# Complete group classification of shallow water equations

by

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# Abstract

The thesis is devoted to the symmetry analysis of the shallow water equations with variable bottom topography in dimensions one and two. We find the generalized equivalence groups for the classes of one- and two-dimensional shallow water equations with variable bottom topography using the automorphism-based version of the algebraic method. It turns out that for both the classes, the generalized equivalence groups coincide with the corresponding usual equivalence groups. In the case of dimension one, we also compute the generalized equivalence group of the natural reparametrization of the class and then compare the computation with the analogous calculations for the original class. Specific attention is paid to the class of two-dimensional shallow water equations with variable bottom topography. We carry out the complete group classification for this class up to its equivalence transformations using the modern method of furcate splitting. This class is neither normalized in the usual sense nor in the generalized sense. In other words, it possesses admissible transformations that are not induced by elements of the equivalence group, and such admissible transformations establish additional equivalences among classification cases. For a number of pairs of classification cases we either prove that these cases are not equivalent to each other with respect to point transformations or indicate the associated point transformations for pairs of equivalent cases.

# Acknowledgements

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# Statement of contributions

The general topic of research was suggested by my supervisor Dr. Alexander Bihlo. The subject of research was specified in the course of my studies, and Chapters 3 and 4 of the thesis contain only original results. The results in Chapter 3 were derived by myself independently, except for Section 3.3 which was developed in the course of discussion with Dr. Bihlo and Professor Popovych. The main results of Chapter 4, which are Theorems 34–36, were also derived by myself independently. Section 4.4 was written in collaboration with Dr. Bihlo and Professor Popovych.

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# Chapter 1

# Introduction

Differential equations are widely used to model real world phenomena. They are invaluable tools in a variety of disciplines such as, for example, physics, engineering, chemistry, biology and economics. Consequently, there are many techniques for obtaining exact solutions of differential equations. The symmetry methods are one of the most powerful ones, because they can be applied to differential equations of arbitrary type. Classical examples of symmetries are translations, rotations and scalings. The symmetry group of a system of differential equations consists of the point transformations of independent and dependent variables of the system, which transform solutions of the system to solutions of the same system.

The famous Norwegian mathematician Sophus Lie noticed that different methods for solving unrelated types of ordinary differential equations have a common feature: they use symmetries of these differential equations. This observation helped him to develop a new theory for integrating ordinary differential equations. Lie showed that a known one-parameter symmetry group of a given ordinary differential equation, which consists of locally defined parameterized point transformations, allows one to reduce the order of the equation by one [34]. In general, if an ordinary differential equation of order $n$ is invariant with respect to an $r$-parameter solvable Lie group, then it can be reduced to an ordinary differential equation of order $n-r$. The application of Lie theory to differential equations is well explained in [26, 38, 39].

Given a system of differential equations there are various methods for computing its complete point symmetry group. One of them is the direct method [7, 30–32, 47] in which the point symmetries are computed directly from the condition of invariance of the system under the action of point transformations. In general, this leads to a cumbersome nonlinear system of partial differential equations for the components of point symmetry transformations, which is typically difficult to solve. To simplify the computations, one can consider the infinitesimal counterparts of the invariance condition only. This results in the associated Lie algebra of infinitesimal generators of one-parameter point symmetry groups of the system. The main drawback of the infinitesimal method is that it misses discrete symmetry transformations. To obtain the complete point symmetry group, which consists of both discrete and continuous symmetry transformations, one can also use the so-called algebraic method [4,6]. This method was suggested by Hydon and is based on the idea that each point symmetry transformation of a system of differential equations induces an automorphism of the corresponding Lie algebra [24–26]. It leads to restrictions on the form of point symmetry transformations before invoking the direct method and thus it significantly simplifies the further computations within the framework of the direct method. The shortcoming of this method is that for an infinite-dimensional Lie algebra, the computation of the entire automorphism group can be impossible. The refinement of the algebraic method that is based on the notion of megaideals was proposed in [7].

In practice, one often has to deal with a system of differential equations that is parameterized by arbitrary functions or constants and thus is a class of systems of differential equations rather than a single system. The above arbitrary functions and constants are commonly referred to as the arbitrary elements of the class. The problem of group classification for the class is the classification of Lie symmetry properties of systems from this class depending on values of the arbitrary elements. The group classification allows one to choose the most relevant system from the class using symmetry as a selection criterion. The point transformations that map each system from the class to a system from the same class are called equivalence transformations of the class. They play a central role in group classification since they can be used to choose the simplest representative among similar systems. For this reason, there is an extension of the algebraic method for finding the complete point symmetry group of a system of differential equations to computing the complete equivalence group of a class of such systems [4]. Group classification is a quite complicated problem since it usually requires solving an overdetermined system of partial differential equations with respect to the arbitrary elements and the components of infinitesimal symmetry generators simultaneously.

In the thesis, we consider the classes of one- and two-dimensional shallow water equations with variable bottom topography. These classes have been studied earlier in [1, 13–16, 18, 27, 33, 48, 49, 52]. The shallow water equations describe a thin layer of fluid of constant density in hydrostatic balance, bounded by the rigid bottom and by the free surface. As an example of real world application, the shallow water equations can be used to model the propagation of a tsunami as long as the wave is far from the shore. This topic was studied in [11, 22, 53].

The main goal of the thesis is the group classification of the class of two-dimensional shallow water equations with variable bottom topography up to equivalence transformations of this class. The one-dimensional case was considered within the framework of symmetry analysis in [2,3], where Lie symmetry, zero order conservation laws were found without involving the corresponding equivalence transformations. In the present work we compute the generalized equivalence group of the class of one-dimensional shallow water equations, which can be used for rewriting the previously obtained results from [2] in a simplified form. The group classification problem for the two-dimensional shallow water equations is solved in two steps. Firstly, we compute the generalized equivalence group of this class. Secondly, we classify, up to equivalence transformations, Lie symmetries of systems of the class depending on values of the arbitrary element, which is the bottom topography.

The structure of this thesis is as follows.

In Chapter 2 we present some theoretical results on Lie groups and Lie algebras (Section 2.1) and symmetries of differential equations (Section 2.2) following the classical textbooks [39, 42, 43]. We describe the algebraic method of computing the complete point symmetry group (Section 2.3) and explain the notion of equivalence group (Section 2.4).

Chapter 3 is devoted to the construction of the generalized equivalence group of the class of one-dimensional shallow water equations with variable bottom topography. In Section 3.1, we find this group using the algebraic method. We repeat the computations for the naturally re-parameterized class in Section 3.2 and carry out the comparison of these two approaches in Section 3.3.

In Chapter 4 we carry out the group classification of the class of two-dimensional shallow water equations with variable bottom topography. The generalized equivalence group of this class is computed in Section 4.1. Preliminary analysis of Lie symmetries of systems from the class and a complete list of inequivalent Lie symmetry extensions are presented in Section 4.2. In Section 4.3, we provide the proof of the classification results. Section 4.4 is devoted to the computation of additional equivalence transformations among listed classification cases for the class of two-dimensional shallow water equations.

In the last chapter we discuss the results of the thesis and outline possible directions of further study within the symmetry analysis of shallow water equations.

# Chapter 2

# Theoretical background

## 2.1 General notions of Lie theory

To begin with we will introduce the notion of a Lie group, which plays a key role in the symmetry analysis of differential equations. In this section we closely follow [39].

Let us assume that the reader is familiar with basic concepts of differential geometry and basic notions of group theory. In the following $M$ denotes a $C^\infty$ manifold, and $TM|_x$ is the tangent space to $M$ at $x \in M$.

**Definition 1.** A Lie group, $G$, is a group endowed with the structure of a $C^\infty$ manifold such that the inversion map and multiplication map

$$s\colon G \to G, \quad g \mapsto g^{-1},$$
$$m\colon G \times G \to G, \quad (g,h) \mapsto gh$$

are smooth.

**Remark.** It suffices to require that the single map $(g,h) \mapsto gh^{-1}$ is smooth. For example, the inversion map is smooth because it is the restriction of this map to $\{e\} \times G$, where $e$ is the identity element.

**Definition 2.** Suppose $G$ is a Lie group and $M$ is a $C^\infty$ manifold. An action of $G$ on $M$ is a map $G \times M \to M$ such that

$$g \cdot (h \cdot l) = (gh) \cdot l \quad \text{for all} \quad g, h \in G,\ l \in M.$$

The symmetry group of a system of differential equations is a particular case of a Lie group. In practical applications one often deals with local Lie groups, which contain only elements sufficiently close to the identity element, instead of global ones. The advantage of local Lie groups is that one can define all operations in local coordinates only, and does not have to resort to the theory of manifolds.

**Definition 3.** (see [39]) An $s$-parameter local Lie group consists of connected open subsets $V_0 \subset V \subset \mathbb{R}^s$ containing the origin 0, and smooth maps

$$m\colon\ V \times V \to \mathbb{R}^s,$$
$$i\colon\ V_0 \to V,$$

defining the group multiplication and inversion, respectively, with the following properties.

- Associativity.
  $\forall x, y, z, \in V, \quad m(x, y),\, m(y, z) \in V, \quad m(x, m(y, z)) = m(m(x, y), z).$
- Identity element.
  $\forall x \in V, \quad m(0, x) = x = m(x, 0).$
- Inverse element.
  $\forall x \in V_0, \quad m(x, i(x)) = 0 = m(i(x), x).$

To simplify the further work we define the notion of Lie algebras. It allows one to study the non-linear structure of a Lie group by the linear structure of its Lie algebra, i.e. to switch from differential geometry to linear algebra. This makes the computation of symmetries of differential equations more algorithmic.

The main tool in the theory of Lie groups and transformation groups are so-called "infinitesimal transformations". Recall that one can define the tangent vector $\mathbf{v}|_x$ at the point $x$, via derivations on the space $C^\infty(M)$, which is the space of smooth functions on $M$. A derivation satisfies both the linearity property and the Leibniz rule evaluated at the point $x$:

$$\mathbf{v}(f + g) = \mathbf{v}(f) + \mathbf{v}(g), \quad \mathbf{v}(f \cdot g)\mathbf{v}(f) \cdot g + \mathbf{v}(g) \cdot f, \quad \forall f, g, \in C^\infty(M),$$

and applying it to a smooth function results in a real number $\mathbf{v}(f)(x)$ at $x$. This way of defining tangent vectors is entirely coordinate-free. A vector field on $M$ is a derivation of $C^\infty(M)$, however the following definition is more convenient.

**Definition 4.** (see [39]) A vector field $\mathbf{v}$ on a smooth $n$-dimensional manifold $M$ assigns a tangent vector $\mathbf{v}|_x \in TM|_x$ to each point $x \in M$, with $\mathbf{v}|_x$ varying smoothly from point to point. In local coordinates $(x_1, \ldots, x_n)$, a vector field has the form

$$\mathbf{v}|_x = \xi^1(x)\partial_{x_1} + \xi^2(x)\partial_{x_2} + \cdots + \xi^n(x)\partial_{x_n},$$

where each $\xi^i(x)$ is a smooth function of $x$, and $\partial_{x_i}$ stands for $\frac{\partial}{\partial x_i}$.

**Definition 5.** An integral curve of a vector field $\mathbf{v}$ is a smooth curve parameterized by $x = \phi(\varepsilon)$, such that at any point the tangent vector to that curve at this point coincides with the value of $\mathbf{v}$ at the same point: $\dot{\phi}(\varepsilon) = \mathbf{v}|_\varepsilon$ for all $\varepsilon$.

Given a smooth vector field $\mathbf{v}$ and a point $x \in M$, there exists a neighborhood $M' \subset M$ containing $x$ and a smooth map $\phi\colon (-\varepsilon, \varepsilon) \times M' \to M$ such that the curve $\phi(t, x)$ is the unique integral curve passing through $x$ with

$$\frac{\mathrm{d}\phi(t,x)}{\mathrm{d}t} = \mathbf{v}(\phi(t,x)), \quad \phi(0,x) = x. \tag{2.1}$$

This curve is called the flow generated by the vector field $\mathbf{v}$.

For each vector field $\mathbf{v}$ we define the flow generated by $\mathbf{v}$, denoted by $\phi(\varepsilon, x)$, to be the parameterized maximal integral curve passing through $x$ in $M$. Note that the flow of a vector field has the properties that

$$\phi(\delta, \phi(\varepsilon, x)) = \phi(\delta + \varepsilon, x), \quad x \in M,$$

for all $\delta, \varepsilon \in \mathbb{R}$ such that the involved expressions are defined,

$$\phi(0, x) = x,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\phi(\varepsilon, x) = \mathbf{v}|_{\phi(\varepsilon,x)}.$$

Thus, the flow generated by a vector field coincides with the local group action of the Lie group $\mathbb{R}$ on the manifold $M$, which is called a one-parameter group of transformations. We say that the vector field $\mathbf{v}$ is the infinitesimal generator of the action. The computation of the flow or one-parameter group generated by a given vector field $\mathbf{v}$ is equivalent to solving system (2.1) of ordinary differential equations and is also called the exponentiation of this vector field, and indicated as

$$\phi(\varepsilon, x) = \exp(\varepsilon \mathbf{v})x.$$

This notation is justified because of the properties of the exponential function.

Suppose we are given two vector fields $\mathbf{v}$ and $\mathbf{w}$ on $M$, then we define their Lie bracket $[\mathbf{v}, \mathbf{w}]$ to be the unique vector field satisfying

$$[\mathbf{v}, \mathbf{w}](f) = \mathbf{v}(\mathbf{w}(f)) - \mathbf{w}(\mathbf{v}(f)),$$

for all smooth functions $f \colon M \to \mathbb{R}$.

Now we can define the notion of Lie algebras.

**Definition 6.** A Lie algebra is a vector space $\mathfrak{g}$ endowed with a bilinear operation

$$[\cdot,\cdot] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

called the Lie bracket (or the commutator), such that the following properties are satisfied:

1. For all $\mathbf{v}, \mathbf{w} \in \mathfrak{g}$ the Lie bracket is skew-symmetric, i.e. $[\mathbf{v}, \mathbf{w}] = -[\mathbf{w}, \mathbf{v}]$.
2. For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathfrak{g}$ the Jacobi identity is satisfied, i.e.

$$[\mathbf{u}, [\mathbf{v}, \mathbf{w}]] + [\mathbf{w}, [\mathbf{u}, \mathbf{v}]] + [\mathbf{v}, [\mathbf{w}, \mathbf{u}]] = 0.$$

A vector subspace $\mathfrak{h} \subset \mathfrak{g}$ that is closed under the Lie bracket is called a subalgebra of the Lie algebra $\mathfrak{g}$.

We need to present a few more definitions to show the relation between a given Lie group $G$ and its associated Lie algebra. The right multiplication map $R_g \colon G \to G$, $R_g(h) = hg$, is a Lie group diffeomorphism with inverse $R_{g^{-1}} = (R_g)^{-1}$.

**Definition 7.** A vector field on $G$ is said to be right-invariant if $\mathrm{d}R_g(\mathbf{v}|_h) = \mathbf{v}|_{R_g(h)} = \mathbf{v}|_{hg}$, for all $g, h \in G$.

**Remark.** The set of right-invariant vector fields forms a vector space.

**Definition 8.** The Lie algebra $\mathfrak{g}$ of the Lie group $G$ is the vector space of all right-invariant vector fields on $G$.

It is important to note that a right-invariant vector field is uniquely determined by its value at the identity transformation. The converse statement holds as well. Besides, the commutator of two right-invariant vector fields is also right-invariant. Therefore, right multiplication respects the Lie bracket and the vector space of right-invariant vector fields $\mathfrak{g}$ indeed has the structure of a Lie algebra.

**Theorem 9.** *(see [50])*
*Given a Lie group $G$ and the associated Lie algebra $\mathfrak{g}$, for each $s \leq \dim G$ there is a one-to-one correspondence between $s$-parameter connected subgroups of $G$ and the $s$-dimensional subalgebras of $\mathfrak{g}$.*

This theorem generalizes the correspondence between Lie subalgebras and Lie subgroups. Suppose we are given the Lie algebra $\mathfrak{g}$ of the Lie group $G$. There is a one-to-one correspondence between one-parameter subgroups of $G$ and one-dimensional subspaces of $\mathfrak{g}$. Each one-dimensional subspace of $\mathfrak{g}$ generates a one-parameter subgroup of the associated one-parameter Lie group $G$ via the exponentiation procedure.

## 2.2 Symmetries of differential equations

In this section we will introduce the basic theory that is necessary for understanding the algebraic method for finding the complete point symmetry group of a system of differential equations. The main references in this section are [26, 39].

Consider a system $\mathcal{L}$ of differential equations for $m$ unknown functions $u = (u^1, \dots, u^m)$ of $n$ independent variables $x = (x_1, \dots, x_n)$. The solutions of the system will be of the form $u = f(x)$, i.e. $u^i = f^i(x_1, \dots, x_n)$, $i = 1, \dots, m$. Let $X = \mathbb{R}^n$, with coordinates $x = (x_1, \dots, x_n)$, be the space representing the independent variables, and let $U = \mathbb{R}^m$, with the coordinates $u = (u^1, \dots, u^m)$, represent the dependent variables, and $\Omega \subset X$ is the domain of $f$.

The explanation of how a given transformation $g$ in the Lie group $G$ transforms a function $u = f(x)$ is essential here.

To begin with one can identify the function $u = f(x)$ with its graph

$$\Gamma_f = \{(x, f(x)) : x \in \Omega\} \subset X \times U.$$

The graph $\Gamma_f$ is an $n$-dimensional submanifold of $X \times U$. If the graph is in the domain of definition of the group transformation $g$, then $\Gamma_f$ transformed by $g$ is

$$g \cdot \Gamma_f = \{(\tilde{x}, \tilde{u}) = g \cdot (x, u) : (x, u) \in \Gamma_f\}.$$

For elements $g$ near the identity, the transformation $\Gamma_{\tilde{f}} = g \cdot \Gamma_f$ is the graph of some single-valued smooth function $\tilde{u} = \tilde{f}(x)$, since the identity element of $G$ leaves $\Gamma_f$ unchanged and $G$ acts smoothly. In practice, Lie groups arise as groups of transformations of some manifold $M$. The group transformations may neither be defined for all elements of the group nor for all points on the manifold, i.e. the group acts only locally.

**Definition 10.** Let $M$ be a smooth manifold. A (local) Lie group $G$ is a local group of transformations if $\mathcal{U}$ is an open subset such that

$$\{e\} \times M \subset \mathcal{U} \subset G \times M,$$

which is the domain of definition of the group action, and a smooth map $\Psi \colon \mathcal{U} \to M$, which satisfies the following properties:

- $\Psi(g, \Psi(h, x)) = \Psi(gh, x)$ for all $(h, x)$, $(g, \Psi(h, x))$, and $(gh, x) \in \mathcal{U}$;
- $\Psi(e, x) = x$, for all $x \in M$;
- If $(g, x) \in \mathcal{U}$, then $(g^{-1}, \Psi(g, x)) \in \mathcal{U}$ and $\Psi(g^{-1}, \Psi(g, x)) = x$,

where $e$ is the identity transformation in $G$.

In the following definition the word "solution" means any smooth solution of the system defined on any subdomain $\Omega \subset X$.

**Definition 11.** A symmetry group of the system $\mathcal{L}$ is a local group of transformations $G$ acting on an open subset $M$ of the space of independent and dependent variables for the system with the property that whenever $u = f(x)$ is a solution of $\mathcal{L}$, and whenever $g \cdot f$ is defined for $g \in G$, then $u = g \cdot f(x)$ is a solution of $\mathcal{L}$.

It is necessary to extend the base space $X \times U$ to introduce the geometric formulation of the derivatives of $u$. The resulting space is called the jet space and it includes both the independent and the dependent variables, as well as all the derivatives of $u$ with respect to $x$ up to some fixed order. To rigorously define the notion of a jet space we need to introduce the following, first.

Let $f(x) = f(x_1, \ldots, x_n)$ be a smooth real-valued function of $n$ independent variables. There exist $n_k$ different $k$-th order partial derivatives of $f$, where

$$n_k = \begin{pmatrix} n+k-1 \\ k \end{pmatrix}.$$

Here and in the following we will use the notation $J = (j_1, \ldots, j_k)$ for an unordered $k$-tuple of integers and

$$\partial_J f(x) = \frac{\partial^k f(x)}{\partial x_{j_1} \partial x_{j_2} \cdots \partial x_{j_k}}$$

for the corresponding derivative of order $k$. Let us define an Euclidean $m^{(r)}$ dimensional space $U^{(r)}$, with its coordinates being all possible partial derivatives of $u$ of orders from 0 (just $u$) to $r$. We denote $m^{(r)}$ to be the total number of partial derivatives of all orders from 0 to $r$, i.e.

$$m^{(r)} := m\binom{n+r}{r}.$$

Since for a given smooth function $f \colon X \to U$, where $X = \mathbb{R}^n$ and $U = \mathbb{R}^m$ there exist $mn_k$ different $k$-th order partial derivatives $u^i_J = \partial_J f^i(x)$ of components of $f$ at a given point $x$, the above can be defined. To be more precise:

**Definition 12.** The total space $X \times U^{(r)}$ is called the $r$-th order jet space of the underlying space $X \times U$ if its coordinates represent the independent variables, the dependent variables and the derivatives of the dependent variables up to order $r$.

Note that some differential equations are not defined on the whole space $X \times U$ but on some open subsets of the underlying space. Having introduced the notion of jet spaces, we have to adopt the notion of the solution of the system to it. In the following we will discuss how the group action $G$ acting on $X \times U$ can be extended to a group action acting on $X \times U^{(r)}$. Given a smooth function $u = f(x)$, $\Gamma_f \subset X \times U$ we define its $r$-th prolongation

$$u^{(r)} = \mathrm{pr}^{(r)} f(x),$$

where $\mathrm{pr}^{(r)} f(x)$ is the $r$-jet of $f$, also known as the prolongation of $f$. It is defined by the equations

$$u^i_J = \partial_J f^i(x), \quad i = 1, \ldots, m.$$

Hence, the prolongation of $f$ is a function from $X$ to $U^{(r)}$ and for each $x \in X$, the function $\mathrm{pr}^{(r)} f(x)$ is a vector with entries representing the values of $f$ and all its derivatives up to order $r$ at the point $x$. Now, when we have the prolongation of the group transformation on the $r$-th jet space, we can consider the infinitesimal action of the prolonged group action by differentiating the group transformation with respect to the group parameter at the identity element.

