Quantum dynamics with classical noise

by

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Department of Mathematics and Statistics
Memorial University

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Abstract

In this thesis, we study the evolution of qubits evolving according to the Schrödinger equation with a Hamiltonian containing noise terms, modeled by random diagonal and off-diagonal matrix elements. For a single qubit exposed to such noise, we show that the noise-averaged qubit density matrix converges to a specific final state, in the limit of large time t. We find that the convergence speed is polynomial in $1/t$, with a power that depends on the regularity and the low frequency behaviour of the noise probability density. We evaluate the final state explicitly in the regimes of weak and strong off-diagonal noise. We show that the process implements the well-known dephasing channel in the localized and delocalized basis, respectively.

Furthermore, we consider the evolution of the entanglement of two (or more) qubits subject to Gaussian noises with varying means and variances. We consider two different cases: individual noise where each qubit feels an independent noise, and common noise where all qubits are subjected to the same noise. We find the following characteristics of entanglement, measured by the concurrence of qubits. Initially entangled states lose their amount of entanglement in time due to the presence of the noise. The decay of entanglement happens more quickly for common noise than for individual noise. We also detect creation of entanglement due to the common noise: for some initially disentangled states, entanglement is created for intermediate times and then decays to zero in the long time again.

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Chapter 1

Guide to main results

The goal of this chapter is to give a list of our main results. The underlying concepts will be explained in the thesis.

- **Results on decoherence.** In Chapter 3 we consider a single qubit evolving in a noisy environment. The noise is modeled by a random Hamiltonian containing a diagonal part (energy basis) and an off-diagonal part. We show the following results.
  - (Theorem 4.1) The noise-averaged qubit density matrix converges to a final state in the limit of large times t. If the diagonal noise has a probability density which is n times continuously differentiable ($n \geq 0$ an integer), then the speed of convergence is (at least) $\propto t^{-n}$. Moreover, the final state has an explicit form. It depends on the characteristics of the noises *as well as* on the initial state.
  - (Theorem 4.2) In absence of diagonal noise, the pure off-diagonal noise still drives the diagonal density matrix elements (called the populations) to final values as $t \to \infty$, at a speed $1/t$. If the noise does not contain a strong low frequency component, that is, if its probability density vanishes at frequency $\omega = 0$ as a power ω^k, where $k \geq 1$ is an integer, then the averaged density matrix still converges to a final state (the same as in Theorem 4.1), at a speed $t^{-(k+1)/2}$. This result shows that intuitively, the higher frequency noise modes speed up the convergence.
  - (Theorem 4.3) The final state has the following properties: For weak off-diagonal noise, it is close to the state obtained simply by setting the off-diagonal density matrix elements of the initial state, when represented in the energy basis, equal to zero. (Dephasing in the energy basis.) For strong off-diagonal noise, the final state is close to the one obtained from the initial one by removing the off-diagonal density matrix elements, when represented in the delocalized (adiabatic) basis. (Dephasing in the delocalized basis.)
- **Results on entanglement.** In Chapter 4 we analyze the evolution of entanglement of two or more qubits subjected to individual ('local') noises and/or common ('global')

noises. The qubits are not coupled directly, but in the common noise case, they do interact indirectly. We are concerned with two questions: (i) does the noise suppress initially existing entanglement in the course of time, and (ii) can the noise create entanglement in an initially disentangled state?

- In Section 4.2 we take two qubits initially in a Bell state, which is a (maximally) entangled state, subjected to a Gaussian noise with variance σ^2. We show that the concurrence $\mathcal{C}(t)$ evolves as follows: for individual noises, $\mathcal{C}(t) = e^{-\sigma^2 t^2}$, while for common noise, $\mathcal{C}(t) = e^{-2\sigma^2 t^2}$. (The concurrence is independent of the mean.) This confirms the intuitive picture that common noise accelerates the loss of quantum properties (entanglement = concurrence), relative to individual noise decoherence.
- In Section 4.3 we take two qubits in an arbitrary pure product state (no entanglement). We show that even though at any time $t > 0$, the noise causes the two qubit state to be a mixed one, the latter will always stay separable (not entangled), $\forall t \geq 0$. Local noise cannot create entanglement. However, we show that for common noise, the qubit pair *does* become entangled for intermediate times, and the amount of entanglement can become sizable (up to at least $\approx 75\%$ of the maximally possible value). Creation of entanglement by the common noise is especially large for intermediate sizes of the mean of the noise distribution. (For noises with mean zero the amount is very small.) For increasing variance, the maximal amount of entanglement produced decreases.
- In Section 4.3.2 we consider N qubits in local and collective noises. Again, given an initially arbitrary pure product state of the N qubits, we find that the local noise cannot create entanglement at any time $t \geq 0$. Finally we consider a system consisting of two M-level systems ($M = 2$ represents the qubit situation before) and we show that again, local noises can never create entanglement in an initially disentangled state.

Chapter 2

Introduction

Quantum theory plays a central role of our present understanding of the laws of physics and mathematics. It contains the research of quantum systems subjected to classical (commutative) noises which are often used to model the effects of an environment (a reservoir or a bath) on a relatively small system [1]. In nature, very many systems are open, that is, subjected to a coupling with an uncontrollable environment which influences it in a non-negligible way. The theory of open quantum systems thus becomes very important in many applications of quantum physics since it is impossible for us to isolate quantum systems [5]. An important phenomenon happening in open quantum systems is *decoherence*, which can be viewed as the loss of information from a system into the environment. Mathematically, decoherence means that the off-diagonal elements of the system density matrix (of an open quantum system), in the energy basis, decay to zero as time goes to infinity [15]. Another important aspect in open quantum systems is the influence of noise on the *entanglement* between sub-systems. Entanglement is a quantum mechanical property that Schrödinger singled out many decades ago as "the characteristic trait of quantum mechanics" [32]. Quantum entanglement usually occurs when two (or more) particles become inextricably linked, and whatever happens to one immediately affects the other, regardless of how far apart they are.

2.1 The postulates of quantum theory

Quantum mechanics is a mathematical framework for the development of physical theories. We will introduce the postulates of quantum mechanics which provide a connection between the physical world and the mathematical formalism of quantum mechanics [27].

2.1.1 Postulate 1: Pure state space

Associated to any isolated physical system $\mathcal{S}$ is a complex Hilbert space $\mathcal{H}_{\mathcal{S}}$, called the pure state space. The system is completely described by its state vector, which is a unit vector in the system's space. That is,

$$|\Psi\rangle \in \mathcal{H}_{\mathcal{S}}, \quad ||\Psi|| = 1. \tag{2.1}$$

This pure state vector is called a **ket** (or a wave function). We can also define the dual space of $\mathcal{H}_{\mathcal{S}}$, denoted by $\mathcal{H}_{\mathcal{S}}^*$. Any $\langle\Phi| \in \mathcal{H}_{\mathcal{S}}^*$, known as a **bra**, is a linear function acting on the Hilbert space $\mathcal{H}_{\mathcal{S}}$, given by

$$\langle\Phi|\big(|\Psi\rangle\big) = \langle\Phi|\Psi\rangle, \tag{2.2}$$

where the right hand side representing the inner product of Φ and Ψ.

In the sequel, we will often write $\mathcal{H}$ instead of $\mathcal{H}_{\mathcal{S}}$ for the pure state Hilbert space.

Example. The state of the simplest quantum mechanical system having two degrees of freedom, that is, a spin or a qubit. A qubit has a two-dimensional Hilbert space $\mathcal{H}$. Suppose $|+\rangle$ and $|-\rangle$ form an orthonormal basis for this space $\mathcal{H}$, which may be written as

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.3}$$

Then any state vector $|\Psi\rangle \in \mathcal{H}$ can be written as

$$|\Psi\rangle = \alpha|+\rangle + \beta|-\rangle, \tag{2.4}$$

where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$ as we have the condition that $||\Psi||^2 = \langle\Psi|\Psi\rangle = 1$.

2.1.2 Postulate 2: Evolution (Schrödinger equation)

The time evolution of the state of a closed quantum system is described by the Schrödinger equation,

$$i\partial_t|\Psi(t)\rangle = H_0|\Psi(t)\rangle \tag{2.5}$$

where H_0 is the Hamiltonian, a fixed hermitian operator acting on the Hilbert space $\mathcal{H}_{\mathcal{S}}$. The subindex 0 indicates that, this is a 'noiseless' Hamiltonian. Given any initial state $|\Psi(0)\rangle \in \mathcal{H}_{\mathcal{S}}$, the equation (2.5) has a unique solution given by

$$|\Psi(t)\rangle = e^{-itH_0}|\Psi(0)\rangle. \tag{2.6}$$

Example. Suppose the Hamiltonian is given by

$$H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}, \tag{2.7}$$

with real eigenvalues E_1, E_2 and associated eigenvectors (2.4), making up the so-called energy-, localized- or diabatic basis. Denoting the spectral projections by $P_1 = |+\rangle\langle+|$ and $P_2 = |-\rangle\langle-|$, the evolution (2.6) takes the form

$$|\Psi(t)\rangle = \sum_{k=1,2} e^{-itE_k} P_k |\Psi(0)\rangle. \tag{2.8}$$

2.1.3 Postulate 3: Composite systems

The state space of a composite physical system is the tensor product of the state spaces of the individual physical systems. That is, if we have N systems with Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_N$, respectively, then

$$\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N \tag{2.9}$$

is the space of pure states of the composite system. Suppose we know that system j is in the state $|\phi_j\rangle$, $j = 1, \ldots, N$. Then the state of the composite system is

$$|\Psi\rangle = |\phi_1\rangle \otimes \cdots \otimes |\phi_N\rangle \equiv |\phi_1 \cdots \phi_N\rangle. \tag{2.10}$$

However, not all elements $|\Psi\rangle \in \mathcal{H}$ are of product form (2.10). A general element of $\mathcal{H}$, (2.9) is given by

$$|\Psi\rangle = \sum_{i_1,\ldots,i_N} \Psi^1_{i_1} \cdots \Psi^N_{i_N} |\phi^1_{i_1} \cdots \phi^N_{i_N}\rangle, \tag{2.11}$$

where, $\{|\phi^j_k\rangle\}_k$ is an orthonormal basis of $\mathcal{H}_j$ and the $\Psi^k_{i_k}$ are complex numbers. Relation (2.11) gives the decomposition of the vector $|\Psi\rangle$ in the orthonormal basis $|\phi^1_{i_1} \cdots \phi^N_{i_N}\rangle$ of $\mathcal{H}$.

Example. Suppose we have two systems $\mathcal{A}$ and $\mathcal{B}$ with associated Hilbert space $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$, respectively. The composite system of $\mathcal{A}$ and $\mathcal{B}$ is described by the tensor product

$$\mathcal{H} = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}. \tag{2.12}$$

For any $|\varphi_1\rangle \in \mathcal{H}_\mathcal{A}$ and $|\varphi_2\rangle \in \mathcal{H}_\mathcal{B}$, the state of the composite system $\mathcal{A} \cup \mathcal{B}$ in which each subsystem is in the prescribed state just given, is

$$|\varphi\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle. \tag{2.13}$$

2.1.4 Postulate 4: Quantum measurements

Quantum measurements are described by a collection $\{M_m\}$ of *measurement operators* which are acting on the state space $\mathcal{H}$ of the system being measured and satisfying the *completeness relation*,

$$\sum_m M_m^\dagger M_m = I, \tag{2.14}$$

where I is an identity operator on $\mathcal{H}$. The index m refers to the measurement outcomes that may occur. If the state of the quantum system is $|\psi\rangle$ immediately before the measurement then the probability that the result m occurs is given by

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle. \tag{2.15}$$

If the outcome m is measured, then the state of the system immediately after the measurement is

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}. \tag{2.16}$$

This fact is called the *collapse of the wave function*. Namely, which outcome will be observed is a random process, but once the outcome m has happened (was measured), the state immediately after the measurement is known precisely, it is (2.16). In this context, the state (2.16) is called the *post measurement state*. The completeness relation expresses the fact that probabilities sum to one:

$$1 = \sum_m p(m) = \sum_m \langle\psi|M_m^\dagger M_m|\psi\rangle. \tag{2.17}$$

Example. We consider the measurement of a qubit in the computational basis $\{|+\rangle, |-\rangle\}$. Let the two measurement operators be given by

$$M_0 = |+\rangle\langle+| \quad \text{and} \quad M_1 = |-\rangle\langle-|. \tag{2.18}$$

It is easily to check that

$$M_0^\dagger M_0 + M_1^\dagger M_1 = 1, \tag{2.19}$$

satisfying the completeness equation. Suppose the state being measured is

$$|\psi\rangle = a|+\rangle + b|-\rangle, \quad a, b \in \mathbb{C}. \tag{2.20}$$

Then the probabilities of obtaining measurement outcomes 0 and 1 are

$$p(0) = \langle\psi|M_0^\dagger M_0|\psi\rangle = \langle\psi|M_0|\psi\rangle = |a|^2, \tag{2.21}$$
$$p(1) = \langle\psi|M_1^\dagger M_1|\psi\rangle = \langle\psi|M_1|\psi\rangle = |b|^2. \tag{2.22}$$

The associated post-measurement states are:

$$\text{If } m = 0 \text{ is the outcome} \quad \Rightarrow \quad \frac{M_0|\psi\rangle}{|a|} = \frac{a}{|a|}|+\rangle, \tag{2.23}$$
$$\text{If } m = 1 \text{ is the outcome} \quad \Rightarrow \quad \frac{M_1|\psi\rangle}{|b|} = \frac{b}{|b|}|-\rangle. \tag{2.24}$$

A special and frequently considered case is that of a *von Neumann projective measurement* associated to an observable A (a Hermitian operator on the sate space of the system). A has a spectral decomposition,

$$A = \sum_m a_m P_m, \tag{2.25}$$

where the a_m are the distinct eigenvalues and the P_m are the associated eigenprojections (of rank ≥ 1). Then $M_m = P_m$ is a collection of measurement operators. Suppose we want to measure the state $|\psi\rangle$. According to (2.15), the probability of getting the result a_m is

$$p(m) = \langle\psi|P_m|\psi\rangle = \|P_m\psi\|^2$$

and the associated post-measurement state is (see (2.16))

$$\frac{P_m|\psi\rangle}{\|P_m\psi\|}.$$

The expectation value of observable A associated to a pure state $|\psi\rangle$ is denoted by $\langle A\rangle$,

$$\langle A\rangle = \sum_m a_m p(m) = \sum_m a_m \langle\psi|P_m|\psi\rangle = \langle\psi|A|\psi\rangle = \mathrm{Tr}(|\psi\rangle\langle\psi|A). \tag{2.26}$$

2.1.5 The postulates of quantum mechanics phrased for mixed states

Mixed states

Let $|\phi_j\rangle \in \mathcal{H}$, $0 \leq p_j \leq 1$, $j = 1, \ldots, N$, be a collection of pure states and probabilities, respectively. The family

$$\Big\{|\phi_j\rangle, p_j\Big\}_{j=1}^{N}$$

is called an *ensemble of pure states.* We define the associated *mixed state* by

$$\rho = \sum_j p_j |\phi_j\rangle\langle\phi_j|, \quad \sum_j p_j = 1. \tag{2.27}$$

From (2.26), we know that the average of an observable A associated to a pure state $|\psi\rangle$ is $\langle\psi|A|\psi\rangle$. The expectation value of A associated to ρ is defined by

$$\langle A\rangle = \sum_j p_j \langle\phi_j|A|\phi_j\rangle = \sum_j p_j \mathrm{Tr}(|\phi_j\rangle\langle\phi_j|A) = \mathrm{Tr}\Big(\sum_j p_j |\phi_j\rangle\langle\phi_j|A\Big) = \mathrm{Tr}(\rho A). \tag{2.28}$$

An operator acting on the Hilbert space $\mathcal{H}$ is called a *density matrix* if it is satisfying the following properties:

- ρ is self-adjoint, that is, $\rho^\dagger = \rho$.
- ρ is positive semi-definite, which means that the eigenvalues of ρ are all non-negative.
- ρ has unit trace one, namely, the sum of all the diagonal elements of ρ is one, *i.e.* $\mathrm{Tr}\rho = 1$.

The density matrix in (2.27) is a mixed state which is a statistical ensemble of pure states. If it has rank one, then ρ is a pure state, denoted by $\rho = |\psi\rangle\langle\psi|$, see (2.26).

