Initial Value Problems in General Relativity

by

© *Sharmila Gunasekaran*

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Abstract

The initial value formulation viewpoint is one of the main foci of research in general relativity. This thesis establishes results for two problems pertinent to it.

In the first of the problems which form the contents of Chapters 3 and 4, the focus is on a class of five dimensional stationary asymptotically flat spacetimes. These naturally arise in high energy physics as possible microstates of black hole spacetimes. In Chapter 3, several spacetimes with non-trivial topology in the domain of outer communication are considered, of which the soliton spacetime is one such example. The mass variation formula previously established for such spacetimes is used to compute energy, angular momenta and charge for these spacetimes. It is shown that regularity is essential for the formula relating them to hold. In Chapter 4, the decay of the wave equation in a family of soliton spacetimes is studied and a slow decay rate is established, hinting at nonlinear instability.

The second problem is establishing a horizon-based initial boundary value formulation with the goal of studying the near-horizon spacetime. The problem is addressed in the setting of four-dimensional general relativity. In Chapter 5, we establish that data specified on the horizon and a future null boundary determine the near horizon geometry and illustrate this for spherically symmetric spacetimes with a massless scalar field. In Chapter 6 we conclude with directions for future research.

Key words : general relativity, initial value formulation, linear waves, boundary conditions, near horizon geometry

Lay Summary

Partial differential equations (PDEs) are used to model the evolution of a wide range of physical systems. The equations governing gravitational physics which are given by the theory of General Relativity (GR) are a set of PDEs which relate the curvature of spacetime to the concentration of matter and energy in the Universe. These equations, called the Einstein equations, were not immediately pliant to analysis using the known tools from PDEs. These were later reformulated appropriately to represent a "viable" system. This reformulation showed that the Einstein equations govern the evolution of a classical, deterministic system i.e., a system where the future state can be derived from the knowledge of the present or initial conditions of the system. We accordingly call this the initial value formulation. There are two aspects of the Einstein equations addressed in this thesis. In the first we use the standard (re)formulation to understand the long time behaviour of simple perturbations. This is an important aspect as the Einstein equations are nonlinear and solutions can become singular in the future even though they evolved from well-behaved initial conditions. We conclude here that the solution we have looked at might be unstable for generic (nonlinear) perturbations. The second aspect we look at is a formulation of the Einstein equations that is applicable for understanding the dynamics near a

black hole horizon. The standard initial value formulation addresses the evolution of a gravitational system starting from a "moment of time". This cannot be easily adapted to understand near horizon phenomena in black holes. We have hence developed a formulation that is more suited for understanding the near horizon dynamics and evolution of spherical black holes.

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Chapter 1

Introduction

This thesis is written in a manuscript format. The manuscripts in this thesis are contained in Chapters 3, 4 and 5. They are connected through the initial value formulation of General Relativity (GR). This chapter will introduce GR and the initial value formulation and elaborate on the connection between chapters.

1.1 Background

1.1.1 Lorentzian geometry

General Relativity is a geometric theory describing gravity in which the universe is modeled as a four dimensional *Lorentzian manifold* (M, g). In this section we will collect some basic results on Lorentzian geometry from [1], [2] and [3].

Definition 1.1.1 (Lorentzian manifold)**.** *A Lorentzian manifold* (M, g) *is a differentiable manifold equipped with the metric* $g : T_pM \times T_pM \to \mathbb{R}$ *which is a symmetric, bilinear and non-degenerate form of signature* $(-, +, +, +)$ *such that in an orthonor-*

mal basis $\{e_0, e_1, e_2, e_3\}$

$$g(e_\alpha, e_\beta) = \eta_{\alpha\beta}, \tag{1.1}$$

where $\eta = \mathrm{diag}(-1, 1, 1, 1)$.

This definition extends in an obvious way when there are additional spatial dimensions. Hence, each tangent space is isometric to the four dimensional Minkowski space, $\mathbb{M}^{1,3}$. The Minkowski space is the Lorentzian analogue of Euclidean space $\mathbb{R}^{3+1}$. It can also be seen as $\mathbb{R}^4$ endowed with the Minkowski metric η defined below. For vectors X and $Y \in T_p\mathbb{R}^{3+1}$ given in Cartesian coordinates by

$$X = X^i \frac{\partial}{\partial x^i} \quad \text{and} \quad Y = Y^i \frac{\partial}{\partial x^i}, \tag{1.2}$$

we define the Minkowski metric η by

$$\eta(X, Y) = -X^0 Y^0 + \sum_{i=1}^{3} X^i Y^i = \eta_{ij} X^i Y^j, \tag{1.3}$$

where $\eta_{ij} = \varepsilon_i \delta_{ij}$ and $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3) = (-1, 1, 1, 1)$.

From (1.1), it can be seen that the 'norm' of a vector X can be positive, negative or zero. This leads to the causal character of vectors. At any point $p \in M$, $X \in T_pM$ can be classified into timelike, null or spacelike as follows,

$$X \quad \text{is} \quad \begin{cases} \text{timelike if } \ g(X,X) < 0 \\ \text{null if } \ g(X,X) = 0 \\ \text{spacelike if } \ g(X,X) > 0 \end{cases} . \tag{1.4}$$

We can extend this notion to causal curves $\gamma : (a, b) \to M$ as follows :

$$\gamma \quad \text{is} \begin{cases} \text{timelike if } \gamma^{'}(t) \text{ is timelike } \forall t \in (a, b) \\ \text{null if } \gamma^{'}(t) \text{ is null } \forall t \in (a, b) \\ \text{spacelike if } \gamma^{'}(t) \text{ is spaceline } \forall t \in (a, b) \end{cases} \quad . \tag{1.5}$$

γ is called a causal curve if it is timelike or null. The collection of null vectors forms a double cone N_p in T_pM. One half cone may be designated as the future cone (N_p^+) and the other as the past light cone (N_p^-) at p. At each point $p \in M$ the set of timelike vectors forms two disjoint open cones, which we will denote as I_p^+ and I_p^-. These are the interiors of the future and past light cones respectively. The causal structure at every point is illustrated in the figure below.

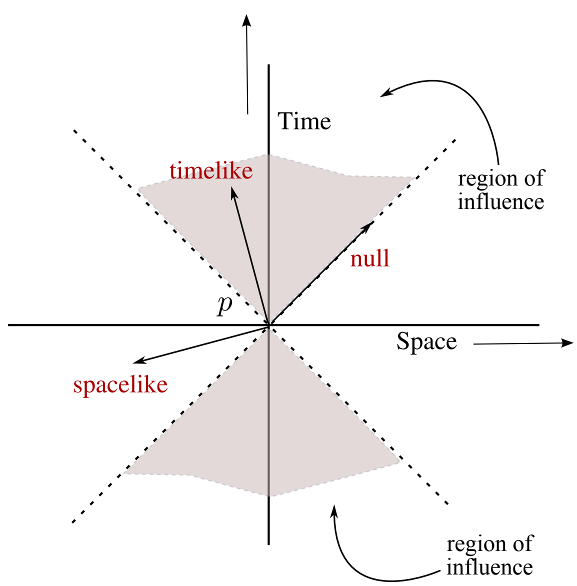

Figure 1.1: Causal structure in a Lorentzian manifold

If the assignment of a future light cone at each point can be carried out in a continuous manner, then M is time-orientable. This results in the following - a Lorentzian manifold (M, g) is time orientable if and only if it admits a smooth timelike vector field T. The choice of such a T fixes a time-orientation on M. Time-orientability is analogous to but distinct from the notion of orientability of a topological manifold [4].

Definition 1.1.2 (Spacetime)**.** *A spacetime (M, g) is a connected, time-orientable Lorentzian manifold.*

In this thesis, our discussion of Lorentzian manifolds is restricted to spacetimes. Time orientation gives a classification for causal vectors as future/past pointing. If T is the vector field fixing the time orientation on (M, g), then, for any nonzero causal vector $V \in T_pM$, $g(V, T)$ is either positive or negative. If $g(V, T)$ is negative we say that V is future directed and it would lie in I_p^+. If $g(V, T)$ is positive, we say that V is past directed and V then lies in I_p^-. Similarly a causal curve γ is said to be future directed if γ' is future directed at each point along γ.

We say $p \ll q$ if there is a future pointing timelike curve in M from p to q, and $p < q$ if there is a future pointing causal curve in M from p to q. $p \leq q$ means that either $p = q$ or $p < q$.

Definition 1.1.3. *Let $A \subset M$.*

$$I^+(A) = \{p \in M : q \ll p \textit{ for some } q \in A\} \tag{1.6}$$

$$J^+(A) = \{p \in M : q \leq p \textit{ for some } q \in A\} \tag{1.7}$$

$I^+(A)$ is called the chronological future of A and $J^+(A)$ is called the causal future of A. The past sets $I^-(A)$ and $J^-(A)$ are similarly defined. The sets $I^\pm(p)$ are open. We have the following fundamental causality result.

Proposition 1.1.4. *If $q \in J^+(p) \setminus I^+(p)$, i.e., if q is in the causal future of p but not in the timelike future of p, then any future-directed causal curve from p to q must be a null geodesic.*

On physical grounds, one needs to impose an appropriate causality condition on our spacetimes in order to prohibit pathologies (such as closed timelike curves). One such condition that rules out both closed and almost closed causal curves is the *strong causality condition.* The strong causality condition holds at $p \in M$ if, given any neighbourhood U of p, there is a neighbourhood $V \subset U$ of p such that every causal curve segment with endpoints in V lies entirely in U. A spacetime (M, g) is said to be strongly causal if strong causality holds at each point $p \in M$. Strong causality implies the following :

Lemma 1.1.5. *Suppose that strong causality holds in a spacetime (M, g). Let K be a compact subset of M. If $\gamma : [0, b) \to M$ is a future inextendible causal curve that starts in K, then it eventually leaves K and does not return, i.e., $\exists t_0 \in [0, b)$ such that $\gamma(t) \notin K \, \forall \, t \in [t_0, b)$.*

What the above lemma means is that in a spacetime on which strong causality holds, a future or past inextendible causal curve cannot be imprisoned forever within a compact set. In particular this excludes closed timelike curves. In Riemannian geometry geodesically complete manifolds have some nice properties. In Lorentzian geometry, *global hyperbolicity* plays a similar role.

Definition 1.1.6. *(M, g) is globally hyperbolic if it is strongly causal and for every pair $p < q$, the set $J(p, q) = J^+(p) \cap J^-(q)$ is compact ("called internal compactness").*

This property is very important for the solvability of hyperbolic PDEs. Internal compactness means that $J^+(p) \cap J^-(q)$ does not contain any singularities or points on the edge of spacetime i.e., infinity. Global hyperbolicity guarantees the existence of maximal timelike geodesic segments joining timelike separated points.

Theorem 1.1.7. *If M is globally hyperbolic and $q \in I^+(p)$, then,*

1. *there exists a timelike geodesic segment γ from p to q*

2. *γ is maximal i.e., $L(\gamma) \geq L(\sigma)$ for all future directed causal curves σ from p to q, where $L(\sigma)$ stands for the length of curve σ.*

Global hyperbolicity is also connected to the strong cosmic censorship conjecture introduced by Roger Penrose [4], which says that generically (globally hyperbolic) solutions to the Einstein equations do not admit observable singularities. Some of the consequences of global hyperbolicity are :

Theorem 1.1.8. *Let (M, g) be a globally hyperbolic spacetime. Then*

1. *The sets $J^\pm(A)$ are closed, for all compact subsets $A \subset M$.*

2. *The sets $J^+(A) \cap J^-(B)$ are compact, for all compact subsets $A, B \subset M$.*

3. *If we have convergent sequences on M, $p_n \to p$ and $q_n \to q$ and $p_n \leq q_n$, then $p \leq q$, i.e., the causality relation $\leq$ is closed on M.*

$\Sigma \subset M$ is called *achronal* if there is no pair of points $p, q \in \Sigma$ that can be connected by a timelike curve. Let $\Sigma \subset M$ be achronal, we define the future ($D^+(\Sigma)$) and past domains of dependence ($D^-(\Sigma)$) (also called Cauchy developments) of Σ as follows :

$$D^+(\Sigma) = \{p \in M\colon \text{every past inextendible causal curve from } p \text{ meets } \Sigma\} \tag{1.8}$$

$$D^-(\Sigma) = \{p \in M\colon \text{every future inextendible causal curve from } p \text{ meets } \Sigma\}\,. \tag{1.9}$$

The domain of dependence of Σ is $D(\Sigma) = D^+(\Sigma) \cup D^-(\Sigma)$. Since information travels along causal curves, $D(\Sigma)$ consist of the set of points in spacetime which are (potentially) influenced by every point in the set Σ, to either the past or the future.

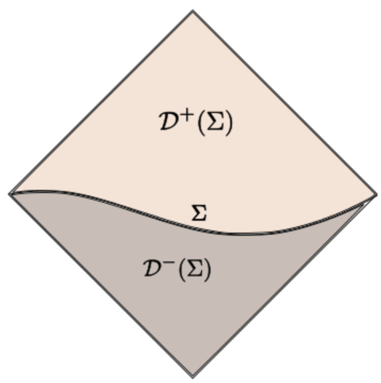

Figure 1.2: Domains of dependence of an achronal surface Σ

If physics is to be deterministic then initial data on Σ should completely determine the state of the fields on all of $D(\Sigma)$. Domains of dependence are tied to global hyperbolicity through the following proposition.

Proposition 1.1.9. *Let $\Sigma \subset M$ be achronal.*

1. *Strong causality holds on int $D(\Sigma)$.*

2. *Internal compactness holds on int* $D(\Sigma)$*, i.e., for all* $p, q \in \text{int}D(\Sigma)$*, with* $p < q$ $J^+(p) \cap J^-(q)$ *is compact.*

We wish to find a condition on an achronal subset Σ that will insure that the domain of dependence of Σ is all of M i.e., $D(\Sigma) = M$. The significance of this will be clear when we try to approach an analytical theory (namely the Einstein field equations) via an evolutionary perspective by prescribing initial data on Σ and try to determine the spacetime metric by solving a system of PDEs in $D(\Sigma)$.

Definition 1.1.10. *A Cauchy surface S is an achronal subset of M which is met exactly once by every inextendible causal curve in M.*

If Σ is a Cauchy surface for M then $\Sigma = \partial I^+(\Sigma) = \partial I^-(\Sigma)$ which means that Σ is a closed C^0 hypersurface [4]. The existence of Cauchy surfaces and global hyperbolicity for the entire spacetime are closely connected.

Theorem 1.1.11. *Let* M *be a spacetime.*

1. *If* M *is globally hyperbolic then it admits a Cauchy surface.*

2. *If* Σ *is a Cauchy surface for* M *then* M *is homeomorphic to* $\mathbb{R} \times \Sigma$*.*

Thus we see that for globally hyperbolic spacetimes, the topology of a Cauchy surface Σ determines the topology of the entire spacetime. We have the following proposition :

Proposition 1.1.12. *If a spacetime has a Cauchy surface* Σ *then* $D(\Sigma) = M$*.*

In summary a spacetime M is globally hyperbolic if and only if it admits a Cauchy surface Σ. Moreover, a globally hyperbolic spacetime has topology is $\mathbb{R} \times \Sigma$ and

$D(\Sigma) = M$, where Σ is any Cauchy surface for M. A Cauchy surface Σ in a spacetime (M, g) inherits a natural geometry as a submanifold.

1.1.2 Einstein equations and the 3+1 formulation

We will now turn to studying evolution in GR by prescribing data on a Cauchy surface. The relevant geometric data on M is the induced metric h (Riemannian as S is spacelike) and second fundamental form K. The Einstein Field Equations (or simply the Einstein equations) given below encode gravitational physics in a Lorentzian manifold :

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{1.10}$$

Here $g_{\mu\nu}$ should be thought of as the unknown Lorentzian metric and $T_{\mu\nu}$ represents the contribution of matter in the universe and obeys a conservation equation

$$\nabla^{\mu} T_{\mu\nu} = 0, \tag{1.11}$$

which follows from the Bianchi identity. $R_{\mu\nu}$ and R are the Ricci tensor and Ricci scalar respectively. Equation (1.10) along with equations governing the matter $T_{\mu\nu}$ form a closed set of equations. Equation (1.10) is a system of coupled second order nonlinear Partial Differential Equations (PDEs) for the metric functions $g_{\mu\nu}$. GR is a diffeomorphism invariant theory which means that it takes the same form in any coordinate system. This is required of any physical theory as phenomena arising from the theory should not depend on the choice of coordinates on the spacetime. More precisely, if $\mathcal{F} : M \to M$ is a diffeomorphism, then (M, g) and $(M, \mathcal{F}^{*} g)$ represent the same spacetime. Hence, the uniqueness of a solution to the system (1.10) does not hold in a straightforward way, as a given solution has different coordinate representations.

The initial value formulation resolves this uniqueness issue. It establishes local existence and uniqueness for the system (1.10). The idea of determinism captured by any initial value formulation is the following - given initial conditions for a system, if the system is allowed to evolve without outside interference, the dynamical evolution of the system is completely determined by the theory. We expect any physically viable theory to have an initial value formulation. As a simpler example of a system of PDEs, let us consider electromagnetism. The Maxwell equations read,

$$\nabla \cdot \mathbb{E} = \frac{\rho}{\varepsilon_0}\,, \tag{1.12}$$

$$\nabla \cdot \mathbb{B} = 0\,, \tag{1.13}$$

$$\nabla \times \mathbb{E} = -\frac{\partial \mathbb{B}}{\partial t}\,, \tag{1.14}$$

$$\nabla \times \mathbb{B} = \mu_0 \left(\mathbb{J} + \varepsilon_0 \frac{\partial \mathbb{E}}{\partial t} \right). \tag{1.15}$$

Here, $\mathbb{B}$ and $\mathbb{E}$ are the magnetic and electric fields respectively. ρ is the electric charge distribution and $\mathbb{J}$ is the total current density. The physical constants μ_0 and ε_0 are the magnetic permeability and vacuum permittivity respectively. (1.12),(1.13), (1.14) and (1.15) are 8 coupled first order PDEs for six unknown functions (components of $\mathbb{B}$ and $\mathbb{E}$). Out of the 8 equations, only (1.14) and (1.15), a total of 6 equations describe dynamics i.e., evolution with time. These are appropriately called evolution equations. The remaining two scalar equations involving the divergence of $\mathbb{B}$ and $\mathbb{E}$ only restrict the fields at a given time and hence are called constraint equations. It can be shown that the constraint equations need to be satisfied only at an instant of time (say the initial time) and then the evolution equations guarantee that they are satisfied at all later times. Using (1.14) and (1.15) and their divergence,

$$\frac{\partial(\nabla\cdot\mathbb{B})}{\partial t} = -\nabla\cdot(\nabla\times\mathbb{E}) = 0 \tag{1.16}$$

$$\frac{\partial(\nabla\cdot\mathbb{E})}{\partial t} = \frac{1}{\varepsilon_0\mu_0}\left(\mu_0\nabla\cdot\mathbb{J} - \nabla\cdot(\nabla\times\mathbb{B})\right)$$

$$\Longrightarrow \frac{\partial\left(\nabla\cdot\mathbb{E} - \dfrac{\rho}{\varepsilon_0}\right)}{\partial t} = 0 \tag{1.17}$$

where in the last step we have used the continuity equation,

$$\nabla\cdot\mathbb{J} = -\frac{\partial\rho}{\partial t}. \tag{1.18}$$

This is usually described by the phrase "the constraints propagate".

The first step to studying any PDE is to classify the system as elliptic, parabolic or hyperbolic. For simplicity, we will consider the vacuum Einstein equations, where $T = 0$. By taking the trace, one sees that $R(g) = 0$, so that the Einstein equations reduce to

$$R_{\mu\nu} = 0. \tag{1.19}$$

We would like to understand this as a system of PDEs for the unknown metric g by trying to understand equation (1.19) as an equation consisting of derivatives for the metric components relative to a coordinate basis $\{\partial_\mu\}$ of T_pM. We recall that the Christoffel symbols are defined by,

$$\nabla_{\partial_\alpha}\partial_\beta = \Gamma^\gamma_{\alpha\beta}\partial_\gamma, \tag{1.20}$$

where ∇ is the Levi-Civita connection (covariant derivative). In terms of the metric,

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}g^{\gamma\delta}\left(\partial_\beta g_{\alpha\delta} + \partial_\alpha g_{\beta\delta} - \partial_\delta g_{\alpha\beta}\right). \tag{1.21}$$

The Riemann curvature is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{1.22}$$

Relative to a coordinate basis, components of the Riemann curvature tensor are defined by

$$R^{\alpha}_{\beta\gamma\delta} = \langle \mathrm{d}x^{\alpha}, R(\partial_\gamma, \partial_\delta)\partial_\beta \rangle. \tag{1.23}$$

This leads to the formula

$$R_{\alpha\beta\gamma\delta} = \partial_\gamma \Gamma^{\alpha}_{\beta\delta} - \partial_\delta \Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\alpha}_{\sigma\gamma}\Gamma^{\sigma}_{\beta\delta} - \Gamma^{\alpha}_{\sigma\delta}\Gamma^{\sigma}{}_{\beta\gamma}\,, \tag{1.24}$$

with the components of Ricci tensor given by

$$R_{\alpha\beta} = R^{\gamma}_{\alpha\gamma\beta}. \tag{1.25}$$

This shows that the Ricci tensor is linear in the second derivatives of the metric, with coefficients which are rational in the components of the metric, and quadratic in the first derivatives of the metric, with coefficients which are rational in g. Thus the vacuum Einstein equations are a second order system of quasi-linear (linear in the highest order derivatives with coefficients depending on the independent variables as well as the functions $g_{\mu\nu}$) partial differential equations for the unknown metric g. Explicitly these equations are

$$R_{\mu\nu} = \frac{g^{\alpha\beta}}{2}\left(\partial_\mu\partial_\alpha g_{\beta\nu} + \partial_\nu\partial_\alpha g_{\beta\mu} - \partial_\mu\partial_\nu g_{\alpha\beta} - \partial_\alpha\partial_\beta g_{\mu\nu}\right) + \mathcal{N}(g, \partial g) = 0. \tag{1.26}$$

$\mathcal{N}(g, \partial g)$ collects lower order terms. However, in contrast to the Maxwell equations, the Einstein equations do not have any obvious structure (parabolic, hyperbolic or elliptic) in an arbitrary coordinate system. The most significant difficulty with (1.26) from the PDE point of view is the high degree of non-uniqueness owing to diffeomorphism invariance. In the language of physics, one says that the diffeomorphism group expresses the gauge freedom of the Einstein field equations. Quite remarkably, in 1952, [5], Yvonne Choquet-Bruhat proved that there is an underlying system of hyperbolic PDE governing the behaviour of (1.26). The proof involves the introduction of a special set of coordinates (which in particular, breaks the diffeomorphism invariance) and the exploitation of the Bianchi identity together with the Einstein constraint equations to obtain a solution of the geometric equation.

We will briefly review the proof here. We start by understanding the geometry of spacelike hypersurfaces. Let (M, g) be a spacetime and let

$$i : \Sigma \hookrightarrow M \tag{1.27}$$

be an embedded spacelike hypersurface. This means that the induced metric $h = i^*(g)$ on Σ is Riemannian (positive definite signature). Let τ denote the timelike future-pointing unit normal vector field to Σ. If we let ∇ be the Levi-Civita connection on (M, g) and ∇^Σ be the Levi-Civita connection on (Σ, h) the second fundamental form, K on Σ, is defined by considering vector fields X and Y tangent to Σ and setting

$$\nabla_X Y = \nabla^\Sigma_X Y + K(X, Y)\tau \tag{1.28}$$

so that for each $p \in \Sigma$

$$K : T_p\Sigma \times T_p\Sigma \to \mathbb{R}. \tag{1.29}$$

Note that, using the fact that ∇ is torsion free and compatible with g one can see that

$$K(X, Y) = g(\nabla_X \tau, Y) \implies K(X, Y) = K(Y, X) \tag{1.30}$$

so K is symmetric. A time function t on (M, g) is said to be adapted to Σ if Σ is a level set of t. If $x = \{x^i\}$ are local coordinates on Σ then (x^i, t) form adapted local coordinates for M near Σ. With respect to such a coordinate system, the lapse-shift form for the vector field is

$$\tau = N^{-1}\left(\partial_t - \mathcal{V}^i \partial_{x^i}\right), \tag{1.31}$$

where N is called the lapse and $\mathcal{V}$ is called the shift vector field. This freedom is a consequence of the diffeomorphism invariance in the theory. Let $T = \partial_t$. The relation between T, N, $\mathcal{V}$ and the unit normal to the hypersurface τ is $T^a = N\tau^a + \mathcal{V}^a$ (depicted in Fig.1.3).

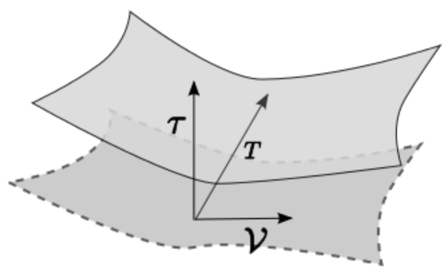

Figure 1.3: Relation of the evolution vector T to the lapse and shift

In terms of h, N and $\mathcal{V}$, the ambient metric on M is expressed in these coordinates by

$$g = -N\mathrm{d}t^2 + h_{ij}\left(\mathrm{d}x^i + \mathcal{V}^i \mathrm{d}t\right)\left(\mathrm{d}x^j + \mathcal{V}^j \mathrm{d}t\right) \tag{1.32}$$

and the second fundamental form is given by

$$K_{ij} = K(\partial_{x^i}, \partial_{x^j}) = \frac{1}{2}N^{-1}\left(\frac{\partial h_{ij}}{\partial t} - \mathcal{L}_{\mathcal{V}} h_{ij}\right) \tag{1.33}$$

where $\mathcal{L}_{\mathcal{V}} h_{ij}$ is the Lie derivative of the spatial metric h in the direction $\mathcal{V}$. In particular this gives a formula for the time derivative of the spatial metric

$$\frac{\partial}{\partial t} h_{ij} = 2NK_{ij} + \mathcal{L}_{\mathcal{V}} h_{ij}. \tag{1.34}$$

In the special case when $N = 1$ and $\mathcal{V} = 0$, we have

$$\frac{\partial}{\partial t} h_{ij} = 2K_{ij}. \tag{1.35}$$

We also give the evolution equation for K_{ij} here.

$$\begin{aligned} \frac{\partial}{\partial t} K_{ij} =& \nabla^{\Sigma}{}_i \nabla^{\Sigma}_j N - N({}^{\Sigma}R_{ij} + KK_{ij} - 2K_i^k K_{jk}) \\ &+ \mathcal{L}_{\mathcal{V}} K_{ij} + \left(e_i^{\alpha} e_i^{\beta} - \frac{1}{2} h_{ij} g^{\alpha\beta} \right) G_{\alpha\beta}. \end{aligned} \tag{1.36}$$

The Gauss and Codazzi equations for $\Sigma \hookrightarrow M$ tell us that the ambient Einstein equations on M impose a relationship between the intrinsic and extrinsic curvatures of $(\Sigma, h) \hookrightarrow (M, g)$ and the components of the stress-energy-momentum tensor $T_{\mu\nu}$ in a local adapted frame.

Proposition 1.1.13 (Einstein Constraint Equations)**.** *If (M, g) is a spacetime satisfying the Einstein field equations and $\Sigma \hookrightarrow M$ is a spacelike hypersurface with induced Riemannian metric h and second fundamental form K then*

$$R(h) - |K|_h^2 - (\mathrm{tr}_h K)^2 = 16\pi T_{00} = 2G_{00} = 2\rho \tag{1.37}$$

$$(\mathrm{div} K)_i - \nabla_i{}^{\Sigma}(\mathrm{tr}_h K) = 8\pi T_{0i} = G_{0i} = J_i \tag{1.38}$$

where $(\mathrm{div} K)_i = \nabla^{\Sigma}_j K_i^j$.

The scalar function ρ is the local mass density and the vector field J is the local current density of the initial data set (Σ, h, K). (1.37) is the Hamiltonian constraint equation and (1.38) is the momentum constraint equation.

1.1.3 Einstein equations as hyperbolic PDEs

Shortly we will see that the Einstein equations are just a system of hyperbolic equations. So, we review the existence and uniqueness result for the most basic hyperbolic PDE i.e., the wave equation on a curved background. For a scalar function ϕ on a spacetime (M, g) the wave operator associated to the metric g is the operator given by the trace of the Hessian :

$$\begin{aligned}\Box_g \phi &= \nabla_\mu \nabla^\mu \phi \\ &= \frac{1}{\sqrt{\text{-det} g_{\alpha\beta}}} \partial_\mu (\sqrt{\text{-det} g_{\alpha\beta}} g^{\mu\nu} \partial_\nu \phi).\end{aligned} \tag{1.39}$$

We have the following existence result :

Theorem 1.1.14. *Given an open set $U \in \Sigma$ and smooth functions f_1, f_2 on U, there exists a unique smooth solution defined on $D(U)$ for the problem*

$$\Box_g \phi = 0, \quad \phi|_U = f_1, \quad \left.\frac{\partial \phi}{\partial t}\right|_U = f_2. \tag{1.40}$$

The Einstein equations are not manifestly hyperbolic, but, we can identify the principal part of the operator which looks like the wave operator applied to the metric. Let us suppose that we already know the metric g in a spacetime neighbourhood $O(\Sigma)$ of a spacelike hypersurface Σ. We introduce "wave" or "harmonic coordinates" $\{x^\alpha\}$ by setting

$$H^\alpha := \Box_g x^\alpha = 0 \quad \text{in} \quad O(\Sigma) \tag{1.41}$$

$$x^0 = t = 0, \; x^i = \bar{x}^i \quad \text{and} \quad \frac{\partial x^\alpha}{\partial t} = 0 \quad \text{on} \quad \Sigma. \tag{1.42}$$

Locally, such coordinates are guaranteed to exist by Theorem 1.1.14. By specifying coordinates, we break the gauge symmetry imposed by the diffeomorphism invariance of the geometric equations. We end up with the *reduced Einstein equations*,

$$R_{\alpha\beta} = R^H_{\alpha\beta} - H_{(\alpha,\beta)}, \tag{1.43}$$

where

$$R^H_{\alpha\beta} = -\frac{1}{2} g^{\gamma\delta} g_{\alpha\beta,\gamma\delta} + Q(g, \partial g) \tag{1.44}$$

$$= -\frac{1}{2} \Box_g g_{\alpha\beta} + Q(g, \partial g) \tag{1.45}$$

is the harmonic part, $H_{(\alpha,\beta)}$ vanishes in wave coordinates and Q collects lower order terms. The reduced vacuum Einstein Equations are

$$R^H_{\alpha\beta} = 0. \tag{1.46}$$

This is a second order quasi linear hyperbolic system for the metric g, so we can solve this provided we specify Cauchy data $g_{\alpha\beta}$ and $\partial_t g_{\alpha\beta}$ on Σ.

Definition 1.1.15. *An initial data set for the* $(n+1)$*-dimensional vacuum Einstein Equations is a set* (Σ, h, K) *where* (Σ, h) *is an* n*-dimensional Riemannian manifold and* K *is a symmetric* $(0, 2)$ *tensor on* Σ*.*

We need to define the Cauchy data for the reduced Einstein equations from a given initial data set as above. First define

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & h_{ij} \end{pmatrix} \text{at } t = 0 \tag{1.47}$$

which gives

$$\frac{\partial g_{ij}}{\partial t} = 2K_{ij} \quad \text{at} \quad t = 0 \tag{1.48}$$

This amounts to the following choices for lapse and shift.

$$N = 1 \quad \text{and} \quad \mathcal{V}^i = 0. \tag{1.49}$$

We are still free to choose $\partial_t g_{0\beta}$ which amounts to choosing the rate of change of lapse and shift. We will choose this so that

$$H_\alpha = 0 \quad \text{initially on} \;\; \Sigma. \tag{1.50}$$

The contracted second Bianchi identity implies that the Einstein tensor is divergence free

$$\nabla^\beta G_{\alpha\beta} = 0 \tag{1.51}$$

implying the following evolution equation for H_α

$$\Box_g H_\alpha + \text{l.o.t} = 0 \tag{1.52}$$

where l.o.t stands for lower order terms linear in H_α. We have chosen $\partial_t g_{0\beta}$ so that $H_\alpha = 0$ initially. If we can also ensure that $\partial_t H_\alpha = 0$, then by uniqueness for solutions to (1.52), we must also have

$$H_\alpha \equiv 0 \quad \text{on } O(\Sigma) \tag{1.53}$$

This implies that the solution to the reduced Einstein equations is actually a solution to the full vacuum Einstein equations

$$R_{\mu\nu} = 0 \tag{1.54}$$

We still need to ensure that $\partial_t H_\alpha = 0$. This is where the constraint equations come in. We have the following proposition

Proposition 1.1.16. *The vacuum constraint equations for* (Σ, h, K) *imply that* $\partial_t H_\alpha = 0$.

The momentum constraint equation $G_{0i} = \mathrm{div}K - \nabla^\Sigma(\mathrm{tr}K) = 0$ implies that

$$\frac{\partial H_i}{\partial t} = 0 \quad \text{for } i = 1, 2 \text{ and } 3. \tag{1.55}$$

From the Hamiltonian constraint equation

$$G_{00} = -\frac{\partial H_0}{\partial t} = 0. \tag{1.56}$$

Therefore $H_\alpha = 0$ so that $H_\alpha \equiv 0$ on $O(\Sigma)$ i.e., the coordinates we obtain are actually wave coordinates for the spacetime metric evolved from h on Σ by solving the reduced Einstein equations. This metric therefore satisfies the vacuum Einstein equations.

In summary, the constraint equations together with the second Bianchi identity, ensure that the wave coordinate gauge is evolved in time as one solves the reduced Einstein equations, yielding a solution of the full geometric equations.

The very first local existence result for vacuum Einstein equations from the PDE perspective was established by Choquet-Bruhat in [5] which we state here.

Theorem 1.1.17 (Choquet-Bruhat, 1952)**.** *Given an initial data set* (Σ, h, K) *satisfying the vacuum constraint equations there exists a spacetime* (M, g) *satisfying the vacuum Einstein equations* $R_{\mu\nu}(g) = 0$ *where* $\Sigma \hookrightarrow M$ *is a spacelike surface with induced metric* h *and second fundamental form* K*.*

This is a local existence result. Hence, the resulting spacetime may break down or develop singularities. We cannot ascertain the size of (M, g) from this theorem and one would like to know more about the global solution. 17 years later this existence theorem was improved to the following result.

Theorem 1.1.18 (Choquet Bruhat and Geroch, 1969)**.** *Given an initial data set* (Σ, h, K) *satisfying the vacuum constraint equations there exists a unique (up to diffeomorphism), globally hyperbolic, maximal* [1], *spacetime* (M, g) *satisfying the vacuum Einstein equations* $R_{\mu\nu}(g) = 0$ *where* $\Sigma \hookrightarrow M$ *is a Cauchy surface with induced metric* h *and second fundamental form* K*.*

These local existence results provide the mathematical foundation for an analysis of the Einstein field equations. Despite the assertion in Theorem 1.1.18 of the existence of a maximal, globally hyperbolic development, the question of global existence is left unresolved and this is a very active area of research in mathematical relativity.

In this thesis we are interested in the following two initial value problems :

(i) Investigate gravitational solitons with a particular focus on decay estimates for the wave equation $\Box_g \psi = 0$ where $\Box_g$ is the wave operator associated to a family of gravitational solitons

(ii) Determine the near horizon geometry of a spacetime in a horizon based initial value formulation

Chapters 3 and 4 discuss results in (i) and (ii) is addressed in Chapter 5. We start with an introduction to both the problems from the initial value problem point of view.

[1] There is a partial ordering $\leq$ defined on the set of all developments of initial data. For two developments M and $M^{'}$, $M \leq M^{'}$ if $M^{'}$ is an extension of M.

1.2 Decay of waves in gravitational solitons

This work is covered in Chapters 3 and 4 and is an investigation of the stability of a specific family of spacetimes called *gravitational solitons* denoted by $\mathcal{S}$. The main result here is quantifying the decay rate of solutions to the wave equation in the fixed background of $\mathcal{S}$. Here the notion of stability is as follows - if we have a nonlinear PDE $\mathcal{N}[\phi] = 0$ with a stationary solution ϕ_0, we would like to know if perturbations of ϕ_0 converge as $t \to \infty$. We understand stability from the viewpoint of the initial value formulation of GR which is the appropriate way to study the Einstein equations as a system of PDEs.

Gravitational solitons are smooth, globally stationary, asymptotically flat globally hyperbolic spacetimes with positive energy. They are termed solitons due to qualitative similarities with standing-wave stationary solutions of other nonlinear wave equations. The family $\mathcal{S}$ of soliton spacetimes that we are interested in are solutions to a supergravity theory. Supergravity theories are higher dimensional classical theories that describe low-energy dynamics in string theory. The bosonic sector of their action typically consists of a metric g coupled to $p-$form gauge fields and scalar fields. The action has the nice property of being invariant under supersymmetry transformations. Einstein-Maxwell theory is one of the simplest examples of a supergravity theory. These higher dimensional classical gravitational theories are studied via dimensional reduction on a compact manifold. This results in a supergravity theory in the remaining set of macroscopic dimensions. These are expected to reflect observable dynamics and are of interest in string theory. There a couple of ways in which this reduction can be done :

1. By reducing a $10d$ theory reduced on a $6d$ Calabi-Yau manifold, one gets a $4d$ theory. This reduction has the nice property that the $4d$ theory is supersymmetric. Whether this reduction is itself stable is an important question. The nonlinear stability for spacetimes with supersymmetric compactifications which in particular includes this reduction on a Calabi–Yau manifold was recently established by [6].

2. One can also reduce a $10d$ type IIB supergravity on a five torus to a $5d$ theory called $5d$ minimal supergravity, which is what is the focus of our work.

The spacetime we study (a gravitational soliton) is a solution to this $5d$ minimal supergravity theory. The action for this theory is

$$S = \frac{1}{16}\int_M \left(\star R - 2F\wedge\star F - \frac{8}{3\sqrt{3}}F\wedge F\wedge A\right). \tag{1.57}$$

In addition to the usual Einstein Maxwell system, we have a nonlinear term called a Chern-Simons term. We hence have a self-sourced electromagnetic field F as can be seen from the equations of motion here,

$$\mathrm{Ric}(g)_{\mu\nu} = 2\left(F_\mu{}^\rho F^{\nu\rho} - \frac{1}{6}g_{\mu\nu}F^2\right), \quad \mathrm{d}\star F + \frac{2}{\sqrt{3}}F\wedge F = 0. \tag{1.58}$$

The significance of soliton spacetimes arises in theoretical high energy physics where they are interpreted as classical microstate geometries corresponding to black holes carrying the same conserved charges and so their stability is interesting from this point of view. Nevertheless, solutions to supergravity theory are also rich from a purely gravitational point of view as they possess non-trivial 2-cycles which give rise to their mass, charge and angular momenta by a supporting flux. Such examples are considered in Chapter 3 and expressions for the angular momenta and charge for such spacetimes are derived.

To understand how supergravity theories admit these smooth non-trivial solutions, let us first try to understand solitons in the example of a stationary Einstein-Maxwell system. Consider an asymptotically flat, n dimensional globally hyperbolic spacetime $(M, \hat{g}, F)$ invariant under the everywhere timelike Killing vector field K.

$$\mathrm{Ric}(\hat{g})_{\mu\nu} = 2\left(F_\mu{}^\rho F^{\nu\rho} - \frac{1}{2(n-2)} g_{\mu\nu} F^2\right) \tag{1.59}$$

$$\mathrm{d}F = 0, \quad \mathrm{d} \star F = 0. \tag{1.60}$$

From $\mathrm{d}F = 0$, we observe that $i_K F$ is exact, hence, $\mathrm{d}\psi = -i_K F$. This defines a globally defined electric potential ψ. From $\mathrm{d} \star F = 0$, we obtain a closed $(n-3)$ form $\Theta = i_K \star F$. Θ is not necessarily exact if H^{n-3} is non-trivial. Topological censorship [7] states that the domain of outer communications of a spacetime is simply connected (i.e., $\pi_1(M)$ is trivial). This automatically implies a trivial $H^1(M)$. Hence at least in four dimensions, there exists a potential ψ and an exact Θ. This implies that there are no solitons in four dimensions [8] from the following argument. Let us start with an application of Stokes' theorem to the Komar formula :

$$M = -\frac{1}{8\pi}\int_{S^2_\infty} \star \mathrm{d}K = -\frac{1}{8\pi}\int_\Sigma \mathrm{d} \star \mathrm{d}K = \frac{1}{4\pi}\int_\Sigma \star \mathrm{Ric}(K).$$

From the field equations,

$$\star \mathrm{Ric}(K) = 2\left[\frac{1}{2}\mathrm{d} \star (\psi F) + \frac{1}{2}(\Theta \wedge F)\right]. \tag{1.61}$$

The mass can be rewritten as

$$M = \frac{1}{4\pi}\int_\Sigma \mathrm{d} \star (\psi F) + \Theta \wedge F.$$

Using the boundary conditions for asymptotically flat spacetimes, $F \to 0$ as $r \to \infty$ which makes the first term in the integral vanish if there is no inner boundary (horizon). As $\Theta = \mathrm{d}\mu$ in four dimensions, we also have

$$M = \frac{1}{4\pi} \int_{\Sigma} \mathrm{d}(\mu F)$$

which again vanishes by a similar argument. The rigidity of the positive mass theorem [9–11] along with $M = 0$ from above leaves four dimensional Minkowski spacetime $\mathbb{M}^{1,3}$ as the only possibility. In higher dimensions, topological censorship is much less of a constraint. In particular it does not eliminate $(n-3)$ cycles in the spacetime and so the mass need not vanish by the previous argument. Hence solitons are admissible in higher dimensions. Known examples in supergravity theories are self-sourced by a nonlinear Chern-Simons term in the Maxwell equation which gives them a charge

$$d \star F = -\frac{2}{\sqrt{3}} F \wedge F. \tag{1.62}$$

It turns out that one can rule out static [2] solutions in pure Einstein-Maxwell theory in $n > 4$, but there are no known examples of stationary non-static solitons. Their stability is interesting from the microstate interpretation point of view in theoretical high energy physics.