**Definition 13.** Let $M \subset X \times U$ be open and suppose $\mathbf{v}$ is a vector field on $M$, with corresponding one-parameter group $\exp(\varepsilon \mathbf{v})$. The $r$-th prolongation of $\mathbf{v}$, denoted $\mathrm{pr}^{(r)}[\exp(\varepsilon \mathbf{v})]$ is a vector field on the $r$-th jet space $J^{(r)}$. It is defined as the infinitesimal generator of the corresponding prolonged one-parameter group $\mathrm{pr}^{(r)}[\exp(\varepsilon \mathbf{v})]$. Suppose that the vector $\mathbf{v}$ is of the form

$$\mathbf{v} = \sum_{i=1}^{n} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{m} \phi_\alpha(x,u) \frac{\partial}{\partial u^\alpha},$$

then the prolongation of $\mathbf{v}$ is

$$\mathrm{pr}^{(r)} \mathbf{v} = \sum_{i=1}^{n} \xi^i(x,u) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \sum_{J} \phi^J_\alpha \frac{\partial}{\partial u^\alpha}. \tag{2.2}$$

The components $\phi_\alpha^J$ in (2.2) are determined as

$$\phi_\alpha^J\left(x,u^{(r)}\right) = \mathrm{D}_J\left(\phi_\alpha - \sum_{i=1}^{n}\xi^i u_i^\alpha\right) + \sum_{i=1}^{n}\xi^i u_{J,i}^\alpha, \tag{2.3}$$

where $\mathrm{D}_J = \mathrm{D}_{j_1}\cdots\mathrm{D}_{j_r}$ is the $J$-th total derivative with

$$\mathrm{D}_i P = \frac{\partial P}{\partial x_i} + \sum_{\alpha=1}^{m}\sum_J u_{J,i}^\alpha \frac{\partial P}{\partial u_J^\alpha},$$

being the total derivative with respect to $x_i$, and

$$u_{J,i}^\alpha = \frac{\partial u_J^\alpha}{\partial x_i}.$$

The following definition is necessary to explain the notion of a symmetry group of a system of differential equations.

**Definition 14.** Let

$$\Delta_\nu(x,u^{(r)}) = 0, \quad \nu = 1,\ldots,l,$$

be a system of differential equations. The system is said to be of maximal rank if its $l\times(n+mn_r)$-dimensional Jacobian matrix

$$J_\Delta(x,u^{(r)}) = \left(\frac{\partial\Delta_\nu}{\partial x^i}, \frac{\partial\Delta_\nu}{\partial u_\alpha^J}\right)$$

of $\Delta$ with respect to all the variables $(x,u^{(r)})$ is of rank $l$ whenever $\Delta(x,u^{(r)}) = 0$.

Finally, we can present the following theorem.

**Theorem 15.** *(see [39])*
*Suppose*

$$\Delta_\nu(x,u^{(r)}) = 0, \quad \nu = 1,\ldots,l,$$

*is a system of differential equations of maximal rank defined over $M\subset X\times U$. If $G$ is a local group of transformations acting on $M$ and,*

$$\mathrm{pr}^{(r)}\mathbf{v}[\Delta_\nu(x,u^{(r)})] = 0, \quad \nu = 1,\ldots,l,$$

*whenever $\Delta(x, u^{(r)}) = 0$, for every infinitesimal generator $\mathbf{v}$ of $G$, then $G$ is a symmetry group of the system.*

## 2.3 Algebraic method for computing complete point symmetry groups

A complete point symmetry group consists of both discrete and continuous point symmetry transformations. A standard way for computing continuous symmetry transformations that form a connected Lie group is the so-called infinitesimal method. It allows one to compute the continuous component of complete point symmetry groups but misses discrete symmetry transformations.

In the literature, there are two methods for computing complete point symmetry groups of systems of differential equations, the direct method [7, 30, 40, 47] and the algebraic method [7, 12, 24–26].

The most universal is the direct method, which is based on the definition of a point symmetry transformation. The application of this method results in a system of PDEs (nonlinear highly-coupled and thus difficult to solve), which are called the determining equations.

In this thesis we compute complete point symmetry groups of systems of differential equations (resp. complete equivalence groups of classes of such systems) using the algebraic method, which was suggested by Hydon [24–26] and extended to the case of infinite-dimensional point symmetry groups in [7, 12] and to the construction of complete equivalence groups in [4]. This method is simpler than the direct method because it allows one to avoid solving nonlinear systems of partial differential equations. The main idea of the algebraic method for the computation of discrete symmetries of differential equations is the following: Let $\mathfrak{g}_{\mathcal{L}}$ be the maximal Lie invariance algebra of a system of differential equations $\mathcal{L}$. To restrict the form of those point transformations that can be symmetries of $\mathcal{L}$ we use the following property: Push-forwards of vector fields defined on the corresponding space of independent and dependent variables of point symmetries of $\mathcal{L}$ induce automorphisms of $\mathfrak{g}_{\mathcal{L}}$. The above property requires the explicit computation of the entire automorphism group of $\mathfrak{g}_{\mathcal{L}}$. Then the restricted form can be substituted into the system $\mathcal{L}$ and from this point one proceeds with the direct method. It is important to note that the computation of the entire automorphism group for infinite-dimensional Lie algebras is sometimes an impossible task. Therefore, for a proper application of the described method (that is called automorphism-based version of the algebraic method) the algebra $\mathfrak{g}_{\mathcal{L}}$ has to be finite-dimensional. If this is not the case, the computation of the automorphism group $\mathrm{Aut}(\mathfrak{g}_{\mathcal{L}})$ becomes a complicated problem. For this reason the megaideal-based version of the algebraic method was proposed and well explained in [4, 7, 12].

In this section we will illustrate the algebraic method with the computation of the complete point symmetry group of the Harry Dym equation.

Consider the Harry Dym equation of the form

$$u_t = u^3 u_{xxx}. \tag{2.4}$$

First of all we have to find the maximal Lie invariance algebra $\mathfrak{g}$ of this equation by the infinitesimal invariance criterion. Equation (2.4) is of third order with two independent variables $x$ and $t$, and one dependent variable $u$, so $n = 2$ and $m = 1$ in our notation from Section 2.2. The Harry Dym equation can be identified with the linear subvariety in $X \times U^{(3)}$ determined by the vanishing of $\Delta(t, x, u^{(3)}) = u_t - u^3 u_{xxx}$.

Let

$$\mathbf{v} = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \phi(t, x, u)\partial_u$$

be a vector field on $X \times U$. We have to compute all possible coefficient functions $\tau$, $\xi$ and $\phi$ such that the corresponding one-parameter group $\exp(\varepsilon\mathbf{v})$ is a symmetry group of the Harry Dym equation. According to Theorem 15, we need to compute the third prolongation

$$\begin{aligned}\mathrm{pr}^{(3)}\mathbf{v} = \mathbf{v} &+ \phi^t\partial_{u_t} + \phi^x\partial_{u_x} + \phi^{tt}\partial_{u_{tt}} + \phi^{xt}\partial_{u_{xt}} + \phi^{xx}\partial_{u_{xx}} \\ &+ \phi^{ttt}\partial_{u_{ttt}} + \phi^{ttx}\partial_{u_{ttx}} + \phi^{txx}\partial_{u_{txx}} + \phi^{xxx}\partial_{u_{xxx}}\end{aligned}$$

of $\mathbf{v}$. Applying $\mathrm{pr}^{(3)}\mathbf{v}$ to (2.4), we find the infinitesimal invariance criterion to be

$$\phi^t = 3u^2 u_{xxx}\phi + u^3\phi^{xxx}, \tag{2.5}$$

which must be satisfied whenever $u_t - u^3 u_{xxx} = 0$.

In the criterion above $\phi^t$ and $\phi^{xxx}$ are computed from the following formulas

$$\phi^t = \mathrm{D}_t(\phi - \tau u_t - \xi u_x) + \xi u_{xt} + \tau u_{tt},$$
$$\phi^{xxx} = \mathrm{D}_x^3(\phi - \tau u_t - \xi u_x) + \tau u_{txxx} + \xi u_{xxxx},$$

where the operators $\mathrm{D}_t$ and $\mathrm{D}_x$ denote the total derivatives with respect to $t$ and $x$

$$\mathrm{D}_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + \cdots\,,$$
$$\mathrm{D}_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_t} + u_{xx}\frac{\partial}{\partial u_x} + \cdots\,.$$

Expanding the condition (2.5), substituting $u^3u_{xxx}$ for $u_t$ and splitting the resulting equation with respect to the derivatives $u_x$, $u_{xx}$ and $u_{xxx}$, we find the system of determining equations,

$$\tau_x = \tau_u = 0, \quad \xi_u = 0, \quad 3\phi = (3\xi_x - \tau_t)u,$$
$$\xi_t + u^3(\phi_{xxu} - \xi_{xxx}) = 0, \quad \xi_{xx} - \phi_{xu} = 0, \quad \phi_t - u^3\phi_{xxx} = 0.$$

The solution of this system gives us the most general infinitesimal symmetry of the Harry Dym equation with the coefficient functions of the form

$$\tau = c_1 + c_5 t, \quad \xi = c_4 + c_2 x + \frac{c_3}{2}x^2, \quad \phi = \left(c_2 - \frac{1}{3}c_5 + c_3 x\right)u,$$

where $c_1, \dots, c_5$ are arbitrary constants. Thus the maximal Lie invariance algebra $\mathfrak{g}$ of (2.4) is spanned by five vector fields

$$P^x = \partial_x, \quad D^x = x\partial_x + u\partial_u, \quad \Pi = x^2\partial_x + 2xu\partial_u,$$
$$P^t = \partial_t, \quad D^t = t\partial_t - \frac{u}{3}\partial_u.$$

Each point symmetry transformation $\mathcal{T}$

$$\mathcal{T}\colon\ (\tilde{t}, \tilde{x}, \tilde{u}) = (T, X, U)(t, x, u)$$

in the space of $(x, u^{(r)})$ of a system of differential equations $\mathcal{L}$ induces an automorphism of the maximal Lie invariance algebra $\mathfrak{g}$ of $\mathcal{L}$ via push-forwarding of vector fields in the space of system variables. This condition implies constraints for $\mathcal{T}$ which are then taken into account in further calculations using the direct method. If $\dim \mathfrak{g} = l < \infty$, then the above means that

$$\mathcal{T}_* e_i = \sum_{j=1}^{l} a^i_j e_j, \quad i = 1, \dots, l, \tag{2.6}$$

where $\mathcal{T}_*$ is the push-forward of the vector fields induced by $\mathcal{T}$, and $A = (a^i_j)^l_{i,j=1}$ is the matrix of an automorphism of $\mathfrak{g}$ in the chosen basis $(e_1, \dots, e_l)$.

Our next goal is to compute the general form $(a^i_j)$ of the automorphism matrices, that correspond to the linear operators of which $\mathrm{Aut}(\mathfrak{g})$ consists. We fix $\mathcal{B} = (P^x, D^x, \Pi, P^t, D^t)$ to be the basis of $\mathfrak{g}$. One can obtain the general form of $(a^i_j)^5_{i,j=1}$ via solving the system of algebraic equations

$$c^{k'}_{i'j'} a^{i'}_i a^{j'}_j = c^k_{ij} a^{k'}_k, \tag{2.7}$$

where $c^k_{ij}$ are the structure constants of $\mathfrak{g}$ in the fixed basis $\mathcal{B}$ and repeated indices in subscript and superscript means the summation over them. In the case of the Harry Dym equation it is useful to investigate the structure of the maximal Lie invariance algebra $\mathfrak{g}$ first. According to the Levi–Mal'tzev theorem, any finite-dimensional real Lie algebra is the semidirect product of a maximal solvable ideal (radical) and a semisimple subalgebra (Levi subalgebra). This theory is well explained in [23, 51].

**Theorem 16.** *The radical $\mathfrak{r}$ of $\mathfrak{g}$ is spanned by the vector fields $P^t$, $D^t$ and the Levi factor $\mathfrak{f}$ is spanned by $P^x$, $D^x$, $\Pi$,*

$$\mathfrak{r} = \langle P^t, D^t \rangle, \quad \mathfrak{f} = \langle P^x, D^x, \Pi \rangle.$$

One can notice that $\mathfrak{f}$ is isomorphic to the special linear Lie algebra $\mathrm{sl}(2, \mathbb{R})$, and thus, $\mathrm{Aut}(\mathfrak{f})$ is isomorphic to the group of diagonal matrices of the form $\mathrm{diag}(\varepsilon, 1, \varepsilon)$, $\varepsilon = \pm 1$. Finally, we have that the general form of automorphism matrices of $\mathfrak{g}$ is the following

$$\mathcal{A} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & a_4^4 & a_5^4 \\ 0 & 0 & 0 & a_4^5 & a_5^5 \end{pmatrix},$$

where $a_j^i$'s are arbitrary nonzero constants and $a_4^4 a_5^5 \neq 0$.

We have obtained a restricted form of the matrix $A$, and so can expand the condition (2.6) for $\mathcal{T}_*$ that will give us constraints for the transformation $\mathcal{T}$. These constraints are to be used within the direct method afterwards. Consequently equating the corresponding vector-field components of right- and left-hand sides of the equations (2.6) produces a system of differential equations for components of the transformation $\mathcal{T}$. Integrating this system gives us the following intermediate form

$$T = a_5^4 t + a_5^4 - a_5^5, \quad X = \varepsilon x, \quad U = cu, \quad \varepsilon = \pm 1,\ c \in \mathbb{R}$$

for $\mathcal{T}$. The set of transformations found in the way described above constitutes the complete point symmetry group of the system (2.4) including both continuous and discrete transformations. To produce the final form for $\mathcal{T}$ we solve the system of constraints for $\mathcal{T}$ within the framework of the direct method. This produces the condition $c^3 a_5^4 = \varepsilon$. All the above proves the following corollary.

**Corollary 17.** *A complete list of discrete symmetry transformations of the Harry Dym equation* (2.4) *that are independent up to combining with continuous symmetry transformations of the equation and with each other is exhausted by two transformations alternating signs of variables,*

$$(t, x, u) \mapsto (-t, x, -u) \quad \textit{and} \quad (t, x, u) \mapsto (t, -x, -u).$$

## 2.4 Equivalence group

Differential equations are used to analyze and understand a variety of real-world problems. Most of them contain numerical or functional parameters (so-called arbitrary elements), which are often determined experimentally. Differential equations that contain parameters are called classes of differential equations. The following definition makes this more precise (the main references here are [4, 47]).

Let

$$\mathcal{L}_\theta\colon\ \Delta_\nu(x, u^{(p)}, \theta(x, u^{(p)})) = 0, \quad \nu = 1, \dots, l,$$

denote a system of differential equations parameterized by the tuple of arbitrary elements $\theta(x, u^{(p)}) = (\theta^1(x, u^{(p)}), \dots, \theta^k(x, u^{(p)}))$ running through the set $\mathcal{S}$ of solutions of the auxiliary system $S(x, u^{(p)}, \theta_{(q)}(x, u^{(p)})) = 0$. Often the set $\mathcal{S}$ is additionally constrained by the non-vanishing condition $\Sigma(x, u^{(p)}, \theta_{(q)}(x, u^{(p)})) \neq 0$ with another tuple $\Sigma$ of differential functions. In the above notations, by $\theta_{(q)}$ we denote the partial derivatives of the arbitrary element $\theta$ of order not greater than $q$ for which both $x$ and $u^{(p)}$ act as independent variables.

**Remark.** The above definition of a class of systems of differential equations is not complete, since different values of arbitrary elements may correspond to the same system. The values $\theta$ and $\tilde\theta$ of arbitrary elements are called gauge equivalent ($\theta \sim_g \tilde\theta$) if $\mathcal{L}_\theta$ and $\mathcal{L}_{\tilde\theta}$ are the same system of differential equations.

The set $\mathcal{S}$ of arbitrary elements should be factorized with respect to the gauge equivalence relation, to avoid ambiguity in the following correspondence $\theta \to \mathcal{L}_\theta$. The above discussion allows us to present the following definition:

**Definition 18.** The set $\{\mathcal{L}_\theta \,|\, \theta \in \mathcal{S}\}$ denoted by $\mathcal{L}|_\mathcal{S}$ is called a class of differential equations defined by parameterized systems $\mathcal{L}_\theta$ and the set $\mathcal{S}$ of arbitrary elements $\theta$.

The main aim of this section is to define the notion of the equivalence group of the class $\mathcal{L}|_\mathcal{S}$. It is convenient to first introduce the notion of admissible transformations and the equivalence groupoid. The set of admissible transformations $T(\theta, \tilde\theta)$ from $\mathcal{L}_\theta$ into $\mathcal{L}_{\tilde\theta}$ is the set of point transformations which map the system $\mathcal{L}_\theta$ into the system $\mathcal{L}_{\tilde\theta}$. The maximal point symmetry group $G_\theta$ of the system $\mathcal{L}_\theta$ coincides with $T(\theta, \theta)$. If the systems $\mathcal{L}_\theta$ and $\mathcal{L}_{\tilde\theta}$ are equivalent with respect to point transformations then $T(\theta, \tilde\theta) = \phi^0 \circ G_\theta = G_{\tilde\theta} \circ \phi^0$, where $\phi^0$ is a fixed transformation from $T(\theta, \tilde\theta)$. Otherwise, $T(\theta, \tilde\theta) = \varnothing$. Analogously, the set $T(\theta, \mathcal{L}|_\mathcal{S}) = \{\tilde\theta \mid \tilde\theta \in \mathcal{S}, \phi \in T(\theta, \tilde\theta)\}$ is called the set of admissible transformations of the system $\mathcal{L}_\theta$ into the class $\mathcal{L}|_\mathcal{S}$.

**Definition 19.** $T(\mathcal{L}|_{\mathcal{S}}) = \{(\theta, \tilde{\theta}, \phi) \mid \theta, \tilde{\theta} \in \mathcal{S}, \phi \in T(\theta, \tilde{\theta})\}$ is called the set of admissible transformations in $\mathcal{L}|_{\mathcal{S}}$.

Each admissible transformation is invertible $(\theta, \tilde{\theta}, \phi)^{-1} = (\tilde{\theta}, \theta, \phi^{-1})$ and the partial binary operation of composition is naturally defined for pairs of admissible transformations for which the target system of the first admissible transformation and the source system of the second admissible transformation coincide, $(\theta, \tilde{\theta}, \phi) \circ (\tilde{\theta}, \bar{\theta}, \tilde{\phi})$. The set of admissible transformations of the class $\mathcal{L}|_{\mathcal{S}}$ that is endowed with the operations of composition and taking the inverse is called the equivalence groupoid of this class and denoted by $\mathcal{G}^{\sim} = \mathcal{G}^{\sim}(\mathcal{L}|_{\mathcal{S}})$. Recall that a point transformation $\phi$: $\tilde{z} = \phi(z)$ in the space of variables $z = (z_1, \ldots, z_k)$ is projectable on the space of variables $z' = (z_{i_1}, \ldots, z_{i_k})$, with $1 \leq i_1 \leq \cdots \leq i_k \leq k$ if the expressions for the transformed variables $\tilde{z}'$ depend only on $z'$. The projection of $\phi$ to the $z'$-space is denoted by $\phi|_{z'}$, $\tilde{z}' = \phi|_{z'}(z')$.

**Definition 20.** The (usual) equivalence group, denoted by $G^{\sim} = G^{\sim}(\mathcal{L}|_{\mathcal{S}})$, of the class $\mathcal{L}|_{\mathcal{S}}$ is the group of point transformations in the space of $(x, u^{(p)}, \theta)$, each element $\Phi$ of which satisfies the following properties: It is projectable to the space of $(x, u^{(p')})$ for any $p'$ with $0 \leq p' \leq p$. The projection $\Phi|_{(x,u^{(p')})}$ is the $p'$-th order prolongation of $\Phi|_{(x,u)}$. For any $\theta$ from $\mathcal{S}$ its image $\Phi\theta$ also belongs to $\mathcal{S}$. Finally, $\Phi|_{(x,u)} \in T(\theta, \Phi\theta)$.

Elements of $G^{\sim}$ are called equivalence transformations of the class $\mathcal{L}|_{\mathcal{S}}$. Each equivalence transformation $\Phi$ induces a family of admissible transformations parameterized by the arbitrary elements, $\{(\theta, \Phi\theta, \Phi|_{(x,u)}) \mid \theta \in \mathcal{S}\}$. If the entire equivalence groupoid is induced by the equivalence group $G^{\sim}$ in the above way, then transformational properties of the class $\mathcal{L}|_{\mathcal{S}}$ are particularly nice and the class is called normalized [47].

The equivalence group of the class of equations may contain transformations which act only on arbitrary elements and do not really change systems, i.e. which generate gauge admissible transformations.

**Definition 21.** (See [47].) An element $(\theta, \tilde{\theta}, \phi)$ from the set of admissible transformations of the given $\mathcal{L}_\theta$ is called a gauge admissible transformation in the class $\mathcal{L}_S$ if $\theta$ is gauge-equivalent to $\tilde{\theta}$ and $\phi$ is the identical transformation.

In general, transformations of this type can be considered as trivial (gauge) equivalence transformations and form a gauge subgroup of the equivalence group. In some cases, if the arbitrary elements $\theta$ depend on $x$ and $u$ only, we can neglect the condition that the transformation components for $(x,u)$ of equivalence transformations do not involve $\theta$, which gives the generalized equivalence group $G^\sim_{\rm gen}=G^\sim_{\rm gen}(\mathcal L|_{\mathcal S})$ of the class $\mathcal L|_{\mathcal S}$. Each element $\Phi$ of $G^\sim_{\rm gen}$ is a point transformation in the $(x,u,\theta)$-space such that for any $\theta$ from $\mathcal S$ its image $\Phi\theta$ also belongs to $\mathcal S$, and $\Phi(\cdot,\cdot,\theta(\cdot,\cdot))|_{(x,u)}\in T(\theta,\Phi\theta)$.

The algebraic method for computing complete point symmetry groups from Section 2.3 can be easily extended to the framework of equivalence transformations.