Example. We consider a quantum system associated with Hilbert space $\mathbb{C}^2$. A pure state can be a *superposition* of energy basis $\{|+\rangle, |-\rangle\}$ in (2.4), *i.e.*,

$$\rho = |\Psi\rangle\langle\Psi|$$

with $|\Psi\rangle = a_1|+\rangle + a_2|-\rangle$ and $|a_1|^2 + |a_2|^2 = 1$. The associated density matrix reads

$$\rho = \begin{pmatrix} |a_1|^2 & a_1\bar{a}_2 \\ \bar{a}_1 a_2 & |a_2|^2 \end{pmatrix}. \tag{2.29}$$

Generally, a mixed state is of the form

$$\rho = p|+\rangle\langle+| + (1-p)|-\rangle\langle-| + z|+\rangle\langle-| + \bar{z}|-\rangle\langle+| \tag{2.30}$$

with $p \in [0,1]$, and where $z \in \mathbb{C}$, $|z|^2 \leq p(1-p)$, is called the coherence of ρ. When $z = 0$, we call the state ρ, (2.30), an *incoherent superposition* of the energy states. For $z \neq 0$ it is called a *coherent* one.[1] The density matrix associated to (2.30), written in the energy basis, reads

$$\rho = \begin{pmatrix} p & z \\ \bar{z} & 1-p \end{pmatrix}. \tag{2.31}$$

This ρ is a density matrix associated with a pure state if and only if $\text{rank}(\rho) = 1$. (In that case, it is of the form (2.29).)

Postulates for mixed states

We have introduced the postulates for pure states. Now, we introduce the postulates of quantum theory for mixed states.

(P1) The state of a system is given by a density matrix ρ on a Hilbert space $\mathcal{H}$, which satisfies the conditions that

$$\rho = \rho^\dagger \quad \text{and} \quad \text{Tr}\rho = 1.$$

(P2) The dynamics of state ρ is given by

$$i\partial_t \rho_t = [H, \rho_t], \tag{2.32}$$

where H is the Hamiltonian acting on the Hilbert space $\mathcal{H}$. Given any initial condition ρ_0,

[1] This notion should not be confused with that of a “coherent state”, say, of an oscillator or similar.

the equation (2.32) has the unique solution

$$\rho_t = e^{-itH}\rho_0 e^{itH}.$$

(P3) If $\rho_1, \ldots, \rho_N$ are states on Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_N$, then the composite state where subsystem i is in state ρ_i, is given by

$$\rho = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_N.$$

(P4) When measuring $M = \sum_k m_k P_k$ on ρ, the possible outcomes are the m_k. The probability of finding the outcome m_k is

$$p(m_k) = \mathrm{Tr}(P_k \rho).$$

If m_k is the outcome, then the post-measurement state is

$$\frac{P_k \rho P_k}{\mathrm{Tr}(P_k \rho)}.$$

The postulates (P1)-(P4) reduce to the four postulates for pure states given in Section 2.1 when ρ is a pure state, $\rho = |\Psi\rangle\langle\Psi|$.

2.2 Open quantum systems

2.2.1 The reduced density matrix

Suppose we have physical systems $\mathcal{A}$ and $\mathcal{B}$ with corresponding Hilbert space $\mathcal{H}_1$ and $\mathcal{H}_2$. Let ρ^{12} be a density matrix of the composite quantum system, *i.e.*, ρ^{12} is a density matrix on $\mathcal{H}_1 \otimes \mathcal{H}_2$. The *reduced density matrix* for system $\mathcal{A}$ is defined by

$$\rho^1 \equiv \mathrm{Tr}_2(\rho^{12}),$$

where Tr_2 is a map of operators known as the *partial trace* over system $\mathcal{B}$. The partial trace is defined by

$$\mathrm{Tr}_2\big(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|\big) \equiv |a_1\rangle\langle a_2| \; \mathrm{Tr}\big(|b_1\rangle\langle b_2|\big),$$

where $|a_1\rangle, |a_2\rangle \in \mathcal{H}_1$ and $|b_1\rangle, |b_2\rangle \in \mathcal{H}_2$ and by extending its action by linearity to all operators on $\mathcal{H}_1 \otimes \mathcal{H}_2$. The trace operation of the right hand side is the usual trace operation for system $\mathcal{B}$, *i.e.*, $\mathrm{Tr}\big(|b_1\rangle\langle b_2|\big) = \langle b_2|b_1\rangle$. Of course, one defines $\rho^2 = \mathrm{Tr}_1(\rho^{12})$ analogously.

The average of an observable A of system $\mathcal{A}$ is

$$\mathrm{Tr}_{12}\big(\rho^{12}(A \otimes \mathbb{1}_2)\big) = \mathrm{Tr}_1(\rho^1 A), \tag{2.33}$$

$\mathbb{1}_2 \in \mathcal{H}_2$ is a identity matrix. This means that if we are interested in one component of a multi-component system, we can use the reduced density matrix (which contains information on the other parts as well).

Example. Suppose we have a two qubit quantum system associated with the Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$. Consider the Bell state $|\phi\rangle \in \mathcal{H}$, given by

$$|\phi\rangle = \frac{1}{\sqrt{2}}\big(|{+}{+}\rangle + |{-}{-}\rangle\big),$$

where $|{+}{+}\rangle = |+\rangle \otimes |+\rangle$ and $|{-}{-}\rangle = |-\rangle \otimes |-\rangle$. We let $\rho^{12} = |\phi\rangle\langle\phi|$ be its density matrix, then the reduced density matrix for the first qubit is

$$\rho^1 = \mathrm{Tr}_2(\rho^{12}) = \frac{1}{2}\Big(|+\rangle\langle+| + |-\rangle\langle-|\Big) = \frac{1}{2}\mathbb{1}$$

which, having rank two, is a mixed state.

Suppose we measure the quantity A on the first qubit, the average outcome is

$$\mathrm{Tr}_{12}\big(\rho^{12}(A \otimes \mathbb{1}_2)\big) = \langle\phi|(A \otimes \mathbb{1}_2)\phi\rangle.$$

We let $\{e_1, e_2\}$ be the orthonormal basis of $\mathbb{C}^2$, then

$$\phi = \sum_{i,k} c_{i,k} e_i \otimes e_k, \quad c_{i,k} \in \mathbb{C}.$$

Then we have

$$\begin{aligned}
\langle\phi|(A \otimes \mathbb{1}_2)\phi\rangle &= \sum_{i,k,j,l} \bar{c}_{i,k} c_{j,l} \langle e_i \otimes e_k|(A \otimes \mathbb{1}_2) e_j \otimes e_l\rangle \\
&= \sum_{i,k,j,l} \bar{c}_{i,k} c_{j,l} \langle e_i|A e_j\rangle \delta_{kl} \\
&= \sum_{i,k,j} \bar{c}_{i,k} c_{j,k} \mathrm{Tr}(|e_j\rangle\langle e_i|A) \\
&= \mathrm{Tr}_1\big(\sum_{i,k,j} \bar{c}_{i,k} c_{j,k} |e_j\rangle\langle e_i|A\big) \\
&= \mathrm{Tr}_1\big(\mathrm{Tr}_2 \rho^{12} A\big) = \mathrm{Tr}_1(\rho^1 A).
\end{aligned} \tag{2.34}$$

This example also proves the relation (2.33).

2.2.2 Kraus representation

Suppose our quantum system is associated with the Hilbert space $\mathcal{H}$. A linear opeator T: $\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ acting on bounded operators on $\mathcal{H}$, is said to be a super-operator.

- T is called *completely positive (CP)* if for all integers $n \geq 1$, the map $T \otimes \mathbb{1}$ acting on $\mathcal{B}(\mathcal{H}\otimes\mathbb{C}^n)$, the bounded operators on $\mathcal{H}\otimes\mathbb{C}^n$, maps positive operators to positive operators.
- T is called *trace preserving* if $\mathrm{Tr}(T(X)) = \mathrm{Tr}(X)$ for all $X \in \mathcal{B}(\mathcal{H})$.
- T is called CPT, or CPTP (completely positive trace preserving) if it is completely positive and trace preserving.

The following result says that CPT maps are precisely those which have a specific, so-called 'Kraus form'.

Theorem 2.1 (Kraus representation [27]). *Suppose T is a CPT map on $\mathcal{B}(\mathcal{H})$, with $d = \dim \mathcal{H} < \infty$. Then it can be written as*

$$T(X) = \sum_{k=1}^{K} M_k X M_k, \quad \textit{with} \quad \sum_{k=1}^{K} M_k^{\dagger} M_k = 1, \tag{2.35}$$

where $K \leq d$ is called the Kraus number. Conversely, any map of the form (2.35) *is CPT.*

2.2.3 The master equation

Markovian approximation [29].

The evolution of an open system A is not unitary (that is, given by the Schrödinger equation), even though it is obtained by reducing (partial trace) the evolution of the *total* system AE (system plus environment), which is supposed to be a closed combined system and hence does have unitary evolution (Postulate 2 of Quantum Mechanics). The dynamics of A is not Markovian (even though that of the whole AE is). Markovianity here means, intuitively, that the evolution from time t_1 to t_2 only depends on the initial and final states (at times t_1 and t_2). A more precise definition and measurements of non-Markovianity are discussed for instance in [5]. The reason is that the information can flow from A to E and then return at a later time. This results in a non-Markovian dynamics of the system.

Non-markovian effects are inevitable for any open system, and an exact Markovian description of quantum dynamics is impossible. However, in certain physical regimes, one may hope that a Markovian *approximation* of the system dynamics may be possible. This might work if there is a clean separation between typical correlation times of non-Markovian fluctuations in the system and the time scale of the evolution that we want to follow. Let $(\Delta t)_E$ denote the time that the environment takes to "forget" information obtained from the system (reservoir correlation time). We can regard that the information is lost forever after time $(\Delta t)_E$ and we can neglect the possibility that the information may return to influence the subsequent evolution of the system.

To describe the evolution we "coarse-grain" in time, perceiving the dynamics through a filter that screens out the high frequencies ω present in the motion with $\omega \gg ((\Delta t)_{coarse})^{-1}$. An approximately Markovian description should be possible for $(\Delta t)_E \ll (\Delta t)_{coarse}$, because on the time scale $(\Delta t)_{coarse}$, the environment has lost its memory already. This Markovian approximation is useful if the time scale of the dynamics that one considers is long compared to $(\Delta t)_{coarse}$.

Markovian master equation. The system of interest is coupled to an 'environment', a very large other quantum system (having many degrees of freedom). The complex (system + environment) is regarded as a closed system and evolves according to the Schrödinger equation, governed by a Hamiltonian which describes the system, the environment and the interaction between the two. In a sense, this is the most fundamental description of a noisy system, but at the same time it is enormously complicated, because the dynamics describes all details about the system and all degrees of freedom of the reservoir [5, 8, 11, 14, 16, 17, 25].

Upon restricting the full dynamics to just the system, by 'tracing out the environment degrees of freedom', one arrives at an effective equation for the system alone. In the absence of interaction with the environment, this equation reduces to the system Schrödinger equation, but it is much more complicated in the presence of interactions. In certain approximative regimes (weak coupling, fast reservoir dynamics or dissipation), this effective equation takes the form of the ubiquitous *Markovian master equation*. A rigorous derivation of the master equation has recently been given in [22].

The Markovian master equation resulting from the procedure explained above is of the form

$$\rho(t) = e^{t\mathcal{L}}\rho(0).$$

Here, the Lindblad operator $\mathcal{L}$ (which acts on density matrices) is not hermitian and has complex eigenvalues $-x+iy$ (with $x \geq 0$), leading to time decay $\sim e^{-tx}$, thus describing irreversible effects. Moreover, since $e^{t\mathcal{L}}$ is a group of completely positive, trace preserving maps, its generator $\mathcal{L}$ is constrained to have a standard form (Lindblad, Gorini-Kossakovski-Sudarshan) [2, 6, 9]. This standard form is often taken as the *starting point* for modeling an open system dynamics. Namely, one specifies the components in the standard form to build a generator $\mathcal{L}$ without deriving it from a microscopic model. Then one analyzes the Markovian dynamics resulting from $\mathcal{L}$ [12].

In this thesis, we take a slightly different view. We model the noise by adding to the Hamiltonian a *random* part. In a sense, this strategy is similar to that of the famous Anderson model of condensed matter, where particles are moving in a random potential landscape. For a fixed realization of the random variables, the Hamiltonian is just that of a closed system; however, what counts is the average over the noise degrees of freedom, which result in a non-Hamiltonian dynamics of the system. Taking the expectation over the noise is the analogous action to taking the 'partial trace' in deterministic models.

Chapter 3

Results on decoherence

In this chapter, we first present the background and the mathematical models we use to induce the dynamics of the quantum system, then we discuss our main results based on weak and strong noise regimes and illustrate some theories about quantum theory. At the end of this chapter, we show the proofs of these theories. This chapter is is based on the prepublication [13].

3.1 Background

According to postulate 2, states evolves according to Schrödinger equation. The associated density matrix evolves entirely *coherently*, namely as a superposition of time-periodic functions. This is the fate of any (finite-dimensional) quantum system evolving according to the Schrödinger equation. But this is *not* what is often observed in nature: commonly systems undergo irreversible processes, such as the exchange and transport of excitation and charges within molecules, or generally, equilibration and decoherence. These phenomena are caused by noise effects induced by the contact with external agents.

As an example, *decoherence*, equivalently called *dephasing*, is an important phenomenon in modern quantum theory and in the quantum information sciences in particular. To explain it, we denote the density matrix elements in the energy basis by

$$\rho_{ij}(t) = \big\langle \Phi_i, \rho(t)\Phi_j \big\rangle. \tag{3.1}$$

The average outcome when measuring an observable $\mathcal{O}$ in the state ρ, (2.30), is given by

$$\langle \mathcal{O} \rangle = \mathrm{Tr}\rho\mathcal{O} = p\langle \mathcal{O} \rangle_1 + (1-p)\langle \mathcal{O} \rangle_2 + 2\mathrm{Re}\, z\langle \Phi_2, \mathcal{O}\Phi_1 \rangle. \tag{3.2}$$

(Here, we denote the basis element $|\pm\rangle$ by $\Phi_{1,2}$.) For an incoherent ρ, the last term on the right side of (3.2) vanishes ($z = 0$) and the measurement process has the characteristics of a classical dynamical system, where two states are mixed with probabilities p and $1-p$. However, for coherent ρ, the cross term $2\mathrm{Re}\, z\langle \Phi_2, \mathcal{O}\Phi_1 \rangle$ gives an additional term, which reflects an interplay of Φ_1 and Φ_2 within the state ρ. This term is due to the quantum nature of ρ. Note that for observables $\mathcal{O}$ commuting with H_0, this effect is not visible, as $\langle \Phi_2, \mathcal{O}\Phi_1 \rangle = 0$. Coherence is thus a basis dependent notion.[1]

The process of *decoherence* is then defined to be the transition of $\rho \mapsto \rho'$ were ρ' is a decoherent superposition. In quantum information theory, a *channel* $\mathcal{E}$ is defined as a completely positive, trace preserving map on density matrices. The *dephasing channel* is given by [29]

$$\mathcal{E}\rho = \mathcal{E}\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} \rho_{11} & (1-\eta)\rho_{12} \\ (1-\eta)\rho_{21} & \rho_{22} \end{pmatrix}, \tag{3.3}$$

where $0 \leq \eta \leq 1$ controls the reduction of the off-diagonals (in the energy basis). For $\eta = 1$ the coherences are entirely suppressed by the action of $\mathcal{E}$ and the state undergoes full decoherence. It is well known (and discussed in a huge amount of literature) that a gradual reduction of coherences is a generic effect that noises induce on a quantum system [5,15,28,34]. In this setup, decoherence is the dynamical process, which using (3.1), is expressed as

$$\rho_{12}(t) \to 0 \quad \text{as } t \to \infty.^2 \tag{3.4}$$

Decoherence plays a central role in quantum theory, particularly in quantum information and computation, where coherence is a resource exploited in the design of fast algorithms, and decoherence is to be avoided as much as possible. A core task is to establish mathematical models which enable to uncover mechanisms leading to, and quantify the details of, the decoherence process, including, *e.g.*, the speed of convergence in (3.4). This is a first step in designing countermeasures to protect systems from losing quantum features because of noise.

3.2 Mathematical models

The Hamiltonian of the system is

$$H = H_0 + H_{\text{noise}}, \tag{3.5}$$

where

$$H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \quad \text{and} \quad H_{\text{noise}} = \begin{pmatrix} \xi_1 & \xi_o \\ \xi_o & \xi_2 \end{pmatrix}. \tag{3.6}$$

Here, $E_1, E_2 \in \mathbb{R}$ are constants and ξ_o, ξ_1 and ξ_2 are real valued, independent random variables representing the noise. Figure 3.1 shows the possible shapes of these three random variables.

We assume that the Bohr energy satisfies

$$\varepsilon = E_1 - E_2 > 0 \tag{3.7}$$

[1] Every density matrix ρ can be diagonalized and is an incoherent superposition of its eigenprojections.

[2] If the limit of the off-diagonals converge to zero we say there is *full decoherence*. Depending on the models, only *partial decoherence* is also observed [24,28].

