Hyperbolically Embedded Subgroups in terms of Fine graphs

by

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A thesis submitted to the School of Graduate Studies in partial fulfilment of the requirements for the degree of Master of Science.

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August 31, 2020

Abstract

Since the 1980s, mathematicians have been studying hyperbolic groups, and hyperbolic geometry has been considered one of the most useful tools in geometric group theory. In the 2000s, a generalization of hyperbolic groups was introduced by Dahmani, Guiraldel and Osin, known as hyperbolically embedded subgroups. This notion has several applications in group theory and low dimensional topology. In this thesis, we introduce an alternative definition of hyperbolic embedded subgroup modelled on a characterization of a relatively hyperbolic group by Bowditch [2] and prove that our notion is equivalent to the original definition.

General Summary

Group theory is a mathematical abstraction to study the collection of symmetries of an object. The collection of transformations that leave a geometric structure invariant is the called the group of isometries of the object. Here we study the group of symmetries of infinite geometric objects. The theory is rich when the objects resemble non-positive curvature. The thesis deals with a particular class of non-positively curved discrete groups known as groups with a hyperbolically embedded subgroup. The thesis shows that this particular class of groups can be described in an alternative way generalizing previous results.

Acknowledgments

I wish to show my most profound gratitude first to my supervisor, Dr. Eduardo Martinez Pedroza, who continuously supported me throughout my research with patience and encouragement. His unwavering enthusiasm and immense knowledge for mathematics kept me motivated in generating some useful results. I could not have imagined a better advisor and mentor for my research. He is not only a smart mathematician but also a very kind and humble human being.

Then I would like to pay my special regards to the Mathematics Department of Memorial University for giving me a chance and providing me with some financial support that helped me in focussing on my thesis.

I would also like to show my gratitude towards the examiners for providing comments that helped me to improve the quality of this thesis.

Last but not least, I thank my family and friends for the moral support they always provided in every part of life.

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Introduction

Groups acting on hyperbolic spaces properly and cocompactly represent the class of hyperbolic groups. Mathematicians started giving attention to this direction in the late 1980s when Gromov introduced the notion of hyperbolic groups in his paper [9]. After that, hyperbolic geometry has proved to be one of the most useful tools in geometric group theory.

In the 90s, the notion of a relatively hyperbolic group was developed independently in the Ph.D. thesis of Benson Farb [8], under the direction of Thurston, and by Bowditch [2]. It is a generalization of the notion of hyperbolic group, replacing the condition of properness with a certain finiteness condition. In the early 2000s, a more comprehensive study of relatively hyperbolic groups started in series of articles by Denis Osin in [16] and [17], as wells as works of other researchers like Groves and Manning in [10]. There are different equivalent definitions of relatively hyperbolic groups. In this thesis, we review the definition by Bowditch in [2] and a definition based on one by Farb [7], which is implicit in the work of Martinez-Pedroza and Wise [14]. In Chapter 3, we review the equivalence of these two definitions based on results in [14].

The theory of relatively hyperbolic groups has found several applications in group theory and low dimensional topology. A highlight of the theory of relatively hyperbolic groups is a generalization of Thurston's Dehn filling of hyperbolic knot complements in the framework of group theory, which can be found in [17] and [10]. This tool has had important applications for example its role in the solution of the virtual Haken conjecture on 3-manifolds by Agol in [1].

The notion of hyperbolically embedded subgroup generalizes the notion of relatively hyperbolic group. In this context, a generalization of Thurston Dehn filling has been developed by Dahmani, Guiraldel and Osin [5]. A group admitting a non-degenerate hyperbolically embedded subgroup is also known as an acylindrically hyperbolic group, see [18]. The theory of acylindrically hyperbolic groups started with the first non-trivial example that is the result of Bowditch, who showed that the action of the mapping class group of a hyperbolic surface on its curve complex is acylindrical, see [3]. Bowditch used this result to obtain many interesting applications and results about the mapping class group [3].

In this thesis, we introduce an alternative definition of hyperbolically embedded subgroup modelled on a definition of relatively hyperbolic groups by Bowdictch [2] and prove that our notion is equivalent to the idea introduced by Osin [5].

In order to state our result, we introduce some definitions. Let G be a group, let S be a subset of G, let H be a subgroup, let $\Gamma(G, H \cup S)$ be the Cayley graph of G with respect $H \cup S$. We will define the *angle-metric* on the subgroup H as follows, for $h, k \in H$, $\angle_H(h, k)$ is the minimal length of an edge-path in $\Gamma(G, H \cup S)$ between h and k that does not contain a vertex of $\Gamma(H, H \setminus \{e\})$ other than h and k, and let $\angle_H(h, k) = \infty$ if such a path does not exist, see Chapter 4 for precise definitions. The subgroup H is *hyperbolically embedded* in G with respect to S if G is generated by $H \cup S$, the Cayley graph $\Gamma(G, H \cup S)$ is hyperbolic, and H is a locally finite metric space with respect to the angle-metric.

A graph Γ is a (G, H)-graph if G acts on Γ while satisfying the following conditions:

- Γ is connected and hyperbolic,
- Γ is fine at a vertex v if the metric on set of vertices adjacent to v induced by the path-metric on Γ_v is locally finite, see Chapter 2 for a precise definition,
- for all $w \in V_\infty(\Gamma) = \{\text{Vertices with infinite stabilizers}\}$, G_w acts on $T_w\Gamma$ with finitely many G-orbits,
- number of G-orbits of vertices are finite,
- stabilizer subgroups of edges are finite,
- stabilizer subgroups of vertices are either finite or a conjugate of H,
- and there exists a vertex that has the stabilizer subgroup equals to H.

Our main result is the following:

Theorem 0.1. *A subgroup H is hyperbolically embedded in G for some subset S if and only if there exists a (G, H)-graph.*

Table 1 summarizes the structure of the thesis. Two well known equivalent notions of hyperbolic groups are in the first row. The equivalence between a_1 and a_2 is Proposition 3.1. Whereas, the second row provides equivalent definitions of relative hyperbolic groups, their equivalence is explained in Chapter 3. The equivalence between b_0 and b_1 is Corollary 4.10 and the equivalence between b_1 and b_2 is Proposition 3.5. Then we stated the characterizations of acylindrically hyperbolic groups in the third row. The equivalence between c_0 and c_1 is Proposition 4.8, and the equivalence between c_0 and c_2 is the main result of the thesis. The table uses the following notions and their precise definitions can be found in the following chapters:

- $\Gamma(G, H)$: Cayley graph (Chapter 1.1).
- $\hat{\Gamma}(G, H, S)$: Coned-off Cayley graph (Chapter 1.6).
- G_v: G-stabilizer of v (Chapter 1).
- $H \hookrightarrow_h G$: H is a hyperbolically embedded subgroup (Chapter 4).
- $V_\infty(\Gamma)$: Set of vertices with infinite stabilizers (Chapter 4).
- $T_\infty\Gamma$: Set of vertices adjacent to v (Chapter 2).

In this thesis, we have divided the chapters as follows: Chapter 1 is of preliminaries, which is for those readers who have minimal knowledge of geometric group theory. Chapter 2 is introducing the fine graph and its properties, which is used to define a (G, H)-graph. The main result in this chapter is Theorem 2.11 which guarantees that a particular finiteness property (fineness) on graph is preserved under certain extensions of the graph. The first and second rows of the table are discussed in Chapter 3. Then in Chapter 4, our main result is proved. At last, Chapter 5 explains a non-trivial example of a (G, H)-graph, that is, $SL_2(\mathbb{Z})$ acting on the Farey Graph.

Figure 1: Main structure of thesis

a_0: Hyperbolic Group. G is hyperbolic $\iff$ $\exists S \subseteq G$ finite such that • $\Gamma(G, S)$ is connected • $\Gamma(G, S)$ is hyperbolic	a_1: Hyperbolic Group. G is hyperbolic $\iff$ $\exists S \subseteq G$ finite such that • $\Gamma(G, S)$ is connected • $\Gamma(G, S)$ is hyperbolic	a_2: Hyperbolic Group. G is hyperbolic $\iff$ $\exists \Gamma$ such that • Γ is connected and hyperbolic • G-action on Γ is cocompact and proper
b_0: Relative Hyperbolic Group $H \leq G$. G is hyperbolic relative to $H \iff \exists S \subseteq G$ finite such that • $\Gamma(G, H \cup S)$ is connected • $\Gamma(G, H \cup S)$ is hyperbolic • H is locally finite with respect to the angle metric $\angle_H$.	b_1: Relative Hyperbolic Group $H \leq G$. G is hyperbolic relative to $H \iff \exists S \subseteq G$ finite such that • $\hat{\Gamma}(G, H, S)$ is connected • $\hat{\Gamma}(G, H, S)$ is hyperbolic • $\hat{\Gamma}(G, H, S)$ is fine.	b_2: Relative Hyperbolic Group $H \leq G$. G is hyperbolic relative to $H \iff \exists \Gamma$ such that • Γ is connected and hyperbolic • $\forall e \in E(\Gamma), G_e$ is finite • $\forall v \in V(\Gamma), G_v$ is either finite or a conjugate of H • $\exists v \in V(\Gamma)$ such that $G_v = H$ • G-action on Γ is cocompact • Γ is fine
c_0: Hyperbolically Embedded Subgroup $H \leq G$. $H \hookrightarrow_h G$ wrt $S \iff$ $\exists S \subseteq G$ such that • $\Gamma(G, H \cup S)$ is connected • $\Gamma(G, H \cup S)$ is hyperbolic • H is locally finite with respect to the angle metric $\angle_H$.	c_1: Hyperbolically Embedded Subgroup $H \leq G$. $H \hookrightarrow_h G$ wrt $S \iff$ $\exists S \subseteq G$ such that • $\hat{\Gamma}(G, H, S)$ is connected • $\hat{\Gamma}(G, H, S)$ is hyperbolic • $\hat{\Gamma}(G, H, S)$ is fine at cone vertices.	c_2: Hyperbolically Embedded Subgroup $H \leq G$. $H \hookrightarrow_h G \iff \exists \Gamma$ such that • Γ is connected and hyperbolic • $\forall e \in E(\Gamma), G_e$ is finite • $\forall v \in V(\Gamma), G_v$ is either finite or a conjugate of H • $\exists v \in V(\Gamma)$ such that $G_v = H$ • $\lvert \{Gv \mid v \in V(\Gamma)\} \rvert < \infty$ • Γ is fine at $V_\infty(\Gamma)$. • $\forall w \in V_\infty(\Gamma), G_w$ acts on $T_\infty\Gamma$ with finitely many orbits.

Chapter 1

Preliminaries

A *graph* is an ordered pair (V, E), where V is a set, and E is a relation on V that is anti-reflexive and symmetric. Elements of set V are called *vertices*, and elements of set E are called *edges*. For a graph Γ, we denote $V(\Gamma)$ and $E(\Gamma)$ its vertex and edge set, respectively. For a vertex $w \in V(\Gamma)$, $\Gamma - m$ is a graph with vertex set $V(\Gamma) - \{w\}$ and edge set $E(\Gamma) - \{\{v, w\} \mid v \in V(\Gamma)\}$. If a vertex is an element of an edge, it is called an *incident*. Two vertices that are in the same edge are called *adjacent*. And the number of adjacent vertices of a vertex is the *degree* of it.

A *path* from a vertex v_0 to a vertex v_n of Γ is a sequence of vertices $[v_0, v_1 \ldots, v_n]$, where v_i and v_{i+1} are adjacent for all $i \in \{0, \ldots, n-1\}$. Its *reverse-path* would be $[v_n, v_{n-1} \ldots, v_0]$. The length of a path is one less than the total number of vertices in the sequence. If no vertex on a path appears in the sequence more than once, the path is called an *embedded path*. Most metrics considered in this thesis are *path metrics* on the set of vertices of graphs, which are defined such that the distance between any two vertices is the length of the shortest embedded path between them if such exists, otherwise infinity.

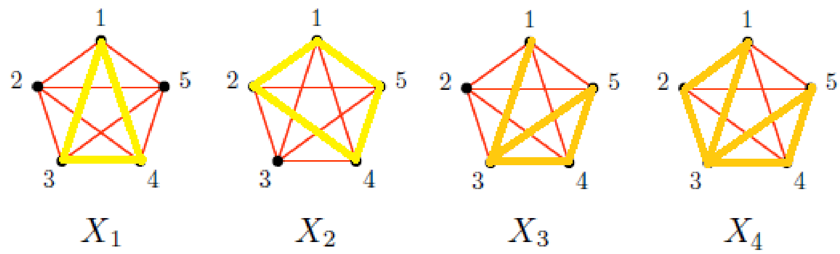

Figure 1.1: Circuits

A path $[v_0, v_1 \ldots, v_n]$ is *closed* if $v_0 = v_n$, and a closed path is called a *circuit* if $[v_0, v_1 \ldots, v_{n-1}]$ is an embedded path. And a path is *escaping* if for every $i \in \{0, 1, \ldots, n\}$, $v_i = v_0$ implies $i = 0$. See Figure 1.1, the paths in X_1 and X_2 are circuits, whereas the paths in X_3 and X_4 are not as the vertex 3 is repeating. Also note that in Figure 1.2, X_3 can not have a circuit. The concatenation of two paths $\alpha = [u_0, u_1 \ldots, u_n]$ and $\beta = [v_0, v_1 \ldots, v_m]$ is $[\alpha, \beta] = [u_0, u_1 \ldots, u_n, v_0, v_1 \ldots, v_m]$. Analogously, the concatenation of a vertex a and the path α would be $[a, \alpha] = [a, u_0, u_1 \ldots, u_n]$.

A graph is *connected* if, for any two vertices $u, v \in V$, there exists a path between them. See Figure 1.2, X_1 is not connected as there does not exist any path from vertex 4 to any other vertex, while X_2 and X_3 are connected. A *connected component* of a graph is a maximal subgraph that is connected. A graph in which any two distinct vertices are adjacent is called a *complete graph*. See Figure 1.2, X_2 is a complete graph.

A *group homomorphism* $\phi : G \to H$ is a function from a group G to a group H with the property that $(xy) = (x)(y)$ for all $x, y \in G$. An *automorphism* $\phi : V(\Gamma) \to V(\Gamma)$ of Γ is a bijection such that it preserves the adjacency relation. The collection of automorphisms of Γ forms a group under composition denoted as $Aut(\Gamma)$. An *action*

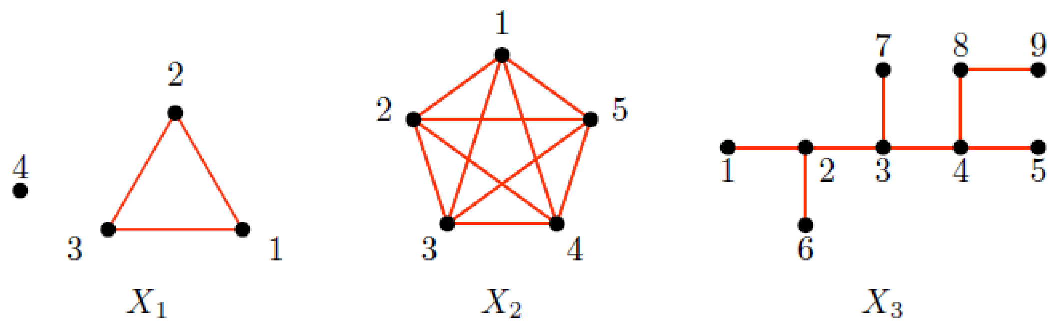

Figure 1.2: Connected and disconnected graphs

of a group G on a graph Γ is a group homomorphism $\phi : G \to Aut(\Gamma)$. Such a graph is called a G-graph.