A classic result of Bardeen, Carter, and Hawking [12] are the mass and mass variation formulae for stationary, axisymmetric asymptotically flat black hole solutions. They established laws relating the mass variation of black holes to variations in angular momenta, area and charge of a four dimensional black hole. The former follows from using Stokes' theorem and the definitions of Komar mass and angular momenta

[2] A stationary spacetime is said to be static if the stationary Killing vector field ξ is hypersurface orthogonal, i.e. $\xi \wedge \mathrm{d}\xi = 0$

which goes by the name of the Smarr relation. The variation law follows from studying stationary, axisymmetric linearized solutions of the field equations representing variations close to a fixed background black hole solution. In Einstein-Maxwell theory the Smarr relation is

$$M = \frac{3\kappa A_H}{16\pi} + \frac{3}{2}\Omega J + \Phi_H Q \,, \tag{1.63}$$

and the first law of black hole mechanics is

$$\delta M = \frac{\kappa \delta A_H}{8\pi} + \Omega \delta J + \Phi_H \delta Q. \tag{1.64}$$

Here, κ is the surface gravity, A_H is the area of the horizon H, Ω is the angular velocity, J is the angular momentum and Q is the electric charge and Φ is the electric potential where δM represents an infinitesimal variation of M. In addition to the area increase law $\delta A \geq 0$ these theorems are collectively known as the law of black hole mechanics in analogy to the empirical formula describing a macroscopic thermodynamic system. These results were extended to the case of $5d$ supergravity by [13] but with the assumption of a trivial topology in the domain of outer communication. This problem was revisited recently in [14] with the derivation allowing for a general $H_2(\Sigma)$. Here, both solitons and black holes with non-trivial topology in the exterior region were considered. The mass and mass variation read

$$M = \frac{3\kappa A_H}{16\pi} + \frac{3}{2}\Omega_i J_i + \Phi_H Q + \frac{1}{2}\sum_{[C]} \mathcal{Q}[C]\Phi[C] + \frac{1}{2}\sum_{[D]} \mathcal{Q}[D]\Phi[D] \tag{1.65}$$

and the first law of black hole mechanics is

$$\delta M = \frac{\kappa \delta A_H}{8\pi} + \Omega_i \delta J_i + \Phi_H \delta Q + \sum_{[C]} \mathcal{Q}[C]\delta\Phi[C] + \sum_{[D]} \mathcal{Q}[D]\delta\Phi[D] \,. \tag{1.66}$$

Definitions of the additional terms can be found in [14] and $[C]$ and $[D]$ represent a basis for the 2-cycles and disc topology surfaces in the spacetime respectively. This generalized mass and mass variation result is applied to three different spacetimes with non-trivial topology in Chapter 3. The first is an asymptotically flat non-supersymmetric soliton. The second example is a supersymmetric asymptotically flat spacetime containing two solitons. The final example we consider is an asymptotically flat dipole ring [15]. The generalized mass and variation formula applicable to these solutions is (3.2) and (3.3). With these three examples we show the extra terms that arise in the first law (as a result of the non-trivial spacetime topology).

The stability of the non-supersymmetric 1-parameter family of soliton spacetimes considered in Chapter 3 is investigated in detail in Chapter 4. We refer to the elaborate introduction in Chapter 4 for an outline of those results.

1.3 Horizon based initial value problem

The standard initial value formulation for (1.10) consists of data on a spacelike surface at a "moment of time". In the characteristic initial value problem the initial spacelike slice is replaced by two intersecting null hypersurfaces $\mathcal{N}_1$ and $\mathcal{N}_2$ (see Fig. 1.4).

What we obtain from this initial value formalism is the information about a future spacelike slice from the evolution of initial data. It turns out that the initial value formulation is suitable for physical problems that involve spacelike infinity whereas the characteristic formulations are relevant for problems involving null infinity like gravitational wave observations.

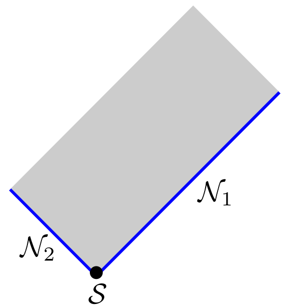

Figure 1.4: Characteristic initial value problem

Another possibility for an initial or boundary value formulation of the Einstein equations is the choice of a compact spatial/null slice with data on an intersecting null surface. This possibility allows for including different types of horizons as initial surfaces. This scenario is considered here with a section of the horizon as the choice for the compact slice (see Fig. 1.5).

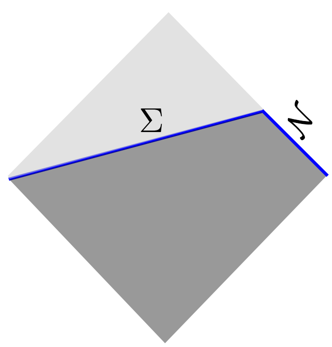

Figure 1.5: Horizon based initial value problem

By a horizon, we mean a non-degenerate isolated or dynamical trapping horizon (marginally outer trapped tubes). We briefly review their definitions here.

Definition 1.3.1. *A marginally outer trapped surface (MOTS) is a closed, spacelike,* *two-surface for which the outgoing null expansion* $\theta_{(\ell)} = 0$*. A three-surface* $\mathcal{H}$ *which can be entirely foliated with MOTSs is called a marginally outer trapped tube (MOTT).*

Definition 1.3.2. *A three-dimensional submanifold* Δ *equipped with an equivalence class of null tangent vectors* $[\ell]$ *is called an isolated horizon if it respects the following conditions:*

1. Δ *is null and topologically* $S^2 \times \mathbb{R}$;

2. *Along any null normal field* l *tangent to* Δ*, the outgoing expansion rate* $\theta_{(l)} := h^{ab}\nabla_a l_b$ *vanishes on* Δ*;*

3. *All field equations hold on* Δ*, and the stress–energy tensor* T_{ab} *on* Δ *is such that* $V^a := -T^a_b l^b$ *is a future-directed causal vector* $(V^a V_a \leq 0)$ *for any future-directed null normal* l^a*.*

4. *The commutator* $[\mathcal{L}_\ell, \mathcal{D}_a] = 0$*, where* $\mathcal{D}_a$ *denotes the induced connection on the horizon.*

It was shown by Rendall in [16] that the characteristic initial value problem for (1.10), where data is given on two intersecting null hypersurfaces $\mathcal{N}_1$ and $\mathcal{N}_2$ (see Fig. 1.4) is also well-posed with the local existence of the solution in the neighbourhood of the intersection $\mathcal{S} = \mathcal{N}_1 \cap \mathcal{N}_2$. The region of local existence was improved in [17] to a neighbourhood of $\mathcal{N}_1 \cup \mathcal{N}_2$ rather than just a neighbourhood of $\mathcal{S}$.

The idea of a characteristic formulation itself started much earlier with the study of gravitational waves in [18, 19]. In [18], *Bondi coordinates* were used to study the radiation from an isolated system. With this coordinate choice it was possible to calculate expansions appropriate to large distances. The metric was asymptotically expanded in powers of a radial coordinate and the structure of the Einstein equations and the Bianchi identity were studied. It was shown here that together with initial data on a future light cone, a single function called the *news function* fully defines the flow of information at infinity. Unlike the standard initial value formulation, no constraint equations come up in the system.

In [20], the analysis of the equations for a characteristic formulation was carried out in more detail. The initial data were considered on a system similar to Fig. 1.4. The data were given on a pair of intersecting null surfaces, $\mathcal{N}_1$ and $\mathcal{N}_2$ and their intersection $\mathcal{S}$. The data required for solving the equations are the conformal inner metric of $\mathcal{N}_1$ and $\mathcal{N}_2$, the intrinsic metric of $\mathcal{S}$, the two mean extrinsic curvatures of $\mathcal{S}$ and an additional extrinsic curvature quantity for $\mathcal{S}$. Here the Einstein equations and the Bianchi identity were divided into four groups of equations and sequentially solved. This hierarchical approach to solving equation was also taken in more recent works [21–23] in addition to our own formalism in [24].

In [25], the approach in [20] was analyzed in more detail and it was proved that the characteristic initial value problem can be seen as a symmetric hyperbolic system lending itself to the techniques used in the analysis of PDEs. The analogue of a rigorous well-posedness result similar to [5, 26] was established in [16] where it was proved that the characteristic initial value problem could be reduced to the standard Cauchy problem where the existing results for well-posedness previously established for Einstein equations could be used.

What we start in [24] is a metric based treatment of Einstein equations with the following motivations :

1. using data from the horizon as a boundary condition

2. a formalism that works for both isolated and dynamical horizons, or more ambitiously for hypersurfaces of any signature

Such a formulation is of intrinsic interest in GR and it is also relevant to numerical relativity simulations. In particular, it would mathematically quantify how horizon geometry constraints the full spacetime. Physically it would allow one to study the connection between an evolving horizon geometry and any gravitational wave signal at ∞ [27].

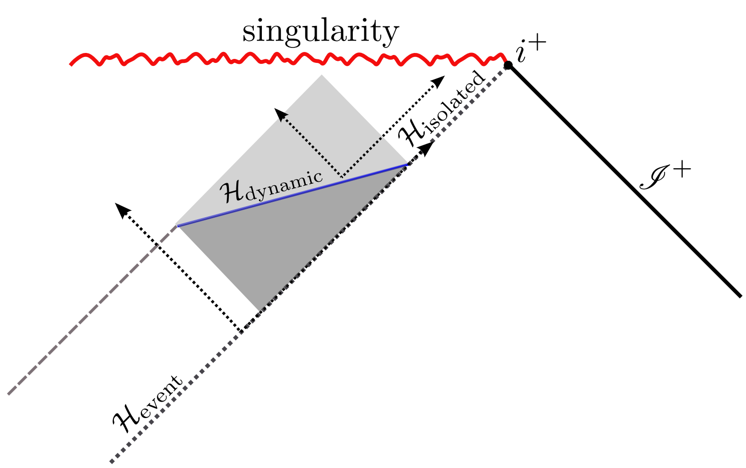

Figure 1.6: Horizon based data

We can see that data on the horizon, $\mathcal{H}_{\text{dynamic}}$ predict the shaded region which lies entirely inside the event horizon. By definition nothing inside or on the event horizon can send signals to infinity and apparent horizons live inside the event horizon. So, even if the standard initial value formulation is adaptable to dynamical horizons, the solution is completely irrelevant to an external observer. So, to make the horizon data relevant, we need to appeal to a spacetime region that is causally connected to null infinity.

The spacetime region near the horizon can send signals to infinity. An example is a timelike surface just outside the event horizon dubbed *the stretched horizon.* So, it is good strategy to determine the geometry of the stretched horizon using data on the horizon. Unfortunately, we just saw that horizon data are not sufficient by itself to determine anything observable. But with some data from a transverse null surface $\mathcal{N}$, one can determine the spacetime region near the horizon.

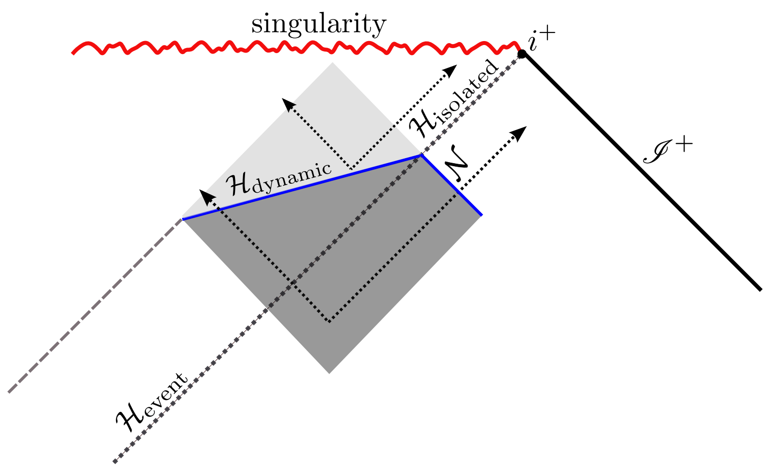

Figure 1.7: Horizon based data + data from $\mathcal{N}$

The aim of this program is to develop a formulation that mathematically relates horizons, the near horizon spacetime and infinity. Such a formulation would accommodate data on an isolated horizon and is meaningful to an external observer. Here and in [24] we address this for the scalar field in spherical symmetry as a model problem for gravitational waves. The broad goal here is to identify the free and constrained data on a finite section of the horizon $\bar{\mathcal{H}}$ and $\bar{\mathcal{N}}$. The idea is illustrated in

Fig. 1.8.

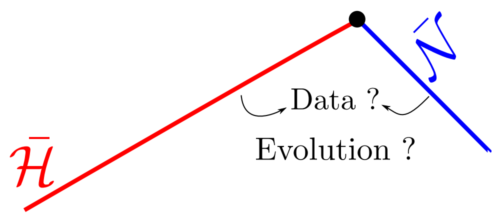

Figure 1.8: Formulation using data on $\mathcal{H}$ and $\mathcal{N}$

We have determined that there are no constrained data on either of the surfaces.

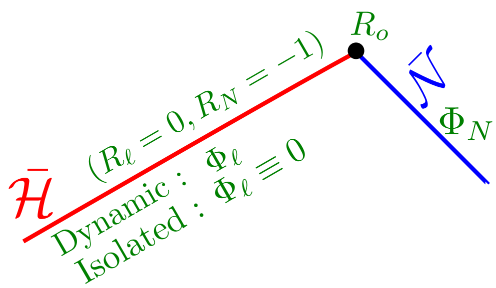

Figure 1.9: Required data on $\mathcal{H}$ and $\mathcal{N}$

The data that is required to fully determine the past domain of dependence are the fluxes through $\bar{\mathcal{H}}$ and $\bar{\mathcal{N}}$, Φ_ℓ and Φ_N respectively and the areal radius R_o at the intersection of the two surfaces. In the general vacuum gravitational case without symmetries, we expect the fluxes to be replaced by the shears.

Chapter 2

Statement of contributions

This chapter summarizes the contributions of various authors. The abbreviations are Dr. Ivan Booth (IB), Dr. Hari Kunduri (HK), Dr. Uzair Hussain (UH) and Sharmila Gunasekaran (SG)

- Project 1 : Soliton mechanics
- Project 2 : Slow decay of waves in gravitational solitons
- Project 3 : Horizons as boundary conditions in spherical symmetry

Collaborative aspect (↓)	Project 1	Project 2	Project 3
Design and identification of the research proposal	HK	HK	IB
Practical aspects of research	HK, SG and UH	HK and SG	IB and SG
Data analysis and numerics	HK, SG and UH	HK and SG	IB and SG
Manuscript preparation	HK, SG and UH	HK and SG	IB and SG
Status	Published	Undergoing revisions	Published

Chapter 3

Soliton mechanics

This chapter is based on work published in [28] which appeared as :

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3.1 Abstract

The domain of outer communication of five-dimensional asymptotically flat stationary spacetimes may possess non-trivial 2-cycles (bubbles). Spacetimes containing such 2-cycles can have non-zero energy, angular momenta, and charge even in the absence of horizons. A mass variation formula has been established for spacetimes containing bubbles and possibly a black hole horizon. This 'first law of black hole and soliton mechanics' contains new intensive and extensive quantities associated to each 2-cycle. We consider examples of such spacetimes for which we explicitly calculate these quantities and show how regularity is essential for the formulae relating them to hold. We also derive new explicit expressions for the angular momenta and charge

for spacetimes containing solitons purely in terms of fluxes supporting the bubbles.

3.2 Introduction

A striking feature of Einstein-Maxwell theory in four dimensions is the absence of globally stationary, asymptotically flat solutions with non-zero energy - that is, there are 'no solitons without horizons' [29]. This property is closely linked to uniqueness theorems for black holes, and indeed it fails to hold in Einstein-Yang Mills theory for which 'hairy' black holes exist (see, e.g. [30]). In five and higher dimensions, however, non-trivial topology in the spacetime can support the existence of such horizonless solitons even in Einstein-Maxwell supergravity theories. For an asymptotically flat solution, the topological censorship theorem [7] asserts that the domain of outer communication of a spacetime must be simply connected. In four dimensions, that is sufficient to ensure the absence of any cycles in the exterior. In five dimensions, simple connectedness is a weaker constraint, and in particular does not exclude the possibility of 2-cycles ('bubbles'). Physically, these cycles are supported by magnetic flux supplied by Maxwell fields and contribute to both the energy and angular momenta of the spacetime.

In this note we will focus on five-dimensional asymptotically flat stationary spacetimes with two commuting rotational Killing fields, possibly containing a single black hole. In this case it has been shown that the topology of the domain of outer communication is $\mathbb{R} \times \Sigma$, where[1]

$$\Sigma \cong \Big(\mathbb{R}^4 \# n(S^2 \times S^2) \# n'(\pm\mathbb{CP}^2)\Big) \backslash B, \tag{3.1}$$

for some $n, n' \in \mathbb{N}_0$ and B is the black hole region, where the horizon $H = \partial B$ must topologically be one of $S^3, S^1 \times S^2$ or $L(p,q)$ [31–34]. The integers n, n' determine the 2-cycle structure of Σ.

In the absence of black holes, soliton spacetimes with 2-cycles supported by flux are known to exist, with a large number of supersymmetric (see the review [35]) and non-supersymmetric examples [36–38]. The largest known family of solutions to our knowledge of these two types appeared in [39] and [40] respectively. These spacetimes carry positive energy. The relationship between the mass of these spacetimes and their fluxes is expressed in a Smarr-type formula, as observed for BPS solitons in supergravity theories by Gibbons and Warner [41]. Subsequently, it was shown that under stationary, $U(1)^2$-invariant variations satisfying the linearized field equations, variations of the mass and magnetic fluxes for general soliton spacetimes are governed by a 'first law' formula [14] (see (3.11) below).

Furthermore, one can derive a generalised mass and mass variation formula for $\mathbb{R} \times U(1)^2$-invariant spacetimes containing a black hole with an arbitrary number of 2-cycles in the exterior region. Similar to the soliton case it was found that on top of the familiar terms for a black hole, extra terms due to the bubbles are present. However, unlike the pure soliton case, these additional terms are most naturally expressed in terms of variations of an intensive quantity (a potential), as opposed to an extensive

[1] In fact, the statement regarding Σ is still true if only one rotational Killing field is assumed, although then there are more possibilities for the horizon topology [31].

quantity (a flux). For Einstein-Maxwell theory, possibly with a Chern-Simons term, the mass formula is [14]

$$M = \frac{3\kappa A_H}{16\pi} + \frac{3}{2}\Omega_i J_i + \Phi_H Q + \frac{1}{2}\sum_{[C]} \mathcal{Q}[C]\Phi[C] + \frac{1}{2}\sum_{[D]} \mathcal{Q}[D]\Phi[D] \tag{3.2}$$

and the first law of black hole mechanics is

$$\delta M = \frac{\kappa \delta A_H}{8\pi} + \Omega_i \delta J_i + \Phi_H \delta Q + \sum_{[C]} \mathcal{Q}[C]\delta\Phi[C] + \sum_{[D]} \mathcal{Q}[D]\delta\Phi[D] \; . \tag{3.3}$$

In the above $[C]$ is a basis for the second homology of Σ, $[D]$ are certain disc topology surfaces which extend from the horizon, Φ are magnetic potentials and $\mathcal{Q}$ are certain 'electric' fluxes defined on these surfaces which we will define precisely below. This shows that non-trivial spacetime topology plays an important role in black hole thermodynamics, thus providing further motivation to study such objects beyond the obvious implications for black hole non-uniqueness [42].

It should be noted that most explicitly known examples of soliton spacetimes are supersymmetric, in which case the mass variation formula simply follows from the BPS relation. The same is true for the supersymmetric solution describing a rotating black hole with a soliton in the exterior region [42]. Indeed quite generally for BPS black hole solutions one can show that the additional terms arising in (3.2) and (3.3) vanish identically. This is analogous to the fact that for BPS black holes in these theories, the surface gravity and angular velocities also vanish identically. However, for non-supersymmetric solutions describing black holes with exterior bubbles, these terms would generically contribute. Examples of such solutions are not explicitly known, although there seems to be no obstruction to their existence, even in the vacuum.

The purpose of this paper is to apply the formalism developed in [14] to explicitly compute the various potentials and fluxes appearing above for some known spacetimes with non-trivial Σ. In so doing we will verify the first variation formula above. We will also derive some new relations that show how the angular momenta and total electric charge of a spacetime may arise solely from the presence of flux through the 2-cycles. Finally, we will reexamine the singly-rotating dipole black ring [15]. The solution is characterized by a local dipole 'charge' resulting from magnetic flux through the S^2 of the ring horizon. The first law for black rings derived in [43] contains additional terms due to the dipole charge and we show how this is recovered using the general formalism of [14]. This will use in a crucial way the disc topology region that lies in the domain of outer communication of the black ring.

3.3 First law for black holes and solitons in supergravity

The mass and mass variation formulae for asymptotically flat, stationary spacetimes invariant under two commuting rotational symmetries has been established for a general five-dimensional theory of gravity coupled to an arbitrary set of Maxwell fields and uncharged scalars. We will be concerned with specific soliton and black hole solutions to five-dimensional minimal supergravity, whose bosonic action is (setting Newton's constant $G_5 = 1$)

$$S = \frac{1}{16\pi} \int_{\mathcal{M}} \left(\star R - 2F \wedge \star F - \frac{8}{3\sqrt{3}} F \wedge F \wedge A \right) \tag{3.4}$$

Here $F = \mathrm{d}A$ and A is a locally defined gauge potential. The existence of a non-trivial second homology H_2 implies that F is closed but not exact. The theory can be recovered from the general theory considered in [14] upon setting $I = 1$, $g_{IJ} = 2$ and $C_{IJK} = 16/\sqrt{3}$. We will follow this convention throughout when appealing to the construction of potentials and fluxes used in [14]. The equations of motion are

$$R_{ab} = \frac{4}{3}F_{ac}F_b{}^c + \frac{1}{3}G_{acd}G_b{}^{cd}\,, \qquad \mathrm{d}\star F + \frac{2}{\sqrt{3}}F \wedge F = 0 \tag{3.5}$$

where $G = \star F$. The central observation of [41] was that the non-triviality of the second homology H_2 makes it more natural to work with G rather than the gauge potential A which cannot be globally defined.

Let ξ be the stationary Killing field normalized so that $|\xi|^2 \to -1$ at spatial infinity (in the case of a spacetime containing a black hole, ξ is instead identified with the Killing field which is the null generator of the event horizon). Using the fact that F is closed and invariant under this action, we have a globally defined potential Φ_ξ defined by

$$\mathrm{d}\Phi_\xi \equiv i_\xi F \tag{3.6}$$

and the requirement $\Phi_\xi \to 0$ at spatial infinity. From the Maxwell equation one may define a closed two-form

$$\Theta = 2 i_\xi G - \frac{8}{\sqrt{3}} F \Phi_\xi \tag{3.7}$$

If, in addition to being stationary, the spacetime is invariant under a $U(1)^2$ isometry generated by the Killing fields $m_i = (m_1, m_2)$ (normalized to have 2π-periodic orbits), we also have globally defined magnetic potentials

$$\mathrm{d}\Phi_i = i_{m_i} F \tag{3.8}$$

and we also fix the freedom by requiring these vanish at an asymptotically flat end. Together (ξ, m_i) generate an $\mathbb{R}\times U(1)^2$ action acting as isometries on $(\mathcal{M}, g, F)$. Using these potentials one can finally deduce the existence of globally defined potentials U_i

$$\mathrm{d}U_i = i_{m_i}\Theta + \frac{8}{\sqrt{3}}\mathrm{d}\Phi_i\Phi_\xi^H \tag{3.9}$$

which are again fixed by requiring they vanish at the asymptotically flat end. Here Φ_ξ^H is the pullback of Φ_ξ to the horizon if a black hole is present in the spacetime; for a pure soliton spacetime this term is ignored. The potentials and fluxes defined above can be thought of as functions on a 2d orbit space $\mathcal{B} \cong \Sigma/U(1)^2$ [32]. The rank of the matrix $\lambda_{ij} = m_i \cdot m_j$ divides the space into two dimensional interior points, one dimensional boundary segments ($\partial\mathcal{B}$) called rods and zero dimensional points that lie on 'corners' where the segments intersect. A black hole is represented by a compact rod $I_H \cong H/U(1)^2$ where the timelike Killing field goes null. There are two non-compact semi-infinite rods corresponding to the two asymptotic axes of rotation extending out to spatial infinity. The rest of $\partial\mathcal{B}$ contains finite rods I_i where an integer linear combination $v^i m_i, v^i \in \mathbb{Z}$ of the rotational Killing fields vanishes. This orbit space data thus encodes the action of the isometry group and determines the full spacetime topology up to diffeomorphism [32]. In particular finite rods represent two-dimensional submanifolds which may have the topology of either S^2, or a closed disc D if the corresponding rod is adjacent to I_H. We will discuss specific examples of spacetimes containing such 2-cycles and discs below.

For purely soliton spacetimes (i.e. without black holes), the Smarr formula and mass variation reduce to [14]

$$M = \frac{1}{2} \sum_{[C]} \Psi[C] q[C] \tag{3.10}$$

$$\delta M = \sum_{[C]} \Psi[C] \delta q[C] \tag{3.11}$$

where

$$q[C] = \frac{1}{4\pi} \int_C F \quad \text{and} \quad \Psi[C] = \pi v^i U_i \tag{3.12}$$

represent the magnetic flux and magnetic potential associated to each element of $[C]$. Note that in (3.11) the extensive variable $q[C]$ appears naturally in the first law in contrast to (3.3).

Before discussing specific examples, we would like to present new Smarr-type formulae for the angular momenta and electric charge for purely soliton spacetimes as a sum over fluxes through the 2-cycles. These are useful as they demonstrate how a spacetime can possess such conserved charges in the absence of horizons.

Firstly, consider the angular momenta J_i associated to the rotational Killing field m_i defined by the Komar integrals

$$J[m_i] = \frac{1}{16\pi} \int_{S^3_\infty} \star \mathrm{d} m_i \ . \tag{3.13}$$

The Maxwell equation and Killing property of the m_i imply the existence of two closed (though not necessarily exact) two-forms Υ_i defined by

$$\Upsilon_i \equiv 2 i_{m_i} G - \frac{8}{\sqrt{3}} F \Phi_i \ . \tag{3.14}$$

Cartan's formula immediately implies the existence of global potential functions χ_{ij} satisfying $\mathrm{d}\chi_{ij} = i_{m_i} \Upsilon_j$. Note that we can always choose the integration constant so that $\chi_{ij} = 0$ on an interval on which m_i vanishes for fixed j. Now using Stokes' theorem

$$J[m_i] = \frac{1}{8\pi}\int_\Sigma \star\text{Ric}(m_i) = \frac{1}{8\pi}\int_\Sigma \left(-\frac{1}{3}\right)\Upsilon_i \wedge F + \frac{4}{3}\text{d}\star(F\Phi_i) \tag{3.15}$$

The final term above may be shown to vanish by converting it to an integral over S^3_∞ where Φ_i vanishes. We can evaluate this integral over the orbit space $\mathcal{B}$, giving

$$J[m_i] = \frac{\pi}{6}\int_{\mathcal{B}} \eta^{jk}\text{d}\chi_{ji}\wedge \text{d}\Phi_k = \frac{\pi}{6}\int_{\mathcal{B}} \text{d}[\eta^{jk}\chi_{ji}\wedge\text{d}\Phi_k] \tag{3.16}$$

where η^{ij} is the antisymmetric symbol with $\eta^{12} = 1$. The final term can be converted to a boundary term on $\partial\mathcal{B}$, and using the fact that the potentials vanish on the semi-infinite rods $I_\pm$, we are left with

$$J[m_i] = \frac{\pi}{6}\sum_i \int_{I_i} \eta^{jk}\chi_{ji}\text{d}\Phi_k \tag{3.17}$$

This can be further simplified by using the fact that each rod is specified by a pair of integers v^i, so that $v^i m_i$ vanishes By definition $v^i\text{d}\Phi_i = 0$ on the rod, so that $\Phi[C] \equiv v^i\Phi_i$ is constant. By an $SL(2,\mathbb{Z})$ change of basis let us define a new basis $(\hat{m}_1, \hat{m}_2)$ for the $U(1)^2$ generators such that $\hat{m}_1 = v^i m_i$. The other Killing field $\hat{m}_2$ is non-vanishing on the rod except at the endpoints (these correspond to topologically S^2 submanifolds in the spacetime). Note that in the obvious notation, $\hat{\chi}_{1i}, \hat{\Phi}_1$ are constants on the rod. Using $SL(2,\mathbb{Z})$-invariance, $\eta^{jk}\chi_{ji}\text{d}\Phi_k = \eta^{jk}\hat{\chi}_{ji}\text{d}\hat{\Phi}_k$. Putting the above facts together we arrive at

$$J[m_i] = \frac{1}{3}\sum_{[C]} \chi_i[C]q[C] \tag{3.18}$$

where $q[C]$ are the magnetic fluxes associated to a given cycle C and $\chi_i[C] \equiv -\pi\hat{\chi}_{1i} = -\pi v^j\chi_{ji}$ is a constant associated to each cycle. It is natural to interpret the $\chi_i[C]$

as *magnetic angular momenta potentials* as they encode how the magnetic flux $q[C]$ contribute to the total angular momenta of the spacetime.

Now let us turn to an expression for the total electric charge Q, defined by

$$Q \equiv \frac{1}{4\pi} \int_{S^3_\infty} \star F = -\frac{1}{2\sqrt{3}\pi} \int_\Sigma F \wedge F \tag{3.19}$$

It may appear counterintuitive that magnetic fluxes contribute to the electric charge, but it should be noted that the Maxwell equation in supergravity is self-sourced. We now proceed to evaluate this over the boundary of the orbit space. Using the definition of the magnetic potentials, we have

$$Q = \frac{\pi}{\sqrt{3}} \int_{\mathcal{B}} \eta^{ij} \mathrm{d}\Phi_i \wedge \mathrm{d}\Phi_j = \frac{\pi}{\sqrt{3}} \int_{\partial\mathcal{B}} \eta^{ij} \Phi_i \mathrm{d}\Phi_j \ . \tag{3.20}$$

We can now express this as a sum over the 2-cycles using the argument used above for the angular momenta. The result is

$$Q = -\frac{4\pi}{\sqrt{3}} \sum_{[C]} \Phi[C] q[C] \tag{3.21}$$

where $\Phi[C] = v^i \Phi_i$ are constant magnetic potentials associated to each 2-cycle with corresponding rod vector v^i.

3.4 Examples

3.4.1 Single soliton spacetime

Our first example is a charged, non-supersymmetric gravitational soliton with spatial slices $\Sigma \cong \mathbb{R}^4 \# \mathbb{CP}^2$ which was concisely analyzed in [41] (see also [37] for a discussion of a generalization which is asymptotically AdS_5). In the following we will use a different parametrization which is convenient for our purposes. The equations of motion (3.5) admit the following local solution, invariant under an $\mathbb{R} \times SU(2) \times U(1)$ isometry:

$$\mathrm{d}s^2 = -\frac{r^2 W(r)}{4b(r)^2}\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{W(r)} + \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2) + b(r)^2(\sigma_3 + f(r)\mathrm{d}t)^2 \tag{3.22}$$

$$F = \frac{\sqrt{3}q}{2}\mathrm{d}\left[\left(\frac{1}{r^2}\right)\left(\frac{j}{2}\sigma_3 - \mathrm{d}t\right)\right] \tag{3.23}$$

where σ_i are left-invariant one-forms on $SU(2)$:

$$\sigma_1 = -\sin\psi\mathrm{d}\theta + \cos\psi\sin\theta\mathrm{d}\phi\,, \quad \sigma_2 = \cos\psi\mathrm{d}\theta + \sin\psi\sin\theta\mathrm{d}\phi\,, \quad \sigma_3 = \mathrm{d}\psi + \cos\theta\mathrm{d}\phi \tag{3.24}$$

which satisfy $\mathrm{d}\sigma_i = \frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k$ and $\psi \sim \psi + 4\pi$, $\phi \sim \phi + 2\pi$, $\theta \in [0, \pi]$ is required for asymptotic flatness. The functions appearing in the metric are given by

$$W(r) = 1 - \frac{2}{r^2}(p - q) + \frac{q^2 + 2pj^2}{r^4} \tag{3.25}$$

$$f(r) = -\frac{j}{2b(r)^2}\left(\frac{2p - q}{r^2} - \frac{q^2}{r^4}\right) \tag{3.26}$$

$$b(r)^2 = \frac{r^2}{4}\left(1 - \frac{j^2q^2}{r^6} + \frac{2j^2p}{r^4}\right) \tag{3.27}$$

where $p, q, j \in \mathbb{R}$. We will take $m_i = (\partial_{\hat{\psi}}, \partial_\phi)$, $\hat{\psi} = \psi/2$, to be our basis for the generators of the $U(1)^2$ action with 2π-periodic orbits.

The parameters (p, q, j) in the above local metric can be chosen to describe asymptotically flat, charged rotating black holes. However we may obtain a regular soliton spacetime by requiring that the S^1 parameterized by the coordinate ψ degenerates smoothly at some $r = r_0$ in the spacetime, leaving an S^2 bolt, or bubble. We therefore require $g_{\psi\psi} = b(r)^2$ vanishes at r_0. Regularity of the spacetime metric imposes that $W(r_0) = 0$. The existence of a simultaneous root fixes

$$p = \frac{r_0^4(r_0^2 - j^2)}{2j^4} \qquad q = -\frac{r_0^4}{j^2} \tag{3.28}$$

In order for $\partial_{\hat{\psi}}$ to degenerate smoothly and avoid a conical singularity at $r = r_0$ requires $W'(r_0)(b^2(r_0))' = 1$, or equivalently

$$(1-x)(2+x)^2 = 1 \tag{3.29}$$

for $x = x_* = r_0^2/j^2$. This cubic has a unique positive solution at $x \approx 0.870385$, and in particular $r_0^2 < j^2$.

With this inequality it is easy to check that $W(r), b(r)^2 > 0$ for $r > r_0$ and the spacetime metric is globally regular. Further

$$g^{tt} = -\frac{4b(r)^2}{r^2 W(r)} < 0 \tag{3.30}$$

so the spacetime is stably causal, and in particular the t =constant hypersurfaces are Cauchy surfaces. It can be verified that $g_{tt} < 0$ everywhere, so $\partial/\partial t$ is globally timelike and in particular there are no ergoregions. However, if one uplifts the soliton to six dimensions, we expect it will suffer from the instability discussed in [44].

We thus obtain a 1-parameter family of $\mathbb{R} \times SU(2) \times U(1)$-invariant soliton spacetime.

The S^2 at $r = r_0$ has a round metric

$$\mathrm{d}s_2^2 = \frac{r_0^2}{4}(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \tag{3.31}$$

and carries a magnetic flux

$$q[C] = \frac{1}{4\pi}\int_{S^2} F = \frac{\sqrt{3}r_0^2}{4j} \tag{3.32}$$

It is straightforward to read off

$$\Phi_\xi = \frac{\sqrt{3}q}{2r^2}\,, \qquad \Phi_{\hat{\psi}} = -\frac{\sqrt{3}qj}{2r^2}\,, \qquad \Phi_\phi = -\frac{\sqrt{3}qj\cos\theta}{4r^2}\,. \tag{3.33}$$

A long but straightforward calculation yields, using (3.7) and (3.9):

$$\mathrm{d}U_{\hat{\psi}} = \left[\frac{2\sqrt{3}jq}{r^3} - \frac{4\sqrt{3}jq^2}{r^5}\right]\mathrm{d}r \tag{3.34}$$

$$\mathrm{d}U_{\phi} = \left[-\frac{2\sqrt{3}jq^2\cos\theta}{r^5} + \frac{\sqrt{3}jq\cos\theta}{r^3}\right]\mathrm{d}r + \left[-\frac{\sqrt{3}jq^2\sin\theta}{2r^4} + \frac{\sqrt{3}jq\sin\theta}{2r^2}\right]\mathrm{d}\theta \tag{3.35}$$

which leads to

$$U_{\hat{\psi}} = \frac{\sqrt{3}jq}{r^2}\left(\frac{q}{r^2} - 1\right) , \qquad U_{\phi} = \frac{\sqrt{3}jq\cos\theta}{2r^2}\left(\frac{q}{r^2} - 1\right) \tag{3.36}$$

where the integration constants have been fixed so that the potentials vanish as $r \to \infty$.

On the S^2 'bolt' at $r = r_0$, the Killing field $\partial_{\hat{\psi}} = 2\partial_{\psi}$ degenerates smoothly. The interval structure of the orbit space is given below in the basis of rotational Killing fields orthogonal at infinity $(\partial_{\phi_1}, \partial_{\phi_2})$ where $\partial_{\phi_1} = \partial_{\psi} - \partial_{\phi}$ and $\partial_{\phi_2} = \partial_{\phi} + \partial_{\psi}$. In this basis the two semi-infinite rods can be manifestly seen as axes of rotation with vanishing ∂_{ϕ_1} or ∂_{ϕ_2}.

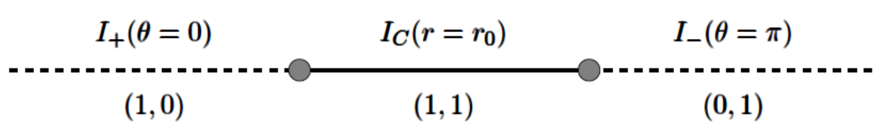

Figure 3.1: Rod structure for single soliton spacetime in (ϕ_1, ϕ_2) basis.

We now turn to the computation of the potentials associated to the soliton. Firstly,

$$\Psi[C] = \pi U_{\hat{\psi}}(r_0) = \frac{\sqrt{3}\pi r_0^2(j^2 + r_0^2)}{j^3} \tag{3.37}$$

We then find

$$\frac{\Psi[C]q[C]}{2} = \frac{3\pi}{8}\left(\frac{r_0}{j}\right)^4 (j^2 + r_0^2) \tag{3.38}$$

which is indeed the ADM mass of the spacetime, which can easily be read off from the expansion

$$g_{tt} = -1 + \frac{8M}{3\pi r^2} + O(r^{-4}) \tag{3.39}$$

Finally the first law of soliton mechanics asserts that

$$\mathrm{d}M = \Psi[C]\mathrm{d}q[C] \tag{3.40}$$

In our explicit example,

$$\mathrm{d}M - \Psi[C]\mathrm{d}q[C] = \frac{3\pi r_0^5}{4j^5}(j\mathrm{d}r_0 - r_0\mathrm{d}j) \tag{3.41}$$

and the right hand side vanishes as a consequence of the regularity condition $r_0^2/j^2 = x_*$. We emphasize that the Smarr-type relation for the mass does not require regularity of the spacetime to hold, whereas the first law is in fact a finer probe of regularity. Finally one can explicitly check that the electric charge is indeed given by

$$Q = -\frac{4\pi}{\sqrt{3}}\Phi[C]q[C] = -\frac{\sqrt{3}\pi r_0^4}{2j^2}\ . \tag{3.42}$$

To compute the magnetic angular momentum potentials χ_{ij}, it is convenient to work in the $U(1)^2$ basis $(\partial_\psi, \partial_\phi)$ and then convert to the basis $(\partial_{\phi_1}, \partial_{\phi_2})$ which is orthogonal at the asymptotically flat end, in order to fix integration constants. A long but straightforward calculation yields

$$\begin{aligned} \chi_{\psi\psi} &= -\frac{\sqrt{3}q^2 j^2}{4r^4} + \frac{\sqrt{3}q}{4}\ , \qquad \chi_{\phi\psi} = \frac{\sqrt{3}q\cos\theta}{4}\left(1 - \frac{qj^2}{r^4}\right) \\ \chi_{\phi\phi} &= -\frac{\sqrt{3}q^2 j^2\cos^2\theta}{4r^4} - \frac{\sqrt{3}q}{4}\ , \qquad \chi_{\psi\phi} = -\frac{\sqrt{3}q\cos\theta}{4}\left(1 + \frac{qj^2}{r^4}\right) \end{aligned} \tag{3.43}$$

Since the 2-cycle is specified by the vanishing of $\partial_{\hat{\psi}}$, using the formula (3.18) we find

$$J_\psi = \frac{\pi r_0^6}{4j^3}\ , \qquad J_\phi = 0 \tag{3.44}$$

where in the second equality we observe that $\chi_{\psi\phi} = 0$ on C using (3.28) . It is easy to check that these expressions agree with the standard ADM angular momenta computed from the asymptotic fall-off of the metric. As expected, the $SU(2) \times U(1)$-invariant solution has equal angular momenta in orthogonal 2-planes, $J_1 = J_2 = J_\psi$. Note that $J_\psi \neq 0$ for the soliton; indeed, we have the constraint

$$J_\psi = -\frac{2Qq[C]}{3} = \frac{16\pi q[C]^3}{3\sqrt{3}} \ . \tag{3.45}$$

3.4.2 Double soliton spacetime

Our second example is a supersymmetric, asymptotically flat spacetime containing two non-homologous two-cycles. The spatial slices $\Sigma \cong \mathbb{R}^4\#(S^2 \times S^2)$ where the connected sum with $\mathbb{R}^4$ corresponds to removing a point. The solution is originally given in the more general $U(1)^3$ five-dimensional supergravity [45]. We will quickly review this double soliton solution to the minimal supergravity theory (3.4) as this particular case does not seem to be reproduced explicitly in the literature. Note that it belongs to the general family of solutions with Gibbons-Hawking base space first analyzed in detail in [46].