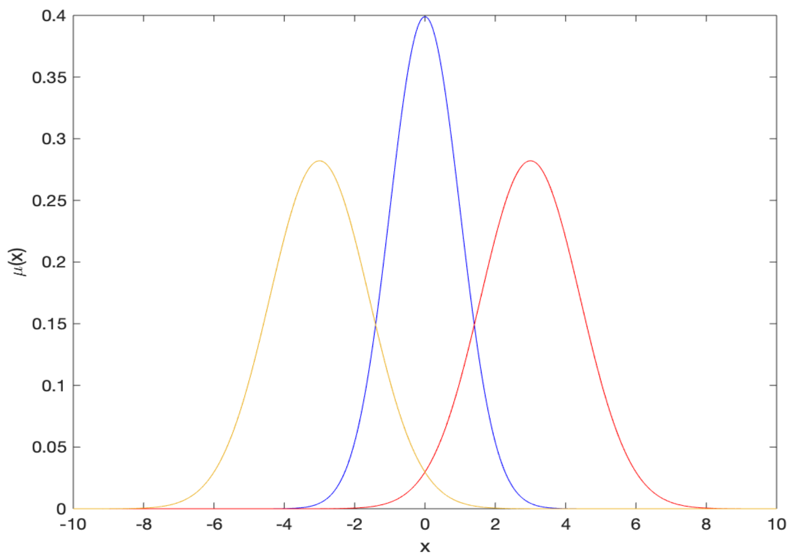

Figure 3.1: Possible shapes of the distributions of ξ_0, ξ_1 and ξ_2.

and define the quantity

$$\xi_{\mathrm{d}} = \xi_1 - \xi_2. \tag{3.8}$$

We call ξ_{d} and ξ_{o} the diagonal and the off-diagonal (or, tunneling) noise, respectively. To those random variables are associated the probability densities[3]

$$\xi_{\mathrm{d}} \leftrightarrow \mu_{\mathrm{d}}(y)^4 \qquad \text{and} \qquad \xi_{\mathrm{o}} \leftrightarrow \mu_{\mathrm{o}}(x). \tag{3.9}$$

The eigenbasis of H_0 is denoted by (2.4). Shifting the Hamiltonian H by adding a matrix $\alpha\mathbb{1}$, where α is a real number (or a random variable) and $\mathbb{1}$ is the 2×2 identity matrix, does not alter the evolution of quantum states, as $e^{-it(H+\alpha\mathbb{1})}\rho\, e^{it(H+\alpha\mathbb{1})} = e^{-itH}\rho\, e^{itH}$. It is then apparent from (3.6) that only the quantities (3.7) and (3.8) will play a role in the dynamics.

Open two-level (two state) systems are ubiquitous in quantum theory. Despite being of mathematically simplest form (two-dimensional), they represent diverse physical systems, ranging from qubits, spins or atoms interacting with radiation in quantum information theory and quantum optics [12, 18] to donor-acceptor systems in quantum chemical and quantum biological processes [23, 33]. The analysis of open two-level systems is far from trivial [19] and new results are emerging regularly [10, 16, 17]. One possible realization of such a two-level system is given by a quantum particle in a double well potential, having minima (say at spatial locations x_1 and x_2). It is assumed that the wells are deep enough so that it makes sense to talk about the states Φ_1 and Φ_2 representing the particle being located in the respective well. The associated

[3]The probability of the event $\{\xi \in [a, b]\}$ is given by $\int_a^b \mu(x)dx$.

[4]We assume that the probability density functions of ξ_1 and ξ_2 are μ_1 and μ_2 respectatively, then $\mu_d(y) = \int_{\mathbb{R}} \mu_1(x)\mu_2(x-y)dx$.

energies are E_1 and E_2 and they are shifted by ξ_1 and ξ_2 due to external noise. The particle can tunnel between the wells due to environmental effects, which corresponds to the tunneling matrix element $\xi_{\rm o}$ in (3.6). In contrast to the localized basis (2.4), one introduces the 'delocalized' or 'adiabatic' basis [20] given by

$$\Phi_{\pm} = \frac{1}{\sqrt{2}}(\Phi_1 \pm \Phi_2). \tag{3.10}$$

If $\Phi_{1,2}$ represent states with relatively well localized positions, then $\Phi_{\pm}$ are those having the largest position uncertainty (variance). Then following diagram illustrates what we mean above.

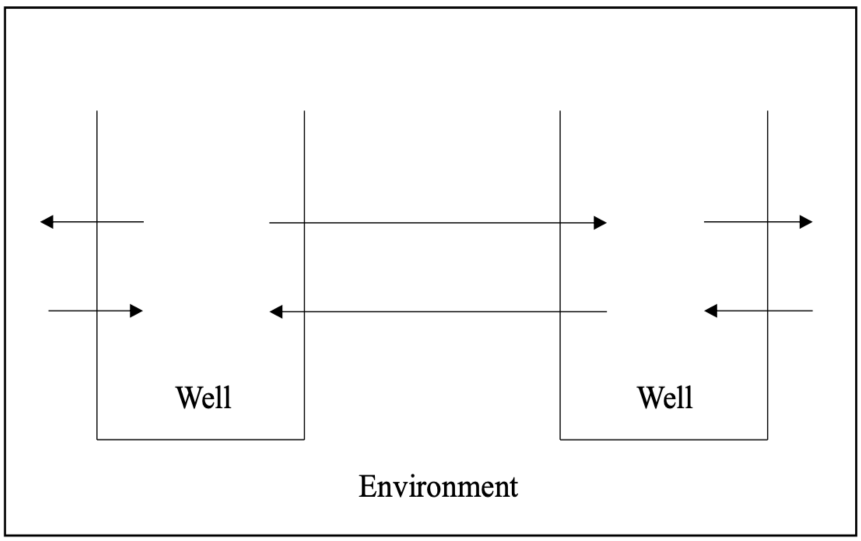

Figure 3.2: Two-level open quantum system.

3.3 Discussion of main results

To derive our rigorous results, we assume that the probability densities $\mu_{\rm d}$ and $\mu_{\rm o}$ in (3.9) are compactly supported within intervals $(-\eta_{\rm d}, \eta_{\rm d})$ and $(-\eta_{\rm o}, \eta_{\rm o})$, respectively. We also assume that $\eta_o, \eta_d > 0$ and $\eta_{\rm d} < \varepsilon$.

3.3.1 Analysis of final state

Convergence to a final state. We show in Theorem 4.1 that if $\mu_{\rm d}$ is $n = 1, 2, \ldots$ times continuously differentiable, then for $t \geq 0$, we have

$$\left\|\mathbb{E}[\rho(t)] - \bar{\rho}\ \right\| \leq \frac{C}{1+t^n}. \tag{3.11}$$

Here, $\mathbb{E}[\rho(t)]$ means that we take the expectation of $\rho(t)$; more generally, for a quantity $F(\xi_{\rm d}, \xi_{\rm o})$ depending on the two noises (c.f. (3.9)), we set

$$\mathbb{E}[F] = \int_{\mathbb{R}^2} F(y,x)\mu_{\rm d}(y)\mu_{\rm o}(x)dxdy.$$

In (3.11), $\bar\rho$ is an explicit final state which depends on the noises and on the initial state $\rho(0)$. The result also holds in absence of the off-diagonal noise $\xi_{\rm o}$. We conclude that the diagonal noise drives the state to a final one, at a speed that depends on the smoothness of the noise distribution.

In absence of diagonal noise, when only $\xi_{\rm o}$ is present, our Theorem 4.2 shows that the diagonals $\mathbb{E}[\rho_{jj}(t)]$ converge to $\bar\rho_{jj}$ at speed $1/t$. Moreover, if the initial coherence vanishes, $\rho_{12}(0) = 0$, then (3.11) holds with $n = 1$. Increased regularity of $\mu_{\rm o}$ does not speed up the convergence, however. The hindrance to a speedup are the slow noise modes (frequencies close to 0). We show that when the latter are suppressed, meaning that $\mu_{\rm o}(\omega)$ vanishes at the origin as $\mu_{\rm o}(\omega) \sim \omega^k$ for some $k = 1, 3, 5, \dots$ then the convergence (3.11) is valid with $n = \frac{k+1}{2}$.

Properties of the final state. We establish the explicit form of the final state $\bar\rho$ in all parameter regimes. We show in Theorem 4.3 that in the

– weak off-diagonal noise regime $\eta_{\rm o} \ll \varepsilon$ (at fixed $\eta_{\rm d}$) and the
– strong off-diagonal noise regime $\varepsilon \ll \mu_{\rm o}^{\rm min} \equiv \min\{|\omega| \,:\, \mu_{\rm o}(\omega) \neq 0\}$,

the final state is given by

$$\bar\rho = \begin{cases} \rho_{11}(0)\,|\Phi_1\rangle\langle\Phi_1| + \rho_{22}(0)\,|\Phi_2\rangle\langle\Phi_2| + O(\eta_{\rm o}/\varepsilon) & \text{weak off-diagonal noise} \\ \rho_{++}(0)\,|\Phi_+\rangle\langle\Phi_+| + \rho_{--}(0)\,|\Phi_-\rangle\langle\Phi_-| + O(\varepsilon/\mu_{\rm o}^{\rm min}) & \text{strong off-diagonal noise} \end{cases} \tag{3.12}$$

where $\Phi_{1,2}$ is the localized basis (2.4) and $\Phi_\pm$ is the delocalized basis (3.10). We also set $\rho_{++} = \langle\Phi_+, \rho\,\Phi_+\rangle$ and analogously for ρ_{--}. This shows that the noise implements the *dephasing channel* (3.3) (with $\eta = 1$) in the localized basis (weak noise) or in the delocalized basis (strong noise). The speed at which the channel is implemented depends on the properties of $\mu_{\rm d}$ and $\mu_{\rm o}$, as specified in the results on convergence.

3.3.2 Heuristic analysis of strong and weak noise regime

As we show in Lemma 5.1, the density matrix elements are of the form

$$\rho_{kl}(t) = \bar\rho_{kl}(x/\varepsilon, y/\varepsilon) + p_{kl}(x/\varepsilon, y/\varepsilon)\, e^{\pm it\varepsilon\Phi(x/\varepsilon, y/\varepsilon)} \tag{3.13}$$

with $\xi_{\rm o}(x) = x$ and $\xi_{\rm d}(y) = y$ and where the dynamical phase is given by

$$\Phi(x/\varepsilon, y/\varepsilon) = (1 + y/\varepsilon)\sqrt{1 + 4\frac{(x/\varepsilon)^2}{(1 + y/\varepsilon)^2}}\ . \tag{3.14}$$

Here, $\bar{\rho}_{kl}$ and p_{kl} are both quantities depending on the initial state $\rho(0)$ and the noises, but they are time independent. The $\pm$ in (3.13) indicates that a linear combination can occur (with different p_{kl}). The average over x and y of the exponential carrying the dynamical phase in (3.13) determines the time decay properties of $\rho_{kl}(t)$.

Heuristics of the weak noise regime

This is the regime in which the probability densities $\mu_{\rm o}(x)$ and $\mu_{\rm d}(y)$ yield the restrictions x/ε, $y/\varepsilon \ll 1$. Up to order three in the noise, we have

$$\Phi(x/\varepsilon, y/\varepsilon) \ \sim \ 1 + y/\varepsilon + 2(x/\varepsilon)^2. \tag{3.15}$$

Depending on the values of k, l, the p_{kl} have the lowest order expansions (Lemma 5.2)

$$p_{kl}(x/\varepsilon, y/\varepsilon) \sim (x/\varepsilon)^n, \qquad n = 0, 1, 2. \tag{3.16}$$

This implies

$$\mathbb{E}\Big[p_{kl}(x/\varepsilon, y/\varepsilon) e^{-it\varepsilon\Phi(x/\varepsilon,y/\varepsilon)}\Big] \ \sim \ e^{-it\varepsilon}\, \mathbb{E}\big[e^{-it\varepsilon(y/\varepsilon)}\big]\, \mathbb{E}\big[(x/\varepsilon)^n e^{-2it\varepsilon(x/\varepsilon)^2}\big]. \tag{3.17}$$

The contribution coming from the diagonal noise is given by the Fourier transform of the probability density,

$$\mathbb{E}\big[e^{-it\varepsilon(y/\varepsilon)}\big] = \int_{\mathbb{R}} e^{-ity}\mu_{\rm d}(y)dy = \widehat{\mu}_{\rm d}(t). \tag{3.18}$$

The decay for large values of t is determined by the smoothness of $\mu_{\rm d}(y)$. If μ_d is k times continuously differentiable then for large t we have $\widehat{\mu}_{\rm d}(t) \sim t^{-k}$; for a Gaussian $\mu_{\rm d}(x) \propto e^{-x^2/2\sigma^2}$, the decay is $\widehat{\mu}_{\rm d}(t) \propto e^{-\sigma^2 t^2/2}$.

The contribution to (3.17) coming from the off-diagonal noise is

$$\begin{aligned}\mathbb{E}\big[(x/\varepsilon)^n e^{-2it\varepsilon(x/\varepsilon)^2}\big] \ &= \ \varepsilon \int_{\mathbb{R}} x^n e^{-2i(\sqrt{\varepsilon t}x)^2}\mu_{\rm o}(\varepsilon x)dx \\ &= \ \varepsilon t^{-(n+1)/2}\int_{\mathbb{R}} x^n e^{-2ix^2}\mu_{\rm o}\big(x\sqrt{\varepsilon/t}\big)dx \\ &\sim \ t^{-(n+1)/2}\mu_{\rm o}(0), \quad t \gg \varepsilon. \end{aligned} \tag{3.19}$$

This contribution decays as an inverse power of $\sqrt{t}$, a power which does *not* depend on the shape (smoothness) of $\mu_{\rm o}$, but only on the value $\mu_{\rm o}(0)$. The slowest decay is for terms with $n = 0$, and is given by $1/\sqrt{t}$. (Even though quicker decay can be achieved by suppressing slow noise modes, *i.e.*, if $\mu_{\rm o}(0) = 0$.) This heuristic analysis shows the following picture:

- In the weak noise regime, both the diagonal and the off-diagonal noises contribute to the convergence of the density matrix to a final state $\bar{\rho}$,

$$\big\| \, \mathbb{E}[\rho(t)] - \bar{\rho} \, \big\| \;\sim\; \big| \widehat{\mu}_{\rm d}(t) \, \mu_{\rm o}(0) \big| \, t^{-1/2}. \tag{3.20}$$

The diagonal noise contribution depends on the smoothness of the probability density $\mu_{\rm d}$, while the off-diagonal noise contribution $\mu_{\rm o}(0)\, t^{-1/2}$ is insensitive to the shape and smoothness of $\mu_{\rm o}$.

Heuristics of the strong off-diagonal noise regime

This is the regime in which $\mu_{\rm d}(y)$ and $\mu_{\rm o}(x)$, (3.9), are such that $|x|/\varepsilon \gg 1 + y/\varepsilon$. (Note that $1 + y/\varepsilon > 0$ due to (3.7) and $\eta_{\rm d} < \varepsilon$.) The dynamical phase (3.14) is

$$\Phi(x/\varepsilon, y/\varepsilon) \sim |x|/\varepsilon. \tag{3.21}$$

According to Lemma 5.1, in this regime we have

$$p_{kl}(x/\varepsilon, y/\varepsilon) \sim \text{constant}. \tag{3.22}$$

Let us discuss the situation when $\mu_{\rm o}$ is supported essentially around some average value $x_* \gg \varepsilon > 0$ (the general case is treated in the same way). Then $|x|/\varepsilon \sim x/\varepsilon$ and

$$\mathbb{E}\Big[p_{kl}(x/\varepsilon, y/\varepsilon) e^{-it\varepsilon\Phi(x/\varepsilon, y/\varepsilon)} \Big] \;\sim\; e^{-it\varepsilon} \, \mathbb{E}\big[e^{-2itx} \big] = e^{-it\varepsilon} \, \widehat{\mu}_{\rm o}(2t). \tag{3.23}$$

In contrast to the weak coupling regime, here the decay depends on the smoothness of $\mu_{\rm o}$, while the diagonal noise does not contribute. This heuristic analysis shows the following picture:

- In the strong off-diagonal noise regime, only the off-diagonal noise contributes to the decay of the density matrix towards a final state $\bar{\rho}$,

  $$\big\| \, \mathbb{E}[\rho(t)] - \bar{\rho} \, \big\| \;\sim\; \big| \widehat{\mu}_{\rm o}(2t) \big|.$$

  The time-decay of the Fourier transform $\widehat{\mu}_{\rm o}(2t)$ depends on the smoothness of the probability density $\mu_{\rm o}(x)$.

A heuristic identification the final state $\bar{\rho}$ is seems more difficult. In particular, the final state depends on the initial condition $\rho(0)$. Nevertheless, for this simple 2×2 system, it can be calculated explicitly, see Theorems 4.1 and 4.3.

