The L_p John ellipsoids for general measures

by

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Abstract

This thesis aims to develop the L_p John ellipsoids related to general measures. Our L_p John ellipsoids contain many well-known ellipsoids constructed from given convex bodies as special cases, including but not limited to the classical John ellipsoid, the L_p John ellipsoid, the Lutwak-Yang-Zhang ellipsoid, the Petty ellipsoid, etc.

Let μ be an α-homogeneous measure on $\mathbb{R}^n$ for $\alpha > 0$. Our L_p John ellipsoids for the general measure μ for $p > 0$ are defined as the solutions to the following optimization problem:

$$\max_{E \in \mathcal{E}_o^n} V(E) \quad \text{subject to} \quad V_{\mu,p}(K, E) \leq \mu(K), \tag{1}$$

where $\mathcal{E}_o^n$ denotes the set of all origin-symmetric ellipsoids, K is a compact convex set in $\mathbb{R}^n$ containing the origin in its interior, V is the volume function, and

$$V_{\mu,p}(K, E) = \frac{1}{\alpha\mu(K)} \int_{S^{n-1}} h_E^p(v) dS_{\mu,p}(K, v),$$

with h_E the support function of E and $dS_{\mu,p}(K, \cdot)$ the L_p-surface μ-area measure of K.

In this thesis, for $p > 0$, we establish the existence and uniqueness of the L_p John ellipsoid for μ. A characterization of the L_p John ellipsoid for μ is obtained. We also investigate the case for $p = 0$, which is related to the logarithmic function. Besides, the inclusion for the L_p John ellipsoid for μ is provided. The convex bodies with identical John and L_p John ellipsoids for the general measure μ are characterized. Finally, we provide a study for another arguably more general family of L_p John ellipsoids, defined in a way similar to the one in (1) but with $V_{\mu,p}(K, E)$ replaced by $\int_{S^{n-1}} h_E^p(v) d\nu(v)$ and with $\mu(K)$ repalced by $\nu(S^{n-1})$, respectively.

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Lay summary

Ellipsoids, the images of the Euclidean balls under linear transforms, are special convex bodies (i.e., compact convex sets in $\mathbb{R}^n$ with nonempty interiors) with elegant properties. In affine geometry, ellipsoids can be viewed as the balls and hence play fundamental roles in applications, such as in the affine isoperimetric inequalities where ellipsoids are often the extreme points.

Constructing ellipsoids from a given convex body has a long history. A classical example is the John ellipsoid introduced by John in his elegant work [30] in the 1940s. Later the John ellipsoid has been extended to others including the L_p John ellipsoid, the Lutwak-Yang-Zhang ellipsoid, the Petty ellipsoid, etc. These ellipsoids have found many applications in other areas of science and information theory.

In this thesis, we aim to define the L_p John ellipsoids in the (arguably) most general setting. By studying the optimization problems regarding some general measures, we are able to show the existence and uniqueness of the L_p John ellipsoids for general measures. We also provide the characterization of the L_p John ellipsoids, and establish some inequalities and properties related to these ellipsoids.

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Chapter 1

Introduction

This thesis aims to develop the L_p John ellipsoids related to general measures. Our L_p John ellipsoids contain many well-known ellipsoids constructed from given convex bodies as the special cases, including but not limited to the classical John ellipsoid, the L_p John ellipsoid, the Lutwak-Yang-Zhang ellipsoid, the Petty ellipsoid, etc.

The John ellipsoid, introduced by John in his elegant work [30] in the 1940s, is a classical concept in convex geometry with lots of applications. Let K be a convex body (i.e., a convex compact set in $\mathbb{R}^n$ with nonempty interior) and $V(K)$ denotes the volume of K. Denote by JK the John ellipsoid of convex body K, which is the solution to the following optimization problem:

$$\max_{E\in\mathcal{E}^n} V(E) \quad \text{subject to} \quad E\subseteq K, \tag{1.1}$$

where $\mathcal{E}^n$ denotes the set of all ellipsoids in $\mathbb{R}^n$. John [30] proved that the optimization problem (1.1) admits one and only one solution, which of course established the existence and uniqueness of JK. By the nature of the optimization problem (1.1), JK is actually the ellipsoid contained in K with maximal volume. The John ellipsoid has many nice properties, which make it pretty useful in applications in many areas of mathematics, such as, convex geometry, functional analysis, and partial differential equations, and receives a lot of attention (see, e.g., [1–3, 17–20, 23, 31, 37, 43, 55]). For instance, if the convex body K is a symmetric polytope (the smallest convex body containing finitely many points) in $\mathbb{R}^n$, the solutions to some problems in experimental designs exhibit equivalence to finding the John ellipsoid of K [48]. When one needs to estimate the size of K or manage the shape of K, the John ellipsoid JK becomes even more important since JK is an easier object to analyze. For example, the John ellipsoid has been used in the seminal works of Huang, Lutwak, Yang and Zhang [24], Huang and Zhao [25], and Böröczky, Lutwak, Yang, Zhang and Zhao [9] to solve the dual Minkowski problem. We would like to mention that the dual Minkowski problem provides an elegant connection between the Brunn-Minkowski theory and its dual, and hence quickly becomes a central problem of interest in convex geometry.

The John ellipsoid is a special case of the L_p John ellipsoid introduced by Lutwak, Yang and Zhang [37]. The L_p John ellipsoid is developed based on the p-Firey addition (usually called the L_p addition) of convex bodies [15]. Here, for $p \geq 1$, two constants $a, b \geq 0$, and for two convex bodies K, L containing the origin in its interior, the L_p addition of K and L, denoted by $a \cdot K +_p b \cdot L$, is defined by

$$h^p_{a\cdot K+_p b\cdot L}(x) = ah^p_K(x) + bh^p_L(x),$$

for all $x \in \mathbb{R}^n$, where $h_K : \mathbb{R}^n \to [0, \infty)$ denotes the support function of convex body K, namely, for $x \in \mathbb{R}^n$,

$$h_K(x) = \max\{x \cdot y : y \in K\}.$$

When $p = 1$, the L_p addition reduces to the classical Minkowski addition, namely $K + L = \{x + y : x \in K, y \in L\}$.

From the classical Brunn-Minkowski theory of convex bodies to its L_p theory took more than 30 years until Lutwak in his prominent work [34] derived a variational formula related to the L_p addition. That is, for two convex bodies K, L containing the origin in their interiors, the L_p mixed volume of K and L is given by

$$V_p(K, L) = \frac{1}{n}\int_{S^{n-1}} h^p_L(v)h^{1-p}_K(v)dS(K, v),$$

where $S(K, \cdot)$ is the surface area measure of K defined on the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ and is formulated by

$$S(K, \eta) = \mathcal{H}^{n-1}(\nu^{-1}_K(\eta)),$$

with $\eta \subset S^{n-1}$ a Borel set, $\mathcal{H}^{n-1}$ the $(n-1)$ dimensional Hausdorff measure, and ν^{-1}_K the inverse Gauss image. It is often convenient to define the L_p surface area measure of K on S^{n-1} by:

$$dS_p(K, \cdot) = h^{1-p}_K\, dS(K, \cdot).$$

We would like to mention that the L_p surface area measure is one of the crucial ingredients in the rapidly developing L_p Brunn-Minkowski theory for convex bodies. Many important concepts in convex geometry are related to the L_p surface area measure. An example is the L_p Minkowski problem [34] with great impact in convex geometry and many other areas of mathematics, see e.g., [5–7, 12, 13, 26, 36, 45–47, 52–54].

The L_p John ellipsoid [37] is another example based on the L_p surface area measure and is defined as solutions to the following problem.

Problem S_p. *Given a convex body $K \subset \mathbb{R}^n$ that contains the origin in its interior and $0 < p \leq \infty$, find an origin-symmetric ellipsoid E which solves the following constrained extremal problem:*

$$\max_{E \in \mathcal{E}_o^n} V(E) \quad \text{subject to} \quad \overline{V}_p(K, E) \leq 1, \tag{1.2}$$

where $\overline{V}_p(K,E) = (V_p(K,E)/V(K))^{\frac{1}{p}}$ and $\overline{V}_\infty(K,E) = \lim_{p\to\infty} \overline{V}_p(K,E)$.

Lutwak, Yang and Zhang [37] proved that there is one and only one origin-symmetric ellipsoid which solves Problem S_p. The L_p John ellipsoid of convex body K, denoted by E_pK, is just the unique solution of Problem S_p. Moreover, the following characterization result was given in [37]: *E_pK solves Problem S_p if and only if it satisfies*

$$V(K) h^2_{E_p^* K}(v) = \int_{S^{n-1}} |x \cdot v|^2 h^{p-2}_{E_pK}(v) dS_p(K, v) \tag{1.3}$$

for any $x \in \mathbb{R}^n$, where E_p^*K denotes the polar body of E_pK and is defined as

$$E_p^* K = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \quad \textit{for all } y \in E_pK\}.$$

Note that the L_p John ellipsoids unify several fundamental ellipsoids constructed from a given convex body. These ellipsoids include the John ellipsoid for $p = \infty$ with some additional conditions on K, the Petty ellipsoid for $p = 1$ [42], the Lutwak-Yang-Zhang ellipsoid for $p = 2$ [35] (in some sense dual to the Legendre ellipsoid of inertia in classical mechanics [40]). We would like to mention that the inclusion between the LYZ ellipsoid and the Legendre ellipsoid is the geometric analogue of the Cramer-Rao inequality, a fundamental inequality in the information theory. The L_p John ellipsoids have been extended to, for instance, the logarithmic John ellipsoid and Orlicz John ellipsoid in [23, 55].

In view of (1.2), one sees that the volume and the L_p mixed volumes play an essential role. The volume of a convex body is its Lebesgue measure, and hence has many nice properties. In particular, $V(K)$ is invariant under volume-preserving linear transforms (this property is usually called the $\mathrm{SL}(n)$-invariance). This $\mathrm{SL}(n)$-invariance plays essential roles in establishing many properties for the L_p John ellipsoids, especially the argument for characterizing the L_p John ellipsoid (see (1.3)). Although the L_p mixed volume is used in (1.2), what is important for (1.2) is indeed the L_p surface area measure $dS_p(K,\cdot)$. This measure is the central object in the L_p Minkowski problem [34]. A crucial property of the L_p surface area measure used in the theory of the L_p John ellipsoid is that the L_p surface area measure is not concentrated on any closed hemisphere for $p>0$. We would like to mention that this condition for the L_p surface area measure is necessary (and very often is sufficient) for the L_p Minkowski problem.

In this thesis, we aim to investigate the L_p John ellipsoid in its most possible general setting. This requires us to find good substitutions for $V(\cdot)$ and the L_p mixed volume in (1.2). Although one can replace $V(\cdot)$ by any other non-degenerate homogeneous $\mathrm{SL}(n)$-invariant geometric invariants, they differ from each other only by multiplicative constants and power indices. So replacing $V(\cdot)$ by other affine-invariants is meaningless and hence only the L_p surface area measures will be changed. As discussed in the previous paragraph, to replace the L_p surface area measure, one should have some measures which are based on $p\in\mathbb{R}$ and are also not concentrated on any closed hemisphere. Besides the L_p surface area measures, there are also many measures having the above mentioned properties, such as the L_q p-capacitary measures [14, 22, 27, 28, 57] and the L_p dual curvature measure [38]. In the main context of this thesis, we will mainly focus on $S_{\mu,p}(K,\cdot)$, a measure obtained by an α-homegeneous measure μ. This measure is related to the following variational formula (Livshyts [33] for $p=1$ and Wu [49] for $p>1$):

$$V_{\mu,p}(K,L)=\frac{p}{\alpha}\lim_{\varepsilon\to 0^+}\frac{\mu(K+_p\varepsilon\cdot L)-\mu(K)}{\varepsilon}=\frac{1}{\alpha}\int_{S^{n-1}}h_L^p(v)dS_{\mu,p}(K,v),$$

where $dS_{\mu,p}(K,\cdot)$ is the L_p-surface μ-area measure of convex body K. The quantity $V_{\mu,p}(K,L)$ is called the L_p mixed volume of convex bodies K and L with respect to the measure μ. Our L_p John ellipsoids for general measures are then defined as the solution to the following optimization problem.

Problem $M_{\mu,p}$. *Given a convex body K in $\mathbb{R}^n$ that contains the origin in its interior and $p \geq 0$, find an origin-symmetric ellipsoid E which solves the following optimization problem:*

$$\max_{E \in \mathcal{E}_o^n} V(E) \quad \text{subject to} \quad \overline{V}_{\mu,p}(K,E) \leq 1,$$

where $\overline{V}_{\mu,p}(K,E) = (V_{\mu,p}(K,E)/\mu(K))^{\frac{1}{p}}$ *for* $p > 0$,

$$\overline{V}_{\mu,\infty}(K,E) = \lim_{p\to\infty} \overline{V}_{\mu,p}(K,E) = \max\left\{\frac{h_E(u)}{h_K(u)} : u \in \operatorname{supp}(S_{\mu,1}(K,\cdot))\right\},$$

and

$$\overline{V}_{\mu,0}(K,E) = \lim_{p\to 0^+} \overline{V}_{\mu,p}(K,E) = \exp\left\{\frac{1}{\alpha\mu(K)} \int_{S^{n-1}} \log\left(\frac{h_E(u)}{h_K(u)}\right) h_K(u) dS_{\mu,1}(K,u)\right\}.$$

This thesis is organized as follows. In Chapter 3, we prove the existence and uniqueness of the solution to Problem $M_{\mu,p}$; and such a solution, as usual, will be named as the L_p John ellipsoid of K for μ and denoted by $E_{\mu,p}K$. Especially, we call $E_{\mu,0}K$ the logarithmic John ellipsoid of K for μ. We also show that $E_{\mu,p}K$ solves Problem $M_{\mu,p}$ for $p > 0$ if and only if for all $x \in \mathbb{R}^n$,

$$\frac{\alpha}{n}\mu(K) h^2_{E^*_{\mu,p}K}(x) = \int_{S^{n-1}} |x \cdot v|^2 h^{p-2}_{E_{\mu,p}K}(v) dS_{\mu,p}(K,v).$$

Moreover, under some extra conditions, $E_{\mu,0}K$ solves Problem $M_{\mu,0}$ if and only if for all $x \in \mathbb{R}^n$,

$$\frac{\alpha}{n}\mu(K) h^2_{E^*_{\mu,0}K}(x) = \int_{S^{n-1}} \left(\frac{|x \cdot v|}{h_{E_{\mu,0}K}(v)}\right)^2 h_K(v) dS_{\mu,1}(K,v).$$

In Chapter 4, we investigate the properties related to $E_{\mu,p}K$. The main result is the John's inclusion, that is,

$$n^{\frac{1}{2}-\frac{1}{p}} E_{\mu,p}K \subseteq \Gamma_{\mu,-p}K \subseteq E_{\mu,p}K \qquad \text{for } 0 < p \leq 2,$$

$$E_{\mu,p}K \subseteq \Gamma_{\mu,-p}K \subseteq n^{\frac{1}{2}-\frac{1}{p}} E_{\mu,p}K \qquad \text{for } 2 \leq p \leq \infty,$$

where $\Gamma_{\mu,-p}K$ is a star body, see Chapter 4 for its definition.

In Chapter 5, we study the conditions of convex bodies with identical John and L_p John ellipsoids for general measures. Let $\alpha > 0$ and $K \in \mathcal{K}_o^n$. Suppose that $g : \mathbb{R}^n \to \mathbb{R}_+$ is the $(\alpha - n)$-homogeneous density of the α-homogeneous measure μ on $\mathbb{R}^n$ such that g is positive on $\widetilde{\partial K}$. In addition, assume that the measure $h_K dS_{\mu,1}(K, \cdot)$ satisfies the subspace concentration condition and for $0 \le p < \infty$, there exists a $\phi_p \in \mathrm{GL}(n)$ such that $E_{\mu,p}K = \phi_p B^n$ and $\int_{S^{n-1}} v/|\phi_p^t v| dS_{\mu,1}(K, v) = o$. Then the following statements are equivalent:

(1) $E_{\mu,p}K = E_{\mu,\infty}K$;

(2) $E_{\mu,p}K \subseteq K$;

(3) $\phi_p^{-1}K$ is the tangential body of B^n with respect to the measure μ^{ϕ_p} and the measure $S_{\mu^{\phi_p},p}(\phi_p^{-1}K, \cdot)$ is isotropic.

The results in Chapters 3-5 are all related to $S_{\mu,p}(K, \cdot)$. In Chapter 6, we turn to an even more general setting and consider the following optimization problem.

Problem M_p^ν. *Given ν a probability measure which is not concentrated on any great subsphere and $p \ge 0$, find an origin-symmetric ellipsoid E which solves the following optimization problem:*

$$\max_{E \in \mathcal{E}_o^n} V(E) \quad \text{subject to} \quad \|h_E\|_{L_\nu^p} \le 1,$$

where

$$\|h_E\|_{L_\nu^p}^p = \int_{S^{n-1}} h_E^p(u) d\nu(u).$$

We would like to mention that ν is in fact more general than $S_{\mu,p}(K, \cdot)$. Moreover, ν also contains many other measures as its special cases (after normalization), such as the L_q p-capacitary measures [14,22,27,28,57] and the L_p dual curvature measure [38]. Again, we show the existence and uniqueness of the solution to Problem M_p^ν, and hence define the general L_p John ellipsoid for ν. Such an ellipsoid is denoted by E_p^ν. A characterization for E_p^ν is provided as well, that is, E_p^ν is a solution to Problem M_p^ν for $p > 0$ if and only if for all $x \in \mathbb{R}^n$,

$$\frac{h^2_{(E_p^\nu)^*}(x)}{n} = \int_{S^{n-1}} |x \cdot v|^2 h_{E_p^\nu}^{p-2}(v) d\nu(v).$$

Moreover, under extra conditions, E_0^ν solves Problem M_0^ν if and only if for all $x \in \mathbb{R}^n$,

$$\frac{h^2_{(E_0^\nu)^*}(x)}{n} = \int_{S^{n-1}} \left(\frac{|x \cdot v|}{h_{E_0^\nu}(v)} \right)^2 d\nu(v).$$

Chapter 2

Notations and Preliminaries

For quick reference, we collect some background material in this chapter. The books of Gardner [16] and Schneider [44] are recommended for more details.

Throughout this paper, $\mathbb{R}^n$ ($n \geq 2$) denotes the n-dimensional Euclidean space. For $x \in \mathbb{R}^n$, let $|x|$ be the Euclidean norm of x and denote $\bar{x} = x/|x|$ if $x \neq o$. The unit ball of $\mathbb{R}^n$ is denoted by B^n, and the unit sphere by $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. Write ω_n for the volume of B^n. The great subsphere of S^{n-1} is the intersection of S^{n-1} and the $(n-1)$-dimensional hyperplane passing through the origin. By ∂K we mean the boundary of K.