The spacetime metric takes the canonical form of a timelike fibration over a hyperKähler 'base space'

$$\mathrm{d}s^2 = -f^2(\mathrm{d}t + \omega)^2 + f^{-1}\mathrm{d}s^2_M \ , \tag{3.46}$$

where $V = \partial/\partial t$ is the supersymmetric, timelike Killing vector field and $\mathrm{d}s^2_M$ is a hyperKähler base [46]. The solution has a Gibbons-Hawking hyperKähler base

$$\mathrm{d}s^2_M = H^{-1}(\mathrm{d}\psi + \chi)^2 + H(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)) \ , \tag{3.47}$$

where (r,θ,ϕ) are spherical coordinates on $\mathbb{R}^3$, the function H is harmonic on $\mathbb{R}^3$ and χ is a 1-form on $\mathbb{R}^3$ satisfying $\star_3 \mathrm{d}\chi = \mathrm{d}H$.

The analysis of [46] shows a general technique for constructing solutions of the above form. Defining the following harmonic functions on $\mathbb{R}^3$ [45]

$$H = \frac{1}{r} - \frac{1}{r_1} + \frac{1}{r_2}, \qquad K = \frac{k_0}{r} + \frac{k_1}{r_1} + \frac{k_2}{r_2}, \tag{3.48}$$

$$L = 1 + \frac{\ell_0}{r} + \frac{\ell_1}{r_1} + \frac{\ell_2}{r_2}, \qquad M = m + \frac{m_1}{r_1} + \frac{m_2}{r_2}, \tag{3.49}$$

with

$$r_1 = \sqrt{r^2 + a_1^2 - 2ra_1\cos\theta}, \qquad r_2 = \sqrt{r^2 + a_2^2 - 2ra_2\cos\theta} \tag{3.50}$$

where we assume $0 < a_1 < a_2$, we arrive at a solution provided

$$f^{-1} = H^{-1}K^2 + L\ , \qquad \omega = \omega_\psi(\mathrm{d}\psi + \chi) + \hat{\omega}\ , \tag{3.51}$$

where

$$\omega_\psi = H^{-2}K^3 + \frac{3}{2}H^{-1}KL + M\ , \tag{3.52}$$

$$\star_3\mathrm{d}\hat{\omega} = H\mathrm{d}M - M\mathrm{d}H + \frac{3}{2}(K\mathrm{d}L - L\mathrm{d}K)\ . \tag{3.53}$$

The Maxwell field is then

$$F = \frac{\sqrt{3}}{2}\mathrm{d}\left[f(\mathrm{d}t + \omega) - KH^{-1}(\mathrm{d}\psi + \chi_i\mathrm{d}x^i) - \xi_i\mathrm{d}x^i\right]\ , \tag{3.54}$$

where the 1-form ξ satisfies $\star_3\mathrm{d}\xi = -\mathrm{d}K$. For the above choice of harmonic functions one finds

$$\chi = \left[\cos\theta - \frac{r\cos\theta - a_1}{r_1} + \frac{r\cos\theta - a_2}{r_2}\right]\mathrm{d}\phi\ , \tag{3.55}$$

and

$$\xi = -\left[k_0\cos\theta + \frac{k_1(r\cos\theta - a_1)}{r_1} + \frac{k_2(r\cos\theta - a_2)}{r_2}\right]\mathrm{d}\phi\ , \tag{3.56}$$

where we have absorbed the integration constant in χ by suitably shifting ψ. One may also integrate explicitly for $\hat{\omega} = \hat{\omega}_\phi \mathrm{d}\phi$.

For a suitable choice of constants this solution is asymptotically flat provided $\Delta\psi = 4\pi$, $\Delta\phi = 2\pi$ and $0 \leq \theta \leq \pi$. In particular setting $r = \rho^2/4$ and sending $\rho \to \infty$ one finds

$$\mathrm{d}s_M^2 \sim \mathrm{d}\rho^2 + \frac{\rho^2}{4}\left[(\mathrm{d}\psi + \cos\theta \mathrm{d}\phi)^2)^2\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right] \tag{3.57}$$

with $O(\rho^{-2})$ corrections in the associated Cartesian chart. Finally, choosing

$$m = -\frac{3}{2}(k_0 + k_1 + k_2) \tag{3.58}$$

and suitably fixing the integration constant in $\hat{\omega}_\phi$, we find $f = 1 + \mathcal{O}(\rho^{-2})$, $\omega_\psi = \mathcal{O}(\rho^{-2})$ and $\hat{\omega}_\phi = \mathcal{O}(\rho^{-2})$. Thus the spacetime is asymptotically Minkowski $\mathbb{R}^{1,4}$.

The free parameters characterizing these local 'three-centre' solutions may be chosen so that globally, the spacetime describes a two-soliton spacetime (see, e.g. [41]). It is clear that the spacetime metric is regular apart from possible singularities at the 'centres' which lie at the points $\mathbf{x_0} = (0, 0, 0)$, $\mathbf{x_1} = (0, 0, a_1)$, and $\mathbf{x_2} = (0, 0, a_2)$ in the usual Cartesian coordinates on the ambient $\mathbb{R}^3$ on the base space. To ensure that the spacetime metric degenerates smoothly at these points, it is sufficient to first require that the base space be smooth. It can be shown that this is in fact the case without any further restriction of parameters (the base space metric approaches, up to an overall sign, the Euclidean metric near the origin of $\mathbb{R}^4$). Note that on the base space, ∂_ψ degenerates smoothly at the centres.

Next to ensure that the spacetime metric is well behaved and has the correct signature, we must have $f \neq 0$ ($f = 0$ would correspond to an event horizon).

Equivalently we must ensure f^{-1} does not diverge, which fixes

$$\ell_2 = -k_2^2\,, \quad \ell_1 = k_1^2\,, \quad \ell_0 = -k_0^2\,. \tag{3.59}$$

Further, since ∂_ψ degenerates on the base, near the centres we have

$$|\partial_\psi|^2 = -f^2\omega_\psi^2 \leq 0 \tag{3.60}$$

which immediately implies that ω_ψ must *vanish* at these points. It turns out generically ω_ψ actually has simple poles at these points. Removing these requires

$$m_1 = \frac{k_1^3}{2}\,, \quad m_2 = \frac{k_2^3}{2}\,, \quad k_0 = 0\,. \tag{3.61}$$

Actually imposing that $\omega_\psi = 0$ leads to the so called 'bubble equations'

$$a_2 k_1^3 + a_1 k_2^3 - 3a_1 a_2(k_1 + k_2) = 0 \tag{3.62}$$

$$a_1(k_1 + k_2)^3 + (a_2 - a_1)(k_1^3 - 3a_1(2k_1 + k_2)) = 0 \tag{3.63}$$

$$a_2(k_1 + k_2)^3 - (a_2 - a_1)(k_2^3 + 3a_2 k_1) = 0 \tag{3.64}$$

which correspond to the enforcing regularity at $r = 0, r = a_1$, and $r = a_2$ respectively. This leaves a one-parameter family of 2-soliton spacetimes parameterized by (a_1, a_2, k_1, k_2) subject to the three regularity constraints. An analysis of the geometry shows that the spacetime is stably causal ($g^{tt} \leq 0$) [41].

Let us now consider the boundary structure of the orbit space $\mathcal{B} = \Sigma/U(1)^2$, which determines the topology of the spacetime. There is a semi-infinite rod I_+ corresponding to one of axes of symmetry in the asymptotically flat region. The appropriately normalized Killing field which vanishes on this rod is $v_+ = \partial_\psi - \partial_\phi$. In terms of the spherical coordinates on the ambient $\mathbb{R}^3$ associated to the Gibbons-Hawking space, $I_+ = \{r > a_2, \theta = 0\}$. Next, there is a finite rod $I_{C_2} = \{a_1 < r <$

$a_2, \theta = 0\}$ with associated vanishing Killing field $v_2 = -(\partial_\phi + \partial_\psi)$. Note that the Killing field ∂_ψ is non vanishing on C_2 and degenerates smoothly at the endpoints $r = a_1, a_2$ implying that C_2 is a topologically S^2-submanifold in the spacetime. The second 'bubble' corresponds to the interval $I_{C_1} = \{0 < r < a_1, \theta = 0\}$ with associated Killing field $v_1 = -\partial_\phi + \partial_\psi$. The Killing field ∂_ψ is again non-vanishing on this interval and degenerates smoothly at the endpoints $r = 0, r = a_1$. Finally, there is a second semi-infinite rod $I_- = \{r > 0, \theta = \pi\}$ with associated Killing field $v_- = \partial_\phi + \partial_\psi$.

The rod structure is most naturally expressed in terms of the basis of Killing fields $m_1 = v_+, m_2 = v_-$ which have 2π periodic orbits:

$$v_+ = (1,0)\,, \quad v_2 = (0,-1)\,, \quad v_1 = (1,0)\,, \quad v_- = (0,1) \tag{3.65}$$

from which it is easy to check that the compatibility condition $|\det(v_i^T v_{i+1}^T)| = 1$ is satisfied for adjacent rods.

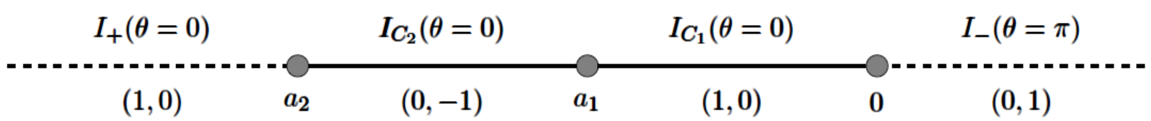

Figure 3.2: Rod structure for double soliton spacetime in (ϕ_1, ϕ_2) basis. Here, $\partial_{\phi_1} = \partial_\psi - \partial_\phi$ and $\partial_{\phi_2} = \partial_\phi + \partial_\psi$.

We now turn to a computation of the various intensive and extensive quantities appearing in the first law. The magnetic fluxes through the bubbles C_1, C_2 are found to be

$$q[C_2] = \frac{1}{4\pi}\int_{S_2^2} F = -\frac{\sqrt{3}}{2}(k_1 + k_2)\,, \qquad q[C_1] = \frac{1}{4\pi}\int_{S_1^2} F = \frac{\sqrt{3}}{2}k_1 \tag{3.66}$$

The computation of the 'electric' potentials U_i requires some more work. For a general supersymmetric solution in the timelike class, one can derive the relation

$$i_\xi \star F = \frac{\sqrt{3}}{2} f^2 \star_4 \mathrm{d}\omega - \frac{fG^+}{\sqrt{3}} \tag{3.67}$$

where $\star_4$ is the Hodge dual taken with respect to the base space and $G^+ = \frac{f}{2}(\mathrm{d}\omega + \star_4 \mathrm{d}\omega)$ is a self-dual 2 form. Using this, and the general form of the Maxwell field leads to the simple expression

$$\Theta = \sqrt{3}\mathrm{d}(f^2(\mathrm{d}t + \omega)) - 4F \tag{3.68}$$

from which it is manifest that Θ is closed, though not exact, as expected. We then have

$$U_\psi = -\sqrt{3} f^2 \omega_\psi + 4A_\psi + 2\sqrt{3}(k_1 + k_2) \tag{3.69}$$

$$U_\phi = -\sqrt{3} f^2 \omega_\phi + 4A_\phi \tag{3.70}$$

where A_ψ, A_ϕ are the components of the gauge field and integration constants have been chosen so that U_i vanish at spatial infinity. As discussed above, $v^i_{C_2} U_i$ and $v^i_{C_1} U_i$ must be constant on the two-cycles C_2 and C_1 respectively. In order to demonstrate this, one must make use of the regularity constraints (3.62). We find

$$\Psi[C_2] = \pi U_{C_2} \equiv -\pi(U_\psi + U_\phi)|_{I_{C_2}} = -4\sqrt{3}k_1 \tag{3.71}$$

$$\Psi[C_1] = \pi U_{C_1} \equiv \pi(U_\psi - U_\phi)|_{I_{C_1}} = 4\pi\sqrt{3}(k_1 + k_2) \tag{3.72}$$

Using this we can indeed verify that

$$\frac{1}{2}\sum_C \Psi[C] q[C] = 6\pi k_1(k_1 + k_2) = M \tag{3.73}$$

The first law

$$\delta M = \Psi[C_1]\delta q[C_1] + \Psi[C_2]\delta q[C_2] \tag{3.74}$$

can then be verified explicitly (we emphasize this is independent from (3.73)). Note that it is straightforward to check that the magnetic potentials are

$$\Phi[C_1] = -\sqrt{3}(k_1 + k_2) = -\frac{1}{4\pi}\Psi[C_1]\ , \qquad \Phi[C_2] = \sqrt{3}k_1 = -\frac{1}{4\pi}\Psi[C_2] \tag{3.75}$$

and inserting these into (3.42) for the total electric charge expressed as sum over the basis of 2-cycles, one recovers the usual BPS relation $M = \sqrt{3}Q/2$. The variational formula (3.74) is surprising as it represents a genuine 'first law' for BPS geometries, whereas for BPS black holes, the first law trivially follows from the BPS condition (i.e. $\delta M = \sqrt{3}\delta Q/2$).

The calculation of angular momenta from the general formula (3.18) is less straightforward. The difficulty arises from the complexity of the solution, and although it is possible to show that $\mathrm{d}\chi_{ij} = 0$, obtaining the integrated potentials in closed form has proved difficult. However, it should be noted that the asymptotic conditions $v^i_+\chi_{ij} = 0$ on I_+ and $v^i_-\chi_{ij}$ on I_-, as well as the evaluation of $\chi_i[C]$ on each cycle, only require knowledge of χ_{ij} on the 'axes' $\theta = 0, \pi$. Hence we need only integrate for $\chi_{ij}(r, 0)$ and $\chi_{ij}(r, \pi)$ on each segment on the axis (i.e. $I_\pm, I_{C_i}$). Since the χ_{ij} must be continuous functions of r along the axes across the rod points at $r = a_2, r = a_1$, and $r = 0$, the integration constants arising from integrating separately over each segment are determined completely by the asymptotic conditions. Carrying this out carefully one finds

$$\chi_\phi[C_2] = 2\sqrt{3}k_1(k_1 + 2k_2)\ , \qquad \chi_\phi[C_1] = -2\sqrt{3}(k_2^2 - k_1^2) \tag{3.76}$$

and

$$\chi_\psi[C_2] = -2\sqrt{3}k_1(3k_1 + 2k_2) \ , \qquad \chi_\psi[C_1] = 2\sqrt{3}(3k_1^2 + 4k_1k_2 + k_2^2) \tag{3.77}$$

where we have used the regularity constraints (3.62) to significantly simplify these expressions. Using the expressions for the fluxes (3.66) we obtain the angular momenta

$$J_\psi = 3\pi k_1(k_1 + k_2)(2k_1 + k_2) \ , \qquad J_\phi = -3\pi k_1 k_2(k_1 + k_2) \ , \tag{3.78}$$

which do in fact agree with the standard ADM angular momenta provided that (3.62) is used to simplify the latter.

Using the above expressions for the charges (J_ψ, J_ϕ, Q) and fluxes $q[C_i]$, we can derive

$$J_\psi \;=\; = \frac{Q}{2}(q[C_1] - q[C_2]) = \frac{8\pi}{\sqrt{3}} q[C_1]q[C_2]\left(q[C_2] - q[C_1]\right) \ , \tag{3.79}$$

$$J_\phi \;=\; \frac{Q}{2}(q[C_2] + q[C_1]) = -\frac{8\pi}{\sqrt{3}} q[C_1]q[C_2]\left(q[C_2] + q[C_1]\right) \ . \tag{3.80}$$

The angular momenta about the $\psi-$ and $\phi-$ directions thus are a measure of the difference and sum of the magnetic fluxes out of the two bubbles.

3.4.3 Dipole black ring

As a last example, we consider asymptotically flat dipole black rings[15] where the horizon topology is $S^1 \times S^2$ and $\Sigma \cong \mathbb{R}^4 \# (S^2 \times D^2)$ [47,48] . The rings are a solution to five dimensional Einstein-Maxwell theory (and also the minimal supergravity theory because the Chern-Simons term is of no consequence to the solutions). For convenience to match with the conventions used in [15], in this section we take $g_{IJ} = 1/2$

in the general formalism of [14]. The metric is given by

$$ds^2 = -\frac{F(y)}{F(x)}\left(\frac{H(x)}{H(y)}\right)\left(\mathrm{d}t + C(\nu,\lambda)\,R\,\frac{1+y}{F(y)}\,\mathrm{d}\psi\right)^2 \tag{3.81}$$

$$+\frac{R^2}{(x-y)^2}\,F(x)\left(H(x)H(y)^2\right)\left[-\frac{G(y)}{F(y)H(y)^3}\mathrm{d}\psi^2 - \frac{\mathrm{d}y^2}{G(y)} + \frac{\mathrm{d}x^2}{G(x)} + \frac{G(x)}{F(x)H(x)^3}\mathrm{d}\varphi^2\right]$$

with the gauge potential,

$$A_\varphi = \sqrt{3}C(\nu,-\mu)R\frac{1+x}{H(x)} \tag{3.82}$$

The functions in the metric are defined as follows,

$$F(\xi) = 1+\lambda\xi, \quad G(\xi) = (1-\xi^2)(1+\nu\xi), \quad H(\xi) = 1-\mu\xi \tag{3.83}$$

$$\text{with } 0<\nu\leq\lambda<1\,, 0\leq\mu<1 \text{ and } C(\alpha,\beta) = \sqrt{\beta(\beta-\alpha)\frac{1+\beta}{1-\beta}},$$

where α and β are any two of the parameters μ,ν and λ.

The following relations remove conical singularities at $y=-1$, $x=-1$ and $x=+1$.

$$\Delta\psi = \Delta\varphi = 2\pi\frac{(1+\mu)^{3/2}\sqrt{1-\lambda}}{1-\nu}\,, \quad \frac{1-\lambda}{1+\lambda}\left(\frac{1+\mu}{1-\mu}\right)^3 = \left(\frac{1-\nu}{1+\nu}\right)^2 \tag{3.84}$$

Thermodynamic quantities for (3.81) were calculated in [15]. Here, we specifically focus on rederiving the the extra terms that contribute to the mass using the results in [14]. These extra terms arise from disc topology surfaces denoted by D that meet the horizon. The fluxes and potentials evaluated on these surfaces can be done so on any other surface that is homologous to D with the same boundary as D. Studying the rod structure of the solution reveals a disc topology surface at $x=1$.

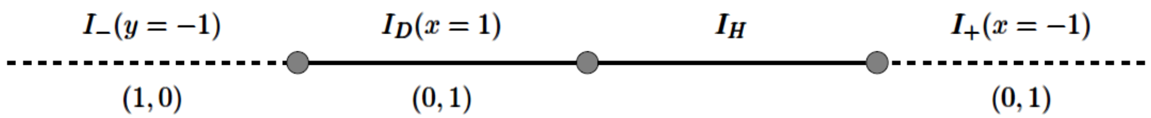

Figure 3.3: Rod structure for dipole ring

The disc D is parametrized by (y, ψ) at constant t, ϕ and $x = 1$. The flux $\mathcal{Q}[D]$ is given by

$$\mathcal{Q}[D] = \int_{[D]} \Theta = -\frac{\sqrt{3}\pi(\mu+1)R\sqrt{\mu(1-\lambda)(1-\mu)}}{4\sqrt{(\mu+\nu)}} \tag{3.85}$$

(For usual Einstein-Maxwell theory $g_{IJ} = \frac{1}{2}$ and $C_{IJK} = 0$). ∂_φ vanishes at $x = 1$. $(v^1, v^2) = (0, 1)$ in the $(\hat{\partial}_\psi, \hat{\partial}_\varphi)$ basis, where the Killing fields are normalized to have 2π periodic orbits.

$$\Phi[D] = v^i \Phi_i = -\frac{2\sqrt{3}(1+\mu)R\sqrt{\mu(1-\lambda)(\mu+\nu)}}{\sqrt{(1-\mu)}(1-\nu)} \tag{3.86}$$

It is easily checked that the potential $\Phi[D] = -2\mathcal{D}$ and flux $\mathcal{Q}[D] = -\frac{1}{2}\hat{\Phi}$ where $\mathcal{D}$ is the local dipole charge and $\hat{\Phi}$ is the magnetic potential introduced[2] in [15]. Therefore, we see that the Smarr relation and first law given in [15]

$$M = \frac{3}{16\pi}\kappa A_H + \frac{3}{2}\Omega_H J + \frac{1}{2}\mathcal{D}\hat{\Phi}\ , \qquad \delta M = \frac{\kappa \delta A_H}{8\pi} + \Omega_H \delta J + \hat{\Phi}\delta\mathcal{D} \tag{3.87}$$

match precisely with the derived expressions in (3.2) and (3.3). An important point to emphasize is that, although the local dipole charge $\mathcal{D}$ arises as a *flux* integral of F over the S^2 of the black ring [15], in our formalism it arises as the constant value of Φ evaluated on the equipotential disc surface D which ends on the horizon. Hence,

[2]The quantities $\mathcal{D}$ and $\hat{\Phi}$ are referred to as $\mathcal{Q}$ and Φ respectively in the notation of [15]. We are using different symbols to avoid confusion with the notation of [14].

although it seems counterintuitive that variations of an 'intensive' variable such as $\Phi[D]$ appear in the general first law, we see that at least in the present case, it is more naturally interpreted as an extensive variable (the dipole charge). Indeed if one looks at the fall-off of the gauge field A at the asymptotically flat region [49], this quantity can be interpreted as producing a dipole contribution. The fact that $\Phi[D]$ captures, in an invariant way, the dipole charge has also been observed in the context of black lenses [50–52]. In the case of black lenses, there is in fact no natural 2-cycle in the spacetime on which to define a dipole charge as there is for a ring [51].

3.5 Discussion

We have explicitly computed the additional terms in the Smarr relation and first law arising from non-trivial spacetime topology in three different geometries, two describing solitons and another describing a black ring. For purely soliton spacetimes, we have complemented the results in [14] with a Smarr type formula for J and Q. These expressions also demonstrate the presence of conserved charges in the absence of a horizon. We have seen that spacetime regularity is crucial for the first law to be satisfied for all examples.

A conjectured relation [53] between dynamical and thermodynamic instability has been established by Hollands and Wald [54]. They have shown that the black p-brane spacetime $M \times \mathbb{T}^p$ associated to a thermodynamically unstable black hole M is itself dynamically unstable. This result of course applies to spacetimes with horizons only, and does not pertain to the soliton spacetimes considered here. Very recently, the linear stability of supersymmetric soliton geometries has been investigated [55] (see also [56] for a rigorous analysis of the scalar wave equation). In particular the authors of [55] have produced evidence that these solutions suffer from a non-linear instability associated with the slow decay of linear waves. It would be interesting if a connection could be found between these studies of dynamical instability and an analogue of thermodynamic instability using the laws of soliton mechanics discussed in this work.

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Chapter 4

Slow decay of waves in gravitational solitons

This chapter is based on the arXiv preprint [57] :

S. Gunasekaran and H. K. Kunduri, *"Slow decay of waves in gravitational solitons,"* (arXiv:2007.04283) *(submitted to Annales Henri Poincaré, currently undergoing revisions)*

4.1 Abstract

We consider a family of globally stationary (horizonless), asymptotically flat solutions of five-dimensional supergravity. We prove that massless linear scalar waves in such soliton spacetimes cannot have a uniform decay rate faster than inverse logarithmically in time. This slow decay can be attributed to the stable trapping of null geodesics. Our proof uses the construction of quasimodes which are time periodic approximate solutions to the wave equation. The proof is based on previous work to

prove an analogous result in Kerr-AdS$_4$ black holes [58]. We remark that this slow decay is suggestive of an instability at the nonlinear level.

4.2 Introduction

Gravitational solitons are globally stationary, asymptotically flat spacetimes with positive energy. A classic result of Lichnerowicz [8] demonstrates that there are no such vacuum solutions in four dimensions. The result can be obtained more directly from the modern viewpoint by an application of the positive mass theorem along with Stokes' theorem and identities related to the stationary Killing field. Intuitively, the result states that an isolated self gravitating system in equilibrium with positive energy must contain a black hole [29]. The result extends to Einstein-Maxwell theory and vacuum general relativity in dimensions greater than four. However, within the supergravity theories that govern the low-energy dynamics in string theory, gravitational solitons arise naturally (we note that *static* solitons can be ruled out in pure Einstein-Maxwell theory in $D > 4$ [59], and there are no known stationary examples). In fact several large families of such supergravity solutions have been obtained explicitly (e.g. see the review [35]). The solitons obtained in these constructions are typically characterized by their mass, angular momenta, global electric charges, and non-trivial spacetime topology. They have received considerable interest, as it has been suggested that they represent classical 'microstate geometries' corresponding to black holes carrying the same conserved charges, thus providing a resolution to the information paradox [60].

Quite independently of these considerations, gravitational solitons possess a number of novel features that distinguish them from black holes. In particular, certain supersymmetric examples contain 'evanescent ergosurfaces', which are timelike hypersurfaces upon which the stationary Killing field can become null [41]. It has been proved that such spacetimes suffer from nonlinear instabilities [55, 56] and exhibit a certain kind of linear instability [61]. On the other hand, soliton spacetimes satisfy a mass variation formula which is analogous to the familiar first law of black hole mechanics [14]. Moreover, solutions have been explicitly constructed that physically correspond to bound state configurations of black holes and solitons (i.e. they have 2-cycles in the domain of outer communication) [42, 62]. Somewhat surprisingly, these solutions have been shown to lead to a continuous failure of black hole uniqueness in higher dimensions even in the supersymmetric setting [63].

A natural question to consider is whether these globally stationary solutions are actually stable in some precise sense. There is, of course, presently a rich body of results concerning the analogous problem for stationary black holes. This stability problem can be posed at increasing levels of complexity. As is well known, the Einstein equations in a suitable gauge reduce to the following schematic form,

$$\Box_g g_{\mu\nu} = Q_{\mu\nu}(g, \partial g) + T_{\mu\nu} \tag{4.1}$$

where Q is quadratic in ∂g. One of the important questions concerning explicit solutions to (4.1) is the analysis of their nonlinear stability in a similar vein as the groundbreaking work of Christodoulou-Klainerman [64] [1]. In this work, it was made clear that perturbations propagate as waves. A natural associated problem to consider is the coupled set of equations governing gravitational perturbations, namely those

[1] Alternate proofs for this nonlinear stability result have been obtained in [65] and [66].

obtained by linearizing (4.1) about an explicit solution. Hence the equation for a single massless scalar field Φ, in a fixed background (M, g), is a good starting point:

$$\Box_g \Phi = 0. \tag{4.2}$$

Though (4.2) is the simplest version of the gravitational perturbation equations, it still preserves many geometric features of the spacetime through the metric, g. Hence understanding the properties of solutions to (4.2) is a useful precursor to the problem of nonlinear stability in the spacetime. The study of linear scalar waves in spacetimes has a well established history; [67–71] are essential reviews on the subject. The study of linear wave equations on explicit stationary solutions has also seen remarkable advancements for spacetimes with other asymptotics and dimensions greater than four. We present a non-exhaustive review below with a marked focus on the methods and results most pertinent to the present work. Our work falls under the domain of stability results in stationary asymptotically flat backgrounds in five spacetime dimensions.

In the realm of stationary asymptotically flat black hole spacetimes, two central unresolved problems are to confirm or disprove the nonlinear stability of the Schwarzschild and Kerr solutions. The initial investigations into stability were focused on mode analysis which confirms the absence of certain exponentially growing modes (in the subextremal case for Kerr) [72, 73]. However these results do not address any boundedness or decay of perturbations. The first step in this direction was the proof of boundedness of scalar waves on Schwarzschild spacetime by Kay–Wald [74, 75] with stronger results subsequently obtained using more universal and robust techniques [67, 76–81]. The black hole case presents a number of challenges, most notably the degeneracy of energy at the horizon, the trapping of null geodesics [82] and superradiance. These problems have been addressed with significant progress in quantitative decay rates [83–86]. These efforts culminated in the proof for decay of linear waves in sub-extremal Kerr spacetime by Dafermos–Rodnianski–Shlapentokh-Rothman [87] (see also [88–90]). In contrast, extremal black holes are affected by an instability discovered by Aretakis (non-decay along the horizon) which also affects long-time decay as discussed in [91–93]. This also implies that the extremal Kerr solution is unstable to linearized gravitational perturbations as shown by Lucietti–Reall in [94]. For the Schwarzschild case, linear stability under the full set of gravitational perturbations (i.e., the linearization of (4.1)) was proved by Dafermos–Holzegel–Rodnianski [95] (see also [96]). It is now known due to Klainerman–Szeftel that Schwarzschild is *nonlinearly* stable to the class of polarized perturbations [97]. See [98, 99] for the recent annoucement of the full finite-codimension non-linear asymptotic stability of the Schwarzschild family. The authors of [95] have further established boundedness and polynomial decay for the spin-2 Teukolsky equation on the Kerr spacetime, which is required to prove the full linearized stability of Kerr to gravitational perturbations [100]. Hafner–Hintz–Vasy in [101] proved linear stability for slowly rotating Kerr black holes using spectral methods.

One may consider stability problems that are asymptotically Anti-de Sitter (AdS) or de Sitter (dS) which are the two other maximally symmetric constant curvature backgrounds. In particular, vacuum AdS, which has a timelike boundary, has been conjectured to be unstable under perturbations of its initial data leading to the formation of a black hole. Numerical work strongly supporting this claim was given in the seminal work of Bizon–Rostworowski [102]. The rigorous results by Moschidis

[103, 104] give further strong evidence for the instability. Recent progress in this problem was announced in [105]. The decay of Klein Gordon fields in AdS was investigated in [106–108] and the global dynamics of solutions to the massive wave equation in AdS black hole spacetimes have been investigated in [106, 109]. In particular as we discuss below, they exhibit a slow decay rate. Finally, on general asymptotically dS spacetimes, waves decay exponentially fast, in contrast with the asymptotically flat case where the decay is at most polynomial. For results on the nonlinear stability of the dS spacetime see [25, 110] with extensions in [111, 112]. Remarkably, the nonlinear stability of slowly rotating Kerr-dS spacetime has been proved by Hintz–Vasy in [113] and extended by Hintz in [114].

The investigation of the stability for higher-dimensional black holes has also received much recent attention. The problem is motivated both for intrinsic mathematical reasons and by connections to high energy physics (see the review [115]). Unsurprisingly, the presence of extra spatial dimensions allows for various novel geometric and topological features, such as the gravitational solitons discussed here and black holes with non-spherical topology. An important rigorous result is that of Schlue, who proved robust quantitative energy decay estimates for solutions of (4.2) in the Schwarzschild family in $D > 4$ spacetime dimensions (see also [116–119]). There is a rather vast literature on mode instabilities associated to rotating Myers-Perry black holes (the natural generalization of the Kerr solution) that arise at sufficiently high angular momenta, as well as numerical analyses on the dynamical evolution [120–122]. Like the Kerr solution, in the Myers-Perry background, (4.2) admits separable solutions, which is particularly useful in the above studies. The black ring family of solutions that describe rotating, asymptotically flat black holes with $S^1 \times S^2$ topology [123, 124], in contrast, are not presented in coordinates which admit a similar separation of variables. This has impeded progress on stability outside of robust numerical strategies [125]. Nonetheless, Benomio [126] has recently proved that the uniform decay rate is slow for generic solutions to (4.2). This provides strong evidence that black rings must be nonlinearly unstable. Quite recently, the nonlinear stability of higher dimensional spacetimes that arise in supersymmetric compactifications of string theory was investigated in [6, 127].

One of the main geometric obstructions to proving a strong decay statement (i.e., fast decay) for solutions to (4.2) is the phenomenon of *trapping* - the confinement of null geodesics in a bounded region of space. The rates of decay of solutions to (4.2) are characterized as *fast* or *slow* depending on their applicability in nonlinear problems. Polynomial decay is robust enough to give hope for nonlinear stability whereas logarithmic decay is not and is hence considered slow. A well-known example of trapping occurs at the photon sphere ($r = 3M$) of Schwarzschild spacetime. Here, initially trapped geodesics are not trapped when perturbed and this structure is characterized as *unstable trapping*. The trapping in Kerr black holes is another such example. Since the propagation of high-frequency waves can be approximated by null geodesics, intuitively one expects energy to clump in a trapped region, leading to slower decay. When trapping is the only obstruction, how strongly the geodesics are trapped is a factor that ultimately dictates whether there is slow or fast decay. The unstable trapping in the Schwarzschild solution roughly leads to sufficiently fast decay. In contrast, the structure of trapping in the soliton geometry to be considered here is *stable*.

The question of whether waves decay at all was answered in the affirmative for a general class of stationary asymptotically flat spacetimes due to a powerful result of Moschidis [128]. The general decay result he established is restated here :

Theorem 4.2.1 (Moschidis, 2015)**.** *Let $(\mathcal{M}^{d+1}, g)$, $d \geq 3$, be a globally hyperbolic spacetime, which is stationary and asymptotically flat, and which can possibly contain black holes with a non-degenerate horizon and a small ergoregion. Moreover, suppose that an energy boundedness statement is true for solutions Φ of the linear wave equation (4.2) on the domain of outer communications $\mathcal{D}$ of the spacetime. Then the local energy of Φ on $\mathcal{D}$ decays at least with a logarithmic rate :*

$$E_{loc}[\Phi](t) \leq C_m \frac{1}{\{\log(2+t)\}^{2m}} E_w^m[\Phi](0) \tag{4.3}$$

where t is a suitable time function on $\mathcal{D}$ and $E_w^m[\Phi](0)$ is an initial energy based on the first m derivatives of Φ.

We note in particular that the above result establishes an upper bound on decay for solutions to (4.2) for a wide class of spacetimes (that is, it is a statement asserting that waves must decay at least inverse logarithmically).

The results in this paper, following closely the strategy of [56, 58, 126, 129] follow from an investigation of slow decay rates for certain stationary spacetimes. In these spacetimes, there are families of trapped null geodesics that have the property that perturbed null geodesics will still be trapped. Hence this structure of trapping is *stable*. Examples of spacetimes exhibiting stable trapping are Kerr-AdS$_4$ black holes [58], ultracompact neutron stars [129], black strings and black rings (mentioned above) [126] and the supersymmetric[2] *microstate geometries* analyzed in [56]. We recall that

[2] Supersymmetric spacetimes admit Killing spinors, i.e., non-trivial spinor fields which are covariantly constant with respect to an appropriate connection.

microstates are stationary, asymptotically flat horizonless solutions of supergravity, and hence in our terminology above, are examples of gravitational solitons. Physically, the mechanism of stable trapping at work in Kerr-AdS$_4$ black holes is the combined effect of lack of dispersion at the asymptotic end and the usual unstable trapping outside the horizon [109], whereas in the case of ultracompact neutron stars and microstates, stable trapping is a result of the coupling between the lack of horizon and trapping. The mechanism behind stable trapping for black rings appears related to the topology of the domain of outer communication. The slow decay result pertaining to stable trapping in supersymmetric solitons proved in [56] is clearly most relevant to our problem, and is restated here:

Theorem 4.2.2 (Keir, 2017)**.** *Let Φ be a solution to the wave equation (4.2) on a two-charge geometry. Let Ω be an open set containing the trapped region. Then for all $k \geq 1$, there exist positive constants C_k such that,*

$$\limsup_{t\to\infty} \sup_{\Phi\neq 0} \left(\frac{\log(2+t)}{\log\log((2+t))} \right)^{2k} \frac{E_{\Omega}[\Phi](t)}{E_{k+1}[\Phi](0)} \geq C_k. \tag{4.4}$$

We remark here that these solutions are the 'closest analogue' to extremal black holes for horizonless solutions bearing in mind that there is no notion of surface gravity here. There are some similarities to extremal black holes with regards to the kind of instability these solutions exhibit as noted by Keir in [61] - namely, that the solutions to the linear wave equation in these backgrounds have a quantity that is non-decaying on a particular surface, though in this case the surface is a 'evanescent ergosurface'. This is a timelike submanifold on which an otherwise everywhere timelike Killing field becomes null. For solutions of the linear wave equation, Keir has shown that generically in spacetimes containing such evanescent surfaces, either there is a concentration of a finite amount of energy into an arbitrarily small spatial region, or the energy of solutions measured by a stationary observers can be amplified by an arbitrarily large amount [61].

In contrast to the family of supersymmetric solitons studied in [56], the soliton spacetimes we examine are non-supersymmetric, possess a globally timelike Killing vector field and hence are devoid of an ergoregion or evanescent ergosurface. Therefore the energy of solutions to the massless wave equation i.e., (4.2) is easily seen to be uniformly bounded. The solutions we study have isometry group $\mathbb{R}\times SU(2)\times U(1)$ and are in fact subfamilies of a larger family of non-supersymmetric solitons with isometry group $\mathbb{R}\times U(1)\times U(1)$ first found by [130]. The latter contain ergoregions, and hence must suffer from the Friedman instability [131] which was recently rigorously proved by Moschidis [132]. (For an analysis of unstable modes for these general solitons, see [44]).

Hence unlike [56] and the solutions discussed in [44], the spacetime we investigate satisfies the conditions for the application of the upper bound stated in Theorem 4.2.1. In this paper we prove a *lower bound* for the decay rate. More precisely, our main result is

Theorem 4.2.3. *Let Φ be a solution to the wave equation (4.2) on a soliton spacetime. Let Ω be an open set containing the trapped region. Then for all $k \geq 1$, there exist positive constants C_k such that,*

$$\limsup_{t\to\infty} \sup_{\Phi\neq 0} (\log(2+t))^{2k} \frac{E_\Omega[\Phi](t)}{E_{k+1}[\Phi](0)} \geq C_k \tag{4.5}$$

where the supremum is taken over all functions Φ in the completion of the set of smooth, compactly supported functions with respect to the norm defined by the higher order energy, E_{k+1}. See (4.40) and (4.44) for the definition of energy.

An immediate consequence of this result, in conjunction with Moschidis' Theorem 4.2.1, is that the bound given by (4.3) is sharp for this class of spacetimes. Furthermore, our result strongly suggests that decay in the fully nonlinear regime is unlikely. As mentioned in [129], one expects the end point of such a nonlinear instability to be gravitational collapse, intuitively caused by the trapping of waves. In light of this result, it would be interesting to study the stability of more general families of nonsupersymmetric solitons that were constructed in [130]. Furthermore, it would be natural to extend the investigations here to investigate the stability of spacetimes containing both a black hole and soliton [42], or a black lens [50] (an asymptotically flat black hole with horizon topology $S^3/\mathbb{Z}_2$), which contains both a horizon and an evanescent ergosurface in the domain of outer communcations. One might expect that the presence of a horizon might influence the stability.

This chapter is organized as follows. We introduce solitons and review the properties of the spacetime in §4.3. We understand trapping by studying null geodesics. More specifically, we prove that, from the geodesic point of view there is a region of phase space exhibiting stable trapping. The uniform boundedness argument in this spacetime is quite straightforward and we give this in §4.4 to present a complete discussion on stability. In §4.5, after a separation of the wave equation into a one variable Schrödinger type equation, we see how geodesic trapping manifests in high frequency waves. This Schrödinger type equation is a nonlinear eigenvalue problem and establishing the existence of eigenavalues to this problem is central to proving the lower bound on the uniform decay rate. Here, we also state the nonlinear eigenvalue problem $(\mathcal{P}_\beta)$ and the corresponding linear eigenvalue problem $(\mathcal{P}_0)$ that will be studied first. In §4.6, $\mathcal{P}_0$ is examined and the existence of eigenvalues to this problem is proved using a version of Weyl's law. In §4.7, we move to the actual nonlinear problem of interest $\mathcal{P}_\beta$. We start by examining the properties of the 'nonlinear potential' and restore the setting of $\mathcal{P}_0$ for $\mathcal{P}_\beta$. Using the bounds on the eigenvalues and the implicit function theorem, we will establish the existence of eigenvalues to $\mathcal{P}_\beta$. The remaining part of the paper contains the details of how these eigenfunctions prove a logarithmic lower bound on the uniform decay rate. In §4.8, we use Agmon estimates to quantitatively measure the solution (eigenfunctions) in the cut-off region. This estimate decays exponentially in a certain parameter n. Quasimodes are constructed by smoothly cutting off the solution near the boundary of a set containing the trapped region. The corresponding wave function Ψ is shown to be an approximate solution to the wave equation (4.2) with an exponentially small error in n. This in conjunction with Duhamel's formula will give the logarithmic lower bound. Our work is heavily indebted to the clear exposition given by Benomio [126].

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4.3 A Class of Nonsupersymmetric Gravitational Solitons

4.3.1 Metric and properties of the solution

We consider an asymptotically flat, globally stationary family of non-supersymmetric soliton spacetimes. The underlying manifold has topology $\mathbb{R} \times \Sigma$ with the spatial slices $\Sigma \cong \mathbb{R}^4 \# \mathbb{CP}^2$. It is analyzed in detail in [41] and [28]. The spacetimes are solutions to five-dimensional minimal supergravity whose action is

$$S = \frac{1}{16\pi} \int_{\mathcal{M}} \left(\star R - 2F \wedge \star F - \frac{8}{3\sqrt{3}} F \wedge F \wedge A \right) . \tag{4.6}$$

Here $F = \mathrm{d}A$ is a smooth 2-form on the spacetime describing the Maxwell field and A is a locally defined gauge potential. The local solution (g, F) is

$$\begin{aligned} \mathrm{d}s^2 &= \frac{-r^2 W(r)}{4b(r)^2}\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{W(r)} + \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2) + b(r)^2(\sigma_3 + f(r)\mathrm{d}t)^2 \,, \\ F &= \frac{\sqrt{3}q}{2}\mathrm{d}\left[r^{-2}\left(\frac{j}{2}\sigma_3 - \mathrm{d}t \right) \right] . \end{aligned} \tag{4.7}$$

The functions appearing in the metric are given below :

$$\begin{aligned} W(r) = 1 - \frac{2}{r^2}(p-q) + \frac{1}{r^4}\left(q^2 + 2pj^2\right) \,, \quad b(r)^2 = \frac{r^2}{4}\left(1 - \frac{j^2 q^2}{r^6} + \frac{2j^2 p}{r^4}\right) \\ f(r) = \frac{-j}{2b(r)^2}\left(\frac{2p-q}{r^2} - \frac{q^2}{r^4} \right) \,, \end{aligned} \tag{4.8}$$

and the σ_i are left-invariant one-forms on $SU(2)$ given by

$$\begin{aligned} \sigma_1 = -\sin\psi \mathrm{d}\theta + \cos\psi \sin\theta \mathrm{d}\phi, \ \sigma_2 = \cos\psi \mathrm{d}\theta + \sin\psi \sin\theta \mathrm{d}\phi \\ \sigma_3 = \mathrm{d}\psi + \cos\theta \mathrm{d}\phi \end{aligned} \tag{4.9}$$

which satisfy $\mathrm{d}\sigma_i = \frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k$. In order to describe an asymptotically flat metric in the region $r \to \infty$, we must periodically identify $\psi \sim \psi + 4\pi$, $\phi \sim \phi + 2\pi$ and $\theta \in (0, \pi)$.