3.4 Main results, rigorous

In order to make the analysis rigorous, we make the following assumption.

Assumption (A) The probability densities $\mu_{\rm o}$ and $\mu_{\rm d}$, (3.9), have compact support in the open intervals $(-\eta_{\rm o}, \eta_{\rm o})$ and $(-\eta_{\rm d}, \eta_{\rm d})$, respectively, where $0 < \eta_{\rm o} < \infty$ and $0 < \eta_{\rm d} < \varepsilon$.

Theorem 4.1 (Convergence)**.** *Suppose assumption* (A) *holds and that $\mu_{\rm d}$ is n times continuously differentiable for some $n \in \mathbb{N} \cup \{0\}$. Then there is a constant C_n* s.t. *for all $t \geq 0$,*

$$\big\| \, \mathbb{E}[\rho(t)] - \bar{\rho} \, \big\| \leq \frac{C_n}{1+t^n}. \tag{3.24}$$

The final state $\bar{\rho}$ is given by

$$\begin{aligned} \bar{\rho}_{11} &= \alpha + \beta\rho_{11}(0) - 2\gamma \mathrm{Re}\rho_{12}(0) \\ \bar{\rho}_{12} &= \gamma(1 - 2\rho_{11}(0)) + 2\alpha \mathrm{Re}\rho_{12}(0), \end{aligned} \tag{3.25}$$

where $\alpha, \beta \geq 0$, $\gamma \in \mathbb{R}$, are explicit constants depending on the noises but not on the initial state $\rho(0)$. Moreover, if the off-diagonal noise satisfies $\mu_{\rm o}(-x) = \mu_{\rm o}(x)$, then $\gamma = 0$.

The bound (3.24) is consistent with the heuristic estimate (3.20). We note though, that decay of non-integer powers (such as $t^{-1/2}$ as in (3.20)) is not detected in Theorem 4.1. This is because we derive the result using integration by parts, which only yields decay of integer inverse powers of t.

Theorem 4.1 is also valid in case the off-diagonal noise vanishes, *i.e.*, for $\xi_{\rm o} = 0$ in (3.6). We conclude that the diagonal noise drives the convergence to a final state ρ, at a speed depending on the smoothness of the noise distribution. Our next result examines the situation when $\xi_{\rm d} = 0$.

Theorem 4.2 (Purely off-diagonal noise)**.** *Suppose that $\xi_{\rm d} = 0$ and that $\mu_{\rm o}$ is compactly supported and continuously differentiable.*

1. *There is a constant C such that for all $t \geq 0$ and $j = 1, 2$,*

   $$\big| \, \mathbb{E}[\rho_{jj}(t)] - \bar{\rho}_{jj} \, \big| \leq \frac{C}{1+t},$$

   where $\bar{\rho}_{jj}$ are the diagonal elements of $\bar{\rho}$, (3.25)*. Moreover, if the initial density matrix is incoherent, $\rho_{12}(0) = 0$, then there is a constant C such that for all $t \geq 0$,*

   $$\big\| \, \mathbb{E}[\rho(t)] - \bar{\rho} \, \big\| \leq \frac{C}{1+t},$$

   where $\bar{\rho}$ is given in (3.25)*.*

2. *Let $k = 1, 3, 5 \ldots$ be a fixed odd number and assume that $\mu_{\rm o}$ is k times continuously differentiable and has a zero of order at least k at the origin, meaning that*

   $$\lim_{\omega \to 0} \frac{\mu_{\rm o}(\omega)}{|\omega|^k} < \infty.$$

Then there is a constant C *such that for all* $t \geq 0$*,*

$$\big\| \, \mathbb{E}[\rho(t)] - \bar{\rho} \, \big\| \leq \frac{C}{1+t^{\frac{k+1}{2}}} \, . \tag{3.26}$$

The final state $\bar{\rho}$ *is given by* (3.25)*.*

Theorem 4.2 shows that the dynamical process is slowed down by slow noise modes. Namely, $\mu_{\rm o}$ has to vanish quickly at $\omega = 0$ to increase the speed.

We point out that the final state $\bar{\rho}$ (the same in both Theorems 4.1 and 4.2) is known for all parameter regimes. The following result is obtained by expanding the coefficients α, β, γ (see (3.62)) in two regimes, where the off-diagonal noise is either small or large.

Theorem 4.3 (Final state)**.** *Assume the setting of Theorem 4.1.*

1. **Weak noise.** *In the weak off-diagonal noise regime*

$$\nu_1 \equiv \frac{\eta_{\rm o}}{\varepsilon} \frac{1}{1 - \eta_{\rm d}/\varepsilon} \ll 1, \tag{3.27}$$

we have

$$\begin{aligned} \alpha &= 2\mathbb{E}\big[(x/\varepsilon)^2\big]\,\mathbb{E}\Big[\frac{1}{(1+y/\varepsilon)^2}\Big] + O(\nu_1^4) \\ \beta &= 1 + O(\nu_1^2) \\ \gamma &= -\mathbb{E}[x/\varepsilon]\,\mathbb{E}\Big[\frac{1}{1+y/\varepsilon}\Big] + 4\mathbb{E}\Big[\big(x/\varepsilon\big)^3\Big]\,\mathbb{E}\Big[\frac{1}{(1+y/\varepsilon)^3}\Big] + O(\nu_1^5). \end{aligned} \tag{3.28}$$

2. **Strong off-diagonal noise.** *Suppose* $\mu_{\rm o}$ *is supported in* $|x| > \mu_0^{\rm min}$*, for some* $\mu_0^{\rm min} > 0$*. In the strong off-diagonal noise regime*

$$\nu_2 \equiv \frac{\varepsilon}{\mu_{\rm o}^{\rm min}} \ll 1, \tag{3.29}$$

we have

$$\begin{aligned} \alpha &= \frac{1}{2} + O\big(\nu_2^2\big) \\ \beta &= \frac{1}{4}\mathbb{E}\Big[\frac{1}{(x/\varepsilon)^2}\Big]\,\mathbb{E}\Big[(1+y/\varepsilon)^2\Big] + O\big(\nu_2^3\big) \\ \gamma &= -\mathbb{E}\Big[\frac{1}{x/\varepsilon}\Big]\,\mathbb{E}\Big[1+y/\varepsilon\Big] + O\big(\nu_2^2\big). \end{aligned} \tag{3.30}$$

Remark. The result for the strong noise regime holds also (approximately) if $\mu_{\rm o}$ is not strictly supported in $|x| > \mu_0^{\rm min}$. It suffices that most of the support of $\mu_{\rm o}$ be in that region. This modification is easy to quantify.

Discussion. Relations (3.25) and (3.28), (3.30) show that

$$\bar{\rho} = \begin{cases} \rho_{11}(0)\,|\Phi_1\rangle\langle\Phi_1| + \rho_{22}(0)\,|\Phi_2\rangle\langle\Phi_1| + O(\nu_1) & \text{weak noise} \\ \rho_{++}(0)\,|\Phi_+\rangle\langle\Phi_+| + \rho_{--}(0)\,|\Phi_-\rangle\langle\Phi_-| + O(\nu_2) & \text{strong noise} \end{cases} \tag{3.31}$$

where Φ_{12} is the canonical (localized, diabatic) basis (2.4) and $\Phi_\pm = \frac{1}{\sqrt{2}}(\Phi_1 \pm \Phi_2)$ is the delocalized (adiabatic) basis. We also set $\rho_{++} = \langle \Phi_+, \rho\,\Phi_+\rangle$ and analogously for ρ_{--}. This shows that the noise implements the *dephasing channel* in the localized basis (weak noise) or in the delocalized basis (strong noise), at a speed which is at least $\propto t^{-n}$.

3.5 Proofs

The proofs are based on the explicit diagonalization of the Hamiltonian. Let $z \in \mathbb{C}$ and $a, b \in \mathbb{R}$. We diagonalize of the 2×2 self-adjoint matrix to find its spectral representation,

$$H = \begin{pmatrix} a & z \\ \bar{z} & b \end{pmatrix} = \sum_{j=1,2} \lambda_j |\Psi_j\rangle\langle\Psi_j|. \tag{3.32}$$

We consider the case where

$$(a-b)^2 + |z|^2 \neq 0, \tag{3.33}$$

which is equivalent to $\lambda_1 \neq \lambda_2$ (if equality holds in (3.33) then H is a multiple of the identity – this situation holds on a set of measure zero with respect to the noise probability measures and is not relevant for the dynamics). By considering (2.4), then the eigenvalues and eigenvectors have the explicit expressions

$$\Psi_j = c_1^{(j)}\Phi_1 + c_2^{(j)}\Phi_2, \qquad j = 1, 2, \tag{3.34}$$

with

$$\lambda_j = \frac{a + b + (-1)^{j+1}\sqrt{(a-b)^2 + 4|z|^2}}{2} \tag{3.35}$$

and

$$c_1^{(j)} = \frac{z}{\sqrt{|z|^2 + (a-\lambda_j)^2}}, \qquad c_2^{(j)} = \frac{\lambda_j - a}{\sqrt{|z|^2 + (a-\lambda_j)^2}}. \tag{3.36}$$

The functional calculus implies $e^{-itH} = \sum_{j=1,2} e^{-it\lambda_j}|\Psi_j\rangle\langle\Psi_j|$. Upon setting $a = E_1 + \xi_1$, $b = E_2 + \xi_2$ and $z = \xi_0$ in (3.32)-(3.36), a direct calculation yields the expressions of the density matrix elements (3.1), as shown in Lemma 5.1 below. To express them, we define the following functions of a (for now real) variable $P \neq 0$.

$$Q(P) \;=\; \frac{1 + \sqrt{1 + P^2}}{P}, \tag{3.37}$$

$$\begin{aligned} h(P) &= Q\frac{(2\rho_{11}(0)-1)Q+\rho_{21}(0)-\rho_{12}(0)Q^2}{(1+Q^2)^2}, \\ g_1(P) &= -Q^2\frac{(2\rho_{11}(0)-1)Q-\rho_{21}(0)-\rho_{12}(0)Q^2}{(1+Q^2)^2}, \\ g_2(P) &= \frac{(2\rho_{11}(0)-1)Q-\rho_{21}(0)Q^2+\rho_{12}(0)}{(1+Q^2)^2}. \end{aligned} \tag{3.38}$$

Here, $\rho_{ij}(0)$ are the matrix elements of the initial density matrix $\rho(0)$. We introduce real variables x, y and set

$$P(x,y)=\frac{2x}{\varepsilon+y}, \tag{3.39}$$

so that P, Q and h, g_1, g_2 become functions of x, y.

Lemma 5.1. *Consider a realization where the random variables are 'frozen',* i.e., $\xi_{\rm o}=x$ *and* $\xi_{\rm d}=\xi_1-\xi_2=y$*, for some fixed* x, y *satisfying* $x\neq 0$ *or* $y\neq -\varepsilon$ *(so that* (3.33) *holds). Then the density matrix elements* $\rho_{11}(t)$ *and* $\rho_{12}(t)$*, defined in* (3.1)*, are given by*

$$\begin{aligned} \rho_{11}(t) = \frac{2Q^2}{(1+Q^2)^2}+\Big(\frac{1-Q^2}{1+Q^2}\Big)^2\rho_{11}(0)-2\frac{Q(1-Q^2)}{(1+Q^2)^2}\mathrm{Re}\rho_{12}(0) \\ +2\mathrm{Re}\ e^{-it(\varepsilon+y)\sqrt{1+P^2}}h(x,y) \end{aligned} \tag{3.40}$$

and

$$\begin{aligned} \rho_{12}(t) = \frac{Q(1-Q^2)}{(1+Q^2)^2}\big(1-2\rho_{11}(0)\big)+\frac{4Q^2}{(1+Q^2)^2}\ \mathrm{Re}\rho_{12}(0) \\ +e^{-it(\varepsilon+y)\sqrt{1+P^2}}g_1(x,y)+e^{it(\varepsilon+y)\sqrt{1+P^2}}g_2(x,y). \end{aligned} \tag{3.41}$$

Remark. We have used that $\varepsilon+y\geq 0$ to arrive the above formulas.

Since $\rho(t)$ is self-adjoint and has unit trace, Lemma 5.1 specifies $\rho(t)$ entirely, which are $\rho_{22}(t)=1-\rho_{11}(t)$ and $\rho_{21}(t)=\bar\rho_{12}(t)$.

The time decay properties of the expectation of (3.40) and (3.41) are determined by the oscillating phases and the smoothness of the functions h and $g_{1,2}$.

Lemma 5.2. *The functions* h*,* g_1 *and* g_2 *have analytic extensions to* $P\in\mathbb{C}\backslash\{\pm i[1,\infty)\}$*. Their Taylor series at the origin (radius of convergence* 1*), satisfy*

$$h(P) = -\frac{1}{2}\rho_{12}(0)P-\frac{1}{4}\big(1-2\rho_{11}(0)\big)P^2+O(P^3), \tag{3.42}$$

$$g_1(P) = \rho_{12}(0)+\frac{1}{2}\big(1-2\rho_{11}(0)\big)P+O(P^2), \tag{3.43}$$

$$g_2(P) = -\frac{1}{4}\rho_{21}(0)P^2-\frac{1}{8}\big(1-2\rho_{11}(0)\big)P^3+O(P^4). \tag{3.44}$$

It follows in particular from (3.42)-(3.44) *that*

$$g_1(P) = -4\,\overline{\frac{g_2(\bar{P})}{P^2}} + O(P^2) = -2\frac{h(P)}{P} + O(P^2). \tag{3.45}$$

Proof of Lemma 5.2. The square root in (3.37) extends analytically to $P \in \mathbb{C}$ for P such that $1 + P^2 \notin (-\infty, 0]$ to avoid the branch cut. According to (3.37), Q is meromorphic in this region for P, with a simple pole at the origin,

$$Q = \frac{2}{P} + \frac{P}{2} + O(P^3). \tag{3.46}$$

The relations (3.42)-(3.44) follow then in a simple way from (3.38). ■

3.5.1 Proof of Theorem 4.1

Decay of the time-dependent parts

Proposition 5.3. *Suppose the conditions of Theorem 4.1 hold. Denote by* $F(x, y)$ *any of* $h(P(x, y))$, $g_1(P(x, y))$ *or* $g_2(P(x, y))$, *where* $P(x, y) = \frac{2x}{\varepsilon+y}$ *(see* (3.39)*). Then there is a constant* C_n s.t. *for all* $t \geq 0$,

$$\mathbb{E}\big[e^{-it(\varepsilon+\xi_{\rm d})\sqrt{1+P(\xi_{\rm o},\xi_{\rm d})^2}}F(\xi_{\rm o}, \xi_{\rm d})\big] \;\leq\; \frac{C_n}{1+t^n}. \tag{3.47}$$

Remark. We do not make any assumptions on the smoothness or the size of the support of $\mu_{\rm o}(x)$.

Proof of Proposition 5.3. We write $F(x, y) = F(P(x, y))$ and start by noticing that $F(P(x, y))$ is an analytic function of two variables [31] in the domain

$$\mathcal{D} = \big\{(x, y) \in \mathbb{C}^2 \;:\; |{\rm Im} x| < \varepsilon - \eta_{\rm d}, \quad |y| < \varepsilon\big\}.$$

F is analytic in a bigger domain but $\mathcal{D}$ suffices for our purposes as we only need it to contain ${\rm supp}\mu_{\rm o} \times {\rm supp}\mu_{\rm d} \in \mathbb{R} \times \mathbb{R}$. It is obvious that $P(x, y) = \frac{2x}{\varepsilon+y}$ is analytic in $\mathcal{D}$. Then according to Lemma 5.2, F is analytic in $x, y \in \mathbb{C}$ satisfying $P(x, y) \notin \pm i[1, \infty)$. As one readily verifies, this latter condition is satisfied in for $(x, y) \in \mathcal{D}$. Since F is analytic on $\mathcal{D}$, so are all the derivatives $\partial_x^k \partial_y^\ell F(x, y)$. Moreover, $\partial_x^k \partial_y^\ell F(x, y)$ is bounded on any compact set inside $\mathcal{D}$, for arbitrary $k, \ell \in \mathbb{N} \cup \{0\}$.