A subset $K \subseteq \mathbb{R}^n$ is convex if for any two points $x, y \in K$, the line segment connecting x and y is contained in K. A convex body is a compact convex set with nonempty interior. Let $\mathcal{K}^n$ be the set of convex bodies in $\mathbb{R}^n$ and $\mathcal{K}^n_o$ the set of convex bodies with origin in their interiors. For $K \in \mathcal{K}^n$, its support function $h_K : \mathbb{R}^n \to \mathbb{R}$ is given by, for $x \in \mathbb{R}^n$,

$$h_K(x) = \max\{x \cdot y : y \in K\},$$

where $x \cdot y$ denotes the inner product of $x, y \in \mathbb{R}^n$. It is easy to verify that the support function is positively homogeneous of degree 1, i.e., for any $c > 0$,

$$h_K(cx) = ch_K(x). \tag{2.1}$$

For $K_i, K \in \mathcal{K}^n$, K_i converges to K in the Hausdorff metric if it satisfies

$$d_H(K_i, K) = \max_{u \in S^{n-1}} |h_{K_i}(u) - h_K(u)| \to 0 \quad \text{as } i \to \infty, \tag{2.2}$$

where $d_H(K_i, K)$ is the Hausdorff distance between K_i and K.

For each $K \in \mathcal{K}_o^n$, there exists a unique convex body, called the polar body of K, which will be defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}.$$

A subeset $L \subseteq \mathbb{R}^n$ is said to be star-shaped with respect to the origin if the intersection of L and each ray starting from the origin is a line segment. Its radial function $\rho_L : \mathbb{R}^n \backslash \{o\} \to \mathbb{R}$ is given by,

$$\rho_L(x) = \max\{\lambda \geq 0 : \lambda x \in L\}.$$

The set L is called a star body if the radial function restricted on S^{n-1} is positive and continuous. Obviously, every convex body with the origin in its interior is a star body. It can be verified in [44] that for $K \in \mathcal{K}_o^n$ and $u \in S^{n-1}$,

$$h_{K^*}(u) = \rho_K^{-1}(u) \quad \text{and} \quad \rho_{K^*}(u) = h_K^{-1}(u). \tag{2.3}$$

Denote by $\mathrm{GL}(n)$ the group of all invertible linear transforms on $\mathbb{R}^n$ and $\mathrm{SL}(n)$ the subgroup of $\mathrm{GL}(n)$ with $\det \phi = 1$, where $\det \phi$ means the determinant of $\phi \in \mathrm{GL}(n)$. Denote by ϕ^t the transpose of ϕ and ϕ^{-1} the inverse of ϕ. For $K \subset \mathbb{R}^n$, let $\phi K = \{\phi x : x \in K\}$ be the image of K under ϕ. It is easy to see that for $x \in \mathbb{R}^n$ and $\phi \in \mathrm{GL}(n)$,

$$h_{\phi K}(x) = h_K(\phi^t x). \tag{2.4}$$

In particular, for $c > 0$ and $x \in \mathbb{R}^n$,

$$h_{cK}(x) = c h_K(x),$$

where $cK = \{cx : x \in K\}$. For $K \in \mathcal{K}_o^n$, it is obvious that for $x \in \mathbb{R}^n \backslash \{o\}$ and $\phi \in \mathrm{GL}(n)$,

$$\rho_{\phi K}(x) = \rho_K(\phi^{-1} x).$$

We say K is origin-symmetric if for $x \in K$, then $-x \in K$. An origin-symmetric ellipsoid is ϕB^n for some $\phi \in \mathrm{GL}(n)$. Denote by $\mathcal{E}_o^n$ the set of origin-symmetric ellipsoids in $\mathbb{R}^n$.

Let $\nu_K : \partial K \to S^{n-1}$ be the Gauss map of K, that is, $\nu_K(x)$ is a unit outer normal vector of K at $x \in \partial K$. Let $\widetilde{\partial K} \subseteq \partial K$ be the set of boundary points which have a unique outer normal vector. By ν_K^{-1} we mean the inverse Gauss map which maps the unit vectors to the boundary points of K. For any Borel set $\eta \subset S^{n-1}$, the surface area measure $S(K,\cdot)$ of $K \in \mathcal{K}^n$ defined on S^{n-1} is formulated by

$$S(K,\eta) = \mathcal{H}^{n-1}(\nu_K^{-1}(\eta)), \tag{2.5}$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$ dimensional Hausdorff measure. Clearly,

$$V(K) = \frac{1}{n}\int_{S^{n-1}} h_K(u)dS(K,u).$$

Moreover, for $c > 0$ and $\phi \in \mathrm{GL}(n)$,

$$V(cK) = c^n V(K) \text{ and } V(\phi K) = |\det(\phi)|V(K).$$

For $p \geq 1$, two constants $a, b \geq 0$, and for two convex bodies $K, L \in \mathcal{K}_o^n$, the L_p addition of K and L, denoted by $a \cdot K +_p b \cdot L$, is defined by

$$h^p_{a\cdot K +_p b\cdot L}(x) = a h^p_K(x) + b h^p_L(x), \tag{2.6}$$

for all $x \in \mathbb{R}^n$. In particular, one has

$$h^p_{a\cdot K}(\cdot) = a h^p_K(\cdot),$$

together with (2.6),

$$a \cdot K +_p a \cdot L = a \cdot (K +_p L). \tag{2.7}$$

The L_p mixed volume $V_p(K,L)$ of K, L is defined by

$$\begin{aligned} V_p(K,L) &= \frac{p}{n}\lim_{\varepsilon\to 0^+}\frac{V(K +_p \varepsilon\cdot L) - V(K)}{\varepsilon} \\ &= \frac{1}{n}\int_{S^{n-1}} h^p_L(v)dS_p(K,v), \end{aligned}$$

where $dS_p(K,\cdot)$ is the L_p surface area measure of K on S^{n-1} and denoted by

$$dS_p(K,\cdot) = h_K^{1-p}dS(K,\cdot).$$

Note that

$$V_p(K,K) = V(K).$$

Let μ be an α-homogeneous measure on $\mathbb{R}^n$ for some constant $\alpha > 0$, that is,

$$\mu(cA) = c^{\alpha}\mu(A), \tag{2.8}$$

for any $c > 0$ and measurable set $A \subset \mathbb{R}^n$. Let $p \geq 1$ and $K \in \mathcal{K}_o^n$. Assume that in addition μ has the density function $g : \mathbb{R}^n \to \mathbb{R}$ which is nonnegative and continuous on the support of μ, i.e. $d\mu(x) = g(x)dx$. (Note that g is $(\alpha - n)$ homogeneous, namely, $g(cx) = c^{\alpha-n}g(x)$ for all $c > 0$ and $x \in \mathbb{R}^n$). The L_p-surface μ-area measure $S_{\mu,p}(K,\cdot)$ introduced by Livshyts [33] ($p = 1$) and Wu [49] ($p > 1$) is defined as: for every Borel set $\eta \subset S^{n-1}$,

$$S_{\mu,p}(K,\eta) = \int_{\nu_K^{-1}(\eta)\bigcap\widetilde{\partial K}} h_K^{1-p}(\nu_K(x))g(x)d\mathcal{H}^{n-1}(x). \tag{2.9}$$

In particular, $dS_{\mu,p}(K,\cdot) = h_K^{1-p}dS_{\mu,1}(K,\cdot)$ and

$$S_{\mu,p}(cK,\cdot) = c^{\alpha-n}S_{\mu,p}(K,\cdot) \tag{2.10}$$

holds for all $c > 0$.

It has been proved by Livshyts [33] for $p = 1$ and Wu [49] for $p > 1$ that, for $K, L \in \mathcal{K}_o^n$,

$$\begin{aligned} V_{\mu,p}(K,L) &= \frac{p}{\alpha}\lim_{\varepsilon\to 0^+}\frac{\mu(K +_p \varepsilon\cdot L) - \mu(K)}{\varepsilon} \\ &= \frac{1}{\alpha}\int_{S^{n-1}} h_L^p(v)\, dS_{\mu,p}(K,v). \end{aligned} \tag{2.11}$$

Although $V_{\mu,p}(K,L)$ in (2.11) is defined for $p \geq 1$, we showed in the Appendix that it can be extended to all $p > 0$ as follows:

$$V_{\mu,p}(K,L) = \frac{1}{\alpha}\int_{S^{n-1}} h_E^p(v)\, dS_{\mu,p}(K,v). \tag{2.12}$$

We would like to mention that an Orlicz analogue can be found in [41].

By (2.7) and (2.8), it follows that for all $p > 0$,

$$\begin{aligned} V_{\mu,p}(K,K) &= \frac{p}{\alpha} \lim_{\varepsilon\to 0^+} \frac{\mu(K +_p \varepsilon \cdot K) - \mu(K)}{\varepsilon} \\ &= \frac{p}{\alpha} \lim_{\varepsilon\to 0^+} \frac{\mu((1+\varepsilon)^{\frac{1}{p}} K) - \mu(K)}{\varepsilon} \\ &= \frac{p}{\alpha}\mu(K) \lim_{\varepsilon\to 0^+} \frac{(1+\varepsilon)^{\frac{\alpha}{p}} - 1}{\varepsilon} \\ &= \mu(K). \end{aligned} \tag{2.13}$$

Clearly, (2.11), (2.12) and (2.13) imply that for any $p > 0$,

$$\frac{1}{\alpha} \int_{S^{n-1}} h_K^p(v)\, dS_{\mu,p}(K,v) = V_{\mu,p}(K,K) = \mu(K). \tag{2.14}$$

It follows from (2.12) that for $p > 0$ and $c > 0$,

$$V_{\mu,p}(K, cL) = c^p V_{\mu,p}(K,L). \tag{2.15}$$

By (2.10) and (2.12), one gets for $p > 0$ and $c > 0$,

$$V_{\mu,p}(cK, L) = c^{\alpha-p} V_{\mu,p}(K,L). \tag{2.16}$$

Let $p > 0$, μ be an α-homogeneous measure on $\mathbb{R}^n$, and $K, L \in \mathcal{K}_o^n$. Define $\overline{V}_{\mu,p}(K,L)$ by

$$\begin{aligned} \overline{V}_{\mu,p}(K,L) &= \left[\frac{V_{\mu,p}(K,L)}{\mu(K)}\right]^{\frac{1}{p}} \\ &= \left[\frac{1}{\alpha\mu(K)} \int_{S^{n-1}} \left(\frac{h_L(u)}{h_K(u)}\right)^p h_K(u) dS_{\mu,1}(K,u)\right]^{\frac{1}{p}}. \end{aligned} \tag{2.17}$$

It follows from (2.14) that $h_K dS_{\mu,1}(K,\cdot)/\alpha\mu(K)$ is a probability measure on $\mathrm{supp}(S_{\mu,1}(K,\cdot))$, where $\mathrm{supp}(\omega)$ denotes the support of a measure ω. When $p = \infty$, we define

$$\overline{V}_{\mu,\infty}(K,L) = \lim_{p\to\infty} \overline{V}_{\mu,p}(K,L) = \max\left\{\frac{h_L(u)}{h_K(u)} : u \in \mathrm{supp}(S_{\mu,1}(K,\cdot))\right\}. \tag{2.18}$$

When $p = 0$, let

$$\begin{aligned}\overline{V}_{\mu,0}(K,L) &= \lim_{p\to 0^+} \overline{V}_{\mu,p}(K,L)\\ &= \exp\left\{\frac{1}{\alpha\mu(K)}\int_{S^{n-1}} \log\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u)dS_{\mu,1}(K,u)\right\}.\end{aligned} \tag{2.19}$$

By (2.15), (2.16), (2.17), (2.18) and (2.19), one gets for real $p \geq 0$,

$$\overline{V}_{\mu,p}(cK,L) = \frac{1}{c}\overline{V}_{\mu,p}(K,L), \quad \text{for } c > 0,$$

and

$$\overline{V}_{\mu,p}(K,cL) = c\overline{V}_{\mu,p}(K,L), \quad \text{for } c > 0. \tag{2.20}$$

We say that a finite Borel measure ν on S^{n-1} satisfies the subspace concentration condition if the following assertions hold:

(i) for every subspace Ω of $\mathbb{R}^n$, such that $1 < \dim(\Omega) < n$,

$$\nu(\Omega \cap S^{n-1}) \leq \frac{\dim(\Omega)}{n}\nu(S^{n-1});$$

(ii) if there is a subspace Ω such that

$$\nu(\Omega \cap S^{n-1}) = \frac{\dim(\Omega)}{n}\nu(S^{n-1}),$$

then there exists a subspace Ω', which is complementary to Ω in $\mathbb{R}^n$, so that also

$$\nu(\Omega' \cap S^{n-1}) = \frac{\dim(\Omega')}{n}\nu(S^{n-1}).$$

Chapter 3

The L_p John ellipsoids for general measures

In this chapter, we investigate the existence and the uniqueness of the L_p John ellipsoids for general measures. We also provide a characterization result for the L_p John ellipsoids for general measures.

Throughout this chapter, for $\alpha > 0$, let μ be an α-homogeneous measure on $\mathbb{R}^n$ such that its density function $g : \mathbb{R}^n \to \mathbb{R}$ is nonnegative and continuous on the support of μ, i.e., $d\mu(x) = g(x)dx$.

The L_p John ellipsoid for general measures is related to the following optimization problem.

Problem $M_{\mu,p}$**.** *Given a convex body $K \in \mathcal{K}_o^n$ and $p \geq 0$, find an origin-symmetric ellipsoid E which solves the following optimization problem:*

$$\max_{E \in \mathcal{E}_o^n} V(E) \quad \text{subject to} \quad \overline{V}_{\mu,p}(K, E) \leq 1. \tag{3.1}$$

Another closely related problem is stated as follows.

Problem $\overline{M}_{\mu,p}$**.** *Given a convex body $K \in \mathcal{K}_o^n$ and $p \geq 0$, find an origin-symmetric ellipsoid E which solves the following optimization problem:*

$$\min_{E \in \mathcal{E}_o^n} \overline{V}_{\mu,p}(K, E) \quad \text{subject to} \quad V(E) \geq \omega_n. \tag{3.2}$$

We would like to mention that to find a solution to (3.1) is equivalent to solve the following problem

$$\max_{E\in\mathcal{E}_o^n} V(E) \text{ subject to } \overline{V}_{\mu,p}(K,E)=1.$$

This is obvious because, if E_0 solves (3.1) but $\overline{V}_{\mu,p}(K,E_0)<1$, one can let

$$E=\frac{E_0}{\overline{V}_{\mu,p}(K,E_0)}\supsetneq E_0$$

and hence $V(E)>V(E_0)$. By (2.20), one has $\overline{V}_{\mu,p}(K,E)=1$ and this leads to a contradiction with the maximality of $V(E_0)$. Similarly, finding a solution to (3.2) is equivalent to solving the following problem

$$\min_{E\in\mathcal{E}_o^n} \overline{V}_{\mu,p}(K,E) \text{ subject to } V(E)=\omega_n.$$

For convenience, let

$$\operatorname{vrad}(L)=\left(\frac{V(L)}{\omega_n}\right)^{\frac{1}{n}}. \tag{3.3}$$

Clearly,

$$\operatorname{vrad}(cL)=c\cdot\operatorname{vrad}(L)\quad\text{for } c>0. \tag{3.4}$$

With the help of $\operatorname{vrad}(\cdot)$, (3.1) can be reformulated as:

$$\max_{E\in\mathcal{E}_o^n} \operatorname{vrad}(E)\quad\text{subject to}\quad \overline{V}_{\mu,p}(K,E)=1. \tag{3.5}$$

Similarly, (3.2) can be reformulated as:

$$\min_{E\in\mathcal{E}_o^n} \overline{V}_{\mu,p}(K,E)\quad\text{subject to}\quad \operatorname{vrad}(E)=1. \tag{3.6}$$

The following lemma shows that the solutions to Problems $M_{\mu,p}$ and $\overline{M}_{\mu,p}$ only differ by a scalar factor.

Lemma 3.1. *Let $p\geq 0$, $\alpha>0$ and $K\in\mathcal{K}_o^n$. Suppose that μ is an α-homogeneous measure on $\mathbb{R}^n$. If E_M is a solution to Problem $M_{\mu,p}$, then $E_M/\operatorname{vrad}(E_M)$ is a solution to Problem $\overline{M}_{\mu,p}$. Conversely, if E_m is a solution to Problem $\overline{M}_{\mu,p}$, then $\overline{V}_{\mu,p}(K,E_m)^{-1}E_m$ is a solution to Problem $M_{\mu,p}$.*

Proof. Suppose that E_M is a solution to Problem $M_{\mu,p}$. For any origin-symmetric ellipsoid E, from (2.20), we have, for all $p \geq 0$,

$$\overline{V}_{\mu,p}(K, \overline{V}_{\mu,p}(K,E)^{-1}E) = 1.$$

Since E_M is a solution to Problem $M_{\mu,p}$, it follows from (3.4) and (3.5) that

$$\mathrm{vrad}(E_M) \geq \mathrm{vrad}(\overline{V}_{\mu,p}(K,E)^{-1}E) = \mathrm{vrad}(E)/\overline{V}_{\mu,p}(K,E). \tag{3.7}$$

From (2.20), (3.7), the fact that $\overline{V}_{\mu,p}(K,E_M) \leq 1$, we have, for any $E \in \mathcal{E}_o^n$ such that $\mathrm{vrad}(E) \geq 1$,

$$\begin{aligned}\overline{V}_{\mu,p}\left(K, \frac{E_M}{\mathrm{vrad}(E_M)}\right) &= \frac{\overline{V}_{\mu,p}\left(K, E_M\right)}{\mathrm{vrad}(E_M)} \\ &\leq \frac{1}{\mathrm{vrad}(E_M)} \\ &\leq \frac{\overline{V}_{\mu,p}(K,E)}{\mathrm{vrad}(E)} \\ &\leq \overline{V}_{\mu,p}\left(K,E\right).\end{aligned}$$

On the other hand, it follows from (3.4) that $\mathrm{vrad}(E_M/\mathrm{vrad}(E_M)) = 1$. Thus, the ellipsoid $E_M/\mathrm{vrad}(E_M)$ is a solution to Problem $\overline{M}_{\mu,p}$.

