Group Gradings on Classical Lie Superalgebras

by

© Caio De Naday Hornhardt

A thesis submitted to the School of Graduate Studies
in partial fulfillment of the requirements for the degree
of Doctor of Philosophy.

Department of Mathematics and Statistics
Memorial University

November, 2020

St. John's, Newfoundland and Labrador, Canada

Abstract

Assuming the base field is algebraically closed, we classify, up to isomorphism, gradings by arbitrary groups on non-exceptional classical simple Lie superalgebras, excluding those of type $A(1,1)$, and on finite dimensional superinvolution-simple associative superalgebras. We assume the characteristic to be 0 in the Lie case, and different from 2 in the associative case. Our approach is based on a version of Wedderburn Theorem for graded-simple associative superalgebras satisfying a descending chain condition, which allows us to classify superinvolutions using nondegenerate supersymmetric sesquilinear forms on graded modules over a graded-division superalgebra. To transfer the results from the associative case to the Lie case, we use the duality between G-gradings and $\widehat{G}$-actions for finite dimensional universal algebras.

To Ruth Roy and Dauto Kean Dos Santos,

awesome friends who exemplified the circle of life during the production of this work.

Lay summary

One of the most basic concepts in high school mathematics is *polynomials in one variable*. Each polynomial is a sum of *monomials*, which are expressions of the form ax^n, where a is a real number, x is the variable and n is a nonnegative integer. If $a \neq 0$, we say that the number n is the *degree* of the monomial. One simple but crucial fact is that a monomial of degree n times a monomial of degree m gives a monomial of degree $n+m$.

In simple terms, an *algebra*[1] is a set whose elements can be added, multiplied by numbers and multiplied with each other. For example, polynomials form an algebra, and so do $k \times k$ matrices. A *grading*[2] on an algebra is a choice of "building blocks" in it such that

1. Every element of the algebra can be expressed as a sum of these building blocks;
2. To each building block is associated a *degree*, in a way that the product of building blocks of degrees m and n is either zero or a building block of degree $m+n$.

We will not impose that the degree is an integer. It could be, for example, an element of the set $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ with addition given by $\bar{0}+\bar{0}=\bar{0}$, $\bar{0}+\bar{1}=\bar{1}$, $\bar{1}+\bar{0}=\bar{1}$ and $\bar{1}+\bar{1}=\bar{0}$. One way of thinking about $\mathbb{Z}_2$ is that $\bar{0}$ means "even" and $\bar{1}$ means "odd", so the addition rules are nothing but the usual behaviour of even and odd numbers under addition.

One example of an algebra with degrees in $\mathbb{Z}_2$ is the set of complex numbers. A complex number is a sum $a + b\mathbf{i}$ where a is a real number, b is a real number and $\mathbf{i}$ is

[1]A precise definition can be found, for example, in [Bre19, Chapter 1].
[2]We define it in the Introduction, Section 0.1.

a symbol subject to the rule $\mathbf{i}^2 = -1$. We can take as building blocks the elements of the form $a + 0\mathbf{i}$ (real numbers), to which we assign degree $\bar{0}$, and the elements of the form $0 + b\mathbf{i}$ (purely imaginary numbers), to which we assign degree $\bar{1}$. It is easy to see that this satisfies conditions 1 and 2 above.

More generally, we will consider the degrees to be elements of a *group*[3], which is a set with only one operation.

Some important algebras arising in Mathematics and Physics have a natural grading with degrees in $\mathbb{Z}_2$; they are called *superalgebras*. This work is about the *additional* gradings that we can put on certain superalgebras, namely, *classical Lie superalgebras*. These superalgebras can be modeled using matrices with the operation of *supercommutator*[4] instead of the usual product. Note that matrices, unlike polynomials, do not have canonical building blocks and degrees, but there are many natural and interesting choices that we can make. In this work we classify all possible choices.

[3]Again, we refer to [Bre19, Chapter 1] for a precise definition.
[4]Definition 0.11.

Acknowledgements

Given that this thesis is the final requirement for the highest academic degree, I first want to thank all teachers and professors I had during my long educational journey. I thank my parents, Marta De Naday and Nelson Hornhardt, for everything they have done for me. I am grateful to the tax payers of Brazil and Canada, specially of the state of São Paulo and the province of Newfoundland and Labrador, for financially supporting not only the universities I studied in, but also myself, through scholarships. I want to thank Memorial University of Newfoundland for all the financial, academic and extra-academic support, and also the Natural Sciences and Engineering Research Council of Canada for funding my research. Finally, I thank Yuri Bahturin, for suggesting the topic and working with me at the beginning of my program, and Mikhail Kochetov, for his active approach to supervising over all these years.

Statement of contribution

In this thesis we present a classification of group gradings on the classical Lie superalgebras of series A, B, C, D, P and Q, except type $A(1,1)$, over an algebraically closed field $\mathbb{F}$ of characteristic 0 (see Theorems 5.7, 5.10, 5.20, 5.26 and 5.38). To this end, we also classify group gradings on the finite dimensional superinvolution-simple associative superalgebras over an algebraically closed field of characteristic different from 2 (see Theorems 4.29, 4.55, 4.56, 4.70 and 4.74).

In previous works (together with Helen Samara Dos Santos and Mikhail Kochetov in [HSK19], and with the same coauthors and Yuri Bahturin in [BHSK17]), we already gave a complete classification of group gradings on the Lie superalgebras of series P and Q and a partial classification (Type I gradings) for series A. Also, the Ph.D. thesis [San19] of Helen Samara Dos Santos includes a classification of group gradings on the Lie superalgebras of series B. Nevertheless, the techniques developed in this work, in collaboration with Mikhail Kochetov, allow us to give a complete classification for all series A, B, C, D, P and Q in a uniform fashion.

For some important results (Theorems 3.18, 3.27 and 3.37 and Corollary 4.8), no assumptions on the base field are needed, and the conditions of finite dimensionality and (superinvolution-)simplicity of the superalgebra is weakened to the descending chain condition on graded one-side superideals and graded-(superinvolution-)simplicity.

Also, we present some known results in greater generality than found in the literature: in Section 1.3, we develop the correspondence between G-gradings and $\widehat{G}$-actions (see, for example, [EK13, Section 1.4]) for universal algebras (assuming the base field is algebraically closed of characteristic 0 and G is a finitely generated abelian group); in Section 3.6, we give a classification of finite dimensional graded-simple (rather than simple as algebras, as was assumed in [HSK19]) associative superalgebras over an algebraically closed field in terms of the group G.

Contents

Introduction

The use of group gradings in Lie theory can be traced back to 1888 (see [Kil88]), when W. Killing introduced the root space decomposition of a complex semisimple Lie algebra L, which gives us a $\mathbb{Z}^n$-grading, where n is the rank of L. Later, $\mathbb{Z}_2$-gradings appeared in the work of E. Cartan on real semisimple Lie algebras (see [Car14]). The interest in gradings increased in the 1960s, in connection with the works of J. Tits, I.L. Kantor, and M. Koecher (see [Tit62, Kan64, Koe67]). V. Kac classified gradings by cyclic groups on complex semisimple Lie algebras and used them in the theory of symmetric spaces in differential geometry in [Kac68], and later for the construction of the so called twisted loop algebras, which are fundamental for the theory of the famous affine Kac–Moody Lie algebras (see, e.g., [Kac90]). A systematic classification of group gradings on Lie algebras started with [PZ89] and remains an active area in the theory of Lie algebras and their representations.

The first appearances of Lie superalgebras were related to cohomology (see [FN56, Ger63, Ger64, MM65]), but they were independently introduced in physics, in relation to the so called supersymmetries (see [GN64, Miy68, Mic70]). They became a mainstream topic in theoretical physics with the development of string theory in the 1970s.

In the present work, our main goal is to classify, up to isomorphism, the group gradings (see Definition 0.4) on nonexceptional classical Lie superalgebras (see Subsection 0.3.2) over an algebraically closed field of characteristic 0. These Lie superalgebras are defined in terms of associative superalgebras and superinvolutions. In Chapter 1, we will give the basic definitions and tools we will use throughout the thesis. In particular, in Section 1.3, we will present a result that will allow us to transfer the classification of gradings on associative superalgebras with superinvolution to Lie superalgebras (see Theorem 1.30). Chapter 2 is devoted to graded-simple associative superalgebras, which not only will give us a model for some of the gradings on Lie superalgebras in series A and Q, but will also be the foundation for Chapters 3 and 4. In the former, we will study super-anti-automorphisms and, in particular, superinvolutions on graded-simple associative superalgebras (Theorems 3.18, 3.27 and 3.37); in the latter, we will classify gradings on superinvolution-simple superalgebras (Theorems 4.29, 4.55, 4.56, 4.70 and 4.74). Finally, in Chapter 5, we prove that, except for type $A(1,1)$, the classification of group gradings on nonexceptional classical Lie superalgebras is the same as on the finite dimensional superinvolution-simple associative superalgebras (see Corollary 5.4), and we work it out for each series of Lie superalgebras, giving explicit models for the gradings (Theorems 5.7, 5.10, 5.20, 5.26 and 5.38).

0.1 Gradings and superalgebras

The main concepts in this work are group gradings and superalgebras, so let us define these first. Gradings are usually defined for algebras, but it is useful to consider, more generally, gradings on vector spaces. All vector spaces under consideration will be over a fixed field $\mathbb{F}$.

Definition 0.1. Let G be a group. A *G-grading on a vector space V* is a direct sum decomposition

$$\Gamma : V = \bigoplus_{g\in G} V_g, \tag{0.1}$$

indexed by the elements of G. When endowed with a fixed G-grading Γ, V will be called a *G-graded vector space*.

For each $g \in G$, the subspace V_g is called the *homogeneous component of degree g*. An element $v \in V$ is said to be *homogeneous* if it belongs to a homogeneous component. Clearly, a nonzero homogeneous element v belongs to a unique component V_g and, in this case, we say that g is the *degree* of v and write $\deg v = g$. Whenever we refer to the degree of an element, we will assume that it is a nonzero homogeneous element. Given G-graded spaces V and W, we say that a linear map $\psi\colon V \to W$ is *degree-preserving* or a *homomorphism of graded vector spaces* if $\psi(V_g) \subseteq W_g$. A subspace $U \subseteq V$ is said to be a *graded subspace* if $U = \bigoplus_{g\in G}(U \cap V_g)$, i.e., if every element in U is a sum of homogeneous elements in U.

Definition 0.2. Let G be a group. A *G-grading on an algebra A* is a grading $\Gamma : A = \bigoplus_{g\in G} A_g$ on the vector space underlying A such that

$$\forall g, h \in G, \quad A_g A_h \subseteq A_{gh}.$$

When endowed with a fixed G-grading Γ, A is called a *G-graded algebra*.

A *homomorphism of G-graded algebras* is a homomorphism of algebras that is also a degree-preserving map. If V is a finite dimensional graded vector space, then the endomorphism algebra $\mathrm{End}(V)$ can be considered as a G-graded algebra by setting

$$\mathrm{End}(V)_g \coloneqq \{T \in \mathrm{End}(V) \mid \forall h \in G,\, T(V_h) \subseteq V_{gh}\}. \tag{0.2}$$

(Compare with Definitions 1.1 and 1.3). Note that $\mathrm{End}(V)$ consists of all linear maps and $\mathrm{End}(V)_e$ consists of the degree-preserving linear maps.

Definition 0.3. A *super vector space* or *superspace* is a $\mathbb{Z}_2$-graded vector space. A (general) *superalgebra* is a $\mathbb{Z}_2$-graded algebra.

It should be noted that this definition can be misleading, since it does not account for different *varieties* of superalgebras; for example, a Lie superalgebra is not simply a $\mathbb{Z}_2$-graded Lie algebra. We will expand on this in the next section.

The $\mathbb{Z}_2$-grading in Definition 0.3 will be called the *canonical $\mathbb{Z}_2$-grading*. We will index the homogeneous components of the canonical $\mathbb{Z}_2$-grading by superscripts, i.e., for a superspace V, we will write Equation (0.1) as $V = V^{\bar{0}} \oplus V^{\bar{1}}$. The elements of $V^{\bar{0}}$ are said to be *even* and the elements of $V^{\bar{1}}$ are said to be *odd*; we may use the word "*parity*" instead of "degree" and write $|v|$ instead of $\deg v$. Also, in this situation, we refer to graded subspaces as *subsuperspaces*. The subalgebras of a superalgebra that are also subsuperspaces are called *subsuperalgebras*. This special notation and nomenclature will serve to distinguish the canonical $\mathbb{Z}_2$-grading from an additional group grading we may consider on A:

Definition 0.4. Let G be a group. A *G-grading on a superalgebra $A = A^{\bar{0}} \oplus A^{\bar{1}}$* is a G-grading $\Gamma : A = \bigoplus_{g\in G} A_g$ on the underlying algebra such that Γ is compatible with the canonical $\mathbb{Z}_2$-grading, i.e., all homogeneous components A_g are subsuperspaces or, equivalently, $A^{\bar{0}}$ and $A^{\bar{1}}$ are G-graded subspaces. When endowed with a fixed grading

Γ, A is called a *G-graded superalgebra*. Two G-gradings, Γ and Γ', on a superalgebra A are said to be *isomorphic* if (A, Γ) and (A, Γ') are isomorphic as graded superalgebras.

Given a G-graded superalgebra A, we can combine the G-grading with the canonical $\mathbb{Z}_2$-grading: we set $G^\# := G \times \mathbb{Z}_2$ and define a $G^\#$-grading on the algebra underlying A by setting $A_{(g,i)} := A_g \cap A^i$, for all $g \in G$ and $i \in \mathbb{Z}_2$. Conversely, it is clear that any $G^\#$-graded algebra can be seen as a G-graded superalgebra.

0.2 Varieties of superalgebras

A *variety of algebras* is a class of algebras defined by polynomial identities or, equivalently, a class of algebras closed under subalgebras, homomorphic images and arbitrary products (see, e.g., [GZ05, Coh81]). One can define a *variety of superalgebras* in a similar fashion by considering $\mathbb{Z}_2$-graded polynomial identities (see, e.g., [Bah87, BMPZ92]).

We now define the variety of superalgebras we are mainly interested in:

Definition 0.5. A superalgebra $L = L^{\bar{0}} \oplus L^{\bar{1}}$, with product denoted by $[\cdot, \cdot]\colon L \times L \to L$, is said to be a *Lie superalgebra* if, for all nonzero homogeneous elements $a, b, c \in L$, we have[5]:

(i) $[a, b] = -(-1)^{|a||b|}[b, a]$ (*super-anti-commutativity*);

(ii) $[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]]$ (*super Jacoby identity*).

Note that if $L^{\bar{0}} = L$, then we have the usual definition of a Lie algebra.

It is easier to define the variety of associative superalgebras:

Definition 0.6. A superalgebra R is said to be an *associative superalgebra* if the algebra underlying R is associative.

The reader may be curious why we introduce signs in Definition 0.5 but not in Definition 0.6. This has to do with the so called *rule of signs*. Roughly speaking, every time two elements a and b exchange positions in a product in one of the identities that

[5] Some authors require additional conditions if $\operatorname{char} \mathbb{F} = 2$ or 3 (e.g., [BMPZ92, Subsection 1.2]).

define the variety, we have to introduce the sign $(-1)^{|a||b|}$. To make this more precise, we will use the notion of Grassmann envelope.[6]

Definition 0.7. Let V be a vector space. The *exterior* or *Grassmann superalgebra* of V is the algebra

$$\mathcal{G}(V) = \bigoplus_{k=0}^{+\infty} \bigwedge^k V,$$

with product given by $\wedge$ and $\mathbb{Z}_2$-grading given by

$$\mathcal{G}(V)^{\bar{0}} = \bigoplus_{i=0}^{+\infty} \bigwedge^{2i} V \text{ and } \mathcal{G}(V)^{\bar{1}} = \bigoplus_{i=0}^{+\infty} \bigwedge^{2i+1} V.$$

Note that, if $a, b \in \mathcal{G}(V)$ are homogeneous elements, then $a \wedge b = (-1)^{|a||b|} b \wedge a$. Superalgebras with this property are called *(super)commutative*.

Definition 0.8. Let V be a fixed vector space with a countably infinite basis. Given a superalgebra $A = A^{\bar{0}} \oplus A^{\bar{1}}$, we define the *Grassmann envelope* of A to be the algebra $(A^{\bar{0}} \otimes \mathcal{G}(V)^{\bar{0}}) \oplus (A^{\bar{1}} \otimes \mathcal{G}(V)^{\bar{1}})$. If $\mathfrak{V}$ is a class of algebras, we say that A is a $\mathfrak{V}$-*superalgebra* if the Grassmann envelope of A belongs to $\mathfrak{V}$.

One can easily see that, if $\mathfrak{V}$ is the class of all Lie (respectively associative, commutative) algebras, then the $\mathfrak{V}$-superalgebras are precisely the Lie (respectively associative, commutative) superalgebras. This approach can be used to define other varieties of superalgebras (e.g., Jordan).

Another kind of object that plays a major role in this work is associative superalgebras with superinvolution. By an *involution* on an algebra A we mean an involutive anti-automorphism, i.e., a linear map $\varphi\colon A \to A$ such that $\varphi(ab) = \varphi(b)\varphi(a)$, for all $a, b \in A$, and $\varphi^2 = \mathrm{id}_A$.

Definition 0.9. Let $A = A^{\bar{0}} \oplus A^{\bar{1}}$ be a superalgebra. We say that a bijective linear map $\varphi\colon A \to A$ is a *super-anti-automorphism* if $\varphi(A^{\bar{0}}) = A^{\bar{0}}$, $\varphi(A^{\bar{1}}) = A^{\bar{1}}$ and

$$\forall a, b \in A^{\bar{0}} \cup A^{\bar{1}}, \quad \varphi(ab) = (-1)^{|a||b|} \varphi(b)\varphi(a). \tag{0.3}$$

If, further, $\varphi^2 = \mathrm{id}_A$, we say that φ is a *superinvolution*.

[6]Another framework in which the rule of signs can be formulated is the symmetric monoidal category of super vector spaces (see, e.g., [Var04, Chapter 3]).

For example, the identity map $\mathrm{id}\colon \mathcal{G}(V) \to \mathcal{G}(V)$ is a superinvolution. For comparison with Definition 0.8, note that a parity-preserving bijective linear map $\varphi\colon A \to A$ satisfies Equation (0.3) if, and only if, the map $\varphi \otimes \mathrm{id}\colon (A^{\bar 0} \otimes \mathcal{G}(V)^{\bar 0}) \oplus (A^{\bar 1} \otimes \mathcal{G}(V)^{\bar 1}) \to (A^{\bar 0} \otimes \mathcal{G}(V)^{\bar 0}) \oplus (A^{\bar 1} \otimes \mathcal{G}(V)^{\bar 1})$ is an anti-automorphism.

Definition 0.10. Let (R, φ) be a superalgebra R endowed with a super-anti-automorphism φ. A G-grading on (R, φ) is a G-grading $\Gamma : R = \bigoplus_{g \in G} R_g$ on the superalgebra R such that $\varphi(R_g) = R_g$, for all $g \in G$.

We can always get a Lie superalgebra from an associative one by considering the supercommutator:

Definition 0.11. Let R be an associative superalgebra. We define the *supercommutator* $[\cdot,\cdot]\colon R \to R$ to be the bilinear map such that

$$\forall a, b \in R^{\bar 0} \cup R^{\bar 1}, \quad [a, b] = ab - (-1)^{|a||b|} ba.$$

The superalgebra $R^{(-)}$ is defined to be the superspace R endowed with the product $[\cdot,\cdot]$. If $\varphi\colon R \to R$ is a super-anti-automorphism, then we define

$$\mathrm{Skew}(R, \varphi) \coloneqq \{a \in R^{(-)} \mid \varphi(a) = -a\}.$$

It is straightforward to check that $R^{(-)}$ is a Lie superalgebra and that $\mathrm{Skew}(R, \varphi)$ is a subsuperalgebra of $R^{(-)}$. If G is an abelian group, then a G-grading on R is also a G-grading on $R^{(-)}$. Moreover, if (R, φ) is G-graded, then $\mathrm{Skew}(R, \varphi)$ is a graded subsuperalgebra of $R^{(-)}$.

0.3 Simple superalgebras

Definition 0.12. Let $A = A^{\bar 0} \oplus A^{\bar 1}$ be a superalgebra. A *superideal* of A is an ideal $I \subseteq A$ that is also a subsuperspace. We say that A is a *simple superalgebra* if $A \cdot A \neq 0$ and the only superideals are 0 and A.

We are mainly interested in simple Lie superalgebras in this work, but many of them are closely related to simple associative superalgebras, so we start with these. If $\mathbb{F}$ is algebraically closed, the finite dimensional simple associative superalgebras are:

- the matrix superalgebras $M(m,n)$;
- the queer superalgebras $Q(n)$.

This is a well known result, obtained in [Wal64]. We will also deduce it in Chapter 2 (Theorem 2.43) as a special case of the classification of graded-simple associative algebras. The definitions of $M(m,n)$ and $Q(n)$ are given in Subsection 0.3.1, below.

If $\mathbb{F}$ is algebraically closed and $\operatorname{char}\mathbb{F} = 0$, the simple finite dimensional Lie superalgebras (that are not Lie algebras) were classified by V. G. Kac (see [Kac77b] and [Sch79b]). They are divided into two big classes according to the action of $L^{\bar{0}}$ on $L^{\bar{1}}$ (note that, for any Lie superalgebra $L = L^{\bar{0}} \oplus L^{\bar{1}}$, $L^{\bar{0}}$ is a Lie algebra and $L^{\bar{1}}$ is an $L^{\bar{0}}$-module).

Definition 0.13. Let $L = L^{\bar{0}} \oplus L^{\bar{1}}$ be a simple Lie superalgebra.

(i) We say that L is *classical* if $L^{\bar{1}}$ is a semisimple $L^{\bar{0}}$-module.

(ii) We say that L is of *Cartan type* if $L^{\bar{1}}$ has a nonzero largest proper submodule, i.e., a proper submodule that contains all proper submodules.

Every simple finite dimensional Lie superalgebra (that is not a Lie algebra) is either classical or of Cartan type. The classical ones are, in their turn, divided as follows:

- 4 series, $A(m,n)$, $B(m,n)$, $C(n)$, $D(m,n)$, which are analogous to the corresponding series of simple Lie algebras;
- 2 series, $P(n)$ and $Q(n)$, which are called the *strange Lie superalgebras*;
- 3 exceptional cases: $F(4)$, $G(3)$ and the family $D(2,1,\alpha)$, $\alpha \in \mathbb{F} \setminus \{0,-1\}$.

The Lie superalgebras of Cartan type are analogs of the corresponding simple (infinite dimensional) Lie algebras of Cartan type, as well as the simple restricted Lie algebras of Cartan type. They are divided in the following series:

- the Witt superalgebras $W(n)$;
- the special superalgebras $S(n)$ and their deformations $\tilde{S}(n)$;

- the Hamiltonian superalgebras $H(n)$.

It is important to mention that there are restrictions on the parameters m and n above. We are going to make them explicit when we give the corresponding definitions in Subsections 0.3.2 and 0.3.3. Also note that the symbol $Q(n)$ denotes both an associative superalgebra and a Lie superalgebra; we will explicitly use the words "associative" and "Lie" if there is a chance of confusion.

To define some of the simple Lie superalgebras listed above we will need superinvolutions on simple associative superalgebras (which will be described in Subsection 0.3.1) and also the following concepts:

Definition 0.14. Let L be a Lie superalgebra. The *(super)center* of L is the superideal $Z(L) \coloneqq \{x \in L \mid [x, L] = 0\}$. The *derived superalgebra* of L is $L^{(1)} \coloneqq [L, L]$. In the case $L = R^{(-)}$ for an associative superalgebra R, we may also denote $L^{(1)}$ by $R^{(1)}$.

The notions of simplicity for superalgebras endowed with a super-anti-automorphism and/or a grading will play a major role in this work:

Definition 0.15. Let $A = A^{\bar{0}} \oplus A^{\bar{1}}$ be a superalgebra endowed with a super-anti-automorphism $\varphi\colon A \to A$. A superideal I is said to be *φ-invariant* if $\varphi(I) \subseteq I$. We say that A is *simple as a superalgebra with super-anti-automorphism* if $A \cdot A \neq 0$ and the only φ-invariant superideals in A are 0 and A. In the case φ is a superinvolution, we say that A is *superinvolution-simple.*

The classification of superinvolution-simple associative superalgebras is well known (see, e.g., [Rac98]) and will be proved in Section 4.2 as a special case of the theory we develop in this work.

Definition 0.16. Let A be a G-graded superalgebra. A *graded superideal* is a superideal that is also a G-graded subspace. We say that A is a *graded-simple superalgebra* if $A \cdot A \neq 0$ and the only graded superideals in A are 0 and A. If A is endowed with a super-anti-automorphism $\varphi\colon A \to A$, we say that A is *simple as a graded superalgebra with super-anti-automorphism* if the only φ-invariant graded superideals in A are 0 and A. In the case φ is a superinvolution, we say that A is *graded-superinvolution-simple.*

0.3.1 Simple associative superalgebras

The series $M(m,n)$

Let $m, n \geq 0$ be integers that are not both zero. The *matrix superalgebra* $M(m,n)$ is defined to be the matrix algebra $M_{m+n}(\mathbb{F})$ endowed with the following $\mathbb{Z}_2$-grading:

$$M(m,n)^{\bar{0}} := \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array} \right) \mid A \in M_m(\mathbb{F}),\ D \in M_n(\mathbb{F}) \right\},$$

$$M(m,n)^{\bar{1}} := \left\{ \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array} \right) \mid B \in M_{m\times n}(\mathbb{F}),\ D \in M_{n\times m}(\mathbb{F}) \right\}.$$

Note that this is nothing but the matrix representation of the superalgebra $\mathrm{End}(\mathbb{F}^{m|n})$, as in Equation (0.2), where $\mathbb{F}^{m|n}$ is defined to be the superspace $V = V^{\bar{0}} \oplus V^{\bar{1}}$ with $V^{\bar{0}} := \mathbb{F}^m$ and $V^{\bar{1}} := \mathbb{F}^n$.

Clearly, $M(m,n)$ is simple. We note that $M(m,n) \simeq M(m',n')$ if, and only if, $m = m'$ and $n = n'$, or $m = n'$ and $n = m'$ (see Theorem 2.43).

There is an important super-anti-automorphism on $M(m,n)$:

Definition 0.17. We define the *supertranspose* of a matrix in $M(m,n)$ to be

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)^{s\top} := \left(\begin{array}{c|c} A^\top & -C^\top \\ \hline B^\top & D^\top \end{array} \right).$$

Note that, if $\operatorname{char}\mathbb{F} \neq 2$, the map $X \mapsto X^{s\top}$ is not a superinvolution, it has order 4. We also note that some authors define supertranspose differently, by putting the negative sign in the bottom left block (see, e.g., [CW12, Subsection 1.1.2]).

If there is an element $\mathbf{i} \in \mathbb{F}$ such that $\mathbf{i}^2 = -1$, there is a variation of supertranspose that will be useful when working with $Q(n)$ and $A(n,n)$:

Definition 0.18. We define the *queer supertranspose* of a matrix in $M(m,n)$ to be

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)^{s\top_Q} := \left(\begin{array}{c|c} A^\top & \mathbf{i}C^\top \\ \hline \mathbf{i}B^\top & D^\top \end{array} \right).$$

Superinvolutions on $M(m,n)$ only exist for certain values of m and n: either $m = n$ or at least one of them is even (see, e.g., Proposition 4.19). They can be described in matrix terms (see Definition 4.18), but it is worth describing them more abstractly, in the model $M(m,n) = \mathrm{End}(\mathbb{F}^{m|n})$. To simplify notation, we will write V for $\mathbb{F}^{m|n}$, as above.

Definition 0.19. Let $\langle\cdot,\cdot\rangle\colon V \times V \to \mathbb{F}$ be a bilinear form. We say that $\langle\cdot,\cdot\rangle$ is *supersymmetric* if

$$\forall u, v \in V^{\bar{0}} \cup V^{\bar{1}}, \quad \langle u, v\rangle = (-1)^{|u||v|}\langle v, u\rangle,$$

and *super-skew-symmetric* if

$$\forall u, v \in V^{\bar{0}} \cup V^{\bar{1}}, \quad \langle u, v\rangle = -(-1)^{|u||v|}\langle v, u\rangle.$$

Also, we say that $\langle\cdot,\cdot\rangle$ is *even* if $\langle V^{\bar{0}}, V^{\bar{1}}\rangle = \langle V^{\bar{1}}, V^{\bar{0}}\rangle = 0$, *odd* if $\langle V^{\bar{0}}, V^{\bar{0}}\rangle = \langle V^{\bar{1}}, V^{\bar{1}}\rangle = 0$, and *homogeneous* if it is either even or odd. If $\langle\cdot,\cdot\rangle$ is homogeneous and nondegenerate, we define the *superadjunction* to be the unique linear map $\varphi\colon M(m,n) \to M(m,n)$ such that

$$\forall u, v \in V^{\bar{0}} \cup V^{\bar{1}}, \forall T \in M(m,n)^{\bar{0}} \cup M(m,n)^{\bar{1}}, \quad \langle T(u), v\rangle = (-1)^{|T||u|}\langle u, \varphi(T)(v)\rangle.$$

The superadjunction is always a super-anti-automorphism, and it is a superinvolution if, and only if, $\langle\cdot,\cdot\rangle$ is supersymmetric or super-skew-symmetric (see, e.g., Theorem 3.37). In fact, all super-anti-automorphisms on $M(m,n)$ arise in this way. We note that using the isomorphism $M(m,n) \simeq M(n,m)$ if necessary, we can avoid super-skew-symmetric bilinear forms altogether. This is because if we exchange the degrees of the components $V^{\bar{0}}$ and $V^{\bar{1}}$ (i.e., change $\mathbb{F}^{m|n}$ to $\mathbb{F}^{n|m}$), a super-skew-symmetric form becomes supersymmetric.

The (associative) series $Q(n)$

Let $n > 0$ be an integer. The *queer associative superalgebra* $Q(n)$ is the subsuperalgebra of $M(n,n)$ defined by

$$Q(n) := \left\{ \left(\begin{array}{c|c} A & B \\ \hline B & A \end{array} \right) \mid A, B \in M_n(\mathbb{F}) \right\}.$$

Another model of $Q(n)$ is $M_n(\mathbb{F}) \oplus u\, M_n(\mathbb{F})$, where $M_n(\mathbb{F})$ is the even part and u is an odd element commuting with all elements of $M_n(\mathbb{F})$ and satisfying $u^2 = 1$. In the first model, u corresponds to the matrix $\left(\begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array} \right)$.

While $Q(n)$ is not simple as an algebra, it is simple as a superalgebra. Also, $Q(n) \simeq Q(n')$ if, and only if, $n = n'$.

The queer supertranspose on $M(n,n)$ restricts to $Q(n)$, but, if $\operatorname{char} \mathbb{F} \neq 2$, $Q(n)$ does not admit any superinvolution (see, e.g., Corollary 4.13).

0.3.2 Classical Lie superalgebras

The series A

Let $m, n > 0$. We define the *supertrace* of a matrix in $M(m,n)$ by

$$\operatorname{str} \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) := \operatorname{tr} A - \operatorname{tr} D.$$

The *general linear Lie superalgebra* $\mathfrak{gl}(m|n)$ is defined to be $M(m,n)^{(-)}$. The *special linear Lie superalgebra* $\mathfrak{sl}(m|n)$ is the derived superalgebra of $\mathfrak{gl}(m|n)$ or, equivalently,

$$\mathfrak{sl}(m|n) := \{ T \in \mathfrak{gl}(m|n) \mid \operatorname{str} T = 0 \}.$$

If $m \neq n$ then $\mathfrak{sl}(m|n)$ is a simple Lie superalgebra. However, if $m = n$, then

$$Z(\mathfrak{sl}(m|n)) = \mathbb{F}1 = \left\{ \left(\begin{array}{c|c} \lambda I & 0 \\ \hline 0 & \lambda I \end{array} \right) \mid \lambda \in \mathbb{F} \right\}$$

is a nontrivial superideal. The quotient $\mathfrak{sl}(n|n)/\mathbb{F}1$ is a simple superalgebra if, and only if, $n > 1$.

For $m, n \geq 0$, the Lie superalgebra $A(m, n)$ is defined to be $\mathfrak{sl}(m+1\,|\,n+1)$ if $m \neq n$, and $\mathfrak{psl}(n+1\,|\,n+1) \coloneqq \mathfrak{sl}(n+1\,|\,n+1)/\mathbb{F}1$ if $m = n$.

Since $A(m, n) \simeq A(n, m)$, one may impose $m \geq n$ to avoid repetition.

The series B, C and D

The *orthosymplectic Lie superalgebra* $\mathfrak{osp}(m|n)$ is defined to be $\mathrm{Skew}(M(m, n), \varphi)$, where φ is the superadjunction with respect to an *even* nondegenerate supersymmetric bilinear form. Since we are assuming that $\mathbb{F}$ is algebraically closed and $\operatorname{char}\mathbb{F} = 0$, we do not need to specify the form: this is well defined up to isomorphism (see, e.g., Proposition 4.19).

Since $\langle V^{\bar{0}}, V^{\bar{1}} \rangle = 0$, any even nondegenerate supersymmetric bilinear form $\langle \cdot, \cdot \rangle$ restricts to a nondegenerate symmetric bilinear form on $V^{\bar{0}} = \mathbb{F}^m$ and to a nondegenerate skew-symmetric bilinear form on $V^{\bar{1}} = \mathbb{F}^n$. Hence the name "orthosymplectic": the superinvolution φ is a hybrid between orthogonal and symplectic involutions on matrix algebras.

If $m, n > 0$, the Lie superalgebra $\mathfrak{osp}(m|n)$ is simple. We define:

- $B(m, n) \coloneqq \mathfrak{osp}(2m+1\,|\,2n)$, for $m \geq 0$ and $n \geq 1$;
- $C(n) \coloneqq \mathfrak{osp}(2\,|\,2n-2)$, for $n \geq 2$;
- $D(m, n) \coloneqq \mathfrak{osp}(2m\,|\,2n)$, for $m \geq 2$ and $n \geq 1$.

Since $C(2) \simeq A(1, 0)$, one may impose $n \geq 3$ in the $C(n)$ case to avoid repetition.

The series P

The *periplectic Lie superalgebra* $\mathfrak{p}(n)$ is defined to be $\mathrm{Skew}(M(n, n), \varphi)$, where φ is the superinvolution with respect to an *odd* nondegenerate supersymmetric bilinear form.

As in the orthosymplectic case, the isomorphism class of $\mathfrak{p}(n)$ does not depend on the choice of the bilinear form, but in this case it is easier to prove and does not depend on the hypothesis that $\mathbb{F}$ is algebraically closed. Let $\langle \cdot, \cdot \rangle$ be an odd nondegenerate supersymmetric bilinear form on $V = \mathbb{F}^{m|n}$. Since $\langle V^{\bar{0}}, V^{\bar{0}} \rangle = \langle V^{\bar{1}}, V^{\bar{1}} \rangle = 0$, we must have $V^{\bar{1}} \simeq (V^{\bar{0}})^*$ via the map $v \mapsto \langle v, \cdot \rangle$, hence, in particular, $m = n$. Using this isomorphism to identify $V^{\bar{1}}$ with $(V^{\bar{0}})^*$, we see that V is isomorphic to $V^{\bar{0}} \oplus (V^{\bar{0}})^*$ endowed with the following form:

$$\forall u, v \in V^{\bar{0}}, \forall u^*, v^* \in (V^{\bar{0}})^*, \quad \langle u + u^*, v + v^* \rangle \coloneqq u^*(v) + v^*(u).$$

Using the canonical basis of $V^{\bar{0}} = \mathbb{F}^n$, we can identify $(V^{\bar{0}})^* = (\mathbb{F}^n)^*$ with $\mathbb{F}^n$ and obtain:

$$\mathfrak{p}(n) = \left\{ \left(\begin{array}{c|c} A & B \\ \hline C & -A^\top \end{array} \right) \in M(n,n)^{(-)} \mid B = B^\top \text{ and } C = -C^\top \right\}.$$

The superalgebra $\mathfrak{p}(n)$ is not simple. We define $P(n)$ to be the derived superalgebra of $\mathfrak{p}(n+1)$. In the model above, we have:

$$P(n) = \left\{ \left(\begin{array}{c|c} A & B \\ \hline C & -A^\top \end{array} \right) \in M(n+1,n+1)^{(-)} \mid \operatorname{tr} A = 0,\ B = B^\top \text{ and } C = -C^\top \right\}.$$

It is known that $P(n)$ is simple if, and only if, $n \geq 2$.

The (Lie) series Q

Let R denote the associative superalgebra $Q(n+1)$. Its derived Lie superalgebra is

$$R^{(1)} = \left\{ \left(\begin{array}{c|c} A & B \\ \hline B & A \end{array} \right) \in R^{(-)} \mid \operatorname{tr} A = \operatorname{tr} B = 0 \right\}.$$

Since $Z(R^{(1)}) = \mathbb{F}1$, we define the *queer Lie superalgebra* $Q(n)$ to be $R^{(1)}/\mathbb{F}1$. It is simple if, and only if, $n \geq 2$.

Exceptional Lie Superalgebras

Since defining $F(4)$, $G(3)$ and $D(2,1,\alpha)$ here would be a long detour, we refer the reader to [Kac77b] or [FSS00]. We do not consider them in this work except $D(2,1,\alpha)$

for $\alpha \in \{1, -\frac{1}{2}, -2\}$, which are isomorphic to $D(2,1)$.

0.3.3 Lie superalgebras of Cartan type

Lie superalgebras of Cartan type are not considered in this work either. Since their definitions do not require much background, we will briefly introduce them here.

The series W

Consider the Grassmann superalgebra $\mathcal{G}(\mathbb{F}^n)$, as in Definition 0.7. A homogeneous map $D \in \mathrm{End}(\mathcal{G}(\mathbb{F}^n))$ is said to be a *superderivation* if

$$\forall a, b \in \mathcal{G}(\mathbb{F}^n)^{\bar{0}} \cup \mathcal{G}(\mathbb{F}^n)^{\bar{1}}, \quad D(a \wedge b) = D(a) \wedge b + (-1)^{|D||a|} a \wedge D(b).$$

One can check that, if D_1 and D_2 are superderivations, then the supercommutator $[D_1, D_2]$ is also a superderivation.

We define the *Witt superalgebra* $W(n)$ to be the linear span of the superderivations in $\mathrm{End}(\mathcal{G}(\mathbb{F}^n))^{(-)}$. It is simple for $n \geq 2$, but $W(2) \simeq A(1,0) \simeq C(2)$, so we impose $n \geq 3$ in the list of simple Lie superalgebras of Cartan type.

To see what $W(n)$ is in more concrete terms, let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{F}^n$. For $i \in \{1, \ldots, n\}$, we define $\partial_i \in W(n)$ to be the unique (odd) superderivation such that

$$\forall j \in \{1, \ldots, n\}, \quad \partial_i(e_j) := \delta_{ij}.$$

It is easy to see that every element of $W(n)$ is of the form $\sum_{i=1}^{n} f_i \wedge \partial_i$, where $f_i \in \mathcal{G}(\mathbb{F}^n)$ for all $i \in \{1, \ldots, n\}$.

The series S

We define the *special superalgebra* $S(n)$ to be the subsuperalgebra of $W(n)$ given by

$$S(n) := \left\{ \sum_{i=1}^{n} f_i \wedge \partial_i \in W(n) \mid \sum_{i=1}^{n} \partial_i(f_i) = 0 \right\}.$$

$S(n)$ is simple for $n \geq 3$ but $S(3) \simeq P(2)$, so we impose $n \geq 4$.

The series $\tilde{S}$

Let n be even and fix $\omega = 1 - e_1 \wedge e_2 \wedge \ldots \wedge e_n$. We define $\tilde{S}(n)$ by

$$\tilde{S}(n) = \left\{ \sum_{i=1}^{n} f_i \wedge \partial_i \in W(n) \mid \sum_{i=1}^{n} \partial_i(\omega \wedge f_i) = 0 \right\}.$$

$\tilde{S}(n)$ is simple for $n \geq 4$.

The series H

Let us define another multiplication on $\mathcal{G}(\mathbb{F}^n)$, the *Poisson bracket*:

$$\forall f, g \in \mathcal{G}(\mathbb{F}^n)^{\bar{0}} \cup \mathcal{G}(\mathbb{F}^n)^{\bar{1}}, \quad \{f, g\} = (-1)^{|f|} \sum_{i=1}^{n} \partial_i(f) \wedge \partial_i(g).$$

This bracket turns $\mathcal{G}(\mathbb{F}^n)$ into a Lie superalgebra, which we denote by $\tilde{H}(n)$. We have $Z(\tilde{H}(n)) = \mathbb{F}1$ and, hence, define the *Hamiltonian superalgebra* $H(n)$ to be the derived superalgebra of $\tilde{H}(n)/\mathbb{F}1$.

We can see $H(n)$ as a subsuperalgebra of $W(n)$: we send every homogeneous $f \in \tilde{H}(n)$ to $(-1)^{|f|} \sum_{i=1}^{n} \partial_i(f) \wedge \partial_i$, extend this map linearly and induce an embedding $H(n) \to W(n)$.

Though the superalgebra $H(n)$ is simple for $n \geq 4$, we have $H(4) \simeq A(1,1)$, so we impose $n \geq 5$ in this case.

0.4 An overview of known results

The classification of gradings is best understood over algebraically closed fields, so in this section we assume that $\mathbb{F}$ is algebraically closed, unless stated otherwise.

In the case $\operatorname{char} \mathbb{F} = 0$, there is a bijective correspondence between gradings by a finitely generated abelian group G and actions of the character group $\widehat{G}$ by automorphisms (see Section 1.3). Using this, a classification of *fine abelian group gradings* up to *equivalence* (see Definitions 1.37 and 1.39) on the associative algebra $M_n(\mathbb{F})$ was obtained in [HPP98], in terms of *maximal abelian diagonalizible (MAD)* *subgroups* of $\mathrm{PGL}_n(\mathbb{F}) \simeq \mathrm{Aut}(M_n(\mathbb{F}))$. In [BSZ01, BZ02], G-gradings on $M_n(\mathbb{F})$ were described intrinsically, and those descriptions were extended to arbitrary characteristic in [BZ03]. Degree-preserving involutions on graded matrix algebras were described in [BZ07, BG08] and classified up to isomorphism in [BK10].

For classical simple Lie algebras, assuming $\operatorname{char} \mathbb{F} = 0$, a description of fine gradings was obtained in [HPP98], and an incomplete description of G-gradings was obtained in [BSZ05, BZ06]. The questions of equivalence and isomorphism of gradings were left open. The equivalence problem for fine gradings was solved in [Eld10]. The description of G-gradings was completed in [BK10] in any characteristic different from 2; the isomorphism problem was also solved there. It is worth mentioning that [BSZ05] introduced the idea of obtaining all gradings on non-exceptional simple Lie and Jordan algebras from associative algebras. In this work we follow the same idea for Lie superalgebras.

The classification of fine group gradings (up to equivalence) on finite dimensional simple Lie algebras has recently been completed in characteristic 0 by the efforts of many authors: see [EK13, Chapters 3–6] and the references therein, [Eld16] and [Yu16] for types E_6, E_7 and E_8; an overview can be found in [DE16].

The classification of all G-gradings (up to isomorphism) is complete for the classical simple Lie algebras and also for types G_2 and F_4, in characteristic different from 2: see [EK13, Chapters 3–6] and the references therein, also [EK15b] for type D_4. Note that, in positive characteristic, there are many more finite dimensional simple Lie algebras than in characteristic 0, including Lie algebras of Cartan type (see [Str04, Str09, Str13]). A classification of gradings in the restricted case for the Witt and special series was obtained in [BK11] (see also [EK13, Chapter 7]).

Gradings have been classified for some non-simple algebras. For example, the classification of G-gradings on semisimple algebras reduces to the classification of graded-simple algebras, and this latter, if G is abelian, can be obtained using the generalization of loop algebra construction introduced in [ABFP08] if we know the gradings by quotients of G on simple algebras (see, e.g., [CE18]). Other examples include the upper triangular matrices, considered as an associative [VZ07], Lie [KY17b] or Jordan algebra [KY17a], and certain nilpotent Lie algebras [BGR16].

The situation is more complicated if the base field is not algebraically closed. For real closed fields (for example, the field of real numbers), all group gradings on classical central simple and type G_2 Lie algebras were classified in [BKR18] and [EK18].

Abelian group gradings on finite dimensional simple associative superalgebras were described in [BS06], and those on superinvolution-simple but not simple associative superalgebras were described in [BTT09]. Both papers imposed some restrictions on characteristic and did not consider the isomorphism problem.

The $\mathbb{Z}$-gradings on classical Lie superalgebras where classified in [Kac77a]. In [Ser84], gradings by finite cyclic groups were considered and the corresponding twisted loop superalgebras were classified. Fine gradings on the exceptional Lie superalgebras were classified up to equivalence in [DEM11].

For a given graded associative or Lie algebra, there is a natural concept of graded module (see Definition 1.5). A classification of graded-simple modules over semisimple graded Lie algebras was obtained in [EK15a, EK15b, DEK17] and further studied in [EK17]. Gradings on a Lie superalgebra $L = L^{\bar{0}} \oplus L^{\bar{1}}$ can be approached by considering $L^{\bar{1}}$ as a graded module over the graded algebra $L^{\bar{0}}$. This was used in [BHSK17] for the series Q and in [HSK19] for the series P and A (only Type I gradings for the latter, see Definition 5.12). As already mentioned, here we will follow a different approach to all series of classical Lie superalgebras, namely, the reduction of the problem to a suitable associative superalgebra with superinvolution (see Section 5.1). The easiest case for this approach is series B, which was treated in [San19].

0.5 Some applications of gradings

Given the role of Lie superalgebras in Physics, gradings on Lie superalgebras have applications in this field. In a quantum system, quantum numbers are eigenvalues of operators that commute with the Hamiltonian. If the symmetries of the system are described by a Lie (super)algebra L, then a grading on L gives rise to *additive quantum numbers* (see [Jeu88, Pat89, PPS02]).

Physicists are also interested in *contractions* of Lie (super)algebras, to compare phenomena in system with different symmetries (see [IW53]). Many interesting contractions arise from gradings (see [MP91]): if $L = \bigoplus_{g \in G} L_g$ is a G-graded (super)algebra, with product denoted by $[\cdot, \cdot]$, and $\sigma\colon G \times G \to \mathbb{F}$ is any map, we define L^σ to be

G-graded (super)algebra with product determined by

$$\forall g, h \in G,\ \forall a \in L_g,\ \forall b \in L_h, \quad [a,b]^\sigma \coloneqq \sigma(g,h)\,[a,b].$$

If L is a Lie (super)algebra, G is abelian and $\sigma\colon G \times G \to \mathbb{F}^\times$ is a symmetric 2-cocycle (see Definition 4.32), then L^σ is again a Lie (super)algebra. In this case, the operation is invertible: $(L^\sigma)^{\sigma^{-1}} = L$; it is known as *cocycle twist*.

The twist with a non-symmetric 2-cocycle can be used to transform color Lie algebras into Lie superalgebras. Given an abelian group G and a skew-symmetric bicharacter $\epsilon\colon G \times G \to \mathbb{F}^\times$ (see Definition 2.31), a *color Lie algebra* with commutation factor ϵ is a G-graded algebra, with product $[\cdot,\cdot]$, such that, for all $g, h \in G$, $a \in L_g$, $b \in L_h$ and $c \in L$,

(i) $[a,b] = -\epsilon(g,h)\,[b,a]$;

(ii) $[a,[b,c]] = [[a,b],c] + \epsilon(g,h)\,[b,[a,c]]$.

Note that if G is trivial, we get a Lie algebra, and if $G = \mathbb{Z}_2$ and $\epsilon(i,j) = (-1)^{ij}$ for all $i, j \in \mathbb{Z}_2$, we get a Lie superalgebra. It was proven in [Sch79a] (for the case G is finitely generated) and in [BM99] (in general) that, for any color Lie algebra L, there exists a 2-cocycle $\sigma\colon G \times G \to \mathbb{F}^\times$ such that L^σ is a Lie superalgebra. Hence, one possible approach to classify simple color Lie algebras is using gradings on Lie superalgebras (see [BP09]).

The identities defining Lie superalgebras and color Lie algebras are examples of the so called *graded polynomial identities*, which are polynomial identities that hold for all elements of specified degrees in a graded algebra. Recently, the theory of such identities and their combinatorial characteristics have been extensively studied, especially in the associative case (see, e.g., [BD02, GZ05, AK10, KZ10, AG13, Gor13, AH14, Gor15, YK18, BD19, BY19]). Graded identities for certain gradings on $P(n)$ were recently considered in [RZ17]. A classification of all gradings on simple Lie superalgebras opens paths for further research in this area.

0.6 Future Research

As mentioned before, in this work we classify, up to isomorphism, all group gradings on the 6 series of classical simple Lie superalgebras, with the exception of type $A(1,1)$. The most difficult case turns out to be gradings on the series $A(m,n)$, which contain, in a sense, all gradings on the other 5 series (see Subsections 5.3.2 and 5.3.4). The remaining cases to complete the classification of gradings up to isomorphism on finite dimensional simple Lie superalgebras are types $A(1,1)$, $F(4)$, $G(3)$, $D(2,1,\alpha)$ and the Cartan types. Finally, the classification of fine gradings up to equivalence is known only for series B, P, Q and types $F(4)$, $G(3)$, $D(2,1,\alpha)$.

Chapter 1

Generalities on gradings

The purpose of this chapter is to introduce the basic notions and constructions involving gradings and also to fix notation and terminology.

Let G be a group. In Section 1.1, we introduce the concepts of homogeneous linear maps, elementary G-gradings on matrix (super)algebras, G-graded modules, tensor product of G-graded vector spaces and (super)algebras, and supercenter of an associative superalgebra. In Section 1.2, we define universal algebra, so we can handle gradings on different structures (such as algebras, superalgebras and superalgebras with superinvolution) in a uniform manner. In Section 1.3, under the assumptions that G is abelian, $\mathbb{F}$ is algebraically closed and $\operatorname{char}\mathbb{F} = 0$, we present the duality between G-gradings and $\widehat{G}$-actions for universal algebras. This allows us to transfer G-gradings between universal algebras with different signatures (see Definition 1.15 and Theorem 1.30); we will use this in Chapter 5 to get a classification of gradings on classical Lie superalgebras from a classification of gradings on associative superinvolution-simple superalgebras. Finally, Section 1.4 is devoted to concepts related to refinement, coarsening and fine gradings, where we cannot keep the grading group fixed. We introduce set gradings on universal algebras and define equivalence of gradings and universal grading group in this context. We warn the reader that some terms appearing in Section 1.4 are not used consistently in the literature (see discussion in [GS19, Section 2.7]); here we follow [EK13].

1.1 Basic concepts

In this section, we will expand on concepts discussed in the Introduction and present some basic constructions.

Let V be a vector space. The *support of a G-grading* $\Gamma : V = \bigoplus_{g\in G} V_g$ is the subset of G given by

$$\operatorname{supp}\Gamma \coloneqq \{g \in G \mid V_g \neq 0\}.$$

If Γ is fixed, then we may refer to $\operatorname{supp}\Gamma$ as the *support of* V and denote it by $\operatorname{supp} V$.

Definition 1.1. Let $V = \bigoplus_{g\in G} V_g$ and $W = \bigoplus_{g\in G} W_g$ be G-graded vector spaces. A linear map $T\colon V \to W$ is said to be *homogeneous of degree g* if

$$\forall h \in G, \quad T(V_h) \subseteq W_{gh}.$$

The subspace of $\operatorname{Hom}(V, W)$ consisting of all linear maps of degree g is denoted by $\operatorname{Hom}(V, W)_g$, and we define the graded vector space $\operatorname{Hom}^{\mathrm{gr}}(V, W)$ by

$$\operatorname{Hom}^{\mathrm{gr}}(V, W) \coloneqq \bigoplus_{g\in G} \operatorname{Hom}(V, W)_g.$$

In the case $V = W$, we denote $\operatorname{Hom}(V, W)_g$ by $\operatorname{End}(V)_g$ and $\operatorname{Hom}^{\mathrm{gr}}(V, W)$ by $\operatorname{End}^{\mathrm{gr}}(V)$.

If V is finite dimensional, it is easy to see that $\operatorname{Hom}^{\mathrm{gr}}(V, W) = \operatorname{Hom}(V, W)$. Also, if U is another graded vector space and $S\colon U \to V$ and $T\colon V \to W$ are homogeneous linear maps of degrees h and g, respectively, then $T \circ S$ is a homogeneous map of degree gh. In particular, $\operatorname{End}^{\mathrm{gr}}(V)$ is a G-graded algebra (compare with Equation (0.2)).

We emphasize that by a homomorphism of graded vector spaces we mean a degree preserving linear map, i.e., an element of $Hom(V, W)_e$.

The following is an easy result that will be used in Chapter 4:

Lemma 1.2. *Let V be a G-graded vector space. If $T\colon V \to V$ is a degree preserving map, then its eigenspaces are graded subspaces of V.*

Proof. Let $v \in V$ be an eigenvector with eigenvalue $\lambda \in \mathbb{F}$ and write $v = \sum_{g\in G} v_g$, where $v_g \in V_g$. On the one hand, $T(v) = \lambda v = \sum_{g\in G} \lambda v$. On the other hand, $T(v) = \sum_{g\in G} T(v_g)$. Since the sum of V_g, $g \in G$, is direct, we must have that $T(v_g) = \lambda v_g$ for all $g \in G$, and the result follows. $\square$

As we saw above, if V is a finite dimensional vector space, then any G-grading on V gives rise to a grading on the associative algebra $\mathrm{End}(V)$; gradings of this form are called *elementary*. An elementary grading can be described in matrix terms. Given a n-tuple $\gamma = (g_1, \ldots, g_n)$ of elements in G, we can define a G-grading on $\mathbb{F}^n$ by setting $\deg e_i = g_i$, where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{F}^n$. Clearly, any finite dimensional G-graded vector space is isomorphic to $\mathbb{F}^n$ endowed with such a grading. The grading on $M_n(\mathbb{F})$ induced from $\mathbb{F}^n$ is the following:

Definition 1.3. The *elementary grading defined by an n-tuple $\gamma = (g_1, \ldots, g_n) \in G^n$* on $M_n(\mathbb{F})$ is the G-grading determined by

$$E_{ij} = g_i g_j^{-1}, \quad 1 \leq i, j \leq n,$$

where E_{ij} denotes the matrix with 1 in the (i, j)-entry and 0 in every other entry.

Recall that the superspace $V = \mathbb{F}^{m|n}$ is defined by setting $V^{\bar{0}} = \mathbb{F}^m$ and $V^{\bar{1}} = \mathbb{F}^n$ (see Subsection 0.3.1). Given an m-tuple $\gamma_{\bar{0}}$ and an n-tuple $\gamma_{\bar{1}}$ of elements in G, we can consider the gradings defined as above on $\mathbb{F}^m$ and $\mathbb{F}^n$, and, hence, a grading on $\mathbb{F}^{m|n}$. The corresponding grading on $M(m, n)$ is the following:

Definition 1.4. Let $\gamma_{\bar{0}}$ be an m-tuple and $\gamma_{\bar{1}}$ be an n-tuple of elements of G. The *elementary grading defined by $\gamma_{\bar{0}}$ and $\gamma_{\bar{1}}$* on the superalgebra $M(m, n)$ is the elementary grading on its underlying algebra $M_{m+n}(\mathbb{F})$ defined by the concatenation of $\gamma_{\bar{0}}$ and $\gamma_{\bar{1}}$.

It is useful to consider gradings not only on vector (super)spaces and (super)algebras, but also on modules:

Definition 1.5. Let $R = \bigoplus_{g \in G} R_g$ be a graded associative algebra and let $V = \bigoplus_{g \in G} V_g$ be a graded vector space. If V has a structure of left R-module, we say that V is a *graded left module* if

$$\forall g, h \in G, \quad R_g \cdot V_h \subseteq V_{gh},$$

i.e., if the image of the representation $\rho\colon R \to \mathrm{End}(V)$ is in $\mathrm{End}^{\mathrm{gr}}(V)$ and ρ is degree-preserving. One can define graded right modules and graded bimodules analogously.

In the case R is a superalgebra and V is a superspace with their canonical $\mathbb{Z}_2$-gradings, we say that V is a *left R-supermodule*. A *G-graded left R-supermodule* is a $G^{\#}$-graded R-module (recall the definition of $G^{\#}$ in Section 0.1).

Example 1.6. Any graded vector space V is a graded left $\mathrm{End}^{\mathrm{gr}}(V)$-module, with the natural action. Also $\mathrm{Hom}^{\mathrm{gr}}(V,W)$ is a graded $(\mathrm{End}^{\mathrm{gr}}(W),\mathrm{End}^{\mathrm{gr}}(V))$-bimodule, with the action given by map composition.

Definition 1.7. Let $\Gamma : V = \bigoplus_{h\in G} V_h$ be a grading on a vector space V. Given an element $g \in G$, the *right shift of* Γ *by* g, denoted $\Gamma^{[g]}$, is the grading obtained by replacing every index $h \in G$ by hg, i.e., $\Gamma^{[g]} : V = \bigoplus_{h\in G} V'_h$ where $V'_h \coloneqq V_{hg^{-1}}$, for all $h \in G$. Analogously, the *left shift of* Γ *by* g is defined to be ${}^{[g]}\Gamma : V = \bigoplus_{h\in G} V''_h$ where $V''_h \coloneqq V_{g^{-1}h}$, for all $h \in G$. If Γ is fixed, we define the right (respectively, left) shift of the graded vector space V to be the underlying vector space endowed with $\Gamma^{[g]}$ (respectively, ${}^{[g]}\Gamma$) and denote it by $V^{[g]}$ (respectively, ${}^{[g]}V$).

If V is a graded left module over a graded algebra R, then $V^{[g]}$ is a graded left R-module. Furthermore, ${}^{[g]}R^{[g^{-1}]}$ is a graded algebra and ${}^{[g]}V$ is a graded left module over ${}^{[g]}R^{[g^{-1}]}$. Of course, similar statements hold for graded right modules.

We are now going to define tensor product of graded spaces:

Definition 1.8. Let $V = \bigoplus_{g\in G} V_g$ and $W = \bigoplus_{g\in G} W_g$ be G-graded vector spaces. The *tensor product* of V and W is the vector space $V \otimes W$ endowed with the G-grading $\Gamma : V \otimes W = \bigoplus_{g\in G}(V \otimes W)_g$ where

$$\forall g \in G, \quad (V \otimes W)_g = \bigoplus_{h\in G} V_h \otimes W_{h^{-1}g}.$$

If G is an abelian group and R and S are G-graded algebras, then it follows that $R \otimes S$ is a graded algebra with the usual multiplication: $(r \otimes s)(r' \otimes s') = rr' \otimes ss'$. Note that, in this case, the notion of graded bimodule can be reduced to the notion of left module, as in the ungraded case.

For superalgebras, though, following the rule of signs mentioned in Section 0.2, one often considers a different multiplication on the tensor product:

Definition 1.9. Let $R = R^{\bar 0} \oplus R^{\bar 1}$ and $S = S^{\bar 0} \oplus S^{\bar 1}$ be superalgebras. We define the superalgebra $R \underline{\otimes} S$ to be the $\mathbb{Z}_2$-graded tensor product $R \otimes S$ endowed with the multiplication determined by

$$\forall r, r' \in R^{\bar 0} \cup R^{\bar 1},\ s, s' \in S^{\bar 0} \cup S^{\bar 1}, \quad (r \otimes s)(r' \otimes s') = (-1)^{|s||r'|}\, rr' \otimes ss'.$$

If one of the superalgebras has trivial canonical $\mathbb{Z}_2$-grading, i.e., if $R = R^{\bar{0}}$ or $S = S^{\bar{0}}$, then the tensor product of superalgebras coincides with the tensor product of algebras. We will encounter this situation in Chapter 2 (see Remark 2.57).

Definition 1.10. Let R be an associative superalgebra. The *center* of R is the set

$$Z(R) = \{c \in R \mid cr = rc \text{ for all } r \in R\},$$

i.e., the center of R seen as an algebra, and the *supercenter* of R is the set $sZ(R) \coloneqq sZ(R)^{\bar{0}} \oplus sZ(R)^{\bar{1}}$, where

$$sZ(R)^i = \{c \in R^i \mid cr = (-1)^{i|r|} rc \text{ for all } r \in R^{\bar{0}} \cup R^{\bar{1}}\}, \quad i \in \mathbb{Z}_2.$$

Note that $sZ(R)$ is, by definition, a subsuperalgebra. It is easy to see (and follows from Lemma 1.13, below, by taking trivial G) that $Z(R)$ is also a subsuperalgebra.

Example 1.11. Let V be a vector space and consider the Grassmann superalgebra $\mathcal{G}(V)$ (Definition 0.7). Then $Z(\mathcal{G}(V)) = \mathcal{G}(V)^{\bar{0}}$ while $sZ(\mathcal{G}(V)) = \mathcal{G}(V)$.

Example 1.12. Consider the associative superalgebra $Q(n) = M_n(\mathbb{F}) \oplus u\, M_n(\mathbb{F})$. Then $Z(Q(n))$ is the subspace spanned by 1 and u, so $Z(Q(n)) \simeq Q(1)$, while $sZ(Q(n)) = \mathbb{F}1$.

Lemma 1.13. *Let G be an abelian group and let R be an associative G-graded superalgebra. Then the center $Z(R)$ and the supercenter $sZ(R)$ are G-graded subsuperalgebras of R.*

Proof. We consider R as a $G^{\#}$-graded algebra. Let $c \in Z(R)$ and write $c = \sum_{g \in G^{\#}} c_g$, where $c_g \in R_g$ for all $g \in G^{\#}$. For every homogeneous $r \in R$, we have

$$\Big(\sum_{g \in G^{\#}} c_g\Big) r = r \Big(\sum_{g \in G^{\#}} c_g\Big).$$

Comparing the components of degree $gh = hg$, where $h = \deg r$, we conclude that $rc_g = c_g r$ for all $g \in G^{\#}$. By linearity, $rc_g = c_g r$ for all $r \in R$, hence $c_g \in Z(R)$.

The same argument works to show that $sZ(R)^{\bar{0}}$ is graded and, with straightforward modifications, to show that $sZ(R)^{\bar{1}}$ is graded. $\square$

1.2 Gradings on universal algebras

In the Introduction, we defined different graded structures: vector spaces, algebras, superspaces, superalgebras and superalgebras endowed with a super-anti-automorphism. The language of *universal algebra* allows us to consider all these in a uniform fashion. This language will be particularly convenient in Section 1.3 to formulate Theorem 1.30, which allows us to transfer gradings between different structures. It will also be used in Section 1.4 to define fine gradings and universal grading groups in a uniform way (Definitions 1.39 and 1.40) rather than *ad hoc* for each structure.

We note that universal algebras are usually defined in the category of sets (see, e.g., [Coh81]), but we will work in the category of vector spaces over $\mathbb{F}$. This approach was used in [Raz94] and has recently been applied to gradings (and graded identities) in [BY19].

Definition 1.14. An *n-ary operation* on a vector space V is a linear map $V^{\otimes n} \to V$, where $V^{\otimes n} := \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}$.

In other words, an n-ary operation is a multilinear map $V^n \to V$. In the case $n = 0$, we will follow the convention that $V^{\otimes 0} := \mathbb{F}$. In particular, a 0-ary operation is determined by its value on $1 \in \mathbb{F}$ and, hence, 0-ary operations are constants in V.

Definition 1.15. A *signature* Ω is a set with a partition $\Omega = \bigcup_{n \geq 0} \Omega_n$. An *$\Omega$-algebra* or a *universal algebra with signature* Ω is a vector space A endowed with n-ary operations ω^A for each $\omega \in \Omega_n$, for all $n \geq 0$. A *homomorphism between Ω-algebras* A and B is a linear map such that for every $\omega \in \Omega_n$ we have

$$\forall a_1, \ldots, a_n \in A, \quad \psi(\omega^A(a_1 \otimes \cdots \otimes a_n)) = \omega^B(\psi(a_1) \otimes \cdots \otimes \psi(a_n)).$$

Notation 1.16. When dealing with a fixed Ω-algebra A, we will usually drop the superscript A in the operations ω, i.e., we will identify the signature Ω with its corresponding set of operations on A.

Example 1.17. A vector space is an Ω-algebra with $\Omega = \emptyset$.

Example 1.18. An algebra A in the usual sense, with a bilinear product $\cdot : A \otimes A \to A$, is an Ω-algebra with $\Omega = \Omega_2 = \{\cdot\}$.

Example 1.19. A unital algebra A with product $\cdot$ and unity element $1_A \in A$ is an Ω-algebra with $\Omega = \Omega_0 \cup \Omega_2$ where $\Omega_0 = \{\omega_0\}$, with $\omega_0 \colon \mathbb{F} \to A$ determined by $\omega_0(1) \coloneqq 1_A$, and $\Omega_2 = \{\cdot\}$. An algebra A with product $\cdot$ and involution $\varphi \colon A \to A$ is an Ω-algebra with $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 = \{\varphi\}$ and $\Omega_2 = \{\cdot\}$. A unital algebra A with involution φ is an Ω-algebra with $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$, where $\Omega_0 = \{\omega_0\}$, $\Omega_1 = \{\varphi\}$ and $\Omega_2 = \{\cdot\}$.

Example 1.20. A superspace $V = V^{\bar{0}} \oplus V^{\bar{1}}$ can be seen as an Ω-algebra by taking $\Omega = \Omega_1 = \{\pi_{\bar{0}}, \pi_{\bar{1}}\}$, where $\pi_{\bar{0}}, \pi_{\bar{1}} \colon V \to V$ are the projections onto the components $V^{\bar{0}}$ and $V^{\bar{1}}$, respectively. A universal algebra V with this signature is a superspace if, and only if, for all $x \in V$, we have:

(i) $\pi_{\bar{0}}(x) + \pi_{\bar{1}}(x) = x$;

(ii) $\pi_{\bar{0}}(\pi_{\bar{1}}(x)) = \pi_{\bar{1}}(\pi_{\bar{0}}(x)) = 0$.

Example 1.21. A superalgebra $A = A^{\bar{0}} \oplus A^{\bar{1}}$ is an Ω-algebra with $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_2 = \{\cdot\}$, $\Omega_1 = \{\pi_{\bar{0}}, \pi_{\bar{1}}\}$. An algebra A with this signature is a superalgebra if, and only if, we have identities (i) and (ii) as above and, for all $x, y \in A$ and $i, j \in \mathbb{Z}_2$,

(iii) $\pi_i(x) \cdot \pi_j(y) = \pi_{i+j}(\pi_i(x) \cdot \pi_j(y))$.

Example 1.22. Similarly to Example 1.19, a superalgebra $A = A^{\bar{0}} \oplus A^{\bar{1}}$ with super-involution $\varphi \colon A \to A$ is an Ω-algebra with $\Omega = \Omega_1 \cup \Omega_2$ where $\Omega_1 = \{\pi_{\bar{0}}, \pi_{\bar{1}}, \varphi\}$ and $\Omega_2 = \{\cdot\}$.

Remark 1.23. The signatures in Examples 1.20 and 1.21 can be generalized for G-graded spaces/algebras, and, if G is finite, the axioms can be stated as identities, which allows one to define these objects as varieties of algebras (see [BY19, Section 2]), but this is not the approach we are going to follow.

Definition 1.24. A *G-grading on an Ω-algebra* A is a G-grading on its underlying vector space such that, for all $\omega \in \Omega$, the operation $\omega^A \colon A^{\otimes n} \to A$ is degree preserving if we consider G-gradings on tensor products $A^{\otimes n}$ induced by the G-grading on A (see Definition 1.8). If Γ is fixed, we say that A is a *G-graded Ω-algebra.* A *homomorphism of G-graded Ω-algebras* is a degree-preserving homomorphism of the underlying Ω-algebras. Given two G-gradings Γ and Δ on a fixed Ω-algebra A, we say that Γ is *isomorphic* to Δ if $(A, \Gamma) \simeq (A, \Delta)$.

It is straightforward to check that, for each of the Examples 1.17 to 1.22, the notions of homomorphism and G-grading as Ω-algebras coincide with the notions of homomorphism and G-grading we had before. Note that Definition 1.24 entails that, in a unital algebra, the unity element is homogeneous of degree e, but this is automatic according to Definition 0.2: if we write $1 = \sum_{g\in G} a_g$, with $a_g \in A_g$, then, for any $h \in G$, $a_h = a_h 1 = \sum_{g\in G} a_h a_g \in A_h$ and, since $a_h a_g \in A_{hg}$, the only nonzero element in this sum $a_h a_e$. It follows that $a_h = a_h a_e$ and, hence, $1 = \sum_{g\in G} a_g = \sum_{g\in G} a_g a_e = 1 a_e = a_e \in A_e$.

1.3 G-gradings and $\widehat{G}$-actions

In this section, G will be assumed to be an *abelian* group. We will introduce an important tool in the theory of gradings: the duality between G-gradings and $\widehat{G}$-actions. For the purposes of this work, it will be of special importance in Chapter 5, where we will use Theorem 1.30 to transfer our classification of G-gradings on superinvolution-simple superalgebras (achieved in Chapter 4) to classical Lie superalgebras.

Here $\widehat{G}$ denotes the *group of characters* of G, i.e., $\widehat{G}$ is the group whose elements are the group homomorphisms $\chi\colon G \to \mathbb{F}^\times$, with point-wise multiplication of maps. We will also assume that $\mathbb{F}$ is algebraically closed and $\operatorname{char}\mathbb{F} = 0$, since this is the only case we need in this work. The reader interested in arbitrary fields can refer to [EK13].

Definition 1.25. Given a G-grading $\Gamma : V = \bigoplus_{g\in G} V_g$ on a vector space V, we define a $\widehat{G}$-action by

$$\forall \chi \in \widehat{G},\, g \in G,\, v_g \in V_g, \quad \chi \cdot v_g \coloneqq \chi(g) v_g.$$

The corresponding representation map will be denotes by $\eta_\Gamma\colon \widehat{G} \to \mathrm{GL}(V)$.

With our assumptions on G and $\mathbb{F}$, it is well known that $\widehat{G}$ separates points, i.e., given distinct elements $g, g' \in G$, there is a character $\chi \in \widehat{G}$ such that $\chi(g) \neq \chi(g')$. Hence, we have

$$\forall g \in G, \quad V_g = \{v \in V \mid \forall\, \chi \in \widehat{G},\, \chi \cdot v = \chi(g) v\},$$

Thus, we can recover the G-grading Γ from its corresponding $\widehat{G}$-action η_Γ.

Proposition 1.26. *Let A be an Ω-algebra and Γ be a G-grading on its underlying vector space. Then Γ is a G-grading on A if, and only if, $\eta_\Gamma(\widehat{G}) \subseteq \operatorname{Aut}(A)$.*

Proof. Let $\omega \in \Omega_n$ and let $a_1, \ldots, a_n \in A$ be homogeneous elements of degrees $g_1, \ldots, g_n \in G$, respectively. Note that $a_1 \otimes \cdots \otimes a_n \in A^{\otimes n}$ has degree $g_1 \cdots g_n$.

The element $\omega^A(a_1 \otimes \cdots \otimes a_n)$ has degree $g_1 \cdots g_n$ if, and only if,

$$\forall \chi \in \widehat{G}, \quad \chi \cdot \omega^A(a_1 \otimes \cdots \otimes a_n) = \chi(g_1 \cdots g_n)\, \omega^A(a_1 \otimes \cdots \otimes a_n). \tag{1.1}$$

Since we have

$$\begin{aligned} \chi(g_1 \cdots g_n)\, \omega^A(a_1 \otimes \cdots \otimes a_n) &= \chi(g_1) \cdots \chi(g_n)\, \omega^A(a_1 \otimes \cdots \otimes a_n) \\ &= \omega^A\Big(\chi(g_1)a_1 \otimes \cdots \otimes \chi(g_n)a_n\Big) \\ &= \omega^A(\chi \cdot a_1 \otimes \cdots \otimes \chi \cdot a_n), \end{aligned}$$

Equation (1.1) holds for all homogeneous $a_1, \ldots, a_n \in A$ if, and only if, $\eta_\Gamma(\chi)$ is an automorphism for every $\chi \in \widehat{G}$. □

The following is straightforward:

Proposition 1.27. *Two G-gradings Γ and Δ on an Ω-algebra A are isomorphic if, and only if, there is an automorphism $\psi \in \operatorname{Aut}(A)$ such that $\eta_\Delta(\chi) = \psi \circ \eta_\Gamma(\chi) \circ \psi^{-1}$, for all $\chi \in \widehat{G}$.* □

If G is a finite abelian group, it is well known (see, e.g., [FH91, §1.2]) that every action by $\widehat{G}$ on a finite dimensional vector space V is *diagonalizable*, i.e., V can be written as a direct sum of subspaces in which each $\chi \in \widehat{G}$ acts as a nonzero scalar λ_χ. It follows that the map $\chi \mapsto \lambda_\chi$ is a character of $\widehat{G}$ and, by duality, there is a unique $g \in G$ such that $\lambda_\chi = \chi(g)$, for every $\chi \in \widehat{G}$. In summary, every $\widehat{G}$-action corresponds to a G-grading.

This can be extended to the case of finitely generated G by considering actions of *algebraic groups* (for a background on algebraic groups, we refer to [OV90], [Arz07] or [EK13, Appendix A]). Both $\widehat{G}$ and $\operatorname{GL}(V)$ have natural structures of algebraic groups (assuming $\dim V < \infty$) and the representation $\eta_\Gamma \colon \widehat{G} \to \operatorname{GL}(V)$ is a homomorphism of algebraic groups. The algebraic group $\widehat{G}$ is a *quasitorus*, i.e., $\widehat{G} \simeq (\mathbb{F}^\times)^n \times G_f$ where

G_f is a finite abelian group. Every algebraic representation of a quasitorus H on a finite dimensional vector space V is diagonalizable (see, e.g., [OV90, Chapter 3, §2, Theorem 3] or [Arz07, Theorem 1.6.13]), i.e., V can be written as a direct sum of subspaces in which each $h \in H$ acts as a nonzero scalar λ_h. It can be shown that the map $h \mapsto \lambda_h$ is an *algebraic* character, i.e., a homomorphism of algebraic groups $H \to \mathbb{F}^\times$. The duality can be extended to this case: for every algebraic character $\lambda\colon \widehat{G} \to \mathbb{F}^\times$, there is a unique element $g \in G$ such that $\lambda_\chi = \chi(g)$. We then have the following:

Proposition 1.28. *Let V be a finite dimensional vector space and assume G is finitely generated. Then the mapping $\Gamma \mapsto \eta_\Gamma$ is a one-to-one correspondence between G-gradings on V and homomorphisms of algebraic groups $\widehat{G} \to \mathrm{GL}(V)$.* □

For an Ω-algebra A, it is easy to see that $\mathrm{Aut}(A)$ is a (Zariski) closed subgroup of $\mathrm{GL}(A)$, hence an algebraic group. Therefore, Propositions 1.26 and 1.28 imply that the following is well defined:

Definition 1.29. Let A and B be finite dimensional universal algebras, not necessarily with the same signature, and assume G is finitely generated. Given a homomorphism of algebraic groups $\theta\colon \mathrm{Aut}(A) \to \mathrm{Aut}(B)$ and a G-grading Γ on A, we define $\theta(\Gamma)$ to be the G-grading on B corresponding to the homomorphism $\theta \circ \eta_\Gamma\colon \widehat{G} \to \mathrm{Aut}(B)$.

Using Proposition 1.27, we get that if Γ and Δ are isomorphic G-gradings on A, then $\theta(\Gamma)$ and $\theta(\Delta)$ are isomorphic G-gradings on B. Finally, we note that we can drop the hypothesis that G is finitely generated: every G-grading Γ on A can be seen as a grading by the subgroup generated by $\operatorname{supp}\Gamma$, which can be used to define $\theta(\Gamma)$. We summarize these considerations in the following:

Theorem 1.30. *Suppose $\mathbb{F}$ is an algebraically closed field of characteristic 0. Let G be an abelian group and let A and B be finite dimensional universal algebras, not necessarily with the same signature. If there is a isomorphism of algebraic groups $\mathrm{Aut}(A) \to \mathrm{Aut}(B)$, then there is a bijection between the G-gradings on A and the G-gradings on B preserving the isomorphism classes.* □

We note that this sort of transfer, between algebras with different signatures, has been used in other works, but without having the result stated formally (see, e.g., [EK13, Remark 1.40]).

1.4 Refinement, coarsening and equivalence

In this section, we will introduce some concepts that do not involve a fixed grading group. For these, it is useful to have a "group free" notion of grading:

Definition 1.31. A *set grading* Γ *on a vector space* V is a vector space decomposition indexed by elements of a set S, i.e., $\Gamma : V = \bigoplus_{s\in S} V_s$. If V is a superspace, we further impose that each component V_s is a subsuperspace. When endowed with a fixed set grading Γ, we say that V is a *set graded vector space*.

Definition 1.32. Let $\Gamma : V = \bigoplus_{s\in S} V_s$ and $\Delta : V = \bigoplus_{t\in T} V_t$ be set gradings on a vector space V. We say that Γ is a *refinement* of Δ, or that Δ is a *coarsening* of Γ, if for every $s \in S$ there is $t \in T$ such that $V_s \subseteq V_t$. If, for some $s \in S$, this inclusion is strict, we say that the refinement/coarsening is *proper*.

As in Section 1.1, we define the *support* of a set grading $\Gamma : V = \bigoplus_{s\in S} V_s$ to be the set $\operatorname{supp}\Gamma \coloneqq \{s \in S \mid V_s \neq 0\}$. Note that we can always replace S by $\operatorname{supp}\Gamma$. If Δ is a coarsening of Γ as above and $s \in \operatorname{supp}\Gamma$, then there is a unique element $t \in T$ such that $V_s \subseteq V_t$. This motivates the following:

Definition 1.33. Let V be a vector space and let $\Gamma : V = \bigoplus_{s\in S} V_s$ be set grading. Given a set T and a map $\alpha\colon S \to T$, the *coarsening of* Γ *induced by* α is the set grading

$$^{\alpha}\Gamma : V = \bigoplus_{t\in T} V_t,$$

where

$$V_t \coloneqq \bigoplus_{s\in\alpha^{-1}(t)} V_s.$$

Before defining set gradings on Ω-algebras, we need the following:

Definition 1.34. Let $V = \bigoplus_{s\in S} V_s$ and $W = \bigoplus_{t\in T} W_t$ be set graded vector spaces. A linear map $f\colon V \to W$ is said to be *graded* if for any $s \in S$, there is $t \in T$ such that $f(V_s) \subseteq W_t$.

Note that, by definition, a grading Γ is a refinement of a grading Δ on a vector space V if, and only if, the identity map seen as $(V, \Gamma) \to (V, \Delta)$ is a graded map.