**Theorem 22.** *(See [4].) Let $\mathcal L|_{\mathcal S}$ be a class of (systems) of differential equations with $G^\sim$ and $\mathfrak g^\sim$ being the equivalence group and the equivalence algebra of this class. Any transformation $\mathcal T$ from $G^\sim$ induces an automorphism of $\mathfrak g^\sim$ via pushing forward vector fields in the relevant space of independent variables, derivatives of unknown functions and arbitrary elements of the class.*

The extension works properly if the corresponding equivalence algebra $\mathfrak g^\sim$ is of finite, nonzero, and moreover low dimension since the knowledge of the entire automorphism group is needed. The new feature is that one should consider appropriate point transformations and vector fields in the relevant extended vector space of independent variables, derivatives of unknown functions and arbitrary elements.

# Chapter 3

# Equivalence group of one-dimensional shallow water equations

The class of systems of one-dimensional shallow water equations with variable bottom topography in dimensionless variables is given by

$$\begin{aligned} &u_t + uu_x + \eta_x = 0, \\ &\eta_t + \big((\eta + h)u\big)_x = 0. \end{aligned} \tag{3.1}$$

Here $h(x)$ is the bottom profile, $u = u(t, x)$ is the vertically averaged horizontal velocity, and $\eta = \eta(t, x)$ is the deviation of the free surface.

Each system of the form (3.1) is an inhomogeneous one-dimensional system of hydrodynamic type equations. In general, such systems for $n$ dependent variables are of the form

$$u^i_t = \sum_{j=1}^{n} v^{ij}(t, x, u^1, \dots, u^n)u^j_x + v^{i0}(t, x, u^1, \dots, u^n), \quad i = 1, \dots, n,$$

where the coefficients $v^{ij}$ are sufficiently smooth functions of $(t, x, u^1, \dots, u^n)$; see [19, 54]. For systems of the form (3.1) we have $n = 2$ and $(u^1, u^2) = (u, h)$.

The system of one-dimensional shallow water equations of the general form (3.1) was studied in [2, 3]. We will rewrite this system in different notations.

Let us denote $b(x) := h(x)$ and $\hat{h}(x) := \eta(t,x)+b(x)$, and further we will write $h(x)$ instead of $\hat{h}(x)$. As a result systems of the form (3.1) reduce to

$$\begin{aligned} &u_t + uu_x + h_x = b_x, \\ &h_t + hu_x + uh_x = 0, \end{aligned} \tag{3.2}$$

where $x$ is the horizontal coordinate, $t$ is the time variable, $u(t,x)$ and $h(t,x)$ are the horizontal component of the velocity and the depth of the free surface at the point $x$ at the time $t$, respectively.

The systems of the form (3.2) involve the parameter function $b$, which is referred to as the arbitrary element of the class (3.2). According to the interpretation of $b$ as arbitrary element or a fixed function, we will refer to (3.2) as a class of systems or as a fixed system.

The complete system of auxiliary equations and inequalities for the arbitrary element $b$ of the class (3.2) is given by

$$b_t = 0, \quad b_u = 0, \quad b_h = 0, \quad b_{u_t} = 0, \quad b_{u_x} = 0, \quad b_{h_t} = 0, \quad b_{h_x} = 0.$$

## 3.1 Computation without re-parametrizing the class

Since the arbitrary element $b$ depends only on $(t,x)$ and does not depend on $u$ and $h$ we will treat the arbitrary element $b = b(t,x)$ as a dependent variable. The generalized equivalence group $G^{\sim}$ for the class (3.2) coincides with the complete point symmetry group $G$ of the extended system

$$\begin{aligned} &u_t + uu_x + h_x = b_x, \\ &h_t + hu_x + uh_x = 0, \\ &b_t = 0. \end{aligned} \tag{3.3}$$

Similarly, the equivalence algebra $\mathfrak{g}^{\sim}$ can be identified with the maximal Lie invariance algebra $\mathfrak{g}$ of the system (3.3). In order to compute the maximal Lie invariance algebra $\mathfrak{g}$ of a given system of differential equations we invoke the infinitesimal method.

The infinitesimal generators of one-parameter Lie symmetry groups for the system (3.3) are of the form

$$\mathbf{v} = \tau\partial_t + \xi\partial_x + \eta^1\partial_u + \eta^2\partial_h + \eta^3\partial_b,$$

where the components $\tau$, $\xi$, $\eta^1$, $\eta^2$ and $\eta^3$, depend on $t$, $x$, $u$, $h$ and $b$. Recall that we use the notation $\partial_\alpha$ instead of $\partial/\partial\alpha$, $\alpha \in \{t, x, u, h, b\}$.

The infinitesimal invariance criterion (Theorem 15) requires that

$$\begin{aligned}
&\mathrm{pr}^{(1)}\mathbf{v}(w_t^1 + ww_x^1 + w_x^2 - w_x^3) = 0,\\
&\mathrm{pr}^{(1)}\mathbf{v}(w_t^2 + w^2w_x^1 + w^1w_x^2) = 0,\\
&\mathrm{pr}^{(1)}\mathbf{v}(w_t^3) = 0,
\end{aligned}$$

whenever the system (3.3) holds. Here $w = (w^1, w^2, w^3) := (u, h, b)$ and the vector field $\mathrm{pr}^{(1)}\mathbf{v}$ is the first prolongation of $\mathbf{v}$ of the form

$$\mathrm{pr}^{(1)}\mathbf{v} = \mathbf{v} + \sum_{i=1}^{3}\left(\eta^{it}\frac{\partial}{\partial w_t^i} + \eta^{ix}\frac{\partial}{\partial w_x^i}\right).$$

The components $\eta^{it}$ and $\eta^{ix}$ of the prolonged vector field $\mathrm{pr}^{(1)}\mathbf{v}$ are determined by the following expressions

$$\begin{aligned}
&\eta^{it} = \mathrm{D}_t(\eta^i - \tau w_t^i - \xi w_x^i) + \tau w_{tt}^i + \xi w_{tx}^i\\
&\eta^{ix} = \mathrm{D}_x(\eta^i - \tau w_t^i - \xi w_x^i) + \tau w_{tx}^i + \xi w_{xx}^i,\\
&i = 1, 2, 3.
\end{aligned}$$

Here $\mathrm{D}_t$ and $\mathrm{D}_x$ are the total derivative operators with respect to $t$ and $x$, respectively. The criterion of invariance of the system, with respect to the vector field $\mathbf{v}$ is

$$\begin{aligned}
&\eta^{1t} + u_x\eta^1 + u\eta^{1x} + \eta^{2x} - \eta^{3x} = 0,\\
&\eta^{2t} + u_x\eta^2 + h\eta^{1x} + \eta^1 h_x + u\eta^{2x} = 0,\\
&\eta^{3t} = 0,
\end{aligned} \tag{3.4}$$

which must be satisfied whenever the system (3.3) holds.

Substituting the expressions for the derivatives $u_t$, $h_t$ and $b_t$ in view of the system (3.3) into conditions (3.4) and splitting with respect to $u_x$, $h_x$ and $b_x$ results in the system of determining equations

$$
\begin{aligned}
&\tau=\tau(t,x), \quad \xi=\xi(x), \quad \eta^3=\eta^3(x,b),\\
&h\eta^1_x+\eta^2_t+u\eta^2_x=0, \quad \eta^1_t+u\eta^1_x+\eta^2_x-\eta^3_x=0,\\
&-h\tau_x+\eta^2_u+u\eta^2_b+h\eta^3_b=0, \quad u\tau_x+u\eta^1_b+\eta^2_h+\eta^2_b-\eta^3_b=0,\\
&\tau_t+2u\tau_x-\xi_x-\eta^1_u+\eta^2_h=0,\\
&u\tau_t+u^2\tau_x-u\xi_x+\eta_1+h\eta^1_h+h\eta^1_b+u\eta^2_b,\\
&h\tau_t+uh\tau_x-h\xi_x+h\eta^1_u+uh\eta^1_b+\eta_2+u\eta^2_u+u^2\eta^2_b,\\
&h\tau_x+\eta_1+u\eta^1_u-h\eta^1_h+u^2\eta^1_b+u\eta^2_b-u\eta^3_b=0.
\end{aligned}
$$

Integrating this system, we obtain the exact form of the components of the vector field $\mathbf{v}$,

$$
\begin{aligned}
&\tau=c_3t+c_4, \quad \xi=c_1x+c_2,\\
&\eta^1=(c_1-c_3)u, \quad \eta^2=2(c_1-c_3)h, \quad \eta^3=2(c_1-c_3)b+c_5,
\end{aligned}
$$

where $c_1,\dots,c_5$ are arbitrary real constants. A complete set of linearly independent generators of one-parameter Lie symmetry groups for the extended system (3.3), which span the maximal Lie invariance algebra $\mathfrak{g}$ of this system, consists of the vector fields

$$
\begin{aligned}
&P^t=\partial_t, \quad P^x=\partial_x, \quad P^b=\partial_b,\\
&D^t=t\partial_t-u\partial_u-2h\partial_h-2b\partial_b, \quad D^x=x\partial_x+u\partial_u+2h\partial_h+2b\partial_b.
\end{aligned}
$$

We fix the basis $\mathcal{B}=(P^t,\ P^x,\ P^b,\ D^t,\ D^x)$ of the algebra $\mathfrak{g}$. Up to anticommutativity of the Lie bracket of vector fields, the nonzero commutation relations between the basis elements of $\mathfrak{g}$ are exhausted by

$$
[P^t,D^t]=P^t, \quad [P^x,D^x]=P^x, \quad [P^b,D^t]=-2P^b, \quad [P^b,D^x]=2P^b,
$$

i.e., the only nonzero structure constants of $\mathfrak{g}$ in the basis $\mathcal{B}$ are

$$
c^1_{14}=1, \quad c^2_{25}=1, \quad c^3_{34}=-2, \quad c^3_{35}=2.
$$

The automorphism group $\mathrm{Aut}(\mathfrak{g}^{\sim})$ of $\mathfrak{g}$ consists of the linear operators on $\mathfrak{g}$ whose matrices, in the chosen basis $\mathcal{B}$, are nondegenerate $5\times 5$ matrices of the form $(a^i_j)^5_{i,j=1}$ with entries $a^i_j$ satisfying the system of algebraic equations

$$c^{k'}_{i'j'}a^{i'}_i a^{j'}_j = c^k_{ij}a^{k'}_k .$$

Here we assume the summation over repeated indices. The solution of the system is given by

$$\begin{aligned}
&a^1_2 = a^1_3 = a^1_5 = 0,\\
&a^2_1 =^2_3= a^2_4 = 0,\\
&a^3_1 = a^3_2 = 0, \quad a^3_4 = -a^3_5,\\
&a^4_1 = a^4_2 = a^4_3 = a^4_5 = 0, \quad a^4_4 = 1,\\
&a^5_1 = a^5_2 = a^5_3 = a^5_4 = 0, \quad a^5_1 = 1,
\end{aligned}$$

where $a^1_1$, $a^1_4$, $a^2_2$, $a^2_5$, $a^3_3$ and $a^3_5$ are free parameters with $a^1_1a^2_2a^3_3 \neq 0$.

Hence the group $\mathrm{Aut}(\mathfrak{g})$ can be identified with the matrix group that consists of the matrices of the general form

$$\mathcal{A} = \begin{pmatrix} a^1_1 & 0 & 0 & a^1_4 & 0\\ 0 & a^2_2 & 0 & 0 & a^2_5\\ 0 & 0 & a^3_3 & -a^3_5 & a^3_5\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where all nonzero $a^i_j$'s are arbitrary real constants and $a^1_1a^2_2a^3_3 \neq 0$.

By construction, a point symmetry transformation of the system (3.3) is a point transformation in the joint space of the independent variables $(t, x)$ and the dependent variables $u$, $h$ and $b$. In order to compute the complete point symmetry group of the system (3.3) we consider a point symmetry transformation of independent and dependent variables of the form

$$\mathcal{T}\colon \quad (\tilde{t}, \tilde{x}, \tilde{u}, \tilde{h}, \tilde{b}) = (T, X, U, H, B)(t, x, u, h, b).$$

Each point transformation $\mathcal{T}$ of the class induces an automorphism of the maximal Lie invariance algebra $\mathfrak{g}$. Since the algebra $\mathfrak{g}$ is finite-dimensional, $\dim\mathfrak{g} = 5$, the above conditions mean that

$$\mathcal{T}_* e_j = \sum_{i=1}^{5} a^i_j e_i, \quad j = 1, \dots, 5, \tag{3.5}$$

where $\mathcal{T}_*$ is the push-forward of vector fields induced by $\mathcal{T}$ and $(a^i_j)^5_{i,j=1}$ is the matrix of an automorphisms in the chosen basis $\mathcal{B}$, where $e_1 = P^t$, $e_2 = P^x$, $e_3 = P^b$, $e_4 = D^t$ and $e_5 = D^x$. We expand (3.5) for each basis element and get the following constraints for the transformation $\mathcal{T}$:

$$\mathcal{T}_* P^t = T_t\partial_{\tilde t} + X_t\partial_{\tilde x} + U_t\partial_{\tilde u} + H_t\partial_{\tilde h} + B_t\partial_{\tilde b} = a^1_1\partial_{\tilde t}, \tag{3.6a}$$

$$\mathcal{T}_* P^x = T_x\partial_{\tilde t} + X_x\partial_{\tilde x} + U_x\partial_{\tilde u} + H_x\partial_{\tilde h} + B_x\partial_{\tilde b} = a^2_2\partial_{\tilde x}, \tag{3.6b}$$

$$\mathcal{T}_* P^b = T_b\partial_{\tilde t} + X_b\partial_{\tilde x} + U_b\partial_{\tilde u} + H_b\partial_{\tilde h} + B_b\partial_{\tilde b} = a^3_3\partial_{\tilde b}, \tag{3.6c}$$

$$\begin{aligned}\mathcal{T}_* D^t &= (tT_t - uT_u - 2hT_h - 2bT_b)\partial_{\tilde t} + (tX_t - uX_u - 2hX_h - 2bX_b)\partial_{\tilde x}\\ &\quad + (tU_t - uU_u - 2hU_h - 2bU_b)\partial_{\tilde u} + (tH_t - uH_u - 2hH_h - 2bH_b)\partial_{\tilde h}\\ &\quad + (tB_t - uB_u - 2hB_h - 2bB_b)\partial_{\tilde b}\\ &= a^1_4\partial_{\tilde t} - a^3_5\partial_{\tilde b} + T\partial_{\tilde t} - U\partial_{\tilde u} - 2H\partial_{\tilde h} - 2B\partial_{\tilde b},\end{aligned} \tag{3.6d}$$

$$\begin{aligned}\mathcal{T}_* D^x &= (xT_x + uT_u + 2hT_h + 2bT_b)\partial_{\tilde t} + (xX_x + uX_u + 2hX_h + 2bX_b)\partial_{\tilde x}\\ &\quad + (xU_x + uU_u + 2hU_h + 2bU_b)\partial_{\tilde u} + (xH_x + uH_u + 2hH_h + 2bH_b)\partial_{\tilde h}\\ &\quad + (xB_x + uB_u + 2hB_h + 2bB_b)\partial_{\tilde b}\\ &= a^2_5\partial_{\tilde x} + a^3_5\partial_{\tilde b} + X\partial_{\tilde x} + U\partial_{\tilde u} + 2H\partial_{\tilde h} + 2B\partial_{\tilde b}.\end{aligned} \tag{3.6e}$$

The first three equations of the last system imply $T_t = a^1_1$, $T_x = T_b = 0$, $X_x = a^2_2$ and $X_t = X_b = 0$. In view of (3.6d) and restrictions on $T$ we compute that $T = a^1_1 t - a^4_1$, and similarly from the equation (3.6e) we get $X = a^2_2 x - a^5_2$. We also derive that $U = U(u,h)$, $H = H(u,h)$, $B_b = a^3_3$ and $B_t = B_x = 0$ from the equations (3.6a), (3.6b), (3.6c). Two differential equations $U_u + 2hU_h = U$, $H_u + 2hH_h = 2H$, are obtained from the equations (3.6a) and (3.6b). We integrate these equations and get that $U = uF\left(hu^{-2}\right)$ and $H = hG\left(hu^{-2}\right)$, where $F$ and $G$ are arbitrary smooth functions of $hu^{-2}$. Besides, from (3.6d) and (3.6e) we compute that $B = a^3_3 b - a^5_3/2$.

To sum up, using the algebraic method we obtain the general form for the components $T$, $X$, $U$, $H$ and $B$ to be

$$\begin{aligned}&T = a_1^1 t - a_1^4, \quad X = a_2^2 x - a_2^5,\\&U = uF(\omega), \quad H = hG(\omega), \quad B = a_3^3 b - a_3^5/2,\end{aligned}$$

where $F$ and $G$ are smooth functions of $\omega := hu^{-2}$. Further we continue the computations of the complete point symmetry group $G$ of the system (3.3) within the framework of the direct method. We express all required transformed derivatives $\tilde{w}^i_{\tilde{t}}$, $\tilde{w}^i_{\tilde{x}}$, $i = 1, 2, 3$, in terms of the initial coordinates using the chain rule

$$\begin{aligned}&\tilde{u}_{\tilde{t}} = \frac{1}{a_1^1}\left(u_t(F - 2\omega F_\omega) + h_t\frac{F_\omega}{u}\right), \quad \tilde{u}_{\tilde{x}} = \frac{1}{a_2^2}\left(u_x(F - 2\omega F_\omega) + h_x\frac{F_\omega}{u}\right),\\&\tilde{h}_{\tilde{t}} = \frac{1}{a_1^1}\left(-2u_t u\omega^2 G_\omega + h_t(G + \omega G_\omega)\right),\\&\tilde{h}_{\tilde{x}} = \frac{1}{a_2^2}\left(-2u_x u\omega^2 G_\omega + h_x(G + \omega G_\omega)\right),\\&\tilde{b}_{\tilde{t}} = \frac{a_3^3}{a_1^1}b_t, \quad \tilde{b}_{\tilde{x}} = \frac{a_3^3}{a_2^2}b_x.\end{aligned}$$

We substitute the obtained expressions into the copy of the system (3.3) in the new coordinates

$$\begin{aligned}&\tilde{u}_{\tilde{t}} + \tilde{u}\tilde{u}_{\tilde{x}} + \tilde{h}_{\tilde{x}} - \tilde{b}_{\tilde{x}} = 0,\\&\tilde{h}_{\tilde{t}} + \tilde{h}\tilde{u}_{\tilde{x}} + \tilde{u}\tilde{h}_{\tilde{x}} = 0,\\&\tilde{b}_{\tilde{t}} = 0.\end{aligned}$$

Each solution of the system (3.3) should identically satisfy the expanded system of equations. Thus, the final form of the symmetry transformation components is

$$T = a_1^1 t - a_1^4, \quad X = a_2^2 x - a_2^5, \quad U = \frac{a_3^3 a_1^1}{a_2^2}u, \quad H = a_3^3 h, \quad B = a_3^3 b - a_3^5/2,$$

where additionally $(a_1^1)^2 a_3^3/(a_2^2)^2 = 1$.

Recall that the generalized equivalence group $G^\sim$ of the class of one-dimensional shallow water equations (3.2) coincides with the complete point symmetry group $G$ of the extended system (3.3). Summing up, we proved the following theorem.

**Theorem 23.** *The generalized equivalence group $G^{\sim}$ of the class of one-dimensional shallow water equations* (3.3) *consists of the point transformations in the space of the coordinates* $(t, x, u, h, b)$ *whose components are*

$$\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \delta_3 x + \delta_4, \quad \tilde{u} = \frac{\delta_3}{\delta_1} u, \quad \tilde{h} = \frac{\delta_3^2}{\delta_1^2} h, \quad \tilde{b} = \frac{\delta_3^2}{\delta_1^2} b + \delta_5,$$

*where* $\delta_i$*,* $i = 1, \ldots, 5$ *are arbitrary real constants with* $\delta_1 \delta_3 \neq 0$*.*

The generalized equivalence point transformations $\tilde{t}$, $\tilde{x}$, $\tilde{u}$ and $\tilde{h}$ of the class (3.2) do not depend on the arbitrary element $b$, and thus coincide with the usual equivalence transformations of this class.

**Corollary 24.** *The generalized equivalence group $G^{\sim}$ of the class* (3.2) *coincides with the usual equivalence group of this class.*

**Corollary 25.** *The class of one-dimensional shallow water equations* (3.2) *possesses, up to combining with each other and with continuous symmetries, two independent discrete equivalence transformations given by alternating signs of variables*

$$(t, x, u, h, b) \mapsto (-t, x, -u, h, b),$$
$$(t, x, u, h, b) \mapsto (t, -x, -u, h, b).$$

**Corollary 26.** *The factor group of the generalized equivalence group $G^{\sim}$ of the class* (3.2)*, with respect to its identity component is isomorphic to the group* $\mathbb{Z}_2 \times \mathbb{Z}_2$*.*

## 3.2 Computation with re-parametrizing the class

The arbitrary element $b$ appears in the general form in systems from the class (3.2) only as derivative $b_x$. This is why we can re-parametrize the class (3.2) assuming $b_x$ itself as the arbitrary element of this class, re-denoting for convenience $b_x$ by $b$. As a result, we obtain the class of one-dimensional shallow water equations of the form

$$\begin{aligned} &u_t + uu_x + h_x + b = 0, \\ &h_t + hu_x + uh_x = 0, \end{aligned} \tag{3.7}$$

where $b = b(x)$ is the arbitrary element of the class.

