted by the transformations of the form* (5.6) *is an effective generalized equivalence group of this class.*

Proof. The set of the transformations of the form (5.6), which is temporarily denoted by M, is closed with respect to the transformation composition and contains the identity transformation. Each transformation from M is invertible by definition. So, M is a group. The components of transformations from M are of the same form as the components of admissible transformations and the formulas relating the initial and target arbitrary elements. This is why the group M generates the equivalence groupoid $\bar{\mathcal{K}}_2$ and, moreover, it is minimal among subgroups with such property. Therefore, M is an effective generalized equivalence group of the class $\bar{\mathcal{K}}_2$. □

The generalized equivalence group $\bar{G}^{\sim}_{\bar{\mathcal{K}}_2}$ of $\bar{\mathcal{K}}_2$ is much wider than its effective part $\breve{G}^{\sim}_{\bar{\mathcal{K}}_2}$.

Corollary 5.13. *The generalized equivalence group $\bar{G}^{\sim}_{\bar{\mathcal{K}}_2}$ of the class $\bar{\mathcal{K}}_2$ consists of the transformations of the modified form* (5.6), *where $T = T(t)$, $X^1 = X^1(t)$, $X^0 = X^0(t, C)$, $U^1 = U^1(t, C)$ and $U^{00} = U^{00}(t, C)$ are arbitrary smooth functions of their arguments with $T_t X^1 (C U^1_C - U^1) \neq 0$, and the partial derivatives of X^0, U^1 and U^{00} in t should be replaced by the corresponding restricted total derivatives in t with $\bar{\mathrm{D}}_t = \partial_t + Z^0 \partial_C$.*

Proof. Theorem 5.11 implies that elements of $\bar{G}^{\sim}_{\bar{\mathcal{K}}_2}$ are of the modified form (5.6), where the group parameters T, X^1, X^0, U^1 and U^{00} may depend on t and the arbitrary elements $\bar{\theta}$. Hence partial derivatives of these parameter functions should be replaced by the corresponding total derivatives in t with

$$\mathrm{D}_t = \partial_t + \sum_{\alpha} u_{\alpha+\delta_1}\partial_{u_\alpha} + \sum_{k=0}^{r} A^k_t \partial_{A^k} + B_t\partial_B + C_t\partial_C + Z^0_t\partial_{Z^0} + \cdots .$$

After substituting Z^0 for the derivative C_t, the transformation components can be split with respect to the other derivatives of arbitrary elements in t. The splitting implies that in fact the group parameters do not depend on A's, B and Z^0, and, moreover, the parameters T and X^1 do not depend on C. The nondegeneracy condition for elements of $\bar{G}^{\sim}_{\bar{\mathcal{K}}_2}$ is modified in comparison with that for elements of the effective part $\breve{G}^{\sim}_{\bar{\mathcal{K}}_2}$ in view of the parameter function U^1 becoming dependent on C. This condition takes the form $T_tX^1U^1(C/U^1)_C \neq 0$ and reduces to the condition given in the statement of the theorem. □

Remark 5.14. Given a class of differential equations with nontrivial effective generalized equivalence group, this group is in general not defined in a unique way. Indeed, consider the class $\bar{\mathcal{K}}_2$. The effective generalized equivalence group $\breve{G}^{\sim}_{\bar{\mathcal{K}}_2}$ defined in Corollary 5.12 is not a normal subgroup of the entire generalized equivalence group $\bar{G}^{\sim}_{\bar{\mathcal{K}}_2}$ of the class $\bar{\mathcal{K}}_2$. Each subgroup of $\bar{G}^{\sim}_{\bar{\mathcal{K}}_2}$ that is conjugate to $\bar{G}^{\sim}_{\bar{\mathcal{K}}_2}$ is an effective generalized equivalence group of the class $\bar{\mathcal{K}}_2$. In other words, the class $\bar{\mathcal{K}}_2$ possesses a wide family of conjugate effective generalized equivalence groups. The similar fact is even more obvious for the class $\bar{\mathcal{K}}_3$ studied below.

To have the required subclass $\mathcal{K}_3$ of equations from the class (5.1) whose coefficients depend at most on t, we now only need to impose a more restrictive constraint on B, replacing the additional auxiliary equation $B_{xx} = 0$ by $B_x = 0$, which can be implemented by gauging B within the class $\mathcal{K}_2$ using its equivalence transformations. Unfortunately, this deteriorates the normalization property since then the function X^1 parameterizing elements of the equivalence groupoid $\mathcal{G}^{\sim}_{\mathcal{K}_3}$ of the class $\mathcal{K}_3$ depends on the initial arbitrary elements C and A^0 in a nonlocal way via the equation

$$\left(\frac{X^1_t}{C(X^1)^2}\right)_t = A^0\frac{X^1_t}{C(X^1)^2}. \tag{5.7}$$

At the same time, the usual equivalence group $G^\sim_{\mathcal{K}_3}$ of the subclass $\mathcal{K}_3$ coincides with the group $G^\sim_{\mathcal{K}_2}$. The computation of the generalized equivalence group of the subclass $\mathcal{K}_3$ gives the same group, which is a trivial situation from the point of view of generalized equivalence. As a result, the class $\mathcal{K}_3$ is definitely not normalized in both the usual and the generalized senses. This is why we construct the extended generalized equivalence group of the subclass $\mathcal{K}_3$ in a rigorous way. In fact, this is the first construction of such kind in the literature.