Conversely, suppose that E_m is a solution to Problem $\overline{M}_{\mu,p}$. Let E be any origin-symmetric ellipsoid, it follows from (3.4) that

$$\mathrm{vrad}(E/\mathrm{vrad}(E)) = 1.$$

Since E_m is a solution to Problem $\overline{M}_{\mu,p}$, it follows from (2.20) and (3.2) that, for all $p \geq 0$,

$$\overline{V}_{\mu,p}(K,E_m) \leq \overline{V}_{\mu,p}(K, E/\mathrm{vrad}(E)) = \overline{V}_{\mu,p}(K,E)/\mathrm{vrad}(E). \tag{3.8}$$

From (3.4), (3.8), the facts that $\mathrm{vrad}(E_m) \geq 1$ and $\overline{V}_{\mu,p}(K,E) \leq 1$, we have, for any $E \in \mathcal{E}_o^n$ such that $\overline{V}_{\mu,p}(K,E) \leq 1$,

$$\begin{aligned}\mathrm{vrad}(\overline{V}_{\mu,p}(K,E_m)^{-1}E_m) &= \frac{\mathrm{vrad}(E_m)}{\overline{V}_{\mu,p}\left(K,E_m\right)} \\ &\geq \frac{1}{\overline{V}_{\mu,p}\left(K,E_m\right)}\end{aligned}$$

$$\geq \frac{\text{vrad}(E)}{\overline{V}_{\mu,p}(K,E)}$$
$$\geq \text{vrad}(E).$$

On the other hand, it follows from (2.20) that $\overline{V}_{\mu,p}(\overline{V}_{\mu,p}(K,E_m)^{-1}E_m)) = 1$. Thus, the ellipsoid $\overline{V}_{\mu,p}(K,E_m)^{-1}E_m$ is a solution to Problem $M_{\mu,p}$. □

Lemma 3.2. *Let $K, L \in \mathcal{K}_o^n$. Suppose that g, the density of the measure μ, is positive on $\widetilde{\partial K}$. Then $\overline{V}_{\mu,\infty}(K,L) \leq 1$ implies that $L \subseteq K$.*

Proof. Note that $L \subseteq K$ means $h_L(u) \leq h_K(u)$ for $u \in \text{supp}(S(K,\cdot))$. By (2.9), for $p = 1$ and any Borel set $\eta \subseteq S^{n-1}$,

$$S_{\mu,1}(K,\eta) = \int_{\nu_K^{-1}(\eta)\bigcap\widetilde{\partial K}} g(x)d\mathcal{H}^{n-1}(x).$$

Let $\eta_0 \subset S^{n-1}$ be such that $S(K,\eta_0) = 0$. Note that g is continuous in supp$\,(g)$, and hence continuous on the compact set supp$\,(g) \cap K$. Together with the hypothesis that g is positive on $\widetilde{\partial K}$, we have that g has a finite upper bound on $\widetilde{\partial K}$, say $M < \infty$. By (2.5),

$$\begin{aligned} S_{\mu,1}(K,\eta_0) &= \int_{\nu_K^{-1}(\eta_0)\bigcap\widetilde{\partial K}} g(x)d\mathcal{H}^{n-1}(x) \\ &\leq M \int_{\nu_K^{-1}(\eta_0)\bigcap\widetilde{\partial K}} d\mathcal{H}^{n-1}(x) = M \cdot S(K,\eta_0) = 0. \end{aligned}$$

On the other hand, for $j \in \mathbb{N}$, the set of all natural numbers, one can check that

$$S_{\mu,1}(K,\eta) = \int_{\nu_K^{-1}(\eta)\bigcap\widetilde{\partial K}} g(x)d\mathcal{H}^{n-1}(x) \geq \frac{1}{j}\int_{\{x\in\nu_K^{-1}(\eta)\bigcap\widetilde{\partial K}: g(x)\geq\frac{1}{j}\}} d\mathcal{H}^{n-1}(x).$$

So if $\eta_1 \subset S^{n-1}$ is such that $S_{\mu,1}(K,\eta_1) = 0$, then

$$0 = \int_{\{x\in\nu_K^{-1}(\eta_1)\bigcap\widetilde{\partial K}: g(x)\geq\frac{1}{j}\}} d\mathcal{H}^{n-1}(x)$$

holds for all $j \in \mathbb{N}$. The monotone convergence theorem together with the fact that

$\{x \in \nu_K^{-1}(\eta_1) \bigcap \widetilde{\partial K} : g(x) \geq \frac{1}{j}\}$ increases to $\nu_K^{-1}(\eta_1) \bigcap \widetilde{\partial K}$ yields that

$$S(K, \eta_1) = \lim_{j \to \infty} \int_{\{x \in \nu_K^{-1}(\eta_1) \bigcap \widetilde{\partial K} : g(x) \geq \frac{1}{j}\}} d\mathcal{H}^{n-1}(x) = 0.$$

Hence, $\text{supp}(S_{\mu,1}(K,\cdot))$ and $\text{supp}\,(S(K,\cdot))$ are identical. Therefore, from (2.18), $\overline{V}_{\mu,\infty}(K, L)$ can be rewritten as

$$\overline{V}_{\mu,\infty}(K, L) = \max\left\{\frac{h_L(u)}{h_K(u)} : u \in \text{supp}\,(S(K,\cdot))\right\}.$$

This proves the desired result. □

It follows from Lemma 3.2 that if g, the density of measure μ, is positive on $\nu_K^{-1}(\text{supp}\,(S(K,\cdot)))$, solving Problem $M_{\mu,\infty}$ actually requires to find an origin-symmetric ellipsoid contained in a given convex body with maximal volume. Therefore, the solution to Problem $M_{\mu,\infty}$ is exactly the John ellipsoid JK; i.e., the unique ellipsoid contained in K with maximal volume. Consequently, the solutions to Problem $M_{\mu,p}$ may be called the L_p John ellipsoids for general measures. In fact, for each $p \geq 0$, this ellipsoid is unique under some additional conditions, which will be stated later. Particularly, the new L_p John ellipsoid for the general measure μ reduces to the L_p John ellipsoid introduced by Lutwak, Yang and Zhang [37] when μ is the Lebesgue measure on $\mathbb{R}^n$.

Before we show the existence of solutions to Problem $M_{\mu,p}$, we provide the following lemmas, which are crucial in our proof. More details can be found in [44, Theorem 1.8.7] and [8, Lemma 5.1 and Theorem 5.4], respectively.

Lemma 3.3. (Blaschke selection theorem) *Every bounded sequence of convex bodies has a subsequence that converges to a compact convex set.*

Lemma 3.4. *Let ϑ be a finite measure on S^{n-1} which satisfies the subspace concentration condition. Then there exists a solution to the following problem:*

$$\inf_{T \in \text{SL}(n)} \left\{\int_{S^{n-1}} \log|Tu| d\vartheta(u)\right\}.$$

Now we investigate the existence of solutions to Problem $M_{\mu,p}$ for $p > 0$. For a function g, let $\text{supp}(g)$ mean the support of g. By $\mathbb{R}_+$, we mean the set of nonnegative real numbers.

Lemma 3.5. *Let $0 < p \leq \infty$, $\alpha > 0$ and $K \in \mathcal{K}_o^n$. Suppose that $g : \mathbb{R}^n \to \mathbb{R}_+$ is the $(\alpha - n)$-homogeneous density of the α-homogeneous measure μ on $\mathbb{R}^n$ such that $\nu_K(\mathrm{supp}\,(g) \cap \widetilde{\partial K})$ is not contained in any great subsphere of S^{n-1}. Then there exists a solution to Problem $M_{\mu,p}$.*

Proof. For $0 < p < \infty$. Let $r = \alpha - n$. Since g is $(\alpha - n)$-homogeneous, μ is α-homogeneous with $\alpha = n + r$. As $\nu_K(\mathrm{supp}\,(g) \cap \widetilde{\partial K})$ is not contained in any great subsphere of S^{n-1}, it follows from (2.9) that $\mathrm{supp}\,(S_{\mu,p}(K, \cdot))$ is not contained in any great subsphere of S^{n-1}. Indeed, by (2.9), $u \in \mathrm{supp}\,(S_{\mu,p}(K, \cdot))$ means that $\nu_K^{-1}(u) \in \widetilde{\partial K}$ and $g(\nu_K^{-1}(u)) > 0$, which is equivalent to $\nu_K^{-1}(u) \in \mathrm{supp}\,(g) \cap \widetilde{\partial K}$ and $u \in \nu_K(\mathrm{supp}\,(g) \cap \widetilde{\partial K})$. Hence, $\mathrm{supp}\,(S_{\mu,p}(K, \cdot)) = \nu_K(\mathrm{supp}\,(g) \cap \widetilde{\partial K})$. So, the hypothesis that $\nu_K(\mathrm{supp}\,(g) \cap \widetilde{\partial K})$ is not contained in any great subsphere of S^{n-1} is equivalent to that $\mathrm{supp}\,(S_{\mu,p}(K, \cdot))$ is not contained in any great subsphere of S^{n-1}. Therefore, there exists a constant $c_1 > 0$ such that

$$\min_{v \in S^{n-1}} \frac{1}{\alpha} \int_{S^{n-1}} |v \cdot u|^p dS_{\mu,p}(K, u) > c_1, \tag{3.9}$$

as the integral in this inequality is positive and continuous on $v \in S^{n-1}$.

From (3.5), we shall assume that $\{E_j\}_{j=1}^{\infty}$ is the maximizing sequence of origin-symmetric ellipsoids to Problem $M_{\mu,p}$ such that $\overline{V}_{\mu,p}(K, E_j) = 1$ and

$$\lim_{j \to \infty} V(E_j) = \max_{E \in \mathcal{E}_o^n} \left\{ V(E) : \overline{V}_{\mu,p}(K, E) = 1 \right\}. \tag{3.10}$$

Next, we show that the sequence of E_j is bounded. For every E_j, there exists a $v_j \in S^{n-1}$ such that

$$\frac{1}{2}\mathrm{diam}(E_j)|v_j \cdot u| \leq h_{E_j}(u), \tag{3.11}$$

where $\mathrm{diam}(E_j)$ is the diameter of E_j. It follows from (3.10) that

$$V_{\mu,p}(K, E_j) = \mu(K). \tag{3.12}$$

Combining (2.12), (3.11) and (3.12) , we have that for $j \in \mathbb{N}$,

$$\left[\frac{1}{2}\mathrm{diam}(E_j)\right]^p \min_{v \in S^{n-1}} \frac{1}{\alpha} \int_{S^{n-1}} |v_j \cdot u|^p dS_{\mu,p}(K, u) \leq V_{\mu,p}(K, E_j) = \mu(K).$$

Together with (3.9), we obtain that the maximizing sequence $\{E_j\}_{j=1}^{\infty}$ of ellipsoids to Problem $M_{\mu,p}$ is bounded. By Lemma 3.3, there exists a subsequence of this maximazing sequence of ellipsoids, which will be denoted by $\{E_i\}$, with $\lim_{i\to\infty} E_i = E_0$. Let $c' = \overline{V}_{\mu,p}(K,B^n)^{-1}$, and then $\overline{V}_{\mu,p}(K,c'B^n) = 1$ by (2.20). Therefore, it follows from (3.10) that

$$V(E_0) \geq V(c'B^n) > 0,$$

which implies $E_0 \in \mathcal{K}^n_o$. In particular, $E_0 \in \mathcal{E}^n_o$ since it is the limit of the ellipsoids.

Now we show that $V_{\mu,p}(K,\cdot)$ is continuous. Since $\lim_{i\to\infty} E_i = E_0$, it follows from (2.2) that h_{E_i} converges uniformly to h_{E_0} as $i \to \infty$. Together with (2.12), one has

$$\begin{aligned}\lim_{i\to\infty} V_{\mu,p}(K,E_i) &= \lim_{i\to\infty} \frac{1}{\alpha}\int_{S^{n-1}} h^p_{E_i}(u)dS_{\mu,p}(K,u)\\ &= \frac{1}{\alpha}\int_{S^{n-1}} \lim_{i\to\infty} h^p_{E_i}(u)dS_{\mu,p}(K,u)\\ &= \frac{1}{\alpha}\int_{S^{n-1}} h^p_{E_0}(u)dS_{\mu,p}(K,u)\\ &= V_{\mu,p}(K,E_0).\end{aligned}$$

Consequently, $\lim_{i\to\infty} \overline{V}_{\mu,p}(K,E_i) = \overline{V}_{\mu,p}(K,E_0) = 1$ by (2.17). Hence E_0 is a solution to Problem $M_{\mu,p}$ for $0 < p < \infty$.

Now let $p = \infty$. Similar to the proof for the case $0 < p < \infty$, let $\{E_j\}_{j=1}^\infty$ be the maximizing sequence of origin-symmetric ellipsoids to Problem $M_{\mu,\infty}$ such that $\overline{V}_{\mu,\infty}(K,E_j) = 1$ and

$$\lim_{j\to\infty} V(E_j) = \max_{E\in\mathcal{E}^n_o} \left\{V(E) : \overline{V}_{\mu,\infty}(K,E) = 1\right\}.$$

Now we show that the sequence of ellipsoids E_j is bounded. Let $\{v_j\}_{j=1}^\infty$ be the sequence of unit vectors obeying (3.11). Without loss of generality, due to the compactness of S^{n-1}, let $v_j \to v_0 \in S^{n-1}$. Since $\operatorname{supp}(S_{\mu,1}(K,\cdot))$ is not contained in any great subsphere of S^{n-1}, there exists $u_0 \in \operatorname{supp}(S_{\mu,1}(K,\cdot))$ such that $|u_0 \cdot v_0| > 0$. Hence there exist $j_0 \in \mathbb{N}$ and $c_2 > 0$ such that $|v_j \cdot u_0| \geq c_2$ for $j \geq j_0$. It follows from (3.11) that for all $j \geq j_0$,

$$\frac{c_2}{2}\operatorname{diam}(E_j) \leq \frac{1}{2}\operatorname{diam}(E_j)|v_j \cdot u_0| \leq h_{E_j}(u_0) = h_K(u_0) \leq C.$$

Since each E_j is a compact set, there exists $C' > 0$ such that $E_j \subset C'B^n$. Therefore, the maximizing sequence $\{E_j\}_{j=1}^\infty$ of ellipsoids to Problem $M_{\mu,\infty}$ is bounded. By

Lemma 3.3, there exists a convergent subsequence, which will be denoted by $\{E_i\}$, such that $\lim_{i\to\infty} E_i = E_0$. Again, it is easily checked that $E_0 \in \mathcal{K}_o^n$ and hence E_0 is an origin-symmetric ellipsoid with $V(E_0) = \lim_{j\to\infty} V(E_j)$. Note that

$$\left|\overline{V}_{\mu,\infty}(K,E_i) - \overline{V}_{\mu,\infty}(K,E_0)\right| = \left|\max_{u\in\text{supp}(S_{\mu,1}(K,\cdot))} \frac{h_{E_i}(u)}{h_K(u)} - \max_{u\in\text{supp}(S_{\mu,1}(K,\cdot))} \frac{h_{E_0}(u)}{h_K(u)}\right|$$
$$\leq \max_{u\in\text{supp}(S_{\mu,1}(K,\cdot))} \left|\frac{h_{E_i}(u) - h_{E_0}(u)}{h_K(u)}\right|.$$

Therefore $\overline{V}_{\mu,\infty}(K,\cdot)$ is continuous on E_i. In particular, $\overline{V}_{\mu,\infty}(K,E_0) = 1$. This completes the proof of the existence of solutions to Problem $M_{\mu,\infty}$. □

Next we show the existence of solutions to Problem $M_{\mu,0}$.

Lemma 3.6. *Suppose that the measure $h_K dS_{\mu,1}(K,\cdot)$ satisfies the subspace concentration condition. Then there exists a solution to Problem $M_{\mu,0}$.*

Proof. Recall that the subspace concentration condition is presented in Chapter 2. It follows from Lemma 3.1 that the solutions to Problems $M_{\mu,0}$ and $\overline{M}_{\mu,0}$ only differ by a scalar factor. We show the existence of solutions to Problem $M_{\mu,0}$ with the aid of Problem $\overline{M}_{\mu,0}$. By (3.6), we will search for an origin-symmetric ellipsoid E_0 to solve the following problem

$$\min_{E\in\mathcal{E}_o^n} \overline{V}_{\mu,0}(K,E) \quad \text{subject to } \; V(E) = V(B^n).$$

Since V is $\mathrm{SL}(n)$-invariant, there exists $T \in \mathrm{SL}(n)$ such that $E = T^t B^n$. Thus, $h_E(u) = h_{T^tB^n}(u) = |Tu|$ for $T \in \mathrm{SL}(n)$. According to (2.19), $\overline{V}_{\mu,0}(K,E)$ can be replaced by (without changing the solutions)

$$\int_{S^{n-1}} \log\left(\frac{h_E(u)}{h_K(u)}\right) h_K(u)dS_{\mu,1}(K,u) = \int_{S^{n-1}} \log(h_E(u))h_K(u)dS_{\mu,1}(K,u)$$
$$- \int_{S^{n-1}} \log(h_K(u))h_K(u)dS_{\mu,1}(K,u). \tag{3.13}$$

Since the second term of the right hand side of (3.13) is a constant, it suffices to consider $\int_{S^{n-1}} \log(h_E(u))h_K(u)dS_{\mu,1}(K,u)$. By the hypothesis that the finite measure $h_K(u)dS_{\mu,1}(K,u)$ satisfies the subspace concentration condition, Lemma 3.4 implies the existence of solutions to Problem $\overline{M}_{\mu,0}$. □

It follows from Lemma 3.1 that there exist solutions to Problem $\overline{M}_{\mu,p}$ for $0 \leq p \leq \infty$. Before we provide a characterization of the L_p John ellipsoid for the general measure μ, we establish the following theorem, in which the uniqueness of solutions to Problem $\overline{M}_{\mu,p}$ for $0 \leq p < \infty$ will also be discussed.

Theorem 3.7. *Let $p > 0$, $\alpha > 0$ and $K \in \mathcal{K}_o^n$. Suppose that $g : \mathbb{R}^n \to \mathbb{R}_+$ is the $(\alpha - n)$-homogeneous density of the α-homogeneous measure μ on $\mathbb{R}^n$ such that $\nu_K(\operatorname{supp}(g) \cap \widetilde{\partial K})$ is not contained in any great subsphere of S^{n-1}. Then Problem $\overline{M}_{\mu,p}$ has a unique solution. Moreover, an origin-symmetric ellipsoid E solves Problem $\overline{M}_{\mu,p}$ if and only if the following holds: for all $x \in \mathbb{R}^n$,*

$$\frac{\alpha}{n} V_{\mu,p}(K, E)\, h_{E^*}^2(x) = \int_{S^{n-1}} |x \cdot u|^2 h_E^{p-2}(u) dS_{\mu,p}(K, u). \tag{3.14}$$

In addition, if the measure $h_K dS_{\mu,1}(K, \cdot)$ satisfies the subspace concentration condition, Problem $\overline{M}_{\mu,0}$ has a unique solution. Moreover, an origin-symmetric ellipsoid E solves Problem $\overline{M}_{\mu,0}$ if and only if the following holds: for all $x \in \mathbb{R}^n$,

$$\frac{h_{E^*}^2(x)}{n} \int_{S^{n-1}} h_K(u) dS_{\mu,1}(K, u) = \int_{S^{n-1}} \left(\frac{|x \cdot u|}{h_E(u)}\right)^2 h_K(u) dS_{\mu,1}(K, u). \tag{3.15}$$

Proof. Let $r = \alpha - n$. Since g is r-homogeneous, μ is α-homogeneous with $\alpha = n + r$. As $\nu_K(\operatorname{supp}(g) \cap \widetilde{\partial K})$ is not contained in any great subsphere of S^{n-1}, it follows from (2.9) that $\operatorname{supp}(S_{\mu,p}(K, \cdot))$ is not contained in any great subsphere of S^{n-1}. Therefore, the existence of solutions for Problem $M_{\mu,p}$ is guaranteed by Lemma 3.5. The existence of solutions for Problem $M_{\mu,0}$ is guaranteed by Lemma 3.6, if in addition, the measure $h_K dS_{\mu,1}(K, \cdot)$ satisfies the subspace concentration condition. Thus, Lemma 3.1 yields the existence of solutions for Problems $\overline{M}_{\mu,p}$ and $\overline{M}_{\mu,0}$.

