MEMORIAL UNIVERSITY

Reservoir dynamics induced by coupling to a quantum system

by

© **Abed Alsalam AbuMoise**

A thesis submitted to the School of Graduate Studies in partial fulfillment of the requirements for the degree of Master of Science.

Department of Mathematics and Statistics

Memorial University

September 2019

St. John's, Newfoundland and Labrador, Canada

Abstract

We consider the dynamics of fixed size subsystems of an open quantum system, in which N particles interact via a common quantum noise (reservoir). We show that correlations among the particles and between the particles and the reservoir, which are brought about through the interaction for finite N, vanish completely in the high complexity limit $N \to \infty$. We investigate the effect of the particle system on the reservoir, which itself is a large quantum system. For each fixed time, we find the explicit construction of a Hilbert space representation of the asymptotic ($N \to \infty$) reservoir state and analyze the relation between those representations at different times.

To my loving parents and sweet wife.

Acknowledgements

I would like to take this opportunity to thank all those that, in one way or another, helped and supported me in writing this thesis.

I would like to begin with thanking with all my heart my parents, Ahmad and Meyasser, for all their love, support, encouragement, and guidance. Without them, I may have never reached where I am today. I would also like to extend my deep thanks to my wife, Razan, for her love, patience, and support now and for years to come. I also thank my sisters (Dalal, Shatha, and Shaima'a) and brothers (Belal, Mohammed, and Nedal) for their encouragement and kindness.

I would like to thank my supervisor, Dr. Marco Merkli, for his suggestions, advice, contributions, and help in this research project. I thank the two reviewers Dr. M. Kotchetov and Dr. T. Baird for carefully reading the thesis and giving helpful suggestions for improvement.

I wish to acknowledge the financial assistance provided by the School of Graduate Studies and Department of Mathematics and Statistics.

Finally, thank you to all my friends and student fellows for all their help, conversations, and good wishes through this process.

Statement of contribution

This thesis is a collaboration of work by Abed Alsalam AbuMoise and Dr. Marco Merkli. All results included were developed by both parties. Supervision and editing of the thesis was done by Dr. Marco Merkli.

Table of contents

List of figures

Chapter 1

Introduction

The theory of open quantum system, which we consider in this thesis, is an important topic in physics and mathematics [19]. It contains in particular the study of 'generic dynamical effects' imposed on a system in contact with a 'noise'. A prime example of an open quantum system is the so-called *spin-Boson* model, where a single spin (the simplest quantum mechanical object possible) is coupled to a 'noise' modeled by a large collection of oscillating degrees of freedom (a quantum field of oscillators). Generic properties the noise imposes on the spin are thermalization (if the oscillators are in thermal equilibrium initially, then the spin will inherit that temperature as time goes on) and decoherence (loss of quantum coherences in the spin state) [20, 17, 19].

In this thesis, we focus on *complex* open systems, where not one single particle (or spin, or 'qubit') is coupled to a noisy environment, but *many* of them are. Namely, we analyze the dynamics of N quantum particles (idealized 'atoms'), all interacting with a common quantum field (the reservoir, for example the quantized electromagnetic field). The particles do not experience direct coupling with each other, but interact indirectly via the reservoir. Our main question is: what is the dynamics of the particles and the reservoir in the limit $N \to \infty$? In a sense, the particle system itself becomes a large quantum system ($N \to \infty$) and so its interaction with the reservoir, another large system, may actually change the latter. This reaction of a 'large' system on a 'reservoir' is not traditionally studied as much as that of the open system itself. However, it is of interest from a mathematical as well as physical point of view. For instance, one might be interested in analyzing if photons are emitted into the field, starting with some energetically excited atoms. [14, 21].

Describing the time evolution of the coupled system-reservoir complex is a hard problem, and generally only heuristic methods are available (like the famous Markovian Master Equation approximation). To be able to arrive at mathematically rigorous results, we will make two simplifications in our models:

S1 We consider *energy conserving* models. They are characterized by the fact that the energy of the system is conserved during the evolution. So there is no energy exchange between the system and the reservoir. Nevertheless, it is well known that information between the system and reservoir can be still exchanged even in energy conserving systems, which still typically show irreversible effects in the system dynamics [18, 19, 15].

S2 Each one of the N particles is of the *same* kind, and each one is coupled to the common reservoir in the *same*, mean field way. This symmetry helps the mathematical analysis and ultimately produces an effective independence of the particle dynamics, in which each particle evolves according to its own evolution equation.

To be a bit more specific (but leaving the mathematical details for the following sections), we present here the structure of the Hamiltonian of our model,

$$H = \sum_{j=1}^{N} A_j + H_R + \frac{\lambda}{\sqrt{N}} \sum_{j=1}^{N} V_j.$$

The A_j is the Hamiltonian of particle j, it is really a fixed operator A, but acting on particle j (symmetry, assumption S2). H_R is the Hamiltonian of the reservoir. The interaction of particle j with the reservoir is given by an operator V_j. Again, V_j is really a fixed operator V, describing the interaction between one particle and the field, but the index j means the interaction acts on the jth particle and the reservoir. The assumption S1 of energy conservation is expressed by the fact that the operators A_j and V_j commute. Above, $\lambda \in \mathbb{R}$ is a coupling constant and the interaction part (the second sum) is scaled in the *mean field* way, with $1/\sqrt{N}$. The reason for this scaling is this: The uncoupled particle energy is of the order $O(N)$ (N large), simply because it is the sum of N terms A_j. The effective interaction between two particles, since mediated only via the reservoir, is given by the *square* of interaction term in the expression of H, so the square of a term of the order $O(\sqrt{N})$. It is thus comparable to the size of the non-interacting energy. In this mean field scaling, and as $N \to \infty$, the dynamics will contain competing contributions from both the non-interacting evolution and the interaction, since they are both of the same order.

We now describe the main results of the thesis without going into any technical details. The mathematical statements follow in the chapters below. We have two types of results,

- Results on the dynamics of the system and the reservoir,
- Results on the Hilbert space representation of the limiting state as $N \to \infty$.

Results on the dynamics. We take initial states of N particles and the field in which all subsystems are *not entangled*, that is, they are of the form

$$\rho_{\text{initial}} = \rho_S \otimes \cdots \otimes \rho_S \otimes \rho_R,$$

where ρ_S is a single particle state and ρ_R is that of the reservoir. (Think of equilibrium states, for example.) We then consider observables $\mathcal{O}_n$ of n particles and the reservoir. Here, n is an arbitrary but fixed number. Think of such an observable as, for example, the energy of the first n particles plus that of the field. Or the 'position' of particle number three. Since there are a total of N particles, the time evolution of $\mathcal{O}_n$,

$$\langle \mathcal{O}_n \rangle_N(t) = \mathrm{Tr}\Big(e^{-itH} \rho_{\text{initial}} e^{itH} \mathcal{O}_n \Big)$$

is a function of N. We are asking what the limit is, as $N \to \infty$, for n and t fixed. In other words, we consider *a fixed part* (given by n) of the whole system, but this part interacts with an increasing number of other particles and a reservoir, and we consider the limit when the complexity $N \to \infty$.

Our main results are Theorems 1 and 3 below. They show that, for fixed t and n, but in the limit $N \to \infty$, the state of n particles and the reservoir is a *disentangled* state, namely, a product of n independent single-particle states and a reservoir state. Of course, we have started off with a disentangled initial state, but as soon as the coupled dynamics is at work, all components (all particles and the reservoir) become immediately *correlated* (entangled). *The point is that in the limit of large complexity, $N \to \infty$, these correlations disappear, and this for all times t!* Each independent factor in the asymptotic ($N \to \infty$) state undergoes an independent evolution. This evolution, for a single particle, contains the effects of all other particles plus the reservoir. For the reservoir, the asymptotic evolution contains the effects of the particle system. We point out that we can calculate the asymptotic state and its dynamics (as it varies as a function of t) explicitly.

Results on the Hilbert space representation. As happens often when taking 'thermodynamic' limits, which is here the limit $N \to \infty$, the notion of Hilbert space in quantum theory is lost, and a suitable Hilbert space has to be recreated. To illustrate this, we can consider an observable $\mathcal{O}$ which pertains purely to the reservoir. (This is a special case of the above $\mathcal{O}_n$.) Then our theorems on the dynamics provide us with a limit

$$\langle \mathcal{O} \rangle_\infty(t) = \lim_{N \to \infty} \langle \mathcal{O} \rangle_N(t).$$

The question is now: how can we represent the asymptotic state, which is defined by all the values $\langle \mathcal{O} \rangle_\infty(t)$ (as $\mathcal{O}$ runs through all possible reservoir observables (making up a C^*-algebra))? Let t be fixed. We want to find a *new* Hilbert space $\mathcal{H}_t$, a representation π_t and a vector Ω_t satisfying (for all $\mathcal{O}$)

$$\langle \mathcal{O} \rangle_\infty(t) = \langle \Omega_t, \pi_t(\mathcal{O})\Omega_t \rangle.$$

The triple $(\mathcal{H}_t, \pi_t, \Omega_t)$ is called the GNS (Gelfand–Naimark–Segal) representation of the state $\langle \cdot \rangle_\infty(t)$. Our main theorem in this regard is Theorem 3, in which we construct the GNS representation explicitly. The next question then is how two representations at different times t and t', are related to each other. In Theorem 4 we show that π_t and $\pi_{t'}$ are unitarily equivalent, up to multiplicity, for any t and t'.

Organization of the thesis. In Chapter 1 we review some basic concepts of quantum physics and functional analysis, useful to understand the mathematical phrasing of our main theorems. In Chapter 2, all our results and their proofs are discussed in detail. Finally, in Chapter 3, we end with our conclusion.

Chapter 2

Some quantum theory

2.1 The basic postulates of quantum mechanics

Quantum mechanics is a mathematical framework for the development of physical theories. In this chapter we give a brief description of the basic postulates of quantum mechanics. These postulates provide a connection between the physical world and the mathematical formalism of quantum mechanics [6].

2.1.1 Postulate 1: Space of pure states

Any isolated physical system is described by a complex Hilbert space $\mathcal{H}$, known as the (pure) state space of the system. The system is completely described by its state vector, which is a unit vector in the system state space,

$$|\varphi\rangle \in \mathcal{H}, \qquad \|\varphi\| = 1.$$

Such a state (vector) is also called a "**ket**" (or a wave function).

To any ket $|\varphi\rangle$ is associated the "**bra**", denoted by $\langle\varphi|$, defined to be the element in the dual space $\mathcal{H}^*$ of $\mathcal{H}$ acting as

$$\langle\varphi|\Big(|\psi\rangle\Big) = \langle\varphi, \psi\rangle, \tag{2.1}$$

where the r.h.s. is the inner product of φ and ψ in $\mathcal{H}$.