We extend the arbitrary-element tuple θ to $\bar\theta = (A^0,\dots,A^r,B,C,Y^1,Y^2)$ with two more arbitrary elements, Y^1 and Y^2, which are functions of t only and satisfy the auxiliary equations

$$Y^1_t = A^0,\quad Y^2_t = C\mathrm{e}^{Y^1}. \tag{5.8}$$

Thus, we also implicitly impose the auxiliary equations $Y^i_{u_\alpha} = Y^i_x = 0$, $|\alpha|\leqslant r$, $i=1,2$. Each value of $\bar\theta$ satisfying all auxiliary equations of the class $\mathcal{K}_3$ as well as the above equations for Y^1 and Y^2 is associated with an equation of the form (5.1) with the corresponding value of θ. We formally denote this equation by $\bar{\mathcal{L}}_{\bar\theta}$ and the class of such equations by $\bar{\mathcal{K}}_3$. It is obvious that the equations $\bar{\mathcal{L}}_{\bar\theta^1}$ and $\bar{\mathcal{L}}_{\bar\theta^2}$ coincide if $\theta^1=\theta^2$. This defines a gauge equivalence relation on the value set of arbitrary-element tuple $\bar\theta$. We show below that this gauge equivalence gives rise to a nontrivial gauge equivalence group of the class $\bar{\mathcal{K}}_3$. (See Sections 2.1 and 2.5 of [129] for notions related to gauge equivalence, which is called trivial equivalence in [83].) Since the set of point transformations from $\bar{\mathcal{L}}_{\bar\theta^1}$ to $\bar{\mathcal{L}}_{\bar\theta^2}$ coincides with that from $\mathcal{L}_{\theta^1}$ to $\mathcal{L}_{\theta^2}$, the equivalence groupoid of $\mathcal{K}_3$ is isomorphic to the equivalence groupoid of $\bar{\mathcal{K}}_3$ factorized with respect to the gauge equivalence. In the class $\bar{\mathcal{K}}_3$, the constraint (5.7) can be solved with respect to X^1 in terms of Y^2,

$$X^1=\frac{1}{\varepsilon_1Y^2+\varepsilon_0}, \tag{5.9}$$

where ε_1 and ε_0 are arbitrary constants with $(\varepsilon_1,\varepsilon_0)\neq(0,0)$. Using this solution and the auxiliary equations (5.8), we prolong the relation (5.6b)–(5.6d) between initial and transformed arbitrary elements to Y^1 and Y^2. Thus, the equality chain

$$\tilde Y^1_{\tilde t}=\tilde Y^1_{\tilde t}T_t=\tilde A^0T_t=A^0+\frac{U^1_t}{U^1}+\frac{X^1_t}{X^1}=Y^1_t+\frac{U^1_t}{U^1}+\frac{X^1_t}{X^1}$$

implies $\tilde Y^1=Y^1+\ln|U^1X^1|+\delta'$ for some constant δ'. Considering the equality chain

$$\tilde Y^2_{\tilde t}=\tilde Y^2_{\tilde t}T_t=\tilde C\mathrm{e}^{\tilde Y^1}T_t=\frac{X^1}{T_tU^1}C\mathrm{e}^{Y^1}U^1X^1\delta T_t=\frac{\delta Y^2_t}{(\varepsilon_1Y^2+\varepsilon_0)^2},$$

where $\delta=\mathrm{e}^{\delta'}\operatorname{sgn}(U^1X^1)\neq0$, we derive for some constants ε_1' and ε_0' with $\varepsilon_0\varepsilon_1'-\varepsilon_0'\varepsilon_1=\delta$ that

$$\tilde Y^2=\frac{\varepsilon_1'Y^2+\varepsilon_0'}{\varepsilon_1Y^2+\varepsilon_0},\quad\text{and hence}\quad \tilde Y^1=Y^1+\ln(\delta U^1X^1). \tag{5.10}$$

We use parentheses instead of vertical bars in the logarithm since $\delta U^1X^1>0$. This completes the description of the equivalence groupoid $\mathcal G^{\sim}_{\bar{\mathcal K}_3}$. Note that here

$$U^{01}=\frac{X^1_tU^1}{X^1C}=-\varepsilon_1U^1X^1\mathrm{e}^{Y^1},\quad U^{01}_t=U^{01}\left(A^0+\frac{U^1_t}{U^1}-\varepsilon_1CX^1\mathrm{e}^{Y^1}\right)=T_tU^{01}\tilde A^0.$$

Theorem 5.15. *Let $\mathcal K_3$ be the subclass of equations from the class* (5.1) *with coefficients depending at most on t, which is singled out from the class* (5.1) *by the constraints $A^k_x=C_x=B_x=0$, $k=0,\dots,r$. The class $\bar{\mathcal K}_3$ of the same equations, where the arbitrary-element tuple is formally extended with the virtual arbitrary elements Y^1 and Y^2 defined by* (5.8)*, is normalized in the generalized sense. Its generalized equivalence group $\bar G^{\sim}_{\bar{\mathcal K}_3}$ consists of the transformations of the form*

$$\tilde t=\bar T(t,Y^1,Y^2),\quad \tilde x=\bar X^1x+\bar X^0(t,Y^1,Y^2),\quad \bar X^1:=\frac{1}{\varepsilon_1Y^2+\varepsilon_0},$$

$$
\begin{aligned}
&\tilde u=\bar U^1(t,Y^1,Y^2)\big(u-\varepsilon_1\bar X^1\mathrm{e}^{Y^1}x\big)+\bar U^{00}(t,Y^1,Y^2),\\
&\tilde A^j=\frac{(\bar X^1)^j}{\bar{\mathrm D}_t\bar T}A^j,\quad \tilde A^1=\frac{\bar X^1}{\bar{\mathrm D}_t\bar T}\left(A^1+\frac{\bar U^{00}}{\bar U^1}C-\frac{\bar{\mathrm D}_t\bar X^0}{\bar X^1}\right),\\
&\tilde A^0=\frac{1}{\bar{\mathrm D}_t\bar T}\left(A^0+\frac{\bar{\mathrm D}_t\bar U^1}{\bar U^1}-\varepsilon_1C\bar X^1\mathrm{e}^{Y^1}\right),\\
&\tilde B=\frac{\bar U^1}{\bar{\mathrm D}_t\bar T}B+\frac{\bar{\mathrm D}_t\bar U^{00}}{\bar{\mathrm D}_t\bar T}+\varepsilon_1\bar U^1\bar X^1\mathrm{e}^{Y^1}\frac{A^1}{\bar{\mathrm D}_t\bar T}-\bar U^{00}\tilde A^0,\quad \tilde C=\frac{\bar X^1}{\bar U^1\bar{\mathrm D}_t\bar T}C,\\
&\tilde Y^1=Y^1+\ln(\delta\bar U^1\bar X^1),\quad \tilde Y^2=\frac{\varepsilon_1'Y^2+\varepsilon_0'}{\varepsilon_1Y^2+\varepsilon_0},
\end{aligned}
$$

where $j=2,\dots,r$; $\bar T$, $\bar X^0$, $\bar U^1$ and $\bar U^{00}$ are arbitrary smooth functions of t, Y^1 and Y^2 with $\bar T_t\bar U^1\neq0$; ε_0, ε_1, ε_0' and ε_1' are arbitrary constants with $\delta:=\varepsilon_0\varepsilon_1'-\varepsilon_0'\varepsilon_1\neq0$ and, moreover, $\delta\bar U^1\bar X^1>0$; $\bar{\mathrm D}_t=\partial_t+A^0\partial_{Y^1}+C\mathrm{e}^{Y^1}\partial_{Y^2}$ is the restricted total derivative operator with respect to t.