$t \in \mathbb{R}$ is the time coordinate. The range of the radial coordinate is $0 < r_0 < r < \infty$ where r_0 is a parameter that characterizes the size of an S^2 'bolt' as described below. The paramaters p, q, r_0 and j are related by,

$$p = \frac{r_0^4(r_0^2 - j^2)}{2j^4}, \quad q = \frac{-r_0^4}{j^2} \tag{4.10}$$

The spacetime is invariant under an $\mathbb{R} \times SU(2) \times U(1)$ isometry generated by ∂_t, ∂_ψ and the vector fields R_i that leave the σ_i invariant. The above solutions are a subfamily of a more general set of $\mathbb{R} \times U(1)^2$ invariant nonsupersymmetric solitons (see [37, 130]). Surfaces of constant $r > r_0$ are timelike hypersurfaces with spatial geometry of S^3 with a homogeneously squashed metric. An analysis of the metric shows that it is smooth everywhere (apart from standard coordinate singularities at $\theta = 0, \pi$ corresponding to fixed points of $U(1)$ isometries on S^3). However, the parameters p and q have been chosen above so that the functions $W(r)$, $b(r)$ have simple zeroes at $r = r_0$. In particular the Killing field ∂_ψ degenerates at r_0. The degeneration is smooth i.e., there are no conical singularities, provided we require that $W'(r)b^2(r)' = 1$ at $r = r_0$ or

$$(1 - \alpha^2)(2 + \alpha^2)^2 = 1 \tag{4.11}$$

where, $\alpha = r_0/j$. This cubic has a unique positive solution at $\alpha^2 \approx 0.870385$, and in particular $r_0^2 < j^2$. With these relationships between the parameters, it can be checked that, $W(r)$, $b(r)^2 > 0$ for $r > r_0$ and the spacetime metric is globally regular. Further

$$g^{tt} = -\frac{4b(r)^2}{r^2 W(r)} < 0 \tag{4.12}$$

so the spacetime is stably causal, and in particular the $t =$ constant hypersurfaces are Cauchy surfaces. Using the relationships between the parameters, it can also be checked that

$$g_{tt} = -\frac{r^2 W(r)}{4b(r)^2} + b(r)^2 f(r)^2 < 0 \quad \text{everywhere.} \tag{4.13}$$

Hence, $\partial/\partial t$ is globally timelike and there are no ergoregions. Hence the solutions to the wave equation do not suffer from Friedman's ergosphere instability recently proved in [132]. In summary the above metric extends globally to a complete, asymptotically flat metric. Near $r = r_0$, the geometry of the manifold is that of $\mathbb{R} \times \mathbb{R}^2 \times S^2$ (∂_ψ degenerates at the origin of the $\mathbb{R}^2$ in the (r, ψ) coordinates) and the S^2 has radius r_0 and is parameterized by (θ, ϕ).

The ADM mass and angular momenta of the soliton are

$$M = \frac{3\pi}{8}\left(\frac{r_0}{j}\right)^4 (j^2 + r_0^2), \qquad J_\psi = \frac{\pi r_0^6}{4j^3}, \qquad J_\phi = 0\,. \tag{4.14}$$

In terms of angular momenta (J_1, J_2) measured with respect to two orthogonal independent planes of rotation at infinity, this class of solitons is 'self-dual' i.e., $J_1 = J_2$. We note that more general solutions exist with $J_1 \neq J_2$, in which case the isometry group is broken to $\mathbb{R} \times U(1) \times U(1)$. Physically, the 2-cycle $[C]$ is prevented from collapse by a 'dipole' flux

$$\mathcal{Q} := \frac{1}{4\pi}\int_{S^2} F = \frac{\sqrt{3} r_0^2}{4j}\,, \tag{4.15}$$

and these variables satisfy a 'first law' of soliton mechanics $\mathrm{d}M = \Psi[C]\mathrm{d}\mathcal{Q}$ where $\Psi[C]$ is a certain intensive thermodynamical variable conjugate to $\mathcal{Q}$ [28].

4.3.2 Trapping of null geodesics

Let us now consider the properties of null geodesics in this spacetime. We will prove here that there is a region in the phase space of parameters for which null geodesics are stably trapped. A similar analysis was carried out for supersymmetric microstate goemetries in [133].

We start with the fact that the Hamilton-Jacobi function for null geodesics in (4.7) is separable due to the existence of a reducible Killing tensor. In other words, the equations describing null geodesics are integrable. We write the Hamilton-Jacobi function S in the separable form

$$S = -Et + R(r) + \Theta(\theta) + \psi p_\psi + \phi p_\phi \tag{4.16}$$

where E, p_ψ and p_ϕ are conserved quantities associated to the three commuting Killing vector fields ∂_t, ∂_ψ, ∂_ϕ. We have another conserved quantity C which is a separation constant arising from a reducible Killing tensor. Altogether, we have four constants of motion from the isometries of the solution. The conserved momenta can be obtained from the Hamiltonian $H = g^{ab} p_a p_b$:

$$\begin{aligned} p_t = -E &= \left(\frac{-r^2 W(r)}{2b(r)^2} + 2b(r)^2 f(r)^2 \right) \dot{t} + 2b(r)^2 f(r) \left(\dot{\psi} + \cos\theta \dot{\phi} \right) , \\ p_\psi &= 2b(r)^2 (\dot{\psi} + f(r)\dot{t} + \cos\theta \dot{\phi}) \\ p_\phi &= \frac{r^2}{2} \sin^2\theta \dot{\phi} + \cos\theta p_\psi , \quad C = \left(\cot\theta p_\psi - \frac{1}{\sin\theta} p_\phi \right)^2 + \frac{r^4 \dot{\theta}^2}{4} . \end{aligned}$$

The Hamilton-Jacobi function satisfies

$$\frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} g^{\mu\nu} = 0 ,$$

which gives

$$\frac{-4b(r)^2}{r^2 W(r)} \left(E + f(r) p_\psi\right)^2 + R'(r)^2 W(r) + \frac{4C}{r^2} + \frac{p_\psi^2}{b(r)^2} = 0 . \tag{4.17}$$

We can relate $R'(r)$ and $\Theta'(\theta)$ to $\dot{r}$ and $\dot{\theta}$ by,

$$\dot{x^\mu} = g^{\mu\nu} \frac{\partial S}{\partial x^\nu} \tag{4.18}$$

which gives,

$$R^{'}(r) = \frac{\dot{r}}{2W(r)}\,, \quad \Theta^{'}(\theta) = \frac{r^2\dot{\theta}}{8}. \tag{4.19}$$

This allows (4.17) to be rewritten as,

$$\frac{-4b(r)^2}{r^2W(r)}\left(E + f(r)p_\psi\right)^2 + \frac{\dot{r}^2}{4W(r)} + \frac{4C}{r^2} + \frac{p_\psi^2}{b(r)^2} = 0\,. \tag{4.20}$$

In summary, the equations for null geodesic $x^\alpha(\lambda)$ are given by

$$\dot{r}^2 = -\frac{4W(r)p_\psi^2}{b(r)^2} + \frac{16b(r)^2}{r^2}\left(E + f(r)p_\psi\right)^2 - \frac{16W(r)C}{r^2} \tag{4.21}$$

$$\dot{\theta}^2 = \frac{64}{r^4}\left[C - \left(\cot\theta p_\psi - \csc\theta p_\phi\right)^2\right]\,,\ \dot{t} = \frac{8b(r)^2}{r^2W(r)}\left(E + f(r)p_\psi\right) \tag{4.22}$$

$$\dot{\phi} = \frac{8\csc\theta}{r^2}(\csc\theta p_\phi - \cot\theta\, p_\psi)\,, \quad \dot{\psi} = -f(r)\dot{t} + \frac{2p_\psi}{b(r)^2} + \frac{8\cot\theta}{r^2}\left(\cot\theta p_\psi - \csc\theta\, p_\phi\right)\,. \tag{4.23}$$

From (4.21), we can see that close to $r = r_0$ the first term dominates over the others making $\dot{r}^2$ negative. This means null geodesics with non-zero p_ψ approaching the 'origin' must turn around at some $r > r_0$. To simplify the analysis it is sufficient to restrict to motion in a plane with constant θ. Such null geodesics confined to a plane are solutions to $\ddot{\theta} = 0$ with $\dot{\theta} = 0$. For example, from the equation for $\dot{\theta}^2$ i.e., (4.22), we can see that $C = 0$ corresponds to geodesics confined in the $\theta = \pi/2$ equatorial plane.

Stable trapping occurs when there is a region $[r_1, r_2]$ in which $\dot{r}^2 > 0$ in the interior and vanishes at r_2 with $\dot{r}^2 < 0$ immediately outside the closed interval. Hence r_1, r_2 are turning points. Hence, stable trapping occurs when (4.21) has more than one turning point as depicted in Figs. 4.2a and 4.2b.

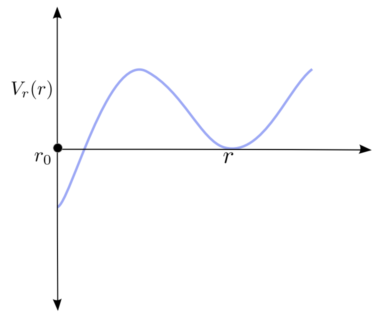
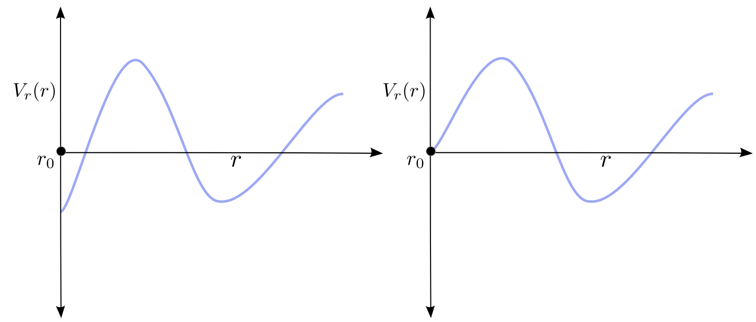


Figure 4.1: Unstable trapping



(a) Case 1 (for $C = 0$) (b) Case 2 (for $C \neq 0$)

Figure 4.2: Stable trapping

Claim 4.3.1 (Existence of stably trapped null geodesics)**.** *There exists a region in the phase space of parameters (of geodesic motion) for which the 1-parameter family of spacetimes given by* (4.7) *exhibits stable trapping of null geodesics.*

Proof. There are two possible cases to consider : $C = 0$ and $C \neq 0$. We examine each of the cases below.

(a) For $C = 0$, we rewrite (4.21) as

$$\dot{r}^2 = V_r(r) := -\frac{4W(r)p_\psi^2}{b(r)^2} + \frac{16b(r)^2}{r^2}\left(E + f(r)p_\psi\right)^2. \tag{4.24}$$

Stable trapping corresponds to $V_r(r)$ having at least two zeros. It is useful to work with dimensionless quantities and scale out the dependence of j, so we use the following scaling for coordinates and parameters,

$$r = j \cdot x, \qquad r_0 = \alpha \cdot j \qquad \text{and} \qquad E = \frac{\widetilde{E}}{j} \tag{4.25}$$

We only need positive roots greater than α^2 to the equation. We recall that

$\alpha^2 \approx 0.870385$. With the following definitions,

$$\eta := \frac{p_\psi}{\widetilde{E}} \quad \text{and } y := x^2 \tag{4.26}$$

(4.24) becomes,

$$\begin{aligned} -\frac{y^3 j^2}{4\widetilde{E}^2} V_r(y) = & - y^3 + 4\,\eta^2 y^2 + \left(4\,\alpha^6\eta - \alpha^6 - 4\,\alpha^6\eta^2 + \alpha^4 - 4\,\alpha^4\eta^2\right) y \\ & - 4\,\alpha^8\eta + \alpha^8 + 4\,\alpha^8\eta^2 \end{aligned} \tag{4.27}$$

For $\eta \in (-1.33, -1.24)$, we have three turning points all bigger than r_0 indicating the existence of stably trapped null geodesics. This case is pictorially depicted in Fig.4.2a.

(b) For $C \neq 0$, fix $\theta = \pi/2$. From $\dot{\theta} = 0$ we have $C = p_\phi^2$ and $\ddot{\theta} = 0$ gives $p_\psi = 0$. (4.24) becomes

$$\dot{r}^2 = \frac{16b(r)^2}{r^2}E^2 - \frac{16W(r)p_\phi^2}{r^2} \tag{4.28}$$

Rewriting using (4.25) and (4.26), (4.28) becomes,

$$\begin{aligned} \frac{j^4 y^3 V_r(y)}{\widetilde{E}^2} = & \, y^3 - 4\,\eta^2 y^2 + \alpha^4\left(\alpha^2 - 1 + 4\eta^2\alpha^2 + 4\eta^2\right) y \\ & + \alpha^4\left(4\eta^2 - 4\eta^2\alpha^2 - \alpha^4 - 4\eta^2\alpha^4\right) \end{aligned}$$

We know that $V_r(\alpha^2) = 0$ and $V_r(y) \to 1$ as $y \to \infty$. So, $V_r(y)$ is positive for large values of y. The two roots of $V(y)$ are

$$y_1 = -\frac{1}{2}\,\alpha^2 + 2\,\eta^2 + \frac{1}{2}\,\sqrt{\alpha^4 + 8\,\alpha^2\eta^2 + 16\,\eta^4 - 4\,\alpha^6 - 16\,\eta^2\alpha^6 - 16\,\eta^2\alpha^4}$$
$$y_2 = -\frac{1}{2}\,\alpha^2 + 2\,\eta^2 - \frac{1}{2}\,\sqrt{\alpha^4 + 8\,\alpha^2\eta^2 + 16\,\eta^4 - 4\,\alpha^6 - 16\,\eta^2\alpha^6 - 16\,\eta^2\alpha^4}$$

There is a double root (unstable trapping) if $(\alpha^2+4\eta^2)^2 - 4\alpha^4(\alpha^2+4\eta^2\alpha^2+4\eta^2) = 0$. The real values of η that solve this equation are

$$\eta = \pm\sqrt{\frac{\alpha}{4}\left(2\,\alpha^2 + 2\,\sqrt{\alpha^3\,(\alpha+2)} + 2\,\alpha - 1\right)} \tag{4.29}$$

Hence there is a region of phase space corresponding to unstable trapping. With the solved value of η^2, we can also verify that y_1 and y_2 (in this case $y_1 = y_2$) are greater than α^2. This trapping structure is depicted in Fig. 4.1. For stable trapping, $V_r(y)$ should have three distinct positive roots. Clearly, $V_r(\alpha^2) = 0$. We require that y_2 is real and $y_2 - \alpha^2 > 0$. This will hold provided η satisfies

$$\eta^2 < \frac{\alpha^4 + 2\alpha^2}{8 - 4\,\alpha^4 - 4\,\alpha^2}\,. \tag{4.30}$$

Also $y_1 > y_2 > \alpha^2$ automatically. Hence there is a range of η in phase space for which null geodesics are stably trapped. This is depicted in Fig.4.2b.

$\square$

As discussed in the introduction, the above result suggests that waves with sufficiently high frequency will not decay rapidly enough to guarantee inverse polynomial decay for nonlinear applications.

4.4 Uniform boundedness

In this section we collect some basic results on solutions to the wave equation in this spacetime. Consider a solution Φ to the linear wave equation (4.2). The energy momentum tensor associated with the field Φ is

$$Q_{\mu\nu} = \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \Phi \nabla_\alpha \Phi, \tag{4.31}$$

which satisfies the conservation equation $\nabla^\mu Q_{\mu\nu} = 0$. We introduce an orthonormal frame of one forms so that the spacetime metric (4.7) can be expanded as $g = \eta_{ab}\omega^a\omega^b$

where $\eta = \text{diag}(-1,1,1,1,1)$:

$$\begin{gathered}\omega^0 = \frac{r\sqrt{W}}{2b}\mathrm{d}t, \qquad \omega^r = \frac{\mathrm{d}r}{\sqrt{W}}, \qquad \omega^1 = \frac{r}{2}\sigma^1, \\ \omega^2 = \frac{r}{2}\sigma^2, \qquad \omega^3 = b(\sigma^3 + f\mathrm{d}t)\end{gathered} \tag{4.32}$$

The dual orthonormal frame of vector fields satisfying $g^{-1} = \eta^{ab}e_a e_b$ is

$$\begin{gathered}e_0 = \frac{2b}{r\sqrt{W}}(\partial_t - fL_3), \qquad e_r = \sqrt{W}\partial_r, \qquad e_1 = \frac{2}{r}L_1, \\ e_2 = \frac{2}{r}L_2, \qquad e_3 = \frac{L_3}{b}.\end{gathered} \tag{4.33}$$

where L_i are the vector fields dual to the left-invariant one-forms σ^i, i.e. $\sigma^i(L_j) = \delta^i_j$, $i = 1,2,3$. The unit normal to a $t =$ constant surface is $n = -\omega^0$. As a vector field the unit future-pointing normal is $N = e_0$. Note that $n \propto -\mathrm{d}t$. The timelike Killing vector field $T = \partial_t$ is, in this frame,

$$T = \frac{r\sqrt{W}}{2b}e_0 + fbe_3\,. \tag{4.34}$$

The current $J^T[\Phi]_a = Q_{ab}T^b$ associated to this vector field is

$$J^T[\Phi] = \left(\frac{r\sqrt{W}}{2b}e_0(\Phi) + fbe_3(\Phi)\right)d\Phi + \frac{1}{2}\left(\frac{r\sqrt{W}}{2b}\omega^0 - fb\omega^3\right)|\mathrm{d}\Phi|^2 \tag{4.35}$$

where

$$|\mathrm{d}\Phi|^2 = -(e_0(\Phi))^2 + \sum_{i=1}^{4}(e_i(\Phi))^2 \tag{4.36}$$

Since T, N are future directed, timelike vector fields, the scalar $Q(T,N)$ must be positive definite. We can observe this quite explicitly by computing

$$J^T(N)[\Phi] := Q(T,N) = \frac{r\sqrt{W}}{4b}(e_0(\Phi))^2 + fbe_3(\Phi)e_0(\Phi) + \frac{r\sqrt{W}}{4b}\sum_{i=1}^{4}(e_i(\Phi))^2 \tag{4.37}$$

and then using Young's inequality

$$\begin{aligned}J^T(N)[\Phi] &\geq \frac{r\sqrt{W}}{4b}(e_0(\Phi))^2 + \frac{r\sqrt{W}}{4b}\sum_{i=1}^{4}(e_i(\Phi))^2 - \frac{fb}{2}(e_0(\Phi))^2 - \frac{fb}{2}(e_3(\Phi))^2 \\ &\geq C\sum_{\alpha=0}^{4}(e_\alpha(\Phi))^2\end{aligned} \tag{4.38}$$

where we have noted that $g_{tt} < 0$ implies that

$$\frac{r\sqrt{W}}{2b} > |fb|. \tag{4.39}$$

Let Σ_t denote a spatial hypersurface defined by $t = \text{constant}$ with induced metric h. From the above, the following first-order energy associated to Σ_t is non-negative:

$$E[\Phi](t) := \int_{\Sigma_t} J^T(N)[\Phi] \, \mathrm{dVol}_h \sim \int_{\Sigma_t} \sum_{\alpha=0}^{4} (e_\alpha(\Phi))^2 \, \mathrm{dVol}_h \tag{4.40}$$

and in the following we show that it is controlled by the energy of the initial data. We will use the symbol $E_\Omega[\Phi](t)$ to represent the same integral as above with the region of integration replaced with $\Omega \cap \Sigma_t$ where Ω is a spacetime region. If T is a timelike Killing vector field, one finds that the current is conserved. Using the fact that $Q_{\mu\nu}$ is divergence-free, it is easy to see that

$$\nabla^\mu J^T_\mu(\Phi) = 0. \tag{4.41}$$

Let Σ_0 and Σ_t be two homologous surfaces with a common boundary. Integrating $J^T_\mu(\Phi)$ over the region enclosed by Σ_0 and Σ_t, whose normals are n_0^μ and n_t^μ respectively, and using the divergence theorem, we get,

$$\int_{\Sigma_t} J^T_\mu(\Phi) n_t^\mu = \int_{\Sigma_0} J^T_\mu(\Phi) n_0^\mu \tag{4.42}$$

This holds as long as T is timelike. For the soliton spacetime (4.7), we have a global timelike Killing vector field. No part of Σ_t or Σ_0 is null and hence the control on Φ and its derivatives does not degenerate anywhere. We thus quite straightforwardly obtain the following uniform energy bound.

$$E[\Phi](t) = E[\Phi](0). \tag{4.43}$$

Finally, we define higher-order energies

$$E_k[\Phi](t) := \sum_{0\le|\alpha|\le k-1} \int_{\Sigma_t} J^T(N)[\partial_\alpha \Phi] \, \mathrm{dVol}_h = \sum_{0\le|\alpha|\le k-1} E_t[\partial_\alpha \Phi] \tag{4.44}$$

These energies are roughly equivalent to the sum of the homogeneous seminorms $\dot{H}^k$ on Σ_t with $s \in [1, k]$.

4.5 Separation of variables and eigenvalue problems

4.5.1 Separation of variables

A preliminary step towards the construction of quasimodes is the separation of variables of the wave equation to reduce the problem to a one-dimensional Schrödinger type equation. The advantage to the class of geometries we are considering is that, due to the $\mathbb{R} \times SU(2) \times U(1)$ isometries, apart from a single radial equation, the remaining parts of the wave equation can be solved explicitly, and in particular the spectrum is completely understood. This simplification also allows us to observe how trapping manifests at the wave equation level by studying an effective potential in the radial equation. In the metric given by (4.7) we set

$$\widetilde{\psi} = \frac{\psi}{2} \implies \frac{\partial}{\partial \psi} = \frac{1}{2}\frac{\partial}{\partial \widetilde{\psi}}, \qquad \hat{b}(r)^2 = 4b(r)^2, \tag{4.45}$$

so that $\widetilde{\psi} \sim \widetilde{\psi} + 2\pi$. This normalization is consistent with the conventions used in [134]. We can rewrite (4.7) as

$$\mathrm{d}s^2 = \frac{-r^2 W(r)}{\hat{b}(r)^2}\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{W(r)} + \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2) + \hat{b}(r)^2 \left(\mathrm{d}\widetilde{\psi} + \frac{\cos\theta}{2}\mathrm{d}\phi + f(r)\mathrm{d}t \right)^2 \tag{4.46}$$

For later reference we record the inverse metric:

$$\left(\frac{\partial}{\partial s}\right)^2 = -\frac{\hat{b}(r)^2}{r^2 W(r)}\left(\frac{\partial}{\partial t} - \frac{f(r)}{2}\frac{\partial}{\partial \widetilde{\psi}}\right)^2 + W(r)\left(\frac{\partial}{\partial r}\right)^2 + \frac{4}{r^2}\left(\frac{\partial}{\partial \theta}\right)^2 + \frac{4}{r^2}\left(\frac{\cot\theta}{2}\frac{\partial}{\partial \widetilde{\psi}} - \frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right)^2 + \frac{1}{\hat{b}(r)^2}\left(\frac{\partial}{\partial \widetilde{\psi}}\right)^2, \tag{4.47}$$

and volume form is given by,

$$\mathrm{dVol}_g = \frac{r^3}{4}\sin\theta \mathrm{d}t \wedge \mathrm{d}r \wedge \mathrm{d}\widetilde{\psi} \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi \tag{4.48}$$

The wave equation can be explicitly written out as,

$$\Box_g \Phi = \frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3 W(r)\frac{\partial \Phi}{\partial r}\right) + \frac{4}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + g^{AB}\frac{\partial^2 \Phi}{\partial x^A \partial x^B} \tag{4.49}$$

where $A, B = t, \phi, \widetilde{\psi}$ run over the ignorable coordinates and

$$g^{AB}\frac{\partial^2}{\partial x^A \partial x^B} = -\frac{\hat{b}(r)^2}{r^2 W(r)}\left(\frac{\partial}{\partial t} - \frac{f(r)}{2}\frac{\partial}{\partial \widetilde{\psi}}\right)^2 + \frac{4}{r^2}\left(\frac{\cot\theta}{2}\frac{\partial}{\partial \widetilde{\psi}} - \frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right)^2 + \frac{1}{\hat{b}(r)^2}\frac{\partial^2}{\partial \widetilde{\psi}^2}$$

The isometry group suggests we seek separable solutions of the form

$$\Phi(t, r, \theta, \widetilde{\psi}, \phi) = e^{-i\hat{\omega}t} e^{in\widetilde{\psi}} R(r) Y(\theta, \phi). \tag{4.50}$$

With the separation ansatz (4.50),

$$g^{AB}\frac{\partial^2}{\partial x^A \partial x^B} = \frac{\hat{b}}{r^2 W}\left(\hat{\omega} + \frac{fn}{2}\right)^2 \Phi - \frac{n^2}{\hat{b}^2}\Phi + \frac{4}{r^2}\left(\frac{\cot\theta}{2}\frac{\partial}{\partial \widetilde{\psi}} - \frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right)^2 \Phi \tag{4.51}$$

Consider the following round metric on S^2:

$$\hat{g}_{\hat{i}\hat{j}} \mathrm{d}x^{\hat{i}} \mathrm{d}x^{\hat{j}} = \frac{1}{4}\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) \tag{4.52}$$

normalized so that $\mathrm{Ric}(\hat{g}) = 4\hat{g}$. Define the 1-form $A = \frac{\cos\theta}{2}\mathrm{d}\phi$ which is locally defined on S^2 which is easily seen to be a potential for the Kähler form on $\mathbb{CP}^1 \cong S^2$. Let

$$D := \nabla_{S^2} - inA$$

We have

$$D^2 = \hat{g}^{\hat{i}\hat{j}} D_{\hat{i}} D_{\hat{j}} = g^{\hat{i}\hat{j}} \left((\nabla_{S^2})_{\hat{i}} - inA_{\hat{i}}\right) \left((\nabla_{S^2})_{\hat{j}} - inA_{\hat{j}}\right)$$
$$= \Delta_{S^2} - 2inA_{\hat{i}}\hat{g}^{\hat{i}\hat{j}}(\nabla_{S^2})_{\hat{j}} - in\mathrm{div}_{\hat{g}}A - n^2\hat{g}^{\hat{i}\hat{j}}A_{\hat{i}}A_{\hat{j}}$$

Since $\mathrm{div}_{\hat{g}}A = \hat{g}^{\hat{i}\hat{j}}(\nabla_{S^2})_{\hat{i}}A_{\hat{j}} = 0$,

$$D^2 = \Delta_{S^2} - 2inA_{\hat{i}}\hat{g}^{\hat{i}\hat{j}}(\nabla_{S^2})_{\hat{j}} - n^2\hat{g}^{\hat{i}\hat{j}}A_{\hat{i}}A_{\hat{j}} \tag{4.53}$$

We now compute D^2 explicitly. The Laplacian on S^2 is

$$\Delta_{S^2} = \frac{4}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\right) + \frac{4}{\sin^2\theta}\partial_\phi^2 \tag{4.54}$$

and the remaining terms are

$$-2inA_{\hat{i}}\hat{g}^{\hat{i}\hat{j}}(\nabla_{S^2})_{\hat{j}} = -4in\frac{\cos\theta}{\sin^2\theta}\partial_\phi$$
$$n^2\hat{g}^{\hat{i}\hat{j}}A_{\hat{i}}A_{\hat{j}} = n^2\frac{4}{\sin^2\theta}\frac{\cos^2\theta}{4} = n^2\cot^2\theta \tag{4.55}$$

which gives

$$D^2 = \frac{4}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{4}{\sin^2\theta}\partial_\phi^2 - n^2\cot^2\theta - 4in\frac{\cos\theta}{\sin^2\theta}\partial_\phi \tag{4.56}$$

The operator D^2 is the charged Laplacian on S^2 and its spectrum has been analyzed in detail in the context of $U(1)$ monopoles. Its eigenfunctions $Y(\theta,\phi)$ (suppressing the eigenvalue labels) are similar to the standard spherical harmonics.

$$D^2Y(\theta,\phi) = -\mu Y(\theta,\phi) \tag{4.57}$$

$\mu \geq 0$ are a discrete family of eigenvalues with corresponding eigenfunctions $Y(\theta,\phi)$ [134, 135]. The values taken by μ are,

$$\mu = \ell(\ell+2) - n^2\,, \text{ where } \ell = 2K + |n| \text{ with } K = 0,1,2,3... \tag{4.58}$$

For simplicity throughout this work we will suppress the eigenvalue labels that characterize the eigenfunctions; generally we will work with individual modes with eigenvalue μ. We can concisely write the wave operator on g as

$$\Box_g \Phi = \frac{1}{r^3}\partial_r\left(r^3 W \partial_r \Phi\right) + \frac{D^2}{r^2}\Phi + \frac{\hat{b}}{r^2 W}\left(\hat{\omega} + \frac{nf}{2}\right)^2 \Phi - \frac{n^2}{\hat{b}^2}\Phi \tag{4.59}$$

The wave equation with the separation ansatz (4.50) becomes

$$\begin{aligned}\frac{1}{r^3}e^{-i\hat{\omega}t}e^{in\widetilde{\psi}}Y(\theta,\phi)\frac{\mathrm{d}R(r)}{\mathrm{d}r}\left(r^3 W(r)\frac{\mathrm{d}R}{\mathrm{d}r}\right) + e^{-i\hat{\omega}t}e^{in\widetilde{\psi}}\frac{R(r)}{r^2}D^2 Y(\theta,\phi)\\ + e^{-i\hat{\omega}t}e^{in\widetilde{\psi}}\frac{\hat{b}(r)^2}{r^2 W(r)}\left(\hat{\omega} + \frac{nf(r)}{2}\right)^2 R(r)Y(\theta,\phi) - \frac{n^2}{\hat{b}(r)^2}e^{-i\hat{\omega}t}e^{in\widetilde{\psi}}R(r)Y(\theta,\phi) = 0\end{aligned} \tag{4.60}$$

which finally reduces to,

$$\frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^3 W(r)\frac{\mathrm{d}R(r)}{\mathrm{d}r}\right) + \left[-\frac{\mu}{r^2} + \frac{\hat{b}(r)^2}{r^2 W(r)}\left(\hat{\omega} + \frac{nf(r)}{2}\right)^2 - \frac{n^2}{\hat{b}(r)^2}\right]R(r) = 0 \tag{4.61}$$

To recast this into a Schödinger-like form, we make the following transformations.

$$R(r) = \frac{u}{r\sqrt{\hat{b}(r)}}, \qquad w = \int_{r_0}^{w}\frac{\hat{b}(s)}{sW(s)}\mathrm{d}s \tag{4.62}$$

after which (4.61) becomes,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^3 W(r)\frac{\mathrm{d}R(r)}{\mathrm{d}r}\right) = \frac{\hat{b}(r)^{3/2}}{W(r)}\frac{\mathrm{d}^2 u}{\mathrm{d}w^2} - \left[\frac{rW(r)}{\sqrt{\hat{b}(r)}} + \frac{1}{2}\frac{r^2 W(r)}{\hat{b}(r)^{3/2}}\frac{\mathrm{d}\hat{b}(r)}{\mathrm{d}r}\right]u \tag{4.63}$$

which can be rewritten as,

$$\begin{aligned}-\frac{\mathrm{d}^2 u}{\mathrm{d}w^2} + \Bigg[\frac{W(r)}{\hat{b}(r)^{3/2}}\left(\frac{rW(r)}{\sqrt{\hat{b}(r)}} + \frac{1}{2}\frac{r^2 W(r)}{\hat{b}(r)^{3/2}}\frac{\mathrm{d}\hat{b}(r)}{\mathrm{d}r}\right)\\ + \frac{W(r)}{\hat{b}(r)}\left(\mu - \frac{\hat{b}(r)^2}{W(r)}\left(\hat{\omega} + \frac{nf(r)}{2}\right)^2 - \frac{n^2 r^2}{\hat{b}(r)^2}\right)\Bigg]u = 0\end{aligned} \tag{4.64}$$

Comparing with a Schrödinger equation of the form

$$-\frac{\mathrm{d}^2u}{\mathrm{d}w^2}+\widetilde{V}u=0 \tag{4.65}$$

we can read off the potential as $\widetilde{V}$,

$$\widetilde{V}=\frac{W}{\hat{b}^{\frac{3}{2}}}\partial_r\left[\frac{rW}{\hat{b}^{\frac{1}{2}}}+\frac{1}{2}\frac{r^2W\partial_r\hat{b}}{\hat{b}^{\frac{3}{2}}}+\right]+\frac{W}{\hat{b}^2}\left[\mu+\frac{n^2r^2}{\hat{b}^2}-\frac{\hat{b}^2}{W}\left(\hat{\omega}+\frac{nf}{2}\right)^2\right] \tag{4.66}$$

In summary we have shown that not only can the wave equation be separated, but we can obtain explicit, analytic solutions for the separated solution apart from a single radial Schrödinger equation. This is in contrast with other stationary non-static solutions for which the angular part of the wave equation cannot be solved explicitly (e.g. Kerr or generic Myers-Perry black holes). This nice property characteristic of cohomogeneity-one rotating black holes has been used in the study of linearized gravitational perturbations (see, e.g. [134])

4.5.2 Trapping of high frequency waves

We look at the qualitative behaviour of waves in one spatial dimension by studying a model problem. Consider solutions to the wave equation $\Box_g\Phi=0$ which are of the form, $\Phi(\mathbf{y},t)=e^{-i\hat{\omega}t}U(\mathbf{y})$ where $\mathbf{y}$ refers to a spatial variable. Let $U(\mathbf{y})$ solve the following Schrödinger type equation,

$$-\frac{\mathrm{d}^2U}{\mathrm{d}\mathbf{y}^2}+(\mathbf{V}-\hat{\omega}^2)U=0 \tag{4.67}$$

where $\mathbf{V}:=\mathbf{V}(\mathbf{y})$. Consider a structure of the potential $\mathbf{V}$ as depicted in Fig.4.3.

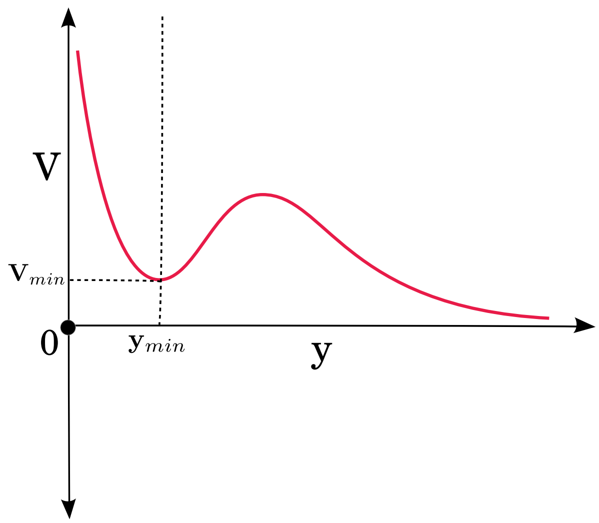


Figure 4.3: Structure of potential in stable trapping

A minimum in the potential $\mathbf{V}_{min}$ indicates that high frequency waves with *suitable energy* (ω^2) which roughly travel along null geodesics remain localized in the the region about $\mathbf{y}_{min}$. In other words, we say that high frequency waves are *trapped*. One can intuitively see that this trapping ultimately leads to a slow decay of waves. The purpose of this work is to prove this rigorously.

In comparison to the discussion above, $\widetilde{V}$ has a term dependent on $\hat{\omega}$ viz., $-nf\hat{\omega}$. Here, we will understand how to analyze the structure of $\widetilde{V}$. The expression for $\widetilde{V}$, (4.66) has two kinds of terms involving $\hat{\omega}$, namely (a) $\hat{\omega}^2$ which is the eigenvalue and (b) $-nf\hat{\omega}$ which is a nonlinear term in the potential $\widetilde{V}$. We define $\widehat{V}$ as the part of the potential independent of $\hat{\omega}$ which is the analogue of $\mathbf{V}$ above.

$$
\begin{aligned}
\widehat{V} =& \widetilde{V} + \hat{\omega}^2 + nf\hat{\omega} \\
=& \frac{W^2}{\hat{b}^2} + \frac{rW(\partial_r W)}{\hat{b}^2} - \frac{rW^2}{2}\left(\frac{\partial_r \hat{b}}{\hat{b}^3}\right) + \frac{r^2 W(\partial_r W)}{2}\left(\frac{\partial_r \hat{b}}{\hat{b}^3}\right) + rW^2\left(\frac{\partial_r \hat{b}}{\hat{b}^3}\right) \\
&+ \frac{W^2 r^2}{2}\left(\frac{\partial_r^2 \hat{b}}{\hat{b}^3}\right) - \frac{3W^2 r^2 \hat{b}^2}{4}\left(\frac{\partial_r \hat{b}}{\hat{b}^3}\right)^2 + \frac{W}{\hat{b}^2}\left[\mu + \frac{n^2 r^2}{\hat{b}^2} - \left(\frac{nf}{2}\right)^2\right]
\end{aligned} \tag{4.68}
$$

We use the following definitions to simplify the expressions:

$$
Y_1(r) = \frac{\partial_r \hat{b}}{\hat{b}^3}, \quad Y_2(r) = \frac{\partial_r^2 b}{\hat{b}^3} = \partial_r(Y_1(r)) - 3\hat{b}^2(Y_1(r))^2. \tag{4.69}
$$

Rewriting the potential in terms of $Y_1(r)$ and $Y_2(r)$ (chiefly to avoid the appearance of odd powers of $\hat{b}(r)$), we get the following expression,

$$
\begin{aligned}
\widehat{V} =& \frac{W^2}{\hat{b}^2} + \frac{rW(\partial_r W)}{\hat{b}^2} - \frac{rW^2}{2}Y_1(r) + \frac{r^2 W(\partial_r W)}{2}Y_1(r) + rW^2 Y_1(r) \\
&+ \frac{W^2 r^2}{2}Y_2(r) - \frac{3W^2 r^2 \hat{b}^2}{4}Y_1(r)^2 \frac{\mu W}{\hat{b}^2} + \frac{Wn^2 r^2}{\hat{b}^4} - \frac{n^2 f^2}{4}.
\end{aligned} \tag{4.70}
$$

$\widehat{V}$ has terms which depend on n and μ and terms independent of these charged Laplacian eigenvalues. Hence, we decompose $\widehat{V} = \widehat{V}_{dom} + \widehat{V}_j$ where $\widehat{V}_{dom}$ is the dominant part of the potential. This reflects the fact that for large n or μ, $\widehat{V}_{dom}$ would be the term dictating the behaviour of the potential i.e., for sufficiently large n and μ, $\widehat{V} \lesssim \widehat{V}_{dom}$.

$$
\widehat{V}_{dom} = \frac{\mu W}{\hat{b}^2} + \frac{Wn^2 r^2}{\hat{b}^4} - \frac{n^2 f^2}{4} \tag{4.71}
$$

$$
\begin{aligned}
\widehat{V}_j = \widehat{V} - \widehat{V}_{dom} =& \frac{W^2}{\hat{b}^2} + \frac{rW(\partial_r W)}{\hat{b}^2} - \frac{rW^2}{2}Y_1(r) + \frac{r^2 W(\partial_r W)}{2}Y_1(r) \\
&+ rW^2 Y_1(r) + \frac{W^2 r^2}{2}Y_2(r) - \frac{3W^2 r^2 \hat{b}^2}{4}Y_1(r)^2
\end{aligned} \tag{4.72}
$$

We recall that the eigenvalues n and μ are related as in (4.58) and K can be independently chosen and here we choose it to vary as n. With this, the dependence of μ

on n is :

$$\mu = 8n^2 + 6n \tag{4.73}$$

We can see that only the terms proportional to n^2 in $\widehat{V}_{dom}$ matter when n is large. This happens to be the regime of n we are interested in for reasons which will be given in the next section. V_{dom} can be split up as,

$$\widehat{V}_{dom} = n^2 \widehat{V}_{\sigma_1} + n \widehat{V}_{\sigma_2} \tag{4.74}$$

Explicitly,

$$\widehat{V}_{\sigma_1} = \frac{8W}{\hat{b}^2} + \frac{W r^2}{\hat{b}^4} - \frac{f^2}{4} \quad \text{and} \quad \widehat{V}_{\sigma_2} = \frac{6W}{\hat{b}^2} \tag{4.75}$$

Hence, the ODE of interest is

$$-\frac{\mathrm{d}^2 u}{\mathrm{d}w^2} + (n^2 \widehat{V}_{\sigma_1} - n f \hat{\omega} - \hat{\omega}^2) u = 0 \quad \text{with appropriate boundary conditions.} \tag{4.76}$$

The results for (4.76) will carry over for $(\widehat{V}_{\sigma_1} - n f \hat{\omega} - \hat{\omega}^2)$ replaced by the effective potential $\widetilde{V}$. Here, we rewrite the differential equation in terms of dimensionless variables as we did while analyzing trapping of null geodesics. With the following rescalings, $w = jx$, $r_0 = \alpha j$ and $\omega = \hat{\omega}/j$, (noting that w scales the same way as r) (4.76) becomes,

$$\begin{aligned} -\frac{1}{j^2}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{1}{j^2}\left(n^2 V_{\sigma_1} - n\tilde{f}\omega - \omega^2\right) u = 0 \\ \implies -\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \left(V_{\sigma_1} - n\tilde{f}\omega - \omega^2\right) u = 0 \end{aligned} \tag{4.77}$$

where, $V_{\sigma_1} = j^2 \widehat{V}_{\sigma_1}$ and $\tilde{f} = jf$. V_{σ_1} is explicitly given below,

$$\begin{aligned} V_{\sigma_1} = \frac{\left(x^2 - \alpha^2\right)^{-1}}{16\left(\alpha^6 + \alpha^2 x^2 + x^4\right)^2} \Big[\, & 129\alpha^{14} - \left(129x^2 - 128\right)\alpha^{12} - 128\alpha^{10} + 128\,\alpha^8 x^2 \\ & - \left(144\,x^6 - 128\,x^4 + 128\,x^2\right)\alpha^6 - \left(144x^6 + 128\,x^4\right)\alpha^4 \\ & + 144x^6\alpha^2 + 144x^8\Big] \end{aligned} \tag{4.78}$$

As the first step, we confirm that the spacetime exhibits the structure for stable trapping with a plot of V_{σ_1} in Fig.4.4. The minimum characterizes the stably trapped region and the region in the neighbourhood of the minimum, which is devoid of any local maxima will be denoted by $[x_-, x_+]$.