Analogously to the previous case, we can treat the arbitrary element $b$ as one more dependent variable, and replace the class (3.7) by the extended system,

$$\begin{aligned} &u_t + uu_x + h_x + b = 0,\\ &h_t + hu_x + uh_x = 0, \\ &b_t = 0. \end{aligned} \tag{3.8}$$

The computation of the complete point symmetry group of the system (3.8) is similar to that for the system (3.3). Using the classical infinitesimal approach we construct the maximal Lie invariance algebra $\mathfrak{h} = \langle P^t, D^t, P^x, D^x \rangle$ of the system (3.8), where

$$\begin{aligned} &P^t = \partial_t, \quad D^t = t\partial_t - u\partial_u - 2h\partial_h - 2b\partial_b,\\ &P^x = \partial_x, \quad D^x = x\partial_x + u\partial_u + 2h\partial_h + b\partial_b. \end{aligned}$$

Let us fix the basis $\mathcal{H} = (P^t, D^t, P^x, D^x)$ of $\mathfrak{h}$. All nonzero Lie brackets of basis elements, with respect to anticommutativity, are

$$[P^t, D^t] = P^t, \quad [P^x, D^x] = P^x.$$

It is essential to know the automorphism group $\mathrm{Aut}(\mathfrak{h})$[1] to invoke the algebraic method for computing complete point symmetry group of the system (3.8). Automorphism matrices of the algebra $\mathfrak{h}$ in the chosen basis $\mathcal{H}$ are of the following two forms, which were presented in [45]:

$$\mathcal{A}_1 = \begin{pmatrix} a^1_1 & a^1_2 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & a^3_3 & a^3_4\\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} 0 & 0 & a^1_3 & a^1_4\\ 0 & 0 & 0 & 1\\ a^3_1 & a^3_2 & 0 & 0\\ 0 & 1 & 0 & 0 \end{pmatrix},$$

where $a^1_1a^3_3 \neq 0$ for $\mathcal{A}_1$ and $a^1_3a^3_1 \neq 0$ for $\mathcal{A}_2$.

[1] For many finite-dimensional Lie algebras, the automorphism groups have been computed and presented in the literature. This includes all semi-simple Lie algebras [29], and for Lie algebras of dimension not greater than six, [17, 20, 21, 45]. The algebra $\mathfrak{h}$ is solvable and decomposable. More specifically it is isomorphic to the Lie algebra being the direct sum of two copies of the two-dimensional noncommutative algebras, which is denoted by e.g., $2\mathfrak{g}_2$ [36] or $2A_{2.1}$ [45]. The automorphism group of this algebra was computed in [17, 45].

Consider a point symmetry transformation $\mathcal{T}$ of the system (3.8),

$$\mathcal{T}\colon\quad (\tilde t,\tilde x,\tilde u,\tilde h,\tilde b)=(T,X,U,H,B)(t,x,u,h,b).$$

This transformation induces an automorphism of $\mathfrak{h}$, which means that

$$\mathcal{T}_*e_j=\sum_{i=1}^{4}a^i_je_i,\quad j=1,\dots,4, \tag{3.9}$$

where $\mathcal{T}_*$ is the push-forward of vector fields associated with $\mathcal{T}$ and $(a^i_j)^4_{i,j=1}$ is the matrix of an automorphism of $\mathfrak{h}$ in the chosen basis $\mathcal{H}=(e_1,e_2,e_3,e_4)=(P^t,D^t,P^x,D^x)$.

First, let us consider the automorphism matrix of the form $\mathcal{A}_1$. The system (3.9) is expanded to

$$\begin{aligned}
\mathcal{T}_*P^t&=T_t\partial_{\tilde t}+X_t\partial_{\tilde x}+U_t\partial_{\tilde u}+H_t\partial_{\tilde h}+B_t\partial_{\tilde b}=a^1_1\partial_{\tilde t},\\
\mathcal{T}_*D^t&=(tT_t-uT_u-2hT_h-2bT_b)\partial_{\tilde t}+(tX_t-uX_u-2hX_h-2bX_b)\partial_{\tilde x}\\
&\quad+(tU_t-uU_u-2hU_h-2bU_b)\partial_{\tilde u}+(tH_t-uH_u-2hH_h-2bH_b)\partial_{\tilde h}\\
&\quad+(tB_t-uB_u-2hB_h-2bB_b)\partial_{\tilde b}\\
&=a^1_2\partial_{\tilde t}+T\partial_{\tilde t}-U\partial_{\tilde u}-2H\partial_{\tilde h}-2B\partial_{\tilde b},\\
\mathcal{T}_*P^x&=T_x\partial_{\tilde t}+X_x\partial_{\tilde x}+U_x\partial_{\tilde u}+H_x\partial_{\tilde h}+B_x\partial_{\tilde b}=a^3_3\partial_{\tilde x},\\
\mathcal{T}_*D^x&=(xT_x+uT_u+2hT_h+bT_b)\partial_{\tilde t}+(xX_x+uX_u+2hX_h+bX_b)\partial_{\tilde x}\\
&\quad+(xU_x+uU_u+2hU_h+bU_b)\partial_{\tilde u}+(xH_x+uH_u+2hH_h+bH_b)\partial_{\tilde h}\\
&\quad+(xB_x+uB_u+2hB_h+bB_b)\partial_{\tilde b}\\
&=a^3_4\partial_{\tilde x}+X\partial_{\tilde x}+U\partial_{\tilde u}+2H\partial_{\tilde h}+B\partial_{\tilde b}.
\end{aligned} \tag{3.10}$$

System (3.10) leads to the following collection of equations

$$\begin{aligned}
&T_x=0,\quad T_t=a^1_1,\quad tT_t-bT_b=T+a^1_2,\quad uT_u+2hT_h+bT_b=0,\\
&X_t=0,\quad X_x=a^3_3,\quad xX_x-bX_b=X+a^3_4,\quad uX_u+2hX_h+2bT_b=0,\\
&U_t=U_x=U_b=0,\quad uU_u+2hU_h=U,\\
&H_t=H_x=H_b=0,\quad uH_u+2hH_h=2H,\\
&B_t=B_x=0,\quad bB_b=B,\quad uB_u+2hB_h=0.
\end{aligned}$$

The general solution to the last system is

$$T = a_1^1 t - a_2^1 + \frac{u}{b} F^1\left(\frac{h}{u^2}\right), \quad X = a_3^3 x - a_4^3 + \frac{u}{b} F^2\left(\frac{h}{u^2}\right),$$
$$U = uF^3\left(\frac{h}{u^2}\right), \quad H = hF^4\left(\frac{h}{u^2}\right), \quad B = bF^5\left(\frac{h}{u^2}\right), \tag{3.11}$$

where $F^1$, …, $F^5$, are arbitrary smooth functions of $hu^{-2}$. To obtain the final form of $\mathcal{T}$ we will continue computations within the framework of the direct method. In order to find the explicit forms of the functions $F^i$, $i = 1, \ldots, 5$, we express all required derivatives $(\tilde{u}_{\tilde{t}}, \tilde{u}_{\tilde{x}}, \tilde{h}_{\tilde{t}}, \tilde{h}_{\tilde{x}}, \tilde{b}_{\tilde{t}})$ in terms of the initial coordinates,

$$\tilde{u}_{\tilde{t}} = \frac{(\mathrm{D}_t U)\mathrm{D}_x X - (\mathrm{D}_x U)\mathrm{D}_t X}{(\mathrm{D}_t T)\mathrm{D}_x X - (\mathrm{D}_x T)\mathrm{D}_t X}, \quad \tilde{u}_{\tilde{x}} = \frac{(\mathrm{D}_t T)\mathrm{D}_x U - (\mathrm{D}_x T)\mathrm{D}_t U}{(\mathrm{D}_t T)\mathrm{D}_x X - (\mathrm{D}_x T)\mathrm{D}_t X},$$
$$\tilde{h}_{\tilde{t}} = \frac{(\mathrm{D}_t H)\mathrm{D}_x X - (\mathrm{D}_x H)\mathrm{D}_t X}{(\mathrm{D}_t T)\mathrm{D}_x X - (\mathrm{D}_x T)\mathrm{D}_t X}, \quad \tilde{h}_{\tilde{x}} = \frac{(\mathrm{D}_t T)\mathrm{D}_x H - (\mathrm{D}_x T)\mathrm{D}_t H}{(\mathrm{D}_t T)\mathrm{D}_x X - (\mathrm{D}_x T)\mathrm{D}_t X}, \tag{3.12}$$
$$\tilde{b}_{\tilde{t}} = \frac{(\mathrm{D}_t B)\mathrm{D}_x X - (\mathrm{D}_x B)\mathrm{D}_t X}{(\mathrm{D}_t T)\mathrm{D}_x X - (\mathrm{D}_x T)\mathrm{D}_t X}, \quad \tilde{b}_{\tilde{x}} = \frac{(\mathrm{D}_t T)\mathrm{D}_x B - (\mathrm{D}_x T)\mathrm{D}_t B}{(\mathrm{D}_t T)\mathrm{D}_x X - (\mathrm{D}_x T)\mathrm{D}_t X},$$

where

$$\mathrm{D}_t = \partial_t + u_t\partial_u + h_t\partial_h + b_t\partial_b,$$
$$\mathrm{D}_x = \partial_x + u_x\partial_u + h_x\partial_h + b_x\partial_b.$$

Then we substitute the expressions (3.12) (expanded using the form (3.11) for the components of $\mathcal{T}$) into the system (3.8) rewritten in the new variables $(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{h}, \tilde{b})$

$$\tilde{u}_{\tilde{t}} + \tilde{u}\tilde{u}_{\tilde{x}} + \tilde{h}_{\tilde{x}} - \tilde{b} = 0,$$
$$\tilde{h}_{\tilde{t}} + \tilde{h}\tilde{u}_{\tilde{x}} + \tilde{u}\tilde{h}_{\tilde{x}} = 0, \tag{3.13}$$
$$\tilde{b}_{\tilde{t}} = 0.$$

This gives a collection of equations, which are identically satisfied for each solution of the system (3.8). We substitute the expressions for the derivatives of $u$, $h$ and $b$ with respect to $t$ in view of the system (3.8) into these equations and collect the coefficients of the derivatives $u_x$, $h_x$ and $b_x$. Integrating the derived system of the determining equations for the parameter functions $F^1$, ..., $F^5$ and substituting the found expressions in (3.11), we get the final form of $T$, $X$, $U$, $V$, $H$ and $B$,

$$T = a_1^1 t - a_1^2, \quad X = a_3^3 x - a_4^3, \quad U = \frac{a_3^3}{a_1^1} u, \quad H = \left(\frac{a_3^3}{a_1^1}\right)^2 h, \quad B = \frac{a_3^3}{(a_1^1)^2} b.$$

We also need to look for point symmetry transformations of the system (3.8) that are associated with the second form of the automorphism matrix $\mathcal{A}_2$. The conditions (3.9) are expanded for $\mathcal{A}_2$ to

$$\begin{aligned}
\mathcal{T}_* P^t &= T_t \partial_{\tilde t} + X_t \partial_{\tilde x} + U_t \partial_{\tilde u} + H_t \partial_{\tilde h} + B_t \partial_{\tilde b} = a_1^3 \partial_{\tilde x}, \\
\mathcal{T}_* D^t &= (tT_t - uT_u - 2hT_h - 2bT_b)\partial_{\tilde t} + (tX_t - uX_u - 2hX_h - 2bX_b)\partial_{\tilde x} \\
&\quad + (tU_t - uU_u - 2hU_h - 2bU_b)\partial_{\tilde u} + (tH_t - uH_u - 2hH_h - 2bH_b)\partial_{\tilde h} \\
&\quad + (tB_t - uB_u - 2hB_h - 2bB_b)\partial_{\tilde b} \\
&= a_2^3 \partial_{\tilde x} + X\partial_{\tilde x} + U\partial_{\tilde u} + 2H\partial_{\tilde h} + B\partial_{\tilde b}, \\
\mathcal{T}_* P^x &= T_x \partial_{\tilde t} + X_x \partial_{\tilde x} + U_x \partial_{\tilde u} + H_x \partial_{\tilde h} + B_x \partial_{\tilde b} = a_3^1 \partial_{\tilde t}, \\
\mathcal{T}_* D^x &= (xT_x + uT_u + 2hT_h + bT_b)\partial_{\tilde t} + (xX_x + uX_u + 2hX_h + bX_b)\partial_{\tilde x} \\
&\quad + (xU_x + uU_u + 2hU_h + bU_b)\partial_{\tilde u} + (xH_x + uH_u + 2hH_h + bH_b)\partial_{\tilde h} \\
&\quad + (xB_x + uB_u + 2hB_h + bB_b)\partial_{\tilde b} \\
&= a_4^1 \partial_{\tilde t} + T\partial_{\tilde t} - U\partial_{\tilde u} - 2H\partial_{\tilde h} - 2B\partial_{\tilde b}.
\end{aligned} \tag{3.14}$$

The system (3.14) leads to the following system of determining equations for the components of the transformation $\mathcal{T}$:

$$\begin{aligned}
&T_t = 0, \quad T_x = a_3^1, \quad xT_x - bT_b = T + a_4^1, \quad uT_u + 2hT_h + 2bT_b = 0 \\
&X_x = 0, \quad X_t = a_1^3, \quad tX_t - bX_b = X + a_2^3, \quad uX_u + 2hX_h + bX_b = 0, \\
&U_t = U_x = U_b = 0, \quad uU_u + 2hU_h = -U, \\
&H_t = H_x = H_b = 0, \quad uH_u + 2hH_h = -2H, \\
&B_t = B_x = 0, \quad bB_b = B, \quad uB_u + 2hB_h = -3B.
\end{aligned}$$

The general solution of the last system is

$$\begin{aligned} &T = a^1_3 x - a^1_4 + \frac{u}{b}F^1\left(\frac{h}{u^2}\right), \quad X = a^3_1 t - a^3_2 + \frac{u}{b}F^2\left(\frac{h}{u^2}\right), \\ &U = \frac{1}{u}F^3\left(\frac{h}{u^2}\right), \quad H = \frac{1}{u^2}F^4\left(\frac{h}{u^2}\right), \quad B = \frac{b}{u^3}F^5\left(\frac{h}{u^2}\right), \end{aligned} \tag{3.15}$$

where $F^i$, $i = 1, \ldots, 5$, are arbitrary smooth functions of $h/u^2$.

We compute the final form of the transformation $\mathcal{T}$ using the direct method. We express all the required transformed derivatives $\tilde{u}_{\tilde{t}}$, $\tilde{u}_{\tilde{x}}$, $\tilde{h}_{\tilde{t}}$, $\tilde{h}_{\tilde{x}}$ and $\tilde{b}_{\tilde{t}}$ in terms of the initial coordinates using (3.12) and substitute the obtained expressions into the copy (3.13) of the system (3.8). The resulting system should be identically satisfied by each solution of the system (3.8) and processing this system in a way similar to the previous case, we get the equations $a^1_3 = a^3_1 = 0$, which are in contradiction to the nondegeneracy of the matrix $\mathcal{A}_2$. Thus, we obtain no point symmetry transformations in this case.

Recall that the generalized equivalence group $G^\sim$ for the class (3.7) can be identified with the complete point symmetry group $G$ of the extended system (3.8).

**Theorem 27.** *The generalized equivalence group $G$ of the class (3.7) consists of the transformations*

$$T = \delta_1 t + \delta_2, \quad X = \delta_3 x + \delta_4, \quad U = \frac{\delta_3}{\delta_1}u, \quad H = \left(\frac{\delta_3}{\delta_1}\right)^2 h, \quad B = \frac{\delta_3}{\delta_1^2}b,$$

*where $\delta_1$, …, $\delta_4$ are arbitrary real constants with $\delta_1\delta_3 \neq 0$.*

Since the components $T$, $X$, $U$ and $H$ of elements of the generalized equivalence group of the class (3.7) do not depend on the arbitrary element $b$, then these elements are usual equivalence transformations of the class (3.7).

**Corollary 28.** *The generalized equivalence group $G^\sim$ of the class (3.7) coincides with the usual equivalence group of this class.*

**Corollary 29.** *A complete list of discrete equivalence transformations of the class (3.7), which are independent up to combining with each other and with continuous symmetries is exhausted by the two reflections*

$$(t, x, u, h, b) \mapsto (-t, x, -u, h, b), \quad (t, x, u, h, b) \mapsto (t, -x, -u, h, -b).$$

**Corollary 30.** *The factor group of the generalized equivalence group $G^{\sim}$ of the class of one-dimensional shallow water equations* (3.7)*, with respect to its identity component is isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

## 3.3 Comparison of two approaches

We compare the computations of the generalized equivalence groups of the classes (3.2) and (3.7), which in fact coincide up to re-parametrization. Each of the two computational approaches has its advantages and disadvantages.

In the course of the preliminary analysis, it seems that finding the equivalence group of the re-parameterized class (3.7) is more effortless because the equivalence algebra $\mathfrak{h}^{\sim}$ of the class (3.7) is of lower dimension than the algebra $\mathfrak{g}^{\sim}$ of the class (3.2), $\dim \mathfrak{h}^{\sim} = 4 < \dim \mathfrak{g}^{\sim} = 5$. The dimension of the algebra $\mathfrak{g}^{\sim}$ is greater since the class (3.2), which is not re-parameterized, admits equivalence transformations that change the form of the arbitrary element $b$ but leave any system of the class (3.2) unchanged. This can be explained by the fact that the arbitrary element $b$ of the class (3.2) appears in a system only as a derivative $b_x$, and thus the equivalence transformations of shifts with respect to $b$ do not change any system from the class (3.2). Such transformations are called gauge equivalence transformations [47]. The gauge transformations of the class (3.2) disappear after the re-parametrization to the class (3.7). Despite this preliminary analysis, the full examination shows that the first way (Section 3.1) is more convenient than the second one (Section 3.2).

The equivalence algebra $\mathfrak{g}^{\sim}$ of the class (3.2) is isomorphic to the algebra $\mathrm{g}_{5,33}$ with $\beta = -2$ and $\gamma = 2$ from Mubarakzianov's classification [35] of real five-dimensional Lie algebras. Note that, the algebra is solvable and non-decomposable. The computation of the automorphism group of lower dimensional Lie algebras is algorithmic and, therefore, can easily be carried out with the help of computer algebra systems. The automorphism group of the algebra $\mathfrak{g}^{\sim}$ can be found in [20], where the algebra $\mathrm{g}_{5,33}$ is denoted by $A_{5,33}$ with $\beta = v$ and $\gamma = u$. According to the procedure of the algebraic method, we have to consider five conditions for pushforwards of the basis vector fields, which increases the amount of computations. On the other hand, these conditions lead to the nice intermediate form for the components of equivalence transformations. As a result the further application of the direct method is easy. More specifically, the obtained form is twice fiber-preserving, i.e., the transformation components for the independent variables do not depend on the dependent variables, as well as the $h$-component does not depend on the arbitrary element $b$. All these restrictions make the successive computation within the framework of the direct method less complicated and cumbersome.

Unlike the equivalence algebra $\mathfrak{g}^{\sim}$ of the class (3.2), the algebra $\mathfrak{h}^{\sim}$ of the class (3.7) is solvable, decomposable and four-dimensional because the gauge equivalence transformations of the class (3.2) are factored out due to the re-parametrization. Its automorphism group has been computed and presented earlier in [17, 45]. Unfortunately, the general form of the automorphisms matrices of the algebra $\mathfrak{h}^{\sim}$ cannot be written uniformly. This is the drawback that doubles the amount of necessary computations since we need to repeat the entire procedure of computations within the framework of the algebraic method separately for each of the two forms of automorphism matrices. The computations for matrices of the form $\mathcal{A}_1$ lead to the equivalence transformations of the class (3.7), while the same computations with matrices of the form $\mathcal{A}_2$ end up with a contradiction and thus we get no equivalence transformations. Moreover, the intermediate form of the components of equivalence transformations that is constructed in the course of purely algebraic steps of the procedure is not fiber-preserving at all. Thus, the further application of the direct method is essentially more tricky.

# Chapter 4

# Group classification of two-dimensional shallow water equations

Consider the class of systems of two-dimensional shallow water equations

$$
\begin{aligned}
&u_t + uu_x + vu_y + h_x = b_x,\\
&v_t + uv_x + vv_y + h_y = b_y,\\
&h_t + (uh)_x + (vh)_y = 0.
\end{aligned}
\tag{4.1}
$$

Here $(u, v)$ is the horizontal fluid velocity averaged over the height of the fluid column, $h$ is the thickness of a fluid column and $b = b(x, y)$ is a parameter function that is the bottom topography measured downward with respect to a fixed reference level. In this class, $(t, x, y)$ is the tuple of independent variables, $(u, v, h)$ is the tuple of the dependent variables and $b$ is considered to be the arbitrary element of the class. These values are graphically represented in Figure 4.1. According to the interpretation of $b$ as a varying arbitrary element or a fixed function, we will refer to (4.1) as to a class of systems of differential equations or to a fixed system.

The complete system of auxiliary equations for the arbitrary element $b$ of the class (4.1) consists of the equations

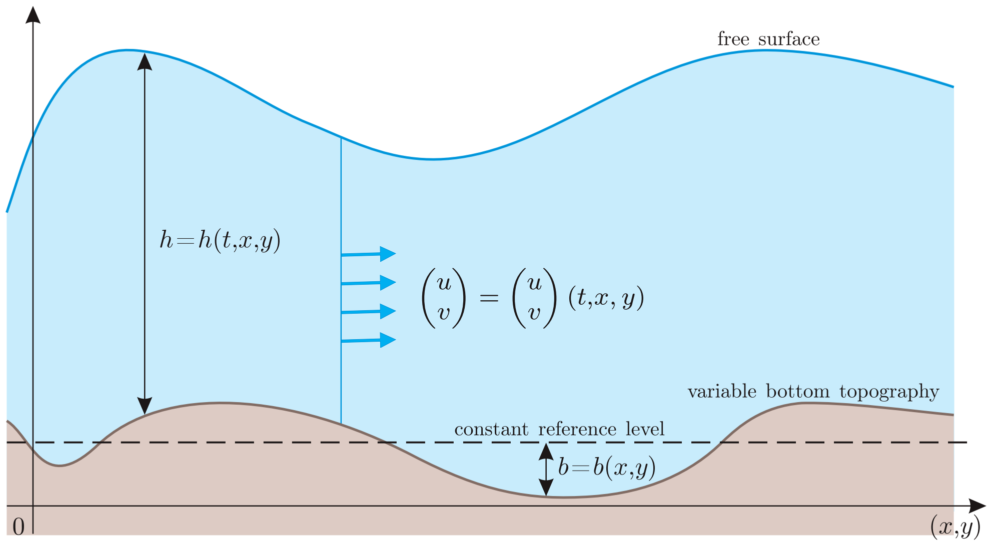


Figure 4.1: The shallow water model.

$$
\begin{aligned}
&b_t = 0, \\
&b_u = b_{u_t} = b_{u_x} = b_{u_y} = 0, \\
&b_v = b_{v_t} = b_{v_x} = b_{v_y} = 0, \\
&b_h = b_{h_t} = b_{h_x} = b_{h_y} = 0.
\end{aligned}
$$

Note that there are no auxiliary inequalities for the arbitrary element $b$.