Proof. In view of the above description of the equivalence groupoid $\mathcal G^\sim_{\bar{\mathcal K}_3}$ of the class $\bar{\mathcal K}_3$, elements of $\bar G^\sim_{\bar{\mathcal K}_3}$ have the general form

$$
\begin{aligned}
&\tilde t=\bar T(t,\bar\theta),\quad \tilde x=\bar X^1(t,\bar\theta)x+\bar X^0(t,\bar\theta),\\
&\tilde u=\bar U^1(t,\bar\theta)u+\bar U^{01}(t,\bar\theta)x+\bar U^{00}(t,\bar\theta),\quad \tilde{\bar\theta}=\bar\Theta(t,x,u,\bar\theta).
\end{aligned}
$$

The computation of $\bar G^\sim_{\bar{\mathcal K}_3}$ by the direct method is quite similar to the computation of $\mathcal G^\sim_{\bar{\mathcal K}_3}$ and, after splitting with respect to x and parametric derivatives of u, gives similar expressions for transformation components for the variables (t,x,u) and similar constraints for parameter functions. The relations between the initial and target arbitrary elements in the equivalence groupoid just convert to the transformation components for arbitrary elements in the equivalence group. But there are several differences, which we are going to discuss.

In particular, the total derivative operators should be prolonged to the arbitrary elements. Since the arbitrary elements of the class $\bar{\mathcal K}_3$ depend at most on t, the prolongation is essential only for $\mathrm D_t$,

$$
\mathrm D_t=\partial_t+\sum_\alpha u_{\alpha+\delta_1}\partial_{u_\alpha}+\sum_{k=0}^r A^k_t\partial_{A^k}+B_t\partial_B+C_t\partial_C+Y^1_t\partial_{Y^1}+Y^2_t\partial_{Y^2}+\cdots.
$$

The expression for D_x is formally preserved, $\mathrm{D}_x = \partial_x + \sum_\alpha u_{\alpha+\delta_2}\partial_{u_\alpha}$. As a result, all partial derivatives with respect to t in the expressions derived after splitting with respect to x and parametric derivatives of u are converted to the total derivatives with respect to t.

The second difference is the possibility of splitting with respect to arbitrary elements and their derivatives. After substituting for the constrained derivatives Y^1_t and Y^2_t in view of (5.8) into the constraint for $\bar X^1$,

$$\mathrm{D}_t^2\frac{1}{\bar X^1} = \left(\frac{C_t}{C} + A^0\right)\mathrm{D}_t\frac{1}{\bar X^1},$$

we can split the resulting equation with respect to A^0_{tt}, …, A^r_{tt}, B_{tt}, C_{tt}, A^0_t and C_t. This leads to the system $\bar X^1_{A^0} = \dots = \bar X^1_{A^r} = 0$, $\bar X^1_B = 0$, $\bar X^1_C = 0$, $\bar X^1_{Y^1} = 0$, $\bar X^1_t = 0$ and $(1/\bar X^1)_{Y^2Y^2} = 0$, whose general solution is of the form (5.3). The expressions for the transformed arbitrary elements $\tilde A^0$, …, $\tilde A^r$, $\tilde B$ and $\tilde C$ can also be split with respect to unconstrained derivatives of arbitrary elements in t, implying that the derivatives of $\bar T$, $\bar X^0$, $\bar U^1$ and $\bar U^{00}$ with respect to A^0, …, A^r, B and C are zero. Hence the operator D_t can be replaced by the restricted total derivative operator $\bar{\mathrm{D}}_t$. In particular, the parameter function $\bar U^{01}$ is defined by $\bar U^{01} = (\bar U^1\bar{\mathrm{D}}_t\bar X^1)/(\bar X^1 C)$.

The additional auxiliary equations (5.8) are also treated in a different way. We substitute the expressions for Y^1_t and Y^2_t given by these equations into their expanded version for transformed arbitrary elements. Splitting the resulting equations with respect to the other derivatives of arbitrary elements leads to the system of determining equations for the (Y^1, Y^2)-components of equivalence transformations

$$\begin{gathered}\tilde Y^2_t = \tilde Y^2_{Y^1} = \tilde Y^i_{A^k} = \tilde Y^i_B = \tilde Y^i_C = 0, \quad i = 1,2, \quad k = 0,\dots,r,\\ \tilde Y^1_t = \frac{U^1_t}{U^1}, \quad \tilde Y^1_{Y^1} = \frac{U^1_{Y^1}}{U^1} + 1, \quad \tilde Y^1_{Y^2} = \frac{U^1_{Y^2}}{U^1} - \frac{\varepsilon_1}{\varepsilon_1 Y^2 + \varepsilon_0}, \quad \tilde Y^2_{Y^2} = \frac{\bar X^1}{U^1}\mathrm{e}^{\tilde Y^1 - Y^1},\end{gathered}$$

whose general solution is of the form presented in the statement of the theorem. □

Remark 5.16. Each element of the generalized equivalence group $\bar G^\sim_{\bar{\mathcal K}_3}$ generates a family of admissible transformations of the class $\bar{\mathcal K}_3$ with sources at those values of $\bar\theta$ where the evaluation of $\bar{\mathrm{D}}_t\bar{T}$ does not vanish,

$$\bar{G}^{\sim}_{\bar{\mathcal{K}}_3} \ni \mathcal{T} \mapsto \big\{(\bar\theta^1,\bar\theta^2,\varphi) \mid \bar\theta^1 \in \bar{\mathcal{S}}_3,\ (\bar{\mathrm{D}}_t\bar{T})|_{\bar\theta=\bar\theta^1} \neq 0,\ \bar\theta^2 = \mathcal{T}\bar\theta^1,\ \varphi = (\mathcal{T}|_{\bar\theta=\bar\theta^1})\big|_{(t,x,u)}\big\} \subset \mathcal{G}^{\sim}_{\bar{\mathcal{K}}_3}.$$

Here $\bar{\mathcal{S}}_3$ is the value set of the arbitrary-element tuple $\bar\theta$ of the class $\bar{\mathcal{K}}_3$.