First, we show that if the origin-symmetric ellipsoid E is a solution to Problem $\overline{M}_{\mu,p}$, then (3.14) holds for all $x \in \mathbb{R}^n$. To this end, take $\hat{\phi} \in \mathrm{GL}(n)$ such that for $y, z \in \mathbb{R}^n$, $\hat{\phi} = z \otimes z$ is the rank 1 linear operator which maps y to $(y \cdot z)z$. Choose $\varepsilon_{\hat{\phi}} > 0$ sufficiently small so that $I_n + \varepsilon\hat{\phi}$ is invertible for all $\varepsilon \in (-\varepsilon_{\hat{\phi}}, \varepsilon_{\hat{\phi}})$. Let

$$T_\varepsilon = \frac{I_n + \varepsilon\hat{\phi}}{\det(I_n + \varepsilon\hat{\phi})^{1/n}} \in \mathrm{SL}(n). \tag{3.16}$$

Assume $E = \phi^t B^n$ for some $\phi \in \mathrm{GL}(n)$. Since V is $\mathrm{SL}(n)$-invariant and E solves Problem $\overline{M}_{\mu,p}$, from (3.3) and (3.6), we have $\omega_n = V(\phi^t B^n) = V(\phi^t T_\varepsilon^t B^n)$. Moreover, for all $\varepsilon \in (-\varepsilon_{\hat{\phi}}, \varepsilon_{\hat{\phi}})$,

$$\overline{V}_{\mu,p}(K, \phi^t B^n) \le \overline{V}_{\mu,p}(K, \phi^t T_\varepsilon^t B^n).$$

By (2.17), it can be equivalently written as

$$V_{\mu,p}(K, \phi^t B^n) \le V_{\mu,p}(K, \phi^t T_\varepsilon^t B^n).$$

Note that $h_{\phi^t T_\varepsilon^t B^n}(x) = h_{(T_\varepsilon \phi)^t B^n}(x) = h_{B^n}(T_\varepsilon \phi x) = |T_\varepsilon \phi x|$. Therefore, by (2.12), we have

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_{S^{n-1}} |T_\varepsilon \phi u|^p \, dS_{\mu,p}(K,u) = 0.$$

That is,

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_{S^{n-1}} \frac{(\phi u \cdot \phi u + 2\varepsilon \phi u \cdot \hat{\phi}\, \phi u + \varepsilon^2 \hat{\phi}\, \phi u \cdot \hat{\phi}\, \phi u)^{\frac{p}{2}}}{\det(I_n + \varepsilon \hat{\phi})^{\frac{p}{n}}} dS_{\mu,p}(K,u) = 0. \tag{3.17}$$

Note that

$$\phi u \cdot \hat{\phi}\, \phi u = |z \cdot \phi u|^2 \quad \text{and} \quad \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \det(I_n + \varepsilon \hat{\phi}) = |z|^2, \tag{3.18}$$

for $z \in \mathbb{R}^n$. Since $|T_\varepsilon \phi u|$ is smooth, it implies that the integrand depends smoothly on sufficiently small ε. Thus, (3.17) yields that

$$\int_{S^{n-1}} (\phi u \cdot \hat{\phi}\, \phi u) |\phi u|^{p-2} dS_{\mu,p}(K,u) = \frac{|z|^2}{n} \int_{S^{n-1}} |\phi u|^p dS_{\mu,p}(K,u). \tag{3.19}$$

For $E = \phi^t B^n$, it follows that for any $y \in \mathbb{R}^n$,

$$h_E(y) = |\phi y| \quad \text{and} \quad h_{E^*}(y) = h_{\phi^{-1} B^n}(y) = |\phi^{-t} y|. \tag{3.20}$$

Thus, from (2.12), (3.18), (3.19), (3.20) and $x = \phi^t z$, we have

$$\int_{S^{n-1}} |x \cdot u|^2 h_E^{p-2}(u) dS_{\mu,p}(K,u) = \frac{\alpha h_{E^*}^2(x)}{n} V_{\mu,p}(K,E),$$

which is the desired result (3.14).

Conversely, suppose that (3.14) holds for an origin-symmetric ellipsoid E. Let $E = \phi^t B^n$ for some $\phi \in \mathrm{GL}(n)$. Due to (2.17) and (3.2), in order to show that E solves Problem $\overline{M}_{\mu,p}$, it is enough to show that if E' is an origin-symmetric ellipsoid such that $V(E') = V(E)$, then

$$\overline{V}_{\mu,p}(K, E') \geq \overline{V}_{\mu,p}(K, E),$$

which can be equivalently expressed as

$$V_{\mu,p}(K, E') \geq V_{\mu,p}(K, E), \tag{3.21}$$

with equality if and only if $E' = E$.

Indeed, one can let $E' = \phi^t T^t B^n$ for some $T \in \mathrm{SL}(n)$ as $V(E') = V(E) = V(\phi^t B^n) = V(\phi^t T^t B^n)$ due to the SL(n)-invariance of V. From (2.12) and (3.20), we have

$$\begin{aligned} V_{\mu,p}(K, E') &= \frac{1}{\alpha} \int_{S^{n-1}} h^p_{E'}(u) dS_{\mu,p}(K, u) \\ &= \frac{1}{\alpha} \int_{S^{n-1}} h^p_{\phi^t T^t B^n}(u) dS_{\mu,p}(K, u) \\ &= \frac{1}{\alpha} \int_{S^{n-1}} |T\phi u|^p dS_{\mu,p}(K, u) \\ &= \frac{1}{\alpha} \int_{S^{n-1}} |T\,\overline{\phi u}|^p |\phi u|^p dS_{\mu,p}(K, u), \end{aligned}$$

where $\overline{\phi u} = \phi u/|\phi u|$. Here $\phi u \neq 0$ for all $u \in S^{n-1}$, otherwise it implies $u = 0$. Note that for $T \in \mathrm{SL}(n)$, it can be represented by $T = PDQ$, where P and Q are orthogonal matrices and $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is diagonal and positive definite.

Thus, from inequality (3.21), we shall prove

$$\left[\frac{1}{\alpha V_{\mu,p}(K, E)} \int_{S^{n-1}} |DQ\,\overline{\phi u}|^p |\phi u|^p dS_{\mu,p}(K, u)\right]^{\frac{1}{p}} \geq 1, \tag{3.22}$$

with equality if and only if $D = I_n$. Note that

$$V_{\mu,p}(K, E) = \frac{1}{\alpha} \int_{S^{n-1}} |\phi u|^p dS_{\mu,p}(K, u).$$

Thus, using the Jensen's inequality, we have

$$\left[\frac{1}{\alpha V_{\mu,p}(K, E)} \int_{S^{n-1}} |DQ\,\overline{\phi u}|^p |\phi u|^p dS_{\mu,p}(K, u)\right]^{\frac{1}{p}}$$

$$\geq \exp\left[\frac{1}{\alpha V_{\mu,p}(K,E)}\int_{S^{n-1}} \log|DQ\,\overline{\phi u}||\phi u|^p dS_{\mu,p}(K,u)\right], \tag{3.23}$$

with equality if and only if there exists $c > 0$ such that $|DQ\,\overline{\phi u}| = c$ for all $u \in \operatorname{supp}(S_{\mu,p}(K,\cdot))$. Hence, to establish (3.22), we only need to prove that

$$\int_{S^{n-1}} \log|DQ\,\overline{\phi u}||\phi u|^p dS_{\mu,p}(K,u) \geq 0. \tag{3.24}$$

Taking $x = \phi^t Q^t e_i$ in (3.14), where $\{e_1, \cdots, e_n\}$ is the canonical basis of $\mathbb{R}^n$, together with (3.20) and the assumption that $E = \phi^t B^n$, we immediately get

$$\int_{S^{n-1}} |e_i \cdot Q\,\overline{\phi u}|^2 |\phi u|^p dS_{\mu,p}(K,u) = \frac{\alpha}{n} V_{\mu,p}(K,E). \tag{3.25}$$

Let $v = Q\,\overline{\phi u}$ and $v_i = e_i \cdot Q\,\overline{\phi u}$. Together with the fact that D is diagonal, the concavity of the log function, and (3.25) with $v_i = e_i \cdot Q\,\overline{\phi u}$, we can see

$$\begin{aligned}
\int_{S^{n-1}} \log|DQ\,\overline{\phi u}||\phi u|^p dS_{\mu,p}(K,u) &= \int_{S^{n-1}} \log|Dv||\phi u|^p dS_{\mu,p}(K,u)\\
&= \frac{1}{2}\int_{S^{n-1}} \log(\lambda_1^2 v_1^2 + \cdots + \lambda_n^2 v_n^2)|\phi u|^p dS_{\mu,p}(K,u)\\
&\geq \frac{1}{2}\int_{S^{n-1}} (v_1^2 \log\lambda_1^2 + \cdots + v_n^2 \log\lambda_n^2)|\phi u|^p dS_{\mu,p}(K,u)\\
&= \frac{\alpha}{n} V_{\mu,p}(K,E)\log(\lambda_1 \cdots \lambda_n) = 0,
\end{aligned} \tag{3.26}$$

which gives (3.24) as desired.

Now let us characterize the equality of (3.22). The strict concavity of the log function implies that the equality in (3.26) holds only if $v_{i_1} \cdots v_{i_N} \neq 0$ then $\lambda_{i_1} = \ldots = \lambda_{i_N}$ for all $u \in \operatorname{supp}(S_{\mu,p}(K,\cdot))$. Therefore, $|Dv| = \lambda_i$ when $v_i \neq 0$ for $u \in \operatorname{supp}(S_{\mu,p}(K,\cdot))$. Furthermore, the equality of (3.23) yields that $|Dv| = |DQ\,\overline{\phi u}| = c$ for all $u \in \operatorname{supp}(S_{\mu,p}(K,\cdot))$. Since $\operatorname{supp}(S_{\mu,p}(K,\cdot))$ is not contained in any great subsphere of S^{n-1}, it follows that $v = Q\,\overline{\phi u}$ is not contained in any great subsphere of S^{n-1} for all $u \in \operatorname{supp}(S_{\mu,p}(K,\cdot))$. Consequently, we have $\lambda_i = c$ for all $i = 1, \ldots, n$. However, the fact that $\lambda_1 \cdots \lambda_n = 1$ forces $D = I_n$, which is exactly the equality condition of (3.22). This also shows the uniqueness of solution to Pronblem $\overline{M}_{\mu,p}$ for $p > 0$.

The assertion for Problem $\overline{M}_{\mu,0}$ follows from a similar approach to the proof when

$p > 0$. For one direcrtion, define T_ε as in (3.16). Suppsoe $E = \phi^t B^n$ that solves Problem $\overline{M}_{\mu,0}$. It follows from the SL(n)-invariance of V and (3.2) that for all $\varepsilon \in (-\varepsilon_{\hat{\phi}}, \varepsilon_{\hat{\phi}})$,

$$\overline{V}_{\mu,0}(K, \phi^t B^n) \le \overline{V}_{\mu,0}(K, \phi^t T_\varepsilon^t B^n).$$

From (2.19), we have

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_{S^{n-1}} 2\log|T_\varepsilon \phi u|\, h_K(u) dS_{\mu,1}(K,u) = 0.$$

That is, for $\hat{\phi} = z \otimes z$,

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_{S^{n-1}} \log\left(\frac{\phi u \cdot \phi u + 2\varepsilon \phi u \cdot \hat{\phi}\phi u + \varepsilon^2 \hat{\phi}\phi u \cdot \hat{\phi}\phi u}{\det(I_n + \varepsilon\hat{\phi})^{\frac{2}{n}}}\right) h_K(u) dS_{\mu,1}(K,u) = 0.$$

which is equivalent to

$$\int_{S^{n-1}} (\phi u \cdot \hat{\phi}\phi u)|\phi u|^{-2} h_K(u) dS_{\mu,1}(K,u) = \frac{|z|^2}{n}\int_{S^{n-1}} h_K(u) dS_{\mu,1}(K,u).$$

Together with (3.18) and $x = \phi^t z$, the desired result (3.15) follows. For the other direction, it is already included in Problem $\overline{M}_{\mu,p}$ for $p > 0$, see (3.23) and (3.24). Therefore, we also obtain that there exists the unique solution to Problem $\overline{M}_{\mu,0}$. □

Combining Lemma 3.1 and Theorem 3.7, we can get the main result of this chapter, the characterization of the L_p John ellipsoid for μ.

Theorem 3.8. *Let $p > 0$, $\alpha > 0$ and $K \in \mathcal{K}_o^n$. Suppose that $g : \mathbb{R}^n \to \mathbb{R}_+$ is the $(\alpha - n)$-homogeneous density of the α-homogeneous measure μ on $\mathbb{R}^n$ such that $\nu_K(\operatorname{supp}(g) \cap \widetilde{\partial K})$ is not contained in any great subsphere of S^{n-1}. Then Problem $M_{\mu,p}$ has a unique solution. Moreover, an origin-symmetric ellipsoid E solves Problem $M_{\mu,p}$ if and only if the following holds: for all $x \in \mathbb{R}^n$,*

$$\frac{\alpha}{n}\mu(K) h_{E^*}^2(x) = \int_{S^{n-1}} |x \cdot v|^2 h_E^{p-2}(v) dS_{\mu,p}(K,v). \tag{3.27}$$

In addition, if the measure $h_K dS_{\mu,1}(K,\cdot)$ satisfies the subspace cocentration condition, Problem $M_{\mu,0}$ has a unique solution. Moreover, an ellipsoid E solves Problem $M_{\mu,0}$ if *and only if the following holds: for all $x \in \mathbb{R}^n$,*

$$\frac{\alpha}{n}\mu(K)h^2_{E^*}(x) = \int_{S^{n-1}} \left(\frac{|x \cdot v|}{h_E(v)}\right)^2 h_K(v)dS_{\mu,1}(K,v). \tag{3.28}$$

Theorems 3.7 and 3.8 are equivalent. Indeed, from Lemma 3.1, if E_m solves Problem $\overline{M}_{\mu,p}$ for $p \geq 0$, $\overline{V}_{\mu,p}(K,E_m)^{-1}E_m$ is a solution to Problem $M_{\mu,p}$. Together with Theorem 3.8, we get Theorem 3.7. We have studied the uniqueness of solutions to Problem $M_{\mu,p}$ for $p \geq 0$ in Theorem 3.8, now we turn to consider the case for $p = \infty$.

Theorem 3.9. *Let $\alpha > 0$ and $K \in \mathcal{K}^n_o$. Suppose that $g : \mathbb{R}^n \to \mathbb{R}_+$ is the $(\alpha - n)$-homogeneous density of the α-homogeneous measure μ on $\mathbb{R}^n$ such that $\nu_K(\mathrm{supp}\,(g) \cap \widetilde{\partial K})$ is not contained in any great subsphere of S^{n-1}. Then there exists a unique solution to Problem $M_{\mu,\infty}$.*

Proof. We prove it by contradiction. Suppose that there exist two different origin-symmetric ellipsoids E_1 and E_2 solving Problem $M_{\mu,\infty}$. From (3.5), we have $V(E_1) = V(E_2)$, $\overline{V}_{\mu,\infty}(K,E_1) = 1$ and $\overline{V}_{\mu,\infty}(K,E_2) = 1$.

As $E_1 \neq E_2$ and $V(E_1) = V(E_2)$, the Brunn-Minkowski inequality and its equality condition [44, Theorem 7.1.1] imply that

$$V\left(\frac{E_1 + E_2}{2}\right)^{\frac{1}{n}} > \frac{V(E_1)^{\frac{1}{n}} + V(E_2)^{\frac{1}{n}}}{2}. \tag{3.29}$$

Let $E_3 = (E_1 + E_2)/2$. One has $V(E_3) > V(E_1) = V(E_2)$.

It follows from (2.6) and (2.18) that

$$\begin{aligned}
\overline{V}_{\mu,\infty}(K,E_3) &= \max\left\{\frac{h_{E_3}(u)}{h_K(u)} : u \in \mathrm{supp}(S_{\mu,1}(K,\cdot))\right\} \\
&= \max\left\{\frac{h_{E_1}(u) + h_{E_2}(u)}{2h_K(u)} : u \in \mathrm{supp}(S_{\mu,1}(K,\cdot))\right\} \\
&\leq \frac{1}{2}\max\left\{\frac{h_{E_3}(u)}{h_K(u)} : u \in \mathrm{supp}(S_{\mu,1}(K,\cdot))\right\} \\
&\quad + \frac{1}{2}\max\left\{\frac{h_{E_3}(u)}{h_K(u)} : u \in \mathrm{supp}(S_{\mu,1}(K,\cdot))\right\} \\
&= \frac{1}{2}\overline{V}_{\mu,\infty}(K,E_1) + \frac{1}{2}\overline{V}_{\mu,\infty}(K,E_1) = 1.
\end{aligned} \tag{3.30}$$

This further implies that $V(E_3) \leq V(E_1) = V(E_2)$ (as E_1 and E_2 solve (3.1)), which leads to a contradiction to (3.29). Therefore, there exists a unique solution to Problem $M_{\mu,\infty}$. □

Motivated by Theorems 3.7, 3.8 and 3.9, we can propose the definition of the L_p John ellipsoid for general measure as follows.

Definition 3.10. Let $K \in \mathcal{K}_o^n$ and $0 \leq p \leq \infty$. Suppose that $g : \mathbb{R}^n \to \mathbb{R}_+$ is the $(\alpha - n)$-homogeneous density of the α-homogeneous measure μ on $\mathbb{R}^n$ such that $\nu_K(\mathrm{supp}\,(g) \cap \widetilde{\partial K})$ is not contained in any great subsphere of S^{n-1} for $0 < p \leq \infty$ and the measure $h_K dS_{\mu,1}(K, \cdot)$ satisfies the subspace concentration condition for $p = 0$. Define $E_{\mu,p}K$, the L_p John ellipsoid of K for the general measure μ, to be the unique solution of the following optimization problem:

$$\max_{E \in \mathcal{E}_o^n} V(E) \quad \text{subject to} \quad \overline{V}_{\mu,p}(K, E) = 1.$$

Similarily, define $\overline{E}_{\mu,p}K$, the normalized L_p John ellipsoid of K for the general measure μ, to be the unique solution of the following optimization problem:

$$\min_{E \in \mathcal{E}_o^n} \overline{V}_{\mu,p}(K, E) \quad \text{subject to} \qquad V(E) = \omega_n.$$

Chapter 4

Properties of $E_{\mu,p}K$

A basic property of the John ellipsoid is the following John's inclusion [30], which states that if K is an origin-symmetric convex body in $\mathbb{R}^n$, then

$$JK \subseteq K \subseteq \sqrt{n}JK, \tag{4.1}$$

where JK is the John ellipsoid of K. Later, the John's inclusion has been extended to the L_p setting due to Lutwak, Yang and Zhang [37]. We will establish the L_p John's inclusion for our new L_p John ellipsoids. Some other inequalities related to $E_{\mu,p}K$ will be presented as well.

For $\alpha > 0$, let μ be an α-homogeneous measure on $\mathbb{R}^n$ which has a continuous density g with respect to dx, i.e., $d\mu(x) = g(x)dx$. For $\phi \in \mathrm{GL}(n)$, let μ^ϕ be the measure which has a density $g \circ \phi$ with respect to dx. It can be easily verified that for $K \in \mathcal{K}_o^n$ and $\phi \in \mathrm{GL}(n)$,

$$\mu(\phi K) = |\det \phi| \mu^\phi(K). \tag{4.2}$$

Before presenting the L_p John's inclusion for $E_{\mu,p}K$, some preparations are needed.