Definition 1.35. Let $V = \bigoplus_{s\in S} V_s$ and $W = \bigoplus_{t\in T} W_t$ be set graded vector spaces. The *tensor product* of V and W is the the vector space $V \otimes W$ endowed with the grading

$$\Gamma : V \otimes W = \bigoplus_{(s,t)\in S\times T} V_s \otimes W_t.$$

We note that the G-grading on the tensor product of two vector spaces (Definition 1.8) is the coarsening of the set grading in Definition 1.34 induced by the map $\alpha\colon G \times G \to G$ given by $\alpha(g, h) := gh$, for all $g, h \in G$.

Definition 1.36. A *set grading on an* Ω*-algebra* A is a set grading $\Gamma : A = \bigoplus_{s\in S} A_s$ on the underlying vector space of A such that $\omega^A\colon A^{\otimes n} \to A$ is a graded linear map for all $\omega \in \Omega$.

In particular, if A is an algebra in the usual sense, a set grading on A is a vector space decomposition $\Gamma : A = \bigoplus s_{s\in S} A_s$ such that, for any $s_1, s_2 \in S$ there exists $s_3 \in S$ such that $A_{s_1} A_{s_2} \subseteq A_{s_3}$.

Definition 1.37. Let A and B be Ω-algebras endowed, respectively, with set gradings $\Gamma : A = \bigoplus_{s\in S} A_s$ and $\Delta = \bigoplus_{t\in T} A_t$. An *equivalence* $\psi\colon A \to B$ is an isomorphism of Ω-algebras such that both ψ and ψ^{-1} are graded maps. If $A = B$ and there is an equivalence $\psi\colon (A, \Gamma) \to (A, \Delta)$, we say that Γ and Δ are *equivalent gradings.*

Note that an equivalence $\psi\colon A \to B$ determines a bijection $\alpha\colon \operatorname{supp}\Gamma \to \operatorname{supp}\Delta$ by $\varphi(A_s) = B_{\alpha(s)}$.

We will now bring groups and group gradings back to the picture:

Definition 1.38. We say that a set grading Γ on an Ω-algebra A can be *realized as an (abelian) group grading* if there is an (abelian) group G and a injective map $\alpha\colon \operatorname{supp}\Gamma \to G$ such that ${}^{\alpha}\Gamma$ is a G-grading on A.

Definition 1.39. Let A be an Ω-algebra and let Γ be a set grading on A. We say that Γ is a *fine (abelian) group grading* if it can be realized as an (abelian) group grading but no proper refinement of Γ can.

When a grading can be realized as an (abelian) group grading, it can be done in many different ways. But there is a special realization that has a universal property:

Definition 1.40. Let A be an Ω-algebra and let Γ be a set grading on A. A *universal (abelian) group* of Γ is a group G together with a map $\iota\colon \operatorname{supp}\Gamma \to G$ such that ${}^{\iota}\Gamma$ is a G-grading and, for every (abelian) group G' and map $\iota'\colon \operatorname{supp}\Gamma \to G'$ such that ${}^{\iota'}\Gamma$ is a G'-grading, there is a unique group homomorphism $\alpha\colon G \to G'$ such that $\iota' = \alpha \circ \iota$, i.e., the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\iota"] \arrow[dr, "\iota'"'] & G \arrow[d, dashed, "\alpha"] \\ & G' \end{tikzcd}$$

Clearly, Γ can be realized as an (abelian) group grading if, and only if, the map ι above is injective. Also, we can construct a universal (abelian) group using generators and relations: we take $\operatorname{supp}\Gamma$ as the set of generators and, for each $n \geq 0$ and $\omega \in \Omega_n$, we consider relations $s_1 \cdots s_n = s_{n+1}$ for all $s_1, \ldots, s_n, s_{n+1} \in \operatorname{supp}\Gamma$ such that $0 \neq \omega^A(A_{s_1} \otimes \cdots \otimes A_{s_n}) \subseteq A_{s_{n+1}}$.

Remark 1.41. Let A be an Ω-algebra, G and G' be groups and $\alpha\colon G \to G'$ be a group homomorphism. If $\Gamma : A = \bigoplus_{g\in G} A_g$ is a G-grading, then it is easy to see that ${}^{\alpha}\Gamma$ is a G'-grading. We note that, by Definition 1.40, if G is the universal group of Γ, then every G'-grading that is a coarsening of Γ is obtained this way.

By means of the duality between gradings and actions outlined in Section 1.3, the fine abelian group gradings on a finite dimensional algebra A over an algebraically closed field of characteristic 0 correspond to maximal quasitori in the algebraic group $\operatorname{Aut}(A)$. Moreover, the group of algebraic characters of a maximal quasitorus is the universal abelian group of the corresponding grading.

We conclude this chapter with some comments about the two types of classification mentioned in the Introduction: fine gradings up to equivalence and G-gradings up to isomorphism. Any group grading on a finite dimensional algebra A is a coarsening of a fine group grading. So, if we have a classification of fine group gradings on A up to equivalence and know their universal groups, we can obtain any G-grading as ${}^{\alpha}\Gamma$ for some fine grading Γ and homomorphism α from the universal group of Γ to G (see Definition 1.40). However, Γ and α are not unique and in practice it is difficult to determine when two such induced G-gradings ${}^{\alpha}\Gamma$ and ${}^{\beta}\Delta$ are isomorphic.

On the other hand, if we know all G-gradings on A, for any G, we can try to determine which of them are fine and compute their universal groups. This was done for simple Lie superalgebras of series Q, P and B in [BHSK17, HSK19, San19].

Chapter 2

Graded-Simple Associative Superalgebras

In this chapter we review some results about graded-simple associative algebras and extend them to superalgebras. The resulting theory will serve as a foundation for Chapters 3 and 4. Also, in Chapter 5, gradings on the associative superalgebras $M(m,n)$ and $Q(n)$ will give us the so called Type I gradings (Definition 5.12) on the Lie superalgebras of series A and Q. Throughout this chapter, G is a fixed group and the term *algebra* means *associative algebra*.

In Section 2.1 we recall the theory of graded-simple algebras satisfying the descending chain condition (d.c.c.) on graded left ideals (Definition 2.1), following closely [EK13, Chapter 2], although with some differences in notation (see Definitions 2.18 and 2.20 and Remark 2.19) and proofs. In Subsection 2.1.1, we consider graded-division algebras (Definition 2.2) and classify their graded modules of finite rank. In Subsection 2.1.2, we reduce the classification of graded-simple algebras satisfying d.c.c. on graded left ideals to these objects: Theorem 2.23 states that every such graded algebra is isomorphic to $\mathrm{End}_{\mathcal{D}}(\mathcal{U})$, where $\mathcal{D}$ is a graded-division algebra and $\mathcal{U}$ is a graded right $\mathcal{D}$-module of finite rank, while Theorem 2.27 describes isomorphisms between these endomorphism algebras. In Subsections 2.1.3 and 2.1.4, we specialize to finite dimensional algebras over an algebraically closed field. In the former we parametrize the graded-division algebras (Proposition 2.32) and give the construction of the so called standard realizations (Definition 2.36), which are models of graded-division algebras that are simple as algebras; in the latter, we extend the parametrization to graded-simple algebras to get a classification result (Theorem 2.40), and use it to recover the well-known classifications of gradings on matrix algebras (Corollary 2.42) and of simple associative superalgebras (Theorem 2.43).

In Section 2.2, we extend this theory to graded-simple superalgebras, by seeing them as $G^{\#}$-graded algebras, where $G^{\#} = G \times \mathbb{Z}_2$ (see Section 0.1). The subsections follow the same pattern as in Section 2.1, adapting the definitions and results to the "super" setting. In Subsection 2.2.1 we define graded-division superalgebras (Definition 2.46), which can be even or odd (Definition 2.50), and parametrize their graded supermodules of finite rank in each case. In Subsection 2.2.2, we apply Theorem 2.23 to graded-simple superalgebras satisfying d.c.c. on graded left ideals; these graded superalgebras can also be even or odd depending on the graded-division superalgebra involved (Definition 2.55). In Subsection 2.2.3, we parametrize finite dimensional graded-division superalgebras over an algebraically closed field and give standard realizations of type Q (Definition 2.63), which are graded-division superalgebras that are simple as superalgebras but not as algebras. In Subsection 2.2.4, we extend this parametrization to classify even and odd graded-simple superalgebras (Theorems 2.73 and 2.74), and obtain as corollaries the classification of gradings on $M(m,n)$ (Corollaries 2.76 and 2.78) and on $Q(n)$ (Corollary 2.80).

The parameters used to classify even gradings in Section 2.2 are in terms of the group G, but the parameters used for odd gradings are in terms of the group $G^{\#}$ (except when the underlying superalgebra is $Q(n)$). In Section 2.3, we present a parametrization of odd gradings in terms of G (see Definition 2.81). Subsection 2.3.1 handles the general case (Corollary 2.86), while Subsection 2.3.2 handles the case where the underlying superalgebra is $M(m,n)$, giving necessary and sufficient conditions on the parameters for this to be the case (Lemma 2.96 and Proposition 2.99), and also simplifying the parametrization (Theorem 2.103).

We note that in [BS06], a description of gradings on $M(m,n)$ and $Q(n)$ was given, but the isomorphism problem was not considered. A classification of gradings on $M(m,n)$ was obtained in [HSK19] (see also [San19]).

2.1 Graded-simple associative algebras

In this section, we will recall the classification of gradings on matrix algebras [BSZ01, BZ02, BK10] and, more generally, that of graded-simple algebras satisfying the d.c.c. on one-sided graded ideals (see below). We will follow the exposition of [EK13, Chapter 2] but use a slightly different notation for parameters, which will be extended to superalgebras in Section 2.2.

Our main interest is in finite dimensional graded algebras. It is clear that they satisfy the following condition:

Definition 2.1. We say that a graded algebra R satisfies the *descending chain condition* (or *d.c.c.*) on graded left ideals if, for every sequence $\{I_k\}_{k\in\mathbb{N}}$ of graded left ideals such that

$$k \leq \ell \implies I_k \supseteq I_\ell,$$

there is $n \in \mathbb{N}$ such that

$$n \leq \ell \implies I_n = I_\ell.$$

As we will see in Theorem 2.23, a graded-simple algebra satisfying this condition can be described, up to isomorphism, by a graded-division algebra $\mathcal{D}$ and a graded right $\mathcal{D}$-module of finite rank. This is the graded analog of a classical result of Wedderburn.

2.1.1 Graded-division algebras and their modules

It is easy to see that if a G-graded algebra R has the unit element 1, then $1 \in R_e$. Also, if an element $r \in R_g$ is invertible, then $r^{-1} \in R_{g^{-1}}$.

Definition 2.2. A *graded-division algebra* is a unital graded algebra $\mathcal{D}$ such that every nonzero homogeneous element has an inverse. In this case, we may also refer to the G-grading on the algebra $\mathcal{D}$ as a *division grading*.

Example 2.3. The group algebra $\mathbb{F}G$ can be regarded as a graded algebra if we declare $\mathbb{F}g$ to be the homogeneous component of degree g, for all $g \in G$. It is a graded-division algebra since $(\lambda g)^{-1} = \frac{1}{\lambda} g^{-1}$, for all $0 \neq \lambda \in \mathbb{F}$.

Example 2.4. Another example of a graded-division algebra is the matrix algebra $M_2(\mathbb{F})$, $\operatorname{char}\mathbb{F} \neq 2$, equipped with the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading, sometimes called *Pauli grading*, defined by

$$\deg\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = (\bar{0},\bar{0}),\quad \deg\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = (\bar{0},\bar{1}),$$
$$\deg\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = (\bar{1},\bar{0}),\quad \deg\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = (\bar{1},\bar{1}).$$

Example 2.5. Example 2.4 can be generalized to define a $\mathbb{Z}_\ell \times \mathbb{Z}_\ell$-grading on $M_\ell(\mathbb{F})$, given that there exists a primitive ℓ^{th}-root of unity $\xi \in \mathbb{F}$. Let

$$A \coloneqq \begin{pmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & \xi & 0 & \cdots & 0\\ 0 & 0 & \xi^2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \xi^{\ell-1}\end{pmatrix} \quad\text{and}\quad B \coloneqq \begin{pmatrix}0 & 0 & 0 & \cdots & 0 & 1\\ 1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 1 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & 0\end{pmatrix}.$$

Note that $AB = \xi BA$ and $A^\ell = B^\ell = 1$. One can check that setting $\deg A = (\bar{1},\bar{0})$ and $\deg B = (\bar{0},\bar{1})$ defines a grading making $M_\ell(\mathbb{F})$ a graded-division algebra.

For what follows, let us fix a graded-division algebra $\mathcal{D}$. We note that $\mathcal{D}_e$ is a division algebra in the usual sense and that, if $0 \neq X_t \in \mathcal{D}_t$, then $\mathcal{D}_t = X_t\,\mathcal{D}_e$.

Lemma 2.6. *The support $T \coloneqq \operatorname{supp}\mathcal{D}$ is a subgroup of G.*

Proof. As noted in Section 1.2, $1 \in \mathcal{D}_e$, so $e \in T$. Given $s, t \in T$, take $0 \neq X_s \in \mathcal{D}_s$ and $0 \neq X_t \in \mathcal{D}_t$. Since X_sX_t is invertible, it is a nonzero element in $\mathcal{D}_{st}$, hence $st \in T$. Finally, write $X_t^{-1} = \sum_{g\in G} d_g$. Then $1 = X_tX_t^{-1} = \sum_{g\in G} X_td_g$ and $1 = X_t^{-1}X_t = \sum_{g\in G} d_gX_t$. Since $1 \in \mathcal{D}_e$, it follows that $1 = X_td_{t^{-1}}$ and $1 = d_{t^{-1}}X_t$, so $X_t^{-1} = d_{t^{-1}} \in \mathcal{D}_{t^{-1}}$. □

Lemma 2.7. *The group T is the universal group of the grading on $\mathcal{D}$.*

Proof. Denote the division grading on $\mathcal{D}$ by $\Gamma : \mathcal{D} = \bigoplus_{t\in T}\mathcal{D}_t$, considered as a set grading, and set $\iota\colon T \to T$ to be the identity map. Let G' be a group and let $\iota'\colon T \to G'$ be a (set-theoretic) map such that ${}^{\iota'}\Gamma$ is a G'-grading. It is straightforward that T, together with ι, satisfies the universal property in Definition 1.40 if, and only if, ι' is a group homomorphism. To prove the latter, let $0 \neq X_s \in \mathcal{D}_s$ and $0 \neq X_t \in \mathcal{D}_t$ and note that X_sX_t is a nonzero element of $\mathcal{D}_{st}$. When considering the grading ${}^{\iota'}\Gamma$, the elements X_s, X_t and X_sX_t have degrees $\iota'(s), \iota'(t), \iota'(st) \in G'$, respectively. Since ${}^{\iota'}\Gamma$ is a group grading, it also follows that X_sX_t has degree $\iota'(s)\iota'(t)$, so we conclude that $\iota'(st) = \iota'(s)\iota'(t)$, as desired. $\square$

Graded $\mathcal{D}$-modules will play an important role in this work. We will now recall their classification up to isomorphism.

Consider a graded right $\mathcal{D}$-module $\mathcal{U} = \bigoplus_{g\in G} \mathcal{U}_g$. Note that a homogeneous component $0 \neq \mathcal{U}_g$ is a $\mathcal{D}_e$-module but, unless $\mathcal{D} = \mathcal{D}_e$, it is not a $\mathcal{D}$-submodule. It is easy to see that the $\mathcal{D}$-span of $\mathcal{U}_g$ is $\mathcal{U}_{gT} := \bigoplus_{t\in T} \mathcal{U}_{gt}$.

Definition 2.8. Given a left coset $x \in G/T$, the *isotypic component* $\mathcal{U}_x$ of a graded right $\mathcal{D}$-module $\mathcal{U} = \bigoplus_{g\in G} \mathcal{U}_g$ is the $\mathcal{D}$-submodule given by:

$$\mathcal{U}_x := \bigoplus_{g\in x} \mathcal{U}_g.$$

Clearly, $\mathcal{U} = \bigoplus_{x\in G/T} \mathcal{U}_x$.

Remark 2.9. The use of the terminology "isotypic component" here is consistent with its common use. Recall that, in representation theory, an isotypic component is defined as the sum of all simple submodules of a given isomorphism type. Since $\mathcal{D}$ is a graded-division algebra, the right modules ${}^{[g]}\mathcal{D}$ are simple $\mathcal{D}$-modules. It is easy to see that all simple $\mathcal{D}$-modules are of this form. Indeed, if $\mathcal{V}$ is any graded right $\mathcal{D}$-module and $0 \neq v \in \mathcal{V}_g$, then the map ${}^{[g]}\mathcal{D} \to v\mathcal{D}$ given by $d \mapsto vd$ is an isomorphism. It is also easy to see when ${}^{[g]}\mathcal{D}$ and ${}^{[g']}\mathcal{D}$ are isomorphic. Of course, since $\operatorname{supp} {}^{[g]}\mathcal{D} = gT$ and $\operatorname{supp} {}^{[g']}\mathcal{D} = g'T$, a necessary condition for this to happen is $g'g^{-1} \in T$. Conversely, if $g'g^{-1} \in T$, then we can pick $0 \neq c \in \mathcal{D}_{g'g}$ and define an isomorphism ${}^{[g]}\mathcal{D} \to {}^{[g']}\mathcal{D}$ by $d \mapsto cd$.

Lemma 2.10. *Two graded right $\mathcal{D}$-modules $\mathcal{U}$ and $\mathcal{V}$ are isomorphic if, and only if, the isotypic component $\mathcal{U}_x$ is isomorphic to $\mathcal{V}_x$ for all $x \in G/T$.*

Proof. If $\psi\colon \mathcal{U} \to \mathcal{V}$ is a homomorphism of graded right $\mathcal{D}$-modules, then it is clear that $\psi(\mathcal{U}_x) \subseteq \mathcal{V}_x$, for all $x \in G/T$, and that ψ is determined by the maps $\psi\restriction_{\mathcal{U}_x}\colon \mathcal{U}_x \to \mathcal{V}_x$, $x \in G/T$. Moreover, ψ is an isomorphism if, and only if, each $\psi\restriction_x$ is an isomorphism. $\square$

Lemma 2.10 reduces the problem of classifying graded $\mathcal{D}$-modules up to isomorphism to classifying their isotypic components. We will now reduce the latter to (ungraded) modules over the division algebra $\mathcal{D}_e$. Note that $\mathcal{D}$ can be regarded as a left $\mathcal{D}_e$-module.

Definition 2.11. Let $\mathcal{U}$ be a graded $\mathcal{D}$-module. A $\mathcal{D}_e$-*form* of $\mathcal{U}$ is a graded $\mathcal{D}_e$-submodule $\tilde{\mathcal{U}} \subseteq \mathcal{U}$ such that the map $\tilde{\mathcal{U}} \otimes_{\mathcal{D}_e} \mathcal{D} \to \mathcal{U}$ given by $u \otimes d \mapsto ud$ is an isomorphism of graded right $\mathcal{D}$-modules. If this is the case, we will use this map to identify $\tilde{\mathcal{U}} \otimes_{\mathcal{D}_e} \mathcal{D}$ with $\mathcal{U}$.

Proposition 2.12. *Let $\mathcal{U}$ be a graded right $\mathcal{D}$-module and let $g \in G$. The homogeneous component $\mathcal{U}_g$ is a $\mathcal{D}_e$-form for the isotypic component $\mathcal{U}_{gT}$.*

Proof. Since the $\mathcal{D}$-span of $\mathcal{U}_g$ is $\mathcal{U}_{gT}$, the map $\psi\colon \mathcal{U}_g \otimes_{\mathcal{D}_e} \mathcal{D} \to \mathcal{U}$ given by $\psi(u \otimes d) = ud$ is surjective. To see that ψ is injective, pick $0 \neq X_t \in \mathcal{D}_t$ for every $t \in T$. Let $u \in \mathcal{U}_g \otimes_{\mathcal{D}_e} \mathcal{D}$. It is easy to see that $\{X_t\}_{t \in T}$ is a $\mathcal{D}_e$-basis for $\mathcal{D}$, so we can write $u = \sum_{t \in T} u^t \otimes X_t$, with $u^t \in \mathcal{U}_g$ for all $t \in T$ and $u^t = 0$ for all but finitely many $t \in T$. We then have that $\psi(u) = \sum_{t \in T} u^t X_t$ and hence, if $\psi(u) = 0$, we have $u^t X_t = 0$, for every $t \in T$. Since X_t is invertible, we conclude that $u^t = 0$ for every $t \in T$ and, therefore, $u = 0$. □

Corollary 2.13. *Let $\mathcal{U}$ and $\mathcal{V}$ be graded right $\mathcal{D}$-modules and fix $g \in G$. Then $\mathcal{U}_{gT}$ is isomorphic to $\mathcal{V}_{gT}$ if, and only if, $\mathcal{U}_g$ and $\mathcal{V}_g$ are isomorphic as $\mathcal{D}_e$-modules.* □

Since $\mathcal{D}_e$ is an (ungraded) division algebra, every right $\mathcal{D}_e$-module has a $\mathcal{D}_e$-basis. By Proposition 2.12, a $\mathcal{D}_e$-basis for $\mathcal{U}_g$ is a $\mathcal{D}$-basis for $\mathcal{U}_{gT}$. It follows that every isotypic component has a $\mathcal{D}$-basis consisting of elements of the same degree. Since $\mathcal{U}$ is the direct sum of its isotypic components, we conclude that $\mathcal{U}$ has a graded basis:

Definition 2.14. Let $\mathcal{U}$ be a graded right $\mathcal{D}$-module. A $\mathcal{D}$-basis $\mathcal{B}$ of $\mathcal{U}$ is said to be a *graded basis of* $\mathcal{U}$ if all the elements in $\mathcal{B}$ are homogeneous (of various degrees).

Remark 2.15. Alternatively, the existence of a graded basis can be proved with the same arguments as in the ungraded setting (using Zorn's Lemma).

Proposition 2.16. *Let $\mathcal{U}$ be a graded right $\mathcal{D}$-module. Let $\mathcal{B}$ be a graded basis and set $\mathcal{B}_x := \mathcal{B} \cap \mathcal{U}_x$ for any $x \in G/T$. Then the sets $\mathcal{B}_x$ form a partition of $\mathcal{B}$ and the cardinality $|\mathcal{B}_x|$ is independent of the choice of $\mathcal{B}$ and equal to* $\dim_{\mathcal{D}_e} \mathcal{U}_g$ *for any $g \in x$.*

Proof. Let $\mathcal{B}$ be any graded basis of $\mathcal{U}$ and set $\mathcal{B}_x \coloneqq \mathcal{B} \cap \mathcal{U}_x$ for all $x \in G/T$. Since every element in $\mathcal{B}$ is homogeneous, $\mathcal{B} = \bigcup_{x \in G/T} \mathcal{B}_x$, and, hence, $\mathcal{B}_x$ is a graded basis for $\mathcal{U}_x$, for all $x \in G/T$.

We claim that each $\mathcal{B}_x$ has cardinality $\dim_{\mathcal{D}_e} \mathcal{U}_g$, where g is an arbitrary representative of the coset x. Indeed, let $\mathcal{B}_x = \{u_\lambda\}_{\lambda \in \Lambda_x}$ For every $\lambda \in \Lambda_x$, choose $d_\lambda \in \mathcal{D}$ to be a nonzero element of degree $(\deg u_\lambda)^{-1} g \in T$. Then, clearly $\mathcal{B}'_x = \{u_\lambda d_\lambda\}_{\lambda \in \Lambda_x}$ is also a graded basis of $\mathcal{U}_{gT}$, of the same cardinality as $\mathcal{B}_x$, but with all elements having degree g. It is easy to see that $\mathcal{B}'_x$ is a $\mathcal{D}_e$-basis of $\mathcal{U}_g$ and, therefore, it has cardinality $\dim_{\mathcal{D}_e} \mathcal{U}_g$. $\square$

Definition 2.17. We define the *rank* or $\mathcal{D}$*-dimension* of a graded right module $\mathcal{U}$ to be the cardinality of one (and hence any) graded basis of $\mathcal{U}$, and denote it by $\dim_{\mathcal{D}}(\mathcal{U})$.

We will now restrict ourselves to graded right $\mathcal{D}$-modules of finite rank.

It is clear that a graded right $\mathcal{D}$-module $\mathcal{U}$ has finite rank if, and only if, all isotypic components have finite rank and only finitely many of them are nonzero. In view of Lemma 2.10, Corollary 2.13, and Proposition 2.16, this means that an isomorphism class of such modules is determined by the map $\kappa\colon G/T \to \mathbb{Z}_{\geq 0}$ defined by $\kappa(x) \coloneqq \dim_{\mathcal{D}}(\mathcal{U}_x)$, which has finite support. In other words, κ is a finite *multiset* in G/T, where $\kappa(x)$ is viewed as the multiplicity of the point x. As usually done for multisets, we define $|\kappa| \coloneqq \sum_{x \in \operatorname{supp} \kappa} \kappa(x)$. Clearly $|\kappa| = \dim_{\mathcal{D}}(\mathcal{U})$.

Definition 2.18. Given a map $\kappa\colon G/T \to \mathbb{Z}_{\geq 0}$ with finite support, we say that a k-tuple $\gamma = (g_1, \ldots, g_k) \in G^k$, where $k \coloneqq |\kappa|$, *realizes* κ if the number of entries g_i with $g_i \in x$ is equal to $\kappa(x)$ for every $x \in G/T$.

Clearly, we can always choose a k-tuple $\gamma = (g_1, \ldots, g_k)$ realizing κ. We can use any such γ to construct the graded $\mathcal{D}$-module $\mathcal{U} \coloneqq {}^{[g_1]}\mathcal{D} \oplus \cdots \oplus {}^{[g_k]}\mathcal{D}$, which is a representative of the isomorphism class of graded $\mathcal{D}$-modules determined by κ.

Remark 2.19. Both parameters, the map κ and the the k-tuple γ realizing it, have been used in previous works. Sometimes both are present, as in [EK13] (where the tuple is defined so that all elements belong to different left cosets and the multiplicities are recorded as a separate tuple) and [BHSK17, HSK19]; sometimes only the tuple, as in [BK10]; sometimes only the multiset, as in [BKR18, KY19]. Here we follow the notation of the latter.

A systematic way to obtain γ from κ is using a set-theoretic section $\xi\colon G/T \to G$ of the natural homomorphism $\pi\colon G \to G/T$ (i.e., a map such that $\pi(\xi(x)) = x$ for all $x \in G/T$) and a total order $\leq$ on the set G/T.

Definition 2.20. Let $\xi\colon G/T \to G$ be a set-theoretic section of the natural homomorphism $\pi\colon G \to G/T$ and let $\leq$ be a total order on G/T. Given a map $\kappa\colon G/T \to \mathbb{Z}_{\geq 0}$ with finite support, set $k \coloneqq |\kappa|$ and let γ be the k-tuple given by putting the elements of $\{\xi(x) \mid x \in \operatorname{supp}\kappa\} \subseteq G$ following the order $\leq$ and repeating $\kappa(x)$ times each element $\xi(x)$. We will call γ the *k-tuple realizing κ according to ξ and $\leq$*.

Let $\mathcal{U}$ and $\mathcal{V}$ be graded right $\mathcal{D}$-modules. We will denote by $\operatorname{Hom}_{\mathcal{D}}(\mathcal{U}, \mathcal{V})$ the vector space of all homomorphisms from $\mathcal{U}$ to $\mathcal{V}$ as $\mathcal{D}$-modules, not as graded $\mathcal{D}$-modules. In other words, the elements of $\operatorname{Hom}_{\mathcal{D}}(\mathcal{U}, \mathcal{V})$ do not necessarily preserve degrees. As usual, we also denote $\operatorname{Hom}_{\mathcal{D}}(\mathcal{U}, \mathcal{U})$ by $\operatorname{End}_{\mathcal{D}}(\mathcal{U})$.

Of course, $\operatorname{Hom}_{\mathcal{D}}(\mathcal{U}, \mathcal{V}) \subseteq \operatorname{Hom}_{\mathbb{F}}(\mathcal{U}, \mathcal{V})$. If $\dim_{\mathcal{D}}(\mathcal{U}) < \infty$, we can say more:

Proposition 2.21. *Let $\mathcal{U}$ and $\mathcal{V}$ be graded right $\mathcal{D}$-modules and suppose $\dim_{\mathcal{D}} \mathcal{U} < \infty$. Then $\operatorname{Hom}_{\mathcal{D}}(\mathcal{U}, \mathcal{V}) = \operatorname{Hom}^{\mathrm{gr}}_{\mathcal{D}}(\mathcal{U}, \mathcal{V})$ is a graded subspace of $\operatorname{Hom}^{\mathrm{gr}}_{\mathbb{F}}(\mathcal{U}, \mathcal{V})$ (see Definition 1.1).*

Proof. Let $\mathcal{B} = \{u_1, \ldots, u_\ell\}$ be a graded basis for $\mathcal{U}$ and $\mathcal{C} = \{v_i\}_{i\in I}$ be a graded basis for $\mathcal{V}$, set $g_j \coloneqq \deg u_j$ and $h_i \coloneqq \deg v_i$, for $1 \leq j \leq \ell$ and $i \in I$.

Since $\mathcal{B}$ is, in particular, a basis for $\mathcal{D}$ as a free $\mathcal{D}$-module, a $\mathcal{D}$-linear map $f\colon \mathcal{U} \to \mathcal{U}$ can be defined by its values on the elements of $\mathcal{B}$. Given $1 \leq j \leq \ell$, $i \in I$, $t \in T$ and $0 \neq d \in \mathcal{D}_t$, define $f_{i,j,d}\colon \mathcal{U} \to \mathcal{V}$ to be the $\mathcal{D}$-linear map defined by $f_{i,j,d}(u_r) = \delta_{jr} v_i d$, for every $r \in \{1, \ldots, \ell\}$. It is easy to see that $f_{i,j,d}\colon \mathcal{U} \to \mathcal{V}$ is a homogeneous element in $\operatorname{Hom}^{\mathrm{gr}}_{\mathbb{F}}(\mathcal{U}, \mathcal{V})$ with

$$\deg f_{i,j,d} = h_i t g_j^{-1}. \tag{2.1}$$

Since $\mathcal{B}$ is finite, it is clear that every map in $\operatorname{Hom}_{\mathcal{D}}(\mathcal{U}, \mathcal{V})$ is a finite sum of maps of the form $f_{i,j,d}$, concluding the proof. □

Under the conditions of Proposition 2.21, we will always consider $\operatorname{Hom}_{\mathcal{D}}(\mathcal{U}, \mathcal{V})$ as a graded vector space with the grading restricted from $\operatorname{Hom}^{\mathrm{gr}}_{\mathbb{F}}(\mathcal{U}, \mathcal{V})$. Note that, in the case $\mathcal{U} = \mathcal{V}$, this makes $\operatorname{End}_{\mathcal{D}}(\mathcal{U})$ a graded algebra and $\mathcal{U}$ a graded left $\operatorname{End}_{\mathcal{D}}(\mathcal{U})$-module.

As usual, if both $\mathcal{U}$ and $\mathcal{V}$ have finite rank over $\mathcal{D}$, we can represent the elements of $\operatorname{Hom}_{\mathcal{D}}(\mathcal{U},\mathcal{V})$ as matrices with entries in $\mathcal{D}$. More precisely, given graded bases $\mathcal{B} = \{u_1,\ldots,u_\ell\}$ and $\mathcal{C} = \{v_1,\ldots,v_k\}$ for $\mathcal{U}$ and $\mathcal{V}$, respectively, we have an isomorphism of vector spaces $\operatorname{Hom}_{\mathcal{D}}(\mathcal{U},\mathcal{V}) \to M_{k\times\ell}(\mathcal{D})$ given by $f \mapsto (d_{ij})$, where $f(u_j) = \sum_i v_i d_{ij}$. Under this isomorphism, the map $f_{i,j,d}$ defined in the proof of Proposition 2.21, corresponds to the matrix $E_{ij}(d) \in M_{k\times\ell}(\mathcal{D})$, i.e., the matrix which has d in the position (i,j) and 0 elsewhere.

Remark 2.22. Note that we can identify $M_k(\mathcal{D}) = M_k(\mathbb{F}) \otimes_{\mathbb{F}} \mathcal{D}$, with $E_{ij}(d)$ corresponding to $E_{ij} \otimes d$. In the case G is abelian, Equation (2.1) implies that the grading on $M_k(\mathbb{F}) \otimes_{\mathbb{F}} \mathcal{D}$ is the one of the tensor product of graded algebras (see Section 1.1), where $M_k(\mathbb{F})$ has an elementary grading (see Definition 1.3).

2.1.2 Graded Wedderburn theory

Let $\mathcal{U}$ be a nonzero graded right $\mathcal{D}$-module of finite rank. In this subsection we will study the endomorphism algebra $\operatorname{End}_{\mathcal{D}}(\mathcal{U})$. The main reason for this is Theorem 2.23, below. Special cases of this result appeared in several works, e.g., [BSZ01, NO04, BZ02]. Here we follow [EK13, Theorem 2.6], the converse of which also holds (see [EK13, page 31]).

Theorem 2.23. *Let G be a group and let R be a G-graded graded-simple associative algebra satisfying the d.c.c. on graded left ideals. Then there are a G-graded division algebra $\mathcal{D}$ and a nonzero graded right $\mathcal{D}$-module $\mathcal{U}$ of finite rank such that $R \simeq \operatorname{End}_{\mathcal{D}}(\mathcal{U})$ as graded algebras.* □

The next question is: when are two graded algebras $\operatorname{End}_{\mathcal{D}}(\mathcal{U})$ and $\operatorname{End}_{\mathcal{D}'}(\mathcal{U}')$ isomorphic? We will need the following two definitions:

Definition 2.24. Let $d \in \mathcal{D}$ be a nonzero homogeneous element. We define the *inner automorphism* $\operatorname{Int}_d\colon \mathcal{D} \to \mathcal{D}$ by $\operatorname{Int}_d(c) := dcd^{-1}$, for all $c \in \mathcal{D}$.

Definition 2.25. Let $\psi_0\colon \mathcal{D}' \to \mathcal{D}$ be a homomorphism of graded algebras and let $\mathcal{U}$ be a graded right $\mathcal{D}$-module. The *module induced by* ψ_0 is the graded right $\mathcal{D}'$-module $\mathcal{U}^{\psi_0}$ which is $\mathcal{U}$ as a vector space, but with $\mathcal{D}'$-action defined by $u \cdot d := u\,\psi_0(d)$, for all $u \in \mathcal{U}$ and $d \in \mathcal{D}'$. In the case $\psi_0\colon \mathcal{D} \to \mathcal{D}$ is an automorphism, $\mathcal{U}^{\psi_0}$ is again a graded right $\mathcal{D}$-module, called the *twist of* $\mathcal{U}$ *by* ψ_0.

Remark 2.26. Note that, if ψ_0 is surjective, a map $f\colon \mathcal{U}^{\psi_0} \to \mathcal{U}^{\psi_0}$ is $\mathcal{D}'$-linear if, and only if, f is $\mathcal{D}$-linear as a map $\mathcal{U} \to \mathcal{U}$. In other words, $\mathrm{End}_{\mathcal{D}'}(\mathcal{U}^{\psi_0})$ is the same set as $\mathrm{End}_{\mathcal{D}}(\mathcal{U})$. Nevertheless, the matrix representation of f can be different in each case. To be more precise, note that a subset $\mathcal{B} \subseteq \mathcal{U}$ is a graded $\mathcal{D}'$-basis of $\mathcal{U}^{\psi_0}$ if, and only if, it is a graded $\mathcal{D}$-basis of $\mathcal{U}$. Such a basis gives rise to isomorphisms $M_k(\mathcal{D}') \simeq \mathrm{End}_{\mathcal{D}'}(\mathcal{U}^{\psi_0})$ and $M_k(\mathcal{D}) \simeq \mathrm{End}_{\mathcal{D}}(\mathcal{U})$, and it is easy to see that the matrix representing f in $M_k(\mathcal{D}')$ is equal to the result of applying ψ_0 to every entry of the matrix representing f in $M_k(\mathcal{D})$.

The next result is [EK13, Theorem 2.10] with a slightly different notation:

Theorem 2.27. *Let $R \coloneqq \mathrm{End}_{\mathcal{D}}(\mathcal{U})$ and $R' \coloneqq \mathrm{End}_{\mathcal{D}'}(\mathcal{U}')$, where $\mathcal{D}$ and $\mathcal{D}'$ are graded-division algebras, and $\mathcal{U}$ and $\mathcal{U}'$ are nonzero right graded modules of finite rank over $\mathcal{D}$ and $\mathcal{D}'$, respectively. Given an isomorphism $\psi\colon R \to R'$, there is a triple (g, ψ_0, ψ_1), where $g \in G$, $\psi_0\colon {}^{[g^{-1}]}\mathcal{D}^{[g]} \to \mathcal{D}'$ is an isomorphism of graded algebras, $\psi_1\colon \mathcal{U}^{[g]} \to (\mathcal{U}')^{\psi_0}$ is an isomorphism of graded right $\mathcal{D}$-modules, such that*

$$\forall r \in R, \quad \psi(r) = \psi_1 \circ r \circ \psi_1^{-1}. \tag{2.2}$$

Conversely, given a triple (g, ψ_0, ψ_1) as above, Equation (2.2) *defines an isomorphism of graded algebras $\psi\colon R \to R'$. Another triple (g', ψ_0', ψ_1') defines the same isomorphism ψ if, and only if, there are $t \in \operatorname{supp} \mathcal{D}'$ and $0 \neq d \in \mathcal{D}'_t$ such that $g' = gt$, $\psi_0' = \mathrm{Int}_{d^{-1}} \circ \psi_0$ and $\psi_1'(u) = \psi_1(u)d$ for all $u \in \mathcal{U}$.* $\square$

For the remainder of this subsection, fix a graded-division algebra $\mathcal{D}$ and a nonzero graded right $\mathcal{D}$-module of finite rank $\mathcal{U}$, and set $R \coloneqq \mathrm{End}_{\mathcal{D}}(\mathcal{U})$. Thus, $\mathcal{U}$ is a graded $(R, \mathcal{D})$-bimodule.

First, we state here a well-known result about ungraded algebras, for future reference:

Proposition 2.28. *The (ungraded) algebra $R = \mathrm{End}_{\mathcal{D}}(\mathcal{U})$ is simple if, and only if, the (ungraded) algebra $\mathcal{D}$ is simple.* $\square$

The following result is essentially [EK13, Exercise 3 on page 60].

Lemma 2.29. *As a left R-module, $\mathcal{U}$ is graded-simple. Also, the right action of $\mathcal{D}$ on $\mathcal{U}$ gives an isomorphism $\rho\colon \mathcal{D} \to \mathrm{End}_R(\mathcal{U})$.*

Proof. By the definition of a graded module, ρ is a homomorphism of graded algebras $\mathcal{D} \to \mathrm{End}^{\mathrm{gr}}_R(\mathcal{U})$. Since $\mathcal{D}$ is graded-simple and $\rho(1) = \mathrm{id}_{\mathcal{U}} \neq 0$, ρ is injective.

Let $\{u_1, \dots, u_k\}$ be a graded $\mathcal{D}$-basis of $\mathcal{U}$. Given $u \in \mathcal{U}$, define $r \in R = \mathrm{End}_{\mathcal{D}}(\mathcal{U})$ to be the $\mathcal{D}$-linear map such that $ru_1 = u$ and $ru_i = 0$ for $1 < i \leq k$. Let $f \in \mathrm{End}_R(\mathcal{U})$ and write $u_1 f = \sum_i u_i d_i$ for $d_1, \dots, d_k \in \mathcal{D}$. Then

$$uf = (ru_1)f = r(u_1 f) = r\Big(\sum_i u_i d_i\Big) = ud_1.$$

Therefore $f = \rho(d_1)$, proving that the image of ρ is the whole $\mathrm{End}_R(\mathcal{U})$.

Finally, since any nonzero homogeneous element $u_1 \in \mathcal{U}$ can be included in a graded $\mathcal{D}$-basis $\{u_1, \dots, u_k\}$, the maps r defined as above show that $\mathcal{U} = Ru_1$ and, hence, $\mathcal{U}$ is graded-simple as an R-module. □

Recall that, if G is abelian, the center of a graded algebra is a graded subalgebra (Lemma 1.13).

Proposition 2.30. *Suppose G is abelian. Then the map $\iota\colon Z(\mathcal{D}) \to Z(R)$ given by $\iota(d)(u) := ud$, for all $d \in Z(\mathcal{D})$ and $u \in \mathcal{U}$, is an isomorphism of graded algebras.*

Proof. First of all, ι is well-defined. Indeed, $\iota(d)\colon \mathcal{U} \to \mathcal{U}$ is $\mathcal{D}$-linear for all $d \in Z(\mathcal{D})$, i.e., $\iota(d) \in R = \mathrm{End}_{\mathcal{D}}(\mathcal{U})$. Now, if $r \in R$ and $u \in \mathcal{U}$, then $r(\iota(d)(u)) = r(ud) = r(u)d = \iota(d)(r(u))$. Hence $\iota(d) \in Z(R)$. Clearly, $\deg \iota(d) = \deg d$.

To show that ι is an isomorphism, we identify $\mathcal{D}$ with $\mathrm{End}_R(\mathcal{U})$ via ρ as in Lemma 2.29. Computations analogous to the ones above show that the map $\iota'\colon Z(R) \to Z(\mathcal{D})$ given by $\iota'(r)(u) := r(u)$ is well-defined, and it is straightforward that ι' is the inverse of ι. □

2.1.3 Finite dimensional graded-division algebras over an algebraically closed field

The classification of graded-division algebras over $\mathbb{F}$ involves the classification of usual division algebras and certain cohomology sets of G (see [Kar73]), which is unattainable in general. Fortunately, for our purposes, we only need this classification in a very special case. For gradings on (super)involution-simple associative (super)algebras or simple Lie (super)algebras, we may assume that G is abelian (see Propositions 3.1 and 5.1). Also, we will restrict ourselves to finite dimensional algebras over an algebraically closed field. Hence, for the remainder of this subsection, we will assume that G is abelian and that $\mathbb{F}$ is algebraically closed.

Let $\mathcal{D}$ be a finite dimensional graded-division algebra and set $T \coloneqq \operatorname{supp} \mathcal{D}$, so T is a finite subgroup of G. For every $t \in T$, let us fix $0 \neq X_t \in D_t$. Since $\mathcal{D}_e$ is a finite dimensional division algebra and $\mathbb{F}$ is algebraically closed, $\mathcal{D}_e = \mathbb{F}$ and, hence, $\dim_{\mathbb{F}} \mathcal{D}_t = \dim_{\mathbb{F}} X_t \mathcal{D}_e = 1$ for all $t \in T$.

Definition 2.31. A map $b\colon T \times T \to \mathbb{F}^\times$ is said to be a *bicharacter* if, for every $t \in T$, both maps $b(t,\cdot)\colon T \to \mathbb{F}^\times$ and $b(\cdot,t)\colon T \to \mathbb{F}^\times$ are characters. We say that a bicharacter b is

- *symmetric* if $b(t,s) = b(s,t)$ for all $t,s \in T$;
- *skew-symmetric* if $b(t,s) = b(s,t)^{-1}$ for all $t,s \in T$;
- *alternating* if $b(t,t) = 1$ for all $t \in T$.

Clearly, every alternating bicharacter is skew-symmetric. The *radical* of a (skew-)symmetric bicharacter is the subgroup of T defined by

$$\operatorname{rad} b \coloneqq \{t \in T \mid b(t,T) = 1\}.$$

If $\operatorname{rad} b = \{e\}$, we say that b is *nondegenerate*.

Since T is abelian and every homogeneous component is one-dimensional, for any $t,s \in T$, there exists a nonzero scalar $\beta(t,s)$ such that

$$X_t X_s = \beta(t,s) X_s X_t. \tag{2.3}$$

Note that $\beta(t,s)$ does not depend on the choice of X_t and X_s. It is easy to see that the map $\beta\colon T \times T \to \mathbb{F}^\times$ is an alternating bicharacter and that $\operatorname{rad} \beta$ is the support of $Z(\mathcal{D})$. We will say that (T,β) is the *pair associated to the graded-division superalgebra* $\mathcal{D}$, or that $\mathcal{D}$ is a *graded-superalgebra associated to the pair* (T,β).

The following is a consequence of a well-known result in group cohomology (see [EK15b, Section 2.2]), but we give a different proof here for completeness (see [BZ18, Section 4]).

Proposition 2.32. *The pair (T, β) determines the isomorphism class of the graded-division algebra $\mathcal{D}$.*

Proof. Write $T = \langle t_1 \rangle \times \cdots \times \langle t_k \rangle$ and let n_i denote the order of t_i, for all $1 \le i \le k$. Since $\mathbb{F}$ is algebraically closed, scaling X_{t_i} if necessary, we can assume $X_{t_i}^{n_i} = 1$.

Let $\mathcal{F}$ be the free associative algebra generated by the symbols $Y_{t_1}, \ldots, Y_{t_k}$. We can make $\mathcal{F}$ a T-graded algebra by assigning $\deg Y_{t_i} := t_i$. Let $\mathfrak{D}$ denote the quotient of $\mathcal{F}$ by the ideal generated by

$$Y_{t_i}^{n_i} - 1 \text{ and } Y_{t_i} Y_{t_j} - \beta(t_i, t_j) Y_{t_j} Y_{t_i},$$

for all $1 \le i, j \le k$. Since the relators are homogeneous, $\mathfrak{D}$ is also a T-graded algebra. Note that $\mathfrak{D}$ depends only on T, β and the choice of the elements $t_1, \ldots, t_k \in T$. Also, it is clear from the relators that $\mathfrak{D}$ is spanned by the elements of the form $Y_{t_1}^{m_1} Y_{t_2}^{m_2} \cdots Y_{t_k}^{m_k}$, where $0 \le m_i \le n_i - 1$. In particular $\dim \mathfrak{D} \le |T| = \dim \mathcal{D}$.

Clearly, there is a unique surjective algebra homomorphism $\psi \colon \mathfrak{D} \to \mathcal{D}$ such that $\psi(Y_{t_i}) = X_{t_i}$, which is degree preserving. We then must have that $\dim \mathfrak{D} \ge \dim \mathcal{D}$, so $\dim \mathfrak{D} = \dim \mathcal{D}$ and, therefore, ψ is an isomorphism of T-graded algebras. $\square$

There remains the question of existence of a graded-division algebra $\mathcal{D}$ for a given (T, β). This, again, follows from cohomology. We give another proof for completeness.

Lemma 2.33. *Let T be a finite abelian group and let $\beta \colon T \times T \to \mathbb{F}^\times$ be an alternating bicharacter. Suppose that $T = A \times B$ for subgroups $A, B \subseteq T$ and that there are graded-division algebras $\mathcal{A}$ and $\mathcal{B}$ associated to $(A, \beta \restriction_{A \times A})$ and $(B, \beta \restriction_{B \times B})$, respectively. Then there is a graded-division algebra associated to (T, β). Further, if $\beta(A, B) = 1$, then this graded-division algebra is isomorphic to $\mathcal{A} \otimes \mathcal{B}$ (with its usual product).*

Proof. Choose elements $0 \ne X_a \in \mathcal{A}_a$ and $0 \ne X_b \in \mathcal{B}_b$, for all $a \in A$ and $b \in B$. On the graded vector space $\mathcal{A} \otimes \mathcal{B}$, define a product by

$$(X_a \otimes X_b)(X_{a'} \otimes X_{b'}) := \beta(b, a')(X_a X_{a'}) \otimes (X_b X_{b'}),$$

for all $a, a' \in A$ and $b, b' \in B$ (this construction is called *colour tensor product* in [BMPZ92, page 88], and is a special case of the concept of *twisted tensor product* in [CSV95]). Clearly, this makes $\mathcal{A} \otimes \mathcal{B}$ a graded algebra, and it is associative (see

[BMPZ92]) with identity element $1_A \otimes 1_B$. Since the homogeneous elements in $A \otimes B$ are scalar multiples of $X_a \otimes X_b$, $a \in A$, $b \in B$, we see that it is a graded-division algebra, and that it is associated to (T, β). □

Proposition 2.34. *For every pair (T, β), there is a graded-division algebra associated to it.*

Proof. We write $T = \langle t_1 \rangle \times \cdots \times \langle t_k \rangle$ and proceed by induction on k. If $k = 1$, then β must be trivial: $\beta(t_1^i, t_1^j) = \beta(t_1, t_1)^{ij} = 1$, for all $i, j \in \mathbb{Z}$. Hence, the group algebra $\mathbb{F}T$ is a graded-division algebra associated to (T, β). The induction step follows from the case $k = 1$ and Lemma 2.33. □

If we suppose that β is nondegenerate, than we can construct a graded-division algebra associated to (T, β) using matrices. For that, we follow [EK15a, Remark 18] (see also [EK13, Remark 2.16]).

First of all, we can decompose the group T as $A \times B$, where the restrictions of β to each of the subgroups A and B are trivial (see [EK13, page 36]) and, hence, A and B are in duality by β, i.e., the map $A \to \widehat{B}$ given by $a \mapsto \beta(a, \cdot)$ is an isomorphism of groups (note that, in particular, $|T|$ is a perfect square).

Let V be the vector space with basis $\{e_b\}_{b \in B}$ (i.e., V is the vector space underlying the group algebra $\mathbb{F}B$). For each $a \in A$, define $X_a \in \operatorname{End}(V)$ by

$$\forall b' \in B, \quad X_a(e_{b'}) \coloneqq \beta(a, b') e_{b'},$$

and, for each $b \in B$, define $X_b \in \operatorname{End}(V)$ by

$$\forall b' \in B, \quad X_b(e_{b'}) \coloneqq e_{bb'}.$$

Finally, we define $X_{ab} \coloneqq X_a X_b$, for all $a \in A$ and $b \in B$.

Proposition 2.35. *The operators X_t, $t \in T$ form a basis of the algebra* $\operatorname{End}(V)$ *and define a division grading associated to (T, β).*

Proof. Clearly, the operators X_a and X_b (and, hence, X_{ab}) are invertible, for all $a \in A$ and $b \in B$. It is easy to see that $X_a X_{a'} = X_{aa'}$ and $X_b X_{b'} = X_{bb'}$, for all $a, a' \in A$ and $b, b' \in B$. Also, $X_a(X_b(e_{b'})) = \beta(a, bb') e_{bb'} = \beta(a, b)\beta(a, b') e_{bb'}$ and

$X_b(X_a(e_{b'})) = \beta(a,b')e_{bb'}$, so $X_aX_b = \beta(a,b)X_bX_a$. It follows that, for all $t,s \in T$, $X_tX_s = \beta(t,s)X_sX_t$.

It remains to show that the operators X_t, $t \in T$, form a basis of $\mathrm{End}(V)$. Since $|T| = |A||B| = |B|^2$ and $\dim V = |B|$, it suffices to prove linear independence. Suppose $\sum_{a,b} \lambda_{ab}X_{ab} = 0$, for some $\lambda_{ab} \in \mathbb{F}$. Then for each $b' \in B$, we have that

$$\sum_{a,b} \lambda_{ab}X_{ab}(e_{b'}) = \lambda_{ab}\beta(a,b)\beta(a,b')e_{bb'} = 0.$$

Since $\{e_{bb'}\}$ is linearly independent, we have

$$\sum_{a} \lambda_{ab}\beta(a,b)\beta(a,b') = 0,$$

for all $b,b' \in B$. It follows that

$$\sum_{a} \lambda_{ab}\beta(a,b)\beta(a,\cdot) = 0,$$

for all $b \in B$. Since β is nondegenerate, the maps $\beta(a,\cdot) \in \widehat{B}$ are all distinct characters, and since distinct characters are linearly independent, we have $\lambda_{ab}\beta(a,b) = 0$, for all $a \in A$ and $b \in B$. Therefore, $\lambda_{ab} = 0$, as desired. $\square$

Definition 2.36. We will refer to these matrix models of $\mathcal{D}$ as its *standard realizations*.

Note that Example 2.5 is a standard realization for (T,β) where $T = \mathbb{Z}_\ell \times \mathbb{Z}_\ell$ and $\beta((i,j),(i',j')) = \xi^{ij'-i'j}$, for all $i,i',j,j' \in \mathbb{Z}_\ell$.

Corollary 2.37. *The graded-division algebra $\mathcal{D}$ is simple as an (ungraded) algebra if, and only if, β is nondegenerate.*

Proof. If β is nondegenerate, then the construction of a standard realization above shows that (T,β) has a model with a simple algebra and, hence, $\mathcal{D}$ must be simple by Proposition 2.32.

Conversely, if $\mathcal{D}$ is simple as an algebra, then, since $\mathbb{F}$ is algebraically closed, $\mathcal{D}$ must be central (i.e., $Z(\mathcal{D}) = \mathbb{F}$), so $\operatorname{rad}\beta = \{e\}$. $\square$

We finish with a well-known result for future reference.

Lemma 2.38. *An $\mathbb{F}$-linear map $\psi_0 \colon \mathcal{D} \to \mathcal{D}$ is an automorphism of the graded algebra $\mathcal{D}$ if, and only if, there is $\chi \in \widehat{T}$ such that $\psi_0(X_t) = \chi(t)X_t$ for all $t \in T$.*

Proof. Any invertible degree-preserving linear map $\psi \colon \mathcal{D} \to \mathcal{D}$ is determined by a map $\chi \colon T \to \mathbb{F}^\times$ such that $\psi(X_t) = \chi(t)X_t$, for all $t \in T$. It is easy to see that ψ is an automorphism if, and only if, χ is a group homomorphism, i.e., $\chi \in \widehat{T}$. $\square$

2.1.4 Finite dimensional graded-simple algebras over an algebraically closed field

We continue assuming that $\mathbb{F}$ is algebraically closed and that G is abelian.

If R is a graded-simple algebra then, by Theorem 2.23, $R \simeq \operatorname{End}_{\mathcal{D}}(\mathcal{U})$, where $\mathcal{D}$ is a graded-division algebra and $\mathcal{U}$ is graded right $\mathcal{D}$-module of finite rank, say, k. Since $\operatorname{End}_{\mathcal{D}}(\mathcal{U}) \simeq M_k(\mathcal{D})$, we have that R is finite dimensional if, and only if, $\mathcal{D}$ is finite dimensional if, and only if, $\mathcal{D}$ is associated to a pair (T, β) as in the previous subsection.

Definition 2.39. Let $\mathcal{D}$ be a finite dimensional graded-division algebra over an algebraically closed field $\mathbb{F}$ and let $\mathcal{U}$ be a graded right $\mathcal{D}$-module of finite rank. If $\mathcal{D}$ is associated to (T, β) and $\mathcal{U}$ is associated to $\kappa \colon G/T \to \mathbb{Z}_{\geq 0}$ (see Subsection 2.1.1), we say that (T, β, κ) are the *parameters* of the pair $(\mathcal{D}, \mathcal{U})$.

It is easy to see that, since G is abelian, $\mathcal{U}^{[g]}$ is associated to $g \cdot \kappa$, where the G-action on functions $G/T \to \mathbb{Z}_{\geq 0}$ is defined as usual: $(g \cdot \kappa)(x) := \kappa(g^{-1}x)$ for all $x \in G/T$. Note that if $(g_1, \ldots, g_k)$ is a k-tuple realizing κ (see Definition 2.18), then $(gg_1, \ldots, gg_k)$ is a k-tuple realizing $g \cdot \kappa$.

If $\psi_0 \colon \mathcal{D} \to \mathcal{D}'$ is an isomorphism of graded algebras and $\mathcal{U}'$ is a graded right $\mathcal{D}'$-module associated to κ', it is clear that $\dim_{\mathcal{D}'} \mathcal{U}'_x = \dim_{\mathcal{D}} (\mathcal{U}'_x)^{\psi_0}$, for all $x \in G/T$, and, hence, the graded $\mathcal{D}$-module $(\mathcal{U}')^{\psi_0}$ is also associated to κ'.

Thus, Theorem 2.27 becomes the following:

Theorem 2.40. *Let $(\mathcal{D}, \mathcal{U})$ and $(\mathcal{D}', \mathcal{U}')$ be pairs as in Definition 2.39, and let (T, β, κ) and (T', β', κ') be their parameters. Then $\operatorname{End}_{\mathcal{D}}(\mathcal{U}) \simeq \operatorname{End}_{\mathcal{D}'}(\mathcal{U}')$ if, and only if, $T = T'$, $\beta = \beta'$, and κ and κ' belong to the same G-orbit.* $\square$

It should be noted that, combining Proposition 2.28 and Corollary 2.37, we have that $\mathrm{End}_{\mathcal{D}}(\mathcal{U})$ is simple as an (ungraded) algebra if, and only if, β is nondegenerate. If this is the case, Theorem 2.40 gives us the classification of abelian group gradings on matrix algebras up to isomorphism:

Definition 2.41. Let $n > 0$ be a natural number. Given a finite subgroup $T \subseteq G$, a nondegenerate bicharacter $\beta\colon T \times T \to \mathbb{F}^\times$ and a map $\kappa\colon G/T \to \mathbb{Z}_{\geq 0}$ with finite support such that $|\kappa|\sqrt{|T|} = n$, consider

(i) a standard realization $\mathcal{D}$ (see Definition 2.36) of a matrix algebra with a division grading associated to (T, β);

(ii) the elementary grading (see Definition 1.3) on $M_k(\mathbb{F})$ defined by a k-tuple γ of elements of G realizing κ, where $k \coloneqq |\kappa|$ (see Definition 2.18).

We define $\Gamma_M(T, \beta, \kappa)$ to be the grading on $M_n(\mathbb{F})$ given by identifying $M_n(\mathbb{F})$ with the graded algebra $M_k(\mathbb{F}) \otimes \mathcal{D}$ via Kronecker product, i.e.,

$$\deg\left(E_{ij} \otimes d\right) = g_i g_j^{-1} t,$$

for all $1 \leq i, j \leq k$, $t \in T$ and $0 \neq d \in \mathcal{D}_t$. The algebra $M_n(\mathbb{F})$ endowed with the grading $\Gamma_M(T, \beta, \kappa)$ will be denoted by $M(T, \beta, \kappa)$.

Note that we are abusing notation in Definition 2.41. The grading $\Gamma_M(T, \beta, \kappa)$ actually depends on the choices of the standard realization $\mathcal{D}$ and of the k-tuple γ realizing κ. Nevertheless, its isomorphism class depends only on (T, β, κ).

Corollary 2.42 ([BK10, Theorem 2.6],[EK13, Theorem 2.27])**.** *Every G-grading on $M_n(\mathbb{F})$ is isomorphic to $\Gamma_M(T, \beta, \kappa)$ as in Definition 2.41. Two such gradings $\Gamma_M(T, \beta, \kappa)$ and $\Gamma_M(T', \beta', \kappa')$ are isomorphic if, and only if, $T = T'$, $\beta = \beta'$ and there is $g \in G$ such that $g \cdot \kappa = \kappa'$.* $\square$

As an application of Theorems 2.23 and 2.40, we can obtain the classification of finite dimensional simple superalgebras over an algebraically closed field.

Theorem 2.43 ([Wal64])**.** *Let R be a finite dimensional simple superalgebra over an algebraically closed field $\mathbb{F}$. Then R is isomorphic to either $M(m, n)$ or $Q(n)$, where $m, n \in \mathbb{Z}_{\geq 0}$. Moreover,* *(i)* $M(m,n) \not\simeq Q(n')$;

(ii) $M(m,n) \simeq M(m',n')$ *if, and only if, either* $m = m'$ *and* $n = n'$, *or* $m = n'$ *and* $n = m'$;

(iii) $Q(n) \simeq Q(n')$ *if, and only if,* $n = n'$.

Proof. Since R is simple as a $\mathbb{Z}_2$-graded algebra, so, by Theorem 2.23, $R \simeq \operatorname{End}_{\mathcal{D}}(\mathcal{U})$. Let (T, β, κ) be the parameters of $(\mathcal{D}, \mathcal{U})$. Since $T \subseteq \mathbb{Z}_2$, we either have $T = \{\bar{0}\}$ or $T = \mathbb{Z}_2$.

If $T = \{\bar{0}\}$, then $\mathcal{D} = \mathbb{F}$ and $R \simeq \operatorname{End}_{\mathbb{F}}(\mathcal{U})$. The isomorphism class of $\mathcal{U}$ is determined by the map $\kappa\colon \mathbb{Z}_2/\{\bar{0}\} = \mathbb{Z}_2 \to \mathbb{Z}_{\geq 0}$ defined by $\kappa(i) = \dim_{\mathbb{F}}(\mathcal{U}^i)$ for all $i \in \mathbb{Z}_2$. In other words, the isomorphism class of $\mathcal{U}$ is determined by the numbers $m := \kappa(\bar{0})$ and $n := \kappa(\bar{1})$. By choosing bases for $\mathcal{U}^{\bar{0}}$ and $\mathcal{U}^{\bar{1}}$, we get $R \simeq M(m,n)$. By Theorem 2.40, we conclude that $M(m,n) \simeq M(m',n')$ if, and only if, either $m = m'$ and $n = n'$ (for $g = \bar{0}$), or $m = n'$ and $n = m'$ (for $g = \bar{1}$).

If $T = \mathbb{Z}_2$, we first note that the only alternating bicharacter $\beta\colon \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{F}^\times$ is the trivial one. Hence, $\mathcal{D} \simeq \mathbb{F}\mathbb{Z}_2 \simeq Q(1)$ and $R \simeq \operatorname{End}_{Q(1)}(\mathcal{U})$. The isomorphism class of $\mathcal{U}$ is determined by a map $\kappa\colon \mathbb{Z}_2/\mathbb{Z}_2 \to \mathbb{Z}_{\geq 0}$, i.e., by the single number $n := \kappa(\mathbb{Z}_2) = \dim_{Q(1)}(\mathcal{U})$.

To conclude the proof, we will show that $\operatorname{End}_{Q(1)}(\mathcal{U}) \simeq Q(n)$. By Proposition 2.12, we can take a graded basis $\mathcal{B}$ for $\mathcal{U}$ with all elements having degree $\bar{0}$. We, then, can write $\operatorname{End}_{Q(1)}(\mathcal{U}) \simeq M_n(Q(1)) \simeq M_n(\mathbb{F}) \otimes Q(1)$ as graded algebras (Remark 2.22), where the grading on $M_n(\mathbb{F})$ is trivial. Finally, using the Kronecker product, we get $M_n(\mathbb{F}) \otimes Q(1) \simeq Q(1) \otimes M_n(\mathbb{F}) \simeq Q(n)$. $\square$

Definition 2.44. Let R be a finite dimensional simple associative superalgebra. If $R \simeq M(m,n)$, for some $m, n \geq 0$, we say that R is of type M. If $R \simeq Q(n)$, for some $n \geq 0$, we say that R is of type Q.

2.2 Graded-simple associative superalgebras

We will now adapt the results of the previous section to graded-simple superalgebras. For any abelian group G, we obtain a classification of G-graded graded-simple superalgebra over an algebraically closed field of any characteristic, following the approach used in [HSK19] for $M(m,n)$. A different (and more complicated) approach was used in [BS06] to obtain a description of G-gradings (with some restrictions on characteristic), but the isomorphism problem was not solved there.

Let G be a group. Recall that a G-graded superalgebra R can be seen as a $G^\# := G \times \mathbb{Z}_2$-graded algebra (see Section 0.1) by defining $R_{(g,i)} = R_g \cap R^i$, for all $g \in G$ and $i \in \mathbb{Z}_2$. We identify G with $G \times \{\bar{0}\} \subseteq G^\#$. Clearly, R is graded-simple as a G-graded superalgebra if, and only if, it is graded-simple as a $G^\#$-graded algebra. This allows us to easily transfer the results of the previous section to gradings on superalgebras, but at the cost of working in the group $G^\#$ instead of G.

Remark 2.45. If the canonical $\mathbb{Z}_2$-grading is a coarsening of the G-grading by means of a homomorphism $p\colon G \to \mathbb{Z}_2$ (referred to as the *parity homomorphism*), then we could work with the pair (G,p) instead of the group $G^\#$. Note that, in this case we would have another isomorphic copy of G in $G^\#$, namely, the image of the embedding $g \mapsto (g, p(g))$, which contains the support of the $G^\#$-grading.

2.2.1 Graded-division superalgebras and their supermodules

Definition 2.46. A G-graded superalgebra $\mathcal{D}$ is said to be a *graded-division superalgebra* if every nonzero element which is homogeneous with respect to both the G-grading and the canonical $\mathbb{Z}_2$-grading is invertible. In this case, we may also refer to the G-grading on the superalgebra $\mathcal{D}$ as a *division grading.*

In other words, the graded-division superalgebras with respect to the G-grading are precisely the graded-division algebras with respect to the $G^\#$-grading. Recall that $T := \operatorname{supp} \mathcal{D}$ is a subgroup of $G^\#$, so we can see $\mathcal{D}$ as a T-graded algebra and the canonical $\mathbb{Z}_2$-grading is the coarsening by the group homomorphism $p\colon T \subseteq G^\# \to \mathbb{Z}_2$ given by $p(g,i) = i$ for all $(g,i) \in G^\#$. We will denote the kernel of p by T^+, i.e., $T^+ := T \cap (G \times \{\bar{0}\}) = \operatorname{supp} \mathcal{D}^{\bar{0}}$. Similarly, we define $T^- := T \cap (G \times \{\bar{1}\}) = \operatorname{supp} \mathcal{D}^{\bar{1}}$.

Notation 2.47. For graded-division superalgebras we will use subscripts to refer to the T-grading, i.e., $\mathcal{D}_t$ refers to the homogeneous component $\mathcal{D}^i_g$ where $t = (g,i) \in T \subseteq G^\#$.

Example 2.48. Let $\langle u \rangle$ be a cyclic group of order 2 and let $G = \langle h \rangle$ where h has order at most 2. The group algebra $\mathcal{D} := \mathbb{F}\langle u \rangle$, with canonical $\mathbb{Z}_2$-grading given by

$\mathcal{D}^{\bar{0}} := \mathbb{F}1$ and $\mathcal{D}^{\bar{1}} = \mathbb{F}u$, becomes a G-graded graded-division superalgebra if we declare the G-degree of u to be h. In this case $G^\# = \langle h\rangle \times \mathbb{Z}_2$ and $T = \langle (h, \bar{1})\rangle \simeq \mathbb{Z}_2$ (compare with Example 2.3). Note that $\mathcal{D} \simeq Q(1)$ as a superalgebra.

Example 2.49. Example 2.4 can be seen as a division $\mathbb{Z}_2$-grading on $M(1,1)$, $\operatorname{char}\mathbb{F} \neq 2$. In this case, $G^\# = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Let $\mathcal{U} = \mathcal{U}^{\bar{0}} \oplus \mathcal{U}^{\bar{1}}$ be a graded right $\mathcal{D}$-supermodule of finite rank. The isomorphism class of the graded right $\mathcal{D}$-supermodule $\mathcal{U}$ is, as in Subsection 2.1.1, determined by the map $\kappa\colon G^\#/T \to \mathbb{Z}_{\geq 0}$ given by $\kappa(x) = \dim_{\mathcal{D}} \mathcal{U}_x$ for all $x \in G^\#/T$. We also have a description only in terms of the group G, but for that we separate the graded-division superalgebras in two classes:

Definition 2.50. If $\mathcal{D} = \mathcal{D}^{\bar{0}}$ (i.e., $\mathcal{D}^{\bar{1}} = 0$) we say that $\mathcal{D}$ is an *even* graded-division superalgebra, and if $\mathcal{D} \neq \mathcal{D}^{\bar{0}}$ (i.e., $\mathcal{D}^{\bar{1}} \neq 0$) we say that $\mathcal{D}$ is an *odd* graded-division superalgebra.

Note that if $\mathcal{D}$ is odd and finite dimensional, then $\dim_{\mathbb{F}} \mathcal{D}^{\bar{0}} = \dim_{\mathbb{F}} \mathcal{D}^{\bar{1}}$.

Lemma 2.51. *If $\mathcal{D}$ is odd and $\mathcal{D} \simeq M(m,n)$ as a superalgebra, then $m = n$.*

Proof. We have that $\dim M(m,n)^{\bar{0}} = m^2 + n^2$ and $\dim M(m,n)^{\bar{1}} = 2mn$. Hence $\dim M(m,n)^{\bar{0}} = \dim M(m,n)^{\bar{1}}$ if, and only if, $m = n$. □

Assume that $\mathcal{D}$ is an even graded-division superalgebra. Then both $\mathcal{U}^{\bar{0}}$ and $\mathcal{U}^{\bar{1}}$ are graded $\mathcal{D}$-submodules, hence we can describe their isomorphism classes, respectively, by maps $\kappa_{\bar{0}}, \kappa_{\bar{1}}\colon G/T \to \mathbb{Z}_{\geq 0}$ given by $\kappa_i(x) = \dim \mathcal{U}^i_x$ for all $i \in \mathbb{Z}_2$ and $x \in G/T$. Note that, in this case, $G^\#/T = G^\#/T^+$ is the disjoint union of G/T and $(e, \bar{1}) \cdot G/T$ and, clearly, $\kappa((g,i)T) = \kappa_i(gT)$, for all $i \in \mathbb{Z}_2$ and $g \in G$.

Now assume that $\mathcal{D}$ is odd. In this case, unless $\mathcal{U} = 0$, the graded subspaces $\mathcal{U}^{\bar{0}}$ and $\mathcal{U}^{\bar{1}}$ are not $\mathcal{D}$-submodules. But we can follow a different approach.

Definition 2.52. A graded basis of a graded supermodule $\mathcal{U}$ is said to be an *even basis* if it consists only of even elements.

Let $\mathcal{B} = \{u_\lambda\}_{\lambda\in\Lambda}$ be any basis for $\mathcal{U}$, and let $0 \neq d_1 \in \mathcal{D}^{\bar{1}}$. For every $\lambda \in \Lambda$, if u_λ is an odd element, let us replace it by $u_\lambda d_1$. The resulting set is, clearly, an even basis of $\mathcal{U}$.

Convention 2.53. If $\mathcal{D}$ is an odd graded-division superalgebra, we choose the graded basis $\mathcal{B}$ to be an even basis.

It follows that the canonical $\mathbb{Z}_2$-grading on $\mathcal{U}$ is determined only by $\mathcal{D}$. To see that, take any even basis of $\mathcal{U}$ and let $\tilde{\mathcal{U}}$ be its $\mathcal{D}_e$-span. Clearly, $\tilde{\mathcal{U}} \subseteq \mathcal{U}^{\bar{0}}$ and it is a $\mathcal{D}_e$-form of $\mathcal{U}$, i.e., we can identify $\mathcal{U} = \tilde{\mathcal{U}} \otimes_{\mathcal{D}_e} \mathcal{D}$ and, hence, $\mathcal{U}^{\bar{0}} = \tilde{\mathcal{U}} \otimes_{\mathcal{D}_e} \mathcal{D}^{\bar{0}}$ and $\mathcal{U}^{\bar{1}} = \tilde{\mathcal{U}} \otimes_{\mathcal{D}_e} \mathcal{D}^{\bar{1}}$.

This can also be seen from the point of view of the map κ. Note that, every coset $x \in G^{\#}/T$ has an even representative. This entails that the map $\iota\colon G/T^+ \to G^{\#}/T$ given by $\iota(gT^+) = (g, \bar{0})T$ is a bijection, so we can work with $\kappa \circ \iota$, which we will, by abuse of notation, also denote by κ.

It should be noted that, even though $\mathcal{U}^{\bar{0}}$ and $\mathcal{U}^{\bar{1}}$ are not $\mathcal{D}$-submodules, they are $\mathcal{D}^{\bar{0}}$-submodules. Clearly, the map $\kappa\colon G/T^+ \to \mathbb{Z}_{\geq 0}$ above is the map associated to both $\mathcal{U}^{\bar{0}}$ and $\mathcal{U}^{\bar{1}}$, and an even graded $\mathcal{D}$-basis for $\mathcal{U}$ is a graded $\mathcal{D}^{\bar{0}}$-basis for $\mathcal{U}^{\bar{0}}$.

Remark 2.54. Set $R \coloneqq \mathrm{End}_{\mathcal{D}}(\mathcal{U})$. If $\mathcal{D}$ is even, then we can identify $R^{\bar{0}}$ with $\mathrm{End}_{\mathcal{D}}(\mathcal{U}^{\bar{0}}) \oplus \mathrm{End}_{\mathcal{D}}(\mathcal{U}^{\bar{1}})$. If $\mathcal{D}$ is odd, then we can identify $R^{\bar{0}}$ with $\mathrm{End}_{\mathcal{D}^{\bar{0}}}(\mathcal{U}^{\bar{0}})$.

2.2.2 Graded Wedderburn theory for superalgebras

Let R be a G-graded superalgebra. Note that the graded (left) superideals of R are precisely the graded (left) ideals of R as a $G^{\#}$-graded superalgebra. Hence, Theorems 2.23 and 2.27 translate to graded superalgebras by simply attaching the "super" prefix wherever it is needed.

Let R be a graded-simple superalgebra satisfying the d.c.c. on graded left superideals and write, as in Theorem 2.23, $R \simeq \mathrm{End}_{\mathcal{D}}(\mathcal{U})$ where $\mathcal{D}$ is a graded-division superalgebra and $\mathcal{U}$ is a graded right $\mathcal{D}$-supermodule of finite rank. Since the property of being an even or an odd a graded-division superalgebra is invariant under isomorphism, by Theorem 2.27, we can extend Definition 2.50 to R:

Definition 2.55. If $\mathcal{D}$ is an even graded-division superalgebra, we say that the grading on R is *even*; if $\mathcal{D}$ is an odd graded-division superalgebra, we say that the grading on R is *odd*.

For the remaining part of this subsection, fix a graded-division algebra $\mathcal{D}$ and a nonzero graded right $\mathcal{D}$-module of finite rank $\mathcal{U}$, and set $R \coloneqq \mathrm{End}_{\mathcal{D}}(\mathcal{U})$.

By the discussion on Subsection 2.2.1, if the grading on R is even, all the information related to the canonical $\mathbb{Z}_2$-grading on R is encoded in the graded right $\mathcal{D}$-supermodule $\mathcal{U} = \mathcal{U}^{\bar{0}} \oplus \mathcal{U}^{\bar{1}}$. More explicitly,

$$R^{\bar{0}} = \mathrm{End}_{\mathcal{D}}(\mathcal{U}^{\bar{0}}) \oplus \mathrm{End}_{\mathcal{D}}(\mathcal{U}^{\bar{1}}) \text{ and } R^{\bar{1}} = \mathrm{Hom}_{\mathcal{D}}(\mathcal{U}^{\bar{0}}, \mathcal{U}^{\bar{1}}) \oplus \mathrm{Hom}_{\mathcal{D}}(\mathcal{U}^{\bar{1}}, \mathcal{U}^{\bar{0}}).$$

This can also be seen by fixing a graded basis $\{u_1, \ldots, u_{k_{\bar{0}}}\}$ of $\mathcal{U}^{\bar{0}}$ and a graded basis $\{u_{k_{\bar{0}}+1}, \ldots, u_{k_{\bar{0}}+k_{\bar{1}}}\}$ of $\mathcal{U}^{\bar{1}}$. Then R can be identified with $M_{k_{\bar{0}}|k_{\bar{1}}}(\mathcal{D})$ as a graded superalgebra. If G is an abelian group, following Remark 2.22, we can identify R with the tensor product of $G^{\#}$-graded algebras $M_{k_{\bar{0}}|k_{\bar{1}}}(\mathbb{F}) \otimes \mathcal{D}$. So, $R^{\bar{0}} = M_{k_{\bar{0}}|k_{\bar{1}}}(\mathbb{F})^{\bar{0}} \otimes \mathcal{D}$ and $R^{\bar{1}} = M_{k_{\bar{0}}|k_{\bar{1}}}(\mathbb{F})^{\bar{1}} \otimes \mathcal{D}$.

Remark 2.56. Note that, if the G-grading on R is even, we can refine the $G^{\#}$-grading to a $G \times \mathbb{Z}$-grading by defining $R^{-1} := \mathrm{Hom}_{\mathcal{D}}(\mathcal{U}^{\bar{1}}, \mathcal{U}^{\bar{0}})$, $R^0 := R^{\bar{0}} = \mathrm{End}_{\mathcal{D}}(\mathcal{U}^{\bar{0}}) \oplus \mathrm{End}_{\mathcal{D}}(\mathcal{U}^{\bar{1}})$ and $R^1 := \mathrm{Hom}_{\mathcal{D}}(\mathcal{U}^{\bar{0}}, \mathcal{U}^{\bar{1}})$.

If the grading on R is odd, then, again by the discussion on Subsection 2.2.1, all the information about the canonical $\mathbb{Z}_2$-grading is encoded in $\mathcal{D}$. To see it more explicitly, let us follow Convention 2.53 and recall the maps $f_{i,j,d} \in R = \mathrm{End}_{\mathcal{D}}(\mathcal{U})$ defined in the beginning of Subsection 2.1.2. Those maps generate R and, by Equation (2.1), their parity depend only on $d \in \mathcal{D}$.

If G is an abelian group, again, this can also be seen from the identification $R = M_k(\mathbb{F}) \otimes \mathcal{D}$ in Remark 2.22. The grading on $M_k(\mathbb{F})$ is elementary grading defined by a tuple of elements in $G = G \times \{\bar{0}\}$ and, hence, $R^{\bar{0}} = M_k(\mathcal{D}) \otimes \mathcal{D}^{\bar{0}}$ and $R^{\bar{1}} = M_k(\mathcal{D}) \otimes \mathcal{D}^{\bar{1}}$.

Remark 2.57. Recall that, for superalgebras, the tensor product is defined with a different multiplication on the superspace $M_k(\mathbb{F}) \otimes \mathcal{D}$ (Definition 1.9). But since we are following Convention 2.53, either $M_k(\mathbb{F})$ or $\mathcal{D}$ will have trivial canonical $\mathbb{Z}_2$-grading, hence the tensor product of superalgebras coincides with that of algebras.

Note that we already have a description of even gradings only in terms of the group G. For odd gradings, though, we still need to describe $\mathcal{D}$ only in terms of the group G, which is a harder task, and we are going to do that only for the case R is a finite dimensional superalgebra over an algebraically closed field (see Subsection 2.2.3 and Section 2.3).

Proposition 2.58. *The superalgebra* $R = \mathrm{End}_{\mathcal{D}}(\mathcal{U})$ *is simple if, and only if, the superalgebra* $\mathcal{D}$ *is simple.*

Proof. Pick a graded basis for $\mathcal{U}$ following Convention 2.53 and use it to identify R with $M_k(\mathcal{D}) = M_k(\mathbb{F}) \otimes \mathcal{D}$, as in Remark 2.57.

It is well known that the ideals of $M_k(\mathcal{D})$ are precisely the sets of the form $M_k(I)$ for I an ideal of $\mathcal{D}$. We will prove an analog of this for superideals.