The gauge equivalence group of the class $\bar{\mathcal{K}}_3$ is the subgroup of $\bar{G}^{\sim}_{\bar{\mathcal{K}}_3}$ that is singled out by the constraints $\varepsilon_0 = 1$, $\varepsilon_1 = 0$, $\bar{T} = t$, $\bar{X}^0 = 0$, $\bar{U}^1 = 1$, $\bar{U}^{00} = 0$. In other words, all the components of gauge equivalence transformations are identities, except the components for Y^1 and Y^2, for which we get $\tilde{Y}^1 = Y^1 + \ln\varepsilon_1'$, $\tilde{Y}^2 = \varepsilon_1'Y^2 + \varepsilon_0'$ with $\varepsilon_1' > 0$. The usual equivalence group of the class $\bar{\mathcal{K}}_3$ is singled out from $\bar{G}^{\sim}_{\bar{\mathcal{K}}_3}$ by the constraints

$$\varepsilon_1 = 0, \quad \bar{T}_{Y^i} = \bar{X}^0_{Y^i} = \bar{U}^1_{Y^i} = \bar{U}^{00}_{Y^i} = 0, \quad i = 1,2,$$

and its quotient group with respect to the gauge equivalence group of the class $\bar{\mathcal{K}}_3$ is isomorphic to the usual equivalence group of the class $\mathcal{K}_3$.

It is obvious that the generalized equivalence group $\bar{G}^{\sim}_{\bar{\mathcal{K}}_3}$ of the class $\bar{\mathcal{K}}_3$ generates the whole equivalence groupoid of this class. At the same time, functions parameterizing the group depend on two more arguments, Y^1 and Y^2, than functions parameterizing the groupoid. If we omit the arguments Y^1 and Y^2 in the parameter functions, the corresponding set of transformations still generates the equivalence groupoid but it is not a group with respect to the transformation composition. This shows that the class $\bar{\mathcal{K}}_3$ may possess an effective generalized equivalence group being a proper subgroup of $\bar{G}^{\sim}_{\bar{\mathcal{K}}_3}$, and its construction needs a more delicate consideration than, e.g., for the class $\mathcal{K}_2$.

Corollary 5.17. *The class $\mathcal{K}_3$ is normalized in the extended generalized sense. Its extended generalized equivalence group $\hat{G}^{\sim}_{\mathcal{K}_3}$ can be identified with the effective generalized equivalence group of the class $\bar{\mathcal{K}}_3$ that consists of the transformations of the form*

$$\begin{aligned}
&\tilde t = T(t), \quad \tilde x = X^1\big(x + X^{01}(t)Y^2 + X^{00}(t)\big), \quad X^1 := \frac{1}{\varepsilon_1 Y^2 + \varepsilon_0},\\
&\tilde u = V(t)\left(\frac{u}{X^1} - \mathrm{e}^{Y^1}(\varepsilon_1 x - \varepsilon_0 X^{01} + \varepsilon_1 X^{00})\right),
\end{aligned}$$

$$
\begin{aligned}
&\tilde A^j=\frac{(X^1)^j}{T_t}A^j,\quad \tilde A^1=\frac{X^1}{T_t}\left(A^1-X^{01}_tY^2-X^{00}_t\right),\quad \tilde A^0=\frac{1}{T_t}\left(A^0+\frac{V_t}{V}\right),\\
&\tilde B=\frac{V}{T_t}\left(\frac{B}{X^1}-\mathrm{e}^{Y^1}(\varepsilon_1A^1-\varepsilon_0X^{01}_t+\varepsilon_1X^{00}_t)\right),\quad \tilde C=\frac{(X^1)^2}{T_tV}C,\\
&\tilde Y^1=Y^1+\ln(\delta V),\quad \tilde Y^2=\frac{\varepsilon_1'Y^2+\varepsilon_0'}{\varepsilon_1Y^2+\varepsilon_0},
\end{aligned}
$$

where $j=2,\dots,r$; and T, X^{00}, X^{01} and V are arbitrary smooth functions of t with $T_tV\neq 0$; ε_0, ε_1, ε_0' and ε_1' are arbitrary constants with $\delta:=\varepsilon_0\varepsilon_1'-\varepsilon_0'\varepsilon_1\neq 0$ and, moreover, $\delta V>0$.

Proof. We temporarily denote by M the set of the transformations of the above form. This set is a subset of the group $G^{\sim}_{\bar{\mathcal K}_3}$. It is singled out from $G^{\sim}_{\bar{\mathcal K}_3}$ by setting the following values for group parameters:

$$
\begin{aligned}
&\bar T=T(t),\quad \bar X^0=X^1\big(X^{01}(t)Y^2+X^{00}(t)\big),\quad \bar U^1=\frac{V(t)}{X^1},\\
&\bar U^{01}=-\varepsilon_1V(t)\mathrm{e}^{Y^1},\quad \bar U^{00}=V(t)\mathrm{e}^{Y^1}\big(\varepsilon_0X^{01}(t)-\varepsilon_1X^{00}(t)\big).
\end{aligned}
$$

The set M is closed with respect to the transformation composition, i.e., M is a subgroup of the group $G^{\sim}_{\bar{\mathcal K}_3}$. The subgroup M generates the entire equivalence groupoid $\mathcal G^{\sim}_{\bar{\mathcal K}_3}$ of the class $\bar{\mathcal K}_3$ and thus the entire equivalence groupoid of the class $\mathcal K_3$. Indeed, let us fix any equation $\bar{\mathcal L}_{\bar\theta}$ from the class $\bar{\mathcal K}_3$. The set $\mathrm T_{\bar\theta}$ of all admissible transformations with source at $\bar\theta$ is parameterized by the arbitrary smooth functions T, X^0, U^1 and U^{00} of t and the arbitrary constants ε_0, ε_1, ε_0' and ε_1' with $T_tU^1\neq 0$, $\delta:=\varepsilon_0\varepsilon_1'-\varepsilon_0'\varepsilon_1\neq 0$ and $\delta U^1X^1>0$, where X^1 is defined by (5.3) for the fixed value of the arbitrary element Y^2, $Y^2=Y^2(t)$. Each admissible transformation from $\mathrm T_{\bar\theta}$ is generated by the equivalence transformation from M with the same values of T, ε_0, ε_1, ε_0' and ε_1', and the values of X^{00}, X^{01} and V defined by

$$
\begin{aligned}
&X^{00}=\varepsilon_0X^0(t)-Y^2(t)\frac{U^{00}(t)}{U^1(t)}\mathrm{e}^{-Y^1(t)},\quad X^{01}=\varepsilon_1X^0(t)+\frac{U^{00}(t)}{U^1(t)}\mathrm{e}^{-Y^1(t)},\\
&V=\frac{U^1(t)}{\varepsilon_1Y^2(t)+\varepsilon_0}.
\end{aligned}
$$

This establishes a one-to-one correspondence between M and $\mathrm{T}_{\bar\theta}$, and thus the subgroup M is minimal among the subgroups of $G^{\sim}_{\bar{\mathcal K}_3}$ that generate the groupoid $\mathcal{G}^{\sim}_{\bar{\mathcal K}_3}$.