Examples. (1) A single spin has a Hilbert space $\mathcal{H} = \mathbb{C}^2$ of dimension 2, it has basis $B = \{|\uparrow\rangle, |\downarrow\rangle\}$, where

$$|\uparrow\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |\downarrow\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Any state can be written as a linear combination of the basis elements,

$$|\psi\rangle \in \mathbb{C}^2 \qquad |\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle, \qquad \alpha, \beta \in \mathbb{C}. \tag{2.2}$$

The interpretation of the complex numbers α, β is that $|\alpha|^2$, $|\beta|^2$ are probabilities of finding the spin in the state up or down, respectively (upon measurement, see Postulate 4 below). The normalization $\|\psi\|^2 = 1$ is consistent with this probability interpretation of the coordinates.

(2) A single particle in three dimensional space is described by the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, d^3x)$. A (pure) state is given by a square integrable, normalized function ψ. The physical interpretation of the 'component' $\psi(x)$ is this: $|\psi(x)|^2 d^3x$ is the probability density of finding the particle at location $x \in \mathbb{R}^3$.

2.1.2 Postulate 2: Dynamics (Schrödinger equation)

The state of a quantum system evolves in time according to an evolution equation, the Schrödinger equation. Namely, the orbit $t \mapsto |\varphi(t)\rangle$ satisfies the first-order linear differential equation

$$i\hbar \frac{d|\varphi(t)\rangle}{dt} = H|\varphi(t)\rangle. \tag{2.3}$$

Here $\hbar$ is the *Planck constant* and H is a self-adjoint operator acting on the pure state Hilbert space $\mathcal{H}$, called the *Hamiltonian*. Equation (2.3) can be written as

$$|\varphi(t)\rangle = e^{-itH}|\varphi(0)\rangle, \tag{2.4}$$

where we have "set" $\hbar = 1$ (this is customary in the mathematical literature and amounts to a rescaling of physical scales). The unitary group

$$t \mapsto U(t) = e^{-itH} \tag{2.5}$$

is often called the *propagator*, as it pushes the initial condition to the state at time t.

Examples. If a spin is initially in the state $|\psi(0)\rangle = \alpha_0|\uparrow\rangle + \beta_0|\downarrow\rangle$, then according to the Schrödinger equation (2.4), with Hamiltonian

$$H = \tfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{2.6}$$

the state at time t is

$$|\psi(t)\rangle = \alpha_0 e^{-it/2}|\uparrow\rangle + \beta_0 e^{it/2}|\downarrow\rangle. \tag{2.7}$$

2.1.3 Postulate 3: Composition of systems

If two systems have Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ then the joint, composite system is described by the tensor product,

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2. \tag{2.8}$$

Examples. (1) The Hilbert space describing N particles is given by

$$\bigotimes_{\mathrm{i}=1}^{N} \mathcal{H}_{\mathrm{i}} = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_N,$$

where for each $1 \leq \mathrm{i} \leq N$, $\mathcal{H}_{\mathrm{i}} = L^2(\mathbb{R}^3, d^3x)$.

(2) The composite space $\mathcal{H}$ of a spin and a single particle is

$$\mathcal{H} = \mathbb{C}^2 \otimes L^2(\mathbb{R}^3, d^3x).$$

2.1.4 Postulate 4: Measurements

To every physical observable (energy, position, momentum,....) is associated a self-adjoint operator $A = A^*$. The Hamiltonian H (see Postulate 2) is the observable of *energy*. Suppose the spectral decomposition of A is given by

$$A = \sum_j \lambda_j P_j, \tag{2.9}$$

where the P_j are the spectral projections and λ_j the eigenvalues. When measuring the observable A in any state, the possible measurement outcomes are one of $\{\lambda_1, \lambda_2, \dots\}$. When the measurement is performed on the state $|\psi\rangle$, the outcome λ_i will occur with probability

$$p_\text{i} = \|P_j|\psi\rangle\|^2 = \langle\psi, P_j\psi\rangle. \tag{2.10}$$

If the measurement reveals the outcome λ_j, then the state of the system *immediately after measurement* is

$$\psi_\text{post} = \frac{P_j|\psi\rangle}{\|P_j\psi\|}. \tag{2.11}$$

This part of the postulate is called the "wave function collapse" and (2.11) is called the post measurement state.

Examples. (1) Consider the spin with Hamiltonian (2.6),

$$H = \tfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \tfrac{1}{2}P_+ - \tfrac{1}{2}P_- \tag{2.12}$$

(with obvious notation for the spectral projections). The measurement outcomes for the energy are $\pm 1/2$ in any state. Upon measurement of the energy in the state $|\psi\rangle$, (2.2), the measurement value $+1/2$ occurs with probability

$$p_+ = \|P_+\psi\|^2 = |\alpha|^2.$$

(2) Let A be an observable and $|\psi\rangle$ a state. The expectation value (statistical average) of A with respect to state $|\psi\rangle$ is denoted by $\langle A\rangle$. From Postulate 4 we know that the possible measurement outcomes of A are its eigenvalues, where each eigenvalue λ_j will occur with probability p_j. Thus the average of A is

$$\langle A\rangle = \sum_j \lambda_j p_j. \tag{2.13}$$

Using the probability formula in equation (2.10) and the spectral decomposition of A

in (2.9) we have

$$\begin{aligned} \langle A \rangle &= \sum_j \lambda_j \langle \psi, P_j \psi \rangle \\ &= \langle \psi, \sum_j \lambda_j P_j \psi \rangle \\ &= \langle \psi, A\psi \rangle \\ &= \mathrm{Tr}(|\psi\rangle\langle\psi|A). \end{aligned} \tag{2.14}$$

The trace of an operator X (if it exists) is given by

$$\mathrm{Tr}(X) = \sum_{n\in\mathbb{N}} \langle e_n, X e_n \rangle, \tag{2.15}$$

for any orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$. The definition of trace is independent of the choice of the orthonormal basis.

2.2 Mixed states

2.2.1 Density matrix

The average of an observable $\mathcal{O}$ in the pure state $|\varphi\rangle$ is $\langle \varphi, \mathcal{O}\varphi \rangle$, see (2.14). Suppose now that our knowledge on the state is not perfect, namely, that we only know that with probabilities p_j our state is $|\varphi_j\rangle$. The collection $\{|\varphi_j\rangle, p_j\}$ is called an *ensemble of pure states*. The average of the observable $\mathcal{O}$ associated to that ensemble is naturally defined to be

$$\langle \mathcal{O} \rangle = \sum_j p_j \langle \varphi_j, \mathcal{O}\varphi_j \rangle. \tag{2.16}$$

By defining the *density matrix*

$$\rho := \sum_j p_j |\varphi_j\rangle\langle\varphi_j|, \tag{2.17}$$

we see that

$$\langle \mathcal{O} \rangle = \mathrm{Tr}(\rho\, \mathcal{O}). \tag{2.18}$$

The density matrix ρ, (2.17), is called a *mixed state* [1]. If ρ has rank one, then it is just equivalent to a pure state, $\rho = |\varphi\rangle\langle\varphi|$ (see (2.14)). Pure states are defined as the states whose density matrices have rank one (projections). More generally, any operator ρ acting on $\mathcal{H}$ satisfying the following properties is a density matrix:

- $\rho \geq 0$ (positivity, in particular self-adjoint),
- $\mathrm{Tr}\rho = 1$ (normalized).

Examples. (1) For any $0 \leq p \leq 1$, the following is a family of density matrices of a spin,

$$\rho = p|\uparrow\rangle\langle\uparrow| + (1-p)|\downarrow\rangle\langle\downarrow| = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix}.$$

Here, ρ is pure if and only if $p \in \{0, 1\}$.

(2) Let ψ be a general pure state of a spin, (2.2). The associated density matrix (written in the basis $|\uparrow\rangle$, $|\downarrow\rangle$) reads

$$\rho = |\psi\rangle\langle\psi| = \begin{bmatrix} |\alpha|^2 & \bar{\alpha}\beta \\ \alpha\bar{\beta} & |\beta|^2 \end{bmatrix}. \tag{2.19}$$

2.2.2 Reduced density matrix and partial trace

Let $X \otimes Y$ be an operator on the composite system $\mathcal{H}_1 \otimes \mathcal{H}_2$, formed by two separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. We define the partial trace over $\mathcal{H}_2$ by

$$\mathrm{Tr}_2(X \otimes Y) = X\mathrm{Tr}(Y). \tag{2.20}$$

Tr_2 extends by linearity and countinuity to a linear map from $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ to $\mathcal{B}(\mathcal{H}_1)$ [10]. The partial trace is important when we study the physical state for a subsystem of the composite system. In other words, if ρ_{12} is a density matrix of the composite system $\mathcal{H}_1 \otimes \mathcal{H}_2$, then the *reduced states* are

$$\rho_1 = \mathrm{Tr}_2(\rho_{12}), \qquad \rho_2 = \mathrm{Tr}_1(\rho_{12}). \tag{2.21}$$

ρ_1 and ρ_2 are called the **reduced density operators** for the system 1 and 2, respectively. The point of this construction is the following.

Suppose ρ_{12} is as above, and we want to find the average of an observable $\mathcal{O}_1$ of system 1 only. This average is

$$\mathrm{Tr}_{\mathcal{H}_1\otimes\mathcal{H}_2}\Big(\rho_{12}\left(\mathcal{O}_1\otimes\mathbb{1}_2\right)\Big)=\mathrm{Tr}_{\mathcal{H}_1}(\rho_1\mathcal{O}_1). \tag{2.22}$$

This means we can use the reduced density matrix of a composite system if we are interested in properties of a subsystem only.

Example. The Hilbert space $\mathcal{H}=\mathbb{C}^2\otimes\mathbb{C}^2$ describes the pure states of two spins. Consider the pure state (Bell state)

$$|\psi\rangle=\frac{|\uparrow\uparrow\rangle+|\downarrow\downarrow\rangle}{\sqrt{2}}, \tag{2.23}$$

where $|\uparrow\uparrow\rangle=|\uparrow\rangle\otimes|\uparrow\rangle$, $|\downarrow\downarrow\rangle=|\downarrow\rangle\otimes|\downarrow\rangle$ and call its density matrix $\rho_{12}=|\psi\rangle\langle\psi|$. The reduction to the first spin is

$$\rho_1=\mathrm{Tr}_2\rho_{12}=\tfrac{1}{2}(|\uparrow\rangle\langle\uparrow|+|\downarrow\rangle\langle\downarrow|). \tag{2.24}$$

This example shows that the reduced state of a pure state can actually be a mixed state. (Note, the rank of ρ_1 is two.)

2.3 Evolution of closed and open system

2.3.1 Postulate 2, again

According to Postulate 2, the dynamics of the pure initial state $|\psi(0)\rangle$ is given by $|\psi(t)\rangle=e^{-itH}|\psi(0)\rangle$. Equivalently, the propagator

$$U(t)=e^{-itH}, \tag{2.25}$$

satisfies the evolution equation

$$i\frac{dU(t)}{dt}=HU(t). \tag{2.26}$$

This is the setup of Postulate 2, which implicitly assumes that the system considered is *closed*, meaning that it is not in contact with 'external agents'. (Strictly speaking, thus, the only closed system is the whole universe, since in reality, *any* system is in contact with its surroundings.)