Let Γ be a G-graph. In this case, when the action is understood from the context, for $g \in G$, v a vertex and e an edge of Γ we use $g.v$ to denote $\rho(g)(v)$ and $g.e$ to denote $\rho(g)(e)$. We will use G_v to denote the set $\{g \in G | g.v = v\}$, which is called the *G-stabilizer* of the vertex v. And $G.v$ denotes the set $\{g.v \mid g \in G\}$ which is called the *G-orbit* of the vertex v. The G-stabilizer and G-orbit of an edge are defined similarly. The G-action on a graph Γ is *proper* if G-stabilizers of all the vertices in Γ are finite. Note that this also implies that the stabilizer subgroups of edges are finite. The G-action of a graph Γ is *cocompact* if there are finitely many G-orbits of vertices and edges. If there is only one G-orbit of vertices, the G-action is called *vertex transitive*.

A metric space in which every ball of finite radius contains finitely many elements is called a *locally finite metric space*. X is a *geodesic metric space* if for any $x, y \in X$, there exists a path $\gamma : [0, d] \to X$ such that $d = \mathsf{dist}(x, y)$ and for all $t, s \in [0, d]$, $\mathsf{dist}(\gamma(s), \gamma(t)) = |s - t|$ and $\gamma(0) = x$ and $\gamma(d) = y$.

Let G be a group. A *G-action* on a metric space X is a group homomorphism $\varphi : G \to Isom(X)$. The action of a group G on a graph Γ is also an action by isometries on the graph considered as a metric space. The G-action on a metric space X is *proper* if for all $\Gamma \subseteq X$ compact, $\{g \in G | \Gamma \cap g.\Gamma \neq \phi\}$ is finite. The G-action on a metric space X is *cocompact* if there exist $\Gamma \subseteq X$ compact such that $G.\Gamma = X$, in other words, $\bigcup_{g \in G} g.\Gamma = X$.

1.1 Cayley Graphs

Let G be a group and let S be a subset of G. Then, the *Cayley graph* $\Gamma(G, S)$ of G with respect to S is the graph which has

- $V(\Gamma(G, S)) := G$
- $E(\Gamma(G, S)) := \{\{g, gs\} | g \in G, s \in S \setminus \{e\}\}$.

Remark 1.1. *The Cayley graphs, $\Gamma(G, S)$ and $\Gamma(G, S \cup S^{-1})$, are the same graphs; here S^{-1} denotes the set of inverses of elements of S.*

The group G acts on the Cayley graph $\Gamma(G, S)$ by multiplication as follows: for $g \in G$, $v \in V(\Gamma(G, S))$ and $\{v, vs\} \in E(\Gamma(G, S))$,

1. $g.v = gv$ which is a vertex of $\Gamma(G, S)$, and
2. $g.\{v, vs\} = \{gv, gvs\}$ which is an edge of $\Gamma(G, S)$.

The following results until Proposition 1.6 can be found in the book [4], which is a classical reference in geometric group theory, and they can also be found in [13].

Lemma 1.2. *Let G be a group and let $S \subset G$ be a generating set of G. Then the Cayley graph $\Gamma(G, S)$ is a connected graph.*

Proof. Without loss of generality, assume that S is closed under inverses. Let $g \in G$. As S generates G, $g = s_1 s_2 \dots s_l$ where $s_i \in S$, for all $i \in \{1, 2, \dots, l\}$. Then the sequence $[e, s_1, s_1 s_2, s_1 s_2 s_3, \dots, s_1 s_2 \dots s_l]$ is a path from the identity e, to g. Hence, $\Gamma(G, S)$ is connected. □

The path-metric metric on the set of vertices of $\Gamma(G, S)$ is also called the word metric induced by S on G.

Proposition 1.3. *The following statements are true about the G-action on a Cayley graph $\Gamma(G, S)$:*

1. *It is vertex transitive.*

2. *It is cocompact if S is finite.*

3. *It is proper.*

4. *The stabilizer subgroups of edges are finite.*

Proof.

1. Note that $g.1 = g$, hence all vertices are in the G-orbit of the identity element. In particular, $\Gamma(G, S)$ is vertex transitive.

2. The number of G-orbits of edges is equal to the cardinality of S as all the edges are the translates of the edges $\{1, s\}$, for $s \in S$.

3. Since for all $g, v \in G$, $gv = g$ implies $g = 1$, the G-stabilizers of vertices are trivial.

4. The stabilizer subgroups of edges are of order at most two. Note that the edge $e = \{g, gs\}$, for $g \in G$ and $s \in S$, has G-stabilizer of order two if s has order two in G; otherwise e has trivial G-stabilizer. Note that the element gsg^{-1} is in the stabilizer of e.

$\square$

1.2 Svark-Milnor's Lemma

An action of a group on a metric space is called a *geometric action* if it is proper, cocompact and by isometries. A metric space X is *proper* if closed and bounded subsets of X are compact.

Definition 1.4. *(Quasi-Isometry) Let $a \geq 1$, $b \geq 0$ and $c \geq 0$. A function $f : X \to Y$ is a (a, b, c)-quasi-isometry if*

1. *for every $x_1, x_2 \in X$*
   $\frac{1}{a} \mathsf{dist}_X(x_1, x_2) - b \leq \mathsf{dist}_Y(f(x_1), f(x_2)) \leq a \, \mathsf{dist}_X(x_1, x_2) + b$
2. *and for every $y \in Y$ there is $x \in X$ such that $\mathsf{dist}_Y(f(x), y) \leq c$.*

Two metric spaces X and Y are said to be quasi-isometric is there exists a quasi isometry between them.

Proposition 1.5. *(**Svark-Milnor's Lemma**) Let X be a geodesic and proper metric space. If a group G acts on X by a geometric action then G is finitely generated*

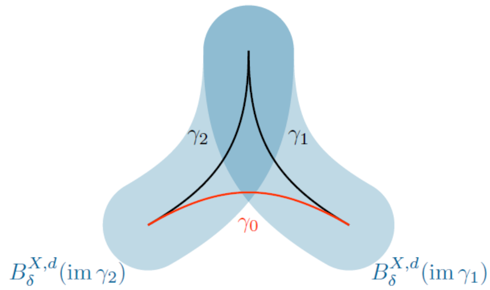

Figure 1.3: [13] A δ-slim triangle.

and for any $x_0 \in X$, the map

$$G \longrightarrow X$$

$$g \longmapsto g.x_0$$

is a quasi-isometry.

1.3 Hyperbolic Metric Spaces

Let (X, d) be a geodesic metric space and let $\delta \geq 0$. Then X is a *δ-hyperbolic space* if for any triangle with sides γ_0, γ_1 and γ_2, the union of the δ-neighbourhoods of γ_1 and γ_2 contains γ_0. See Figure 1.3.

Examples. *[13]*

- *Every geodesic metric space X of finite diameter is diam(X)-hyperbolic.*
- *The Euclidean plane $\mathbb{R}^2$ is not hyperbolic because, for $\delta \in \mathbb{R}_{\geq 0}$, the Euclidean triangle with vertices $(0,0), (0,3\delta)$, and $(3\delta, 0)$ is not δ-slim. See Figure 1.4.*
- *A tree is a connected graph without embedded closed paths. Trees are hyperbolic with respect to the path-metric.*

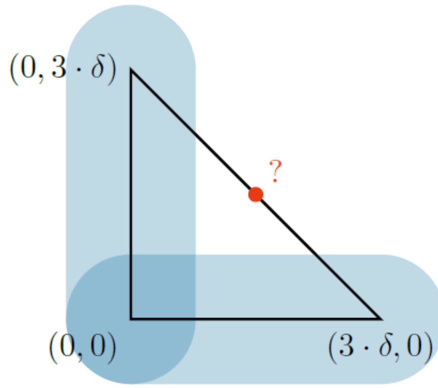

Figure 1.4: [13] The Euclidean plane is not hyperbolic.

Proposition 1.6. *[13, Corollary 7.2.13] Let X, Y be geodesic metric spaces. If there exists a quasi-isometry between X and Y, and Y is hyperbolic, then so is X.*

To a graph, we associate a geodesic metric space by considering all the edges as intervals of length one and defining the distance between any two points as the infimum of the lengths of continuous rectifiable paths between the points. The distance between two vertices in this metric coincides with the length of the shortest embedded path in the graph between them. Note that the definition of a cocompact action on a metric space agrees with the definition of cocompact action on a graph when considering the graph as a metric space with the path-metric.

A graph is said to be hyperbolic if the graph is hyperbolic with respect to the path-metric. Two graphs are quasi-isometric if there exists a quasi-isometry between the graphs with respect to the path-metric. Hence, Proposition 1.6 also holds for two graphs: the quasi-isometry between two graphs preserves hyperbolicity.

1.4 Operations on Graphs

Let Γ be a G-graph. Each of the following four operations constructs a new G-graph Γ' as described:

- *(Edge G-attachment)* Attaching to Γ the G-orbit of an edge with a representative incident to $u, v \in V(\Gamma)$. Observe that G-stabilizer equal to $G_u \cap G_v$.

$$V(\Gamma') = V(\Gamma),$$
$$E(\Gamma') = E(\Gamma) \cup \{\{g.u, g.v\} \mid g \in G\}.$$

- *(Edge G-removal)* Removing the G-orbit of an edge of Γ with a representative $e \in E(\Gamma)$.

$$V(\Gamma') = V(\Gamma),$$
$$E(\Gamma') = E(\Gamma) - \{g.e \mid g \in G\}.$$

- Attaching a G-orbit of a vertex of degree one of Γ adjacent to $v \in V(\Gamma)$ with trivial G-stabilizer. Assuming that $G \cap V(\Gamma)$ is empty,

$$V(\Gamma') = V(\Gamma) \cup G,$$
$$E(\Gamma') = E(\Gamma) \cup \{\{g.v, g\} \mid g \in G\}.$$

1.5 Important results about group actions

Let S be a subset of a group G. Then G is *generated* by S if $G = \langle S \rangle$, where $\langle S \rangle$ is the intersection of all the subgroups of G containing S. If S is finite, then we say, G is *finitely generated* by S.

Lemma 1.7. *Let G be a group acting cocompactly on a graph Γ. Let $v \in V(\Gamma)$ and let $T_v\Gamma = \{u \in V(\Gamma) \mid \ u$ is adjacent to $v\}$. Then G_v acts on $T_v\Gamma$ with finitely many G-orbits.*

Proof. Suppose there are infinitely many G_v-orbits in $T_v\Gamma$. Let $u_1, u_2, \dots$ be representatives of distinct orbits in $T_v\Gamma$. Since G is acting cocompactly on Γ, there are finitely many G-orbits of edges, say $w_1, \dots, w_l$ are representatives. Note that $E(\Gamma) = G.w_1 \cup G.w_2 \cup \dots G.w_l$. By the pigeon-hole argument, passing to a subcollection, we can assume that $\{v, u_i\} \in G.w_1$ for all i. For each i, there is $g_i \in G$ such that $g_i.\{v, u_1\} = \{v, u_i\}$. As u_1 and u_i are in distinct G_v-orbits, $g_i.u_1 = u_i$ and $g_i.v = v$ are not possible. Hence, the only possibility is that $g_i.v = u_i$ and $g_i.u_1 = v$ for each i. Note that for each i,

$$g_i g_2^{-1}.v = g_i.u_1 = v \tag{1.1}$$

$$g_i g_2^{-1}.u_2 = g_i.v = u_i. \tag{1.2}$$

Equation 1.1 implies that $g_i g_2^{-1} \in G_v$. Then expression 1.2 implies that u_2 and u_i are in the same G_v-orbit for all $i's$. This contradicts the assumption that u_i's represent distinct G_v-orbits of elements in $T_v\Gamma$. Hence, there are finitely many G_v-orbits in $T_v\Gamma$. □

Lemma 1.8. *Let Γ be a connected G-graph with finitely many G-orbits of vertices.*

Then there exists a finite subgraph $\Gamma_0 \subseteq \Gamma$ such that $G.V(\Gamma_0) = V(\Gamma)$ and

1. *$T = \{g \in G \mid \mathsf{dist}(\Gamma_0, g.\Gamma_0) \leq 1\}$ generates G,*

2. *If in addition, there are finitely many G-orbits of edges, then there exists a finite set of vertices F such that $V(\Gamma_0) \subset F$, and a finite set $W \subset G$ such that $G = \langle W \cup \bigcup_{v \in F} G_v \rangle$.*

Proof. Let Γ_0 be a subgraph of Γ consisting of a collection of representatives of G-orbits of vertices; in particular, Γ_0 is a finite graph without edges.

1. Fix $x_0 \in V(\Gamma_0)$ and let $g \in G$ such that $g.x_0 \neq x_0$. Otherwise, $gx_0 = x_0$ implies $g \in T$. Since Γ is connected, there is a path $E_g = [x_0, x_1, x_2, ..., x_n]$, where $x_n = g.x_0$. Note that $n \geq 1$ since $g.x_0 \neq x_0$. For each $x_i \in E_{x_0}$, there is $g_i \in G$ such that $x_i \in g_i.\Gamma_0$. Then $\mathsf{dist}(g_i.\Gamma_0, g_{i+1}.\Gamma_0) \leq 1$. Therefore, ${g_i}^{-1}g_{i+1} \in T$ for $0 \leq i < n$. Observe that

$$g = g_0(g_0^{-1}g_1)(g_1^{-1}g_2)(g_2^{-1}g_3)...(g_{n-1}^{-1}g) \in \langle T \rangle.$$

   Hence, $G = \langle T \rangle$.

2. Let $\{v_0, \dots, v_m\} = V(\Gamma_0)$. For each $0 \leq i \leq m$, let $\{w_0^i, \dots, w_{\ell_i}^i\}$ be a collection of representatives of G_{v_i}-orbits of vertices adjacent to v_i. Lemma 1.7 implies that this collection is finite by compactness. Let

$$F = V(\Gamma_0) \cup \bigcup_{i=0}^{m} \{w_0^i, \dots, w_{\ell_i}^i\} = \{u_0, \dots, u_\ell\}$$

   For each ordered pair of indexes $0 \leq i, j \leq \ell$, let $g_{i,j} \in G$ such that $g_{i,j}.u_i = u_j$, if there exists such an element; otherwise let $g_{i,j}$ be the identity element. Let

$$W = \{g_{i,j} : 0 \leq i, j \leq m\}.$$

Let $G_0 = \langle W \cup \bigcup_{u \in F} G_u \rangle$. Note that $G_0 \subseteq G$. We just have to prove that $G \subseteq G_0$, for which it is sufficient to prove that $T \subseteq G_0$. Let Γ_0^+ be a subgraph of Γ such that

- $V(\Gamma_0^+) = F$, and
- $E(\Gamma_0^+) = E(\Gamma_0) \cup \bigcup_{i=0}^{m} \bigcup_{k=0}^{\ell_i} \{[v_i, w_k^i]\}$.