The expectation value (3.47) reads

$$\mathbb{E}\big[e^{-it(\varepsilon+\xi_d)\sqrt{1+P^2}}F(\xi_{\rm o}, \xi_{\rm d})\big] = \int_{\mathbb{R}^2} e^{-it(\varepsilon+y)q(x,y)} F(x, y)\mu_{\rm o}(x)\mu_{\rm d}(y)dxdy, \tag{3.48}$$

where

$$q(x,y)=\sqrt{1+\frac{4x^2}{(\varepsilon+y)^2}}. \tag{3.49}$$

We show below that

$$\sup_{x\in\mathrm{supp}\mu_{\mathrm{o}}}\sup_{t\geq 0}\left|t^n\int_{\mathbb{R}}e^{-it(\varepsilon+y)q(x,y)}F(x,y)\mu_{\mathrm{d}}(y)dy\right|<\infty. \tag{3.50}$$

Here, suppμ_o denotes the support of μ_o. The result (3.47) then follow from (3.48) and (3.50), as we have the assumptions that μ_o and μ_d are compactly supported. To prove (3.50) we start by noting that

$$q_y'(x,y)=\frac{-P^2}{\sqrt{1+P^2}}\cdot\frac{1}{\varepsilon+y} \tag{3.51}$$

and

$$\partial_y e^{-it(\varepsilon+y)q(x,y)}=-ite^{-it(\varepsilon+y)q(x,y)}\big((\varepsilon+y)q(x,y)\big)_y'. \tag{3.52}$$

By considering (3.51) and (3.52) we have the following equation

$$te^{-it(\varepsilon+y)q(x,y)}=iq(x,y)\partial_y e^{-it(\varepsilon+y)q(x,y)}. \tag{3.53}$$

We integrate by parts n times in (3.50), using (3.53) and the fact that $\mu_{\mathrm{d}}(y)$ is compactly supported (which makes the boundary terms vanish), to get

$$\begin{aligned}&\left|t^n\int_{\mathbb{R}}e^{-it(\varepsilon+y)q(x,y)}F(x,y)\mu_{\mathrm{d}}(y)dy\right|\\ &=\left|t^{n-1}\int_{\mathbb{R}}q(x,y)F(x,y)\mu_d(y)de^{-it(\varepsilon+y)q(x,y)}\right|\\ &=\left|\int_{\mathbb{R}}e^{-it(\varepsilon+y)q(x,y)}\big(\partial_y\circ q(y)\big)^n\big(F(x,y)\mu_{\mathrm{d}}(y)\big)dy\right|\\ &\leq\int_{\mathbb{R}}\left|\big(\partial_y\circ q(y)\big)^n\big(F(x,y)\mu_{\mathrm{d}}(y)\big)\right|dy,\end{aligned} \tag{3.54}$$

where $\big(\partial_y\circ q(y)\big)^n$ is viewed as the *operator* acting by n times applying $\partial_y\circ q(y)$. Namely, for a function f of y,

$$\big(\partial_y\circ q(y)\big)^n f(y)=\partial_y\Big(q(y)\partial_y\big(q(y)\cdots\partial_y(q(y)f(y))\cdots\big)\Big).$$

Here, for notational simplicity, we consider x fixed and simply write $q(y)$ instead of $q(x,y)$.

We are going to show that

$$\sup_{x\in\mathrm{supp}\mu_{\mathrm{o}}}\sup_{y\in\mathrm{supp}\mu_{\mathrm{d}}}\left|\big(\partial_y\circ q(y)\big)^n\big(F(x,y)\mu_{\mathrm{d}}(y)\big)\right|<\infty, \tag{3.55}$$

where suppμ is the support of μ. Then, since $\mu_{\rm d}$ has compact support, which constricts the integration domain of (3.54) to a compact set, the result (3.50) follows from (3.55). To show (3.55), we expand the operator $(\partial_y \circ q(y))^n$ using the product law for derivatives, giving that $\big(\partial_y \circ q(y)\big)^n \big(F(x,y)\mu_{\rm d}(y)\big)$ is a sum of $\frac{1}{2}(n+2)!$ terms, each one of the form

$$q^{(i_1)}(y)\cdots q^{(i_\ell)}(y)\mu_{\rm d}^{(j)}(y)\partial_y^k F(x,y), \tag{3.56}$$

where $(\cdot)^{(r)}$ denotes the rth derivative w.r.t. y, and where the indices satisfy $1 \le \ell \le n$ and $0 \le i_1, i_2, \ldots, i_\ell, j, k \le n$. So it is enough to show that $q^{(k)}(y)$, $\partial_y^k F(x,y)$ and $\mu_{\rm d}^{(k)}(y)$, $0 \le k \le n$, are all bounded.

• As discussed above, $q(y)(\equiv q(x,y))$, given in (3.49), is analytic in $\mathcal{D}$. Thus all y derivatives are bounded, uniformly in any compact subset of $\mathcal{D}$. It follows that

$$\sup_{x\in{\rm supp}\mu_{\rm o}}\ \sup_{y\in{\rm supp}\mu_{\rm d}}\ \max_{0\le k\le n} |q^{(k)}(y)| < \infty. \tag{3.57}$$

• Again, the analyticity of F in $\mathcal{D}$ and the ensuing boundedness of all its derivatives on any compact subset in $\mathcal{D}$ immediately gives

$$\sup_{x\in{\rm supp}\mu_{\rm o}}\ \sup_{y\in{\rm supp}\mu_{\rm d}}\ \max_{0\le k\le n} |\partial_y^k F(x,y)| < \infty. \tag{3.58}$$

• Finally, $\mu_{\rm d}(y)$ is n times continuously differentiable with compact support and so the derivatives are bounded,

$$\sup_{y\in{\rm supp}\mu_{\rm d}}\ \max_{0\le k\le n} |\mu_{\rm d}^{(k)}(y)| < \infty. \tag{3.59}$$

Keeping in mind that the left hand side of (3.55) is a sum of terms of the form (3.56), we see that the estimates (3.57), (3.58) and (3.59) show the bound (3.55). We have thus shown (3.47). This completes the proof of Proposition 5.3. ■

Final state

According to Lemma 5.1 and Proposition 5.3, the final state is

$$\lim_{t\to\infty} \mathbb{E}[\rho_{11}(t)] = 2\mathbb{E}\Big[\frac{Q^2}{(1+Q^2)^2}\Big] + \mathbb{E}\Big[\Big(\frac{1-Q^2}{1+Q^2}\Big)^2\Big]\rho_{11}(0) - 2\mathbb{E}\Big[\frac{Q(1-Q^2)}{(1+Q^2)^2}\Big]{\rm Re}\rho_{12}(0) \tag{3.60}$$

and

$$\lim_{t\to\infty} \mathbb{E}[\rho_{12}(t)] = \mathbb{E}\Big[\frac{Q(1-Q^2)}{(1+Q^2)^2}\Big]\big(1-2\rho_{11}(0)\big) + 4\mathbb{E}\Big[\frac{Q^2}{(1+Q^2)^2}\Big]{\rm Re}\rho_{12}(0). \tag{3.61}$$

Combining (3.60) and (3.61) with the definitions

$$\alpha = 2\mathbb{E}\Big[\frac{Q^2}{(1+Q^2)^2}\Big], \quad \beta = \mathbb{E}\Big[\Big(\frac{1-Q^2}{1+Q^2}\Big)^2\Big], \quad \gamma = \mathbb{E}\Big[\frac{Q(1-Q^2)}{(1+Q^2)^2}\Big] \tag{3.62}$$

yields the result (3.25). Note that for fixed y, $Q \propto x$ is an odd function of x and hence so is $Q(1-Q^2)$. It follows from (3.62) that if $\mu_{\rm o}$ is even, then $\gamma = 0$.

3.5.2 Proof of Theorem 4.2

We need to analyze the decay in t of $\mathbb{E}\big[e^{-it\varepsilon\sqrt{1+P^2}}F(x)\big]$, where $F(x)$ is either of $h(x,0)$, $g_1(x,0)$ or $g_2(x,0)$ and $P = \frac{2x}{\varepsilon}$. We start by noticing that

$$e^{-it\varepsilon\sqrt{1+P^2}} = \frac{i}{2t}\frac{\sqrt{1+P^2}}{P}\partial_x\big(e^{-it\varepsilon\sqrt{1+P^2}}\big). \tag{3.63}$$

Upon integrating n times by parts and using that the boundary term vanish (as $\mu_{\rm o}$ has compact support), we get

$$\begin{aligned}\mathbb{E}\big[e^{-it\varepsilon\sqrt{1+P^2}}F(x)\big] &= \int_{\mathbb{R}} e^{-it\varepsilon\sqrt{1+P^2}}\mu_{\rm o}(x)F(x)dx \\ &= \Big(-\frac{i}{2t}\Big)^n\int_{\mathbb{R}} e^{-it\varepsilon\sqrt{1+P^2}}\partial_x\tfrac{\sqrt{1+P^2}}{P}\Big(\cdots\partial_x\big(\tfrac{\sqrt{1+P^2}}{P}\mu_{\rm o}(x)F(x)\big)\Big)dx,\end{aligned} \tag{3.64}$$

where we apply n times the operator $\partial_x\frac{\sqrt{1+P^2}}{P}$ to $\mu_{\rm o}(x)F(x)$. Once again, it is clear from (3.37), (3.38) that $P\mapsto F(2P/\varepsilon)$ is a C^∞ function on $\mathbb{R}$. So all derivatives of $F(x)$ are bounded on the support of $\mu_{\rm o}$. By expanding the n fold action of the x derivatives inside the last integral of (3.64) (or, simply counting powers), we obtain the following result: If $\mu_{\rm o}(x)$ has a zero of order $k = 1,2,\ldots$ at $x=0$, then the last integral in (3.64) is finite (integrable at the origin) provided $k \geq 2n-1$.

Proof of statement 1. in the theorem: According to (3.42)-(3.44), $h(P)\sim P \propto x$ for $x\sim 0$ and moreover, $\rho_{11}(t)$ only depends on $F = h$, see (3.40). We also have

$$\Big|\int_{\mathbb{R}} e^{-it\varepsilon\sqrt{1+P^2}}h(x)\mu_o(x)dx\Big| \leq \frac{1}{2t}\int_{\mathbb{R}}\Big|\partial_x\big(\frac{\sqrt{1+P^2}}{P}h(x)\mu_o(x)\big)\Big|dx \tag{3.65}$$

and the fact that

$$\frac{\sqrt{1+P^2}}{P}\sim\frac{1}{P}\propto\frac{1}{x}\quad\text{for}\quad x\sim 0. \tag{3.66}$$

So the diagonal of $\mathbb{E}[\rho(t)]$ converges at speed $1/t$ even if $\mu_{\rm o}(0)\neq 0$. Furthermore, if $\rho_{12}(0)=0$, then again due to (3.42)-(3.44), all of $F = h, g_1, g_2$ are $O(x)$ for $x\sim 0$ and then all matrix elements of $\mathbb{E}[\rho(t)]$ converge at speed $1/t$. In either of these cases, we get

$$\mathbb{E}\big[e^{-it\varepsilon\sqrt{1+P^2}}F(x)\big] \leq \frac{C}{1+t}. \tag{3.67}$$

Proof of statement 2. in the theorem: For $\mu_{\rm o}(x)\sim x^k$ at $x\sim 0$, the integral (3.64) is finite provided $k\geq 2n-1$ and so $\mathbb{E}\big[e^{-it\varepsilon\sqrt{1+P^2}}F(x)\big]\leq\frac{C}{1+t^{\frac{k+1}{2}}}$. ■

Chapter 4

Results on entanglement

In this chapter, we first introduce some notions associated with entanglement. Then we consider the dynamics of two qubits with an initially entangled state and an initially separable state, respectively. In the latter case, we also analyze N-qubit systems and M-level bipartite systems.

4.1 Definition

Suppose we have a Hilbert space $\mathcal{H}$ in (2.9), we now introduce some definitions related to quantum entanglement [3].

Definition 4.1 (Separable pure states)**.** *A pure state* $|\Psi\rangle \in \mathcal{H}_{\mathcal{S}}$ *is* **separable** *if it is of the form*

$$|\Psi\rangle = |\Psi_1\rangle \otimes \cdots \otimes |\Psi_N\rangle, \tag{4.1}$$

for some $|\Psi_i\rangle \in \mathcal{H}_i$*,* $1 \leq i \leq N$*.*

Example. Take $N = 2$ in (2.9) and $\mathcal{H}_1 = \mathcal{H}_2$ two Hilbert spaces of dimension two. The following state is separable,

$$|\Psi\rangle = (\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle) \otimes (\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle), \tag{4.2}$$

where

$$|+\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad |-\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{4.3}$$

is the computational basis.

Definition 4.2 (Entangled pure states)**.** *A pure state is* **entangled** *if it is not separable.*

Example. Take again $N = 2$ in (2.9) and $\mathcal{H}_1 = \mathcal{H}_2$ of dimension two. The following state is one of the *Bell states*,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|++\rangle + \frac{1}{\sqrt{2}}|--\rangle, \tag{4.4}$$

where

$$|{+}{+}\rangle = |{+}\rangle \otimes |{+}\rangle \quad \text{and} \quad |{-}{-}\rangle = |{-}\rangle \otimes |{-}\rangle, \tag{4.5}$$

is an entangled pure state. It is standard and easy to check that (4.4) is indeed entangled: by trying to solve $|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle$ one quickly reaches a contradiction.

Definition 4.3 (Separable mixed states)**.** *A mixed state ρ on $\mathcal{H}_\mathcal{S}$ is* **separable** *if it can be expressed as a convex combination of pure states, namely,*

$$\rho = \sum_\mu p_\mu |\Psi_{\mu 1}\rangle\langle\Psi_{\mu 1}| \otimes \cdots \otimes |\Psi_{\mu N}\rangle\langle\Psi_{\mu N}|, \tag{4.6}$$

for some $|\Psi_{\mu j}\rangle \in \mathcal{H}_j$, and $0 < p_\mu \leq 1$ such that $\sum_\mu p_\mu = 1$.

Example. In the setting of the last example, the state

$$\rho = \frac{1}{2}|{+}{+}\rangle\langle{+}{+}| + \frac{1}{2}|{-}{-}\rangle\langle{-}{-}|, \tag{4.7}$$

is a mixed, separable state.

Definition 4.4 (Entangled mixed states)**.** *A mixed state ρ on $\mathcal{H}_\mathcal{S}$ is* **entangled** *if it is not separable.*

Example. In the setting of the previous example,

$$\rho = \frac{1}{2}|{+}{+}\rangle\langle{+}{+}| + \frac{1}{2}|{-}{-}\rangle\langle{-}{-}| + \frac{1}{5}|{+}{+}\rangle\langle{-}{-}| + \frac{1}{5}|{-}{-}\rangle\langle{+}{+}|, \tag{4.8}$$

is a mixed, entangled state. The spectral representation of ρ reads

$$\rho = \frac{7}{10}|\Psi_1\rangle\langle\Psi_1| + \frac{3}{10}|\Psi_2\rangle\langle\Psi_2|, \tag{4.9}$$

where

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}|{+}{+}\rangle + \frac{1}{\sqrt{2}}|{-}{-}\rangle, \tag{4.10}$$

$$|\Psi_2\rangle = -\frac{1}{\sqrt{2}}|{+}{+}\rangle + \frac{1}{\sqrt{2}}|{-}{-}\rangle. \tag{4.11}$$

It is easy to check that there are no vectors $|\Psi_{i1}\rangle \in \mathcal{H}_1$ and $|\Psi_{i2}\rangle \in \mathcal{H}_2$ such that $|\Psi_i\rangle = |\Psi_{i1}\rangle \otimes |\Psi_{i2}\rangle$, $i \in \{1, 2\}$. So this ρ can not be expressed of the form in (4.6).

Now, we know the basic concepts of entanglement. It is important to find measures to quantify the 'amount of entanglement' in a state. There are three basic such measures: *concurrence*, *negativity* and *quantum discord* [4]. In this thesis, we consider concurrence. This is a notion which quantifies the degree of entanglement in two-qubit states [32].

Definition 4.5 (Concurrence of pure states)**.** *The* **concurrence** *of a pure two-qubit state* $|\Psi\rangle$ *is defined by*

$$\mathcal{C}(\Psi) = |\langle\Psi|\tilde{\Psi}\rangle|, \tag{4.12}$$

where $|\tilde{\Psi}\rangle = (\sigma_y \otimes \sigma_y)|\Psi^*\rangle$*,* Ψ^* *is the vector obtained from* Ψ *by taking the complex conjugate of the coordinates in the computational basis (eigenbasis* $\{|+\rangle, |-\rangle\}$ *of* σ_z*)* Ψ *and*

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{4.13}$$

is the Pauli y matrix (written in the same basis).

We know from [21,32] that the concurrence of two qubits satisfies $0 \leq \mathcal{C}(\rho) \leq 1$. It will vanish if and only if ρ is separable.

Example. Take the pure state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|{+}{+}\rangle + |{-}{-}\rangle)$ as in (4.4). We calculate its concurrence,

$$\begin{aligned}
\mathcal{C}(\Psi) =& |\langle\Psi|\tilde{\Psi}\rangle| \\
=& \frac{1}{2}\Big|\big(\langle{+}{+}| + \langle{-}{-}|\big)\big(\sigma_y \otimes \sigma_y|{+}{+}\rangle + \sigma_y \otimes \sigma_y|{-}{-}\rangle\big)\Big| \\
=& \frac{1}{2}\Big|\big(\langle{+}{+}| + \langle{-}{-}|\big)\big((-1)|{-}{-}\rangle + (-1)|{+}{+}\rangle\big)\Big| \\
=& 1.
\end{aligned} \tag{4.14}$$

This shows that $|\Psi\rangle$ is an entangled pure state.