How does the dynamics look for a closed system in a mixed state? Suppose the system is described by the density matrix $\rho(0)$ at time zero. To get the equation of motion for this state, we use the definition of the density matrix in (2.17),

$$\rho(0) = \sum_j p_j |\varphi_j(0)\rangle\langle\varphi_j(0)|. \tag{2.27}$$

Now the evolution of $|\varphi_j\rangle$ is given by $|\varphi_j(t)\rangle = U(t)|\varphi_j(0)\rangle$ and so the density matrix at time t is

$$\begin{aligned} \rho(t) &= \sum_j p_j U(t)|\varphi_j(0)\rangle\langle\varphi_j(0)|U^*(t) \\ &= U(t)\rho(0)U^*(t), \end{aligned} \tag{2.28}$$

where $U^*(t)$ is the adjoint of $U(t)$. With (2.25) this becomes

$$\rho(t) = e^{-itH}\rho(0)e^{itH}. \tag{2.29}$$

Equation (2.29) is called the **Liouville-von Neumann equation** [2]. In differential form, it takes the shape

$$\frac{d}{dt}\rho(t) = -i[H, \rho(t)]. \tag{2.30}$$

Let $\mathcal{O}$ be an observable. Its average in the state $\rho(t)$ is

$$\begin{aligned} \mathrm{Tr}\Big(\rho(t)\mathcal{O}\Big) &= \mathrm{Tr}(e^{-itH}\rho(0)e^{itH}\mathcal{O}) \\ &= \mathrm{Tr}(\rho(0)\mathcal{O}(t)), \end{aligned} \tag{2.31}$$

where

$$\mathcal{O}(t) = e^{itH}\mathcal{O}e^{-itH}. \tag{2.32}$$

The map $t \mapsto \mathcal{O}(t)$ called the **Heisenberg evolution** of observable $\mathcal{O}$ [4].

2.3.2 Open systems

An open system is a system in contact with an 'environment', with which the system can exchange energy, matter, information... The following diagram illustrates what we mean by an open quantum system.

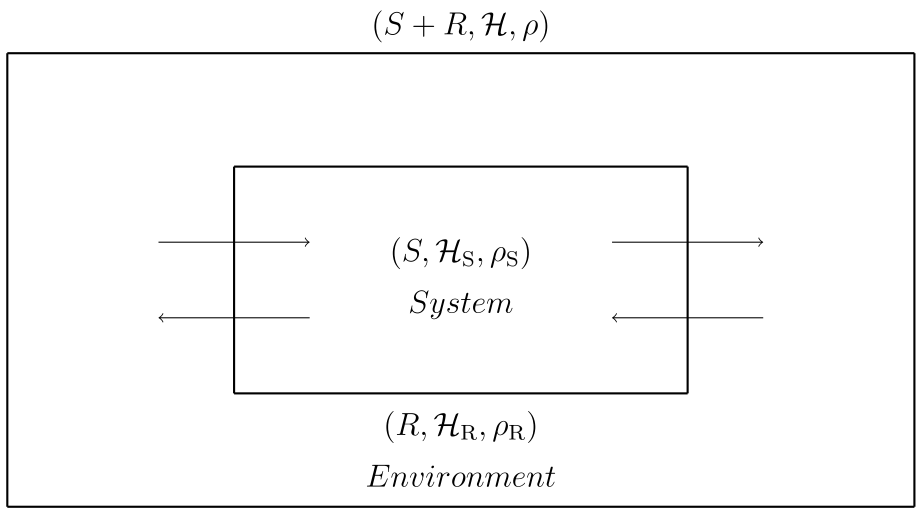

Figure 2.1: Open Quantum system

In the above figure the system S is described by a Hilbert space $\mathcal{H}_{\mathrm{S}}$ and a state ρ_{S}. It is coupled with the environment R ("reservoir") which is described by a Hilbert space $\mathcal{H}_{\mathrm{R}}$ and a state ρ_{R}.

Postulate 3 tells us that the total system $S+R$ is given by the tensor product $\mathcal{H} = \mathcal{H}_{\mathrm{S}} \otimes \mathcal{H}_{\mathrm{R}}$. A state of the joint system, in which the system and reservoir parts are not correlated (no entanglement) is given by $\rho = \rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}$. The total Hamiltonian H for the composite system has the form

$$H = H_{\mathrm{S}} \otimes \mathbb{1}_{\mathrm{R}} + \mathbb{1}_{\mathrm{S}} \otimes H_{\mathrm{R}} + H_{\mathrm{C}}, \tag{2.33}$$

where H_{S} and H_{R} are the Hamiltonians of the system and environment. H_C is the Hamiltonian of the interaction between the system and environment, which acts on the total system $\mathcal{H}$. Even though the evolution of the total complex S+R is given by the Schrödinger equation (unitary propagator), the time evolution of the open subsystem S is not, in general, unitary. The non unitary dynamics of the open system comes from the interaction between the system and the environment. It reflects that the system can lose energy, matter,....

Let $\rho(0)$ be the initial state of the complex $S+R$. The reduced density matrix of S at time t is given by

$$\rho_{\mathrm{S}}(t) = \mathrm{Tr}_{\mathrm{R}}\{U(t)\rho(0)U^*(t)\}, \tag{2.34}$$

where we take the partial trace over the reservoir degrees of freedom. In equation (2.34), $U(t)\rho(0)U^*(t)$ is the (closed, unitary) evolution of the *total* complex $S+R$.

The differential form of (2.34) is

$$\frac{d}{dt}\mathrm{Tr}_{\mathrm{R}}\rho(t) = \frac{d}{dt}\rho_{\mathrm{S}}(t) = -i\mathrm{Tr}_{\mathrm{R}}[H, \rho(t)], \tag{2.35}$$

where H is as in (2.33).

An observable $\mathcal{O}_{\mathrm{S}}$ of the open system S has the form

$$\mathcal{O}_{\mathrm{S}} = \mathcal{O}_{\mathrm{S}} \otimes \mathbb{1}_{\mathrm{R}}, \tag{2.36}$$

where operator $\mathcal{O}_{\mathrm{S}}$ acting on $\mathcal{H}_{\mathrm{S}}$ and $\mathbb{1}_{\mathrm{R}}$ stands for the identity operator of $\mathcal{H}_{\mathrm{R}}$. The average value of $\mathcal{O}_{\mathrm{S}}$ is given by

$$\langle \mathcal{O}_{\mathrm{S}}(t) \rangle = \mathrm{Tr}_{\mathrm{S}}(\rho_{\mathrm{S}}(t)\mathcal{O}_{\mathrm{S}}), \tag{2.37}$$

where $\rho_{\mathrm{S}}(t)$ as is in (2.34).

2.4 Quantum field

In this section we will define Fock space, creation and annihilation opertors and Weyl operators [7].

2.4.1 Fock space

Given a Hilbert space $\mathcal{H}$, its n-fold tensor product is

$$\mathcal{H}^{\otimes n} = \mathcal{H} \otimes \ldots \otimes \mathcal{H}. \tag{2.38}$$

For example, if $\mathcal{H} = L^2(\mathbb{R}^3, d^3k)$, then according to the third postulate the above expression (2.38) describes a system of n particles.

Definition 1. **Fock space** over Hilbert space $\mathcal{H}$ is the direct sum Hilbert space

$$\widetilde{\mathcal{F}}(\mathcal{H}) = \bigoplus_{n\geq 0} \mathcal{H}^{\otimes n}, \tag{2.39}$$

where $\mathcal{H}^0 = \mathbb{C}$ is called the zero sector or vacuum sector. An element $\psi \in \widetilde{\mathcal{F}}(\mathcal{H})$ is a sequence $\psi = \{\psi_n\}_{n\geq 0}$ where $\psi_n \in \mathcal{H}^{\otimes n}$. The scalar product of two elements $\psi, \varphi \in \widetilde{\mathcal{F}}(\mathcal{H})$ is given by

$$\langle \psi, \varphi \rangle = \sum_{n\geq 0} \langle \psi_n, \varphi_n \rangle_{\mathcal{H}^{\otimes n}}, \tag{2.40}$$

where $\langle . , . \rangle_{\mathcal{H}^{\otimes n}}$ is the scalar product of $\mathcal{H}^{\otimes n}$ which is defined by

$$\langle \psi_1 \otimes \ldots \otimes \psi_n, \phi_1 \otimes \ldots \otimes \phi_n \rangle = \langle \psi_1, \phi_1 \rangle \ldots \langle \psi_n, \phi_n \rangle \tag{2.41}$$

for $\psi_1, \ldots, \psi_n, \phi_1, \ldots, \phi_n \in \mathcal{H}$.

The vector $f_1 \otimes \ldots \otimes f_n \in \mathcal{H}^{\otimes n}$ is the state of n 'particles' (subsystems) where the particle labelled by j is in the state f_j. If the n particles are *indistinguishable* then the state describing the system is given by the symmetric state vector

$$\frac{1}{n!} \sum_{\sigma \in \Delta_n} f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(n)}, \tag{2.42}$$

where Δ_n is the group of all permutations σ of n objects.

Definition 2. Let $\{f_j\}_{j=1}^n \subset \mathcal{H}$, $n \geq 1$. Define the **symmetrization** operator P on $\mathcal{F}(\mathcal{H})$ by linear extension and sector wise action of

$$P f_1 \otimes \ldots \otimes f_n = \frac{1}{n!} \sum_{\sigma \in \Delta_n} f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(n)}, \tag{2.43}$$

P is a self-adjoint projection operator satisfying $||P||= 1$.

The symmetrization operator provides a powerful structure for dealing with the symmetries of states and operators for systems with many identical, indistinguishable particles.

Applying the symmetrization operator to Fock space $P\mathcal{F}(\mathcal{H})$ we obtain **bosonic**

Fock space,

$$\mathcal{F}(\mathcal{H}) \equiv P\widetilde{\mathcal{F}}(\mathcal{H}) \equiv \bigoplus_{n\geq 0} P\mathcal{H}^{\otimes n}. \tag{2.44}$$

In this thesis we keep the notation $\mathcal{F}(\mathcal{H})$ to present the bosonic Fock space, for simplicity of notation.

2.4.2 Creation and annihilation operators

Let $\mathcal{H}$ be a Hilbert space and consider the Fock space $\mathcal{F}(\mathcal{H})$. We define the **vacuum vector** to be the vector $\Omega = (1, 0, 0, \ldots) \in \mathcal{F}(\mathcal{H})$.

Definition 3. Let $\{f_j\}_{j=1}^n \subset \mathcal{H}$, $n \geq 1$.

- The **annihilation** operator $a(f)$ is a linear map $\mathcal{H}^{\otimes n} \mapsto \mathcal{H}^{\otimes(n-1)}$, defined by

$$a(f)f_1 \otimes \cdots \otimes f_n = \sqrt{n}\, \langle f, f_1\rangle f_2 \otimes \ldots \otimes f_n \tag{2.45}$$

  for $n \geq 1$ and $a(f)\Omega = 0$.