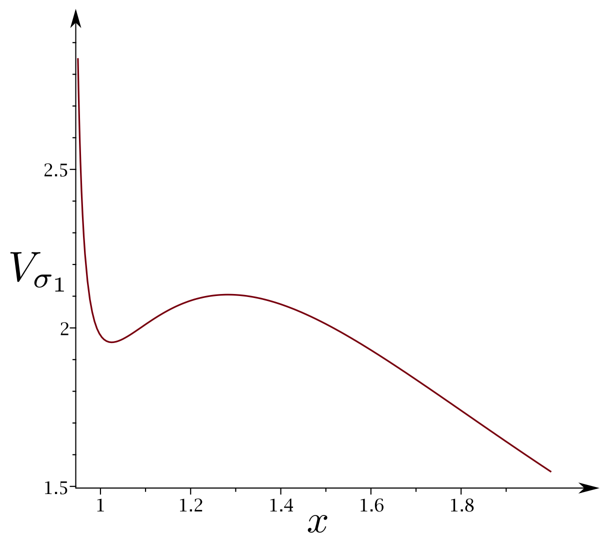


Figure 4.4: Plot of V_{σ_1} against x

The aim of studying (4.77) is the construction of eigenfunctions in $[x_-, x_+]$ with Dirichlet conditions which will be seen in subsequent sections. To give more relevance to this construction here, we give an informal introduction to quasimodes (see [126]).

Consider time periodic functions of the form $\Psi_n(t,x) = e^{-i\omega_n t}u_n(x)$ where ω_n are real. Quasimodes are approximate solutions of the form Ψ_n to the wave equation satisfying the following properties:

1. Ψ_n belongs to an appropriate energy space.

2. They are localized in frequency and space i.e.,

$$\left\|\partial^2\Psi_n\right\| \approx \omega_n^2 \left\|\Psi_n\right\|$$

and Ψ_n are compactly supported.

3. They are an approximate solution to the wave equation i.e.,

$$\Box_g \Psi_n = F_n(\Psi_n)$$

where $F_n(\Psi_n) \to 0$ as $n \to \infty$. Intuitively, the error can be made small in an appropriate limit.

In particular, consider the case where $F_n(\Psi_n) \sim e^{-Cn}$ where C is any constant. By constructing an appropriate sequence of the approximate solutions Ψ_n one can establish that there are slow decaying solutions to the wave equation which contradicts any uniform fast decay statement.

4.5.3 Linear and nonlinear eigenvalue problems

The main eigenvalue problem that we study is the Schrödinger type wave equation along with Dirichlet boundary conditions at x_- and x_+. The problem is stated below

$$\begin{gathered} -\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \left(n^2 V_{\sigma_1} - n\widetilde{f}\omega - \omega^2\right) u = 0 \\ u(x_-) = u(x_+) = 0 \end{gathered} \tag{4.79}$$

As mentioned previously the "potential term" appearing here has a nonlinear dependence on ω, which constitutes a nonlinear eigenvalue problem. This makes a straightforward analysis of (4.79) difficult. Here, by linear we mean linear in ω^2. We define $\mathcal{P}_\beta$ to be the following family of eigenvalue problems labeled by $\beta \in [0,1]$.

$$\mathcal{P}_\beta \quad : \quad \begin{aligned} -\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \left(n^2 V_{\sigma_1} - \beta n \widetilde{f}\omega - \omega^2\right) u = 0 \\ u(x_-) = u(x_+) = 0 \end{aligned} \tag{4.80}$$

We can identify $\mathcal{P}_0$ as the linear eigenvalue problem (since the potential does not depend on ω).

$$\mathcal{P}_0 \quad : \quad \begin{aligned} -\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + n^2 V_{\sigma_1} u = \omega^2 u \\ u(x_-) = u(x_+) = 0 \end{aligned} \tag{4.81}$$

and $\mathcal{P}_1$ is the nonlinear eigenvalue problem (4.79) that we want to solve. Hence β is a nonlinear parameter that represents a transition from the linear eigenvalue problem $\mathcal{P}_0$ to the nonlinear eigenvalue problem $\mathcal{P}_1$.

Before continuing with the analysis of these problems, we pause to note similarities in the soliton and Kerr-AdS$_4$ case for the construction of quasimodes, the most fundamental being the phenomenon of stable trapping occuring in both. A key difference arises in the extension of results from $\mathcal{P}_0$ to $\mathcal{P}_1$. In the Kerr-AdS$_4$ case [58], the potential has a nonlinear term which is proportional to ω^2, so the whole eigenvalue equation is quadratic in ω^2. In our case the nonlinearity is $\sim \omega n$. The difficulty arises from the presence of the eigenvalue n with ω and the fact that we have terms proportional to both ω^2 and ω in the equation. Such problems were encountered in the analysis of quasimodes and stable trapping in black ring spacetimes [126], and we will follow the strategy developed there. We also note here that the phenomenon of nonlinear terms in the eigenvalue problem which are linear in ω was also encountered in [56]. But the effective scaling with n of the potential V is different leading to a different approach being taken there.

4.6 Eigenvalues for the linear problem

In this section we use a suitable version of Weyl's law to establish the existence of eigenfunctions for the linear problem $\mathcal{P}_0$ defined by (4.81). Here, we essentially follow the approach used in [58] and [56]. We start by defining a semi classical parameter $h^2 = n^{-2}$ and express the problem in the form

$$
\begin{aligned}
-h^2 \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + V_{\sigma_1} u &= \kappa u \\
u(x_-) = u(x_+) &= 0
\end{aligned}
\tag{4.82}
$$

where we have defined κ to be the eigenvalue i.e., $\kappa := h^2\omega^2$. We identify the region $\Omega := [x_-, x_+]$ for the eigenvalue problem through the following lemma.

Lemma 4.6.1. *Let $V_{\sigma_1}^{min}$ be the local minimum of the potential and let $x_{min} \in (\alpha, \infty)$ be the point where this minimum is attained i.e., $V_{\sigma_1}(x_{min}) = V_{\sigma_1}^{min}$. Let $c > 0$ be sufficiently small so that there exist x_- and x_+ with $x_- < x_{min} < x_+$ for which, $V_{\sigma_1}^{min} + c = V_{\sigma_1}(x_-) = V_{\sigma_1}(x_+)$ and there are no local maxima of V_{σ_1} in $[x_-, x_+]$. Let $E > V_{\sigma_1}^{min}$ such that $E - V_{\sigma_1}^{min} < c$. Then for any sufficiently small constants $\delta, \delta' > 0$ there exists some constant $c' > 0$ such that*

$$
|x_\pm - x| < \delta' \implies V_{\sigma_1}(x) - \kappa > c' \tag{4.83}
$$

for all $\kappa \in [E - \delta, E + \delta]$.

Proof. The idea behind the above lemma is illustrated in Fig.4.5. We can fix a sufficiently small constant δ such that $E + \delta < V_{\sigma_1}^{min} + c$. $V_{\sigma_1}(x)$ is continuous at x_-. In the following we will establish the result for x_- and the proof for x_- replaced by x_+ follows by a similar argument. For a given $\tilde{\epsilon}$, one can find a δ' such that,

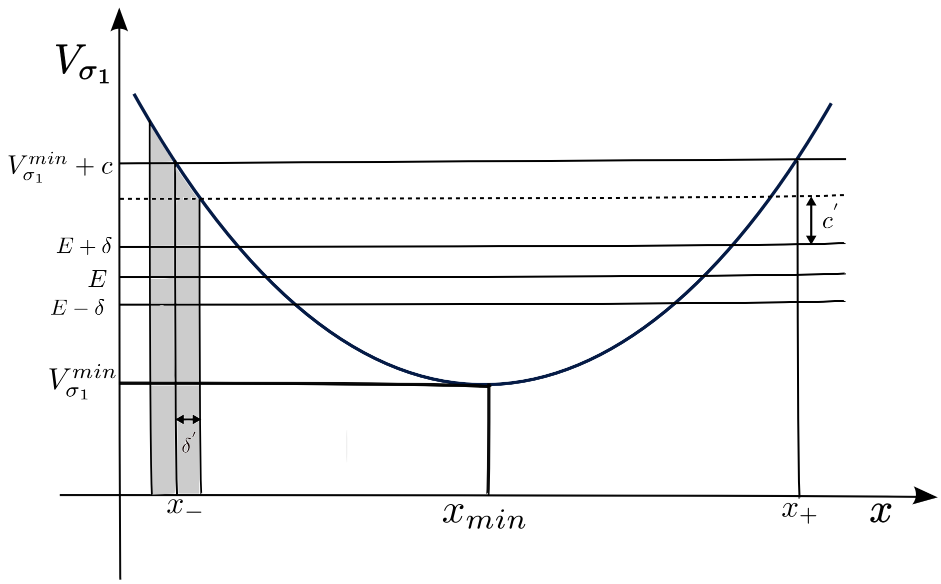


Figure 4.5: Domain for the linear eigenvalue problem

$|V_{\sigma_1}(x) - V_{\sigma_1}(x_-)| < \tilde{\epsilon}$ whenever $|x - x_-| < \delta'$. Choose

$$\tilde{\epsilon} = \frac{V_{\sigma_1}(x_-) - (E + \delta)}{3}$$

This means, whenever $|x_- - x| < \delta'$, $|V_{\sigma_1}(x_-) - V_{\sigma_1}(x)| < \tilde{\epsilon}$

$$V_{\sigma_1}(x_-) - \tilde{\epsilon} < V_{\sigma_1}(x) < V_{\sigma_1}(x_-) + \tilde{\epsilon}$$

For $\kappa \in [E - \delta, E + \delta]$ we have

$$\begin{aligned} V_{\sigma_1}(x) - \kappa &> V_{\sigma_1}(x_-) - \tilde{\epsilon} - \kappa \\ &> V_{\sigma_1}(x_-) - \tilde{\epsilon} - (E + \delta) \\ &= 3\tilde{\epsilon} - \tilde{\epsilon} = 2\tilde{\epsilon} \end{aligned}$$

Setting $c' = 2\tilde{\epsilon}$ completes the proof. □

We now state and prove Weyl's law. This allows us to establish the existence of eigenfunctions for the Dirichlet problem in the domain $[x_-, x_+]$. More precisely, this statement proves that the number of eigenvalues κ in some small neighbourhood scale as h^{-1}. The eigenvalue problem with Dirichlet conditions will be denoted by $\mathcal{P}_D(x_-, x_+)$ and $N_{\leq E}(\widetilde{\mathcal{P}})$ denotes the number of eigenvalues of the problem $\widetilde{\mathcal{P}}$ which are less than or equal to E.

Theorem 4.6.2 (**Weyl's law**)**.** *Consider the eigenvalue problem $\mathcal{P}_D(x_-, x_+)$. Let E be an energy level such that $E - V_{\sigma_1}^{min}$ is sufficiently small and $E - V_{\sigma_1}^{min} > \delta$ for some fixed positive constant δ such that $E + \delta < V_{\sigma_1}^{min} + c$ with the constant $c > 0$ introduced in Lemma 4.6.1. Then the number of eigenvalues of the problem $\mathcal{P}_D(x_-, x_+)$ less than E, denoted by $N_{\leq E}(\mathcal{P}_D(x_-, x_+))$, satisfies the following estimate called Weyl's law.*

$$N_{\leq E}(\mathcal{P}_D(x_-, x_+)) \sim \mathcal{Q}_{E,h} \tag{4.84}$$

where

$$\mathcal{Q}_{E,h} := \frac{1}{h\pi} \int_{x_-}^{x_+} \sqrt{E - V_{\sigma_1}(x^*)}\chi_{\{V_{\sigma_1} \leq E\}} \mathrm{d}x^*.$$

We will also establish the following result which estimates the number of eigenvalues for the problem for $\mathcal{P}_D(x_-, x_+)$ lying in a δ interval of E.

Theorem 4.6.3. *Let $N[E-\delta, E+\delta]$ denote the number of eigenvalues of $\mathcal{P}_D(x_-, x_+)$ lying in the interval $[E - \delta, E + \delta]$. Then $N[E - \delta, E + \delta]$ satisfies Weyl's law i.e.,*

$$N[E - \delta, E + \delta] \sim \mathcal{Q}_{E+\delta,h} - \mathcal{Q}_{E-\delta,h} \tag{4.85}$$

We will prove this with the following two lemmas which give upper and lower bounds for $N_{\leq E}(\mathcal{P}_D(x_-, x_+))$. These bounds will be explicitly calculated. We first partition $[x_-, x_+]$ into k intervals $[x_-^i, x_+^i]$ where

$$x_-^i = x_- + (i-1)\gamma \ \text{ and } \ x_+^i = x_- + i\gamma \qquad \text{where } \gamma = \frac{x_+ - x_-}{k}\,, \tag{4.86}$$

and define k Dirichlet problems in each $[x_-^i, x_+^i]$. The Dirichlet problems $\mathcal{P}_D^i$ for $i = 1, 2, ..., k$ are stated below

$$\begin{aligned} -h^2 \frac{\mathrm{d}^2 u}{dx^2} + V_{\sigma_1} u &= \kappa u \\ u(x_-^i) = u(x_+^i) &= 0 \end{aligned} \tag{4.87}$$

We next define k Neumann problems $\mathcal{P}_N^i$ analogously. $\mathcal{P}_D^i$ and $\mathcal{P}_N^i$ will serve as two comparison problems for estimating $N_{\leq E}(\mathcal{P}_D(x_-, x_+))$ through lower and upper bounds respectively. We start with the following lemma which gives a lower bound through the k Dirichlet problems $\mathcal{P}_D^i$.

Lemma 4.6.4 (Lower bound). *The number of eigenvalues of the problem $\mathcal{P}_D(x_-, x_+)$ less than E i.e., $N_{\leq E}(\mathcal{P}_D(x_-, x_+))$ satisfies*

$$\sum_i^k N_{\leq E}(\mathcal{P}_D^i) \leq N_{\leq E}(\mathcal{P}_D(x_-, x_+))\,. \tag{4.88}$$

Proof. The proof relies on the variational characterization of eigenvalues using the min-max principle. The smallest eigenvalue of the problem $\mathcal{P}_D(x_-, x_+)$ can be characterized by

$$\kappa_1 = \inf_{\substack{u \in H_0^1([x_-, x_+]) \\ \|u\|_{L^2} \neq 0}} \frac{\int_{x_-}^{x_+} (h^2 |\partial_x u|^2 + V_{\sigma_1}(x)|u|^2)\mathrm{d}x}{\|u\|_{L^2}^2} \tag{4.89}$$

The n-th eigenvalue of $\mathcal{P}_D(x_-, x_+)$ (this is not to be confused with the integer n appearing in the separation of variables (4.50)) can be characterized by

$$\kappa_n = \inf_{\substack{\{u_1, u_2, ..., u_n\},\, u_m \in H_0^1([x_-, x_+]) \\ \|u_m\|_{L^2} \neq 0,\, \langle u_m, u_j \rangle = 0\ \forall m \neq j}} \max_{m \leq n} \frac{\int_{x_-}^{x_+} (h^2 |\partial_x u_m|^2 + V_{\sigma_1}(x)|u_m|^2)\mathrm{d}x}{\|u_m\|_{L^2}^2} \tag{4.90}$$

Similarly, we can characterize the eigenvalues for $\mathcal{P}_D^i$, denoted by λ_n^i as

$$\lambda_n^i = \inf_{\substack{\{u_1,u_2,\ldots,u_n\},\, u_m\in H_0^1([x_-^i,x_+^i])\\ \|u_m\|_{L^2}\neq 0,\, \langle u_m,u_j\rangle=0\ \forall m\neq j}} \max_{m\leq n} \frac{\int_{x_-^i}^{x_+^i}(h^2|\partial_x u_m|^2 + V_{\sigma_1}(x)|u_m|^2)\mathrm{d}x}{\|u_m\|_{L^2}^2} \tag{4.91}$$

We can see from the variational characterization that $\kappa_n \leq \lambda_n^i$. By arranging all the eigenvalues λ_n^i into a single non-decreasing sequence λ_n, we can deduce the following :

$$\kappa_n \leq \lambda_n. \tag{4.92}$$

To see this, let f_n be the eigenfunctions corresponding to λ_n. f_n can be extended to $[x_-, x_+]$ by setting them to vanish outside the corresponding $[x_-^i, x_+^i]$. These n functions are orthogonal in $H_0^1[x_-, x_+]$ either because they are eigenfunctions supported in different regions or because they are different eigenfunctions (with the same or different eigenvalues) to the same problem, which makes them orthogonal [129]. Hence we have $\kappa_n \leq \lambda_n$ which proves the inequality. □

From the k Neumann problems $\mathcal{P}_N^i$ and their corresponding eigenvalues μ_n^i, we have the following lemma.

Lemma 4.6.5 (Upper bound). *The number of eigenvalues of the problem $\mathcal{P}_D(x_-, x_+)$ less than E i.e., $N_{\leq E}(\mathcal{P}_D(x_-, x_+))$ satisfies*

$$N_{\leq E}(\mathcal{P}_D(x_-, x_+)) \leq \sum_i^k N_{\leq E}(\mathcal{P}_N^i). \tag{4.93}$$

Proof. The eigenvalues μ_n^i can be characterized as

$$\mu_n^i = \inf_{\substack{\{u_1,u_2,\ldots,u_n\},\, u_m\in \widetilde{H}^1([x_-,x_+])\\ \|u_m\|_{L^2}\neq 0,\, \langle u_m,u_j\rangle=0\ \forall m\neq j}} \max_{m\leq n} \frac{\sum_{i=1}^k \int_{x_-^i}^{x_+^i}(h^2|\partial_x u_m|^2 + V_{\sigma_1}(x)|u_m|^2)\mathrm{d}x}{\|u_m\|_{L^2}^2} \tag{4.94}$$

where,

$$\widetilde{H}^1\left([x_-, x_+]\right) = \left\{u_m \in L^2([x_-, x_+]) | u_m \in H^1([x_-^i, x_+^i]) \text{ for all } i\right\}$$

Similar to the previous case, we arrange them in a single non-decreasing sequence μ_n. We observe that $H_0^1([x_-, x_+]) \subset \widetilde{H}^1([x_-, x_+])$ which implies that $\mu_n^i \leq \kappa_n$. In particular this means, $\mu_n \leq \kappa_n$ which completes the proof. Since we are in one dimension, the H^1 spaces mentioned here in fact embed into Holder spaces $C^{0,1/2}$. □

Proof of Theorem 4.6.2 (Weyl's law). To compute explicit bounds for $N_{\leq E}(\mathcal{P}_D(r_-, r_+))$ we consider the following sets of problems.

- $\widetilde{\mathcal{P}}_D^i$: Problems $\mathcal{P}_D^i$ where the potential V_{σ_1} is replaced by its maximum value (say V_+^i) in the interval $[x_-^i, x_+^i]$.
- $\widetilde{\mathcal{P}}_N^i$: Problems $\mathcal{P}_N^i$ where the potential V_{σ_1} is replaced by its minimum value (say V_-^i) in the interval $[x_-^i, x_+^i]$.

The bounds for $N_{\leq E}(\mathcal{P}_D(x_-, x_+))$ in Lemmas 4.6.4 and 4.6.5 hold when $\mathcal{P}_D^i$ and $\mathcal{P}_N^i$ are replaced by $\widetilde{\mathcal{P}}_D^i$ and $\widetilde{\mathcal{P}}_N^i$ respectively. These problems can be solved exactly as the potential is just a constant in the interval. The number of eigenvalues of $\widetilde{\mathcal{P}}_D^i$ with energy less than or equal to E is given by,

$$\begin{aligned} N_{\leq E}(\widetilde{\mathcal{P}}_D^i) &= \left\lfloor \frac{\gamma\sqrt{E - V_+^i}}{h\pi} \chi_{\{V_+^i \leq E\}} \right\rfloor \\ \sum_{i=1}^{k} N_{\leq E}(\widetilde{\mathcal{P}}_D^i) &= \sum_{i=1}^{k} \left\lfloor \frac{\gamma\sqrt{E - V_+^i}}{h\pi} \chi_{\{V_+^i \leq E\}} \right\rfloor \\ &= \sum_{i=1}^{k} \left(\frac{\gamma\sqrt{E - V_+^i}}{h\pi} \chi_{\{V_+^i \leq E\}} \right) + \mathcal{O}(k) \end{aligned} \tag{4.95}$$

Similarly for $\widetilde{\mathcal{P}}_N^i$, we have,

$$N_{\leq E}(\widetilde{\mathcal{P}}_N^i) = \left\lfloor \frac{\gamma\sqrt{E-V_-^i}}{h\pi}\chi_{\{V_-^i \leq E\}} \right\rfloor + 1$$

$$\sum_{i=1}^{k} N_{\leq E}(\widetilde{\mathcal{P}}_N^i) = \sum_{i=1}^{k} \left\lfloor \frac{\gamma\sqrt{E-V_-^i}}{h\pi}\chi_{\{V_-^i \leq E\}} \right\rfloor + k \tag{4.96}$$

$$= \sum_{i=1}^{k} \left(\frac{\gamma\sqrt{E-V_-^i}}{h\pi}\chi_{\{V_-^i \leq E\}} \right) + \mathcal{O}(k)$$

Based on Lemmas 4.6.4 and 4.6.5, $N_{\leq E}(\mathcal{P}_D(x_-, x_+))$ satisfies

$$\sum_{i=1}^{k} N_{\leq E}(\widetilde{\mathcal{P}}_D^i) \leq N_{\leq E}(\mathcal{P}_D(r_-, r_+)) \leq \sum_{i=1}^{k} N_{\leq E}(\widetilde{\mathcal{P}}_N^i) \tag{4.97}$$

which becomes,

$$\sum_{i=1}^{k} \left(\frac{\gamma\sqrt{E-V_+^i}}{h\pi}\chi_{\{V_+^i \leq E\}} \right) + \mathcal{O}(k) \leq N_{\leq E}(\mathcal{P}_D(x_-, x_+)) \leq \sum_{i=1}^{k} \left(\frac{\gamma\sqrt{E-V_-^i}}{h\pi}\chi_{\{V_-^i \leq E\}} \right) + \mathcal{O}(k).$$

If we let the number of partitions go to infinity as $h \to 0$ such that $k(h) = o(1/h)$, the sums converge as a Riemann sum and the error terms are of order $o(1/h)$. We can then express $N_{\leq E}(\mathcal{P}_D(x_-, x_+))$ as

$$N_{\leq E}(\mathcal{P}_D(x_-, x_+)) \sim \frac{1}{h\pi} \int_{x_-}^{x_+} \sqrt{E - V_{\sigma_1}(x^*)}\chi_{\{V_{\sigma_1} \leq E\}} \mathrm{d}x^*. \tag{4.98}$$

This proves Theorem 4.6.2. This technique is commonly referred to as Dirichlet-Neumann bracketing [136]. □

Proof of Theorem 4.6.3. This follows by computing $N_{\leq E+\delta}(\mathcal{P}_D(x_-, x_+))$ and $N_{\leq E-\delta}(\mathcal{P}_D(x_-, x_+))$ from (4.98). □

4.7 Eigenvalues for the nonlinear problem

We now turn to establishing the existence of eigenvalues for the radial equation, which as discussed earlier is nonlinear in the 'energy' ω, namely the ODE $\mathcal{P}_1$ which reads,

$$-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + Vu = 0, \qquad u(x_-) = u(x_+) = 0. \tag{4.99}$$

Here V is the nonlinear potential defined by $V := n^2 V_{\sigma_1} - n\widetilde{f}\omega - \omega^2$. The strategy is to prove the existence of eigenvalues of (4.99) through continuity arguments. The potential is a complicated rational function of the rescaled radial variable x explicitly given by

$$\begin{aligned} V = \frac{\left(x^2-\alpha^2\right)^{-1}}{16\,\left(\alpha^6+\alpha^2x^2+x^4\right)^2} \{ \ & (129 - 8\,n\omega)\,\alpha^{14} + \left(128 - (129 - 8\,n\omega)\,x^2\right)\alpha^{12} \\ & - \left(8\,n\omega\,x^2 + 128\right)\alpha^{10} + 128\ x^2\alpha^8 \\ & + \left((8\,n\omega - 144)\,x^6 + 128\,x^4 - 128\,x^2\right)\alpha^6 \\ & - \left(144\,x^6 + 128\,x^4\right)\alpha^4 + 144\,x^6\alpha^2 + 144\,x^8 \} - \omega^2. \end{aligned} \tag{4.100}$$

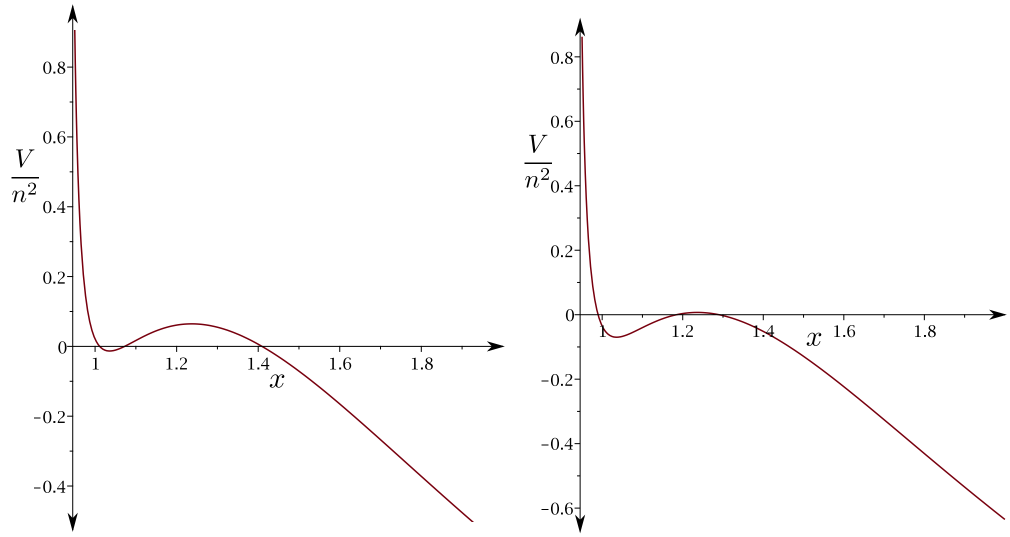
For the nonlinear problem we want to reproduce the setting of the linear problem $\mathcal{P}_0$. We begin by verifying that there is still a trapped region. From the definition of V above, this would amount to checking that there is a region where V has a negative minimum. Here we are interested in determining the existence of eigenvalues close to 0 (as opposed to eigenvalues close to E in the linear case). Lemma 4.6.1 identified such a region for $\mathcal{P}_0$. Here we state a nonlinear version i.e., Lemma 4.7.2 (following [126]) which identifies the corresponding Ω for $\mathcal{P}_1$. In the following proposition, we list some properties of V which will be useful in proving Lemma 4.7.2. Elements of the proofs in Proposition 4.7.1 and Lemma 4.7.2 which involve the structure of V will be illustrated with some plots owing to the complicated expression of V. To emphasize the dependence of V on ω and n, we denote the nonlinear potential as $V_{(\omega,n)}$ rather than V.

Proposition 4.7.1 (Properties of $V_{(\omega,n)}$). *Consider $\omega \in \mathbb{R}$ and $n \in \mathbb{Z}$.*

1. *If $\omega = 0$, $V_{(0,n)}$ is always positive and does not admit any real roots.*
2. *There exists a pair (ω_0, n_0) such that $V_{(\omega_0,n_0)}$ admits three distinct real roots $x_1^{\omega_0}, x_2^{\omega_0}$ and $x_3^{\omega_0}$ such that $V_{(\omega_0,n_0)}$ has a local minimum at $x_{min}^{\omega_0}$ with $x_1^{\omega_0} < x_{min}^{\omega_0} < x_2^{\omega_0} < x_3^{\omega_0}$.*
3. *Consider a pair (ω_0, n_0) for which $V_{(\omega_0,n_0)}$ admits three distinct real roots. There exist $\mathcal{E}^-$ and $\mathcal{E}^+$ such that*
   (a) *$\omega_0 \in (\mathcal{E}^-, \mathcal{E}^+)$ and for any $\omega \in (\mathcal{E}^-, \mathcal{E}^+)$ $V_{(\omega,n_0)}$ admits three distinct real roots.*
   (b) *Let ω_1 and ω_2 be two such values with $\{x_1^{\omega_1}, x_2^{\omega_1}, r_3^{\omega_1}\}$ and $\{x_1^{\omega_2}, x_2^{\omega_2}, x_3^{\omega_2}\}$ being the corresponding roots. If $\omega_1 > \omega_2$, then $(x_1^{\omega_2}, x_2^{\omega_2}) \subsetneq (x_1^{\omega_1}, x_2^{\omega_1})$.*

Proof. We know that $\lim\limits_{x\to\alpha} V = \infty$ and $\lim\limits_{x\to\infty} V = -\omega^2$. Hence the potential admits at least one real root. We note that the potential is invariant under $(\omega, n) \to (-\omega, -n)$. Hence we assume that $\omega > 0$ and only discuss the cases $n \in \mathbb{Z}^+$ and $n \in \mathbb{Z}^-$ where needed.

1. With $\omega = 0$, we have $V_{(0,n)} = n^2 V_{\sigma_1} > 0$ as can be seen from Fig.4.4.
2. (a) Case 1 : $n \in \mathbb{Z}^+$. For $\omega \in [1.465n, 1.485n]$, $V_{(\omega,n)}$ admits two roots. This can be seen in the plots below (Figs.4.6a and 4.6b).



(a) $\dfrac{V}{n^2}$ for $\omega = 1.465n$

(b) $\dfrac{V}{n^2}$ for $\omega = 1.485n$

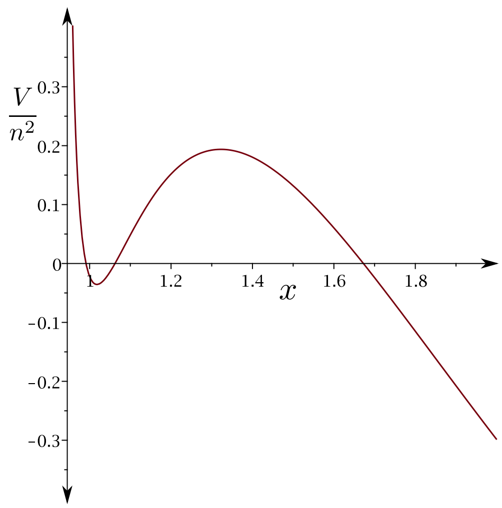
Figure 4.6: Nonlinear potential for $n \in \mathbb{Z}^+$

(b) Case 2 : $n \in \mathbb{Z}^-$. For $\omega \in [1.35|n|, 1.415|n|]$, we can see from Figs.4.7a and 4.7b that $V_{(\omega,n)}$ admits two roots.

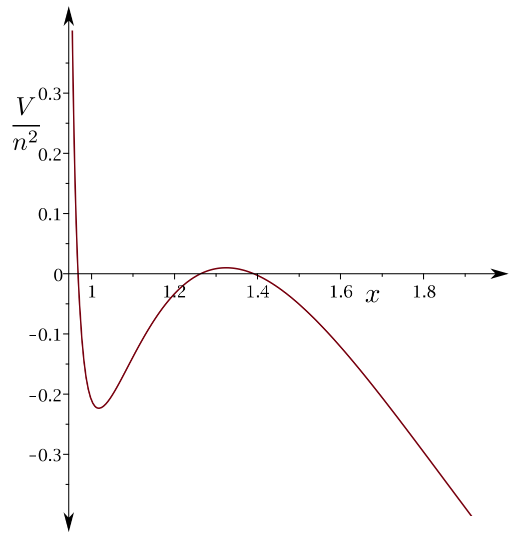
3. As a consequence of (2) above, for $n \in \mathbb{Z}^+$, the choice $\mathcal{E}^- = 1.47$ and $\mathcal{E}^+ = 1.48$ satisfies the condition (a). For (b), in the following plot (Fig.4.8a), as ω increases in $(\mathcal{E}^-, \mathcal{E}^+)$, the corresponding interval (x_-^ω, x_+^ω) also increases. For the case $n \in \mathbb{Z}^-$, we observe that $\widetilde{f} < 0$. Hence, from the following expression for the nonlinear potential,

$$V = n^2 V_{\sigma_1} - n\widetilde{f}\omega - \omega^2 \tag{4.101}$$

the increase (decrease) of the interval (x_-^ω, x_+^ω) with increase (decrease) in ω follows. This can also be seen in Fig.4.8b

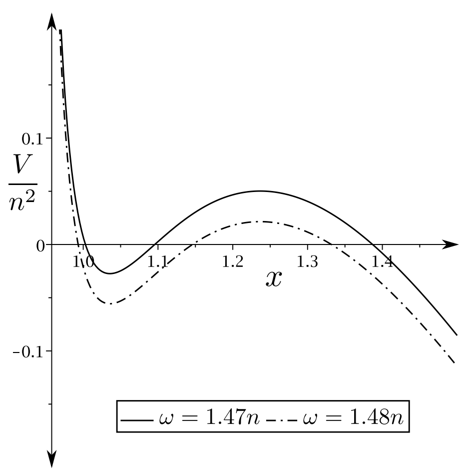


(a) $\frac{V}{n^2}$ for $\omega = 1.35|n|$

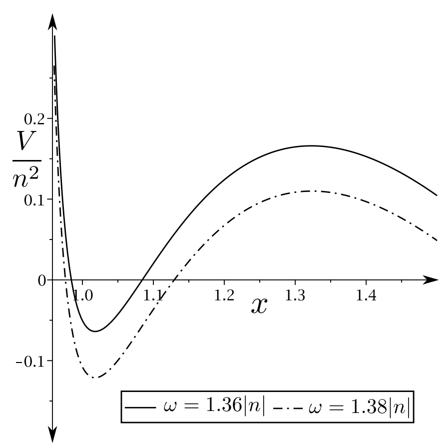


(b) $\frac{V}{n^2}$ for $\omega = 1.415|n|$

Figure 4.7: Nonlinear potential for $n \in \mathbb{Z}^-$



(a) $n \in \mathbb{Z}^+$



(b) $n \in \mathbb{Z}^-$

Figure 4.8: Properties of (x_-^ω, x_+^ω)

$\square$

Lemma 4.7.2. *Let V_{min} be the minimum of the nonlinear potential $V_{(\omega,n)}$. Let $x_{min} \in (\alpha, \infty)$ such that $V_{(\omega,n)}(x_{min}) = V_{min}$. Consider some constant $\mathcal{E} > 0$ such that $V_{(\mathcal{E}n,n)}$ has a local minimum and there exists $x_-^{\mathcal{E}}$ and $x_+^{\mathcal{E}}$ satisfying $x_-^{\mathcal{E}} < x_{min} < x_+^{\mathcal{E}}$ for which $V_{(\mathcal{E}n,n)}(x_-^{\mathcal{E}}) = V_{(\mathcal{E}n,n)}(x_+^{\mathcal{E}}) = 0$ and there are no local maxima of $V_{(\mathcal{E}n,n)}$ in $(x_-^{\mathcal{E}}, x_+^{\mathcal{E}})$. Let $E > 0$ be an energy level such that $E < \mathcal{E}$ and $V_{(En,n)}$ has a local minimum and there exists constants x_-^E and x_+^E with the same properties as $x_-^{\mathcal{E}}$ and $x_+^{\mathcal{E}}$ respectively but now with respect to $V_{(En,n)}$. Then for sufficiently small constants $\delta, \delta' > 0$ there exists a constant $c > 0$ such that*

$$|x - x_\pm^{\mathcal{E}}| < \delta' \implies \frac{1}{n^2} V_{(\kappa n,n)} > c \tag{4.102}$$

for all $\kappa \in \mathbb{R}$ satisfying $|\kappa^2 - E^2| \leq \delta$. In addition, for the linear problem Lemma 4.6.1 holds for $\left.\frac{n^2 V_{\sigma_1} - \omega^2}{n^2}\right|_{\omega = En}$.

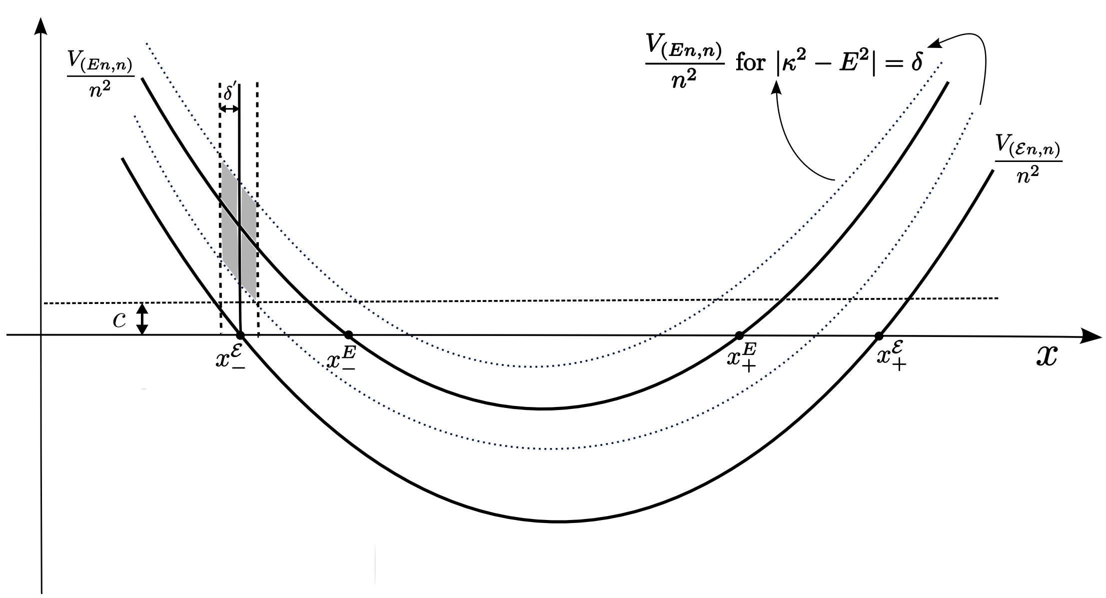


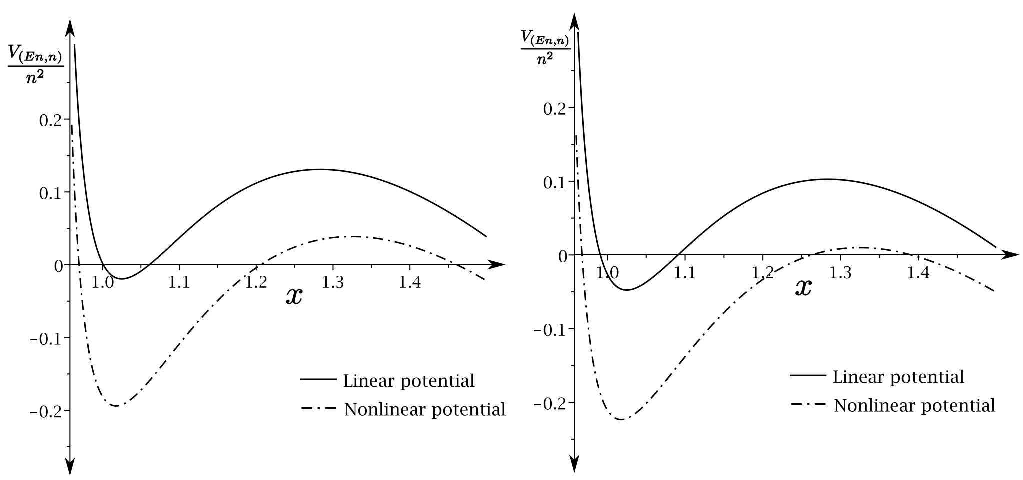
Figure 4.9: Domain for the nonlinear eigenvalue problem

Proof. We observe the following as consequences of Proposition 4.7.1.

1. There exists E and $\mathcal{E}$ for which the potential admits three distinct real roots with a local minimum as shown in Fig 4.9.

2. $E < \mathcal{E} \implies (x_-^E, x_+^E) \subsetneq (x_-^{\mathcal{E}}, x_+^{\mathcal{E}})$.