Claim: Let $x \in G$. Then $\mathsf{dist}(\Gamma_0^+, x.\Gamma_0^+) = 0$ implies $x \in G_0$.

In this case, Γ_0^+ and $x.\Gamma_0^+$ have a vertex in common, say u_j. Then there is a vertex u_i of Γ_0^+ such that $x.u_i = u_j$; note that u_i might be equal to u_j. By definition of W, the element $g_{i,j}$ satisfies that $g_{i,j}.u_i = u_j$. Observe that $g_{i,j}^{-1}x$ stabilizes the vertex u_i, and hence $g_{i,j}^{-1}x$ is in G_0. Since $g_{i,j}$ is also in G_0, it follows that $x \in G_0$ as well.

Let $x \in T$. There are two cases to consider.

A. $\mathsf{dist}(\Gamma_0, x.\Gamma_0) = 0$: Observe that $\mathsf{dist}(\Gamma_0, x.\Gamma_0) = 0$ implies $\mathsf{dist}(\Gamma_0^+, x.\Gamma_0^+) = 0$. Hence, $x \in G_0$.

B. $\mathsf{dist}(\Gamma_0, x.\Gamma_0) = 1$: Under the assumption, there are vertices u of Γ_0 and v of $x.\Gamma_0$ such that u and v are adjacent. By definition of F there is an element $g \in G_u$ such that $g.v \in F$. It follows that Γ_0^+ and $gx.\Gamma_0$ have a vertex in common, namely $g.v$. Hence, $\mathsf{dist}(\Gamma_0^+, gx.\Gamma_0) = 0$ which also means $\mathsf{dist}(\Gamma_0^+, gx.\Gamma_0^+) = 0$, and therefore $gx \in G_0$. Since $g \in G_0$, it follows that $x \in G_0$.

The statements A and B imply that for any $x \in T$, if $\mathsf{dist}(\Gamma_0, x.\Gamma_0) \leq 1$ then $x \in G_0$. Therefore, $G = G_0 = \langle W \cup \bigcup_{u \in F} G_u \rangle$.

$\square$

1.6 Coned-off Cayley Graphs

For this section, let G be a group, let S be a subset of G and let H be a subgroup of G.

Let $\Gamma(G, S)$ be the Cayley graph of G. Now we will construct a new graph from $\Gamma(G, S)$, as follows:

1. For each left coset gH of H in G, add a vertex $v(gH)$ in $\Gamma(G, S)$, which we will regard as a *cone vertex*.
2. For each $g \in G$ add an edge $e(gh) = \{gh, v(gH)\}$ between gh and $v(gH)$, where $gh \in gH$.

The new graph is called the *Coned–off Cayley graph* of G with respect to H. It is denoted by $\hat{\Gamma} = \hat{\Gamma}(G, H, S)$. Note that

$$V(\hat{\Gamma}) = G \cup \{v(gH) \mid gH \in G/H\},$$
$$E(\hat{\Gamma}) = E(\Gamma(G, S)) \cup \{e(gh) \mid gh \in gH\}.$$

Remark 1.9. *Note that all the following sets are representing* $\{e(gh) \mid gh \in gH\}$*:*

$$\{\{x, v(gH)\} : g \in G, x \in gH\},$$
$$\{\{g, v(gH)\} : g \in G\},$$
$$\{\{gh, v(gH)\} : gh \in gH\}.$$

The group G acts on the Coned-off Cayley graph $\hat{\Gamma}(G,H,S)$ by multiplication as follows: for $g \in G$,

- If $v \in V(\hat{\Gamma})$, then $g.v = gv$ which is a vertex in $\hat{\Gamma}$.
- If $v(gH) \in V(\hat{\Gamma})$, then $g.v(gH) = gv(gH)$ which is a vertex in $\hat{\Gamma}$.
- If $\{v, vs\} \in E(\hat{\Gamma})$, then $g.\{v, vs\} = \{gv, gvs\}$ which is an edge of $\hat{\Gamma}$.
- If $[x, v(gH)] \in E(\hat{\Gamma})$, then $g.\{x, v(gH)\} = \{gx, gv(gH)\}$ which is an edge of $\hat{\Gamma}$.

Lemma 1.10. *Let $S \subset G$ and let $g \in G$. The following statements are equivalent:*

1. *$g = h_0 s_0 h_1 s_1 h_2 s_2 \ldots h_m s_m$ where $h_i \in H$ and $s_i \in S$, for $i \in \{1, 2, \ldots, m\}$.*
2. *$[e, h_0, h_0 s_0, h_0 s_0 h_1, \ldots, g]$ is a path in $\Gamma(G, S \cup H)$ from 1 to g.*
3. *$[e, v(H), h_0, h_0 s_0, v(h_0 s_0 H), h_0 s_0 h_1, h_0 s_0 h_1 s_1, v(h_0 s_0 h_1 s_1 H), h_0 s_0 h_1 s_1 h_2, \ldots, g]$ is a path in the Coned-off Cayley graph $\hat{\Gamma}(G,H,S)$ from 1 to g.*

Proof. $1 \leftrightarrow 2$: It is trivial to observe that this holds.
$2 \rightarrow 3$: Indeed, an edge of the form $[g, gh]$ in $\Gamma(G, H \cup S)$ where $h \in H$, corresponds to the path of length two in $\hat{\Gamma}(G,H,S)$ given by $[g, v(gH), gh]$.
$3 \rightarrow 2$: As cone vertices are not the vertices of $\Gamma(G, S \cup H)$, we can replace the sub-path $[g, v(gH), gh]$ to the edge incident to g and gh. □

Lemma 1.11. *$G = \langle S \cup H \rangle$ if and only if $\hat{\Gamma}(G,H,S)$ is connected if and only if $\Gamma(G, H \cup S)$ is connected.*

Proof. Let $g \in G$. Note that if G is generated by H and S, then g can be written as a word in the elements of H and S. Then, Lemma 1.10 concludes the proof. □

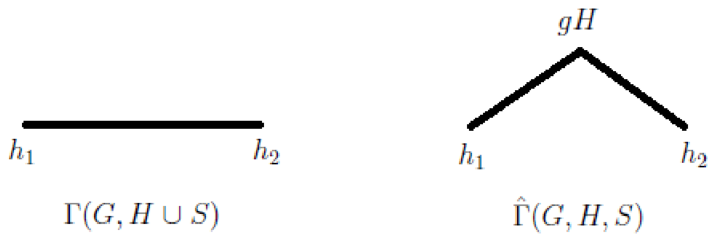

Figure 1.5: Addition of a vertex

Lemma 1.12. *Suppose* $G = \langle S \cup H \rangle$. $\hat{\Gamma}(G, H, S)$ *is quasi-isometric to the Cayley graph* $\Gamma(G, H \cup S)$.

Proof. Consider the inclusion map $i : \Gamma(G, H \cup S) \hookrightarrow \hat{\Gamma}(G, H, S)$.

Claim 1: Every vertex of $\hat{\Gamma}$ is at a distance at most one from a vertex in the image of i. Indeed, every cone vertex is adjacent to a vertex of Γ, and all other vertices are vertices of Γ.

Claim 2: $\mathsf{dist}_{\hat{\Gamma}}(x, y) \leq 2\,\mathsf{dist}_{\Gamma}(x, y)$ for any pair of vertices $x, y \in V(\Gamma)$. Note that if two vertices of Γ are adjacent, then their images in $\hat{\Gamma}$ are at a distance at most two, see Figure 1.5. Consider a path in Γ from the identity element e to g. Then by Lemma 1.10, there is a corresponding path from e to g in $\hat{\Gamma}$ of at most twice the length.

Claim 3: $\mathsf{dist}_{\Gamma}(x, y) \leq \mathsf{dist}_{\hat{\Gamma}}(x, y)$. Observe that in Lemma 1.5, the path in statement 2 is shorter or equal to the path in statement 3. Hence, the inequality holds.

Therefore, the inclusion map $i : \Gamma(G, H \cup S) \hookrightarrow \hat{\Gamma}(G, H, S)$ is a $(2, 0, 1)$-quasi-isometry.

□

Proposition 1.13. *Consider the* G*-action on* $\hat{\Gamma}(G, H, S)$*, the following statements hold:*

1. *There are finitely many G-orbits of vertices.*

2. *The action is cocompact if S is finite.*

3. *The stabilizer subgroups of edges are finite.*

4. *The stabilizer subgroups of vertices are either trivial or a conjugate of H.*

5. *There exists a vertex that has the stabilizer subgroup equals to H.*

Proof. *Claim 1:* $\hat{\Gamma}(G, H, S)$ has finitely many G-orbits of vertices. There are exactly two G-orbits of vertices, all the cone vertices are the G-translates of the cone vertex $v(H)$, and all other vertices are the elements of G.

Claim 2: $\hat{\Gamma}(G, H, S)$ has finitely many G-orbits of edges if S is finite. There are only two types of edges:

- Edges incident to only elements of G of the form $\{g, gs\}$. These edges are in the G-orbit of $\{1, s\}$, for the corresponding $s \in S$.

- Edges incident to a cone vertex of the form $\{g, v(gH)\}$. These edges are in the G-orbit of $\{1, v(H)\}$, for $h \in H$,

Hence the number of G-orbits of edges is $|S| + 1$.

Claim 3: Stabilizer subgroups of edges are finite. The stabilizer subgroups of edges with cone vertices are trivial while the stabilizer subgroups of edges other than that are of order at most two by the same argument as in Proposition 1.3.

Claim 4: Stabilizer subgroups of vertices are trivial or a conjugate of H. Every cone vertex is a translate of $v(H)$; hence stabilizers of cone vertices are conjugates of H. Whereas, the stabilizer subgroup of all the other vertices is the identity element as all the vertices are elements of G. Note that the stabilizer of the cone vertex $v(H)$ is H. $\square$

Chapter 2

Fineness in Graphs

Let Γ be a graph and $v \in V(\Gamma)$. Recall that the set of the adjacent vertices to v is denoted as:

$$T_v\Gamma = \{w \in V(\Gamma) \mid \{v, w\} \in E(\Gamma)\}.$$

Observe that the cardinality of $T_v\Gamma$ is the degree of v. For $x, y \in T_v\Gamma$, denote by $\angle_v(x, y)$ the minimal length of a path in $\Gamma \setminus \{v\}$ between x and y, and let $\angle_v(x, y) = \infty$ if such a path does not exist. The function $\angle_v$ is a metric on $T_v\Gamma$ that we shall refer to as the *angle metric* on $T_v\Gamma$.

Let v be a vertex of a graph Γ. Then Γ is *fine at* v if $(T_v\Gamma, \angle_v)$ is a locally finite metric space. Let $C \subset V(\Gamma)$. Then Γ is *fine at* C if Γ is fine at v, for all $v \in C$. Subsequently, Γ is a *fine graph* if it is fine at $V(\Gamma)$.

In the case that Γ is a G-graph and $u, v \in V(\Gamma)$, observe that the vertex stabilizer G_v acts on $T_v\Gamma$ by isometries, and if u and v are in the same G-orbit then $T_u\Gamma$ and $T_v\Gamma$ are isomorphic as metric spaces. In particular, if Γ is fine at v then Γ is fine at

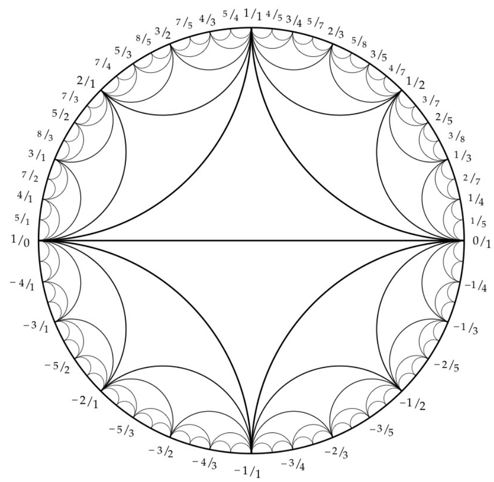

Figure 2.1: Farey Graph

every vertex in the orbit of v.

Examples.

- *Every finite graph is fine.*
- *Every locally-finite graph is fine. Indeed, for all vertices there will be finitely many adjacent vertices.*
- *Every Cayley graph of a finitely generated group is fine since it is locally-finite.*
- *The Farey graph, Figure 2.1, is fine but not locally-finite. We will prove this in chapter 5.*

2.1 Characterizations of Fineness

Proposition 2.1 and Corollary 2.2 are results which appear in [2] and [14], respectively. The generalization of these results is done in the remaining part of the section. Lemma 2.7, at the end of the section, will be used later in the proof of Proposition 4.8.

Proposition 2.1. *[2, Proposition 2.1] Let* Γ *be a graph. The following statements are equivalent:*

1. *For every edge $e \in E(\Gamma)$ and for every positive integer k, there are finitely many circuits of length at most k containing e.*

2. *For every two distinct vertices in $V(\Gamma)$ and for every positive integer k, there are finitely many embedded paths of length at most k that connect each other.*

3. Γ *is a fine graph.*

Proof. $1 \to 2$: Bowditch has proved this in [2, Proposition 2.1].

$1 \leftarrow 2$: This follows from the observation that if c is a circuit of length k, containing the edge $e = \{u, v\}$, then $c \setminus \{e\}$ is a path of length $k-1$ between u and v.

$2 \to 3$ By contrary, suppose $T_v\Gamma$ is not locally finite for $v \in V(\Gamma)$. Suppose $u \in T_v\Gamma$ and the ball $B(u, r) = \{w \in T_v\Gamma \mid \angle_v(w, u) \leq r\} \subset (T_v\Gamma, \angle_v)$ is infinite. Note that, for all $w, w' \in B(u, r)$, there exists a path α in $\Gamma \setminus \{v\}$, of length at most r, from w to w'. Hence, $\alpha \cup \{u, w'\}$ is a path of length at most $r+1$ between v and w. As there are infinitely many elements in $B(u, r)$, there are infinitely many paths from v to w, which contradicts 2.