- The **creation** operator $a^*(f)$ is the linear map $\mathcal{H}^{\otimes n} \mapsto \mathcal{H}^{\otimes(n+1)}$ defined by

$$a^*(f)f_1 \otimes \cdots \otimes f_n = \sqrt{n+1}\, f \otimes f_1 \otimes \ldots \otimes f_n, \tag{2.46}$$

  The map $f \mapsto a(f)$ is antilinear, while $f \mapsto a^*(f)$ is linear. We extend the action of a and a^* by linearity to D^n for all n, where

$$D^n = \left\{ \sum_{k=1}^{K} f_1^{(k)} \otimes \cdots \otimes f_n^{(k)} | K \in \mathbb{N}, f_l^{(k)} \in \mathcal{H} \right\} \subset \mathcal{H}^{\otimes n}. \tag{2.47}$$

  The operators $a(f)$ and $a^*(f)$ thus defined are *closable* and we denote their closure again by the same symbol.

In (2.45), (2.46) we have defined $a^*(f)$ and $a(f)$ on the non-symmetrized Fock space $\mathcal{F}(\mathcal{H})$. The creation and annihilation operators on $P\mathcal{F}(\mathcal{H})$ are defined simply by $Pa^*(f)P$ and $Pa(f)P$.

Note that the span of $\{a^*(f_1)\ldots a^*(f_n)\Omega : f_i \in \mathcal{H}, n \in \mathbb{N}\}$ is dense in $\mathcal{F}(\mathcal{H})$.

The canonical commutation relations (CCR) are given by

$$\begin{aligned} [a(g), a^*(f)] &= \langle g, f\rangle \mathbb{1}_{\mathcal{F}(\mathcal{H})}, \\ [a(f), a(g)] &= [a^*(f), a^*(g)] = 0, \quad \forall f, g \in \mathcal{H}, \end{aligned} \tag{2.48}$$

where the $\mathbb{1}_{\mathcal{F}(\mathcal{H})}$ is the identity operator acting on bosonic Fock space and

$$[x, y] = xy - yx$$

is the commutator.

2.4.3 Weyl operators

The bosonic creation and annihilation are unbounded operators and the *field operator*

$$\varphi(f) = \frac{a(f) + a^*(f)}{\sqrt{2}} \tag{2.49}$$

is a self-adjoint, unbounded operator. The mathematicians prefer to replace them by bounded operators, called *Weyl operators*.

Definition 4. For $f \in \mathcal{H}$, we define

$$W(f) = e^{i\varphi(f)}. \tag{2.50}$$

This is a unitary operator on bosonic Fock space $\mathcal{F}(\mathcal{H})$. Using the Taylor expansion of the exponential in (2.50),

$$W(f) = \sum_{n\geq 0} \frac{i^n}{n!} \varphi(f)^n,$$

together with the canonical commutation relations (2.48) one can easily deduce the formula

$$\langle \Omega, W(f)\Omega\rangle = e^{-\|f\|^2/4}. \tag{2.51}$$

The set of all Weyl operators generates a unital C^*-algebra of operators (see the definition in the next section), called the *Weyl algebra*, and denoted by $\mathcal{W}$. The unit is $W(0) = \mathbb{1}$.

The CCR (2.48) take the form

$$W(f)W(g) = e^{-\frac{i}{2}\mathrm{Im}\langle f,g\rangle}W(f+g). \tag{2.52}$$

Theorem I. [8] *Let $f_n \to f$ in $\mathcal{H}$, then $W(f_n) \to W(f)$ in strong sense on $\mathcal{F}(\mathcal{H})$,* i.e., *for any $\psi \in \mathcal{F}(\mathcal{H})$ we have*

$$\lim_{n\to\infty} \|W(f_n)\psi - W(f)\psi\|_{\mathcal{F}(\mathcal{H})} = 0. \tag{2.53}$$

2.4.4 The Weyl algebra

Let $\mathfrak{H}$ be a Hilbert space. Its interpretation is that normalized vectors in $\mathfrak{H}$ are single particle states (wave functions). Weyl operators over $\mathfrak{H}$ form an abstract C^*-algebra. They are denoted by $W(f)$, $f \in \mathfrak{H}$ and satisfy the properties

$$\begin{aligned} W(f)^* &= W(-f), \qquad \forall f \in \mathfrak{H}, \\ W(f)W(g) &= e^{-\frac{i}{2}\mathrm{Im}\langle f,g\rangle}W(f+g), \qquad \forall f,g \in \mathfrak{H}. \end{aligned}$$

The second relation is called the *canonical commutation relation* (in Weyl form) and the inner product $\langle \cdot, \cdot \rangle$ is that of $\mathfrak{H}$. A typical Hilbert space representation of the Weyl algebra is given in the previous section in (2.50), where the abstract element $W(f)$ of the C^* algebra is represented as the unitary operator $e^{i\varphi(f)}$ on Fock space $\mathcal{F}(\mathcal{H})$ (here, $\mathcal{H} = \mathfrak{H}$). Often, people take the same notation $W(f)$ for the element in the C^* algebra and the represented operator. For more detail, we refer to [16, 13].

2.5 Algebraic approach

In the algebraic approach to quantum theory the Hilbert space loses its primary importance. The primary object one starts with is an abstract C^*-algebra containing an algebra of quantum observables. The Hilbert space is a secondary concept which may be derived by constructing particular representation in the spirit of GNS construction [3, 13]. The necessity of such an approach comes from physically natural limiting procedures. For example, one may consider a system of particles in equilibrium, within a confined, compact region of position space $\Lambda \subset \mathbb{R}^3$. One might then want to construct a 'thermodynamic limit', where the size of Λ becomes infinite. Now, while the Hilbert space for the system at all finite Λ is well defined, it is not clear what the 'right' Hilbert space is to describe the infinitely extended particle system. Physical quantities depending on Λ (like, local energy density or so) have a well defined limit as $|\Lambda| \to \infty$, but the Hilbert space *per se* does not. Generally, *observables* of 'local' nature are well defined in the thermodynamic limit, and so it is natural to consider the observables as the core quantities (which do not change even in such limiting procedures).

Definition 5. An *associative algebra* is a complex vector space V equipped with a multiplication $V \times V \to V : (u, v) \to uv$ satisfying the following conditions. $\forall u, v, w \in V$ and scalars $\lambda, \mu \in \mathbb{C}$:

- $(uv)w = u(vw)$ (associativity),
- $(\lambda u + \mu v)w = \lambda(uw) + \mu(vw)$ and $w(\lambda u + \mu v) = \lambda(wu) + \mu(wv)$ (bi-linearity).

One says that V is **unital** algebra if V has unit *i.e,* if there is a $\mathbb{1} \in V$ so that $ev = ve = v$ for all $v \in V$.

Definition 6. An involution over an algebra V is a map $v \to v^*$ from V to itself so that $\forall u, v \in V$ and $\lambda \in \mathbb{C}$ we have

- $u^{**} = u$,
- $(u + v)^* = u^* + v^*$,
- $(\lambda u)^* = \overline{\lambda} u^*$,
- $(uv)^* = v^* u^*$.

In view of the definitions 5 and 6 we define $*$-*algebra* to be an algebra equipped with an involution.

Example. Consider $\mathcal{H} = \mathbb{C}^d$, then the algebra of linear operators $\mathcal{B}(\mathcal{H})$ is $*$-algebra, with the star operation given by the adjoint of an operator i.e, for any operator a on $\mathcal{H}$, its adjoint is defined by the equation

$$\langle a^* \psi, \phi \rangle = \langle \psi, a\phi \rangle, \quad \forall \psi, \phi \in \mathcal{H}.$$

This is the algebra of $d \times d$ complex matrices $\mathcal{M}^d$.

Definition 7. A C^*-algebra is a complex Banach space $\mathcal{A}$ which at the same time is a $*$-algebra, such that for all $x, y \in \mathcal{A}$ we have

- $\|xy\| \leq \|x\| \, \|y\|$,
- $\|x^*x\| = \|x\|^2$.

The structure of a C^*-algebra allows us to introduce a collection of concepts related to operators on a Hilbert space:

Definition 8. Suppose $\mathcal{A}$ is a C^*-algebra and $x \in \mathcal{A}$.

- x is normal iff $xx^* = x^*x$,
- x is self adjoint (Hermitian) iff $x^* = x$,
- x is unitary iff $xx^* = x^*x = \mathbb{1}$,
- x is positive iff $x = y^*y$ for some $y \in \mathcal{A}$,
- x is a projection iff $x^* = x = x^2$.

The importance of normal operators is that the spectral theorem holds for them: every normal operator on a finite-dimensional Hilbert space is diagonalizable by a unitary operator.

Definition 9. Let $\mathcal{A}$ be a C^*-algebra and let ω be a linear functional on $\mathcal{A}$. Then

- ω is Hermitian if $\omega(x^*) = \overline{\omega(x)}$, for all $x \in \mathcal{A}$,
- ω is positive if $\omega(x) \geq 0$, whenever x is positive.

We note that ω positive implies ω Hermitian [13]. As a consequence of the above definitions we define a quantum **state** to be a positive linear functional ω on a unital $*$-algebra $\mathcal{A}$ with $\omega(\mathbb{1}) = 1$. For every $*$-algebra $\mathcal{A}$, we denote by $\mathcal{S}(\mathcal{A})$ the set of all states on $\mathcal{A}$.

Definition 10. [11] A *representation* a of C^*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a complex linear map $\pi : \mathcal{A} \to B(\mathcal{H})$ such that

- $\pi(xy) = \pi(x)\pi(y)$ for all $x, y \in \mathcal{A}$,
- $\pi(x^*) = \pi(x)^*$ for all $x \in \mathcal{A}$.

A representation π is automatically continuous satisfying the bound $||\pi(x)|| \leq ||x||$, [13].

Definition 11. A representation is **irreducible** if there is no proper, nontrivial subspace of $\mathcal{H}$ that is invariant under π.

2.5.1 GNS Construction

The GNS (Gelfand–Naimark–Segal) construction shows that every C^*-algebra is isomorphic to a C^*-subalgebra of bounded operators acting on some Hilbert space $\mathcal{H}$ [12]. To establish the GNS construction we need the following propositions following from [12].

Proposition 1. *Suppose ω is a state on a C^*-algebra $\mathcal{A}$ and set*

$$\mathcal{L}_\omega = \{x \in \mathcal{A} : \omega(x^*x) = 0\}.$$

Then $\mathcal{L}_\omega$ is a closed left ideal in $\mathcal{A}$. Moreover, $\omega(x^*y) = 0$ whenever x or y is in $\mathcal{L}_\omega$.

Now define the quotient space $\mathcal{H}_\omega^\circ = \mathcal{A}/\mathcal{L}_\omega$ where the quotient is relative to the equivalence relation

$$x \sim y \quad \Leftrightarrow \quad x - y \in \mathcal{L}_\omega.$$

Note that $\mathcal{H}_\omega^\circ$ is a pre-Hilbert space with inner product

$$\langle [x], [y] \rangle = \omega(x^*y), \quad \forall x, y \in \mathcal{A} \tag{2.54}$$

and $[x], [y]$ are cosets in the quotient space. Define $\mathcal{H}_\omega$ to be the completion of $\mathcal{H}_\omega^\circ$ with respect to inner product. Then $\mathcal{H}_\omega$ is the GNS Hilbert space.