Hence, $\mathcal{E}$ and E with the desired properties exist. We can see that $V_{(En,n)}$ has no local maxima in $(x_-^{\mathcal{E}}, x_+^{\mathcal{E}})$. $V_{(En,n)}(x_-^E) = 0$, so for $x \in [x_-^{\mathcal{E}}, x_-^E)$, $\dfrac{V_{(En,n)}(x)}{n^2} > 0$. Then it follows that there exists $\delta' > 0$ such that for $|x - x_-^{\mathcal{E}}| < \delta'$, $\dfrac{V_{(En,n)}(x)}{n^2} > 0$. Since $\dfrac{V_{(En,n)}(x)}{n^2}$ is also continuous as a function of E, there exists some $\delta > 0$ such that for $|\kappa^2 - E^2| \leq \delta$, $\dfrac{V_{(\kappa n,n)}(x)}{n^2} > c$ for some constant $c > 0$. For the final part of the lemma, let us refer to Fig. 4.10a and Fig. 4.10b.



(a) $E = 1.405$ (b) $E = 1.415$

Figure 4.10: Continuity for linear and nonlinear potentials

For the relation to the linear potential, we observe that for $n \in \mathbb{Z}^-$ there exists

$E \in [1.405, 1.415]$ such that Lemma 4.6.1 holds. □

Hence from this stage onwards, we will only consider $n \in \mathbb{Z}^-$ as it is sufficient for our construction of quasimodes. We use $-|n|$ instead of n in expressions to indicate this sign choice.

4.7.1 Lower bound for ω^2

In this section, we establish a lower bound for the eigenvalues of $\mathcal{P}_1$. Consider the following family of eigenvalue problems

$$\mathcal{Q}(\beta, \omega)u = \Lambda(\beta, \omega)$$
$$u(x_-) = u(x_+) = 0 \tag{4.103}$$
$$\text{where } \mathcal{Q}(\beta, \omega) := -\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + n^2 V_{\sigma_1} + \beta|n|\widetilde{f}\omega - \omega^2$$

We prove that if the j^{th} eigenvalue of $\mathcal{Q}(\beta, \omega)$ i.e., $\Lambda_j(\beta, \omega)$ is zero, then the corresponding $\omega_{\beta,n}$ satisfies certain properties. Let $u_j(\beta, \omega)$ be a normalized eigenfunction in $[x_-, x_+]$ associated to the eigenvalue $\Lambda_j(\beta, \omega)$. Then

$$\Lambda_j(\beta, \omega) = \int_{x_-}^{x_+} u_j(\beta, \omega)\, \mathcal{Q}(\beta, \omega)\, u_j(\beta, \omega)\, \mathrm{d}x = 0. \tag{4.104}$$

We have the following lemma which gives a lower bound on $\omega_{\beta,n}$.

Lemma 4.7.3. *Let $u_j(\beta, n)$ be a nontrivial eigenfunction of (4.103), then the following hold for sufficiently large n and $\beta \in [0, 1]$:*

1. $\omega_{\beta,n} \neq 0$

2. $\omega_{\beta,n} \neq o(n)$.

Proof. 1. If $\omega_{\beta,n} = 0$ we have

$$\begin{aligned}
&\int_{x_-}^{x_+} u_j(\beta,0)\mathcal{Q}(\beta,\omega)u_j(\beta,\omega)\,\mathrm{d}x \\
&= \int_{x_-}^{x_+} -u_j(\beta,0)\frac{\mathrm{d}^2 u_j(\beta,0)}{\mathrm{d}x^2} + \left.(n^2 V_{\sigma_1} u_j^2(\beta,0) + \beta|n|f\omega u_j^2(\beta,0) - \omega^2 u_j^2(\beta,0))\right|_{\omega=0}\,\mathrm{d}x \\
&= \int_{x_-}^{x_+} -u_j(\beta,0)\frac{\mathrm{d}^2 u_j(\beta,0)}{\mathrm{d}x^2} + (n^2 V_{\sigma_1} u_j^2(\beta,0))\,\mathrm{d}x \\
&= \int_{x_-}^{x_+} \left|\frac{\mathrm{d}u_j(\beta,0)}{\mathrm{d}x}\right|^2 \mathrm{d}x + \int_{x_-}^{x_+} n^2 V_{\sigma_1} u_j^2(\beta,0)\mathrm{d}x.
\end{aligned} \tag{4.105}$$

The first integral is positive and $V_{\sigma_1} > 0$. Hence $\Lambda_j(\beta,0) \neq 0$ which concludes the proof of the first part.

2. Suppose that $\omega_{\beta,n} = o(n)$. We proceed with the same steps as above and arrive at,

$$\begin{aligned}
&\int_{x_-}^{x_+} u_j(\beta,0)\mathcal{Q}(\beta,\omega)\,u_j(\beta,\omega)\,\mathrm{d}x \qquad (4.106)\\
&\qquad = \int_{x_-}^{x_+} \left|\frac{\mathrm{d}u_j(\beta,0)}{\mathrm{d}x}\right|^2 \mathrm{d}x + \int_{x_-}^{x_+} (n^2 V_{\sigma_1} + \beta|n|\widetilde{f}\omega - \omega^2)u_j^2(\beta,\omega)\,\mathrm{d}x
\end{aligned}$$

With $\omega_{\beta,n} = o(n)$, we have

$$n^2 V_{\sigma_1} + \beta|n|\widetilde{f}\omega - \omega^2 \geq n^2 V_{\sigma_1} - C\left[1 - \beta\widetilde{f}\right] n^2 \tag{4.107}$$

The right hand side is positive when C is sufficiently small making the second term in the integral positive implying that $\omega \neq o(n)$. In particular we note here that the above result holds for $\beta = 0$ which is the case for the linear eigenvalue problem.

□

Corollary 4.7.4. *As a consequence of Lemma 4.7.3, we conclude that given the existence of eigenvalues $\omega_{\beta,n}$ for sufficiently large n^2,*

$$\omega_{\beta,n}^2 \geq \mathcal{O}(n^2)$$

i.e., there exists a positive constant C_β independent of n^2 such that

$$\omega_{\beta,n}^2 \geq C_\beta \, n^2.$$

4.7.2 Eigenvalues for $\beta \neq 0$

Lemma 4.7.5. *Let $\beta_0 \in [0,1]$, $\omega_{\beta_0,n} > 0$ and $n \in \mathbb{Z}^-$ be such that the j^{th} eigenvalue of $\mathcal{Q}(\beta_0, \omega_{\beta_0,n})$ is zero. Then for sufficiently large n^2, there exists a constant $\epsilon > 0$ (independent of β_0) such that there is a differentiable function $\omega_{\beta,n}(\beta)$ such that the n^{th} eigenvalue of $\mathcal{Q}(b, \omega_{\beta,n})$ is zero for any $\beta \in (\max(0, \beta_0 - \epsilon), \beta_0 + \epsilon)$.*

Proof. We start with the expression for the j^{th} eigenvalue $\Lambda_j(\beta,\omega)$:

$$\Lambda_j(\beta,\omega) = \int_{x_-}^{x_+} u_j(\beta,\omega)\, \mathcal{Q}(\beta,\omega)\, u_j(\beta,\omega)\, \mathrm{d}x. \tag{4.108}$$

We assume that the j^{th} eigenvalue is zero. $\Lambda_j(\beta_0, \omega_{\beta_0,n}) = 0$ gives an implicit relation between β and $\omega_{\beta,n}$. In a neighbourhood of β_0, the implicit function theorem provides necessary and sufficient conditions for the existence of $\omega_{\beta,n}(\beta)$. We have,

$$\begin{aligned}
\frac{\partial \Lambda_j}{\partial \omega}(\beta_0, \omega_{\beta_0,n}) &= \int_{x_-}^{x_+} u_j(\beta,\omega)\, \frac{\partial}{\partial \omega}\left(n^2 V_{\sigma_1} + \beta |n| \widetilde{f} \omega - \omega^2\right)\, u_j(\beta,\omega)\, \mathrm{d}x \\
&= \int_{x_-}^{x_+} u_j(\beta,\omega)\, \left(\beta |n| \widetilde{f} - 2\omega\right)\, u_j(\beta,\omega)\, \mathrm{d}x. \\
\frac{\partial \Lambda_j}{\partial \beta}(\beta_0, \omega_{\beta_0,n}) &= \int_{x_-}^{x_+} u_j(\beta,\omega)\, \frac{\partial}{\partial \beta}\left(n^2 V_{\sigma_1} + \beta |n| \widetilde{f} \omega - \omega^2\right)\, u_j(\beta,\omega)\, \mathrm{d}x \\
&= \int_{x_-}^{x_+} u_j(\beta,\omega)\, \left(|n| \widetilde{f} \omega\right)\, u_j(\beta,\omega)\, \mathrm{d}x.
\end{aligned} \tag{4.109}$$

Since $\widetilde{f} < 0$, we have,

$$\beta|n|\widetilde{f} - 2\omega \leq -2\omega \text{ for } n \in \mathbb{Z}^- \text{ and } \omega \in \mathbb{R}^+ \tag{4.110}$$

This holds for all $\beta \in [0, 1]$ and we have from Lemma 4.7.3 that $\omega \geq nC_\beta$. Hence we have a uniform constant $B := \inf_{\beta\in[0,1]} C_\beta$ such that

$$\beta|n|\widetilde{f} - 2\omega \leq -2Bn \tag{4.111}$$

This means that $\frac{\partial \Lambda_j}{\partial \omega}(\beta_0, \omega_{\beta_0,n})$ is bounded away from zero. By the implicit function theorem, this proves the existence of $\omega_{\beta,n}(\beta)$ in a neighbourhood of β_0. We can compute the derivative of $\omega_{\beta,n}(\beta)$ at β_0 using,

$$\frac{\mathrm{d}\omega_{\beta,n}}{\mathrm{d}\beta}(\beta_0) = -\frac{\frac{\partial \Lambda_j}{\partial \beta}(\beta_0, \omega_{\beta_0,n})}{\frac{\partial \Lambda_j}{\partial \omega}(\beta_0, \omega_{\beta_0,n})}, \qquad \frac{\partial \Lambda_j}{\partial \omega}(\beta_0, \omega_{\beta_0,n}) = |n|\widetilde{f}\omega_{\beta_0,n}. \tag{4.112}$$

Similar to the argument above, we have that $\frac{\partial \Lambda_j}{\partial \beta}(\beta_0, \omega_{\beta_0,n})$ is bounded away from zero. We hence arrive at

$$-|n|\widetilde{C}_\beta \leq \frac{\mathrm{d}\omega_{\beta,n}}{\mathrm{d}\beta}(\beta_0) < 0 \tag{4.113}$$

for some $\widetilde{C}_\beta > 0$ and $\beta \in [0, 1]$.

The bound here is uniform in β_0. ϵ is independent of β_0 and this ensures that finite applications of Lemma 4.7.5 covers the whole interval $\beta \in [0, 1]$. In particular one can extend the results to $\beta = 1$. □

4.7.3 Existence of eigenvalues for the nonlinear problem

We conclude this section by demonstrating the existence of eigenvalues for the nonlinear problem. Note that along the same lines as Remark 8.20 in [126], it is clear from the final part of Lemma 4.7.2 that the energy level $E \in [1.405, 1.415]$ which is an 'appropriate value' for the nonlinear problem also works for the linear problem in the sense that Lemma 4.6.1 holds for the chosen value of E.

Theorem 4.7.6. *Consider fixed energy levels E and $\mathcal{E}$ as in Lemma 4.7.2. Let $n \in \mathbb{Z}^-$. Given eigenvalues $\omega^2_{\text{lin},n}$ for the linear eigenvalue problem where $\omega^2_{\text{lin},n} > 0$, there exists an eigenvalue ω_n^2 and corresponding eigenfunction u_n to the nonlinear eigenvalue problem for large enough n. Furthermore $\omega_n > 0$ and the following bound holds for any $\delta > 0$,*

$$\mathcal{C} \leq \frac{\omega_n^2}{n^2} \leq E^2 + \delta \tag{4.114}$$

where the constant $\mathcal{C}$ is independent of n.

Proof. We start by looking at the eigenvalue problem for $\beta = 0$. We know from the linear eigenvalue problem that for large n^2 there exists a $\omega_{0,n}$ such that $\mathcal{Q}(0, \omega_{0,n})$ admits a zero eigenvalue i.e., $\Lambda_j(0, \omega_{0,n}) = 0$ for some j. By Lemma 4.7.3, $\omega_{0,n} \neq 0$.

Let $\omega_{0,n} > 0$. From Lemma 4.7.5, for some $\epsilon > 0$ there exists a continuous function $\omega_{\beta,n}(\beta)$ such that for any $\beta \in [0, \epsilon)$ the nonlinear eigenvalue problem admits a zero eigenvalue i.e., $\Lambda_j(\beta, \omega_{\beta,n}) = 0$ for some j. By (4.113),

$$\omega_n^2 = \omega_{1,n}^2(1) \leq \omega_{0,n}^2(0) \leq Cn^2 \tag{4.115}$$

Here C does not depend on n. The bound $\omega_{0,n}(0) \leq Cn^2$ comes from conditions on appropriate energy levels E from the assumptions of Lemma 4.6.1. In conjunction with Lemma 4.7.3, we have,

$$C_1 \leq \frac{\omega_n^2}{n^2} \leq C_2 \tag{4.116}$$

for constants C_1 and C_2 independent of n. For $\beta \in [0,1]$, let us consider the problems

$$\overline{Q}_\beta u = E_j^2(\beta) u \tag{4.117}$$

where

$$\begin{aligned} \overline{Q}_\beta u &:= -\frac{1}{n^2}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{1}{n^2}(n^2 V_{\sigma_1} + \beta |n| \widetilde{f}\omega) \\ E_j(\beta) &:= \frac{\omega_{\beta,n}^2(\beta)}{n^2} \end{aligned} \tag{4.118}$$

We have from Weyl's law for the linear problem in conjunction with Lemma 4.7.2 that,

$$E_j^2(0) \in [E^2 - \delta, E^2 + \delta] \tag{4.119}$$

for any arbitrary small δ and sufficiently large n^2. For $n \in \mathbb{Z}^-$ and $\beta \in [0,1]$ we have the estimate

$$0 \leq \int_{x_-}^{x_+} u(\overline{Q}_0 - \overline{Q}_\beta) u \,\mathrm{d}x = \int_{x_-}^{x_+} -|n|\widetilde{f}\omega_{\beta,n}\,\mathrm{d}x \tag{4.120}$$

which means

$$\int_{x_-}^{x_+} u\overline{Q}_\beta u \,\mathrm{d}x \leq \int_{x_-}^{x_+} u\overline{Q}_0 u\,\mathrm{d}x \tag{4.121}$$

implying

$$E_j^2(\beta) \leq E_j^2(0). \tag{4.122}$$

In particular this means

$$E_j^2(1) \leq E_j^2(0) \leq E^2 + \delta \tag{4.123}$$

Combining the bounds we have

$$\begin{aligned} \mathcal{C} &\leq E_j^2(1) \leq E^2 + \delta \\ \mathcal{C} &\leq \frac{\omega_n^2}{n^2} \leq E^2 + \delta \end{aligned} \tag{4.124}$$

□

4.8 Lower bound on the uniform energy decay rate

The purpose of this section is to prove an energy estimate for solutions to the eigenvalue problem discussed in the previous sections. This is the main step towards establishing the desired lower bound on energy decay. Here, we closely follow the proof in [58] and [56]. We begin with the following basic lemma which can be proved using integration by parts.

Lemma 4.8.1. *Let $x_- < x_+$, $h > 0$ be a constant and W and ϕ be smooth functions on $[x_-, x_+]$. Then, for all smooth functions u defined on $[x_-, x_+]$,*

$$\int_{x_-}^{x_+} \left(\left| \frac{\mathrm{d}}{\mathrm{d}x} \left(e^{\phi/h} u \right) \right|^2 + h^{-2} \left(W - \left(\frac{d\phi}{dx} \right)^2 \right) e^{2\phi/h} |u|^2 \right) \mathrm{d}x$$
$$= \int_{x_-}^{x_+} \left(-\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + h^{-2} W u \right) u \, e^{2\phi/h} \mathrm{d}x$$

Proof. We start by expanding the expression on the left hand side.

$$\begin{aligned}
&\int_{x_-}^{x_+} \left(\left| \frac{\mathrm{d}}{\mathrm{d}x} \left(e^{\phi/h} u \right) \right|^2 + h^{-2} \left(W - \left(\frac{\mathrm{d}\phi}{\mathrm{d}x} \right)^2 \right) e^{2\phi/h} |u|^2 \right) \mathrm{d}x \\
&\quad = \int_{x_-}^{x_+} \left(\left| u h^{-1} e^{\phi/h} \frac{\mathrm{d}\phi}{\mathrm{d}x} + e^{\phi/h} \frac{\mathrm{d}u}{\mathrm{d}x} \right|^2 + h^{-2} W e^{2\phi/h} |u|^2 - h^{-2} \left(\frac{\mathrm{d}\phi}{\mathrm{d}x} \right)^2 e^{2\phi/h} |u|^2 \right) \mathrm{d}x \\
&\quad = \int_{x_-}^{x_+} \left(u^2 h^{-2} e^{2\phi/h} \left(\frac{\mathrm{d}\phi}{\mathrm{d}x} \right)^2 + e^{2\phi/h} \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right)^2 + \frac{2|u| e^{2\phi/h}}{h} \left(\frac{\mathrm{d}\phi}{\mathrm{d}x} \right) \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right) \right. \\
&\qquad\qquad \left. + h^{-2} W e^{2\phi/h} |u|^2 - h^{-2} \left(\frac{\mathrm{d}\phi}{\mathrm{d}x} \right)^2 e^{2\phi/h} |u|^2 \right) \mathrm{d}x \\
&\quad = \int_{x_-}^{x_+} \left(e^{2\phi/h} \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right)^2 + \frac{2u e^{2\phi/h}}{h} \left(\frac{\mathrm{d}\phi}{\mathrm{d}x} \right) \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right) + h^{-2} W e^{2\phi/h} |u|^2 \right) \mathrm{d}x
\end{aligned}$$

Using integration by parts on the second term

$$\int_{x_-}^{x_+} \left(\left| \frac{\mathrm{d}}{\mathrm{d}x} \left(e^{\phi/h} u \right) \right|^2 + h^{-2} \left(W - \left(\frac{\mathrm{d}\phi}{\mathrm{d}x} \right)^2 \right) e^{2\phi/h} |u|^2 \right) \mathrm{d}x$$

$$= e^{2\phi/h} u \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right) \Bigg|_{x_-}^{x_+} - \int_{x_-}^{x_+} e^{2\phi/h} \frac{\mathrm{d}}{\mathrm{d}x} \left(u \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right) \right)$$

$$+ \int_{x_-}^{x_+} \left(e^{2\phi/h} \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right)^2 + h^{-2} W e^{2\phi/h} |u|^2 \right) \mathrm{d}x$$

$$= \int_{x_-}^{x^+} \left(-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + h^{-2} W u \right) u e^{2\phi/h} \mathrm{d}x$$

□

4.8.1 Agmon distance

Consider the effective potential

$$V_{\text{eff}}^{h,E} = h^2 V_{(En,n)}$$

where we recall the previously defined semi-classical parameter $h > 0$

$$h^2 = n^{-2}$$

(note that, since we have taken $n \in \mathbb{Z}^-$, $h = -1/n$) and the energy level E is chosen as in Lemma 4.7.2. The Agmon distance between two points x_1 and x_2 associated to the energy level E and potential V_{eff} is defined as

$$d(x_1, x_2) := \left| \int_{x_1}^{x_2} \sqrt{V_{\text{eff}}^{h,E}(x)} \chi_{\left\{ V_{\text{eff}}^{h,E} \geq 0 \right\}} \mathrm{d}x \right|. \tag{4.125}$$

Physically the Agmon distance is a measure of distance between two points in the classically forbidden region. Agmon distance is the distance associated with the Agmon metric, $V_+ \mathrm{d}x^2$ where $V_+ := \max(0, V)$. The Agmon distance satisfies the bound [58]

$$|\nabla_x d(x, x_2)|^2 \leq \max \left\{ V_{\text{eff}}^{h,E}(x), 0 \right\}. \tag{4.126}$$

For a given E, using Agmon distance, one can define the distance to the classically allowed region as

$$d_E(x) := \inf_{x_1 \in \left\{ V_{\text{eff}}^{h,E} \leq 0 \right\}} d(x_1, x). \tag{4.127}$$

We recall that $\Omega := [x_-, x_+]$. For $\epsilon \in (0, 1)$, we define

$$\Omega_\epsilon^+(E) := \left\{ x : V_{\text{eff}}^{h,E} > \epsilon \right\} \cap \Omega \tag{4.128}$$

with its complement in Ω defined by

$$\Omega_\epsilon^-(E) := \left\{ x : V_{\text{eff}}^{h,E} \leq \epsilon \right\} \cap \Omega \tag{4.129}$$

We can now state and prove the following exponentially weighted energy estimate.

Lemma 4.8.2 (Energy estimate)**.** *Let u be a solution to the nonlinear eigenvalue problem (4.103). Let κ be a eigenvalue satisfying $|\kappa^2 - E^2| \leq \delta$. For $\epsilon \in (0, 1)$, define*

$$\phi_{E,\epsilon}(x) := (1 - \epsilon) d_E(x) \quad \text{and} \quad a_E(\epsilon) := \sup_{\Omega_\epsilon^-(E)} d_E. \tag{4.130}$$

Then for sufficiently small ϵ and h and sufficiently small δ (depending on ϵ and h), u satisfies

$$\int_\Omega h^2 \left| \frac{d}{dx} \left(e^{\phi_{E,\epsilon}/h} u \right) \right|^2 \mathrm{d}x + \frac{1}{2}\epsilon^2 \int_{\Omega_\epsilon^+} e^{2\phi_{E,\epsilon}/h} |u|^2 \mathrm{d}x \leq C \left(\kappa^2 + \frac{1}{2}\epsilon \right) e^{2a_E(\epsilon)/h} \|u\|_{L^2(\Omega)}^2 \tag{4.131}$$

where the constant C depends only on the parameters of the soliton spacetime and Ω.

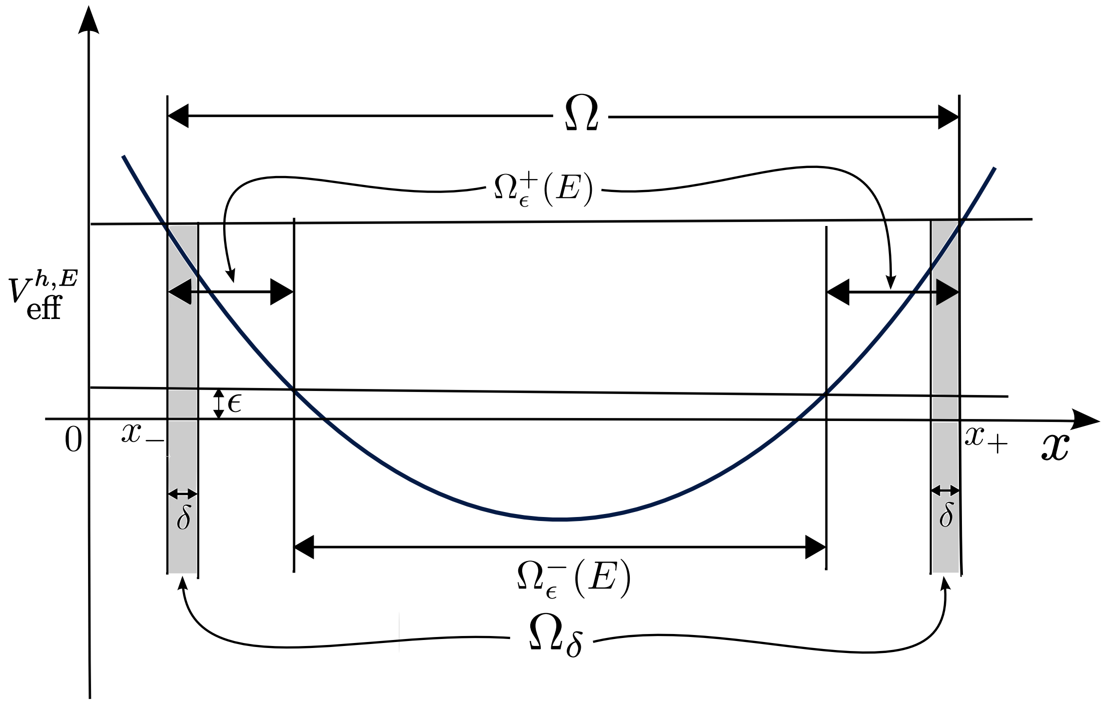


Figure 4.11: Classical and forbidden regions

Proof. We apply Lemma 4.8.1 to u with the following identifications,

$$W = V_{\text{eff}}^{h,\kappa}, \quad \phi = \phi_{E,\epsilon} \tag{4.132}$$

Since u is a solution to the eigenvalue problem, the right hand side vanishes which gives

$$\int_\Omega \left(\left| \frac{\mathrm{d}}{\mathrm{d}x} \left(e^{\phi_{E,\epsilon}/h} u \right) \right|^2 + h^{-2} \left(V_{\text{eff}}^{h,\kappa} - \left(\frac{\mathrm{d}\phi_{E,\epsilon}}{\mathrm{d}x} \right)^2 \right) e^{2\phi_{E,\epsilon}/h} |u|^2 \right) \mathrm{d}x = 0. \tag{4.133}$$

This can be rewritten as

$$\int_\Omega h^2 \left| \frac{\mathrm{d}}{\mathrm{d}x} \left(e^{\phi_{E,\epsilon}/h} u \right) \right|^2 \mathrm{d}x + \int_{\Omega_\epsilon^+(E)} \left(V_{\text{eff}}^{h,\kappa} - \left(\frac{\mathrm{d}\phi_{E,\epsilon}}{\mathrm{d}x} \right)^2 \right) e^{2\phi_{E,\epsilon}/h} |u|^2 \mathrm{d}x$$
$$= \int_{\Omega_\epsilon^-(E)} \left(-V_{\text{eff}}^{h,\kappa} + \left(\frac{\mathrm{d}\phi_{E,\epsilon}}{\mathrm{d}x} \right)^2 \right) e^{2\phi_{E,\epsilon}/h} |u|^2 \mathrm{d}x. \tag{4.134}$$

We make the following observations :

1. By definition, we have $\phi_{E,\epsilon}|_{\Omega_\epsilon^-(E)} \leq a_E(\epsilon)$.

2. $\|u\|^2_{L^2(\Omega_\epsilon^-(E))} \leq \|u\|^2_{L^2(\Omega)}$.

3. We have in $\Omega_\epsilon^-(E)$,

$$\begin{aligned}\left(\frac{\mathrm{d}\phi_{E,\epsilon}}{\mathrm{d}x}\right)^2 &= (1-\epsilon)^2|\nabla_x d_E|^2 \\ &\leq (1-\epsilon)^2\epsilon \quad \text{(by definition)} \\ &\leq (1-\epsilon)\epsilon \quad (\text{since } \epsilon < 1)\end{aligned} \tag{4.135}$$

We also note that $-V_{\text{eff}}^{h,\kappa} \leq \kappa^2$ which finally gives

$$-V_{\text{eff}}^{h,\kappa} + \left(\frac{\mathrm{d}\phi_{E,\epsilon}}{\mathrm{d}r}\right)^2 \leq \kappa^2 + \epsilon(1-\epsilon) \tag{4.136}$$

We can thus estimate the integral on the right hand side as

$$\begin{aligned}&\int_{\Omega_\epsilon^-(E)} \left(-V_{\text{eff}}^{h,\kappa} + \left(\frac{\mathrm{d}\phi_{E,\epsilon}}{\mathrm{d}x}\right)^2\right) e^{2\phi_{E,\epsilon}/h}|u|^2\mathrm{d}x \\ &\leq (\kappa^2 + \epsilon(1-\epsilon)) \int_{\Omega_\epsilon^-(E)} e^{2\phi_{E,\epsilon}/h}|u|^2\mathrm{d}x \\ &\leq (\kappa^2 + \epsilon(1-\epsilon))e^{2a_E(\epsilon)/h}\,\|u\|^2_{L^2(\Omega)} \\ &\leq \left(\kappa^2 + \frac{1}{2}\epsilon\right) e^{2a_E(\epsilon)/h}\,\|u\|^2_{L^2(\Omega)} \qquad \text{for } \epsilon < \frac{1}{2}\end{aligned} \tag{4.137}$$

4. Consider the region $\Omega_\epsilon^+(E)$. The potential $V_{\text{eff}}^{h,\kappa}$ is continuous in κ. Hence given a $\delta' > 0$ one can find a δ such that $|\kappa^2 - E^2| \leq \delta \implies |V_{\text{eff}}^{h,\kappa} - V_{\text{eff}}^{h,E}| < \delta'$. Hence we have

$$V_{\text{eff}}^{h,E} - \delta' < V_{\text{eff}}^{h,\kappa} < V_{\text{eff}}^{h,E} + \delta' \tag{4.138}$$

With this we can estimate the second integrand on the left hand side as

$$\begin{aligned} V_{\text{eff}}^{h,\kappa} - \left(\frac{\mathrm{d}\phi_{E,\epsilon}}{\mathrm{d}x}\right)^2 &\geq V_{\text{eff}}^{h,E} - \delta^{'} - (1-\epsilon)^2 V_{\text{eff}}^{h,E} \\ &\geq V_{\text{eff}}^{h,E} - \delta^{'} - (1-\epsilon) V_{\text{eff}}^{h,E} \quad (\text{since } \epsilon < 1) \\ &\geq \epsilon V_{\text{eff}}^{h,E} - \delta^{'} \\ &\geq \epsilon^2 - \delta^{'} \quad (\text{from the definition of } \Omega_\epsilon^+(E)) \\ &\geq \frac{\epsilon^2}{2} \quad \text{for any } \delta^{'} < \frac{\epsilon^2}{2} \end{aligned} \tag{4.139}$$

Hence for $\epsilon \leq \frac{1}{2}$ there exists a sufficiently small $\delta^{'}$ such that estimate in the theorem holds. □

Quasimodes, as explained above, are functions that solve the wave equation everywhere except in the cut-off region. Hence, we require estimates for u in the cut-off region to approximate this deviation and determine how the resulting error depends on the frequency parameter n. We first define the cut-off region Ω_δ as

$$\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \leq \delta\}. \tag{4.140}$$

The following theorem estimates u on Ω_δ.

Theorem 4.8.3. *From Theorem 4.7.6, we have that for $\mathcal{E}, E$ and sufficiently small $\delta^{'}$, there exist eigenvalues $\kappa_n := \frac{\omega_n^2}{n^2}$ and corresponding eigenfunctions u_n to the nonlinear eigenvalue problem for large enough $|n|$ such that $\omega_n > 0$ and*

$$\mathcal{C} \leq \kappa_n^2 \leq E^2 + \delta^{'} \tag{4.141}$$

where the constant $\mathcal{C}$ is independent of n. Then for any sufficiently small δ, there holds

$$\int_{\Omega_\delta} \left(\left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \right) \mathrm{d}x \leq C e^{-C|n|} \|u\|_{L^2(\Omega)} \tag{4.142}$$

for a constant C independent of n.

Proof. From Lemma 4.8.2, we have

$$\begin{aligned}\int_\Omega h^2 \left|\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\phi_{E,\epsilon}/h}u\right)\right|^2 \mathrm{d}x + \frac{1}{2}\epsilon^2 \int_{\Omega_\epsilon^+} e^{2\phi_{E,\epsilon}/h}|u|^2\mathrm{d}x \\ \leq C\left(\kappa^2 + \frac{1}{2}\epsilon\right) e^{2a_E(\epsilon)/h} \|u\|^2_{L^2(\Omega)}\end{aligned} \tag{4.143}$$

Both terms in the left hand side are positive, so the inequality applies to each, that is

$$\int_\Omega h^2 \left|\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\phi_{E,\epsilon}/h}u\right)\right|^2 \mathrm{d}x \leq C\left(\kappa^2 + \frac{1}{2}\epsilon\right) e^{2a_E(\epsilon)/h} \|u\|^2_{L^2(\Omega)}, \tag{4.144}$$

$$\frac{1}{2}\epsilon^2 \int_{\Omega_\epsilon^+} e^{2\phi_{E,\epsilon}/h}|u|^2\mathrm{d}x \leq C\left(\kappa^2 + \frac{1}{2}\epsilon\right) e^{2a_E(\epsilon)/h} \|u\|^2_{L^2(\Omega)}. \tag{4.145}$$

Since $\Omega_\delta \subset \Omega_\epsilon^+(E)$, (4.145) becomes

$$\int_{\Omega_\delta} e^{2\phi_{E,\epsilon}/h}|u|^2\mathrm{d}x \leq \frac{C}{\epsilon^2}\left(\kappa^2 + \frac{1}{2}\epsilon\right) e^{2a_E(\epsilon)/h} \|u\|^2_{L^2(\Omega)} \tag{4.146}$$

With $\frac{\epsilon}{2} < \frac{1}{2}$ and by the definition of $\phi_{E,\epsilon}$, we see that there is a uniform constant c such that, $\phi_{E,\epsilon} \geq c$ for any $x \in \Omega_\delta$ and $|\kappa^2 - E^2| \leq \delta'$. By definition, we have

$$a_E(\epsilon) := \sup_{\Omega_\epsilon^-(E)} \inf_{x_1 \in \left\{V_{\text{eff}}^{h,E} \leq 0\right\}} d(x_1, x). \tag{4.147}$$

For $x \in \Omega_\epsilon^-(E)$,

$$\inf_{x_1 \in \left\{V_{\text{eff}}^{h,E} \leq 0\right\}} d(x_1, x) \leq \sqrt{\epsilon}\Delta \tag{4.148}$$

where $\Delta = \max_{x \in \left\{V_{\text{eff}}^{h,E} \leq 0\right\}} d(x_E^{\pm}, x)$. Hence $a_E(\epsilon) \to 0$ as $\epsilon \to 0$ and there exists ϵ small enough such that $a_E(\epsilon) \leq c/2$. We note that $a_E(\epsilon) \to 0$ as $\epsilon \to 0$ and $\epsilon \to 0$

independently of $h \to 0$. Putting these together, we have

$$e^{2c/h} \int_{\Omega_\delta} |u|^2 \mathrm{d}x \le \frac{C}{\epsilon^2} \left(\kappa^2 + \frac{1}{2}\epsilon \right) e^{2a_E(\epsilon)/h} \|u\|^2_{L^2(\Omega)} , \tag{4.149}$$

$$\int_{\Omega_\delta} |u|^2 \mathrm{d}x \le e^{-c/h} \frac{C}{\epsilon^2} \left(\kappa^2 + \frac{1}{2}\epsilon \right) \|u\|^2_{L^2(\Omega)} . \tag{4.150}$$

There exists a constant C such that,

$$\int_{\Omega_\delta} |u|^2 \le C h^{-2} e^{-C/h} \|u\|^2_{L^2(\Omega)} \tag{4.151}$$

Since $\epsilon \to 0$ uniformly in h, we can absorb h^{-2} in C giving

$$\int_{\Omega_\delta} |u|^2 \le C e^{-C/h} \|u\|^2_{L^2(\Omega)} . \tag{4.152}$$

Now, similarly, the left hand side of (4.144) becomes

$$\begin{aligned} &\int_{\Omega_\delta} h^2 \left| \frac{\mathrm{d}}{\mathrm{d}x} \left(e^{\phi_{E,\epsilon}/h} u \right) \right|^2 \mathrm{d}x \\ &\quad = \int_{\Omega_\delta} h^2 \left(e^{\phi_{E,\epsilon}/h} \frac{u}{h} \frac{\mathrm{d}\phi_{E,\epsilon}}{\mathrm{d}x} + e^{\phi_{E,\epsilon}/h} \frac{\mathrm{d}u}{\mathrm{d}x} \right)^2 \mathrm{d}x \\ &\quad = \int_{\Omega_\delta} h^2 e^{2\phi_{E,\epsilon}/h} \left(\frac{u}{h} \frac{\mathrm{d}\phi_{E,\epsilon}}{\mathrm{d}x} + \frac{\mathrm{d}u}{\mathrm{d}x} \right)^2 \mathrm{d}x \\ &\quad = \int_{\Omega_\delta} h^2 e^{2\phi_{E,\epsilon}/h} \left[\frac{u^2}{h^2} \left(\frac{\mathrm{d}\phi_{E,\epsilon}}{\mathrm{d}x} \right)^2 + \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right)^2 + 2\frac{u}{h} \left(\frac{\mathrm{d}\phi_{E,\epsilon}}{\mathrm{d}x} \right) \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right) \right] \mathrm{d}x \end{aligned} \tag{4.153}$$

Discarding the first term which is positive, we have

$$\begin{aligned} &\int_{\Omega_\delta} h^2 e^{2\phi_{E,\epsilon}/h} \left[\left(\frac{\mathrm{d}u}{\mathrm{d}x} \right)^2 + 2\frac{u}{h} \left(\frac{\mathrm{d}\phi_{E,\epsilon}}{\mathrm{d}x} \right) \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right) \right] \mathrm{d}x \\ &\qquad \le C \left(\kappa^2 + \frac{1}{2}\epsilon \right) e^{2a_E(\epsilon)/h} \|u\|^2_{L^2(\Omega)} \end{aligned} \tag{4.154}$$

Applying Young's inequality to the second term we get

$$\int_{\Omega_\delta} h^2 e^{2\phi_{E,\epsilon}/h}\left[\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 + 2\frac{u}{h}\left(\frac{\mathrm{d}\phi_{E,\epsilon}}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\right]\mathrm{d}x$$
$$\geq \int_{\Omega_\delta} h^2 e^{2\phi_{E,\epsilon}/h}\left[\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - \frac{2u^2}{h^2}\left(\frac{\mathrm{d}\phi_{E,\epsilon}}{\mathrm{d}x}\right)^2 - \frac{1}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\right]\mathrm{d}x \quad (4.155)$$
$$= \int_{\Omega_\delta} h^2 e^{2\phi_{E,\epsilon}/h}\left[\frac{1}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - \frac{2u^2}{h^2}\left(\frac{\mathrm{d}\phi_{E,\epsilon}}{\mathrm{d}x}\right)^2\right]\mathrm{d}x$$

Hence, we have

$$\int_{\Omega_\delta} h^2 e^{2\phi_{E,\epsilon}/h}\left[\frac{1}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - \frac{2u^2}{h^2}\left(\frac{\mathrm{d}\phi_{E,\epsilon}}{\mathrm{d}x}\right)^2\right]\mathrm{d}x \leq C\left(\kappa^2 + \frac{1}{2}\epsilon\right)e^{2a_E(\epsilon)/h}\|u\|^2_{L^2(\Omega)}$$
$$\int_{\Omega_\delta} h^2 e^{2\phi_{E,\epsilon}/h}\frac{1}{2}\left(\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\right)\mathrm{d}x \leq C\left(\kappa^2 + \frac{1}{2}\epsilon\right)e^{2a_E(\epsilon)/h}\|u\|^2_{L^2(\Omega)} + \int_{\Omega_\delta} 2u^2\left(\frac{\mathrm{d}\phi_{E,\epsilon}}{\mathrm{d}x}\right)^2$$

The second term on the right hand side can be absorbed by redefining the constant C. Hence, we have

$$\int_{\Omega_\delta} h^2 e^{2\phi_{E,\epsilon}/h}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\mathrm{d}x \leq C\left(\kappa^2 + \frac{1}{2}\epsilon\right)e^{2a_E(\epsilon)/h}\|u\|^2_{L^2(\Omega)} \quad (4.156)$$

Similar to (4.145), we use bounds on $a_E(\epsilon)$ and $\phi_{E,\epsilon}$ to get the following,

$$\int_{\Omega_\delta}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\mathrm{d}x \leq Ch^{-2}e^{-C/h}\|u\|^2_{L^2(\Omega)} \quad (4.157)$$

Absorbing the h^{-2} in C, we get

$$\int_{\Omega_\delta}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\mathrm{d}x \leq Ce^{-C/h}\|u\|^2_{L^2(\Omega)} \quad (4.158)$$

Combining (4.152) and (4.158) above proves the theorem. □

4.8.2 Quasimodes and an upper bound on the error

Quasimodes are defined as functions $\Psi_n(t, r, \theta, \phi, \widetilde{\psi}) : \mathcal{D} \to \mathbb{C}$ defined by

$$\Psi_n(t, r, \theta, \phi, \widetilde{\psi}) := \chi(r)e^{-i\omega_n t}e^{in\widetilde{\psi}}R(r)Y(\theta, \phi) \quad (4.159)$$

where $\chi : \mathcal{D} \to \mathbb{R}$ is a smooth cut-off function defined by the radial function

$$\chi(r) = \begin{cases} 1 & \text{if } r \in \Omega \setminus \Omega_\delta \\ 0 & \text{if } r \notin \Omega \end{cases} \tag{4.160}$$

We recall the relation between $u(r)$ and $R(r)$ and the coordinates r, w and x here for clarity.

$$R(r) = \frac{u}{r\sqrt{\hat{b}(r)}}, \quad w = \int_{r_0}^{w} \frac{\hat{b}(s)}{sW(s)} \mathrm{d}s \ \text{ and } \ w = jx \tag{4.161}$$

These quasimodes are clearly approximate solutions, defined to be extensions of the solutions to the wave equation on Ω to the whole spacetime $\mathcal{D}$. They fail to solve the wave equation because of the smooth extension in the cut-off region outside of which they are trivial solutions (because they vanish). The error, i.e. $\Box_g \Psi_n$, is supported on Ω_δ. The following lemma estimates the error, which is exponentially small as $|n| \to \infty$.