Therefore, M is the effective generalized equivalence group of the class $\bar{\mathcal K}_3$. □

Our second example of an extended generalized equivalence group also comes from studying a subclass of (5.1), but this time $r = 2$ and thus (generalized) Burgers equations are under investigation. The class had arisen when we classified the class of variable-coefficient Burgers equations [111].

The class $\hat{\mathcal L}_0$ consists of Burgers equations of the form

$$u_t + uu_x = A^2(t,x)u_{xx} + A^{11}(t)x + A^{10}(t),$$

with the arbitrary elements being the sufficiently smooth functions of their arguments and $A^2 \neq 0$.

Proposition 5.18. *The equivalence groupoid $\hat{\mathcal G}^{\sim}_0$ of the class $\hat{\mathcal L}_0$ is constituted by the triples $(\theta,\varphi,\tilde\theta)$, where θ and $\tilde\theta$ denote the tuples of arbitrary elements of the source and the target equations in the class $\hat{\mathcal L}_0$, and φ is a point transformation of the form*

$$\tilde t = T,\quad \tilde x = T_tU^1x + X^0,\quad \tilde u = U^1u - U^1_tx + U^0, \tag{5.11a}$$

where T, X^0, U^1 and U^0 are smooth functions of t, satisfying $T_tU^1 \neq 0$ and

$$U^1_{tt} = A^{11}U^1_t,\quad U^0_t = -A^{10}U^1_t. \tag{5.11b}$$

In turn, the arbitrary elements of the source and target equations are related as follows

$$\begin{aligned}&\tilde A^2 = (U^1)^2T_tA^2,\quad \tilde A^{11} = \frac{1}{T_t}\left(A^{11} - \frac{T_{tt}}{T_t} - \frac{2U^1_t}{U^1}\right),\\ &\tilde A^{10} = U^1A^{10} - \frac{X^0_t}{T_t} + U^0 - \frac{X^0}{T_t}\left(A^{11} - \frac{T_{tt}}{T_t} - \frac{2U^1_t}{U^1}\right).\end{aligned} \tag{5.11c}$$

To construct the usual equivalence group $\hat G^{\sim}_0$ of the class $\hat{\mathcal L}_0$, we split the classifying conditions (5.11b) for admissible transformations with respect to the arbitrary elements A^{10} and A^{11} and find U^1 and U^0 to be constants. This means that the group $\hat{G}_0^\sim$ consists of the point transformations in the space with coordinates $(t,x,u,A^{10},A^{11},A^2)$ whose components are of the form (5.11a), (5.11c), where T and X^0 are smooth functions of t and U^1 and U^0 are arbitrary constants with $T_tU^1\neq 0$. Therefore, the group $\hat{G}_0^\sim$ also coincides with $\hat{G}^\sim$ but the class $\hat{\mathcal{L}}_0$ is not normalized in the usual sense. On the other hand, introducing the virtual nonlocal arbitrary elements Y^0, Y^1 and Y^2 defined by the equations

$$Y^0_t = A^{11},\quad Y^1_t = \mathrm{e}^{Y^0},\quad Y^2_t = A^{10}\mathrm{e}^{Y^0}, \tag{5.12}$$

we construct a covering of the auxiliary system for the arbitrary elements of the class $\hat{\mathcal{L}}_0$. The form of these nonlocal arbitrary elements is implied by solutions of the determining equations on the parameter-functions U^0 and U^1. (This is an application of techniques from the theory of nonlocal symmetries of differential equations [26, Section 5] and [25] in the context of classes of differential equations.) By $\bar{\mathcal{L}}_0$ we denote the class obtained by reparameterizing the class $\hat{\mathcal{L}}_0$ with the extended tuple of the arbitrary elements $\bar\theta=(A^{10},A^{11},A^2,Y^0,Y^1,Y^2)$. The class $\bar{\mathcal{L}}^0$ will be shown to be normalized in the generalized sense.

Corollary 5.19. *The equivalence groupoid of the class* $\bar{\mathcal{L}}_0$ *consists of the triples* $(\bar\theta,\varphi,\tilde{\bar\theta})$*, where the arbitrary-element tuples* $\bar\theta$ *and* $\tilde{\bar\theta}$ *of the source and the target equations are related by* (5.11c) *and*

$$\begin{aligned}
&\tilde Y^0 = Y^0 + \ln\frac{\delta}{T_t(c_1Y^1+c_0)^2},\quad \tilde Y^1 = \frac{c_1'Y^1+c_0'}{c_1Y^1+c_0},\\
&\tilde Y^2 = \frac{\delta Y^2}{c_1Y^1+c_0} - \frac{\delta X^0\mathrm{e}^{Y^0}}{T_t(c_1Y^1+c_0)^2} - c_2\frac{c_1'Y^1+c_0'}{c_1Y^1+c_0} + c_3,
\end{aligned}\tag{5.13}$$

and the components of the point transformation φ *are the form* (5.11a) *with*

$$U^1 = c_1Y^1 + c_0,\quad U^0 = c_2 - c_1Y^2, \tag{5.14}$$

$\delta = c_1'c_0 - c_1c_0'$*,* T *and* X^0 *being arbitrary smooth functions of* t *and* c*'s being arbitrary constants such that* $\delta T_t>0$*.*