Proposition 2. *Let ω be a state on a C^*-algebra $\mathcal{A}$. For any $x \in \mathcal{A}$ define an operator $F_x : \mathcal{H}_\omega^\circ \to \mathcal{H}_\omega^\circ$ by*

$$F_x([y]) = [xy], \ \textit{for all } y \in \mathcal{A}, \tag{2.55}$$

then F_x *is well-defined and extends to a bounded linear operator on* $\mathcal{H}_\omega$ *with* $||F_x|| \leq ||x||$.

Proposition 3. Suppose $\mathcal{A}$ *is a* C^**-algebra and* ω *is a state on* $\mathcal{A}$*, then the mapping* $\pi : \mathcal{A} \to B(\mathcal{H}_\omega)$ *defined by* $\pi(x) = F_x,\ \forall x \in \mathcal{A}$ *is a representation of* $\mathcal{A}$.

The proofs of Propositions 1 - 3 are not hard, see for instance [13] and the conclusion is the following. Let ω be a state on a C^*-algebra $\mathcal{A}$ and let $x \in \mathcal{A}$. Then

$$\omega(x) = \langle [\mathbb{1}], [x\mathbb{1}] \rangle = \langle [\mathbb{1}], F_x[\mathbb{1}] \rangle = \langle [\mathbb{1}], \pi(x)[\mathbb{1}] \rangle$$

where the first equality follows from (2.54), the second one from (2.55) and the third one from Proposition 3. Setting $u = [\mathbb{1}] \in \mathcal{H}_\omega$ we thus have $\omega(x) = \langle u, \pi(x)u \rangle$. This is the skeleton of a proof of the following result.

Theorem II. (GNS representation [12]) *Let* ω *be a state of a* C^**-algebra* $\mathcal{A}$*, then there is a representation* $(\mathcal{H}, \pi, u)$ *of* $\mathcal{A}$*, where* u *is a unit vector in* $\mathcal{H}$ *such that*

- $\omega(x) = \langle u, \pi(x)u \rangle$ *for all* $x \in \mathcal{A}$*,*
- $\Big\{ \pi(x)u \mid x \in \mathcal{A} \Big\}$ *is dense in* $\mathcal{H}$.

Furthermore, the representation $(\mathcal{H}, \pi, u)$ *is unique up to unitary equivalence.*

Note that we say $(\mathcal{H}_1, \pi_1, u_1)$ and $(\mathcal{H}_2, \pi_2, u_2)$ are unitarily equivalent if there is a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that for all $x \in \mathcal{A}$

$$U\pi_1(x) = \pi_2(x)U, \tag{2.56}$$

and

$$Uu_1 = u_2.$$

Chapter 3

Results

3.1 Statement of the problem

We consider a quantum system of N particles interacting with a collective thermal environment (reservoir). Each single particle is described by a complex Hilbert space $\mathcal{H}_{\mathrm{S}}$. We can assume that $\mathcal{H} = \mathbb{C}^d$. The Hilbert space of the reservoir is the Fock space

$$\mathcal{F} = \bigoplus_{n \geq 0} L^2_{symm}(\mathbb{R}^{3n}, d^{3n}k)$$

Here, $L^2_{symm}(\mathbb{R}^{3n}, d^{3n}k)$ is the space of square-integrable complex-valued functions which are symmetric in n arguments from $\mathbb{R}^3$. The direct summand for $n = 0$ is interpreted to be $\mathbb{C}$, it is called the vacuum sector. The one for $n \geq 1$ is called the n-particle sector.

We denote the field operator as

$$\varphi(f) = \frac{1}{\sqrt{2}}[a^*(f) + a(f)] \quad \text{for all } f \in L^2(\mathbb{R}^3, d^3k), \tag{3.1}$$

where $a^*(f)$ and $a(f)$ are the annihilation and creation operators respectively.

The Hilbert space of the total system-reservoir complex is given by

$$\mathcal{H}_N = \mathcal{H}_{\mathrm{S}}^{\otimes N} \otimes \mathcal{F}. \tag{3.2}$$

According to the principles of quantum theory, (3.2) is the Hilbert space of pure states of N particles plus the reservoir.

The *dynamics* is generated by a self-adjoint Hamiltonian H_N, acting on $\mathcal{H}_N$, of the form

$$\begin{aligned} H_N &= H^0 + \lambda I, \\ H^0 &= H_{\mathrm{S}} + H_{\mathrm{R}}. \end{aligned} \tag{3.3}$$

The Hamiltonian H^0 is the sum of the individual system and reservoir Hamiltonians, which generate the dynamics of the system alone and the reservoir alone, respectively. The term λI in (3.3) is the interaction operator, including a *coupling constant* $\lambda \in \mathbb{R}$.

In our model, we take

$$H_{\mathrm{S}} = \sum_{j=1}^{N} A_j, \tag{3.4}$$

$$H_{\mathrm{R}} = \int_{\mathbb{R}^3} \omega(k) a^*(k) a(k) d^3k, \tag{3.5}$$

$$I = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} Q_j \otimes \varphi(h). \tag{3.6}$$

In (3.4), A_j is short form for the operator

$$A_j = \mathbb{1} \otimes \cdots \otimes A \otimes \cdots \otimes \mathbb{1}, \tag{3.7}$$

where A, which is a fixed self-adjoint operator on $\mathcal{H}$, stands in the jth position in the N-fold tensor product on the right side. The operator A represents the Hamiltonian (energy operator) of a single particle. The real valued function $k \mapsto \omega(k)$ is called the *dispersion relation* of the reservoir particles, it gives the energy associated to the wave vector $k \in \mathbb{R}^3$. For instance, in the case of the quantized electromagnetic field, one has

$$\omega(k) = |k|.$$

The form (3.5) of H_{R} is a notation (called in physics 'second quantization'), it is equivalent to the following action (on field operators (3.1), for example),

$$e^{itH_{\rm R}}\varphi(f)e^{-itH_{\rm R}} = \varphi(e^{i\omega t}f), \tag{3.8}$$

where $(e^{i\omega t}f)(k) = e^{it\omega(k)}f(k) \in L^2(\mathbb{R}^3, d^3k)$. Similarly to A_j, the Q_j in (3.6) is interpreted as a fixed operator Q acting nontrivially on the jth tensor factor only. Physically, $Q\otimes\varphi(h)$ encodes the way a single particle is coupled to the reservoir. The function $h \in L^2(\mathbb{R}^3, d^3k)$ in (3.6) is called the *form factor*. The size of $h(k)$, for a given $k \in \mathbb{R}^3$, determines how strongly the mode k is coupled to the particle system.

It is important to point out the scaling factor $1/\sqrt{N}$ in the interaction I. The motivation for this scaling is the following. Since the particles do not interact directly, but only via contact with the reservoir, the 'effective particle interaction' is of the size of the interaction squared, I^2. In terms of N, this effective interaction without the prefactor $1/\sqrt{N}$ would be $O(N^2)$ (considering $N \to \infty$). However, the 'free particle' energy, $\sum_{j=1}^N A_j$ is only of $O(N)$. To have both the free energy and the interaction energy of the same order in N (namely, $O(N)$), we thus introduce the factor $1/\sqrt{N}$. In this way, interaction effects and free dynamics effects occur at the same strength. Our crucial assumption is

(A) The operators A and Q commute, $AQ = QA$.

Physically, this means that the energy of each particle is conserved during the dynamics. There is no energy exchange between the particles and the reservoir. Such models are called *energy conserving models*. The great advantage is that often, one can 'explicitly' calculate the dynamics for them.

To sum it up, our interacting Hamiltonian reads

$$H_N = \sum_{j=1}^N A_j + H_{\rm R} + \frac{\lambda}{\sqrt{N}}\sum_{j=1}^N Q_j\otimes\varphi(h) \tag{3.9}$$

and we have $[A, Q] = 0$ (commutator).

We consider initial states of the product (non-entangled) form

$$\rho_N(0) = \rho_{\rm S}\otimes\cdots\otimes\rho_{\rm S}\otimes\rho_{\rm R}, \tag{3.10}$$

in which each particle is in the same state $\rho_{\rm S}$ and the reservoir density matrix is $\rho_{\rm R}$.

According to the Schrödinger equation, the state at time t is given by

$$\rho_N(t) = e^{-itH_N}\rho_N(0)e^{itH_N}. \tag{3.11}$$

Due to the interaction term, $\rho_N(t)$ will *not* be of product form for $t \neq 0$. Given any system-reservoir observable $\mathcal{A} \in \mathcal{B}(\mathcal{H}_N)$, its average at time t is

$$\langle \mathcal{A} \rangle_N(t) \equiv \mathrm{Tr}(\rho_N(t)\mathcal{A}), \tag{3.12}$$

where Tr denotes the trace and the symbol $\equiv$ means equivalent by definition (i.e., $\langle \mathcal{A} \rangle_N(t)$ is defined to be $\mathrm{Tr}(\rho_N(t)\mathcal{A})$). A general $\mathcal{A} \in \mathcal{B}(\mathcal{H}_N)$ is a (possibly infinite) sum of factorized operators of the form

$$\mathcal{P} = \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_N \otimes W(f), \tag{3.13}$$

where the $\mathcal{O}_j$ are arbitrary single particle operators and $W(f)$ is an arbitrary Weyl operator. When we are only interested in properties of the first n particles and the reservoir, for a fixed number $n \geq 1$, then we only need to consider operators of the form

$$\mathcal{O} = \mathcal{O}_1 \otimes \cdots \mathcal{O}_n \otimes \mathbb{1}_{\mathrm{S}} \cdots \mathbb{1}_{\mathrm{S}} \otimes W(f). \tag{3.14}$$

Our goal is to describe the limit $N \to \infty$ of the dynamics

$$t \mapsto \langle \mathcal{O} \rangle_N(t), \tag{3.15}$$

where $\mathcal{O}$ is of the form (3.14) for a *fixed* n.

Let

$$A = \sum_{m=1}^{d} a^{(m)} P^{(m)} \tag{3.16}$$

be the spectral decomposition of A, where $a^{(m)}$ are its eigenvalues and $P^{(m)}$ its (rank-one) projections. We introduce the notation

$$p_m = \mathrm{Tr}_{\mathrm{S}}(P^{(m)}\rho_{\mathrm{S}}). \tag{3.17}$$

The p_m are probabilities, i.e.,

$$0 \le p_m \le 1, \qquad \sum_m p_m = 1. \tag{3.18}$$

We denote also

$$\langle Q \rangle = \mathrm{Tr}(\rho_S Q), \qquad \mathrm{var}(Q) = \langle Q^2 \rangle - \langle Q \rangle^2 \tag{3.19}$$

for the expectation value and the variance of Q in the initial single particle state ρ_S. We also define the (time dependent) single particle density matrix

$$\widetilde{\rho}(t) = e^{2i\lambda^2 Q\langle Q \rangle S(t)} \rho_S \, e^{-2i\lambda^2 Q \langle Q \rangle S(t)} \tag{3.20}$$

where

$$S(t) = \frac{1}{2} \int_{\mathbb{R}^3} |h(k)|^2 \, \frac{\omega t - \sin \omega t}{\omega^2} d^3k. \tag{3.21}$$

In (3.21) h is the form factor appearing in the interaction (3.6), and $\omega \equiv \omega(k)$ is the dispersion relation.