Definition 4.1. Suppose $p > 0$ and let ϑ be a Borel measure on S^{n-1}. Given $\phi \in \mathrm{GL}(n)$, define $\phi_{p^\dashv}\vartheta$, the L_p image of ϑ under ϕ, to be a Borel measure such that

$$\int_{S^{n-1}} f(u) d\phi_{p^\dashv}\vartheta(u) = \int_{S^{n-1}} \left|\phi^{-1}u\right|^p f\left(\overline{\phi^{-1}u}\right) d\vartheta(u), \tag{4.3}$$

for each Borel $f : S^{n-1} \to \mathbb{R}$.

The following lemma (see e.g., [38, Lemma 2.1]), will be used in the proof of Propsition 4.3.

Lemma 4.2. *For each real number q, the set*

$$\left\{ch_K^{\bar{q}} - ch_{B^n}^{\bar{q}} : K \in \mathcal{K}_o^n, c > 0, u \in S^{n-1}\right\}$$

is dense in $C(S^{n-1})$, the set of continuous functions on S^{n-1}, where

$$t^{\bar{q}} = \begin{cases} \frac{1}{q}t^q & \text{for} \quad q \neq 0 \\ \log t & \text{for} \quad q = 0. \end{cases}$$

We now prove the following result.

Proposition 4.3. *Suppose $\alpha > 0$. Let μ be an α-homogeneous measure on $\mathbb{R}^n$ which has a continuous density g with respect to dx. Then, for $0 < p < \infty$, $K \in \mathcal{K}_o^n$ and $\phi \in \mathrm{GL}(n)$,*

$$S_{\mu,p}(\phi K, u) = |\det \phi| \phi^t_{p^{-1}} S_{\mu^\phi,p}(K, u). \tag{4.4}$$

Proof. Let $K, L \in \mathcal{K}_o^n$ and $\phi \in \mathrm{GL}(n)$. It follows from (2.4) and (2.6) for $p \geq 1$ that

$$\begin{aligned} h^p_{\phi K +_p \varepsilon \cdot \phi L}(x) &= h^p_{\phi K}(x) + \varepsilon h^p_{\phi L}(x) \\ &= h^p_K(\phi^t x) + \varepsilon h^p_L(\phi^t x) \\ &= h^p_{K +_p \varepsilon \cdot L}(\phi^t x) \\ &= h^p_{\phi(K +_p \varepsilon \cdot L)}(x). \end{aligned}$$

Therefore, $\phi K +_p \varepsilon \cdot \phi L = \phi(K +_p \varepsilon \cdot L)$ for $p \geq 1$. Moreover, by (2.11) and (4.2), one has, for $p \geq 1$,

$$\begin{aligned} V_{\mu,p}(\phi K, \phi L) &= \frac{p}{\alpha} \lim_{\varepsilon \to 0^+} \frac{\mu(\phi K +_p \varepsilon \cdot \phi L) - \mu(\phi K)}{\varepsilon} \\ &= \frac{p}{\alpha} \lim_{\varepsilon \to 0^+} \frac{\mu(\phi(K +_p \varepsilon \cdot L)) - \mu(\phi K)}{\varepsilon} \\ &= |\det \phi| \frac{p}{\alpha} \lim_{\varepsilon \to 0^+} \frac{\mu^\phi(K +_p \varepsilon \cdot L) - \mu^\phi(K)}{\varepsilon} \\ &= |\det \phi| V_{\mu^\phi,p}(K, L). \end{aligned} \tag{4.5}$$

Indeed, (4.5) holds for all $0 < p < \infty$. To see this, by (2.4), (2.9) and (2.12),

$$
\begin{aligned}
V_{\mu,p}(\phi K, \phi L) &= \frac{1}{\alpha} \int_{S^{n-1}} \left(\frac{h_{\phi L}(v)}{h_{\phi K}(v)} \right)^p g(\nu_{\phi K}^{-1}(v)) h_{\phi K}(v)\, dS(\phi K, v) \\
&= \frac{|\det \phi|}{\alpha} \int_{S^{n-1}} \left(\frac{h_L(\overline{\phi^t v})}{h_K(\overline{\phi^t v})} \right)^p h_K(\overline{\phi^t v})(g \circ \phi)(\nu_K^{-1}(\overline{\phi^t v}))\, dS(K, \overline{\phi^t v}) \\
&= |\det \phi| V_{\mu^\phi, p}(K, L), \qquad (4.6)
\end{aligned}
$$

where we have used $\nu_{\phi K}^{-1}(v) = \phi \nu_K^{-1}(\overline{\phi^t v})$.

By (2.12) and (4.6), one gets

$$
\int_{S^{n-1}} h_{\phi L}^p(u) dS_{\mu,p}(\phi K, u) = |\det \phi| \int_{S^{n-1}} h_L^p(u) dS_{\mu^\phi, p}(K, u),
$$

or equivalently, by replacing ϕL with L (hence L with $\phi^{-1} L$),

$$
\int_{S^{n-1}} h_L^p(u) dS_{\mu,p}(\phi K, u) = |\det \phi| \int_{S^{n-1}} h_{\phi^{-1} L}^p(u) dS_{\mu^\phi, p}(K, u).
$$

Then, it follows from (2.1), (2.4) and (4.3) that

$$
\begin{aligned}
|\det \phi| \int_{S^{n-1}} h_L^p(u) d\phi^t_{p^{-1}} S_{\mu^\phi, p}(K, u) &= |\det \phi| \int_{S^{n-1}} \left| \phi^{-t} u \right|^p h_L^p \left(\overline{\phi^{-t} u} \right) dS_{\mu^\phi, p}(K, u) \\
&= |\det \phi| \int_{S^{n-1}} h_L^p \left(\phi^{-t} u \right) dS_{\mu^\phi, p}(K, u) \\
&= |\det \phi| \int_{S^{n-1}} h_{\phi^{-1} L}^p(u) dS_{\mu^\phi, p}(K, u) \\
&= \int_{S^{n-1}} h_L^p(u) dS_{\mu,p}(\phi K, u), \qquad (4.7)
\end{aligned}
$$

where ϕ^{-t} is the inverse of ϕ^t. The desired formula (4.4) is a direct consequence of Lemma 4.2 and (4.7). □

Lemma 4.4. *Suppose $\alpha > 0$. Let μ be an α-homogeneous measure on $\mathbb{R}^n$ which has a continuous density g with respect to dx. Then, for $0 \le p \le \infty$, $K, L \in \mathcal{K}_o^n$ and $\phi \in \mathrm{GL}(n)$, one has,*

$$
\overline{V}_{\mu,p}(\phi K, \phi L) = \overline{V}_{\mu^\phi, p}(K, L).
$$

Proof. Suppose $K, L \in \mathcal{K}_o^n$. Let $\phi \in \mathrm{GL}(n)$. For $0 < p < \infty$, it follows from (2.17), (4.2) and (4.6) that

$$\overline{V}_{\mu,p}(\phi K,\phi L)=\left[\frac{V_{\mu,p}(\phi K,\phi L)}{\mu(\phi K)}\right]^{\frac{1}{p}}=\left[\frac{|\det\phi|\,V_{\mu^\phi,p}(K,L)}{|\det\phi|\,\mu^\phi(K)}\right]^{\frac{1}{p}}=\overline{V}_{\mu^\phi,p}(K,L). \tag{4.8}$$

Let $\phi\in\mathrm{GL}(n)$. For $p=0$ and $p=\infty$, it follows from (2.18), (2.19) and (4.8) that

$$\overline{V}_{\mu,0}(\phi K,\phi L)=\lim_{p\to0^+}\overline{V}_{\mu,p}(\phi K,\phi L)=\lim_{p\to0^+}\overline{V}_{\mu^\phi,p}(K,L)=\overline{V}_{\mu^\phi,0}(K,L),$$

$$\overline{V}_{\mu,\infty}(\phi K,\phi L)=\lim_{p\to\infty}\overline{V}_{\mu,p}(\phi K,\phi L)=\lim_{p\to\infty}\overline{V}_{\mu^\phi,p}(K,L)=\overline{V}_{\mu^\phi,\infty}(K,L).$$

This completes the proof. □

Next, we establish the following fundamental property for the newly introduced L_p John ellipsoid for general measures.

Theorem 4.5. *Let $\alpha>0$ and $K\in\mathcal{K}_o^n$. Suppose that $g:\mathbb{R}^n\to\mathbb{R}_+$ is the $(\alpha-n)$-homogeneous density of the α-homogeneous measure μ on $\mathbb{R}^n$ such that $\nu_K(\mathrm{supp}\,(g)\cap\widetilde{\partial K})$ is not contained in any great subsphere of S^{n-1} for $0<p\le\infty$, and $h_K dS_{\mu,1}(K,\cdot)$ satisfies the subspace concentration condition for $p=0$. Then for $\phi\in\mathrm{GL}(n)$ and $0\le p\le\infty$,*

$$E_{\mu,p}\phi K=\phi E_{\mu^\phi,p}K. \tag{4.9}$$

Proof. Let $K,L\in\mathcal{K}_o^n$. It follows from (3.6) and Lemma 4.4 that for $0\le p\le\infty$, $\phi\in\mathrm{GL}(n)$,

$$1=\overline{V}_{\mu,p}(\phi K,E_{\mu,p}\phi K)=\overline{V}_{\mu^\phi,p}(K,\phi^{-1}E_{\mu,p}\phi K).$$

By (3.3) and (3.5), we have

$$V(\phi^{-1}E_{\mu,p}\phi K)\le V(E_{\mu^\phi,p}K). \tag{4.10}$$

By (3.6) and Lemma 4.4, we also have for $0\le p\le\infty$, $\phi\in\mathrm{GL}(n)$,

$$1=\overline{V}_{\mu^\phi,p}(K,E_{\mu^\phi,p}K)=\overline{V}_{\mu,p}(\phi K,\phi E_{\mu^\phi,p}K). \tag{4.11}$$

Then, it follows from (3.3) and (3.5) that

$$V(\phi E_{\mu^\phi,p}K)\le V(E_{\mu,p}\phi K). \tag{4.12}$$

Therefore, (4.10), together with (4.12), implies that

$$|\det\phi|V(E_{\mu^\phi,p}K) = V(E_{\mu,p}\phi K).$$

Moreover,

$$V(\phi E_{\mu^\phi,p}K) = V(E_{\mu,p}\phi K).$$

By (4.11) and the uniqueness of the L_p John ellipsoid for the general measure μ, the desired result (4.9) follows. □

Next we will define a star body regarding the convex body $K \in \mathcal{K}_o^n$ and the general measure μ. This star body plays an essential role in establishing the new L_p John's inclusion.

Definition 4.6. Let $K \in \mathcal{K}_o^n$ and $p > 0$, define $\Gamma_{\mu,-p}K$ to be a star body whose radial function is given by, for $x \in \mathbb{R}^n \setminus \{o\}$,

$$\rho_{\Gamma_{\mu,-p}K}(x)^{-p} = \frac{n}{\alpha\mu(K)} \int_{S^{n-1}} |x \cdot v|^p \, dS_{\mu,p}(K,v). \tag{4.13}$$

For $p = \infty$, let

$$\Gamma_{\mu,-\infty}K = \lim_{p\to\infty} \Gamma_{\mu,-p}K. \tag{4.14}$$

It follows from (4.13) that

$$\begin{aligned} \rho_{\Gamma_{\mu,-\infty}K}(x)^{-1} &= \lim_{p\to\infty} \rho_{\Gamma_{\mu,-p}K}(x)^{-1} \\ &= \lim_{p\to\infty} n^{1/p}\Big[\frac{1}{\alpha\mu(K)} \int_{S^{n-1}} \left(\frac{|x\cdot v|}{h_K(v)}\right)^p h_K(v) dS_{\mu,1}(K,v)\Big]^{1/p} \\ &= \max\left\{\frac{|x\cdot v|}{h_K(v)} : v \in \operatorname{supp}(S_{\mu,1}(K,\cdot))\right\}. \end{aligned} \tag{4.15}$$

Clearly, we have

$$E_{\mu,2}K = \Gamma_{\mu,-2}K.$$

Indeed, it follows from (2.3) and (3.27) that

$$h^2_{E^*_{\mu,2}K}(x) = \frac{n}{\alpha\mu(K)} \int_{S^{n-1}} |x\cdot v|^2 \, dS_{\mu,2}(K,v) = \rho_{\Gamma_{\mu,-2}K}(x)^{-2} = h^2_{\Gamma^*_{\mu,-2}K}(x).$$

Thus, we have $E^*_{\mu,2}K = \Gamma^*_{\mu,-2}K$, which further implies $E_{\mu,2}K = \Gamma_{\mu,-2}K$.

Lemma 4.7. *Suppose $\alpha > 0$. Let μ be an α-homogeneous on $\mathbb{R}^n$ which has a continuous $(\alpha - n)$-homogeneous density g with respect to dx. Then, for $0 < p \le \infty$, $K \in \mathcal{K}^n_o$ and $\phi \in \mathrm{GL}(n)$,*

$$\Gamma_{\mu,-p}\phi K = \phi\Gamma_{\mu^\phi,-p}K.$$

Proof. From (4.2), (4.4), (4.13) and Proposition 4.3, it follows that for $\phi \in \mathrm{GL}(n)$ and $u \in S^{n-1}$,

$$\begin{aligned}
\rho_{\Gamma_{\mu,-p}\phi K}(u)^{-p} &= \frac{n}{\alpha\mu(\phi K)}\int_{S^{n-1}} |u\cdot v|^p \, dS_{\mu,p}(\phi K, v)\\
&= \frac{n}{\alpha\mu^\phi(K)}\int_{S^{n-1}} |u\cdot v|^p \, d\phi^t_{p^\dashv}S_{\mu^\phi,p}(K, v)\\
&= \frac{n}{\alpha\mu^\phi(K)}\int_{S^{n-1}} \left|\phi^{-t}v\right|^p \left|u\cdot \overline{\phi^{-t}v}\right|^p dS_{\mu^\phi,p}(K, v)\\
&= \frac{n}{\alpha\mu^\phi(K)}\int_{S^{n-1}} \left|\phi^{-1}u\cdot v\right|^p dS_{\mu^\phi,p}(K, v)\\
&= \rho_{\Gamma_{\mu^\phi,-p}K}(\phi^{-1}u)^{-p}.
\end{aligned}$$

The result for $p = \infty$ follows from (4.14). □

Lemma 4.8. *Let $\alpha > 0$ and $K \in \mathcal{K}^n_o$. Suppose that $g : \mathbb{R}^n \to \mathbb{R}_+$ is the $(\alpha - n)$-homogeneous density of the α-homogeneous measure μ on $\mathbb{R}^n$ such that $\nu_K(\mathrm{supp}\,(g) \cap \widetilde{\partial K})$ is not contained in any great subsphere of S^{n-1}. Then there exists a $\phi_p \in \mathrm{GL}(n)$ such that*

$$\begin{aligned}
\Gamma_{\mu^{\phi_p},-p}\phi_p^{-1}K &\subseteq E_{\mu^{\phi_p},p}\phi_p^{-1}K \qquad \text{for} \quad 0 < p \le 2,\\
E_{\mu^{\phi_p},p}\phi_p^{-1}K &\subseteq \Gamma_{\mu^{\phi_p},-p}\phi_p^{-1}K \qquad \text{for} \quad 2 \le p \le \infty.
\end{aligned}$$

Proof. For $p > 0$, $E_{\mu,p}K$ is an origin-symmetric ellipsoid. There exists a $\phi_p \in \mathrm{GL}(n)$ such that $E_{\mu,p}K = \phi_p B^n$. Together with Theorem 4.5, one has

$$E_{\mu^{\phi_p},p}\phi_p^{-1}K = \phi_p^{-1}E_{\mu,p}K = B^n. \tag{4.16}$$

Applying Theorem 3.8 with $E_{\mu^{\phi_p},p}\phi_p^{-1}K = B^n$ and $x = u \in S^{n-1}$, one gets

$$1 = \frac{n}{\alpha\mu^{\phi_p}(\phi_p^{-1}K)}\int_{S^{n-1}} |u\cdot v|^2 dS_{\mu^{\phi_p},p}(\phi_p^{-1}K, v). \tag{4.17}$$

For $0 < p \le 2$, from (4.13), (4.16) and (4.17), we have, for $u \in S^{n-1}$,

$$\begin{aligned}\rho_{\Gamma_{\mu^{\phi_p},-p}\phi_p^{-1}K}(u)^{-p} &= \frac{n}{\alpha\mu^{\phi_p}(\phi_p^{-1}K)}\int_{S^{n-1}}|u\cdot v|^p\,dS_{\mu^{\phi_p},p}(\phi_p^{-1}K,v)\\ &\ge \frac{n}{\alpha\mu^{\phi_p}(\phi_p^{-1}K)}\int_{S^{n-1}}|u\cdot v|^2\,dS_{\mu^{\phi_p},p}(\phi_p^{-1}K,v)\\ &= 1 = \rho_{E_{\mu^{\phi_p},p}\phi_p^{-1}K}(u)^{-p}.\end{aligned}$$

Therefore, $\rho_{\Gamma_{\mu^{\phi_p},-p}\phi_p^{-1}K}(u) \le \rho_{E_{\mu^{\phi_p},p}\phi_p^{-1}K}(u)$ for $u \in S^{n-1}$. That is,

$$\Gamma_{\mu^{\phi_p},-p}\phi_p^{-1}K \subseteq E_{\mu^{\phi_p},p}\phi_p^{-1}K,$$

for $0 < p \le 2$.

For $2 \le p < \infty$, the above inequaluty will be reversed. Therefore

$$E_{\mu^{\phi_p},p}\phi_p^{-1}K \subseteq \Gamma_{\mu^{\phi_p},-p}\phi_p^{-1}K,$$

for $2 \le p < \infty$. Due to (4.14), the desired inclusion also holds for $p = \infty$. □

Theorem 4.9. *Let $\alpha > 0$ and $K \in \mathcal{K}_o^n$. Suppose that $g : \mathbb{R}^n \to \mathbb{R}_+$ is the $(\alpha - n)$-homogeneous density of the α-homogeneous measure μ on $\mathbb{R}^n$ such that $\nu_K(\operatorname{supp}(g)\cap \widetilde{\partial K})$ is not contained in any great subsphere of S^{n-1}. Then there exists a $\phi_p \in \mathrm{GL}(n)$ such that*

$$\begin{aligned} n^{\frac{1}{2}-\frac{1}{q}}E_{\mu^{\phi_p},p}\phi_p^{-1}K &\subseteq \Gamma_{\mu^{\phi_p},-q}\phi_p^{-1}K \qquad \text{for} \quad 0 < q \le p \le 2,\\ \Gamma_{\mu^{\phi_p},-q}\phi_p^{-1}K &\subseteq n^{\frac{1}{2}-\frac{1}{q}}E_{\mu^{\phi_p},p}\phi_p^{-1}K \qquad \text{for} \quad 2 \le p \le q \le \infty.\end{aligned}$$

Proof. As in Lemma 4.8, for every $p > 0$, there exists a $\phi_p \in \mathrm{GL}(n)$ such that $E_{\mu,p}K = \phi_p B^n$ and $E_{\mu^{\phi_p},p}\phi_p^{-1}K = \phi_p^{-1}E_{\mu,p}K = B^n$ by Theorem 4.5. According to (3.5), we have

$$\overline{V}_{\mu^{\phi_p},p}(\phi_p^{-1}K, B^n) = 1.$$

It follows from (2.12) and (2.17) that

$$\frac{1}{\alpha}\int_{S^{n-1}} dS_{\mu^{\phi_p},p}(\phi_p^{-1}K,v) = V_{\mu^{\phi_p},p}(\phi_p^{-1}K,B^n) = \mu^{\phi_p}(\phi_p^{-1}K), \tag{4.18}$$

which means $dS_{\mu^{\phi_p},p}(\phi_p^{-1}K,\cdot)/(\alpha\mu^{\phi_p}(\phi_p^{-1}K))$ is a probability measure on S^{n-1}. Moreover, it follows from (2.13) and (2.14) that

$$\frac{1}{\alpha}\int_{S^{n-1}} h_{\phi_p^{-1}K}(v)dS_{\mu^{\phi_p},1}(\phi_p^{-1}K,v)=V_{\mu^{\phi_p},p}(\phi_p^{-1}K,\phi_p^{-1}K)=\mu^{\phi_p}(\phi_p^{-1}K),$$

which means $h_{\phi_p^{-1}K}dS_{\mu^{\phi_p},1}(\phi_p^{-1}K,\cdot)/(\alpha\mu^{\phi_p}(\phi_p^{-1}K))$ is a probability measure on S^{n-1}.