Proof. On introducing the virtual arbitrary elements, we can solve the equations (5.11b) for U^1 and U^0 in terms of Y's. The expression for the transformed nonlocal arbitrary element $\tilde Y^0$ follows from the chain of identities

$$\partial_t\tilde Y^0=\tilde Y^0_{\tilde t}T_t=\tilde A^{11}T_t=A^{11}-\frac{T_{tt}}{T_t}-\frac{2c_1Y^1_t}{c_1Y^1+c_0}=\left(Y^0+\ln\frac{1}{|T_t|(c_1Y^1+c_0)^2}\right)_t.$$

For Y^1 and Y^2, the procedure is similar. □

Remark 5.20. There is a nontrivial gauge equivalence amongst equations in the reparameterized class $\bar{\mathcal{L}}_0$ stemming from the indeterminacy in defining the virtual arbitrary elements. More specifically, the arbitrary-element tuples $\bar\theta$ and $\tilde{\bar\theta}$ are associated with the same equation in the class $\bar{\mathcal{L}}_0$ if and only if

$$\begin{aligned}&\tilde A^{10}=A^{10},\quad \tilde A^{11}=A^{11},\quad \tilde A^2=A^2,\\&\tilde Y^0=Y^0+\ln c_1',\quad \tilde Y^1=c_1'Y^1+c_0',\quad \tilde Y^2=c_1'Y^2+c_3,\end{aligned}\tag{5.15}$$

where c's are arbitrary constants with $c_1'>0$. The equations (5.15) jointly with the equations $\tilde t=t$, $\tilde x=x$ and $\tilde u=u$ represent the components of the gauge equivalence transformations in $\bar{\mathcal{L}}_0$, which constitute the gauge equivalence group $G^{\mathrm{g}\sim}_{\bar{\mathcal{L}}_0}$ of $\bar{\mathcal{L}}_0$. This group is a normal subgroup of the usual equivalence group $G^{\sim}_{\bar{\mathcal{L}}_0}$ of $\bar{\mathcal{L}}_0$, and the quotient group $G^{\sim}_{\bar{\mathcal{L}}_0}/G^{\mathrm{g}\sim}_{\bar{\mathcal{L}}_0}$ is isomorphic to the usual equivalence group of the subclass $\hat{\mathcal{L}}_0$ of $\hat{\mathcal{L}}$, which coincides with the usual equivalence group of the entire class $\hat{\mathcal{L}}$.

Theorem 5.21. *The class $\bar{\mathcal{L}}_0$ is normalized in the generalized sense. Its generalized equivalence group $\bar G^{\sim}_0$ consists of the point transformations of the form*

$$\tilde t=\bar T,\ \tilde x=(\bar{\mathrm D}_t\bar T)(c_1Y^1{+}c_0)x+\bar X^0,\ \tilde u=(c_1Y^1{+}c_0)u-c_1\mathrm e^{Y^0}x{+}c_2-c_1Y^2,\tag{5.16a}$$

$$\tilde A^2=(\bar{\mathrm D}_t\bar T)(c_1Y^1+c_0)^2A^2,\quad \tilde A^{11}=\frac{1}{\bar{\mathrm D}_t\bar T}\left(A^{11}-\frac{\bar{\mathrm D}_t^2\bar T}{\bar{\mathrm D}_t\bar T}-\frac{2c_1\mathrm e^{Y^0}}{c_1Y^1+c_0}\right),\tag{5.16b}$$

$$\tilde A^{10}=(c_1Y^1{+}c_0)A^{10}-\frac{\bar{\mathrm D}_t\bar X^0}{\bar{\mathrm D}_t\bar T}{+}c_2{-}c_1Y^2-\frac{\bar X^0}{\bar{\mathrm D}_t\bar T}\left(A^{11}-\frac{\bar{\mathrm D}_t^2\bar T}{\bar{\mathrm D}_t\bar T}-\frac{2c_1\mathrm e^{Y^0}}{c_1Y^1{+}c_0}\right),\tag{5.16c}$$

$$\tilde Y^0=Y^0+\ln\frac{\delta}{(\bar{\mathrm D}_t\bar T)(c_1Y^1+c_0)^2},\quad \tilde Y^1=\frac{c_1'Y^1+c_0'}{c_1Y^1+c_0},\tag{5.16d}$$

$$\tilde Y^2=\frac{\delta Y^2}{c_1Y^1+c_0}-\frac{\delta\bar X^0\mathrm e^{Y^0}}{(\bar{\mathrm D}_t\bar T)(c_1Y^1+c_0)^2}-c_2\frac{c_1'Y^1+c_0'}{c_1Y^1+c_0}+c_3. \tag{5.16e}$$

Here $\bar{\mathrm D}_t=\partial_t+A^{11}_t\partial_{A^{11}}+A^{10}_t\partial_{A^{10}}+A^{11}\partial_{Y^0}+\mathrm e^{Y^0}\partial_{Y^1}+A^{10}\mathrm e^{Y^0}\partial_{Y^2}+A^2_t\partial_{A^2}$ *is the restricted total derivative operator with respect to* t*,* $\delta:=c_1'c_0-c_0'c_1$*,* $\bar T$ *and* $\bar X^0$ *are smooth functions of* (t,Y^1) *and* (t,Y^0,Y^1,Y^2)*, respectively, and* c*'s are arbitrary constants with* $\delta\bar{\mathrm D}_t\bar T>0$*.*

Proof. Elements of the group $\bar G_0^{\sim}$ are point transformations in the space with the coordinates $(t,x,u,A^{10},A^{11},A^2,Y^0,Y^1,Y^2)$. Each of these transformations, $\mathcal T$, generates a family of admissible transformations of the class $\bar{\mathcal L}_0$ with the following properties:

- they are smoothly and pointwise parameterized by the source arbitrary-element tuple $\bar\theta$,
- their transformational parts are of the general form (5.11a),
- their target and the source arbitrary-element tuples are related according to (5.11c) and (5.13),
- and the parameters U^1 and U^0 in them are necessarily of the form (5.14).