3.2 Results on the dynamics

3.2.1 Dynamics of observables

Our main result about the dynamics of observables is the following.

Theorem 1 (Dynamics of observables) *Consider the observable*

$$\mathcal{O} = \mathcal{O}_1 \otimes \cdots \mathcal{O}_n \otimes \mathbb{1}_S \cdots \mathbb{1}_S \otimes W(f), \tag{3.22}$$

where the $\mathcal{O}_j$, $j = 1, \ldots, n$ *and* $f \in L^2(\mathbb{R}^3, d^3k)$ *are arbitrary. For each fixed* $t \in \mathbb{R}$, *we have*

$$\begin{aligned} &\lim_{N\to\infty} e^{i\sqrt{N}\lambda \langle Q \rangle \mathrm{Im}\langle h, \frac{e^{i\omega t}-1}{i\omega} f \rangle} \langle \mathcal{O} \rangle_N(t) \\ &= e^{-\frac{1}{2}\lambda^2 \mathrm{var}(Q) (\mathrm{Im}\langle h, \frac{e^{i\omega t}-1}{i\omega} f\rangle)^2} \, \mathrm{Tr}_R\Big(\rho_R W(e^{i\omega t} f)\Big) \prod_{j=1}^n \mathrm{Tr}_S\Big(e^{-itA} \widetilde{\rho}(t) \, e^{itA} \mathcal{O}_j\Big). \end{aligned} \tag{3.23}$$

Recall that h *is the form factor in* (3.9), $\omega \equiv \omega(k)$ *and* $\widetilde{\rho}(t)$ *is given in* (3.20).

Discussion.

(1) The result holds for all dispersion relations $\omega(k)$ and coupling 'form factors' $h \in L^2(\mathbb{R}^3)$.

(2) Relation (3.23) shows that $\langle\mathcal{O}\rangle_N(t)$ alone does not converge as $N \to \infty$ unless $\langle Q\rangle = 0$. In special case $f = 0$, we get also convergence. In particular, when $\mathcal{O}$ is an observable of the particle system alone, then we do get convergence. What is the meaning of the fast oscillating 'correction factor' $e^{i\sqrt{N}\lambda\langle Q\rangle \mathrm{Im}\langle h, \frac{e^{i\omega t}-1}{i\omega} f\rangle}$ in (3.23)? It is created by the action of the particle system on reservoir observables (in the sense that if $f = 0$ (no reservoir observable), then the factor is not present). This means that the particle system induces fast oscillations in the reservoir (with frequency $\propto \sqrt{N}$). That these oscillations do not die off as $N \to \infty$ may be attributed to the fact that the system is *not dispersive*. This is in contrast to the effect the reservoir has on the particle system, which is, to induce irreversible (dispersive) dynamics.

(3) In the special case when $\mathcal{O}_j = \mathbb{1}_{\mathrm{S}}$ for all j, we obtain the reservoir dynamics,

$$\lim_{N\to\infty} e^{i\sqrt{N}\lambda\langle Q\rangle \mathrm{Im}\langle h, \frac{e^{i\omega t}-1}{i\omega} f\rangle} \langle W(f)\rangle_N(t) = e^{-\frac{1}{2}\lambda^2 \mathrm{var}(Q)(\mathrm{Im}\langle h, \frac{e^{i\omega t}-1}{i\omega} f\rangle)^2}\ \mathrm{Tr}_{\mathrm{R}}\Big(\rho_{\mathrm{R}} W(e^{i\omega t} f)\Big).$$

Proof of Theorem 1.

According to (3.10), (3.11), (3.12) and (3.14), we have

$$\begin{aligned} \langle\mathcal{O}\rangle_N(t) &= \mathrm{Tr}\Big(\rho_N(t)\ \mathcal{O}_1 \otimes \cdots \mathcal{O}_n \otimes \mathbb{1}_{\mathrm{S}} \cdots \mathbb{1}_{\mathrm{S}} \otimes W(f)\Big) \\ \rho_N(t) &= e^{-itH_N}(\rho_{\mathrm{S}}^{\otimes N} \otimes \rho_{\mathrm{R}})e^{itH_N}. \end{aligned} \tag{3.24}$$

We write Q in its diagonal form,

$$Q = \sum_{m=1}^{d} q^{(m)} P^{(m)}, \tag{3.25}$$

where $q^{(m)}$ are the (real) eigenvalues of Q and $P^{(m)}$ are its rank-one spectral projections, satisfying $\sum_m P^{(m)} = \mathbb{1}$ and $P^{(m)}P^{(n)} = \delta_{mn}P^{(m)}$ (Kronecker). Since A and Q commute (see the assumption (A) before (3.9)), we may assume that the projection

$P^{(m)}$ also diagonalize the operator A, namely,

$$A = \sum_{m=1}^{d} a^{(m)} P^{(m)}, \tag{3.26}$$

where the $a^{(m)} \in \mathbb{R}$ form the spectrum of A. Using that

$$\sum_{m_1,\ldots,m_N} P^{(m_1)} \otimes \ldots \otimes P^{(m_N)} = \mathbb{1}_{\mathrm{S}} \otimes \ldots \otimes \mathbb{1}_{\mathrm{S}} \tag{3.27}$$

we get from (3.9)

$$H_N = \sum_{m_1,\ldots,m_N} P^{(m_1)} \otimes \ldots \otimes P^{(m_N)} \Big[\sum_{j=1}^{N} A_j + H_{\mathrm{R}} + \frac{\lambda}{\sqrt{N}} \sum_{j=1}^{N} q^{(m_j)} \varphi(h) \Big]. \tag{3.28}$$

Notation. In (3.28), the product of the projections is actually a short form for the expression $P^{(m_1)} \otimes \ldots \otimes P^{(m_N)} \otimes \mathbb{1}_{\mathcal{F}}$, however, we leave out in the notation the trivial factor $\otimes \mathbb{1}_{\mathcal{F}}$ and we hope no confusion will arise by doing so. Also, note that in (3.28), one may replace A_j by $a^{(m_j)} \mathbb{1}_{\mathrm{S}}$ due to the presence of the projections $P^{(m_j)}$.

It follows from (3.28) that

$$\begin{aligned} & e^{-itH_N} \qquad (3.29) \\ & = \sum_{m_1,\ldots,m_N} P^{(m_1)} \otimes \ldots \otimes P^{(m_N)} \exp -it \Big[\sum_{j=1}^{N} A_j + H_{\mathrm{R}} + \frac{\lambda}{\sqrt{N}} \sum_{j=1}^{N} q^{(m_j)} \varphi(h) \Big]. \end{aligned}$$

Using the expansion (3.29) for the propagator in (3.24) gives

$$\begin{aligned} \rho_N(t) = & \sum_{m_1,\ldots,m_N} P^{(m_1)} \otimes \ldots \otimes P^{(m_N)} e^{-it(\sum_{j=1}^{N} A_j + H_R + I)} (\rho_{\mathrm{S}}^{\otimes N} \otimes \rho_R) \\ & \times \sum_{m'_1,\ldots,m'_N} P^{(m'_1)} \otimes \ldots \otimes P^{(m'_N)} e^{it(\sum_{j=1}^{N} A_j + H_R + I')} \end{aligned} \tag{3.30}$$

where we have defined

$$\begin{aligned} I &= \frac{\lambda}{\sqrt{N}} \sum_{j=1}^{N} q^{(m_j)} \varphi(h) \\ I' &= \frac{\lambda}{\sqrt{N}} \sum_{j=1}^{N} q^{(m'_j)} \varphi(h). \end{aligned} \tag{3.31}$$

In the next step, we find the reduced density matrix of the first n particles and the reservoir,

$$\rho_{n,N}(t) = \mathrm{Tr}_{[n+1,N]}(\rho_N(t)), \tag{3.32}$$

where $\mathrm{Tr}_{[n+1,N]}$ is the partial trace over all particles $j = n+1, \ldots, N$.

We have $\mathrm{Tr}_{[n+1,N]}(F_1 \otimes \ldots \otimes F_N) = F_1 \otimes \ldots \otimes F_n \cdot \mathrm{Tr}(F_{n+1} \otimes \ldots \otimes F_N)$ (where the F_j is any operator acting on the jth particle) and $\mathrm{Tr}(P^{(m)} \rho_{\mathrm{S}} P^{(m')}) = \delta_{m,m'} p_m$ where p_m defined in (3.17). H_{R}, I and I' are operators acting non-trivially on the space of the reservoir only, so (3.32) becomes

$$\rho_{n,N}(t) = e^{-it(A_1+\cdots+A_n)} \Big[\sum_{\substack{m_1,\ldots,m_N \\ m'_1,\ldots,m'_n}} (\Pi_{j=n+1}^{N} p_{m_j}) \Big(\bigotimes_{j=1}^{n} P^{(m_j)} \rho_{\mathrm{S}} P^{(m'_j)} \Big) \otimes \Big(e^{-it(H_R+I)} \rho_{\mathrm{R}} e^{it(H_R+I'')} \Big) \Big] e^{it(A_1+\cdots+A_n)} \tag{3.33}$$

(note that the $A_{n+1}, \ldots A_N$ disappear due to the cyclicity of the trace) where

$$I'' = \Big(\frac{\lambda}{\sqrt{N}} \sum_{j=1}^{n} q^{(m'_j)} + \frac{\lambda}{\sqrt{N}} \sum_{j=n+1}^{N} q^{(m_j)} \Big) \varphi(h). \tag{3.34}$$

From equation (3.33) and (3.24) and by cyclicity of trace we have

$$\langle \mathcal{O} \rangle_N(t) = \mathrm{Tr} \Big[e^{-it(A_1+\cdots+A_n)} \sum_{\substack{m_1,\ldots,m_N \\ m'_1,\ldots,m'_n}} (\Pi_{j=n+1}^{N} p_{m_j}) \bigotimes_{j=1}^{n} P^{(m_j)} \rho_{\mathrm{S}} P^{(m'_j)} e^{it(A_1+\cdots+A_n)} \mathcal{O}_1 \otimes \ldots \otimes \mathcal{O}_n \otimes \Big(e^{it(H_R+I'')} W(f) e^{-it(H_{\mathrm{R}}+I)} \rho_{\mathrm{R}} \Big) \Big]. \tag{3.35}$$

The trace in (3.35) is over the particle spaces with indices $1, \ldots, n$ and the reservoir.

As the argument is a product, we first evaluate the part over the reservoir,

$$\mathrm{Tr}_{\mathrm{R}} e^{it(H_{\mathrm{R}}+I'')} W(f) e^{-it(H_{\mathrm{R}}+I)} \rho_{\mathrm{R}}. \tag{3.36}$$

To evaluate this, the following result is useful.