Remark 4.8.4. *In the following sections, we will take Ω_δ and Ω to refer to the spacetime regions : $\Omega_\delta = \Omega_\delta \times [0, \infty) \times (0, \pi) \times [0, 2\pi) \times [0, 2\pi)$ and $\Omega = \Omega \times [0, \infty) \times (0, \pi) \times [0, 2\pi) \times [0, 2\pi)$ respectively. Here $[0, \infty)$ is the time domain.*

Lemma 4.8.5. *Consider quasimodes which satisfy*

$$\Box_g \Psi_n = \mathrm{err}(\Psi_n) \tag{4.162}$$

where $\mathrm{err}(\Psi_n)$ is the error. Then for sufficiently large $|n|, n < 0$, we have

$$\|\Box_g \Psi_n\|_{H^k(\Sigma_t)} \leq C_k e^{-C_k|n|} \|\Psi_n\|_{L^2(\Sigma_0)} \tag{4.163}$$

Proof. For functions, $\mathcal{G}$ and χ, we have

$$\Box_g(\chi\mathcal{G}) = \chi(\Box_g\mathcal{G}) + 2g^{\mu\nu}(\partial_\mu\chi)(\partial_\nu\mathcal{G}) + \mathcal{G}(\Box_g\chi) \tag{4.164}$$

If we let $\mathcal{G} = e^{-i\omega_n t}e^{in\widetilde{\psi}}R(r)Y(\theta,\phi)$, then the first term vanishes everywhere since $\mathcal{G}$ solves the wave equation in Ω. Using the fact that χ is smooth and therefore controlled in L^∞, we have from (4.164),

$$\begin{aligned} \|\Box_g\Psi_n\|_{L^2(\Sigma_t\cap\Omega_\delta)} &\lesssim \|u_n\|_{H^1(\Sigma_t\cap\Omega_\delta)} \\ &\lesssim \|u_n\|_{H^1(\Sigma_0\cap\Omega_\delta)} \end{aligned} \tag{4.165}$$

Note that the L^2-norm of all the other eigenfunctions in (4.159) can be bounded. Using this with Theorem 4.8.3, we get

$$\|\Box_g\Psi_n\|_{L^2(\Sigma_t\cap\Omega_\delta)} \leq Ce^{-C|n|}\|\Psi_n\|_{L^2(\Sigma_0\cap\Omega)} \tag{4.166}$$

which can be written as

$$\|\Box_g\Psi_n\|_{L^2(\Sigma_t\cap\Omega_\delta)} \leq Ce^{-C|n|}\|\Psi_n\|_{L^2(\Sigma_0)} \tag{4.167}$$

owing to the spatial localization of quasimodes. To get bounds on the higher derivatives, let us make the following observations

1. We need only be concerned with $r-$derivatives of $\Box_g(\Psi_n)$ as other eigenfunctions are bounded in L^∞ (as they are analytic).
2. $\partial_r(\Box_g\mathcal{G})$ vanishes and hence the first term vanishes.
3. The second and third term contain higher derivatives of u. This can be bounded using the eigenvalue equation.

We hence deduce that

$$\|\Box_g \Psi_n\|_{H^k(\Sigma_t\cap\Omega_\delta)} \leq C_k e^{-C_k|n|} \|\Psi_n\|_{L^2(\Sigma_0)} . \tag{4.168}$$

□

4.8.3 Duhamel's principle

Here we adapt the standard construction of an inhomogeneous solution to the wave equation from a homogeneous one. Suppose $P(t, \mathbf{x}; s)$ is the solution to the following initial value problem for $t > s$.

$$\begin{aligned} &\Box_g P(t, \mathbf{x}; s)(f_1, f_2) = 0, \\ &P(t, \mathbf{x}; s)(f_1, f_1)|_{\Sigma_s} = f_1\, ,\ \partial_t P(t, \mathbf{x}, ; s)(f_1, f_1)|_{\Sigma_s} = f_2 \end{aligned} \tag{4.169}$$

In other words, $P(t, \mathbf{x}; s)(u_0, u_1)$ is the solution of the homogeneous wave equation with initial data (u_0, u_1) prescribed on the spatial hypersurface $t = s$. Note that it is sufficient that $u_0, u_1 \in H^1_{loc}(\Sigma)$ for a solution to exist and be unique. Now consider the function

$$\Psi(t, \mathbf{x}) = P(t, \mathbf{x}; 0)(\Phi_0, \Phi_1) + \frac{1}{2}\int_0^t P(t, \mathbf{x}; s)(0, (g^{00})^{-1}F(s, \mathbf{x}))\,\mathrm{d}s \tag{4.170}$$

Claim 4.8.6. $\Psi(t, \mathbf{x})$ *solves the following initial value problem (inhomogeneous wave equation).*

$$\begin{aligned} &\Box_g \Psi(t, \mathbf{x}) = F(t, \mathbf{x}) \\ &\Psi(\mathbf{x}, 0) = \Phi_0(\mathbf{x}), \quad \partial_t \Psi(0, \mathbf{x}) = \Phi_1(\mathbf{x}) \end{aligned} \tag{4.171}$$

Proof.

$$
\begin{aligned}
\Box_g \Psi &= g^{a0}(\partial_a \partial_0 \Psi) + g^{ai}(\partial_a \partial_i \Psi) - g^{ab}\Gamma^0_{ab}\partial_0 \Phi - g^{ab}\Gamma^i_{ab}\partial_i \Psi \\
&= \frac{g^{00}}{2}\partial_t P(t,\mathbf{x};t) + \frac{g^{0i}}{2}\partial_i P(t,\mathbf{x};t) + \frac{g^{00}}{2}\partial_t P(t,\mathbf{x};t) + \frac{g^{00}}{2}\int_0^t \partial_t^2 P \mathrm{d}s \\
&\quad + \frac{g^{0i}}{2}\int_0^t \partial_i \partial_0 P \mathrm{d}s + \frac{g^{0i}}{2}\partial_i P(t,\mathbf{x};t) + \frac{g^{0i}}{2}\int_0^t \partial_0 \partial_i P \mathrm{d}s + \frac{g^{ij}}{2}\int_0^t \partial_i \partial_j P \mathrm{d}s \\
&\quad - \frac{1}{2}\Gamma^0_{ab} g^{ab} P(t,\mathbf{x};t) - \frac{1}{2} g^{ab}\Gamma^c_{ab}\int_0^t \partial_c P \mathrm{d}s \\
&= g^{00}\partial_t P(t,\mathbf{x};t) + g^{0i}\partial_i P(t,\mathbf{x};t) - \frac{1}{2} g^{ab}\Gamma^0_{ab} P(t,\mathbf{x};t) \\
&\quad + \frac{1}{2}\int_0^t \Box_g P\left(0, (g^{00})^{-1} F(s,\mathbf{x})\right) \mathrm{d}s \\
&= F(t,\mathbf{x}),
\end{aligned}
$$

where we used that $\Box_g P = 0$, $P(t,\mathbf{x};t) = 0$, and $\partial_t P(t,\mathbf{x};t) = (g^{00})^{-1} F(t,\mathbf{x})$. □

4.8.4 Bound on the uniform decay rate

We have constructed quasimodes, namely, approximate solutions to the wave equation $\Box_g \Psi_n = \mathrm{err}_n(\Psi_n)$ with compactly supported initial data $(\Psi_n(0,\mathbf{x}), \partial_t \Psi_n(0,\mathbf{x})$. We have also seen that the error can be made exponentially small as $|n| \to \infty$. Now consider a solution of the homogeneous wave equation with the same initial data

$$
\Box \Phi_n = 0, \qquad \Phi_n(0,\mathbf{x}) = \Psi_n(0,\mathbf{x}), \qquad \partial_t \Phi_n(0,\mathbf{x}) = \partial_t \Psi_n(0,\mathbf{x}). \tag{4.172}
$$

Using Duhamel's principle we have

$$
\Psi_n(t,\mathbf{x}) = \Phi_n(t,\mathbf{x}) + \frac{1}{2}\int_0^t P(t,\mathbf{x};s)(0, (g^{00})^{-1}\mathrm{err}_n(\Psi_n))\, \mathrm{d}s \tag{4.173}
$$

where $P(t,\mathbf{x};s)$ is a solution to the homogeneous wave equation described above. In terms of the 'local' energy integral $E_{t,\Omega}[\Phi]$ measured over Ω (recall this is quadratic in derivatives of Φ)

$$E_{t,\Omega}[\Psi_n - \Phi_n] = E_{t,\Omega}\left[\int_0^t P(t,\mathbf{x};s)\,\mathrm{d}s\right] \tag{4.174}$$

We use the fact that $P(t,\mathbf{x};t) = 0$ and

$$\int_0^t \partial_\alpha P(t,\mathbf{x};s)(0,(g^{00})^{-1}\mathrm{err}_n(\Psi_n))\,\mathrm{d}s \leq t \sup_{s\in[0,t]} |\partial_\alpha P(s,\mathbf{x};s)(0,(g^{00})^{-1}\mathrm{err}_n(\Psi_n))| \tag{4.175}$$

to get

$$\left(E_{t,\Omega}[\Psi_n - \Phi_n]\right)^{1/2} \leq \frac{t}{2}\sup_{s\in[0,t]}\left(E_{t,\Omega}[P]\right)^{1/2} \leq Ct\left(E_{0,\Omega}[P]\right)^{1/2} \tag{4.176}$$

where we used the uniform boundedness of the energy to express the estimate in terms of the energy at $t = 0$. Evaluating the energy of $P(t,\mathbf{x};s)$ at $t = 0$, we see that

$$\left(E_{0,\Omega}[P]\right)^{1/2} \sim \left\|(g^{00})^{-1}\mathrm{err}_n(\Psi_n(0))\right\|_{L^2(\Omega)} \leq Ce^{-C|n|}\left\|\Psi_n(0)\right\|_{L^2(\Omega)} \tag{4.177}$$

where we used the above estimate. Applying the Poincaré inequality we arrive at

$$\left(E_{t,\Omega}[\Psi_n - \Phi_n]\right)^{1/2} \leq Cte^{-C|n|}\left(E_{0,\Omega}[\Psi_n]\right)^{1/2}. \tag{4.178}$$

Using the reverse triangle inequality we find

$$\left|\left(E_{t,\Omega}[\Psi_n]\right)^{1/2} - \left(E_{t,\Omega}[\Phi_n]\right)^{1/2}\right| \leq \left(E_{t,\Omega}[\Psi_n - \Phi_n]\right)^{1/2}. \tag{4.179}$$

Therefore for all $t \leq \frac{1}{2C}e^{C|n|}$ there holds

$$\frac{1}{2}\left(E_{0,\Omega}[\Psi_n]\right)^{1/2} \leq \left(E_{t,\Omega}[\Phi_n]\right)^{1/2}. \tag{4.180}$$

Of course, since by construction Ψ_n vanishes outside of Ω, we can write this as

$$\left(E_{t,\Omega}[\Phi_n]\right)^{1/2} \geq \frac{1}{2}\left(E_0[\Psi_n]\right)^{1/2}. \tag{4.181}$$

We now bound the energy of the homogeneous solution Φ_n from below by a higher order energy. Note that

$$E_2[\Psi_n](0) = E[\Psi_n](0) + \sum_{\alpha=0}^{4} E[\partial_\alpha \Psi_n] \tag{4.182}$$

Taking derivatives with respect to $t, \widetilde{\psi}$ will simply pull down Cn, n respectively (since $\omega = Cn$ for some C). Derivatives with respect to θ, ϕ will simply yield linear combinations of the charged spherical harmonics Y. Thus all the energies associated to these values of α will at most be of order $E[\Psi_n]$. However, $e_1(\partial_r \Psi_n) \sim Y(\theta, \phi) e^{-\omega t + in\widetilde{\psi}} u''$. Using the equation satisfied by u we can rewrite $u'' = V(r)u$. On the other hand $e_1(\Psi) \sim e^{-\omega t + in\widetilde{\psi}} u'$. Using the Poincaré inequality we know $\|u\|_{L^2(\Omega)} \leq C \|u'\|_{L^2(\Omega)}$. These considerations imply

$$E_2[\Psi_n](0) \leq (C_1 + C_2 n^2) E[\Psi_n](0) \tag{4.183}$$

and in particular for $|n|$ sufficiently large

$$(E[\Psi_n])^{1/2} \geq \frac{C}{|n|} (E_2[\Psi_n](0))^{1/2} \tag{4.184}$$

Similar inequalities will apply with $(n, E_2[\Psi_n](0))$ replaced with $(n^{k-1}, E_k[\Psi_n](0))$ for $k > 2$ (essentially, additional derivatives will pull down factors of n). Now because by construction Φ_n has the same initial data as Ψ_n, we have for sufficiently large $|n|$ and $0 < t \leq e^{C|n|}/2C$, $k > 2$,

$$(E_\Omega[\Phi_n](t))^{1/2} \geq \frac{C}{|n|^{k-1}} (E_k[\Phi_n](0))^{1/2}. \tag{4.185}$$

The above result prevents the possibility of a local uniform logarithmic decay statement of the form

$$\limsup_{t \to \infty} \delta(t) E_\Omega[\Phi](t) \leq C E[\Phi](0) \tag{4.186}$$

where $\delta(t)$ encodes the rate of decay, for all solutions Φ to the wave equation with suitably regular initial data. Here 'uniform' implies that such a decay must hold for any smooth solution. Setting $\tau_n = e^{C|n|}/2C$, we obtain

$$(\log(2+\tau_n))^2 \frac{E_\Omega[\Phi_n](\tau_n)}{E_2[\Phi_n](0)} \geq C \tag{4.187}$$

This produces a sequence $\{(\tau_n, \Phi_n)\}$, $|n| \geq N$ for N sufficiently large, of solutions to the homogeneous wave equation. We conclude that for some positive constant C,

$$\limsup_{\tau\to\infty} \sup_{\Phi\neq 0} (\log(2+\tau))^2 \frac{E_\Omega[\Phi](\tau)}{E_2[\Phi](0)} \geq C \tag{4.188}$$

where the supremum is taken over all Φ that lie in the completion of the set of smooth solutions to the wave equation with compactly supported initial data on the hypersurface Σ_0, with respect to the norm defined by E_2. An analogous statement holds for $k \in \mathbb{N}$, there are positive constants C_k such that

$$\limsup_{\tau\to\infty} \sup_{\Phi\neq 0} (\log(2+\tau))^{2k} \frac{E_\Omega[\Phi](\tau)}{E_{k+1}[\Phi](0)} \geq C_k. \tag{4.189}$$

This completes the proof of Theorem 4.2.3. As emphasized by Keir [56], it should be noted that a given smooth solution could decay faster than logarithmically, and indeed it could be the case that *all* smooth solutions decay faster than logarithmically. However, there cannot be a *uniform* decay bound, that applies to all smooth solutions, and which yields decay at a rate faster than logarithmic.

Chapter 5

Horizons as boundary conditions in spherical symmetry

This chapter is based on work published in [24] which appeared as :

S. Gunasekaran and I. Booth, *"Horizons as boundary conditions in spherical symmetry,"*, Phys. Rev. D **100** no.6, 064019 (2019) (arXiv:1609.08500)

5.1 Abstract

We initiate the development of a horizon-based initial (or rather final) value formalism to describe the geometry and physics of the near-horizon spacetime: data specified on the horizon and a future ingoing null boundary determine the near-horizon geometry. In this initial paper we restrict our attention to spherically symmetric spacetimes made dynamic by matter fields. We illustrate the formalism by considering a black hole interacting with a) inward-falling, null matter (with no outward flux) and b) a massless scalar field. The inward-falling case can be exactly solved from horizon

data. For the more involved case of the scalar field we analytically investigate the near slowly evolving horizon regime and propose a numerical integration for the general case.

5.2 Introduction

This paper begins an investigation into what horizon dynamics can tell us about external black hole physics. At first thought this might seem obvious: if one watches a numerical simulation of a black hole merger and sees a post-merger horizon ringdown (see for example [137]) then it is natural to think of that oscillation as a source of emitted gravitational waves. However this cannot be the case. Neither event nor apparent horizons can actually send signals to infinity: apparent horizons lie inside event horizons which in turn are the boundary for signals that can reach infinity [2]. It is not horizons themselves that interact but rather the "near horizon" fields. This idea was (partially) formalized as a "stretched horizon" in the membrane paradigm [138].

Then the best that we can hope for from horizons is that they act as a proxy for the near horizon fields with horizon evolution reflecting some aspects of their dynamics. As explored in [27, 139–142] there should then be a correlation between horizon evolution and external, observable, black hole physics.

Robinson-Trautman spacetimes (see for example [143]) demonstrate that this correlation cannot be perfect. In those spacetimes there can be outgoing gravitational (or other) radiation arbitrarily close to an isolated (equilibrium) horizon [144]. Hence our goal is two-fold: both to understand the conditions under which a correlation will exist and to learn precisely what information it contains.

The idea that horizons should encode physical information about black hole physics is not new. The classical definition of a black hole as the complement of the causal past of future null infinity [2] is essentially global and so defines a black hole spacetime rather than a black hole *in* some spacetime. However there are also a range of geometrically defined black hole boundaries based on outer and/or marginally trapped surfaces that seek to localize black holes. These include apparent [2], trapping [145], isolated [144, 146–148] and dynamical [149] horizons as well as future holographic screens [150]. These quasilocal definitions of black holes have successfully localized black hole mechanics to the horizon [145–147, 149–151] and been particularly useful in formalizing what it means for a (localized) black hole to evolve or be in equilibrium. They are used in numerical relativity not only as excision surfaces (see, for example the discussions in [152, 153]) but also in interpreting physics (for example [27, 139–142, 154–158]).

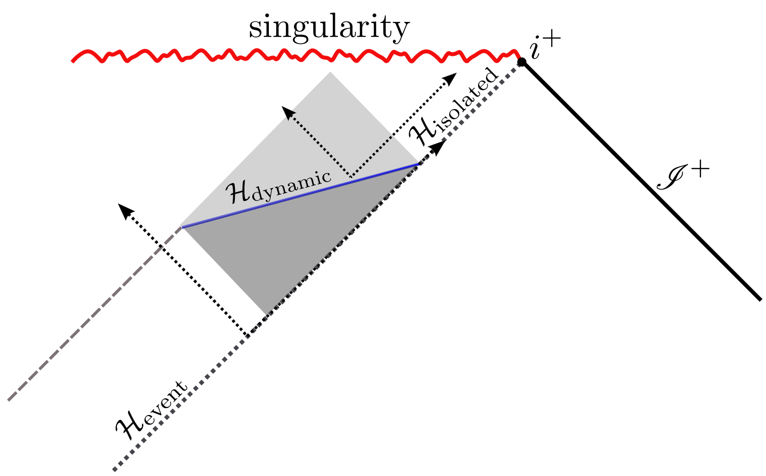


Figure 5.1: Future and past domains of dependence for $\mathcal{H}_{\text{dynamic}}$: standard (3+1) IVP

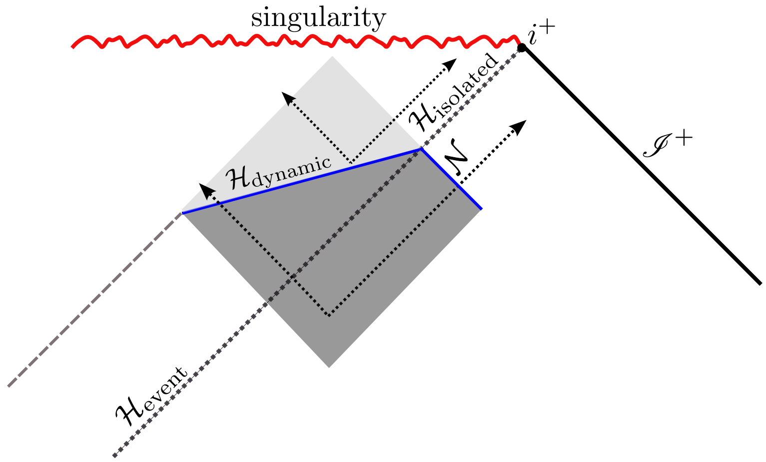


Figure 5.2: Future and past domains of dependence for $\mathcal{H}_{\text{dynamic}} \cup \mathcal{N}$: characteristic IVP

In this thesis we work to quantitatively link horizon dynamics to observable black hole physics. To establish an initial framework and build intuition we for now restrict our attention to spherically symmetric marginally outer trapped tubes (MOTTs) in similarly symmetric spacetimes. Matter fields are included to drive the dynamics. Our primary approach is to take horizon data as a (partial) final boundary condition that is used to determine the fields in a region of spacetime in its causal past. In particular these boundary conditions constrain the geometry and physics of the associated "near horizon" spacetime. The main application that we have in mind is interpreting the physics of evolving horizons that have been generated by either numerical simulations or theoretical considerations.

Normally, data on a MOTT by itself is not sufficient to specify any region of the external spacetime. As shown in FIG.5.1 even for a spacelike MOTT (a dynamical horizon) the region determined by a standard (3+1) initial value formulation would lie entirely within the event horizon. More information is needed to determine the near-horizon spacetime and hence in this paper we work with a characteristic initial value formulation [16, 21–23, 159] where extra data are specified on a null surface $\mathcal{N}$ that is transverse to the horizon (FIG. 5.2). Intuitively the horizon records inward-moving information while $\mathcal{N}$ records the outward-moving information. Together they are sufficient to reconstruct the spacetime.

There is an existing literature that studies spacetimes near horizons, however it does not exactly address this problem. Most works focus on isolated horizons. [160] and [161] examine spacetime near an isolated extremal horizon as a Taylor series expansion of the horizon while [162] and [163] study spacetime near more general isolated horizons but in a characteristic initial value formulation with the extra information specified on a transverse null surface. [164] studied both the isolated and dynamical case though again as a Taylor series expansion off the horizon. In the case of the Taylor expansions, as one goes to higher and higher orders one needs to know higher and higher order derivatives of metric quantities at the horizon to continue the expansion. While the current paper instead investigates the problem as a final value problem, it otherwise closely follows the notation of and uses many results from [164].

It is organized as follows. We introduce the final value formulation of spherically symmetric general relativity in Sec.5.3. We illustrate this for infalling null matter in 5.4 and then the much more interesting massless scalar field in Sec.5.5. We conclude with a discussion of results in Sec.5.6.

5.3 Formulation

5.3.1 Coordinates and metric

We work in a spherically symmetric spacetime $(\mathcal{M}, g)$ and a coordinate system whose non-angular coordinates are ρ (an ingoing affine parameter) and v (which labels the ingoing null hypersurfaces and increases into the future). Hence, $g_{\rho\rho} = 0$ and the curves tangent to the future-oriented inward-pointing

$$N = \frac{\partial}{\partial \rho} \tag{5.1}$$

are null. We then scale v so that $\mathcal{V} = \frac{\partial}{\partial v}$ satisfies

$$\mathcal{V} \cdot N = -1. \tag{5.2}$$

One coordinate freedom still remains: the scaling of the affine parameter on the individual null geodesics

$$\tilde{\rho} = f(v)\rho\,. \tag{5.3}$$

In subsection 5.3.3 we will fix this freedom by specifying how N is to be scaled along the $\rho = 0$ surface Σ (which we take to be a black hole horizon H).

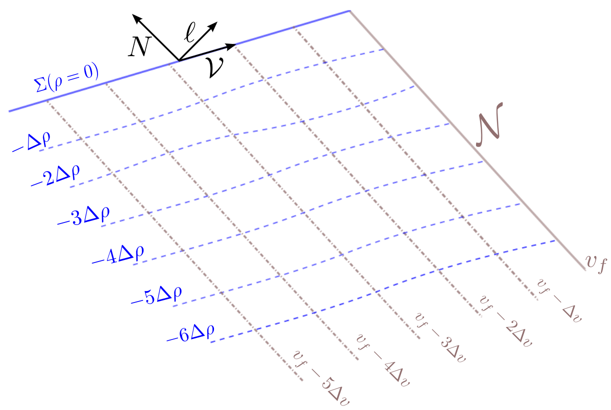


Figure 5.3: Coordinate system for characteristic evolution. We work with final boundary conditions so that in the region of interest $\rho < 0$.

Next we define the future-oriented outward-pointing null normal to the spherical surfaces $S_{(v,\rho)}$ as ℓ^a and scale so that

$$\ell \cdot N = -1 \,. \tag{5.4}$$

With this choice the four-metric g_{ab} and induced two-metric $\tilde{q}_{ab}$ on the $S_{(v,\rho)}$ are related by

$$g^{ab} = \tilde{q}^{ab} - \ell^a N^b - N^a \ell^b \,. \tag{5.5}$$

Further for some function C we can write

$$\mathcal{V} = \ell - CN \,. \tag{5.6}$$

The coordinates and normal vectors are depicted in FIG.5.3 and give the following form of the metric:

$$ds^2 = 2C(v,\rho)\mathrm{d}v^2 - 2\mathrm{d}v\,\mathrm{d}\rho + R(v,\rho)^2\mathrm{d}\Omega^2 \tag{5.7}$$

where $R(v,\rho)$ is the areal radius of the $S_{(v,\rho)}$ surfaces. Note the similarity to ingoing Eddington-Finkelstein coordinates for a Schwarzschild black hole. However $\frac{\partial}{\partial \rho}$ points inwards as opposed to the outward oriented $\frac{\partial}{\partial r}$ in those coordinates (hence the negative sign on the $\mathrm{d}v\mathrm{d}\rho$ cross-term).

Typically, as shown in FIG.5.3 we will be interested in regions of spacetime that are bordered in the future by a surface Σ of indeterminate sign on which $\rho = 0$ and a null $\mathcal{N}$ which is one of the v=constant surfaces (and so $\rho < 0$ in the region of interest). We will explore how data on those surfaces determines the region of spacetime in their causal past.

5.3.2 Equations of motion

In this section we break up the Einstein equations relative to these coordinates, beginning by defining some geometric quantities that appear in the equations.

First the null expansions for the ℓ^a and N^a congruences are

$$\theta_{(\ell)} = \tilde{q}^{ab}\nabla_a \ell_b = \frac{2}{R}\mathcal{L}_\ell R \quad \text{and} \tag{5.8}$$

$$\theta_{(N)} = \tilde{q}^{ab}\nabla_a N_b = \frac{2}{R}\mathcal{L}_N R = \frac{2}{R}R_{,\rho}. \tag{5.9}$$

while the inaffinities of the null vector fields are

$$\kappa_N = -N^a N_b \nabla_a \ell^b = 0 \quad \text{and} \tag{5.10}$$

$$\kappa_{\mathcal{V}} = \kappa_\ell - C\kappa_N = -\ell^a N_b \nabla_a \ell^b \,. \tag{5.11}$$

By construction $\kappa_N = 0$ and so we can drop it from our equations and henceforth write

$$\kappa \equiv \kappa_{\mathcal{V}} = \kappa_\ell \,. \tag{5.12}$$

Finally the Gaussian curvature of $S_{(v,\rho)}$ is:

$$\tilde{K} = \frac{1}{R^2} \,. \tag{5.13}$$

Then these curvature quantities are related by constraint equations along the surfaces of constant ρ

$$\mathcal{L}_{\mathcal{V}} R = \mathcal{L}_{\ell} R - C \mathcal{L}_{N} R \quad \text{(by definition)}\,, \tag{5.14}$$

$$\begin{aligned} \mathcal{L}_{\mathcal{V}} \theta_{(\ell)} =& \kappa\, \theta_{(\ell)} + C \left(\frac{1}{R^2} + \theta_{(N)}\theta_{(\ell)} - G_{\ell N} \right) \\ & - \left(G_{\ell\ell} + \frac{1}{2}\theta_{(\ell)}^2 \right) \,, \end{aligned} \tag{5.15}$$

$$\begin{aligned} \mathcal{L}_{\mathcal{V}} \theta_{(N)} =& - \kappa\, \theta_{(N)} - \left(\frac{1}{R^2} + \theta_{(N)}\theta_{(\ell)} - G_{\ell N} \right) \\ & + C \left(G_{NN} + \frac{1}{2}\theta_{(N)}^2 \right) \,, \end{aligned} \tag{5.16}$$

and "time" derivatives in the ρ direction

$$\mathcal{L}_{N} \theta_{(N)} = - \frac{\theta_{(N)}^2}{2} - G_{NN}, \tag{5.17}$$

$$\mathcal{L}_{N} \theta_{(\ell)} = - \frac{1}{R^2} - \theta_{(N)}\theta_{(\ell)} + G_{\ell N}, \tag{5.18}$$

$$\mathcal{L}_{N} \kappa = \frac{1}{R^2} + \frac{1}{2}\theta_{(N)}\theta_{(\ell)} - \frac{1}{2} G_{\tilde{q}} - G_{\ell N}, \tag{5.19}$$

where by the choice of the coordinates

$$\kappa = \mathcal{L}_{N} C \,. \tag{5.20}$$

These equations can be derived from the variations for the corresponding geometric quantities (see, for example, [165] and [164]) and of course are coupled to the matter content of the system through the Einstein equations

$$G_{ab} = 8\pi T_{ab} \,. \tag{5.21}$$

Using (5.8) and (5.9) we can rewrite the constraint and evolution equations in terms of the metric coefficients and coordinates as:

$$R_{,v} = R_\ell - CR_N\,, \tag{5.22}$$

$$R_{\ell,v} = \kappa R_\ell + \frac{C\,(1+4R_\ell R_N)}{2R} - \frac{R}{2}\,(G_{\ell\ell} + CG_{\ell N})\,, \tag{5.23}$$

$$R_{N,v} = -\,\kappa R_N - \frac{(1+4R_\ell R_N)}{2R} + \frac{R}{2}\,(G_{\ell N} + CG_{NN}). \tag{5.24}$$

and

$$R_{,\rho\rho} = -\frac{R}{2}\,G_{NN}, \tag{5.25}$$

$$(RR_\ell)_{,\rho} = -\frac{1}{2} + \frac{R^2}{2}G_{\ell N}, \tag{5.26}$$

$$C_{,\rho\rho} = \frac{1}{R^2} + \frac{2R_\ell R_N}{R^2} - \frac{1}{2}G_{\tilde{q}} - G_{\ell N}, \tag{5.27}$$

where

$$\kappa = C_{,\rho}. \tag{5.28}$$

For those who don't want to work through the derivations of [165] and [164], these can also be derived fairly easily (thanks to the spherical symmetry) from an explicit calculation of the Einstein tensor for (5.7).

5.3.3 Final Data

We will focus on the case where $\rho = 0$ is an isolated or dynamical horizon H. Thus

$$\theta_{(\ell)} \stackrel{H}{=} 0 \quad \Longleftrightarrow \quad R_\ell \stackrel{H}{=} 0\,. \tag{5.29}$$

The notation $\stackrel{H}{=}$ indicates that the equality holds on H (but not necessarily anywhere else). Further we can use the coordinate freedom (5.3) to set

$$R_N \stackrel{H}{=} R_{,\rho}| \stackrel{H}{=} -1\,. \tag{5.30}$$

On H, the constraints (5.22)-(5.24) fix three of

$$\{C, \kappa, R, R_\ell, R_N, G_{\ell\ell}, G_{\ell N}, G_{NN}\} \tag{5.31}$$

given the other five quantities. For example if $R_\ell \stackrel{H}{=} 0$ and $R_N \stackrel{H}{=} -1$ then (5.22) and (5.23) give

$$R_{,v} \stackrel{H}{=} C \stackrel{H}{=} \frac{R^2 G_{\ell\ell}}{1 - R^2 G_{\ell N}} \tag{5.32}$$

and (5.24) gives

$$\kappa = C_\rho \stackrel{H}{=} \frac{1}{2R} - \frac{R}{2}\left(G_{\ell N} + C G_{NN}\right) . \tag{5.33}$$

Thus if $G_{\ell\ell}$ and $G_{\ell N}$ are specified for $v_i \leq v \leq v_f$ on H and $R(v_f) \stackrel{H}{=} R_f$ then one can solve (5.32) to find R over the entire range. Equivalently one could take R and one of $G_{\ell\ell}$ or $G_{\ell N}$ as primary and then solve for the other component of the stress-energy.

Of course, in general the matter terms will also be constrained by their own equations; these will be treated in later sections. Further data on $\rho = 0$ will generally not be sufficient to fully determine the regions of interest and data will also be needed on an $\mathcal{N}$. Again this will depend on the specific matter model.

Nevertheless if there is a MOTT at $\rho = 0$ then the constraints provide significant information about the horizon. If $G_{\ell\ell} = 0$ (no flux of matter through the horizon) then we have an isolated horizon with $C = 0$, a constant R and a null H. This is independent of other components of the stress-energy.

Alternatively if $G_{\ell\ell} > 0$ (the energy conditions forbid it to be negative) and $G_{\ell N} < 1/R^2$ then we have a dynamical horizon with $C > 0$, increasing R and spacelike H[1]. Note that this growth doesn't depend in any way on G_{NN}: there is no sense in

[1] $G_{\ell N} > 1/R^2$ signals that another marginally outer trapped surface (MOTS) has formed outside the original one and so a numerical simulation would see an apparent horizon "jump" [150, 166]. In

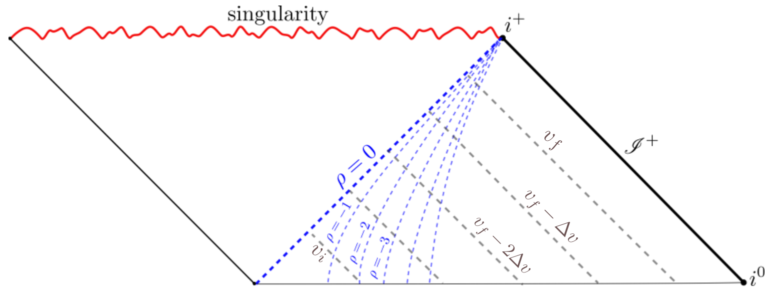


Figure 5.4: Isolated horizon : $d\rho$ is timelike for all values of ρ

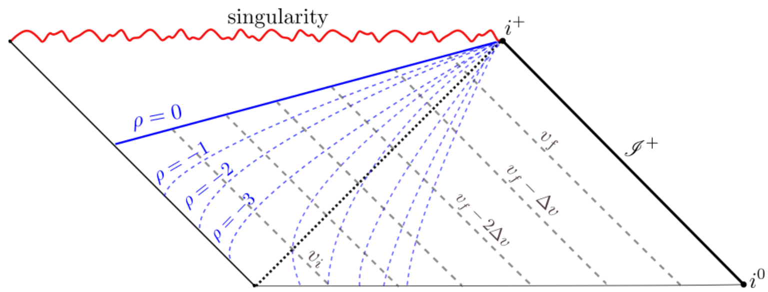


Figure 5.5: Dynamical horizon : $d\rho$ is spacelike for small values of ρ and eventually becomes timelike for large values of ρ

which the growing horizon "catches" outward moving matter and hence grows even faster. The behaviour of the coordinates relative to isolated and dynamical horizons at $(\rho = 0)$ along with $\mathscr{I}^+$ is illustrated in FIGS.5.5 and 5.4.

The evolution equations are more complicated and depend on the matter field equations. We examine two such cases in the following sections.

5.4 Traceless inward flowing null matter

As our first example consider matter that falls entirely in the inward N-direction with no outward ℓ-flux. Then data on the horizon should be sufficient to entirely determine

the current paper all matter satisfies $G_{\ell N} < 1/R^2$ and so this situation does not arise.

the region of spacetime traced by the horizon-crossing inward null geodesics: there are no dynamics that don't involve the horizon.

Translating these words into equations, we assume that

$$T_{ab}N^aN^b = 0 \tag{5.34}$$

(no matter flows in the outward ℓ-direction). Further, for simplicity we also assume that it is trace-free

$$g^{ab}T_{ab} = 0 \quad \Leftrightarrow \quad T_{\tilde{q}} = 2T_{\ell N} \; . \tag{5.35}$$

Then we can solve for the metric using only the Bianchi identities

$$\nabla_a G^{ab} = 0, \tag{5.36}$$

without any reference to detailed equations of motion for the matter field. Keeping spherical symmetry but temporarily suspending the other simplifying assumptions they may be written as:

$$\begin{aligned} \mathcal{L}_\ell(R^2 G_{NN}) + \mathcal{L}_N(R^2 G_{\ell N}) + R^2(2\kappa_\ell G_{NN}) & \\ + \frac{1}{2}R^2\theta_{(N)}G_{\tilde{q}} = 0, & \end{aligned} \tag{5.37}$$

$$\begin{aligned} \mathcal{L}_N(R^2 G_{\ell\ell}) + \mathcal{L}_\ell(R^2 G_{\ell N}) + R^2(-2\kappa_N G_{\ell\ell}) & \\ + \frac{1}{2}R^2\theta_{(\ell)}G_{\tilde{q}} = 0\,. & \end{aligned} \tag{5.38}$$

In terms of metric coefficients with $\kappa_N = 0$ plus (5.34) and (5.35) these reduce to:

$$(R^4 G_{\ell N})_{,\rho} = 0 \;\; \text{and} \tag{5.39}$$

$$(R^2 G_{\ell\ell})_{,\rho} + \frac{1}{R^2}(R^4 G_{\ell N})_{,v} = 0. \tag{5.40}$$

As we shall see, this class of matter includes interesting examples like Vaidya-Reissner-Nordström (charged null dust).

We now demonstrate that given knowledge of $G_{\ell\ell}$ and $G_{\ell N}$ over a region of horizon $\bar{H} = \{H : v_i \leq v \leq v_f\}$ as well as $R(v_f) \stackrel{H}{=} R_f$ we can determine the spacetime everywhere out along the horizon-crossing inward null geodesics.

5.4.1 On the horizon

First consider the constraints on $\bar{H}$. In this case it is tidier to take R and $G_{\ell N}$ as primary. Then we can specify

$$R \stackrel{H}{=} R_H(v) \quad \text{and} \quad G_{\ell N} \stackrel{H}{=} \frac{Q(v)}{R_H^4} \tag{5.41}$$

for some functions $R_H(v)$ (dimensions of length) and $Q_H(v)$ (dimensions of length squared) where the form of the latter is chosen for future convenience. Then

$$C \stackrel{H}{=} R_{H,v} \tag{5.42}$$

and by (5.32)

$$G_{\ell\ell} \stackrel{H}{=} R_{H,v} \left(\frac{1}{R_H^2} - \frac{Q}{R_H^4} \right) \tag{5.43}$$

Finally by (5.33),

$$\kappa \stackrel{H}{=} C_\rho \stackrel{H}{=} \frac{1}{2R_H} \left(1 - \frac{Q}{R_H^2} \right) . \tag{5.44}$$

5.4.2 Off the horizon

Next, integrate away from $\bar{H}$. First with $G_{NN} = 0$ (5.25) can be integrated with initial condition (5.30) to give

$$R(v, \rho) = R_H(v) - \rho \, . \tag{5.45}$$

Then with (5.41) we can integrate (5.39) to find

$$G_{\ell N} = \frac{Q}{R^4} \tag{5.46}$$

and use this result and (5.43) to integrate(5.40) to get

$$G_{\ell\ell} = \frac{(R_H^2 - Q)\, R_{H,v}}{R_H^2 R^2} + \frac{\rho\, Q_{,v}}{R_H R^3}\,. \tag{5.47}$$

With these results in hand and initial condition $R_\ell \stackrel{H}{=} 0$ we integrate (5.26) to get

$$R_\ell = \frac{\rho(Q - R_H^2 + \rho R_H)}{2R^2 R_H} \tag{5.48}$$

and finally with initial conditions (5.32) and (5.33) we can integrate (5.27) to find

$$C = R_{H,v} - R_\ell\,. \tag{5.49}$$

5.4.3 Comparison with Vaidya-Reissner-Nordström

We can now compare this derivation to a known example. The Vaidya-Reissner-Nordström (VRN) metric takes the form

$$\mathrm{d}s^2 = -\left(1 - \frac{2m(v)}{r} + \frac{q(v)^2}{r^2}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}\Omega^2 \tag{5.50}$$

where the apparent horizon $r_H = m + \sqrt{m^2 - q^2}$ and r is an affine parameter of the ingoing null geodesics. To put it into the form of (5.7) where the affine parameter measures distance off the horizon we make the transformation

$$r = r_H - \rho \tag{5.51}$$

whence the metric takes the form

$$\begin{aligned}\mathrm{d}s^2 = &-\left(2r_{H,v} - \frac{\rho\,(q^2 - r_H(r_H - \rho))}{r_H(r_H - \rho)^2}\right)\mathrm{d}v^2 \\ &- 2\mathrm{d}v\mathrm{d}\rho + (r_H - \rho)^2\mathrm{d}\Omega^2\,.\end{aligned} \tag{5.52}$$

That is

$$C = r_{H,v} - \frac{\rho\,(q^2 - r_H(r_H-\rho))}{2r_H(r_H-\rho)^2} \tag{5.53}$$

$$R = r_H - \rho \tag{5.54}$$

and on the horizon

$$C \overset{H}{=} r_{H,v} \qquad \text{and} \qquad R \overset{H}{=} r_H \tag{5.55}$$

as expected.

To do a complete match we calculate the rest of the quantities. First appropriate null vectors are

$$\ell = \frac{\partial}{\partial v} + \left(r_{H,v} - \frac{\rho\,(q^2 - r_H(r_H-\rho))}{2r_H(r_H-\rho)^2} \right) \frac{\partial}{\partial \rho} \tag{5.56}$$

$$N = \frac{\partial}{\partial \rho}\,. \tag{5.57}$$

Then direct calculation shows that

$$R_\ell = -\frac{\rho\,(q^2 - r_H(r_H-\rho))}{2r_H(r_H-\rho)^2} \tag{5.58}$$

$$R_N = -1 \tag{5.59}$$

and

$$G_{\ell\ell} = \frac{(r_H^2 - q^2)\,r_{H,v}}{r_H^2 r^2} + \frac{2\rho q q_{,v}}{r_H r^3} \tag{5.60}$$

$$G_{\ell N} = \frac{q^2}{(r_H-\rho)^2} \tag{5.61}$$

$$G_{NN} = 0 \tag{5.62}$$

$$G_q = \frac{2q^2}{(r_H-\rho)^2}\;. \tag{5.63}$$

It is clear that with $R_H = r_H$ and $Q = q^2$ our general results (5.41)-(5.49) give rise to the VRN spacetime (as they should).