Assume $0<q\le p\le 2$. Applying Jensen's inequality to (4.13), together with (4.17), one gets that for $u\in S^{n-1}$,

$$\begin{aligned}
\rho_{\Gamma_{\mu^{\phi_p},-q}\phi_p^{-1}K}(u)^{-1} &= \left[\frac{n}{\alpha\mu^{\phi_p}(\phi_p^{-1}K)}\int_{S^{n-1}}|u\cdot v|^q\,dS_{\mu^{\phi_p},q}(\phi_p^{-1}K,v)\right]^{\frac{1}{q}}\\
&= n^{\frac{1}{q}}\left[\frac{1}{\alpha\mu^{\phi_p}(\phi_p^{-1}K)}\int_{S^{n-1}}\left(\frac{|u\cdot v|}{h_{\phi_p^{-1}K}(v)}\right)^q h_{\phi_p^{-1}K}(v)dS_{\mu^{\phi_p},1}(\phi_p^{-1}K,v)\right]^{\frac{1}{q}}\\
&\le n^{\frac{1}{q}}\left[\frac{1}{\alpha\mu^{\phi_p}(\phi_p^{-1}K)}\int_{S^{n-1}}\left(\frac{|u\cdot v|}{h_{\phi_p^{-1}K}(v)}\right)^p h_{\phi_p^{-1}K}(v)dS_{\mu^{\phi_p},1}(\phi_p^{-1}K,v)\right]^{\frac{1}{p}}\\
&= n^{\frac{1}{q}}\left[\frac{1}{\alpha\mu^{\phi_p}(\phi_p^{-1}K)}\int_{S^{n-1}}|u\cdot v|^p\,dS_{\mu^{\phi_p},p}(\phi_p^{-1}K,v)\right]^{\frac{1}{p}}\\
&\le n^{\frac{1}{q}}\left[\frac{1}{\alpha\mu^{\phi_p}(\phi_p^{-1}K)}\int_{S^{n-1}}|u\cdot v|^2\,dS_{\mu^{\phi_p},p}(\phi_p^{-1}K,v)\right]^{\frac{1}{2}}\\
&= n^{\frac{1}{q}-\frac{1}{2}} = n^{\frac{1}{q}-\frac{1}{2}}\rho_{E_{\mu^{\phi_p},p}\phi_p^{-1}K}(u)^{-1}.
\end{aligned}$$

Thus $\rho_{\Gamma_{\mu^{\phi_p},-q}\phi_p^{-1}K}(u)\ge n^{\frac{1}{2}-\frac{1}{q}}\rho_{E_{\mu^{\phi_p},p}\phi_p^{-1}K}(u)$ for $u\in S^{n-1}$. Therefore

$$n^{\frac{1}{2}-\frac{1}{q}}E_{\mu^{\phi_p},p}\phi_p^{-1}K\subseteq\Gamma_{\mu^{\phi_p},-q}\phi_p^{-1}K,$$

for $0<q\le p\le 2$.

For $2\le p\le q<\infty$, the above inequality will be reversed. Thus

$$\Gamma_{\mu^{\phi_p},-q}\phi_p^{-1}K\subseteq n^{\frac{1}{2}-\frac{1}{q}}E_{\mu^{\phi_p},p}\phi_p^{-1}K,$$

for $2\le p\le q<\infty$. For $p=\infty$, due to (4.14), the desired result holds as well. $\square$

Note that it follows from Theorem 4.5 and Lemma 4.7 that for $\phi_p \in \mathrm{GL}(n)$,

$$\Gamma_{\mu^{\phi_p},-p}\phi_p^{-1}K = \phi_p^{-1}\Gamma_{\mu,-p}K \quad \text{and} \quad E_{\mu^{\phi_p},p}\phi_p^{-1}K = \phi_p^{-1}E_{\mu,p}K.$$

Together with Lemmas 4.8 and 4.9 with $p = q$, we can obtain the following L_p John's inclusion for our new L_p John ellipsoids.

Theorem 4.10. *Let $\alpha > 0$ and $K \in \mathcal{K}_o^n$. Suppose that $g : \mathbb{R}^n \to \mathbb{R}_+$ is the $(\alpha - n)$-homogeneous density of the α-homogeneous measure μ on $\mathbb{R}^n$ such that $\nu_K(\mathrm{supp}\,(g) \cap \widetilde{\partial K})$ is not contained in any great subsphere of S^{n-1}. Then*

$$n^{\frac{1}{2}-\frac{1}{p}}E_{\mu,p}K \subseteq \Gamma_{\mu,-p}K \subseteq E_{\mu,p}K \qquad \textit{for } \ 0 < p \leq 2,$$

$$E_{\mu,p}K \subseteq \Gamma_{\mu,-p}K \subseteq n^{\frac{1}{2}-\frac{1}{p}}E_{\mu,p}K \qquad \textit{for } \ 2 \leq p \leq \infty.$$

Actually, this new L_p John's inclusion contains the classical John's inclusion as a special case. Indeed, if g, the density of the measure μ is positive on $\widetilde{\partial K}$, then $\mathrm{supp}(S_{\mu,1}(K,\cdot))$ and $\mathrm{supp}(S(K,\cdot))$ are identical. It follows from (2.18) and Lemma 3.2 that

$$E_{\mu,\infty}K = JK.$$

On the other hand, if K is an origin-symmetric convex body in $\mathbb{R}^n$, then it follows from (4.15) that

$$\begin{aligned}
\rho_{\Gamma_{\mu,-\infty}K}(u)^{-1} &= \max\left\{\frac{|u\cdot v|}{h_K(v)} : v \in \mathrm{supp}\,(S_{\mu,1}(K,\cdot))\right\}\\
&= \max\left\{\frac{|u\cdot v|}{h_K(v)} : v \in \mathrm{supp}\,(S(K,\cdot))\right\}\\
&= \max\{|u\cdot v|\rho_{K^*}(v) : v \in \mathrm{supp}\,(S(K,\cdot))\}\\
&= h_{K^*}(u) = \rho_K(u)^{-1}.
\end{aligned}$$

Thus $\Gamma_{\mu,-\infty}K = K$. Consequently, the inclusion for $p = \infty$ in Theorem 4.10 reduces to the classical John's inclusion (4.1).

Proposition 4.11. *Let $\alpha > 0$ and $K \in \mathcal{K}_o^n$. Suppose that $g : \mathbb{R}^n \to \mathbb{R}_+$ is the $(\alpha - n)$-homogeneous density of the α-homogeneous measure μ on $\mathbb{R}^n$ such that $\nu_K(\mathrm{supp}\,(g) \cap \widetilde{\partial K})$ is not contained in any great subsphere of S^{n-1} for $0 < p \leq \infty$ and $h_K dS_{\mu,1}(K,\cdot)$*

satisfies the subspace concentration condition for $p = 0$. Then for $0 \le p \le q \le \infty$,

$$V(E_{\mu,q}K) \le V(E_{\mu,p}K).$$

Proof. It follows from (2.14) that $h_K dS_{\mu,1}(K,\cdot)/(\alpha\mu(K))$ is a probability measure on S^{n-1}. Together with (2.17), (2.19), Jensen's inequality and the fact that the logarithmic function is concave, we have for $p > 0$,

$$\begin{aligned}
\overline{V}_{\mu,0}(K,E) &= \exp\left\{\frac{1}{\alpha\mu(K)}\int_{S^{n-1}}\log\left(\frac{h_E(u)}{h_K(u)}\right)h_K(u)dS_{\mu,1}(K,u)\right\}\\
&= \exp\left\{\frac{1}{p}\frac{1}{\alpha\mu(K)}\int_{S^{n-1}}\log\left(\frac{h_E(u)}{h_K(u)}\right)^p h_K(u)dS_{\mu,1}(K,u)\right\}\\
&\le \exp\left\{\frac{1}{p}\log\left(\int_{S^{n-1}}\frac{1}{\alpha\mu(K)}\left(\frac{h_E(u)}{h_K(u)}\right)^p h_K(u)dS_{\mu,1}(K,u)\right)\right\}\\
&= \left(\frac{1}{\alpha\mu(K)}\int_{S^{n-1}}\left(\frac{h_E(u)}{h_K(u)}\right)^p h_K(u)dS_{\mu,1}(K,u)\right)^{\frac{1}{p}} = \overline{V}_{\mu,p}(K,E).
\end{aligned}$$

For $0 < p \le q < \infty$, we see

$$\begin{aligned}
\overline{V}_{\mu,p}(K,E) &= \left[\frac{1}{\alpha\mu(K)}\int_{S^{n-1}}\left(\frac{h_E(u)}{h_K(u)}\right)^p h_K(u)dS_{\mu,1}(K,u)\right]^{\frac{1}{p}}\\
&\le \left[\frac{1}{\alpha\mu(K)}\int_{S^{n-1}}\left(\frac{h_E(u)}{h_K(u)}\right)^q h_K(u)dS_{\mu,1}(K,u)\right]^{\frac{1}{q}} = \overline{V}_{\mu,q}(K,E).
\end{aligned}$$

For $p < \infty$, we obtain

$$\begin{aligned}
\overline{V}_{\mu,p}(K,E) &= \left[\frac{1}{\alpha\mu(K)}\int_{S^{n-1}}\left(\frac{h_E(u)}{h_K(u)}\right)^p h_K(u)dS_{\mu,1}(K,u)\right]^{\frac{1}{p}}\\
&\le \max\left\{\frac{h_E(u)}{h_K(u)} : v \in \operatorname{supp}(\,S_{\mu,1}(K,\cdot))\right\} = \overline{V}_{\mu,\infty}(K,E).
\end{aligned}$$

For $0 \le p \le \infty$, denote by $\Omega_p = \left\{E \in \mathcal{E}_o^n : \overline{V}_{\mu,p}(K,E) \le 1\right\}$. Together with the above inequalities, we have that $\Omega_\infty \subseteq \Omega_q \subseteq \Omega_p \subseteq \Omega_0$ for $0 \le p \le q \le \infty$. Then the desired result follows from (3.1). □

We shall use the following lemma due to Wu [49, Corollary 4.4], Milman, Rotem [39] and Livshyts [33] for $p = 1$.

Lemma 4.12. *Let $p \geq 1$ and $r > 0$. Let $g : \mathbb{R}^n \to \mathbb{R}_+$ be an r-homogeneous and $1/r$-concave density of a measure μ on $\mathbb{R}^n$, continuous on its support. Let $s = 1/(n+r)$. If $K, L \in \mathcal{K}_o^n$, then*

$$V_{\mu,p}(K, L) \geq p\mu(K)^{1-s}\mu(L)^s + (1-p)\mu(K). \tag{4.19}$$

Theorem 4.13. *Let $r > 0$, $\alpha > 0$ and $K \in \mathcal{K}_o^n$. Suppose that $g : \mathbb{R}^n \to \mathbb{R}_+$ is the r-homogeneous and $1/r$-concave density of the α-homogeneous measure μ on $\mathbb{R}^n$ such that $\operatorname{supp}(g) \cap S^{n-1}$ is not contained in any great subsphere of S^{n-1}. Then for $p \geq 1$,*

$$\mu(K) \geq \mu(E_{\mu,p}K). \tag{4.20}$$

Proof. It follows from the definition of the $E_{\mu,p}K$ that $\overline{V}_{\mu,p}(K, E_{\mu,p}K) = 1$. From (2.17), we have $V_{\mu,p}(K, E_{\mu,p}K) = \mu(K)$. Therefore, inequality (4.19) with $L = E_{\mu,p}K$ yields

$$\mu(K) = V_{\mu,p}(K, E_{\mu,p}K) \geq p\mu(K)^{1-s}\mu(E_{\mu,p}K)^s + (1-p)\mu(K).$$

Simplifying this inequality gives the desired inequality (4.20). $\square$

Chapter 5

Convex bodies with identical John and L_p John ellipsoids for general measures

The setting of this chapter is the same as the one in Chapter 4. For $\alpha > 0$, let μ be an α-homogeneous measure on $\mathbb{R}^n$ which has a continuous density g with respect to dx. For $\phi \in \mathrm{GL}(n)$, let μ^{ϕ} be the measure which has the density $g \circ \phi$ with respect to dx.

This chapter is devoted to the following theorem. In particular, our result reduces to the corresponding results in [23, 56] if $g \equiv 1$.

Theorem 5.1. *Let $\alpha > 0$ and $K \in \mathcal{K}_o^n$. Suppose that $g : \mathbb{R}^n \to \mathbb{R}_+$ is the $(\alpha - n)$-homogeneous density of the α-homogeneous measure μ on $\mathbb{R}^n$ such that g is positive on $\widetilde{\partial K}$. In addition, assume that the measure $h_K dS_{\mu,1}(K,\cdot)$ satisfies the subspace concentration condition and for $0 \le p < \infty$, there exists a $\phi_p \in \mathrm{GL}(n)$ such that $E_{\mu,p}K = \phi_p B^n$ and $\int_{S^{n-1}} v/|\phi_p^t v| dS_{\mu,1}(K,v) = o$. Then the following statements are equivalent:*

(1) $\quad E_{\mu,p}K = E_{\mu,\infty}K$;

(2) $\quad E_{\mu,p}K \subseteq K$;

(3) $\quad \phi_p^{-1}K$ *is the tangential body of B^n with respect to the measure μ^{ϕ_p} and the measure $S_{\mu^{\phi_p},p}(\phi_p^{-1}K,\cdot)$ is isotropic.*

Note that, if g is positive on $\widetilde{\partial K}$, then $\nu_K(\mathrm{supp}\,(g) \cap \widetilde{\partial K})$ is not contained in any great subsphere of S^{n-1}. According to Lemma 3.2, the measures $S_p(K,\cdot)$ and

$S_{\mu,p}(K,\cdot)$ share the same supports. In particular,

$$\overline{V}_{\mu,\infty}(K,L)=\max\left\{\frac{h_L(u)}{h_K(u)}:u\in\operatorname{supp}\left(S(K,\cdot)\right)\right\}, \tag{5.1}$$

for $K,L\in\mathcal{K}_o^n$. Hence, $E_{\mu,\infty}K$ is exactly the John ellipsoid JK.

For convenience, let

$$dS_{\mu^{\phi_0},0}(\phi_0^{-1}K,\cdot)=h_{\phi_0^{-1}K}dS_{\mu^{\phi_0},1}(\phi_0^{-1}K,\cdot).$$

For $K,L\in\mathcal{K}^n$ and $L\subseteq K$, K is a tangential body of L with respect to the measure μ if it satisfies

$$h_K(u)=h_L(u)\quad\text{for }\ u\in\operatorname{supp}(S_{\mu,1}(K,\cdot)). \tag{5.2}$$

A finite positive Borel measure ν on S^{n-1} is said to be isotropic if for all $x\in\mathbb{R}^n$,

$$\int_{S^{n-1}}|x\cdot u|^2d\nu(u)=\frac{|x|^2}{n}|\nu|, \tag{5.3}$$

where $|\nu|$ is the total mass of the measure ν, i.e., $\nu(S^{n-1})$. The centroid of a finite positive Borel measure ν on S^{n-1} is

$$\frac{1}{|\nu|}\int_{S^{n-1}}ud\nu(u).$$

For the sake of brevity, we divide the proof of Theorem 5.1 into two parts. The equivalence between (1) and (3) follows from Theorem 5.5, and the equivalence between (1) and (2) follows from Theorem 5.6. In order to establish Theorem 5.5, we first establish the following lemma.

Lemma 5.2. *Let $\alpha>0$ and $K\in\mathcal{K}_o^n$. Suppose that $g:\mathbb{R}^n\to\mathbb{R}_+$ is the $(\alpha-n)$-homogeneous density of the α-homogeneous measure μ on $\mathbb{R}^n$ such that g is positive on $\widetilde{\partial K}$ and suppose that the measure $h_KdS_{\mu,1}(K,\cdot)$ satisfies the subspace concentration condition. Then for $p\geq 0$, there exists a $\phi_p\in\mathrm{GL}(n)$ such that $E_{\mu^{\phi_p},p}\phi_p^{-1}K=B^n$ if and only if the measure $S_{\mu^{\phi_p},p}(\phi_p^{-1}K,\cdot)$ is isotropic.*

Proof. Since $E_{\mu,p}K$ is an origin-symmetric ellipsoid, there exists a $\phi_p\in\mathrm{GL}(n)$ such that $E_{\mu,p}K=\phi_pB^n$. It follows from Theorem 4.5 that for $p\geq 0$,

$$E_{\mu^{\phi_p},p}\phi_p^{-1}K=\phi_p^{-1}E_{\mu,p}K=B^n.$$

Together with Theorem 3.8 and (4.2), one has, $E_{\mu^{\phi_p},p}\phi_p^{-1}K = B^n$ for $p \geq 0$ if and only if

$$\frac{|x|^2}{n} = \frac{1}{\alpha\mu^{\phi_p}(\phi_p^{-1}K)} \int_{S^{n-1}} |x \cdot v|^2 dS_{\mu^{\phi_p},p}(\phi_p^{-1}K, v),$$

for all $x \in \mathbb{R}^n$. It follows from (4.18) that $dS_{\mu^{\phi_p},p}(\phi_p^{-1}K, \cdot)/(\alpha\mu^{\phi_p}(\phi_p^{-1}K))$ is a probability measure. Then by (5.3), the measure $S_{\mu^{\phi_p},p}(\phi_p^{-1}K, \cdot)$ is isotropic. This completes the proof. □

Following the similar steps of [56, Lemma 3.2], we obtain the following lemma.