Lemma 1 *For any α, $\beta \in \mathbb{R}$ and functions $f, h \in L^2(\mathbb{R}^3, d^3k)$, we have*

$$\begin{aligned} & e^{it(H_{\mathrm{R}}+\alpha\varphi(h))} W(f) e^{-it(H_{\mathrm{R}}+\beta\varphi(h))} \\ & \quad = e^{-\frac{i}{2}(\alpha+\beta)\mathrm{Im}\langle h, \frac{e^{it\omega}-1}{i\omega} f\rangle} e^{-i(\alpha+\beta)(\alpha-\beta)S(t)} W\Big((\alpha-\beta)\frac{e^{it\omega}-1}{i\omega}h + e^{it\omega} f\Big), \end{aligned} \tag{3.37}$$

where $S(t)$ is given in (3.21)*.*

We give a proof of Lemma 1 below. For now, we continue the proof of Theorem 1.

To analyze (3.36) we use (3.37) with

$$\alpha = \frac{\lambda}{\sqrt{N}} \sum_{j=1}^{n} q^{(m'_j)} + \frac{\lambda}{\sqrt{N}} \sum_{j=n+1}^{N} q^{(m_j)}, \quad \beta = \frac{\lambda}{\sqrt{N}} \sum_{j=1}^{N} q^{(m_j)}.$$

Letting

$$c = \mathrm{Im}\Big\langle h, \frac{e^{i\omega t}-1}{i\omega} f\Big\rangle \tag{3.38}$$

we obtain

$$\begin{aligned} & e^{it(H_{\mathrm{R}}+I'')} W(f) e^{-it(H_{\mathrm{R}}+I)} \\ & = \exp\Big\{\frac{-i}{2}\frac{\lambda c}{\sqrt{N}}\Big[\sum_{j=1}^{n}(q^{(m_j)}+q^{(m'_j)}) + 2\sum_{j=n+1}^{N} q^{(m_j)}\Big]\Big\} \\ & \quad \times \exp\Big\{i\frac{\lambda^2}{N}\Big[\sum_{j=1}^{n}(q^{(m_j)}+q^{(m'_j)}) + 2\sum_{j=n+1}^{N} q^{(m_j)}\Big]\Big[\sum_{j=1}^{n}(q^{(m_j)}-q^{(m'_j)})\Big]S(t)\Big\} \\ & \quad \times W\Big(\frac{\lambda}{\sqrt{N}}\sum_{j=1}^{n}(q^{(m_j)}-q^{(m'_j)})\frac{e^{it\omega}-1}{i\omega}h + e^{it\omega} f\Big). \end{aligned} \tag{3.39}$$

The equation (3.35) becomes

$$\begin{aligned}
\langle\mathcal{O}\rangle_N(t) &= \mathrm{Tr}\Big[e^{-it(A_1+\cdots+A_n)}\sum_{\substack{m_1,\ldots,m_n\\ m'_1,\ldots,m'_n}}\zeta\,\Big(\bigotimes_{j=1}^n P_j^{(m_j)}\rho P_j^{(m'_j)}\Big)e^{it(A_1+\cdots+A_n)} \\
&\qquad \mathcal{O}_1\otimes\ldots\otimes\ \mathcal{O}_n\otimes\rho_{\mathrm{R}}W\Big(\frac{\lambda}{\sqrt{N}}\sum_{j=1}^n(q^{(m_j)}-q^{(m'_j)})\frac{e^{it\omega}-1}{i\omega}h+e^{it\omega}f\Big) \\
&\qquad \times\exp\Big\{-\frac{i\lambda c}{2\sqrt{N}}\sum_{j=1}^n q^{(m_j)}+q^{(m'_j)}\Big\} \\
&\qquad \times\exp\Big\{i\frac{\lambda^2}{N}\Big[\sum_{j=1}^n q^{(m_j)}+q^{(m'_j)}\Big]\Big[\sum_{j=1}^n(q^{(m_j)}-q^{(m'_j)})\Big]S(t)\Big\}, \qquad (3.40)
\end{aligned}$$

where we have defined

$$\begin{aligned}
\zeta &\equiv \zeta(m_1,\ldots,m_n,m'_1,\ldots,m'_n) \qquad (3.41)\\
&= \sum_{m_{n+1},\ldots,m_N}(\Pi_{j=n+1}^N p_{m_j})\exp\Big\{-i\frac{\lambda c}{\sqrt{N}}\sum_{j=n+1}^N q^{(m_j)}\Big\} \\
&\quad \times\exp\Big\{i\frac{\lambda^2}{N}\Big[\sum_{j=1}^n\{q^{(m_j)}+q^{(m'_j)}\}+2\sum_{j=n+1}^N q^{(m_j)}\Big]\Big[\sum_{j=1}^n q^{(m_j)}-q^{(m'_j)}\Big]S(t)\Big\}.
\end{aligned}$$

We set

$$Z=\sum_{j=1}^n q^{(m_j)}-q^{(m'_j)}. \qquad (3.42)$$

Then (3.41) becomes

$$\begin{aligned}
\zeta &= \sum_{m_{n+1},\ldots,m_N}(\Pi_{j=n+1}^N p_{m_j})\exp\Big\{-i\frac{\lambda}{\sqrt{N}}c\sum_{j=n+1}^N q^{(m_j)}\Big\}\exp\Big\{2i\frac{\lambda^2}{N}ZS(t)\sum_{j=n+1}^N q^{(m_j)}\Big\} \\
&= \sum_{m_{n+1},\ldots,m_N}(\Pi_{j=n+1}^N p_{m_j})\exp\Big\{-i\frac{\lambda}{\sqrt{N}}\Big[c-2\frac{\lambda}{\sqrt{N}}ZS(t)\Big]\sum_{j=n+1}^N q^{(m_j)}\Big\} \\
&= \Big(\sum_m p_m\exp\Big\{-i\frac{\lambda}{\sqrt{N}}\Big[c-2\frac{\lambda Z}{\sqrt{N}}S(t)\Big]q^{(m)}\Big\}\Big)^{N-n} \\
&= \Big(\sum_m p_m e^{ix_m(N)/N}\Big)^{N-n}, \qquad (3.43)
\end{aligned}$$

where we have introduced

$$x_m(N) = \Big(-\sqrt{N}\lambda c + 2\lambda^2 Z S(t) \Big) q^{(m)}. \tag{3.44}$$

We write

$$\begin{aligned} \zeta &= \exp\Big\{ (N-n) \ln \Big(\sum_m p_m e^{i x_m(N)/N} \Big) \Big\} \\ &= \exp\Big\{ (N-n) \ln \Big(1 + \sum_m p_m (e^{i x_m(N)/N} - 1) \Big) \Big\}, \end{aligned} \tag{3.45}$$

where we have used that $\sum_m p_m = 1$. The sum in the logarithm of (3.45) is small for N large, it is of the size $x_m(N)/N \propto 1/\sqrt{N}$. Set

$$\epsilon = \sum_m p_m (e^{i x_m(N)/N} - 1) = O(1/\sqrt{N}) \qquad (N \to \infty). \tag{3.46}$$

We use the expansion

$$\log(1+\epsilon) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \epsilon^k}{k} \qquad (|\epsilon| < 1)$$

to obtain

$$\zeta = \exp\Big\{ (N-n) \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\epsilon^k}{k} \Big\} = \exp\Big\{ (N-n) \Big(\epsilon - \tfrac{1}{2}\epsilon^2 + O(N^{-3/2}) \Big) \Big\}. \tag{3.47}$$

Now we expand

$$\begin{aligned} \epsilon &= \sum_m p_m \Big(i \frac{x_m(N)}{N} - \tfrac{1}{2} \frac{x_m(N)^2}{N^2} \Big) + O(N^{-3/2}) \\ \tfrac{1}{2}\epsilon^2 &= \tfrac{1}{2} \Big(\sum_m p_m i \frac{x_m(N)}{N} \Big)^2 + O(N^{-3/2}). \end{aligned} \tag{3.48}$$

Using (3.48) in (3.47) gives

$$\zeta = \exp{(N-n)}\Big[\sum_m p_m\Big(i\frac{x_m(N)}{N} - \frac{x_m(N)^2}{2N^2}\Big) + \tfrac{1}{2}\Big(\sum_m p_m\frac{x_m(N)}{N}\Big)^2 + O(N^{-3/2})\Big]$$
$$= \exp{(N-n)}[\frac{i}{N}\langle x(N)\rangle - \frac{1}{2N^2}\text{var}(x(N)) + O(N^{-3/2})], \tag{3.49}$$

where (see (3.44) and (3.19))

$$\begin{aligned}\langle x(N)\rangle &= \Big(-\sqrt{N}\lambda c + 2\lambda^2 ZS(t)\Big)\langle Q\rangle \\ \text{var}(x(N)) &= \Big(-\sqrt{N}\lambda c + 2\lambda^2 ZS(t)\Big)^2\text{var}(Q).\end{aligned} \tag{3.50}$$

In view of definitions in (3.50) we write

$$\zeta = \exp\Big\{-i\sqrt{N}\lambda\langle Q\rangle c + 2i\lambda^2 ZS(t)\langle Q\rangle - \tfrac{1}{2}\lambda^2\text{var}(Q)c^2\Big\}\exp\Xi, \tag{3.51}$$

where

$$\begin{aligned}\Xi &= \frac{1}{N}\Big(-2\sqrt{N}\lambda^3 ZS(t)c + 2\lambda^4 Z^2 S^2(t)\Big)\text{var}(Q) \\ &\quad -i\frac{n}{N}\langle x(N)\rangle + \frac{n}{2N^2}\text{var}(x(N)) + O(N^{-3/2}).\end{aligned}$$

Substituting the value of ζ given in (3.51) into (3.40) and doing some rearrangement, we get

$$\begin{aligned}\langle\mathcal{O}\rangle_N(t) &= e^{-i\sqrt{N}\lambda\langle Q\rangle c}\ \text{Tr}\Big[e^{-it(A_1+\cdots+A_n)}\sum_{\substack{m_1,\ldots,m_n\\ m'_1,\ldots,m'_n}}\exp\Xi \\ &\quad \Big(\bigotimes_{j=1}^n P^{(m_j)}e^{2i\lambda^2 ZS(t)\langle Q\rangle}\rho_{\rm S}P^{(m'_j)}\Big)e^{it(A_1+\cdots+A_n)}\mathcal{O}_1\otimes\ldots\otimes\ \mathcal{O}_n \\ &\quad \otimes e^{-\frac{1}{2}\lambda^2\text{var}(Q)c^2}\rho_{\rm R}W\Big(\frac{\lambda}{\sqrt{N}}\sum_{j=1}^n(q^{(m_j)} - q^{(m'_j)})\frac{e^{it\omega}-1}{i\omega}h + e^{it\omega}f\Big) \\ &\quad \times\exp\Big\{-\frac{i\lambda c}{2\sqrt{N}}\sum_{j=1}^n(q^{(m_j)} + q^{(m'_j)})\Big\} \\ &\quad \times\exp\Big\{i\frac{\lambda^2}{N}\Big[\sum_{j=1}^n q^{(m_j)} + q^{(m'_j)}\Big]\Big[\sum_{j=1}^n q^{(m_j)} - q^{(m'_j)}\Big]S(t)\Big\}\Big].\end{aligned} \tag{3.52}$$