In what follows, we will represent the density matrix of the two qubit system always as a matrix written in the orthonormal, ordered basis

$$\{|{+}{+}\rangle, |{+}{-}\rangle, |{-}{+}\rangle, |{-}{-}\rangle\}.$$

Definition 4.6 (Concurrence of mixed states)**.** *The* ***concurrence*** *of a mixed state* ρ *of two qubits is defined as*

$$\mathcal{C}(\rho) \equiv \max\big(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\big), \tag{4.15}$$

where $\lambda_1 \geq \cdots \geq \lambda_4$ *are the eigenvalues, in decreasing order, of the Hermitian matrix*

$$R = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}. \tag{4.16}$$

Here,

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y), \tag{4.17}$$

where

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

is the Pauli y matrix (written here in the computational basis) and ρ^* *is obtained from* ρ *by taking the entry-wise complex conjugation of* ρ *in that same basis.*

The square roots in (4.16) are those of non-negative hermitian operators (denoted ≥ 0). Since $\rho \geq 0$ it is clear from the spectral theorem that $\sqrt{\rho} \geq 0$ as well: the spectral representation of ρ reads

$$\rho = \lambda_1 |P_1\rangle\langle P_1| + \lambda_2 |P_2\rangle\langle P_2| + \lambda_3 |P_3\rangle\langle P_3| + \lambda_4 |P_4\rangle\langle P_4|, \tag{4.18}$$

where the $\lambda_i \geq 0$ are the non-negative eigenvalues and P_i are associated eigenprojections. Then by the spectral theorem, we have

$$\sqrt{\rho} = \sqrt{\lambda_1}|P_1\rangle\langle P_1| + \sqrt{\lambda_2}|P_2\rangle\langle P_2| + \sqrt{\lambda_3}|P_3\rangle\langle P_3| + \sqrt{\lambda_4}|P_4\rangle\langle P_4|. \tag{4.19}$$

So $\sqrt{\rho}$ is hermitian and non-negative. Next we explain why $\sqrt{\rho}\tilde{\rho}\sqrt{\rho}$ is hermitian and non-negative as well. Note that

$$(\tilde{\rho})^\dagger = (\sigma_y \otimes \sigma_y)^\dagger (\rho^*)^\dagger (\sigma_y \otimes \sigma_y)^\dagger = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y) = \tilde{\rho}. \tag{4.20}$$

We have used in (4.20) that $(\rho^*)^\dagger = \rho^\dagger$, which follows since the operation of taking $*$ and taking the adjoint commute, $(\rho^*)^\dagger = (\rho^\dagger)^* = \rho^*$. It is then immediate that $\sqrt{\rho}\tilde{\rho}\sqrt{\rho}$ is hermitian. Next we show that $\sqrt{\rho}\tilde{\rho}\sqrt{\rho}$ is non-negative. For $\psi \in \mathcal{H}$, we have $\langle \psi, \sqrt{\rho}\tilde{\rho}\sqrt{\rho}\psi\rangle = \langle (\sqrt{\rho}\psi), \tilde{\rho}(\sqrt{\rho}\psi)\rangle$, so it is enough to show that $\tilde{\rho} \geq 0$. From (4.18) and the definition of $\tilde{\rho}$, we find the spectral decomposition of $\tilde{\rho}$ to be

$$\tilde{\rho} = \lambda_1 |Q_1\rangle\langle Q_1| + \lambda_2 |Q_2\rangle\langle Q_2| + \lambda_3 |Q_3\rangle\langle Q_3| + \lambda_4 |Q_4\rangle\langle Q_4| \tag{4.21}$$

with

$$|Q_i\rangle = (\sigma_y \otimes \sigma_y)|P_i^*\rangle, \tag{4.22}$$

where $|P_i^*\rangle$ is obtained from $|P_i\rangle$ by taking component wise complex conjugates in the computational basis $\{|+\rangle, |-\rangle\}$. Thus $\tilde{\rho}$ is a non-negative operator as all the eigenvalues of $\tilde{\rho}$ are non-negative.

Remark: It is shown in [32] that the eigenvalues of Hermitian matrix $R = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$ are the square roots of the eigenvalues of the non-Hermitian matrix $\rho\tilde{\rho}$, namely,

$$\text{The spectrum of the two matrices } \sqrt{\rho}\tilde{\rho}\sqrt{\rho} \text{ and } R^2 \text{ coincide.} \tag{4.23}$$

Since we are going to use this fact, we will give a proof of it now.

First suppose that $\rho > 0$ strictly, *i.e.*, all eigenvalues are strictly positive. Then ρ and $\sqrt{\rho}$ are invertible and

$$(\sqrt{\rho})^{-1}\rho\tilde{\rho}\sqrt{\rho} = \sqrt{\rho}\tilde{\rho}\sqrt{\rho}. \tag{4.24}$$

This shows that $\sqrt{\rho}\tilde{\rho}\sqrt{\rho}$ and $\rho\tilde{\rho}$ are *similar*. So in particular, they have the same eigenvalues. Now we do the general case. For any $\eta > 0$, $\rho + \eta \equiv \rho + \eta\mathbb{1}$ is strictly positive. Thus

$$\rho\tilde{\rho} = (\rho+\eta)\tilde{\rho} - \eta\tilde{\rho} = \sqrt{\rho+\eta}\big(\sqrt{\rho+\eta}\tilde{\rho}\sqrt{\rho+\eta}\big)\frac{1}{\sqrt{\rho+\eta}} - \eta\tilde{\rho}. \tag{4.25}$$

Denoting by $\mathrm{spec}(X)$ the spectrum of a matrix X, and noting that for matrices X, Y and small $\epsilon \in \mathbb{C}$, $\mathrm{spec}(X + \epsilon Y) = \mathrm{spec}(X) + O(\epsilon)$, we have from (4.25):

$$\mathrm{spec}(\rho\tilde{\rho}) = \mathrm{spec}\Big(\sqrt{\rho+\eta}\big(\sqrt{\rho+\eta}\tilde{\rho}\sqrt{\rho+\eta}\big)\frac{1}{\sqrt{\rho+\eta}}\Big) + O(\eta). \tag{4.26}$$

By similarity, we have $\forall\eta > 0$

$$\mathrm{spec}\Big(\sqrt{\rho+\eta}\big(\sqrt{\rho+\eta}\tilde{\rho}\sqrt{\rho+\eta}\big)\frac{1}{\sqrt{\rho+\eta}}\Big) = \mathrm{spec}(\sqrt{\rho+\eta}\tilde{\rho}\sqrt{\rho+\eta}). \tag{4.27}$$

Combining this with (4.26) gives

$$\mathrm{spec}(\rho\tilde{\rho}) = \mathrm{spec}(\sqrt{\rho+\eta}\tilde{\rho}\sqrt{\rho+\eta}) + O(\eta). \tag{4.28}$$

Next, denoting the projection onto the kernel of ρ by P_0, and by $P_0^\perp$ its complement, we have $\sqrt{\rho+\eta} = \sqrt{\rho+\eta}\,P_0^\perp + \sqrt{\eta}\,P_0 = \sqrt{\rho}P_0^\perp + O(\sqrt{\eta}) = \sqrt{\rho} + O(\eta)$. Using this last equality in (4.28) shows that

$$\mathrm{spec}(\rho\tilde{\rho}) = \mathrm{spec}(\sqrt{\rho}\tilde{\rho}\sqrt{\rho}) + O(\sqrt{\eta}). \tag{4.29}$$

Upon taking $\eta \to 0$ we get $\mathrm{spec}(\rho\tilde{\rho}) = \mathrm{spec}(\sqrt{\rho}\tilde{\rho}\sqrt{\rho})$.

Example. We first consider the mixed separable state ρ in (4.7), which has the matrix representation

$$\rho = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{4.30}$$

in the standard basis $\{|{+}{+}\rangle, |{+}{-}\rangle, |{-}{+}\rangle, |{-}{-}\rangle\}$. We can also find the matrix form of $\sigma_y \otimes \sigma_y$ in the same basis

$$\sigma_y \otimes \sigma_y = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{4.31}$$

From the above quantities and Definition 4.6, we have

$$R^2 = \sqrt{\rho}\tilde{\rho}\sqrt{\rho} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \tag{4.32}$$

Two of the eigenvalues of R are 0 and the other two equal 1/2. By the definition of concurrence, we conclude that $\mathcal{C}(\rho) = 0$.

Example. Consider the entangled mixed state ρ in (4.8). Considering (4.9), by the spectral theorem, we have

$$\sqrt{\rho} = \sqrt{\frac{7}{10}}|\Psi_1\rangle\langle\Psi_1| + \sqrt{\frac{3}{10}}|\Psi_2\rangle\langle\Psi_2|. \tag{4.33}$$

Combing the matrix form of $\tilde{\rho}$, we can get R^2 of matrix form in the same basis

$$R^2 = \sqrt{\rho}\tilde{\rho}\sqrt{\rho} = \begin{pmatrix} \frac{29}{100} & 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{5} & 0 & 0 & \frac{29}{100} \end{pmatrix}. \tag{4.34}$$

The eigenvalues of R^2 are 0, 0, $\frac{1}{4}$ and $\frac{33}{100}$. Hence, $\mathcal{C}(\rho) = \frac{\sqrt{33}}{10} - \frac{1}{2} > 0$. This concludes that ρ is entangled. This argument again shows that the state in example 4.4 is entangled, this time by calculating its concurrence.

4.2 Dynamics of an initially entangled state

We consider two qubits with Hamiltonian

$$H = H_1 \otimes \mathbb{1} + \mathbb{1} \otimes H_2 \tag{4.35}$$

where

$$H_1 = \begin{pmatrix} \varepsilon_1 + \zeta_1 & 0 \\ 0 & \varepsilon_2 + \zeta_2 \end{pmatrix}, \quad H_2 = \begin{pmatrix} \eta_1 + \nu_1 & 0 \\ 0 & \eta_2 + \nu_2 \end{pmatrix}, \tag{4.36}$$

and $\mathbb{1}$ is the 2×2 identity matrix.

Here, ε_1, ε_2, η_1 and η_2 are constants and ζ_1, ζ_2, ν_1 and ν_2 are real-valued random variables, representing the noise.

Let

$$\varepsilon = \varepsilon_1 - \varepsilon_2, \quad \zeta = \zeta_1 - \zeta_2, \quad \eta = \eta_1 - \eta_2 \quad \text{and} \quad \nu = \nu_1 - \nu_2. \tag{4.37}$$

From postulate 2, we know that the solution of the Liouville - von Neumann equation is

$$\rho(t) = e^{-itH}\rho(0)e^{itH}, \tag{4.38}$$

where $\rho(0)$ is the initial condition.

We can apply the same idea as in Chapter 2: Let α_1 and α_2 be some constants (or, random variables), then we have

$$e^{-it\big((H_1+\alpha_1\mathbb{1})\otimes\mathbb{1}+\mathbb{1}\otimes(H_2+\alpha_2\mathbb{1})\big)}\rho(0)e^{it\big((H_1+\alpha_1\mathbb{1})\otimes\mathbb{1}+\mathbb{1}\otimes(H_2+\alpha_2\mathbb{1})\big)} = e^{-it(H_1\otimes\mathbb{1}+\mathbb{1}\otimes H_2)}\rho(0)e^{it(H_1\otimes\mathbb{1}+\mathbb{1}\otimes H_2)}. \tag{4.39}$$

Due to the (4.39), it is equivalent, from the point of view of the dynamics, to take instead of (4.36)

$$H_1 = \begin{pmatrix} \varepsilon+\zeta & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad H_2 = \begin{pmatrix} \eta+\nu & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.40}$$

We assume the initial condition is the pure state

$$\rho(0) = |\varphi_0\rangle\langle\varphi_0|. \tag{4.41}$$

Using $e^{it(H_1\otimes\mathbb{1}+\mathbb{1}\otimes H_2)} = e^{itH_1}\otimes e^{itH_2}$ we obtain

$$\begin{aligned} \rho(t) &= \big(e^{-itH_1}\otimes e^{-itH_2}\big)|\varphi_0\rangle\langle\varphi_0|\big(e^{itH_1}\otimes e^{itH_2}\big) \\ &= |(e^{-itH_1}\otimes e^{-itH_2})\varphi_0\rangle\langle(e^{-itH_1}\otimes e^{-itH_2})\varphi_0| \\ &= |\varphi_t\rangle\langle\varphi_t|, \end{aligned} \tag{4.42}$$

where

$$|\varphi_t\rangle = (e^{-itH_1}\otimes e^{-itH_2})|\varphi_0\rangle. \tag{4.43}$$

We first take the initial state to be entangled,

$$|\varphi_0\rangle = \frac{1}{\sqrt{2}}|{+}{+}\rangle + \frac{1}{\sqrt{2}}|{-}{-}\rangle. \tag{4.44}$$

Taking into account (4.43), we have the following result

$$|\varphi_t\rangle \;=\; (e^{-itH_1}\otimes e^{-itH_2})|\varphi_0\rangle = \frac{1}{\sqrt{2}}e^{-it(\varepsilon+\eta+\zeta+\nu)}|{+}{+}\rangle + \frac{1}{\sqrt{2}}|{-}{-}\rangle. \tag{4.45}$$

According to (4.42) and (4.45), we can get the expectation of $\rho(t)$,

$$\begin{aligned} \bar{\rho}(t) \equiv& \mathbb{E}[\rho(t)] = \mathbb{E}\big[|\varphi_t\rangle\langle\varphi_t|\big] \\ =& \frac{1}{2}\mathbb{E}\big[(e^{-it(\varepsilon+\eta+\zeta+\nu)}|{+}{+}\rangle + |{-}{-}\rangle)(e^{it(\varepsilon+\eta+\zeta+\nu)}\langle{+}{+}| + \langle{-}{-}|)\big] \end{aligned}$$

$$=\frac{1}{2}e^{-it(\varepsilon+\eta)}|{+}{+}\rangle\langle{-}{-}|\mathbb{E}[e^{-it(\zeta+\nu)}]+\frac{1}{2}e^{it(\varepsilon+\eta)}|{-}{-}\rangle\langle{+}{+}|\mathbb{E}[e^{it(\zeta+\nu)}]$$
$$+\frac{1}{2}|{+}{+}\rangle\langle{+}{+}|+\frac{1}{2}|{-}{-}\rangle\langle{-}{-}|. \tag{4.46}$$

Let

$$b_t = e^{-it(\varepsilon+\eta)}\mathbb{E}[e^{-it(\zeta+\nu)}]. \tag{4.47}$$

We now calculate the concurrence $\mathcal{C}(\bar{\rho}(t))$. In the standard basis $\{|{+}{+}\rangle, |{+}{-}\rangle, |{-}{+}\rangle, |{-}{-}\rangle\}$, $\bar{\rho}(t)$ reads

$$\bar{\rho}(t) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & b_t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \bar{b}_t & 0 & 0 & 1 \end{pmatrix}. \tag{4.48}$$

We then use ρ to represent $\bar{\rho}(t)$.