Lemma 5.3. *Let $K \in \mathcal{K}_o^n$ and $\lambda B_n \subseteq K$ for some $\lambda > 0$. Suppose that μ is an α-homogeneous measure on $\mathbb{R}^n$ with a continuous density g which is positive on $\widetilde{\partial K}$. If $h_K(u) = \lambda$ for $S_{\mu,1}(K, \cdot)$-almost all $u \in \operatorname{supp}(S_{\mu,1}(K, \cdot))$, then*

$$h_K|_{\operatorname{supp}(S_{\mu,1}(K,\cdot))} = \lambda.$$

The following lemma (see [56, Lemma 3.4]) is needed for the proof of Theorem 5.5.

Lemma 5.4. *Assume the finite Borel measure σ on S^{n-1} is isotropic and its centroid is at the origin. Let*

$$K = \bigcap_{u \in \operatorname{supp}\sigma} \{x \in \mathbb{R}^n : x \cdot u \leq 1\}.$$

Then $JK = B^n$.

Theorem 5.5. *Let $\alpha > 0$ and $K \in \mathcal{K}_o^n$. Suppose that $g : \mathbb{R}^n \to \mathbb{R}_+$ is the $(\alpha - n)$-homogeneous density of the α-homogeneous measure μ on $\mathbb{R}^n$ such that g is positive on $\widetilde{\partial K}$. In addition, assume that the measure $h_K dS_{\mu,1}(K, \cdot)$ satisfies the subspace concentration condition and for $0 \leq p < \infty$, there exists a $\phi_p \in \mathrm{GL}(n)$ such that $E_{\mu,p}K = \phi_p B^n$ and $\int_{S^{n-1}} v/|\phi_p^t v| dS_{\mu,1}(K, v) = o$. Then $E_{\mu,p}K = E_{\mu,\infty}K$ if and only if $\phi_p^{-1}K$ is the tangential body of B^n with respect to the measure μ^{ϕ_p} and the measure $S_{\mu^{\phi_p},p}(\phi_p^{-1}K, \cdot)$ is isotropic.*

Proof. By (2.9), it can be easily verified that, for $v \in S^{n-1}$ and $\phi \in \mathrm{GL}(n)$,

$$v \in \operatorname{supp}(S_{\mu,1}(K, \cdot)) \;\Leftrightarrow\; \overline{\phi^t v} \in \operatorname{supp}(S_{\mu^\phi,1}(\phi^{-1}K, \cdot)). \tag{5.4}$$

Suppose that $E_{\mu,p}K = E_{\mu,\infty}K$ for $0 \leq p < \infty$. There exists a $\phi_p \in \mathrm{GL}(n)$ such that

$E_{\mu,p}K = E_{\mu,\infty}K = \phi_p B^n$. It follows from Theorem 4.5 that

$$B^n = \phi_p^{-1} E_{\mu,p}K = \phi_p^{-1} E_{\mu,p}\phi_p\phi_p^{-1}K = E_{\mu^{\phi_p},p}\phi_p^{-1}K.$$

It works for $E_{\mu,\infty}K$ as well. Therefore, we have

$$E_{\mu^{\phi_p},p}\phi_p^{-1}K = E_{\mu^{\phi_p},\infty}\phi_p^{-1}K = B^n.$$

By Lemma 5.2, the measure $S_{\mu^{\phi_p},p}(\phi_p^{-1}K,\cdot)$ is isotropic due to $E_{\mu^{\phi_p},p}\phi_p^{-1}K = B^n$ for $0 \le p < \infty$.

Next, we show that $\phi_p^{-1}K$ is the tangential body of B^n. Take $p = 0$ for example, we have $E_{\mu^{\phi_0},0}\phi_0^{-1}K = E_{\mu^{\phi_0},\infty}\phi_0^{-1}K = B^n$. It follows from Definition 3.10 that

$$\overline{V}_{\mu^{\phi_0},0}(\phi_0^{-1}K, B^n) = \overline{V}_{\mu^{\phi_0},\infty}(\phi_0^{-1}K, B^n) = 1.$$

By (2.18) and (2.19), the above equations can be rewritten as

$$\begin{aligned} 1 &= \exp\left\{\frac{1}{\alpha\mu^{\phi_0}(\phi_0^{-1}K)}\int_{S^{n-1}}\log\left(\frac{1}{h_{\phi_0^{-1}K}(u)}\right)h_{\phi_0^{-1}K}(u)dS_{\mu^{\phi_0},1}(\phi_0^{-1}K,u)\right\} \\ &= \max\left\{\frac{1}{h_{\phi_0^{-1}K}(u)},\ \ u \in \operatorname{supp}\left(S_{\mu^{\phi_0},1}(\phi_0^{-1}K,\cdot)\right)\right\}. \end{aligned}$$

It follows from the equality condition of Jensen's inequality that $h_{\phi_0^{-1}K}(u) = 1$ for almost all $u \in \operatorname{supp}(S_{\mu^{\phi_0},1}(\phi_0^{-1}K,\cdot))$. By Lemma 5.3, we have $h_K|_{\operatorname{supp}(S_{\mu^{\phi_0},1}(\phi_0^{-1}K,\cdot))} = 1$. Since g is positive on $\widetilde{\partial K}$, it follows from (2.5) and (2.9) that $\operatorname{supp}(S(\phi_0^{-1}K,\cdot))$ and $\operatorname{supp}(S_{\mu^{\phi_0},1}(\phi_0^{-1}K,\cdot))$ are indentical. Together with $\overline{V}_{\mu^{\phi_0},\infty}(\phi_0^{-1}K, B^n) = 1$, one has $B^n \subseteq \phi_0^{-1}K$. Therefore, we obtain that $\phi_0^{-1}K$ is the tangential body of B^n with respect to the measure μ^{ϕ_0} by (5.2).

Conversely, suppose that $\phi_p^{-1}K$ is the tangential body of B^n with respect to the measure μ^{ϕ_p} and the measure $S_{\mu^{\phi_p},p}(\phi_p^{-1}K,\cdot)$ is isotropic for $0 \le p < \infty$. From (5.2), we see $h_{\phi_p^{-1}K}(u) = h_{B^n}(u) = 1$ for $u \in \operatorname{supp}\left(S_{\mu^{\phi_p},1}(\phi_p^{-1}K,\cdot)\right)$, which means $h_{\phi_p^{-1}K}|_{\operatorname{supp}\left(S_{\mu^{\phi_p},1}(\phi_p^{-1}K,\cdot)\right)} = 1$. Therefore, we have that for $x \in \mathbb{R}^n$ and letting $v = \overline{\phi_p^t u}$,

$$\phi_p^{-1}K = \bigcap_{u \in \operatorname{supp}\left(S_{\mu,1}(K,\cdot)\right)} \{\phi_p^{-1}y :\ y \cdot u \le h_K(u)\}$$

$$= \bigcap_{u \in \text{supp}\,(S_{\mu,1}(K,\cdot))} \{x : \phi_p x \cdot u \le h_K(u)\}$$
$$= \bigcap_{u \in \text{supp}\,(S_{\mu,1}(K,\cdot))} \left\{x : x \cdot \overline{\phi_p^t u} \le \frac{h_K(u)}{|\phi_p^t u|}\right\}$$
$$= \bigcap_{v \in \text{supp}\,(S_{\mu^{\phi_p},1}(\phi_p^{-1}K,\cdot))} \left\{x : x \cdot v \le h_{\phi_p^{-1}K}(v)\right\}$$
$$= \bigcap_{v \in \text{supp}\,(S_{\mu^{\phi_p},1}(\phi_p^{-1}K,\cdot))} \{x : x \cdot v \le 1\}.$$

By the hypothesis that $\int_{S^{n-1}} v/|\phi_p^t v| dS_{\mu,1}(K,v) = o$ and (5.4), we have

$$o = \frac{1}{|S_{\mu^{\phi_p},1}(\phi_p^{-1}K,\cdot)|} \int_{S^{n-1}} u\, dS_{\mu^{\phi_p},1}(\phi_p^{-1}K,u).$$

Therefore, the centroid of $S_{\mu^{\phi_p},1}(\phi_p^{-1}K,\cdot)$ is at the origin. It follows from Lemma 5.4 that $J(\phi_p^{-1}K) = B^n$. By the affine property of the John ellipsoid JK (see [37, Lemma 2.5]), we have

$$JK = \phi_p B^n.$$

Since the measure $S_{\mu^{\phi_p},1}(\phi_p^{-1}K,\cdot)$ is isotropic, we obtain $E_{\mu^{\phi_p},p}\phi_p^{-1}K = B^n$ for $0 \le p < \infty$ from Lemma 5.2. By Theorem 4.5, we have

$$E_{\mu,p}K = \phi_p B^n.$$

Consequently, we conclude that $E_{\mu,p}K = JK = \phi_p B^n$ for $0 \le p < \infty$. Since g is positive on $\widetilde{\partial K}$ and $E_{\mu,\infty}K = JK$, we get $E_{\mu,p}K = E_{\mu,\infty}K$ for $0 \le p < \infty$. $\square$

Theorem 5.6. *Let $\alpha > 0$ and $K \in \mathcal{K}_o^n$. Suppose that $g : \mathbb{R}^n \to \mathbb{R}_+$ is the $(\alpha - n)$-homogeneous density of the α-homogeneous measure μ on $\mathbb{R}^n$ such that g is positive on $\widetilde{\partial K}$. In addition, assume that the measure $h_K dS_{\mu,1}(K,\cdot)$ satisfies the subspace cocentration condition. Then $E_{\mu,p}K = E_{\mu,\infty}K$ if and only if $E_{\mu,p}K \subseteq K$ for $0 \le p < \infty$.*

Proof. Assume $E_{\mu,p}K = E_{\mu,\infty}K$ for $0 \le p < \infty$. The facts that the John ellipsoid $JK \subseteq K$ and $E_{\mu,\infty}K = JK$, we get $E_{\mu,p}K \subseteq K$ for $0 \le p < \infty$.

Conversely, suppose $E_{\mu,p}K \subseteq K$ for $0 \le p < \infty$. It follows that $h_{E_{\mu,p}K}(u) \le h_K(u)$ for all $u \in S^{n-1}$. Together with (5.1), we have $E_{\mu,p}K \in \left\{E \in \mathcal{E}_o^n : \overline{V}_{\mu,\infty}(K,E) \le 1\right\}$.

It follows from Definition 3.10 that for $0 \leq p < \infty$,

$$V(E_{\mu,p}K) \leq V(E_{\mu,\infty}K). \tag{5.5}$$

Proposition 4.11 yields that for $0 \leq p < \infty$,

$$V(E_{\mu,p}K) \geq V(E_{\mu,\infty}K). \tag{5.6}$$

Combining (5.5) and (5.6), we see $V(E_{\mu,p}K) = V(E_{\mu,\infty}K)$. Thus $E_{\mu,p}K = E_{\mu,\infty}K$ for $0 \leq p < \infty$ follows from the uniqueness of John ellipsoid JK and the fact that $E_{\mu,\infty}K = JK$. □

Chapter 6

General L_p John ellipsoids

Recall that for $p > 0$,

$$V_{\mu,p}(K, L) = \frac{1}{\alpha} \int_{S^{n-1}} h_L^p(u) dS_{\mu,p}(K, u).$$

Note that $S_{\mu,p}(K, \cdot)$ is a measure constructed from a given convex body K. The Minkowski type problem [4,9–11,21,24,25,29,32,50,51] aims to characterize $S_{\mu,p}(K, \cdot)$ (and its special case when μ is the Lebesgue measure), namely, for a given nonzero finite measure ν on S^{n-1}, can we find a convex body $K \in \mathcal{K}_o^n$ such that $\nu = S_{\mu,p}(K, \cdot)$? Such a Minkowski problem has found solutions under certain conditions on ν for $p \geq 0$. However, in general, it is neither known whether there is a solution to the equation $\nu = S_{\mu,p}(K, \cdot)$ nor is it known whether $K \in \mathcal{K}_o^n$ if such a $K \in \mathcal{K}^n$ does exist. Hence, to study problems similar to (but somehow different from) Problems $M_{\mu,p}$ and $\overline{M}_{\mu,p}$ for ν is interesting in its own right and, as we have discussed above, is even more general than Problems $M_{\mu,p}$ and $\overline{M}_{\mu,p}$. In this chapter, we will give a brief discussion on these problems for ν.

For $p > 0$, and a nonzero Borel probability measure ν on S^{n-1}, let

$$\|h_L\|_{L_\nu^p} = \left[\int_{S^{n-1}} h_L^p(u) d\nu(u) \right]^{\frac{1}{p}}. \tag{6.1}$$

When $p = \infty$, let

$$\|h_L\|_{L_\nu^\infty} = \lim_{p \to \infty} \|h_L\|_{L_\nu^p} = \max \{h_L(u), u \in \text{supp}(\nu)\},$$

and when $p = 0$, let

$$\|h_L\|_{L^0_\nu} = \lim_{p \to 0^+} \|h_L\|_{L^p_\nu} = \exp\left\{ \int_{S^{n-1}} \log h_L(u) d\nu(u) \right\}.$$

For $c > 0$, it is easy to verify that

$$\|h_{cL}\|_{L^p_\nu} = c\|h_L\|_{L^p_\nu}. \tag{6.2}$$

When $p \geq 1$, $\|\cdot\|_{L^p_\nu}$ defines a norm. Note that ν is chosen to be a probability measure mainly for technical reasons, and it can be certainly replaced by any finite nonzero Borel measure on S^{n-1}.

Motivated by Problems $M_{\mu,p}$ and $\overline{M}_{\mu,p}$ in Chapter 3, we consider the following closely related optimization problems.

Problem M^ν_p. *Given a probability measure ν which is not concentrated on any great subsphere and $p \geq 0$, find an origin-symmetric ellipsoid E which solves the following optimization problem:*

$$\max_{E \in \mathcal{E}^n_o} V(E) \quad \text{subject to} \quad \|h_E\|_{L^p_\nu} \leq 1. \tag{6.3}$$

Problem $\overline{M}^\nu_p$. *Given a probability measure ν which is not concentrated on any great subsphere and $p \geq 0$, find an origin-symmetric ellipsoid E which solves the following optimization problem:*

$$\min_{E \in \mathcal{E}^n_o} \|h_E\|_{L^p_\nu} \quad \text{subject to} \quad V(E) \geq \omega_n. \tag{6.4}$$

Again, to find a solution to (6.3) is equivalent to solve the following problem:

$$\max_{E \in \mathcal{E}^n_o} \text{vrad}(E) \quad \text{subject to} \quad \|h_E\|_{L^p_\nu} = 1. \tag{6.5}$$

To find a solution to (6.4) is equivalent to solve the following problem:

$$\min_{E \in \mathcal{E}^n_o} \|h_E\|_{L^p_\nu} \quad \text{subject to} \quad \text{vrad}(E) = 1. \tag{6.6}$$

Following the proof of Lemma 3.1, by (3.4) and (6.2), one can prove the following lemma.

Lemma 6.1. *Let $p \geq 0$. Suppose that ν is a probability measure which is not concentrated on any great subsphere. If E_M solves Problem M_p^ν, then $E_M/\mathrm{vrad}(E_M)$ is a solution to Problem $\overline{M}_p^\nu$. Conversely, if E_m solves Problem $\overline{M}_p^\nu$, then $\|h_{E_m}\|_{L_\nu^p}^{-1} E_m$ is a solution to Problem M_p^ν.*

Next, we establish the existence of the solutions to Problem M_p^ν for $0 < p \leq \infty$ and Problem M_0^ν.

Lemma 6.2. *Let $0 < p \leq \infty$. Suppose that ν is a probability measure which is not concentrated on any great subsphere. Then there exists a solution to Problem M_p^ν.*

Proof. For $0 < p < \infty$. From (6.5), we can choose $\{E_j\}_{j=1}^\infty$, the maximizing sequence of origin-symmetric ellipsoids to Problem M_p^ν, such that $\|h_{E_j}\|_{L_\nu^p} = 1$ and

$$\lim_{j\to\infty} V(E_j) = \max_{E\in\mathcal{E}_o^n} \left\{V(E) : \|h_E\|_{L_\nu^p} = 1\right\}. \tag{6.7}$$

It can be checked that the sequence $\{E_j\}_{j=1}^\infty$ is bounded by the same approach of Lemma 3.5. From Lemma 3.3, there exists a convergent subsequence of $\{E_j\}_{j=1}^\infty$, which will be denoted by $\{E_i\}$, with $\lim_{i\to\infty} E_i = E_0$. It follows from (6.2) that $\|h_{B^n}\|_{L_\nu^p} = 1$. Therefore $V(E_0) \geq V(B^n) > 0$ by (6.5). It can be easily checked that $\|h_K\|_{L_\nu^p}$ is continuous on $K \in \mathcal{K}_o^n$. Together with (6.7), one gets that E_0 is a solution to Problem M_p^ν. As for the existence of solutions to Problem M_∞^ν, it follows a similar approach for Problem $M_{\mu,\infty}$ in Lemma 3.5. □

The following lemma shows the existence of solutions to Problem M_0^ν.

Lemma 6.3. *Suppose that the measure ν satisfies the subspace concentration condition. Then there exists a solution to Problem M_0^ν.*

Proof. Similar to Lemma 3.6, we show the existence of solution to Problem M_0^ν with the aid of Problem $\overline{M}_0^\nu$. Since ν satisfies the subspace concentration condition, the existence of solution to Problem $\overline{M}_0^\nu$ is a direct result of Lemma 3.4. □

The following theorem provides an argument for the uniqueness of solutions to Problem $\overline{M}_p^\nu$, as well as the characterization for the solutions to Problem $\overline{M}_p^\nu$.

Theorem 6.4. *Suppose $p > 0$. Let ν be a probability measure which is not concentrated on any great subsphere. Then Problem $\overline{M}_p^\nu$ has a unique solution. Moreover, an ellipsoid E solves Problem $\overline{M}_p^\nu$ if and only if the following holds: for all $x \in \mathbb{R}^n$,*

$$\frac{h_{E^*}^2(x)}{n}\|h_E\|_{L_\nu^p}^p = \int_{S^{n-1}} |x\cdot u|^2 h_E^{p-2}(u)d\nu(u). \tag{6.8}$$

In addition, if the measure ν satisfies the subspace concentration condition, Problem $\overline{M}_0^\nu$ has a unique solution. Moreover, an ellipsoid E solves Problem $\overline{M}_0^\nu$ if and only if the following holds: for all $x \in \mathbb{R}^n$,

$$\frac{h_{E^*}^2(x)}{n} = \int_{S^{n-1}} \left(\frac{|x\cdot u|}{h_E(u)}\right)^2 d\nu(u). \tag{6.9}$$

Proof. Combining the hypothesis with Lemmas 6.2 and 6.3, we see that there exist solutions to Problems M_p^ν and M_0^ν. This implies, together with Lemma 6.1, the existence of solutions to Problems $\overline{M}_p^\nu$ and $\overline{M}_0^\nu$.