Therefore, the components of $\mathcal T$ are of the form (5.16), where the parameters $\bar T$, $\bar X^0$ and c's are considered as smooth functions of the above coordinates that satisfy the equations

$$\begin{gathered}\bar{\mathrm D}_x\bar T=\bar{\mathrm D}_u\bar T=0,\quad \bar{\mathrm D}_x\bar X^0=\bar{\mathrm D}_u\bar X^0=0,\\ \bar{\mathrm D}_tc_i=\bar{\mathrm D}_xc_i=\bar{\mathrm D}_uc_i=0,\ i=0,\dots,3,\quad \bar{\mathrm D}_tc_j'=\bar{\mathrm D}_xc_j'=\bar{\mathrm D}_uc_j'=0,\ j=0,1,\end{gathered}$$

with $\bar{\mathrm D}_t$ defined in the theorem's statement, $\bar{\mathrm D}_x:=\partial_x+A^2_x\partial_{A^2}$ and $\bar{\mathrm D}_u:=\partial_u$. Successively splitting these equations with respect to A^2_x, and then the equations for c's with respect to A^{10}_t, A^{11}_t, A^{10}, A^{11} and Y^0 (the last three splittings are allowed in view of equations derived in the course of the previous splittings), we get that $\bar T$ and $\bar X^0$ are smooth functions of $(t,A^{10},A^{11},Y^0,Y^1,Y^2)$, and c's are constants. After this, we also split the equations (5.16b) and (5.16c) with respect to A^{10}_t and A^{11}_t, obtaining $\bar T_{A^{10}}=\bar T_{A^{11}}=\bar T_{Y^0}=\bar T_{Y^2}=0$ and $\bar X^0_{A^{10}}=\bar X^0_{A^{11}}=0$, which completes the proof. □

Note that should we merely omit the dependence of the group parameters $\bar T$ and

$\bar{X}^0$ in $\bar{G}_0^\sim$ on the nonlocal arbitrary elements Y's, we would obtain the set of equivalence transformations that is not a group as it is not closed under the composition of transformations although this set still generates the entire equivalence groupoid of $\bar{\mathcal{L}}_0$. In particular, the value $\bar{X}^{0,3}$ of the parameter function $\bar{X}^0$ for the composition $\mathcal{T}^3$ of transformations $\mathcal{T}^1, \mathcal{T}^2 \in \bar{G}_0^\sim$ would be of the form

$$\bar{X}^{0,3} = \mathrm{D}_{\tilde{t}}\bar{T}^{,2}(c_{1,2}\tilde{Y}^1 + c_{0,2})\bar{X}^{0,1} + \bar{X}^{0,2}, \tag{5.17}$$

where an index after comma indicates the number of the transformation the parameters are associated with. Thus, the dependence of $\bar{X}^0$ on Y^1 is necessary for closedness with respect to the composition of the transformations. In a similar way, we can show that the parameter $\bar{X}^0$ should depend on Y^0. At the same time, the dependence of $\bar{T}$ on the virtual arbitrary elements as well as the dependence of $\bar{X}^0$ on Y^2 are superfluous. Guided by inspection and intuition, we look for transformations with the parameter $\bar{X}^0$ of the form $\bar{X}^0 = T_t \exp(\alpha Y^0)(c_1Y^1 + c_0)^\beta \breve{X}^0(t)$ for some constants α and β. The substitution of the ansatz into (5.17) readily produces $\alpha = -1/2$ and $\beta = 1$.

Corollary 5.22. *An effective generalized equivalence group $\breve{G}_0^\sim$ of the class $\bar{\mathcal{L}}_0$ is constituted by the point transformations*

$$\begin{aligned}
&\tilde{t} = T, \quad \tilde{x} = T_t(c_1Y^1 + c_0)\left(x + \mathrm{e}^{-Y^0/2}\breve{X}^0\right),\\
&\tilde{u} = (c_1Y^1 + c_0)u - c_1\mathrm{e}^{Y^0}x + c_2 - c_1Y^2,\\
&\tilde{A}^{10} = (c_1Y^1 + c_0)\left(A^{10} - \frac{1}{2}\mathrm{e}^{-Y^0/2}\breve{X}^0A^{11} - \mathrm{e}^{-Y^0/2}\breve{X}^0_t\right) + c_1\mathrm{e}^{Y^0/2+c_2-c_1Y^1},\\
&\tilde{A}^{11} = \frac{1}{T_t}\left(A^{11} - \frac{T_{tt}}{T_t} - \frac{2c_1\mathrm{e}^{Y^0}}{c_1Y^1 + c_0}\right), \quad \tilde{A}^2 = T_t(c_1Y^1 + c_0)^2A^2,\\
&\tilde{Y}^0 = Y^0 + \ln\frac{\delta}{T_t(c_1Y^1 + c_0)^2}, \quad \tilde{Y}^1 = \frac{c_1'Y^1 + c_0'}{c_1Y^1 + c_0},\\
&\tilde{Y}^2 = \frac{\delta Y^2}{c_1Y^1 + c_0} - \frac{\delta\breve{X}^0\mathrm{e}^{Y^0/2}}{c_1Y^1 + c_0} - c_2\frac{c_1'Y^1 + c_0'}{c_1Y^1 + c_0} + c_3,
\end{aligned}$$

where $\delta := c_1'c_0 - c_0'c_1$, T and $\breve{X}^0$ are smooth functions of t and c's are arbitrary constants with $\delta T_t > 0$.

Proof. To prove that the set of transformations from the corollary's statement is an effective generalized equivalence group of the class $\bar{\mathcal{L}}_0$, one should show that it is indeed a group under the composition of transformations, it induces the entire equivalence groupoid of the class $\bar{\mathcal{L}}_0$, and it is a minimal group with this property. The first statement is proved by mere inspection, while the second (two-part) statement is more involved. Given an equation $\bar{\mathcal{L}}_0^{\bar\theta}$ in the class $\bar{\mathcal{L}}_0$ with a fixed value of the tuple of arbitrary elements $\bar\theta$, the set $\mathrm{T}_{\bar\theta}$ of admissible transformations with the source $\bar\theta$ is parameterized by arbitrary smooth functions T and X^0 of t and arbitrary constants c_0, $\dots$, c_3, c_0' and c_1' such that $(c_1'c_0-c_1c_0')T_t>0$. At the same time, each admissible transformation in $\mathrm{T}_{\bar\theta}$ is generated by the element from $\breve G_0^\sim$ with the same values of all the parameters except $\breve X^0$ whose value is defined by $\breve X^0=X^0\mathrm{e}^{Y^0/2}/(T_t(c_1Y^1+c_0))$ with the fixed values of the arbitrary elements Y^0 and Y^1. This establishes a one-to-one correspondence between the group $\bar G_0^\sim$ and $\mathrm{T}_{\bar\theta}$, completing the proof. □

Corollary 5.23. *The class $\hat{\mathcal{L}}_0$ is normalized in the extended generalized sense. Its equivalence groupoid is generated by the group $\breve G_0^\sim$.*

5.4 Conclusion

For a long time after the first discussion of the notion of generalized equivalence groups in [87, 88], no examples of nontrivial generalized equivalence groups were known in the literature, except classes for which some of arbitrary elements are constants and thus some of components of equivalence transformations associated with system variables depend on such arbitrary elements; see, e.g., [129, Section 6.4], [156, Section 2] and [158, Section 3]. Note that in all these papers, effective generalized equivalence groups were given instead of the corresponding generalized equivalence groups. This is why certain doubts started to circulate in the symmetry community whether this notion is valuable at all.