If I is a superideal, $M_k(I) = M_k(\mathbb{F}) \otimes I$ is also a superideal since it is spanned by a set of $\mathbb{Z}_2$-homogeneous elements, namely, the elements of the form $E_{ij} \otimes d$ where $1 \leq i, j \leq k$ and $d \in I^{\bar{0}} \cup I^{\bar{1}}$. Conversely, if $J = M_k(I)$ is a superideal, then we can write $I = \{d \in \mathcal{D} \mid E_{11} \otimes d \in J\}$. For every $d \in I$, write $d = d_{\bar{0}} + d_{\bar{1}}$, where $d_\alpha \in \mathcal{D}^\alpha$, $\alpha \in \mathbb{Z}_2$. Since the $\mathbb{Z}_2$-homogeneous components of $E_{11} \otimes d$ are $E_{11} \otimes d_{\bar{0}}$ and $E_{11} \otimes d_{\bar{1}}$ and they belong to J, we have $d_{\bar{0}}, d_{\bar{1}} \in I$. $\square$

We also could easily adapt the correspondence between the centers of $\mathrm{End}_{\mathcal{D}}(\mathcal{U})$ and $\mathcal{D}$ (Proposition 2.30) to a correspondence between supercenters. It turns out, though, that the correspondence between centers is more useful even in the case of superalgebras. For example, if $\operatorname{char} \mathbb{F} \neq 2$, then $sZ(M(m,n)) = sZ(Q(n)) = \mathbb{F}$ while $Z(M(m,n)) \neq Z(Q(n))$, so we can use the centers to distinguish the types of superalgebra (see Proposition 4.16).

2.2.3 Finite dimensional graded-division superalgebras over an algebraically closed field

We will now focus on the case where G is abelian and $\mathbb{F}$ is algebraically closed. For many of the results in this section we will also assume that $\operatorname{char} \mathbb{F} \neq 2$, but this hypothesis is not necessary for the classification results in Subsection 2.2.4.

Let $\mathcal{D}$ be a finite dimensional graded-division superalgebra and consider the pair (T, β) associated to it, where $T \subseteq G^\#$ is a finite subgroup of $G^\#$ and β is an alternating bicharacter on T. As we did in Subsection 2.2.1, let $p\colon T \to \mathbb{Z}_2$ be the restriction to T of the projection on the second component of $G^\# = G \times \mathbb{Z}_2 \to \mathbb{Z}_2$, and set $T^+ = \ker p = T \cap (G \times \{\bar{0}\})$ and $T^- = T \cap (G \times \{\bar{1}\})$. We will also denote by β^+ the restriction of β to $T^+ \times T^+$. As done in Subsection 2.1.3, for every $t \in T$ we fix $0 \neq X_t \in D_t$.

It is useful to consider another bicharacter on T. We define $\tilde{\beta}\colon T \times T \to \mathbb{F}^\times$ by

$$\forall t, s \in T, \quad \tilde{\beta}(t, s) := (-1)^{p(t)p(s)} \beta(t, s). \tag{2.4}$$

In other words, we have that

$$\forall t, s \in T, \quad X_t X_s = (-1)^{p(t)p(s)} \tilde{\beta}(t, s) X_s X_t$$

(compare with Equation (2.3)). Clearly, the support of $sZ(\mathcal{D})$, the supercenter of $\mathcal{D}$, is $\operatorname{rad} \tilde{\beta}$.

The following result is the first step in the reduction of Lie colour algebras to Lie superalgebras, due to M. Scheunert (see [Sch79a]).

Proposition 2.59. *Suppose* $\operatorname{char} \mathbb{F} \neq 2$. *Let T be an abelian group and $\tilde{\beta}$ be a skew-symmetric bicharacter on T. Then there is a alternating bicharacter β on T and a group homomorphism $p\colon T \to \mathbb{Z}_2$ such that $\tilde{\beta}(t, s) = (-1)^{p(t)p(s)} \beta(t, s)$ for all $t, s \in T$.*

Proof. Since $\tilde{\beta}$ is skew-symmetric, $\tilde{\beta}(t, t) = \tilde{\beta}(t, t)^{-1}$, so $\tilde{\beta}(t, t) \in \{\pm 1\}$. Define $h\colon T \to \{\pm 1\}$ by $h(t) := \tilde{\beta}(t, t)$, for all $t \in T$. We have that

$$h(ts) = \tilde{\beta}(ts, ts) = \tilde{\beta}(t, t)\tilde{\beta}(t, s)\tilde{\beta}(s, t)\tilde{\beta}(s, s) = h(t)h(s),$$

for all $t, s \in T$, so h is a group homomorphism. We define $p\colon T \to \mathbb{Z}_2$ as the unique group homomorphism such that $h(t) = (-1)^{p(t)}$ for all $t \in T$.

Finally, we define $\beta\colon T \times T \to \mathbb{F}^\times$ by $\beta(t, s) := (-1)^{p(t)p(s)} \tilde{\beta}(t, s)$ for all $t, s \in T$. Clearly, β is a bicharacter and $\tilde{\beta}(t, s) = (-1)^{p(t)p(s)} \beta(t, s)$. It remains to show that β is alternating: $\beta(t, t) = (-1)^{p(t)p(t)} \tilde{\beta}(t, t) = (-1)^{p(t)} \tilde{\beta}(t, t) = h(t)\tilde{\beta}(t, t) = 1$, for all $t \in T$. □

Proposition 2.59 tells us that a pair $(T, \tilde{\beta})$, where $\tilde{\beta}$ is a skew-symmetric bicharacter on T, carries the same information as a triple (T, β, p), where β is an alternating bicharacter and $p\colon T \to \mathbb{Z}_2$ is a group homomorphism. Throughout this work we decided to use the triples (T, β, p) to parametrize finite dimensional graded-division superalgebras over an algebraically closed field, since we refer to the bicharacter β and the homomorphism p frequently and also because this parametrization is valid even in the case $\operatorname{char} \mathbb{F} = 2$. We we will say that the graded-division superalgebra $\mathcal{D}$

is *associated* to triple (T, β, p) if the graded-division algebra $\mathcal{D}$ is associated to (T, β) and p is its parity homomorphism.

Remark 2.60. If $\mathcal{D}$ is considered as a $G^\#$-graded algebra, the parameter p is redundant. Nevertheless, it is convenient to keep it since there are situations where we may want to regard $\mathcal{D}$ as a T-graded algebra or as a G-graded superalgebra.

Lemma 2.61. *Suppose* $\operatorname{char}\mathbb{F} \neq 2$. *Then* $\operatorname{rad}\tilde{\beta} = (\operatorname{rad}\beta) \cap T^+$ *and, therefore,* $sZ(\mathcal{D}) = Z(\mathcal{D}) \cap \mathcal{D}^{\bar{0}}$.

Proof. Let $t \in T^+$. In this case $\tilde{\beta}(t, \cdot) = \beta(t, \cdot)$, so $\tilde{\beta}(t, T) = 1$ if, and only if, $\beta(t, T) = 1$. Hence $(\operatorname{rad}\tilde{\beta}) \cap T^+ = (\operatorname{rad}\beta) \cap T^+$.

Now let $t \in T^-$. In this case $\tilde{\beta}(t, t) = (-1)^{|t|} = -1$, so $t \notin \operatorname{rad}\tilde{\beta}$. Therefore $\operatorname{rad}\tilde{\beta} = (\operatorname{rad}\tilde{\beta}) \cap T^+$, concluding the proof. $\square$

Note that if $\operatorname{char}\mathbb{F} = 2$, then $\tilde{\beta} = \beta$. Hence, regardless of the characteristic, we have that if β is nondegenerate, then so is $\tilde{\beta}$. The converse, however, is not true. In view of Lemma 2.61, if $\operatorname{char}\mathbb{F} \neq 2$, the next result characterizes the case where $\tilde{\beta}$ is nondegenerate but β is degenerate.

Lemma 2.62. *Suppose* β *is degenerate. Then* β^+ *is nondegenerate if, and only if,* $(\operatorname{rad}\beta) \cap T^+ = \{e\}$. *If this is the case, then* $\operatorname{rad}\beta = \langle t_1 \rangle$ *for an element* $t_1 \in T^-$ *of order* 2.

Proof. By definition of radical, it is clear that $(\operatorname{rad}\beta) \cap T^+ \subseteq \operatorname{rad}\beta^+$, hence if β^+ is nondegenerate, then $(\operatorname{rad}\beta) \cap T^+ = \{e\}$.

Conversely, suppose $(\operatorname{rad}\beta) \cap T^+ = \{e\}$. Since $\operatorname{rad}\beta \neq \{e\}$, there is $t_1 \in (\operatorname{rad}\beta) \cap T^-$. For any $t_1' \in (\operatorname{rad}\beta) \cap T^-$, we have $t_1 t_1' \in (\operatorname{rad}\beta) \cap T^+ = \{e\}$, so $t_1' = t_1^{-1}$ and, hence, t_1 has order 2 and $\operatorname{rad}\beta = \langle t_1 \rangle$. To show that β^+ is nondegenerate, let $t_0 \in \operatorname{rad}\beta^+$. Then $\beta(t_0, T^+) = 1$ and, clearly, $\beta(t_0, t_1) = 1$. It follows that $t_0 \in \operatorname{rad}\beta$ and, since $t_0 \in T^+$, $t_0 = \{e\}$, concluding the proof. $\square$

In Definition 2.36 we introduced the standard realizations for finite dimensional graded-division algebras that are simple as algebras. We are now going to extend this definition for finite dimensional graded-division superalgebras that are simple as superalgebras, i.e., to include superalgebras of type Q.

If $\mathcal{D} \simeq Q(n)$, then $\mathcal{D}^{\bar{0}} \simeq M(n)$ is a graded-division algebra that is simple as an algebra. Clearly, $\mathcal{D}^{\bar{0}}$ is associated to the pair (T^+, β^+), so by Corollary 2.37 and Lemma 2.62, we have that $\operatorname{rad}\beta = \langle t_1 \rangle$ for an element $t_1 \in T^-$ of order 2, hence $t_1 = (h, \bar{1}) \in G^\#$ for an element $h \in G$ of order at most 2.

Conversely, given a finite subgroup $T^+ \subseteq G$, a nondegenerate alternating bicharacter $\beta^+ \colon T^+ \times T^+ \to \mathbb{F}^\times$ and an element $h \in G$ of order at most 2, let $\mathcal{D}^{\bar{0}}$ be a standard realization (see Definition 2.36) of a matrix algebra with a division grading associated to (T^+, β^+), set $t_1 \coloneqq (h, \bar{1})$ and $T \coloneqq \langle t_1 \rangle \times T^+$, and define $\beta \colon T \times T \to \mathbb{F}^\times$ by $\beta(st_1^i, tt_1^j) \coloneqq \beta^+(s,t)$. It is clear that β is an alternating bicharacter and that $\operatorname{rad}\beta = \langle t_1 \rangle$. If we consider $Q(1) = \mathbb{F}\langle u \rangle$ as in Example 2.48, with h the G-degree of u, then $Q(1)$ is the graded-division superalgebra associated to $(\langle t_1 \rangle, \beta \restriction_{\langle t_1 \rangle \times \langle t_1 \rangle})$. By Lemma 2.33, $\mathcal{D} \coloneqq Q(1) \otimes \mathcal{D}^{\bar{0}}$ is the $G^\#$-graded graded-division algebra associated to (T, β), and, via Kronecker product, it is easy to see that $\mathcal{D}$ is a superalgebra of type Q.

Definition 2.63. The G-graded superalgebra $\mathcal{D} \coloneqq Q(1) \otimes \mathcal{D}^{\bar{0}} = \mathcal{D}^{\bar{0}} \oplus u\mathcal{D}^{\bar{0}}$, where we declare the G-degree of u to be h, will be referred to as a *standard realization* of type Q superalgebra with division grading associated to (T^+, β^+, h).

Corollary 2.64. *The graded-division superalgebra $\mathcal{D}$ is simple as a superalgebra if, and only if,* $(\operatorname{rad}\beta) \cap T^+ = \{e\}$. *More precisely, if this is the case, then*

(i) $\mathcal{D}$ is a superalgebra of type M if, and only if, β is nondegenerate;

(ii) $\mathcal{D}$ is a superalgebra of type Q if, and only if, if β is degenerate.

Proof. If $\mathcal{D}$ is simple as a superalgebra, then by Theorem 2.43, $\mathcal{D} \simeq M(m,n)$ or $\mathcal{D} \simeq Q(n)$ as superalgebra. In both cases, $sZ(\mathcal{D}) = \mathbb{F}$, so $\tilde{\beta}$ is nondegenerate.

Conversely, suppose $\tilde{\beta}$ is nondegenerate. If β is nondegenerate, then $\mathcal{D}$ is simple as an algebra, by Corollary 2.37. If β is degenerate, then combining Lemma 2.62 with Lemma 2.33, we get that, as a $G^\#$-graded algebra, $\mathcal{D} \simeq \mathcal{A} \otimes \mathcal{B}$, where $\mathcal{A}$ is the graded-division algebra associated to $(\operatorname{rad}\beta, \beta \restriction_{(\operatorname{rad}\beta) \times (\operatorname{rad}\beta)})$ and $\mathcal{B}$ is the graded-division algebra associated to $(T^+, \beta \restriction_{T^+ \times T^+})$.

By Lemma 2.62, it is easy to see that $\mathcal{A} \simeq Q(1)$, where we declare the elements of $Q(1)^{\bar{1}}$ to have degree t_1. By Lemma 2.62 and Corollary 2.37, $\mathcal{B}$ is a $G^\#$-graded matrix algebra, say $M_n(\mathbb{F})$, with $\operatorname{supp}\mathcal{B} = T^+$. Hence, as a superalgebra, $\mathcal{D} \simeq \mathcal{A} \otimes \mathcal{B} \simeq$

$Q(1) \otimes M_n(\mathbb{F}) \simeq Q(n)$, where the second isomorphism is given by the Kronecker product. $\square$

Remark 2.65. Clearly, if $\operatorname{char}\mathbb{F} \neq 2$, Lemma 2.61 and Corollary 2.64 imply that $\mathcal{D}$ is simple as a superalgebra if, and only if, $\tilde{\beta}$ is nondegenerate (compare with Corollary 2.37).

Recall that, any nonzero $G^\#$-homogeneous element $d \in \mathcal{D}$ gives rise to the inner automorphism $\operatorname{Int}_d \colon \mathcal{D} \to \mathcal{D}$ defined by $\operatorname{Int}_d(c) := dcd^{-1}$, for all $c \in \mathcal{D}$ (Definition 2.24). We will now generalize this to graded superalgebras:

Definition 2.66. Let $d \in \mathcal{D}$ be a nonzero $G^\#$-homogeneous element. We define the *superinner automorphism* $\operatorname{sInt}_d \colon \mathcal{D} \to \mathcal{D}$ by $\operatorname{sInt}_d(c) := (-1)^{|c||d|}dcd^{-1}$, for all $c \in \mathcal{D}$.

Proposition 2.67. *Let $\psi_0 \colon \mathcal{D} \to \mathcal{D}$ be an automorphism that restricts to the identity on $Z(\mathcal{D}) \cap \mathcal{D}^{\bar{0}}$. Then $\psi_0 = \operatorname{sInt}_{X_t}$ for some $t \in T$.*

Proof. By Lemma 2.38, there is a character $\chi \in \widehat{T}$ such that $\psi_0(X_t) = \chi(t)X_t$, for all $t \in T$. Since ψ_0 is the identity on $sZ(\mathcal{D})$, we have that $\chi(t) = 1$ for all $t \in \operatorname{rad}\tilde{\beta} = (\operatorname{rad}\beta) \cap T^+ = \operatorname{rad}\tilde{\beta}$.

Set $\overline{T} := T/\operatorname{rad}\tilde{\beta}$ and let $\pi \colon T \to \overline{T}$ denote the quotient homomorphism. It is clear that $b \colon \overline{T} \times \overline{T} \to \mathbb{F}^\times$ given by $b(\pi(s), \pi(t)) := \tilde{\beta}(s,t)$ is well-defined and that b is a nondegenerate skew-symmetric bicharacter on $\overline{T}$. Therefore the map $\overline{T} \to \widehat{\overline{T}}$ given by $t \mapsto b(t, \cdot)$ is a group isomorphism.

Since χ is trivial on $\operatorname{rad}\tilde{\beta}$, there is $\bar{\chi} \in \widehat{\overline{T}}$ such that $\chi = \bar{\chi} \circ \pi$ and, since b is nondegenerate, there is an element $t \in T$ such that $b(\pi(t), \cdot) = \bar{\chi}$. A straightforward computation shows that $\psi_0 = \operatorname{sInt}_{X_t}$. $\square$

Definition 2.68. We say that $t_p \in T$ is *a parity element* if $\tilde{\beta}(t_p, t) = (-1)^{p(t)}$ for all $t \in T$.

Clearly, t_p is a parity element if, and only if, $\operatorname{sInt}_{X_{t_p}} = \nu$, where $\nu \colon \mathcal{D} \to \mathcal{D}$ is the parity automorphism $\nu(X_t) = (-1)^{p(t)}X_t$ for every $t \in T$. For Example 2.48, the parity automorphism is given by sInt_u, hence $\bar{1} = \deg u \in T^-$ is a parity element. For Example 2.49, the parity automorphism is given by $\operatorname{sInt}_d = \operatorname{Int}_d$ where

$$d := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

hence $\deg d = (\bar{1}, \bar{0}) \in T^+$ is a parity element.

Corollary 2.69. *There always exists a parity element $t_p \in T$, and the set T_p of all parity elements in T is the coset $t_p(\operatorname{rad}\tilde{\beta})$.*

Proof. Existence follows from Proposition 2.67, since the parity automorphism is trivial on $Z(\mathcal{D}) \cap \mathcal{D}^{\bar{0}}$. The second assertion is clear from the definition. $\square$

We can describe the set $T_p = t_p(\operatorname{rad}\tilde{\beta})$ differently:

Lemma 2.70. *Suppose* $\operatorname{char}\mathbb{F} \neq 2$. *If $T^- = \emptyset$, then the set of all parity elements is* $\operatorname{rad}\tilde{\beta} = \operatorname{rad}\beta = \operatorname{rad}\beta^+$. *Otherwise, $T_p \cap T^+ = (\operatorname{rad}\beta^+) \smallsetminus (\operatorname{rad}\beta)$ and $T_p \cap T^- = (\operatorname{rad}\beta) \cap T^-$.*

Proof. If $T^- = \emptyset$, then $(-1)^{p(t)} = 1$ for all $t \in T$, so $T_p = \operatorname{rad}\tilde{\beta}$ and $\tilde{\beta} = \beta = \beta^+$. We will now suppose $T^- \neq \emptyset$.

If t_p is an even parity element, then clearly $t_p \in \operatorname{rad}\beta^+$ but $t_p \notin \operatorname{rad}\beta$. Conversely, if $t_p \in \operatorname{rad}\beta^+ \smallsetminus \operatorname{rad}\beta$, let $t_1 \in T^-$. Since $\beta(t_p, T^+) = 1$, we have $\beta(t_p, T^-) = \beta(t_p, t_1T^+) = \beta(t_p, t_1) \neq 1$. Since $t_1^2 \in T^+$, $\beta(t_p, t_1)^2 = \beta(t_p, t_1^2) = 1$ and, hence, $\beta(t_p, t_1) = -1$, proving that t_p is a parity element.

Finally, an odd element t_p is a parity element if, and only if, $\tilde{\beta}(t_p, t) = (-1)^{p(t)} = (-1)^{p(t_p)p(t)}$ for all $t \in T$ which, by the definition of $\tilde{\beta}$, is equivalent to $\beta(t_p, t) = 1$ for all $t \in T$. The result follows. $\square$

Note that, if $\operatorname{char}\mathbb{F} \neq 2$, then, by Lemma 2.61, all elements in T_p have the same parity, i.e., either $T_p \cap T^+ = \emptyset$ or $T_p \cap T^- = \emptyset$.

Corollary 2.71. *Let $t_p \in T$ be a parity element. If $t_p \in T^+$, then* $\operatorname{rad}\beta = \operatorname{rad}\tilde{\beta}$ *and* $\operatorname{rad}\beta^+ = (\operatorname{rad}\tilde{\beta}) \cup t_p(\operatorname{rad}\tilde{\beta})$. *If $t_p \in T^-$, then* $\operatorname{rad}\beta = (\operatorname{rad}\tilde{\beta}) \cup t_p(\operatorname{rad}\tilde{\beta})$ *and* $\operatorname{rad}\beta^+ = \operatorname{rad}\tilde{\beta}$. $\square$

2.2.4 Finite dimensional graded-simple superalgebras over an algebraically closed field

We are now going to specialize the main results of Subsection 2.1.4 to the superalgebra case, so we continue assuming that $\mathbb{F}$ is algebraically closed and that G is abelian.

Recall from Subsection 2.2.1 that our classification of graded $\mathcal{D}$-modules of finite rank depend on $\mathcal{D}$ being even or odd.

Definition 2.72. Let $\mathcal{D}$ be a finite dimensional graded-division superalgebra over an algebraically closed field $\mathbb{F}$, associated to the triple (T, β, p), and let $\mathcal{U}$ be a graded right $\mathcal{D}$-supermodule of finite rank. We define the *parameters* of the pair $(\mathcal{D}, \mathcal{U})$ as:

(i) the quadruple $(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}})$, if $\mathcal{D}$ is even and $\mathcal{U}$ is associated to the maps $\kappa_{\bar{0}}, \kappa_{\bar{1}} \colon G/T \to \mathbb{Z}_{\geq 0}$;

(ii) the quadruple (T, β, p, κ), if $\mathcal{D}$ is odd and $\mathcal{U}$ is associated to the map $\kappa \colon G/T^+ \to \mathbb{Z}_{\geq 0}$.

Recall that for every map $\kappa \colon G/T \to \mathbb{Z}_{\geq 0}$ with finite support, we define $|\kappa| := \sum_{x \in G/T} \kappa(x)$. We, then, have that Theorem 2.40 translates the following result in the "super" case:

Theorem 2.73. *Let $(\mathcal{D}, \mathcal{U})$ and $(\mathcal{D}', \mathcal{U}')$ be pairs as in Definition 2.72, with both $\mathcal{D}$ and $\mathcal{D}'$ even. Let $(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}})$ and $(T', \beta', \kappa'_{\bar{0}}, \kappa'_{\bar{1}})$ be the parameters of $(\mathcal{D}, \mathcal{U})$ and $(\mathcal{D}', \mathcal{U}')$, respectively. Then $\operatorname{End}_{\mathcal{D}}(\mathcal{U}) \simeq \operatorname{End}_{\mathcal{D}'}(\mathcal{U}')$ if, and only if, $T = T'$, $\beta = \beta'$, $p = p'$ and there is $g \in G$ such that either $g \cdot \kappa_{\bar{0}} = \kappa'_{\bar{0}}$ and $g \cdot \kappa_{\bar{1}} = \kappa'_{\bar{1}}$, or $g \cdot \kappa_{\bar{0}} = \kappa'_{\bar{1}}$ and $g \cdot \kappa_{\bar{1}} = \kappa'_{\bar{0}}$.* $\square$

Theorem 2.74. *Let $(\mathcal{D}, \mathcal{U})$ and $(\mathcal{D}', \mathcal{U}')$ be pairs as in Definition 2.72, with both $\mathcal{D}$ and $\mathcal{D}'$ odd. Let (T, β, p, κ) and $(T', \beta', p', \kappa')$ be the parameters of $(\mathcal{D}, \mathcal{U})$ and $(\mathcal{D}', \mathcal{U}')$, respectively. Then $\operatorname{End}_{\mathcal{D}}(\mathcal{U}) \simeq \operatorname{End}_{\mathcal{D}'}(\mathcal{U}')$ if, and only if, $T = T'$, $\beta = \beta'$, $p = p'$ and there is $g \in G$ such that $\kappa' = g \cdot \kappa$.* $\square$

Gradings on simple associative superalgebras will be needed in Chapter 5 and are interesting in their own right. Proposition 2.58 and Remark 2.65 together imply that $\operatorname{End}_{\mathcal{D}}(\mathcal{U})$ is simple as a superalgebra if, and only if, $(\operatorname{rad} \beta) \cap T^+ = \{e\}$, which, in the case $\operatorname{char} \mathbb{F} \neq 2$, is equivalent to $\tilde{\beta}$ being nondegenerate (Lemma 2.61). If this is the case, Theorems 2.73 and 2.74 give a classification of abelian group gradings on finite dimensional simple superalgebras, as follows. We note that, in the same way it as with Definition 2.41, there is an abuse of notation in Definitions 2.75, 2.77 and 2.79.

We start with even gradings on superalgebras of type M:

Definition 2.75. Let $m, n \in \mathbb{Z}_{\geq 0}$, not both zero. Given a finite subgroup $T \subseteq G$, a nondegenerate bicharacter $\beta\colon T \times T \to \mathbb{F}^\times$ and maps $\kappa_{\bar{0}}, \kappa_{\bar{1}}\colon G/T \to \mathbb{Z}_{\geq 0}$ with finite support such that $|\kappa_{\bar{0}}|\sqrt{|T|} = m$ and $|\kappa_{\bar{1}}|\sqrt{|T|} = n$, consider:

(i) a standard realization $\mathcal{D}$ (see Definition 2.36) of a matrix algebra with a division grading associated to (T, β);

(ii) the elementary grading (see Definition 1.4) on $M(k_{\bar{0}}, k_{\bar{1}})$ defined by $(\gamma_{\bar{0}}, \gamma_{\bar{1}})$, where $\gamma_{\bar{0}}$ and $\gamma_{\bar{1}}$ are a $k_{\bar{0}}$-tuple and a $k_{\bar{1}}$-tuple of elements of G realizing $\kappa_{\bar{0}}$ and $\kappa_{\bar{1}}$, respectively, where $k_{\bar{0}} \coloneqq |\kappa_{\bar{0}}|$ and $k_{\bar{1}} \coloneqq |\kappa_{\bar{1}}|$ (see Definition 2.18).

We define $\Gamma_M(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}})$ to be the even grading on $M(m, n)$ given by identifying $M(m, n)$ with the graded superalgebra $M(k_{\bar{0}}, k_{\bar{1}}) \otimes \mathcal{D}$ via Kronecker product, i.e.,

$$\deg\left(E_{ij} \otimes d\right) = g_i g_j^{-1} t,$$

for all $1 \leq i, j \leq k_{\bar{0}} + k_{\bar{1}}$, $t \in T$ and $0 \neq d \in \mathcal{D}_t$. We will denote the superalgebra $M(m, n)$ endowed with $\Gamma_M(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}})$ by $M(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}})$.

Corollary 2.76. *Every even G-grading on $M(m, n)$ is isomorphic to $\Gamma_M(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}})$ as in Definition 2.75. Two such gradings $\Gamma_M(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}})$ and $\Gamma_M(T', \beta', \kappa'_{\bar{0}}, \kappa'_{\bar{1}})$ are isomorphic if, and only if, $T = T'$, $\beta = \beta'$ and there is $g \in G$ such that either $g \cdot \kappa_{\bar{0}} = \kappa'_{\bar{0}}$ and $g \cdot \kappa_{\bar{1}} = \kappa'_{\bar{1}}$, or $g \cdot \kappa_{\bar{0}} = \kappa'_{\bar{1}}$ and $g \cdot \kappa_{\bar{1}} = \kappa'_{\bar{0}}$.* □

Note that if $m \neq n$, then $k_{\bar{0}} \neq k_{\bar{1}}$ and, hence, only the case $g \cdot \kappa_{\bar{0}} = \kappa'_{\bar{0}}$ and $g \cdot \kappa_{\bar{1}} = \kappa'_{\bar{1}}$ is possible.

Now let us consider the odd gradings on $M(m, n)$. Note that in this case, by Lemma 2.51, we have that $m = n$.

Definition 2.77. Let $n > 0$ be a natural number. Given a finite subgroup $T \subseteq G^\#$, $T \nsubseteq G$, a nondegenerate alternating bicharacter $\beta\colon T \times T \to \mathbb{F}^\times$ and a map $\kappa\colon G/T^+ \to \mathbb{Z}_{\geq 0}$ with finite support such that $|\kappa|\sqrt{|T|} = n$, consider

(i) a standard realization $\mathcal{D}$ (see Definition 2.36) of a matrix algebra with a division grading associated to (T, β), viewed as a G-graded superalgebra with canonical $\mathbb{Z}_2$-grading given by the projection on the second entry $p\colon T \subseteq G^\# \to \mathbb{Z}_2$;

(ii) the elementary grading (see Definition 1.3) on $M_k(\mathbb{F})$ defined by a k-tuple γ of elements of G realizing κ, where $k \coloneqq |\kappa|$ (see Definition 2.18).

We define $\Gamma_M(T, \beta, p, \kappa)$ to be the odd grading on $M(n, n)$ given by identifying $M(n, n)$ with the graded superalgebra $M_k(\mathbb{F}) \otimes \mathcal{D}$ via Kronecker product. We will denote the superalgebra $M(n, n)$ endowed with $\Gamma_M(T, \beta, p, \kappa)$ by $M(T, \beta, p, \kappa)$.

Corollary 2.78. *Every odd G-grading on $M(n, n)$ is isomorphic to $\Gamma_M(T, \beta, p, \kappa)$ as in Definition 2.41. Two such gradings $\Gamma_M(T, \beta, p, \kappa)$ and $\Gamma_M(T', \beta', p, \kappa')$ are isomorphic if, and only if, $T = T'$, $\beta = \beta'$, $p = p'$ and there is a $g \in G$ such that $g \cdot \kappa = \kappa'$.* □

Finally, we classify the gradings on the superalgebra $Q(n)$. Note that we only have odd gradings in this case.

Definition 2.79. Let $n > 0$ be a natural number. Given a finite subgroup $T^+ \subseteq G$, a nondegenerate bicharacter $\beta\colon T^+ \times T^+ \to \mathbb{F}^\times$, an element $h \in G$ such that $h^2 = 1$ and a map $\kappa\colon G/T^+ \to \mathbb{Z}_{\geq 0}$ with finite support such that $|\kappa|\sqrt{|T^+|} = n$, consider

(i) a standard realization $\mathcal{D}$ (see Definition 2.63) of a superalgebra of type Q with a division grading associated to (T^+, β^+, h);

(ii) the elementary grading (see Definition 1.3) on $M_k(\mathbb{F})$ defined by a k-tuple γ of elements of G of elements of G realizing κ, where $k \coloneqq |\kappa|$ (see Definition 2.18).

We define $\Gamma_Q(T^+, \beta^+, h, \kappa)$ to be the grading on $Q(n)$ given by identifying $Q(n)$ with the graded superalgebra $M_k(\mathbb{F}) \otimes \mathcal{D}$ via Kronecker product. We will denote the superalgebra $Q(n)$ endowed with $\Gamma_Q(T^+, \beta^+, h, \kappa)$ by $Q(T^+, \beta^+, h, \kappa)$.

Corollary 2.80. *Every G-grading on $Q(n)$ is isomorphic to $\Gamma_Q(T^+, \beta^+, h, \kappa)$ as in Definition 2.41. Two such gradings $\Gamma_Q(T^+, \beta^+, h, \kappa)$ and $\Gamma_Q(T'^+, \beta'^+, h', \kappa')$ are isomorphic if, and only if, $T^+ = T'^+$, $\beta^+ = \beta'^+$, $h = h'$ and there is a $g \in G$ such that $g \cdot \kappa = \kappa'$.* □

2.3 Odd gradings on simple associative superalgebras in terms of G

In Section 2.2, our results about both odd graded-simple and odd simple G-graded superalgebras were in terms of the group $G^\#$. We will now present a classification in term of the group G. Throughout this section, $\mathbb{F}$ will be assumed algebraically closed.

2.3.1 Odd graded-simple superalgebras

Let $\mathcal{D}$ be an odd finite dimensional graded-division superalgebra associated to (T, β, p). It is clear that $\mathcal{D}^{\bar{0}}$ is a graded-division algebra associated to (T^+, β^+). Also, given any $t_1 \in T^-$, then $t_1 = (h, \bar{1})$ for some $h \in G$ such that $h^2 \in T^+$ and the map $\chi\colon T^+ \to \mathbb{F}^\times$ given by $\chi(t) := \beta(t_1, t)$ is a character in $\widehat{T^+}$ such that $\chi(h^2) = 1$ and $\chi^2 = \beta^+(h^2, \cdot)$.

Definition 2.81. We say that the odd graded-division superalgebra $\mathcal{D}$ is *associated* to the quadruple (T^+, β^+, h, χ). If $\mathcal{U}$ is a graded right $\mathcal{D}$-supermodule of finite rank associated to the map $\kappa\colon G/T^+ \to \mathbb{F}^\times$, we say that $(T^+, \beta^+, h, \chi, \kappa)$ are the *G-parameters* of the pair $(\mathcal{D}, \mathcal{U})$.

Any quadruple (T^+, β^+, h, χ) associated to $\mathcal{D}$ has enough information to recover T and β, since $T^- = (h, \bar{1})T^+$ and

$$\forall s, t \in T^+,\ \forall i, j \in \mathbb{Z}, \quad \beta(s\, t_1^i, t\, t_1^j) = \beta^+(s, t)\, \chi(s)^{-j}\, \chi(t)^i, \tag{2.5}$$

and, therefore, (T^+, β^+, h, χ) determines the isomorphism class of the graded-division superalgebra $\mathcal{D}$.

Definition 2.82. Let T^+ be any finite subgroup of G and let β^+ be an alternating bicharacter on T^+. A pair $(h, \chi) \in G \times \widehat{T^+}$ is said to be (T^+, β^+)-*admissible* if $h^2 \in T^+$, $\chi(h^2) = 1$ and $\chi^2 = \beta^+(h^2, \cdot)$. We will denote the set of all (T^+, β^+)-admissible pairs by $\mathbf{O}(T^+, \beta^+)$.

For each (T^+, β^+)-admissible pair (h, χ), we can construct a triple (T, β, p) corresponding to a odd graded-division superalgebra. First, we set $t_1 := (h, \bar{1})$ and define $T := T^+ \cup t_1 T^-$. Since $h^2 \in T^+$, T is a subgroup of $G^\#$. We take $\beta\colon T \times T \to \mathbb{F}^\times$ to be as in the following:

Lemma 2.83. *There is a unique alternating bicharacter β on T such that $\beta \restriction_{T^+ \times T^+} = \beta^+$ and $\beta(t_1, t) = \chi(t)$, for all $t \in T^+$.*

Proof. Clearly, a bicharacter β satisfies the conditions above if, and only if, Equation (2.5) is satisfied. It follows that there is at most one such bicharacter.

Let $\widetilde{T}$ be the direct product of T^+ and the infinite cyclic group generated by a new symbol τ, and let $\phi\colon \widetilde{T} \to T$ be the unique group homomorphism such that $\phi(t) = t$ for all $t \in T^+$ and $\phi(\tau) = t_1$. It is easy to see that $\ker\phi = \langle h^2\tau^{-2}\rangle$ and that ϕ is surjective.

Let us define $b\colon \widetilde{T} \times \widetilde{T} \to \mathbb{F}^\times$ by $b(s\tau^i, t\tau^j) := \beta^+(s,t)\,\chi(s)^{-j}\,\chi(t)^i$, for all $s, t \in T^+$ and $i, j \in \mathbb{Z}$. Clearly, b is an alternating bicharacter. We claim that $h^2\tau^{-2} \in \operatorname{rad} b$. Indeed, for all $t \in T^+$ and $j \in \mathbb{Z}$,

$$b(h^2\tau^{-2}, t\tau^j) = \beta^+(h^2, t)\,\chi(h^2)^{-j}\,\chi(t)^{-2} = \chi(t)^2\,\chi(h^2)^{-j}\,\chi(t)^{-2} = \chi(h^2)^{-j} = 1.$$

It follows that $\ker\phi \subseteq \operatorname{rad} b$ and, hence, b induces an alternating bicharacter $\beta\colon T \times T \to \mathbb{F}$ such that $b = \beta \circ (\phi \times \phi)$. From the definition of b, it follows that β satisfies Equation (2.5). $\square$

Combining this with Lemma 2.33, we obtain:

Proposition 2.84. *Let $(h,\chi) \in \mathbf{O}(T^+, \beta^+)$. Then there exists an odd graded-division superalgebra $\mathcal{D}$ associated to the quadruple (T^+, β^+, h, χ).* $\square$

We now have a parametrization of odd graded-division superalgebras in terms of the group G. It remains to see when different quadruples determine isomorphic graded-division superalgebras.

We define a T^+-action on $\mathbf{O}(T^+, \beta^+)$ by

$$\forall\, t \in T^+,\ \forall\,(h,\chi) \in \mathbf{O}(T^+, \beta^+), \quad t \cdot (h,\chi) := (th, \beta^+(t,\cdot)\,\chi)\,. \tag{2.6}$$

To see it is well defined, note that $(th)^2 = t^2h^2 \in T^+$, $\beta^+(t,(th)^2)\chi((th)^2) = \beta^+(t,h^2)\chi(t^2)\chi(h^2) = \chi^{-2}(t)\chi^2(t) = 1$ and $\beta^+((th)^2,\cdot) = \beta^+(t,\cdot)\chi$, so, indeed, $t\cdot(h,\chi) \in \mathbf{O}(T^+,\beta^+)$. It is straightforward that the axioms of action are satisfied.

Theorem 2.85. *Let $\mathcal{D}$ and $\mathcal{D}'$ be finite dimensional odd graded-division superalgebras associated to quadruples (T^+, β^+, h, χ) and $(T'^+, \beta'^+, h', \chi')$, respectively. Then $\mathcal{D} \simeq \mathcal{D}'$ as graded algebras if, and only if, $T^+ = T'^+$, $\beta^+ = \beta'^+$, and the pairs (h,χ) and (h',χ') are in the same T^+-orbit in $\mathbf{O}(T^+, \beta^+)$.*

Proof. Let (T,β,p) and (T',β',p') be the triples associated to $\mathcal{D}$ and $\mathcal{D}'$, respectively. By Proposition 2.32, $\mathcal{D} \simeq \mathcal{D}'$ if, and only if, $T = T'$ and $\beta = \beta'$.

Let $t_1 \coloneqq (h, \bar{1})$ and $t_1' \coloneqq (h', \bar{1})$. By definition, $T = T'$ if, and only if, $T^+ = T'^+$ and $t_1 T^+ = t_1' T^+$. Clearly, the latter condition is equivalent to $h' = th$ for some $t \in T^+$.

Now suppose $T = T'$ and let $t \in T^+$ be such that $h' = th$. By Lemma 2.83, $\beta = \beta'$ if, and only if, $\beta^+ = \beta'^+$ and $\beta(t_1, \cdot) = \beta'(t_1, \cdot)$. Since $t_1' = tt_1$, the latter condition is equivalent to $\chi' = \beta^+(t, \cdot)\chi$. $\square$

Combining this with Theorem 2.74, we obtain:

Corollary 2.86. *Let $(\mathcal{D}, \mathcal{U})$ and $(\mathcal{D}', \mathcal{U}')$ be pairs as in Definition 2.81 and let $(T^+, \beta^+, h, \chi, \kappa)$ and $(T'^+, \beta'^+, h', \chi', \kappa)$ be the G-parameters of $(\mathcal{D}, \mathcal{U})$ and $(\mathcal{D}', \mathcal{U}')$, respectively. Then $\mathrm{End}_{\mathcal{D}}(\mathcal{U}) \simeq \mathrm{End}_{\mathcal{D}'}(\mathcal{U}')$ if, and only if, $T^+ = T'^+$, $\beta^+ = \beta'^+$, the pairs (h, χ) and (h', χ') are in the same T^+-orbit in $\mathbf{O}(T^+, \beta^+)$, and there is $g \in G$ such that $\kappa' = g \cdot \kappa$.* $\square$

2.3.2 Odd G-gradings on simple superalgebras

For specific cases, the parametrization obtained in the previous subsection can be simplified using the following general observation about group actions, which will also be useful in the following chapters:

Lemma 2.87. *Let H be a group, let X and Y be H-sets, and let $\pi\colon X \to Y$ be an H-equivariant map. For every $y \in Y$, the H-action on X restricts to a $\mathrm{Stab}_H(y)$-action on $\pi^{-1}(y)$ and, if the H-action on Y is transitive, the inclusion map $\pi^{-1}(y) \hookrightarrow X$ induces a bijection between $\mathrm{Stab}_H(y)$-orbits in $\pi^{-1}(y)$ and H-orbits in X.*

Proof. The first assertion is straightforward: if $x \in \pi^{-1}(y)$ and $g \in \mathrm{Stab}_H(y)$, then $g \cdot x \in \pi^{-1}(y)$.

Now suppose that the H-action on Y is transitive. It is clear that the $\mathrm{Stab}_H(y)$-orbit of any $x \in \pi^{-1}(y)$ is contained in the H-orbit of x. Conversely, given $x \in X$, by transitivity, there is $g \in H$ such that $g \cdot \pi(x) = y$ and, hence, $g \cdot x \in \pi^{-1}(y)$. We claim that the intersection of the H-orbit of x with $\pi^{-1}(y)$ is precisely the $\mathrm{Stab}_H(y)$-orbit of $g \cdot x$. Indeed, if for some $g' \in H$ we have $g' \cdot x \in \pi^{-1}(y)$, then $y = \pi(g' \cdot x) = (g'g^{-1}) \cdot \pi(g \cdot x) = (g'g^{-1}) \cdot y$, so $g'g^{-1} \in \mathrm{Stab}_H(y)$. $\square$

As an application of this lemma, we will revisit the classification of graded-division superalgebras of type Q. Recall that in this case β^+ is nondegenerate.

Let $\pi\colon \mathbf{O}(T^+, \beta^+) \to \widehat{T^+}$ be the projection on the second entry, which is T^+-equivariant if we define the action of T^+ on $\widehat{T^+}$ by $t \cdot \chi = \beta^+(t, \cdot)\chi$, for all $t \in T$ and $\chi \in \widehat{T^+}$. Since β^+ is nondegenerate, the map $T \to \widehat{T^+}$ sending $s \to \beta^+(s, \cdot)$ is an isomorphism of T^+-sets, hence this T^+-action on $\widehat{T^+}$ is simply transitive.

By Lemma 2.87, it follows that the T^+-orbits in $\mathbf{O}(T^+, \beta^+)$ are in bijection with points in $\pi^{-1}(1)$, where 1 denotes the trivial character of T^+. By Definition 2.82, $(h, 1) \in \mathbf{O}(T^+, \beta^+)$, $(h, 1)$ if, and only if, $h^2 \in T^+$ and $h^2 \in \ker \beta^+$, that is, if, and only if, $h^2 = e$.

We note that, in this case, $(h, \bar{1}) \in \ker \beta$, hence, in view of Lemma 2.62, we have $\ker \beta = \langle (h, \bar{1}) \rangle$ and $\mathcal{D}$ is isomorphic to a standard realization of a graded-division superalgebra of type Q associated to (T^+, β^+, h) (see Definition 2.63). Thus, we recover Corollary 2.80 from Corollary 2.86.

We will now focus on the case of graded superalgebras of type M. Let us fix a finite dimensional odd graded-division superalgebra $\mathcal{D}$ associated to a triple (T, β, p). Since $\mathcal{D}$ is odd, T^- is a non-empty coset of T^+ and, hence, $|T| = |T^+| + |T^-| = 2|T^+|$.

Lemma 2.88. *Suppose* $\operatorname{char} \mathbb{F} = p > 0$. *If p divides $|T|$, then β is degenerate.*

Proof. Let $t \in T$ be an element of order p. For every $s \in T$, we have $1 = \beta(e, s) = \beta(t^p, s) = \beta(t, s)^p$, hence, since $\operatorname{char} \mathbb{F} = p$, $\beta(t, s) = 1$, i.e., $t \in \operatorname{rad} \beta$. $\square$

Since we are interested in the case when $\mathcal{D}$ is of type M, i.e., that β is nondegenerate, from now on we will assume that $\operatorname{char} \mathbb{F} \neq 2$.

We start with a necessary condition on the pair (T^+, β^+) for β be nondegenerate:

Proposition 2.89. *If β is nondegenerate, then there is an element $t_p \in T^+$ of order* 2 *such that* $\operatorname{rad} \beta^+ = \langle t_p \rangle$. *Moreover,* $\beta(t_p, T^-) = -1$.

Proof. Consider $\chi \in \widehat{T}$ given by $\chi(t) = (-1)^{p(t)}$. Since β is nondegenerate, there is $t_p \in T$ of order 2 such that $\chi = \beta(t_p, \cdot)$. Since $\beta(t_p, t_p) = 1 = (-1)^{p(t_p)}$, t_p is even. Hence $t_p \in \operatorname{rad} \beta^+$.

If $0 \neq t \in \operatorname{rad} \beta^+$, using again that β is nondegenerate, there is $t_1 \in T^-$ such that $\beta(t, t_1) \neq 1$. Since $t_1^2 \in T^+$, $\beta(t_p, t_1)^2 = \beta(t, t_1^2) = 1$. It follows that $\beta(t, t_1) = -1$, and since $T^- = t_1 T^+$, $\beta(t,s) = -1$ for all $s \in T^-$. Hence $\beta(t,s) = (-1)^{p(s)}$ and, therefore, $t = t_p$. □

We note that t_p is the (unique) parity element of (T, β, p) (see Definition 2.68).

As we will see in Example 2.95, not every pair (T^+, β^+) with $|\operatorname{rad}\beta^+| = 2$ is a restriction of a triple (T, β, p) with β nondegenerate (see Example 2.95). Roughly speaking, this is because some information about the *squares* of elements in T^- is encoded in (T^+, β^+). To make this precise, we will need the following:

Definition 2.90. Let H be an abelian group and b be a symmetric or skew-symmetric bicharacter on H. For every subgroup $A \subseteq H$, we define the *orthogonal complement to A with respect to b* to be the subgroup

$$A^\perp \coloneqq \{h \in H \mid b(h, A) = 1\}.$$

Lemma 2.91. *Let H and b be as in Definition 2.90, and suppose that H is finite and b is nondegenerate. Then for every subgroup $A \subseteq H$, we have that $|A^\perp| = [H : A]$ and $(A^\perp)^\perp = A$.*

Proof. It is easy to see that the map $A^\perp \to \widehat{\left(\frac{H}{A}\right)}$ given by $t \mapsto \beta(t, \cdot)$ is an isomorphism of groups. Also, since β is nondegenerate, by Lemma 2.88, $\operatorname{char}\mathbb{F}$ does not divide $[H : A]$. In this case, it is well-known that $\widehat{\left(\frac{H}{A}\right)} \simeq \frac{H}{A}$, proving the first assertion.

Since b is symmetric or skew-symmetric, it is clear that $A \subseteq (A^\perp)^\perp$. Using the first assertion, we have that $|A| = |(A^\perp)^\perp|$ and, therefore, $A = (A^\perp)^\perp$. □

Definition 2.92. Let A be an abelian group and $m \in \mathbb{Z}$. We define $A^{[m]} \coloneqq \{a^m \mid a \in A\}$ and $A_{[m]} \coloneqq \{a \in A \mid a^m = e\}$.

Note that $(T^+)^{[2]} \subseteq T^{[2]} \subseteq T^+$, but we can have $(T^+)^{[2]} \neq T^{[2]}$.

Lemma 2.93. *Let H and b be as in Definition 2.90, and suppose that H is finite and b is nondegenerate. Then, for every $m \in \mathbb{Z}$, $(H_{[m]})^\perp = H^{[m]}$.*

Proof. By Lemma 2.91, it suffices to prove that $(H^{[m]})^{\perp} = H_{[m]}$:

$$\begin{aligned}(H^{[m]})^{\perp} &= \{h \in H \mid b(h, g^m) = 1 \text{ for all } g \in H\}\\ &= \{h \in H \mid b(h^m, g) = 1 \text{ for all } g \in H\}\\ &= \{h \in H \mid h^m = e\}\\ &= H_{[m]},\end{aligned}$$

where we are using that b is nondegenerate in the third line. □

Now suppose β is nondegenerate, let t_p be as in Proposition 2.89, set $\overline{G} := G/\langle t_p\rangle$, let $\theta\colon G \to \overline{G}$ be the natural homomorphism, set $\overline{T^+} := \theta(T^+)$ and let $\overline{\beta^+}$ be the bicharacter on $\overline{T^+}$ induced by β^+. Note that $\overline{\beta^+}$ is nondegenerate. Next result explains what was meant by "some information about the squares of elements in T^-" above. It is similar to Lemma 2.93, but does not follow from it.

Proposition 2.94. *Consider $\theta(T^{[2]})$ and $\theta(T^+_{[2]})$ as subgroups of $\overline{T^+}$. We have that $\theta(T^{[2]})$ is the orthogonal complement of $\theta(T^+_{[2]})$ with respect to $\overline{\beta^+}$, i.e., $\theta(T^+_{[2]})^{\perp} = \theta(T^{[2]})$. In particular, $\theta(T^+_{[2]})^{\perp} \subseteq \overline{G}^{[2]}$.*

Proof. By Lemma 2.91, it suffices to prove that $\theta(T^{[2]})^{\perp} = \theta(T^+_{[2]})$:

$$\begin{aligned}\theta(T^{[2]})^{\perp} &= \{\theta(t) \mid t \in T^+ \text{ and } \overline{\beta^+}(\theta(t), \theta(s^2)) = 1 \text{ for all } s \in T\}\\ &= \{\theta(t) \mid t \in T^+ \text{ and } \beta^+(t, s^2) = 1 \text{ for all } s \in T\}\\ &= \{\theta(t) \mid t \in T^+ \text{ and } \beta(t^2, s) = 1 \text{ for all } s \in T\}\\ &= \{\theta(t) \mid t \in T^+ \text{ and } t^2 = e\}\\ &= \theta(T^+_{[2]}),\end{aligned}$$

where we are using that β is nondegenerate in the fourth line. □

Example 2.95. Take $G := T^+ := \mathbb{Z}_2 \times \mathbb{Z}_4$ and define $\beta^+\colon T^+ \times T^+ \to \mathbb{F}^{\times}$ by $\beta^+((i,j),(i',j')) := (-1)^{ij'-i'j}$, for all $i, j, i', j' \in \mathbb{Z}$. It is easy to see that β^+ is an alternating bicharacter and that $\operatorname{rad}\beta^+ = \langle(\bar{0}, \bar{2})\rangle$. We have that $T^+_{[2]} = \langle(\bar{1}, \bar{0}), (\bar{0}, \bar{2})\rangle$ and, hence, $\theta(T^+_{[2]}) = \langle\theta(\bar{1}, \bar{0})\rangle$. Since $\theta(T^+_{[2]})$ is cyclic, $\theta(\bar{1}, \bar{0}) \in \theta(T^+_{[2]})^{\perp}$. But it is clear that $\theta(\bar{1}, \bar{0}) \notin \overline{G}^{[2]}$, so the pair (T^+, β^+) cannot be obtained from a triple (T, β, p) with β nondegenerate.

From now on, let T^+ be a finite subgroup of G and β^+ be an alternating bicharacter on T^+ such that $\operatorname{rad}\beta^+ = \langle t_p \rangle$, where $t_p \in T^+$ is an element of order 2, and define $\overline{G}$, θ, $\overline{T^+}$ and $\overline{\beta^+}$ as above. In Proposition 2.99, we will see that the condition $\theta(T^+_{[2]})^\perp \subseteq \overline{G}^{[2]}$ is sufficient to extend (T^+, β^+) to a triple (T, β, p) with β nondegenerate.

Lemma 2.96. *Let $(h, \chi) \in \mathbf{O}(T^+, \beta^+)$ and let $\mathcal{D}$ be a graded-division superalgebra associated to (T^+, β^+, h, χ). Then $\mathcal{D}$ is of type M if, and only if, $\chi(t_p) = -1$.*

Proof. Let (T, β, p) be the triple constructed from (T^+, β^+, h, χ) and let $t_1 = (h, \bar{1})$.

If β is nondegenerate, then $\chi(t_p) = \beta(t_1, t_p) = -1$ by Proposition 2.89.

Conversely, assume $\chi(t_p) = -1$. Then $\beta(t_p, T^-) = \beta(t_p, t_1 T^+) = \beta(t_p, t_1) = \chi(t_p)^{-1} = -1$. We conclude that $(\operatorname{rad}\beta) \cap T^- = \emptyset$ and that $t_p \notin \operatorname{rad}\beta$. Also, it is clear that $(\operatorname{rad}\beta) \cap T^+ \subseteq \operatorname{rad}\beta^+ = \langle t_p \rangle$, hence $(\operatorname{rad}\beta) \cap T^+ = \{e\}$ and, therefore, $\operatorname{rad}\beta = \{e\}$. □

We define $\mathbf{O}_M(T^+, \beta^+) := \{(h, \chi) \in \mathbf{O}(T^+, \beta^+) \mid \chi(t_p) = -1\}$. Since $t_p \in \operatorname{rad}\beta^+$, we see that the T^+-action on $\mathbf{O}(T^+, \beta^+)$ restricts to $\mathbf{O}_M(T^+, \beta^+)$.

Lemma 2.97. *The set $Y := \{\chi \in \widehat{T^+} \mid \chi(t_p) = -1\}$ is non-empty.*

Proof. Since $\overline{\beta^+}$ is a nondegenerate bicharacter, by Lemma 2.88, $\operatorname{char}\mathbb{F}$ does not divide $|\overline{T^+}|$. Using that $|T^+| = 2|\overline{T^+}|$ and $\operatorname{char}\mathbb{F} \neq 2$, it follows that $\operatorname{char}\mathbb{F}$ does not divide $|T^+|$. It is well-known that, in this case, every character on a subgroup of T^+ can be extended to a character of T^+. We can define a character χ on $\langle t_p \rangle$ by $\chi(t_p) = -1$, and, extending χ, we conclude that $Y \neq \emptyset$. □

Lemma 2.98. *Given $\chi \in Y$, there is a unique element $a \in T^+$ such that $\chi^2 = \beta^+(a, \cdot)$ and $\chi(a) = 1$. Further, an element $h \in G$ is such that $(h, \chi) \in \mathbf{O}_M(T^+, \beta^+)$ if, and only if, $h^2 = a$.*

Proof. Since $\chi^2(t_p) = 1$, χ^2 can be seen as a character on $\overline{T^+}$. It follows that there is a unique element $\bar{a} \in \overline{T^+}$ such that $\chi^2(\bar{t}) = \bar{\beta}(\bar{a}, \bar{t})$, for all $\bar{t} \in \overline{T^+}$. We have that $\theta^{-1}(\bar{a}) = \{a, at_p\}$ for some $a \in T^+$. Since $\chi^2(a) = \chi^2(\bar{a}) = 1$, $\chi(a) = \pm 1$. Relabeling if necessary, we can assume $\chi(a) = 1$ and $\chi(at_p) = -1$. This shows existence and uniqueness. The "further" part follows directly from Definition 2.82. □

Proposition 2.99. *The set $\mathbf{O}_M(T^+, \beta^+)$ is non-empty if, and only if, $\theta(T^+_{[2]})^\perp \subseteq \overline{G}^{[2]}$.*

Proof. The "only if" part follows from Proposition 2.94. For the "if" part, let us fix $\chi \in Y$, which exists by Lemma 2.97, and take $a \in T^+$ as in Lemma 2.98. We want to find $h \in G$ such that $(h, \chi) \in \mathbf{O}_M(T^+, \beta^+)$, i.e., $h^2 = a$.

By the definition of a, for any $b \in T^+_{[2]}$ we have that

$$\overline{\beta^+}(\theta(a), \theta(b)) = \beta^+(a, b) = \chi^2(b) = \chi(b^2) = \chi(e) = 1,$$

so $\theta(a) \in \theta(T^+_{[2]})^\perp$ and, hence, $\theta(a) \in \overline{G}^{[2]}$.

Let $h \in G$ be such that $\theta(a) = \theta(h)^2$, so either $a = h^2$ or $a = h^2 t_p$. If $t_p \in G^{[2]}$, we can change h if necessary to make $a = h^2$. If $t_p \notin G^{[2]}$, it is straightforward that $\theta(T^+_{[2]}) = \theta(T^+)_{[2]}$, so, using Lemma 2.93, we have

$$a \in \theta(T^+_{[2]})^\perp = (\theta(T^+)_{[2]})^\perp = \theta(T^+)^{[2]},$$

and, hence, we can take $h \in T^+$. In this case, h^2 satisfies the same conditions as a in Lemma 2.98:

$$\forall t \in T^+, \quad \chi^2(t) = \beta^+(a, t) = \beta^+(h^2, t)$$

and

$$\chi(h^2) = \chi^2(h) = \beta^+(a, h) = \overline{\beta^+}(\theta(a), \theta(h)) = \overline{\beta^+}(\theta(h)^2, \theta(h)) = 1.$$

By the uniqueness of a, it follows that $h^2 = a$, concluding the proof. □

Combining Corollary 2.86 and Lemma 2.96, we can get a classification result for finite dimensional odd graded superalgebras of type M in terms of orbits of the T^+-action on $\mathbf{O}_M(T^+, \beta^+)$. But, as we did for Q in the end of Subsection 2.3.2, we can simplify this classification using Lemma 2.87. To this end, define a T^+-action on Y by $t \cdot \chi = \beta^+(t, \cdot)\chi$, which is well defined since $t_p \in \operatorname{rad} \beta^+$.

Lemma 2.100. *The T^+-action on Y is transitive.*

Proof. Let $\chi, \chi' \in Y$. We have $(\chi'\chi^{-1})(t_p) = 1$, hence $(\chi'\chi^{-1})$ can be seen as a character on $\overline{T^+} = T^+/\langle t_p \rangle$. By nondegeneracy of $\overline{\beta^+}$, there is $t \in T^+$ such that $(\chi'\chi^{-1}) = \overline{\beta^+}(\bar{t}, \cdot)$ as a character on $\overline{T^+}$. It follows that $(\chi'\chi^{-1}) = \beta^+(t, \cdot)$ as a character on T^+, so $\chi' = \beta(t, \cdot)\chi$. We conclude that $\chi' = t \cdot \chi$, as desired. □

In the case of superalgebras of type Q, we were able to fix the character to be the trivial one. In the present case, we can also fix the character $\chi \in Y$, but there is no canonical choice for it.

Definition 2.101. Given $\chi \in \widehat{T^+}$ such that $\chi(t_p) = -1$, let $a \in T^+$ be the unique element as in Lemma 2.98. We define $\mathbf{O}_M(T^+, \beta^+)_\chi \coloneqq \{h \in G \mid h^2 = a\}$.

The projection onto the second entry $\pi \colon \mathbf{O}_M(T^+, \beta^+) \to Y$ is a T^+-equivariant map and, clearly, $\mathbf{O}_M(T^+, \beta^+)_\chi$ is in bijection with $\pi^{-1}(\chi)$. In what follows, for each pair (T^+, β^+) we fix an arbitrary character $\chi \in Y$ and its corresponding element $a \in T^+$ as above.

Definition 2.102. Given a finite subgroup $T^+ \subseteq G$, an alternating bicharacter $\beta^+ \colon T \times T \to \mathbb{F}^\times$ such that $\operatorname{rad} \beta^+$ is generated by an order 2 element $t_p \in T^+$, an element $h \in \mathbf{O}_M(T^+, \beta^+)_\chi$ and a map $\kappa \colon G/T^+ \to \mathbb{Z}_{\geq 0}$ with finite support, we define $\Gamma_M(T^+, \beta^+, h, \kappa)$ to be the G-grading $\Gamma_M(T, \beta, p, \kappa)$ on $M(n, n)$ (see Definition 2.77), where $n \coloneqq |\kappa| \sqrt{2|T^+|}$ and (T, β, p) is constructed from T^+, β^+, h and χ as in Subsection 2.3.1. The superalgebra $M(n, n)$ endowed with $\Gamma_M(T^+, \beta^+, h, \kappa)$ will be denoted by $M(T^+, \beta^+, h, \kappa)$.

Theorem 2.103. *Every odd G-grading on $M(n, n)$ is isomorphic to $\Gamma_M(T^+, \beta^+, h, \kappa)$ as in Definition 2.102. Two such gradings $\Gamma_M(T^+, \beta^+, h, \kappa)$ and $\Gamma_M(T'^+, \beta'^+, h', \kappa')$ are isomorphic if, and only if, $T^+ = T'^+$, $\beta^+ = \beta'^+$, $h' \in \{h, ht_p\}$ and there is $g \in G$ such that $\kappa' = g \cdot \kappa$.*

Proof. From Corollary 2.78, every grading is isomorphic to one of this form. The isomorphism condition follows from Corollary 2.86 and Lemma 2.87, by noting that, considering the action of T^+ on Y, $\operatorname{Stab}_{T^+}(\chi) = \operatorname{rad} \beta^+ = \{e, t_p\}$. $\square$

Chapter 3

Super-anti-automorphisms on Graded-Simple Associative Superalgebras

In this and in the next chapter we will study graded associative superalgebras with superinvolution. Here we will focus on the graded-simple case, adapting the approach of [Eld10] (see also [EK13, Section 2.4]) to superalgebras. We also take inspiration from [BKR18], even though we do not use results from there. As we will see in Chapter 5 (assuming that $\mathbb{F}$ is algebraically closed and $\operatorname{char}\mathbb{F} = 0$), the gradings on superinvolution-simple associative superalgebras making them graded-simple give us all gradings on the Lie superalgebras of series B, C, D and P and the Type II gradings (Definition 5.12) on the Lie superalgebras of series A and Q, except for type $A(1,1)$.

Throughout this chapter, the term *(super)algebra* will mean *associative (super)algebra*. We start in a very general setting, adding more assumptions as the chapter develops. In Section 3.1, we introduce the concept of superdual for graded supermodules over a graded-division superalgebra $\mathcal{D}$ and for $\mathcal{D}$-linear maps between them. In Sections 3.2 to 3.4, we deal with graded-simple superalgebras satisfying d.c.c. on graded left superideals, over an arbitrary field. Sections 3.2 and 3.3 handle the problem more abstractly: in the former, we describe super-anti-automorphisms in terms of sesquilinear forms (see Definition 3.13 and Theorem 3.18) and, in the latter, we solve the isomorphism problem for graded superalgebras with super-anti-automorphism in terms of this description (see Theorem 3.27 and Corollary 3.29). In Section 3.4, we express super-anti-automorphisms in the language of matrices with entries in $\mathcal{D}$ (Proposition 3.32). In Section 3.5, we study necessary and sufficient conditions on the sesquilinear form for the super-anti-automorphism to be involutive, assuming that the homogeneous components of the graded-division superalgebra are 1-dimensional (Theorem 3.37). Finally, Section 3.6 is devoted to classification results (up to isomorphism) assuming the base field is algebraically closed with characteristic different from 2. It is divided into three subsections: Subsection 3.6.1 deals with graded-division superalgebras, Subsection 3.6.2 with graded supermodules with sesquilinear forms and Subsection 3.6.3 with graded-simple superalgebras with superinvolution. The main classification result is Theorem 3.54.

We will assume the grading group G to be abelian. We can do this without loss of generality because of the next result, which is a generalization of [BSZ05, Theorem 1] (see also [EK13, Proposition 2.49] and its proof, which works in this more general situation).

Proposition 3.1. *Let (R, φ) be an associative G-graded superalgebra with super-anti-automorphism. If R does not have nonzero proper graded φ-invariant superideals, then the subgroup of G generated by the support of R is abelian.* □

3.1 The superdual

Let $\mathcal{D}$ be a graded-division superalgebra. We start with the definition of the superdual for G-graded $\mathcal{D}$-supermodules or, equivalently, the dual for $G^{\#}$-graded $\mathcal{D}$-modules (see [EK13, Definition 2.56]).

Definition 3.2. Let $\mathcal{U}$ be a graded right $\mathcal{D}$-supermodule of finite rank. The *superdual of* $\mathcal{U}$ is defined to be $\mathcal{U}^\star := \mathrm{Hom}_{\mathcal{D}}(\mathcal{U}, \mathcal{D})$, with its usual $G^{\#}$-grading (see Proposition 2.21). We give $\mathcal{U}^\star$ the structure of a graded *left* $\mathcal{D}$-module: if $d \in \mathcal{D}$ and $f \in \mathcal{U}^\star$, then we define $(df)(u) = d\,f(u)$ for all $u \in \mathcal{U}$.

We have used right supermodules over a graded-division superalgebra to classify graded-simple superalgebras (see Subsections 2.1.1 and 2.2.1). Hence, it is convenient to see the superdual of a right supermodule again as a right supermodule over a suitable graded-division superalgebra.

Definition 3.3. Let R be a G-graded superalgebra. We define the *superopposite G-graded superalgebra* R^{sop} to be R as a graded superspace, but with a different product. When an element $r \in R$ is regarded as an element of R^{sop}, we will denote it by $\bar{r}$. The product on R^{sop} is defined by $\bar{r}\,\bar{s} = (-1)^{|r||s|}\,\overline{sr}$ for all $r, s \in R^{\bar{0}} \cup R^{\bar{1}}$.

Note that R^{sop} being a G-graded superalgebra depends on the assumption that G is abelian.

Remark 3.4. If $\varphi\colon R \to S$ is a super-anti-isomorphism between the G-graded superalgebras R and S, then the map φ can be viewed as an isomorphism $R \to S^{\text{sop}}$ or $R^{\text{sop}} \to S$.

It is easy to see that $\mathcal{D}^{\text{sop}}$ is also a graded-division superalgebra. If $\mathcal{V}$ is a graded left $\mathcal{D}$-supermodule, then we can regard $\mathcal{V}$ as a graded right $\mathcal{D}^{\text{sop}}$-supermodule by means of the action $v\bar{d} := (-1)^{|d||v|}dv$, for all $d \in \mathcal{D}^{\bar{0}} \cup \mathcal{D}^{\bar{1}}$ and $v \in \mathcal{V}^{\bar{0}} \cup \mathcal{V}^{\bar{1}}$. In particular, if $\mathcal{U}$ is as in Definition 3.2, then the left $\mathcal{D}$-supermodule $\mathcal{U}^\star$ can be regarded as a graded right $\mathcal{D}^{\text{sop}}$-supermodule.

Definition 3.5. Let $\mathcal{U}$ and $\mathcal{V}$ be graded right $\mathcal{D}$-modules of finite rank. Given a homogeneous $\mathcal{D}$-linear map $L : \mathcal{U} \to \mathcal{V}$, we define the *superdual of* L to be the $\mathbb{F}$-linear map $L^\star\colon \mathcal{V}^\star \to \mathcal{U}^\star$ defined by

$$L^\star(f) = (-1)^{|L||f|} f \circ L,$$

for all $f \in (\mathcal{V}^\star)^{\bar{0}} \cup (\mathcal{V}^\star)^{\bar{1}}$. It is easy to see that $L^\star$ is $\mathcal{D}^{\text{sop}}$-linear. We extend the definition of superdual to every map in $\operatorname{Hom}_{\mathcal{D}}(\mathcal{U}, \mathcal{V})$ by linearity.

Definition 3.6. If $\mathcal{B} = \{u_1, \dots, u_k\}$ is a graded basis, we can consider its two *superdual bases* ${}^\star\mathcal{B} = \{{}^\star u_1, \dots, {}^\star u_k\}$ and $\mathcal{B}^\star = \{u_1^\star, \dots, u_k^\star\}$ in $\mathcal{U}^\star$, where ${}^\star u_i\colon \mathcal{U} \to \mathcal{D}$ is defined by ${}^\star u_i(u_j) = \delta_{ij}$ and $u_i^\star\colon \mathcal{U} \to \mathcal{D}$ is defined by $u_i^\star(u_j) = (-1)^{|u_i||u_j|}\delta_{ij}$. Clearly, $\deg({}^\star u_i) = \deg(u_i^\star) = (\deg u_i)^{-1}$.

Remark 3.7. In the case $\mathcal{D} = \mathbb{F}$, if we denote by $[L]$ the matrix of L with respect to the graded bases $\mathcal{B}$ of $\mathcal{U}$ and $\mathcal{C}$ of $\mathcal{V}$, then the supertranspose $[L]^{s\top}$ is the matrix corresponding to $L^\star$ with respect to the superdual bases $\mathcal{C}^\star$ and $\mathcal{B}^\star$.

Since $\mathcal{U}^\star$ is a graded right $\mathcal{D}^{\text{sop}}$-supermodule of finite rank, we can define $\mathcal{U}^{\star\star} := \operatorname{Hom}_{\mathcal{D}^{\text{sop}}}(\mathcal{U}^\star, \mathcal{D}^{\text{sop}})$, which is a graded left $\mathcal{D}^{\text{sop}}$-supermodule and, hence, a graded right $\mathcal{D}$-supermodule. As with finite dimensional vector spaces, there is a canonical isomorphism $\mathcal{U} \to \mathcal{U}^{\star\star}$:

Lemma 3.8. *The $\mathbb{F}$-linear map $\epsilon_{\mathcal{U}} \colon \mathcal{U} \to \mathcal{U}^{\star\star}$ defined by $\epsilon_{\mathcal{U}}(u)(f) = (-1)^{|u||f|}\,\overline{f(u)}$, for all $u \in \mathcal{U}^{\bar{0}} \cup \mathcal{U}^{\bar{1}}$ and $f \in (\mathcal{U}^\star)^{\bar{0}} \cup (\mathcal{U}^\star)^{\bar{1}}$, is an isomorphism of graded right $\mathcal{D}$-supermodules.*

Proof. For brevity, we will write ϵ for $\epsilon_{\mathcal{U}}$. First we check that $\epsilon(u)$ belongs to $\mathcal{U}^{\star\star}$, i.e., it is a $\mathcal{D}^{\mathrm{sop}}$-linear map. Let $f \in (\mathcal{U}^\star)^{\bar{0}} \cup (\mathcal{U}^\star)^{\bar{1}}$ and $d \in \mathcal{D}^{\bar{0}} \cup \mathcal{D}^{\bar{1}}$. Then,

$$\begin{aligned}
\epsilon(u)(f\bar{d}) &= (-1)^{|u||f\bar{d}|}\,\overline{f\bar{d}(u)} = (-1)^{|u|(|f|+|d|)}\,(-1)^{|f||d|}\,\overline{df(u)} \\
&= (-1)^{|u||f|+|u||d|+|f||d|}(-1)^{(|f|+|u|)|d|}\,\overline{f(u)}\,\bar{d} \\
&= (-1)^{|u||f|}\,\overline{f(u)}\bar{d} = \epsilon(u)(f)\bar{d}.
\end{aligned}$$

It is easy to see that $\epsilon(u)$ has the same $G^{\#}$-degree as u, for all $u \in \mathcal{U}$.

To show $\mathcal{D}$-linearity of ϵ, we compute:

$$\begin{aligned}
\epsilon(ud)(f) &= (-1)^{|ud||f|}\,\overline{f(ud)} = (-1)^{(|u|+|d|)|f|}\,\overline{f(u)d} \\
&= (-1)^{|u||f|+|d||f|}(-1)^{(|f|+|u|)|d|}\bar{d}\,\overline{f(u)} \\
&= (-1)^{|u||f|+|d||u|}\bar{d}\,\overline{f(u)} = (-1)^{|d||u|}\bar{d}\,\epsilon(u)(f),
\end{aligned}$$

for all $f \in (\mathcal{U}^\star)^{\bar{0}} \cup (\mathcal{U}^\star)^{\bar{1}}$, so

$$\epsilon(ud) = (-1)^{|d||u|}\bar{d}\epsilon(u) = \epsilon(u)d.$$

To see that ϵ is an isomorphism, let $\{u_1, \ldots, u_k\}$ be a graded basis of $\mathcal{U}$. We have that $\epsilon(u_i)(u_j^\star) = (-1)^{|u_i||u_j|}\,\overline{u_j^\star(u_i)} = \delta_{ij}$, i.e., $\epsilon(u_i) = {}^\star(u_i^\star) = ({}^\star u_i)^\star$. Therefore, ϵ sends a graded basis to a graded basis, concluding the proof. $\square$

We note in passing that the sign in the definition of $\epsilon_{\mathcal{U}}$ is essential: without it, $\epsilon_{\mathcal{U}}$ would not be well-defined in the case of odd $\mathcal{D}$.

Definition 3.9. Let $\mathcal{U}$ and $\mathcal{V}$ be graded right $\mathcal{D}$-supermodules of finite rank. For every $\mathcal{D}$-linear map $L \colon \mathcal{V}^\star \to \mathcal{U}^\star$, we define ${}^\star L \colon \mathcal{U} \to \mathcal{V}$ to be the map $\epsilon_{\mathcal{V}}^{-1} \circ L^\star \circ \epsilon_{\mathcal{U}}$, where $\epsilon_{\mathcal{U}}$ and $\epsilon_{\mathcal{V}}$ are as in Lemma 3.8.

The following result is a straightforward computation:

Proposition 3.10. *Let $\mathcal{U}$ be a nonzero graded right $\mathcal{D}$-supermodule of finite rank. The map* $\operatorname{End}_{\mathcal{D}}(\mathcal{U}) \to \operatorname{End}_{\mathcal{D}^{\mathrm{sop}}}(\mathcal{U}^\star)$ *defined by $L \mapsto L^\star$ is a degree-preserving super-anti-isomorphism and its inverse is given by $L \mapsto {}^\star L$.* □

Since G is abelian, $G^\#/T$ is a group and, hence, the map $G^\#/T \to G^\#/T$ given by $x \mapsto x^{-1}$ is well-defined. From the construction of the superdual basis, it is easy to see that $\dim_{\mathcal{D}} \mathcal{U}_x = \dim_{\mathcal{D}^{\mathrm{sop}}} \mathcal{U}^\star_{x^{-1}}$, for all $x \in G^\#/T$. Hence, if $\kappa\colon G^\#/T \to \mathbb{Z}_{\geq 0}$ is the map associated to $\mathcal{U}$ as a $G^\#$-graded right $\mathcal{D}$-module (see Subsection 2.1.1), then $\kappa^\star\colon G^\#/T \to \mathbb{Z}_{\geq 0}$ defined by $\kappa^\star(x) = \kappa(x^{-1})$ is the map associated to $\mathcal{U}^\star$ as a $G^\#$-graded right $\mathcal{D}^{\mathrm{sop}}$-module.

It is straightforward to translate this to the maps $G/T \to \mathbb{Z}_{\geq 0}$ (even $\mathcal{D}$) and $G/T^+ \to \mathbb{Z}_{\geq 0}$ (odd $\mathcal{D}$) associated to the G-graded supermodule $\mathcal{U}$, as in Subsection 2.2.1. If $\mathcal{D}$ is even and $\kappa_{\bar 0}, \kappa_{\bar 1}$ are the maps associated to $\mathcal{U}$, then $\kappa^\star_{\bar 0}, \kappa^\star_{\bar 1}$ are the maps associated to $\mathcal{U}^\star$. If $\mathcal{D}$ is odd and κ is the map associated to $\mathcal{U}$, then $\kappa^\star$ is the map associated to $\mathcal{U}^\star$.

Finally, let us assume that $\mathbb{F}$ is algebraically closed and $\mathcal{D}$ is finite dimensional. If (T, β, p) is the triple associated to $\mathcal{D}$, then it is clear that $\operatorname{supp} \mathcal{D}^{\mathrm{sop}} = T$ and that the parity map for $\mathcal{D}^{\mathrm{sop}}$ is also p. Moreover, if $s, t \in T$, $0 \neq X_s \in \mathcal{D}_s$ and $0 \neq X_t \in \mathcal{D}_t$, then, following the notation in Definition 3.3, we have:

$$\overline{X_s}\,\overline{X_t} = (-1)^{|s||t|}\,\overline{X_t X_s} = (-1)^{|s||t|}\,\beta(t,s)\overline{X_s X_t} = \beta(t,s)\overline{X_t}\,\overline{X_s} = \beta(s,t)^{-1}\,\overline{X_t}\,\overline{X_s}.$$

We conclude that (T, β^{-1}, p) is the triple associated to $\mathcal{D}^{\mathrm{sop}}$.

3.2 Super-anti-automorphisms and sesquilinear forms

Let $\mathcal{D}$ be a graded-division superalgebra and let $\mathcal{U}$ be a nonzero graded right $\mathcal{D}$-supermodule of finite rank. Consider the graded superalgebra $R := \operatorname{End}_{\mathcal{D}}(\mathcal{U})$. Then $\mathcal{U}$ is an $(R, \mathcal{D})$-superbimodule. By Lemma 2.29, we have a natural identification between $\mathcal{D}$ and $\operatorname{End}_R(\mathcal{U})$.

As we saw on Section 3.1, $\mathcal{U}^\star = \mathrm{Hom}_{\mathcal{D}}(\mathcal{U}, \mathcal{D})$ is a graded right $\mathcal{D}^{\mathrm{sop}}$-supermodule, through $f\bar{d} = (-1)^{|f||d|} df$. Also, we can make $\mathcal{U}^\star$ a graded right R-supermodule by defining $(fr)(u) := f(ru)$ for all $f \in \mathcal{U}^\star$, $r \in R$ and $u \in \mathcal{U}$ (i.e., $fr = f \circ r$, since $R = \mathrm{End}_{\mathcal{D}}(\mathcal{U})$). Hence, we can consider $\mathcal{U}^\star$ as a graded left R^{sop}-supermodule via $\bar{r}f = (-1)^{|r||f|} fr$, and, in this way, $\mathcal{U}^\star$ becomes an $(R^{\mathrm{sop}}, \mathcal{D}^{\mathrm{sop}})$-superbimodule.

Lemma 3.11. *As a left R^{sop}-supermodule, $\mathcal{U}^\star$ is graded-simple. Also, the right action of $\mathcal{D}^{\mathrm{sop}}$ on $\mathcal{U}^\star$ gives an isomorphism $\mathcal{D}^{\mathrm{sop}} \to \mathrm{End}_{R^{\mathrm{sop}}}(\mathcal{U}^\star)$.*

Proof. By definition of the superdual of an operator, the left R^{sop}-action on $\mathcal{U}^\star$ corresponds to the representation $R^{\mathrm{sop}} \to \mathrm{End}_{\mathcal{D}^{\mathrm{sop}}}(\mathcal{U}^\star)$ given by $r \mapsto r^\star$, which is an isomorphism since $\mathcal{U}$ is a graded $\mathcal{D}$-supermodule of finite rank (Proposition 3.10). Identifying R^{sop} and $\mathrm{End}_{\mathcal{D}^{\mathrm{sop}}}(\mathcal{U}^\star)$ via this isomorphism, we can apply Lemma 2.29. □

Now let $\varphi : R \to R$ be a super-anti-automorphism that preserves the G-degree (i.e., is homogeneous of degree e with respect to the G-grading). We can see it as an isomorphism $R \to R^{\mathrm{sop}}$, $r \mapsto \overline{\varphi(r)}$, and use it to identify R with R^{sop}. In particular, we will make the left R^{sop}-action on $\mathcal{U}^\star$ into a left R-action via $r \cdot f = \overline{\varphi(r)} f$, for all $r \in R$ and all $f \in \mathcal{U}^\star$. In other words,

$$r \cdot f = (-1)^{|r||f|} f \circ \varphi(r). \tag{3.1}$$

We will now consider $\mathcal{U}^\star$ as an $(R, \mathcal{D}^{\mathrm{sop}})$-superbimodule. In particular, the superalgebra $\mathrm{End}_R(\mathcal{U}^\star)$ should be understood as the set of R-linear maps with respect to the R-action on the left (which is the same set as $\mathrm{End}_{R^{\mathrm{sop}}}(\mathcal{U}^\star)$). Also, since the action is on the left, we will follow the convention of writing R-linear maps on the right.