As expected the data on the horizon are sufficient to determine the spacetime everywhere back out along the ingoing null geodesics: we simply solve a set of (coupled) ordinary differential equations along each curve. With the matter providing the only dynamics and that matter only moving inwards along the geodesics the problem is quite straightforward. In this case there is no need to specify extra data on $\mathcal{N}$.

We now turn to the more interesting case where the dynamics are driven by a scalar field for which there will be both inward and outward fluxes of matter.

5.5 Massless scalar field

Spherical spacetimes containing a massless scalar field $\phi(v, \rho)$ are governed by the stress energy tensor given by,

$$T_{ab} = \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla^c \phi \nabla_c \phi \tag{5.64}$$

This system has nonvanishing inward and outward fluxes which are

$$T_{\ell\ell} = (\phi_\ell)^2 \tag{5.65}$$

$$T_{NN} = (\phi_N)^2. \tag{5.66}$$

Here and in the following keep in mind that $N = \frac{\partial}{\partial \rho}$ and so $\phi_N = \phi_{,\rho}$. We also observe from (5.64) that

$$T_{\ell N} = 0. \tag{5.67}$$

These fluxes are related by the wave equation

$$\Box_g \phi := \nabla^\alpha \nabla_\alpha \phi = 0 \implies \left(R \phi_\ell\right)_{,\rho} = -R_\ell \phi_{,\rho}. \tag{5.68}$$

For our purposes we are not particularly interested in the value of ϕ itself but rather in the associated net flux of energies in the ingoing and outgoing null direction. Hence we define

$$\Phi_\ell = \sqrt{4\pi} R \phi_\ell \quad \text{and} \quad \Phi_N = \sqrt{4\pi} R \phi_N \, . \tag{5.69}$$

Respectively these are the square roots of the scalar field energy fluxes in the N and ℓ directions. That is, over a sphere of radius R, Φ_ℓ is the square root of the total integrated flux in the N-direction and Φ_N is the square root of the total integrated flux in the ℓ-direction. Though not strictly correct, we will often refer to Φ_ℓ and Φ_N themselves as fluxes.

Then (5.68) becomes

$$\Phi_{\ell,\rho} = -\frac{R_\ell \Phi_N}{R} \tag{5.70}$$

or, making use of the fact that $\phi_{,v\rho} = \phi_{,\rho v}$,

$$\Phi_{N,v} = -\kappa \Phi_N - C \Phi_{N,\rho} - \frac{R_N \Phi_\ell}{R} \, . \tag{5.71}$$

These can usefully be understood as advection equations with sources. Recall that a general homogeneous advection equation can be written in the form

$$\frac{\partial \psi}{\partial t} + C \frac{\partial \psi}{\partial x} = 0 \tag{5.72}$$

where C is the speed of flow of ψ: if C is constant then this has the exact solution

$$\psi = \psi(x - Ct) \tag{5.73}$$

and so any pulse moves with speed $\frac{dx}{dt} = C$. Any non-homogeneous term corresponds to a source which adds or removes energy from the system. Then (5.70) tells us that the flux in the N-direction (Φ_ℓ) is naturally undiminished as it flows along a (null) surface of constant v and increasing ρ. However the interaction with the flux in the ℓ direction can cause it to increase or decrease. Similarly (5.71) describes the flow of the flux in the ℓ-direction (Φ_N) along a surface of constant ρ and increasing v. Rewriting with respect to the affine derivative (see Appendix 5.8) $D_v = \partial_v + \kappa$ it becomes

$$D_v \Phi_N + C\Phi_{N,\rho} = -\frac{R_N \Phi_\ell}{R} \,. \tag{5.74}$$

Then, as might be expected, Φ_N naturally flows with coordinate speed C (recall that $\ell = \frac{\partial}{\partial v} + C\frac{\partial}{\partial \rho}$ so this is the speed of outgoing light relative to the coordinate system) but its strength can be augmented or diminished by interactions with the outward flux.

5.5.1 System of first order PDEs

Together (5.70) and (5.71) constitute a first-order system of partial differential equations for the scalar field. We now restructure the gravitational field equations in the same way.

First with respect to Φ_ℓ and Φ_N the constraint equations (5.14)-(5.16) on constant ρ surfaces become:

$$R_{,v} = R_\ell - CR_N \tag{5.75}$$

$$R_{\ell,v} = \kappa R_\ell + \frac{C\left(1 + 2R_\ell R_N\right)}{2R} - \frac{\Phi_\ell^2}{R} \tag{5.76}$$

$$R_{N,v} = -\kappa R_N - \frac{\left(1 + 2R_\ell R_N\right)}{2R} + \frac{C\Phi_N^2}{R} \tag{5.77}$$

while the “time”-evolution equations (5.17)-(5.19) are:

$$R_{,\rho\rho} = -\frac{\Phi_N^2}{R} \tag{5.78}$$

$$\left(RR_\ell\right)_{,\rho} = -\frac{1}{2} \tag{5.79}$$

$$C_{,\rho\rho} = \frac{1+2R_\ell R_N}{R^2} - \frac{2\Phi_\ell \Phi_N}{R^2}\,. \tag{5.80}$$

Two of these equations can be simplified. First, integrating (5.79) from $\rho = 0$ on which $R_\ell \stackrel{H}{=} 0$ we find

$$R_\ell = -\frac{\rho}{2R}\,. \tag{5.81}$$

This can be substituted into (5.76) to turn it into an algebraic constraint

$$C = 2\Phi_\ell^2 - 2R_\ell\left(\kappa R + R_\ell\right)\,. \tag{5.82}$$

Despite these simplifications, the presence of interacting outward and inward matter fluxes means that in contrast to the dust examples, this is truly a set of coupled partial differential equations. Hence we can expect that the matter and spacetime dynamics will be governed by off-horizon data in addition to data at $\rho = 0$.

We reformulate as a system of first order PDEs in the following way. First designate

$$\{R, R_N, \kappa, \Phi_\ell, \Phi_N\} \tag{5.83}$$

as the *primary variables*. The *secondary variables* $\{R_\ell, C\}$ are defined by (5.81) and (5.82) in terms of the primaries.

Next on $\rho =$ constant surfaces the primary variables are constrained by

$$R_{,v} = R_\ell - CR_N \;\;\text{and} \tag{5.84}$$

$$R_{N,v} = -\kappa R_N - \frac{1}{2R}\left(1 + 2R_\ell R_N - 2C\Phi_N^2\right) \tag{5.85}$$

along with scalar flux equation (5.71) while their time evolution is governed by

$$R_{,\rho} = R_N \tag{5.86}$$

$$R_{N,\rho} = -\frac{\Phi_N^2}{R} \tag{5.87}$$

$$\kappa_{,\rho} = \frac{1}{R^2}\left(1 + 2R_\ell R_N - 2\Phi_\ell \Phi_N\right) \tag{5.88}$$

$$\Phi_{\ell,\rho} = -\frac{R_\ell \Phi_N}{R}\,. \tag{5.89}$$

We now consider how all of these equations may be used to integrate final data. The scheme is closely related to that used in [21].

5.5.2 Final data on $\bar{H}$ and $\bar{\mathcal{N}}$

In line with the depiction in FIG.5.2, we specify final data on $H \cup \mathcal{N}$ or rather on the sections $\bar{H} \cup \bar{\mathcal{N}}$ where

$$\bar{H} = \{(0,v) \in H : v_i \leq v \leq v_f\} \ \text{ and} \tag{5.90}$$

$$\bar{\mathcal{N}} = \{(\rho, v_f) \in \mathcal{N} : \rho_i \leq \rho \leq 0\}\,.$$

Their intersection sphere is $\bar{H} \cap \bar{\mathcal{N}} = (0, v_f)$. Here and in what follows we suppress the angular coordinates. The final data are

$$\bar{H} : \ \Phi_\ell \tag{5.91}$$

$$\bar{\mathcal{N}} : \ \Phi_N \ \text{ and}$$

$$\bar{H} \cap \bar{\mathcal{N}} : \ R = R_o\,.$$

Φ_ℓ on $\bar{H}$ is a function of v while Φ_N on $\bar{\mathcal{N}}$ is a function of ρ. R_o is a single number.

Further on H we have

$$R_\ell \stackrel{H}{=} 0 \ \text{ and } \ R_N \stackrel{H}{=} -1 \tag{5.92}$$

where the null vectors are scaled in the usual way and, as before, the notation $\stackrel{H}{=}$ indicates that all quantities on both sides of the equality are evaluated on H.

These data can be used to evaluate C and R on $\bar{H}$. From (5.82) and (5.84)

$$C \stackrel{H}{=} 2\Phi_\ell^2 \ \text{ and} \tag{5.93}$$

$$R \stackrel{H}{=} R_o + 2\int_{v_f}^{v} \Phi_\ell^2 \, \mathrm{d}v \, . \tag{5.94}$$

To find Φ_N on $\bar{H}$ we would need to solve

$$\Phi_{N,v} + \frac{1}{2R}\left(1 - 4\Phi_\ell^2\Phi_N^2\right)\Phi_N \stackrel{H}{=} -2\Phi_\ell^2\Phi_{N,\rho} + \frac{\Phi_\ell}{R} \tag{5.95}$$

which comes from (5.71) combined with the above results. However at this stage $\Phi_{N,\rho}$ isn't known and so this can only be solved directly in the isolated $\Phi_\ell \stackrel{H}{=} 0$ case. There

$$\Phi_N^{\text{iso}} \stackrel{H}{=} \Phi_{N_f} e^{-(v-v_f)/2R_o} \tag{5.96}$$

where $\Phi_{N_f} = \Phi_N(0, v_f)$. Equivalently (see Appendix 5.8) Φ_N is affinely constant on an isolated horizon.

With $R_N = -1$, (5.77) tells us that

$$\kappa \stackrel{H}{=} \frac{1}{2R}\left(1 - 2C\Phi_N^2\right) \, , \tag{5.97}$$

and so without Φ_N on $\bar{H}$ we also can't determine this (away from isolation). However the corner $\bar{H} \cap \bar{\mathcal{N}}$ is an exception to that rule. There we know Φ_ℓ, Φ_N and R_o and so

$$\kappa \stackrel{\bar{H}\cap\bar{\mathcal{N}}}{=} \frac{1}{2R_o}\left(1 - 4\Phi_\ell^2\Phi_N^2\right) \, . \tag{5.98}$$

The situation is less complicated on $\bar{\mathcal{N}}$. There with Φ_N as known data and final values known for all quantities on $\bar{H} \cap \bar{\mathcal{N}}$ all other quantities can be calculated in order

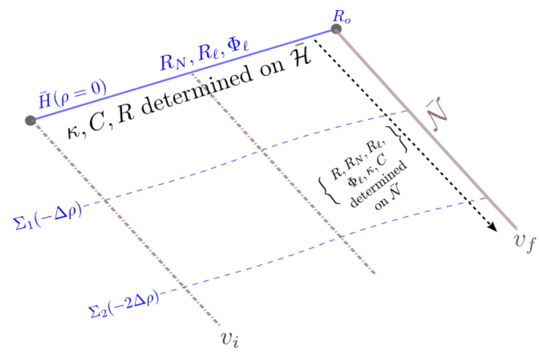


Figure 5.6: The constraint equations along with initial conditions on the horizon, i.e. $R_\ell \stackrel{H}{=} 0, R_N \stackrel{H}{=} -1$ determine κ, C and R on $\bar{H}$

i) Solve (5.86) and (5.87) for R and R_N.

ii) Calculate R_ℓ from (5.81).

iii) Solve (5.89) for Φ_ℓ.

iv) Solve (5.88) for κ.

v) Calculate C from (5.82).

We then have all data on $\bar{\mathcal{N}}$.

5.5.3 Integrating from the final data

We now consider how the data can be integrated into the causal past of $\bar{H} \cup \bar{\mathcal{N}}$. The basic steps in the integration scheme are demonstrated in a simple numerical integration based on Euler approximations. This scheme alternates between using steps i))-v)) to integrate data down the characteristics of constant v followed by an application of (5.71) to calculate Φ_N on the next characteristic.

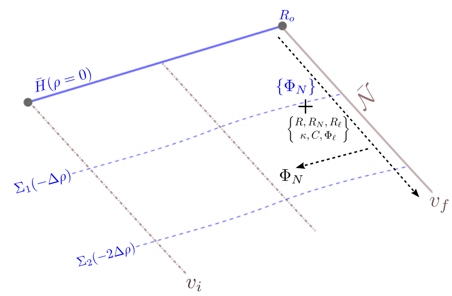


Figure 5.7: Evolving Φ in the $-\frac{\partial}{\partial v}$ direction

In more detail, assume a discretization $\{v_m, \rho_n\}$ (with m and n at their maxima along the final surfaces) by steps Δv and $\Delta\rho$. Then if all data are known along a surface v_{m+1} and R and Φ_ℓ are known everywhere on $\bar{H}$:

a) Use the knowledge of Φ_N on v_{m+1} to calculate $\Phi_{N,\rho}$.

b) Use (5.71) at (v_m, ρ_n) to find $\Phi_{N,v}$. Then

$$\Phi_N(v_m, \rho_n) \approx \Phi_N(v_{m+1}, \rho_n) - \Phi_{N,v}(v_{m+1}, \rho_n)\Delta v \tag{5.99}$$

c) Apply (5.97) to calculate κ at $(v_m, 0)$.

d) Use (5.86)-(5.89) to integrate the values of $R_{N,\rho}$, $\kappa_{,\rho}$ and $\Phi_{\ell,\rho}$ out along the $v = v_m$ characteristic as for the initial data.

This can then be repeated marching all the way along $\bar{H}$ as shown in FIG.5.7.

This is how we would proceed for general cases. However those general studies will be left for a future paper. Here instead we will focus on spacetime near a slowly evolving horizon. There, as will be seen in the next section, $\Phi_{N,\rho}$ is negligible and it becomes possible to integrate along surfaces of constant v.

It may not be immediately obvious how this integration scheme obeys causality and what restricts it to determining points inside the domain of dependence. This is briefly discussed in Appendix 5.7.

5.5.4 Spacetime near a slowly evolving horizon

We now apply the formalism to a concrete example: weak scalar fields near the horizon. Physically the black hole will be close to equilibrium and hence the horizon slowly evolving in the sense of [151, 165].

"Near horizon" means that we expand all quantities as Taylor series in ρ and keep terms up to order ρ^2. "Weak scalar field" means that we assume

$$\Phi_N, \Phi_\ell \sim \frac{\varepsilon}{R} \tag{5.100}$$

and then expand the terms of the Taylor series up to order ϵ^2. To order ϵ^0 the spacetime will be vacuum (and Schwarzschild), order ϵ^1 will be a test scalar field propagating on the Schwarzschild background and order ϵ^2 will include the back reaction of the scalar field on the geometry.

5.5.4.1 Expanding the equations

We expand all quantities as Taylor series in ρ. That is for $X \in \{R, R_N, R_\ell, \kappa, C, \Phi_\ell, \Phi_N\}$

$$X(v, \rho) = \sum_{n=0}^{\infty} \frac{\rho^n X^{(n)}(v)}{n!} \tag{5.101}$$

with

$$R_N^{(n)} = R^{(n+1)} \text{ and } \kappa^{(n)} = C^{(n+1)} . \tag{5.102}$$

The free final data are $\Phi_\ell^{(0)}$ on $\bar{H}$, R_o on $\bar{H} \cap \bar{\mathcal{N}}$ and the Taylor expanded

$$\Phi_{N_f}(\rho) = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} \Phi_{N_f}^{(n)} \tag{5.103}$$

on $\bar{\mathcal{N}}$. Following [167] we give names to special cases of this free data:

i) *out-modes*: no flux through $\bar{H}$ ($\Phi_\ell^{(0)} = 0$),
non-zero flux through $\bar{\mathcal{N}}$ ($\Phi_N^{(n)} \neq 0$ for some n)

ii) *down-modes*: non-zero flux through $\bar{H}$ ($\Phi_\ell^{(0)} \neq 0$),
zero flux through $\bar{\mathcal{N}}$ ($\Phi_N^{(n)} = 0$ for all n)

From the free data we construct the rest of the final data on $\bar{H}$. Equations (5.93) and (5.97) give

$$C^{(0)} = 2{\Phi_\ell^{(0)}}^2 \tag{5.104}$$

$$C^{(1)} = \kappa^{(0)} \approx \frac{1}{2R^{(0)}} \,. \tag{5.105}$$

Here and in what follows the $\approx$ indicates that terms of order ϵ^3 or higher have been dropped. Further by our gauge choice

$$R_N^{(0)} = R^{(1)} = -1 \tag{5.106}$$

and so from (5.94)

$$R^{(0)} = R_o + \int_{v_f}^{v} C^{(0)} \, \mathrm{d}v \,. \tag{5.107}$$

This is an order ϵ^2 correction as long as the interval of integration is small relative to $1/\epsilon$.

The last piece of final data on $\bar{H}$ is $\Phi_N^{(0)}$ and comes from the first order differential equation (5.95)

$$\frac{\mathrm{d}\Phi_N^{(0)}}{\mathrm{d}v} + \frac{\Phi_N^{(0)}}{2R_o} \approx \frac{\Phi_\ell^{(0)}}{R_o} \tag{5.108}$$

which has the solution

$$\Phi_N^{(0)} = \Phi_{N_f}^{(0)} e^{(v_f - v)/2R_o} + e^{-v/2R_o} \int_{v_f}^{v} e^{\tilde{v}/2R_o} \Phi_\ell^{(0)} \mathrm{d}\tilde{v} \tag{5.109}$$

in which the free data $\Phi_{N_f}^{(0)}$ came in as a boundary condition. Note that scalar fields that start small on the boundaries remain small in the interior, again as long as the integration time is short compared to $1/\epsilon$. We assume that this is the case.

From the final data, the black hole is close to equilibrium and the horizon is slowly evolving to order ϵ^2. That is, the expansion parameter [151, 165]:

$$C\left(\frac{1}{2}\theta_{(N)}^2 + G_{ab}N^aN^b\right) \approx \left(\frac{4\Phi_\ell^2}{R^2}\right) \sim \frac{4\epsilon^2}{R^2} \,. \tag{5.110}$$

Further we already have the first order expansion of C:

$$C \approx 2{\Phi_\ell^{(0)}}^2 + \frac{\rho}{2R_o} \,. \tag{5.111}$$

That is (to first order) there is a null surface at

$$\rho_{\text{EHC}} \approx -4R_o{\Phi_\ell^{(0)}}^2 \,. \tag{5.112}$$

This null surface is the event horizon candidate discussed in [164]: if the horizon remains slowly evolving throughout its future evolution and ultimately transitions to isolation then the event horizon candidate is the event horizon.

Moving off the horizon to calculate up to second order in ρ^2, from (5.86) and (5.87) we find

$$R_N^{(1)} = R^{(2)} \approx -\frac{{\Phi_N^{(0)}}^2}{R_o} \tag{5.113}$$

$$R_N^{(2)} \approx -\frac{\Phi_N^{(0)}\left(\Phi_N^{(0)} + 2R_o\Phi_N^{(1)}\right)}{R_o^2} \tag{5.114}$$

and so from (5.81)

$$R_\ell^{(0)} = 0 \tag{5.115}$$

$$R_\ell^{(1)} = -\frac{1}{2R^{(0)}} \tag{5.116}$$

$$R_\ell^{(2)} = -\frac{1}{{R^{(0)}}^2}\,. \tag{5.117}$$

Note that the last two terms will include terms of order ϵ^2 once the (5.107) integration is done to calculate $R^{(0)}$.

From (5.89) we can rewrite $\Phi_\ell^{(n)}$ terms with respect to $\Phi_N^{(n)}$ ones:

$$\Phi_\ell^{(1)} = 0 \tag{5.118}$$

$$\Phi_\ell^{(2)} \approx \frac{\Phi_N^{(0)}}{2R_o^2}\,. \tag{5.119}$$

The vanishing linear-order term reflects the fact that close to the horizon (where $R_\ell = 0$) the inward flux decouples from the outward (5.89) and so freely propagates into the black hole. Physically this means that (to first order in ρ near the horizon) the horizon flux is approximately equal to the "near-horizon" flux.

Next, from (5.88)

$$\kappa^{(1)} = C^{(2)} \approx \frac{1}{{R^{(0)}}^2} - \frac{2\Phi_\ell^{(0)}\Phi_N^{(0)}}{R_o^2} \text{ and} \tag{5.120}$$

$$\kappa^{(2)} \approx \frac{3}{{R^{(0)}}^2} - \frac{2\Phi_\ell^{(0)}\left(2\Phi_N^{(0)} + R_o\Phi_N^{(1)}\right)}{R_o^2}\,. \tag{5.121}$$

Again keep in mind that the $R^{(0)}$ terms will be corrected to order ϵ^2 from (5.107).

Finally these quantities may be substituted into (5.71) to get differential equations for the $\Phi_N^{(n)}$:

$$\frac{\mathrm{d}\Phi_N^{(1)}}{\mathrm{d}v} + \frac{\Phi_N^{(1)}}{R_o} \approx \frac{\Phi_\ell^{(0)}}{R_o^2} - \frac{\Phi_N^{(0)}}{R_o^2} \tag{5.122}$$

$$\frac{\mathrm{d}\Phi_N^{(2)}}{\mathrm{d}v} + \frac{3\Phi_N^{(2)}}{2R_o} \approx \frac{2\Phi_\ell^{(0)}}{R_o^3} - \frac{5\Phi_N^{(0)}}{2R_o^3} - \frac{3\Phi_N^{(1)}}{R_o^2} \,. \tag{5.123}$$

Like (5.109) these are easily solved with an integrating factor and respectively have $\Phi_{N_f}^{(1)}$ and $\Phi_{N_f}^{(2)}$ as boundary conditions.

Note the important simplification in this regime that enables these straightforward solutions. The fact that $R_\ell \sim \rho$ has raised the ρ-order of the $\Phi_{N,\rho}$ terms. As a result we can integrate directly across the ρ = constant surfaces rather than having to pause at each step to first calculate the ρ-derivative. The $\Phi_{N_f}^{(n)}$ are final data for these equations. They can be solved order-by-order and then substituted back into the other expressions to reconstruct the near-horizon spacetime.

It is also important that the matter and geometry equations decompose cleanly in orders of ϵ: we can solve the matter equations at order ϵ relative to a fixed background geometry and then use those results to solve for the corrections to the geometry at order ϵ^2.

5.5.4.2 Constant inward flux

We now consider the concrete example of an affinely constant flux through $\bar{H}$ along with an analytic flux through $\bar{\mathcal{N}}$. Then by Appendix 5.8

$$\Phi_\ell^{(0)} = \Phi_{\ell_f}^{(0)} e^V \,, \tag{5.124}$$

where $\Phi_{\ell_f}^{(0)}$ is the value of $\Phi_\ell^{(0)}$ at v_f and $V = \frac{v-v_f}{2R_o}$ while Φ_{N_f} retains its form from (5.103).

We solve the equations for this data up to second order in ρ and ϵ. First for $\Phi_N^{(n)}$ equations we find:

$$\Phi_N^{(0)} \approx \left(e^V - e^{-V}\right)\Phi_{\ell_f}^{(0)} + e^{-V}\Phi_{N_f}^{(0)} \tag{5.125}$$

$$\Phi_N^{(1)} \approx \frac{2\Phi_{\ell_f}^{(0)}}{R_o}\left(1 - e^{-2V}\right) + \frac{2\Phi_{N_f}^{(0)}}{R_o}\left(e^{-2V} - e^{-V}\right) + \Phi_{N_f}^{(1)} e^{-2V} \tag{5.126}$$

$$\begin{aligned}\Phi_N^{(2)} \approx &-\frac{\Phi_{\ell_f}^{(0)}}{4R_o^2}\left(e^V + 14e^{-V} - 48e^{-2V} + 33e^{-3V}\right)\\ &+ \frac{\Phi_{N_f}^{(0)}}{2R_o^2}\left(7e^{-V} - 24e^{-2V} + 17e^{-3V}\right)\\ &+ \frac{6\Phi_{N_f}^{(1)}}{R_o}\left(e^{-3V} - e^{-2V}\right) + \Phi_{N_f}^{(2)} e^{-3V}\end{aligned} \tag{5.127}$$

and so

$$\Phi_\ell^{(0)} = e^V \Phi_{\ell_f}^{(0)} \tag{5.128}$$

$$\Phi_\ell^{(1)} = 0 \tag{5.129}$$

$$\Phi_\ell^{(2)} \approx \frac{\Phi_{\ell_f}^{(0)}}{2R_o^2}\left(e^V - e^{-V}\right) + \frac{\Phi_{N_f}^{(0)}}{2R_o^2} e^{-V}\,. \tag{5.130}$$

The scalar field equations are linear and so it is not surprising that to this order in ϵ each solution can be thought of as a linear combination of down and out modes.

However for the geometry at order ϵ^2, down and out modes no longer combine in a linear way. These quantities can be found simply by substituting the $\Phi_\ell^{(n)}$ and $\Phi_N^{(n)}$ into the expression for $R^{(n)}$, $R_N^{(n)}$, $R_\ell^{(n)}$, $C^{(n)}$ and $\kappa^{(n)}$ given in the last section. They are corrected at order ϵ^2 by flux terms that are quadratic in combinations of $\Phi_{\ell_f}^{(m)}$

and $\Phi^{(n)}_{N_f}$. The terms are somewhat messy and the details not especially enlightening. Hence we do not write them out explicitly here.

5.5.4.3 $\bar{H} - \bar{\mathcal{N}}$ correlations

From the preceding sections it is clear that there does not need to be any correlation between the scalar field flux crossing $\bar{H}$ and that crossing $\bar{\mathcal{N}}$. These fluxes are actually free data. Any correlations will result from appropriate initial configurations of the fields. In this final example we consider a physically interesting case where such a correlation exists.

Consider quadratic affine final data (Appendix 5.8) on $\bar{H} = \{(v, 0) : v_i < v < v_f\}$:

$$\Phi^{(0)}_{\ell} = a_0 e^{V} + a_1 e^{2V} + a_2 e^{3V} \tag{5.131}$$

for $V = {}^{v-v_f}/_{2R_o}$ along with similarly quadratic affine data on $\bar{\mathcal{N}}$:

$$\Phi_{N_f} = \Phi^{(0)}_{N_f} + \rho \Phi^{(1)}_{N_f} + \frac{\rho^2}{2} \Phi^{(2)}_{N_f}. \tag{5.132}$$

A priori these are uncorrelated but let us restrict the initial configuration so that $\Phi^{(n)}_{N}(v_i) = 0$. That is, there is no Φ_N flux through $v = v_i$.

Then the process to apply these conditions is, given the free final data on $\bar{H}$:

i) Solve for the $\Phi^{(n)}_{N}$ from (5.108), (5.122) and (5.123).

ii) Solve $\Phi^{(n)}_{N}(v_i) = 0$ to find the $\Phi^{(n)}_{N_f}$ in terms of the a_n. These are linear equations and so the solution is straightforward.

iii) Substitute the resulting expressions for $\Phi^{(n)}_{N}$ into results from the previous sections to find all other quantities.

These calculations are straightforward but quite messy. Here we only present the final results for Φ_{N_f}:

$$\Phi^{(0)}_{N_f} \approx (1 - e^{2V_i})a_0 + \frac{2a_1(1 - e^{3V_i})}{3} + \frac{a_2(1 - e^{4V_i})}{2} \tag{5.133}$$

$$\Phi^{(1)}_{N_f} \approx \frac{2a_0(e^{2V_i} - e^{3V_i})}{R_o} + \frac{a_1(1 + 8e^{3V_i} - 9e^{4V_i})}{6R_o} + \frac{a_2(1 + 5e^{4V_i} - 6e^{5V_i})}{5R_o} \tag{5.134}$$

$$\Phi^{(2)}_{N_f} \approx -\frac{a_0(1 + 14e^{2V_i} - 48e^{3V_i} + 33e^{4V_i})}{4R_o^2} - \frac{a_1(1 + 35e^{3V_i} - 135e^{4V_i} + 99e^{5V_i})}{15R_o^2} + \frac{a_2(1 - 35e^{4V_i} + 144e^{5V_i} - 110e^{6V_i})}{20R_o^2} \tag{5.135}$$

where $V_i = V(v_i)$. If V_i is sufficiently negative that we can neglect the exponential terms:

$$\begin{aligned} \Phi^{(0)}_{N_f} &\approx a_0 + \frac{2a_1}{3} + \frac{a_2}{2} \\ \Phi^{(1)}_{N_f} &\approx \frac{a_1}{6R_o} + \frac{a_2}{5R_o} \\ \Phi^{(2)}_{N_f} &\approx -\frac{a_0}{4R_o^2} + -\frac{a_1}{15R_o^2} + \frac{a_2}{20R_o^2} \ . \end{aligned} \tag{5.136}$$

In either case the flux through $\bar{H}$ fully determines the flux through $\bar{\mathcal{N}}$. The constraint at v_i is sufficient to determine the Taylor expansion of the flux through $\bar{\mathcal{N}}$ relative to the expansion of the flux through $\bar{H}$. Though we only did this to second order in ρ/v we expect the same process to fix the expansions to arbitrary order.

5.6 Discussion

In this paper we have begun building a formalism that constructs spacetimes in the causal past of a horizon $\bar{H}$ and an intersecting ingoing null surface $\bar{\mathcal{N}}$ using final data on those surfaces. It can be thought of as a specialized characteristic initial value formulation and is particularly closely related to that developed in [22]. Our main interest has been to use the formalism to better understand the relationship between horizon dynamics and off-horizon fluxes. So far we have restricted our attention to spherical symmetry and so included matter fields to drive the dynamics.

One of the features of characteristic initial value problems is that they isolate free data that may be specified on each of the initial surfaces. Hence it is no surprise that the corresponding data in our formalism are also free and uncorrelated. We considered two types of data: inward flowing null matter and massless scalar fields.

For the inward-flowing null matter, data on the horizon actually determines the entire spacetime running backwards along the ingoing null geodesics that cross $\bar{H}$. Physically this makes sense. This is the only flow of matter and so there is nothing else to contribute to the dynamics.

More interesting are the massless scalar field spacetimes. In that case, matter can flow both inwards and outwards and further inward moving radiation can scatter outwards and vice versa. For the weak field near-horizon regime that we studied most closely, the free final data is the scalar field flux through $\bar{H}$ and $\bar{\mathcal{N}}$ along with the value of R at their intersection. Hence, as noted, these fluxes are uncorrelated. However we also considered the case where there was no initial flux of scalar field travelling "up" the horizon. In this case the coefficients of the Taylor expansion of the inward flux on $\bar{H}$ fully determined those on $\bar{\mathcal{N}}$ (though in a fairly complicated way). This constraint is physically reasonable: one would expect the dominant matter fields close to a black hole horizon to be infalling as opposed to travelling (almost) parallel to the horizon. It is hard to imagine a mechanism for generating strong parallel fluxes.

While we have so far worked in spherical symmetry the current work still suggests ways to think about the horizon-$\mathscr{I}^+$ correlation problem for general spacetimes. For a dynamic non-spherical vacuum spacetime, gravitational wave fluxes will be the analogue of the scalar field fluxes of this paper and almost certainly they will also be free data. Then any correlations will necessarily result from special initial configurations. However as in our example these may not need to be very exotic. It may be sufficient to eliminate strong outward-travelling near horizon fluxes. In future works we will examine these more general cases in detail.

Acknowledgements

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5.7 Appendix A - Causal past of $\bar{H} \cup \bar{\mathcal{N}}$

In this appendix we consider how the general integration scheme for the scalar field spacetimes of Section 5.5 "knows" how to stay within the past domain of dependence of $\bar{H} \cup \bar{\mathcal{N}}$.

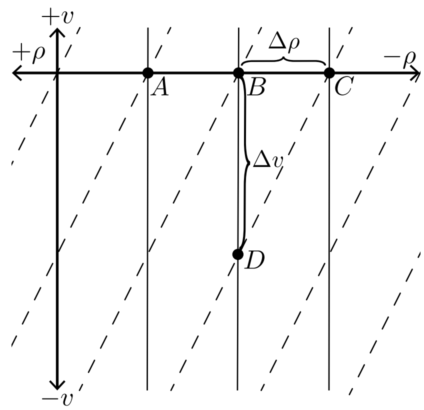

Figure 5.8: Causality restrictions on Δv: the CFL condition restricts the choice of Δv to ensure that attempted numerical evolutions respect causality. In this figure the ρ and v coordinates are drawn to be perpendicular to clarify the connection with the usual advection equation: to compare to other diagrams rotate about 45° clockwise and skew so coordinate curves are no longer perpendicular. The dashed lines are null and have slope C in this coordinate system. If data at points A, B and C are used to determine $\Phi_{N,\rho}$ then the size of the discrete v-evolution is limited to lie inside the null line from point C. The largest Δv allowed by the restriction evolves to D.

First, it is clear how the process develops spacetime up to the bottom left-hand null boundary ($v = v_i$) of the past domain of dependence. The bottom right-hand boundary is a little more complicated but follows from the advection form of the $\Phi_{N,v}$ equation (5.74). Details will depend on the exact numerical scheme but the general picture is as follows.

Assume that we have discretized the problem so that we are working at points (v_j, ρ_k). Then in using (5.74) to move from a surface v_i to v_{i-1}, the Courant-Friedrichs-Lewy (CFL) condition (common to many hyperbolic equations) tells us that the maximum allowed Δv is

$$\Delta v < \frac{\Delta\rho}{C}\,, \tag{5.137}$$

where $\Delta\rho$ is the coordinate separation of the points that we are using to calculate the right-hand side of (5.74).

Then, as shown in FIG. 5.8, the discretization progressively loses points of the bottom right of the diagram: they are outside of the domain of dependence of the individual points being used to determine them. For example if we are using a centred derivative so that

$$\Phi_{N,\rho} \approx \frac{\Phi_N(v_j, \rho_{k+1}) - \Phi_N(v_j, \rho_{k-1})}{2\Delta\rho} \tag{5.138}$$

then we need adjacent points as shown in FIG. 5.8.

The lower-right causal boundary of FIG. 5.1 and FIG. 5.2 is then enforced by a combination of the endpoints of $\bar{\mathcal{N}}$ and the CFL condition as shown in FIG. 5.9. Points are progressively lost as they require greater than the maximum allowed Δv. The numerical past-domain of dependence necessarily lies inside the analytic domain. The coarseness of the discretization in the figure dramatizes the effect: a finer discretization would keep the domains closer.

5.8 Appendix B - Affine derivatives and final data

The off-horizon ρ-coordinate in our coordinate system is affine while v is not. However, as seen in the main text, when considering the final data on $\bar{H}$ it is more natural to work relative to an affine parameter. This is somewhat complicated because Φ_ℓ and Φ_N are respectively linearly dependent on ℓ and N and the scaling of those vectors is

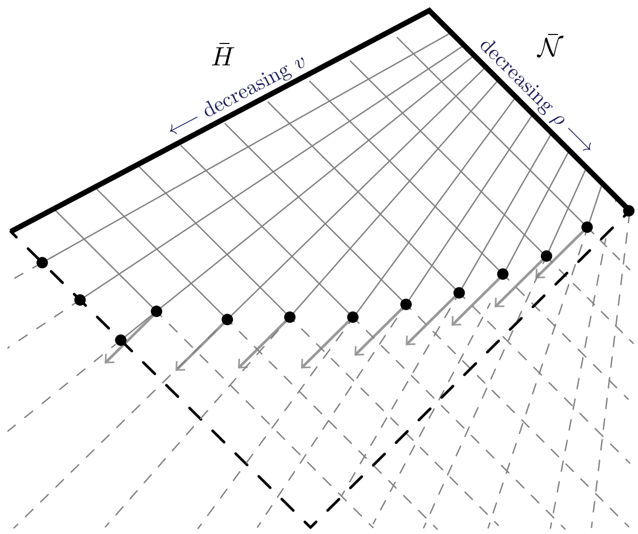

Figure 5.9: A cartoon showing the CFL-limited past domain of dependence of $\bar{H} \cup \bar{\mathcal{N}}$. Null lines are now drawn at 45° so the analytic past domain of dependence is bound by the heavy dashed null lines running back from the ends of $\bar{H}$ and $\bar{\mathcal{N}}$. A (very coarse) discretization is depicted by the gray lines and the region that cannot be determined with dashed lines. The boundary points of that region are heavy dots.

also tied to coordinates via (5.1), (5.2) and (5.6). In this appendix we will discuss the affine parameterization of the horizon and the associated affine derivatives for various quantities.

Restricting our attention to an isolated horizon $\bar{H}$ with $\kappa = \frac{1}{2R_o}$, consider a reparameterization

$$\tilde{v} = \tilde{v}(v)\,. \tag{5.139}$$

Then

$$\frac{\partial}{\partial v} = \frac{\mathrm{d}\tilde{v}}{\mathrm{d}v}\frac{\partial}{\partial \tilde{v}} \tag{5.140}$$

and so

$$\ell = e^V \tilde{\ell} \text{ and } N = e^{-V}\tilde{N} \tag{5.141}$$

where we have defined V so that $e^V = \dfrac{\mathrm{d}\tilde{v}}{\mathrm{d}v}$. Hence

$$\tilde{\kappa} = -\tilde{N}_b \tilde{\ell}^a \nabla_a \tilde{\ell}^b = e^{-V}\left(\kappa - \frac{\mathrm{d}V}{\mathrm{d}v}\right) \tag{5.142}$$

and so for an affine parameterization ($\kappa = \partial_v V$):

$$e^V = \exp\left(\frac{v - v_f}{2R_o}\right) \tag{5.143}$$

for some v_f and

$$\tilde{v} - \tilde{v}_o = 2R_o e^V \tag{5.144}$$

for some $\tilde{v}_o$. The v_f freedom corresponds to the freedom to rescale an affine parameterization by a constant multiple while the $\tilde{v}_o$ is the freedom to set the zero of $\tilde{v}$ wherever you like.

Now consider derivatives with respect to this affine parameter. For a regular scalar field

$$\frac{\mathrm{d}f}{\mathrm{d}\tilde{v}} = e^{-V}\frac{\mathrm{d}f}{\mathrm{d}v}\,. \tag{5.145}$$

However in this paper we are often interested in scalar quantities that are defined with respect to the null vectors:

$$\Phi^{(0)}_{\ell} = e^V \Phi^{(0)}_{\tilde{\ell}} \text{ and } \Phi^{(0)}_{N} = e^{-V}\Phi^{(0)}_{\tilde{N}}\,. \tag{5.146}$$

Then

$$\frac{\mathrm{d}\Phi^{(0)}_{\tilde{\ell}}}{\mathrm{d}\tilde{v}} = e^{-V}\frac{\mathrm{d}}{\mathrm{d}v}\left(e^{-V}\Phi^{(0)}_{\ell}\right) = e^{-2V}\left(\frac{\mathrm{d}\Phi^{(0)}_{\ell}}{\mathrm{d}v} - \kappa\Phi^{(0)}_{\ell}\right) \tag{5.147}$$

$$\frac{\mathrm{d}\Phi^{(0)}_{\tilde{N}}}{\mathrm{d}\tilde{v}} = e^{-V}\frac{\mathrm{d}}{\mathrm{d}v}\left(e^{V}\Phi^{(0)}_{N}\right) = \frac{\mathrm{d}\Phi^{(0)}_{N}}{\mathrm{d}v} + \kappa\Phi^{(0)}_{N}\,. \tag{5.148}$$

That is these quantities are affinely constant if

$$\Phi_\ell = e^V \Phi^{(0)}_{\ell_f} \text{ and } \Phi_N = e^{-V} \Phi^{(0)}_{N_f} \tag{5.149}$$

for some constants $\Phi^{(0)}_{\ell_f}$ and $\Phi^{(0)}_{N_f}$.

In the main text we write this affine derivative on $\bar{H}$ as D_v with its exact form depending on the ℓ or N dependence of the quantity being differentiated.

Finally at (5.131) we consider a Φ_ℓ that is "affinely quadratic". By this we mean that:

$$\begin{aligned} \Phi_{\tilde{\ell}} &= A_o + A_1 \tilde{v} + A_2 \tilde{v}^2 \\ &\Updownarrow \\ \Phi_\ell &= a_o e^V + a_1 e^{2V} + a_2 e^{3V}\,, \end{aligned} \tag{5.150}$$

where for simplicity we have set $\tilde{v}_o$ to zero (so that $v = 0$ is $\tilde{V} = 2R_o$) and absorbed the extra $2R_o$s into the a_n.

Chapter 6

Summary

In this thesis, we have investigated two problems which used the initial value viewpoint of GR. The first part is devoted to the study of linear wave equations in asymptotically flat gravitational solitons. In Chapter 3, we applied the first law for black hole and soliton mechanics to spacetimes with non-trivial topology. We considered three examples here - a single soliton, a supersymmetric double soliton and a dipole ring. We computed the extra terms arising from non-trivial topology and showed how they are essential for the mass and mass variation formulas to hold. For the specific case of the single soliton spacetime, we saw how spacetime regularity is essential for the first law to hold. In Chapter 4, the single soliton spacetime was investigated in more detail. A slow decay result for massless scalar waves was established here. As the wave equation for a single scalar field is a toy model for gravitational perturbations, this result suggests an instability at the nonlinear level. The main obstruction to decay here is stable trapping. A natural question to ask is whether the presence of a horizon would change the decay rate for spacetimes containing a soliton. An example of a black hole in equilibrium with a soliton was discovered in [42]. It would be interesting to study the structure of trapping here and conclude how it competes with the effects associated to a horizon.

The second part of the thesis focuses on initial value problems near the horizon. In Chapter 5, we set up a spherically symmetric formalism to study spacetime dynamics as constrained by horizon geometry. The main motivation for studying the scalar field here was to gain some intuition on gravitational waves. We are currently working on extending this formalism by removing symmetry assumptions to include gravitational waves. This formalism is a part of a larger body of work on useful formulation for investigating gravitational waves and the correlations with the horizon geometry. The well-posedness of these initial-boundary value problems is an open question which is of vital importance to numerical relativity.

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