In definition 4.6, we know that

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y), \tag{4.49}$$

where

$$\sigma_y \otimes \sigma_y = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{4.50}$$

in the basis $\{|{+}{+}\rangle, |{+}{-}\rangle, |{-}{+}\rangle, |{-}{-}\rangle\}$. Then we have

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y) \tag{4.51}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & \bar{b}_t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b_t & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & b_t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \bar{b}_t & 0 & 0 & 1 \end{pmatrix}. \tag{4.52}$$

According to (4.23), the eigenvalues of Hermitian matrix $R = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$ are the square roots of the eigenvalues of $\rho\tilde{\rho}$ which is in our case equals

$$\rho\tilde{\rho} = \frac{1}{4}\begin{pmatrix} 1+|b_t|^2 & 0 & 0 & 2b_t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2\bar{b}_t & 0 & 0 & 1+|b_t|^2 \end{pmatrix}. \tag{4.53}$$

The eigenvalues of the matrix (4.53) are

$$\begin{aligned} \lambda_1' &= \frac{1}{4}(1+|b_t|)^2, \\ \lambda_2' &= \frac{1}{4}(1-|b_t|)^2, \\ \lambda_3' &= \lambda_4' = 0. \end{aligned} \tag{4.54}$$

According to Definition 4.6 (and noticing that by (4.47), $|b_t| \leq 1$) the concurrence of $\bar{\rho}(t)$ is

$$\begin{aligned} \mathcal{C}\big(\bar{\rho}(t)\big) &= \max\big(0, \sqrt{\lambda_1'} - \sqrt{\lambda_2'} - \sqrt{\lambda_3'} - \sqrt{\lambda_4'}\big) \\ &= \sqrt{\lambda_1'} - \sqrt{\lambda_2'} && (4.55) \\ &= |b_t| && (4.56) \\ &= \big|\,\mathbb{E}[e^{-it(\zeta+\nu)}]\,\big|. && (4.57) \end{aligned}$$

Case 1: Individual noise

Now we consider the situation where ζ and ν are independent random variables with probability density functions

$$\zeta \leftrightarrow \mu_1(x) \quad \text{and} \quad \nu \leftrightarrow \mu_2(y). \tag{4.58}$$

In order to obtain explicit expressions, we take

$$\mu_1(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-w)^2}{2\sigma^2}} \quad \text{and} \quad \mu_2(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-w)^2}{2\sigma^2}} \tag{4.59}$$

to be normal distributions with mean w and variance σ^2. According to (4.47),

$$b_t = e^{-it(\varepsilon+\eta)}\mathbb{E}[e^{-it(\zeta+\nu)}] = e^{-it(\varepsilon+\eta)} \int_{\mathbb{R}^2} e^{-it(x+y)}\mu_1(x)\mu_2(y)dxdy = e^{-it(\varepsilon+\eta)}e^{-2iwt-t^2\sigma^2}. \tag{4.60}$$

It follows from (4.55) that

$$\mathcal{C}\big(\bar{\rho}(t)\big) = \big|\,\mathbb{E}[e^{-it(\zeta+\nu)}]\,\big| = e^{-t^2\sigma^2}. \tag{4.61}$$

This shows that the concurrence decreases monotonically in time from its maximal value 1 at $t = 0$ to zero for large times.

Case 2: Common noise

In this part, we consider common noise, which means that ζ and ν are dependent. We let $\zeta = \nu$ and the probability density function $\mu(x)$ of ζ be normal distribution as in (4.59). In this case, we have

$$b_t = e^{-it(\varepsilon+\eta)}\mathbb{E}[e^{-it(\zeta+\nu)}] = e^{-it(\varepsilon+\eta)} \int_{\mathbb{R}} e^{-2itx}\mu(x)dx = e^{-it(\varepsilon+\eta)}e^{-2iwt-2t^2\sigma^2}. \tag{4.62}$$

It follows from (4.55) that

$$\mathcal{C}\big(\bar{\rho}(t)\big) = \big|\,\mathbb{E}[e^{-it(\zeta+\nu)}]\,\big| = e^{-2t^2\sigma^2}. \tag{4.63}$$

Comparing (4.61) with (4.63), we conclude that the concurrence is destroyed much more quickly in presence of common noise, as compared to individual noises. Moreover, the concurrence depends only on the *variance* σ^2 of the noise, not on its mean w.

Remark. It is equally possible to carry out the analysis if the two individual noises are Gaussian with different means and variances. Let the probability density functions of the two noisees be given by

$$\mu_1(x) = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(x-w_1)^2}{2\sigma_1^2}} \quad \text{and} \quad \mu_2(y) = \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(y-w_2)^2}{2\sigma_2^2}}. \tag{4.64}$$

According to (4.47), we have

$$b_t = e^{-it(\varepsilon+\eta)}\mathbb{E}[e^{-it(\zeta+\nu)}] = e^{-it(\varepsilon+\eta)} \int_{\mathbb{R}^2} e^{-it(x+y)}\mu_1(x)\mu_2(y)dxdy = e^{-it(\varepsilon+\eta)} e^{-i(w_1+w_2)t-\frac{t^2}{2}(\sigma_1^2+\sigma_2^2)}. \tag{4.65}$$

It follows from (4.55) that

$$\mathcal{C}\big(\bar{\rho}(t)\big) = \big|\,\mathbb{E}[e^{-it(\zeta+\nu)}]\,\big| = e^{-\frac{t^2}{2}(\sigma_1^2+\sigma_2^2)}. \tag{4.66}$$

For $\sigma_1 = \sigma_2 = \sigma$, this reduces correctly to (4.63).

4.3 Dynamics of initially separable states

4.3.1 Two qubits

The Hamiltonian of two-qubit system is again

$$H = H_1 \otimes \mathbb{1} + \mathbb{1} \otimes H_2, \tag{4.67}$$

where

$$H_1 = \begin{pmatrix} \varepsilon+\zeta & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad H_2 = \begin{pmatrix} \eta+\nu & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.68}$$

We consider the most general pure separable initial state, given by

$$|\varphi_0\rangle = \big(\alpha_+|+\rangle + \alpha_-|-\rangle\big) \otimes \big(\beta_+|+\rangle + \beta_-|-\rangle\big), \tag{4.69}$$

where $\alpha_\pm, \beta_\pm \in \mathbb{C}$ satisfy

$$|\alpha_+|^2 + |\alpha_-|^2 = 1 \qquad \text{and} \qquad |\beta_+|^2 + |\beta_-|^2 = 1.$$

The dynamics of the initial state (4.69) is given by

$$\begin{aligned}|\varphi_t\rangle &= \big(e^{-itH_1}\otimes e^{-itH_2}\big)|\varphi_0\rangle \\ &= \big(e^{-itH_1}\otimes e^{-itH_2}\big)\big\{(\alpha_+|+\rangle+\alpha_-|-\rangle)\otimes(\beta_+|+\rangle+\beta_-|-\rangle)\big\} \\ &= \big(e^{-itH_1}(\alpha_+|+\rangle+\alpha_-|-\rangle)\big)\otimes\big(e^{-itH_2}(\beta_+|+\rangle+\beta_-|-\rangle)\big) \\ &= \big(\alpha_+e^{-it(\varepsilon+\zeta)}|+\rangle+\alpha_-|-\rangle\big)\otimes\big(\beta_+e^{-it(\eta+\nu)}|+\rangle+\beta_-|-\rangle\big). \end{aligned} \tag{4.70}$$

The expectation of $\rho(t)=|\varphi_t\rangle\langle\varphi_t|$ is

$$\begin{aligned}\bar\rho(t) \equiv&\mathbb{E}\big[\rho(t)\big] \\ =&\mathbb{E}\Big[\big\{(\alpha_+e^{-it(\varepsilon+\zeta)}|+\rangle+\alpha_-|-\rangle)\otimes(\beta_+e^{-it(\eta+\nu)}|+\rangle+\beta_-|-\rangle)\big\} \\ &\big\{(\bar\alpha_+e^{it(\varepsilon+\zeta)}\langle+|+\bar\alpha_-\langle-|)\otimes(\bar\beta_+e^{it(\eta+\nu)}\langle+|+\bar\beta_-\langle-|)\big\}\Big] \\ =&\mathbb{E}\Big[\big(|\alpha_+|^2\,|+\rangle\langle+|+\alpha_+\bar\alpha_-e^{-it(\varepsilon+\zeta)}|+\rangle\langle-|+\bar\alpha_+\alpha_-e^{it(\varepsilon+\zeta)}|-\rangle\langle+|+|\alpha_-|^2\,|-\rangle\langle-|\big) \\ &\otimes\big(|\beta_+|^2\,|+\rangle\langle+|+\beta_+\bar\beta_-e^{-it(\eta+\nu)}|+\rangle\langle-|+\bar\beta_+\beta_-e^{it(\eta+\nu)}|-\rangle\langle+|+|\beta_-|^2\,|-\rangle\langle-|\big)\Big]. \end{aligned} \tag{4.71}$$

• For the case of **individual noises**, the random variables ζ and ν are independent, and we have

$$\begin{aligned}\bar\rho(t) =&\Big(|\alpha_+|^2\,|+\rangle\langle+|+\bar\alpha_-\alpha_+\mathbb{E}\big[e^{-it(\varepsilon+\zeta)}\big]|+\rangle\langle-|+\bar\alpha_+\alpha_-\mathbb{E}\big[e^{it(\varepsilon+\zeta)}\big]|-\rangle\langle+|+|\alpha_-|^2\,|-\rangle\langle-|\Big) \\ &\otimes\Big(|\beta_+|^2\,|+\rangle\langle+|+\bar\beta_-\beta_+\mathbb{E}\big[e^{-it(\eta+\nu)}\big]|+\rangle\langle-|+\bar\beta_+\beta_-\mathbb{E}\big[e^{it(\eta+\nu)}\big]|-\rangle\langle+|+|\beta_-|^2|-\rangle\langle-|\Big). \end{aligned} \tag{4.72}$$

We see that $\bar\rho(t)$, (4.72), is the product of two qubit density matrices. So this state is disentangled (separable) for all times. This shows that the individual noise does not create entanglement in an initially pure separable state. Note, however, that $\bar\rho(t)$ becomes a mixed state for all times t satisfying $t>0$. From (4.71), we can denote $\rho(t)$ by the product of the matrices in the basis $\{|{+}{+}\rangle,|{+}{-}\rangle,|{-}{+}\rangle,|{-}{-}\rangle\}$ as

$$\rho(t)=\begin{pmatrix}|\alpha_+|^2 & \bar\alpha_-\alpha_+e^{-it(\varepsilon+\zeta)} \\ \bar\alpha_+\alpha_-e^{it(\varepsilon+\zeta)} & |\alpha_-|^2\end{pmatrix}\otimes\begin{pmatrix}|\beta_+|^2 & \bar\beta_-\beta_+e^{-it(\eta+\nu)} \\ \bar\beta_+\beta_-e^{it(\eta+\nu)} & |\beta_-|^2\end{pmatrix} \tag{4.73}$$

Then we can get the 4×4 matrix form of $\rho(t)$, by taking the expectation, we have

$$\bar\rho(t)=\mathbb{E}[\rho(t)]=\begin{pmatrix}|\alpha_+|^2|\beta_+|^2 & |\alpha_+|^2|\bar\beta_-\beta_+y_t & \bar\alpha_-\alpha_+|\beta_+|^2x_t & \bar\alpha_-\alpha_+\bar\beta_-\beta_+x_ty_t \\ |\alpha_+|^2|\bar\beta_+\beta_-\bar y_t & |\alpha_+|^2||\beta_-|^2 & \bar\alpha_-\alpha_+\bar\beta_+\beta_-x_t\bar y_t & \bar\alpha_-\alpha_+|\beta_-|^2x_t \\ \bar\alpha_+\alpha_-|\beta_+|^2\bar x_t & \bar\alpha_+\alpha_-\bar\beta_-\beta_+\bar x_ty_t & |\alpha_-|^2|\beta_+|^2 & |\alpha_-|^2\bar\beta_-\beta_+y_t \\ \bar\alpha_+\alpha_-\bar\beta_+\beta_-\bar x_t\bar y_t & \bar\alpha_+\alpha_-|\beta_-|^2\bar x_t & |\alpha_-|^2\bar\beta_+\beta_-\bar y_t & |\alpha_-|^2|\beta_-|^2\end{pmatrix}, \tag{4.74}$$

where $x_t=\mathbb{E}[e^{-it(\varepsilon+\zeta)}]$ and $y_t=\mathbb{E}[e^{-it(\eta+\nu)}]$. Denote by $|v_1\rangle$, $|v_2\rangle$, $|v_3\rangle$ and $|v_4\rangle$ the four column vectors of $\bar{\rho}(t)$, respectively.

When $t = 0$, we see that $x_t = y_t = 1$ and we get that $\text{rank}(\bar{\rho}(t)) = 1$, since

$$\begin{aligned} \bar{\beta}_-|v_1\rangle &= \bar{\beta}_+|v_2\rangle, \\ \bar{\alpha}_-\bar{\beta}_+|v_2\rangle &= \bar{\alpha}_+\bar{\beta}_-|v_3\rangle, \\ \bar{\beta}_-|v_3\rangle &= \bar{\beta}_+|v_4\rangle. \end{aligned} \tag{4.75}$$

When $t > 0$, the rank of $\bar{\rho}(t)$ is ≥ 2. To show this, let's assume by contradiction that $|v_1\rangle$ and $|v_2\rangle$ are linearly dependent, that is, there exists $c \in \mathbb{C}$, such that

$$|v_1\rangle = c|v_2\rangle.$$

Looking at the components of the vectors $|v_1\rangle$ and $|v_2\rangle$, we find that

$$c = \frac{1}{y_t} = \bar{y}_t.$$

This implies that

$$\mathbb{E}[e^{-it(\eta+\nu)}]\mathbb{E}[e^{it(\eta+\nu)}] = 1.$$

However, from (4.60), we know that

$$\mathbb{E}[e^{-it(\eta+\nu)}]\mathbb{E}[e^{it(\eta+\nu)}] = e^{-t^2\sigma^2} \neq 1 \quad \text{when} \quad t > 0.$$

This contradicts our assumption that $|v_1\rangle$ and $|v_2\rangle$ are linearly dependent. Thus $\text{rank}(\bar{\rho}(t)) > 1$ and $\bar{\rho}(t)$ is a mixed state for all $t > 0$.

- For **common noise** we have $\zeta = \nu$. Using the result in (4.60) and (4.62), we have

$$\mathbb{E}[e^{-it(\varepsilon+\eta+\zeta+\nu)}] = e^{-it(\varepsilon+\eta)}e^{-2iwt-2t^2\sigma^2} \tag{4.76}$$

and

$$\mathbb{E}[e^{-it(\varepsilon+\zeta)}]\mathbb{E}[e^{-it(\eta+\nu)}] = e^{-it(\varepsilon+\eta)}e^{-2iwt-t^2\sigma^2}. \tag{4.77}$$

We first represent $\rho(t)$ in (4.71) as a matrix in the standard basis and then we take expectation term by term in this matrix. We also consider the specific choice $\alpha_\pm = \beta_\pm = \frac{1}{\sqrt{2}}$ and $\varepsilon = \eta = 0$. We have

$$\mathbb{E}[e^{-it(\varepsilon+\eta+\zeta+\nu)}] = e^{-2iwt-2t^2\sigma^2}$$

and

$$\mathbb{E}[e^{-it(\varepsilon+\zeta)}] = \mathbb{E}[e^{-it(\eta+\nu)}] = e^{-iwt-\frac{t^2\sigma^2}{2}}.$$

Then we can get $\bar{\rho}(t)$

$$\bar{\rho}(t) = \frac{1}{4}\begin{pmatrix} 1 & \alpha & \alpha & \beta \\ \bar{\alpha} & 1 & 1 & \alpha \\ \bar{\alpha} & 1 & 1 & \alpha \\ \bar{\beta} & \bar{\alpha} & \bar{\alpha} & 1 \end{pmatrix}, \tag{4.78}$$

where $\alpha \equiv \alpha(t) = e^{-iwt-\frac{t^2\sigma^2}{2}}$ and $\beta \equiv \beta(t) = e^{-2iwt-2t^2\sigma^2}$. By definition 4.6, we have

$$(\sigma_y \otimes \sigma_y)\bar{\rho}(t)^*(\sigma_y \otimes \sigma_y) = \frac{1}{4}\begin{pmatrix} 1 & -\bar{\alpha} & -\bar{\alpha} & \bar{\beta} \\ -\alpha & 1 & 1 & -\bar{\alpha} \\ -\alpha & 1 & 1 & -\bar{\alpha} \\ \beta & -\alpha & -\alpha & 1 \end{pmatrix}, \tag{4.79}$$

and we also find

$$\begin{aligned} &\bar{\rho}(t)\big((\sigma_y \otimes \sigma_y)\bar{\rho}(t)^*(\sigma_y \otimes \sigma_y)\big) \\ =&\frac{1}{16}\begin{pmatrix} \beta^2 - 2\alpha^2 + 1 & -\bar{\alpha} + 2\alpha - \alpha\beta & -\bar{\alpha} + 2\alpha - \alpha\beta & \bar{\beta} - 2|\alpha|^2 + \beta \\ \bar{\alpha} - 2\alpha + \alpha\beta & -\bar{\alpha}^2 + 2 - \alpha^2 & -\bar{\alpha}^2 + 2 - \alpha^2 & \bar{\alpha}\bar{\beta} - 2\bar{\alpha} + \alpha \\ \bar{\alpha} - 2\alpha + \alpha\beta & -\bar{\alpha}^2 + 2 - \alpha^2 & -\bar{\alpha}^2 + 2 - \alpha^2 & \bar{\alpha}\bar{\beta} - 2\bar{\alpha} + \alpha \\ \bar{\beta} - 2|\alpha|^2 + \beta & -\bar{\alpha}\bar{\beta} + 2\bar{\alpha} - \alpha & -\bar{\alpha}\bar{\beta} + 2\bar{\alpha} - \alpha & \bar{\beta}^2 - 2\bar{\alpha}^2 + 1 \end{pmatrix}. \end{aligned} \tag{4.80}$$

We fix $\sigma = 1$. By using Matlab, we find values of the concurrence with different w as reported in the following table.