From Lemma 3.11, it follows that $\mathcal{U}^\star$ is a simple graded left R-supermodule and that we can identify $\mathcal{D}^{\mathrm{sop}}$ with $\mathrm{End}_R(\mathcal{U}^\star) = \mathrm{End}_{R^{\mathrm{sop}}}(\mathcal{U}^\star)$. But from [EK13, Lemma 2.7], R has only one graded-simple supermodule up to isomorphism and shift, hence there is an invertible R-linear map $\varphi_1 : \mathcal{U} \to \mathcal{U}^\star$ which is homogeneous of some degree $(g_0, \alpha) \in G^\#$. Fix one such φ_1.

Lemma 3.12. *A map $\varphi_1' : \mathcal{U} \to \mathcal{U}^\star$ is R-linear and homogeneous with respect to the $G^\#$-grading if, and only if, there is an element $\bar{d} \in \mathcal{D}^{\mathrm{sop}} = \mathrm{End}_R(\mathcal{U}^\star)$ homogeneous with respect to the $G^\#$-grading such that $\varphi_1' = \varphi_1 \bar{d}$, where juxtaposition represents composition of maps written on the right.*

Proof. This follows from the fact that $\varphi_1^{-1}\varphi' \in \mathcal{D}^{\mathrm{sop}} = \mathrm{End}_R(\mathcal{U}^\star)$ if, and only if, $\varphi' \in \mathrm{Hom}_R(\mathcal{U}, \mathcal{U}^\star)$. □

We will use the R-linear map φ_1 to construct a nondegenerate sesquilinear form B on $\mathcal{U}$.

Definition 3.13. We say that a map $B\colon \mathcal{U} \times \mathcal{U} \to \mathcal{D}$ is a *sesquilinear form on* $\mathcal{U}$ if it is $\mathbb{F}$-bilinear, $G^{\#}$-homogeneous if considered as a linear map $\mathcal{U} \otimes \mathcal{U} \to \mathcal{D}$, and there is $\varphi_0\colon \mathcal{D} \to \mathcal{D}$ a degree-preserving super-anti-automorphism such that, for all $u, v \in \mathcal{U}$ and $d \in \mathcal{D}$,

(i) $B(u, vd) = B(u, v)d$;

(ii) $B(ud, v) = (-1)^{(|B|+|u|)|d|}\varphi_0(d)B(u, v)$.

If we want to specify the super-anti-automorphism φ_0, we will say that B is *sesquilinear with respect to* φ_0 or that B is φ_0*-sesquilinear*. The *(left) radical* of B is the set $\mathrm{rad}\, B := \{u \in \mathcal{U} \mid B(u, v) = 0 \text{ for all } v \in \mathcal{U}\}$. We say that the form B is *nondegenerate* if $\mathrm{rad}\, B = 0$.

Lemma 3.14. *Let $B \neq 0$ be a sesquilinear form on $\mathcal{U}$. Then there is a unique super-anti-automorphism φ_0 on $\mathcal{D}$ such that B is sesquilinear with respect to φ_0.*

Proof. Since $B \neq 0$, there are homogeneous elements $u, v \in \mathcal{U}$ such that $B(u, v) \neq 0$ and, hence, $B(u, v)$ is an invertible element of $\mathcal{D}$. Suppose B is sesquilinear with respect to super-anti-automorphisms φ_0 and φ_0' on $\mathcal{D}$. Then, for all $d \in \mathcal{D}^{\bar{0}} \cup \mathcal{D}^{\bar{1}}$, we have

$$B(ud, v) = (-1)^{(|B|+|u|)|d|}\varphi_0(d)B(u, v) = (-1)^{(|B|+|u|)|d|}\varphi_0'(d)B(u, v)$$

and, therefore, $\varphi_0(d) = \varphi_0'(d)$. □

Suppose for now that a degree-preserving super-anti-automorphism $\varphi_0 : \mathcal{D} \to \mathcal{D}$ is given. We can use it to define a right $\mathcal{D}$-action on $\mathcal{U}^\star$ by interpreting it as an isomorphism from $\mathcal{D}$ to $\mathcal{D}^{\mathrm{sop}}$ and putting

$$f \cdot d := f\overline{\varphi_0(d)}, \tag{3.2}$$

for all $f \in \mathcal{U}^\star$ and $d \in \mathcal{D}$. Using this action, we have $\mathrm{End}_{\mathcal{D}}(\mathcal{U}^\star) = \mathrm{End}_{\mathcal{D}^{\mathrm{sop}}}(\mathcal{U}^\star)$. We also have the following:

Proposition 3.15. *Let $\varphi_0\colon \mathcal{D} \to \mathcal{D}$ be a a degree-preserving super-anti-automorphism and consider the right $\mathcal{D}$-supermodule structure on $\mathcal{U}^\star$ given by Equation* (3.2)*. Then the φ_0-sesquilinear forms $B : \mathcal{U} \times \mathcal{U} \to \mathcal{D}$ are in a one-to-one correspondence with the homogeneous $\mathcal{D}$-linear maps $\theta\colon \mathcal{U} \to \mathcal{U}^\star$ via $B \mapsto \theta$ where $\theta(u) \coloneqq B(u, \cdot)$, for all $u \in \mathcal{U}$ or, inversely, $\theta \mapsto B$ where $B(u, v) \coloneqq \theta(u)(v)$, for all $u, v \in \mathcal{U}$. Moreover, B is nondegenerate if, and only if, θ is an isomorphism.*

Proof. Suppose B is given. Condition (i) of Definition 3.13 tells us that θ defined this way is, indeed, a map from $\mathcal{U}$ to $\mathcal{U}^\star$. Also, it is easy to check that θ is homogeneous of the same parity and degree as B.

Recalling the left $\mathcal{D}$-action on $\mathcal{U}^\star$, condition (ii) tells us that, for all $u \in \mathcal{U}$ and $d \in \mathcal{D}$,

$$\theta(ud) = (-1)^{|d|(|\theta|+|u|)}\varphi_0(d)\theta(u),$$

which, by the definition of the right $\mathcal{D}^{\text{sop}}$-action on $\mathcal{U}^\star$, is equivalent to $\theta(ud) = \theta(u)\overline{\varphi_0(d)}$, i.e., θ is, indeed, $\mathcal{D}$-linear considering the left $\mathcal{D}$-action on $\mathcal{U}^\star$ given by Equation (3.2).

To show that the correspondence is bijective, note that all the considerations above can be reversed when, given $\theta\colon \mathcal{U} \to \mathcal{U}^\star$, we define $B(u, v) \coloneqq \theta(u)(v)$.

The "moreover" part follows from the fact that $\operatorname{rad} B = \ker\theta$, so the nondegeneracy of B is equivalent to θ being injective. But $\mathcal{U}$ and $\mathcal{U}^\star$ have the same (finite) rank over $\mathcal{D}$, so θ is injective if, and only if, it is bijective. $\square$

Coming back to our map $\varphi_1\colon \mathcal{U} \to \mathcal{U}^\star$ and using the identifications $\mathcal{D} = \operatorname{End}_R(\mathcal{U})$ and $\mathcal{D}^{\text{sop}} = \operatorname{End}_R(\mathcal{U}^\star)$ introduced above, consider the map $\mathcal{D} \to \mathcal{D}^{\text{sop}}$ sending $d \mapsto (-1)^{|d||\varphi_1|}\varphi_1^{-1} d\,\varphi_1$, where juxtaposition denotes composition of maps on the right. It is straightforward to check that this map is an isomorphism and, hence, we can consider it as a super-anti-automorphism $\varphi_0\colon \mathcal{D} \to \mathcal{D}$. Then, for all $u \in \mathcal{U}$ and $d \in \mathcal{D}$, we have

$$(ud)\varphi_1 = u(d\varphi_1) = u(\varphi_1\varphi_1^{-1}d\varphi_1) = (-1)^{|d||\varphi_1|}(u\,\varphi_1)\varphi_0(d). \tag{3.3}$$

Definition 3.16. Let $\mathcal{V}$ and $\mathcal{V}'$ be left R-supermodules and let $\psi : \mathcal{V} \to \mathcal{V}'$ be a $\mathbb{Z}_2$-homogeneous R-linear map. We define $\psi^\circ : \mathcal{V} \to \mathcal{V}'$ to be the following map, written on the left:

$$\forall v \in \mathcal{V}^{\bar 0} \cup \mathcal{V}^{\bar 1}, \quad \psi^\circ(v) = (-1)^{|\psi||v|} v\psi.$$

For example, using the identification $\mathcal{D}^{\mathrm{sop}} = \mathrm{End}_R(\mathcal{U}^\star)$ as before, the left $\mathcal{D}$-action on $\mathcal{U}^\star$ is given by $df = \bar{d}^\circ(f)$, for all $d \in \mathcal{D}$ and $f \in \mathcal{U}^\star$.

Lemma 3.17. *Under the conditions of Definition 3.16, we have that, for all $r \in R^{\bar{0}} \cup R^{\bar{1}}$ and $v \in \mathcal{V}$, $\psi^\circ(rv) = (-1)^{|\psi||r|} r\psi^\circ(v)$. Further, given another $\mathbb{Z}_2$-homogeneous R-linear map $\tau : \mathcal{V}' \to \mathcal{V}''$, we have $(\psi\tau)^\circ = (-1)^{|\psi||\tau|}\tau^\circ\psi^\circ$.* □

Using the notation just introduced, we can rewrite Equation (3.3) as follows:

$$\begin{aligned}\varphi_1^\circ(ud) &= (-1)^{|\varphi_1|(|u|+|d|)}(ud)\varphi_1\\ &= (-1)^{|\varphi_1||u|}(u\,\varphi_1)\overline{\varphi_0(d)} = \varphi_1^\circ(u)\overline{\varphi_0(d)},\end{aligned}$$

which means, considering the right $\mathcal{D}$-action defined via Equation (3.2), that φ_1° is $\mathcal{D}$-linear. (Note, however, that Lemma 3.17 shows that φ_1° is not R-linear, in general.)

Now we define $B : \mathcal{U} \times \mathcal{U} \to \mathcal{D}$ by $B(u, v) = \varphi_1^\circ(u)(v)$. By Proposition 3.15, we have that B is a nondegenerate φ_0-sesquilinear map. Using Lemma 3.17 and Equation (3.1), we have

$$\begin{aligned}B(ru, v) = \varphi_1^\circ(ru)(v) &= (-1)^{|r||\varphi_1|}\Big(r \cdot \varphi_1^\circ(u)\Big)(v)\\ &= (-1)^{|r||\varphi_1|}(-1)^{|r|(|\varphi_1|+|u|)}\Big(\varphi_1^\circ(u) \circ \varphi(r)\Big)(v)\\ &= (-1)^{|r||u|}\varphi_1^\circ(u)\Big(\varphi(r)v\Big) = (-1)^{|r||u|}B(u, \varphi(r)v).\end{aligned}$$

We have proved one direction of Theorem 3.18, below. Recall the superinner automorphism sInt_d (Definition 2.66).

Theorem 3.18. *Let $\mathcal{D}$ be a graded-division superalgebra and let $\mathcal{U}$ be a nonzero right graded module of finite rank over $\mathcal{D}$. If φ is degree-preserving super-anti-automorphism on $R \coloneqq \mathrm{End}_{\mathcal{D}}(\mathcal{U})$, then there is a pair (φ_0, B), where φ_0 is a degree-preserving super-anti-automorphism on $\mathcal{D}$ and $B\colon \mathcal{U} \times \mathcal{U} \to \mathcal{D}$ is a nondegenerate φ_0-sesquilinear form, such that*

$$\forall r \in R^{\bar{0}} \cup R^{\bar{1}}, \forall u, v \in \mathcal{U}^{\bar{0}} \cup \mathcal{U}^{\bar{1}}, \quad B(ru, v) = (-1)^{|r||u|}B(u, \varphi(r)v). \tag{3.4}$$

Conversely, given a pair (φ_0, B) as above, there is a unique degree-preserving super-anti-automorphism φ on R satisfying Equation (3.4). Moreover, another pair (φ_0', B') determines the same super-anti-automorphism φ if, and only if, there is a nonzero

$G^{\#}$-homogeneous element $d \in \mathcal{D}$ such that $B'(u,v) = dB(u,v)$ for all $u, v \in \mathcal{U}$, and, hence, $\varphi_0' = \mathrm{sInt}_d \circ \varphi_0$.

Proof. The first assertion is already proved. For the converse, let φ_0 be a degree-preserving super-anti-automorphism on $\mathcal{D}$, let $B\colon \mathcal{U} \times \mathcal{U} \to \mathcal{D}$ be a nondegenerate φ_0-sesquilinear form and consider θ as in Proposition 3.15. Then Equation (3.4) is equivalent to:

$$\forall r \in R^{\bar{0}} \cup R^{\bar{1}},\, \forall u, v \in \mathcal{U}^{\bar{0}} \cup \mathcal{U}^{\bar{1}}, \qquad \begin{aligned} \theta(ru)(v) &= (-1)^{|r||u|}\theta(u)(\varphi(r)v) \\ &= (-1)^{|r||u|}\Big(\theta(u) \circ \varphi(r)\Big)(v) \end{aligned}$$

and, hence, equivalent to

$$\forall r \in R^{\bar{0}} \cup R^{\bar{1}}, \forall u \in \mathcal{U}^{\bar{0}} \cup \mathcal{U}^{\bar{1}}, \qquad \theta(ru) = (-1)^{|r||u|}\theta(u) \circ \varphi(r). \tag{3.5}$$

Recalling the definition of superadjoint operator, Equation (3.5) becomes

$$\forall r \in R^{\bar{0}} \cup R^{\bar{1}},\, \forall u \in \mathcal{U}^{\bar{0}} \cup \mathcal{U}^{\bar{1}}, \quad \begin{aligned} (\theta \circ r)(u) &= (-1)^{|r||u|}(-1)^{(|\theta|+|u|)|r|}\Big(\varphi(r)\Big)^{\star}\Big(\theta(u)\Big) \\ &= (-1)^{|r||\theta|}\Big(\Big(\varphi(r)\Big)^{\star} \circ \theta\Big)(u), \end{aligned}$$

which is the same as

$$\forall r \in R^{\bar{0}} \cup R^{\bar{1}}, \qquad \theta \circ r = (-1)^{|r||\theta|}\Big(\varphi(r)\Big)^{\star} \circ \theta.$$

In other words, we have

$$\forall r \in R^{\bar{0}} \cup R^{\bar{1}}, \qquad \Big(\varphi(r)\Big)^{\star} = (-1)^{|r||\theta|}\, \theta \circ r \circ \theta^{-1}. \tag{3.6}$$

Since $\mathcal{U}$ has finite rank over $\mathcal{D}$, the superadjunction map $\mathrm{End}_{\mathcal{D}}(\mathcal{U}) \to \mathrm{End}_{\mathcal{D}^{\mathrm{sop}}}(\mathcal{U}^{\star})$ is invertible and, hence, φ is uniquely determined. Also, the properties of superadjunction imply that φ is, indeed, a super-anti-automorphism of R.

For the "moreover" part, let d be a nonzero $G^{\#}$-homogeneous element of $\mathcal{D}$ and consider $\varphi_0' = \mathrm{sInt}_d \circ \varphi_0$ and $B' = dB$. We have that B' is φ_0'-sesquilinear since, for all

$c \in \mathcal{D}^{\bar{0}} \cup \mathcal{D}^{\bar{1}}$ and $u, v \in \mathcal{U}^{\bar{0}} \cup \mathcal{U}^{\bar{1}}$,

$$\begin{aligned} B'(uc, v) = dB(uc, v) &= (-1)^{(|B|+|u|)|c|} d\varphi_0(c) B(u, v) \\ &= (-1)^{(|B|+|u|)|c|} d\varphi_0(c) d^{-1} dB(u, v) \\ &= (-1)^{(|B|+|u|)|c|} (-1)^{|c||d|} (\mathrm{sInt}_d \circ \varphi_0)(c)\, dB(u, v) \\ &= (-1)^{(|B'|+|u|)|c|} (\mathrm{sInt}_d \circ \varphi_0)(c)\, B'(u, v). \end{aligned} \tag{3.7}$$

To show that B' is nondegenerate, note that $dB(u, v) = 0$ implies $B(u, v) = 0$, hence $\operatorname{rad} B' \subseteq \operatorname{rad} B$. Finally, it is straightforward that Equation (3.4) is still true if we replace B by B'.

To prove the other direction, we consider, again, the left R-supermodule structure on $\mathcal{U}^\star$ given by Equation (3.1) and let $\theta \colon \mathcal{U} \to \mathcal{U}^\star$ be as above, i.e., $\theta(u) = B(u, \cdot)$. Similarly, let $\theta' \colon \mathcal{U} \to \mathcal{U}^\star$ be defined by $\theta' \coloneqq B'(u, \cdot)$.

Combining Equations (3.1) and (3.5), we have that

$$\theta(ru) = (-1)^{|\theta||r|} r \cdot \theta(u). \tag{3.8}$$

Define the map $\tilde{\theta} \colon \mathcal{U} \to \mathcal{U}^\star$, written on the right, by $u\,\tilde{\theta} = (-1)^{|u||\theta|}\theta(u)$, for all $u \in \mathcal{U}$ (compare with Definition 3.16 and note that $\theta = (\tilde{\theta})^\circ$). Then Equation (3.8) becomes $(ru)\tilde{\theta} = r \cdot (u\,\tilde{\theta})$, i.e., $\tilde{\theta}$ is R-linear.

All these considerations about θ are also valid for θ', so we define $\tilde{\theta}' \colon \mathcal{U} \to \mathcal{U}^\star$ by $u\,\tilde{\theta}' \coloneqq (-1)^{|u||\theta'|}\theta'(u)$ and we get another R-linear map from $\mathcal{U}$ to $\mathcal{U}^\star$. By Lemma 3.12, there is $\bar{d} \in \mathcal{D}^{\mathrm{sop}}$ such that $\tilde{\theta}' = \tilde{\theta}\bar{d}$. Applying Lemma 3.17, this implies

$$\theta' = (-1)^{|\theta||\bar{d}|} \bar{d}^\circ \theta.$$

But $\bar{d}^\circ \theta(u) = d\theta(u)$, where in the last term we use the left $\mathcal{D}$-action on $\mathcal{U}^\star$. Therefore $B'(u, v) = \theta'(u)(v) = d\theta(u)(v) = (-1)^{|\theta||d|} dB(u, v)$, for all $u, v \in \mathcal{U}$. Replacing d by $(-1)^{|\theta||d|}d$, we get $B' = dB$.

It remains to check that $\varphi_0' = \mathrm{sInt}_d \circ \varphi_0$. Since $B' = dB$, Equation (3.7) is valid, hence B' is $(\mathrm{sInt}_d \circ \varphi_0)$-sesquilinenar. We then have, for all $c \in \mathcal{D}^{\bar{0}} \cup \mathcal{D}^{\bar{1}}$ and $u, v \in \mathcal{U}^{\bar{0}} \cup \mathcal{U}^{\bar{1}}$,

$$\varphi_0'(c)\, B'(u, v) = (-1)^{(|B'|+|u|)|c|} B'(uc, v) = (\mathrm{sInt}_{\mathrm{d}} \circ \varphi_0)(c)\, B'(u, v).$$

The form B' is nondegenerate, so we can choose $G^{\#}$-homogeneous $u, v \in \mathcal{U}$ with $B'(u,v) \neq 0$. Then $B'(u,v)$ is invertible, hence $\varphi_0'(c) = (\mathrm{sInt}_d \circ \varphi_0)(c)$, concluding the proof. $\square$

The "conversely" part of Theorem 3.18 motivates the following:

Definition 3.19. Let $\mathcal{D}$ be a graded-division superalgebra, $\mathcal{U}$ a graded right $\mathcal{D}$-module of finite rank and B a nondegenerate sesquilinear form on $\mathcal{U}$. The unique super-anti-automorphism φ on $\mathrm{End}_{\mathcal{D}}(\mathcal{U})$ defined by Equation (3.4) is called the *superadjunction with respect to* B and, for every $r \in \mathrm{End}_{\mathcal{D}}(\mathcal{U})$, the $\mathcal{D}$-linear map $\varphi(r)$ is called the *superadjoint of* r. We will denote by $E(\mathcal{D}, \mathcal{U}, B)$ the graded superalgebra $\mathrm{End}_{\mathcal{D}}(\mathcal{U})$ endowed with this super-anti-automorphism φ.

Remark 3.20. Under the conditions of Theorem 3.18, if $\mathcal{D}$ is an odd graded division superalgebra (i.e., $\mathcal{D}^{\bar{1}} \neq 0$), then we can choose the form B to be even. This is possible since, by the "moreover" part, we can substitute an odd form B by dB, for some $d \in \mathcal{D}^{\bar{1}}$.

Proposition 3.21. *Recall the isomorphism of $G^{\#}$-graded algebras $\iota\colon Z(\mathcal{D}) \to Z(R)$ given by $\iota(d)(u) := ud$, for all $d \in Z(\mathcal{D})$ and $u \in \mathcal{U}$ (see Proposition 2.30). We have that $\varphi(\iota(d)) = (-1)^{|B||d|}\iota(\varphi_0(d))$.*

Proof. Fix $d \in Z(\mathcal{D})^{\bar{0}} \cup Z(\mathcal{D})^{\bar{1}}$ and let $u, v \in \mathcal{U}^{\bar{0}} \cup \mathcal{U}^{\bar{1}}$. On the one hand,

$$B(ud, v) = B(\iota(d)(u), v) = (-1)^{|d||u|} B(u, \varphi(\iota(d))v).$$

On the other hand,

$$\begin{aligned} B(ud, v) &= (-1)^{(|B|+|u|)|d|}\varphi_0(d)B(u,v) = (-1)^{(|B|+|u|)|d|}B(u,v)\varphi_0(d) \\ &= (-1)^{|B||d|+|u||d|}B(u, v\varphi_0(d)) = (-1)^{|B||d|}(-1)^{|u||d|}B(u, \iota(\varphi_0(d))(v)). \end{aligned}$$

Since B is nondegenerate, we conclude that $\varphi(\iota(d)) = (-1)^{|B||d|}\iota(\varphi_0(d))$, as desired. $\square$

3.3 Isomorphisms of graded-simple superalgebras with super-anti-automorphism

In this section, we are going to describe isomorphisms between G-graded superalgebras with super-anti-automorphism that are graded-simple and satisfy the d.c.c. on graded left ideals. As we have seen, such superalgebras are, up to isomorphism, of the form $\mathrm{End}_{\mathcal{D}}(\mathcal{U})$ where the super-anti-automorphism is given by the superadjunction with respect to a nondegenerate homogeneous sesquilinear form $B\colon \mathcal{U}\times\mathcal{U}\to\mathcal{D}$.

Even though, as superalgebras, $\mathrm{End}_{\mathcal{D}}(\mathcal{U})$ is the same as $\mathrm{End}_{\mathcal{D}}(\mathcal{U}^{[g]})$ for every $g\in G^{\#}$, an extra care should be taken when considering super-anti-automorphisms. If $g\in G^{\#}$ is odd, a φ_0-sesquilinear form B on the $\mathcal{D}$-module $\mathcal{U}$ is not φ_0-sesquilinear if considered on the $\mathcal{D}$-module $\mathcal{U}^{[g]}$, and Equation (3.4) does not determine the same super-anti-automorphism φ. This motivates the following:

Definition 3.22. Let $\mathcal{U}$ be a graded right $\mathcal{D}$-module and $\varphi_0\colon \mathcal{D}\to\mathcal{D}$ be a super-anti-automorphism. Given a homogeneous φ_0-sesquilinear form B on $\mathcal{U}$ and $g\in G^{\#}$, we define the φ_0-sesquilinear form $B^{[g]}$ on $\mathcal{U}^{[g]}$ by $B^{[g]}(u,v):=(-1)^{|u||g|}B(u,v)$, for all $u,v\in\mathcal{U}$.

Remark 3.23. Note that $\deg B^{[g]}=g^{-2}\deg B$ and, in particular, $|B^{[g]}|=|B|$.

Lemma 3.24. *For every $g\in G^{\#}$, $B^{[g]}$ is a homogeneous φ_0-sesquilinear form on $\mathcal{U}^{[g]}$. Further, if B is nondegenerate, then so is $B^{[g]}$ and the superadjunction with respect to both is the same super-anti-automorphism φ on $\mathrm{End}_{\mathcal{D}}(\mathcal{U})=\mathrm{End}_{\mathcal{D}}(\mathcal{U}^{[g]})$.*

Proof. Let $u,v\in\mathcal{U}^{\bar 0}\cup\mathcal{U}^{\bar 1}$. To avoid confusion, we will denote u and v by $u^{[g]}$ and $v^{[g]}$, respectively, when regarded as elements of $\mathcal{U}^{[g]}$. For all $d\in D^{\bar 0}\cup\mathcal{D}^{\bar 1}$, we have:

$$B^{[g]}(u^{[g]},v^{[g]}d)=(-1)^{|g||u|}B(u,vd)=(-1)^{|g||u|}B(u,v)d=B^{[g]}(u^{[g]},v^{[g]})d$$

and

$$\begin{aligned}B^{[g]}(u^{[g]}d,v^{[g]})&=(-1)^{|g||ud|}B(ud,v)=(-1)^{|g|(|u|+|d|)}(-1)^{|d|(|B|+|u|)}\varphi_0(d)B(u,v)\\&=(-1)^{|d|(|B|+|u|+|g|)}\varphi_0(d)(-1)^{|g||u|}B(u,v)\\&=(-1)^{|d|(|B|+|u^{[g]}|)}\varphi_0(d)B^{[g]}(u^{[g]},v^{[g]})d.\end{aligned}$$

Also, for all $r \in R^{\bar{0}} \cup R^{\bar{1}}$, we have:

$$\begin{aligned} B^{[g]}(ru^{[g]}, v^{[g]}) &= (-1)^{|g||ru|} B(ru, v) = (-1)^{|g|(|r|+|u|)}(-1)^{|r||u|} B(u, \varphi(r)v) \\ &= (-1)^{|r|(|g|+|u|)}(-1)^{|u||g|} B(u, \varphi(r)v) \\ &= (-1)^{|r||u^{[g]}|} B^{[g]}(u^{[g]}, \varphi(r)v^{[g]}). \end{aligned}$$

□

Recall the concept of a module induced by a homomorphism of algebras (Definition 2.25).

Lemma 3.25. *Let $\mathcal{D}$ and $\mathcal{D}'$ be graded-division superalgebras and let $\psi_0 \colon \mathcal{D} \to \mathcal{D}'$ be an isomorphism. If $\mathcal{U}'$ is a $\mathcal{D}'$-supermodule and B' is a homogeneous φ_0'-sesquilinear form on it, then $\psi_0^{-1} \circ B'$ is a homogeneous $(\psi_0^{-1} \circ \varphi_0' \circ \psi_0)$-sesquilinear form on the $\mathcal{D}$-supermodule $(\mathcal{U}')^{\psi_0}$ of the same degree as B'. Further, if B' is nondegenerate, then so is $\psi_0^{-1} \circ B'$, and the superadjunction with respect to both is the same super-anti-automorphism φ' on $\mathrm{End}_{\mathcal{D}}((\mathcal{U}')^{\psi_0}) = \mathrm{End}_{\mathcal{D}'}(\mathcal{U}')$.*

Proof. To simplify notation, let us put $B'' \coloneqq \psi_0^{-1} \circ B'$ and $\varphi_0'' \coloneqq \psi_0^{-1} \circ \varphi_0' \circ \psi_0$. It is clear that $\deg B'' = \deg B$ and, hence, $|B''| = |B'|$. Let $u, v \in (\mathcal{U}')^{\bar{0}} \cup (\mathcal{U}')^{\bar{1}}$. To avoid confusion, we will denote u and v by u^{ψ_0} and v^{ψ_0}, respectively, when regarded as elements of $(\mathcal{U}')^{\psi_0}$. For all $d \in D^{\bar{0}} \cup \mathcal{D}^{\bar{1}}$, we have:

$$\begin{aligned} B''(u^{\psi_0}, v^{\psi_0} d) &= \psi_0^{-1}\Big(B'(u, v\,\psi_0(d))\Big) = \psi_0^{-1}\Big(B'(u,v)\psi_0(d)\Big) \\ &= \psi_0^{-1}\Big(B'(u,v)\Big)d = B''(u^{\psi_0}, v^{\psi_0})\, d \end{aligned}$$

and

$$\begin{aligned} B''(u^{\psi_0} d, v^{\psi_0}) &= \psi_0^{-1}\Big(B'(u\,\psi_0(d), v)\Big) \\ &= \psi_0^{-1}\Big((-1)^{(|B'|+|u|)|\psi_0(d)|}\varphi_0'\Big(\psi_0(d)\Big)B'(u,v)\Big) \\ &= (-1)^{(|B'|+|u|)|d|}\psi_0^{-1}\Big(\varphi_0'\Big(\psi_0(d)\Big)\Big)\psi_0^{-1}\Big(B'(u,v)\Big) \\ &= (-1)^{(|B''|+|u|)|d|}\varphi_0''(d) B''(u^{\psi_0}, v^{\psi_0}). \end{aligned}$$

Also, for all $r \in R^{\bar{0}} \cup R^{\bar{1}}$, we have:

$$
\begin{aligned}
B''(ru^{\psi_0}, v^{\psi_0}) &= \psi_0^{-1}\Big(B'(ru, v)\Big) \\
&= \psi_0^{-1}\Big((-1)^{|r||u|}B'(u, \varphi'(r)v)\Big) \\
&= (-1)^{|r||u|}B''(u, \varphi'(r)v).
\end{aligned}
$$

□

Definition 3.26. Let $\mathcal{U}$ and $\mathcal{U}'$ be graded right $\mathcal{D}$-supermodules, and let B and B' be homogeneous sesquilinear forms on $\mathcal{U}$ and $\mathcal{U}'$, respectively. An *isomorphism from* $(\mathcal{U}, B)$ *to* $(\mathcal{U}', B')$ is an isomorphism of graded modules $\theta\colon \mathcal{U} \to \mathcal{U}'$ such that $B'(\theta(u), \theta(v)) = B(u, v)$, for all $u, v \in \mathcal{U}$.

Note that if $(\mathcal{U}, B)$ and $(\mathcal{U}', B')$ are isomorphic, then B and B' have the same degree in $G^{\#}$ and are sesquilinear with respect to the same φ_0.

Theorem 3.27. *Let $R \coloneqq \operatorname{End}_{\mathcal{D}}(\mathcal{U})$ and $R' \coloneqq \operatorname{End}_{\mathcal{D}'}(\mathcal{U}')$, where $\mathcal{D}$ and $\mathcal{D}'$ are graded division superalgebras, and $\mathcal{U}$ and $\mathcal{U}'$ are nonzero right graded supermodules of finite rank over $\mathcal{D}$ and $\mathcal{D}'$, respectively. Let φ and φ' be degree preserving super-anti-automorphisms on R and R' determined, as in Theorem 3.18, by pairs (φ_0, B) and (φ_0', B'), respectively. If $\psi\colon (R, \varphi) \to (R', \varphi')$ is an isomorphism, then there are $g \in G^{\#}$, a homogeneous element $0 \neq d \in \mathcal{D}$, an isomorphism $\psi_0\colon \mathcal{D} \to \mathcal{D}'$, and an isomorphism*

$$\psi_1\colon (\mathcal{U}^{[g]}, dB^{[g]}) \to ((\mathcal{U}')^{\psi_0}, \psi_0^{-1} \circ B') \tag{3.9}$$

such that

$$\forall r \in R, \quad \psi(r) = \psi_1 \circ r \circ \psi_1^{-1}. \tag{3.10}$$

Conversely, for any g, d, ψ_0 and ψ_1 as above, Equation (3.10) *defines an isomorphism $\psi\colon (R, \varphi) \to (R', \varphi')$.*

Proof. Given an isomorphism of graded superalgebras $\psi\colon R \to R'$, define

$$\tilde{\varphi} \coloneqq \psi^{-1} \circ \varphi' \circ \psi.$$

Then ψ is an isomorphism $(R, \varphi) \to (R', \varphi')$ if, and only if, $\varphi = \tilde{\varphi}$.

Since ψ is an isomorphism of $G^{\#}$-graded algebras, we can apply Theorem 2.27 to conclude that there are $g \in G^{\#}$, an isomorphism of graded superalgebras $\psi_0 \colon \mathcal{D} \to \mathcal{D}'$, and an isomorphism of graded modules $\psi_1 \colon \mathcal{U}^{[g]} \to (\mathcal{U}')^{\psi_0}$ such that $\psi(r) = \psi_1 \circ r \circ \psi_1^{-1}$, for all $r \in R$.

As in the proof of Lemma 3.25, consider $\varphi_0'' := \psi_0^{-1} \circ \varphi_0' \circ \psi_0$ and $B'' := \psi_0^{-1} \circ B'$. Then define $\widetilde{B} \colon \mathcal{U}^{[g]} \times \mathcal{U}^{[g]} \to \mathcal{D}$ by

$$\widetilde{B}(u,v) := B''\Big(\psi_1(u), \psi_1(v)\Big)$$

for all $u, v \in \mathcal{U}^{[g]}$. We claim that $\widetilde{B}$ is φ_0''-sesquilinear. Indeed, by Lemma 3.25, we have

$$\begin{aligned}\widetilde{B}(u,vd) &= B''\Big(\psi_1(u), \psi_1(vd)\Big) = B''\Big(\psi_1(u), \psi_1(v)d\Big) \\ &= B''\Big(\psi_1(u), \psi_1(v)\Big)d = \widetilde{B}(u,v)d\end{aligned}$$

and

$$\begin{aligned}\widetilde{B}(ud,v) &= B''\Big(\psi_1(ud), \psi_1(v)\Big) = B''\Big(\psi_1(u)d, \psi_1(v)\Big) \\ &= (-1)^{(|B''|+|\psi_1(u)|)|d|}\varphi_0''(d)B''\Big(\psi_1(u), \psi_1(v)\Big) \\ &= (-1)^{(|\widetilde{B}|+|u|)|d|}\varphi_0''(d)\widetilde{B}(u,v).\end{aligned}$$

Also, $\tilde{\varphi}$ is the superadjunction with respect to $\widetilde{B}$:

$$\begin{aligned}\widetilde{B}(ru,v) &= B''\Big(\psi_1(ru), \psi_1(v)\Big) = B''\Big((\psi_1 \circ r \circ \psi_1^{-1})\psi_1(u), \psi_1(v)\Big) \\ &= B''\Big(\psi(r)\psi_1(u), \psi_1(v)\Big) \\ &= (-1)^{|\psi(r)||\psi_1(u)|}B''\Big(\psi_1(u), \varphi'(\psi(r))\psi_1(v)\Big) \\ &= (-1)^{|r||u|}B''\Big(\psi_1(u), \psi(\tilde{\varphi}(r))\psi_1(v)\Big) \\ &= (-1)^{|r||u|}B''\Big(\psi_1(u), (\psi_1 \circ \tilde{\varphi}(r) \circ \psi_1^{-1})\psi_1(v)\Big) \\ &= (-1)^{|r||u|}B''\Big(\psi_1(u), \psi_1(\tilde{\varphi}(r)v)\Big) \\ &= (-1)^{|r||u|}\widetilde{B}\Big(u, \tilde{\varphi}(r)v\Big),\end{aligned}$$

where we have used Lemma 3.25 in the third line and the definition of $\tilde{\varphi}$ on the fourth line. Hence, applying Theorem 3.18 for $\mathcal{U}^{[g]}$ and Lemma 3.24, we conclude that $\varphi = \tilde{\varphi}$

if, and only if, there is a homogeneous $0 \neq d \in \mathcal{D}$ such that $\widetilde{B} = dB^{[g]}$. The result follows. □

We can interpret Theorem 3.27 in terms of group actions. For that, fix a fixed graded-division superalgebra $\mathcal{D}$. We will define three (left) group actions on the class of pairs $(\mathcal{U}, B)$, where $\mathcal{U} \neq 0$ is a graded $\mathcal{D}$-supermodule and B is a nondegenerate homogeneous sesquilinear form. Recall that, by Lemma 3.14, B is φ_0-sesquilinear for a unique super-anti-automorphism φ_0 of $\mathcal{D}$.

Let $\mathcal{D}_{\mathrm{gr}}^{\times} \coloneqq \left(\bigcup_{g \in G^{\#}} \mathcal{D}_g\right) \setminus \{0\}$, the group of nonzero homogeneous elements of $\mathcal{D}$. Given $d \in \mathcal{D}_{\mathrm{gr}}^{\times}$, we define

$$d \cdot (\mathcal{U}, B) \coloneqq (\mathcal{U}, dB). \tag{3.11}$$

Note that dB is $(\mathrm{sInt}_d \circ \varphi_0)$-sesquilinear by Theorem 3.18.

Let $A \coloneqq \mathrm{Aut}(\mathcal{D})$, the group of automorphisms of $\mathcal{D}$ as a graded superalgebra. Given $\tau \in A$, we define

$$\tau \cdot (\mathcal{U}, B) \coloneqq (\mathcal{U}^{\tau^{-1}}, \tau \circ B).$$

Note that $\tau \circ B$ is $(\tau \circ \varphi_0 \circ \tau^{-1})$-sesquilinear by Lemma 3.25.

Finally, consider the group $G^{\#}$. Given $g \in G^{\#}$, we define

$$g \cdot (\mathcal{U}, B) \coloneqq (\mathcal{U}^{[g]}, B^{[g]}). \tag{3.12}$$

Note that $B^{[g]}$ is φ_0-sesquilinear by Lemma 3.24.

Lemma 3.28. *The three actions defined above give rise to a $(\mathcal{D}_{\mathrm{gr}}^{\times} \rtimes A) \times G^{\#}$-action, where A acts on $\mathcal{D}_{\mathrm{gr}}^{\times}$ by evaluation.*

Proof. Let $d \in \mathcal{D}_{\mathrm{gr}}^{\times}$, $\tau \in A$, $g \in G^{\#}$ and $u, v \in \mathcal{U}$. First note that the action of d does not change $\mathcal{U}$, so we only have to consider its effect on B. Since

$$(\tau \circ dB)(u,v) = \tau\Big(dB(u,v)\Big) = \tau(d)\tau\Big(B(u,v)\Big) = \Big(\tau(d)(\tau \circ B)\Big)(u,v),$$

the $\mathcal{D}_{\mathrm{gr}}^{\times}$-action combined with the A-action gives us a $(\mathcal{D}_{\mathrm{gr}}^{\times} \rtimes A)$-action. The $G^{\#}$-action commutes with the $\mathcal{D}_{\mathrm{gr}}^{\times}$-action since

$$(dB)^{[g]}(u,v) = (-1)^{|g||u|} dB(u,v) = dB^{[g]}(u,v).$$

Finally, the $G^{\#}$-action also commutes with the A-action since $(\mathcal{U}^{\tau^{-1}})^{[g]} = (\mathcal{U}^{[g]})^{\tau^{-1}}$ and

$$(\tau \circ B^{[g]})(u,v) = \tau\Big((-1)^{|g||u|}B(u,v)\Big) \quad = (-1)^{|g||u|}(\tau \circ B)(u,v) = (\tau \circ B)^{[g]}(u,v).$$

$\square$

Corollary 3.29. *Under the assumptions of Theorem 3.27, if $\mathcal{D} \not\simeq \mathcal{D}'$, then $(R,\varphi) \not\simeq (R',\varphi')$. Otherwise, fix an isomorphism $\psi_0 \colon \mathcal{D} \to \mathcal{D}'$. Then $(R,\varphi) \simeq (R',\varphi')$ if, and only if, $\big((\mathcal{U}')^{\psi_0}, \psi_0^{-1} \circ B'\big)$ is isomorphic to an object in the $(\mathcal{D}_{\mathrm{gr}}^{\times} \rtimes A) \times G^{\#}$-orbit of $(\mathcal{U}, B)$.*

3.4 Matrix representation of a super-anti-automorphism

In this section, we are going to express the super-anti-automorphism (not necessarily involutive) φ in terms of matrices with entries in $\mathcal{D}$. One could do that by following Equation (3.6), but we will take a different path.

As before, we suppose $\mathcal{D}$ is a graded division superalgebra, $\mathcal{U}$ is a nonzero right graded module of finite rank over $\mathcal{D}$, $R = \operatorname{End}_{\mathcal{D}}(\mathcal{U})$ and φ is a degree-preserving super-anti-automorphism on R. Also, let φ_0 be a super-anti-automorphism on $\mathcal{D}$ and B be a nondegenerate φ_0-sesquilinear form on $\mathcal{U}$ determinig φ as in Theorem 3.18.

Definition 3.30. Given a graded basis $\{u_1, \dots, u_k\}$ of $\mathcal{U}$, the *matrix representing the form B* is defined to be $\Phi = (\Phi_{ij}) \in M_k(\mathcal{D})$, where $\Phi_{ij} = B(u_i, u_j)$.

From now on, let $\mathcal{B} = \{u_1, \dots, u_k\}$ be a fixed homogeneous $\mathcal{D}$-basis of $\mathcal{U}$, following Convention 2.53 (i.e., if $\mathcal{D}$ is odd, we take $\mathcal{B}$ with only even elements). We will use $\mathcal{B}$ to identify $R = \operatorname{End}_{\mathcal{D}}(\mathcal{U})$ with $M_k(\mathcal{D})$. Also, we will denote $|u_i|$ simply by $|i|$ for all $i \in \{1, \dots, k\}$.

Definition 3.31. Let $X = (x_{ij})$ be a matrix in $M_k(\mathcal{D})$. We define $\varphi_0(X)$ to be the matrix obtained by applying φ_0 in each entry, i.e., $\varphi_0(X) \coloneqq (\varphi_0(x_{ij}))$. We also extend the definition *supertranspose* to matrices over $\mathcal{D}$ by putting $X^{s\top} \coloneqq \Big((-1)^{(|i|+|j|)|i|}x_{ji}\Big)$. Note that, with our choice of $\mathcal{B}$ in the case of odd $\mathcal{D}$, we have that $X^{s\top} = X^{\top}$, the ordinary transpose.

Proposition 3.32. *Let Φ be the matrix representing B. For every $r \in R^{\bar{0}} \cup R^{\bar{1}}$, let $X \in M_k(\mathcal{D})$ be the matrix representing r and let $Y \in M_k(\mathcal{D})$ be the matrix representing $\varphi(r)$. Then, if $\mathcal{D}$ is even, we have that*

$$Y = \Phi^{-1}\, \varphi_0(X^{s\top})\, \Phi, \tag{3.13}$$

and, if $\mathcal{D}$ is odd, following Convention 2.53, we have

$$Y = (-1)^{|B||r|}\, \Phi^{-1}\, \varphi_0(X^{s\top})\, \Phi. \tag{3.14}$$

Proof. First of all, note that Equation (3.4) is equivalent to the following:

$$\forall u_i, u_j \in \mathcal{B}, \qquad B(ru_i, u_j) = (-1)^{|r||i|} B(u_i, \varphi(r)u_j),$$

which, by the definitions of X, Y and Φ, becomes

$$\forall u_i, u_j \in \mathcal{B}, \quad B\left(\sum_{\ell=1}^{k} u_\ell x_{\ell i}, u_j\right) = (-1)^{|r||i|} B\left(u_i, \sum_{\ell=1}^{k} u_\ell y_{\ell j}\right). \tag{3.15}$$

Fix arbitrary $p, q \in \{1, \ldots, k\}$ and suppose that the (p,q)-entry of X is a nonzero $G^{\#}$-homogeneous element of $\mathcal{D}$ and $x_{ij} = 0$ elsewhere, i.e., X represents the map $r \in \mathrm{End}_{\mathcal{D}}(\mathcal{U})$ defined by $ru_i = \delta_{iq} u_p x_{pq}$. By the $\mathbb{F}$-linearity of Equation (3.13), it suffices to consider such X. Note that $|r| = |u_p| + |x_{pq}| - |u_q| = |p| + |q| + |x_{pq}|$. Then, on the one hand,

$$\begin{aligned} B\left(\sum_{\ell=1}^{k} u_\ell x_{\ell i}, u_j\right) = B(u_p x_{pi}, u_j) &= (-1)^{(|B|+|p|)|x_{pi}|} \varphi_0(x_{pi}) B(u_p, u_j) \\ &= (-1)^{(|B|+|p|)|x_{pi}|} \varphi_0(x_{pi}) \Phi_{pj}, \end{aligned}$$

which is only nonzero if $i = q$. On the other hand,

$$\begin{aligned} (-1)^{|r||i|} B\left(u_i, \sum_{\ell=1}^{k} u_\ell y_{\ell j}\right) &= (-1)^{|r||i|} \sum_{\ell=1}^{k} B(u_i, u_\ell) y_{\ell j} \\ &= (-1)^{(|p|+|q|+|x_{pq}|)|i|} \sum_{\ell=1}^{k} \Phi_{i\ell} y_{\ell j}. \end{aligned}$$

Therefore, Equation (3.15) is equivalent to, for all $i, j \in \{1, \ldots, k\}$,

$$(-1)^{(|B|+|p|)|x_{pi}|}\varphi_0(x_{pi})\Phi_{pj} = (-1)^{(|p|+|q|+|x_{pq}|)|i|}\sum_{\ell=1}^{k}\Phi_{i\ell}y_{\ell j}. \tag{3.16}$$

If $\mathcal{D}$ is even, then $|x_{pi}| = \bar{0}$ and this equation reduces to

$$\varphi_0(x_{pi})\Phi_{pj} = (-1)^{(|p|+|q|)|i|}\sum_{\ell=1}^{k}\Phi_{i\ell}y_{\ell j}$$

or, equivalently,

$$\sum_{\ell=1}^{k}\Phi_{i\ell}y_{\ell j} = (-1)^{(|p|+|q|)|i|}\varphi_0(x_{pi})\Phi_{pj}.$$

The left-hand side is the (i, j)-entry of ΦY. The right-hand side is only nonzero if $i = q$, so it can be rewritten as $(-1)^{(|p|+|i|)|i|}\varphi_0(x_{pi})\Phi_{pj}$. Recalling our choice of X, this is equal to $\sum_{\ell=1}^{k}(-1)^{(|\ell|+|i|)|i|}\varphi_0(x_{\ell i})\Phi_{\ell j}$, since $x_{\ell i}$ is only nonzero if $\ell = p$. Hence the right-hand side is the (i, j)-entry of $\varphi_0(X^{s\top})\Phi$, and Equation (3.13) follows.

If $\mathcal{D}$ is odd, by our choice of basis, Equation (3.16) reduces to

$$(-1)^{|B||x_{pi}|}\varphi_0(x_{pi})\Phi_{pj} = \sum_{\ell=1}^{k}\Phi_{i\ell}y_{\ell j},$$

which, by the same reasoning as above, implies $(-1)^{|B||r|}\varphi_0(X^{s\top})\Phi = \Phi Y$. □

Proposition 3.33. *The superalgebra R is φ-simple if, and only if, the superalgebra $\mathcal{D}$ is φ_0-simple.*

Proof. Pick a homogeneous $\mathcal{D}$-basis for $\mathcal{U}$ following Convention 2.53 and use it to identify R with $M_k(\mathcal{D}) = M_k(\mathbb{F}) \otimes \mathcal{D}$. By Proposition 3.32, for every $X \in M_k(\mathcal{D})^{\bar{0}} \cup M_k(\mathcal{D})^{\bar{1}}$, we have $\varphi(X) = (-1)^{|B||X|}\Phi^{-1}\varphi_0(X^{s\top})\Phi$, where $\Phi \in M_k(\mathcal{D})$ is the matrix representing B.

It was proved in Proposition 2.58 that the superideals of $M_k(\mathcal{D}) = M_k(\mathbb{F}) \otimes \mathcal{D}$ are precisely the sets of the form $M_k(I) = M_k(\mathbb{F}) \otimes I$, where I an superideal of $\mathcal{D}$. We are going to show that $M_k(I)$ is φ-invariant if, and only if, I is φ_0-invariant.

Suppose I is φ_0-invariant. Then if $X \in M_k(\mathcal{D})^{\bar{0}} \cup M_k(\mathcal{D})^{\bar{1}}$, it is clear that $\varphi_0(X^{s\top})$ is also in $M_k(I)$. It follows that $\varphi(X) = (-1)^{|B||X|}\Phi^{-1}\varphi_0(X^{s\top})\Phi \in M_k(I)$ since

$M_k(I)$ is an ideal. Conversely, suppose $M_k(I)$ is φ-invariant. Take $d \in I^{\bar{0}} \cup I^{\bar{1}}$ and define $X := E_{11} \otimes d \in M_k(I)^{\bar{0}} \cup M_k(I)^{\bar{1}}$. Then $E_{11} \otimes \varphi_0(d) = \varphi_0(X^{s\top}) = (-1)^{|B||X|}\Phi\, \varphi(X)\, \Phi^{-1} \in M_k(I)$, which shows that $\varphi_0(d) \in I$. □

3.5 Superinvolutions and sesquilinear forms

Our goal now is to specialize the results of Section 3.2 to the case where φ is a superinvolution. To this end, let us investigate what super-anti-automorphism of $\mathcal{D}$ and what sesquilinear form on $\mathcal{U}$ determine the super-anti-automorphism φ^{-1}. Again, we suppose $\mathcal{D}$ is a graded division superalgebra, $\mathcal{U}$ is a nonzero right graded module of finite rank over $\mathcal{D}$ and put $R = \mathrm{End}_{\mathcal{D}}(\mathcal{U})$.

Definition 3.34. Given a super-anti-automorphism φ_0 on $\mathcal{D}$ and a φ_0-sesquilinear form B on $\mathcal{U}$, we define $\overline{B}\colon \mathcal{U} \times \mathcal{U} \to \mathcal{D}$ by $\overline{B}(u,v) := (-1)^{|u||v|}\varphi_0^{-1}(B(v,u))$ for all $u, v \in \mathcal{U}$.

Proposition 3.35. *Under the conditions of Definition 3.34, we have that $\overline{B}$ is a φ_0^{-1}-sesquilinear form of the same degree and parity as B. Further, if B is nondegenerate and φ is the super-anti-automorphism on R determined by (φ_0, B) as in Theorem 3.18, then $\overline{B}$ is nondegenerate and φ^{-1} is determined by $(\varphi_0^{-1}, \overline{B})$, i.e.,*

$$\forall r \in R^{\bar{0}} \cup R^{\bar{1}},\, \forall u, v \in \mathcal{U}^{\bar{0}} \cup \mathcal{U}^{\bar{1}}, \quad \overline{B}(ru, v) = (-1)^{|r||u|}\overline{B}(u, \varphi^{-1}(r)v). \tag{3.17}$$

Proof. Since B is $\mathbb{F}$-bilinear, so is $\overline{B}$. Also, since φ_0 preserves degree and parity, $\overline{B}$ is homogeneous of the same degree and parity as B. Let us check the conditions of Definition 3.13 and Equation (3.17).

Condition (i):

$$\begin{aligned}
\overline{B}(u, vd) &= (-1)^{|u|(|v|+|d|)}\varphi_0^{-1}(B(vd, u)) \\
&= (-1)^{|u|(|v|+|d|)}(-1)^{|d|(|B|+|v|)}\varphi_0^{-1}\Big(\varphi_0(d)B(v,u)\Big) \\
&= (-1)^{|u||v|+|u||d|+|d||B|+|d||v|}(-1)^{|d|(|B|+|v|+|u|)}\varphi_0^{-1}\Big(B(v,u))d \\
&= (-1)^{|u||v|}\varphi_0^{-1}\Big(B(v,u))d = \overline{B}(u,v)d.
\end{aligned}$$

Condition (ii):

$$\begin{aligned}\overline{B}(ud,v) &= (-1)^{(|u|+|d|)|v|}\varphi_0^{-1}\Big(B(v,ud)\Big)\\ &= (-1)^{(|u|+|d|)|v|}\varphi_0^{-1}\Big(B(v,u)d\Big)\\ &= (-1)^{(|u|+|d|)|v|}(-1)^{|d|(|B|+|v|+|u|)}\varphi_0^{-1}(d)\varphi_0^{-1}\Big(B(v,u)\Big)\\ &= (-1)^{|u||v|+|d||B|+|d||u|}\varphi_0^{-1}(d)\varphi_0^{-1}\Big(B(v,u)\Big)\\ &= (-1)^{(|B|+|u|)|d|}\varphi_0^{-1}(d)\overline{B}(u,v).\end{aligned}$$

For Equation (3.17), note that replacing r for $\varphi^{-1}(r)$, Equation (3.4) can be rewritten as

$$B(v,ru) = (-1)^{|r||v|}B(\varphi^{-1}(r)v,u).$$

Hence, we have that

$$\begin{aligned}\overline{B}(ru,v) &= (-1)^{(|r|+|u|)|v|}\varphi_0^{-1}\Big(B(v,ru)\Big)\\ &= (-1)^{(|r|+|u|)|v|}(-1)^{|r||v|}\varphi_0^{-1}\Big(B(\varphi^{-1}(r)v,u)\Big)\\ &= (-1)^{|u||v|}\varphi_0^{-1}\Big(B(\varphi^{-1}(r)v,u)\Big)\\ &= (-1)^{|u||v|}(-1)^{(|r|+|v|)|u|}\overline{B}(u,\varphi^{-1}(r)v)\\ &= (-1)^{|r||u|}\overline{B}(u,\varphi^{-1}(r)v).\end{aligned}$$

Finally, Equation (3.17) together with B being nondegenerate implies that $\overline{B}$ is nondegenerate. To see that, let u be a nonzero homogeneous element in $\operatorname{rad}\overline{B}$. Then for for every $r \in R^{\bar{0}} \cup R^{\bar{1}}$ and $v \in \mathcal{U}^{\bar{0}} \cup \mathcal{U}^{\bar{1}}$, we have that $\overline{B}(u,\varphi^{-1}(r)v) = 0$, hence $\overline{B}(ru,v) = 0$. Since $r \in R^{\bar{0}} \cup R^{\bar{1}}$ and $v \in \mathcal{U}^{\bar{0}} \cup \mathcal{U}^{\bar{1}}$ were arbitrary, this implies $\overline{B}(Ru,\mathcal{U}) = 0$. But $\mathcal{U}$ is simple as a graded R-supermodule, so we would have $\overline{B}(\mathcal{U},\mathcal{U}) = 0$ and then, using that φ_0 is bijective, $B(\mathcal{U},\mathcal{U}) = 0$, a contradiction. □

Lemma 3.36. *Under the conditions of Definition 3.34, let d be a nonzero $G^{\#}$-homogeneous element of $\mathcal{D}$ and consider $\varphi_0' := \operatorname{sInt}_d \circ \varphi_0$ and $B' := dB$. Then $\overline{B'} = (-1)^{|d|}\varphi_0^{-1}(d)\overline{B}$.*

Proof. Note that $(\varphi_0')^{-1} = \varphi_0^{-1} \circ \mathrm{sInt}_d^{-1} = \varphi_0^{-1} \circ \mathrm{sInt}_{d^{-1}}$. Hence, for all $u, v \in \mathcal{U}^{\bar{0}} \cup \mathcal{U}^{\bar{1}}$,

$$\begin{aligned}\overline{B'}(u,v) &= (-1)^{|u||v|}(\varphi_0^{-1} \circ \mathrm{sInt}_{d^{-1}})(dB(v,u)) \\ &= (-1)^{|u||v|}\varphi_0^{-1}\Big((-1)^{|d|(|d|+|B|+|u|+|v|)}\, d^{-1}dB(v,u)d\Big) \\ &= (-1)^{|u||v|}(-1)^{|d|}\, \varphi_0^{-1}(d)\varphi_0^{-1}(B(v,u)) = (-1)^{|d|}\, \varphi_0^{-1}(d)\overline{B}(u,v).\end{aligned}$$

□

We are primarily interested in the case $\mathbb{F}$ is an algebraically closed field and $\mathcal{D}$ is finite dimensional. In this case, we have that $\mathcal{D}_e^{\bar{0}} = \mathbb{F}1$, so we are under the hypothesis of the following theorem, which is a graded version of [Rac98, Theorem 7]:

Theorem 3.37. *Let $\mathcal{D}$ be a graded division superalgebra such that $\mathcal{D}_e = \mathbb{F}1$, let $\mathcal{U}$ be a nonzero right graded module of finite rank over $\mathcal{D}$ and let φ be a degree-preserving super-anti-automorphism on $R := \mathrm{End}_{\mathcal{D}}(\mathcal{U})$. Consider a super-anti-automorphism φ_0 on $\mathcal{D}$ and a nondegenerate φ_0-sesquilinear form B on $\mathcal{U}$ determining φ as in Theorem 3.18. Then φ is a superinvolution if, and only if, $\overline{B} = \pm B$. Moreover, if this is the case, then φ_0 is a superinvolution.*

Proof. Using Proposition 3.35 and Theorem 3.18, we conclude that $\varphi = \varphi^{-1}$ if, and only if, there is a $G^{\#}$-homogeneous element $0 \neq \delta \in \mathcal{D}$ such that $\overline{B} = \delta B$. Hence, it only remains to prove that, in this case, we have $\delta \in \{\pm 1\}$ and $\varphi_0^2 = \mathrm{id}_{\mathcal{D}}$.

Since B and $\overline{B}$ have the same $G^{\#}$-degree, $\overline{B} = \delta B$ implies that $\delta \in \mathcal{D}_e$. By Lemma 3.36, we have that

$$B = \overline{\overline{B}} = \overline{\delta B} = (-1)^{|\delta|}\varphi_0^{-1}(\delta)\overline{B} = \varphi_0^{-1}(\delta)\overline{B} = \varphi_0^{-1}(\delta)\delta B,$$

hence $\varphi_0^{-1}(\delta)\delta = 1$. Since we are assuming $\mathcal{D}_e = \mathbb{F}1$, we have that $\varphi_0^{-1}(\delta) = \delta$, so $\delta^2 = 1$ and, therefore, $\delta \in \{\pm 1\}$.

To see that $\varphi_0^2 = \mathrm{id}_{\mathcal{D}}$, note that, from Theorem 3.18, $\varphi_0^{-1} = \mathrm{sInt}_\delta \circ \varphi_0 = \varphi_0$. □

3.6 Classification of graded-simple superalgebras with superinvolution

In this section we introduce parameters that describe the graded finite dimensional associative superalgebras with superinvolution (R, φ) where R is graded-simple and, then, we give a classification result in terms of these parameters. We follow the ideas used in [BKR18, Sections 2 and 3] to classify graded-simple associative algebras with involution over the field of real numbers. Throughout this section, we will assume that $\mathbb{F}$ is an algebraically closed field with $\operatorname{char}\mathbb{F} \neq 2$.

Recall from Theorems 2.23, 3.18 and 3.37 that, under the above assumptions, we have that $R \simeq \operatorname{End}_{\mathcal{D}}(\mathcal{U})$ and that φ is determined by a superinvolution φ_0 on $\mathcal{D}$ and a nondegenerate homogeneous φ_0-sesquilinear form B on $\mathcal{U}$.

3.6.1 Parametrization of $(\mathcal{D}, \varphi_0)$

Recall, from Subsection 2.2.3, that the isomorphism class of a finite dimensional graded-division superalgebra $\mathcal{D}$ is determined by a triple (T, β, p) where $T \coloneqq \operatorname{supp}\mathcal{D} \subseteq G^{\#}$ is a finite abelian group, $\beta\colon T \times T \to \mathbb{F}^{\times}$ is an alternating bicharacter and $p\colon T \to \mathbb{Z}_2$ is a group homomorphism. Also recall that we define the skew-symmetric bicharacter $\tilde{\beta}\colon T \times T \to \mathbb{F}^{\times}$ by $\tilde{\beta}(a, b) = (-1)^{p(a)p(b)}\beta(a, b)$, for all $a, b \in T$ (see Equation (2.4)).

Recall that, since each component $\mathcal{D}_t$ of $\mathcal{D}$ is one-dimensional, an invertible degree-preserving map $\varphi_0\colon \mathcal{D} \to \mathcal{D}$ is completely determined by a map $\eta\colon T \to \mathbb{F}^{\times}$.

Proposition 3.38. *Let $\varphi_0\colon \mathcal{D} \to \mathcal{D}$ be the invertible degree-preserving map determined by a map $\eta\colon T \to \mathbb{F}^{\times}$ as follows: $\varphi_0(X_t) = \eta(t)X_t$ for all $t \in T$ and $X_t \in \mathcal{D}_t$. Then φ_0 is a super-anti-automorphism if, and only if,*

$$\forall a, b \in T, \quad \eta(ab) = \tilde{\beta}(a, b)\eta(a)\eta(b). \tag{3.18}$$

Moreover, $\mathcal{D}$ admits a super-anti-automorphism if, and only if, $\tilde{\beta}$ (or, equivalently, β) only takes values ± 1.

Proof. For all $a, b \in T$, let $X_a \in \mathcal{D}_a$ and $X_b \in \mathcal{D}_b$. Then:

$$\begin{aligned}
& \varphi_0(X_a X_b) = (-1)^{p(a)p(b)} \varphi_0(X_b)\varphi_0(X_a) \\
\iff\ & \eta(ab) X_a X_b = (-1)^{p(a)p(b)} \eta(a)\eta(b) X_b X_a \\
\iff\ & \eta(ab) X_a X_b = (-1)^{p(a)p(b)} \eta(a)\eta(b)\beta(b,a) X_a X_b \\
\iff\ & \eta(ab) = (-1)^{p(a)p(b)} \eta(a)\eta(b)\beta(b,a) \\
\iff\ & \eta(ab) = \tilde{\beta}(b,a)\eta(a)\eta(b)
\end{aligned}$$

If a and b are switched, since T is abelian, we get $\eta(ab) = \tilde{\beta}(a,b)\eta(a)\eta(b)$, as desired. Also, it follows that $\tilde{\beta}(b,a) = \tilde{\beta}(a,b)$. Using that $\tilde{\beta}$ is skew-symmetric, i.e., $\tilde{\beta}(b,a) = \tilde{\beta}(a,b)^{-1}$, we have that $\tilde{\beta}(a,b)^2 = 1$ and, hence, $\tilde{\beta}$ only takes values ± 1, proving one direction of the "moreover" part. The converse follows from the fact that the isomorphism class of $\mathcal{D}^{\mathrm{sop}}$ is determined by (T, β^{-1}, p), so if β takes only values in $\{\pm 1\}$, there must be an isomorphism from $\mathcal{D}$ to $\mathcal{D}^{\mathrm{sop}}$, which can be seen as a super-anti-automorphism of $\mathcal{D}$. □

Corollary 3.39. *The graded-superalgebra $\mathcal{D}$ admits a super-anti-automorphism if, and only if, $t^2 \in \operatorname{rad} \tilde{\beta}$, for all $t \in T$.* □

Next definition is borrowed from the theory of group cohomology, and allows a compact statement of Equation (3.18). It will also be used in Section 4.4.

Definition 3.40. Let H and K be groups and let $f \colon H \to K$ be any map. We define $\mathrm{d}f \colon H \times H \to K$ to be the map given by $(\mathrm{d}f)(a,b) = f(ab)f(a)^{-1}f(b)^{-1}$ for all $a, b \in H$. The maps $H \times H \to K$ of this form are called 2-*coboundaries*.

Hence Equation (3.18) can be written simply as $\mathrm{d}\eta = \tilde{\beta}$.

If φ_0 is a super-anti-automorphism on $\mathcal{D}$ as in Proposition 3.38, we say that $(\mathcal{D}, \varphi_0)$ is a graded-division superalgebra with super-anti-automorphism *associated* to the quadruple (T, β, p, η). It follows from Lemma 2.33 and Proposition 3.38 that for any finite abelian group T, alternating bicharacter $\beta \colon T \times T \to \mathbb{F}^\times$, group homomorphism $p \colon T \to \mathbb{Z}_2$ and map $\eta \colon T \to \mathbb{F}^\times$ such that $\mathrm{d}\eta = \tilde{\beta}$, there is a graded-division superalgebra with super-anti-automorphism associated to (T, β, p, η). Thus the quadruples (T, β, p, η) parametrize the isomorphism classes of finite dimensional graded-division superalgebra with super-anti-automorphism. It is clear that the corresponding φ_0 is a superinvolution if, and only if, $\eta(t) \in \{\pm 1\}$ for all $t \in T$.

Remark 3.41. By Proposition 2.59, it follows that from (T, η), where T is a finite abelian group and $\eta \colon T \to \mathbb{F}^\times$ is a map such that $\mathrm{d}\eta$ is a skew-symmetric bicharacter, we can recover both β and p.

Corollary 3.42. *Suppose η determines a superinvolution on $\mathcal{D}$, i.e., η takes values in $\{\pm 1\}$. For every element $t \in T$, $\eta(t^2) = (-1)^{p(t)}$ for all $t \in T$. In particular, every element in T^- has order at least 4.*

Proof. By Equation (3.18), we have $\eta(t^2) = \tilde{\beta}(t,t)\eta(t)^2 = (-1)^{p(t)}\beta(t,t) = (-1)^{p(t)}$.

Now let $t \in T^-$. Every odd power of t is also an odd element, so t cannot have an odd order. But $\eta(t^2) = -1$, hence $t^2 \neq e$. □

3.6.2 Parametrization of $(\mathcal{U}, B)$

Let $(\mathcal{D}, \varphi_0)$ be a fixed graded-division superalgebra with super-anti-automorphism associated to (T, β, p, η).

Recall that a G-graded supermodule $\mathcal{U}$ can be regarded as a $G^\#$-graded module and that its isomorphism class is determined by the map $\kappa \colon G^\#/T \to \mathbb{Z}_{\geq 0}$ with finite support (see Subsection 2.1.1). Explicitly, $\kappa(gT) = \dim_{\mathcal{D}} \mathcal{U}_{gT}$, where $\mathcal{U}_{gT}$ is the isotypic component associated to the coset gT.

We will now consider a homogeneous φ_0-sesquilinear form B on $\mathcal{U}$, $\deg B = g_0 \in G^\#$. Since B has degree g_0 and takes values in $\mathcal{D}$, if $B(\mathcal{U}_g, \mathcal{U}_h) \neq 0$ for some $g, h \in G^\#$, then $g_0 gh \in T$. In terms of isotypic components, this means that, given $\mathcal{U}_{gT}$, there is at most one isotypic component $\mathcal{U}_{hT}$ such that $B(\mathcal{U}_{gT}, \mathcal{U}_{hT}) \neq 0$, namely, $\mathcal{U}_{g_0^{-1}g^{-1}T}$. We say that the components $\mathcal{U}_{gT}$ and $\mathcal{U}_{g_0^{-1}g^{-1}T}$ are *paired by B*.

We will now reduce the study of B to the study of $\mathbb{F}$-bilinear forms. Fix a set-theoretic section $\xi \colon G^\#/T \to G^\#$ of the natural homomorphism $G^\# \to G^\#/T$, i.e., $\xi(x) \in x$ for all $x \in G^\#/T$, and fix a nonzero element $X_t \in \mathcal{D}_t$ for all $t \in T$. Note that $\mathcal{U}_{\xi(x)} \otimes \mathcal{D} \simeq \mathcal{U}_x$ via the map $u \otimes d \mapsto ud$ and, hence, an $\mathbb{F}$-basis of $\mathcal{U}_{\xi(x)}$ is a graded $\mathcal{D}$-basis for $\mathcal{U}_x$. In view of Convention 2.53, if $\mathcal{D}$ is odd we choose ξ to take values in $G = G \times \{\bar{0}\}$.

For a given $x \in G^\#/T$, set $y \coloneqq g_0^{-1}x^{-1} \in G^\#/T$ and $t \coloneqq g_0\xi(x)\xi(y) \in T$. Also, set $\mathcal{V}_x \coloneqq \mathcal{U}_x + \mathcal{U}_y$ (so $\mathcal{V}_x = \mathcal{U}_x$ if $x = y$ and $\mathcal{V}_x = \mathcal{U}_x \oplus \mathcal{U}_y$ if $x \neq y$) and $V_x \coloneqq \mathcal{U}_{\xi(x)} + \mathcal{U}_{\xi(y)}$

(so $\mathcal{V}_x \simeq V_x \otimes \mathcal{D}$). We will denote the restriction of B to $\mathcal{V}_x$ by B_x and define the bilinear map $\tilde{B}_x\colon V_x \times V_x \to \mathbb{F}$ by

$$\tilde{B}_x(u,v) := X_t^{-1} B_x(u,v),$$

for all $u, v \in V_x$. It is clear that B is nondegenerate if, and only if, B_x is nondegenerate for every $x \in G^\#/T$. If this is the case, $\mathcal{U}_x$ and $\mathcal{U}_y$ are dual to each other and, hence, $\kappa(x) = \kappa(y)$.

Lemma 3.43. *The φ_0-sesquilinear form B_x is nondegenerate if, and only if, the bilinear form $\tilde{B}_x$ is nondegenerate.*

Proof. Assume B_x is nondegenerate and let $u \in V_x$ be such that $\tilde{B}_x(u,v) = 0$ for all $v \in V_x$. Then $B_x(u,v) = X_t \tilde{B}_x(u,v) = 0$ for all $v \in V_x$ and, hence, $B_x(u,vd) = B_x(u,v)d = 0$ for all $d \in \mathcal{D}$. It follows that $B_x(u,v) = 0$ for all $v \in \mathcal{V}_x = V_x\mathcal{D}$ and, therefore, $u = 0$.

Now assume $\tilde{B}_x$ is nondegenerate and let $u \in \mathcal{V}_x$ be such that $B_x(u,v) = 0$ for all $v \in \mathcal{V}_x$. Let $v \in V_x$ be homogeneous and write $u = \sum_{g\in G^\#} u_g$ where $u_g \in \mathcal{U}_g$. Then $0 = B_x(\sum_{g\in G^\#} u_g, v) = \sum_{g\in G^\#} B_x(u_g, v)$ and, since the summands have pairwise distinct degrees, $B_x(u_g, v) = 0$ for all $g \in G^\#$. Also, since $\xi(gT)^{-1}g \in T$, we have that $u_g = \tilde{u}_g d$ for some homogeneous elements $\tilde{u}_g \in V_x$ and $0 \neq d \in \mathcal{D}$. Then $0 = B_x(u_g, v) = (-1)^{|d|(|B|+|\tilde{u}_g|)}\varphi_0(d) B_x(\tilde{u}_g, v)$, and hence $B_x(\tilde{u}_g, v) = 0$. It follows that $\tilde{B}_x(\tilde{u}_g, v) = 0$, for all $v \in V_x$, which implies $\tilde{u}_g = 0$. Therefore, $u = 0$, concluding the proof. □

We are interested in the case when the superadjunction with respect to B is involutive. By Theorem 3.37, if this is the case, then φ_0 is also involutive. Hence, from now on, we will assume that η takes values in $\{\pm 1\}$.

Lemma 3.44. *Let $\delta \in \{\pm 1\}$. Then $\overline{B_x} = \delta B_x$ if, and only if,*

$$\tilde{B}_x(v,u) = (-1)^{|u||v|}\eta(t)\delta\tilde{B}_x(u,v)$$

for all $u, v \in V_x$, where $t := g_0\xi(x)\xi(g_0^{-1}x^{-1})$.

Proof. Let $u, v \in \mathcal{V}_x$. By definition of $\overline{B_x}$, we have:

$$\begin{aligned}\overline{B_x}(u,v) &= (-1)^{|u||v|}\varphi_0^{-1}(B_x(v,u)) \quad = (-1)^{|u||v|}\varphi_0^{-1}(X_t\tilde{B}_x(v,u)) \\ &= (-1)^{|u||v|}\tilde{B}_x(v,u)\varphi_0^{-1}(X_t) = (-1)^{|u||v|}\tilde{B}_x(v,u)\eta(t)^{-1}X_t,\end{aligned}$$

where we have used the fact that $\tilde{B}_x(v,u) \in \mathbb{F}$. Hence, $\overline{B_x}(u,v) = \delta B_x(u,v)$ if, and only if, $(-1)^{|u||v|}\tilde{B}_x(v,u)\eta(t)^{-1}X_t = \delta X_t\tilde{B}_x(u,v)$, and the result follows. $\square$

Recall the identification $M_k(\mathcal{D}) = M_k(\mathbb{F}) \otimes \mathcal{D}$ (see Remark 2.57). In the next two propositions we consider a component paired to itself and two components paired to one another.

Proposition 3.45. *Let $\delta \in \{\pm 1\}$. Suppose $g_0 x^2 = T$, and set $t := g_0\xi(x)^2 \in T$. Then*

$$\mu_x := (-1)^{|\xi(x)|}\eta(t)\delta \in \{\pm 1\}$$

does not depend on the choice of the section $\xi\colon G^\#/T \to G^\#$. Moreover, the restriction B_x of B to $\mathcal{U}_x$ is nondegenerate and satisfies $\overline{B_x} = \delta B_x$ if, and only if, there is a $\mathcal{D}$-basis of $\mathcal{U}_x$ consisting only of elements of degree $\xi(x)$ such that the matrix representing B_x is given by

(i) $I_{\kappa(x)} \otimes X_t$ *if* $\mu_x = +1$;

(ii) $J_{\kappa(x)} \otimes X_t$ *if* $\mu_x = -1$, *where* $\kappa(x)$ *is even and* $J_{\kappa(x)} := \begin{pmatrix} 0 & I_{\kappa(x)/2} \\ -I_{\kappa(x)/2} & 0 \end{pmatrix}$.

Proof. Let $g := \xi(x)$. If $\xi'\colon G^\#/T \to G^\#$ is another section, then there is $s \in T$ such that $\xi'(x) = gs$. Hence:

$$\begin{aligned}(-1)^{|\xi'(x)|}\eta(g_0\xi'(x)^2) &= (-1)^{|gs|}\eta(g_0 g^2 s^2) \\ &= (-1)^{|g|+|s|}(-1)^{|g_0 g^2||s^2|}\beta(g_0g^2, s^2)\eta(g_0g^2)\eta(s^2) \\ &= (-1)^{|g|+|s|}(-1)^{2|g_0g^2||s|}\beta(g_0g^2, s)^2\eta(g_0g^2)\eta(s^2) \\ &= (-1)^{|g|+|s|}\eta(g_0g^2)\eta(s^2) \\ &= (-1)^{|g|+|s|}\eta(g_0g^2)(-1)^{|s|}\eta(s)^2 = (-1)^{|g|}\eta(g_0g^2),\end{aligned}$$

where we have used Equation (3.18) twice.

For the "moreover" part, it follows from Lemmas 3.43 and 3.44 that B_x is nondegenerate and $\overline{B_x} = \delta B_x$ if, and only if, $\tilde{B}_x$ is nondegenerate and $B_x(u,v) = \mu_x B_x(v,u)$, for all $u, v \in V_x = \mathcal{U}_{\xi(x)}$. Then the result follows from the well-known classification of (skew-)symmetric bilinear forms over an algebraically closed field of characteristic different from 2 and the fact that an $\mathbb{F}$-basis for $\mathcal{U}_{\xi(x)}$ is a $\mathcal{D}$-basis for $\mathcal{U}_x$. $\square$

Remark 3.46. Even though μ_x does not depend on ξ, the element $t = g_0\xi(x)^2$ may depend on ξ.

Proposition 3.47. *Let $\delta \in \{\pm 1\}$. Suppose $g_0 xy = T$ for $x \neq y$ and set $t \coloneqq g_0\xi(x)\xi(y) \in T$. Then the restriction B_x of B to $\mathcal{U}_x \oplus \mathcal{U}_y$ is nondegenerate and satisfies $\overline{B_x} = \delta B_x$ if, and only if, there is a $\mathcal{D}$-basis of $\mathcal{U}_x$ with all elements having degree $\xi(x)$ and a $\mathcal{D}$-basis of $\mathcal{U}_y$ with all elements having degree $\xi(y)$ such that the matrix representing B_x is*

$$\begin{pmatrix} 0 & I_{\kappa(x)} \\ (-1)^{|\xi(x)||\xi(y)|}\eta(t)\delta I_{\kappa(x)} & 0 \end{pmatrix} \otimes X_t.$$

Proof. Assume that B_x is nondegenerate. Then by Lemma 3.43, the bilinear form $\tilde{B}_x$ on $V_x = \mathcal{U}_{\xi(x)} \oplus \mathcal{U}^*_{\xi(y)}$ is nondegenerate, and hence, the map $\mathcal{U}_{\xi(x)} \to \mathcal{U}_{\xi(y)}$ given by $u \mapsto \tilde{B}_x(u, \cdot)$ is an isomorphism of vector spaces. Hence, we can fix a basis $\{u_1, \ldots, u_{\kappa(x)}\}$ for $\mathcal{U}_{\xi(x)}$ and take its dual basis $\{v_1, \ldots, v_{\kappa(x)}\}$ for $\mathcal{U}_{\xi(y)}$, i.e., $\tilde{B}_x(u_i, v_j) = \delta_{ij}$. If we also assume $\overline{B_x} = \delta B_x$, then by Lemma 3.44, we have $\tilde{B}_x(v_i, u_j) = (-1)^{|\xi(x)||\xi(y)|}\eta(t)\delta\delta_{ij}$. This proves the only if part. The converse is clear. $\square$

Definition 3.48. Let $\mathcal{U} \neq 0$ be a graded $\mathcal{D}$-module of finite rank and B be a nondegenerate homogeneous sesquilinear form with respect a superinvolution φ_0 on $\mathcal{D}$ and such that $\overline{B} = \delta B$ for some $\delta \in \{\pm 1\}$. We say that the quadruple $(\eta, \kappa, g_0, \delta)$ is the *inertia of* $(\mathcal{U}, B)$, where η defines φ_0 by $\varphi_0(X_t) = \eta(t) X_t$ (see Subsection 3.6.1), $g_0 = \deg B \in G^\#$ and $\kappa(x) = \dim_{\mathcal{D}} \mathcal{U}_x$ for all $x \in G^\#/T$.

By the results above, the quadruple $(\eta, \kappa, g_0, \delta)$ satisfies the following:

Definition 3.49. Given $\eta \colon T \to \{\pm 1\}$, $\kappa \colon G^\#/T \to \mathbb{Z}_{\geq 0}$, $g_0 \in G^\#$ and $\delta \in \{\pm 1\}$, we say that the quadruple $(\eta, \kappa, g_0, \delta)$ is *admissible* if:

(i) $\mathrm{d}\eta = \tilde{\beta}$ (see Equation (2.4) and Definition 3.40);

(ii) κ has finite support;

(iii) $\kappa(x) = \kappa(g_0^{-1}x^{-1})$ for all $x \in G^\#/T$;

(iv) for any $x \in G^\#/T$, if $g_0x^2 = T$ and $\mu_x = -1$, then $\kappa(x)$ is even (where $\mu_x \coloneqq (-1)^{|g|}\eta(g_0g^2)\delta$ for $g \in x$, see Proposition 3.45).

The set of all admissible quadruples will be denoted by $\mathbf{I}(\mathcal{D})$ or $\mathbf{I}(T, \beta, p)$.

Given a quadruple $(\eta, \kappa, g_0, \delta) \in \mathbf{I}(\mathcal{D})$, we can construct a pair $(\mathcal{U}, B)$ such that $(\eta, \kappa, g_0, \delta)$ is its inertia. To see that, fix a total order $\leq$ on the set $G^\#/T$. We define $W_x \coloneqq (\mathbb{F}^{\kappa(x)})^{[\xi(x)]}$, and $\mathcal{U} = \sum_{x\in G^\#/T} \mathcal{U}_x$ where $\mathcal{U}_x \coloneqq W_x \otimes \mathcal{D}$, for all $x \in G^\#/T$. For all $x, y \in G^\#/T$ such that $g_0xy = T$ and $x \leq y$, we let B_x be the φ_0-sesquilinear on $\mathcal{U}_x + \mathcal{U}_y$ represented, relative to the standard basis of $W_x + W_y$, by the matrices in Proposition 3.45, if $x = y$, or in Proposition 3.47, if $x \neq y$.