In the present chapter we provided some examples of nontrivial generalized equivalence groups such that equivalence-transformation components corresponding to equation variables locally depend on nonconstant arbitrary elements of the corresponding classes.

All related classes are (reparameterized) subclasses of the class (5.1). The most significant consequence of the construction of these examples is that they make evident the necessity of introducing the notion of effective generalized equivalence group. Moreover, they also answer, just by their existence, some theoretical questions, which leads to properly posing further questions. In particular, the entire generalized equivalence group of a class may be effective itself and thus it is a unique effective generalized equivalence group of this class, cf. Theorem 5.6. Nevertheless, there are classes of differential equations admitting multiple effective generalized equivalence groups. This claim is exemplified by classes $\bar{\mathcal{K}}_2$, $\bar{\mathcal{K}}_3$ and $\bar{\mathcal{L}}_0$, for which we have constructed effective generalized equivalence groups that are proper but not normal subgroups of the corresponding generalized equivalence groups. All known examples of generalized equivalence groups that are related to constant arbitrary elements have the same property, see [108] for the El Dorado of such examples. Then the natural question is whether there exists a class of differential equations with effective generalized equivalence group being a proper normal subgroup of the corresponding generalized equivalence group. Furthermore, Corollary 5.17 shows that even merely singling out an effective generalized equivalence group from the already known generalized equivalence group of a class may be a nontrivial problem.

The class $\mathcal{K}_3$ of general Burgers–KdV equations with coefficients depending at most on the time variable is normalized in the extended generalized sense. This property had been found for a number of classes of differential equations (see, e.g., [155, 157, 159]) but one of the main achievements of the thesis is a discovery of a rigorous way to prove it. A principal step is introducing virtual arbitrary elements that are nonlocally related to the native arbitrary elements of a class under study. Similar results were earlier obtained only for classes of linear ordinary differential equations in the preprint version of [27]. The reparameterization technique developed gives hope to us that such construction will be realized soon for many classes of differential equations.

The group $\hat{G}^{\sim}_{\mathcal{F}}$ gives the first nontrivial example of *a finite-dimensional effective generalized equivalence group* in the literature. Moreover, the class $\mathcal{F}$ has another unexpected property formulated in Theorem 5.9: any effective generalized equivalence group of the class $\mathcal{F}$ does not contain the usual equivalence group of this class. This phenomenon had not been observed before for any class of differential equations. Since $\hat{G}^{\sim}_{\mathcal{F}}$ is not a normal subgroup of $\bar{G}^{\sim}_{\mathcal{F}}$, it is obvious that $\hat{G}^{\sim}_{\mathcal{F}}$ is not a unique effective generalized equivalence group of the subclass $\mathcal{F}$. Whether this group is unique up to the subgroup similarity within $\bar{G}^{\sim}_{\mathcal{F}}$ is still an open problem.

In a wider perspective, the most interesting question in the developed theory of generalized equivalence groups is whether the normality of an effective (extended) generalized equivalence group is equivalent to the uniqueness thereof.

Summary of results and future research

Here we emphasize the most important results of the thesis.

• Using the representation of the (1+1)-dimensional Klein–Gordon equation $\mathcal{K}$ in the light-cone variables, we explicitly find its algebra of generalized symmetries and describe it in terms of the universal enveloping algebra of the essential Lie invariance algebra of the Klein–Gordon equation. By choosing a suitable basis of the algebra, we single out variational symmetries of the corresponding Lagrangian, which allow us to compute the space of local conservation laws of this equation via the Noether theorem.

• An isothermal no-slip drift flux model is governed by a hydrodynamic-type partially coupled, non-genuinely nonlinear system $\mathcal{S}$, and the essential subsystem $\mathcal{S}_0$ of $\mathcal{S}$ reduces to (1+1)-dimensional Klein–Gordon equation. These properties allow us to exhaustively describe all generalized symmetries, cosymmetries and conservation laws of $\mathcal{S}$ by finding separately objects stemming from the equation $\mathcal{K}$ and from the double degeneracy of the system $\mathcal{S}$.

• Not all generalized symmetries of the Klein–Gordon equation can be locally prolonged to the entire system $\mathcal{S}$. In view of this we initiate studying of coverings of the system $\mathcal{S}$ in order to find nonlocal prolongation of the above symmetries. Although the positive result is not obtained yet, we hypothesize that a suitable Abelian covering is associated with the conservation laws of the Klein–Gordon equation (3.4a) with characteristics of the form $\mathscr{J}^\kappa \mathrm{e}^{y+z}$, $\kappa \in \mathbb{N}_0$. We plan to study the question in more detail in future research. Likewise, the system $\mathcal{S}_0$ possesses three first-order hydrodynamic Hamiltonian structures, while only one of them locally prolongs to the entire system $\mathcal{S}$. We show that other two operators prolong nonlocally.

- We make a preparatory mathematical step toward geometric parameterization of the (1+2)-dimensional shallow water model by describing the algebra of differential invariants of its point symmetry group using the method of moving frames. The physical step is still necessary to complete the parameterization and we plan to return to this question in future research.

- First nontrivial examples of generalized equivalence groups are given, i.e. equivalence groups whose parameters depend on nonconstant arbitrary elements of a class. Also, for the first time extended generalized equivalence groups are rigourously constructed. It is done via introducing nonlocal *virtual* arbitrary elements of a class, which are connected nonlocally to the arbitrary elements of the class.

- The notion of an effective (extended) generalized equivalence group is introduced. Found are both finite- and infinite-dimensional examples, examples of classes with unique and multiple effective generalized equivalence groups and a class, no effective generalized equivalence group of which contains the usual equivalence group of the class.

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