Following the proof of Theorem 3.7, we first show that if an origin-symmetric ellipsoid E solves Problem $\overline{M}_p^\nu$, then (6.8) holds for all $x \in \mathbb{R}^n$. To this end, we let $T_\varepsilon \in \mathrm{SL}(n)$ be the same as in (3.16). Assume $E = \phi^t B^n$ for some $\phi \in \mathrm{GL}(n)$. Since V is $\mathrm{SL}(n)$-invariant, we can see $V(\phi^t B^n) = V(\phi^t T_\varepsilon^t B^n)$. Due to (6.1) and (6.6), we have

$$\|h_{(\phi^t B^n)}\|_{L_\nu^p}^p \le \|h_{(\phi^t T_\varepsilon^t B^n)}\|_{L_\nu^p}^p$$

for all $\varepsilon \in (-\varepsilon_{\hat{\phi}}, \varepsilon_{\hat{\phi}})$. Note that $h_{\phi^t T_\varepsilon^t B^n}(x) = |T_\varepsilon \phi x|$. Thus by (6.1), we get

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_{S^{n-1}} |T_\varepsilon \phi u|^p \, d\nu(u) = 0. \tag{6.10}$$

Since $|T_\varepsilon \phi u|$ is smooth, it implies that the integrand depends smoothly on sufficiently small ε. The later calculation follows the same steps as in Theorem 3.7. From (3.18), (3.20), (6.10) and $x = \phi^t z$, it follows that

$$\int_{S^{n-1}} |x\cdot u|^2 h_E^{p-2}(u)d\nu(u) = \frac{h_{E^*}^2(x)}{n}\|h_E\|_{L_\nu^p}^p,$$

which is (6.8).

Conversely, assume that (6.8) holds for all $x \in \mathbb{R}^n$. Let $E = \phi^t B^n$ for some $\phi \in \mathrm{GL}(n)$. Due to (6.1), (6.6) and $p > 0$, in order to prove that E solves Problem

$\overline{M}_p^\nu$, we need to prove if $E' \in \mathcal{E}_o^n$ such that $V(E') = V(E)$, then

$$\|h_{E'}\|_{L_\nu^p}^p \geq \|h_E\|_{L_\nu^p}^p, \tag{6.11}$$

with equality if and only if $E' = E$.

Without loss of generality, let $E' = \phi^t T^t B^n$ for some $T \in \mathrm{SL}(n)$. By (3.19) and (6.1), we have

$$\|h_{E'}\|_{L_\nu^p}^p = \int_{S^{n-1}} |T\,\overline{\phi u}|^p |\phi u|^p d\nu(u).$$

Again $\phi u \neq 0$ for all $u \in S^{n-1}$. Note that $T \in \mathrm{SL}(n)$ can be represented by $T = PDQ$, where P and Q are orthogonal matrices and $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is diagonal and positive definite. Thus, by (6.11), we need to prove

$$\left[\frac{1}{\|h_E\|_{L_\nu^p}^p} \int_{S^{n-1}} |DQ\,\overline{\phi u}|^p |\phi u|^p d\nu(u)\right]^{\frac{1}{p}} \geq 1, \tag{6.12}$$

with equality if and only if $D = I_n$. Thus, using Jensen's inequality, we see

$$\left[\frac{1}{\|h_E\|_{L_\nu^p}^p} \int_{S^{n-1}} |DQ\,\overline{\phi u}|^p |\phi u|^p d\nu(u)\right]^{\frac{1}{p}} \geq \exp\left[\frac{1}{\|h_E\|_{L_\nu^p}^p} \int_{S^{n-1}} \log|DQ\,\overline{\phi u}| |\phi u|^p d\nu(u)\right], \tag{6.13}$$

with equality if and only if there exists $c > 0$ such that $|DQ\,\overline{\phi u}| = c$ for all $u \in \mathrm{supp}\,(\nu)$. To establish (6.12), we only need to prove that

$$\int_{S^{n-1}} \log|DQ\,\overline{\phi u}| |\phi u|^p d\nu(u) \geq 0. \tag{6.14}$$

Taking $x = \phi^t Q^t e_i$ in (6.8), together with (3.20) and $E = \phi^t B^n$, we immediately get

$$\int_{S^{n-1}} |e_i \cdot Q\,\overline{\phi u}|^2 |\phi u|^p d\nu(u) = \frac{|\phi^{-t} x|^2}{n} \|h_E\|_{L_\nu^p}^p. \tag{6.15}$$

Let $v = Q\,\overline{\phi u}$ and $v_i = e_i \cdot Q\,\overline{\phi u}$. A calculation similar to (3.26), by (6.15), shows that

$$\begin{aligned}\int_{S^{n-1}} \log|DQ\,\overline{\phi u}| |\phi u|^p d\nu(u) &= \int_{S^{n-1}} \log|Dv| |\phi u|^p d\nu(u)\\ &= \frac{1}{2}\int_{S^{n-1}} \log(\lambda_1^2 v_1^2 + \cdots + \lambda_n^2 v_n^2) |\phi u|^p d\nu(u)\end{aligned}$$

$$\geq \frac{1}{2}\int_{S^{n-1}}(v_1^2\log\lambda_1^2+\cdots+v_n^2\log\lambda_n^2)|\phi u|^p d\nu(u)$$
$$=\frac{\|h_E\|^p_{L^p_\nu}}{n}\log(\lambda_1\cdots\lambda_n)=0, \tag{6.16}$$

which is the desired result (6.14).

Now let us characterize the equality of (6.12). The strict concavity of the log function implies that equality in (6.16) holds only if $v_{i_1}\cdots v_{i_N}\neq 0$ then $\lambda_{i_1}=\ldots=\lambda_{i_N}$ for all $u\in \operatorname{supp}(\nu)$. Therefore, $|Dv|=\lambda_i$ when $v_i\neq 0$ for $u\in\operatorname{supp}(\nu)$. Furthermore, the equality of (6.13) yields that $|Dv|=|DQ\,\overline{\phi u}|=c$ for all $u\in\operatorname{supp}(\nu)$. Since $\operatorname{supp}(\nu)$ is not contained in any great subsphere of S^{n-1}, it follows that $v=Q\,\overline{\phi u}$ is not contained in any great subsphere of S^{n-1} for all $u\in\operatorname{supp}(\nu)$. Consequently, we have $\lambda_i=c$ for all $i=1,\ldots,n$. However, the fact that $\lambda_1\cdots\lambda_n=1$ force $D=I_n$, which is exactly the equality condition of (6.12). This investigates the uniqueness of solutions to Problem $\overline{M}_p^\nu$ when $p>0$.

The assertions for Problem $\overline{M}_0^\nu$ can be approved by a similar approach as $p>0$. Suppose that $E\in\mathcal{E}_o^n$ solves Problem $\overline{M}_0^\nu$. Define T_ε as in (3.16), the desired result (6.9) follows from the same steps as $p>0$. Conversely, from (6.13) and (6.14), its proof is already included in $p>0$. Therefore, the uniqueness of solutions to Problem $\overline{M}_0^\nu$ is guaranteed. □

The following characteristic theorem for the genreal L_p John ellipsoid for the measure ν immediately follows from Lemma 6.1 and Theorem 6.4. It can be stated as follows.

Theorem 6.5. *Suppose $p>0$. Let ν be a probability measure which is not concentrated on any great subsphere. Then Problem M_p^ν has a unique solution. Moreover, an origin-symmetric ellipsoid E solves Problem M_p^ν if and only if the following holds: for all $x\in\mathbb{R}^n$,*
$$\frac{h^2_{E^*}(x)}{n}=\int_{S^{n-1}}|x\cdot u|^2 h_E^{p-2}(u)d\nu(u).$$
In addition, if the measure ν satisfies the subspace concentration condition, Problem M_0^ν has a unique solution. Moreover, an origin-symmetric ellipsoid E solves Problem M_0^ν if and only if the following holds: for all $x\in\mathbb{R}^n$,

$$\frac{h^2_{E^*}(x)}{n}=\int_{S^{n-1}}\left(\frac{|x\cdot u|}{h_E(u)}\right)^2 d\nu(u).$$

Theorems 6.4 and 6.5 are equivalent. Indeed, by Lemma 6.1, one gets that if E_m solves Problem $\overline{M}_p^\nu$ for $p \geq 0$, $\|h_{E_m}\|_{L_\nu^p}^{-1} E_m$ is a solution to Problem M_p^ν. Together with Theorem 6.5, we can get Theorem 6.4.

Theorem 6.5 shows the uniqueness of solutions to Problem M_p^ν for $0 \leq p < \infty$. The case for $p = \infty$ follows from the same lines as the proof for Theorem 3.9. The proof will be omitted.

Theorem 6.6. *Let ν be a probability measure which is not concentrated on any great subsphere. Then Problem M_∞^ν has a unique solution.*

From Theorems 6.4, 6.5 and 6.6, one has the following definition.

Definition 6.7. Let $0 \leq p \leq \infty$. Suppose that ν is a probability measure which is not concentrated on any great subsphere for $0 < p \leq \infty$ and satisfies the subspace concentration condition for $p = 0$. Define E_p^ν, the general L_p John ellipsoid for the measure ν , to be the unique solution of the following optimization problem:

$$\max_{E \in \mathcal{E}_o^n} V(E) \quad \text{subject to} \quad \|h_E\|_{L_\nu^p} = 1.$$

Similarly, define $\overline{E}_p^\nu$, the normalized general L_p John ellipsoid for the measure ν, to be the unique solution of the following optimization problem:

$$\min_{E \in \mathcal{E}_o^n} \|h_E\|_{L_\nu^p} \quad \text{subject to} \quad V(E) = \omega_n.$$

Note that our new general L_p John ellipsoid is just related to a given measure ν, which is different from the L_p John ellipsoid that constructed from a given convex body K. If $d\nu(\cdot) = dS_{\mu,p}(K,\cdot)/|S_{\mu,p}(K,\cdot)|$, this new general L_p John ellipsoid coincides with the L_p John ellipsoid for the general measure μ up to a constant $|S_{\mu,p}(K,\cdot)|^{-\frac{1}{\alpha-n}}$. Hence, it contains the L_p John ellipsoid, the classical John ellipsoid, the Lutwak-Yang-Zhang ellipsoid, and Petty ellipsoid as special cases.

Properties and inequalities, if applicable, should take similar formulas to those in Chapters 4 and 5, and the proofs should follow along the same manner.

Chapter 7

Conclusion and Future Works

In this thesis, we study the existence and uniqueness of solutions to optimization problems ($M_{\mu,p}$, $\overline{M}_{\mu,p}$, M_p^ν and $\overline{M}_p^\nu$) for $0 \le p \le \infty$, as well as the characterization of the solutions. These solutions naturally result in new families of L_p John ellipsoids for $0 \le p \le \infty$ which contain many well-known ellipsoids constructed from given convex bodies as the special cases, including but not limited to the classical John ellipsoid, the L_p John ellipsoid, the Lutwak-Yang-Zhang ellipsoid, the Petty ellipsoid, etc. We also prove some inequalities and properties regarding these newly introduced L_p John ellipsoids, including the L_p John's inclusion. Moreover, convex bodies with identical John and L_p John ellipsoids for general measures are characterized.

In the literature, all results for the L_p John ellipsoids are for $0 \le p \le \infty$ and so are the results in this thesis. A future project of interest is to investigate the L_p John ellipsoids for $p < 0$ for both $S_{\mu,p}(K, \cdot)$ and for ν. It would be nice to have some results even for the case of $S_p(K, \cdot)$.

The Löwner ellipsoid, the ellipsoid containing K with minimal volume, is known to be dual, in some sense, to the John ellipsoid. It will be interesting to study the Löwner ellipsoid for both $S_{\mu,p}(K, \cdot)$ and for ν. In particular, the connections of the Löwner ellipsoid for general measures and our L_p John ellipsoids as well as their applications are in great demand. We leave them as future works.

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Appendix A

Some necessary lemmas

As mentioned in [6], the L_p addition can be extended to $p \geq 0$. Let $K, L \in \mathcal{K}_o^n$ and $\varepsilon > 0$. The L_p addition $K +_p \varepsilon \cdot L$ of K and L for $p > 0$ is definned by

$$K +_p \varepsilon \cdot L = \bigcap_{u \in S^{n-1}} \left\{ x \in \mathbb{R}^n : x \cdot u \leq \left(h_K^p(u) + \varepsilon h_L^p(u)\right)^{1/p} \right\}. \tag{A.1}$$

Note that the function $\left(h_K^p(u) + \varepsilon h_L^p(u)\right)^{1/p}$ is the support function of a convex body $K +_p \varepsilon \cdot L$ when $p \geq 1$. But when $0 < p < 1$, the convex body $K +_p \varepsilon \cdot L$ is the Wulff shape of $\left(h_K^p(u) + \varepsilon h_L^p(u)\right)^{1/p}$.

Suppose that $f_\varepsilon(u) = f(\varepsilon, u) : I \times S^{n-1} \to (0, \infty)$ is a continuous function where $I \subset \mathbb{R}$ is an interval containing 0. For fixed $\varepsilon \in I$, let

$$K_\varepsilon = \bigcap_{u \in S^{n-1}} \left\{ x \in \mathbb{R}^n : x \cdot u \leq f_\varepsilon(u) \right\},$$

be the Wulff shape (or Aleksandrov body) associated with the function f_ε. Clearly,

$$h_{K_\varepsilon} \leq f_\varepsilon.$$

Moreover, it is well-known that

$$h_{K_\varepsilon} = f_\varepsilon \quad \text{a.e. with respect to } S(K_\varepsilon, \cdot). \tag{A.2}$$

Lemma A.1. (Aleksandrov's convergence lemma) *Assume that h_ε converges to h_0 uniformly on S^{n-1} as $\varepsilon \to 0$, then K_ε converges to K_0 in the Hausdorff metric.*

The following result gives a variational formula for $0 < p < 1$. An Orlicz analogue has been given in [41].

Theorem A.2. *Suppose $r > 0$ and $K, L \in \mathcal{K}_o^n$. Suppose that $g : \mathbb{R}^n \to \mathbb{R}_+$ is the r-homogeneous and $\frac{1}{r}$-concave density of the α-homogeneous μ on $\mathbb{R}^n$. For $0 < p < 1$, define the L_p-mixed volume of K and L with respect to measure μ by*

$$V_{\mu,p}(K,L) = \frac{p}{\alpha} \lim_{\varepsilon \to 0^+} \frac{\mu(K +_p \varepsilon \cdot L) - \mu(K)}{\varepsilon}. \tag{A.3}$$

Then

$$V_{\mu,p}(K,L) = \frac{1}{\alpha} \int_{S^{n-1}} h_L^p(u) dS_{\mu,p}(K,u), \tag{A.4}$$

where $dS_{\mu,p}(K,\cdot)$ is the L_p-surface μ-area measure of K and $\alpha = n + r$.

Proof. Consider the Wulff shape $K_\varepsilon = K +_p \varepsilon \cdot L$ defined in (A.1), let

$$f_\varepsilon(u) = \left(h_K^p(u) + \varepsilon h_L^p(u)\right)^{1/p}. \tag{A.5}$$

It follows from (2.13) and (2.14) that

$$\mu(K_\varepsilon) = V_{\mu,1}(K_\varepsilon, K_\varepsilon) \qquad \mu(K) = V_{\mu,1}(K,K). \tag{A.6}$$

Since f_ε converges uniformly to h_K on S^{n-1} as $\varepsilon \to 0$, the Aleksandrov's convergence lemma A.1 yields that K_ε converges to K in the Hausdorff metric. Together with (2.9), one has $S_{\mu,p}(K_\varepsilon,\cdot) \to S_{\mu,p}(K,\cdot)$ weakly on S^{n-1} as $\varepsilon \to 0$.

Applying Lemma 4.12 for $p = 1$, we get

$$V_{\mu,1}(K,L) \geq \mu(K)^{\frac{\alpha-1}{\alpha}} \mu(L)^{\frac{1}{\alpha}}. \tag{A.7}$$

Together with (A.2), (A.3), (A.6) and (A.7), we obtain that

$$\begin{aligned}
\liminf_{\varepsilon \to 0^+} \mu(K_\epsilon)^{\frac{\alpha-1}{\alpha}} \frac{\mu(K_\epsilon)^{\frac{1}{\alpha}} - \mu(K)^{\frac{1}{\alpha}}}{\varepsilon} &= \liminf_{\varepsilon \to 0^+} \frac{\mu(K_\varepsilon) - \mu(K_\varepsilon)^{\frac{\alpha-1}{\alpha}} \mu(K)^{\frac{1}{\alpha}}}{\varepsilon} \\
&\geq \liminf_{\varepsilon \to 0^+} \frac{\mu(K_\varepsilon) - V_{\mu,1}(K_\epsilon, K)}{\varepsilon} \\
&= \liminf_{\varepsilon \to 0^+} \frac{V_{\mu,1}(K_\varepsilon, K_\varepsilon) - V_{\mu,1}(K_\varepsilon, K)}{\varepsilon} \\
&= \liminf_{\varepsilon \to 0^+} \frac{1}{\alpha} \int_{S^{n-1}} \frac{f_\varepsilon(u) - h_K(u)}{\varepsilon} dS_{\mu,1}(K_\varepsilon, u).
\end{aligned}$$

Similarly, one gets

$$\begin{aligned}\limsup_{\varepsilon\to0^+}\mu(K)^{\frac{\alpha-1}{\alpha}}\frac{\mu(K_\epsilon)^{\frac{1}{\alpha}}-\mu(K)^{\frac{1}{\alpha}}}{\varepsilon} &= \limsup_{\varepsilon\to0^+}\frac{\mu(K)^{\frac{\alpha-1}{\alpha}}\mu(K_\varepsilon)^{\frac{1}{\alpha}}-\mu(K)}{\varepsilon}\\ &\le \limsup_{\varepsilon\to0^+}\frac{V_{\mu,1}(K,K_\varepsilon)-\mu(K)}{\varepsilon}\\ &= \limsup_{\varepsilon\to0^+}\frac{V_{\mu,1}(K,K_\varepsilon)-V_{\mu,1}(K,K)}{\varepsilon}\\ &= \limsup_{\varepsilon\to0^+}\frac{1}{\alpha}\int_{S^{n-1}}\frac{f_\varepsilon(u)-h_K(u)}{\varepsilon}\,dS_{\mu,1}(K,u).\end{aligned}$$

Combining the above two inequalities, we get that

$$\mu(K)^{\frac{\alpha-1}{\alpha}}\lim_{\varepsilon\to0^+}\frac{\mu(K_\varepsilon)^{\frac{1}{\alpha}}-\mu(K)^{\frac{1}{\alpha}}}{\varepsilon}=\lim_{\varepsilon\to0^+}\frac{1}{\alpha}\int_{S^{n-1}}\frac{f_\varepsilon(u)-h_K(u)}{\varepsilon}\,dS_{\mu,1}(K,u).$$

Moreover, from (A.5) and the fact that $f_0=h_K$, one has

$$\begin{aligned}\lim_{\varepsilon\to0^+}\frac{\mu(K_\varepsilon)-\mu(K)}{\varepsilon} &= \alpha\mu(K)^{\frac{\alpha-1}{\alpha}}\lim_{\varepsilon\to0^+}\frac{\mu(K_\varepsilon)^{\frac{1}{\alpha}}-\mu(K)^{\frac{1}{\alpha}}}{\varepsilon}\\ &= \lim_{\varepsilon\to0^+}\int_{S^{n-1}}\frac{f_\varepsilon(u)-f_0(u)}{\varepsilon}\,dS_{\mu,1}(K,u)\\ &= \frac{1}{p}\int_{S^{n-1}}h_K^{1-p}(u)h_L^p(u)\,dS_{\mu,1}(K,u),\end{aligned}$$

which is the desired formula. □