Theorem 3.50. *Suppose $\mathbb{F}$ is an algebraically closed field and* $\operatorname{char}\mathbb{F} \neq 2$. *Let $\mathcal{D}$ be a finite dimensional graded-division superalgebra and let φ_0 be a degree preserving superinvolution on $\mathcal{D}$. The assignment of inertia to a pair $(\mathcal{U}, B)$ as in Definition 3.48 gives a bijection between the isomorphism classes of these pairs and the set $\mathbf{I}(\mathcal{D})$.*

Proof. Suppose there is an isomorphism $\psi\colon (\mathcal{U}, B) \to (\mathcal{U}', B')$. Since, in particular, ψ is an isomorphism of graded $\mathcal{D}$-modules, both $\mathcal{U}$ and $\mathcal{U}'$ correspond to the same map $\kappa\colon G^\#/T \to \mathbb{Z}_{\geq 0}$. Also, from the fact that $B'(\psi(u), \psi(v)) = B(u, v)$ for all $u, v \in \mathcal{U}$, it is clear that $\deg B' = \deg B$. Moreover, if $\overline{B} = \delta B$, then

$$\begin{aligned}
\overline{B'}\Big(\psi(u), \psi(v)\Big) &= (-1)^{|\psi(u)||\psi(v)|}\varphi_0^{-1}\Big(B'\Big(\psi(v), \psi(u)\Big)\Big) \\
&= (-1)^{|u||v|}\varphi_0^{-1}\Big(B(v, u)\Big) = \overline{B}(u, v) \\
&= \delta B(u, v) = \delta B'\Big(\psi(u), \psi(v)\Big),
\end{aligned}$$

for all $u, v \in \mathcal{U}$. Since ψ is bijective, it follows that $\overline{B'} = \delta B'$.

Conversely, suppose that $(\mathcal{U}, B)$ and $(\mathcal{U}, B')$ have the same inertia (κ, g_0, δ). To show that $(\mathcal{U}, B)$ and $(\mathcal{U}, B')$ are isomorphic, it suffices to find homogeneous $\mathcal{D}$-bases $\{u_1, \ldots, u_k\}$ of $\mathcal{U}$ and $\{u_1', \ldots, u_k'\}$ of $\mathcal{U}'$ such that $\deg u_i = \deg u_i'$, $1 \leq i \leq k$, and B and B' are represented by the same matrix. Indeed, in this case the $\mathcal{D}$-linear map $\psi\colon \mathcal{U} \to \mathcal{U}'$ defined by $\psi(u_i) = u_i'$, $1 \leq i \leq k$, is degree preserving, and $B(u_i, u_j) = B(u_i', u_j')$, $1 \leq i, j \leq k$, implies $B(u, v) = B(\psi(u), \psi(v))$ for all $u, v \in \mathcal{U}$. The existence of such bases follows from $\dim_{\mathcal{D}}(\mathcal{U}_x) = \dim_{\mathcal{D}}(\mathcal{U}'_x) = \kappa(x)$, for all $x \in G^{\#}/T$, and Propositions 3.45 and 3.47. $\square$

3.6.3 Parametrization of (R, φ)

We start with a definition to have a concise description of the graded superalgebras with superinvolution we are working on:

Definition 3.51. Let $\mathcal{D}$ be a finite dimensional graded-division superalgebra over an algebraically closed field $\mathbb{F}$, $\operatorname{char}\mathbb{F} \neq 2$, let $\mathcal{U} \neq 0$ be a graded right $\mathcal{D}$-module of finite rank and let B be a nondegenerate homogeneous sesquilinear form on $\mathcal{U}$ such that $\overline{B} = \pm B$. By Theorem 3.37, $E(\mathcal{D}, \mathcal{U}, B)$ (see Definition 3.19) is a (finite dimensional) graded superalgebra with superinvolution. If $\mathcal{D}$ is associated to (T, β, p) (see Subsection 2.1.3) and $(\mathcal{U}, B)$ has inertia $(\eta, \kappa, g_0, \delta) \in \mathbf{I}(T, \beta, p)$ (see Definitions 3.48 and 3.49), then we say that $(T, \beta, p, \eta, \kappa, g_0, \delta)$ are the parameters of the triple $(\mathcal{D}, \mathcal{U}, B)$.

By Theorems 2.23, 3.18 and 3.37, any finite dimensional graded-simple superalgebra with superinvolution (R, φ) is isomorphic to $E(\mathcal{D}, \mathcal{U}, B)$ for some triple $(\mathcal{D}, \mathcal{U}, B)$ as in Definition 3.51. We will now classify these graded superalgebras with superinvolution in terms of the parameters $(T, \beta, p, \eta, \kappa, g_0, \delta)$. Let $(\mathcal{D}, \mathcal{U}, B)$ and $(\mathcal{D}', \mathcal{U}', B')$ be triples as in Definition 3.51, and let $(R, \varphi) := E(\mathcal{D}, \mathcal{U}, B)$ and $(R', \varphi') := E(\mathcal{D}', \mathcal{U}', B')$. Consider the corresponding parameters $(T, \beta, p, \eta, \kappa, g_0, \delta)$ and $(T', \beta', p', \eta', g'_0, \delta', \kappa')$. If $(R, \varphi) \simeq (R', \varphi')$, then $\mathcal{D} \simeq \mathcal{D}'$ and, hence, $T = T'$, $\beta = \beta'$ and $p = p'$.

Lemma 3.52. *Let $\psi_0 \colon \mathcal{D} \to \mathcal{D}'$ be a degree preserving isomorphism. Then $(\mathcal{U}', B')$ and $((\mathcal{U}')^{\psi_0}, \psi_0^{-1} \circ B')$ have the same inertia $(\eta', \kappa', g'_0, \delta') \in \mathbf{I}(T, \beta, p)$.*

Proof. By Lemma 3.25, $\psi_0^{-1} \circ B'$ is $(\psi_0^{-1} \circ \varphi'_0 \circ \psi_0)$-sesquilinear. Let $X_t \in \mathcal{D}_t$. Then $\psi_0(X_t) \in \mathcal{D}'_t$ and, hence, $\psi'_0\big(\psi_0(X_t)\big) = \eta'(t)\psi_0(X_t)$. It follows that $(\psi_0^{-1} \circ \varphi'_0 \circ \psi_0)(X_t) = \eta'(t) X_t$, therefore the superinvolution $(\psi_0^{-1} \circ \varphi'_0 \circ \psi_0)$ also corresponds to the map $\eta' \colon T \to \mathbb{F}^{\times}$.

Since $\dim_{\mathcal{D}'} \mathcal{U}_x = \dim_{\mathcal{D}} (\mathcal{U}_x)^{\psi_0^{-1}}$, for all $x \in G^{\#}/T$, the graded $\mathcal{D}$-modules $\mathcal{U}$ and $\mathcal{U}^{\psi_0^{-1}}$ correspond to the same κ. Also, it is clear that $\deg(\psi_0 \circ B) = \deg B$. Finally, using that $\psi_0^{-1} \circ \varphi_0 \circ \psi_0$ and φ_0 are involutive, we have that

$$\begin{aligned}\overline{(\psi_0^{-1} \circ B)}(u,v) &= (-1)^{|u||v|}(\psi_0^{-1} \circ \varphi_0 \circ \psi_0)\Big((\psi_0^{-1} \circ B)(v,u)\Big) \\ &= \psi_0^{-1}\left((-1)^{|u||v|}\varphi_0\Big(B(v,u)\Big)\right) \\ &= \psi_0^{-1}\Big(\overline{B}(u,v)\Big) = \psi_0^{-1}\Big(\delta B(u,v)\Big) = \delta(\psi_0^{-1} \circ B)(u,v),\end{aligned}$$

for all $u, v \in \mathcal{U}^{\bar{0}} \cup \mathcal{U}^{\bar{1}}$. □

By Corollary 3.29, $(R, \varphi) \simeq (R', \varphi')$ if, and only if, $(\eta', g_0', \delta', \kappa')$ is in the orbit of $(\eta, \kappa, g_0, \delta)$ under the action of the group $(\mathcal{D}_{\mathrm{gr}}^\times \rtimes A) \times G^\#$ on $\mathbf{I}(T, \beta, p)$ determined by its action on the isomorphism classes of $(\mathcal{U}, B)$ (see Lemma 3.28). Let us compute this action explicitly.

First, applying Lemma 3.52 in the case $\mathcal{D}' = \mathcal{D}$, we have that the A-action on $\mathbf{I}(T, \beta, p)$ is trivial.

Let us now consider the $\mathcal{D}_{\mathrm{gr}}^\times$-action. Let $t \in T$ and $0 \neq d \in \mathcal{D}_t$ and let $(\eta', \kappa', g_0', \delta')$ be the parameters corresponding to $d \cdot (\mathcal{U}, B) = (\mathcal{U}, dB)$ (see Equation (3.11)). To compute η', recall that dB is $(\mathrm{sInt}_d \circ \varphi_0)$-sesquilinear. Let $s \in T$ and $c \in \mathcal{D}_s$. Then

$$\begin{aligned}(\mathrm{sInt}_d \circ \varphi_0)(c) &= \mathrm{sInt}_d(\eta(s)c) = (-1)^{|t||s|}\eta(s)dcd^{-1} \\ &= (-1)^{|t||s|}\eta(s)\beta(t,s)cdd^{-1} = \tilde{\beta}(t,s)\eta(s)c\end{aligned}$$

and, hence, $\eta'(s) = \tilde{\beta}(t,s)\eta(s)$ for all $s \in T$. Since the action by d does not change $\mathcal{U}$, we have $\kappa' = \kappa$. Clearly $g_0' = \deg(dB) = tg_0$. It remains to compute δ'. Using Lemma 3.36, we have

$$\overline{dB} = (-1)^{|t|}\varphi_0(d)\overline{B} = (-1)^{|t|}\eta(t)d\delta B = (-1)^{|t|}\eta(t)\delta(dB),$$

so $\delta' = (-1)^t\eta(t)\delta$. Note that $(\eta', \kappa', g_0', \delta')$ depends only on t, so the $\mathcal{D}_{\mathrm{gr}}^\times$-action on $\mathbf{I}(T, \beta, p)$ factors through the action of $T \simeq \mathcal{D}_{\mathrm{gr}}^\times / \mathbb{F}^\times$.

Finally, we consider the $G^\#$-action. Let $g \in G$ and let $(\eta'', \kappa'', g_0'', \delta'')$ be the parameters corresponding to $g \cdot (\mathcal{U}, B) = (\mathcal{U}^{[g]}, B^{[g]})$ (see Equation (3.12)). Since $B^{[g]}$ is φ_0-sesquilinear, $\eta'' = \eta$. By the definition of $\mathcal{U}^{[g]}$, $\kappa'' = g \cdot \kappa$ where $(g \cdot \kappa)(x) = \kappa(g^{-1}x)$ for all $x \in G^\#/T$. As noted in Remark 3.23, $g_0'' = \deg B^{[g]} = g_0 g^{-2}$. Also, for all

$u, v \in \mathcal{U}^{\bar{0}} \cup \mathcal{U}^{\bar{1}}$, and writing $u^{[g]}$ and $v^{[g]}$ for u and v when they are considered as elements of $\mathcal{U}^{[g]}$, we compute:

$$\begin{aligned}
\overline{B^{[g]}}(u^{[g]}, v^{[g]}) &= (-1)^{|u^{[g]}||v^{[g]}|}\varphi_0^{-1}\Big(B^{[g]}(v^{[g]}, u^{[g]})\Big)\\
&= (-1)^{(|g|+|u|)(|g|+|v|)}\varphi_0^{-1}\Big((-1)^{|g||v|}B(v,u)\Big)\\
&= (-1)^{|g|+|g||u|+|g||v|+|u||v|}(-1)^{|g||v|}\varphi_0^{-1}\Big(B(v,u)\Big) = (-1)^{|g|+|g||u|}\overline{B}(u,v)\\
&= (-1)^{|g|}(-1)^{|g||u|}\delta B(u,v) = (-1)^{|g|}\delta B^{[g]}(u^{[g]}, v^{[g]}),
\end{aligned}$$

We conclude that $\delta'' = (-1)^{|g|}\delta$.

Definition 3.53. The group $T \times G^{\#}$ acts on $\mathbf{I}(T, \beta, p)$ by

$$t \cdot (\eta, \kappa, g_0, \delta) := (\tilde{\beta}(t, \cdot)\eta, \kappa, tg_0, (-1)^{|t|}\eta(t)\delta)$$

and

$$g \cdot (\eta, \kappa, g_0, \delta) := (\eta, g \cdot \kappa, g_0 g^{-2}, (-1)^{|g|}\delta),$$

for all $t \in T$, $g \in G^{\#}$ and $(\eta, \kappa, g_0, \delta) \in \mathbf{I}(T, \beta, p)$.

In view of these considerations, Corollary 3.29 can be restated as follows:

Theorem 3.54. *Let $(\mathcal{D}, \mathcal{U}, B)$ and $(\mathcal{D}', \mathcal{U}', B')$ be triples as in Definition 3.51, and let $(T, \beta, p, \eta, \kappa, g_0, \delta)$ and $(T', \beta', p', \eta', \kappa', g_0', \delta')$ be their parameters. Then $E(\mathcal{D}, \mathcal{U}, B) \simeq E(\mathcal{D}', \mathcal{U}', B')$ if, and only if, $T = T'$, $\beta = \beta'$, $p = p'$, and $(\eta, \kappa, g_0, \delta)$ and $(\eta', \kappa', g_0', \delta')$ lie in the same orbit of the $T \times G^{\#}$-action in $\mathbf{I}(T, \beta, p)$ given in Definition 3.53.* $\square$

We will now proceed to simplify our parameter set and the group acting on it, by using Lemma 2.87. To this end, consider the following equivalence relation among all possible η.

Definition 3.55. Let $\eta, \eta' \colon T \to \{\pm 1\}$ be maps such that $\mathrm{d}\eta = \mathrm{d}\eta' = \tilde{\beta}$. We say that η and η' are *equivalent*, and write $\eta \sim \eta'$, if there is $t \in T$ such that $\eta' = \tilde{\beta}(t, \cdot)\eta$.

We partition the set $\mathbf{I}(T, \beta, p)$ according to the equivalence class of η and refer to $\{(\eta', \kappa, g_0, \delta) \in \mathbf{I}(T, \beta, p) \mid \eta' \sim \eta\}$ as the η-block of the partition. Note that if $(\eta, \kappa, g_0, \delta), (\eta', \kappa', g_0', \delta') \in \mathbf{I}(T, \beta, p)$ are in the same $T \times G^{\#}$-orbit, then, clearly, $\eta' \sim \eta$. In other words, the $T \times G^{\#}$-action on $\mathbf{I}(T, \beta, p)$ restricts to each η-block of the partition.

We wish to fix η. Given $\eta\colon T \to \{\pm 1\}$ such that $\mathrm{d}\eta = \tilde{\beta}$, we define

$$\mathbf{I}(T, \beta, p)_\eta \coloneqq \{(\kappa, g_0, \delta) \mid (\eta, \kappa, g_0, \delta) \in \mathbf{I}(T, \beta, p)\}.$$

It is clear from Definition 3.53 that the action by $(t, g) \in T \times G^{\#}$ does not change η if, and only if, $t \in \operatorname{rad} \tilde{\beta}$. Thus, the $T \times G^{\#}$-action on $\mathbf{I}(T, \beta, p)$ induces an action of the subgroup $(\operatorname{rad} \tilde{\beta}) \times G^{\#}$ on $\mathbf{I}(T, \beta, p)_\eta$ given by

$$\begin{aligned} t \cdot (\kappa, g_0, \delta) &\coloneqq (\kappa, t g_0, \eta(t)\delta) \quad \text{and} \\ g \cdot (\kappa, g_0, \delta) &\coloneqq (g \cdot \kappa, g_0 g^{-2}, (-1)^{|g|}\delta), \end{aligned}$$

for all $t \in \operatorname{rad} \tilde{\beta} = (\operatorname{rad} \beta) \cap T^+$ (recall Lemma 2.61), $g \in G^{\#}$ and $(\kappa, g_0, \delta) \in \mathbf{I}(T, \beta, p)_\eta$.

Now we wish to fix $\delta = 1$. Note that in every orbit we have a triple with $\delta = 1$ since the action by $(e, \bar{1}) \in G^{\#} = G \times \mathbb{Z}_2$ changes the sign of δ. We define

$$\mathbf{I}(T, \beta, p)^+_\eta \coloneqq \{(\kappa, g_0) \mid (\eta, \kappa, g_0, 1) \in \mathbf{I}(T, \beta, p)\}. \tag{3.19}$$

By the definition of the $(\operatorname{rad} \tilde{\beta}) \times G^{\#}$-action on $\mathbf{I}(T, \beta, p)_\eta$, we see that the action of (t, g) does not change δ if, and only if, $\eta(t) = (-1)^{|g|}$. Note that $\eta \restriction_{\operatorname{rad} \tilde{\beta}}$ is a group homomorphism, since $\mathrm{d}\eta = \tilde{\beta} = 1$ on $\operatorname{rad} \tilde{\beta}$. Hence

$$\mathcal{G} \coloneqq \{(t, g) \in (\operatorname{rad} \tilde{\beta}) \times G^{\#} \mid \eta(t) = (-1)^{|g|}\} \tag{3.20}$$

is a subgroup of $(\operatorname{rad} \tilde{\beta}) \times G^{\#}$. Thus, the $(\operatorname{rad} \tilde{\beta}) \times G^{\#}$-action on $\mathbf{I}(T, \beta, p)_\eta$ induces a $\mathcal{G}$-action on $\mathbf{I}(T, \beta, p)^+_\eta$ given by

$$\begin{aligned} t \cdot (\kappa, g_0) &\coloneqq (\kappa, t g_0) \quad \text{and} \\ g \cdot (\kappa, g_0) &\coloneqq (g \cdot \kappa, g_0 g^{-2}), \end{aligned} \tag{3.21}$$

for all $(t, g) \in \mathcal{G}$ and $(\kappa, g_0) \in \mathbf{I}(T, \beta, p)^+_\eta$.

Lemma 2.87 implies the following:

Proposition 3.56. *Fix a map $\eta\colon T \to \{\pm 1\}$ such that $\mathrm{d}\eta = \tilde{\beta}$. Then the map $\iota\colon \mathbf{I}(T, \beta, p)^+_\eta \to \mathbf{I}(T, \beta, p)$ given by $\iota(\kappa, g_0) \coloneqq (\eta, \kappa, g_0, 1)$ induces a bijection between the $\mathcal{G}$-orbits in $\mathbf{I}(T, \beta, p)^+_\eta$ and the $T \times G^{\#}$-orbits in the η-block of $\mathbf{I}(T, \beta, p)$.* $\square$

Therefore, the classification up to isomorphism of G-graded superalgebras with superinvolution that are finite dimensional and graded-simple over an algebraically closed field $\mathbb{F}$, $\operatorname{char}\mathbb{F} \neq 2$, reduces to the classification of finite subgroups $T \subseteq G^{\#}$, alternating bicharacters $\beta\colon T\times T \to \{\pm 1\}$, equivalence classes of η (see Definition 3.55) and $\mathcal{G}$-orbits in $\mathbf{I}(T,\beta,p)^{+}_{\eta}$. This gives a useful simplification when we have one equivalence class of η, as it will be the case in the next chapter, where we will study gradings on superinvolution-simple associative superalgebras.

In the case $T^{-} \neq \emptyset$, another simplification can be made, but it has to be done before fixing η. As noted in Remark 3.20, we can make the sesquilinear form B even, i.e., we can choose g_0 such that $|g_0| = \bar{0}$. Equivalently, every $T \times G^{\#}$-orbit has a element $(\eta, \kappa, g_0, \delta)$ with $|g_0| = \bar{0}$. We let

$$\mathbf{I}(T,\beta,p)^{\bar{0}} := \{(\eta,\kappa,g_0,\delta) \in \mathbf{I}(T,\beta,p) \mid |g_0| = \bar{0}\}.$$

If we wish to restrict the action to only these elements, we have to act by $T^{+} \times G^{\#}$.

Definition 3.57. Let $\eta, \eta'\colon T \to \{\pm 1\}$ be maps such that $\mathrm{d}\eta = \mathrm{d}\eta' = \tilde{\beta}$. We say that η and η' are *evenly equivalent*, and write $\eta \sim_{\bar{0}} \eta'$, if there is $t \in T^{+}$ such that $\eta' = \tilde{\beta}(t,\cdot)\eta$.

We partition the set $\mathbf{I}(T,\beta,p)^{\bar{0}}$ according to this new equivalence relation: the η-block is now $\{(\eta',\kappa,g_0,\delta) \in \mathbf{I}(T,\beta,p)^{\bar{0}} \mid \eta' \sim_{\bar{0}} \eta\}$. If $(\eta,\kappa,g_0,\delta), (\eta',\kappa',g_0',\delta') \in \mathbf{I}(T,\beta,p)^{\bar{0}}$ are in the same $T^{+} \times G^{\#}$-orbit, then $\eta' \sim_{\bar{0}} \eta$. Hence, the $T^{+} \times G^{\#}$-action on $\mathbf{I}(T,\beta,p)^{\bar{0}}$ restricts to each η-block of the partition.

Remark 3.58. It should be noted that the equivalence class of η according to the relation $\sim_{\bar{0}}$ is, in general, smaller than the equivalence class according to $\sim$. This implies that graded superalgebras corresponding to points in the same η-block of $\mathbf{I}(T,\beta,p)$ may correspond to different η-blocks of $\mathbf{I}(T,\beta,p)^{\bar{0}}$.

As before, we wish to fix η and make $\delta = 1$. Given $\eta\colon T \to \{\pm 1\}$ such that $\mathrm{d}\eta = \tilde{\beta}$, we define

$$\mathbf{I}(T,\beta,p)^{\bar{0},+}_{\eta} := \{(\kappa,g_0) \mid (\eta,\kappa,g_0,1) \in \mathbf{I}(T,\beta,p) \,\text{and}\, |g_0| = \bar{0}\}.$$

Following the same reasoning as above, we get a $\mathcal{G}$-action on $\mathbf{I}(T,\beta,p)^{\bar{0},+}_{\eta}$, where $\mathcal{G}$ is defined by Equation (3.20) and the action by Equation (3.21). Then Lemma 2.87

implies:

Proposition 3.59. *Suppose $T^- \neq \emptyset$ and fix a map $\eta\colon T \to \{\pm 1\}$ such that $\mathrm{d}\eta = \tilde{\beta}$. Then the map $\iota\colon \mathbf{I}(T,\beta,p)^{\bar{0},+}_{\eta} \to \mathbf{I}(T,\beta,p)^{\bar{0}}$ given by $\iota(\kappa, g_0) \coloneqq (\eta, \kappa, g_0, 1)$ induces a bijection between the $\mathcal{G}$-orbits in $\mathbf{I}(T,\beta,p)^{\bar{0},+}_{\eta}$ and the $T^+ \times G^{\#}$-orbits in the η-block of $\mathbf{I}(T,\beta,p)^{\bar{0}}$.* $\square$

Chapter 4

Gradings on Superinvolution-Simple Associative Superalgebras

The main goal of this chapter is to classify (up to isomorphism) the G-gradings on the finite dimensional superinvolution-simple associative superalgebras over an algebraically closed field of characteristic different from 2. Not only is this classification of independent interest, but also, in Chapter 5 (assuming the characteristic is 0), it will be translated to a complete classification of G-gradings on the Lie superalgebras of series A, B, C, D, P and Q, with the sole exception of type $A(1,1)$. As in Chapter 3, the term *(super)algebra* will mean *associative (super)algebra* and the grading group G will be assumed abelian (see Proposition 3.1).

In Section 4.1, we consider more general objects: graded-superinvolution-simple superalgebras, and show that they are either graded-simple or of the form $S \times S^{\mathrm{sop}}$, with exchange superinvolution, where S is graded-simple (Proposition 4.6). The former case was already classified in Chapter 3 (Theorems 3.18, 3.27 and 3.54), so we complete the classification in the latter case (Lemma 4.5 and Theorems 4.9 to 4.11). In Section 4.2, we specialize to trivial G and obtain the well-known classification of superinvolution-simple associative superalgebras, which are of the following 3 types: M, $M \times M^{\mathrm{sop}}$ and $Q \times Q^{\mathrm{sop}}$ (see Definition 4.15).

The gradings on superinvolution-simple superalgebras of type M are classified in Section 4.3 (Theorem 4.29). For superalgebras of types $M \times M^{\mathrm{sop}}$ and $Q \times Q^{\mathrm{sop}}$, we first develop a way to construct division gradings, in Section 4.4, and then use them to classify all gradings, in Section 4.5: Theorems 4.55 and 4.56 consider the gradings on the superalgebras of types $M \times M^{\mathrm{sop}}$ and $Q \times Q^{\mathrm{sop}}$ that do not make them graded-simple, whereas Theorems 4.70 and 4.74 consider the remaining gradings. The latter theorem deals with the case of odd gradings and gives a classification in terms of $G^{\#}$, so we derive Corollaries 4.75 and 4.76 to express it in terms of G. A description of gradings for types $M \times M^{\mathrm{sop}}$ and $Q \times Q^{\mathrm{sop}}$ (assuming that G is finite and $\operatorname{char}\mathbb{F}$ does not divide $|G|$) was obtained in [BTT09, Theorems 4 and 5] using a different method, which, however, does not lend itself to a classification up to isomorphism. We note that elementary gradings on superinvolution-simple superalgebras of type M were described in [TT06, Theorems 5.2 and 5.3], but the description of general gradings in this preprint is incorrect.

4.1 Graded-superinvolution-simple superalgebras

In [Rac98], finite dimensional superinvolution-simple superalgebras are classified over any field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2$. In this section we adapt some of the results there to classify the finite dimensional graded-superinvolution-simple superalgebras in the case of algebraically closed $\mathbb{F}$. Most of the following results are valid for any $\mathbb{F}$.

Definition 4.1. Let S be a G-graded superalgebra and consider the G-graded superalgebra $S \times S^{\mathrm{sop}}$ with the homogeneous component of degree $g \in G$ being $S_g \times S_g^{\mathrm{sop}}$. We define the *exchange superinvolution* on $S \times S^{\mathrm{sop}}$ to be the map $\varphi\colon S \times S^{\mathrm{sop}} \to S \times S^{\mathrm{sop}}$ given by $\varphi(s_1, \bar{s}_2) = (s_2, \bar{s}_1)$ (recall from Section 3.1 that $\bar{s}$ denotes the element $s \in S$ seen as an element of S^{sop}). Unless stated otherwise, we will always consider $S \times S^{\mathrm{sop}}$ to be endowed with exchange superinvolution.

We will now give two examples where G is trivial, which will play a role in Section 4.2.

Example 4.2. The simplest possible example is to take $S = \mathbb{F}$, with superalgebra structure. If $\operatorname{char}\mathbb{F} \neq 2$, then $S \times S^{\mathrm{sop}} = \mathbb{F}[\zeta]$ where $\zeta = (1, -1)$ and the exchange superinvolution is given by $\varphi(1) = 1$ and $\varphi(\zeta) = -\zeta$. Note that $\mathbb{F}[\zeta] \simeq \mathbb{F}\mathbb{Z}_2$ with the trivial superalgebra structure.

Example 4.3. Consider $S = Q(1)$, so $S^{\bar{0}} = \mathbb{F}1$ and $S^{\bar{1}} = \mathbb{F}u$ where $u^2 = 1$. Note that S is isomorphic to $\mathbb{F}\mathbb{Z}_2$, but this time with the superalgebra structure given by its natural $\mathbb{Z}_2$-grading. If $\operatorname{char}\mathbb{F} \neq 2$, we claim that $R := S \times S^{\mathrm{sop}}$ is isomorphic to $\mathbb{F}\mathbb{Z}_4$. Indeed, the element $\omega := (u, \bar{u}) \in S \times S^{\mathrm{sop}}$ has order 4 and generates $S \times S^{\mathrm{sop}}$: $\omega^2 = (1, -1)$, $\omega^3 = (u, -\bar{u})$ and $\omega^4 = (1, 1)$. Hence $R^{\bar{0}} = \mathbb{F}1 \oplus \mathbb{F}\omega^2$ and $R^{\bar{1}} = \mathbb{F}\omega \oplus \mathbb{F}\omega^3$. Also, the exchange superinvolution on R is given by $\varphi(1) = 1$, $\varphi(\omega) = \omega$, $\varphi(\omega^2) = -\omega^2$ and $\varphi(\omega^3) = -\omega^3$.

We are now going to see when $S \times S^{\mathrm{sop}}$ with exchange superinvolution is graded-superinvolution-simple.

Lemma 4.4. *Let S be a graded superalgebra and consider the exchange superinvolution on $S \times S^{\mathrm{sop}}$. Then $S \times S^{\mathrm{sop}}$ is graded-superinvolution-simple if, and only if, S is graded-simple.*

Proof. Suppose $S \times S^{\mathrm{sop}}$ is graded-superinvolution-simple and let $I \subseteq S$ be a graded ideal. We have that $I \times I^{\mathrm{sop}}$ is a superinvolution-invariant graded superideal in $S \times S^{\mathrm{sop}}$, and, hence, either $I \times I^{\mathrm{sop}} = 0$ or $I \times I^{\mathrm{sop}} = S \times S^{\mathrm{sop}}$. In the first case $I = 0$ and in the second $I = S$, hence S is graded-simple.

Conversely, suppose S is graded-simple. It is clear that S^{sop} is also graded-simple and, hence, by a standard argument, the graded superideals of $S \times S^{\mathrm{sop}}$ are 0, $\{0\} \times S^{\mathrm{sop}}$, $S \times \{0\}$ and $S \times S^{\mathrm{sop}}$. Among those, only 0 and $S \times S^{\mathrm{sop}}$ are superinvolution-invariant, concluding the proof. $\square$

Lemma 4.5. *Let S_1 and S_2 be graded-simple superalgebras. Then $S_1 \times S_1^{\mathrm{sop}} \simeq S_2 \times S_2^{\mathrm{sop}}$ as graded superalgebras with superinvolution if, and only if, $S_1 \simeq S_2$ or $S_1 \simeq S_2^{\mathrm{sop}}$ as graded superalgebras.*

Proof. Let $\psi\colon S_1 \times S_1^{\mathrm{sop}} \to S_2 \times S_2^{\mathrm{sop}}$ be an isomorphism of graded superalgebras with superinvolution. Since the only nonzero proper graded superideals of $S_i \times S_i^{\mathrm{sop}}$ are $\{0\} \times S_i^{\mathrm{sop}}$ and $S_i \times \{0\}$, $i = 1, 2$, we have that $\psi(S_1 \times \{0\}) = S_2 \times \{0\}$ or $\psi(S_1 \times \{0\}) = \{0\} \times S_2^{\mathrm{sop}}$.

For the converse, we can suppose $S_1 \simeq S_2$ by relabeling S_2 with S_2^{sop} if necessary. If $\theta\colon S_1 \to S_2$ is an isomorphism of graded superalgebras, it is clear that $\psi\colon S_1 \times S_1^{\mathrm{sop}} \to S_2 \times S_2^{\mathrm{sop}}$ given by $\psi(x, \bar{y}) := (\theta(x), \overline{\theta(y)})$, for all $x, y \in S_1$, is an isomorphism of graded superalgebras with superinvolution. $\square$

Proposition 4.6. *Let (R, φ) be a graded superalgebra with superinvolution. Then (R, φ) is graded-superinvolution-simple if, and only if, either R is a graded-simple* *or (R, φ) is isomorphic to $S \times S^{\mathrm{sop}}$ with the exchange superinvolution, for some graded-simple superalgebra S.*

Proof. Suppose (R, φ) is graded-superinvolution-simple but R is not graded-simple. Let $0 \neq I \subsetneq R$ be a graded superideal. Note that $\varphi(I)$ is also a graded superideal, hence $I \cap \varphi(I)$ and $I + \varphi(I)$ are φ-invariant graded superideals. Since $I \cap \varphi(I) \subseteq I \neq R$, we have $I \cap \varphi(I) = 0$, so we can write $I + \varphi(I) = I \oplus \varphi(I)$. Since $0 \neq I \subseteq I \oplus \varphi(I)$, we conclude that $R = I \oplus \varphi(I)$. Clearly, this implies that (R, φ) is isomorphic to $I \times I^{\mathrm{sop}}$ with exchange superinvolution. By Lemma 4.4, I must be simple as a graded superalgebra.

The converse is obvious if R is graded-simple, and follows from Lemma 4.4 otherwise. □

It is sometimes convenient to replace S^{sop} by an isomorphic graded superalgebra. Explicitly, suppose $\theta\colon S \to S'$ is a super-anti-isomorphism of graded superalgebras. Then $S \times S^{\mathrm{sop}}$ with the exchange superinvolution is isomorphic to $S \times S'$ endowed with the superinvolution $(s_1, s_2) \mapsto (\theta^{-1}(s_2), \theta(s_1))$.

Definition 4.7. Let $\mathcal{D}$ be a graded-division superalgebra and $\mathcal{U}$ be a graded right $\mathcal{D}$-supermodule of finite rank. Recall that $\mathcal{U}^\star := \mathrm{Hom}_{\mathcal{D}}(\mathcal{U}, \mathcal{D})$ (Definition 3.2) is a graded right $\mathcal{D}^{\mathrm{sop}}$-module, and that the map $\mathrm{End}_{\mathcal{D}}(\mathcal{U}) \to \mathrm{End}_{\mathcal{D}^{\mathrm{sop}}}(\mathcal{U}^\star)$ given by $L \mapsto L^\star$ (Definition 3.5) is a super-anti-isomorphism whose inverse is $L \mapsto {}^\star L$ (Proposition 3.10). We define $E^{\mathrm{ex}}(\mathcal{D}, \mathcal{U})$ to be the graded superalgebra $\mathrm{End}_{\mathcal{D}}(\mathcal{U}) \times \mathrm{End}_{\mathcal{D}^{\mathrm{sop}}}(\mathcal{U}^\star)$ endowed with the superivolution $(L_1, L_2) \mapsto ({}^\star L_2, L_1^\star)$.

Combining Proposition 4.6 and Theorems 2.23 and 3.18, we have:

Corollary 4.8. *Let (R, φ) be a graded superalgebra with superinvolution and suppose R satisfies the d.c.c. on graded left superideals. Then (R, φ) is graded-superinvolution-simple if, and only if, there exists a graded-division superalgebra $\mathcal{D}$ and graded right $\mathcal{D}$-supermodule $\mathcal{U}$ of finite rank such that either $(R, \varphi) \simeq E^{\mathrm{ex}}(\mathcal{D}, \mathcal{U})$ or $(R, \varphi) \simeq E(\mathcal{D}, \mathcal{U}, B)$, for some nondegenerate sesquilinear form B on $\mathcal{U}$.* □

Let us now assume that $\mathbb{F}$ is algebraically closed and $\operatorname{char} \mathbb{F} \neq 2$. We are in a position to classify up to isomorphism the finite dimensional graded-superinvolution-simple superalgebras over $\mathbb{F}$. The ones that are graded-simple are classified in Theorem

3.54. The classification of the ones that are not graded-simple is a consequence of Theorems 2.73 and 2.74, Lemma 4.5 and the description of parameters for $\mathcal{D}^{\mathrm{sop}}$ and $\mathcal{U}^\star$ at the end of Section 3.1:

Theorem 4.9. *Let $(\mathcal{D},\mathcal{U})$ and $(\mathcal{D}',\mathcal{U}')$ be pairs as in Definition 2.72, with both $\mathcal{D}$ and $\mathcal{D}'$ even. Let $(T,\beta,\kappa_{\bar 0},\kappa_{\bar 1})$ and $(T',\beta',\kappa'_{\bar 0},\kappa'_{\bar 1})$ be the parameters of $(\mathcal{D},\mathcal{U})$ and $(\mathcal{D}',\mathcal{U}')$, respectively. Then $E^{\mathrm{ex}}(\mathcal{D},\mathcal{U}) \simeq E^{\mathrm{ex}}(\mathcal{D}',\mathcal{U}')$ if, and only if, $T = T'$ and one of the following conditions holds:*

(i) $\beta' = \beta$ *and there is* $g \in G$ *such that either* $\kappa'_{\bar 0} = g \cdot \kappa_{\bar 0}$ *and* $\kappa'_{\bar 1} = g \cdot \kappa_{\bar 1}$, *or* $\kappa'_{\bar 0} = g \cdot \kappa_{\bar 1}$ *and* $\kappa'_{\bar 1} = g \cdot \kappa_{\bar 0}$;

(ii) $\beta' = \beta^{-1}$ *and there is* $g \in G$ *such that either* $\kappa'_{\bar 0} = g \cdot \kappa^\star_{\bar 0}$ *and* $\kappa'_{\bar 1} = g \cdot \kappa^\star_{\bar 1}$, *or* $\kappa'_{\bar 0} = g \cdot \kappa^\star_{\bar 1}$ *and* $\kappa'_{\bar 1} = g \cdot \kappa^\star_{\bar 0}$. □

Theorem 4.10. *Let $(\mathcal{D},\mathcal{U})$ and $(\mathcal{D}',\mathcal{U}')$ be pairs as in Definition 2.72, with both $\mathcal{D}$ and $\mathcal{D}'$ odd. Let (T,β,p,κ) and (T',β',p',κ') be the parameters of $(\mathcal{D},\mathcal{U})$ and $(\mathcal{D}',\mathcal{U}')$, respectively. Then $E^{\mathrm{ex}}(\mathcal{D},\mathcal{U}) \simeq E^{\mathrm{ex}}(\mathcal{D}',\mathcal{U}')$ if, and only if, $T = T'$, $p = p'$ and one of the following conditions holds:*

(i) $\beta' = \beta$ *and there is* $g \in G$ *such that* $\kappa' = g \cdot \kappa$;

(ii) $\beta' = \beta^{-1}$ *and there is* $g \in G$ *such that* $\kappa' = g \cdot \kappa^\star$. □

The case where $\mathcal{D}$ and $\mathcal{D}'$ are odd can also be classified in terms of G-parameters (see Definition 2.81). We recall the set $\mathbf{O}(T^+,\beta^+)$ and the T^+-action on it (see Definition 2.82 and Equation (2.6)).

Theorem 4.11. *Let $(\mathcal{D},\mathcal{U})$ and $(\mathcal{D}',\mathcal{U}')$ be pairs as in Definition 2.81, and let $(T^+,\beta^+,h,\chi,\kappa)$ and $(T'^+,\beta'^+,h',\chi',\kappa')$ be G-parameters of $(\mathcal{D},\mathcal{U})$ and $(\mathcal{D}',\mathcal{U}')$, respectively. Then $E^{\mathrm{ex}}(\mathcal{D},\mathcal{U}) \simeq E^{\mathrm{ex}}(\mathcal{D}',\mathcal{U}')$ if, and only if, $T^+ = T'^+$ and one of the following conditions holds:*

(i) $\beta'^+ = \beta^+$, (h',χ') *is in the same* T^+*-orbit of* (h,χ) *in* $\mathbf{O}(T^+,\beta^+)$ *and there is* $g \in G$ *such that* $\kappa' = g \cdot \kappa$;

(ii) $\beta'^+ = (\beta^+)^{-1}$, (h',χ') *is in the same* T^+*-orbit of* (h^{-1},χ) *in* $\mathbf{O}(T^+,\beta^+)$ *and there is* $g \in G$ *such that* $\kappa' = g \cdot \kappa^\star$.

Proof. Using Equation (2.5), it is straightforward that $\mathcal{D}^{\text{sop}}$ is associated to the quadruple $(T^+, (\beta^+)^{-1}, h^{-1}, \chi)$ and, hence, $(T^+, (\beta^+)^{-1}, h^{-1}, \chi, \kappa^\star)$ are G-parameters for $(\mathcal{D}^{\text{sop}}, \mathcal{U}^\star)$. The result follows from Corollary 2.86. $\square$

4.2 Finite dimensional superinvolution-simple superalgebras

As an application of our results of Sections 3.6 and 4.1, we can obtain the known classification of superinvolution-simple superalgebras over an algebraically closed field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2$.

We start with a general observation:

Lemma 4.12. *Let (R, φ) be a superalgebra with super-anti-automorphism. Then $Z(R)$ is φ-invariant.*

Proof. By Lemma 1.13, with $G = \mathbb{Z}_2$, we have $Z(R) = Z(R)^{\bar{0}} \oplus Z(R)^{\bar{1}}$, so it is sufficient to show that if $c \in Z(R)^{\bar{0}} \cup Z(R)^{\bar{1}}$, then $\varphi(c) \in Z(R)$. Let $r \in R^{\bar{0}} \cup R^{\bar{1}}$. Since $c\varphi^{-1}(r) = \varphi^{-1}(r)c$, we can apply φ on both sides and get $(-1)^{|c||r|}r\varphi(c) = (-1)^{|c||r|}\varphi(c)r$ and, hence, $r\varphi(c) = \varphi(c)r$. $\square$

The following result was achieved by similar methods in [GS98, Theorem 8.1] and [EV08, Theorem 28].

Corollary 4.13. *If* $\operatorname{char}\mathbb{F} \neq 2$*, the associative superalgebra $Q(n)$ does not admit a superinvolution.*

Proof. The center of $Q(n)$ is isomorphic to $\mathbb{F}1 \oplus \mathbb{F}u$, where u is an odd element with $u^2 = 1$. Let φ be a super-anti-automorphism on $Q(n)$. Since u is odd and central, $\varphi(u)$ is odd and central. Hence there is $\lambda \in \mathbb{F}$ such that $\varphi(u) = \lambda u$. Using that $u^2 = 1$, we have $1 = \varphi(1) = \varphi(u^2) = -\varphi(u)^2 = -\lambda^2$. But then $\varphi^2(u) = \lambda^2 u = -u \neq u$, hence $\varphi^2 \neq \mathrm{id}$. $\square$

Remark 4.14. Corollary 4.13 could also be deduced from Lemma 4.20 (assuming $\mathbb{F}$ is algebraically closed), which shows that if $\mathcal{D}$ is a graded-division superalgebra that is simple as a superalgebra and has a nontrivial odd component, then $\mathcal{D}$ does not admit a superinvolution (in the case $\mathcal{D}$ is of type M, this result appeared in [BTT09, Theorem 3], where it is erroneously stated that $\mathcal{D}$ does not admit a super-anti-automorphism).

It follows from Theorem 2.43 and Proposition 4.6 that, over an algebraically closed field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2$, every finite dimensional superinvolution-simple superalgebra is of one of the following 3 types:

Definition 4.15. Let (R, φ) be a finite dimensional superinvolution-simple superalgebra.

(i) If R is of type M, we say that (R, φ) is of type M;

(ii) If $(R, \varphi) \simeq M(m, n) \times M(m, n)^{\text{sop}}$, for some $m, n \geq 0$, we say that (R, φ) is of type $M \times M^{\text{sop}}$;

(iii) If $R \simeq Q(n) \times Q(n)^{\text{sop}}$, for some $n \geq 0$, we say that (R, φ) is of type $Q \times Q^{\text{sop}}$.

We can distinguish these types using the center:

Proposition 4.16. *Let (R, φ) be a superalgebra with superinvolution.*

(i) If (R, φ) is of type M, then $(Z(R), \varphi) \simeq (\mathbb{F}, \mathrm{id})$;

(ii) If (R, φ) is of type $M \times M^{\text{sop}}$, then $(Z(R), \varphi)$ is isomorphic to the superalgebra with superinvolution in Example 4.2;

(iii) If (R, φ) is of type $Q \times Q^{\text{sop}}$, then $(Z(R), \varphi)$ is isomorphic to the superalgebra with superinvolution in Example 4.3.

Proof. Item (i) follows from the fact that $Z(M_{m+n}(\mathbb{F})) \simeq \mathbb{F}$. It is easy to check that $Z(S \times S^{\text{sop}}) = Z(S) \times Z(S)^{\text{sop}}$ for any superalgebra S, so items (ii) and (iii) follow from $Z(M_{m+n}(\mathbb{F})) \simeq \mathbb{F}$ and $Z(Q(n)) \simeq Q(1)$. $\square$

The classification of the superalgebras with superinvolution of types $M \times M^{\text{sop}}$ and $Q \times Q^{\text{sop}}$ is easier. The following result is valid over any field.

Proposition 4.17. *Let $m, m', n, n' \geq 0$. Then*

(i) $M(m, n) \times M(m, n)^{\text{sop}} \simeq M(m', n') \times M(m', n')^{\text{sop}}$ if, and only if, either $m = m'$ and $n = n'$, or $m = n'$ and $n = m'$;

(ii) $Q(n) \times Q(n)^{\mathrm{sop}} \simeq Q(n') \times Q(n')^{\mathrm{sop}}$ *if, and only if,* $n = n'$.

Proof. Let S_1 and S_2 be simple superalgebras. From Lemma 4.5 with trivial G, $S_1 \times S_2^{\mathrm{sop}} \simeq S_1 \times S_2^{\mathrm{sop}}$ if, and only if, $S_1 \simeq S_2$ or $S_1 \simeq S_2^{\mathrm{sop}}$. In our case, we claim that $S_1 \simeq S_2$ or $S_1 \simeq S_2^{\mathrm{sop}}$ if, and only if, $S_1 \simeq S_2$.

Indeed, if S_1 and S_2 are of type M, then $S_1 \simeq S_2^{\mathrm{sop}}$ implies $S_1 \simeq S_2$ since $S_2^{\mathrm{sop}} \simeq S_2$ via supertransposition. If S_1 and S_2 are of type Q, then $S_1 \simeq S_2^{\mathrm{sop}}$ implies $S_1 \simeq S_2$ by dimension count.

The isomorphism condition follows from Theorem 2.43. □

It remains to classify the superinvolution-simple superalgebras of type M. It should be noted that $M(m,n)$ does not admit a superinvolution for all values of m and n. Also, $M(m,n)$ endowed with different superinvolutions may lead to non-isomorphic superalgebras with superinvolution.

Definition 4.18. Let $m, n \in \mathbb{Z}_{\geq 0}$, not both zero, and let $p_0 \in \mathbb{Z}_2$. If $p_0 = \bar{0}$ and n is even, set

$$\Phi := \left(\begin{array}{c|cc} I_m & \multicolumn{2}{c}{0} \\ \hline 0 & 0 & I_{n/2} \\ & -I_{n/2} & 0 \end{array}\right).$$

If $p_0 = \bar{1}$ and $m = n$, let

$$\Phi := \left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right).$$

It is straightforward to see that $\varphi(X) := \Phi^{-1} X^{s\top} \Phi$ defines a superinvolution on $M(m,n)$. We will denote the superalgebra with superinvolution $(M(m,n), \varphi)$ by $M^*(m,n,p_0)$.

We note that the superalgebras with superinvolution $M^*(m,n,p_0)$ are the ones used in the Introduction to define orthosymplectic Lie superalgebras (series B, C and D) and the periplectic superalgebra (series P).

Proposition 4.19. *Every superalgebra with superinvolution of type M is isomorphic to $M^*(m,n,p_0)$ for some $m, n \geq 0$ and $p_0 \in \mathbb{Z}_2$ as in Definition 4.18. Moreover, $M^*(m,n,p_0) \simeq M^*(m',n',p_0')$ if, and only if $m = m'$, $n = n'$ and $p_0 = p_0'$.*

Proof. This is precisely the case of graded-simple superalgebras considered in Section 3.6, but with trivial G (and, hence, with $G^{\#} = \mathbb{Z}_2$). By Subsection 3.6.3, these graded superalgebras are parametrized by $(T, \beta, p, \eta, \kappa, g_0, \delta)$ where (T, β, p) are associated to a graded-division superalgebra (see Subsection 3.6.1), and $(\eta, \kappa, g_0, \delta) \in \mathbf{I}(\mathcal{D})$ (see Definition 3.49).

Superalgebras of type M are of the form $\operatorname{End}_{\mathbb{F}}(U)$ for a finite dimensional superspace U, so we have $\mathcal{D} = \mathbb{F}$ and, hence, T is the trivial group and β, p and η are trivial maps. Since we have only one possible η, we have only one equivalence class of η and, therefore, our parametrization reduces to elements of $\mathbf{I}(\mathbb{F})^+_\eta$ (see Proposition 3.56), i.e., we choose $\delta = 1$ and the only parameters left are $\kappa\colon G^{\#}/T = \mathbb{Z}_2 \to \mathbb{Z}_{\geq 0}$ and $p_0 \coloneqq g_0 \in G^{\#} = \mathbb{Z}_2$.

Then, the isomorphism classes are in bijection with the orbits by $\mathcal{G}$-action (see Equation (3.20)) on $\mathbf{I}(\mathbb{F})^+_\eta$. In the present case, it is clear that $\mathcal{G}$ is trivial, hence the isomorphism classes are in bijection with points in $\mathbf{I}(\mathbb{F})^+_\eta$. Let us describe these points and find a representative for each isomorphism class.

The map $\kappa\colon G^{\#}/T \simeq \mathbb{Z}_2 \to \mathbb{Z}_{\geq 0}$ is determined by the numbers $m \coloneqq \kappa(\bar{0})$ and $n \coloneqq \kappa(\bar{1})$, so our parametrization reduces to triples (m, n, p_0) satisfying some conditions that come from Definition 3.49. The only conditions that are not automatically satisfied are (iii) and (iv). Condition (iii) is tautological if $p_0 = \bar{0}$, and equivalent to $m = n$ if $p_0 = \bar{1}$. Condition (iv) simplifies to the following: if $p_0 = \bar{0}$ and $g = \bar{1}$, then $n = \kappa(g)$ is even. In other words, conditions (iii) and (iv) become equivalent to n being even if $p_0 = \bar{0}$, and $m = n$ if $p_0 = \bar{1}$.

A representative for each point in $\mathbf{I}(\mathbb{F})^+_\eta$ can be found by using the matrices in Propositions 3.45 and 3.47. $\square$

4.3 Gradings on superinvolution-simple superalgebras of type M

For this section we assume $\mathbb{F}$ is algebraically closed and $\operatorname{char} \mathbb{F} \neq 2$. The next result can be found in [TT06, Theorem 4.3] (assuming that G is finite and $\operatorname{char} \mathbb{F} = 0$).

Lemma 4.20. *Let $\mathcal{D}$ be a finite dimensional graded-division superalgebra that is simple as a superalgebra. If $\mathcal{D}$ admits a degree-preserving superinvolution, then $\mathcal{D}$ is even and*

$T \coloneqq \operatorname{supp} \mathcal{D}$ *is an elementary* 2*-group.*

Proof. By Corollary 3.39, we have that $t^2 \in \operatorname{rad} \tilde{\beta} = \{e\}$ for all $t \in T$, so T is an elementary 2-group. Hence, by Corollary 3.42, T^- is empty. $\square$

Let (R, φ) be a finite dimensional graded superalgebra with superinvolution of type M. Since R is simple as a superalgebra, (R, φ) is graded-simple and, hence, by Subsection 3.6.3, $(R, \varphi) \simeq E(\mathcal{D}, \mathcal{U}, B)$ for some triple $(\mathcal{D}, \mathcal{U}, B)$ as in Definition 3.51. Let $(T, \beta, p, \eta, \kappa, g_0, \delta)$ be the parameters of $(\mathcal{D}, \mathcal{U}, B)$ and let φ_0 be the superinvolution on $\mathcal{D}$ determined by η. Since $E(\mathcal{D}, \mathcal{U}, B) = \operatorname{End}_{\mathcal{D}}(\mathcal{U})$ is of type M, by Proposition 2.28, we have that $(\mathcal{D}, \varphi_0)$ is of type M. By Lemma 4.20, $\mathcal{D}$ is even and T is an elementary 2-group. In particular, $\tilde{\beta} = \beta$.

Since $\mathcal{D}$ is even and of type M, we have that $\mathcal{D} \simeq \operatorname{End}_{\mathbb{F}}(N)$ as a superalgebra, where $N = N^{\bar{0}}$ is a finite dimensional superspace. From now on, let us identify $\mathcal{D}$ with $\operatorname{End}_{\mathbb{F}}(N)$. Since N is a left $\mathcal{D}$-supermodule, we can define $U \coloneqq \mathcal{U} \otimes_{\mathcal{D}} N$. Note that U is a superspace, with $U^{\bar{0}} = \mathcal{U}^{\bar{0}} \otimes_{\mathcal{D}} N$ and $U^{\bar{1}} = \mathcal{U}^{\bar{1}} \otimes_{\mathcal{D}} N$, but it is not G-graded since N is not G-graded. Also, U is not a $\mathcal{D}$-supermodule.

Recall that the $\mathbb{F}$-span of any graded $\mathcal{D}$-basis of $\mathcal{U}$ is an $\mathbb{F}$-form of $\mathcal{U}$ (see Definition 2.11 with $\mathcal{D}_e = \mathbb{F}$).

Lemma 4.21. *If $\tilde{\mathcal{U}}$ is an $\mathbb{F}$-form of $\mathcal{U}$, then $\tilde{\mathcal{U}} \otimes_{\mathbb{F}} N$ is canonically isomorphic to $\mathcal{U} \otimes_{\mathcal{D}} N$ by $u \otimes_{\mathbb{F}} v \mapsto u \otimes_{\mathcal{D}} v$, for all $u \in \tilde{\mathcal{U}}$ and $v \in N$.* $\square$

Proof. By Definition 2.11, the map $\tilde{\mathcal{U}} \otimes_{\mathbb{F}} \mathcal{D} \to \mathcal{U}$ given by $u \otimes_{\mathbb{F}} d \mapsto ud$ is an isomorphism of (graded) right $\mathcal{D}$-modules. Hence, we have canonical isomorphisms:

$$\tilde{\mathcal{U}} \otimes_{\mathbb{F}} N \simeq \tilde{\mathcal{U}} \otimes_{\mathbb{F}} (\mathcal{D} \otimes_{\mathcal{D}} N) \simeq (\tilde{\mathcal{U}} \otimes_{\mathbb{F}} \mathcal{D}) \otimes_{\mathcal{D}} N \simeq \mathcal{U} \otimes_{\mathcal{D}} N,$$

whose composition sends $u \otimes_{\mathbb{F}} v$ to $u \otimes_{\mathcal{D}} v$. $\square$

It follows that if $\{u_1, \ldots, u_k\}$ is a graded $\mathcal{D}$-basis of $\mathcal{U}$ and $\{v_1, \ldots, v_\ell\}$ is an $\mathbb{F}$-basis of N, then $\{u_i \otimes_{\mathcal{D}} v_j \mid 1 \leq i \leq k \text{ and } 1 \leq j \leq \ell\}$ is an $\mathbb{F}$-basis of U.

Lemma 4.22. *The map $\psi \colon \operatorname{End}_{\mathcal{D}}(\mathcal{U}) \to \operatorname{End}_{\mathbb{F}}(U)$ given by $\psi(L) \coloneqq L \otimes_{\mathcal{D}} \mathrm{id}_N$ is an isomorphism of superalgebras.*

Proof. It is clear that ψ is a homomorphism and that it preserves the $\mathbb{Z}_2$-degree. To show that ψ is an isomorphism, recall the canonical isomorphism $N \otimes_{\mathbb{F}} N^* \simeq \mathrm{End}_{\mathbb{F}}(N)$, where $N^* := \mathrm{Hom}_{\mathbb{F}}(N, \mathbb{F})$. Since N is a left $\mathcal{D}$-module, N^* is a right $\mathcal{D}$-module. So we obtain canonical isomorphisms:

$$U \otimes_{\mathbb{F}} N^* = (\mathcal{U} \otimes_{\mathcal{D}} N) \otimes_{\mathbb{F}} N^* \simeq \mathcal{U} \otimes_{\mathcal{D}} (N \otimes_{\mathbb{F}} N^*) \simeq \mathcal{U} \otimes_{\mathcal{D}} \mathcal{D} \simeq \mathcal{U}.$$

We claim that $\psi' \colon \mathrm{End}_{\mathbb{F}}(U) \to \mathrm{End}_{\mathcal{D}}(U \otimes_{\mathbb{F}} N^*)$ defined by $\psi'(L) := L \otimes_{\mathbb{F}} \mathrm{id}_{N^*}$ is the inverse of ψ under the canonical isomorphisms above. Indeed,

$$(\psi' \circ \psi)(L) = (L \otimes_{\mathcal{D}} \mathrm{id}_N) \otimes_{\mathbb{F}} \mathrm{id}_{N^*} = L \otimes_{\mathcal{D}} (\mathrm{id}_N \otimes_{\mathbb{F}} \mathrm{id}_{N^*}) = L \otimes_{\mathcal{D}} 1_{\mathcal{D}} = L.$$

The result follows since $\dim_{\mathbb{F}} \mathrm{End}_{\mathcal{D}}(\mathcal{U}) = k^2\ell^2 = \dim_{\mathbb{F}} \mathrm{End}_{\mathbb{F}}(U)$. $\square$

Note that $\psi \circ \varphi \circ \psi^{-1} = \psi \otimes_{\mathcal{D}} \mathrm{id}_N$. We are now going to construct an $\mathbb{F}$-bilinear form on U whose superadjunction is $\psi \otimes_{\mathcal{D}} \mathrm{id}_N$.

It is well-known (and also follows from Theorem 3.18 with trivial grading) that there is a bilinear form $\langle\, , \rangle_N \colon N \times N \to \mathbb{F}$ such that φ_0 is the (super)adjunction with respect to $\langle\, , \rangle_N$. Since φ_0 is a (super)involution, $\langle\, , \rangle_N$ is either symmetric or skew-symmetric. We will write $\delta_N := 1$ in the former case and $\delta_N := -1$ in the latter.

Lemma 4.23. *There is an $\mathbb{F}$-bilinear map $\langle\, , \rangle_U \colon U \times U \to \mathbb{F}$ determined by*

$$\langle u \otimes_{\mathcal{D}} v, u' \otimes_{\mathcal{D}} v' \rangle_U := \langle v, B(u, u')v' \rangle_N,$$

for all $u, u' \in \mathcal{U}$ and $v, v' \in N$. This bilinear map is supersymmetric if $\delta\delta_N = 1$ and super-skew-symmetric if $\delta\delta_N = -1$. Moreover, $\varphi \otimes \mathrm{id}_N$ is the superadjunction with respect to $\langle\, , \rangle_U$.

Proof. For every $u \in \mathcal{U}$, let $C_u \colon \mathcal{U} \times N \to \mathbb{F}$ be defined by $C(u', v') := B(u, u')v'$. Clearly, C_u is $\mathbb{F}$-bilinear. Further, it is $\mathcal{D}$-balanced:

$$C_u(u'd, v') = B(u, u'd)v' = \big(B(u, u')d\big)v' = B(u, u')(dv') = C_u(u', dv').$$

Hence, there is a unique $\mathbb{F}$-linear map $\mathcal{U} \otimes_{\mathcal{D}} N \to N$, which we will also denote by C_u, such that $u' \otimes_{\mathcal{D}} v' \mapsto B(u, u')v'$. It is easy to see that the mapping $\mathcal{U} \to \mathrm{Hom}_{\mathbb{F}}(\mathcal{U} \otimes N, N)$ also is $\mathbb{F}$-linear.

Now $u' \otimes_F v' \in \mathcal{U} \otimes_{\mathcal{D}} N$ fixed, and let $C' \colon \mathcal{U} \times N \to \mathbb{F}$ be given by $C'(u,v) \coloneqq \langle v, C_u(u',v')\rangle_N$, i.e., $C'(u,v) = \langle v, B(u,u')v'\rangle_N$. Clearly, C' is $\mathbb{F}$-bilinear. Further, it is $\mathcal{D}$-balanced:

$$\begin{aligned} C'(ud,v) = \langle v, B(ud,u')v'\rangle_N &= \langle v, (-1)^{(|B|+|u|)|d|}\varphi_0(d)B(u,u')v'\rangle_N \\ &= \langle v, \varphi_0(d)B(u,u')v'\rangle_N = \langle dv, B(u,u')v'\rangle_N, \end{aligned}$$

for all $u \in \mathcal{U}^{\bar{0}} \cup \mathcal{U}^{\bar{1}}$, $v \in N$ and $d \in \mathcal{D} = \mathcal{D}^{\bar{0}}$. It follows that there is a unique $\mathbb{F}$-linear map $\mathcal{U} \otimes \mathcal{V} \to \mathbb{F}$ such that $u \otimes_{\mathcal{D}} v \mapsto C'(u,v)$. We conclude that the $\mathbb{F}$-bilinear map $\langle\,,\rangle_U \colon U \times U \to \mathbb{F}$ is well-defined.

The form $\langle\,,\rangle_U$ is supersymmetric or super-skew-symmetric, depending on $\delta\delta_N$, since

$$\begin{aligned} \langle u' \otimes_{\mathcal{D}} v',\, u \otimes_{\mathcal{D}} v\rangle_U = \langle v', B(u',u)v\rangle_N &= \langle v', (-1)^{|u||u'|}\delta\, \varphi_0(B(u,u'))v\rangle_N \\ &= (-1)^{|u||u'|}\delta \langle B(u,u')v', v\rangle_N = (-1)^{|u||u'|}\delta\delta_N \langle v, B(u,u')v'\rangle_N \\ &= (-1)^{|u||u'|}\delta\delta_N \langle u \otimes_{\mathcal{D}} v, u' \otimes_{\mathcal{D}} v'\rangle_U, \end{aligned}$$

for all $u, u' \in \mathcal{U}^{\bar{0}} \cup \mathcal{U}^{\bar{1}}$ and $v \in N = N^{\bar{0}}$.

For the "moreover" part, recall from Lemma 4.22 that every element in $\mathrm{End}_D(U)$ is of the form $L \otimes_{\mathcal{D}} \mathrm{id}_N$ for some $L \in \mathrm{End}_{\mathcal{D}}(\mathcal{U})$. Also, we have:

$$\begin{aligned} \langle (L \otimes_{\mathcal{D}} \mathrm{id}_N)(u \otimes_D v), u' \otimes_D v'\rangle_U = \langle L(u) \otimes_{\mathcal{D}} v, u' \otimes_D v'\rangle_U &= \langle v, B(L(u), u')v'\rangle_N \\ &= \langle v, (-1)^{|L||u|} B(u, \varphi(L)(u'))v'\rangle_N \\ &= (-1)^{|L||u|}\langle u \otimes_{\mathcal{D}} v, \varphi(L)(u') \otimes_D v'\rangle_U \\ &= (-1)^{|L||u|}\langle u \otimes_D v, (\varphi(L) \otimes_{\mathcal{D}} \mathrm{id}_N)(u' \otimes_D v')\rangle_U, \end{aligned}$$

for all $L \in \mathrm{End}_{\mathcal{D}}(\mathcal{U})^{\bar{0}} \cup \mathrm{End}_{\mathcal{D}}(\mathcal{U})^{\bar{1}}$, $u, u' \in \mathcal{U}^{\bar{0}} \cup \mathcal{U}^{\bar{1}}$ and $v, v' \in N$. □

Proposition 4.24. *As a superalgebra with superivolution, $E(\mathcal{D}, \mathcal{U}, B)$ is isomorphic to $M^*(m,n,p_0)$, where $m \coloneqq \dim \mathcal{U}^{\bar{0}} \otimes_{\mathcal{D}} N$, $n \coloneqq \dim \mathcal{U}^{\bar{1}} \otimes_{\mathcal{D}} N$ and $p_0 \coloneqq |B|$.*

Proof. It is clear that the bilinear form $\langle\,,\rangle_U \colon U \times U \to \mathbb{F}$ defined in Lemma 4.23 has the same parity as B, which is p_0. By Propositions 3.45 and 3.47, we can find a ($\mathbb{Z}_2$-graded) basis of U where the matrix representing $\varphi \otimes_{\mathcal{D}} \mathrm{id}_N$ is the one as in Definition 4.18, so $(\mathrm{End}_{\mathbb{F}}(U), \varphi \otimes_{\mathcal{D}} \mathrm{id}_N) \simeq M^*(m,n,p_0)$. By Lemma 4.22, we have the desired result. □

To state the classification theorem of gradings on $M^*(m,n,p_0)$, we will revisit Definition 2.36. Let T be a finite abelian group, let β be a nondegenerate alternating bicharacter on T. Decompose T as $A \times B$ where $\beta(A,A) = \beta(B,B) = 1$, and let $\mathcal{D} \coloneqq \mathrm{End}_{\mathbb{F}}(N)$ be the corresponding standard realization of a graded-division algebra, where N is a vector space with a fixed basis $\{e_b\}_{b\in B}$.

Lemma 4.25. *If we identify $\mathcal{D}$ with a matrix algebra using the basis $\{e_b\}_{b\in B}$, we have that $X_{ab}^\top = \beta(a,b)X_{ab^{-1}}$, for all $a \in A$ and $b \in B$.*

Proof. Let $\langle\,,\rangle_U \colon U \times U \to \mathbb{F}$ be the bilinear form determined by

$$\langle e_b, e_{b'}\rangle_N \coloneqq \begin{cases} 1, & \text{if } b = b' \\ 0, & \text{if } b \neq b' \end{cases}.$$

Clearly, the adjunction with respect to $\langle\,,\rangle_U$ correspond to the transposition.

Let $a \in A$. Then

$$\langle X_a e_{b'}, e_{b''}\rangle_N = \beta(a,b')\langle X_a e_{b'}, e_{b''}\rangle_N = \beta(a,b'')\langle X_a e_{b'}, e_{b''}\rangle_N = \langle e_{b'}, X_a e_{b''}\rangle_N,$$

for all $b', b'' \in B$, so $X_a^\top = X_a$. Let $b \in B$. Then

$$\langle X_b e_{b'}, e_{b''}\rangle_N = \langle e_{bb'}, e_{b''}\rangle_N = \langle e_{b'}, e_{b^{-1}b''}\rangle_N = \langle e_{b'}, X_{b^{-1}} e_{b''}\rangle_N,$$

for all $b', b'' \in B$, so $X_b^\top = X_{b^{-1}}$. Finally,

$$X_{ab}^\top = (X_a X_b)^\top = X_{b^{-1}} X_a = \beta(b^{-1},a) X_a X_{b^{-1}} = \beta(b,a)^{-1} X_{ab^{-1}} = \beta(a,b) X_{ab^{-1}}.$$

□

Proposition 4.26. *Suppose T is a elementary 2-group. Then there is a unique equivalence class (see Definition 3.55) of maps of the form $\eta \colon T \to \mathbb{F}^\times$ such that $\mathrm{d}\eta = \tilde{\beta} = \beta$.*

Proof. Let $\mathcal{D}$ be the standard realization of (T,β) as before. Since T is an elementary 2-group, by Lemma 4.25, the transposition gives us a degree-preserving involution on $\mathcal{D}$, corresponding to the map $\eta \colon T \to \mathbb{F}^\times$ given by $\eta(ab) = \beta(a,b)$.

Let $\eta'\colon T\to\mathbb{F}^\times$ be a map corresponding to another degree-preserving involution φ_0 on $\mathcal{D}$. Clearly, φ_0 is the transposition composed with a degree-preserving automorphism of $\mathcal{D}$, so, by Lemma 2.38, $\eta'=\chi\eta$, for some $\chi\in\widehat{T}$. Since $\tilde{\beta}=\beta$ is nondegenerate, there is a $t\in T$ such that $\chi=\tilde{\beta}(t,\cdot)$, and hence $\eta'\sim\eta$. $\square$

We are now in a position to define explicit models for gradings on $M^*(m,n,p_0)$. The choice of φ_0 to be the transposition on $\mathcal{D}$ (when $\mathcal{D}$ is realized as a matrix algebra) is convenient if we view a matrix $X\in M_{k_{\bar 0}|k_{\bar 1}}(\mathcal{D})$ as a (block) matrix with entries in $\mathbb{F}$ by interpreting the entries of X as matrices with entries in $\mathbb{F}$, or, in other words, if we identify $M(k_{\bar 0},k_{\bar 1})\otimes\mathcal{D}$ with $M(m,n)$ via Kronecker product. This corresponds to the isomorphism $\mathrm{End}_{\mathcal{D}}(\mathcal{U})\to\mathrm{End}_{\mathbb{F}}(U)$ of Lemma 4.22 if we use a graded $\mathcal{D}$-basis $\{u_1,\dots,u_k\}$ for $\mathcal{U}$ and the basis $u_i\otimes_{\mathcal{D}}e_b$ for U, ordered lexicographically. Then $\varphi_0(X)^{s\top}$ in Equation (3.13) becomes $X^{s\top}$.

We reformulate the conditions on κ of Definition 3.49 in terms of $\kappa_{\bar 0}$ and $\kappa_{\bar 1}$:

Definition 4.27. Let $T\subseteq G$ be a finite subgroup, $\beta\colon T\times T\to\mathbb{F}^\times$ be an alternating bicharacter, $\eta\colon T\to\{\pm1\}$ be a map such that $\mathrm{d}\eta=\beta$, $g_0=(h_0,p_0)\in G^\#$ and $\kappa_{\bar 0},\kappa_{\bar 1}\colon G/T\to\mathbb{Z}_{\ge 0}$ be maps with finite support. If $p_0=\bar 0$, we say that $(\kappa_{\bar 0},\kappa_{\bar 1})$ is g_0-*admissible* if, for all $i\in\mathbb{Z}_2$ and $x\in G/T$,

(i) $\kappa_i(x)=\kappa_i(h_0^{-1}x^{-1})$;

(ii) if $h_0x^2=T$ and, for some (and, hence, any) $g\in x$, $\eta(h_0g^2)=-(-1)^i$, then $\kappa_i(x)$ is even.

If $p_0=\bar 1$, we say that $(\kappa_{\bar 0},\kappa_{\bar 1})$ is g_0-*admissible* if, for all $x\in G/T$,

(iii) $\kappa_{\bar 1}(x)=\kappa_{\bar 0}(h_0^{-1}x^{-1})$.

Under the conditions of Definition 4.27, let $p\colon T\to\mathbb{Z}_2$ be the trivial homomorphism and define $\kappa\colon G^\#/T\to\mathbb{Z}_{\ge 0}$ by $\kappa((g,i)T)=\kappa_i(gT)$, for all $i\in\mathbb{Z}_2$ and $g\in G$. It is easy to see that $(\kappa_{\bar 0},\kappa_{\bar 1})$ is g_0-admissible if, and only if, $(\kappa,g_0)\in\mathbf{I}(T,\beta,p)^+_\eta$, i.e., $(\eta,\kappa,g_0,1)\in\mathbf{I}(T,\beta,p)$ (see Definition 3.49 and Equation (3.19)). In other words, $(\kappa_{\bar 0},\kappa_{\bar 1})$ is g_0-admissible if, and only if, there is a pair $(\mathcal{U},B)$ whose inertia is $(\eta,\kappa,g_0,1)$ (see Definition 3.48). In this case, we say that $(\mathcal{U},B)$ has *inertia determined by* $(\kappa_{\bar 0},\kappa_{\bar 1})$.

In what follows, for each pair (T,β) where T is a finite abelian group and $\beta\colon T\times T\to\mathbb{F}^\times$ is a nondegenerate alternating bicharacter, we fix $\mathcal{D}$ to be a standard realization associated to (T,β), and define $\varphi_0\colon\mathcal{D}\to\mathcal{D}$ to be the transposition on $\mathcal{D}$ and $\eta\colon T\to\mathbb{F}^\times$ to be the corresponding map, which satisfies $\mathrm{d}\eta=\beta\,(=\tilde{\beta})$.

Definition 4.28. Let $T\subseteq G$ be a finite 2-elementary subgroup, let $\beta\colon T\times T\to\mathbb{F}^\times$ be a nondegenerate alternating bicharacter, let $g_0=(h_0,p_0)\in G^\#$ and let $\kappa_{\bar{0}},\kappa_{\bar{1}}\colon G/T\to\mathbb{Z}_{\geq 0}$ be g_0-admissible maps. Choose a graded $\mathcal{D}$-supermodule $\mathcal{U}$ and a nondegenerate sesquilinear form $B\colon\mathcal{U}\times\mathcal{U}\to\mathcal{D}$ such that $(\mathcal{U},B)$ has inertia determined by $(\kappa_{\bar{0}},\kappa_{\bar{1}})$. We define $M^*(T,\beta,\kappa_{\bar{0}},\kappa_{\bar{1}},g_0)$ to be $E(\mathcal{D},\mathcal{U},\mathcal{B})$. With a choice of a graded basis for $\mathcal{U}$, this becomes the graded superalgebra $M(T,\beta,\kappa_{\bar{0}},\kappa_{\bar{1}})$, i.e., the superalgebra $M(m,n)$ endowed with the grading $\Gamma_M(T,\beta,\kappa_{\bar{0}},\kappa_{\bar{1}})$, where $m:=|\kappa_{\bar{0}}|\sqrt{|T|}$ and $n:=|\kappa_{\bar{1}}|\sqrt{|T|}$ (see Definition 2.75), and the superinvolution φ is given by

$$\varphi(X):=\Phi^{-1}X^{s\top}\Phi,$$

for all $X\in M(m,n)$, where Φ is the matrix representing B with respect to the chosen basis (viewed as a matrix with entries in $\mathbb{F}$ rather than $\mathcal{D}$).

Theorem 4.29. *Suppose the superalgebra with superinvolution $M^*(m,n,p_0)$ is endowed with a G-grading. Then it is isomorphic, as a graded superalgebra with superinvolution, to $M^*(T,\beta,\kappa_{\bar{0}},\kappa_{\bar{1}},g_0)$ as in Definition 4.28, where $p_0=|g_0|$. Moreover, $M^*(T,\beta,\kappa_{\bar{0}},\kappa_{\bar{1}},g_0)\simeq M^*(T',\beta',\kappa'_{\bar{0}},\kappa'_{\bar{1}},g'_0)$ if, and only if, $T=T'$, $\beta=\beta'$ and there is $g\in G$ such that $\kappa'_{\bar{0}}=g\cdot\kappa_{\bar{0}}$, $\kappa'_{\bar{1}}=g\cdot\kappa_{\bar{1}}$ and $g'_0=g^{-2}g_0$.*

Proof. The first assertion follows from the discussion above. The isomorphism condition follows from Proposition 3.56 by noting that, in this case, $\mathcal{G}=\{e\}\times G$ (see Equation (3.20)). ◻

4.4 Graded-division superalgebras of types $M\times M^{\mathrm{sop}}$ and $Q\times Q^{\mathrm{sop}}$

In the next section, we will classify all group gradings on the superinvolution-simple superalgebras of types $M\times M^{\mathrm{sop}}$ and $Q\times Q^{\mathrm{sop}}$. To this end, we will first construct division gradings on those.

We will assume $\mathbb{F}$ is algebraically closed and $\operatorname{char}\mathbb{F} \neq 2$. Fix $\mathbf{i} \in \mathbb{F}$ a primitive fourth-root of unity, i.e., $\mathbf{i}^2 = -1$.

Next subsection presents some well-known concepts and results from the theory of group extensions (see, for example, [Mac95]). We put it here for completeness and for fixing notation.

4.4.1 Abelian group extensions

Definition 4.30. Let H, K be groups. A *group extension of H by K* is a group E together with homomorphisms $\iota\colon K \to E$ and $\pi\colon E \to H$ such that

$$0 \to K \xrightarrow{\iota} E \xrightarrow{\pi} H \to 0$$

is a short exact sequence, i.e., ι is injective, π is surjective and $\operatorname{im}\iota = \ker\pi$ (and, hence, $H \simeq \frac{E}{\iota(K)}$). An *equivalence* between two extensions $K \xrightarrow{\iota} E \xrightarrow{\pi} H$ and $K \xrightarrow{\iota'} E' \xrightarrow{\pi'} H$ is a homomorphism $\alpha\colon E \to E'$ such that the diagram

$$\begin{tikzcd} & E \arrow[dd, "\alpha"] \arrow[dr, "\pi"] & \\ K \arrow[ur, "\iota"] \arrow[dr, "\iota'"'] & & H \\ & E' \arrow[ur, "\pi'"'] & \end{tikzcd}$$

commutes. It is easy to see that α is necessarily an isomorphism. An equivalence from an extension to itself will be referred as a *self-equivalence*.

Example 4.31. Let $T = T^+ \cup T^-$ be the support of a finite graded-division superalgebra. Consider

$$E := \Big(\{\pm 1\} \times T^+\Big) \cup \Big(\{\pm\mathbf{i}\} \times T^-\Big) \subseteq \{\pm 1, \pm\mathbf{i}\} \times T,$$

and let $\iota\colon \{\pm 1\} \to E$ and $\pi\colon E \to T$ be the homomorphisms given by $\iota(\delta) := (\delta, e)$, for all $\delta \in \{\pm 1\}$, and $\pi(x,t) := t$, for all $x \in \{\pm 1, \pm\mathbf{i}\}$ and $t \in T$. Then $\{\pm 1\} \xrightarrow{\iota} E \xrightarrow{\pi} T$ is an extension of T by $\{\pm 1\}$.

Let us fix the abelian groups H and K for the remainder of the section.

Definition 4.32. A map $\sigma\colon H \times H \to K$ is said to be a 2-*cocycle* if, for all $a, b, c \in H$, we have $\sigma(a,b)\sigma(ab,c) = \sigma(a,bc)\sigma(b,c)$. If, further, we have that $\sigma(a,b) = \sigma(b,a)$, we say that σ is *symmetric*, and if we have $\sigma(a,e) = \sigma(e,a) = e$, we say that σ is *normalized*. In the case $\sigma(h_1,h_2) = e$ for all $h_1, h_2 \in H$, we say that σ is *trivial*.

Example 4.33. Let $f\colon H \to K$ be any map. Recall (Definition 3.40) that its 2-coboundary is the map $\mathrm{d}f\colon H \times H \to K$ defined by $\mathrm{d}f\,(a,b) = f(ab)f(a)^{-1}f(b)^{-1}$ for all $a, b \in H$. It is easy to see that $\mathrm{d}f$ is a 2-cocycle, which is symmetric since both H and K are abelian. If $f(e) = e$, $\mathrm{d}f$ is also normalized.

Example 4.34. Let T be an abelian group and let $b\colon T \times T \to \mathbb{F}^\times$ be a symmetric bicharacter. It is also easy to see that b is a normalized symmetric 2-cocycle.

Example 4.35. Let $\mathcal{D}$ be a finite dimensional graded-division superalgebra associated to (T, β, p). If β and $\tilde\beta$ only take values in $\{\pm 1\}$, then both are symmetric and, hence, normalized symmetric 2-cocycles. This is not only a particular case of Example 4.34, but also of Example 4.33 (by Proposition 3.38).

Given a normalized symmetric 2-cocycle σ, we define the abelian group $K \times_\sigma H$ to be the set $K \times H$ with product $*_\sigma$ given by:

$$(k_1, h_1) *_\sigma (k_2, h_2) = (\sigma(h_1,h_2)k_1k_2, h_1h_2),$$

for all $k_1, k_2 \in K$ and $h_1, h_2 \in H$. The condition of σ being a symmetric 2-cocycle is precisely what is needed for the associativity and commutativity of $*_\sigma$. The identity element is (e,e) and the inverse is given by $(k,h)^{-1} = (k^{-1}\sigma(h,h^{-1})^{-1}, h^{-1})$. Note that the usual product on $K \times H$ is recovered as the particular case when σ is trivial.