## 4.1 Equivalence group

The arbitrary element $b$ depends only on independent variables. Therefore, we can treat it as one more dependent variable and consider the extended system

$$
\begin{aligned}
&u_t + uu_x + vu_y + h_x = b_x, \\
&v_t + uv_x + vv_y + h_y = b_y, \\
&h_t + (uh)_x + (vh)_y = 0, \\
&b_t = 0.
\end{aligned}
\tag{4.2}
$$

Here we also use the fact that the arbitrary element $b$ does not depend on $t$ as well.

Since the arbitrary element $b$ does not involve derivative of dependent variables, the generalized equivalence group $G^\sim$ of the class (4.1) can be assumed to act in the space with the coordinates $(t, x, y, u, v, h)$ and thus to coincide with the point symmetry group $G$ of the system (4.2). Analogously, the generalized equivalence algebra $\mathfrak{g}^\sim$ of the class (4.1) can be identified with the maximal Lie invariance algebra $\mathfrak{g}$ of the system (4.2). This is why it suffices to find $\mathfrak{g}$ and $G$ instead of $\mathfrak{g}^\sim$ and $G^\sim$, respectively.

To construct the group $G$, we invoke the algebraic method (see Section 2.3), which was suggested in [25] and further developed in [5, 7, 12]. For this, we need first to compute the algebra $\mathfrak{g}$, and the infinitesimal method [8,9,39,42] is relevant here. The algebra $\mathfrak{g}$ consists of the infinitesimal generators of one-parameter point symmetry groups of the system (4.2), which are vector fields in the space with coordinates $(t, x, y, u, v, h, b)$,

$$\mathbf{v} = \tau\partial_t + \xi^1\partial_x + \xi^2\partial_y + \eta^1\partial_u + \eta^2\partial_v + \eta^3\partial_h + \eta^4\partial_b,$$

where the components $\tau$, $\xi^1$, $\xi^2$ and $\eta^i$, $i = 1, 2, 3, 4$, are smooth functions of these coordinates. For convenience, hereafter we simultaneously use the notation $(w^1, w^2, w^3, w^4)$ for $(u, v, h, b)$. The infinitesimal invariance criterion implies that

$$\begin{aligned}
&\mathrm{pr}^{(1)}\,\mathbf{v}(w^1_t + w^1w^1_x + w^2w^1_y + w^3_x - w^4_x) = 0,\\
&\mathrm{pr}^{(1)}\,\mathbf{v}(w^2_t + w^1w^2_x + w^2w^2_y + w^3_y - w^4_y) = 0,\\
&\mathrm{pr}^{(1)}\,\mathbf{v}(w^3_t + (w^1w^3)_x + (w^2w^3)_y) = 0,\\
&\mathrm{pr}^{(1)}\,\mathbf{v}(w^4_t) = 0,
\end{aligned} \tag{4.3}$$

whenever the system (4.2) holds. Here $\mathrm{pr}^{(1)}\,\mathbf{v}$ is the first order prolongation of the vector field $\mathbf{v}$,

$$\mathrm{pr}^{(1)}\,\mathbf{v} = \mathbf{v} + \sum_{i=1}^{4}(\eta^{it}\partial_{w^i_t} + \eta^{ix}\partial_{w^i_x} + \eta^{iy}\partial_{w^i_y}),$$

where $\eta^{it} = \mathrm{D}_t(\eta^i - \tau w^i_t - \xi^1 w^i_x - \xi^2 w^i_y) + \tau w^i_{tt} + \xi^1 w^i_{tx} + \xi^2 w^i_{ty}$, and similarly for $\eta^{ix}$ and $\eta^{iy}$.

We substitute the expressions for $w^i_t$, $i = 1, \dots 4$, in view of the system (4.2) into the expanded equations (4.3), and then split them with respect to the derivatives $w^i_x$ and $w^i_y$, $i = 1, \dots, 4$. This procedure results in the system of differential equations on the components $\tau$, $\xi^1$, $\xi^2$ and $\eta^i$, $i = 1, \dots, 4$, of the vector field $\mathbf{v}$, which are called the determining equations. Integrating this system, we derive the explicit form of the vector field components,

$$\tau = (c_5 - c_7)t + c_1, \quad \xi^1 = c_5 x + c_6 y + c_2, \quad \xi^2 = -c_6 x + c_5 y + c_3,$$
$$\eta^1 = c_7 u + c_6 v, \quad \eta^2 = -c_6 u + c_7 v, \quad \eta^3 = 2c_7 h, \quad \eta^4 = 2c_7 b + c_4,$$

where $c_1, \dots, c_7$ are arbitrary real constants.

Thus, the maximal Lie invariance algebra $\mathfrak{g}$ of the system (4.2) is spanned by the seven vector fields[1]

$$P^t = \partial_t, \quad P^x = \partial_x, \quad P^y = \partial_y, \quad P^b = \partial_b,$$
$$D^1 = x\partial_x + y\partial_y + u\partial_u + v\partial_v + 2h\partial_h + 2b\partial_b,$$
$$D^2 = t\partial_t - u\partial_u - v\partial_v - 2h\partial_h - 2b\partial_b, \quad J = x\partial_y - y\partial_x + u\partial_v - v\partial_u.$$

Let us fix the basis $\mathcal{B} = (P^t, P^x, P^y, P^b, D^1, D^2, J)$ of the Lie algebra $\mathfrak{g}$. Up to anticommutativity of the Lie bracket of vector fields, the only nonzero commutation relations between the basis elements are

$$[P^x, D^1] = P^x, \quad [P^y, D^1] = P^y, \quad [P^b, D^1] = 2P^b,$$
$$[P^t, D^2] = P^t, \quad [P^b, D^2] = -2P^b, \quad [P^x, J] = P^y, \quad [P^y, J] = -P^x.$$

In other words, the complete list of nonzero structure constants of the Lie algebra $\mathfrak{g}$ in the basis $\mathcal{B}$ is exhausted, up to permutation of subscripts, by

$$c^2_{25} = 1, \quad c^3_{35} = 1, \quad c^4_{45} = 2, \quad c^1_{16} = 1, \quad c^4_{46} = -2, \quad c^3_{27} = 1, \quad c^2_{37} = -1.$$

The general form $A = (a^i_j)^7_{i,j=1}$ of automorphism matrices of the algebra $\mathfrak{g}$ in the basis $\mathcal{B}$ can be found via solving the system of algebraic equations

$$c^{k'}_{i'j'} a^{i'}_i a^{j'}_j = c^k_{ij} a^{k'}_k, \quad i, j = 1, \dots, 7, \tag{4.4}$$

[1]The components of vector fields from $\mathfrak{g}$ that correspond to the independent variables $(t, x, y)$ and dependent variables $(u, v, h)$ of the system (4.1) do not depend on the arbitrary element $b$. Interpreting this result in terms of equivalence algebras, we obtain that the generalized equivalence algebra $\mathfrak{g}^\sim$ of the class (4.1) coincides with its usual equivalence algebra.

under the condition $\det A \neq 0$. Here we assume the summation over the repeated indices. As a result, we obtain that the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{g}$ can be identified with the matrix group that consists of the matrices of the general form

$$A = \begin{pmatrix} a^1_1 & 0 & 0 & 0 & 0 & a^1_6 & 0 \\ 0 & a^2_2 & -\varepsilon a^3_2 & 0 & a^2_5 & 0 & a^2_7 \\ 0 & a^3_2 & \varepsilon a^2_2 & 0 & -\varepsilon a^2_7 & 0 & \varepsilon a^2_5 \\ 0 & 0 & 0 & a^4_4 & -a^4_6 & a^4_6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon \end{pmatrix},$$

where $\varepsilon = \pm 1$, and the remaining parameters $a^i_j$'s are arbitrary real constants with

$$a^1_1\big((a^2_2)^2 + (a^3_2)^2\big)a^4_4 \neq 0.$$

**Theorem 31.** *A complete list of discrete symmetry transformations of the extended system (4.2) that are independent up to combining with each other and with continuous symmetry transformations of this system is exhausted by two transformations alternating signs of variables,*

$$(t, x, y, u, v, h, b) \mapsto (-t, x, y, -u, -v, h, b),$$
$$(t, x, y, u, v, h, b) \mapsto (t, x, -y, u, -v, h, b).$$

*Proof.* The maximal Lie invariance algebra $\mathfrak{g}$ of the system (4.2) is finite-dimensional and nontrivial. The complete automorphism group $\mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{g}$ is computed above. It is not much wider than the inner automorphism group $\mathrm{Inn}(\mathfrak{g})$ of $\mathfrak{g}$, which is constituted by the linear operators on $\mathfrak{g}$ with matrices of the form

$$\begin{pmatrix} e^{-\theta_6} & 0 & 0 & 0 & 0 & \theta_1 & 0 \\ 0 & e^{-\theta_5}\cos\theta_7 & e^{-\theta_5}\sin\theta_7 & 0 & \theta_2 & 0 & -\theta_3 \\ 0 & -e^{-\theta_5}\sin\theta_7 & e^{-\theta_5}\cos\theta_7 & 0 & \theta_3 & 0 & \theta_2 \\ 0 & 0 & 0 & e^{2\theta_6-2\theta_5} & 2\theta_4 & -2\theta_4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where the parameters $\theta_1, \dots, \theta_7$ are arbitrary constants. Continuous point symmetries of the system (4.2) can be easily found by composing elements of one-parameter groups generated by basis elements $\mathfrak{g}$. Moreover, such symmetries constitute the connected component of the identity transformation in the group $G$, which induces the entire group $\mathrm{Inn}(\mathfrak{g})$. This is why it suffices to look only for discrete symmetry transformations, and in the course of the related computation within the framework of the algebraic method one can factor out inner automorphisms. The quotient group $\mathrm{Aut}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$ can be identified with the matrix group consisting of the diagonal matrices of the form $\mathrm{diag}(\varepsilon', 1, \varepsilon, \varepsilon'', 1, 1, \varepsilon)$, where $\varepsilon, \varepsilon', \varepsilon'' = \pm 1$. Suppose that the push-forward $\mathcal{T}_*$ of vector fields in the space with the coordinates $(t, x, y, u, v, h, b)$ by a point transformation

$$\mathcal{T}\colon\quad (\tilde t, \tilde x, \tilde y, \tilde u, \tilde v, \tilde h, \tilde b) = (T, X, Y, U, V, H, B)(t, x, y, u, v, h, b)$$

generates the automorphism of $\mathfrak{g}$ with the matrix $\mathrm{diag}(\varepsilon', 1, \varepsilon, \varepsilon'', 1, 1, \varepsilon)$, i.e.,

$$\begin{aligned}
&\mathcal{T}_*P^t = \varepsilon'\tilde P^t, \quad \mathcal{T}_*P^x = \tilde P^x, \quad \mathcal{T}_*P^y = \varepsilon\tilde P^y, \quad \mathcal{T}_*P^b = \varepsilon''\tilde P^b,\\
&\mathcal{T}_*D^1 = \tilde D^1, \quad \mathcal{T}_*D^2 = \tilde D^2, \quad \mathcal{T}_*J = \varepsilon\tilde J.
\end{aligned}$$

Here tildes over vector fields mean that these vector fields are given in the new coordinates. The above conditions for $\mathcal{T}_*$ imply a system of differential equations for the components of $\mathcal{T}$,

$$\begin{aligned}
&T_t = \varepsilon', \quad T = tT_t, \quad T_x = T_y = T_u = T_v = T_h = T_b = 0,\\
&X_x = 1, \quad X = xX_x, \quad X_t = X_y = X_u = X_v = X_h = X_b = 0,\\
&Y_y = \varepsilon, \quad Y = yY_y, \quad Y_t = Y_x = Y_u = Y_v = Y_h = Y_b = 0,\\
&U_t = U_x = U_y = U_b = 0, \quad V_t = V_x = V_y = V_b = 0,\\
&vU_u - uU_v = \varepsilon V, \quad uU_u + vU_v + 2hU_h = U,\\
&vV_u - uV_v = -\varepsilon U, \quad uV_u + vV_v + 2hV_h = V,\\
&H_t = H_x = H_y = H_b = 0, \quad vH_u - uH_v = 0, \quad uH_u + vH_v + 2hH_h = 2H,\\
&B_b = \varepsilon'', \quad B_t = B_x = B_y = 0, \quad vB_u - uB_v = 0,\\
&uB_u + vB_v + 2hB_h = 2B - 2\varepsilon'' b.
\end{aligned}$$

The general solution of the system is

$$
\begin{aligned}
&T=\varepsilon' t, \quad X=x, \quad Y=\varepsilon y,\\
&U=uF_1\left(\frac{h}{u^2+v^2}\right)+\varepsilon vF_2\left(\frac{h}{u^2+v^2}\right),\\
&V=-uF_2\left(\frac{h}{u^2+v^2}\right)+\varepsilon vF_1\left(\frac{h}{u^2+v^2}\right),\\
&H=(u^2+v^2)F_3\left(\frac{h}{u^2+v^2}\right), \quad B=\varepsilon'' b+(u^2+v^2)F_4\left(\frac{h}{u^2+v^2}\right),
\end{aligned}
$$

where $F_1$, $F_2$, $F_3$ and $F_4$ are arbitrary smooth functions of $h/(u^2+v^2)$.

We continue the computations within the framework of the direct method in order to complete the system of constraints for $\mathcal{T}$. Using the chain rule, we express all required transformed derivatives $\tilde w^i_{\tilde t}$, $\tilde w^i_{\tilde x}$, $\tilde w^i_{\tilde y}$, $i=1,\dots,4$, in terms of the initial coordinates. Then we substitute the obtained expressions into the copy of the system (4.2) in the new coordinates. The expanded system should identically be satisfied by each solution of the system (4.2). This condition implies that

$$
T=\varepsilon' t, \quad X=x, \quad Y=\varepsilon y, \quad U=\varepsilon' u, \quad V=\varepsilon\varepsilon' v, \quad H=h, \quad B=b.
$$

Therefore, discrete symmetries of the equation (4.2) are exhausted, up to combining with continuous symmetries and with each other, by the two involutions

$$
\begin{aligned}
&(t,x,y,u,v,h,b)\mapsto(-t,x,y,-u,-v,h,b),\\
&(t,x,y,u,v,h,b)\mapsto(t,x,-y,u,-v,h,b),
\end{aligned}
$$

which are associated with the values $(\varepsilon',\varepsilon)=(-1,1)$ and $(\varepsilon',\varepsilon)=(1,-1)$, respectively. □

**Corollary 32.** *The factor group of the complete point symmetry group $G$ of the extended system (4.2), with respect to its identity component is isomorphic to the group $\mathbb{Z}_2\times\mathbb{Z}_2$.*

The complete point symmetry group $G$ of the extended system (4.2) is generated by one-parameter point transformation groups associated with vector fields from the algebra $\mathfrak{g}$ and two discrete transformations given in Theorem 31.

**Corollary 33.** *The complete point symmetry group $G$ of the extended system (4.2) consists of the transformations*

$$\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \delta_3 x - \varepsilon\delta_4 y + \delta_5, \quad \tilde{y} = \delta_4 x + \varepsilon\delta_3 y + \delta_6,$$

$$\tilde{u} = \frac{\delta_3}{\delta_1}u - \varepsilon\frac{\delta_4}{\delta_1}v, \quad \tilde{v} = \frac{\delta_4}{\delta_1}u + \varepsilon\frac{\delta_3}{\delta_1}v, \quad \tilde{h} = \frac{\delta_3^2 + \delta_4^2}{\delta_1^2}h, \quad \tilde{b} = \frac{\delta_3^2 + \delta_4^2}{\delta_1^2}b + \delta_7,$$

*where $\varepsilon = \pm 1$ and the parameters $\delta_i$, $i = 1, \dots, 7$, are arbitrary constants with*

$$\delta_1({\delta_3}^2 + {\delta_4}^2) \neq 0.$$

Since the generalized equivalence group $G^{\sim}$ of the class of two-dimensional shallow water equations (4.1) coincides with the complete point symmetry group $G$ of the system (4.2), we can rephrase Theorem 31 and Corollary 33 in terms of equivalence transformations of the class (4.1).

**Theorem 34.** *A complete list of discrete equivalence transformations of the class of two-dimensional shallow water equations (4.1) that are independent up to combining with each other and with continuous equivalence transformations of this class is exhausted by two transformations alternating signs of variables,*

$$(t, x, y, u, v, h, b) \mapsto (-t, x, y, -u, -v, h, b),$$

$$(t, x, y, u, v, h, b) \mapsto (t, x, -y, u, -v, h, b).$$

**Theorem 35.** *The generalized equivalence group $G^{\sim}$ of the class of two-dimensional systems of shallow water equations (4.1) coincides with the usual equivalence group of this class and consists of the transformations*

$$\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \delta_3 x - \varepsilon\delta_4 y + \delta_5, \quad \tilde{y} = \delta_4 x + \varepsilon\delta_3 y + \delta_6,$$

$$\tilde{u} = \frac{\delta_3}{\delta_1}u - \varepsilon\frac{\delta_4}{\delta_1}v, \quad \tilde{v} = \frac{\delta_4}{\delta_1}u + \varepsilon\frac{\delta_3}{\delta_1}v, \quad \tilde{h} = \frac{\delta_3^2 + \delta_4^2}{\delta_1^2}h, \quad \tilde{b} = \frac{\delta_3^2 + \delta_4^2}{\delta_1^2}b + \delta_7,$$

*where $\varepsilon = \pm 1$ and the parameters $\delta_i$, $i = 1, \dots, 7$, are arbitrary constants with*

$$\delta_1({\delta_3}^2 + {\delta_4}^2) \neq 0.$$

## 4.2 Preliminary analysis and classification result

Let $\mathcal{L}_b$ be a system from the class (4.1) with a fixed value of the arbitrary element $b$ and suppose that a vector field $\mathbf{v}$ of the general form

$$\mathbf{v} = \tau(t,x,y,u,v,h)\partial_t + \xi^1(t,x,y,u,v,h)\partial_x + \xi^2(t,x,y,u,v,h)\partial_y + \eta^1(t,x,y,u,v,h)\partial_u + \eta^2(t,x,y,u,v,h)\partial_v + \eta^3(t,x,y,u,v,h)\partial_h$$

defined in the space with the coordinates $(t,x,y,u,v,h)$ is the infinitesimal generator of a one-parameter Lie symmetry group for the system $\mathcal{L}_b$. The set of such vector fields is the maximal Lie invariance algebra $\mathfrak{g}_b$ of the system $\mathcal{L}_b$.

The infinitesimal invariance criterion requires that

$$\mathrm{pr}^{(1)}\,\mathbf{v}(\mathcal{L}_b)\big|_{\mathcal{L}_b} = 0. \tag{4.5}$$

The first prolongation $\mathrm{pr}^{(1)}\,\mathbf{v}$ of the vector field $\mathbf{v}$ is computed similarly to the previous section. We expand the condition (4.5) and confine it on the manifold defined by $\mathcal{L}_b$ in the corresponding first-order jet space, assuming the first-order derivatives of the dependent variables $(u,v,h)$ with respect to $t$ as the leading ones and substituting for these derivatives in view of the system $\mathcal{L}_b$,

$$u_t = -uu_x - vu_y - h_x + b_x,$$

$$v_t = -uv_x - vv_y - h_y + b_y,$$

$$h_t = -(uh)_x - (vh)_y.$$

Then we split the obtained equations with respect to the first-order parametric derivatives, which are the first-order derivatives of the dependent variables $(u,v,h)$ with respect $x$ and $y$. After an additional rearrangement and excluding equations which are differential consequences of the other, we derive the system of determining equations for the components of the vector field $\mathbf{v}$,

$$\begin{aligned}
&\tau_x = \tau_y = \tau_u = \tau_v = \tau_h = 0,\\
&\xi^1_u = \xi^1_v = \xi^1_h = 0, \quad \xi^2_u = \xi^2_v = \xi^2_h = 0, \quad \xi^1_x = \xi^2_y, \quad \xi^1_y + \xi^2_x = 0,\\
&\eta^1 = (\xi^1_x - \tau_t)u + \xi^1_y v + \xi^1_t,\\
&\eta^2 = \xi^2_x u + (\xi^2_y - \tau_t)v + \xi^2_t,\\
&\eta^3 = 2(\xi^1_x - \tau_t)h,\\
&\eta^1_t + u\eta^1_x + v\eta^1_y + \eta^3_x + (\eta^1_u - \tau_t)b_x + \eta^1_v b_y = \xi^1 b_{xx} + \xi^2 b_{xy},\\
&\eta^2_t + u\eta^2_x + v\eta^2_y + \eta^3_y + \eta^2_u b_x + (\eta^2_v - \tau_t)b_y = \xi^1 b_{xy} + \xi^2 b_{yy},\\
&\eta^3_t + u\eta^3_x + v\eta^3_y + h\eta^1_x + h\eta^2_y = 0.
\end{aligned} \tag{4.6}$$

Integrating the subsystem of the system (4.6) that consists of the equations not containing the arbitrary element $b$, we get the following form of the components of the vector field $\mathbf{v}$

$$\begin{aligned}
&\tau = 2F^1 - c_1 t,\\
&\xi^1 = F^1_t x + F^0 y + F^2,\\
&\xi^2 = -F^0 x + F^1_t y + F^3,\\
&\eta^1 = (-F^1_t + c_1)u + F^0 v + F^1_{tt} x + F^2_t,\\
&\eta^2 = -F^0 u + (-F^1_t + c_1)v + F^1_{tt} y + F^3_t,\\
&\eta^3 = 2(-F^1_t + c_1)h,
\end{aligned} \tag{4.7}$$

where $F^i$, $i = 1, 2, 3, 4$, are sufficiently smooth functions of $t$, and $c_1$ is a constant. From the last two equations of the system (4.6), we derive as a differential consequence that $F^0_t = 0$. Thus, $F^0$ is a constant, and we will denote $c_2 := F^0$. In other words, for any $b$

$$\mathfrak{g}_b \subset \mathfrak{g}_{\langle\rangle} := \langle D(F^1),\, D^t,\, J,\, P(F^2, F^3)\rangle$$

where the parameters $F^1$, $F^2$ and $F^3$ runs through the set of smooth functions of $t$,

$$\begin{aligned}
D(F^1) := {}& F^1\partial_t + \tfrac{1}{2}F^1_t x\partial_x + \tfrac{1}{2}F^1_t y\partial_y\\
&- \tfrac{1}{2}(F^1_t u - F^1_{tt} x)\partial_u - \tfrac{1}{2}(F^1_t v - F^1_{tt} y)\partial_v - F^1_t h\partial_h,\\
D^t := {}& t\partial_t - u\partial_u - v\partial_v - 2h\partial_h, \quad J := x\partial_y - y\partial_x + u\partial_v - v\partial_u,\\
P(F^2, F^3) := {}& F^2\partial_x + F^3\partial_y + F^2_t\partial_u + F^3_t\partial_v.
\end{aligned}$$

It is convenient to denote $D^x := 2D(t) - 2D^t = x\partial_x + y\partial_y + u\partial_u + v\partial_v + 2h\partial_h$.