$3 \to 2$: By contrary, suppose 2 does not hold. Pick a minimal counterexample; $v, w \in V(\Gamma)$ such that for a positive integer k, there are infinitely many paths from v to w. Note that, by the minimality condition, the adjacent vertices of v in infinitely many of these paths would be distinct. So a ball of radius $2k-2$ in $(T_v\Gamma, \angle_v)$ would contain infinitely many elements, which contradicts 3. $\square$

Corollary 2.2. *[14, Lemma 2.1] Let Γ be a fine G-graph with finite edge stabilizer subgroups. For vertices $u, v \in \Gamma$, the intersection $G_u \cap G_v$ is finite unless $u = v$.*

For vertices u and v of Γ and $k \in \mathbb{Z}_+$, we define:

$$\vec{vu}(k)_\Gamma = \{w \in T_v\Gamma \mid w \text{ belongs to an escaping path from } v \text{ to } u \text{ of length } \leq k\}.$$

Proposition 2.3. *A graph Γ is fine at $a \in V(\Gamma)$ if and only if $\vec{ab}(k)_\Gamma$ is a finite set for every $k \in \mathbb{Z}_+$ and every vertex $b \in V(\Gamma)$.*

Proof. Let Γ be fine at $a \in V(\Gamma)$. Pick a vertex $b \in V(\Gamma)$. Suppose there are infinitely many vertices $\{v_0, v_1, \dots\} \in \vec{ab}(k)_\Gamma$. Note that for any $i, j \in \mathbb{Z}_+$, $\angle_a(v_i, v_j)$ is at most $2k-2$. Hence, $\{v_0, v_1, \dots\} \in \vec{ab}(k)_\Gamma \subset B_{T_a(\Gamma)}(v_0, 2k)$, where $B_{T_a(\Gamma)}(v_0, 2k)$ is a ball in $(T_a(\Gamma), \angle_a)$. This implies that $B_{T_a(\Gamma)}(v_0, 2k)$ is infinite which contradicts the fact that $(T_a(\Gamma), \angle_a)$ is a locally finite metric space due to the fineness of Γ at a.
Conversely, assume that $\vec{ab}(k)_\Gamma$ is a finite set for every $k \in \mathbb{Z}_+$ and for every vertex $b \in V(\Gamma)$. Let $u \in T_a\Gamma$ and pick a ball $B_{T_a(\Gamma)}(u, r)$ for any $r \in \mathbb{Z}_+$. Our goal is to show that $B_{T_a(\Gamma)}(u, r)$ is finite. Note that, $\angle_a(u, w) \leq r$ for any $w \in B_{T_a(\Gamma)}(u, r)$. Hence, by adding the edge $\{a, w\}$ to the path from w to u in $\Gamma \setminus \{a\}$, we can get an escaping path from a to u of length at most $r+1$. Therefore, $w \in B_{T_a(\Gamma)}(u, r) \subset \vec{au}(r+1)_\Gamma$. As $u \in T_a\Gamma \subset V(\Gamma)$, $\vec{au}(r+1)_\Gamma$ is a finite set by the hypothesis which implies $B_{T_a(\Gamma)}(u, r)$ is finite. $\square$

Remark 2.4. *For a G-graph Γ, if Γ is fine at a then Γ is fine at every vertex in the G-orbit of a.*

Corollary 2.5. *Let Γ be a G-graph with finite edge stabilizers. Suppose that Γ is fine at $a \in V(\Gamma)$. Then for all vertices $b \in V(\Gamma)$, $G_a \cap G_b$ is finite unless $a = b$.*

Proof. Suppose $M = G_a \cap G_b$ and P is a minimum length path from a to b. Say, k is the length of P. Let $P, P_1, P_2, P_3 \dots$ be the M-translates of P. Consider P_i as the sequence of vertices, say, $[u_0^i, u_1^i, \dots, u_k^i]$, where $a = u_0$ and $b = u_k^i$.
Claim 1: Either $U_1 = \{u_1^j \mid j \in \mathbb{N}\}$ is infinite or M is finite. Suppose U_1 is finite. Note that M acts on U_1. Indeed, U_1 is the M-orbit of u_1^0. It follows that any vertex $w \in U_1$, the edge $\{a, w\}$ is fixed by a finite index subgroup of M. Since edge stabilizers are

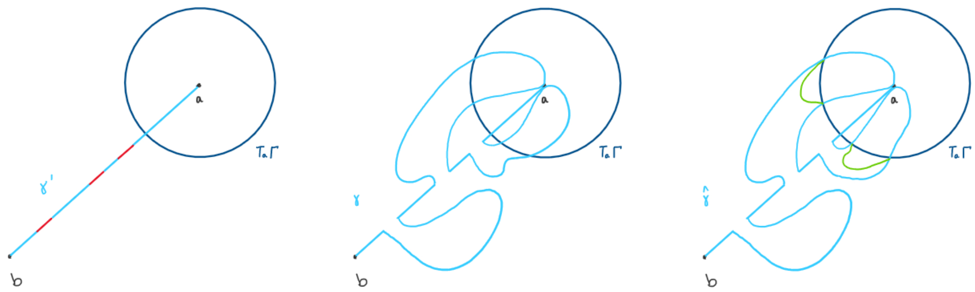

Figure 2.2: Construction of γ and $\hat{\gamma}$.

finite, M is finite.

Claim 2: U_1 is finite. Suppose that U_1 is infinite. Observe that U_1 is contained in the ball $B_{T_a(\Gamma)}(u_1^0, 2k-2)$ of centre u_1^0 and radius $2k-2$ in the metric $(T_a\Gamma, \angle_a)$. Since Γ is fine at a, $(T_a\Gamma, \angle_a)$ is a locally finite metric space and therefore, $B_{T_a(\Gamma)}(u_1^0, 2k-2)$ is finite. This contradicts that U_1 is infinite. □

Remark 2.6. *Note that Corollary 2.2 is a special case of Corollary 2.5.*

The following lemma uses notation that was defined in the introduction of this chapter.

Lemma 2.7. *Let G be a group, let H be a subgroup, and $S \subset G$. Then $\hat{\Gamma}(G, H, S)$ is fine at cone vertices if and only if $(H, \angle_H)$ is a locally finite metric space.*

Proof. Suppose $\hat{\Gamma}(G, H, S)$ is fine at cone vertices $v(gH)$, for all $g \in G$. Then $T_{v(H)}\Gamma$ is locally finite. It is an observation from the structure of $\hat{\Gamma}(G, H, S)$, that $T_{v(H)}\Gamma = H$. Conversely, assume that $(H, \angle_H)$ is a locally finite metric space. As $T_{v(H)}\Gamma = H$, $T_{v(H)}\Gamma$ is also locally finite. Hence, $\hat{\Gamma}(G, H, S)$ is fine at $v(H)$. Then by Remark 2.4, $\hat{\Gamma}(G, H, S)$ is fine at cone vertices $v(gH)$, for all $g \in G$. □

Remark 2.8. *For a G-graph Γ, $(T_v\Gamma, \angle_v)$ and $(T_{gv}\Gamma, \angle_{gv})$ are isometric spaces via $x \mapsto g.x$. In particular, if one is locally finite then other one is too.*

2.2 Fineness and Edge G-attachment

Definition 2.9. *We will call a subpath of length two of a path a corner.*

Theorem 2.10. *[14, Lemma 2.9] [2] Let G act on a graph Δ with finite stabilizer subgroups of edges. Let Γ be a connected G-invariant subgraph of Δ, and let Γ' be obtained from Γ by the edge G-attachment. Then if Γ is fine, Γ' is fine.*

The main result of this chapter is Theorem 2.11 which is a generalization of Theorem 2.10. This is the main technical contribution of the thesis.

Theorem 2.11. *Let Γ be a G-graph such that the G-stabilizers of edges are finite, and let $u, v \in V(\Gamma)$ such that $u \neq v$. Suppose that Γ' is a G-graph obtained from Γ by the edge G-attachment of the orbit of edges with a representative incident to $u, v \in V(\Gamma)$. If $a \in V(\Gamma)$ and Γ is fine at a, then Γ' is fine a.*

Proof. To prove that Γ' is fine at a, by Proposition 2.3, it is enough to show that for every $k \in \mathbb{Z}_+$ and for every vertex $b \in V(\Gamma')$, where $b \neq a$, $\vec{ab}(k)_{\Gamma'}$ is a finite set. First, we define a sequence of sets W_j, Z_j of vertices in $T_a\Gamma$ as follows: Let α be an embedded path from u to v of length l. Let $\hat{\alpha}$ be the reverse path from v to u of length l.

Define subsets $W_1, W_2, \ldots, W_n$ and $Z_1, Z_2, \ldots, Z_{n-1}$ of $T_a\Gamma$ as follows: Let

$$W_n = \vec{ab}(n)_\Gamma \text{ ,where } n = kl.$$

Suppose W_j has been defined for $j \in \{1, 2, \ldots, n\}$, and let

$$Z_{j-1} = W_j \cup \{z \in T_a\Gamma \mid \exists \text{ corner } c \text{ of } \alpha \text{ or } \hat{\alpha} \text{ such that } g.c = [z, a, w] \text{ for } g \in G, w \in W_j\}.$$

$$W_{j-1} = W_j \cup \{w \in T_a\Gamma \mid \exists z \in Z_{j-1} \text{ such that } \angle_{T_a\Gamma}(z, w) \leq n\}.$$

Note that

$$W_j \subseteq Z_{j-1} \subseteq W_{j-1}. \tag{2.1}$$

$W_j \subseteq Z_{j-1}$ holds by definition. Let $z \in Z_{j-1}$. Then $\angle_{T_a\Gamma}(z, z) = 0 < n$. Hence, $z \in W_{j-1}$.

Lemma 2.12. *Then for $j \leq n$, Z_{j-1} and W_j are finite sets.*

Proof. Claim 1: If W_j is finite, then Z_{j-1} is finite. By contrary, assume that there are infinitely many $z \in T_a\Gamma$ such that there exists a corner c of α or $\hat{\alpha}$ for which $g.c = [z, a, w]$, where $w \in W_j$ and $g \in G$. But as W_j is finite, there exists a $w \in W_j$ such that there are infinitely many $z \in Z_{j-1}$ for which there exists a corner c for which $g.c = [z, a, w]$ for $g \in G$. Note that there are infinitely many g's. Pick any two of these g's, say g_1 and g_2. Note that $g_2 g_1^{-1}$ is a stabilizer of $\{a, w\} \in E(\Gamma)$. This implies the stabilizer of the edge $\{a, w\}$ is infinite, which contradicts the hypothesis. Hence, Z_{j-1} is finite.

Claim 2: If W_j is finite then W_{j-1} is finite. As Γ is fine at a, by Proposition 2.3 $\vec{ab}_\Gamma(n) = W_n$. Observe that Z_{n-1} is also finite as W_n is finite by *Claim 1*. This implies that the set $\{w \in T_a\Gamma \mid \exists z \in Z_{n-1} \text{ such that } \angle_{T_a\Gamma}(z, w) \leq n\}$ is finite. Otherwise, $T_a\Gamma$ would not be locally finite which contradicts the fineness of Γ at a.

As Γ is fine at a, W_n is finite. Hence, by *Claim* 2 and then *Claim 1*, W_j and Z_{j-1} are

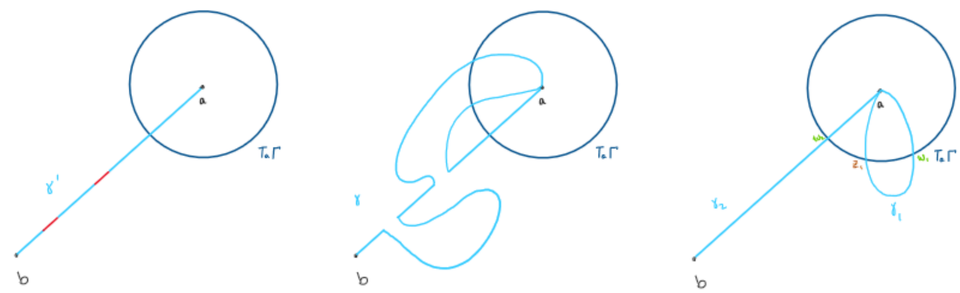

Figure 2.3: Construction of γ.

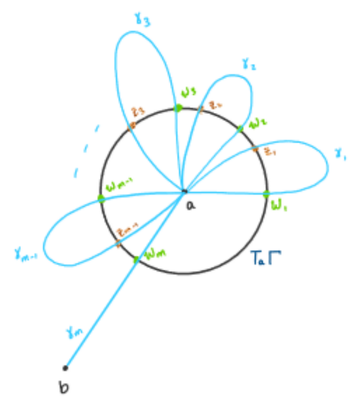

Figure 2.4: γ in terms of w_i and z_i.

finite sets for $j \leq n$, respectively. $\square$

Let γ' be an escaping path in Γ' from a to b of length at most k. Let γ be the path in Γ obtained by replacing each subpath of length one of the form $[g.u, g.v]$ or $[g.v, g.u]$ for $g \in G$ by the path $g.\alpha$ or $g.\hat{\alpha}$, respectively. Then γ is of the form $[a, \gamma_1, a, \gamma_2, a, \ldots, a, \gamma_m]$, where each γ_i does not contain the vertex a. Note that for $m > 1$, γ might not be an escaping path. Let w_i and z_i denote the initial and terminal vertices of γ_i, respectively. See Figure 2.3.

Lemma 2.13. *For $i < m$, $z_i \in Z_i$; and for $i \leq m$, $w_i \in W_i$.*

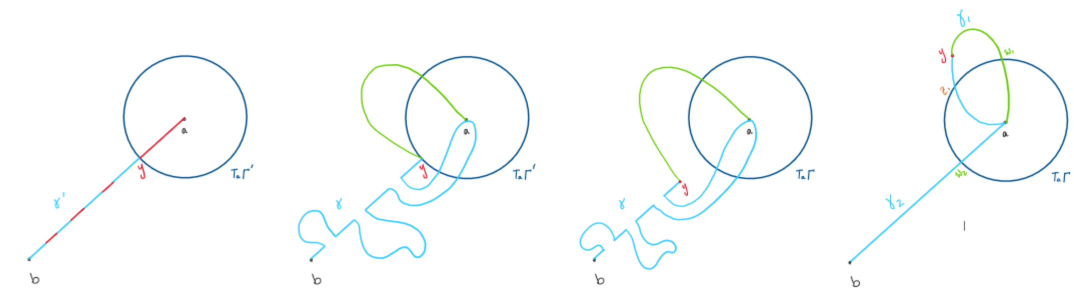

Figure 2.5: Construction of γ.

Proof. Consider Figure 2.4 for this proof.

Claim 1: $w_m \in W_m$ and $z_{m-1} \in Z_{m-1}$. Note that $[a, \gamma_m]$ is an escaping path of length at most n from a to b. Which implies $w_m \in \vec{ab}_\Gamma(n) = W_n \subseteq W_m$ for $n \geq m$. As $[z_{m-1}, a, w_m]$ is the translation of a corner of α or $\hat{\alpha}$ and $w_m \in W_m$, by the definition, $z_{m-1} \in Z_{m-1}$.