$w \setminus t$	0	1/2	1	5	10	100	1000	$\cdots\ \cdots$	∞
0	0	0.03	2.69×10^{-4}	0	0	0	0	$\cdots\ \cdots$	0
1	0	0.2866	0.3539	0	0	0	0	$\cdots\ \cdots$	0
2	0	0.6069	0.3856	0	0	0	0	$\cdots\ \cdots$	0
10	0	0.7369	0.21	0	0	0	0	$\cdots\ \cdots$	0
100	0	0.1369	0.1916	0	0	0	0	$\cdots\ \cdots$	0

Table 4.1: Evolution of concurrence with different w and $\sigma = 1$

As we can see from Table 4.1 that when $t \geq 5$, $\bar{\rho}(t)$ does not create concurrence any more. Besides, for $t = 1/2$ and $t = 1$, the concurrence varies as w does. In particular, we see that large amounts of concurrence (> 0.7) are created by the common noise for certain values of w.

In our next experiment, again, by using Matlab, we fix $w = 1$ and study the concurrence for different σ. In Table 4.2, we can see that for $t = 1/2$ and $t = 1$, the concurrence decreases to zero as σ increases.

$\sigma \setminus t$	0	1/2	1	5	10	100	1000	$\cdots\ \cdots$	∞
1	0	0.2866	0.3539	0	0	0	0	$\cdots\ \cdots$	0
2	0	0.1783	0.025	0	0	0	0	$\cdots\ \cdots$	0
5	0	7.0063×10^{-4}	0	0	0	0	0	$\cdots\ \cdots$	0
10	0	0	0	0	0	0	0	$\cdots\ \cdots$	0
100	0	0	0	0	0	0	0	$\cdots\ \cdots$	0

Table 4.2: Evolution of concurrence with $w = 1$ and different σ

Overall, we conclude that although the two qubits are not entangled in their initial state, they will become entangled for $t > 0$, due to their indirect interaction via the common noise! This is markedly different from the situation with independent (local) noises above.

4.3.2 N qubits

In this part, we consider an N-qubit register with Hamiltonian

$$H = H'_1 + H'_2 + \cdots + H'_N \tag{4.81}$$

acting on $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$, where

$$H'_j = \mathbb{1} \otimes \cdots \otimes H_j \otimes \cdots \otimes \mathbb{1}. \tag{4.82}$$

Here, H_j acts on the jth factor. Each H_j is a 2×2 matrix

$$\begin{pmatrix} \varepsilon_{1j} + \zeta_{1j} & 0 \\ 0 & \varepsilon_{2j} + \zeta_{2j} \end{pmatrix}. \tag{4.83}$$

For example,

$$H'_2 = \mathbb{1} \otimes H_2 \otimes \cdots \otimes \mathbb{1}, \tag{4.84}$$

where

$$H_2 = \begin{pmatrix} \varepsilon_{12} + \zeta_{12} & 0 \\ 0 & \varepsilon_{22} + \zeta_{22} \end{pmatrix}. \tag{4.85}$$

Here, ε_{ij}, $i \in \{1,2\}$ and $j \in \{1, 2, \cdots, N\}$, are constants. ζ_{ij}, $i \in \{1,2\}$ and $j \in \{1, 2, \cdots, N\}$, are real-valued random variables, representing the noise.

Let

$$\varepsilon_j = \varepsilon_{1j} - \varepsilon_{2j} \quad \text{and} \quad \zeta_j = \zeta_{1j} - \zeta_{2j}, \qquad j = 1, \ldots, N. \tag{4.86}$$

The time evolution of an initial density matrix $\rho(0)$ is

$$\rho(t) = e^{-itH} \rho(0) e^{itH}. \tag{4.87}$$

Now, we can apply the same technique as we used to get (4.39) to see that $\rho(t)$ is the same if we replace the H_j above with

$$H_j = \begin{pmatrix} \varepsilon_j + \zeta_j & 0 \\ 0 & 0 \end{pmatrix} \tag{4.88}$$

for all $j \in \{1, 2, \cdots, N\}$.

We take for an initial condition the most general pure separable state, given by

$$|\varphi_0\rangle = |\varphi_0^1\rangle \otimes |\varphi_0^2\rangle \otimes \cdots |\varphi_0^N\rangle, \tag{4.89}$$

where

$$|\varphi_0^j\rangle = \alpha_{j,+}|+\rangle + \alpha_{j,-}|-\rangle \tag{4.90}$$

with

$$|\alpha_{j,+}|^2 + |\alpha_{j,-}|^2 = 1, \quad \alpha_{j,+}, \alpha_{j,-} \in \mathbb{C}. \tag{4.91}$$

for all $j \in \{1, 2, \cdots, N\}$. The associated initial density matrix is

$$\rho(0) = |\varphi_0\rangle\langle\varphi_0|. \tag{4.92}$$

Using $e^{-itH} = e^{-it(H_1' + \cdots + H_N')} = e^{-itH_1} \otimes \cdots \otimes e^{-itH_N}$ we get

$$\begin{aligned} \rho(t) &= \big(e^{-itH_1} \otimes \cdots \otimes e^{-itH_N}\big)|\varphi_0\rangle\langle\varphi_0|\big(e^{itH_1} \otimes \cdots \otimes e^{itH_N}\big) \\ &= |(e^{-itH_1} \otimes \cdots \otimes e^{-itH_N})\varphi_0\rangle\langle(e^{-itH_1} \otimes \cdots \otimes e^{-itH_2})\varphi_0| \\ &= |\varphi_t\rangle\langle\varphi_t|, \end{aligned} \tag{4.93}$$

where

$$|\varphi_t\rangle = (e^{-itH_1} \otimes \cdots \otimes e^{-itH_2})|\varphi_0\rangle.$$

Now

$$\begin{aligned} |\varphi_t\rangle =&(e^{-itH_1} \otimes \cdots \otimes e^{-itH_2})|\varphi_0\rangle \\ =&(e^{-itH_1} \otimes \cdots \otimes e^{-itH_2})\big(|\varphi_0^1\rangle \otimes \cdots \otimes |\varphi_0^N\rangle\big) \\ =&\big(\alpha_{1,+}e^{-it(\varepsilon_1+\zeta_1)}|+\rangle + \alpha_{1,-}|-\rangle\big) \otimes \cdots \otimes \big(\alpha_{N,+}e^{-it(\varepsilon_N+\zeta_N)}|+\rangle + \alpha_{N,-}|-\rangle\big) \\ =&\bigotimes_{j=1}^{N} \big(\alpha_{j,+}e^{-it(\varepsilon_j+\zeta_j)}|+\rangle + \alpha_{j,-}|-\rangle\big) \end{aligned} \tag{4.94}$$

Then we have

$$\begin{aligned} \bar{\rho}(t) \equiv&\mathbb{E}[\rho(t)] = \mathbb{E}\Big[\bigotimes_{j=1}^{N} \big(\alpha_{j,+}e^{-it(\varepsilon_j+\zeta_j)}|+\rangle + \alpha_{j,-}|-\rangle\big)\big(\bar{\alpha}_{j,+}e^{it(\varepsilon_j+\zeta_j)}\langle+| + \bar{\alpha}_{j,-}\langle-|\big)\Big] \\ =&\mathbb{E}\Big[\bigotimes_{j=1}^{N} \big(|\alpha_{j,+}|^2|+\rangle\langle+| + \alpha_{j,+}\bar{\alpha}_{j,-}e^{-it(\varepsilon_j+\zeta_j)}|+\rangle\langle-| \end{aligned}$$

$$+ \bar{\alpha}_{j,+}\alpha_{j,-}e^{it(\varepsilon_j+\zeta_j)}|-\rangle\langle+| + |\alpha_{j,-}|^2|-\rangle\langle-|\Big)\Big]. \tag{4.95}$$

- For individual noise, the random variables $\zeta_1, \ldots, \zeta_N$ are independent. Then (4.95) becomes

$$\bar{\rho}(t) = \bigotimes_{j=1}^{N} \Big(|\alpha_{j,+}|^2|+\rangle\langle+| + \alpha_{j,+}\bar{\alpha}_{j,-}\ \beta_{j,t}\ |+\rangle\langle-| + \bar{\alpha}_{j,+}\alpha_{j,-}\ \bar{\beta}_{j,t}\ |-\rangle\langle+| + |\alpha_{j,-}|^2|-\rangle\langle-|\Big), \tag{4.96}$$

where

$$\beta_{j,t} = e^{-it\varepsilon_j}\,\mathbb{E}[e^{-it\zeta_j}]. \tag{4.97}$$

Clearly, $\bar{\rho}(t)$ given in (4.96) is the product of N single qubit density matrices. So the state $\bar{\rho}(t)$ is separable for all times. This again shows that the individual noise does not create entanglement in any initially pure separable state.

- For the common noise, $\zeta_1 = \cdots = \zeta_N = \zeta$ and we get

$$\begin{aligned}\bar{\rho}(t) =&\mathbb{E}\Big[\bigotimes_{j=1}^{N}\big(\alpha_{j,+}e^{-it(\varepsilon_j+\zeta)}|+\rangle + \alpha_{j,-}|-\rangle\big)\big(\bar{\alpha}_{j,+}e^{it(\varepsilon_j+\zeta)}\langle+| + \bar{\alpha}_{j,-}\langle-|\big)\Big]\\ =&\mathbb{E}\Big[\bigotimes_{j=1}^{N}\big(|\alpha_{j,+}|^2|+\rangle\langle+| + \alpha_{j,+}\bar{\alpha}_{j,-}e^{-it(\varepsilon_j+\zeta)}|+\rangle\langle-|\\ &\qquad + \bar{\alpha}_{j,+}\alpha_{j,-}e^{it(\varepsilon_j+\zeta)}|-\rangle\langle+| + |\alpha_{j,-}|^2|-\rangle\langle-|\big)\Big].\end{aligned} \tag{4.98}$$

Wootters gave the specific formula to calculate the concurrence of the states in a two qubit system [32]. For N-qubit systems with $N \geq 3$, there are also notions of multi-partite entanglement measures, however, their formulation is not explicit, which makes their study intricate [30].

4.3.3 Bipartite M-level system

Now, we consider bipartite M-level quantum system

$$H = H_1 \otimes \mathbb{1} + \mathbb{1} \otimes H_2, \tag{4.99}$$

where

$$H_1 = \begin{pmatrix} \varepsilon_1+\zeta_1 & 0 & 0 & 0 & \cdots & 0\\ 0 & \varepsilon_2+\zeta_2 & 0 & 0 & \cdots & 0\\ 0 & 0 & \varepsilon_3+\zeta_3 & 0 & \cdots & 0\\ 0 & 0 & 0 & \varepsilon_4+\zeta_4 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & \varepsilon_M+\zeta_M \end{pmatrix} \tag{4.100}$$

and

$$H_2 = \begin{pmatrix} \eta_1+\nu_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \eta_2+\nu_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \eta_3+\nu_3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \eta_4+\nu_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \eta_M+\nu_M \end{pmatrix}. \tag{4.101}$$

Here, for $1 \leq i, j \leq M$, the ε_i and η_j are constants and the ζ_i and ν_j are random variables, representing the noise.

We take an initial pure and disentangled state,

$$|\varphi_0\rangle = \big(\sum_{j=1}^{M} \alpha_j |e_j\rangle\big) \otimes \big(\sum_{k=1}^{M} \beta_k |f_k\rangle\big), \tag{4.102}$$

where $\sum_{j=1}^{M} |\alpha_j|^2 = 1$ and $\sum_{k=1}^{N} |\beta_k|^2 = 1$. We denote by $|e_j\rangle$, $j = 1, \dots M$, the orthonormal basis in which the Hamiltonians H_1 and H_2 are expressed as matrices (4.100) and (4.101). The evolution of the initial state $|\varphi_0\rangle$ is given by

$$\begin{aligned} |\varphi_t\rangle &= (e^{-itH_1} \otimes e^{-itH_2})|\varphi_0\rangle \\ &= (e^{-itH_1} \otimes e^{-itH_2})\Big\{\big(\sum_{j=1}^{M} \alpha_j |e_j\rangle\big) \otimes \big(\sum_{k=1}^{M} \beta_k |e_k\rangle\big)\Big\} \\ &= e^{-itH_1} \sum_{j=1}^{M} \alpha_j |e_j\rangle \otimes e^{-itH_2} \sum_{k=1}^{M} \beta_k |e_k\rangle \\ &= \sum_{j=1}^{M} \alpha_j e^{-it(\varepsilon_j+\zeta_j)} |e_j\rangle \otimes \sum_{k=1}^{M} \beta_k e^{-it(\eta_k+\nu_k)} |e_k\rangle. \end{aligned} \tag{4.103}$$

The expectation of the associated density matrix is

$$\begin{aligned} \bar{\rho}(t) \equiv& \mathbb{E}\big[\rho(t)\big] = \mathbb{E}\big[|\varphi_t\rangle\langle\varphi_t|\big] \\ =& \mathbb{E}\Big[\Big(\sum_{m,l=1}^{M} \alpha_m \bar{\alpha}_l e^{it(\varepsilon_l-\varepsilon_m+\zeta_l-\zeta_m)} |e_m\rangle\langle e_l|\Big) \otimes \Big(\sum_{n,h=1}^{M} \beta_n \bar{\beta}_h e^{it(\eta_h-\eta_n+\nu_h-\nu_n)} |e_n\rangle\langle e_h|\Big)\Big]. \end{aligned} \tag{4.104}$$

- Consider the common noise situation, where $\zeta_j = \nu_j = \zeta$ for each j. Then

$$\bar{\rho}(t) = \Big(\sum_{m,l}^{M} \alpha_m \bar{\alpha}_l e^{it(\varepsilon_l-\varepsilon_m)} |e_m\rangle\langle e_l|\Big) \otimes \Big(\sum_{n,h}^{N} \beta_n \bar{\beta}_h e^{it(\eta_h-\eta_n)} |e_n\rangle\langle e_h|\Big).$$

By the definition 4.4, we can see that the state is not entangled, which means $\mathcal{C}(\bar{\rho}(t)) = 0$

for all times t.

- If $\zeta_j = \nu_j$ for each $1 \leq j \leq M$ and they are independent random variables for different j, then we have from (4.104)

$$\bar{\rho}(t) = \mathbb{E}\Big[\Big\{\sum_{m,l}^{M} \alpha_m \bar{\alpha}_l e^{it(\varepsilon_l-\varepsilon_m)} e^{it(\zeta_l-\zeta_m)} |e_m\rangle\langle e_l|\Big\} \otimes \Big\{\sum_{n,h}^{M} \beta_n \bar{\beta}_h e^{it(\eta_h-\eta_n)} e^{it(\zeta_h-\zeta_n)} |f_n\rangle\langle f_h|\Big\}\Big]. \tag{4.105}$$

Lacking an explicit form for the amount of entanglement (such as concurrence), we are unable to calculate explicitly the entanglement of this high dimensional bipartite quantum state. We mention though that it is sometimes possible to find a lower bound on the concurrence, see [7].

Chapter 5

Conclusion and future work

Conclusion. In this thesis, we discuss two important phenomena in modern quantum theory: decoherence and entanglement. For decoherence, we consider the evolution of a qubit evolving according to the Schrödinger equation with a Hamiltonian containing diagonal and off-diagonal random variables, representing the noise. We find the explicit form of the final state (time $t \to \infty$). It depends on the noises and initial state. Besides, we find that the smoothness of the probability density functions of the diagonal and off-diagonal noises determines the convergence speed, which is polynomial in $1/t$. The convergence speed of the dynamical process will increase if the probability density function of diagonal noise becomes smoother. On the other hand, if the diagonal noise is absent, then convergence to the final state still takes place and its speed increses if the low energy modes of the off-diagonal noise are suppressed. For entanglement, we consider a 2 qubit system, as well as an N qubit system and an M-level bipartite system, all coupled to noise. The initial state plays an important role in determining whether the state is entangled or not after a while. For example, for a two qubit system, an independent (individual, local) noise will never create entanglement in an initially separable state. However, the result is different in the case where both qubits are subjected to a common noise. Then typically entanglement is created when $t > 0$.

Future work. In the N qubit system and M-level bipartite system, we only fully analyzed the individual noises case. It turns out that, in this case, the individual noise can never create entanglement with in initial separable state. For the common noise case, we can still find the expectation of the density matrix of the two qubits. However, we were at present unable to calculate concrete measures of entanglement (the "easy" quantity, concurrence, is only defined for two qubits). Nevertheless, some literature gives specific formulas to estimate lower bounds of the entanglement. Since we have the explicit density matrix, in a future work, we could try to analyze these bounds for our N qubit system and/or the M-level bipartite system, under common noise. If the lower bound of the concurrence is positive, the we would be able to conclude that the state of N qubit system and/or M-level bipartite system is entangled when $t > 0$. If not, we would need to find another elegant way to analysis...

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