For elements of $\mathfrak{g}_b$, the parameters $F^1$, $F^2$, $F^3$, $c_1$ and $c_2$ additionally satisfy two equations implied by the last two equations from (4.6), which explicitly involve the arbitrary element $b$ and thus are the classifying equations for the class (4.1). They can be integrated to the single equation

$$\begin{aligned}&(F^1_t x + c_2 y + F^2)b_x + (-c_2 x + F^1_t y + F^3)b_y + 2(F^1_t - c_1)b\\ &\qquad - F^1_{ttt}\frac{x^2+y^2}{2} - F^2_{tt}x - F^3_{tt}y - F^4 = 0,\end{aligned} \tag{4.8}$$

where $F^4$ is one more sufficiently smooth parameter function of $t$. The equation (4.8) can be considered as the only classifying equation instead the above ones. Thus, the group classification problem for the class (4.1) reduces to solving the equation (4.8) up to $G^\sim$-equivalence with respect to the arbitrary element $b$ and the parameters $F^1$, …, $F^4$, $c_1$ and $c_2$.

In the next theorem and in Section 4.3, it is convenient to use, simultaneously with $(x, y)$, the polar coordinates $(r, \varphi)$ on the $(x, y)$-plane,

$$r := \sqrt{x^2 + y^2}, \quad \varphi := \arctan\frac{y}{x}.$$

**Theorem 36.** *The kernel Lie invariance algebra of systems from the class (4.1) is $\mathfrak{g}^\cap = \langle D(1)\rangle$. A complete list of $G^\sim$-inequivalent Lie symmetry extensions within the class (4.1) is exhausted by the following cases, where $f$ denotes an arbitrary smooth function of a single argument, $\alpha$, $\beta$, $\mu$ and $\nu$ are arbitrary constants with $\alpha \geqslant 0 \bmod G^\sim$, $\beta > 0$ and additional constraints indicated in the corresponding cases, $\varepsilon = \pm 1 \bmod G^\sim$ and $\delta \in \{0, 1\} \bmod G^\sim$.*

1. $b = r^\nu f(\varphi + \alpha \ln r)$, $(\alpha, \nu) \neq (0, -2)$, $\nu \neq 0$:
   $\mathfrak{g}_b = \big\langle D(1),\ 4D(t) - (\nu + 2)D^t - 2\alpha J\big\rangle$;

2. $b = f(\varphi + \alpha \ln r) + \nu \ln r$, $\nu \in \{-1, 0, 1\} \bmod G^\sim$: $\mathfrak{g}_b = \big\langle D(1),\ 2D(t) - D^t - \alpha J\big\rangle$;

3. $b = f(r) + \delta\varphi$: $\mathfrak{g}_b = \big\langle D(1),\ J\big\rangle$;

4. $b = f(r)e^{\beta\varphi}$: $\mathfrak{g}_b = \big\langle D(1),\ 2J - \beta D^t\big\rangle$;

5. $b = f(y)e^x$: $\mathfrak{g}_b = \big\langle D(1),\ D^t - P(2, 0)\big\rangle$;

6. (a) $b=r^{-2}f(\varphi)$: $\mathfrak{g}_b=\big\langle D(1),\,D(t),\,D(t^2)\big\rangle$;

   (b) $b=r^{-2}f(\varphi)+\frac12 r^2$: $\mathfrak{g}_b=\big\langle D(1),\,D(e^{2t}),\,D(e^{-2t})\big\rangle$;

   (c) $b=r^{-2}f(\varphi)-\frac12 r^2$: $\mathfrak{g}_b=\big\langle D(1),\,D(\cos 2t),\,D(\sin 2t)\big\rangle$;

7. $b=f(y)+\delta x$: $\mathfrak{g}_b=\big\langle D(1),\,P(1,0),\,P(t,0)\big\rangle$;

8. $b=f(y)+\frac12 x^2$: $\mathfrak{g}_b=\big\langle D(1),\,P(e^t,0),\,P(e^{-t},0)\big\rangle$;

9. $b=f(y)-\frac12 x^2$: $\mathfrak{g}_b=\big\langle D(1),\,P(\cos t,0),\,P(\sin t,0)\big\rangle$;

10. $b=\delta\varphi-\nu\ln r$, $\nu=\pm1 \bmod G^{\sim}$ if $\delta=0$: $\mathfrak{g}_b=\big\langle D(1),\,2D(t)-D^t,\,J\big\rangle$;

11. $b=\varepsilon r^{\nu}e^{\alpha\varphi}$, $\nu\neq-2$, $(\alpha,\nu)\notin\{(0,0),(0,2)\}$:
    $\mathfrak{g}_b=\big\langle D(1),\,4D(t)-(\nu+2)D^t,\,2J-\alpha D^t\big\rangle$;

12. (a) $b=\varepsilon r^{-2}e^{\alpha\varphi}$: $\mathfrak{g}_b=\big\langle D(1),\,D(t),\,D(t^2),\,\alpha D^x+4J\big\rangle$;

    (b) $b=\varepsilon r^{-2}e^{\alpha\varphi}+\frac12 r^2$: $\mathfrak{g}_b=\big\langle D(1),\,D(e^{2t}),\,D(e^{-2t}),\,\alpha D^x+4J\big\rangle$;

    (c) $b=\varepsilon r^{-2}e^{\alpha\varphi}-\frac12 r^2$: $\mathfrak{g}_b=\big\langle D(1),\,D(\cos 2t),\,D(\sin 2t),\,\alpha D^x+4J\big\rangle$;

13. $b=\varepsilon|y|^{\nu}+\delta x$, $\nu\notin\{-2,0,2\}$:
    $\mathfrak{g}_b=\big\langle D(1),\,4D(t)-(\nu+2)D^t-\delta(\nu-1)P(t^2,0),\,P(1,0),\,P(t,0)\big\rangle$;

14. $b=\varepsilon\ln|y|+\delta x$: $\mathfrak{g}_b=\big\langle D(1),\,2D(t)-D^t-\frac12\delta P(t^2,0)),\,P(1,0),\,P(t,0)\big\rangle$;

15. $b=\varepsilon e^y+\delta x$: $\mathfrak{g}_b=\big\langle D(1),\,D^t-P(\delta t^2,2),\,P(1,0),\,P(t,0)\big\rangle$;

16. (a) $b=\varepsilon y^{-2}+\delta x$:
    $\mathfrak{g}_b=\big\langle D(1),\,D(t)+\frac34\delta P(t^2,0),\,D(t^2)+\frac12\delta P(t^3,0),\,P(1,0),\,P(t,0)\big\rangle$;

    (b) $b=\varepsilon y^{-2}+\frac12 r^2$:
    $\mathfrak{g}_b=\big\langle D(1),\,D(e^{2t}),\,D(e^{-2t}),\,P(e^t,0),\,P(e^{-t},0)\big\rangle$;

    (c) $b=\varepsilon y^{-2}-\frac12 r^2$:
    $\mathfrak{g}_b=\big\langle D(1),\,D(\cos 2t),\,D(\sin 2t),\,P(\cos t,0),\,P(\sin t,0)\big\rangle$;

17. $b=\frac12 x^2+\frac12\beta^2y^2$, $0<\beta<1$:
    $\mathfrak{g}_b=\big\langle D(1),\,D^x,\,P(e^t,0),\,P(e^{-t},0),\,P(0,e^{\beta t}),\,P(0,e^{-\beta t})\big\rangle$;

18. $b=\frac12 x^2+\delta y$:
    $\mathfrak{g}_b=\big\langle D(1),\,D^x-\frac12\delta P(0,t^2),\,P(e^t,0),\,P(e^{-t},0),\,P(0,1),\,P(0,t)\big\rangle$;

19. $b=\frac{1}{2}x^2-\frac{1}{2}\beta^2y^2$, $\beta>0$:
$\mathfrak{g}_b=\big\langle D(1),\ D^x,\ P(e^t,0),\ P(e^{-t},0),\ P(0,\cos\beta t),\ P(0,\sin\beta t)\big\rangle$;

20. $b=-\frac{1}{2}x^2+\delta y$:
$\mathfrak{g}_b=\big\langle D(1),\ D^x-\frac{1}{2}\delta P(0,t^2),\ P(\cos t,0),\ P(\sin t,0),\ P(0,1),\ P(0,t)\big\rangle$;

21. $b=-\frac{1}{2}x^2-\frac{1}{2}\beta^2y^2$, $0<\beta<1$:
$\mathfrak{g}_b=\big\langle D(1),\ D^x,\ P(\cos t,0),\ P(\sin t,0),\ P(0,\cos\beta t),\ P(0,\sin\beta t)\big\rangle$;

22. (a) $b=0$:
$\mathfrak{g}_b=\big\langle D(1),\ D(t),\ D(t^2),\ D^x,\ J,\ P(1,0),\ P(t,0),\ P(0,1),\ P(0,t)\big\rangle$;

(b) $b=x$:
$\mathfrak{g}_b=\big\langle D(1),\ D(t)+\frac{3}{4}P(t^2,0),\ D(t^2)+\frac{1}{2}P(t^3,0),\ D^x-\frac{1}{2}P(t^2,0),$
$J-\frac{1}{2}P(0,t^2),\ P(1,0),\ P(t,0),\ P(0,1),\ P(0,t)\big\rangle$;

(c) $b=\frac{1}{2}r^2$:
$\mathfrak{g}_b=\big\langle D(1),D(e^{2t}),D(e^{-2t}),D^x,J,P(e^t,0),P(e^{-t},0),P(0,e^t),P(0,e^{-t})\big\rangle$;

(d) $b=-\frac{1}{2}r^2$:
$\mathfrak{g}_b=\big\langle D(1),\ D(\cos 2t),\ D(\sin 2t),\ D^x,\ J,P(\cos t,0),\ P(\sin t,0)$
$P(0,\cos t),\ P(0,\sin t)\big\rangle$.

**Remark 37.** For Cases 1–9 to really present maximal Lie symmetry extensions, the parameter function $f$ should take only values for which the corresponding values of the arbitrary element $b$ are not $G^{\sim}$-equivalent to ones from the other listed cases.

**Corollary 38.** *The dimension of the maximal Lie invariance algebra of any system from the class (4.1) is not greater than nine. More specifically, for any $b=b(x,y)$ we have* $\dim\mathfrak{g}_b\in\{1,2,3,4,5,6,9\}$*. Moreover,* $\bigcup_b\mathfrak{g}_b\subsetneq\mathfrak{g}_{\langle\rangle}$.

**Corollary 39.** *A system from the class (4.1) is invariant with respect to a six-dimensional Lie algebra if and only if the corresponding value of the arbitrary element $b$ is at most a quadratic polynomial in $(x,y)$.*

## 4.3 Proof of the classification

We solve the group classification problem within the framework of the infinitesimal approach using an optimized version of the *method of furcate splitting*. This method was suggested in [37] in the course of the group classification of the class of nonlinear Schrödinger equations of the form $\mathrm{i}\psi_t + \triangle\psi + F(\psi, \psi^*) = 0$ with an arbitrary number $n$ of space variables. Here $\psi$ is a unknown complex-valued function of real variables $(t, x_1, \dots, x_n)$, and $F$ is an arbitrary sufficiently smooth function of $(\psi, \psi^*)$, which is the arbitrary element of this class. Subsequently, the method of furcate splitting was applied to the group classification of various classes of (1+1)-dimensional variable-coefficient reaction–convection–diffusion equations, where arbitrary elements depend on single but possibly different arguments [28, 41, 46, 56–58].

According to the method of furcate splitting, we fix an arbitrary value of the variable $t$ in the classifying equation (4.8) and obtain the following template form of equations for the arbitrary element $b$:

$$\begin{aligned} &a_1(xb_x + yb_y) + a_2(yb_x - xb_y) + a_3b_x + a_4b_y + a_5b \\ &\qquad + a_6\frac{x^2+y^2}{2} + a_7x + a_8y + a_9 = 0, \end{aligned} \tag{4.9}$$

where $a_1, \dots, a_9$ are constants. For each value of the arbitrary element $b$, we denote by $k = k(b)$ the maximal number of template-form equations with linearly independent coefficient tuples $\bar a^i = (a^i_1, \dots, a^i_9)$, $i = 1, \dots, k$, that are satisfied by this value of $b$. It is obvious that $0 \leqslant k \leqslant 9$. Moreover, for the system of template-form equations

$$\begin{aligned} &a^i_1(xb_x + yb_y) + a^i_2(yb_x - xb_y) + a^i_3b_x + a^i_4b_y + a^i_5b \\ &\qquad + a^i_6\frac{x^2+y^2}{2} + a^i_7x + a^i_8y + a^i_9 = 0, \quad i = 1, \dots, k, \end{aligned} \tag{4.10}$$

with $\operatorname{rank} A = k$ to be consistent with respect to $b$, it is required that $k \leqslant 5$. Here

$$A := (a^i_j)^{i=1,\dots,k}_{j=1,\dots,9}, \quad A_l := (a^i_j)^{i=1,\dots,k}_{j=1,\dots,l}$$

are the matrix of coefficients of the system (4.10) and its submatrix constituted by the first $l$ columns of $A$, respectively. We also have $\operatorname{rank} A_5 = \operatorname{rank} A = k$, and, if $k < 5$, $\operatorname{rank} A_4 = k$ as well. Indeed, if the last condition is not satisfied, the system (4.10) has an algebraic consequence of the form $b = R(x, y) := \beta_3(x^2+y^2) + \beta_{11}x + \beta_{12}y + \beta_0$, where $\beta_0$, $\beta_{11}$, $\beta_{12}$ and $\beta_3$ are constants, and such values of $b$ satisfy five independent template-form equations (see the case $k = 5$ below),

$$xb_x + yb_y = xR_x + yR_y, \quad yb_x - xb_y = yR_x - xR_y, \quad b_x = R_x, \quad b_y = R_y, \quad b = R.$$

For checking the consistency of the system (4.10) with $k>1$, to the $i$th equation of this system for each $i=1,\dots,k$ we associate the vector field

$$\begin{aligned}\mathbf{v}_i = {}& \left(a^i_1x+a^i_2y+a^i_3\right)\partial_x+\left(a^i_1y-a^i_2x+a^i_4\right)\partial_y\\ &-\left(a^i_5b+\tfrac12a^i_6(x^2+y^2)+a^i_7x+a^i_8y+a^i_9\right)\partial_b.\end{aligned} \tag{4.11}$$

Note that $\mathbf{v}_1,\dots,\mathbf{v}_k\in\mathfrak{a}$, where

$$\mathfrak{a}:=\big\langle x\partial_x+y\partial_y,\ -x\partial_y+y\partial_x,\ \partial_x,\ \partial_y,\ b\partial_b,\ (x^2+y^2)\partial_b,\ x\partial_b,\ y\partial_b,\ \partial_b\big\rangle.$$

The span $\mathfrak{a}$ is closed with respect to the Lie bracket of vector fields, i.e., it is a Lie algebra, and thus $[\mathbf{v}_i,\mathbf{v}_{i'}]\in\mathfrak{a}$, $i,i'=1,\dots,k$. More specifically,

$$[\mathbf{v}_i,\mathbf{v}_{i'}]\in[\mathfrak{a},\mathfrak{a}]=\big\langle\partial_x,\ \partial_y,(x^2+y^2)\partial_b,\ x\partial_b,\ y\partial_b,\ \partial_b\big\rangle\subset\mathfrak{a},\quad i,i'=1,\dots,k.$$

In other words, the equation on $b$ that is associated with $[\mathbf{v}_i,\mathbf{v}_{i'}]$ is a differential consequence of the system (4.10) and is of the same template form (4.9). By its definition, the number $k=k(b)$ is equal to the maximal number of linearly independent vector fields associated with template-form equations for the corresponding value of the arbitrary element $b$. Therefore,

$$[\mathbf{v}_i,\mathbf{v}_{i'}]\in\big\langle\mathbf{v}_1,\dots,\mathbf{v}_k\big\rangle,\quad i,i'=1,\dots,k. \tag{4.12}$$

We can also use the counterpart of the condition (4.12) for the projections $\hat{\mathbf{v}}_1$, …, $\hat{\mathbf{v}}_k$ of the vector fields $\mathbf{v}_1$, …, $\mathbf{v}_k$ to the space with the coordinates $(x,y)$,

$$[\hat{\mathbf{v}}_i,\hat{\mathbf{v}}_{i'}]\in\big\langle\hat{\mathbf{v}}_1,\dots,\hat{\mathbf{v}}_k\big\rangle,\quad i,i'=1,\dots,k. \tag{4.13}$$

To simplify the computation, we can gauge coefficients of the system (4.10) by linearly combining its equations and using transformations from $G^\sim$. In particular, we can set $a^i_j=1$, dividing the entire $i$th equation of (4.10) by $a^i_j$ if $a^i_j\neq0$. In the case $(a^i_1,a^i_2)\neq(0,0)$, we can make $a^i_3=a^i_4=0$ with point equivalence transformations of simultaneous shifts with respect to $x$ and $y$. Another possibility is to use these shifts for setting $a^i_7=a^i_8=0$ if $a^i_6\neq0$. Similarly, if $a^i_5\neq0$, then we can shift $b$ to set $a^i_9=0$.

The case with $k=0$ corresponds to the kernel Lie invariance algebra $\mathfrak{g}^\cap$ of systems from the class (4.1), which is also the Lie invariance algebra for a general value of $b$.

For elements of $\mathfrak{g}^\cap$, the classifying equation (4.8) is identically satisfied by $b$. Thus, we can successively split it with respect to $b$ and its derivatives and with respect to $x$ and $y$ to obtain $F^1_t = F^2 = F^3 = 0$ and $c_2 = c_1 = 0$, i.e., $\tau = \text{const}$, $\xi^1 = \xi^2 = \eta^1 = \eta^2 = \eta^3 = 0$. In other words, the algebra $\mathfrak{g}^\cap$ is one-dimensional and spanned by the only basis element $\partial_t$,

$$\mathfrak{g}^\cap = \langle \partial_t \rangle.$$

In the next sections, we separately consider the cases $k = 1, \ldots, k = 5$. For each of these cases, we make the following steps, splitting the consideration into subcases depending on values of the parameters $a$'s:

- find the values of the parameters $a$'s for which the corresponding system (4.10) is compatible and follow from the equation (4.8),
- gauge, if possible, some of the parameters $a$'s by recombining template-form equations and by transformations from the group $G^\sim$ and re-denote the remaining parameters $a$'s,
- integrate the system (4.10) with respect to the arbitrary element $b$,
- gauge, if possible, the integration constant by transformations from $G^\sim$,
- solving the system of determining equations with respect to the parameters $c_1$, $c_2$, $F^1$, $F^2$ and $F^3$, construct the maximal Lie invariance algebras, $\mathfrak{g}_b$, of systems from the class (4.1) with obtained values of $b$.

The order of steps can be varied, and steps can intertwine.

### 4.3.1 One independent template-form equation

In the case $k = 1$, the right-hand side of the equation (4.8) is proportional to the right-hand side of the single equation (4.10) with the proportionality coefficient $\lambda$ that is a sufficiently smooth function of $t$,

$$F^1_t(xb_x + yb_y) + c_2(yb_x - xb_y) + F^2b_x + F^3b_y + 2(F^1_t - c_1)b \\ - F^1_{ttt}\frac{x^2 + y^2}{2} - F^2_{tt}x - F^3_{tt}y - F^4$$

$$= \lambda\Bigg( a_1^1(xb_x + yb_y) + a_2^1(yb_x - xb_y) + a_3^1 b_x + a_4^1 b_y + a_5^1 b + a_6^1\frac{x^2+y^2}{2} + a_7^1 x + a_8^1 y + a_9^1 \Bigg).$$

The function $\lambda$ does not vanish for any vector field from the complement of $\mathfrak{g}^\cap$ in $\mathfrak{g}_b$. We can split the last equation with respect to derivatives of $b$, including $b$ itself, and the independent variables $x$ and $y$. As a result, we obtain the system

$$\begin{aligned} &F^1_t = a_1^1\lambda, \quad c_2 = a_2^1\lambda, \quad F^2 = a_3^1\lambda, \quad F^3 = a_4^1\lambda, \quad 2(F^1_t - c_1) = a_5^1\lambda, \\ &F^1_{ttt} = -a_6^1\lambda, \quad F^2_{tt} = -a_7^1\lambda, \quad F^3_{tt} = -a_8^1\lambda, \quad F^4 = -a_9^1\lambda. \end{aligned} \tag{4.14}$$

The condition $\operatorname{rank} A_4 = k = 1$ means here that $(a_1^1, a_2^1, a_3^1, a_4^1) \neq (0,0,0,0)$. Therefore, the further consideration splits into three cases,

$$a_1^1 \neq 0; \quad a_1^1 = 0,\ a_2^1 \neq 0; \quad a_1^1 = a_2^1 = 0,\ (a_3^1, a_4^1) \neq (0,0).$$

$\boldsymbol{a_1^1 \neq 0}$**.** We can set $a_1^1 = 1$ by rescaling the entire equation (4.10). To simplify the computation we gauge other coefficients of (4.10) by transformations from $G^\sim$. Thus, we can make $a_3^1 = a_4^1 = 0$ with point equivalence transformations of simultaneous shifts with respect to $x$ and $y$. Consequently, we have that $F^2 = 0$ and $F^3 = 0$. Then the system (4.14) implies that $a_7^1 = a_8^1 = 0$. The equation (4.10) reduces in the polar coordinates $(r, \varphi)$ to the form

$$rb_r - a_2^1 b_\varphi + a_5^1 b + \frac{1}{2}a_6^1 r^2 + a_9^1 = 0.$$

The integration of the above equation depends on the values of the parameters $a_j^1$, $j = 2, 5, 6, 9$.