Claim 2: If $z_{i+1} \in Z_{i+1}$, then $w_{i+1} \in W_{i+1}$. Observe that $\angle_{T_a\Gamma}(z_{i+1}, w_{i+1}) \leq n$ and $z_{i+1} \in Z_{i+1}$. Then by the definition, $w_{i+1} \in W_{i+1}$.

Claim 3: If $w_i + 1 \in W_i + 1$ then $z_i \in Z_i$. As $[z_i, a, w_{i+1}]$ is the translation of a corner of α or $\hat{\alpha}$ and $w_{i+1} \in W_{i+1}$, by the definition $z_i \in Z_i$. □

Lemma 2.14. $\gamma \cap T_a\Gamma \subseteq W_1$, *where* $\gamma \cap T_a\Gamma = \{v \in \gamma \mid v \in T_a\Gamma\}$.

Proof. Pick a vertex $v \in \gamma \cap T_a\Gamma$. There are two cases to consider:

Case 1: $v = z_i$ for any integer $i < m$. Then by the previous Lemma, $z_i \in Z_i$ and by Equation 2.1, $Z_i \subseteq Z_1 \subseteq W_1$.

Case 2: $v = w_j$ for any integer $j \leq m$. Then by the previous Lemma, $w_i \in W_i$ and by Equation 2.1, $W_i \subset W_1$. □

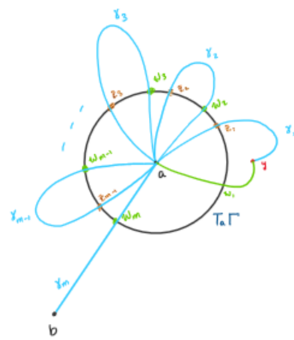

Figure 2.6: γ in terms of w_i and z_i.

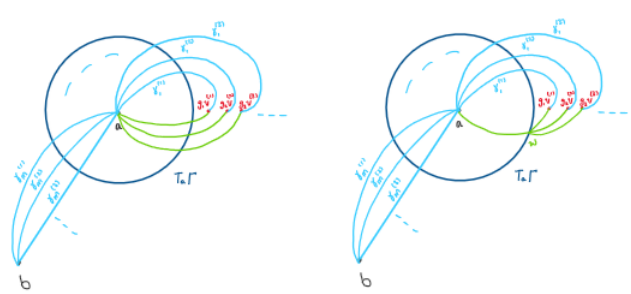

Figure 2.7: Collection of γ's.

Observe that we can write $\vec{ab}(k)_{\Gamma'}$ as follows:

$$\vec{ab}(k)_{\Gamma'} = \{w \in \gamma' \cap T_a\Gamma' \mid \gamma' \text{ is an escaping path in } \Gamma' \text{ between a and b of length at most k}\}.$$

$$\vec{ab}(n)_{\Gamma} = \{w \in \gamma \cap T_a\Gamma \mid \gamma \text{ is an escaping path in } \Gamma \text{ between a and b of length at most } n\}.$$

By Lemma 2.14, $\vec{ab}(n)_\Gamma \subseteq W_1$. Note that, in the process of constructing γ from γ', we are just replacing some subpaths of length one with other paths, while no vertex is eliminated. In fact, we are adding more vertices to γ', which implies $\gamma' \subseteq \gamma$. Now there are two cases to consider:

Case 1: $T_a\Gamma' = T_a\Gamma$. This means $a \neq g.u$ and $a \neq g.v$ for any $g \in G$. Therefore, $\vec{ab}(k)_{\Gamma'} \subseteq \vec{ab}(n)_\Gamma \subseteq W_1$. Then $\vec{ab}(k)_{\Gamma'}$ is finite as by Lemma 2.12, W_1 is finite.

Case 2: $T_a\Gamma \subset T_a\Gamma'$. This means $a = g.u$ or $a = g.v$ for any $g \in G$. There is $g \in G$ such that $g.u = a$ and $g.v \in \vec{ab}(k)_{\Gamma'}$. *Claim:* There are finitely many such g's. Suppose, by contradiction, that there are infinitely many g's such that $g.v \in \vec{ab}(k)_{\Gamma'}$. Let γ'_g be an escaping path from a to b in Γ' such that it contains the subpath $[a, g.v]$. Since $g.v$ might not be in $T_a\Gamma$, we will replace the edge $\{a, g.v\}$ by $g\alpha$ which will give an escaping path in Γ, say γ_g, of length at most n. See Figure 2.5.

Now consider the collection of such γ_g's. As W_1 is finite by Lemma 2.12, there exists a vertex $w \in W_1$ such that there are infinitely many G-translates of α from a to $g.v$ that contains the edge $\{a, w\}$, see Figure 2.7. Therefore, there are infinitely many G-stabilizers of the edge $\{a, w\}$, which contradicts the hypothesis, which is that the edge-stabilizers are finite.

$\square$

Chapter 3

Hyperbolic and Relative Hyperbolic groups

3.1 Hyperbolic Groups

Let G be a group. Then G is a hyperbolic group if there exists a finite generating set S of G such that the Cayley graph $\Gamma(G, S)$ is connected and hyperbolic. Observe that as G is finitely generated, the Svark-Milnor's Lemma implies that the definition of the hyperbolic group is independent of the choice of S.

Proposition 3.1. *G is hyperbolic if and only if there exists a graph Γ such that:*

- *Γ is connected and hyperbolic*
- *and the G-action on Γ is geometric.*

Proof. Let G be a hyperbolic group. Consider the Cayley graph $\Gamma(G, S)$. By the definition of hyperbolic groups, $\Gamma(G, S)$ is connected and hyperbolic. And by Proposition 1.3, the G-action on $\Gamma(G, S)$ is proper and cocompact.

Converse:

Suppose G is acting properly and cocompactly on a connected and hyperbolic graph Γ. As G is finitely generated and quasi-isometric to Γ by Svark-Milnor's Lemma, Proposition 1.5, there exists a finite generating set S such that $\Gamma(G, S)$ is a hyperbolic graph. $\square$

Recall that the G-attachment of an edge $\{u, v\}$ is attaching the G-orbit of $\{u, v\}$ to a graph Γ such that:

$$V(\Gamma') = V(\Gamma),$$
$$E(\Gamma') = E(\Gamma) \cup \{\{g.u, g.v\} \mid g \in G\}.$$

Lemma 3.2. *[14, Lemma 2.7] Let G act on a graph Γ, and let Γ be a connected G-invariant subgraph of Γ. Suppose Γ' is obtained from Γ by a G-attachment of an edge P with at least one of its endpoints in Γ. Then the inclusion $\Gamma \hookrightarrow K'$ is a quasi-isometry. In particular, if Γ is hyperbolic, then Γ' is hyperbolic.*

3.2 Relatively Hyperbolic groups

The notion of a group, G, hyperbolically relative to a subgroup, H, was introduced by Gromov in [9]. There are different but equivalent definitions of a relatively hyperbolic group.

Definition 3.3 is one of the two Bowditch's definitions in [2], which is the elaboration of the idea of Gromov. This definition includes the properties of the group action on a hyperbolic graph, which will be defined in this chapter in detail.

Definition 3.3. *[2] A group G is a hyperbolic group relative to H if and only if there exists a graph Γ such that:*

- *Γ is connected, hyperbolic and fine,*
- *the G-action on Γ is cocompact,*
- *stabilizer subgroups of edges are finite,*
- *stabilizer subgroups of vertices are either finite or a conjugate of H,*
- *and there exists a vertex that has the stabilizer subgroup equals to H.*

We will call such a graph as a cocompact (G, H)-graph.

The second definition, 3.4, of the relative hyperbolicity of a group, is related to the definition stated by Farb in [7].

Definition 3.4. *[14] A group G is a hyperbolic group relative to a subgroup H if there exists a finite subset S of G such that the Coned-off Cayley graph, $\hat{\Gamma}(G, H, S)$, is connected, hyperbolic and fine.*

Examples. *Let $G = H * \mathbb{Z}$ and $S = \{s\}$. Then G is hyperbolic relative to H as $\hat{\Gamma}(G, H, S)$ is quasi-isometric to $\hat{\Gamma}(G, H \cup S)$ by Lemma 1.12 and $\hat{\Gamma}(G, H \cup S)$ is quasi-isometric to a tree, see Figure 3.1.*

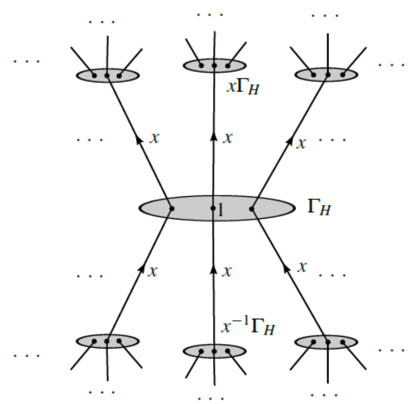

Figure 3.1: $\Gamma(G, H \cup S)$ for $G = H * \mathbb{Z}$

3.3 Equivalence of the Definitions

In this chapter, we will see the equivalence of the two definitions of Bowditch and Martinez-Pedroza and will state some properties of relatively hyperbolic groups.

Proposition 3.5. *Definition 3.4 and 3.3 are equivalent.*

To prove a part of the equivalence, we need a result mentioned below that has appeared in other instances by Dahmani [6], Hruska [12] and Martinez-Pedroza and Wise [14].

Proposition 3.6. *[14, Proposition 4.3] Let Γ be a cocompact (G, H)-graph. Suppose that S is a finite subset of G such that $S \cup H$ generates G. Then there exists a cocompact (G, H)-graph Γ' such that Γ and $\hat{\Gamma}(G, H, S)$ both embed equivariantly and simplicially into Γ'.*

Proof of Proposition 3.5. Suppose there is a finite subset S of G such that the Coned-off Cayley graph $\hat{\Gamma}(G, H, S)$ is connected, hyperbolic and fine. Note that by Proposition 1.13, $\hat{\Gamma}(G, H, S)$ is a cocompact (G, H)-graph.

Conversely, suppose there exists a cocompact (G, H)-graph Γ. Then we have to show that there is a finite $S \subset G$ such that the Coned-off Cayley graph $\hat{\Gamma}(G, H, S)$ is connected, hyperbolic and fine.

Claim 1: There is a finite $S \subset G$ such that $\hat{\Gamma}(G, H, S)$ is connected. Here we will use Lemma 1.8: As the action on Γ is cocompact and Γ is connected, $G = \langle W \bigcup_{u \in F} G_u \rangle$, where W is a finite subset of G and F is a finite subset of vertices of Γ. Now we will construct S such that G is $\langle S \cup H \rangle$. For each $u \in F$, let $S_u = G_u$ if G_u is finite, and otherwise, let $S_u = \{g\}$ for a choice of $g \in G$ such that $G_u = gHg^{-1}$. Let

$$S = W \cup \bigcup_{v \in F} S_v.$$

Observe that S is finite as W and S_v are finite, for all $v \in F$. Hence, by Lemma 1.11, $G = \langle S \cup H \rangle$ implies that $\hat{\Gamma}(G, H, S)$ is connected.

Claim 2: $\hat{\Gamma}(G, H, S)$ is hyperbolic and fine. By Proposition 3.6, there exists a cocompact (G, H)-graph Γ' such that $\hat{\Gamma}(G, H, S)$ equivariantly embeds in it. As $\hat{\Gamma}(G, H, S)$ equivariantly embeds in Γ', by Lemma 3.2 and Lemma 2.10 Γ' is hyperbolic and fine. Therefore, since Γ' is a cocompact (G, H)-graph, $\hat{\Gamma}(G, H, S)$ is also a cocompact (G, H)-graph. $\square$

3.4 Hyperbolic Groups are Relative Hyperbolic

Proposition 3.7. *G is hyperbolic if and only if G is relatively hyperbolic to the trivial subgroup.*

Proof. By Lemma 1.12, $\hat{\Gamma}(G, H, S)$ is quasi-isometric to $\Gamma(G, H \cup S)$. Let H be a trivial subgroup of G. Then $\Gamma(G, H \cup S) = \Gamma(G, S)$. Hence, $\Gamma(G, S)$ is hyperbolic if and only if $\hat{\Gamma}(G, H, S)$ is hyperbolic. $\square$

Chapter 4

Hyperbolically Embedded Subgroups

All the Figures in this chapter are taken from [5].

There are several distinct but equivalent definitions of a relatively hyperbolic group. One characterization of a relatively hyperbolic group is by Bowditch (Definition 3.3), that was discussed in the previous chapter. In this chapter, we recall the notion of hyperbolically embedded subgroup, which is a generalization of the notion of a relatively hyperbolic group. Hyperbolically embedded subgroups were introduced by Dahmani, Guiraldel and Osin to answer questions on mapping class groups [15]. Our main contribution is a characterization of the notion of a hyperbolically embedded subgroup, which extends the definition by Bowditch of a relatively hyperbolic group.

Let H be an infinite subgroup of a group G and let S be a subset of G such that

$H \cup S$ generates G. Consider the Cayley graph $\Gamma(G, H \cup S)$ and observe that the Cayley graph $\Gamma(H, H \setminus \{e\})$ is a *complete* subgraph of it. We will define the *angle-metric* on H as follows, for $h, k \in H$, $\angle_H(h,k)$ is the minimal length of an edge-path in $\Gamma(G, H \cup S)$ between h and k that does not contain a vertex of $\Gamma(H, H \setminus \{e\})$ other than h and k, and let $\angle_H(h,k) = \infty$ if such a path does not exist.

Definition 4.1. *[5] The subgroup H is a hyperbolically embedded subgroup of G with respect to S if:*

1. *G is generated by $H \cup S$,*
2. *the Cayley graph $\Gamma(G, H \cup S)$ is hyperbolic,*
3. *and $(H, \angle_H)$ is a locally finite metric space.*

We write $H \hookrightarrow_h (G, S)$ if H is hyperbolically embedded in G with respect to $S \subseteq G$.