Definition 4.36. The *(abelian) group extension corresponding to* $K \times_\sigma H$ is the group $K \times_\sigma H$ together the group homomorphisms $\iota\colon K \to K \times_\sigma H$ and $\pi\colon K \times_\sigma H \to H$ defined by $\iota(k) = (k,e)$ and $\pi(k,h) = h$, for all $k \in K$ and $h \in H$. If σ is trivial, we say that the extension is *trivial*.

Example 4.37. Under the conditions of Example 4.35, with $\tilde\beta$ only taking values ± 1, we have a group extension $\{\pm 1\} \xrightarrow{\iota} \{\pm 1\} \times_{\tilde\beta} T \xrightarrow{\pi} T$.

The next result tells us that, essentially, these are all abelian group extensions:

Proposition 4.38. *Every abelian group extension $K \xrightarrow{\iota} E \xrightarrow{\pi} H$ is equivalent to the abelian group extension corresponding to $K \times_\sigma H$ for some normalized symmetric 2-cocycle $\sigma \colon H \to K$.*

Proof. Fix $\tau \colon H \to E$ a set- theoretic section of τ such that $\tau(e) = e$. For every $x \in E$, we have that $x(\tau(\pi(x)))^{-1}$ is in $\ker \pi$ and, hence, it is equal to $\iota(k)$ for a unique $k_x \in K$. It follows that $x = \iota(k_x)\tau(\pi(x))$. Define $\alpha \colon E \to K \times H$ by $\alpha(x) = (k_x, \tau(\pi(x)))$. Clearly, α is bijective, so we can make it an isomorphism by using it to define a product on $K \times H$. It is easy to check that this product is the one on $K \times_\sigma H$ where $\sigma \colon H \times H \to K$ is given by $\sigma(h_1, h_2) = \tau(h_1)\tau(h_2)\tau(h_1h_2)^{-1}$ and α is the desired equivalence. $\square$

We will now see when two extensions determined by different normalized symmetric 2-cocycles are equivalent.

Proposition 4.39. *Let $\sigma, \sigma' \colon H \times H \to K$ be normalized symmetric 2-cocycles. A map $\alpha \colon K \times_\sigma H \to K \times_{\sigma'} H$ is an equivalence between the corresponding group extensions if, and only if, there is a map $f \colon H \to K$ such that $\alpha(k, h) = (f(h)k, h)$, for all $k \in K$ and $h \in H$, and $\sigma'\sigma^{-1} = \mathrm{d}f$.*

Proof. The result follows from the following claims:

Claim 1. Suppose α is an isomorphism. Then α is an equivalence if, and only if, there is a map $f \colon H \to K$ such that $\alpha(k, h) = (f(h)k, h)$, for all $k \in K$ and $h \in H$.

Suppose $\alpha(k, h) = (f(h)k, h)$. First, it is clear that $\pi\alpha = \pi$. Also, since $\alpha(e, e) = (e, e)$, it follows that $f(e) = e$ and, hence, $\alpha\iota = \iota$.

Now suppose α is an equivalence. Since for all $h \in H$ $(\pi\alpha)(e, h) = \pi(e, h) = h$, we have that $\alpha(e, h) = (f(h), h)$ for some map $f \colon H \to K$. Also, since for all $k \in K$ $(\alpha\iota)(k) = \iota(k)$, we have that $\alpha(k, e) = (k, e)$. It follows that $\alpha(k, h) = \alpha((k, e) *_\sigma (e, h)) = \alpha(k, e) *_{\sigma'} (e, h) = (f(h)k, h)$.

Claim 2. Suppose there is a map $f \colon H \to K$ such that $\alpha(k, h) = (f(h)k, h)$, for all $k \in K$ and $h \in H$. Then α is an isomorphism if, and only if, $\sigma'\sigma^{-1} = \mathrm{d}f$.

First, it is easy to check that the map $K \times_{\sigma'} H \to K \times_\sigma H$ given by $(k, h) \mapsto (f(h)^{-1}k, h)$ is the inverse of α, so α is bijective.

If $h_1, h_2 \in H$ and $t_1, t_2 \in T$, then, on the one hand,

$$\alpha((h_1, t_1) *_\sigma (h_2, t_2)) = \alpha(\sigma(t_1, t_2)h_1h_2, t_1t_2) = (f(t_1t_2)\sigma(t_1, t_2)h_1h_2, t_1t_2)$$

and, on the other hand,

$$\begin{aligned}\alpha((h_1, t_1)) *_{\sigma'} \alpha((h_2, t_2)) &= (f(t_1)h_1, t_1) *_{\sigma'} (f(t_2)h_2, t_2)\\ &= (\sigma'(t_1, t_2)f(t_1)f(t_2)h_1h_2, t_1t_2).\end{aligned}$$

Comparing both, we conclude that α is a homomorphism if, and only if, $\mathrm{d}f = \sigma'\sigma^{-1}$. □

Corollary 4.40. *The set of self-equivalences of an abelian group extension $K \xrightarrow{\iota} E \xrightarrow{\pi} H$ is in bijection with the set of homomorphisms from H to K.*

Proof. By Proposition 4.38, we can assume our extension is the one corresponding to $K \times_\sigma H$, for some normalized symmetric 2-cocycle σ. The result follows from Proposition 4.39 and the observation that $\mathrm{d}f = 0$ if, and only if, f is a homomorphism. □

4.4.2 Quadratic maps and (super-)anti-automorphisms

Let $\mathcal{D}$ be a finite dimensional graded-division superalgebra associated to (T, β, p) and choose elements $0 \neq X_t \in \mathcal{D}_t$ for all $t \in T$. Recall from Proposition 3.38 that super-anti-automorphisms of $\mathcal{D}$ are in bijection with maps $\eta\colon T \to \mathbb{F}^\times$ such that $\mathrm{d}\eta = \tilde{\beta}$. The same computation with p trivial shows that anti-automorphisms of $\mathcal{D}$ are in bijection with maps $\eta\colon T \to \mathbb{F}^\times$ such that $\mathrm{d}\eta = \beta$ (see also [EK13]).

Even though we are mainly interested in superinvolutions, some super-anti-automorphisms which are not involutive (and even some anti-automorphisms) will be useful in what follows and also in Chapter 5. We will need the ones that correspond to quadratic maps.

Definition 4.41. Let $\mu\colon T \to \mathbb{F}^\times$ be a map. We say that μ is a *(multiplicative) quadratic map* if $\mu(t^{-1}) = \mu(t)$ for all $t \in T$ and the map $\mathrm{d}\mu\colon T \times T \to \mathbb{F}^\times$ is a bicharacter, which is referred to as the *polarization* of μ.

Remark 4.42. In the literature, the additive notation is commonly used in the context of quadratic maps. In general, for a given commutative ring $\mathbb{K}$ and $\mathbb{K}$-modules A and

B, a quadratic map $q\colon A \to B$ is a map satisfying that $q(kx) = k^2x$, for all $k \in \mathbb{K}$ and $x \in A$, and that the map $A \times A \to B$ defined by $(x, y) \mapsto q(x + y) - q(x) - q(y)$ is bilinear. In the case of abelian groups (i.e., $\mathbb{K} = \mathbb{Z}$), the first condition is equivalent to $q(-x) = q(x)$, justifying our nomenclature.

In next example, a quadratic map is used to construct an anti-isomorphism between different graded-division superalgebras:

Example 4.43. Let $\mu_0\colon T \to \mathbb{F}^\times$ be given by $\mu_0(t) = 1$ for all $t \in T^+$ and $\mu_0(t) = \mathbf{i}$ for all $t \in T^-$. It is clear that $\mu_0(t^{-1}) = \mu_0(t)$. Also, we have that $\mathrm{d}\mu_0\,(a, b) = \mu_0(ab)\mu_0(a)^{-1}\mu_0(b)^{-1} = (-1)^{p(a)p(b)}$, which is a symmetric bicharacter. This quadratic map determines an anti-isomorphism $\mathcal{D} \to \mathcal{D}^{\mathrm{sop}}$, $X_t \mapsto \mu_0(t)\overline{X_t}$. Note that, in particular, $\mathcal{D}^{\mathrm{sop}}$ is isomorphic to $\mathcal{D}^{\mathrm{op}}$.

Lemma 4.44. *Let $\mu\colon T \to \mathbb{F}^\times$ be a map and let $\psi\colon \mathcal{D} \to \mathcal{D}$ be the linear map defined by $\psi(X_t) = \mu(t)X_t$ for every $t \in T$ and $X_t \in \mathcal{D}_t$. Then:*

(i) If $\mathrm{d}\mu = \tilde{\beta}$, then:
μ is a quadratic map $\iff$ $\mu(t) \in \{\pm 1\}$ for all $t \in T^+$ and $\mu(t) \in \{\pm \mathbf{i}\}$ for all $t \in T^-$ $\iff$ ψ is a super-anti-automorphism such that ψ^2 is the parity automorphism on $\mathcal{D}$;

(ii) If $\mathrm{d}\mu = \beta$, then:
μ is a quadratic map $\iff$ $\mu(t) \in \{\pm 1\}$ for all $t \in T$ $\iff$ ψ is an involution on $\mathcal{D}$.

Proof. From Equation (3.18), we have $\tilde{\beta}(t, t^{-1}) = \mu(tt^{-1})\mu(t)^{-1}\mu(t^{-1})^{-1}$, which simplifies to $(-1)^{p(t)} = \mu(t)\mu(t^{-1})$. Therefore $\mu(t^{-1}) = \mu(t)$ if, and only if, $\mu(t)^2 = (-1)^{p(t)}$, proving (i).

Item (ii) follows from (i) once we consider $\mathcal{D}$ as an algebra, i.e., if we consider p to be the trivial homomorphism. □

We will now refine the statement about existence of η in Proposition 3.38:

Proposition 4.45. *If β takes values in $\{\pm 1\}$, then there are quadratic maps on T whose polarizations are β and $\tilde{\beta}$.*

Proof. To show there is a quadratic map whose polarization is β, consider the group $\overline{T} := \frac{T}{\operatorname{rad}\beta}$ and let $t \mapsto \bar{t}$ be the natural projection on the quotient. Also, we denote by $\bar{\beta}\colon \overline{T} \times \overline{T} \to \mathbb{F}^\times$ the bicharacter induced by β.

Since $\bar{\beta}$ is nondegenerate and takes values ± 1, $\overline{T}$ is an elementary 2-group. Choose a standard realization $\overline{\mathcal{D}}$ associated to $(\overline{T}, \bar{\beta})$. By Lemma 4.25, the transposition is a degree-preserving involution of $\overline{\mathcal{D}}$, so it is determined by a map $\bar{\mu}\colon T \to \mathbb{F}^\times$. By Lemma 4.44, $\bar{\mu}$ is a quadratic map taking values in $\{\pm 1\}$ whose polarization is $\bar{\beta}$. Define $\mu\colon T \to \mathbb{F}^\times$ by $\mu(t) = \bar{\mu}(\bar{t})$ for all $t \in T$. Then μ takes values in $\{\pm 1\}$ and $\beta = \mathrm{d}\mu$.

To get a quadratic map with polarization $\tilde{\beta}$, simply multiply μ by the quadratic map μ_0 of Example 4.43. □

4.4.3 Division gradings on $\mathcal{D} \times \mathcal{D}^{\mathrm{sop}}$

We are now in position to apply these concepts and results to our original problem.

Theorem 4.46. *Let $\mathcal{D}$ be a graded division superalgebra associated to (T, β, p). Consider $\mathcal{E} := \mathcal{D} \times \mathcal{D}^{\mathrm{sop}}$ with its natural T-grading $\Gamma\colon \mathcal{E} = \bigoplus_{t\in T} \mathcal{D}_t \times \overline{\mathcal{D}_t}$ and let φ be the exchange superinvolution on $\mathcal{E}$. There is a division grading on $(\mathcal{E}, \varphi)$ refining Γ if, and only if, β takes values in $\{\pm 1\}$. If this is the case, then:*

(i) For each such refinement Δ, associated to $(T_\Delta, \beta_\Delta, p_\Delta, \eta_\Delta)$, T_Δ fits into a unique group extension $\{\pm 1\} \xrightarrow{\iota} T_\Delta \xrightarrow{\pi} T$ such that ${}^\pi\Delta = \Gamma$. Moreover, $\beta_\Delta = \beta \circ (\pi \times \pi)$, $p_\Delta = p \circ \pi$ and $\eta_\Delta \circ \iota = \mathrm{id}_{\{\pm 1\}}$.

(ii) If $\tilde{\Delta}$ is another refinement, associated to $(T_{\tilde{\Delta}}, \beta_{\tilde{\Delta}}, p_{\tilde{\Delta}}, \eta_{\tilde{\Delta}})$, there is a unique equivalence between the corresponding group extensions $\alpha\colon T_\Delta \to T_{\tilde{\Delta}}$ such that ${}^\alpha\Delta = \tilde{\Delta}$. Moreover, $\beta_\Delta = \beta_{\tilde{\Delta}} \circ (\alpha \times \alpha)$, $p_\Delta = p_{\tilde{\Delta}} \circ \alpha$ and $\eta_\Delta = \eta_{\tilde{\Delta}} \circ \alpha$.

(iii) There is a refinement Δ_0 corresponding to the extension $\{\pm 1\} \xrightarrow{\iota} \{\pm 1\} \times_{\tilde{\beta}} T \xrightarrow{\pi} T$ (Example 4.37), for which the map $\eta_{\Delta_0}\colon \{\pm 1\} \times_{\tilde{\beta}} T \to \{\pm 1\}$ determining the exchange superinvolution is the projection onto the first component.

Proof. We will first prove the main assertion of (i). Recall that, for a division grading, its support is its universal group (Lemma 2.7). Since Γ is a coarsening of Δ, by the universal property of the universal group, there is a unique homomorphism $\pi\colon T_\Delta \to T$

such that ${}^{\pi}\Delta = \Gamma$. Also, since $|T_\Delta| = 2|T|$, we must have $|\ker \pi| = 2$, and therefore there is a unique monomorphism $\iota\colon \{\pm 1\} \to T_\Delta$ with $\iota(\{\pm 1\}) = \ker \pi$.

We claim that all division gradings on $(\mathcal{E}, \varphi)$ refining Γ must have the same set of subspaces of $\mathcal{E}$ as their homogeneous components. Indeed, for each $t \in T$, let $\mathcal{E}_t$ denote the homogeneous component of degree t of Γ. Clearly, $\mathcal{E}_t$ is 2-dimensional and $(X_t, \overline{X_t})$ and $(X_t, -\overline{X_t})$ are eigenvectors of φ, associated to the eigenvalues 1 and -1, respectively. Hence, the eigenspaces of $\varphi\restriction_{\mathcal{E}_t}$ are 1-dimensional. By Lemma 1.2, it follows that $(X_t, \overline{X_t})$ and $(X_t, -\overline{X_t})$ are homogeneous in any grading Δ on $(\mathcal{E}, \varphi)$ refining Γ. If Δ is a division grading, all the components are 1-dimensional, so they must be the subspaces $\mathcal{E}_{(\delta,t)} := \mathbb{F}(X_t, \delta\overline{X_t})$, for $t \in T$ and $\delta \in \{\pm 1\}$.

Now, for all $a, b \in T$ and $\delta_1, \delta_2 \in \{\pm 1\}$, we have:

$$\begin{aligned}
(X_a, \delta_1\overline{X_a})(X_b, \delta_2\overline{X_b}) &= (X_a X_b, \delta_1\delta_2\overline{X_a}\,\overline{X_b}) \\
&= (X_a X_b, \delta_1\delta_2(-1)^{p(a)p(b)}\overline{X_b X_a}) \\
&= (X_a X_b, \delta_1\delta_2(-1)^{p(a)p(b)}\beta(a,b)\overline{X_a X_b}) \\
&= (X_a X_b, \delta_1\delta_2\tilde{\beta}(a,b)\overline{X_a X_b}).
\end{aligned}$$

On the one hand, if $\beta(a,b) \neq \pm 1$ for some $a, b \in T$, the direct sum decomposition $\mathcal{E} = \bigoplus \mathcal{E}_{(\delta,t)}$ not a grading of $\mathcal{E}$ as a superalgebra. On the other hand, if β takes values in $\{\pm 1\}$, it follows that $\mathcal{E}_{(\delta_1,a)}\mathcal{E}_{(\delta_2,b)} = \mathcal{E}_{(\delta_1\delta_2\tilde{\beta}(a,b),ab)}$. Therefore we have a grading Δ_0 with support $\{\pm 1\} \times_{\tilde{\beta}} T$. Clearly, the exchange superinvolution is determined by the projection on the first entry $\{\pm 1\} \times_{\tilde{\beta}} T \to \{\pm 1\}$, $\pi\colon \{\pm 1\} \times_{\tilde{\beta}} T \to T$ is the projection on the second entry and that $\ker \pi = \{\pm 1\} \times_{\tilde{\beta}} \{e\}$, proving item (iii).

We will use the universal property of the universal group again to prove item (ii). Let $\{\pm 1\} \xrightarrow{\iota} T_\Delta \xrightarrow{\pi} T$ and $\{\pm 1\} \xrightarrow{\tilde{\iota}} T_{\tilde{\Delta}} \xrightarrow{\tilde{\pi}} T$ be the group extensions corresponding to Δ and $\tilde{\Delta}$, respectively. Since they have the same components, Δ is an (improper) refinement of $\tilde{\Delta}$, so there is a unique group homomorphism $\alpha\colon T_\Delta \to T_{\tilde{\Delta}}$ such that ${}^{\alpha}\Delta = \tilde{\Delta}$. By the uniqueness of π, we have $\pi = \tilde{\pi}\alpha$ and, by the uniqueness of $\tilde{\iota}$, we have $\tilde{\iota} = \alpha\iota$, hence α is an equivalence between the corresponding extensions. Since the component of Δ of degree t is the same subspace as the component of $\tilde{\Delta}$ of degree $\alpha(t)$, we get $\beta_\Delta = \beta_{\tilde{\Delta}} \circ (\alpha \times \alpha)$, $p_\Delta = p_{\tilde{\Delta}} \circ \alpha$ and $\eta_\Delta = \eta_{\tilde{\Delta}} \circ \alpha$.

Finally, the "moreover" part of item (i) is trivial for Δ_0, and it follows in general by item (ii). $\square$

Example 4.47. The (trivially) graded superalgebra with superinvolution $\mathbb{F} \times \mathbb{F}^{\mathrm{sop}} = M(1,0) \times M(1,0)^{\mathrm{sop}}$ of Example 4.2 admits a $\mathbb{Z}_2$-grading (refining the trivial grading) that makes it isomorphic to $\mathbb{F}\mathbb{Z}_2$: $\deg(1,1) = \bar{0}$ and $\deg(1,-1) = \bar{1}$. The exchange superinvolution is determined by $\eta(\bar{0}) = 1$ and $\eta(\bar{1}) = -1$.

Example 4.48. The graded superalgebra with superinvolution $Q(1) \times Q(1)^{\mathrm{sop}}$ of Example 4.3 admits a $\mathbb{Z}_4$-grading (refining the trivial grading) that makes it isomorphic to $\mathbb{F}\mathbb{Z}_4$: $\deg(1,1) = \bar{0}$, $\deg(u,\bar{u}) = \bar{1}$, $\deg(1,-1) = \bar{2}$ and $\deg(u,-\bar{u}) = \bar{3}$. The exchange superinvolution is determined by $\eta(\bar{0}) = 1$, $\eta(\bar{1}) = 1$, $\eta(\bar{2}) = -1$ and $\eta(\bar{3}) = -1$. By Theorem 4.46, up to relabeling the components, those are the only possible gradings on these superalgebras with superinvolutions.

It is straightforward that, if Δ is as in item (i), $\{\pm 1\} \to E \to T$ is another group extension and $\alpha\colon T_\Delta \to E$ is an equivalence of extensions, then ${}^\alpha\Delta$ is a refinement corresponding to $\{\pm 1\} \to E \to T$. Hence, we get:

Corollary 4.49. *Under the conditions of Theorem 4.46, the set of all refinements Δ such that* $\operatorname{supp}\Delta = T_\Delta$ *and* ${}^\pi\Delta = \Gamma$ *is in bijection with the group homomorphisms from T to $\{\pm 1\}$.*

Proof. It follows from item (ii) and Corollary 4.40. □

Note that all these refinements are nonisomorphic as graded superalgebras with superinvolution (since they have different η), but they are isomorphic as graded superalgebras (since they have the same β and p).

We can use quadratic forms to replace the extension in Example 4.37 and Theorem 4.46 by an equivalent one, which does not depend on $\tilde{\beta}$.

Lemma 4.50. *Under the conditions of Theorem 4.46, suppose β takes values in $\{\pm 1\}$ and let Δ_0 be the refinement in item* (iii)*. Then every quadratic map $\mu\colon T \to \mathbb{F}^\times$ such that* $\mathrm{d}\mu = \tilde{\beta}$ *gives us an equivalence $\alpha\colon \{\pm 1\} \times_{\tilde{\beta}} T \to E := (\{\pm 1\} \times T^+) \cup (\{\pm \mathbf{i}\} \times T^-)$ between the group extensions of Examples 4.37 and 4.31 given by $\alpha(\delta, t) := (\delta\mu(t), t)$. Further, in the grading ${}^\alpha\Delta_0$, the exchange superinvolution is determined by the map $\eta\colon E \to \{\pm 1\}$ given by $\eta(\lambda, t) = \lambda\mu(t)^{-1}$, for all $(\lambda, t) \in E$.*

Proof. By Lemma 4.44, $\mu(T^+) \subseteq \{\pm 1\}$ and $\mu(T^-) \subseteq \{\pm \mathbf{i}\}$. Hence, by Proposition 4.39, the map $\alpha'\colon \{\pm 1, \pm \mathbf{i}\} \times_{\tilde{\beta}} T \to \{\pm 1, \pm \mathbf{i}\} \times T$ given by $\alpha'(\lambda, t) := (\lambda\mu(t), t)$, for all

$\lambda \in \{\pm 1, \pm \mathbf{i}\}$ and $t \in T$, is an equivalence between the corresponding extensions. Restricting α' to $\{\pm 1\} \times_{\tilde{\beta}} T$, we get precisely α, proving that it is an equivalence. The "further" part follows from item (ii) in Theorem 4.46. $\square$

Lemma 4.50 can be seen as taking a different model for $\mathcal{D} \times \mathcal{D}^{\mathrm{sop}}$. Recall that μ defines a super-anti-automorphism on $\psi\colon \mathcal{D} \to \mathcal{D}$, which can be seen as an isomorphism $\psi\colon \mathcal{D}^{\mathrm{sop}} \to \mathcal{D}$. Then the map $\mathcal{D} \times \mathcal{D}^{\mathrm{sop}} \to \mathcal{D} \times \mathcal{D}$ given by $(d_1, \bar{d_2}) \mapsto (d_1, \psi(d_2))$, for all $d_1, d_2 \in \mathcal{D}$, is an isomorphism of graded superalgebras with superinvolution, where we consider the superinvolution $(d_1, d_2) \mapsto (\psi^{-1}(d_2), \psi(d_1))$ on $\mathcal{D} \times \mathcal{D}$. Under this isomorphism, the element $(X_t, \delta \overline{X_t})$ goes to $(X_t, \delta\mu(t) X_t)$ (compare with the formula for α). If one follows the proof of Theorem 4.46 with $\mathcal{D} \times \mathcal{D}$ instead of $\mathcal{D} \times \mathcal{D}^{\mathrm{sop}}$, the group E would naturally appear in the place of $\{\pm 1\} \times_{\tilde{\beta}} T$.

We will now construct an example of an odd graded-division superalgebra of type $M \times M^{\mathrm{sop}}$, which will play an important role in the next section. Let $\mathcal{D}$ be the graded-division superalgebra of Example 2.49, i.e., $\mathcal{D} = M(1,1)$ with the grading by $T \coloneqq \mathbb{Z}_2 \times \mathbb{Z}_2$ determined by:

$$\deg \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (\bar{0}, \bar{0}), \quad \deg \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (\bar{0}, \bar{1}),$$
$$\deg \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (\bar{1}, \bar{0}), \quad \deg \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (\bar{1}, \bar{1}).$$

Following the proof of Theorem 4.46, we get a $\{\pm 1\} \times_{\tilde{\beta}} T$-grading refining the natural T-grading on $\mathcal{D} \times \mathcal{D}^{\mathrm{sop}}$, given by $\deg(X_t, \delta \overline{X_t}) = (\delta, t)$, for all $\delta \in \{\pm 1\}$ and $t \in T$.

To use Lemma 4.50 and get a simpler model for the group, we need a quadratic map whose polarization is $\tilde{\beta}$. To construct one, we can follow the proof of Proposition 4.45. First, let $\mu'\colon T \to \{\pm 1\}$ be the quadratic map determining the transposition on $\mathcal{D} = M(1,1)$, i.e., $\mu'(\bar{0}, \bar{0}) = 1$, $\mu'(\bar{0}, \bar{1}) = 1$, $\mu'(\bar{1}, \bar{0}) = 1$ and $\mu'(\bar{1}, \bar{1}) = -1$. Then we define $\mu \coloneqq \mu' \mu_0$, where μ_0 is the quadratic map of Example 4.43. In other words, μ is the map determining the queer supertranspose on $\mathcal{D}$ (see Definition 0.18). Then, by Lemma 4.50, we construct a refinement as in Theorem 4.46 with grading group $E = \Big(\{\pm 1\} \times \mathbb{Z}_2 \times \{\bar{0}\} \Big) \cup \Big(\{\pm \mathbf{i}\} \times \mathbb{Z}_2 \times \{\bar{1}\} \Big)$.

Finally, we have a group isomorphism $E \to \mathbb{Z}_2 \times \mathbb{Z}_4$, defined by $(1, \bar{1}, \bar{0}) \mapsto (\bar{1}, \bar{0})$ and $(i, \bar{0}, \bar{1}) \mapsto (\bar{0}, \bar{1})$. We then get:

Example 4.51. Let $\mathcal{O}$ denote the superalgebra $M(1,1)\times M(1,1)^{\text{sop}}$ endowed with the $\mathbb{Z}_2\times\mathbb{Z}_4$-grading determined by:

$$\deg\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},\overline{\begin{pmatrix}1&0\\0&1\end{pmatrix}}\right)=(\bar 0,\bar 0),\quad \deg\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},-\overline{\begin{pmatrix}1&0\\0&1\end{pmatrix}}\right)=(\bar 0,\bar 2),$$
$$\deg\left(\begin{pmatrix}1&0\\0&-1\end{pmatrix},\overline{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}\right)=(\bar 1,\bar 0),\quad \deg\left(\begin{pmatrix}1&0\\0&-1\end{pmatrix},-\overline{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}\right)=(\bar 1,\bar 2),$$
$$\deg\left(\begin{pmatrix}0&1\\1&0\end{pmatrix},\overline{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\right)=(\bar 0,\bar 1),\quad \deg\left(\begin{pmatrix}0&1\\1&0\end{pmatrix},-\overline{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\right)=(\bar 0,\bar 3),$$
$$\deg\left(\begin{pmatrix}0&-1\\1&0\end{pmatrix},\overline{\begin{pmatrix}0&-1\\1&0\end{pmatrix}}\right)=(\bar 1,\bar 3),\quad \deg\left(\begin{pmatrix}0&-1\\1&0\end{pmatrix},-\overline{\begin{pmatrix}0&-1\\1&0\end{pmatrix}}\right)=(\bar 1,\bar 1).$$

Then $\mathcal{O}$ is an odd graded division superalgebra and the exchange superinvolution on it preserves degrees. Thus η takes value 1 on $(\bar 0,\bar 0)$, $(\bar 1,\bar 0)$, $(\bar 0,\bar 1)$ and $(\bar 1,\bar 3)$, and value -1 on $(\bar 0,\bar 2)$, $(\bar 1,\bar 2)$, $(\bar 0,\bar 3)$ and $(\bar 1,\bar 1)$.

We note that, by Corollary 4.49, there are 4 different gradings by $\mathbb{Z}_2\times\mathbb{Z}_4$ on the superalgebra with superinvolution $M(1,1)\times M(1,1)^{\text{sop}}$ refining the $\mathbb{Z}_2\times\mathbb{Z}_2$-grading. The above is one of these 4 gradings.

The next result gives us a characterization of graded-division superalgebras with superinvolution that appear in Theorem 4.46.

Proposition 4.52. *Let $(\mathcal{E},\varphi_0)$ be a finite dimensional graded-division superalgebra with superinvolution associated to a quadruple $(T_{\mathcal{E}},\beta_{\mathcal{E}},p_{\mathcal{E}},\eta_{\mathcal{E}})$. Then there is a graded-division superalgebra $\mathcal{D}$ such that $(\mathcal{E},\varphi_0)$ is isomorphic to $\mathcal{D}\times\mathcal{D}^{\text{sop}}$ with exchange superinvolution and natural grading refined as in Theorem 4.46 if, and only if, there is an order 2 element $f\in\operatorname{rad}\tilde\beta_{\mathcal{E}}$ such that $\eta_{\mathcal{E}}(f)=-1$.*

Proof. For the "only if" direction, let (T,β,p,η) be the quadruple associated to $\mathcal{D}$. By item (i) of Theorem 4.46, the element $f:=\iota(-1)$ has order 2 and satisfies $\eta_{\mathcal{E}}(f)=-1$. Also, $f\in\ker\pi$, so $f\in\operatorname{rad}\tilde\beta_{\mathcal{E}}$.

For the "if" direction, define $T:=T_{\mathcal{E}}/\langle f\rangle$, let $\pi\colon T_{\mathcal{E}}\to T$ be the natural homomorphism, and let $\beta\colon T\times T\to\{\pm1\}$ and $p\colon T\to\mathbb{Z}_2$ be the maps induced by $\beta_{\mathcal{E}}$ and $p_{\mathcal{E}}$, respectively. They are well defined since $f\in\operatorname{rad}\tilde\beta=(\operatorname{rad}\beta)\cap T^+$. Thus, $T_{\mathcal{E}}$ fits into an extension $\{\pm 1\} \xrightarrow{\iota} T_{\mathcal{E}} \xrightarrow{\pi} T$ where $\iota(-1) = f$. For each $t \in T$, there precisely 2 elements of $T_{\mathcal{E}}$, t' and t'', with image t under π, but $\eta(t') = -\eta(t'')$ since $t'' = t'f$. We define a set-theoretic section $\tau\colon T \to T_{\mathcal{E}}$ of π by taking $\tau(t)$ to be the element such that $\eta(\tau(t)) = 1$. Then the map $\alpha\colon T_{\mathcal{E}} \to \{\pm 1\} \times_{\tilde{\beta}} T$ given by $\alpha(\tau(t)) = (1, t)$ and $\alpha(f\tau(t)) = (-1, t)$, for all $t \in T$, is an equivalence from $T_{\mathcal{E}}$ to the extension in item (iii) in Theorem 4.46 (compare with the proof of Proposition 4.38). $\square$

We conclude with a result that will be useful in Chapter 5.

Proposition 4.53. *Let $(\mathcal{E}, \varphi_0)$ be as in Proposition 4.52, with $f \in \ker \tilde{\beta}$ and $\eta(f) = -1$. Let $\mathcal{V}$ be a graded right $\mathcal{E}$-supermodule associated to a map $\kappa\colon G/T_{\mathcal{E}} \to \mathbb{Z}_{\geq 0}$ and suppose there is a superinvolution φ on $R := \operatorname{End}_{\mathcal{E}}(\mathcal{V})$ determined by a nondegenerate φ_0-sesquilinear form on $\mathcal{V}$. Set $\overline{G} := G/\langle f\rangle$, $\overline{G}^{\#} := \overline{G} \times \mathbb{Z}_2 \simeq G^{\#}/\langle f\rangle$, $\overline{T_{\mathcal{E}}} := T_{\mathcal{E}}/\langle f\rangle$, and let $\pi\colon G \to \overline{G}$ be the natural homomorphism and let $\bar{\beta}$ and $\bar{p}$ be the maps induced by β and p, respectively. Then the coarsening $({}^{\pi}R, \varphi)$ is isomorphic to $S \times S^{\mathrm{sop}}$ with exchange superinvolution, where $S \simeq E(\overline{T_{\mathcal{E}}}, \bar{\beta}, \bar{p}, \kappa)$, seeing κ as a map from $\overline{G}^{\#}/\overline{T} \simeq G^{\#}/T$ to $\mathbb{Z}_{\geq 0}$.*

Proof. Recall that $\operatorname{rad}\tilde{\beta} = \operatorname{supp} Z(\mathcal{E})^{\bar{0}}$. Let $\zeta \in Z(\mathcal{E})^{\bar{0}}$ be an element with degree f and, scaling it if necessary, we may assume that $\zeta^2 = 1$. Since $\eta(f) = -1$, $\varphi_0(\zeta) = -\zeta$. By Proposition 3.21, we can identify $(Z(\mathcal{E})^{\bar{0}}, \varphi_0)$ and $(Z(R)^{\bar{0}}, \varphi)$, so $\zeta \in Z(R)^{\bar{0}}$ with $\varphi(\zeta) = -\zeta$ (recall that the ground field $\mathbb{F}$ is algebraically closed). Set $\epsilon := \frac{1+\zeta}{2}$ and $\epsilon' := \frac{1-\zeta}{2}$. Clearly, ϵ and ϵ' are orthogonal central idempotents and $1 = \epsilon + \epsilon'$. It follows that we can write $R = S \oplus S'$, a direct sum of superideals, where $S := \epsilon R$ and $S' := \epsilon' R$. If we consider on R the $\overline{G}^{\#}$-grading, the elements ϵ and ϵ' are homogeneous, so S and S' are graded superideals. Since $\varphi(\epsilon) = \epsilon'$, it follows that $\varphi(S) = S'$ and, hence, $S' \simeq S^{\mathrm{sop}}$ as $\overline{G}$-graded superalgebras. It remains to prove that S is graded-simple and describe its parameters.

Set $\mathcal{U} := \mathcal{V}\epsilon$ and $\mathcal{U}' := \mathcal{V}\epsilon'$. Clearly, $\mathcal{U}$ and $\mathcal{U}'$ are $\overline{G}$-graded right $\mathcal{E}$-supermodules and $\mathcal{V} = \mathcal{U} \oplus \mathcal{U}'$. For any $\mathcal{E}$-linear map $r \in R = \operatorname{End}_{\mathcal{E}}(\mathcal{V})$, it is clear that $\epsilon r(u) = r(u)\epsilon \in \mathcal{U}$ for all $u \in \mathcal{U}$ and, since $\epsilon\epsilon' = 0$, $\epsilon r(u') = 0$ for all $u' \in \mathcal{U}'$. It follows that we can identify $S = \epsilon R$ with $\operatorname{End}_{\mathcal{E}}(\mathcal{U})$.

Set $\mathcal{D} := \epsilon\mathcal{E}$. Restricting the action of $\mathcal{E}$ on $\mathcal{U}$, we can consider $\mathcal{U}$ as a $\mathcal{D}$-supermodule. We claim that if $f\colon \mathcal{U} \to \mathcal{U}$ is $\mathcal{D}$-linear, then it is also $\mathcal{E}$-linear. Indeed,

$u = u\epsilon$ for every $u \in \mathcal{U}$, so $f(ud) = f(u\epsilon d) = f(u)\epsilon d = f(u)d$. We conclude that $\mathrm{End}_{\mathcal{E}}(\mathcal{U}) = \mathrm{End}_{\mathcal{D}}(\mathcal{U})$.

We claim that $\mathcal{D}$ is a graded-division superalgebra. Clearly, $\epsilon \in \mathcal{D}$ is the identity element. For every $t \in T_{\mathcal{E}}$, choose $0 \neq X_t \in \mathcal{E}_t$. Note that, for every $t \in T_{\mathcal{E}}$, we have $X_{ft} \in \mathbb{F}\zeta X_t$ and, hence, $\epsilon X_{ft} \in \mathbb{F}X_t$ since $\epsilon\zeta = \epsilon$. It follows that the $\overline{G}$-homogeneous component $\mathcal{D}_{\bar{t}}$ is spanned by ϵX_t. Since ϵX_t is invertible in $\mathcal{D}$, with inverse ϵX_t^{-1}, we have that $\mathcal{D}$ is a graded-division superalgebra and that $\operatorname{supp}\mathcal{D} = \overline{T_{\mathcal{E}}}$. Also, $\epsilon X_s \epsilon X_t = \epsilon X_s X_t = \beta(s,t)\epsilon X_t X_s = \beta(s,t)\epsilon X_t \epsilon X_s$, for all $s, t \in T$, so $\mathcal{D}$ is associated to $(\overline{T_{\mathcal{E}}}, \bar{\beta}, \bar{p})$.

Finally, to see that the map $\kappa\colon G^{\#}/T_{\mathcal{E}} \simeq \overline{G}^{\#}/\overline{T_{\mathcal{E}}} \to \mathbb{Z}_{\geq 0}$ is associated to the graded right $\mathcal{D}$-supermodule $\mathcal{U}$, let $\{v_1, \dots, v_k\}$ be a G-graded $\mathcal{E}$-basis for $\mathcal{V}$. We claim that $\mathcal{B} = \{v_1\epsilon, \dots, v_k\epsilon\}$ is a $\overline{G}^{\#}$-graded $\mathcal{D}$-basis for $\mathcal{U}$. Indeed, for every $v \in \mathcal{V}$, write $v = v_1 d_1 + \cdots + v_k d_k$, where $d_1, \dots, d_k \in \mathcal{E}$. Then $v\epsilon = v_1 d_1 \epsilon + \cdots + v_k d_k \epsilon = (v_1\epsilon)(d_1\epsilon) + \cdots + (v_k\epsilon)(d_k\epsilon)$. We conclude that $\mathcal{U}$ is the $\mathcal{D}$-span of $\mathcal{B}$. For the $\mathcal{D}$-linear independence, we note that if $(v_1\epsilon)(d_1\epsilon) + \cdots + (v_k\epsilon)(d_k\epsilon) = 0$, then $v_1 d_1 \epsilon + \cdots + v_k d_k \epsilon = 0$, so $d_1\epsilon = \cdots = d_k\epsilon = 0$. $\square$

Remark 4.54. The $\overline{G}$-graded superalgebra $S' \simeq S^{\mathrm{sop}}$ is also isomorphic to $E(\overline{T_{\mathcal{E}}}, \bar{\beta}, \bar{p}, \kappa)$, so S admits a super-anti-automorphism. Nevertheless, we cannot construct one canonically. We will revisit this point in Subsection 5.3.1.

4.5 Gradings on superinvolution-simple superalgebras of types $M \times M^{\mathrm{sop}}$ and $Q \times Q^{\mathrm{sop}}$

As before, we continue assuming that $\mathbb{F}$ is algebraically closed and $\operatorname{char}\mathbb{F} \neq 2$.

Let $R \coloneqq S \times S^{\mathrm{sop}}$ where S is a finite dimensional simple superalgebra, and let φ denote the exchange superinvolution. By Lemma 4.4, (R, φ) is a superinvolution-simple superalgebra. If we endow (R, φ) with a grading, then it becomes, clearly, graded-superinvolution-simple, but not necessarily graded-simple.

In the case R is not graded-simple, by Corollary 4.8, we have $R \simeq E^{\mathrm{ex}}(\mathcal{D}, \mathcal{U})$. Since $S \times \{0\}$ and $\{0\} \times S^{\mathrm{sop}}$ are the only nonzero proper superideals of R, $\mathrm{End}_{\mathcal{D}}(\mathcal{U})$ and, hence, $\mathcal{D}$ must be simple. The isomorphism conditions are given by Theorems 4.9

and 4.11, specialized to the simple case. Recall that gradings on finite dimensional simple superalgebras were classified in Corollaries 2.76 and 2.80 and Theorem 2.103. For odd gradings on $M(n,n)$, also recall that, for each pair (T^+,β^+) with $\operatorname{rad}\beta^+ = \langle t_p\rangle$ of order 2, we fixed a character $\chi \in \widehat{T^+}$ such that $\chi(t_p) = -1$ (see Subsection 2.3.2).

Theorem 4.55. *Let (R,φ) be a superinvolution-simple superalgebra of type $M \times M^{\mathrm{sop}}$ endowed with a G-grading and suppose it is not graded-simple. Then (R,φ) is isomorphic to either $M(T,\beta,\kappa_{\bar 0},\kappa_{\bar 1}) \times M(T,\beta,\kappa_{\bar 0},\kappa_{\bar 1})^{\mathrm{sop}}$ (see Definition 2.75), or to $M(T^+,\beta^+,h,\kappa) \times M(T^+,\beta^+,h,\kappa)^{\mathrm{sop}}$ (see Definition 2.102), but not both, where we consider the exchange superinvolution in both cases.*

Moreover, $M(T,\beta,\kappa_{\bar 0},\kappa_{\bar 1}) \times M(T,\beta,\kappa_{\bar 0},\kappa_{\bar 1})^{\mathrm{sop}} \simeq M(T',\beta',\kappa'_{\bar 0},\kappa'_{\bar 1}) \times M(T',\beta',\kappa'_{\bar 0},\kappa'_{\bar 1})^{\mathrm{sop}}$ if, and only if, $T = T'$ and one of the following conditions holds:

(i) $\beta' = \beta$ and there is $g \in G$ such that either $\kappa'_{\bar 0} = g\cdot\kappa_{\bar 0}$ and $\kappa'_{\bar 1} = g\cdot\kappa_{\bar 1}$, or $\kappa'_{\bar 0} = g\cdot\kappa_{\bar 1}$ and $\kappa'_{\bar 1} = g\cdot\kappa_{\bar 0}$;

(ii) $\beta' = \beta^{-1}$ and there is $g \in G$ such that either $\kappa'_{\bar 0} = g\cdot\kappa^\star_{\bar 0}$ and $\kappa'_{\bar 1} = g\cdot\kappa^\star_{\bar 1}$, or $\kappa'_{\bar 0} = g\cdot\kappa^\star_{\bar 1}$ and $\kappa'_{\bar 1} = g\cdot\kappa^\star_{\bar 0}$;

and $M(T^+,\beta^+,h,\kappa) \times M(T^+,\beta^+,h,\kappa)^{\mathrm{sop}} \simeq M(T'^+,\beta'^+,h',\kappa') \times M(T'^+,\beta'^+,h',\kappa')^{\mathrm{sop}}$ if, and only if, $T^+ = T'^+$ and one of the following conditions holds:

(iii) $\beta'^+ = \beta^+$, $h' \in \{h, ht_p\}$ and there is $g \in G$ such that $\kappa' = g\cdot\kappa$;

(iv) $\beta'^+ = (\beta^+)^{-1}$, $h' \in \{h^{-1}, h^{-1}t_p\}$ and there is $g \in G$ such that $\kappa' = g\cdot\kappa^\star$. □

Theorem 4.56. *Let (R,φ) be a superinvolution-simple superalgebra of type $Q \times Q^{\mathrm{sop}}$ endowed with a G-grading and suppose it is not graded-simple. Then (R,φ) is isomorphic to $Q(T^+,\beta^+,h,\kappa) \times Q(T^+,\beta^+,h,\kappa)^{\mathrm{sop}}$ (see Definition 2.79), where we consider the exchange superinvolution.*

Moreover, $Q(T^+,\beta^+,h,\kappa) \times Q(T^+,\beta^+,h,\kappa)^{\mathrm{sop}} \simeq Q(T^+,\beta^+,h,\kappa) \times Q(T^+,\beta^+,h,\kappa)^{\mathrm{sop}}$ if, and only if, $T^+ = T'^+$, $h = h'$ and one of the following conditions holds:

(i) $\beta'^+ = \beta^+$ and there is $g \in G$ such that $\kappa' = g\cdot\kappa$;

(ii) $\beta'^+ = (\beta^+)^{-1}$ and there is $g \in G$ such that $\kappa' = g\cdot\kappa^\star$. □

Now let us focus on the graded-simple case. By Corollary 4.8, if (R,φ) is a finite dimensional superalgebra with superinvolution with R graded-simple, then $(R,\varphi)\simeq E(\mathcal{D},\mathcal{U},B)$ for some triple $(\mathcal{D},\mathcal{U},B)$ as in Definition 3.51. Let $(T,\beta,p,\eta,\kappa,g_0,\delta)$ be the parameters of $(\mathcal{D},\mathcal{U},B)$ and let φ_0 be the superinvolution on $\mathcal{D}$ determined by η.

Proposition 4.57. *The superalgebra with superinvolution (R,φ) is superinvolution-simple but not simple if, and only if,* $\operatorname{rad}\tilde{\beta}=\langle f\rangle$*, where f has order* 2 *and $\eta(f)=-1$. If this is the case, then (R,φ) has the same type as $(\mathcal{D},\varphi_0)$, which is $M\times M^{\mathrm{sop}}$ if* $\operatorname{rad}\beta=\operatorname{rad}\tilde{\beta}$ *and $Q\times Q^{\mathrm{sop}}$ if* $\operatorname{rad}\beta\neq\operatorname{rad}\tilde{\beta}$.

Proof. Suppose (R,φ) is superinvolution-simple but not simple. Propositions 2.28 and 3.33 imply that $(\mathcal{D},\varphi_0)$ is also superinvolution-simple but not simple. By Proposition 2.30, $Z(R)$ is isomorphic to $Z(\mathcal{D})$, and by Proposition 4.16, different types have nonisomorphic centers, hence (R,φ) has the same type as $(\mathcal{D},\varphi_0)$.

Define $\varphi_0':=\varphi_0$ if B is even and $\varphi_0':=\upsilon\varphi_0$ if B is odd, where υ is the parity automorphism. By Proposition 3.21, $(Z(R),\varphi)$ is isomorphic to $(Z(\mathcal{D}),\varphi_0')$. Again by Proposition 4.16 and by Theorem 4.46, $(Z(\mathcal{D}),\varphi_0')$ must be one of the graded-division superalgebras of Examples 4.47 and 4.48, up to relabeling the homogeneous components. In any case, we get that $\operatorname{supp} Z(\mathcal{D})^{\bar{0}}=(\operatorname{rad}\beta)\cap T^+=\operatorname{rad}\tilde{\beta}$ is a cyclic group of order 2 and η has value -1 on the generator.

For the converse, by Propositions 2.28 and 3.33, it suffices to prove that $\mathcal{D}$ is φ_0-simple. Let $\overline{\mathcal{D}}$ be a graded-division superalgebra associated to $(T/\langle f\rangle,\bar{\beta},\bar{p})$, where $\bar{\beta}$ and $\bar{p}$ are induced by β and p, respectively. Clearly, the bicharacter induced by $\tilde{\beta}$ on $T/\langle f\rangle$ is nondegenerate, so, by Corollary 2.64, $\overline{\mathcal{D}}$ is simple as a superalgebra. By Proposition 4.52, with $\mathcal{D}$ taking the role of $\mathcal{E}$ and $\overline{\mathcal{D}}$ taking the role of $\mathcal{D}$, $(\mathcal{D},\varphi_0)$ is isomorphic to $\overline{\mathcal{D}}\times\overline{\mathcal{D}}^{\mathrm{sop}}$, with exchange superinvolution, endowed with a graded-division refinement of its natural grading. We conclude, by Proposition 4.6, that $\mathcal{D}$ is φ_0-simple, as desired. $\square$

From now on, let us suppose (R,φ) is superinvolution-simple of type $M\times M^{\mathrm{sop}}$ or $Q\times Q^{\mathrm{sop}}$ and get some corollaries from Proposition 4.57.

Corollary 4.58. *If the superalgebra with superinvolution $M(m,n)\times M(m,n)^{\mathrm{sop}}$ admits a grading that makes it graded-simple with $\mathcal{D}$ odd, then $m=n$.*

Proof. If $M(m,n) \times M(m,n)^{\mathrm{sop}}$ has an odd division grading, then $\dim(M(m,n) \times M(m,n)^{\mathrm{sop}})^{\bar{0}} = (M(m,n) \times M(m,n)^{\mathrm{sop}})^{\bar{1}}$. Similar to Lemma 2.51, this only happens when $m = n$.

For the general case, by Proposition 4.57, we have that $M(m,n) \times M(m,n)^{\mathrm{sop}} \simeq M(k) \otimes \mathcal{D}$, where $\mathcal{D} \simeq M(m',n') \times M(m',n')^{\mathrm{sop}}$ and $m = km'$, $n = kn'$. It follows that $m' = n'$ and, hence, $m = n$. $\square$

Corollary 4.59. *The subgroup $T^+ \subseteq T$ is 2-elementary and, for every $t \in T^-$, $t^2 = f$.*

Proof. By Proposition 4.57, we can write $\operatorname{rad}\tilde{\beta} = \{e, f\}$, with $\eta(e) = 1$ and $\eta(f) = -1$. Let $t \in T$. By Corollary 3.39, $t^2 \in \operatorname{rad}\tilde{\beta}$. If t is even, then $\eta(t^2) = \eta(t)^2 = 1$, hence $t^2 = e$. If t is odd, then, $\eta(t^2) = -\eta(t)^2 = -1$, hence $t^2 = f$. $\square$

For the next results, we recall the notion of parity element (Definition 2.68).

Corollary 4.60. *There are precisely 2 parity elements in T, t_p and t'_p, and $\eta(t_p) = -\eta(t'_p)$.*

Proof. From Corollary 2.69, we know that there are 2 parity elements and that $t'_p = f t_p$. Then $\eta(t_p) = -\eta(t'_p)$ since $f \in \operatorname{rad}\tilde{\beta}$ and $\eta(f) = -1$. $\square$

Definition 4.61. We define *the η-parity element* of T to be the unique parity element $t_p \in T$ such that $\eta(t_p) = 1$.

Note that if $T = T^+$, then the η-parity element is $e \in T$.

We will now investigate what happens if we change the map η. Let $\eta' \colon T \to \{\pm 1\}$ be another map such that $\mathrm{d}\eta' = \tilde{\beta}$ and let φ'_0 be the corresponding superinvolution on $\mathcal{D}$.

Lemma 4.62. *The graded-division superalgebra with superinvolution $(\mathcal{D}, \varphi'_0)$ is superinvolution-simple if, and only if, $\eta' \sim \eta$ (see Definition 3.55).*

Proof. By Proposition 4.57, $(\mathcal{D}, \varphi'_0)$ is superinvolution-simple if, and only if, $\eta'(f) = -1$. Suppose $\eta' \sim \eta$. Then there is $t \in T$ such that $\eta' = \tilde{\beta}(t, \cdot)\eta$ and, hence, $\eta'(f) = \tilde{\beta}(t, f)\eta(f) = -1$.

Conversely, if $\eta'(f) = -1$, define $\chi \colon T \to \{\pm 1\}$ by $\chi(t) = \eta(t)\eta'(t)$, for all $t \in T$. Since $\mathrm{d}\eta' = \tilde{\beta} = \mathrm{d}\eta$, $\mathrm{d}\chi = 1$, i.e., χ is a character of T. Also, $\chi(f) = 1$. Hence χ induces a character of $T/\langle f\rangle$ and, since $\tilde{\beta}$ induces a nondegenerate skew-symmetric bicharacter on $T/\langle f\rangle$, there is $t \in T$ such that $\chi = \tilde{\beta}(t,\cdot)$. We conclude that $\eta' = \tilde{\beta}(t,\cdot)\eta$, i.e., $\eta' \sim \eta$. □

The following is straightforward:

Lemma 4.63. *Suppose $\eta' \sim \eta$. Then the η'-parity element of T is the same as the η-parity element of T if, and only if, $\eta' \sim_{\bar{0}} \eta$ (see Definition 3.57).* □

Proposition 4.64. *Let $t_p \in T$ be a parity element of T. Then:*

(a) If (R,φ) is even of type $M \times M^{\mathrm{sop}}$, then $\operatorname{rad}\beta^+ = \operatorname{rad}\beta = \langle f\rangle \simeq \mathbb{Z}_2$;

(b) If (R,φ) is odd of type $M \times M^{\mathrm{sop}}$, then $\operatorname{rad}\beta^+ = \langle f, t_p\rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\operatorname{rad}\beta = \langle f\rangle \simeq \mathbb{Z}_2$;

(c) If (R,φ) is of type $Q \times Q^{\mathrm{sop}}$, then $\operatorname{rad}\beta^+ = \langle f\rangle \simeq \mathbb{Z}_2$, $\operatorname{rad}\beta = \langle t_p\rangle \simeq \mathbb{Z}_4$ and $t_p^2 = f$.

Proof. This follows from Corollaries 2.71 and 4.59 and Proposition 4.57. □

Theorem 4.65. *Let $(\mathcal{D},\varphi_0)$ be a finite dimensional superinvolution-simple graded-division superalgebra, not simple as a superalgebra, associated to (T,β,p,η), and let t_p be its η-parity element. If $\mathcal{D}$ is odd of type $M \times M^{\mathrm{sop}}$, choose $t_1 \in T^-$. Then there are superinvolution-simple graded-division superalgebras $(\mathcal{C},\varphi_{\mathcal{C}})$ and $(\mathcal{M},\varphi_{\mathcal{M}})$ such that $(\mathcal{D},\varphi_0) \simeq (\mathcal{C}\otimes\mathcal{M}, \varphi_{\mathcal{C}}\otimes\varphi_{\mathcal{M}})$, $(\mathcal{M},\varphi_{\mathcal{M}})$ is of type M (and, hence, even) and*

(a) If $\mathcal{D}$ is even of type $M \times M^{\mathrm{sop}}$, $\mathcal{C} = {}^{\alpha}(\mathbb{F}\mathbb{Z}_2)$ considered as an even graded-division superalgebra with superinvolution $\varphi_{\mathcal{C}}$ as in Example 4.47, and $\alpha\colon \mathbb{Z}_2 \to \operatorname{rad}\beta^+$ is the unique group isomorphism;

(b) If $\mathcal{D}$ is odd of type $M \times M^{\mathrm{sop}}$, then $\mathcal{C} = {}^{\alpha}\mathcal{O}$, where $\mathcal{O}$ is the graded-division superalgebra with superinvolution $\varphi_{\mathcal{C}}$ as in Example 4.51, and $\alpha\colon \mathbb{Z}_2\times\mathbb{Z}_4 \to \langle t_p, t_1\rangle$ is the group isomorphism given by $\alpha(\bar{1},\bar{0}) := t_p$ and $\alpha(\bar{0},\bar{1}) := t_1$;

(c) If $\mathcal{D}$ is of type $Q \times Q^{\mathrm{sop}}$, then $\mathcal{C} = {}^{\alpha}(\mathbb{F}\mathbb{Z}_4)$, where $\mathbb{F}\mathbb{Z}_4$ is the odd graded-division superalgebra with superinvolution $\varphi_{\mathcal{C}}$ as in Example 4.48, and $\alpha\colon \mathbb{Z}_4 \to \langle t_p\rangle$ given by $\alpha(\bar{1}) := t_p$.

Proof. Since, by Corollary 4.59, T^+ is an elementary 2-group, it is a vector space over the field with 2 elements. Fix a basis $\mathcal{B}$ for $\operatorname{rad}\beta^+$ and complete it to a basis $\mathcal{B}\cup\mathcal{B}'$ of T^+. In the case $\mathcal{D}$ is odd of type $M\times M^{\mathrm{sop}}$, we can choose $\mathcal{B}'$ such that $\tilde\beta(t_1,b)=1$ for all $b\in\mathcal{B}'$. Indeed, if $\tilde\beta(t_1,b)=-1$, we replace b by t_pb; note that we still get a complement for $\mathcal{B}$ since $t_p\in\operatorname{rad}\beta^+$.

We define $K\subseteq T^+$ to be the subgroup generated by $\mathcal{B}'$. Then $T^+=(\operatorname{rad}\beta^+)\times K$ and, hence, $\beta^+\restriction_{K\times K}=\beta\restriction_{K\times K}$ is a nondegenerate alternating bicharacter. Let $\mathcal{M}$ be a graded division algebra associated to $(K,\beta\restriction_{K\times K})$, and let $\varphi_{\mathcal{M}}$ be the superinvolution determined by $\eta\restriction_K$.

For each of the possibilities for $\mathcal{D}$, we can write $T=C\times K$ where $\beta(C,K)=1$, and then apply Lemma 2.33 to finish the proof. If $\mathcal{D}$ is even of type $M\times M^{\mathrm{sop}}$, take $C:=\operatorname{rad}\beta^+=\operatorname{rad}\beta\simeq\mathbb{Z}_2$. If $\mathcal{D}$ is of type $Q\times Q^{\mathrm{sop}}$, take $C:=\operatorname{rad}\beta=\langle t_p\rangle\simeq\mathbb{Z}_4$. If $\mathcal{D}$ is odd of type $M\times M^{\mathrm{sop}}$, take $C:=\langle\operatorname{rad}\beta^+,t_1\rangle=\langle t_p,t_1\rangle\simeq\mathbb{Z}_2\times\mathbb{Z}_4$ (note that here we are using our choice of $\mathcal{B}'$ to have $\beta(C,K)=1$). In every case, it is easy to see that $(\mathcal{C},\varphi_{\mathcal{C}})$ as in the statement is a graded-division superalgebra with superinvolution associated to $(C,\beta\restriction_{C\times C},p\restriction_C,\eta\restriction_C)$. $\square$

Note that, in cases (a) and (c) in the proof above, we chose K to be an arbitrary complement of $\operatorname{rad}\beta^+$ in T^+. The next result shows that we can do the same in case (b), and then choose the element $t_1\in T^-$ depending on K.

Lemma 4.66. *Let (T,β,p) be as in Theorem 4.65(b), and let $K\subseteq T^+$ be any subgroup such that $T^+=(\operatorname{rad}\beta^+)\times K$. Then the set $S:=\{t_1\in T^-\mid\beta(t_1,K)=1\}$ is a coset of* $\operatorname{rad}\beta^+$.

Proof. Given $t_1\in S$, it is straightforward that $t_1'\in S$ if, and only if, $t_1^{-1}t_1'\in\operatorname{rad}\beta^+$. Hence we only have to show that $S\neq\emptyset$. Let χ be the character of T^+ determined by $\chi(K)=\chi(f)=1$ and $\chi(t_p)=-1$, and extend χ to a character of T. Since $\operatorname{rad}\tilde\beta=\langle f\rangle$, χ induces a character on $T/\operatorname{rad}\tilde\beta$ and, by the nondegeneracy of the bicharacter induced by $\tilde\beta$ on $T/\operatorname{rad}\tilde\beta$, there is $t_1\in T$ such that $\chi=\tilde\beta(t_1,\cdot)$. Since $\tilde\beta(t_1,t_p)=\chi(t_p)=-1$, $t_1\in T^-$, and hence, since $\tilde\beta(t_1,K)=\chi(K)=1$, $t_1\in S$. $\square$

We will now adapt the proof of Theorem 4.65 to construct standard realizations of superinvolution-simple graded-division superalgebras that are not simple as superalgebras.

Definition 4.67. Let $T \subseteq G^\#$ be a finite subgroup, let $\beta\colon T \times T \to \{\pm 1\}$ be an alternating bicharacter and let $p\colon T \subseteq G^\# \to \mathbb{Z}_2$ be the restriction of the projection on the $\mathbb{Z}_2$ component of $G^\# = G \times \mathbb{Z}_2$. Define $T^+, T^- \subseteq T$, $\tilde{\beta}\colon T \times T \to \{\pm 1\}$ and $\beta^+\colon T^+ \times T^+ \to \{\pm 1\}$ as usual (see Subsections 2.2.1 and 2.2.3). Suppose that:

- $\operatorname{rad} \tilde{\beta} = \langle f \rangle$ for an element $e \neq f \in T^+$;
- T^+ is an elementary 2-group;
- for every $t \in T^-$, $t^2 = f$.

Under these assumptions, we have precisely 2 parity elements in T. Let $t_p \in T$ be one of them, taking $t_p \coloneqq e$ in the case $T = T^+$. Also note that:

(a) if $t_p = e$, then $T = T^+$ and $\beta^+ = \beta = \tilde{\beta}$;

(b) if $e \neq t_p \in T^+$, then $\operatorname{rad} \beta^+ = \langle f, t_p \rangle$ and $\operatorname{rad} \beta = \langle f \rangle$;

(c) if $t_p \in T^-$, then $\operatorname{rad} \beta^+ = \langle f \rangle$ and $\operatorname{rad} \beta = \langle t_p \rangle$;

where we have used Corollary 2.71 for (b) and (c). Then choose:

(i) a subgroup $K \subseteq T^+$ such that $(\operatorname{rad} \beta^+) \times K = T^+$;

(ii) a standard realization $\mathcal{M}$ (see Definition 2.36) of a matrix algebra with a division grading associated to $(K, \beta \restriction_{K \times K})$;

(iii) if we are in case (b), an element $t_1 \in T^-$ such that $\beta(t_1, K) = 1$.

A subgroup K in item (i) exists because T^+ is an elementary 2-group, and a standard realization in item (ii) is defined since $\beta \restriction_{K \times K} = \beta^+ \restriction_{K \times K}$ is, clearly, nondegenerate. Finally, in case (b), the element t_1 in (iii) exists by Lemma 4.66.

Let $\varphi_{\mathcal{M}}$ denote the transposition on $\mathcal{M}$, and let $(\mathcal{C}, \varphi_{\mathcal{C}})$ be as in Theorem 4.65: item (a) if $t_p = e$, item (b) if $e \neq t_p \in T^+$, and item (c) if $t_p \in T^-$. We say that $(\mathcal{C} \otimes \mathcal{M}, \varphi_{\mathcal{C}} \otimes \varphi_{\mathcal{M}})$ is a *standard realization* of a graded-division superalgebra with superinvolution associated to $(T, \tilde{\beta}, t_p)$.

For any $t \in T$, we choose X_t to be $X_r \otimes X_s$, where $t = rs$, $r \in \operatorname{supp} \mathcal{C}$, $s \in \operatorname{supp} \mathcal{M} = K$, X_r is the homogeneous element of $\mathcal{C}_r$ as in Examples 4.47, 4.48 and 4.51, and $X_s \in \mathcal{M}_s$ is as in Proposition 2.35.

Remark 4.68. The map $\eta\colon T \to \{\pm 1\}$ corresponding to $\varphi_{\mathcal{C}} \otimes \varphi_{\mathcal{M}}$ is characterized by the following conditions: $\mathrm{d}\eta = \tilde{\beta}$, $\eta\restriction_K$ is the map associated to the transposition on $\mathcal{M}$ and one of

(a) if $t_p = e$, then $\eta(f) = -1$;

(b) if $e \neq t_p \in T^+$, then $\eta(t_p) = 1$ and $\eta(t_1) = 1$;

(c) if $t_p \in T^-$, then $\eta(t_p) = 1$.

In particular, in all cases $\eta(f) = -1$ and the chosen t_p is the η-parity element of T. Also, a different choice t_1' in (iii) leads to $\eta' = \eta$ if $t_1' \in \{t_1, ft_pt_1\}$, and to $\eta' = \tilde{\beta}(t_p, \cdot)\eta$ if $t_1' \in \{ft_1, t_pt_1\}$; in both cases, $\eta'\restriction_{T^+} = \eta\restriction_{T^+}$.

In what follows, for each triple $(T, \tilde{\beta}, t_p)$, we will fix a standard realization $(\mathcal{D}, \varphi_0)$.

Definition 4.69. Let T, β and t_p be as in Definition 4.67 with $T^+ = T$ (so $t_p = e$), let $g_0 \in G^\#$ be any element, and let $\kappa_{\bar{0}}, \kappa_{\bar{1}}\colon G/T \to \mathbb{Z}_{\geq 0}$ be g_0-admissible maps (see Definition 4.27).

Choose a graded $\mathcal{D}$-supermodule $\mathcal{U}$ and a sesquilinear form $B\colon \mathcal{U} \times \mathcal{U} \to \mathcal{D}$ such that $(\mathcal{U}, B)$ has inertia determined by $(\kappa_{\bar{0}}, \kappa_{\bar{1}})$. The graded superalgebra with superinvolution $M^{\mathrm{ex}}(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}}, g_0)$ is defined to be $E(\mathcal{D}, \mathcal{U}, \mathcal{B})$. With a choice of a graded basis for $\mathcal{U}$, this becomes the graded superalgebra $M_{k_{\bar{0}}|k_{\bar{1}}}(\mathcal{D})$, where $k_{\bar{0}} := |\kappa_{\bar{0}}|$ and $k_{\bar{1}} := |\kappa_{\bar{1}}|$, endowed with the superinvolution φ given by

$$\varphi(X) := \Phi^{-1}\varphi_0(X^{s\top})\Phi,$$

for all $X \in M_{k_{\bar{0}}|k_{\bar{1}}}(\mathcal{D})$, where Φ is the matrix representing B with respect to the chosen basis.

Note that, as an ungraded superalgebra with superinvolution, $M^{\mathrm{ex}}(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}}, g_0) \simeq M(m,n) \times M(m,n)^{\mathrm{sop}}$, where $m = k_{\bar{0}}\sqrt{|T|/2}$ and $n = k_{\bar{1}}\sqrt{|T|/2}$.

Theorem 4.70. *Suppose the superalgebra with superinvolution $M(m,n) \times M(m,n)^{\mathrm{sop}}$ is endowed with an even G-grading making it graded-simple. Then it is isomorphic, as a graded superalgebra with superinvolution, to $M^{\mathrm{ex}}(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}}, g_0)$ as in Definition 4.69. Moreover, $M^{\mathrm{ex}}(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}}, g_0) \simeq M^{\mathrm{ex}}(T', \beta', \kappa_{\bar{0}}', \kappa_{\bar{1}}', g_0')$ if, and only if, $T = T'$, $\beta = \beta'$ and there is $g \in G$ such that one of the following conditions holds:*

(i) $\kappa'_{\bar{0}} = g \cdot \kappa_{\bar{0}}$, $\kappa'_{\bar{1}} = g \cdot \kappa_{\bar{1}}$ *and* $g'_0 = g^{-2} g_0$*;*

(ii) $\kappa'_{\bar{0}} = g \cdot \kappa_{\bar{1}}$, $\kappa'_{\bar{1}} = g \cdot \kappa_{\bar{0}}$ *and* $g'_0 = f g^{-2} g_0$*.*

Proof. By Theorems 3.18 and 3.37, our graded superalgebra with superinvolution is isomorphic to $E(\mathcal{D}, \mathcal{U}, B)$, with $(\mathcal{D}, \mathcal{U}, B)$ as in Definition 3.51. Let $(T, \beta, p, \eta, \kappa, g_0, \delta)$ be the parameters of $(\mathcal{D}, \mathcal{U}, B)$, and define $\kappa_{\bar{0}}, \kappa_{\bar{1}}$ from κ as usual. Using Theorem 3.54 and Lemma 4.62, we can assume that η is the one from the fixed standard realization associated to $(T, \tilde{\beta}, t_p)$, where $t_p = e$. Then $E(\mathcal{D}, \mathcal{U}, B)$ is isomorphic to $M^{\mathrm{ex}}(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}}, g_0)$ by Theorem 3.54 and Proposition 3.56.

The isomorphism condition follows from Theorem 3.54 and Proposition 3.56 and the fact that the group $\mathcal{G}$ (see Equation (3.20)) is $\Big(\{e\} \times (G \times \{\bar{0}\})\Big) \cup \Big(\{f\} \times (G \times \{\bar{1}\})\Big)$ in our case. $\square$

We now classify odd graded superalgebras with superinvolution of types $M \times M^{\mathrm{sop}}$ and $Q \times Q^{\mathrm{sop}}$. Recall that in this situation the map $\kappa \colon G^{\#}/T \to \mathbb{Z}_{\geq 0}$ carries the same information as a map $G/T^+ \to \mathbb{Z}_{\geq 0}$ which, by abuse of notation, is also denoted by κ (see Subsection 2.2.1).

Definition 4.71. Let $T^+ \subseteq G$ be a finite subgroup, $\beta^+ \colon T^+ \times T^+ \to \mathbb{F}^\times$ be an alternating bicharacter let $\eta^+ \colon T^+ \to \{\pm 1\}$ be a map such that $\mathrm{d}\eta^+ = \beta^+$ and let $g_0 \in G = G \times \{\bar{0}\} \subseteq G^{\#}$. We say that a map $\kappa \colon G/T^+ \to \mathbb{Z}_{\geq 0}$ with finite support is g_0*-admissible* if

(i) $\kappa(x) = \kappa(g_0^{-1} x^{-1})$ for all $x \in G/T^+$;

(ii) if $g_0 x^2 = T^+$ and for some (and, hence, any) $g \in x$, we have $\eta^+(g_0 g^2) = -1$, then $\kappa(x)$ is even.

For $\eta^+ \coloneqq \eta \restriction_{T^+}$, it is straightforward to check that $\kappa \colon G/T^+ \to \mathbb{Z}_{\geq 0}$ is g_0-admissible if, and only if, $(\kappa, g_0) \in \mathbf{I}(T, \beta, p)_\eta^{\bar{0},+}$ (see Definition 3.49, where κ should be regarded as a function $G^{\#}/T \to \mathbb{Z}_{\geq 0}$). In other words, κ is g_0-admissible if, and only if, there is a pair $(\mathcal{U}, B)$ whose inertia is $(\eta, \kappa, g_0, 1)$ (see Definition 3.48). In this case, we say that $(\mathcal{U}, B)$ has *inertia determined by* κ.

Definition 4.72. Let T, β and t_p be as in Definition 4.67 with $T^+ \neq T$, let $g_0 \in G$ be any element and let $\kappa \colon G/T^+ \to \mathbb{Z}_{\geq 0}$ be a g_0-admissible map. Choose a graded

$\mathcal{D}$-supermodule $\mathcal{U}$ and a nondegenerate sesquilinear form $B\colon \mathcal{U}\times\mathcal{U}\to\mathcal{D}$ such that $(\mathcal{U},B)$ has inertia determined by κ. The graded superalgebra with superinvolution $E(\mathcal{D},\mathcal{U},\mathcal{B})$ will be denoted by $M^{\mathrm{ex}}(T,\beta,t_p,\kappa,g_0)$, if $t_p\in T^+$, and by $Q^{\mathrm{ex}}(T,\beta,t_p,\kappa,g_0)$, if $t_p\in T^-$. With a choice of an even graded basis for $\mathcal{U}$, $E(\mathcal{D},\mathcal{U},\mathcal{B})$ becomes $M_k(\mathcal{D})$, where $k\coloneqq|\kappa|$, endowed with the superinvolution φ given by

$$\varphi(X)\coloneqq\Phi^{-1}\varphi_0(X^{\top})\Phi,$$

for all $X\in M_k(\mathcal{D})$, where Φ is the matrix representing B with respect to the chosen basis.

Remark 4.73. Recall from Definition 4.67(b) that the fixed standard realization $(\mathcal{D},\varphi_0)$ depends on the choice of the element t_1 in item (iii). If t_1' is another such element, then, by Lemma 4.66, $t_1'\in t_1(\operatorname{rad}\beta^+)$. By Remark 4.68, t_1' gives the same map $\eta^+\colon T^+\to\{\pm1\}$ as t_1. Hence, the set of g_0-admissible κ's is the same (see Definition 4.71). Nevertheless, $M^{\mathrm{ex}}(T,\beta,t_p,\kappa,g_0)'$ obtained from t_1' may not be isomorphic to $M^{\mathrm{ex}}(T,\beta,t_p,\kappa,g_0)$ obtained from t_1. More precisely $M^{\mathrm{ex}}(T,\beta,t_p,\kappa,g_0)'\simeq M^{\mathrm{ex}}(T,\beta,t_p,\kappa,g_0)$ for $t_1'\in\{t_1,ft_pt_1\}$, and $M^{\mathrm{ex}}(T,\beta,t_p,\kappa,g_0)'\simeq M^{\mathrm{ex}}(T,\beta,t_p,\kappa,t_pg_0)$ if $t_1'\in\{ft_1,t_pt_1\}$. This follows from Remark 4.68 and Theorem 3.54.

Note that, as an ungraded superalgebra with superinvolution, $M^{\mathrm{ex}}(T,\beta,t_p,\kappa,g_0)\simeq M(n,n)\times M(n,n)^{\mathrm{sop}}$, where $n=|\kappa|\sqrt{|T|/8}$, and $Q^{\mathrm{ex}}(T,\beta,t_p,\kappa,g_0)\simeq Q(n)\times Q(n)^{\mathrm{sop}}$, where $n=|\kappa|\sqrt{|T|/4}$.

Theorem 4.74. *Suppose the superalgebra with superinvolution $M(n,n)\times M(n,n)^{\mathrm{sop}}$ (respectively, $Q(n)\times Q(n)^{\mathrm{sop}}$) is endowed with a G-grading making it graded-simple and odd. Then it is isomorphic, as a graded superalgebra with superinvolution, to $M^{\mathrm{ex}}(T,\beta,t_p,\kappa,g_0)$ (resp., $Q^{\mathrm{ex}}(T,\beta,t_p,\kappa,g_0)$) as in Definition 4.72. Moreover, the graded superalgebras with superinvolution $M^{\mathrm{ex}}(T,\beta,t_p,\kappa,g_0)$ and $M^{\mathrm{ex}}(T',\beta',t_p',\kappa',g_0')$ (resp., $Q^{\mathrm{ex}}(T,\beta,t_p,\kappa,g_0)$ and $Q^{\mathrm{ex}}(T',\beta',t_p',\kappa',g_0')$) are isomorphic if, and only if, $T=T'$, $\beta=\beta'$, $t_p=t_p'$ and there is $g\in G$ such that $\kappa'=g\cdot\kappa$ and $g_0'=g^{-2}g_0$.*

Proof. Suppose $M(n,n)\times M(n,n)^{\mathrm{sop}}$ ($Q(n)\times Q(n)^{\mathrm{sop}}$) is endowed with an odd G-grading making it graded-simple. By Theorems 3.18 and 3.37, it is isomorphic to $E(\mathcal{D},\mathcal{U},B)$, with $(\mathcal{D},\mathcal{U},B)$ as in Definition 3.51. Let $(T,\beta,p,\eta,\kappa,g_0,\delta)$ be the parameters of $(\mathcal{D},\mathcal{U},B)$ and let t_p be its η-parity element. Using Theorem 3.54 and Lemma 4.63, we can assume that η is the one from the fixed standard realization associated to

$(T, \tilde{\beta}, t_p)$. Then $E(\mathcal{D}, \mathcal{U}, B)$ is isomorphic to $M^{\text{ex}}(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}}, g_0)$ $(Q^{\text{ex}}(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}}, g_0))$ by Theorem 3.54 and Proposition 3.56.

To prove the isomorphism condition, let $\tilde{\kappa}\colon G^{\#}/T \to \mathbb{Z}_{\geq 0}$ and $\tilde{\kappa}\colon G^{\#}/T' \to \mathbb{Z}_{\geq 0}$ be the maps in terms of $G^{\#}$ corresponding to κ' and κ', respectively. By Theorem 3.54, Proposition 3.59, and Lemma 4.63, the gradings are isomorphic if, and only if, $T = T'$, $\beta = \beta'$, $t_p = t'_p$ and $(\tilde{\kappa}, g_0)$ and $(\tilde{\kappa}', g'_0)$ are in the same $\mathcal{G}$-orbit in $\mathbf{I}(T, \beta, p)^{\bar{0},+}_{\eta}$. Note that in the present case, $\mathcal{G} = \Big(\{e\} \times (G \times \{\bar{0}\})\Big) \cup \Big(\{f\} \times (G \times \{\bar{1}\})\Big)$ (see Equation (3.20)), so $\mathcal{G} = \Big(\{e\} \times (G \times \{\bar{0}\})\Big) \cup (f, \tilde{t})\Big(\{e\} \times (G \times \{\bar{0}\})\Big)$, for any $\tilde{t} \in T^-$. We claim the action by $(f, \tilde{t})$ is trivial. Indeed, $(f, \tilde{t}) \cdot (\tilde{\kappa}, g_0) = (\tilde{t} \cdot \tilde{\kappa}, f\tilde{t}^{-2} g_0)$ (see Equation (3.21)), and we have that $\tilde{t} \cdot \tilde{\kappa} = \kappa$, since $\tilde{t} \in T$, and $f\tilde{t}^{-2} = e$. Therefore, the $\mathcal{G}$-orbits coincide with the $\Big(\{e\} \times (G \times \{\bar{0}\})\Big)$ orbits. The result follows. $\square$

As done in Chapter 2 for $Q(n)$, gradings on $Q(n) \times Q(n)^{\text{sop}}$ can be easily reduced to gradings on $(Q(n) \times Q(n)^{\text{sop}})^{\bar{0}} \simeq M(n, 0) \times M(n, 0)^{\text{sop}}$. Let $(T, \tilde{\beta}, t_p)$ be as in case (c) in Definition 4.67, and let $h \in G$ be the projection of t_p onto G, i.e., $t_p = (h, \bar{1})$. Then (T^+, β^+, e) is as in case (a) and $h^2 = f$. Conversely, given (T^+, β^+, e) as in case (a) and an element $h \in G$ with $h^2 = f$, we can define $t_p := (h, \bar{1})$, $T := T^+ \cup t_p T^+$ and $\tilde{\beta}\colon T \times T \to \mathbb{F}^{\times}$ by $\tilde{\beta}(st_p^i, tt_p^j) = \beta^+(s, t)$, for all $s, t \in T^+$ and $i, j \in \mathbb{Z}$. Then $(T, \tilde{\beta}, t_p)$ is a triple as in case (c). Hence, we can reparametrize $(T, \tilde{\beta}, t_p)$ by (T^+, β^+, h). Also, $g_0 \in G$ and the g_0-admissibility condition involves only the map $\eta^+ := \eta \restriction_{T^+}$, which is characterized by:

$$\mathrm{d}\eta^+ = \beta^+,\ \eta^+ \restriction_K \text{ is associated to transposition on } \mathcal{M}, \text{ and } \eta^+(f) = -1. \tag{4.1}$$

Therefore, the graded superinvolution-simple superalgebra $Q^{\text{ex}}(T, \beta, t_p, \kappa, g_0)$ in Definition 4.72 will also be denoted by $Q^{\text{ex}}(T^+, \beta^+, h, \kappa, g_0)$, with all parameters in terms of the group G.