If $(a_2^1, a_5^1) = (0, 2)$, then we can set $a_9^1 = 0$ by shifts with respect to $b$ and $a_6^1 \in \{0, -4, 4\}$ by scaling equivalence transformations, which leads to Cases 6a, 6b and 6c of Theorem 36, respectively.

If $(a_2^1, a_5^1) \neq (0, 2)$, in view of the system (4.14) we get that $c_1 = (1 - a_5^1/2)\lambda$, $c_2 = a_2^1\lambda$. Thus, $\lambda$ is a constant, which yields that $F^1_t$ is also constant, and therefore $a_6^1 = 0$. Depending on whether $a_5^1 \neq 0$ or $a_5^1 = 0$, we get Cases 1 and 2, respectively. In the former case, we additionally set $a_9^1 = 0$ by shifts with respect to $b$.

$\boldsymbol{a_1^1 = 0,\ a_2^1 \neq 0.}$ Rescaling the equation (4.10) and using shifts with respect to $x$ and $y$, we can set $a_2^1 = 1$ and $a_3^1 = a_4^1 = 0$. In view of the system (4.14), the above conditions for $a$'s imply $F_t^1 = 0$, $F^2 = F^3 = 0$, $\lambda = c_2$, and thus $a_6^1 = a_7^1 = a_8^1 = 0$. In the polar coordinates $(r, \varphi)$, the equation (4.10) takes the form $b_\varphi = a_5^1 b + a_9^1$. Integrating this equation separately for $a_5^1 = 0$ and for $a_5^1 \neq 0$ we respectively get Cases 3 and 4. Under the former condition, we additionally set $a_9^1 = -1$ by equivalence transformations of scalings and alternating the signs of $(y, v)$ if $a_9^1 \neq 0$. We also can set $a_9^1 = 0$ by shifts with respect to $b$ if $a_5^1 \neq 0$.

$\boldsymbol{a_1^1 = a_2^1 = 0,\ (a_3^1, a_4^1) \neq (0,0).}$ Due to rotation equivalence transformations and the possibility of scaling the entire equation (4.10), we can rotate and scale the vector $(a_3^1, a_4^1)$ to set $a_3^1 = 1$ and $a_4^1 = 0$. From the system (4.14), we derive that $F_t^1 = F^3 = 0$, $c_2 = 0$, $2c_1 = -a_5^1\lambda$, $F^2 = \lambda$ and thus $a_6^1 = a_8^1 = 0$ and $F_{tt}^2 = -a_7^1 F^2$. The template-form equation (4.10) reduces to $b_x + a_5^1 b + a_7^1 x + a_9^1 = 0$.

If $a_5^1 = 0$, then we can set $a_7^1 \in \{0, -1, 1\}$ by using scaling equivalence transformations, which leads to Cases 7, 8 and 9. Note that $a_9^1 = 0 \bmod G^\sim$ if $a_7^1 \neq 0$ and $a_9^1 \in \{0, -1\} \bmod G^\sim$ if $a_7^1 = 0$.

If $a_5^1 \neq 0$, then $\lambda$ is a constant, and thus $a_7^1 = 0$. We can again set $a_9^1 = 0$ by shifts with respect to $b$ as well as $a_5^1 = 1$ up to scaling equivalence transformations and alternating signs of $(x, u)$. This leads to Case 5.

### 4.3.2 Two independent template-form equations

For $k = 2$, the right-hand side of the equation (4.8) is a linear combination of right-hand sides of the first and the second equations of the system (4.10) with coefficients $\lambda^1$ and $\lambda^2$ that depend on $t$,

$$
\begin{aligned}
&F_t^1(xb_x + yb_y) + c_2(yb_x - xb_y) + F^2 b_x + F^3 b_y + 2(F_t^1 - c_1)b \\
&\qquad - F_{ttt}^1 \frac{x^2 + y^2}{2} - F_{tt}^2 x - F_{tt}^3 y - F^4 \\
&= \sum_{i=1}^{2} \lambda^i \bigg( a_1^i(xb_x + yb_y) + a_2^i(yb_x - xb_y) + a_3^i b_x + a_4^i b_y + a_5^i b \\
&\qquad + a_6^i \frac{x^2 + y^2}{2} + a_7^i x + a_8^i y + a_9^i \bigg).
\end{aligned}
$$

**Remark 40.** The coefficients $\lambda^1$ and $\lambda^2$ are not proportional with the same constant multiplier for all vector fields from $\mathfrak{g}_b$ since otherwise there is no additional Lie symmetry extension in comparison with the more general case of Lie symmetry extension with $k = 1$, where the corresponding linear combination of equations of the system (4.10) plays the role of a single template-form equation. In particular, both these coefficients do not vanish identically for some vector fields from $\mathfrak{g}_b$.

Splitting the resulting condition with respect to $b$ and its derivatives $b_x$ and $b_y$, we derive the system

$$\begin{aligned}&F^1_t = a^1_1\lambda^1 + a^2_1\lambda^2, \quad c_2 = a^1_2\lambda^1 + a^2_2\lambda^2,\\ &F^2 = a^1_3\lambda^1 + a^2_3\lambda^2, \quad F^3 = a^1_4\lambda^1 + a^2_4\lambda^2,\\ &2(F^1_t - c_1) = a^1_5\lambda^1 + a^2_5\lambda^2, \quad F^1_{ttt} = -a^1_6\lambda^1 - a^2_6\lambda^2,\\ &F^2_{tt} = -a^1_7\lambda^1 - a^2_7\lambda^2, \quad F^3_{tt} = -a^1_8\lambda^1 - a^2_8\lambda^2, \quad F^4 = -a^1_9\lambda^1 - a^2_9\lambda^2.\end{aligned} \tag{4.15}$$

The further consideration for $k = 2$ is partitioned into different cases depending on the rank of the submatrix $A_2$ that is constituted by the first two columns of $A$. Since $\operatorname{rank} A_2 \leqslant 2$, we have the cases $\operatorname{rank} A_2 = 2$, $\operatorname{rank} A_2 = 1$ and $\operatorname{rank} A_2 = 0$.

**rank $\boldsymbol{A_2 = 2}$.** Linearly re-combining equations of the system (4.10), we can set the matrix $A_2$ to be the identity matrix, i.e., $a^1_1 = a^2_2 = 1$ and $a^1_2 = a^2_1 = 0$. To further simplify the form of the system (4.10), we set $a^1_3 = a^1_4 = 0$ by equivalence transformations of shifts with respect to $x$ and $y$. In view of the condition (4.12), the vector fields $\mathbf{v}_1$ and $\mathbf{v}_2$, which are associated with the first and the second equations of the reduced system (4.10), respectively, commute. This yields the system of algebraic equations with respect to the coefficients $a^i_j$,

$$\begin{aligned}&a^2_3 = a^2_4 = 0, \quad a^1_7 - a^2_8 + a^1_8a^2_5 - a^1_5a^2_8 = 0, \quad a^1_8 + a^2_7 - a^1_7a^2_5 + a^1_5a^2_7 = 0,\\ &2a^2_6 - a^1_6a^2_5 + a^1_5a^2_6 = 0, \quad a^1_9a^2_5 - a^1_5a^2_9 = 0.\end{aligned} \tag{4.16}$$

The reduced form of the system (4.15) is

$$\begin{aligned}&F^1_t = \lambda^1, \quad c_2 = \lambda^2, \quad F^2 = 0, \quad F^3 = 0, \quad (a^1_5 - 2)\lambda^1 = -2c_1 - a^2_5c_2,\\ &\lambda^1_{tt} + a^1_6\lambda^1 + a^2_6\lambda^2 = 0, \quad a^1_7\lambda^1 + a^2_7\lambda^2 = 0, \quad a^1_8\lambda^1 + a^2_8\lambda^2 = 0,\\ &F^4 = -a^1_9\lambda^1 - a^2_9\lambda^2.\end{aligned} \tag{4.17}$$

In view of Remark 40, the seventh and the eighth equations of the system (4.17) imply $a^1_7 = a^2_7 = 0$ and $a^1_8 = a^2_8 = 0$, respectively. The reduced form of the system (4.10) in the polar coordinates $(r, \varphi)$ is

$$rb_r + a^1_5 b + \frac{1}{2}a^1_6 r^2 + a^1_9 = 0, \quad b_\varphi = a^2_5 b + \frac{1}{2}a^2_6 r^2 + a^2_9.$$

The last equation of (4.16), up to equivalence transformations with to respect to $b$ we can set $a^1_9 = a^2_9 = 0$ if $(a^1_5, a^2_5) \neq (0, 0)$.

The further consideration depends on whether or not the parameter $a^1_5$ is equal to 2 and, in the latter case, whether or not the parameter $a^2_5$ is zero.

If $a^1_5 = 2$, then the parameter $a^1_6$ can be assume to belong to $\{0, -4, 4\}$, and $a^2_6 = a^2_5 a^1_6/4$. Integrating the corresponding system (4.10), we find the general form of the arbitrary element $b$,

$$b = b_0 r^{-2} \exp\bigl(a^2_5 \varphi\bigr) - \frac{a^1_6}{8} r^2,$$

where the integration constant $b_0$ is nonzero since otherwise this value of $b$ is associated with the value $k = 5$. This is why we can scale $b_0$ by an equivalence transformation to $\varepsilon = \pm 1$, which leads, depending on the value of $a^1_6$, to Cases 12a, 12b and 12c of Theorem 36.

If $a^1_5 \neq 2$, then $\lambda^1$ is a constant. Then the sixth equation of the system (4.17) takes the form $a^1_6 \lambda^1 + a^2_6 \lambda^2 = 0$, implying according to Remark 40 that $a^1_6 = a^2_6 = 0$. Depending on whether $(a^1_5, a^2_5) \neq (0, 0)$ or $a^1_5 = a^2_5 = 0$, we obtain Cases 10 and 11 of Theorem 36, respectively.

**rank $A_2 = 1$.** Linearly recombining equations of the system (4.10), we reduce the matrix $A_2$ to the form

$$A_2 = \begin{pmatrix} a^1_1 & a^1_2 \\ 0 & 0 \end{pmatrix},$$

i.e., $a^2_1 = a^2_2 = 0$ and $(a^1_1, a^1_2) \neq (0, 0)$. We also have $(a^2_3, a^2_4) \neq (0, 0)$ since $\operatorname{rank} A_4 = 2$. Hence we can set $a^2_3 = 1$, $a^2_4 = 0$ by a rotation equivalence transformation and re-scaling the second equation as well as $a^1_3 = a^1_4 = 0$ by shifts of $x$ and $y$. The condition (4.12) implies that $a^1_2 = 0$ and hence $a^1_1 \neq 0$. This is why we can set

$a^1_1=1$ by rescaling of the first equation. Then the condition (4.12) is equivalent to the commutation relation $[\mathbf{v}_1,\mathbf{v}_2]=-\mathbf{v}_2$, yielding the following system of algebraic equations on the remaining coefficients $a^i_j$:

$$a^2_5=0,\ \ a^2_6(a^1_5+3)=0,\ \ a^2_8(a^1_5+2)=0,\ \ a^2_7(a^1_5+2)=a^1_6,\ \ a^2_9(a^1_5+1)=a^1_7. \quad (4.18)$$

The system (4.15) is simplified to

$$\begin{aligned}&F^1_t=\lambda^1,\quad c_2=0,\quad F^2=\lambda^2,\quad F^3=0,\quad 2c_1=(2-a^1_5)\lambda^1,\\&F^1_{ttt}=-a^1_6\lambda^1-a^2_6\lambda^2,\quad F^2_{tt}=-a^1_7\lambda^1-a^2_7\lambda^2,\quad a^1_8\lambda^1+a^2_8\lambda^2=0,\\&F^4=-a^1_9\lambda^1-a^2_9\lambda^2.\end{aligned} \quad (4.19)$$

In view of Remark 40, the eighth equation of the system (4.19) imply $a^1_8=a^2_8=0$.

Supposing that $a^2_6\neq0$, we successively derive from the second equation of (4.18) and the system (4.19) that $a^1_5=-3$, $\lambda^1$ is a constant, $a^1_6\lambda^1+a^2_6\lambda^2=0$ and, according to Remark 40, $a^1_6=a^2_6=0$, which is a contradiction. Hence $a^2_6=0$.

It is obvious from the fifth equation of (4.19), $2c_1=(2-a^1_5)\lambda^1$, that the value $a^1_5=2$ is special. If $a^1_5\neq2$, repeating the argumentation with constant $\lambda^1$, we derive $a^1_6=0$, and thus the fourth equation of the system (4.18) takes the form $(a^1_5+2)a^2_7=0$, implying $a^2_7=0$ if $a^1_5\neq-2$. Therefore, the value $a^1_5=-2$ is special as well. In the course of integrating the system (4.10) the value $a^1_5=0$ is additionally singled out. Moreover, if $a^1_5\neq0$, we can make $a^1_9=0$ using a shift of $b$. As a result, we need to separately consider each of the above values 2, 0, $-2$ of $a^1_5$ and the case $a^1_5\notin\{-2,0,2\}$.

**1.** $a^1_5=2$. Shifting $b$, we set $a^1_9=0$. The system (4.18) reduces to the equations $a^2_7=a^1_6/4$ and $a^2_9=a^1_7/3$.

Let $a^1_6\neq0$. Then shifting $x$ and $b$ and recombining equations of the system (4.10), we also set $a^1_7=0$, and consequently $a^2_9=0$. The general solution to the system (4.10) is $b=b_0y^{-2}-\frac18a^1_6(x^2+y^2)$, where the integration constant $b_0$ is nonzero since otherwise this value of the arbitrary element $b$ in associated with $k=5$. Using scaling equivalence transformations, we can set $b_0,a^1_6/4\in\{-1,1\}$. Depending on the sign of $a^1_6$, we obtain Cases 16b and 16c of Theorem 36.

If $a^1_6=0$, then $a^2_7$ is also zero. Integrating (4.10), we get $b=b_0y^{-2}-a^2_9x$, where again the integration constant $b_0$ is nonzero since otherwise this value of the arbitrary element $b$ is associated with $k=5$. Using scaling equivalence transformations and alternating the signs of $(x,u)$, we can set $b_0\in\{-1,1\}$ and $a^2_9\in\{0,-1\}$, which gives Case 16a.

**2.** $a^1_5=0$. Then $a^2_7=0$ and $a^1_7=a^2_9$. The system (4.10) integrates to

$$b=-a^1_9\ln|y|-a^2_9x+b_0,$$

where the parameter $a^1_9$ is nonzero since otherwise $k=5$. The integration constant $b_0$ can be set to zero by shifts of $b$, as well as $a^1_6\in\{-1,1\}$ and $a^2_9\in\{-1,0\}$ up to scaling equivalence transformations and alternating the signs of $(x,u)$. This leads to Case 14.

**3.** $a^1_5=-2$. Then $a^1_7=-a^2_9$. We shift $b$ for setting $a^1_9=0$. The general solution of the system (4.10) is $b=b_0y^2-\frac{1}{2}a^2_7x^2-a^2_9x$ but this value of the arbitrary element $b$ is associated with $k>2$.

**4.** $a^1_5\notin\{-2,0,2\}$. Solving the system (4.10), we obtain $b=b_0|y|^{-a^1_5}-a^2_9x$, where the integration constant $b_0$ should be nonzero for $k$ to be equal two. Setting $b_0\in\{-1,1\}$ and $a^2_9\in\{-1,0\}$ by scaling equivalence transformations and alternating the signs of $(x,u)$ results in Case 13.

**rank $\boldsymbol{A_2}=0$.** This means that $a^1_1=a^1_2=a^2_1=a^2_2=0$. Since $\operatorname{rank}A_4=2$, we can linearly recombine equations of the system (4.10) to set $a^1_3=a^2_4=1$ and $a^1_4=a^2_3=0$. The compatibility condition (4.12) means that the vector fields $\mathbf{v}_1$ and $\mathbf{v}_2$ associated to equations of the reduced system (4.10) commutes, $[\mathbf{v}_1,\mathbf{v}_2]=0$, which results to the system

$$\begin{aligned}&a^2_6-a^1_7a^2_5+a^1_5a^2_7=0,\quad a^1_6a^2_5-a^1_5a^2_6=0,\\&a^1_6+a^1_8a^2_5-a^1_5a^2_8=0,\quad a^1_8-a^2_7+a^1_9a^2_5-a^1_5a^2_9=0.\end{aligned}\tag{4.20}$$

Suppose that $a^1_5=a^2_5=0$. The system (4.20) then reduces to $a^1_6=a^2_6=0$ and $a^1_8=a^2_7$. In view of the last equation, we can set $a^1_8=a^2_7=0$ by rotation equivalence transformations. The general solution of the system (4.10) is

$$b=-\frac{a^1_7}{2}x^2-\frac{a^2_8}{2}y^2-a^1_9x-a^2_9y+b_0,$$

where $b_0$ is an integration constant. This form of the arbitrary element $b$ is related to the value $k=3$ if $a^1_7\neq a^2_8$ and to the value $k=5$ if $a^1_7=a^2_8$, which contradicts the supposition $k=2$.

This is why $(a^1_5,a^2_5)\neq(0,0)$, and we set $a^1_5=0$ and $a^2_5=-1$ by equivalence transformations of rotations, scalings and alternating signs. Then the first equation of the system (4.20) implies $a^1_6=0$, and thus the first six equations of the system (4.15) takes the form $F^1_t=0$, $c_2=0$, $F^2=\lambda^1$, $F^3=\lambda^2=2c_1$ and $a^2_6\lambda^2=0$. In view of the last equation, we get $a^2_6=0$. Therefore, the system (4.20) is equivalent to $a^1_7=a^1_8=0$ and $a^2_7=-a^1_9$. The eighth equation of (4.15) gives $a^2_8\lambda^2=0$, i.e., $a^2_8=0$. We can set $a^2_9=0$ up to equivalence transformations of shifts of $b$. The system (4.10) integrates to $b=b_0e^y+a^2_7x$, where the integration constant $b_0$ is nonzero since otherwise $b$ is a linear function, for which $k=5$. Equivalence transformations of scalings and alternating the signs of $(x,u)$ allow us to set $b_0=\pm1$ and $a^2_7\in\{0,1\}$. This results in Case 15 of Theorem 36.

### 4.3.3 More independent template-form equations

We show below that $k>2$ if and only if $b$ is at most quadratic polynomial in $(x,y)$.

$\boldsymbol{k=3.}$ Since $\operatorname{rank}A_4=\operatorname{rank}A=k=3$, we have that $\operatorname{rank}A_2>0$.

Suppose that the submatrix $A_2$ is of rank two. Recombining equations of the system (4.10), we can reduce this submatrix to the form

$$A_2=\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}.$$

Then the projections $\hat{\mathbf v}_1$, $\hat{\mathbf v}_2$ and $\hat{\mathbf v}_3$ of the vector fields $\mathbf v_1$, $\mathbf v_2$ and $\mathbf v_3$ to the space with the coordinates $(x,y)$ are

$$\hat{\mathbf v}_1=(x+a^1_3)\partial_x+(y+a^1_4)\partial_y,\quad \hat{\mathbf v}_2=(y+a^2_3)\partial_x-(x-a^2_4)\partial_y,\quad \hat{\mathbf v}_3=a^3_3\partial_x+a^3_4\partial_y. \quad (4.21)$$

In view of the condition (4.13), the commutator $[\hat{\mathbf v}_2,\hat{\mathbf v}_3]=-a^3_4\partial_x+a^3_3\partial_y$ should belong to the span $\langle\hat{\mathbf v}_1,\hat{\mathbf v}_2,\hat{\mathbf v}_3\rangle$ but this is not the case, which leads to a contradiction.

Therefore, rank $A_2 = 1$, and thus the matrix $A_4$ can be reduced to the form

$$A_4 = \begin{pmatrix} a_1^1 & a_2^1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{where} \quad (a_1^1, a_2^1) \neq (0,0).$$

Then the compatibility condition (4.12) is equivalent to the commutation relations

$$[\mathbf{v}_1, \mathbf{v}_2] = -a_1^1\mathbf{v}_2 + a_2^1\mathbf{v}_3, \quad [\mathbf{v}_1, \mathbf{v}_3] = -a_1^1\mathbf{v}_3 - a_2^1\mathbf{v}_2, \quad [\mathbf{v}_2, \mathbf{v}_3] = 0.$$

From the first two commutation relations, we obtain $a_1^1 a_5^2 - a_2^1 a_5^3 = 0$, $a_2^1 a_5^2 + a_1^1 a_5^3 = 0$, and thus $a_5^2 = a_5^3 = 0$ since $(a_1^1, a_2^1) \neq (0,0)$. Then the last commutation relation yields $a_6^2 = a_6^3 = 0$ and $a_7^3 = a_8^2$. Up to rotation equivalence transformations, we can set $a_8^2 = a_7^3 = 0$. Under this gauge, from the former commutation relations we get the system

$$\begin{aligned} &(2a_1^1 + a_5^1)(a_7^2 - a_8^3) = 0, \quad a_2^1(a_7^2 - a_8^3) = 0, \\ &a_6^1 = (2a_1^1 + a_5^1)a_7^2, \quad a_7^1 = (a_1^1 + a_5^1)a_9^2 - a_2^1 a_9^3, \quad a_8^1 = (a_1^1 + a_5^1)a_9^3 + a_2^1 a_9^2. \end{aligned} \tag{4.22}$$

Since the arbitrary element $b$ satisfies the equations $b_x + a_7^2 x + a_9^2 = 0$ and $b_y + a_8^3 y + a_9^3 = 0$, it is a quadratic function of $(x, y)$. More specifically,

$$b = -\frac{1}{2}a_7^2 x^2 - \frac{1}{2}a_8^3 y^2 - a_9^2 x - a_9^3 y \tag{4.23}$$

up to equivalence transformations of shifts with respect to $b$. Hence $a_7^2 \neq a_8^3$ since otherwise $k = 5$ for this value of $b$, which contradicts the supposition $k = 3$. Then the system (4.22) reduces to

$$a_2^1 = 0, \quad a_5^1 = -2a_1^1, \quad a_6^1 = 0, \quad a_7^1 = -a_1^1 a_9^2, \quad a_8^1 = -a_1^1 a_9^3$$

and guaranties that the above value of $b$ satisfies the corresponding system (4.10).