Examples. *[5]*

- *Finite subgroups are always hyperbolically embedded subgroups. Note that all finite subgraphs are locally finite metric spaces. By taking $S = G$, the Cayley Graph $\Gamma(G, H \cup S)$ has diameter one. Hence, the other conditions hold.*
- *Let $G = H \times \mathbb{Z}$ and $S = \{s\}$, where s is a generator of $\mathbb{Z}$. Let $\Gamma_H = \Gamma(H, H \setminus \{e\})$. $\Gamma(G, H \cup S)$ is hyperbolic as it is quasi-isometric to a line. Note that, $\angle_H(h_1, h_2) \leq 3$ for every $h_1, h_2 \in H$. See Figure 4.1, a path of length three from h_1 to h_2 has two edges between Γ_H and $x\Gamma_H$ and one connecting xh_1 and xh_2 but no edges in Γ_H. So, if H is infinite, then we can find a ball of finite radius containing infinitely many elements. Therefore, for an infinite subgroup, it would not be hyperbolically embedded in G.[5]*

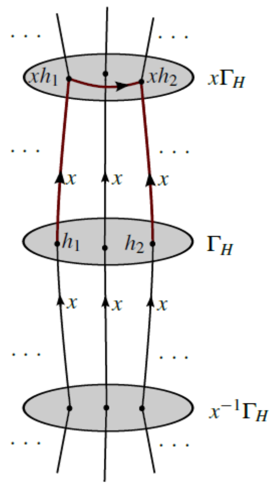

Figure 4.1: $\Gamma(G, H \cup S)$ for $G = H \times \mathbb{Z}$

- *Let $G = H * \mathbb{Z}$ and $S = \{s\}$. $\Gamma(G, H \cup S)$ is hyperbolic as it is quasi-isometric to a tree. Note that $\angle_H(h_1, h_2) = \infty$ if $h_1 \neq h_2$. Hence, H is a hyperbolically embedded subgroup of G. See Figure 3.1.[5]*

Definition 4.2. *Let Γ be a G-graph. We will denote $V_\infty(\Gamma)$ by the set of vertices that have infinite stabilizers,*

$$V_\infty(\Gamma) = \{v \in V(\Gamma) \mid v \text{ has infinite stabilizer}\}.$$

Definition 4.3. *A graph Γ is a (G, H)-graph if G acts on Γ while satisfying the following conditions:*

- *Γ is connected and hyperbolic,*
- *Γ is fine at $V_\infty(\Gamma)$,*
- *for all $w \in V_\infty(\Gamma)$, G_w acts on $T_w\Gamma$ with finitely many orbits,*
- *number of orbits of vertices are finite,*

- *stabilizer subgroups of edges are finite,*
- *stabilizer subgroups of vertices are either finite or a conjugate of H,*
- *and there exists a vertex that has the stabilizer subgroup equals to H.*

A (G, H)-subgraph is a G-subgraph of a (G, H)-graph, which is also a (G, H)-graph.

Remark 4.4. *If Γ is a (G, H)-graph according to Definition 4.3 and G acts cocompactly on Γ, then Γ is a cocompact (G, H)-graph in the sense of Definition 3.3. Indeed, by Lemma 1.7 for $v \in V(\Gamma)$, G_v acts on $T_v\Gamma$ with finitely many orbits, and if v has a finite G-stabilizer then v has finite degree. Therefore, for all $v \in V(\Gamma)$, $T_v\Gamma$ is locally finite. Hence Γ is fine.*

Remark 4.5. *By Remark 4.4, G is a hyperbolic group relative to H if and only if there is a (G, H)-graph on which G acts cocompactly.*

Remark 4.6. *If Γ and Γ' are G-isomorphic and Γ is a (G, H)-graph, then Γ' is a (G, H)-graph.*

The main result of this thesis is the following:

Theorem 4.7. *Let H be an infinite subgroup of G, then $H \hookrightarrow_h (G, S)$ for some subset S if and only if there exists a (G, H)-graph.*

The proof of Theorem 4.7 relies on the following results:

Proposition 4.8. *Let H be an infinite subgroup of G and let S be a subset of G. Then $H \hookrightarrow_h (G, S)$ if and only if the Coned-off Cayley graph $\hat{\Gamma}(G, H, S)$ is a (G, H)-graph.*

Proposition 4.9. *Let H be an infinite subgroup of G. If there is a (G, H)-graph then there is a subset S of G such that the Coned-off Cayley graph $\hat{\Gamma}(G, H, S)$ is a (G, H)-graph.*

Proof of Theorem 4.7. The only if part follows from Proposition 4.8. For the if part, first applies Proposition 4.9 and then Proposition 4.8. □

Corollary 4.10. *[5, Proposition 2.4] A group G is hyperbolic relative to an infinite subgroup H if and only if $H \hookrightarrow_h G$ with respect to a finite subset S.*

Proof. Let G be a hyperbolic group relative to H. By Proposition 3.5, there exists a finite subset S of G such that $\hat{\Gamma}(G, H, S)$ is connected, hyperbolic and fine. By Proposition 1.13, $\hat{\Gamma}(G, H, S)$ is a (G, H)-graph. Then by Proposition 4.8, $H \hookrightarrow_h G$ with respect to the finite subset S.

Conversely, suppose that $H \hookrightarrow G$ with respect to the finite subset S. Then Proposition 4.8 implies that $\hat{\Gamma}(G, H, S)$ is a (G, H)-graph. By the definition of a (G, H)-graph, $\hat{\Gamma}(G, H, S)$ is connected, hyperbolic and fine at the vertices corresponding to left cosets of H. Since S is finite, any vertex of $\hat{\Gamma}(G, H, S)$ that is an element of G has finite degree. Therefore, $\hat{\Gamma}(G, H, S)$ is a connected, hyperbolic and fine graph. Hence, G is hyperbolic relative to H. □

The rest of the chapter is organized as follows: The proof of Proposition 4.8 is in Section 4.1, and the rest of the sections lead towards proving Proposition 4.9.

4.1 Proof of Proposition 4.8

Proof. Let $\Gamma = \Gamma(G, H \cup S)$ be the Cayley graph and let $\hat{\Gamma} = \hat{\Gamma}(G, H, S)$ be the Coned-off Cayley graph of G with respect to H and S.

Claim 1: $\hat{\Gamma}$ is connected if and only if $G = \langle S \cup H \rangle$. Note that, by Lemma 1.11:

$$G = \langle S \cup H \rangle \iff \hat{\Gamma}(G, H, S) \text{ is connected } \iff \Gamma(G, H \cup S) \text{ is connected.}$$

Claim 2: $\hat{\Gamma}$ is hyperbolic if and only if Γ is hyperbolic. By Lemma 1.12, $\hat{\Gamma}$ is quasi-isometric to Γ. Then by Proposition 1.6, quasi-isometry preserves the hyperbolicity.
Claim 3: $\hat{\Gamma}$ is fine at cone vertices if and only if $(H, \angle_H)$ is a locally finite metric space. This holds by Lemma 2.7.
Then by Proposition 1.13, $\hat{\Gamma}$ is a (G, H)-graph.
Therefore, $H \hookrightarrow_h (G, S)$ for some subset S of G if and only if $\hat{\Gamma}$ is a (G, H)-graph. $\square$

4.2 (G, H)-graphs and trivial stabilizers

Definition 4.11. *We say that a (G, H)-graph is a clamped (G, H)-graph if there are no edges incident to two distinct vertices in $V_\infty(\Gamma)$.*

Proposition 4.12. *Let Γ be a clamped (G, H)-graph such that there is $u \in V(\Gamma)$ where G_u is trivial and*

$$V(\Gamma) = V_\infty(\Gamma) \cup G.u.$$

Then there is $S \subseteq G$ such that the coned-off Cayley graph $\hat{\Gamma}(G, H, S)$ and Γ are isomorphic G-graphs.

The proof of this proposition requires some lemmas.

Lemma 4.13. *Let Γ be a G-graph with finite edge stabilizers. Suppose that Γ is fine at $V_\infty(\Gamma)$. For $u, v \in V_\infty(\Gamma)$, if $G_u = G_v$ then $u = v$.*

Proof. This follows directly from Corollary 2.5. $\square$

In [14, Lemma 2.4], an analogous result is proved for relatively hyperbolic groups.

Lemma 4.14. *Let $H \hookrightarrow_h (G, S)$, where H is an infinite subgroup. If Γ if fine at $V_\infty(\Gamma)$, then there are natural G-equivariant bijections*

$$V_\infty(\Gamma) \longrightarrow \{gHg^{-1} \mid g \in G\} \longrightarrow G/H$$

that maps a vertex v to its G-stabilizer and maps a conjugate gHg^{-1} of H to the left coset gH.

Proof. First, we show that $V_\infty(\Gamma) \longrightarrow \{gHg^{-1} \mid g \in G\}$ given by $v \mapsto G_v$ is a bijection:

Claim 1: The range of the map is well-defined. Indeed, by the definition of a (G, H)-graph, vertices with infinite stabilizers have stabilizer subgroups equal to the conjugates of H.

Claim 2: The map is surjective. by the definition of a (G, H)-graph there exists a vertex v that has stabilizer equals to H. Take a conjugate of H, say gHg^{-1} for $g \in G$. Note that it is the stabilizer of $g.v$.

Claim 3: The map is injective. By Lemma 4.13, if the stabilizers of two vertices of $V_\infty(\Gamma)$ are the same then the vertices are equal.

Now we verify that $G/H = \{gH \mid g \in G\} \longrightarrow \{gHg^{-1} \mid g \in G\}$ given by $gH \mapsto gHg^{-1}$ is a bijection:

Claim 1: The range of the map is well-defined. Indeed, if $fH = gH$ then $fHf^{-1} = gHg^{-1}$ for $f, g \in G$.

Claim 2: The map is surjective. As for any subgroup gHg^{-1} , there exists a left coset gH.

Claim 3: The map is injective. Suppose that $g_1Hg_1{}^{-1} = g_2Hg_2{}^{-1}$. This implies

${g_2}^{-1}g_1H{g_1}^{-1}g_2 = H$. Since Γ is a (G,H)-graph, there is a vertex v of Γ such that $G_v = H$. Hence, the stabilizer of v and ${g_2}^{-1}g_1.v$ are equal. And by Lemma 4.13, $v = {g_2}^{-1}g_1.v$. Therefore, ${g_2}^{-1}g_1 \in H$ and $g_1H = g_2H$. □

Proof of Proposition 4.12. We have to find a bijection between $V(\hat{\Gamma})$ and $V(\Gamma)$ such that

- $\forall a, b \in V(\hat{\Gamma})$, $\{a,b\} \in E(\hat{\Gamma}) \Longleftrightarrow \{\phi(a), \phi(b)\} \in E(\Gamma)$.
- $\forall a \in V(\hat{\Gamma})$ and $\forall g \in G$, $\phi(g.a) = g.\phi(a)$.

Recall:

$$V(\hat{\Gamma}) = G \cup \{v(gH) \mid gH \in G/H\}$$
$$E(\hat{\Gamma}) = \{\{g, g.s\} \mid g \in G, s \in S\} \cup \{\{gh, v(gH)\} \mid gh \in gH, gH \in G/H\}.$$

By Lemma 4.14, there is a one-to-one correspondence between $V_\infty(\Gamma)$ and the set of left cosets gH. As there exists a vertex $v \in V_\infty(\Gamma)$ such that $G_v = H$, there is a one-to-one correspondence between the set of left cosets gH and $G.v$. Therefore, $V_\infty(\Gamma) = G.v$. This implies we can map H to v and hence, $v(gH) \in V(\hat{\Gamma})$ to $g.v \in V(\Gamma)$ through a one-to-one correspondence.
Now we have to find $u \in V(\Gamma)$ that is corresponding to $1 \in G$. As Γ is connected, there is a path between v and u. Observe that this path will contain vertices in $G.v$ and $G.u$. Hence, there will be an edge $\{g.v, g'.u\}$ for some $g, g' \in G$, which we can translate to find a vertex in $G.u$ that is adjacent to v. Assume v and u are adjacent.

Then we can create a bijection map such that

$$\begin{aligned}\phi : G \cup G/H &\longrightarrow V(\Gamma) = V_\infty(\Gamma) \cup G.u = G.v \cup G.u \\ g &\longmapsto g.u \\ gH &\longmapsto g.v.\end{aligned}$$

Let

$$S = \{g \in G \mid g.u \text{ and } u \text{ are adjacent}\} \text{ and } S = S^{-1}$$

Claim: ϕ is an isomorphism between $\hat{\Gamma}(G, H, S)$ and Γ. Observe that $g.\phi(x) = \phi(g.x)$. We just have to show that $\{a, b\} \in E(\hat{\Gamma}) \iff \{\phi(a), \phi(b)\} \in E(\Gamma)$, for all $a, b \in V(\hat{\Gamma})$. There are three cases to consider:

1. Suppose $a, b \in G$. Note that $a^{-1}b \in S$ as $\{1, a^{-1}b\} = \{a, b\} \in E(\hat{\Gamma})$. By the definition of ϕ, $\phi(a) = a.u$ and $\phi(b) = b.u$. Then $\{\phi(a), \phi(b)\} = \{a.u, b.u\} = \{u, a^{-1}b.u\} \in E(\Gamma)$, as $a^{-1}b \in S$.

2. Suppose $a \in G$ and $bH \in G/H$. Note that $a^{-1}b \in H$ as $\{1, a^{-1}bH\} = \{a, bH\} \in E(\hat{\Gamma})$. By the definition of ϕ, $\phi(a) = a.u$ and $\phi(bH) = b.v$. Then $\{\phi(a), \phi(bH)\} = \{a.u, b.v\} = \{u, a^{-1}b.v\} \in E(\Gamma)$, as $a^{-1}b \in H$.

3. Suppose $aH, bH \in G/H$. This case does not hold since Γ is clamped.

$\square$

4.3 Existence of Clamped (G, H)-graphs

Proposition 4.15. *If there is a (G, H)-graph Γ, then there exists a clamped (G, H)-graph that has a vertex with trivial stabilizer.*

To prove this proposition, we need to introduce some lemmas.

Lemma 4.16. *The edge G-attachment preserves (G, H)-graph.*

Proof. Let Γ' be a graph after the edge G-attachment to a (G, H)-graph, say Γ.

- Γ' *is connected*: Indeed, $V(\Gamma) = V(\Gamma')$ and every path in Γ is a path in Γ'

- Γ' *is hyperbolic*: Note that the only difference between Γ' and Γ is the addition of the G-orbit of an edge. Suppose the G-orbit of the new edges has representative incident to $u, v \in V(\Gamma)$. As Γ is connected, there exists a path in Γ from u to v. Let P be the shortest path from u to v of length k. Pick any two vertices $a, b \in V(\Gamma')$. As Γ' is connected, there exists a path in Γ' from a to b. But this path might not be in Γ as there might be some edges that are the translates of either $[u, v]$ or $[v, u]$. But we can replace these edges with the corresponding translates of P. The resulting path from a to b will be in Γ and would be of length at most $k\, \mathsf{dist}_{\Gamma'}(a, b)$. Then $V(\Gamma) = V(\Gamma')$ implies that the identity map i is a $(k, 0, 0)$-quasi-isometry from Γ' to Γ such that,

$$\mathsf{dist}_{\Gamma'}(a, b) \leq \mathsf{dist}_{\Gamma}(i(a), i(b)) \leq k\, \mathsf{dist}_{\Gamma'}(a, b),$$
$$\forall u \in V(\Gamma'), \exists a \in V(\Gamma) \text{ such that } \mathsf{dist}_{\Gamma'}(i(a), u) = 0$$

  Then by Proposition 1.6, Γ' is hyperbolic and Γ is hyperbolic.

- Γ' *is fine at $V_\infty(\Gamma')$*: It holds by Theorem 2.11.

- It is easy to see that all the remaining properties of a (G, H)-graph hold for Γ'.