By Theorem 4.74, we have:

Corollary 4.75. *Suppose the superalgebra with superinvolution $Q(n) \times Q(n)^{\text{sop}}$ is endowed with a G-grading making it graded-simple. Then it is isomorphic, as a graded superalgebra with superinvolution, to $Q^{\text{ex}}(T^+, \beta^+, h, \kappa, g_0)$ as above. Moreover, the graded superalgebras with superinvolution $Q^{\text{ex}}(T^+, \beta^+, h, \kappa, g_0)$ and $Q^{\text{ex}}(T'^+, \beta'^+, h', \kappa', g'_0)$ are isomorphic if, and only if, $T^+ = T'^+$, $\beta^+ = \beta'^+$, $h = h'$ and there is $g \in G$ such that $\kappa' = g \cdot \kappa$ and $g'_0 = g^{-2} g_0$.* $\square$

This reparametrization has the following interpretation in terms of gradings. Let $(R,\varphi) \coloneqq Q^{\mathrm{ex}}(T,\beta,t_p,\kappa,g_0)$. Clearly, there is an invertible homogeneous element $\omega \in Z(\mathcal{D})^{\bar{1}}$ of degree t_p such that $\varphi_0(\omega) = \omega$. Scaling ω if necessary, we can suppose $\omega^2 = \zeta$ where $\zeta \coloneqq (1,-1) \in Z(\mathcal{D})^{\bar{0}}$. By Proposition 3.21, we can regard ω as an element of $Z(R)^{\bar{1}}$, and $\varphi(\omega) = \omega$. Then $R = R^{\bar{0}} \oplus \omega R^{\bar{1}}$. Note that $(R^{\bar{0}}, \varphi\restriction_{R^{\bar{0}}}) \simeq M^{\mathrm{ex}}(T^+,\beta^+,\kappa,0,g_0)$, where 0 denotes the identically zero map $G/T^+ \to \mathbb{Z}_{\geq 0}$. Conversely, let $(R^{\bar{0}}, \varphi\restriction_{R^{\bar{0}}}) \coloneqq M^{\mathrm{ex}}(T^+,\beta^+,\kappa,0,g_0)$ then let (R,φ) be the graded superalgebra with superinvolution defined by $R \coloneqq R^{\bar{0}} \oplus \omega R^{\bar{0}}$, where ω is a new symbol of degree $t_p \coloneqq (h,\bar{1})$ that commutes with every element in $R^{\bar{0}}$ and satisfies $\omega^2 = \zeta \in R^{\bar{0}}$ and $\varphi(\omega) = \omega$. Then $(R,\varphi) \simeq Q^{\mathrm{ex}}(T,\beta,t_p,\kappa,g_0)$.

We can also reparametrize the odd graded superalgebras with superinvolution of type $M \times M^{\mathrm{sop}}$ in terms of G. Let $M^{\mathrm{ex}}(T,\beta,t_p,\kappa,g_0)$ be as in Definition 4.72.

As done in Subsection 2.3.1, we can recover the triple (T,β,p) from T^+, β^+, an arbitrary element $t_1 \in T^-$ and the character $\chi \coloneqq \beta(t_1,\cdot) \in \widehat{T^+}$ (see Equation (2.5)). Let $K \subseteq T^+$ be the subgroup chosen in item (i) of Definition 4.67(b), and let S be the coset of $\operatorname{rad}\beta^+$ determined by K as in Lemma 4.66. We can use any representative of S as t_1. Then χ is determined by

$$\chi(K) = \chi(f) = 1 \text{ and } \chi(t_p) = -1. \tag{4.2}$$

Let C be the image of S under the projection $G^\# \to G$. By Corollary 4.59, f is the square of every element in S and, hence, of every element in C. It follows that we can recover $(T,\tilde{\beta},t_p)$ from T^+, β^+, t_p, K and the coset C.

Now let $\mathcal{M}$ be the standard realization chosen in item (ii) of Definition 4.67(b). Clearly, the choice of $t_1 \in S$ in item (iii) is equivalent to the choice of $h \in C$ via $t_1 \coloneqq (h,\bar{1})$, so we have all the data necessary to construct a graded-division superalgebra with superinvolution in terms of G only. In particular, by Remark 4.68, the map $\eta^+ \colon T^+ \to \{\pm 1\}$ is characterized by

$$\begin{gathered} \mathrm{d}\eta^+ = \beta^+,\ \eta^+\restriction_K \text{ is associated to transposition on } \mathcal{M}, \\ \eta^+(f) = -1 \text{ and } \eta^+(t_p) = 1. \end{gathered} \tag{4.3}$$

Thus, $M^{\mathrm{ex}}(T,\beta,t_p,\kappa,g_0)$ can be recovered from T^+, β^+, t_p, K, h, $\mathcal{M}$, g_0 and κ.

Conversely, let $T^+ \subseteq G$ be a 2-elementary subgroup, let $e \neq t_p \in T^+$, let $h \in G$

be such that $f := h^2 \in T^+ \setminus \langle t_p \rangle$, and let $\beta^+ \colon T^+ \times T^+ \to \{\pm 1\}$ be an alternating bicharacter such that $\operatorname{rad}\beta^+ = \langle t_p, f\rangle$. Fix a complement $K \subseteq T^+$ to $\operatorname{rad}\beta^+$ and a standard realization $\mathcal{M}$ of a matrix algebra with division grading associated to $(K, \beta^+ \restriction_{K\times K})$. Let $\chi \in \widehat{T^+}$ be defined by Equation (4.2). It is straightforward to check that $(h, \chi) \in \mathbf{O}(T^+, \beta^+)$, see Definition 2.82. Set $t_1 := (h, \bar{1})$, $T^- := t_1 T^+$ and $T := T^+ \cup T^-$, and define $\beta \colon T \times T \to \{\pm 1\}$ as in Lemma 2.83. It follows that t_p is a parity element and that $f \in \operatorname{rad}\beta$, hence, by Corollary 2.71, we have that $\operatorname{rad}\beta = \operatorname{rad}\tilde{\beta} = \langle f \rangle$. Therefore, $(T, \tilde{\beta}, t_p)$ is as in Definition 4.67(b). Further, since $\beta(t_1, K) = \chi(K) = 1$, we can use K, $\mathcal{M}$ and t_1 as the choices in items (i), (ii) and (iii).

We will denote the graded superalgebra with superinvolution $M^{\mathrm{ex}}(T, \beta, t_p, \kappa, g_0)$, constructed using the choices above, by $M^{\mathrm{ex}}(T^+, \beta^+, t_p, h, \kappa, g_0)$.

Corollary 4.76. *Suppose the superalgebra with superinvolution $M(n, n) \times M(n, n)^{\mathrm{sop}}$ is endowed with a G-grading making it graded-simple and odd. Then it is isomorphic, as a graded superalgebra with superinvolution, to $M^{\mathrm{ex}}(T^+, \beta^+, t_p, h, \kappa, g_0)$ as above. Moreover, $M^{\mathrm{ex}}(T^+, \beta^+, t_p, h, \kappa, g_0)$ and $M^{\mathrm{ex}}(T'^+, \beta'^+, t'_p, h', \kappa', g'_0)$ are isomorphic if, and only if, $T^+ = T'^+$, $\beta^+ = \beta'^+$, $t_p = t'_p$, $h' \in h(\operatorname{rad}\beta^+)$ and there is $g \in G$ such that $\kappa' = g \cdot \kappa$ and*

(i) $g'_0 = g^{-2} g_0$ *if* $h' \in \{h, f t_p h\}$;

(ii) $g'_0 = t_p g^{-2} g_0$ *if* $h' \in \{fh, t_p h\}$.

Proof. The first assertion follows immediately from Theorem 4.74. The second assertion also follows from Theorem 4.74, but we have to take into account that in the isomorphism condition there, the element t_1 was fixed for each triple $(T, \tilde{\beta}, t_p)$. Therefore, when comparing $M^{\mathrm{ex}}(T^+, \beta^+, t_p, h, \kappa, g_0)$ and $M^{\mathrm{ex}}(T^+, \beta^+, t_p, h', \kappa', g'_0)$, we have to apply Remark 4.73 to bring $t'_1 = (h', \bar{1})$ to $t_1 = (h, \bar{1})$. □

Chapter 5

Gradings on Special Linear, Orthosymplectic, Periplectic and Queer Lie Superalgebras

In this chapter we present a classification up to isomorphism of group gradings on the simple Lie superalgebras in series A, B, C, D, P and Q, except for the ones of type $A(1,1)$. We note that gradings for series B were classified in [San19], series P in [HSK19] and series Q in [BHSK17], but the techniques we have developed here allow us to recover those results. We also note that in [HSK19] there is a partial classification of gradings for series A (more precisely, the gradings of Types I, $\mathrm{I_M}$ and $\mathrm{I_Q}$ according to our present terminology, see Definitions 5.15, 5.28 and 5.29).

Section 5.1 shows that the results about gradings on superinvolution-simple associative superalgebras in Chapter 4 can be transferred to the corresponding simple Lie superalgebras: in Subsection 5.1.1, we use Theorem 3.27 to compute the well-known automorphism groups of the superinvolution-simple associative superalgebras (Proposition 5.2), and in Subsection 5.1.2 we compare them to the automorphism groups of the classical Lie superalgebras found in [Ser84, GP04], allowing us to use the tools in Section 1.2 to transfer our classification results (Corollary 5.4). It follows that gradings (and their isomorphism classes) on the simple Lie superalgebras in series B, C, D and P correspond to gradings (and their isomorphism classes) on the associative superalgebras $M(m,n)$ with appropriate superinvolutions that we used to define these Lie superalgebras in Section 0.3. In Section 5.2, we give a classification for these series (Theorems 5.7 and 5.10). For the simple Lie superalgebras in series A and Q, only some gradings come from the simple associative superalgebras $S = M(m,n)$ or $Q(n)$ used to define them, but all gradings come from the superinvolution-simple superalgebras $S \times S^{\mathrm{sop}}$. The goal of Section 5.3 is to present models for all gradings in terms of the Lie superalgebra $S^{(-)}$. In Subsection 5.3.1, we give a general procedure to construct these models, and in Subsections 5.3.2 to 5.3.4 we adapt this procedure to the Lie superalgebras of types $A(m,n)$ (for $m \neq n$), $Q(n)$ and $A(n,n)$, respectively (see Theorems 5.20, 5.26 and 5.38).

As in Chapters 3 and 4, we will assume that the group G is abelian. We can do this without loss of generality because of the next result, which is a generalization of [BZ06, Lemma 2.1] or [DM06, Proposition 1] (see also [EK13, Proposition 1.12] and its proof, which works in this more general situation, as was observed in [HSK19]).

Proposition 5.1. *Let L be a simple Lie superalgebra endowed with a G-grading. Then the subgroup of G generated by the support of L is abelian.* $\square$

5.1 Automorphism groups

In this section, we will present the groups of automorphisms of the finite dimensional simple associative superalgebras (with and without superinvolution), as well as of the classical Lie superalgebras of the 6 infinite series A, B, C, D, P and Q. The goal is to transfer the results about the classification of gradings from the associative to the Lie case (see Corollary 5.4).

The automorphism groups of the simple associative superalgebras, respectively superinvolution-simple associative superalgebras, are easy to compute, and follow from our results in Chapter 2, respectively Chapters 3 and 4. The (outer) automorphism groups of the finite dimensional simple Lie superalgebras were computed in [Ser84]. It is worth mentioning that the algebraic group schemes of automorphisms of these Lie superalgebras are presented in [GP04, Theorem 4.1], and the description there is closer to the one we present here.

5.1.1 Associative superalgebras

The automorphisms of the associative superalgebras $M(m,n)$ and $Q(n)$ are well known and can be computed from Theorem 2.27 with $G = \mathbb{Z}_2$, $\mathcal{D} = \mathcal{D}'$ and $\mathcal{U} = \mathcal{U}'$.

For the case of $M(m,n)$, we have $\mathcal{D} = \mathbb{F}$, which forces $\psi_0 = \mathrm{id}_{\mathcal{D}}$ in Theorem 2.27 and, hence, every automorphism of $M(m,n)$ is the conjugation by an invertible element in $M(m,n)^{\bar{0}} \cup M(m,n)^{\bar{1}}$, namely, ψ_1 in Theorem 2.27. We define $\mathcal{E}(m,n)$ to be the normal subgroup consisting of the conjugations by the invertible elements in $M(m,n)^{\bar{0}}$. In other words, the elements of $\mathcal{E}(m,n)$ are conjugations by matrices of the form

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix},$$

where $x \in M_m(\mathbb{F})$ and $y \in M_n(\mathbb{F})$ are invertible. Clearly,

$$\mathcal{E}(m,n) \simeq (\mathrm{GL}_m \times \mathrm{GL}_n)/\mathbb{F}^\times,$$

where $\mathbb{F}^\times$ is identified with scalar matrices. An odd invertible matrix is necessarily of the form

$$\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix},$$

where $x \in M_{m\times n}(\mathbb{F})$ and $y \in M_{n\times m}(\mathbb{F})$ are invertible and, hence, $m = n$. Therefore, $\mathrm{Aut}(M(m,n)) = \mathcal{E}(m,n)$ if $m \neq n$, and we can write $\mathrm{Aut}(M(n,n)) = \mathcal{E}(m,n) \rtimes \langle \pi \rangle$, where π is the so called *parity transpose*, which is the conjugation by

$$\Pi := \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

To compute the automorphism group of $Q(n)$, let us first consider $n = 1$. We know that $Q(1) \simeq \mathbb{F}\mathbb{Z}_2$ is a graded-division algebra, so we can apply Lemma 2.38 to conclude that there are only 2 automorphisms: the identity and the parity automorphism $\nu\colon Q(1) \to Q(1)$. Now let n be any natural number. Recall that, using Kronecker product, we can write $Q(n) \simeq M_n(\mathcal{D})$ with $\mathcal{D} := Q(1)$, where $Q(n)^i$ corresponds to $M_n(\mathcal{D}^i)$ (see the proof of Theorem 2.43). By Theorem 2.27 and Remark 2.26, an automorphism of $M_n(\mathcal{D})$ is a conjugation by a homogeneous element of $M_n(\mathcal{D})$, followed by applying an automorphism of $\mathcal{D}$ to each entry of the result. In our case, applying the parity automorphism to each entry of a matrix in $M_n(\mathcal{D})$ is the same as applying the parity automorphism to the matrix. Hence, the automorphism group of $Q(n)$ is generated by the conjugations by homogeneous elements of $Q(n)$ and the parity automorphism $\nu\colon Q(n) \to Q(n)$. Since Π is an odd element in the center of $Q(n)$, it is sufficient to take conjugations by even elements only, which are of the form

$$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix},$$

with invertible $x \in M_n(\mathbb{F})$. The parity automorphism commutes with all automorphisms, so we have $\operatorname{Aut}(Q(n)) = \operatorname{Aut}(M_n(\mathbb{F})) \times \langle \nu \rangle \simeq \mathrm{PGL}_n \times \mathbb{Z}_2$. Note that every automorphism of $Q(n)$ is of the form sInt_A for some invertible element $A \in Q(n)^{\bar{0}} \cup Q(n)^{\bar{1}}$, since the parity automorphism is sInt_Π.

We will now focus on the finite dimensional superinvolution-simple superalgebras. If $R = S \times S^{\mathrm{sop}}$, where S is either $M(m,n)$ or $Q(n)$, and $\varphi\colon R \to R$ is the exchange superinvolution, then the automorphisms of (R, φ) can be computed following the proof of Lemma 4.5. Explicitly, an automorphism of (R, φ) is either of the form

$$\forall x, y \in S, \quad \psi_\theta(x, \bar{y}) := (\theta(x), \overline{\theta(y)}), \tag{5.1}$$

where $\theta\colon S \to S$ is an automorphism, or of the form

$$\forall x, y \in S, \quad \psi_\theta(x, \bar{y}) := (\theta(y), \overline{\theta(x)}), \tag{5.2}$$

where $\theta\colon S \to S$ is a super-anti-automorphism. It is straightforward to check that Equations (5.1) and (5.2) define an isomorphism $\overline{\operatorname{Aut}}(S) \to \operatorname{Aut}(R, \varphi)$, where $\overline{\operatorname{Aut}}(S)$ is the group of automorphisms and super-anti-automorphisms of the superalgebra S. Recall that, even though there may be no superinvolution on $S \in \{M(m,n), Q(n)\}$ in general, we always have a super-anti-automorphism (for example, the queer supertranspose of Definition 0.18), so $\operatorname{Aut}(S)$ is a subgroup of index 2 in $\overline{\operatorname{Aut}}(S)$ (excluding the case of $M(1,0)$).

It remains to consider the case $(R, \varphi) = M^*(m, n, p_0)$. Let $\langle\,,\rangle\colon \mathbb{F}^{m|n} \times \mathbb{F}^{m|n} \to \mathbb{F}$ be the nondegenerate symmetric bilinear form with matrix Φ as in Definition 4.18. By Theorem 3.27, we have that every automorphism of $M^*(m, n, p_0)$ is the conjugation by an invertible matrix $A \in M(m,n)^{\bar{0}} \cup M(m,n)^{\bar{1}}$ such that

$$\exists \lambda \in \mathbb{F}^{\times},\, \forall u, v \in (\mathbb{F}^{m|n})^{\bar{0}} \cup (\mathbb{F}^{m|n})^{\bar{1}}, \quad \langle Au, Av \rangle = \lambda (-1)^{|A||u|} \langle u, v \rangle, \tag{5.3}$$

(This comes from Equation (3.9) together with Definition 3.22.) We claim that A must be even. Indeed, let $u, v \in \mathbb{F}^{m|n}$ with $|u| = \bar{0}$ and $|v| = p_0$ such that $\langle u, v\rangle \neq 0$. Then, since the form is supersymmetric, we have $\langle u, v \rangle = \langle v, u\rangle$. Comparing

$$\langle u, v \rangle = \lambda^{-1} \langle Au, Av \rangle$$

and

$$\begin{aligned} \langle v, u \rangle &= \lambda^{-1} (-1)^{|A|\, p_0} \langle Av, Au \rangle \\ &= \lambda^{-1} (-1)^{|A|\, p_0} (-1)^{(|A| + p_0)|A|} \langle Au, Av \rangle \\ &= \lambda^{-1} (-1)^{|A|} \langle Au, Av \rangle, \end{aligned}$$

we conclude that $|A| = \bar{0}$. Hence $\operatorname{Aut}(M^*(m,n,p_0)) \subseteq \mathcal{E}(m,n)$, i.e., $A = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$, where $x \in \mathrm{GL}_n$ and $y \in \mathrm{GL}_m$.

Then Equation (5.3) implies that $\langle u, \varphi(A)Av\rangle = \lambda \langle u, v \rangle$, so $\varphi(A) = \lambda A^{-1}$. Since $\mathbb{F}$ is algebraically closed and we are only interested in the conjugations by A, we can multiply A by $1/\sqrt{\lambda}$ and assume $\varphi(A) = A^{-1}$. We conclude that the group of automorphism is

$$\operatorname{Aut}(M^*(m,n,p_0)) = \frac{\{A \in M(m,n)^{\bar{0}} \mid A \text{ is invertible and } \varphi(A) = A^{-1}\}}{\{\pm I_{m+n}\}}.$$

For $M^*(m,n,\bar{0})$, we have that $x \in \mathrm{O}_m$ and $y \in \mathrm{Sp}_n$, so

$$\operatorname{Aut}(M^*(m,n,\bar{0})) \simeq \frac{\mathrm{O}_m \times \mathrm{Sp}_n}{\{\pm I_{m+n}\}}.$$

For $M(n,n,\bar{1})$, note that $\varphi \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} y^{\top} & 0 \\ 0 & x^{\top} \end{pmatrix}$, so $\varphi(A) = A^{-1}$ if, and only if,

$y = (x^\top)^{-1}$. We conclude that

$$\operatorname{Aut}(M^*(m,n,\bar{1})) \simeq \frac{\mathrm{GL}_n}{\{\pm I_n\}}.$$

To summarize:

Proposition 5.2. *The automorphism groups of the finite dimensional simple or superinvolution-simple associative superalgebras are:*

(i) $\operatorname{Aut}(M(m,n)) = \mathcal{E}(m,n) \simeq (\mathrm{GL}_m \times \mathrm{GL}_n)/\mathbb{F}^\times$, *if* $m \neq n$;

(ii) $\operatorname{Aut}(M(n,n)) = \mathcal{E}(n,n) \rtimes \langle \pi \rangle$;

(iii) $\operatorname{Aut}(Q(n)) = \operatorname{Aut}(M_n(\mathbb{F})) \times \langle \nu \rangle \simeq \mathrm{PGL}_n \times \mathbb{Z}_2$;

(iv) $\operatorname{Aut}(S \times S^{\mathrm{sop}}, \varphi) \simeq \overline{\operatorname{Aut}}(S)$, *where* S *is either* $M(m,n)$ *or* $Q(n)$, *and* φ *is the exchange superinvolution;*

(v) $\operatorname{Aut}(M^*(m,n,\bar{0})) \simeq (\mathrm{O}_m \times \mathrm{Sp}_n)/\{\pm I_{m+n}\}$;

(vi) $\operatorname{Aut}(M^*(n,n,\bar{1})) \simeq \mathrm{GL}_n/\{\pm I_n\}$. □

5.1.2 Lie superalgebras and transfer of gradings

We will now compare the automorphism groups of Proposition 5.2 with the automorphism groups of classical Lie superalgebras.

Let R be any associative superalgebra. Given an automorphism $\psi\colon R \to R$, it is clear that ψ is also an automorphism of the Lie superalgebra $R^{(-)}$. Moreover, ψ can be restricted to $R^{(1)} := [R^{(-)}, R^{(-)}]$ and, also, induces an automorphism on the quotient superalgebra $L := R^{(1)}/Z(R^{(1)})$, since an automorphism maps central elements to central elements. This gives us an algebraic group homomorphism $\operatorname{Aut}(R) \to \operatorname{Aut}(L)$.

In the case R is the associative superalgebra $M(m,n)$ (respectively, $Q(n)$), L is simple of type $A(m-1,n-1)$ (resp. $Q(n-1)$). Comparing [Ser84, Theorem 1] and [GP04, Theorem 4.1] with Proposition 5.2(i, ii, iii), we see that the homomorphism $\operatorname{Aut}(R) \to \operatorname{Aut}(L)$ is injective but not surjective. This means we cannot use Theorem 1.30 to reduce the classification of gradings on L to gradings on R.

This problem can be avoided, for almost all cases, if we consider associative superalgebras with superinvolution. If (R,φ) is one, set $L := \mathrm{Skew}(R,\varphi)^{(1)}/Z(\mathrm{Skew}(R,\varphi)^{(1)})$. The considerations above are still valid, and we have an algebraic group homomorphism $\mathrm{Aut}(R,\varphi) \to \mathrm{Aut}(L)$ given by restriction modulo the center.

If we take $R := S \times S^{\mathrm{sop}}$ with exchange superinvolution φ, where S is the associative superalgebra $M(m,n)$ (respectively, $Q(n)$), then L is simple of type $A(m-1,n-1)$ (resp. $Q(n-1)$). Indeed, $\mathrm{Skew}(R,\varphi) = \{(s,-\bar{s}) \mid s \in S\}$ and, hence, the projection $R \to S$ restricts to an isomorphism of Lie superalgebras $\mathrm{Skew}(R,\varphi) \to S^{(-)}$. Then, by Proposition 5.2(iv), the homomorphism $\mathrm{Aut}(R,\varphi) \to \mathrm{Aut}(L)$ is an isomorphism of algebraic groups, except for the case $L \simeq \mathfrak{psl}(2|2)$ of type $A(1,1)$.

The same reasoning holds for the orthosymplectic and periplectic Lie superalgebras if we take $R := M^*(m,n,p_0)$. To summarize, we have the following:

Proposition 5.3. *Let (R,φ) be a finite dimensional superinvolution-simple associative superalgebra with $R^{\bar{1}} \neq 0$, set $L := \mathrm{Skew}(R,\varphi)^{(1)}/Z(\mathrm{Skew}(R,\varphi)^{(1)})$ and assume that L is not of type $A(1,1)$. Then the map $\mathrm{Aut}(R,\varphi) \to \mathrm{Aut}(L)$, given by restriction and reduction modulo the center, is an isomorphism of algebraic groups.* □

Together with Theorem 1.30, we get:

Corollary 5.4. *Let (R,φ) and L be as in Proposition 5.3. Then every grading on L is a restriction and reduction modulo the center of a grading on (R,φ), and two gradings on L are isomorphic if, and only if, they come from isomorphic gradings on (R,φ).* □

Together with our results from Chapters 2 and 4, this gives us a classification of gradings on the Lie superalgebras in the series A, B, C, D, P and Q, except $A(1,1)$. In the following sections, we will develop this in detail.

5.2 Gradings on simple Lie superalgebras of series B, C, D and P

We will now classify the gradings on orthosymplectic and peripletic Lie superalgebras, i.e., the Lie superalgebras of the form $L := \mathrm{Skew}(R,\varphi)^{(1)}$ where $(R,\varphi) = M^*(m,n,p_0)$ (see Definition 4.18).

By Corollary 5.4, these gradings are precisely the restrictions of the gradings on $M^*(m,n,p_0)$, which are classified in Theorem 4.29. Here we will get a more explicit description of these gradings and, for the Lie superalgebras of types B and P, recover the classification results in [San19] and [HSK19], respectively.

In what follows, for each pair (T,β), where T is a finite abelian group and $\beta\colon T\times T\to\mathbb{F}^\times$ is a nondegenerate alternating bicharacter, we fix a standard realization $\mathcal{D}$ of a matrix algebra with a division grading associated to (T,β) (see Definition 2.36). Recall that this gives us a choice of elements $0\neq X_t\in\mathcal{D}_t$, for all $\bar{t}\in\overline{T}$. We also fix $\varphi_0\colon\mathcal{D}\to\mathcal{D}$ to be the transposition on $\mathcal{D}$ and $\eta\colon T\to\{\pm1\}$ to be the corresponding map, which satisfies $\mathrm{d}\eta=\beta\,(=\tilde{\beta})$.

5.2.1 Gradings on periplectic Lie superalgebras

The gradings on $P(n-1)$ correspond to the gradings on $M^*(n,n,\bar{1})$. By Theorem 4.29, if we endow $M^*(n,n,\bar{1})$ with a grading, then it becomes isomorphic to $M^*(T,\beta,\kappa_{\bar{0}},\kappa_{\bar{1}},g_0)$ as in Definition 4.28, with $|g_0|=\bar{1}$. Let $h_0\in G$ be the element such that $g_0=(h_0,\bar{1})\in G^\#$.

Recall that $(\kappa_{\bar{0}},\kappa_{\bar{1}})$ is g_0-admissible (Definition 4.27), so since $|g_0|=\bar{1}$, $\kappa_{\bar{0}}$ encodes the same information as $(\kappa_{\bar{0}},\kappa_{\bar{1}})$. More precisely, for any map $\kappa_{\bar{0}}\colon G/T\to\mathbb{Z}_{\geq0}$ with finite support, there is a unique map $\kappa_{\bar{1}}\colon G/T\to\mathbb{Z}_{\geq0}$ such that $(\kappa_{\bar{0}},\kappa_{\bar{1}})$ is g_0-admissible, namely, the one defined by $\kappa_{\bar{1}}(x):=\kappa_{\bar{0}}(h_0^{-1}x^{-1})$.

The argument above can be seen from the point of view of the graded $\mathcal{D}$-supermodule $\mathcal{U}=\mathcal{U}^{\bar{0}}\oplus\mathcal{U}^{\bar{1}}$ and the nondegenerate φ_0-sesquilinear form $B\colon\mathcal{U}\times\mathcal{U}\to\mathcal{D}$ that we have to choose in Definition 4.28. Recall that, since $\mathcal{D}$ is even, both $\mathcal{U}^{\bar{0}}$ and $\mathcal{U}^{\bar{1}}$ are G-graded right $\mathcal{D}$-supermodules. Since $g_0=\deg B$ is odd, B defines an isomorphism $(\mathcal{U}^{\bar{1}})^{[h_0]}\to(\mathcal{U}^{\bar{0}})^\star$ by $u\mapsto B(u,\cdot)$. Recall that $(\mathcal{U}^{\bar{0}})^\star$ is a right $\mathcal{D}$-module by $(f\cdot d)(u):=\varphi_0(d)f(u)$, for all $d\in\mathcal{D}$, $f\in(\mathcal{U}^{\bar{0}})^\star$ and $u\in\mathcal{U}^{\bar{0}}$ (see Section 3.1).

This gives us an explicit way to construct the pair $(\mathcal{U},B)$. Let $\gamma=(g_1,\dots,g_k)$, where $k:=|\kappa_{\bar{0}}|$, be a k-tuple of elements in G realizing $\kappa_{\bar{0}}$ (see Definition 2.18). Then define $\mathcal{U}^{\bar{0}}:=\mathcal{D}^{[g_1]}\oplus\cdots\oplus\mathcal{D}^{[g_k]}$, $\mathcal{U}^{\bar{1}}:=((\mathcal{U}^{\bar{0}})^\star)^{[h_0^{-1}]}$, $\mathcal{U}:=\mathcal{U}^{\bar{0}}\oplus\mathcal{U}^{\bar{1}}$ and $B\colon\mathcal{U}\times\mathcal{U}\to\mathcal{D}$ by

$$\forall u_{\bar{0}},u'_{\bar{0}}\in\mathcal{U}^{\bar{0}},\ u_{\bar{1}},u'_{\bar{1}}\in\mathcal{U}^{\bar{1}},\quad B(u_{\bar{0}}+u_{\bar{1}},u'_{\bar{0}}+u'_{\bar{1}}):=u_{\bar{1}}(u'_{\bar{0}})+\varphi_0(u'_{\bar{1}}(u_{\bar{0}})).$$

One can check that $\bar{B} = B$ and $(\mathcal{U}, B)$ has inertia determined by $(\kappa_{\bar{0}}, \kappa_{\bar{1}})$.

Let $\mathcal{B} = \{u_1, \dots, u_k\}$ be the canonical graded basis of $\mathcal{U}^{\bar{0}}$ (i.e., u_i has 1 in the i-th entry and zero elsewhere). Then $\mathcal{B} \cup \mathcal{B}^\star$ is a graded basis for $\mathcal{U}$. Using this basis, the matrix $\Phi \in M_{k|k}(\mathcal{D}) = M(k,k) \otimes (\mathcal{D})$ representing B is

$$\Phi = \left(\begin{array}{c|c} 0 & I_k \\ \hline I_k & 0 \end{array}\right) \otimes 1_{\mathcal{D}}.$$

As in Definition 4.28, we identify $\mathrm{End}_{\mathcal{D}}(\mathcal{U})$ with $M_{k|k}(\mathcal{D})$ and, then, with $M(n,n)$ by considering each entry in $\mathcal{D}$ as a block with entries in $\mathbb{F}$. With this identification, we have that

$$\Phi = \left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right),$$

and, hence, the superinvolution φ of Definition 4.28 becomes precisely the superinvolution of Definition 4.18 (with $p_0 = \bar{1}$).

Definition 5.5. Let $n \geq 2$, let $T \subseteq G$ be a finite 2-elementary subgroup, let $\beta\colon T \times T \to \mathbb{F}^\times$ be a nondegenerate alternating bicharacter, let $h_0 \in G$ and let $\kappa_{\bar{0}}\colon G/T \to \mathbb{Z}_{\geq 0}$ be a map with finite support such that $k\sqrt{|T|} = n+1$, where $k \coloneqq |\kappa_{\bar{0}}|$. Choose a k-tuple $\gamma_{\bar{0}} = (g_1, \dots, g_k)$ of elements in G realizing $\kappa_{\bar{0}}$, and define $\gamma_{\bar{1}} \coloneqq (h_0^{-1} g_1^{-1}, \dots, h_0^{-1} g_k^{-1})$. Consider the elementary grading on $M(k,k)$ determined by $(\gamma_{\bar{0}}, \gamma_{\bar{1}})$ (see Definition 1.4) and let Γ be the grading on $M(n+1, n+1, \bar{1})$ given by identifying it with $M(k,k) \otimes \mathcal{D}$ via Kronecker product. We define $\Gamma_P(T, \beta, \kappa_{\bar{0}}, h_0)$ to be the restriction of Γ to $P(n)$.

Remark 5.6. In [HSK19], the k-tuple $\gamma \coloneqq \gamma_{\bar{0}}$ is taken as a parameter instead of the map $\kappa_{\bar{0}}$, which is denoted by $\Xi(\gamma)$ there, and our element h_0 corresponds to g_0^{-1} there.

The following result is a consequence of Theorem 4.29 (compare with [HSK19, Theorem 6.9]).

Theorem 5.7. *Let $n \geq 2$. Every grading on the simple Lie superalgebra $P(n)$ is isomorphic to some $\Gamma_P(T, \beta, \kappa_{\bar{0}}, h_0)$ as in Definition 5.5. Moreover, $\Gamma_P(T, \beta, \kappa_{\bar{0}}, h_0) \simeq \Gamma_P(T', \beta', \kappa'_{\bar{0}}, h'_0)$ if, and only if, $T = T'$, $\beta = \beta'$ and there is an element $g \in G$ such that $\kappa'_{\bar{0}} = g \cdot \kappa_{\bar{0}}$ and $h'_0 = g^{-2} h_0$.* □

5.2.2 Gradings on orthosymplectic Lie superalgebras

The gradings on $\mathfrak{osp}(m|n)$ correspond to the gradings on $M^*(m,n,\bar{0})$. By Theorem 4.29, if we endow $M^*(m,n,\bar{0})$ with a grading, then it becomes isomorphic to $M^*(T,\beta,\kappa_{\bar{0}},\kappa_{\bar{1}},g_0)$ as in Definition 4.28, with $|g_0|=\bar{0}$.

Unlike in the case of Subsection 5.2.1, our construction of the pair $(\mathcal{U},B)$ will be closer to the one in Subsection 3.6.2. Let $\xi\colon G/T\to G$ be a set-theoretic section of the natural homomorphism, and let $\leq$ be a total order on the set G/T with no elements between x and $g_0^{-1}x^{-1}$. Changing ξ if necessary, we may assume that $\xi(g_0^{-1}x^{-1})=g_0^{-1}\xi(x)^{-1}$ if $x<g_0^{-1}x^{-1}$. For each $i\in\mathbb{Z}_2$, set $k_i:=|\kappa_i|$, and construct tuples $\gamma_{\bar{0}}=(g_1,\dots,g_{k_{\bar{0}}})$ and $\gamma_{\bar{1}}=(h_1,\dots,h_{k_{\bar{1}}})$ realizing $\kappa_{\bar{0}}$ and $\kappa_{\bar{1}}$, respectively, according to ξ and $\leq$ (Definition 2.20). Then define $\mathcal{U}^{\bar{0}}:=\mathcal{D}^{[g_1]}\oplus\cdots\oplus\mathcal{D}^{[g_{k_{\bar{0}}}]}$, $\mathcal{U}^{\bar{1}}:=\mathcal{D}^{[h_1]}\oplus\cdots\oplus\mathcal{D}^{[h_{k_{\bar{1}}}]}$ and $\mathcal{U}:=\mathcal{U}^{\bar{0}}\oplus\mathcal{U}^{\bar{1}}$.

To define the φ_0-sesquilinear form B, we will use the following:

Definition 5.8. Let $i\in\mathbb{Z}_2$ and $x\in G/T$. If $g_0x^2=T$, we put $t:=g_0\xi(x)^2\in T$ and define $\Phi(i,x)$ to be the following $\kappa_i(x)\times\kappa_i(x)$-matrix with entries in $\mathcal{D}$:

(i) $I_{\kappa_i(x)}\otimes X_t$ if $(-1)^i\eta(t_x)=+1$;

(ii) $J_{\kappa_i(x)}\otimes X_t$, where $J_{\kappa_i(x)}:=\begin{pmatrix}0 & I_{\kappa_i(x)/2}\\ -I_{\kappa_i(x)/2} & 0\end{pmatrix}$, if $(-1)^i\eta(t_x)=-1$ (recall that, in this case, $\kappa_i(x)$ is even by Definition 4.27).

If $g_0x^2\neq T$, we define $\Phi(i,x)$ to be the following $2\kappa_i(x)\times 2\kappa_i(x)$-matrix with entries in $\mathcal{D}$:

(iii) $\begin{pmatrix}0 & I_{\kappa_i(x)}\\ (-1)^iI_{\kappa_i(x)} & 0\end{pmatrix}\otimes 1_{\mathcal{D}}$.

Let $\mathcal{B}=\{u_1,\dots,u_{k_{\bar{0}}+k_{\bar{1}}}\}$ be the canonical $G^{\#}$-graded basis of $\mathcal{U}$, and let $x_1<\cdots<x_{\ell_{\bar{0}}}$ be the elements of $\{x\in\operatorname{supp}\kappa_{\bar{0}}\mid x\leq g_0^{-1}x^{-1}\}$ and, similarly, let $y_1<\ldots<y_{\ell_{\bar{1}}}$ be the elements of $\{y\in\operatorname{supp}\kappa_{\bar{1}}\mid y\leq g_0^{-1}y^{-1}\}$.

Let $B\colon\mathcal{U}\times\mathcal{U}\to\mathcal{D}$ be the nondegenerate φ_0-sesquilinear form represented by the following matrix $\Phi \in M_{k_{\bar{0}}|k_{\bar{1}}}(\mathcal{D})$:

$$\Phi := \left(\begin{array}{ccc|ccc} \Phi(\bar{0}, x_1) & & & & & \\ & \ddots & & & 0 & \\ & & \Phi(\bar{0}, x_{\ell_{\bar{0}}}) & & & \\ \hline & & & \Phi(\bar{1}, y_1) & & \\ & 0 & & & \ddots & \\ & & & & & \Phi(\bar{1}, y_{\ell_{\bar{1}}}) \end{array}\right). \tag{5.4}$$

Then $\bar{B} = B$ and $(\mathcal{U}, B)$ has inertia determined by $(\kappa_{\bar{0}}, \kappa_{\bar{1}})$.

As in Definition 4.28, we use the graded basis $\mathcal{B}$ to identify $\mathrm{End}_{\mathcal{D}}(\mathcal{U})$ with $M_{k_{\bar{0}}|k_{\bar{1}}}(\mathcal{D})$ and, then, consider each entry in $\mathcal{D}$ as a block matrix with entries in $\mathbb{F}$, so X and Φ can be seen as elements of $M(m, n)$, and

$$\forall X \in M(m, n), \quad \varphi(X) = \Phi^{-1} X^{s\top} \Phi. \tag{5.5}$$

Note that, unlike in the case $|g_0| = \bar{1}$ (Subsection 5.2.1), this φ does not necessarily correspond to the superinvolution of Definition 4.18 (with $p_0 = \bar{0}$). Nevertheless, disregarding the G-grading, $(M(m, n), \varphi)$ must be isomorphic to $M^*(m, n, \bar{0})$ by Proposition 4.24.

Definition 5.9. Let $T \subseteq G$ be a finite 2-elementary subgroup, let $\beta\colon T \times T \to \mathbb{F}^\times$ be a nondegenerate alternating bicharacter, let $g_0 \in G = G \times \{\bar{0}\}$ and let $\kappa_{\bar{0}}, \kappa_{\bar{1}}\colon G/T \to \mathbb{Z}_{\geq 0}$ be g_0-admissible maps. Set $k_i := |\kappa_i|$, $i \in \mathbb{Z}_2$, and let γ_i the k_i-tuple according to ξ and $\leq$ (Definition 2.20). Consider the elementary grading on $M(k_{\bar{0}}, k_{\bar{1}})$ determined by $(\gamma_{\bar{0}}, \gamma_{\bar{1}})$ (see Definition 1.4), and let $R = M(m, n)$, where $m := k_{\bar{0}}\sqrt{|T|}$ and $n := k_{\bar{1}}\sqrt{|T|}$, be the graded superalgebra with grading given by identifying it with $M(k_{\bar{0}}, k_{\bar{1}}) \otimes \mathcal{D}$ via Kronecker product. Consider on R the superivolution φ defined by Equation (5.5), where Φ is the matrix given by Equation (5.4). We define $\mathfrak{osp}(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}}, g_0)$ to be the graded Lie superalgebra $\mathrm{Skew}(R, \varphi)$.

Note that, disregarding the grading, $\mathfrak{osp}(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}}, g_0)$ is isomorphic to $\mathfrak{osp}(m|n)$. The following result is a consequence of Theorem 4.29, since (R, φ) in the definition above is isomorphic to $M^*(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}}, g_0)$ in Definition 4.28:

Theorem 5.10. *Let L be an orthosymplectic Lie superalgebra endowed with a G-grading. Then L is isomorphic to some $\mathfrak{osp}(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}}, g_0)$ as in Definition 5.9.*

Moreover, $\mathfrak{osp}(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}}, g_0) \simeq \mathfrak{osp}(T', \beta', \kappa'_{\bar{0}}, \kappa'_{\bar{1}}, g'_0)$ *if, and only if,* $T = T'$, $\beta = \beta'$ *and there is an element* $g \in G$ *such that* $\kappa'_{\bar{0}} = g \cdot \kappa_{\bar{0}}$, $\kappa'_{\bar{1}} = g \cdot \kappa_{\bar{1}}$ *and* $g'_0 = g^{-2} g_0$. $\square$

Recall that the Lie superalgebras $\mathfrak{osp}(m|n)$ are separated into series B, C and D: the ones for which m is odd constitute series B, the ones for which $m = 2$ constitute series C and the remaining ones constitute series D (see Subsection 0.3.2). The restriction on the values of m allows us to simplify the classification of G-gradings on the Lie superalgebras of series B.

Consider a graded Lie superalgebra $\mathfrak{osp}(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}}, g_0)$ as in Definition 5.9 and, as before, set $m := k_{\bar{0}}\sqrt{|T|}$ and $n := k_{\bar{1}}\sqrt{|T|}$.

Assume m odd. The group T is 2-elementary, so $\sqrt{|T|}$ is a power of 2. It follows that T must be the trivial group and, hence, $m = k_{\bar{0}}$. We will identify G/T with G and, hence, consider G to be the domain of $\kappa_{\bar{0}}$ and $\kappa_{\bar{1}}$.

We claim that the element $g_0 \in G$ must be a square. Indeed, suppose $g_0 \neq g^2$, for all $g \in G$. Then $g^{-1} \neq g_0^{-1} g$ and, since $(\kappa_{\bar{0}}, \kappa_{\bar{1}})$ is g_0-admissible, $\kappa(g^{-1}) = \kappa(g_0^{-1} g)$. Therefore $k_{\bar{0}} = |\kappa_{\bar{0}}| = \sum_{g \in G} \kappa_{\bar{0}}(g)$ is an even number, a contradiction. By Theorem 5.10, $\mathfrak{osp}(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}}, g_0) \simeq \mathfrak{osp}(T, \beta, g \cdot \kappa_{\bar{0}}, g \cdot \kappa_{\bar{1}}, e)$, where $g^2 = g_0$. In other words, we can restrict ourselves to the case where $g_0 = e$ and, hence, $(\kappa_{\bar{0}}, \kappa_{\bar{1}})$ is e-admissible.

For every e-admissible pair $(\kappa_{\bar{0}}, \kappa_{\bar{1}})$, we define $\Gamma_B(\kappa_{\bar{0}}, \kappa_{\bar{1}})$ to be the grading on $\mathfrak{osp}(m|n)$ given by the isomorphism with $\mathfrak{osp}(\{e\}, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}}, e)$, where β is the trivial bicharacter on $T = \{e\}$. In [San19], this corresponds to Definition 4.4.3, and the following result corresponds to Theorems 4.4.4 and 4.4.6:

Corollary 5.11. *Let* $m \geq 0$ *and* $n > 0$. *Every grading on* $B(m, n) = \mathfrak{osp}(2m + 1|2n)$ *is isomorphic to some* $\Gamma_B(\kappa_{\bar{0}}, \kappa_{\bar{1}})$. *Moreover, two gradings* $\Gamma_B(\kappa_{\bar{0}}, \kappa_{\bar{1}})$ *and* $\Gamma_B(\kappa'_{\bar{0}}, \kappa'_{\bar{1}})$ *are isomorphic if, and only if, there is an element* $g \in G$ *such that* $g^2 = e$, $\kappa'_{\bar{0}} = g \cdot \kappa_{\bar{0}}$ *and* $\kappa'_{\bar{1}} = g \cdot \kappa_{\bar{1}}$. $\square$

We can define gradings on the Lie superalgebras in the series C and D in a similar fashion. If $m = 2$, we can denote by $\Gamma_C(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}}, g_0)$ the grading on $\mathfrak{osp}(2|n)$ given by the isomorphism with $\mathfrak{osp}(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}}, g_0)$. If $m > 2$ is even, we can define $\Gamma_D(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}}, g_0)$ analogously. However, in these cases, we cannot simplify the parameters of these gradings, as we have done for series B.

5.3 Gradings on simple Lie superalgebras of series A and Q

By the discussion in Section 5.1 (Corollary 5.4), we have a classification of G-gradings on the Lie superalgebras of series A and Q in terms of the model $\mathrm{Skew}(R, \varphi)$, where $R = S \times S^{\mathrm{sop}}$ and φ is the exchange superinvolution. We want to describe these gradings in terms of the original model, $S^{(-)}$, where S is either $M(m, n)$ or $Q(n)$ for some positive integers m and n. We use an approach similar to [BKR18, Appendix].

We will also express the parameters of the grading in terms of the group G rather than $G^{\#} = G \times \mathbb{Z}_2$.

5.3.1 Undoubling

Let L be a finite dimensional simple Lie superalgebra in the series A or Q, and suppose L is not of type $A(1,1)$. By definition, we have that $L = S^{(1)}/Z(S^{(1)})$. On the other hand, Corollary 5.4 allows us to classify the gradings on L by considering its isomorphic copy $\tilde{L} \coloneqq \mathrm{Skew}(R, \varphi)^{(1)}/Z(\mathrm{Skew}(R, \varphi)^{(1)})$, where $R = S \times S^{\mathrm{sop}}$ and φ is the exchange superinvolution: the gradings (and their isomorphism classes) on $\tilde{L}$ are in bijection with the gradings (and their isomorphism classes) on (R, φ). The goal of this subsection is to translate the classification of gradings on L from (R, φ) to S. Recall that the isomorphism $\tilde{L} \to L$ is induced by the projection $R \to S$.

Definition 5.12. A grading on L is said to be of *Type I* if it is obtained by restriction and reduction modulo the center from a (unique) grading on S. Otherwise, the grading on L is said to be of *Type II*.

As recalled above, all gradings on $\tilde{L} \simeq L$ come from gradings on (R, φ). Type I and Type II gradings can be distinguished in this model as follows. Recall that gradings on (R, φ) were divided in two classes: those that make R graded-simple, classified in Theorems 4.70 and 4.74, and those that do not, classified in Theorems 4.55 and 4.56. We claim that these classes correspond to the Type II and Type I gradings on $\tilde{L}$, respectively. Indeed, consider a Type I grading on L and the corresponding grading on S. Then (R, φ) is naturally graded so that $S \times \{0\}$ and $\{0\} \times S^{\mathrm{sop}}$ are graded superideals, and the projection $R \to S$ is a homomorphism of graded superalgebras.

This grading on (R,φ) induces a grading on $\tilde{L}$ and the isomorphism of Lie superalgebras $\tilde{L} \to L$ preserves degrees. This proves that the gradings on $\tilde{L}$ corresponding to Type I gradings on L (under the isomorphism $\tilde{L} \to L$) come from the gradings on R that do not make it graded-simple. By Proposition 4.6, every grading on (R,φ) such that R is not graded-simple is of this form, so the converse follows.

By definition, Type I gradings are described in terms of S, so we will now focus on Type II gradings.

Suppose (R,φ) is endowed with a grading Γ making R graded-simple. By Theorem 3.54, there is a triple $(\mathcal{D},\mathcal{U},B)$ with parameters $(T,\beta,p,\eta,\kappa,g_0,\delta)$ as in Definition 3.51 such that $(R,\varphi) \simeq E(\mathcal{D},\mathcal{U},B)$. Recall that, in this case, $\operatorname{rad}\tilde{\beta} = \langle f \rangle$ for an element $f \in T^+$ of order 2 such that $\eta(f) = -1$ (Proposition 4.57).

Set $\overline{G} = G/\langle f \rangle$, let $\pi\colon G \to \overline{G}$ be the natural homomorphism and write $\bar{g} \coloneqq \pi(g)$, for all $g \in G$. By Proposition 4.53, ${}^{\pi}R$ is the sum of two graded-simple superideals, $S \times \{0\}$ and $\{0\} \times S^{\mathrm{sop}}$, so the restriction of ${}^{\pi}\Gamma$ to $S \simeq S \times \{0\}$ induces a $\overline{G}$-grading on L of Type I.

To recover Γ from its coarsening ${}^{\pi}\Gamma$, fix a character $\chi \in \widehat{G^{\#}}$ such that $\chi(f) = -1$ (which exists since $\mathbb{F}$ is algebraically closed) and let $\psi\colon R \to R$ be the automorphism given by the action of χ, i.e., $\psi(r) \coloneqq \chi(g)r$ for every $r \in R_g$. Clearly, ψ restricts to $R_{\bar{g}} = R_g \oplus R_{gf}$ and acts as the multiplication by $\chi(g)$ on R_g and by $\chi(gf) = -\chi(g)$ on R_{gf}. Hence:

$$R_g = \{r \in R_{\bar{g}} \mid \psi(r) = \chi(g)r\}. \tag{5.6}$$

Let $\zeta \coloneqq (1,-1) \in Z(R)^{\bar{0}}$. We have that $\varphi(\zeta) = -\zeta$ and, by Proposition 3.21, ζ is homogeneous of degree f with respect to Γ. Let $\theta\colon R \to R$ be the super-anti-automorphism defined by $\theta \coloneqq \varphi\psi = \psi\varphi$. By the definition of ψ, $\psi(\zeta) = -\zeta$ and, hence, $\theta(\zeta) = \zeta$. It follows that $\theta(1,0) = \theta\left(\frac{1+\zeta}{2}\right) = (1,0)$, so $\theta(S\times\{0\}) = \theta((1,0)R) = S\times\{0\}$. Hence, θ restricts to a super-anti-automorphism on S. Then, it is straightforward to see that

$$\psi(s_1,\overline{s_2}) = (\theta(s_2),\overline{\theta(s_1)}), \tag{5.7}$$

for all $s_1,s_2 \in S$. Since $\operatorname{Skew}(R,\varphi) = \{(s,-\bar{s}) \mid s \in S\}$, combining Equations (5.6) and (5.7), we get that the Type II grading on L corresponding to Γ can be recovered from the restriction of ${}^{\pi}\Gamma$ to S as follows: we define a G-grading $S^{(-)} = \bigoplus_{g\in G} S^{(-)}_g$ by setting

$$\forall g \in G, \quad S_g^{(-)} \coloneqq \{s \in S_{\bar g} \mid \theta(s) = -\chi(g)s\}, \tag{5.8}$$

and induce a G-grading on $L = S^{(1)}/Z(S^{(1)})$. Hence, a Type II grading on L can be described in terms of a $\overline{G}$-grading on S and a super-anti-automorphism on S.

Definition 5.13. We will call L endowed with the G-grading constructed above the *undoubled model of* $\mathrm{Skew}(R,\varphi)/Z(\mathrm{Skew}(R,\varphi))$, or simply the *undoubled model* if (R,φ) is fixed in the context.

Our next goal is to describe the undoubled model without reference to (R,φ). First, we need the parameters of S endowed with the restriction of ${}^{\pi}\Gamma$. By Proposition 4.53, $S \simeq E(\overline{T}, \bar\beta, \bar p, \kappa)$, where $\overline{T} \coloneqq T/\langle f\rangle$, $\bar\beta$ and $\bar p$ are the maps induced by β and p on $\overline{T}$, and $\kappa\colon G^{\#}/T \to \mathbb{Z}_{\geq 0}$ is seen as a map $\kappa\colon \overline{G}^{\#}/\overline{T} \to \mathbb{Z}_{\geq 0}$ via the canonical isomorphism $G^{\#}/T \simeq \overline{G}^{\#}/\overline{T}$.

It remains to describe the super-anti-automorphism $\theta\colon S \to S$. Let us write $\theta\colon R \to R$ in matrix terms. Let $\mathcal{B} = \{u_1, \dots, u_k\}$ be a G-graded basis of $\mathcal{U}$, following Convention 2.53, and use it to identify $\mathrm{End}_{\mathcal{D}}(\mathcal{U})$ with $M_k(\mathcal{D})$. We will also assume that B is even if $\mathcal{D}$ is odd (Remark 3.20). Set $g_i \coloneqq \deg u_i$. By Proposition 3.32, we have $\varphi(X) = \Phi^{-1}\varphi_0(X)^{s\top}\Phi$, for all $X \in M_k(\mathcal{D})$, where $\Phi_{ij} = B(u_i, u_j)$. Also, given $i,j \in \{1,\dots,k\}$ and $0 \neq d \in \mathcal{D}_t$, $t \in T$, we have $\psi(E_{ij}(d)) = \chi(g_i t g_j^{-1})E_{ij}(d) = \chi(g_i)\chi(t)\chi(g_j)^{-1}E_{ij}(d)$. Let $\Lambda \in M_k(\mathcal{D})$ be the diagonal matrix with $\Lambda_{ii} = \chi(g_i)$ for all $i \in \{1,\dots,k\}$, and let ψ_0 denote the action of χ on $\mathcal{D}$. Then $\psi(X) = \Lambda\psi_0(X)\Lambda^{-1}$, for all $X \in M_k(\mathcal{D})$. Define $\theta_0 \coloneqq \psi_0\varphi_0 = \varphi_0\psi_0$. Then:

$$\forall X \in M_k(\mathcal{D}), \quad \theta(X) = (\Lambda\Phi)^{-1}\,\theta_0(X)^{s\top}\,(\Lambda\Phi). \tag{5.9}$$

Clearly, θ_0 is a super-anti-automorphism on $\mathcal{D}$ associated to the map $\mu \coloneqq \eta\chi$. Note that $\mu(f) = 1$ and, hence, we can see μ as a map $\bar\mu\colon \overline{T} \to \mathbb{F}^\times$. Indeed, since $f \in \ker\tilde\beta$, we have

$$\forall t \in T, \quad \mu(tf) = \tilde\beta(t,f)\mu(t)\mu(f) = \mu(t)\mu(f) = \mu(t),$$

so the map $\bar\mu\colon \overline{T} \to \mathbb{F}^\times$ given by

$$\forall \bar t \in \overline{T}, \quad \bar\mu(\bar t) \coloneqq \eta(t)\chi(t) \tag{5.10}$$

is well-defined. Recall, from Corollary 4.59, that T^+ is an elementary 2-group and for every $t \in T^-$, $t^2 = f$. Hence, χ takes values ± 1 on T^+ and $\pm\mathbf{i}$ on T^-. By Lemma 4.44, $\bar{\mu}$ is a quadratic map on $\overline{T}$.

Let $\epsilon := (1, 0) \in Z(R)^{\bar{0}}$ be the identity element of S, and consider it also as an element of $Z(\mathcal{D})^{\bar{0}}$ using Proposition 3.21. Following the proof of Proposition 4.53, with $\mathcal{D}$ playing the role of $\mathcal{E}$ and $\mathcal{U}$ playing the role of $\mathcal{V}$, we have that $S \simeq \operatorname{End}_{\overline{\mathcal{D}}}(\bar{\mathcal{U}})$, where $\overline{\mathcal{D}} := \mathcal{D}\epsilon$ and $\overline{\mathcal{U}} := \mathcal{U}\epsilon$, and that $\mathcal{B}\epsilon := \{u_1\epsilon, \ldots, u_k\epsilon\}$ is a $\overline{G}^{\#}$-graded $\overline{\mathcal{D}}$-basis of $\overline{\mathcal{U}}$. Using this basis, we can identify $\operatorname{End}_{\overline{\mathcal{D}}}(\bar{\mathcal{U}})$ with $M_k(\overline{\mathcal{D}})$.

By definition, if $Y \in M_k(\mathcal{D})$ is the matrix representing an operator $r \in R$ with respect to the basis $\mathcal{B}$, then $ru_j = \sum_{i=1}^k u_i Y_{ij}$, for all $1 \le j \le k$. It follows that

$$(\epsilon r)(u_j\epsilon) = r(u_j)\epsilon = \sum_i^k u_i Y_{ij}\epsilon = \sum_i^k (u_i\epsilon) Y_{ij}\epsilon,$$

i.e., $X \in M_k(\overline{\mathcal{D}})$, given by $X_{ij} := Y_{ij}\epsilon$, is the matrix representing $\epsilon r \in S$ with respect the basis $\mathcal{B}\epsilon$. From Equation (5.9), we now get that

$$\forall X \in M_k(\overline{\mathcal{D}}), \quad \theta(X) = \Theta^{-1}\, \theta_0(X)^{s\top}\, \Theta, \tag{5.11}$$

where

$$\Theta := \Lambda\Phi\epsilon \in M_k(\overline{\mathcal{D}}) \tag{5.12}$$

or, equivalently, $\Theta_{ij} = \chi(g_i)B(u_i, u_j)\epsilon$, and θ_0 is the super-anti-automorphism on $\overline{\mathcal{D}}$ associated to $\bar{\mu}$.

Finally, we note that the refinement obtained in Equation (5.8), from θ given by Equation (5.11), is determined by the values of χ on the subgroup $T \subseteq G^{\#}$.

Lemma 5.14. *Let $\tilde{\chi} \in \widehat{G^{\#}}$ be a character such that $\chi \restriction_T = \tilde{\chi} \restriction_T$, let $\tilde{\psi}\colon R \to R$ be the action by $\tilde{\chi}$, and set $\tilde{\theta} := \tilde{\psi}\varphi = \varphi\tilde{\psi}$. Then, for every $s \in S_{\bar{g}}$, $\tilde{\theta}(s) = -\tilde{\chi}(g)s$ if, and only if, $\theta(s) = -\chi(g)s$.*

Proof. Set $\sigma := \tilde{\chi}\chi^{-1}$, so $\tilde{\chi} = \sigma\chi$. Then σ is a character on $G^{\#}$ with $\sigma(T) = 1$ and, in particular, can be seen as a character on $G^{\#}/\langle f \rangle$. Every element in $S_{\bar{g}}$ is a sum of elements of the form $E_{ij}(d)$, where $1 \le i, j \le k$ and $d \in \overline{\mathcal{D}}_{\bar{t}}$, such that $\bar{g}_i \bar{t} \bar{g}_j^{-1} = \bar{g}$. By

Equation (5.11),

$$\begin{aligned}\tilde{\theta}(E_{ij}(d)) &= \sigma(g_i)\sigma(g_j^{-1})\theta(E_{ij}(d)) = \sigma(\bar{g}_i)\sigma(\bar{g}_j^{-1})\theta(E_{ij}(d)) \\ &= \sigma(\bar{t}^{-1}\bar{g})\theta(E_{ij}(d)) = \sigma(\bar{g})\theta(E_{ij}(d)).\end{aligned}$$

We conclude that $\tilde{\theta}(s) = \sigma(\bar{g})\theta(s)$, for all $s \in S_{\bar{g}}$. $\square$

To summarize, once $\chi \in \widehat{T}$ is fixed, the undoubled model is determined by the $\overline{G}$-graded superalgebra $M_k(\overline{\mathcal{D}})$ with parameters given by Proposition 4.53, the map $\bar{\mu}\colon \overline{T} \to \mathbb{F}^\times$ defined by Equation (5.10), which is associated to a super-anti-automorphism θ_0 on $\overline{\mathcal{D}}$, and the matrix $\Theta \in M_k(\overline{\mathcal{D}})$ defined by Equation (5.12), which is used to define the super-anti-automorphism θ on $M_k(\overline{\mathcal{D}})$ by Equation (5.11).

In the following subsections, we specialize these considerations to the cases of $A(m,n)$ and $Q(n)$. As in Section 5.2, we fix a standard realization of a matrix algebra with a division grading associated to each finite abelian group with a nondegenerate alternating bicharacter (see Definition 2.36).

5.3.2 Gradings on $A(m,n)$ for $m \neq n$

We now discuss the case where $S = M(m+1, n+1)$ and $m \neq n$, so $L = \mathfrak{sl}(m+1|n+1) = S^{(1)}$ is a simple Lie superalgebra of type $A(m,n)$. (As seen before, in this case $Z(S^{(1)}) = \{0\}$.)

We first parametrize the Type I gradings. Since $m \neq n$, by Lemma 2.51, every grading on S is even in this case. In particular, $\kappa\colon G^\#/T \to \mathbb{Z}_{\geq 0}$ corresponds to a pair $(\kappa_{\bar{0}}, \kappa_{\bar{1}})$ (see Subsection 2.2.1).

In the next definition, we allow the possibility of $m = n$ for future reference (in Subsection 5.3.4).

Definition 5.15. Let $m, n \in \mathbb{Z}_{\geq 0}$, not both zero, $T \subseteq G$ be a finite subgroup, $\beta\colon T \times T \to \mathbb{F}^\times$ be a nondegenerate alternating bicharacter, and $\kappa_{\bar{0}}, \kappa_{\bar{1}}\colon G/T \to \mathbb{Z}_{\geq 0}$ be maps with finite support such that $m+1 = |\kappa_{\bar{0}}|\sqrt{|T|/2}$ and $n+1 = |\kappa_{\bar{1}}|\sqrt{|T|/2}$. We will denote by $\Gamma_A^{(\mathrm{I})}(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}})$ the restriction to $S^{(1)}$ of the grading $\Gamma_M(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}})$ on S (see Definition 2.75).

Parametrizing Type II gradings is more involved. As we have seen in Subsection 5.3.1, those correspond to gradings on (R,φ) making R graded-simple, where $R \coloneqq S \times S^{\mathrm{sop}}$ and φ is the exchange superinvolution. Again, since $m \neq n$, these gradings on R are even (see Corollary 4.58). Hence, by Theorem 4.70, a Type II grading on L corresponds to a grading on (R,φ) making it isomorphic to $M^{\mathrm{ex}}(T,\beta,\kappa_{\bar 0},\kappa_{\bar 1},g_0)$ (Definition 4.69), where $T \subseteq G$ is a finite 2-elementary subgroup, $\beta\colon T \times T \to \mathbb{F}^\times$ is an alternating bicharacter with $\operatorname{rad}\beta = \langle f\rangle$ for some $e \neq f \in T$, g_0 is an element in $G^\#$, and $\kappa_{\bar 0},\kappa_{\bar 1}\colon G/T \to \mathbb{Z}_{\geq 0}$ are g_0-admissible maps (see Definition 4.27) such that $|\kappa_{\bar 0}|\sqrt{|T|/2} = m+1$ and $|\kappa_{\bar 1}|\sqrt{|T|/2} = n+1$. Since $m \neq n$, the g_0-admissibility implies that $g_0 \in G$. Recall that the graded algebra $M^{\mathrm{ex}}(T,\beta,\kappa_{\bar 0},\kappa_{\bar 1},g_0)$ corresponds to $(\eta,\kappa,g_0,\delta) \in \mathbf{I}(T,\beta,p)$, where $\eta\colon T \to \{\pm 1\}$ is fixed, $p\colon T \to \mathbb{Z}_2$ is the trivial homomorphism, and $\delta = 1$. The map η was determined by fixing a standard realization $\mathcal{D}$ for (T,β,e) (see Definition 4.67), namely, η is the map associated to the superinvolution $\varphi_{\mathcal{C}} \otimes \varphi_{\mathcal{M}}$ on $\mathcal{D}$.

We will use the parameters $(T,\beta,\kappa_{\bar 0},\kappa_{\bar 1},g_0)$ to construct a representative for the Type II grading in terms of the undoubled model, i.e., we will construct a $\overline{G}$-grading on S, where $\overline{G} \coloneqq G/\langle f\rangle$, and a super-anti-automorphism $\theta\colon S \to S$, as in Subsection 5.3.1. As in that subsection, let $\pi\colon G \to \overline{G}$ denote the natural homomorphism, set $\overline{T} \coloneqq T/\langle f\rangle$, let $\bar\beta$ be the (nondegenerate) bicharacter on $\overline{T}$ induced by β, and consider $\kappa_{\bar 0}$ and $\kappa_{\bar 1}$ as maps defined on $\overline{G}/\overline{T} \simeq G/T$.

Let $\overline{\mathcal{D}}$ be the chosen standard realization associated to $(\overline{T},\bar\beta)$, and let $\bar\mu\colon \overline{T} \to \mathbb{F}^\times$ be the map associated to the transposition map.

Fix a complement K for $\langle f\rangle$ in T, i.e., a subgroup $K \subseteq T$ such that $T = K \times \langle f\rangle$ (it can be done, since T is a elementary 2-group). Let $\chi\colon T \to \mathbb{F}^\times$ be the character defined by $\chi(K) = 1$ and $\chi(f) = -1$, and set $\mu \coloneqq \bar\mu \circ \pi \restriction_T$. Then set

$$\forall t \in T, \quad \eta(t) \coloneqq \mu(t)\chi^{-1}(t). \tag{5.13}$$

Clearly, $\mathrm{d}\eta = \beta$. Note that this definition agrees with Definition 4.67(a) (see the proof of Proposition 5.19).

Extend χ to G. It remains to define a graded right $\overline{\mathcal{D}}$-supermodule $\overline{\mathcal{U}}$ and the matrix Θ. The following is analogous to the construction in Subsection 5.2.2.

Let $\xi\colon G/T \to G$ be a set-theoretic section of the natural homomorphism, and let

$\leq$ be a total order on the set $G/T \simeq \overline{G}/\overline{T}$ with no elements between x and $\bar{g}_0^{-1}x^{-1}$. Changing ξ if necessary, we may assume that $\xi(g_0^{-1}x^{-1}) = g_0^{-1}\xi(x)^{-1}$ if $x < g_0^{-1}x^{-1}$. For each $i \in \mathbb{Z}_2$, set $k_i := |\kappa_i|$, let γ_i be the k_i-tuple of elements in G realizing κ_i according to ξ and $\leq$ (see Definition 2.20), and let $\bar{\gamma}_i$ be the tuple of elements in $\overline{G}$ consisting of the images under $\pi\colon G \to \overline{G}$ of the entries of γ_i (i.e., $\bar{\gamma}_i$ is the k_i-tuple realizing κ according to $\pi \circ \xi$ and $\leq$). Consider on $M_{k_{\bar{0}}|k_{\bar{1}}}(\mathbb{F})$ the elementary grading determined by $(\bar{\gamma}_{\bar{0}}, \bar{\gamma}_{\bar{1}})$ (see Definition 1.4). We identify the $\overline{G}$-graded superalgebra $M_{k_{\bar{0}}|k_{\bar{1}}}(\overline{\mathcal{D}}) = M_{k_{\bar{0}}|k_{\bar{1}}}(\mathbb{F}) \otimes \overline{\mathcal{D}}$ with $S = M(m+1, n+1)$ via Kronecker product.

Definition 5.16. Let $i \in \mathbb{Z}_2$ and $x \in G/T$. If $g_0x^2 = T$, we put $t := g_0\xi(x)^2 \in T$ and let $\bar{t} \in \overline{T}$ be its image under the natural homomorphism $T \to \overline{T}$. We define $\Theta(i, x)$ to be the following $\kappa_i(x) \times \kappa_i(x)$-matrix with entries in $\overline{\mathcal{D}}$:

(i) $I_{\kappa_i(x)} \otimes X_{\bar{t}}$ if $(-1)^i\eta(t) = +1$;

(ii) $J_{\kappa_i(x)} \otimes X_{\bar{t}}$, where $J_{\kappa_i(x)} := \begin{pmatrix} 0 & I_{\kappa_i(x)/2} \\ -I_{\kappa_i(x)/2} & 0 \end{pmatrix}$, if $(-1)^i\eta(t) = -1$ (recall that, in this case, $\kappa_i(x)$ is even by Definition 4.27).

If $g_0x^2 \neq T$, we define $\Theta(i, x)$ to be the following $2\kappa_i(x) \times 2\kappa_i(x)$-matrix:

(iii) $\begin{pmatrix} 0 & I_{\kappa_i(x)} \\ (-1)^i\chi(g_0\xi(x)^2)^{-1}I_{\kappa_i(x)} & 0 \end{pmatrix} \otimes 1_{\overline{\mathcal{D}}}$.

Let $x_1 < \cdots < x_{\ell_{\bar{0}}}$ be the elements of the set $\{x \in \operatorname{supp}\kappa_{\bar{0}} \mid x \leq g_0^{-1}x^{-1}\}$ and, similarly, let $y_1 < \ldots < y_{\ell_{\bar{1}}}$ be the elements of $\{y \in \operatorname{supp}\kappa_{\bar{1}} \mid y \leq g_0^{-1}y^{-1}\}$. Then, we define

$$\Theta := \left(\begin{array}{ccc|ccc} \Theta(\bar{0}, x_1) & & & & & \\ & \ddots & & & 0 & \\ & & \Theta(\bar{0}, x_{\ell_{\bar{0}}}) & & & \\ \hline & & & \Theta(\bar{1}, y_1) & & \\ & 0 & & & \ddots & \\ & & & & & \Theta(\bar{1}, y_{\ell_{\bar{1}}}) \end{array}\right). \tag{5.14}$$

We define the super-anti-automorphism $\theta\colon S \to S$ by Equation (5.11), where $\theta_0\colon \overline{\mathcal{D}} \to \overline{\mathcal{D}}$ denotes the transposition on $\overline{\mathcal{D}}$. Note that $\theta_0(X)^{s\top} \in M_{k_{\bar{0}}|k_{\bar{1}}}(\overline{\mathcal{D}})$ becomes $X^{s\top} \in M(m+1, n+1)$. Hence, Equation (5.11) reduces to

$$\forall X \in M(m+1, n+1), \quad \theta(X) = \Theta^{-1}\, X^{s\top}\, \Theta. \tag{5.15}$$

Finally, we define a G-grading on $L = S^{(1)}$ by

$$\forall g \in G, \quad L_g \coloneqq \{s \in S^{(1)}_{\bar{g}} \mid \theta(s) = -\chi(g)s\}. \tag{5.16}$$

In the next definition, we summarize what has been done for future reference. We also allow the case $m = n$.

Definition 5.17. Let $m, n \in \mathbb{Z}_{\geq 0}$, not both zero. Let $T \subseteq G$ be a finite 2-elementary subgroup and let $\beta\colon T \times T \to \mathbb{F}^\times$ be an alternating bicharacter with $\operatorname{rad}\beta = \langle f\rangle$, for some $e \neq f \in T$. Set $\overline{G} \coloneqq G/\langle f\rangle$ and $\overline{T} \coloneqq T/\langle f\rangle$, and let $\bar{\beta}$ be the nondegenerate alternating bicharacter on $\overline{T}$ induced by β. Consider the chosen standard realization $\overline{\mathcal{D}}$ of a matrix algebra with division grading associated to $(\overline{T}, \bar{\beta})$ and the chosen subgroup $K \subseteq T$ such that $T = K \times \langle f\rangle$, and define $\eta\colon T \to \{\pm 1\}$ by Equation (5.13). Then, let $g_0 \in G$ be any element and let $\kappa_{\bar{0}}, \kappa_{\bar{1}}\colon G/T \to \mathbb{Z}_{\geq 0}$ be g_0-admissible maps (Definition 4.27) such that $m + 1 = |\kappa_{\bar{0}}|\sqrt{|T|/2}$ and $n + 1 = |\kappa_{\bar{1}}|\sqrt{|T|/2}$. Choose:

(i) a set-theoretic section $\xi\colon G/T \to G$ for the natural homomorphism $G \to G/T$;

(ii) a total order $\leq$ on G/T such that there are no elements between x and $\bar{g}_0^{-1}x^{-1}$, for all $x \in G/T$;

and construct tuples $\bar{\gamma}_{\bar{0}}$ and $\bar{\gamma}_{\bar{1}}$ realizing $\kappa_{\bar{0}}$ and $\kappa_{\bar{1}}$, respectively, according to $\pi \circ \xi$ and $\leq$ (Definition 2.20), where $\pi\colon G \to \overline{G}$ is the natural homomorphism. Consider the $\overline{G}$-grading $\Gamma_M(\overline{T}, \bar{\beta}, \kappa_{\bar{0}}, \kappa_{\bar{1}})$ on $S \coloneqq M(m+1, n+1)$ constructed using the choices of $\overline{\mathcal{D}}$, $\bar{\gamma}_{\bar{0}}$ and $\bar{\gamma}_{\bar{1}}$ above (see Definition 2.75), and consider its restriction to $S^{(1)}$. Define $\Theta \in S$ by Equation (5.14) and $\theta\colon S \to S$ by Equation (5.15). Finally, we define $\Gamma^{(\mathrm{II})}_A(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}}, g_0)$ to be the G-grading on $S^{(1)}$ by Equation (5.16).

The following is an easy result that will also be used in Subsections 5.3.3 and 5.3.4.

Lemma 5.18. *Let T be a finite abelian group, $\beta\colon T \times T \to \{\pm 1\}$ be an alternating bicharacter and $K \subseteq T$ be a subgroup such that $T = K \times (\operatorname{rad}\beta)$. Set $\overline{T} \coloneqq T/\operatorname{rad}\beta$ and let $\bar{\beta}\colon \overline{T} \times \overline{T} \to \{\pm 1\}$ be the (nondegenerate) bicharacter induced by β. Then the natural homomorphism $\pi\colon T \to \overline{T}$ induces a bijection between standard realizations $\overline{\mathcal{D}}$ associated to $(\overline{T}, \bar{\beta})$ and standard realizations $\mathcal{M}$ associated to $(K, \beta\restriction_{K\times K})$ (see Definition 2.36). Further, if $\bar{\mu}\colon \overline{T} \to \{\pm 1\}$ is the map associated to the transposition on $\overline{\mathcal{D}}$, then the restriction of $\mu \coloneqq \bar{\mu} \circ \pi$ to K is the map associated to the transposition on $\mathcal{M}$.*

Proof. Recall that a standard realization $\overline{\mathcal{D}}$ associated to $(\overline{T},\bar{\beta})$ is obtained by choosing subgroups $\bar{A}$ and $\bar{B}$ of $\overline{T}$ such that $\overline{T}=\bar{A}\times\bar{B}$ and $\bar{\beta}(\bar{A},\bar{A})=\bar{\beta}(\bar{B},\bar{B})=1$. Since $\pi\restriction_K\colon K\to\overline{T}$ is an isomorphism, and $\beta(s,t)=\bar{\beta}(\pi(s),\pi(t))$, for all $s,t\in K$, the choice of the subgroups $\bar{A}$ and $\bar{B}$ as above is equivalent to a choice of subgroups $A,B\subseteq K$ such that $K=A\times B$ and $\beta(A,A)=\beta(B,B)=1$. The "further" part follows from Lemma 4.25: since $\bar{\mu}(\bar{a}\bar{b})=\bar{\beta}(\bar{a},\bar{b})$, for all $\bar{a}\in\bar{A}$ and $\bar{b}\in\bar{B}$, we have that $\mu(ab)=\bar{\mu}(\bar{a}\bar{b})=\bar{\beta}(\bar{a},\bar{b})=\beta(a,b)$, for all $a\in A$ and $b\in B$. □

Proposition 5.19. *Consider* $(R,\varphi)\coloneqq M^{\mathrm{ex}}(T,\beta,\kappa_{\bar{0}},\kappa_{\bar{1}},g_0)$ *(Definition 4.69). Then* $\operatorname{Skew}(R,\varphi)^{(1)}$ *is isomorphic to* $M(m+1,n+1)^{(1)}$ *endowed with* $\Gamma_A^{(\mathrm{II})}(T,\beta,\kappa_{\bar{0}},\kappa_{\bar{1}},g_0)$.

Proof. We will show how the choices in Definition 5.17 correspond to the choices in Definitions 4.67(a) and 4.69.

In view of Lemma 5.18, the choices of K and $\overline{\mathcal{D}}$ give us the same information as the choices of K in item (i) and $\mathcal{M}$ in item (ii) of Definition 4.67(a), and the map associated to the transposition on $\mathcal{M}$ is $\mu\restriction_K$. Let $(\mathcal{D},\varphi_0)$ denote the standard realization constructed using these choices. Note that the map $\eta\colon T\to\{\pm1\}$ defined in Equation (5.13) is such that $\mathrm{d}\eta=\beta$, $\eta\restriction_K=\mu\restriction_K$ and $\eta(f)=-1$, so, by Remark 4.68(a), η is the map associated to φ_0. In particular, the g_0-admissibility condition for κ is the same in Definitions 4.72 and 5.17.

We can use the choices in items (i) and (ii) in Definition 5.17 to choose the pair $(\mathcal{U},B)$ as in Definition 4.69, by following the same construction as in Subsection 5.2.2. With respect to the graded basis $\mathcal{B}=\{u_1,\ldots,u_k\}$ used there, B is represented by the matrix $\Phi\in M_{k_{\bar{0}}|k_{\bar{1}}}(\mathcal{D})$ as in Equation (5.4).

We will now follow Subsection 5.3.1 to undouble $M^{\mathrm{ex}}(T,\beta,\kappa_{\bar{0}},\kappa_{\bar{1}},g_0)$, constructed with the choices above. Consider the central idempotent $\epsilon\coloneqq\frac{1}{2}(1+X_f)\in\mathcal{D}$ and the $\overline{G}$-graded-division superalgebra $\mathcal{D}\epsilon$. By Proposition 4.53, $\mathcal{D}\epsilon$ has parameters $(\overline{T},\bar{\beta},\bar{p})$, so $\mathcal{D}\epsilon\simeq\overline{\mathcal{D}}$. Specifically, an isomorphism is given by $X_t\epsilon\mapsto X_{\pi(t)}$, for all $t\in T$. From now on, we will identify $\mathcal{D}\epsilon$ with $\overline{\mathcal{D}}$. Also, by Equation (5.13), the map $\bar{\mu}$ as defined in Equation (5.10) coincides with the map $\bar{\mu}$ defined above, i.e., the map associated to the transposition on $\overline{\mathcal{D}}$.

Let $\Lambda\in M_k(\mathcal{D})$ be the diagonal matrix with entries $\Lambda_{ii}\coloneqq\chi(\deg u_i)$, for all $i\in\{1,\ldots,k\}$, as in Subsection 5.3.1. Consider the graded basis $\tilde{\mathcal{B}}=\{\tilde{u}_1,\ldots,\tilde{u}_k\}$ for

$\mathcal{U}$, where

$$\tilde{u}_i := \begin{cases} \sqrt{\Lambda_{ii}^{-1}}u_i, & \text{if } (\deg u_i)T = g_0^{-1}(\deg u_i)^{-1}T; \\ \Lambda_{ii}^{-1}u_i, & \text{if } (\deg u_i)T < g_0^{-1}(\deg u_i)^{-1}T; \\ u_i, & \text{if } (\deg u_i)T > g_0^{-1}(\deg u_i)^{-1}T. \end{cases} \tag{5.17}$$

Let $\tilde{\Phi}$ be the matrix representing B with respect to the graded basis $\tilde{\mathcal{B}}$. Under the identification $\mathcal{D}\epsilon = \overline{\mathcal{D}}$, the matrix $\Lambda\tilde{\Phi}\epsilon$ becomes Θ as in Equation (5.14). Therefore, all the data in the description of the undoubled model of $\mathrm{Skew}(R, \varphi)$ coincide with the data in Definition 5.17, concluding the proof. $\square$

Recall that, for every $\kappa\colon G/T \to \mathbb{Z}_{\geq 0}$, we defined $\kappa^\star\colon G/T \to \mathbb{Z}_{\geq 0}$ by $\kappa^\star(x) = \kappa(x)^{-1}$, for all $x \in G/T$ (see Section 3.1).