Modulo $G^{\sim}$-equivalence, we can assume that $a_7^2 = \pm 1$, $a_9^2 = 0$; $|a_8^3| < |a_7^2|$ if $a_7^2 a_8^3 > 0$; $a_9^3 = 0$ if $a_8^3 \neq 0$; $a_9^3 \in \{-1, 0\}$ if $a_8^3 = 0$. $G^{\sim}$-inequivalent values of $b$ of the form (4.23) with the associated maximal Lie invariance algebras are listed in Cases 17–21 of Theorem 36.

***k* = 4.** Since rank $A_4$ = rank $A = 4$, linearly re-combining the equations (4.10), we can set $A_4$ to be the $4\times4$ identity matrix. In view of the form of the vector fields $\mathbf{v}_1, \dots, \mathbf{v}_4$ associated to the equations of the reduced system (4.10), the compatibility condition (4.12) implies the commutation relations

$$[\mathbf{v}_2, \mathbf{v}_3] = \mathbf{v}_4, \quad [\mathbf{v}_2, \mathbf{v}_4] = -\mathbf{v}_3, \quad [\mathbf{v}_3, \mathbf{v}_4] = 0.$$

From the first two commutation relations, we get that $a^3_5 = a^4_5 = 0$. The last commutation relation together with the previous restrictions on the coefficients $a^i_j$ yields $a^3_6 = a^4_6 = 0$ and $a^3_8 = a^4_7$. Returning to the first two commutation relations, we obtain the equations

$$a^2_6 = -2a^3_8 + a^2_5 a^3_7 = 2a^3_8 + a^2_5 a^4_8, \quad a^3_7 - a^4_8 + a^2_5 a^3_8 = 0$$

implying $a^3_8 = 0$ and $a^3_7 = a^4_8$. Since the arbitrary element $b$ satisfies the equations $b_x + a^3_7 x + a^3_9 = 0$ and $b_y + a^3_7 y + a^4_9 = 0$, it is a quadratic function of $(x, y)$ with the same coefficients of $x^2$ and of $y^2$ and with zero coefficient of $xy$. This means that in fact $k = 5$, which contradicts the supposition $k = 4$.

***k* = 5.** The $5\times5$ matrix $A_5$ of the coefficients of the system (4.10) is of rank 5 and, up to recombining equations of this system, can be assumed to be the $5\times5$ identity matrix. Then the last equation of the system (4.10) implies that $b$ is the specific quadratic polynomial of $(x, y)$,

$$b = -\tfrac{1}{2}a^5_6(x^2 + y^2) - a^5_7 x - a^5_8 y - a^5_9.$$

There are four $G^\sim$-equivalent values of the arbitrary element $b$ among such quadratic polynomials, $b = 0$, $b = x$, $b = \frac{1}{2}(x^2 + y^2)$ and $b = -\frac{1}{2}(x^2 + y^2)$, which correspond to Cases 22a, 22b, 22c and 22d of Theorem 36, respectively.

## 4.4 Additional equivalence transformations and modified classification result

It is obvious that the class (4.1) is not normalized. In other words, it possesses admissible transformations (e.g., those related to Lie symmetries of systems from this class) that are not generated by elements of $G^{\sim}$. See [6, 44, 47] for definitions. Moreover, we will show below that the class (4.1) is not even semi-normalized since some of its admissible transformations cannot be presented as compositions of admissible transformations generated by elements of $G^{\sim}$ and by point symmetries of systems from this class as well. Such admissible transformations may lead to additional point equivalences among classification cases listed in Theorem 36.

Since we do not have the complete description of the equivalence groupoid of the class (4.1), in the course of looking for the above additional equivalences we need to use algebraic tools that do not involve this description. If two systems of differential equations are similar with respect to a point transformation, then the corresponding maximal Lie invariance algebras are isomorphic in the sense of abstract Lie algebras. Moreover, these maximal Lie invariance algebras are similar as realizations of Lie algebras by vector fields with respect to the same point transformation. This gives necessary conditions of similarity for systems of differential equations with respect to point transformations.

Lie algebras of different dimensions are nonisomorphic. Hence we categorize the Lie algebras presented in Theorem 36 according to their dimensions to distinguish the cases that are definitely not equivalent to each other with respect to point transformations,

- $\dim \mathfrak{g}_b = 2$: Cases 1, 2, 3, 4 and 5;
- $\dim \mathfrak{g}_b = 3$: Cases 6a, 6b, 6c, 7, 8, 9, 10 and 11;
- $\dim \mathfrak{g}_b = 4$: Cases 12a, 12b, 12c, 13, 14 and 15;
- $\dim \mathfrak{g}_b = 5$: Cases 16a, 16b and 16c;
- $\dim \mathfrak{g}_b = 6$: Cases 17, 18, 19, 20 and 21;
- $\dim \mathfrak{g}_b = 9$: Cases 22a, 22b, 22c and 22d.

However, the same dimension of algebras does not ensure their isomorphism.

Finding a pair of classification cases with isomorphic maximal Lie invariance algebras and fixing bases of these algebras that are concordant under the found algebra isomorphism, we aim to obtain a point transformation that respectively maps the basis elements of the first algebra to the basis elements of the second one. The existence of such a point transformation hints that the two classification cases may be equivalent with respect to the same point transformation.

In this way, we find three families of $G^{\sim}$-inequivalent admissible transformations for the class (4.1) that are not induced by equivalence transformations of this class and whose target arbitrary elements differ from their source arbitrary elements. In each of these families, the source arbitrary elements are parameterized by an arbitrary function of a single argument. We present these families jointly with the corresponding induced additional equivalences among classification cases of Theorem 36:

1. $b = r^{-2}f(\varphi) - \frac{1}{2}r^2,\ \ \tilde b = \tilde r^{-2}f(\tilde\varphi),$

   $\tilde t = \tan t,\ \ \tilde x = x\sec t,\ \ \tilde y = y\sec t,$

   $\tilde u = u\cos t + x\sin t,\ \ \tilde v = v\cos t + y\sin t,\ \ \tilde h = h\cos^2 t,$

   6c $\to$ 6a, 12c $\to$ 12a, 16b $\to$ 16a$_{\delta=0}$, 22d $\to$ 22a.

2. $b = r^{-2}f(\varphi) + \frac{1}{2}r^2,\ \ \tilde b = \tilde r^{-2}f(\tilde\varphi),$

   $\tilde t = \frac{1}{2}e^{2t},\ \ \tilde x = e^t x,\ \ \tilde y = e^t y,\ \ \tilde u = e^{-t}(u + x),\ \ \tilde v = e^{-t}(v + y),\ \ \tilde h = e^{-2t}h,$

   6b $\to$ 6a, 12b $\to$ 12a, 16c $\to$ 16a$_{\delta=0}$, 22c $\to$ 22a.

3. $b = f(y) + x,\ \ \tilde b = f(\tilde y),$

   $\tilde t = t,\ \ \tilde x = x + \frac{1}{2}t^2,\ \ \tilde y = y,\ \ \tilde u = u + t,\ \ \tilde v = v,\ \ \tilde h = h,$

   22b $\to$ 22a, 7, 13, 14, 15, 16a, 18, 20:[2] $\delta = 1 \to \delta = 0$.

Both the first and second families of admissible transformations can be generalized to the rotating reference frame. The generalization of the transformation with $f = 0$ from the first family to the rotating reference frame was found for the first time for the shallow water equations in cylindrical coordinates in [14, Theorem 1].

Each of the above admissible transformations is $G^{\sim}$-equivalent to no admissible transformations generated by point symmetries of systems from this class. Therefore, the class (4.1) is not semi-normalized.

As by-product, we also prove the following assertions.

[2]For Cases 18 and 20, we should compose the corresponding point transformation with the permutation $(x, u) \leftrightarrow (y, v)$, which is an equivalence transformations of the class (4.1).

**Proposition 41.** *Any system from the class (4.1) that is invariant with respect to a nine-dimensional Lie algebra of vector fields is equivalent, up to point transformations, to the system from the same class with $b = 0$, which is the system of shallow water equations with flat bottom topography.*

**Proposition 42.** *Any system from the class (4.1) with five-dimensional maximal Lie invariance algebra is reduced by a point transformation to the system from the same class with $b = \pm y^{-2}$.*

For the other possible dimensions of maximal Lie invariance algebras of systems from the class (4.1), we prove the inequivalence of the remaining classification cases whenever it is possible to do so using the algebraic technique based on Mubarakzianov's classification of Lie algebras up to dimension four [36] and Turkowski's classification of six-dimensional solvable Lie algebras with four-dimensional nilradials [55]. For convenience, we take these classifications in the form given in [10], where the notation of algebras from Mubarakzianov's classification was modified, in particular, by indicating parameters for families of algebras and where the basis elements of the algebras from Turkowski's classification were renumbered in order to have bases in $K$-canonical forms. For each abstract Lie algebra appearing in the consideration, we present all the nonzero commutation relations among basis elements up to antisymmetry.

Unfortunately, the algebraic criterion of inequivalence with respect to point transformations is not sufficiently powerful for systems with two-dimensional maximal Lie invariance algebra since there are only two nonisomorphic two-dimensional Lie algebras $2A_1$ and $A_{2.1}$, the abelian and the non-abelian ones. Applying this criterion, we can only partition the corresponding classification cases into two sets, Cases $1_{\nu=2}$ and 3 with abelian two-dimensional Lie invariance algebras and Cases $1_{\nu\neq2}$, 2, 4 and 5 with non-abelian two-dimensional Lie invariance algebras. There are definitely no point transformations between cases that belong to different sets.

For the classification cases with maximal Lie invariance algebras of greater dimensions, the algebraic criterion is more advantageous. Thus, the maximal Lie invariance algebras in Cases 6a, $7_{\delta=0}$, 8, 9, 10 and 11 of Theorem 36 are isomorphic to the three-dimensional Lie algebras $\mathrm{sl}(2,\mathbb{R})$, $A_{3.1}$, $A_{3.4}^{-1}$, $A_{3.5}^{0}$, $A_{2.1}\oplus A_1$ and $A_{2.1}\oplus A_1$, respectively. These Lie algebras are defined by the following commutation relations:

$$\begin{array}{ll}
\mathrm{sl}(2,\mathbb{R})\colon & [e_1,e_2]=e_1,\ \ [e_2,e_3]=e_3,\ \ [e_1,e_3]=-2e_2;\\
A_{3.1}\colon & [e_2,e_3]=e_1;\\
A_{3.4}^{-1}\colon & [e_1,e_3]=e_1,\ \ [e_2,e_3]=-e_2;\\
A_{3.5}^{0}\colon & [e_1,e_3]=-e_2,\ \ [e_2,e_3]=e_1;\\
A_{2.1}\oplus A_1\colon & [e_1,e_2]=e_1,
\end{array}$$

and they are well known to be non-isomorphic to each other. This implies the pairwise inequivalence of the above cases of Lie-symmetry extensions with respect to point transformations, except the pair of the last two cases.

In a similar way, we prove that the maximal Lie invariance algebras associated with Cases 12a, $13_{\delta=0}$, $14_{\delta=0}$ and $15_{\delta=0}$, are also not isomorphic to each other although all of them are four-dimensional. The corresponding abstract Lie algebras are $\mathrm{sl}(2,\mathbb{R})\oplus A_1$ (Case 12a) as well as $A_{4,8}^a$ with $a=\nu/(2-\nu)$ if $\nu<1$ and with $a=(2-\nu)/\nu$ if $\nu\geqslant 1$ (Case 13), with $a=0$ (Case 14) and with $a=-1$ (Case 15), where

$$A_{4.8}^a,\ |a|\leqslant 1\colon\quad [e_2,e_3]=e_1,\ [e_1,e_4]=(1+a)e_1,\ [e_2,e_4]=e_2,\ [e_3,e_4]=ae_3.$$

As a result, Cases 12a, 13, 14 and 15 of Theorem 36 are equivalent neither to each other nor to other cases of this theorem with respect to point transformations. Moreover, the parameter $\nu$ in Case 13 cannot be gauged by point transformations.

In Cases 17, $18_{\delta=0}$, 19, $20_{\delta=0}$ and 21, the corresponding maximal Lie invariance algebras are six-dimensional solvable Lie algebras with four-dimensional abelian nilradicals that are isomorphic to the algebras $N_{6.1}^{abcd}$ with

$$(a,b,c,d)=\tfrac{1}{2}(1-\beta,1+\beta,1+\beta,1-\beta),$$

$N_{6.2}^{-1,1,1}$, $N_{6.13}^{-1/\beta,1/\beta,1,1}$, $N_{6.16}^{01}$, $N_{6.18}^{0\beta1}$ from Turkowski's classification, respectively. Here the canonical commutation relations are the following:

$$N_{6.1}^{abcd}\ \ {\scriptstyle \begin{array}{l} ac\neq 0\\ b^2+d^2\neq 0\end{array}}\quad \begin{array}{l}[e_1,e_5]=ae_1,\ \ [e_2,e_5]=be_2,\ \ [e_4,e_5]=e_4,\\ [e_1,e_6]=ce_1,\ \ [e_2,e_6]=de_2,\ \ [e_3,e_6]=e_3\end{array}$$

$N_{6.2}^{abc}$ $\quad a^2+b^2\neq 0$ $\quad [e_1,e_5]=ae_1,\ [e_2,e_5]=e_2,\ [e_4,e_5]=e_3,$
$[e_1,e_6]=be_1,\ [e_2,e_6]=ce_2,\ [e_3,e_6]=e_3,\ [e_4,e_6]=e_4$

$N_{6.13}^{abcd}$ $\quad a^2+c^2\neq 0,\ b^2+d^2\neq 0$ $\quad [e_1,e_5]=ae_1,\ [e_2,e_5]=be_2,\ [e_3,e_5]=e_4,\ [e_4,e_5]=-e_3,$
$[e_1,e_6]=ce_1,\ [e_2,e_6]=de_2,\ [e_3,e_6]=e_3,\ [e_4,e_6]=e_4$

$N_{6.16}^{ab}$ $\quad [e_2,e_5]=e_1,\ [e_3,e_5]=ae_3+e_4,\ [e_4,e_5]=-e_3+ae_4,$
$[e_1,e_6]=e_1,\ [e_2,e_6]=e_2,\ [e_3,e_6]=be_3,\ [e_4,e_6]=be_4$

$N_{6.18}^{abc}$ $\quad b\neq 0$ $\quad [e_1,e_5]=e_2,\ [e_2,e_5]=-e_1,\ [e_3,e_5]=ae_3+be_4,\ [e_4,e_5]=-be_3+ae_4,$
$[e_1,e_6]=e_1,\ [e_2,e_6]=e_2,\ [e_3,e_6]=ce_3,\ [e_4,e_6]=ce_4$

Since the above isomorphisms are not as obvious as for algebras of lower dimensions, we present the necessary basis changes to $(e_1,e_2,e_3,e_4,e_5,e_6)$:

17: $\big(P(0,e^{\beta t}),\ P(0,e^{-\beta t}),\ P(e^t,0),\ P(e^{-t},0),\ \tfrac12 D^x+\tfrac12 D(1),\ \tfrac12 D^x-\tfrac12 D(1)\big);$

18: $\big(P(e^t,0),\ P(e^{-t},0),\ P(0,1),\ P(0,t),\ -D(1),\ D^x\big)\quad$ for $\quad\delta=0;$

19: $\big(P(e^t,0),\ P(e^{-t},0),\ P(0,\cos\beta t),\ P(0,\sin\beta t),\ \beta^{-1}D(1),\ D^x\big);$

20: $\big(P(0,1),\ P(0,t),\ P(\sin t,0),\ P(\cos t,0),\ -D(1),\ D^x\big)\quad$ for $\quad\delta=0;$

21: $\big(P(\cos t,0),\ P(\sin t,0),\ P(0,\cos\beta t),\ P(0,\sin\beta t),\ D(1),\ D^x\big).$

The Lie algebras from Turkowski's classification appearing in the consideration are not isomorphic to each other, including the pairs of algebras from the same series with different values of the parameter $\beta$ within ranges indicated in the corresponding cases of Theorem 36. The claim on such pairs was checked by the direct computation in maple. This is why Cases 17, $18_{\delta=0}$, 19, $20_{\delta=0}$ and 21 are inequivalent with respect to point transformations. Moreover, the parameter $\beta$ in Cases 17, 19 and 21 cannot be gauged further.

Analyzing the classification cases listed in Theorem 36, the additional equivalences among them that are found in this section and the checked necessary algebraic condition for pairs of inequivalent cases, we can suppose that the following assertion holds.

**Conjecture 43.** *A complete list of inequivalent (up to all admissible transformations)*

*Lie symmetry extensions in the class* (4.1) *is exhausted by Cases 1, 2, 3, 4, 5, 6a, $7_{\delta=0}$, 8, 9, 10, 11, 12a, $13_{\delta=0}$, $14_{\delta=0}$, $15_{\delta=0}$, $16a_{\delta=0}$, 17, $18_{\delta=0}$, 19, $20_{\delta=0}$, 21 and 22a of Theorem 36.*

To prove this conjecture, we need to complete the verification of the inequivalence of cases within the sets of cases $\{1_{\nu=2}, 3\}$, $\{1_{\nu\neq2}, 2, 4, 5\}$ and $\{10, 11\}$ as well as the impossibility of further gauging of parameters remaining in some cases. This can be done via the construction of the equivalence groupoid of the class (4.1), which is a nontrivial and cumbersome problem. It is quite difficult to prove even principal properties of admissible transformations of the class (4.1), which can be conjectured after analyzing the form of equivalence transformations of this class, of Lie symmetries of equations from this class and of the three obtained $G^{\sim}$-inequivalent families of admissible transformations. These principal properties include the affineness with respect to the dependent variables, the fiber-preservation, i.e., the projectability to the space with the coordinates $(t, x, y)$, as well as the projectability to the space with the coordinate $t$. We can also conjecture the explicit structure of the equivalence groupoid of the class (4.1).

**Conjecture 44.** *$G^{\sim}$-inequivalent non-identity admissible transformations of the class* (4.1) *that are independent up to inversion, composing with each other and with admissible transformations generated by point symmetries of systems from this class are exhausted by the three families found in this section.*

# Chapter 5

# Conclusion

In the thesis we studied symmetries of systems from the classes (3.2) and (4.1), which are respectively constituted by the one- and two-dimensional shallow water equations with variable bottom topography.

To begin with, the class (3.2) can be re-parameterized since the arbitrary element $b$ arises only as the derivative $b_x$ in systems of this class. In Section 3.1 and Section 3.2 we computed the generalized equivalence groups for both the initial class (3.2) and the re-parameterized class (3.7). These groups are presented in Theorem 23 and Theorem 27, respectively. Since the arbitrary element $b$ does not depend on time, instead of computing the equivalence groups of the classes (3.2) and (3.7), we constructed the complete point symmetry groups of the systems (3.3) and (3.8), respectively. We also showed that for each of the above classes the generalized equivalence group coincides with its usual equivalence group. For computing point symmetry groups of the systems (3.3) and (3.8), we applied the algebraic method. This method is a refinement of the direct method, and it is based on the fact that each point symmetry transformation induces an automorphism of the corresponding Lie algebra. One of the main advantages of the algebraic method is that it provides restrictions on the form of the components of equivalence transformations in an easy way and thus allows to avoid cumbersome computations. In Section 3.3 we analyze and compare the computations of the equivalence groups for the initial class (3.2) and the re-parameterized class (3.7).

The group classification for the class (3.1) of one-dimensional shallow water equations up to equivalence transformations can be obtained from the description of Lie symmetries of these equations that is presented in [2, 3].

Further, in Chapter 4 we solved the group classification problem for the class (4.1) of two-dimensional shallow water equations with variable bottom topography. The result is summed up in Theorem 36.

Applying the algebraic method, similarly to the one-dimensional case, we first construct the generalized equivalence group $G^{\sim}$ of the class (4.1), which is presented in Theorem 35 and is a necessary ingredient for solving the group classification problem. Note that the generalized equivalence group of the class (4.1) coincides with its usual equivalence group. The integration of the system of determining equations for the components of Lie symmetry vector fields, which is quite complicated in this case, required the application of the advanced method of furcate splitting. This method was additionally optimized via reducing the study of compatibility of template-form equations for the arbitrary element $b$ to checking whether the set of vector fields associated to these equations is closed with respect to the Lie bracket of vector fields. In the course of the classification, we continuously used transformations from the equivalence group $G^{\sim}$ for gauging various constants involved in the specific values of the arbitrary element $b$, which leads to a significant simplification of computations. One more complication of the group classification for the class (4.1) is that this class is not normalized and even not semi-normalized. Therefore, this class possesses admissible point transformations which are not generated by its equivalence transformations and point symmetry transformations of systems belonging to it. Such admissible transformations give rise to additional point equivalences among the $G^{\sim}$-inequivalent classification cases listed in Theorem 36. In Section 4.4 we found three families of $G^{\sim}$-inequivalent admissible transformations of the above kind in the class (4.1) and presented the corresponding additional equivalences within the group-classification list for this class up to the $G^{\sim}$-equivalence. Moreover, for all the pairs of listed cases that are possibly inequivalent to each other with respect to point transformations, we checked their inequivalences using algebraic techniques, except the inequivalences within the set of cases with two-dimensional non-abelian maximal Lie invariance algebras. This allowed us to conjecture the group classification of the class (4.1) up to its equivalence groupoid $\mathcal{G}^{\sim}$.

The presented classification of Lie symmetries of two-dimensional shallow water equations with variable bottom topography provides a basis for finding exact solutions of these equations. Such solutions can be used for testing numerical schemes for this model. Moreover, Lie symmetries themselves may be applied for designing invariant numerical schemes.

In further work we plan to continue our studies of shallow water equations to extend and generalize the obtained result. We have already shown that there are admissible transformations of the class of two-dimensional shallow water equations, which are not generated by equivalence transformations of this class and which induce additional equivalences among $G^{\sim}$-inequivalent cases of Lie symmetry extensions in Theorem 36. Nevertheless, the obtained additional equivalences do not guarantee that we have found all admissible transformations in the class (4.1). In other words, we need to construct the equivalence groupoid of the class of two-dimensional shallow water equations to find all such transformations. This construction can be realized via solving the classification problem for conditional equivalence groups of the class (4.1) [47].

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