Therefore, Γ' is a (G, H)-graph. $\square$

Lemma 4.17. *If there is a (G, H)-graph, then there is a (G, H)-graph with a vertex that has trivial G-stabilizer.*

Proof. Let Γ be a (G, H)-graph with no vertex that has trivial G-stabilizer. Now add the G-orbit of a vertex u of trivial G-stabilizer with an edge G-attachment with a representative incident to u and $v \in V_\infty(\Gamma)$. We will denote this new graph as Γ'.

$$V(\Gamma') = V(\Gamma) \bigcup_{g \in G} g.u$$

$$E(\Gamma') = E(\Gamma) \bigcup_{g \in G} g.\{u, v\}, \text{ for } v \in V_\infty(\Gamma).$$

We have to show that Γ' is a (G, H)-graph. Indeed, by Lemma 4.16, the edge G-attachment preserves (G, H)-graph. Therefore, Γ' is a (G, H)-graph. $\square$

Lemma 4.18. *Let Γ be a G-graph and let $u, v \in V(\Gamma)$ such that u and v are adjacent and there is an embedded path from u to v that does not contain the translate of either $[u, v]$ or $[v, u]$, say $P : [u = u_0, u_1, \ldots, u_k = v]$ such that $[u_i, u_{i+1}] \in E(\Gamma)$ for $0 < i < k-1$. Suppose that Γ' is a G-graph obtained from Γ by the edge G-removal of the G-orbit of edges with a representative incident to $u, v \in V(\Gamma)$. If Γ is a (G, H)-graph, then Γ' is a (G, H)-graph.*

Proof. Let Γ' be a graph after the edge G-removal from Γ, such that the hypothesis holds.

- Γ' *is connected and hyperbolic*: Pick any two vertices in Γ'. As Γ is connected, there exists a path in Γ from a to b. But this path might not be in Γ' as there might be some edges that are the translates of either $[u, v]$ or $[v, u]$. But by the hypothesis, we know that we can replace the translates of these edges with the translates of P. The new path from a to b will be in Γ', which implies that Γ' is connected. Note that as $\Gamma' \subseteq \Gamma$ and the length of P is k, the identity map i is a $(k, 0, 0)$-quasi-isometry from Γ' to Γ such that,

$$\mathsf{dist}_{\Gamma'}(a, b) \leq \mathsf{dist}_{\Gamma}(i(a), i(b)) \leq k\, \mathsf{dist}_{\Gamma'}(a, b).$$

Then $V(\Gamma') = V(\Gamma)$ implies that Γ and Γ' are quasi-isometric and hence, Γ' is hyperbolic by Proposition 1.6.

- Γ' *is fine at* $V_\infty(\Gamma')$*:* Observe that $V_\infty(\Gamma) = V_\infty(\Gamma')$. Let $a \in V_\infty(\Gamma')$. Then for $b \in V(\Gamma')$ and $k \in \mathbb{Z}_+$, $\vec{ab}_{\Gamma'}(k) \subset \vec{ab}_{\Gamma}(k)$. Since Γ is fine at $V_\infty(\Gamma)$, $\vec{ab}_{\Gamma}(k)$ is finite. Hence, Γ' is fine at $V_\infty(\Gamma')$.

- It is easy to see that all the remaining properties of a (G, H)-graph hold for Γ'.

Therefore, Γ' is a (G, H)-graph. □

Proof of Proposition 4.15. By Lemma 4.17, if there is a (G, H)-graph, then there is a (G, H)-graph Γ with a vertex that has trivial G-stabilizer. Assume that Γ has a vertex v with trivial G-stabilizer.

By Lemma 4.14, G acts on $V_\infty(\Gamma)$ with a single G-orbit. Let $w \in V_\infty(\Gamma)$. Since Γ is a (G, H)-graph, G_w acts on $T_w\Gamma$ with finitely many orbits. Therefore, G_w acts on $T_w\Gamma \cap V_\infty(\Gamma)$ with finitely many orbits. Hence, we need to remove only finitely many G-orbits of edges represented by w, x with $x \in T_w \cap V_\infty(\Gamma)$. This can be done by applying the following procedure a finite number of times.

Removing the G-orbit of an edge $\{w, x\}$ where $x \in T_w\Gamma \cap V_\infty(\Gamma)$: Add two new G-

orbits of edges with representatives incident to w and v, and x and v. Since $v \notin V_\infty(\Gamma)$, by Lemma 4.16, the new graph is a (G, H)-graph.

Then remove the G-orbit of an edge incident to w and x. Then by Lemma 4.18, the new graph is a (G, H)-graph. $\square$

Remark 4.19. *In the proof of the main result of the thesis, the property of a (G, H)-graph stating that for all $w \in V_\infty(\Gamma)$, G_w acts on $T_w\Gamma$ with finitely many orbits, is only used in the proof of Proposition 4.15.*

4.4 Clamped (G, H)-graphs and Coned-off Cayley Graphs

Proposition 4.20. *If there exists a (G, H)-graph Γ which is clamped and has a vertex with trivial stabilizer, then there is a clamped (G, H)-graph Γ' such that there is $u \in V(\Gamma')$ where G_u is trivial and*

$$V(\Gamma') = V_\infty(\Gamma') \cup G.u.$$

The proof of this proposition requires the following lemma:

Lemma 4.21. *If there exists a clamped (G, H)-graph with a vertex with trivial stabilizer, then there exists a clamped (G,H)-graph with a vertex with trivial stabilizer, and finite connected subgraph Γ_0 such that,*

1. *$V(\Gamma_0) \cap V_\infty(\Gamma) = \phi$.*

2. *For all $u \in V(\Gamma)$, there exists $g \in G$ such that $g.u \in V(\Gamma_0)$.*

3. No two distinct vertices of Γ_0 *are in the same* G*-orbit.*

Proof. Let $\alpha = \{v_0, v_1, \dots, v_n\}$ be a collection of representatives of all the G-orbits of vertices of $V(\Gamma)$ with finite stabilizers such that no two distinct vertices in α represent the same G-orbit. For each $i \in \{1, 2, \dots, n\}$, add a G-orbit of edges with representative $\{v_0, v_i\}$ obtaining a graph Γ'. By Lemma 4.16, Γ' is still a (G, H)-graph. As we are not adding any edges incident to two distinct vertices of $V_\infty(\Gamma)$, Γ' remains a clamped graph. Note that Γ' still contains a vertex with a trivial stabilizer. Consider a subgraph Γ_0 of Γ' such that,

$$V(\Gamma_0) = \alpha$$
$$E(\Gamma_0) = \{\{v_0, v_i\} \mid v_i \in \alpha, \forall i \in \{1, 2, \dots, n\}\}$$

Clearly by the definition of Γ_0, all the three properties are satisfied. □

Proof of Proposition 4.20. We can assume that the (G, H)-graph Γ contains a finite connected subgraph Γ_0 as mentioned in Lemma 4.21. Note that by the definition of $E(\Gamma_0)$, Γ_0 has diameter 2. Let Γ' be a graph such that,

$$V(\Gamma') = \{g.\Gamma_0 \mid g \in G\} \cup V_\infty(\Gamma)$$
$$E(\Gamma') = \{\{g.\Gamma_0, f.\Gamma_0\} \mid g.\Gamma_0 \neq f.\Gamma_0 \text{ and either } g.\Gamma_0 \cap f.\Gamma_0 \neq \phi \text{ or }$$
$$\exists e \in E(\Gamma) \text{ incident to a vertex of } g.\Gamma_0 \text{ and a vertex of } f.\Gamma_0\}$$

Let $\{v_0, \dots, v_n\}$ be the vertices of Γ_0. Note that there is a surjective map from $V(\Gamma)$ to $V(\Gamma')$ such that $g.v_i$ will map on to $g.\Gamma_0$, for all $i \in \{0, 1, \dots, n\}$.

Claim: Γ' is a (G, H)-graph.

- Γ' *is connected and hyperbolic:* Pick a minimal length path γ in Γ from $v_0 \in V(\Gamma_0)$ to $g.v_0 \in V(g\Gamma_0)$. Then by the definition of $E(\Gamma')$, there exists a path in Γ' from Γ_0 to $g.\Gamma_0$ of length less than or equals to the length of γ. Hence, Γ' is connected. Observe that the path in Γ' induced by γ implies the following inequality,

$$\mathsf{dist}_{\Gamma'}(\Gamma_0, g.\Gamma_0) \leq \mathsf{dist}_{\Gamma}(v_0, g.v_0).$$

  Let $g \in G$ and consider a minimal length path γ' in Γ' from Γ_0 to $g\Gamma_0$. By the definition of $E(\Gamma')$, either the edge in γ' belongs to $E(\Gamma)$ incident to vertices of the G-translates of Γ_0 in γ or the G-translates of Γ_0 are disjoint. But as Γ_0 is connected, by adding some paths to γ' we can construct a path γ in Γ from v_0 to gv_0 of length at most $2\,\mathsf{dist}_{\Gamma'}(\Gamma_0, g.\Gamma_0) + 2$. Hence,

$$\mathsf{dist}_{\Gamma}(v_0, g.v_0) \leq 2\,\mathsf{dist}_{\Gamma'}(\Gamma_0, g.\Gamma_0) + 2.$$

  Therefore, Γ' is hyperbolic.

- Γ' *is fine at $V_\infty(\Gamma')$:*. Observe that $V_\infty(\Gamma) = V_\infty(\Gamma')$. Let $a \in V_\infty(\Gamma')$. Then for $b \in V(\Gamma')$ and $k \in \mathbb{Z}_+$, $\vec{ab}_{\Gamma'}(k) \subset \vec{ab}_{\Gamma}(k)$. Since Γ is fine at $V_\infty(\Gamma)$, $\vec{ab}_{\Gamma}(k)$ is finite. Hence, Γ' is fine at $V_\infty(\Gamma')$.

- Γ' *has finitely many G-orbits of vertices:*. Indeed, there are only two G-orbits of vertices, which are $V_\infty(\Gamma')$ and $G.\Gamma_0$.

- *Stabilizers of edges are finite:* Since Γ_0 is finite and all vertices of Γ_0 have finite stabilizers.

- Γ' *has a vertex with trivial stabilizer:* Note that every G-translate of Γ_0 have trivial stabilizers as there exists a vertex in Γ_0 that has trivial stabilizer and no two distinct vertices of Γ_0 are in the same G-orbit.

- *For all $w \in V_\infty(\Gamma')$, G_w acts on $T_w\Gamma'$ with finitely many orbits.* Indeed, the surjective map from $V(\Gamma)$ to $V(\Gamma')$ preserves the G-action.

□

4.5 Proof of Proposition 4.9

Proof of Proposition 4.9. Let Γ be a (G, H)-graph. Then by Proposition 4.15, there exists a clamped (G, H)-graph that has a vertex with trivial stabilizer. By Proposition 4.20, we can construct a clamped (G, H)-graph Γ' such that there is $u \in V(\Gamma')$ where G_u is trivial and

$$V(\Gamma') = V_\infty(\Gamma') \cup G.u.$$

Now we can apply Proposition 4.12, which implies that there is $S \subseteq G$ such that the coned-off Cayley graph $\hat{\Gamma}(G, H, S)$ and Γ' are isomorphic G-graphs. Hence, $\hat{\Gamma}(G, H, S)$ is a (G, H)-graph. □

Chapter 5

The action of $SL_2(\mathbb{Z})$ on the Farey Graph

All the Figures in this chapter are taken from [11].

$SL_2(\mathbb{Z})$ is an example of a relatively hyperbolic group with respect to the cyclic subgroup generated by $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$. This can be seen by the Bowditch's definition as $SL_2(\mathbb{Z})$ acts on the Farey graph the same way as defined in Definition 3.3. In this chapter, first we study the structure of the Farey graph with the help of the book [11] by Allen Hatcher, and then define the action of $SL_2(\mathbb{Z})$ on it.

This chapter has two sections: the first one briefly discusses the structure of the Farey graph, in particular this graph is connected, hyperbolic and fine. The second section explains the action of $SL_2(\mathbb{Z})$ on the Farey graph, by which we can conclude that $SL_2(\mathbb{Z})$ is a relatively hyperbolic group.

5.1 Farey Graph and its properties

The Farey graph Γ is a two-dimensional pictorial representation of rational numbers with some interesting relations between them. The vertex set is $\{\mathbb{Q} \cup \{\infty\}\}$, and we regard each vertex as either a reduced fraction $\frac{p}{q}$ or a 2-vector $\begin{bmatrix} p \\ q \end{bmatrix}$, where p and q are relatively prime. In particular, ∞ is represented by $\frac{1}{0}$ or $\frac{-1}{0}$ and 0 is represented by $\frac{0}{1}$ and $\frac{0}{-1}$. While the edge set can be defined in two equivalent ways:

1. $E(\Gamma) = \{\{\frac{p}{q}, \frac{r}{s}\} \mid det \begin{bmatrix} p & r \\ q & s \end{bmatrix} = \pm 1\}$. We call it $E^{(1)}$.

2. **Mediant Rule:** If $\frac{a}{b}$ and $\frac{c}{d}$ are adjacent then $\frac{a+c}{b+d}$ is adjacent to both of them and $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. So if we want to construct the Farey graph by this rule, we can start with the edge $\{\frac{0}{1}, \frac{1}{0}\}$. This will give the upper half of Figure 2.1, and for the lower half, we can use $\frac{-1}{0}$ instead of $\frac{1}{0}$.
   Take $E_0(\Gamma) = \{\{\frac{0}{1}, \frac{1}{0}\}\}$
   then, $E_1(\Gamma) = \{\{\frac{p}{q}, \frac{r}{s}\} | \frac{r}{s} \in E_0, \frac{u}{v} \in E_0, p = r + u, q = s + v\} = \{\{\frac{0}{1}, \frac{1}{1}\}, \{\frac{1}{0}, \frac{1}{1}\}\}$.
   In general, $E_{n+1}(\Gamma) = \{\{\frac{p}{q}, \frac{r}{s}\} \mid \{\frac{r}{s}, \frac{u}{v}\} \in E_n, p = r + u, q = s + v\}$,
   and let $E(\Gamma) = \bigcup E_i(\Gamma)$. We call it $E^{(2)}$.

Let F^i be the graph obtained using E^i, for $i \in \{1, 2\}$. We will prove that F^1 and F^2 are the same graphs, which we call the Farey graph.