Theorem 5.20. *Let $m, n \in \mathbb{Z}_{\geq 0}$, $m \neq n$. Every grading on $\mathfrak{sl}(m+1|n+1)$ is isomorphic to either $\Gamma_A^{(\mathrm{I})}(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}})$ or $\Gamma_A^{(\mathrm{II})}(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}}, g_0)$ as in Definitions 5.15 and 5.17. Gradings belonging to different types are not isomorphic. Within each type, we have:*

$\boxed{\text{Type I}}$

$\Gamma_A^{(\mathrm{I})}(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}}) \simeq \Gamma_A^{(\mathrm{I})}(T', \beta', \kappa'_{\bar{0}}, \kappa'_{\bar{1}})$ if, and only if, $T' = T$ and one of the following conditions holds:

(i) $\beta' = \beta$ and there is $g \in G$ such that $\kappa'_{\bar{0}} = g \cdot \kappa_{\bar{0}}$ and $\kappa'_{\bar{1}} = g \cdot \kappa_{\bar{1}}$;

(ii) $\beta' = \beta^{-1}$ and there is $g \in G$ such that $\kappa'_{\bar{0}} = g \cdot \kappa^\star_{\bar{0}}$ and $\kappa'_{\bar{1}} = g \cdot \kappa^\star_{\bar{1}}$.

$\boxed{\text{Type II}}$

$\Gamma_A^{(\mathrm{II})}(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}}, g_0) \simeq \Gamma_A^{(\mathrm{II})}(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}}, g_0)$ if, and only if, $T' = T$, $\beta' = \beta$ and there is $g \in G$ such that $\kappa'_{\bar{0}} = g \cdot \kappa_{\bar{0}}$, $\kappa'_{\bar{1}} = g \cdot \kappa_{\bar{1}}$ and $g'_0 = g^{-2}g_0$.

Proof. The result follows from Theorem 4.55 (Type I gradings) and Theorem 4.70 (Type II gradings). Indeed, by Corollary 5.4, the isomorphism classes of gradings on $\mathfrak{sl}(m+1 \,|\, n+1)$ are in bijection with the isomorphism classes of gradings on $M(m+1, n+1) \times M(m+1, n+1)^{\mathrm{sop}}$ endowed with the exchange superinvolution. Type I gradings are already described in terms of $M(m+1, n+1)$, and Type II gradings are described in Proposition 5.19. Note that, since $m \neq n$, the isomorphism conditions in Theorems 4.55 and 4.70 simplify: we cannot have $\kappa'_{\bar{1}} = g \cdot \kappa_{\bar{0}}$ or $\kappa'_{\bar{1}} = g \cdot \kappa^\star_{\bar{0}}$. $\square$

5.3.3 Gradings on $Q(n)$

We will now discuss the case where S is the associative superalgebra $Q(n+1)$, so $L = S^{(1)}/Z(S^{(1)})$ is the Lie superalgebra $Q(n)$.

Let us first parametrize Type I gradings. As seen in Subsection 2.2.4, every grading on S is odd. Nevertheless, we can use parameters in terms of the group G to parametrize the gradings (see Definition 2.79 and Corollary 2.80).

Definition 5.21. Let $n \in \mathbb{Z}_{>0}$, $T^+ \subseteq G$ be a finite subgroup, $\beta^+ \colon T^+ \times T^+ \to \mathbb{F}^\times$ be a nondegenerate bicharacter, $h \in G$ be an element such that $h^2 = 1$, and $\kappa \colon G/T^+ \to \mathbb{Z}_{\geq 0}$ be a map with finite support such that $n+1 = |\kappa|\sqrt{|T^+|}$. We will denote by $\Gamma_Q^{(\mathrm{I})}(T^+, \beta^+, h, \kappa)$ the grading on L induced from the grading $\Gamma_Q(T^+, \beta^+, h, \kappa)$ (see Definition 2.79) by reduction modulo the center.

For Type II gradings on L, set $R \coloneqq S \times S^{\mathrm{sop}}$ and let φ be the exchange superinvolution on it. At the end of Section 4.5, just before Corollary 4.75, we introduced a parametrization of gradings on (R, φ) that make R graded-simple. By Corollary 4.75, (R, φ) endowed with such a grading is isomorphic to $Q^{\mathrm{ex}}(T^+, \beta^+, h, \kappa, g_0)$, where $T^+ \subseteq G$ is a finite 2-elementary subgroup, $\beta^+ \colon T^+ \times T^+ \to \mathbb{F}^\times$ is an alternating bicharacter with $\operatorname{rad} \beta^+ = \langle f \rangle$ for some $e \neq f \in T^+$, $h \in G$ is an element in G such that $h^2 = f$, $g_0 \in G$, and $\kappa \colon G/T^+ \to \mathbb{Z}_{\geq 0}$ is a g_0-admissible map (see Definition 4.71) such that $n+1 = |\kappa|\sqrt{|T^+|/2}$.

We will use the parameters $(T^+, \beta^+, h, \kappa, g_0)$ to construct a representative of the corresponding isomorphism class of Type II gradings directly on the superalgebra L instead of going through $\operatorname{Skew}(R, \varphi)$. Recall that the parametrization above for gradings on $\operatorname{Skew}(R, \varphi)$ was obtained by writing $R = R^{\bar 0} \oplus u R^{\bar 0}$, where $0 \neq u \in Z(R)^{\bar 1}$, and using that $(R^{\bar 0}, \varphi\restriction_{R^{\bar 0}})$ is of type $M \times M^{\mathrm{sop}}$. We will follow an analogous strategy here. Recall that $S = Q(n+1) = S^{\bar 0} \oplus u S^{\bar 0}$, where $u \coloneqq \left(\begin{array}{c|c} 0 & I_{n+1} \\ \hline I_{n+1} & 0 \end{array}\right) \in Z(S)^{\bar 1}$, that $Z(S) = \mathbb{F}1 \oplus \mathbb{F}u$ can be identified with the associative superalgebra $Q(1)$, that $S^{\bar 0}$ can be identified with $M_{n+1}(\mathbb{F}) = M(n+1, 0)$ and that S can be identified with $Q(1) \otimes S^{\bar 0}$ via Kronecker product.

Let $\overline{G} \coloneqq G/\langle f \rangle$. We will, first, construct a $\overline{G}$-grading and a (super-)anti-automorphism on $S^{\bar 0}$, following the same steps as in Subsection 5.3.2, but with T^+ playing the role of T and β^+ playing the role of β, and, then, we will extend the grading and the super-anti-automorphism to S. Let $\pi\colon G \to \overline{G}$ denote the natural homomorphism, set $\overline{T^+} := \pi(T^+)$, let $\overline{\beta^+}\colon \overline{T^+} \times \overline{T^+} \to \mathbb{F}^\times$ be the (nondegenerate) bicharacter on $\overline{T^+}$ induced by β^+, and consider κ as a map defined on $\overline{G}/\overline{T^+} \simeq G/T^+$.

Let $\overline{\mathcal{D}}^{\bar 0}$ be the chosen standard realization associated to $(\overline{T^+}, \overline{\beta^+})$, let $\overline{\mu^+}\colon \overline{T^+} \to \mathbb{F}^\times$ be the map associated to the transposition on $\overline{\mathcal{D}}^{\bar 0}$, and set $\mu^+ := \overline{\mu^+} \circ \pi \restriction_{T^+}$. Fix a subgroup $K \subseteq T^+$ such that $T^+ = K \times \langle f \rangle$, and let $\chi\colon T^+ \to \mathbb{F}^\times$ be the character defined by $\chi(K) = 1$ and $\chi(f) = -1$. Then define $\eta^+\colon T^+ \to \{\pm 1\}$ by

$$\forall t \in T^+, \quad \eta^+(t) := \mu^+(t)\chi^{-1}(t). \tag{5.18}$$

Since $\mathrm{d}\overline{\mu^+} = \overline{\beta^+}$, we get $\mathrm{d}\eta^+ = \beta^+$. In the proof of Proposition 5.25, we will show that η^+ is the map associated to the superinvolution on the even part of a graded-division superalgebra as in Definition 4.67(c).

Extend χ to G and set $k := |\kappa|$. Following the construction after Equation (5.13) in Subsection 5.3.2 with $\kappa_{\bar 0} := \kappa$ and $\kappa_{\bar 1}$ being the zero map, we get an elementary $\overline{G}$-grading on $M_k(\mathbb{F}) = M(k, 0)$. We identify the $\overline{G}$-graded superalgebra $M_k(\overline{\mathcal{D}}^{\bar 0}) = M_k(\mathbb{F}) \otimes \overline{\mathcal{D}}^{\bar 0}$ with $S^{\bar 0} = M_{n+1}(\mathbb{F})$ via Kronecker product, and define

$$\Theta_{\bar 0} := \begin{pmatrix} \Theta(\bar 0, x_1) & & \\ & \ddots & \\ & & \Theta(\bar 0, x_\ell) \end{pmatrix}, \tag{5.19}$$

where $x_1 < \cdots < x_\ell$ are the elements of the set $\{x \in \operatorname{supp}\kappa \mid x \leq g_0^{-1}x^{-1}\}$, and $\Theta(\bar 0, x)$, for $x \in \overline{G}/\overline{T^+}$, is as in Definition 5.16. We, then, define the (super-)anti-automorphism $\theta\colon S^{\bar 0} \to S^{\bar 0}$ by

$$\forall X \in M_{n+1}(\mathbb{F}), \quad \theta(X) := \Theta_{\bar 0}^{-1}\, X^\top\, \Theta_{\bar 0}.$$

We extend the $\overline{G}$-grading on $S^{\bar 0}$ to $S = S^{\bar 0} \oplus uS^{\bar 0}$ by declaring $\deg u = \bar h$, i.e., we define $S^{\bar 1}_{\bar h \bar g} = uS^{\bar 0}_{\bar g}$, for all $\bar g \in \overline{G}$. In terms of the identification $S = Q(1) \otimes S^{\bar 0}$, this corresponds to the usual tensor product of graded superalgebras.

Remark 5.22. Note that $S = Q(1) \otimes S^{\bar 0} = Q(1) \otimes M_k(\mathbb{F}) \otimes \overline{\mathcal{D}}^{\bar 0} \simeq M_k(\mathbb{F}) \otimes Q(1) \otimes \overline{\mathcal{D}}^{\bar 0}$. Hence, this $\overline{G}$-grading is isomorphic to $\Gamma_Q(\overline{T^+}, \overline{\beta^+}, \bar h, \kappa)$ (see Definition 2.79), by choosing the standard realization of type Q (Definition 2.63) to be $\overline{\mathcal{D}} := Q(1) \otimes \overline{\mathcal{D}}^{\bar 0} = \overline{\mathcal{D}}^{\bar 0} \oplus u\overline{\mathcal{D}}^{\bar 0}$.

We also extend the (super-)anti-automorphism θ to S, by declaring $\theta(u) = \mathbf{i}u$. In terms of the identification $S = Q(1) \otimes S^{\bar{0}}$, this corresponds to

$$\forall X \in Q(n+1), \quad \theta(X) \coloneqq \Theta^{-1}\, X^{s\top_Q}\, \Theta, \tag{5.20}$$

where

$$\Theta \coloneqq \left(\begin{array}{c|c} \Theta_{\bar{0}} & 0 \\ \hline 0 & \Theta_{\bar{0}} \end{array}\right). \tag{5.21}$$

Note that, differently from Equation (5.15), we are using the queer supertranspose (Definition 0.18) here.

Finally, we define a G-grading on $S^{(1)}$ by

$$\forall g \in G, \quad S^{(1)}_g \coloneqq \{s \in S^{(1)}_{\bar{g}} \mid \theta(s) = -\chi(g)s\}, \tag{5.22}$$

and reduce it modulo the center to obtain a G-grading on L.

The following definition summarizes the above construction:

Definition 5.23. Let $n \in \mathbb{Z}_{>0}$ and let S denote the associative superalgebra $Q(n+1)$. Let $T^+ \subseteq G$ be a finite 2-elementary subgroup, let $\beta^+ \colon T^+ \times T^+ \to \mathbb{F}^\times$ be an alternating bicharacter with $\operatorname{rad} \beta^+ = \langle f \rangle$, for some element $e \neq f \in T$ and let $h \in G$ be an element such that $h^2 = f$. Set $\overline{G} \coloneqq G/\langle f \rangle$ and let $\pi \colon G \to \overline{G}$ be the natural homomorphism. Set $\overline{T^+} \coloneqq \pi(T^+)$, and let $\overline{\beta^+}$ be the (nondegenerate) alternating bicharacter on $\overline{T^+}$ induced by β^+. Consider the chosen standard realization $\overline{\mathcal{D}}^{\bar{0}}$ of a matrix algebra with division grading associated to $(\overline{T^+}, \overline{\beta^+})$, fix a subgroup $K \subseteq T^+$ such that $T^+ = K \times \langle f \rangle$, and define $\eta^+ \colon T \to \{\pm 1\}$ by Equation (5.18). Then, let $g_0 \in G$ be any element and let $\kappa \colon G/T^+ \to \mathbb{Z}_{\geq 0}$ be a g_0-admissible map (Definition 4.71) such that $n + 1 = |\kappa|\sqrt{|T^+|/2}$. Choose:

(i) a set-theoretic section $\xi \colon G/T^+ \to G$ for the natural homomorphism $G \to G/T^+$;

(ii) a total order $\leq$ on G/T^+ such that there are no elements between x and $\bar{g}_0^{-1}x^{-1}$, for all $x \in G/T^+$;

and construct a tuple $\bar{\gamma}$ realizing κ according to $\pi \circ \xi$ and $\leq$ (Definition 2.20). Consider the $\overline{G}$-grading $\Gamma_M(\overline{T^+}, \overline{\beta^+}, \kappa)$ on $S^{\bar{0}} \simeq M_{n+1}(\mathbb{F})$ constructed using the choices of

$\overline{\mathcal{D}}^{\bar{0}}$ and $\bar{\gamma}$ above (see Definition 2.41), and extend it to S by declaring $\deg u := \bar{h}$. Define $\Theta_{\bar{0}} \in M_{n+1}(\mathbb{F})$ by Equation (5.19), $\Theta \in S$ by Equation (5.21) and $\theta\colon S \to S$ by Equation (5.20) Finally, we define $\Gamma_Q^{(\mathrm{II})}(T^+, \beta^+, h, \kappa, g_0)$ to be the G-grading on $L = S^{(1)}/Z(S^{(1)})$ induced from the G-grading $S^{(1)}$ given by Equation (5.22).

Remark 5.24. We note that in [BHSK17, Theorem 5.1] it is described how to extend a G-grading Γ on the even part of the Lie superalgebra $Q(n)$ to the whole superalgebra, without using a full description of Γ. The construction above corresponds to the case where Γ is of Type II.

Proposition 5.25. *Consider $(R, \varphi) := Q^{\mathrm{ex}}(T^+, \beta^+, h, \kappa, g_0)$, as defined before Corollary 4.75. Then the graded Lie superalgebra* $\mathrm{Skew}(R, \varphi)^{(1)}/Z(\mathrm{Skew}(R, \varphi)^{(1)})$ *is isomorphic to the Lie superalgebra $Q(n)$ endowed with $\Gamma_Q^{(\mathrm{II})}(T^+, \beta^+, h, \kappa, g_0)$.*

Proof. First recall that, by definition, $Q^{\mathrm{ex}}(T^+, \beta^+, h, \kappa, g_0) = Q^{\mathrm{ex}}(T, \beta, t_p, \kappa, g_0)$ (see Definition 4.72), where $t_p := (h, \bar{1})$, $T := T^+ \cup t_p T^+$, $\beta\colon T \times T \to \mathbb{F}^\times$ is the unique alternating bicharacter extending β^+ such that $\operatorname{rad}\beta = \langle t_p \rangle$, and κ is seen as defined on $G^\#/T \simeq G/T^+$.

We will now show how the choices in Definition 5.23 correspond to the choices in Definition 4.67(c) and Definition 4.72.

By Lemma 5.18, with (T^+, β^+) playing the role of (T, β), the choices of K and $\overline{\mathcal{D}}^{\bar{0}}$ give us the same information as the choices of K in item (i) and $\mathcal{M}$ in item (ii) of Definition 4.67(c), and the map associated to the transposition on $\mathcal{M}$ is $\mu^+ \restriction_K$. Let $(\mathcal{D}, \varphi_0)$ denote the standard realization of a graded-division superalgebra with superinvolution constructed using these choices. Note that the map $\eta^+\colon T \to \{\pm 1\}$ defined in Equation (5.18) is such that $\mathrm{d}\eta^+ = \beta^+$, $\eta \restriction_K = \mu^+ \restriction_K$ and $\eta(f) = -1$, so, by Equation (4.1), η^+ is the map associated to $\varphi_0 \restriction_{\mathcal{D}^{\bar{0}}}$. In particular, the g_0-admissibility condition for κ is the same in Definitions 4.72 and 5.23.

In Definition 4.72, we have to choose a graded right $\mathcal{D}$-supermodule $\mathcal{U}$ and a φ_0-sesquilinear form $B\colon \mathcal{U} \times \mathcal{U} \to \mathcal{D}$ such that $(\mathcal{U}, B)$ has inertia determined by κ. We will first construct a graded right $\mathcal{D}^{\bar{0}}$-supermodule $\mathcal{U}^{\bar{0}}$ and a $(\varphi_0 \restriction_{\mathcal{D}^{\bar{0}}})$-sesquilinear form $B^{\bar{0}}\colon \mathcal{U} \times \mathcal{U} \to \mathcal{D}$ such that $(\mathcal{U}^{\bar{0}}, B^{\bar{0}})$ has inertia determined by $(\kappa_{\bar{0}}, \kappa_{\bar{1}})$, where we take $\kappa_{\bar{0}} := \kappa$ (seen as a map defined on G/T^+) and $\kappa_{\bar{1}}$ to be the zero map. To that end, it suffices to follow the same construction as in Subsection 5.2.2. In other words, define $\mathcal{D}^{\bar{0}} := (\mathcal{D}^{\bar{0}})^{[g_1]} \oplus \cdots \oplus (\mathcal{D}^{\bar{0}})^{[g_k]}$, where $(g_1, \ldots, g_k)$ is the k-tuple of elements in G realizing

κ according to ξ and $\leq$, and define B by $B(u_i, u_j) := \Phi_{ij}$, where $\mathcal{B} := \{u_1, \ldots, u_k\}$ is the canonical graded $\mathcal{D}^{\bar{0}}$-basis of $\mathcal{U}^{\bar{0}}$ and $\Phi \in M_k(\mathcal{D}^{\bar{0}})$ is the matrix defined by Equation (5.4). We then define $\mathcal{U} := \mathcal{U}^{\bar{0}} \otimes_{\mathcal{D}^{\bar{0}}} \mathcal{D}$. Note that $\mathcal{B}$ is an even graded $\mathcal{D}$-basis of $\mathcal{U}$, and let $B\colon \mathcal{U} \times \mathcal{U} \to \mathcal{D}$ be the unique φ_0-sesquilinear extension of $B^{\bar{0}}$, which is also represented by Φ, seen as a matrix in $M_k(\mathcal{D})$.

We will now follow Subsection 5.3.1 to undouble $Q^{\mathrm{ex}}(T^+, \beta^+, h, \kappa, g_0)$, constructed with the choices above. Consider the central idempotent $\epsilon := \frac{1}{2}(1 + X_f) \in \mathcal{D}$ and the $\overline{G}$-graded-division superalgebra $\mathcal{D}\epsilon$. By Proposition 4.53, $\mathcal{D}\epsilon$ has parameters $(\overline{T}, \bar{\beta}, \bar{p})$. Also, $\operatorname{rad} \bar{\beta} = \langle \bar{t}_p \rangle$, so $\mathcal{D}\epsilon$ is isomorphic to a standard realization of type Q associated to $(\overline{T^+}, \overline{\beta^+}, \bar{h})$ (see discussion preceding Definition 2.63), specifically, it is isomorphic to $\overline{\mathcal{D}} := Q(1) \otimes \overline{\mathcal{D}}^{\bar{0}}$ as in Remark 5.22. From now on, we will identify $\mathcal{D}\epsilon$ with $\overline{\mathcal{D}}$.

We need to extend χ to $G^{\#}$. Since $\chi(f) = -1$, we can do it in a way such that $\chi(t_p) = \mathbf{i}$. Let $\eta\colon T \to \{\pm 1\}$ be the map associated to φ_0 and let $\bar{\mu}\colon \overline{T} \to \mathbb{F}^{\times}$ be as defined in Equation (5.10). We have already shown that $\eta \restriction_{T^+} = \eta^+$ and, hence, by Equation (5.18), we have that $\bar{\mu} \restriction_{\overline{T^+}} = \overline{\mu^+}$. By Remark 4.68(c), we have $\bar{\mu}(\bar{t}_p) = \eta(t_p)\chi(t_p) = \mathbf{i}$. It follows that $\bar{\mu}$ is the map associated to the queer supertranspose on $\overline{\mathcal{D}}$.

As in the proof of Proposition 5.19, let $\Lambda \in M_k(\mathcal{D})$ be the diagonal matrix with entries $\Lambda_{ii} := \chi(\deg u_i)$, consider a different graded $\mathcal{D}$-basis $\tilde{\mathcal{B}} = \{\tilde{u}_1, \ldots, \tilde{u}_k\}$ of $\mathcal{U}$, where $\tilde{u}_i$ is defined as in Equation (5.17), and let $\tilde{\Phi} \in M_k(\mathcal{D})$ be the matrix representing B with respect to $\tilde{\mathcal{B}}$. Clearly, all the entries of $\tilde{\Phi}$ are in $\overline{\mathcal{D}}^{\bar{0}}$. We will denote $\tilde{\Phi}$ by $\tilde{\Phi}_{\bar{0}}$ when seen as a matrix in $M_k(\mathcal{D}^{\bar{0}})$. In Subsection 5.3.1, $M_k(\mathcal{D})\epsilon$ was identified with $M_k(\overline{\mathcal{D}})$, so we now identify $M_k(\mathcal{D}^{\bar{0}})\epsilon$ with $M_k(\overline{\mathcal{D}}^{\bar{0}})$. Under this identification, the matrix $\Lambda\tilde{\Phi}_{\bar{0}}\epsilon$ becomes $\Theta_{\bar{0}}$ as in Equation (5.19). Also, following the isomorphisms

$$M_k(\overline{\mathcal{D}}) \simeq M_k(\mathbb{F}) \otimes \overline{\mathcal{D}} \simeq M_k(\mathbb{F}) \otimes Q(1) \otimes \overline{\mathcal{D}}^{\bar{0}} \simeq Q(1) \otimes M_k(\mathbb{F}) \otimes \overline{\mathcal{D}}^{\bar{0}} \simeq Q(1) \otimes S^{\bar{0}},$$

the $\overline{G}$-grading on $M_k(\overline{\mathcal{D}})$ becomes the $\overline{G}$-grading we defined on $S = Q(1) \otimes S^{\bar{0}}$, and $\Lambda\tilde{\Phi}\epsilon$ is sent to $I_2 \otimes \Theta_{\bar{0}} = \Theta$. Therefore, all the data in the description of the undoubled model of $\operatorname{Skew}(R, \varphi)$ coincide with the data in Definition 5.23, concluding the proof. $\square$

Theorem 5.26. *Let $n \geq 2$. Every grading on the simple Lie superalgebra $Q(n)$ is isomorphic to either $\Gamma_Q^{(\mathrm{I})}(T^+, \beta^+, h, \kappa)$ or $\Gamma_Q^{(\mathrm{II})}(T^+, \beta^+, h, \kappa, g_0)$ as in Definitions 5.21 and 5.23. Gradings belonging to different types are not isomorphic. Within each type, we have:*

$\boxed{\text{Type I}}$

$\Gamma_Q^{(\mathrm{I})}(T^+,\beta^+,h,\kappa) \simeq \Gamma_Q^{(\mathrm{I})}(T'^+,\beta'^+,h',\kappa')$ if, and only if, $T'^+ = T^+$, $h' = h$ and one of the following conditions holds:

(i) $\beta'^+ = \beta^+$ *and there is* $g \in G$ *such that* $\kappa' = g \cdot \kappa$*;*

(ii) $\beta'^+ = (\beta^+)^{-1}$ *and there is* $g \in G$ *such that* $\kappa' = g \cdot \kappa^\star$*.*

$\boxed{\text{Type II}}$

$\Gamma_A^{(\mathrm{II})}(T^+,\beta^+,h,\kappa,g_0) \simeq \Gamma_A^{(\mathrm{II})}(T'^+,\beta'^+,h',\kappa',g_0')$ if, and only if, $T'^+ = T^+$, $\beta'^+ = \beta^+$, $h' = h$ and there is $g \in G$ such that $\kappa' = g \cdot \kappa$ and $g_0' = g^{-2}g_0$.

Proof. The result follows from Theorem 4.56 (Type I) and Corollary 4.75 (Type II). Indeed, by Corollary 5.4, the isomorphism classes of gradings on the Lie superalgebra $Q(n)$ are in bijection with the isomorphism classes of gradings on the associative superalgebra $Q(n+1) \times Q(n+1)^{\mathrm{sop}}$ endowed with the exchange superinvolution. Type I gradings are already described in terms of the associative superalgebra $Q(n+1)$, and Type II gradings are described in Proposition 5.25. $\square$

5.3.4 Gradings on $A(n,n)$

We now go to our final series of Lie superalgebras, $A(n,n)$. Let $S = M(n+1,n+1)$, so $L = \mathfrak{psl}(n+1,n+1) = S^{(1)}/Z(S^{(1)})$.

Gradings on S are separated into two classes: even gradings, which are similar to what we saw in Subsection 5.3.2, and odd gradings. The gradings on L induced by even gradings on S will be said to be of *Type I_M*, and the gradings on L induced by odd gradings on S will be said to be *Type I_Q*.

Remark 5.27. Our notation is justified by the following fact: every G-grading of Type I on the Lie superalgebra $Q(n)$ gives rise to a $G \times \mathbb{Z}_2$-grading of Type $\mathrm{I_Q}$ on $A(n,n)$. Indeed, the associative superalgebra $Q(n+1)$ consists of the fixed points of the order 2 automorphism π of $M(n+1,n+1)$, and any automorphism of $Q(n+1)$ extends uniquely to an automorphism of $M(n+1,n+1)$ commuting with π (see Section 5.1).

Definition 5.28. Let $n \in \mathbb{Z}_{>0}$, $T \subseteq G$ be a finite subgroup, $\beta\colon T \times T \to \mathbb{F}^\times$ be a nondegenerate alternating bicharacter, and $\kappa_{\bar 0}, \kappa_{\bar 1}\colon G/T \to \mathbb{Z}_{\geq 0}$ be maps with finite support such that $|\kappa_{\bar{0}}|\sqrt{|T|} = |\kappa_{\bar{1}}|\sqrt{|T|} = n+1$. We define $\Gamma_A^{(\mathrm{I_M})}(T,\beta,\kappa_{\bar{0}},\kappa_{\bar{1}})$ to be the grading on L induced from the grading $\Gamma_A^{(\mathrm{I})}(T,\beta,\kappa_{\bar{0}},\kappa_{\bar{1}})$ on $M(n+1,n+1)^{(1)}$ (Definition 5.15) by reduction modulo the center.

To parametrize the gradings of Type $\mathrm{I_Q}$, we recall the parametrization of odd gradings on S in terms of the group G (Subsection 2.3.2). The character $\chi_0 \in \widehat{T^+}$ in the next definition plays the role of χ there.

Definition 5.29. Let $T^+ \subseteq G$ be a finite subgroup, let $\beta^+\colon T^+ \times T^+ \to \mathbb{F}^\times$ be an alternating bicharacter with $\operatorname{rad}\beta^+ = \langle t_p \rangle$, where $t_p \in T^+$ is an element of order 2, and let $\kappa\colon G/T^+ \to \mathbb{Z}_{\geq 0}$ be a map with finite support such that $|\kappa|\sqrt{2|T^+|} = n+1$. Choose a character $\chi_0 \in \widehat{T^+}$ such that $\chi_0(t_p) = -1$. Then let $a \in T^+$ be the unique element such that $\chi_0^2 = \beta^+(a,\cdot)$ and $\chi_0(a) = 1$ (see Lemma 2.98). For each element $h \in G$ such that $h^2 = a$, we set $t_1 \coloneqq (h,\bar{1}) \in G^\#$, $T^- \coloneqq t_1T^+ \subseteq G^\#$ and $T \coloneqq T^+ \cup T^-$. Let $p\colon T \to \mathbb{Z}_2$ be the homomorphism with kernel T^+, and let $\beta\colon T\times T \to \mathbb{F}^\times$ be the unique alternating bicharacter such that $\beta \restriction_{T^+\times T^+} = \beta^+$ and $\beta(t_1,t) = \chi(t)$, for all $t \in T^+$ (see Lemma 2.83). We will denote by $\Gamma_A^{(\mathrm{I_Q})}(T^+,\beta^+,h,\kappa)$ the grading on L induced from the grading $\Gamma_M(T,\beta,p,\kappa)$ on S (see Definition 2.77) by restriction and reduction modulo the center.

Recall that an element $h \in G$ (and, hence, a grading $\Gamma_A^{(\mathrm{I_Q})}(T^+,\beta^+,h,\kappa)$) exists if, and only if, $\theta(T^+_{[2]})^\perp \subseteq \overline{G}^{[2]}$ (see Definitions 2.90 and 2.92 and Proposition 2.99).

The Type II gradings on L will also be subdivided into different types. As in the case $m \neq n$ (Subsection 5.3.2), these gradings correspond to gradings on (R,φ) making R graded-simple, where $R \coloneqq S \times S^{\mathrm{sop}}$ and φ is the exchange superinvolution.

If the grading on (R,φ) is even, then, by Theorem 4.70, R is isomorphic to $M^{\mathrm{ex}}(T,\beta,\kappa_{\bar{0}},\kappa_{\bar{1}},g_0)$, where T is a finite elementary 2-group, $\beta\colon T\times T \to \mathbb{F}^\times$ is an alternating bicharacter with $\operatorname{rad}\beta = \langle f \rangle$ for some $e \neq f \in T$, g_0 is an element in $G^\#$, and $\kappa_{\bar{0}},\kappa_{\bar{1}}\colon G/T \to \mathbb{Z}_{\geq 0}$ are g_0-admissible maps (see Definition 4.27) such that $|\kappa_{\bar{0}}|\sqrt{|T|} = |\kappa_{\bar{1}}|\sqrt{|T|} = n+1$ (see Definition 4.69). Write g_0 as (h_0,p_0), with $h_0 \in G$ and $p_0 \in \mathbb{Z}_2$. If $p_0 = \bar{0}$, we will say that the corresponding grading on L is of Type $\mathrm{II}_{\mathfrak{osp}}$. If $p_0 = \bar{1}$, we will say that the corresponding grading on L is of Type $\mathrm{II_P}$.

Remark 5.30. A G-grading on the Lie superalgebra $\mathfrak{osp}(n+1|n+1)$ (respectively, $P(n)$) gives rise to a $G\times\mathbb{Z}_2$-grading on $A(n,n)$ of Type $\mathrm{II}_{\mathfrak{osp}}$ (respectively, $\mathrm{II_P}$).

Recall that, before Equation (5.13), we fixed a subgroup $K \subseteq T^+$ such that $T^+ = K \times \langle f \rangle$, which was used in Definition 5.17.

Definition 5.31. Let $n \in \mathbb{Z}_{>0}$. We define $\Gamma_A^{(\mathrm{II}_{\mathfrak{osp}})}(T, \beta, \kappa_{\bar 0}, \kappa_{\bar 1}, h_0)$ to be the grading on L induced from the grading $\Gamma_A^{(\mathrm{II})}(T, \beta, \kappa_{\bar 0}, \kappa_{\bar 1}, h_0)$ on $M(n+1, n+1)^{(1)}$ (Definition 5.17) by reduction modulo the center.

To construct a model for gradings of Type II_{P}, we follow a similar approach as for Type $\mathrm{II}_{\mathfrak{osp}}$, but it is simpler since we do not need to choose ξ and $\leq$ to define the matrix Θ (in the same way, these choices were needed in Subsection 5.2.2 but not in Subsection 5.2.1). Also, $\kappa_{\bar 1}$ is determined by $\kappa_{\bar 0}$ and h_0 (recall Definition 4.27).

Definition 5.32. Let $\overline{G} := G/\langle f \rangle$, $T := T/\langle f \rangle$, and let $\bar\beta$ be the nondegenerate alternating bicharacter on $\overline{T}$ induced by β. Choose a k-tuple $\gamma_{\bar 0} = (g_1, \dots, g_k)$ of elements in G realizing $\kappa_{\bar 0}$. Let $\bar\mu \colon \overline{T} \to \mathbb{F}^\times$ be the map associated to the transposition on $\overline{\mathcal{D}}$, let $\chi \in \widehat{T}$ be the character such that $\chi(K) = 1$ and $\chi(f) = -1$, and extend χ to a character of G, which will also be denoted by χ. Then define $\mu := \bar\mu \circ \pi \restriction_T$, where $\pi \colon G \to \overline{G}$ is the natural homomorphism, and fix $\eta \colon T \to \{\pm 1\}$ as in Equation (5.13). Also, set $\gamma_{\bar 1} = (h_0^{-1} g_1^{-1}, \dots, h_0^{-1} g_k^{-1})$. Consider the $\overline{G}$-grading $\Gamma_M(\overline{T}, \bar\beta, \kappa_{\bar 0}, \kappa_{\bar 1})$ on $S := M(n+1, n+1)$ using the choices above (see Definition 2.75), and consider its restriction to $S^{(1)}$. Consider $\Theta \in S$ given by

$$\Theta := \left(\begin{array}{ccc|ccc} & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \\ \hline \chi(h_0^{-1} g_1^{-2}) & & & & & \\ & \ddots & & & 0 & \\ & & \chi(h_0^{-1} g_k^{-2}) & & & \end{array}\right) \otimes 1_{\overline{\mathcal{D}}}.$$

and $\theta \colon S \to S$ as in Equation (5.15). We define $\Gamma_A^{(\mathrm{II}_{\mathrm{P}})}(T, \beta, \kappa_{\bar 0}, h_0)$ to be the G-grading on $L = S^{(1)}/Z(S^{(1)})$ induced from the grading $S^{(1)} = \bigoplus_{g \in G} S^{(1)}_g$, where

$$S^{(1)}_g := \{ s \in S^{(1)}_{\bar g} \mid \theta(s) = -\chi(g) s \},$$

for all $g \in G$.

We now proceed to the last case, the odd gradings of Type II, which will be referred to as gradings of Type II_{Q}.

Remark 5.33. Similar to Remark 5.27, we have that every G-grading of Type II on the Lie superalgebra $Q(n)$ gives rise to a $G\times\mathbb{Z}_2$-grading of Type II_{Q} on $A(n,n)$, which justifies the notation.

At the end of Section 4.5, just before Corollary 4.76, we introduced a parametrization of odd gradings on (R,φ) that make R graded-simple. By Corollary 4.76, (R,φ) endowed with such a grading is isomorphic to $M^{\mathrm{ex}}(T^+,\beta^+,t_p,h,\kappa,g_0)$, where $T^+\subseteq G$ is a finite 2-elementary subgroup, $e\neq t_p\in T^+$, $h\in G$ is such that $f:=h^2\in T^+\smallsetminus\langle t_p\rangle$, $\beta^+\colon T^+\times T^+\to\{\pm1\}$ is an alternating bicharacter such that $\operatorname{rad}\beta^+=\langle t_p,f\rangle$, $g_0\in G$, and $\kappa\colon G/T^+\to\mathbb{Z}_{\geq0}$ is a g_0-admissible map (see Definition 4.71) such that $n+1=|\kappa|\sqrt{|T^+|}/2$.

We will use the parameters $(T^+,\beta^+,t_p,h,\kappa,g_0)$ to construct a representative of the corresponding isomorphism class of Type II_{Q} gradings directly on the superalgebra L instead of going through $\operatorname{Skew}(R,\varphi)$. Let $\tilde{S}$ denote the algebra $M_{n+1}(\mathbb{F})$. Using the Kronecker product, we can identify $S=M(n+1,n+1)$ with $M(1,1)\otimes\tilde{S}$.

Let $\overline{G}:=G/\langle f\rangle$. We will, first, construct a $\overline{G}$-grading and a (super-)anti-automorphism on $\tilde{S}$, in a fashion similar to what we did for $S^{\bar{0}}$ in Subsection 5.3.3, and, then, we will extend the grading and the super-anti-automorphism to S. Fix a subgroup $K\subseteq T^+$ such that $T^+=K\times(\operatorname{rad}\beta^+)$. Let $\pi\colon G\to\overline{G}$ denote the natural homomorphism, set $\overline{T^+}:=\pi(T^+)$ and $\overline{K}:=\pi(K)$, let $\overline{\beta^+}\colon\overline{T^+}\times\overline{T^+}\to\mathbb{F}^\times$ be the bicharacter on $\overline{T^+}$ induced by β^+, and consider κ as a map defined on $\overline{G}/\overline{T^+}\simeq G/T^+$. Also, let $\chi,\chi_0\colon T^+\to\mathbb{F}^\times$ be the characters defined by $\chi(K)=1$, $\chi(f)=-1$ and $\chi(t_p)=1$, and $\chi_0(K)=1$, $\chi_0(f)=1$ and $\chi_0(t_p)=-1$ (we note that χ_0 was denoted by χ in Equation (4.2)).

Since $\overline{\beta^+}\restriction_{\overline{K}\times\overline{K}}$ is nondegenerate, we have a chosen standard realization $\widetilde{\mathcal{D}}$ associated to $(\overline{K},\overline{\beta^+}\restriction_{\overline{K}\times\overline{K}})$. Let $\tilde{\mu}\colon\overline{K}\to\mathbb{F}^\times$ be the map associated to the transposition on $\widetilde{\mathcal{D}}$, and let $\overline{\mu^+}\colon\overline{T^+}\to\mathbb{F}^\times$ be the map extending $\tilde{\mu}$ defined by $\overline{\mu^+}(\bar{t}\bar{t}_p)=\overline{\mu^+}(\bar{t}):=\tilde{\mu}(\bar{t})$, for all $\bar{t}\in\overline{K}$. Set $\mu^+:=\overline{\mu^+}\circ\pi\restriction_{T^+}$ and define $\eta^+\colon T^+\to\{\pm1\}$ by

$$\forall t\in T^+,\quad \eta^+(t):=\mu^+(t)\chi^{-1}(t). \tag{5.23}$$

Since $\mathrm{d}\tilde{\mu}=\overline{\beta^+}\restriction_{\overline{K}\times\overline{K}}$, we get $\mathrm{d}\overline{\mu^+}=\overline{\beta^+}$ and, hence, $\mathrm{d}\eta^+=\beta^+$. In the proof of Proposition 5.37, we will show that η^+ is the map associated to the superinvolution on the even part of a graded-division superalgebra as in Definition 4.67(b).

Now, consider the (division) $\overline{G}$-grading on $M(1,1)$ given by:

$$\begin{aligned} \deg\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right) &= \bar{e}, & \deg\left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right) &= \bar{h}, \\ \deg\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right) &= \bar{t}_p, & \deg\left(\begin{array}{c|c} 0 & -1 \\ \hline 1 & 0 \end{array}\right) &= \bar{t}_p\bar{h}. \end{aligned} \tag{5.24}$$

(Compare with Examples 2.4 and 2.49.) Set $k := |\kappa|$ and fix a set-theoretic section $\xi\colon G/T^+ \to G$ of the natural homomorphism $G \to G/T^+$. As we did in Subsection 5.3.3, we follow the construction after Equation (5.13) in Subsection 5.3.2 with $\kappa_{\bar{0}} := \kappa$ and $\kappa_{\bar{1}}$ being the zero map to construct an elementary $\overline{G}$-grading on $M_k(\mathbb{F}) = M(k,0)$. We identify the $\overline{G}$-graded superalgebra $M_k(\widetilde{\mathcal{D}}) = M_k(\mathbb{F}) \otimes \widetilde{\mathcal{D}}$ with $\tilde{S} = M_{n+1}(\mathbb{F})$ via Kronecker product and, then, get a $\overline{G}$-grading on $S = M(n+1, n+1) = M(1,1) \otimes \tilde{S}$.

Remark 5.34. Note that $S = M(1,1)\otimes\tilde{S} = M(1,1)\otimes M_k(\mathbb{F})\otimes\widetilde{\mathcal{D}} \simeq M_k(\mathbb{F})\otimes M(1,1)\otimes\widetilde{\mathcal{D}}$. Hence, this $\overline{G}$-grading is isomorphic to $\Gamma_M(\overline{T^+}, \overline{\beta^+}, \bar{t}_p, \kappa)$ (see Definition 2.102), where the fixed character is taken to be χ_0.

We will now construct a super-anti-automorphism on S. The next definition is similar to Definition 5.16.

Definition 5.35. Let $x \in G/T^+$. If $g_0x^2 = T^+$, we put $t := g_0\xi(x)^2 \in T^+$ and $\tilde{t} := \pi(\mathrm{pr}_K(t))$, where $\mathrm{pr}_K\colon T^+ \to K$ is the projection on K corresponding to the decomposition $T^+ = K \times (\operatorname{rad}\beta^+)$. We define $\delta_x := \chi_0(t)$ and $\Theta_K(x)$ to be the following $\kappa(x) \times \kappa(x)$-matrix with entries in $\widetilde{\mathcal{D}}$:

(i) $I_{\kappa(x)} \otimes X_{\tilde{t}}$ if $\eta^+(t) = +1$;

(ii) $J_{\kappa(x)} \otimes X_{\tilde{t}}$, where $J_{\kappa(x)} := \begin{pmatrix} 0 & I_{\kappa(x)/2} \\ -I_{\kappa(x)/2} & 0 \end{pmatrix}$, if $\eta^+(t) = -1$ (recall that, in this case, $\kappa(x)$ is even by Definition 4.71).

If $g_0x^2 \neq T^+$, we define $\delta_x = 1$ and $\Theta_K(x)$ to be the following $2\kappa(x) \times 2\kappa(x)$-matrix:

(iii) $\begin{pmatrix} 0 & I_{\kappa(x)} \\ \chi(g_0\xi(x)^2)^{-1} I_{\kappa(x)} & 0 \end{pmatrix} \otimes 1_{\widetilde{\mathcal{D}}}$.

Let $x_1 < \cdots < x_\ell$ be the elements of the set $\{x \in \operatorname{supp}\kappa \mid x \leq g_0^{-1}x^{-1}\}$, and set $\Theta \in S$ to be the matrix

$$\Theta \coloneqq \left(\begin{array}{ccc|ccc} \Theta_K(x_1) & & & & & \\ & \ddots & & & 0 & \\ & & \Theta_K(x_\ell) & & & \\ \hline & & & \delta_{x_1}\Theta_K(x_1) & & \\ & 0 & & & \ddots & \\ & & & & & \delta_{x_\ell}\Theta_K(x_\ell) \end{array}\right). \tag{5.25}$$

We, then, define the super-anti-automorphism $\theta\colon S \to S$ by

$$\forall X \in M(n+1, n+1), \quad \theta(X) \coloneqq \Theta^{-1}\, X^{s\top_Q}\, \Theta. \tag{5.26}$$

Finally, we define a G-grading on $S^{(1)}$ by

$$\forall g \in G, \quad S^{(1)}_g \coloneqq \{s \in S^{(1)}_{\bar g} \mid \theta(s) = -\chi(g)s\}, \tag{5.27}$$

and reduce it modulo the center to obtain a G-grading on L.

In summary:

Definition 5.36. Let $n \in \mathbb{Z}_{>0}$, and denote the associative superalgebra $M(n+1, n+1)$ by S. Let $T^+ \subseteq G$ be a 2-elementary subgroup, let $e \neq t_p \in T^+$, let $h \in G$ be such that $f \coloneqq h^2 \in T^+ \smallsetminus \langle t_p\rangle$, and let $\beta^+\colon T^+ \times T^+ \to \{\pm 1\}$ be an alternating bicharacter such that $\operatorname{rad}\beta^+ = \langle t_p, f\rangle$. Let $\pi\colon G \to \overline{G} \coloneqq G/\langle f\rangle$ be the natural homomorphism, fix a subgroup $K \subseteq T^+$ such that $T^+ = K \times \langle f\rangle$, set $\overline{T^+} \coloneqq \pi(T^+)$ and $\overline{K} \coloneqq \pi(K)$, and let $\overline{\beta^+}$ be the alternating bicharacter on $\overline{T^+}$ induced by β^+. Consider the chosen standard realization $\widetilde{\mathcal{D}}$ of a matrix algebra with division grading associated to $(\overline{K}, \overline{\beta^+}\restriction_{\overline{K}\times\overline{K}})$, and define $\eta^+\colon T \to \{\pm 1\}$ by Equation (5.23). Then, let $g_0 \in G$ be any element and let $\kappa\colon G/T^+ \to \mathbb{Z}_{\geq 0}$ be a g_0-admissible map (Definition 4.71) such that $n+1 = |\kappa|\sqrt{|T^+|}/2$. Choose:

(i) a set-theoretic section $\xi\colon G/T^+ \to G$ for the natural homomorphism $G \to G/T^+$;

(ii) a total order $\leq$ on G/T^+ such that there are no elements between x and $\bar g_0^{-1}x^{-1}$, for all $x \in G/T^+$;

and construct a tuple $\bar{\gamma}$ realizing κ according to $\pi \circ \xi$ and $\leq$ (Definition 2.20). Consider the $\overline{G}$-grading on $M(1,1)$ given by Equation (5.24) and the $\overline{G}$-grading $\Gamma_M(\overline{K}, \overline{\beta^+} \restriction_{\overline{K}\times\overline{K}}, \kappa)$ on $M_{n+1}(\mathbb{F})$ constructed using the choices of $\widetilde{\mathcal{D}}$ and $\bar{\gamma}$ above (see Definition 2.41). We, then, identify $S \coloneqq M(n+1, n+1)$ with the graded superalgebra $M(1,1) \otimes M_{n+1}(\mathbb{F})$. Define $\Theta \in S$ by Equation (5.25) and $\theta\colon S \to S$ by Equation (5.26) Finally, we define $\Gamma_A^{(\mathrm{II_Q})}(T^+, \beta^+, t_p, h, \kappa, g_0)$ to be the G-grading on $L = S^{(1)}/Z(S^{(1)})$ induced from the G-grading $S^{(1)}$ given by Equation (5.27).

Proposition 5.37. *Consider* $(R, \varphi) \coloneqq M^{\mathrm{ex}}(T^+, \beta^+, t_p, h, \kappa, g_0)$*, as defined before Corollary 4.76. Then the graded Lie superalgebra* $\mathrm{Skew}(R,\varphi)^{(1)}/Z(\mathrm{Skew}(R,\varphi)^{(1)})$ *is isomorphic to the Lie superalgebra* $A(n,n)$ *endowed with* $\Gamma_A^{(\mathrm{II_Q})}(T^+, \beta^+, t_p, h, \kappa, g_0)$*.*

Proof. Recall that $M^{\mathrm{ex}}(T^+, \beta^+, t_p, h, \kappa, g_0) = M^{\mathrm{ex}}(T, \beta, t_p, \kappa, g_0)$ (see Definition 4.72), where $t_1 \coloneqq (h, \bar{1})$, $T \coloneqq T^+ \cup t_1 T^+$, $\beta\colon T \times T \to \mathbb{F}^\times$ is the unique alternating bicharacter extending β^+ such that $\operatorname{rad}\beta = \langle f \rangle$ and $\beta(t_1, \cdot) = \chi_0$, and κ is seen as defined on $G^\#/T \simeq G/T^+$.

We will now show how the choices in Definition 5.36 correspond to the choices in Definition 4.67(b) and Definition 4.72.

By Lemma 5.18, with (T^+, β^+) playing the role of (T, β), the choices of K and $\tilde{\mathcal{D}}$ give us the same information as the choices of K in item (i) and $\mathcal{M}$ in item (ii) of Definition 4.67(b), and the map associated to the transposition on $\mathcal{M}$ is $\mu^+ \restriction_K$. Let $(\mathcal{D}, \varphi_0)$ denote the standard realization of a graded-division superalgebra with superinvolution constructed using these choices. Note that the map $\eta^+\colon T \to \{\pm 1\}$ defined in Equation (5.23) is such that $\mathrm{d}\eta^+ = \beta^+$, $\eta^+ \restriction_K = \mu^+ \restriction_K$, $\eta^+(f) = -1$ and $\eta^+(t_p) = 1$, so, by Equation (4.3), η^+ is the map associated to $\varphi_0 \restriction_{\mathcal{D}_{\bar{0}}}$. In particular, the g_0-admissibility condition for κ is the same in Definitions 4.72 and 5.36.

In Definition 4.72, we have to choose a graded right $\mathcal{D}$-supermodule $\mathcal{U}$ and a φ_0-sesquilinear form $B\colon \mathcal{U} \times \mathcal{U} \to \mathcal{D}$ such that $(\mathcal{U}, B)$ has inertia determined by κ. To this end, set $\mathcal{U} \coloneqq \mathcal{D}^{[g_1]} \oplus \cdots \oplus \mathcal{D}^{[g_k]}$, where $(g_1, \ldots, g_k)$ is the k-tuple realizing κ according to ξ and $\leq$, and $B(u_i, u_j) = \Phi_{ij}$, where $\mathcal{B} \coloneqq \{u_1, \ldots, u_k\}$ is the canonical $\mathcal{D}$-basis for $\mathcal{U}$ and $\Phi \in M_k(\mathcal{D})$ is defined by

$$\Phi \coloneqq \begin{pmatrix} \Phi(\bar{0}, x_1) & & \\ & \ddots & \\ & & \Phi(\bar{0}, x_\ell) \end{pmatrix}$$

(see Definition 5.8 and Equation (5.4)).

We will now follow Subsection 5.3.1 to undouble $M^{\text{ex}}(T^+, \beta^+, t_p, h, \kappa, g_0)$, constructed with the choices above. Recall that, by Definition 4.67(b), $\mathcal{D} = {}^{\alpha}\mathcal{O} \otimes \mathcal{M}$, where $\mathcal{O}$ is the graded-division superalgebra of Example 4.51, and $\alpha\colon \mathbb{Z}_2 \times \mathbb{Z}_4 \to \langle t_p, t_1 \rangle$ is the group isomorphism given by $\alpha(\bar{1}, \bar{0}) := t_p$ and $\alpha(\bar{0}, \bar{1}) := t_1$. Consider the central idempotent $\epsilon := \frac{1}{2}(1 + X_f) \in \mathcal{D}$ and the $\overline{G}$-graded-division superalgebra $\mathcal{D}\epsilon$. Clearly, $\epsilon \in \mathcal{O}$. It is straightforward that $\mathcal{O}\epsilon$ is isomorphic to $M(1,1)$ with grading given by Equation (5.24), and that $\mathcal{M}\epsilon$ is isomorphic to $\widetilde{\mathcal{D}}$. It follows that $\mathcal{D}\epsilon \simeq \overline{\mathcal{D}} := M(1,1) \otimes \widetilde{\mathcal{D}}$. We will identify $\mathcal{D}\epsilon$ with $\overline{\mathcal{D}}$ and consider $\overline{\mathcal{D}}$ as a matrix superalgebra via Kronecker product.

We need to extend χ to $G^{\#}$. Since $\chi(f) = -1$, we can do it in a way such that $\chi(t_1) = \mathbf{i}$. Let $\eta\colon T \to \{\pm 1\}$ be the map associated to φ_0 and let $\bar{\mu}\colon \overline{T} \to \mathbb{F}^\times$ be as defined in Equation (5.10). We have already shown that $\eta \restriction_{T^+} = \eta^+$ and, hence, by Equation (5.23), we have that $\bar{\mu} \restriction_{\overline{T^+}} = \overline{\mu^+}$. By Remark 4.68(b), we have $\bar{\mu}(\bar{t}_1) = \eta(t_1)\chi(t_1) = \mathbf{i}$. It follows that $\bar{\mu}$ is the map associated to the queer supertranspose on $\overline{\mathcal{D}}$.

As in the proof of Proposition 5.19, let $\Lambda \in M_k(\mathcal{D})$ be the diagonal matrix with entries $\Lambda_{ii} := \chi(\deg u_i)$, consider a different graded $\mathcal{D}$-basis $\tilde{\mathcal{B}} = \{\tilde{u}_1, \dots, \tilde{u}_k\}$ of $\mathcal{U}$, where $\tilde{u}_i$ is defined as in Equation (5.17), and let $\tilde{\Phi} \in M_k(\mathcal{D})$ be the matrix representing B with respect to $\tilde{\mathcal{B}}$. Note that the entries of $\tilde{\Phi}$ are in $\mathcal{D}^{\bar{0}} = ({}^{\alpha}\mathcal{O})^{\bar{0}} \otimes \mathcal{M}$, and that $\operatorname{supp} \mathcal{D}^{\bar{0}} = (\operatorname{rad} \beta^+) \times K$. Given $t \in T^+$, write $t = rs$, with $r \in \operatorname{rad} \beta^+$ and $s \in K$, and recall that we chose $X_t \in \mathcal{D}^{\bar{0}}$ to be $X_r \otimes X_s$, where $X_r \in ({}^{\alpha}\mathcal{O})^{\bar{0}}$ and $X_s \in \mathcal{M}$. Under the identification $\mathcal{D}\epsilon = \overline{\mathcal{D}}$, the element $X_r\epsilon$ becomes $\begin{pmatrix} 1 & 0 \\ 0 & \chi_0(r) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \chi_0(t) \end{pmatrix} \in M(1,1)^{\bar{0}}$. It follows that $\tilde{\Phi}\epsilon \in M_k(\overline{\mathcal{D}})$ goes to $\Theta \in S = M(1,1) \otimes M_k(\widetilde{\mathcal{D}})$, as defined in Equation (5.25), if we follow the isomorphisms

$$M_k(\overline{\mathcal{D}}) \simeq M_k(\mathbb{F}) \otimes \overline{\mathcal{D}} \simeq M_k(\mathbb{F}) \otimes M(1,1) \otimes \widetilde{\mathcal{D}} \simeq M(1,1) \otimes M_k(\mathbb{F}) \otimes \widetilde{\mathcal{D}} \simeq M(1,1) \otimes M_k(\widetilde{\mathcal{D}}).$$

Therefore, all the data in the description of the undoubled model of $\operatorname{Skew}(R, \varphi)$ coincide with the data in Definition 5.36, concluding the proof. $\square$

Theorem 5.38. *Every grading on the simple Lie superalgebra $\mathfrak{psl}(n+1|n+1)$, $n \geq 2$, is isomorphic to one of $\Gamma_A^{(\mathrm{I_M})}(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}})$, $\Gamma_A^{(\mathrm{I_Q})}(T^+, \beta^+, h, \kappa)$, $\Gamma_A^{(\mathrm{II_{osp}})}(T, \beta, \kappa_{\bar{0}}, \kappa_{\bar{1}}, h_0)$,*

$\Gamma_A^{(\mathrm{II_P})}(T,\beta,\kappa_{\bar 0},h_0)$ or $\Gamma_A^{(\mathrm{II_Q})}(T^+,\beta^+,t_p,h,\kappa,g_0)$, as in Definitions 5.28, 5.29, 5.31, 5.32 and 5.36. Gradings belonging to different types are not isomorphic. Within each type, we have:

$\boxed{\text{Type I}_{\mathrm{M}}}$

$\Gamma_A^{(\mathrm{I_M})}(T,\beta,\kappa_{\bar 0},\kappa_{\bar 1}) \simeq \Gamma_A^{(\mathrm{I_M})}(T',\beta',\kappa'_{\bar 0},\kappa'_{\bar 1})$ if, and only if, $T=T'$ and one of the following conditions holds:

(i) $\beta'=\beta$ *and there is* $g\in G$ *such that either* $\kappa'_{\bar 0}=g\cdot\kappa_{\bar 0}$ *and* $\kappa'_{\bar 1}=g\cdot\kappa_{\bar 1}$*, or* $\kappa'_{\bar 0}=g\cdot\kappa_{\bar 1}$ *and* $\kappa'_{\bar 1}=g\cdot\kappa_{\bar 0}$*;*

(ii) $\beta'=\beta^{-1}$ *and there is* $g\in G$ *such that either* $\kappa'_{\bar 0}=g\cdot\kappa^\star_{\bar 0}$ *and* $\kappa'_{\bar 1}=g\cdot\kappa^\star_{\bar 1}$*, or* $\kappa'_{\bar 0}=g\cdot\kappa^\star_{\bar 1}$ *and* $\kappa'_{\bar 1}=g\cdot\kappa^\star_{\bar 0}$*.*

$\boxed{\text{Type I}_{\mathrm{Q}}}$

$\Gamma_A^{(\mathrm{I_Q})}(T^+,\beta^+,h,\kappa) \simeq \Gamma_A^{(\mathrm{I_Q})}(T'^+,\beta'^+,h',\kappa')$ if, and only if, $T'^+=T^+$ and one of the following conditions holds:

(i) $\beta'^+=\beta^+$*,* $h'\in\{h,ht_p\}$ *and there is* $g\in G$ *such that* $\kappa'=g\cdot\kappa$*;*

(ii) $\beta'^+=(\beta^+)^{-1}$*,* $h'\in\{h^{-1},h^{-1}t_p\}$ *and there is* $g\in G$ *such that* $\kappa'=g\cdot\kappa^\star$*;*

where t_p is the nontrivial element in $\operatorname{rad}\beta^+$.

$\boxed{\text{Type II}_{\mathfrak{osp}}}$

$\Gamma_A^{(\mathrm{II}_{\mathfrak{osp}})}(T,\beta,\kappa_{\bar 0},\kappa_{\bar 1},g_0) \simeq \Gamma_A^{(\mathrm{II}_{\mathfrak{osp}})}(T',\beta',\kappa'_{\bar 0},\kappa'_{\bar 1},g'_0)$ if, and only if, $T=T'$, $\beta=\beta'$ and there is $g\in G$ such that one of the following conditions holds:

(i) $\kappa'_{\bar 0}=g\cdot\kappa_{\bar 0}$*,* $\kappa'_{\bar 1}=g\cdot\kappa_{\bar 1}$ *and* $g'_0=g^{-2}g_0$*;*

(ii) $\kappa'_{\bar 0}=g\cdot\kappa_{\bar 1}$*,* $\kappa'_{\bar 1}=g\cdot\kappa_{\bar 0}$ *and* $g'_0=fg^{-2}g_0$*.*

$\boxed{\text{Type II}_{\mathrm{P}}}$

$\Gamma_A^{(\mathrm{II_P})}(T,\beta,\kappa_{\bar 0},h_0) \simeq \Gamma_A^{(\mathrm{II_P})}(T',\beta',\kappa'_{\bar 0},h'_0)$ if, and only if, $T=T'$, $\beta=\beta'$ and there is $g\in G$ such that one of the following conditions holds:

(i) $\kappa'_{\bar 0}=g\cdot\kappa_{\bar 0}$ *and* $h'_0=g^{-2}h_0$*;*

(ii) $\kappa'_{\bar 0} = gh_0^{-1} \cdot \kappa^{\star}_{\bar 0}$ *and* $h'_0 = fg^{-2}h_0$.

$\boxed{\text{Type II}_{\mathrm{Q}}}$

$\Gamma_A^{(\mathrm{II_Q})}(T^+, \beta^+, t_p, h, \kappa, g_0) \simeq \Gamma_A^{(\mathrm{II_Q})}(T'^+, \beta'^+, t'_p, h', \kappa', g'_0)$ *are isomorphic if, and only if,* $T^+ = T'^+$, $\beta^+ = \beta'^+$, $t_p = t'_p$, $h' \in h(\operatorname{rad}\beta^+)$ *and there is* $g \in G$ *such that* $\kappa' = g \cdot \kappa$ *and*

(i) $g'_0 = g^{-2}g_0$ *if* $h' \in \{h, ft_p h\}$*;*

(ii) $g'_0 = t_p g^{-2} g_0$ *if* $h' \in \{fh, t_p h\}$*.*

Proof. The result follows from Theorem 4.55 (Types $\mathrm{I_M}$ and $\mathrm{I_Q}$), Theorem 4.70 (Types $\mathrm{II}_{\mathfrak{osp}}$ and $\mathrm{II_P}$, taking into account in the latter case that $\kappa_{\bar 1} = h_0^{-1} \cdot \kappa^{\star}_{\bar 0}$) and Corollary 4.76 (Type $\mathrm{II_Q}$). Indeed, by Corollary 5.4, the isomorphism classes of gradings on the Lie superalgebra $A(n,n)$ are in bijection with the isomorphism classes of gradings on the associative superalgebra $M(n+1,n+1) \times M(n+1,n+1)^{\mathrm{sop}}$ endowed with the exchange superinvolution. Gradings of Types $\mathrm{I_M}$ and $\mathrm{I_Q}$ are already described in terms of $M(n+1,n+1)$. Proposition 5.19 gives such a description for gradings of Type $\mathrm{II}_{\mathfrak{osp}}$ and $\mathrm{II_P}$, and Proposition 5.37 for Type $\mathrm{II_Q}$. $\square$

Bibliography

[ABFP08] B. Allison, S. Berman, J. Faulkner, and A. Pianzola, *Realization of graded-simple algebras as loop algebras*, Forum Math. **20** (2008), no. 3, 395–432. MR 2418198

[AG13] E. Aljadeff and A. Giambruno, *Multialternating graded polynomials and growth of polynomial identities*, Proc. Amer. Math. Soc. **141** (2013), no. 9, 3055–3065. MR 3068959

[AH14] E. Aljadeff and D. Haile, *Simple G-graded algebras and their polynomial identities*, Trans. Amer. Math. Soc. **366** (2014), no. 4, 1749–1771. MR 3152711

[AK10] E. Aljadeff and A. Kanel-Belov, *Representability and Specht problem for G-graded algebras*, Adv. Math. **225** (2010), no. 5, 2391–2428. MR 2680170

[Arz07] I. V. Arzhantsev, *Introduction to algebraic groups and invariant theory*, 2007, URL: `http://halgebra.math.msu.su/staff/arzhan/driver.pdf`, visited on 2020-10-13.

[Bah87] Y. A. Bahturin, *Identical relations in Lie algebras*, VNU Science Press, Utrecht, 1987, Translated from the Russian by Bahturin. MR 886063

[BD02] Y. Bahturin and V. Drensky, *Graded polynomial identities of matrices*, Linear Algebra Appl. **357** (2002), 15–34. MR 1935223

[BD19] A. Bianchi and D. Diniz, *Identities and isomorphisms of finite-dimensional graded simple algebras*, J. Algebra **526** (2019), 333–344. MR 3920905

[BG08] Y. Bahturin and A. Giambruno, *Group grading on associative algebras with involution*, Canad. Math. Bull. **51** (2008), no. 2, 182–194. MR 2414206

[BGR16] Y. Bahturin, M. Goze, and E. Remm, *Group gradings on filiform Lie algebras*, Comm. Algebra **44** (2016), no. 1, 40–62. MR 3413670

[BHSK17] Y. Bahturin, C. De Naday Hornhardt, H. S. Dos Santos, and M. Kochetov, *Group gradings on the Lie and Jordan superalgebras $Q(n)$*, Comm. Algebra **45** (2017), no. 5, 1914–1925. MR 3582835

[BK10] Y. Bahturin and M. Kochetov, *Classification of group gradings on simple Lie algebras of types A, B, C and D*, J. Algebra **324** (2010), no. 11, 2971–2989. MR 2732982

[BK11] ———, *Group gradings on restricted Cartan-type Lie algebras*, Pacific J. Math. **253** (2011), no. 2, 289–319. MR 2878812

[BKR18] Y. Bahturin, M. Kochetov, and A. Rodrigo-Escudero, *Gradings on classical central simple real Lie algebras*, J. Algebra **506** (2018), 1–42. MR 3800069

[BM99] Y. Bahturin and S. Montgomery, *PI-envelopes of Lie superalgebras*, Proc. Amer. Math. Soc. **127** (1999), no. 10, 2829–2839. MR 1600092

[BMPZ92] Y. A. Bahturin, A. A. Mikhalev, V. M. Petrogradsky, and M. V. Zaicev, *Infinite-dimensional Lie superalgebras*, De Gruyter Expositions in Mathematics, vol. 7, Walter de Gruyter & Co., Berlin, 1992. MR 1192546

[BP09] Y. Bahturin and D. Pagon, *Classifying simple color Lie superalgebras*, Algebras, representations and applications, Contemp. Math., vol. 483, Amer. Math. Soc., Providence, RI, 2009, pp. 37–54. MR 2497949

[Bre19] M. Brešar, *Undergraduate algebra*, Springer Undergraduate Mathematics Series, Springer, Cham, 2019, A unified approach. MR 3932087

[BS06] Y. Bahturin and I. Shestakov, *Group gradings on associative superalgebras*, Groups, rings and algebras, Contemp. Math., vol. 420, Amer. Math. Soc., Providence, RI, 2006, pp. 1–13. MR 2279228

[BSZ01] Y. Bahturin, S. Sehgal, and M. Zaicev, *Group gradings on associative algebras*, J. Algebra **241** (2001), no. 2, 677–698. MR 1843319

[BSZ05] Y. Bahturin, I. Shestakov, and M. Zaicev, *Gradings on simple Jordan and Lie algebras*, J. Algebra **283** (2005), no. 2, 849–868. MR 2111225

[BTT09] Y. Bahturin, M. Tvalavadze, and T. Tvalavadze, *Group gradings on superinvolution simple superalgebras*, Linear Algebra Appl. **431** (2009), no. 5-7, 1054–1069. MR 2535573

[BY19] Y. Bahturin and F. Yasumura, *Graded polynomial identities as identities of universal algebras*, Linear Algebra Appl. **562** (2019), 1–14. MR 3886336

[BZ02] Y. Bahturin and M. Zaicev, *Group gradings on matrix algebras*, Canad. Math. Bull. **45** (2002), no. 4, 499–508, Dedicated to Robert V. Moody. MR 1941224

[BZ03] ———, *Graded algebras and graded identities*, Polynomial identities and combinatorial methods (Pantelleria, 2001), Lecture Notes in Pure and Appl. Math., vol. 235, Dekker, New York, 2003, pp. 101–139. MR 2021796

[BZ06] ———, *Group gradings on simple Lie algebras of type "A"*, J. Lie Theory **16** (2006), no. 4, 719–742. MR 2270657

[BZ07] ———, *Involutions on graded matrix algebras*, J. Algebra **315** (2007), no. 2, 527–540. MR 2351876

[BZ18] ———, *Graded division algebras over the field of real numbers*, J. Algebra **514** (2018), 273–309. MR 3853066

[Car14] E. Cartan, *Les groupes réels simples, finis et continus*, Ann. Sci. École Norm. Sup. (3) **31** (1914), 263–355. MR 1509178

[CE18] A. S. Córdova-Martínez and A. Elduque, *Gradings on semisimple algebras*, Linear Algebra Appl. **559** (2018), 145–171. MR 3857543

[Coh81] P. M. Cohn, *Universal algebra*, second ed., Mathematics and its Applications, vol. 6, D. Reidel Publishing Co., Dordrecht-Boston, Mass., 1981. MR 620952

[CSV95] A. Cap, H. Schichl, and J. Vanžura, *On twisted tensor products of algebras*, Comm. Algebra **23** (1995), no. 12, 4701–4735. MR 1352565

[CW12] S. Cheng and W. Wang, *Dualities and representations of Lie superalgebras*, Graduate Studies in Mathematics, vol. 144, American Mathematical Society, Providence, RI, 2012. MR 3012224

[DE16] C. Draper and A. Elduque, *An overview of fine gradings on simple Lie algebras*, Note Mat. **36** (2016), no. suppl. 1, 15–34. MR 3601079

[DEK17] C. Draper, A. Elduque, and M. Kochetov, *Gradings on modules over Lie algebras of E types*, Algebr. Represent. Theory **20** (2017), no. 5, 1085–1107. MR 3707906

[DEM11] C. Draper, A. Elduque, and C. Martín, *Fine gradings on exceptional simple Lie superalgebras*, Internat. J. Math. **22** (2011), no. 12, 1823–1855. MR 2872534

[DM06] C. Draper and C. Martín, *Gradings on* $\mathfrak{g}_2$, Linear Algebra Appl. **418** (2006), no. 1, 85–111. MR 2257580

[EK13] A. Elduque and M. Kochetov, *Gradings on simple Lie algebras*, Mathematical Surveys and Monographs, vol. 189, American Mathematical Society, Providence, RI; Atlantic Association for Research in the Mathematical Sciences (AARMS), Halifax, NS, 2013. MR 3087174

[EK15a] ______, *Graded modules over classical simple Lie algebras with a grading*, Israel J. Math. **207** (2015), no. 1, 229–280. MR 3358046

[EK15b] ______, *Gradings on the Lie algebra D_4 revisited*, J. Algebra **441** (2015), 441–474. MR 3391935

[EK17] ______, *Graded simple modules and loop modules*, Groups, rings, group rings, and Hopf algebras, Contemp. Math., vol. 688, Amer. Math. Soc., Providence, RI, 2017, pp. 53–85. MR 3649168

[EK18] ______, *Gradings on the simple real Lie algebras of types G_2 and D_4*, J. Algebra **512** (2018), 382–426. MR 3841529

[Eld10] A. Elduque, *Fine gradings on simple classical Lie algebras*, J. Algebra **324** (2010), no. 12, 3532–3571. MR 2735398

[Eld16] ______, *Gradings on algebras over algebraically closed fields*, Nonassociative and non-commutative algebra and operator theory, Springer Proc. Math. Stat., vol. 160, Springer, Cham, 2016, pp. 113–121. MR 3613822

[EV08] A. Elduque and O. Villa, *The existence of superinvolutions*, J. Algebra **319** (2008), no. 10, 4338–4359. MR 2407903

[FH91] W. Fulton and J. Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics. MR 1153249

[FN56] A. Frölicher and A. Nijenhuis, *Theory of vector-valued differential forms. I. Derivations of the graded ring of differential forms*, Nederl. Akad. Wetensch. Proc. Ser. A. **59** = Indag. Math. **18** (1956), 338–359. MR 0082554

[FSS00] L. Frappat, A. Sciarrino, and P. Sorba, *Dictionary on Lie algebras and superalgebras*, Academic Press, Inc., San Diego, CA, 2000, With 1 CD-ROM (Windows, Macintosh and UNIX). MR 1773773

[Ger63] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. (2) **78** (1963), 267–288. MR 161898

[Ger64] ______, *On the deformation of rings and algebras*, Ann. of Math. (2) **79** (1964), 59–103. MR 171807

[GN64] M. Gell-Mann and Y. Ne'eman, *The eightfold way*, W. A. Benjamin, Inc., New York-Amsterdam, 1964. MR 0180214

[Gor13] A. S. Gordienko, *Amitsur's conjecture for associative algebras with a generalized Hopf action*, J. Pure Appl. Algebra **217** (2013), no. 8, 1395–1411. MR 3030542

[Gor15] ______, *Amitsur's conjecture for polynomial H-identities of H-module Lie algebras*, Trans. Amer. Math. Soc. **367** (2015), no. 1, 313–354. MR 3271263

[GP04] D. Grantcharov and A. Pianzola, *Automorphisms and twisted loop algebras of finite-dimensional simple Lie superalgebras*, Int. Math. Res. Not. (2004), no. 73, 3937–3962. MR 2104290

[GS98] C. Gómez-Ambrosi and I. Shestakov, *On the Lie structure of the skew elements of a simple superalgebra with superinvolution*, J. Algebra **208** (1998), no. 1, 43–71. MR 1643975

[GS19] Y. Ginosar and O. Schnabel, *Groups of central-type, maximal connected gradings and intrinsic fundamental groups of complex semisimple algebras*, Trans. Amer. Math. Soc. **371** (2019), no. 9, 6125–6168. MR 3937320

[GZ05] A. Giambruno and M. Zaicev, *Polynomial identities and asymptotic methods*, Mathematical Surveys and Monographs, vol. 122, American Mathematical Society, Providence, RI, 2005. MR 2176105

[HPP98] M. Havlíček, J. Patera, and E. Pelantová, *On Lie gradings. II*, Linear Algebra Appl. **277** (1998), no. 1-3, 97–125. MR 1624532

[HSK19] C. De Naday Hornhardt, H. S. Dos Santos, and M. Kochetov, *Group gradings on the superalgebras $M(m,n)$, $A(m,n)$ and $P(n)$*, J. Pure Appl. Algebra **223** (2019), no. 4, 1590–1616. MR 3906518

[IW53] E. İnönü and E. P. Wigner, *On the contraction of groups and their representations*, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 510–524. MR 55352

[Jeu88] J. Van der Jeugt, *Finest grading of the Lie superalgebra* $\mathrm{gl}(n/n)$, J. Phys. A **21** (1988), no. 24, L1163–L1167. MR 974693

[Kac68] V. G. Kac, *Graded Lie algebras and symmetric spaces*, Funkcional. Anal. i Priložen. **2** (1968), no. 2, 93–94. MR 0231944

[Kac77a] ______, *Classification of simple Z-graded Lie superalgebras and simple Jordan superalgebras*, Comm. Algebra **5** (1977), no. 13, 1375–1400. MR 0498755

[Kac77b] ______, *Lie superalgebras*, Advances in Math. **26** (1977), no. 1, 8–96. MR 0486011

[Kac90] ______, *Infinite-dimensional Lie algebras*, third ed., Cambridge University Press, Cambridge, 1990. MR 1104219

[Kan64] I. L. Kantor, *Classification of irreducible transitive differential groups*, Dokl. Akad. Nauk SSSR **158** (1964), 1271–1274. MR 0175941

[Kar73] G. Karrer, *Graded division algebras*, Math. Z. **133** (1973), 67–73. MR 325674

[Kil88] W. Killing, *Die Zusammensetzung der stetigen endlichen Transformationsgruppen*, Math. Ann. **33** (1888), no. 1, 1–48. MR 1510529

[Koe67] M. Koecher, *Imbedding of Jordan algebras into Lie algebras. I*, Amer. J. Math. **89** (1967), 787–816. MR 214631

[KY17a] P. Koshlukov and F. Yasumura, *Group gradings on the Jordan algebra of upper triangular matrices*, Linear Algebra Appl. **534** (2017), 1–12. MR 3697052

[KY17b] ______, *Group gradings on the Lie algebra of upper triangular matrices*, J. Algebra **477** (2017), 294–311. MR 3614154

[KY19] M. Kochetov and F. Yasumura, *Group gradings on the Lie and Jordan algebras of block-triangular matrices*, J. Algebra **537** (2019), 147–172. MR 3990040

[KZ10] P. Koshlukov and M. Zaicev, *Identities and isomorphisms of graded simple algebras*, Linear Algebra Appl. **432** (2010), no. 12, 3141–3148. MR 2639274

[Mac95] S. Mac Lane, *Homology*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the 1975 edition. MR 1344215

[Mic70] L. Michel, *Applications of group theory to quantum physics. Algebraic aspects*, Group Representations in Math. and Phys. (Battelle Seattle 1969 Rencontres), Lecture Notes in Physics, Vol. 6, Springer, Berlin, 1970, pp. 36–143. MR 0278639

[Miy68] H. Miyazawa, *Spinor currents and symmetries of baryons and mesons*, Phys. Rev. **170** (1968), 1586–1590.

[MM65] J. Milnor and J. Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) **81** (1965), 211–264. MR 174052

[MP91] M. de Montigny and J. Patera, *Discrete and continuous graded contractions of Lie algebras and superalgebras*, J. Phys. A **24** (1991), no. 3, 525–547. MR 1098200

[NO04] C. Năstăsescu and F. Van Oystaeyen, *Methods of graded rings*, Lecture Notes in Mathematics, vol. 1836, Springer-Verlag, Berlin, 2004. MR 2046303

[OV90] A. L. Onishchik and È. B. Vinberg, *Lie groups and algebraic groups*, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1990, Translated from the Russian and with a preface by D. A. Leites. MR 1064110

[Pat89] J. Patera, *The four sets of additive quantum numbers of* $\mathrm{SU}(3)$, J. Math. Phys. **30** (1989), no. 12, 2756–2762. MR 1025215

[PPS02] J. Patera, E. Pelantová, and M. Svobodová, *The eight fine gradings of* $\mathrm{sl}(4,\mathbb{C})$ *and* $\mathrm{o}(6,\mathbb{C})$, J. Math. Phys. **43** (2002), no. 12, 6353–6378. MR 1940451

[PZ89] J. Patera and H. Zassenhaus, *On Lie gradings. I*, Linear Algebra Appl. **112** (1989), 87–159. MR 976333

[Rac98] M. L. Racine, *Primitive superalgebras with superinvolution*, J. Algebra **206** (1998), no. 2, 588–614. MR 1637088

[Raz94] Y. P. Razmyslov, *Identities of algebras and their representations*, Translations of Mathematical Monographs, vol. 138, American Mathematical Society, Providence, RI, 1994, Translated from the 1989 Russian original by A. M. Shtern. MR 1291603

[RZ17] D. Repovš and M. Zaicev, *Pauli gradings on Lie superalgebras and graded codimension growth*, Linear Algebra Appl. **520** (2017), 134–150. MR 3611462

[San19] H. S. Dos Santos, *Group gradings on the classical Lie superalgebras* $Q(n)$, $P(n)$ *and* $B(m,n)$, Ph.D. Thesis, Memorial University of Newfoundland, April 2019.

[Sch79a] M. Scheunert, *Generalized Lie algebras*, J. Math. Phys. **20** (1979), no. 4, 712–720. MR 529734

[Sch79b] ———, *The theory of Lie superalgebras*, Lecture Notes in Mathematics, vol. 716, Springer, Berlin, 1979, An introduction. MR 537441

[Ser84] V. V. Serganova, *Automorphisms of simple Lie superalgebras*, Izv. Akad. Nauk SSSR Ser. Mat. **48** (1984), no. 3, 585–598. MR 747252

[Str04] H. Strade, *Simple Lie algebras over fields of positive characteristic. I*, De Gruyter Expositions in Mathematics, vol. 38, Walter de Gruyter & Co., Berlin, 2004, Structure theory. MR 2059133

[Str09] ———, *Simple Lie algebras over fields of positive characteristic. II*, De Gruyter Expositions in Mathematics, vol. 42, Walter de Gruyter & Co., Berlin, 2009, Classifying the absolute toral rank two case. MR 2573283

[Str13] ———, *Simple Lie algebras over fields of positive characteristic. III*, De Gruyter Expositions in Mathematics, vol. 57, Walter de Gruyter GmbH & Co. KG, Berlin, 2013, Completion of the classification. MR 3025870

[Tit62] J. Tits, *Une classe d'algèbres de Lie en relation avec les algèbres de Jordan*, Indag. Math. (N.S.) **65** (1962), 530–535. MR 0146231

[TT06] M. Tvalavadze and T. Tvalavadze, *Group gradings on superalgebras*, 2006, Preprint, arXiv:math/0609760v2 [math.RA].

[Var04] V. S. Varadarajan, *Supersymmetry for mathematicians: an introduction*, Courant Lecture Notes in Mathematics, vol. 11, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2004. MR 2069561

[VZ07] A. Valenti and M. Zaicev, *Group gradings on upper triangular matrices*, Arch. Math. (Basel) **89** (2007), no. 1, 33–40. MR 2322777

[Wal64] C. T. C. Wall, *Graded Brauer groups*, J. Reine Angew. Math. **213** (1963/64), 187–199. MR 167498

[YK18] F. Yasumura and P. Koshlukov, *Asymptotics of graded codimension of upper triangular matrices*, Israel J. Math. **223** (2018), no. 1, 423–439. MR 3773068

[Yu16] J. Yu, *Maximal abelian subgroups of compact simple Lie groups of type E*, Geom. Dedicata **185** (2016), 205–269. MR 3570309