Lemma 5.1. *[11] The Farey graph F^2 is connected.*

Sketch of the proof. We will understand the proof with an example. Pick a rational number; let's say $\frac{67}{24}$. Now we will apply the Euclidean algorithm as follows:

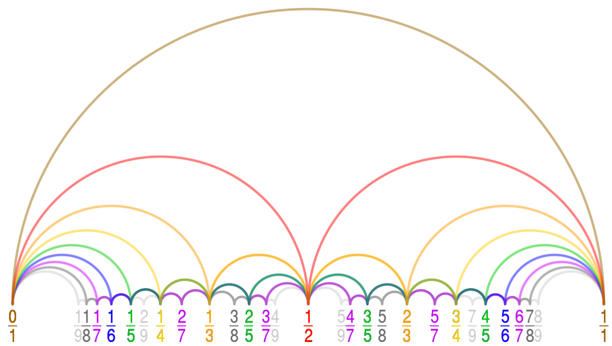

Figure 5.1: Interval [0,1] on the Farey Graph

$$\frac{67}{24} = 2 + \frac{19}{24} = 2 + \frac{1}{\frac{24}{19}} = 2 + \frac{1}{1 + \frac{5}{19}} = 2 + \frac{1}{1 + \frac{1}{\frac{19}{5}}} = 2 + \frac{1}{1 + \frac{1}{3 + \frac{4}{5}}}$$
$$= 2 + \frac{1}{1 + \frac{1}{3 + \frac{1}{1 + \frac{1}{4}}}}$$

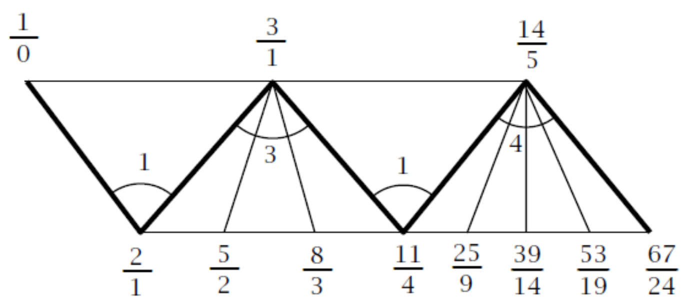

Figure 5.2: Euclidean algorithm

Note that it has 2, 1, 3, 1, 4 as a sequence of partial quotients. We use the last four partial quotients to build a strip of four large triangles, which we can also denote it as fans, subdivided into 1, 3, 1 and 4 smaller triangles, respectively. As in Figure 5.2, we begin labelling the vertices of the strip. Note that the horizontal edges do not play any role in the construction. We always start with $\frac{1}{0}$ for every positive rational number, otherwise with $\frac{-1}{0}$. Now, the adjoint vertex will be the first partial quotient of the sequence, which is $2 = \frac{2}{1}$. Now we can use the mediant rule to compute the remaining vertices in succession as we move from left to right, as shown in Figure 5.2. This sequence of vertices is a path from $\frac{1}{0}$ to $\frac{67}{24}$. As we can find a path from $\frac{1}{0}$ to any rational number, this concludes that the Farey graph is connected. $\square$

Lemma 5.2. *$E^{(2)}$ is a subset of $E^{(1)}$.*

Proof. We argue by induction that $E_i \subseteq E^{(1)}$. Note that $E_0(\Gamma) = \{\{\frac{0}{1}, \frac{1}{0}\}\}$ is a subset of $E^{(1)}$ by definition. By the induction hypothesis, suppose $E_n(\Gamma)$ is contained in $E^{(1)}$. Let $\{\frac{a}{b}, \frac{c}{d}\}$ be an element of $E_n(\Gamma)$, and consider the corresponding elements $\{\frac{a}{b}, \frac{a+c}{b+d}\}$ and $\{\frac{a+c}{b+d}, \frac{c}{d}\}$ of E_{n+1}. See Figure 5.3. Since $\{\frac{a}{b}, \frac{c}{d}\}$ has determinant ± 1, by simple arithmetic, we can compute corresponding determinants and verify that they are ± 1. Hence, E_{n+1} is contained in $E^{(1)}$. Since $E^{(2)} = \bigcup E_i$ we conclude that, $E^{(2)} \subseteq E^{(1)}$.

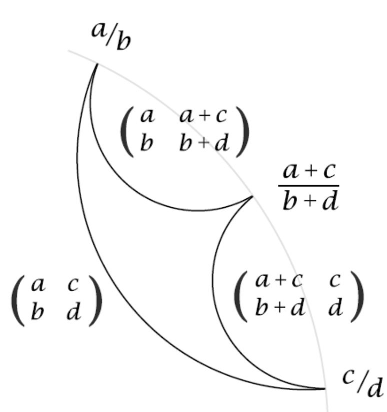

Figure 5.3: Induction on triangles

$\square$

Lemma 5.3. *The Farey graph F^1 is connected.*

Proof. Since $E^{(2)}$ is a subset of $E^{(1)}$, a path in F^2 is a path in F^1. Hence, as F^2 is connected, F^1 is also connected. □

Proposition 5.4. *[11, Theorem 2.2] The two definitions of the edge set are equivalent, and the Farey graph is connected.*

The only thing to prove here is that $E^{(1)} \subseteq E^{(2)}$ which is a consequence of the following lemma. We refer the reader to [11, Page 22] for an argument:

Lemma 5.5. *[11, Lemma 2.4] Suppose a and b are integers with no common divisor greater than 1. If one solution of $ay - bx = n$ is $(x, y) = (c, d)$, then the general solution is $(x, y) = (c + ka, d + kb)$ for k an arbitrary integer.*

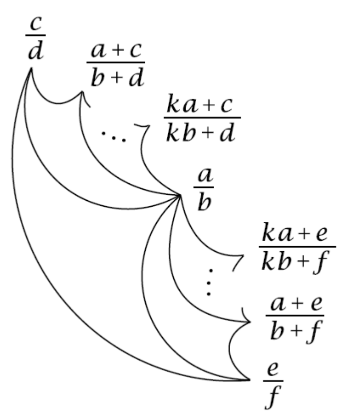

Figure 5.4: Mediant rule

Proposition 5.6. *The Farey graph is a fine graph.*

Sketch of the proof. Using the median rule one can show that there is a natural embedding of Farey graph into the plane; each vertex q maps to $(q, 0)$ and each edge $\{\frac{a}{b}, \frac{c}{d}\}$ maps to the upper semicircle orthogonal to the x-axis with endpoints $(\frac{a}{b}, 0)$ and $(\frac{c}{d}, 0)$. Figure 5.1 illustrates this embedding after composing with the stereographic

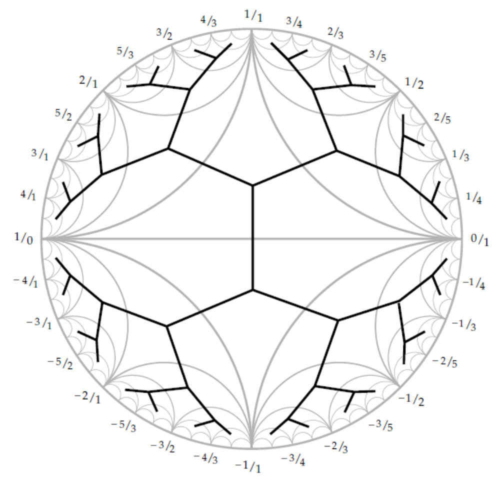

Figure 5.5: Farey tree

projection. In this case, all vertices of the Farey graph are on the unit circle which we refer as the *boundary*. Hence the Farey graph is a planar graph [11], and there are well define subregions that we call triangles. We consider the graph T, whose vertices are triangles of the Farey graph, as shown in Figure 5.5, and two triangles are adjacent if they share a common side. Since the Farey graph is connected, T is connected. One can show that T is a tree by observing that the triangles have all their vertices on the boundary. Since all the vertices of T have degree equal to 3, T is locally finite. First observe that a circuit in the Farey graph encloses a region which is tiled by triangles of the Farey graph. This region induces a connected subgraph of T, not that connectedness follows from the assumption that the boundary of the region is an embedded circle. Note that two distinct circuits induce different subtrees of T, see Figure 5.6. Let n be an arbitrary positive integer and let e be an arbitrary edge in the Farey graph. Note that this edge corresponds to an edge in T. A circuit in the Farey graph that contains e of length at most n induces a finite sub-tree Δ of T having the following properties:

1. e has exactly one vertex of degree 1 in Δ (as no vertex is repeated in a circuit).

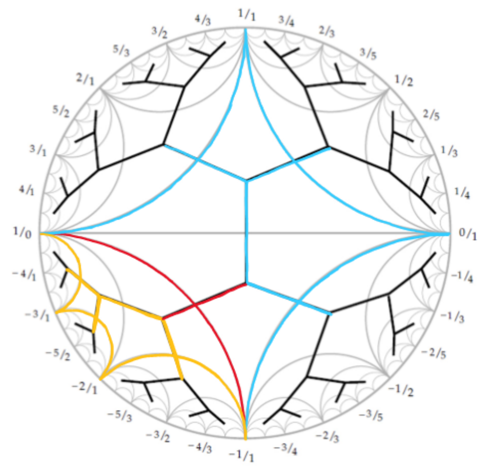

Figure 5.6: Subtrees induced by circuits

2. The degree of all vertices in Δ is either 1 or 3.

3. Δ has at most n vertices of degree 1.

4. Δ is connected.

Since T is locally finite, there are finitely many possible such subtrees Δ for the given edge e; in other words, there are finitely many circuits containing e. Hence, it is a fine graph. $\square$

Proposition 5.7. *The Farey graph is δ-Hyperbolic, where $\delta = 1$.*

Proof. Let u and v be two vertices of the graph, and let p and q be two geodesics from u to v. Suppose that the only vertices in common between p and q are u and v. Let D be the planar region enclosed by p and q. Observe that this region has no interior vertices but only edges. No vertex of p has degree 2 in D other than u and v. Indeed, If we pick a vertex t of p, that has degree two, then there will be an edge between the vertices adjacent to it, that will contradict the geodesy of p. Since every

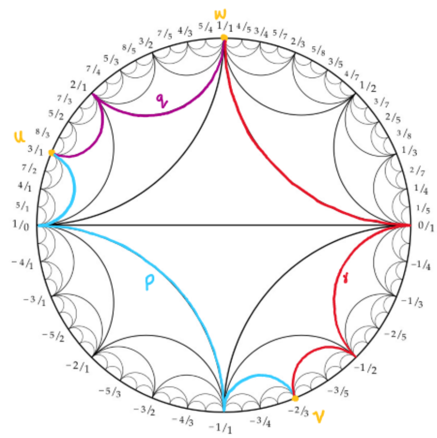

Figure 5.7: 1-thin triangle

vertex has a degree at least three, the third edge would be joining with one of the vertices of q, otherwise it would again contradict the geodesy of p. Therefore, every vertex of p is at distance one from a vertex of q.

Consider a geodesic triangle Δ with vertices u, v and w. Let p, q and r be geodesics between u and v, u and w, and v and w, respectively. See Figure 5.7. Suppose that the only vertices in common between p and q is u; between p and r is v; between q and r is w. Let D be the region enclosed by p, q and r. Recall that this region has no interior vertices. Suppose the vertices of p are $u = p_0, p_1, ..., p_k = v$. Since D has no interior vertices, every vertex of p is adjacent to a vertex of q or r. Note that if p_i is adjacent to a vertex of q, then p_j is adjacent to a vertex of q for every $j \leq i$. Let m be the largest index such that p_m is adjacent to a vertex of q. Then every p_i for $i \leq m$ is at a distance one from a vertex of q. As no edge can intersect each other, by symmetry, every vertex of p is at a distance one from a vertex of q or r. Hence, the triangle is δ-thin, where $\delta = 1$. □

5.2 $SL_2(\mathbb{Z})$ acts on the Farey Graph

Let $\mu = \begin{bmatrix} p & r \\ q & s \end{bmatrix} \in SL_2(\mathbb{Z})$. Let $d = \frac{a}{b} \in \mathbb{Q} \cup \{\infty\}$. Then:

1. $\mu.d = \begin{bmatrix} p & r \\ q & s \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} pa + rb \\ qa + sb \end{bmatrix} = \begin{bmatrix} p\frac{a}{b} + r \\ q\frac{a}{b} + s \end{bmatrix} \in \mathbb{Q} \cup \{\infty\}$
2. $\mu : d \longmapsto \frac{pd+r}{qd+s}$. As μ is invertible, it is an element of $Aut(\mathbb{Q} \cup \{\infty\})$.

Proposition 5.8. *The action of $SL_2(\mathbb{Z})$ on the Farey Graph has the following properties:*

1. *It acts cocompactly,*
2. *The stabilizer subgroups of edges are finite,*
3. *The stabilizer subgroups of vertices are conjugates of $\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle$.*

Proof. 1. Note that every vertex is connected with $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as shown in the proof of Proposition 5.4. For any vertex $\begin{bmatrix} a \\ b \end{bmatrix} \in V(\Gamma)$, $\exists \begin{bmatrix} a & p \\ b & r \end{bmatrix} \in SL_2(\mathbb{Z})$, such that

$$\begin{bmatrix} a & p \\ b & r \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Hence, there is only one orbit of vertices.

Also, the product of two non-singular matrices is non-singular: every edge can be translated to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ by multiplying the edge with its inverse. Hence, there is only one orbit of edges.

2. Let $d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in E(\Gamma)$. Note that every edge in the Farey graph is a translate of d. Also, infinity can be represented as $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Let $\mu \in SL_2(\mathbb{Z})$ be an element of the stabilizer of d such that:

$$\mu\left(\pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\mu\left(\pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \pm \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\mu\left(\pm \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or}$$

$$\mu\left(\pm \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \pm \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This implies, $\mu \in \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$. Hence, the stabilizer subgroup of edges is finite.

3. As all the vertices are the translates of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the stabilizer subgroup of any

vertex would be a conjugate of the stabilizer subgroup of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Note that:

$$\pm \begin{bmatrix} 1 & n \\ 0 & m \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \pm \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \forall n, m \in \mathbb{Z}.$$

Hence, the stabilizer subgroups of vertices are conjugates of $\left\langle \pm \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle$.

$\square$

Now we can conclude the main result of this chapter.

Proposition 5.9. *$SL_2(\mathbb{Z})$ is a relatively hyperbolic group with respect to its subgroup $H = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle$.*

Proof. We will show that the action of $SL_2(\mathbb{Z})$ on the Farey graph satisfies definition of relatively hyperbolic group, recall Definition 3.3 by Bowditch. In Section 5.1, we proved that the Farey graph is connected, hyperbolic and fine. In Section 5.2, we proved that the action of $SL_2(\mathbb{Z})$ on the Farey graph is cocompact, the stabilizer subgroups of edges are finite, the stabilizer subgroups of vertices are conjugates of H. And note that the G-stabilizer of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is H. Hence, the Farey graph is a cocompact $(SL_2(\mathbb{Z}), H)$-graph. $\square$

Chapter 6

Conclusion

The notion of hyperbolically embedded subgroup generalizes the notion of relatively hyperbolic group. Just like relatively hyperbolic groups, hyperbolically embedded subgroups can also be defined in terms of fine graphs: An infinite subgroup H is hyperbolically embedded in a group G for some subset S of G if and only if there exists a (G, H)-graph. A graph Γ is a (G, H)-graph if G acts on Γ while satisfying the following conditions:

- Γ is connected and hyperbolic,
- Γ is fine at $V_\infty(\Gamma)$,
- for all $w \in V_\infty(\Gamma)$, G_w acts on $T_w\Gamma$ with finitely many orbits,
- number of orbits of vertices are finite,
- stabilizer subgroups of edges are finite,
- stabilizer subgroups of vertices are either finite or a conjugate of H,
- and there exists a vertex that has the stabilizer subgroup equals to H.

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