We move $e^{-i\sqrt{N}\lambda\langle Q\rangle c}$ to the left hand side then take the limit as $N \to \infty$ of (3.52). The exponents in the last two factors on the right side of (3.52) vanish in this limit, and also, $\Xi = O(N^{-1/2}) \to 0$. Furthermore, by Theorem I, we have in the strong sense

$$\lim_{N\to\infty} W\Big(\frac{\lambda}{\sqrt{N}}\sum_{j=1}^{n}(q^{(m_j)} - q^{(m'_j)})\frac{e^{it\omega}-1}{i\omega}h + e^{it\omega}f\Big) = W(e^{it\omega}f). \tag{3.53}$$

This limit passes under the trace, as is not difficult to see.[1] We conclude that

$$\lim_{N\to\infty} e^{i\sqrt{N}\lambda\langle Q\rangle c}\langle\mathcal{O}\rangle_N(t) = \mathrm{Tr}\Big[e^{-it(A_1+\cdots+A_n)}\Big(\sum_{\substack{m_1,\ldots,m_n\\ m'_1,\ldots,m'_n}}\bigotimes_{j=1}^{n} P^{(m_j)}e^{2i\lambda^2 ZS(t)\langle Q\rangle}\rho_{\mathrm{S}}P^{(m'_j)}\Big)$$
$$\times\, e^{it(A_1+\cdots+A_n)}\mathcal{O}_1\otimes\ldots\otimes\mathcal{O}_n\otimes e^{-\frac{1}{2}\lambda^2\mathrm{var}(Q)c^2}\rho_{\mathrm{R}}W(e^{it\omega}f)\Big]. \tag{3.54}$$

We use the definition of Z in (3.42) to simplify the sum

$$\begin{aligned}
&\sum_{\substack{m_1,\ldots,m_n\\ m'_1,\ldots,m'_n}}\bigotimes_{j=1}^{n} P^{(m_j)}e^{2i\lambda^2 ZS(t)\langle Q\rangle}\rho_{\mathrm{S}}P^{(m'_j)}\\
&= \sum_{\substack{m_1,\ldots,m_n\\ m'_1,\ldots,m'_n}}\bigotimes_{j=1}^{n} P^{(m_j)}e^{2i\lambda^2 S(t)\langle Q\rangle\sum_{j=1}^{n}q^{(m_j)}}\rho_{\mathrm{S}}e^{-2i\lambda^2 S(t)\langle Q\rangle\sum_{j=1}^{n}q^{(m'_j)}}P^{(m'_j)}\\
&= \bigotimes_{j=1}^{n}\Big[\sum_{m}P^{(m)}e^{2i\lambda^2 S(t)\langle Q\rangle q^{(m)}}\Big]\rho_{\mathrm{S}}\Big[\sum_{m'}e^{-2i\lambda^2 S(t)\langle Q\rangle q^{(m')}}P^{(m'_j)}\Big]\\
&= \bigotimes_{j=1}^{n}\tilde{\rho}.
\end{aligned}\tag{3.55}$$

The last equality holds because of the diagonal form of the operator Q in (3.25) and definition of $\tilde{\rho}$ in (3.20).

Combining (3.54) and (3.55) we obtain the formula (3.23) and thus complete proof

[1] Suppose A_n is a sequence of bounded operators, such that $\|A_n\|\leq a$ and $A_n \to A$ strongly. Then $\mathrm{Tr}\rho A_n \to \mathrm{Tr}\rho A$ as $n\to\infty$. Indeed, let $\{x_n\}$ be an orthonormal basis. Then, given any $\epsilon > 0$ there is an N s.t. $\sum_{k>N}\langle x_k, \rho x_k\rangle < \epsilon/(2a)$. This follows simply from the finiteness of $\mathrm{Tr}\rho$. Now $|\mathrm{Tr}(\rho(A_n - A))| = |\mathrm{Tr}(\sqrt{\rho}(A_n - A)\sqrt{\rho})| \leq \sum_{1\leq k\leq N}|\langle x_n, \sqrt{\rho}(A_n - A)\sqrt{\rho}x_n\rangle| + \sum_{k>N}|\langle x_n, \sqrt{\rho}(A_n - A)\sqrt{\rho}x_n\rangle|$. The summand in the second sum is bounded above by $2a\|x_n\sqrt{\rho}\|^2 = 2a\langle x_n, \rho x_n\rangle$ and hence the value of this sum is bounded above by ϵ. Thus $|\mathrm{Tr}(\rho(A_n - A))| \leq \sum_{1\leq k\leq N}|\langle x_n, \sqrt{\rho}(A_n - A)\sqrt{\rho}x_n\rangle| + \epsilon$. Taking $n\to\infty$ and using the strong convergence to zero of $A_n - A$ shows the result.

of Theorem 1 *modulo* giving a proof of Lemma 1, which we do now.

Proof of Lemma 1. By the Trotter product formula [9],

$$e^{it(H_{\rm R}+\alpha\varphi(h))}W(f)e^{-it(H_{\rm R}+\beta\varphi(h))} = \lim_{n\to\infty}(B_\alpha)^n W(f)(B_\beta^*)^n \tag{3.56}$$

where

$$B_\gamma = e^{\frac{it\gamma\varphi(h)}{n}}e^{\frac{itH_{\rm R}}{n}}, \qquad \gamma\in\{\alpha,\beta\}. \tag{3.57}$$

We have

$$B_\alpha W(f)B_\beta^* = e^{\Phi_1}W(f_1) \tag{3.58}$$

where

$$\begin{aligned}\Phi_1 &= -\tfrac{i}{2}{\rm Im}\Big\langle \tfrac{\alpha t}{n}h, e^{\frac{it\omega}{n}}f\Big\rangle + \tfrac{i}{2}{\rm Im}\Big\langle e^{\frac{it\omega}{n}}f, \tfrac{\beta t}{n}h\Big\rangle\\ f_1 &= (\alpha-\beta)\tfrac{t}{n}h + e^{\frac{it\omega}{n}}f.\end{aligned} \tag{3.59}$$

Relations (3.58) and (3.59) follow directly from (see also (2.52) and (3.8))

$$e^{i\tau H_{\rm R}}W(f)e^{-i\tau H_{\rm R}} = W(e^{i\tau\omega}f) \quad \text{and} \quad W(f)W(g) = e^{\frac{-i}{2}{\rm Im}\langle f,g\rangle}W(f+g). \tag{3.60}$$

Next we look at

$$B_\alpha^2 W(f)(B_\beta^*)^2 = e^{\Phi_1}B_\alpha W(f_1)B_\beta^* = e^{\Phi_2}W(f_2), \tag{3.61}$$

with

$$\begin{aligned}\Phi_2 &= -\tfrac{i}{2}\Big[{\rm Im}\Big\langle \tfrac{\alpha t}{n}h, (\alpha-\beta)e^{\frac{it\omega}{n}}\tfrac{t}{n}h\Big\rangle + {\rm Im}\Big\langle \tfrac{\alpha t}{n}h, e^{\frac{2it\omega}{n}}f\Big\rangle\Big]\\ &\quad +\tfrac{i}{2}\Big[{\rm Im}\Big\langle (\alpha-\beta)e^{\frac{it\omega}{n}}\tfrac{t}{n}h, \frac{\beta t}{n}h\Big\rangle + {\rm Im}\Big\langle e^{\frac{2it\omega}{n}}f, \tfrac{\beta t}{n}h\Big\rangle\Big] + \Phi_1\\ f_2 &= (\alpha-\beta)\tfrac{t}{n}h + (\alpha-\beta)e^{\frac{it\omega}{n}}\tfrac{t}{n}h + e^{\frac{2it\omega}{n}}f.\end{aligned} \tag{3.62}$$

Iterating this procedure n times we obtain

$$B_\alpha^n W(f)(B_\beta^*)^n = e^{\Phi_n}W(f_n), \tag{3.63}$$

where

$$\begin{aligned}\Phi_n &= -\tfrac{i\alpha}{2}\Big[\mathrm{Im}\Big\langle \tfrac{t}{n}h, \sum_{j=1}^{n} e^{\frac{jit\omega}{n}} f + (\alpha-\beta)(\tfrac{t}{n}) \sum_{j=1}^{n}(n-j)e^{\frac{jit\omega}{n}} h\Big\rangle\Big] \\ &\quad + \tfrac{i\beta}{2}\Big[\mathrm{Im}\Big\langle \sum_{j=1}^{n} e^{\frac{jit\omega}{n}} f + (\alpha-\beta)(\tfrac{t}{n}) \sum_{j=1}^{n}(n-j)e^{\frac{jit\omega}{n}} h\rangle, \tfrac{t}{n}h\Big\rangle\Big] \\ f_n &= (\alpha-\beta)\tfrac{t}{n}\sum_{j=0}^{n-1} e^{\frac{jit\omega}{n}} h + e^{it\omega} f. \end{aligned} \tag{3.64}$$

Equation (3.64) can be simplified by using $\frac{-i\alpha}{2}\mathrm{Im}z + \frac{i\beta}{2}\mathrm{Im}\overline{z} = \frac{-i(\alpha+\beta)}{2}\mathrm{Im}z$, it becomes

$$\Phi_n = \tfrac{-i(\alpha+\beta)}{2}\mathrm{Im}\Big\langle h, \frac{t}{n}\sum_{j=1}^{n} e^{\frac{jit\omega}{n}} f + (\alpha-\beta)\Big(\frac{t}{n}\Big)^2 \sum_{j=1}^{n}(n-j)e^{\frac{jit\omega}{n}} h\Big\rangle. \tag{3.65}$$

In view of (3.56) we have to take $n \to \infty$ in (3.63). Clearly, f_n is a Riemann sum, and we obtain

$$\lim_{n\to\infty} f_n = (\alpha-\beta)ht\int_0^1 e^{itx\omega}dx + e^{it\omega}f = (\alpha-\beta)h\frac{e^{it\omega}-1}{i\omega} + e^{it\omega}f. \tag{3.66}$$

The function on the right hand side of (3.66) is indeed the argument of the Weyl operator in (3.37).

To evaluate $\lim_{n\to\infty}\Phi_n$, see (3.65), we again use the Riemann sums,

$$\begin{aligned} \lim_{n\to\infty}\Big(\frac{t}{n}\Big)\sum_{j=1}^{n} e^{\frac{it\omega j}{n}} &= \frac{e^{it\omega}-1}{i\omega}, \\ \lim_{n\to\infty}\Big(\frac{t}{n}\Big)^2\sum_{j=1}^{n}(n-j)e^{\frac{it\omega j}{n}} &= \frac{e^{it\omega}-it\omega-1}{-\omega^2}. \end{aligned} \tag{3.67}$$

We arrive at

$$\lim_{n\to\infty}\Phi_n = -\tfrac{i}{2}(\alpha+\beta)\mathrm{Im}\left\langle h, \frac{e^{it\omega}-1}{i\omega}f\right\rangle - \tfrac{i}{2}(\alpha+\beta)(\alpha-\beta)\mathrm{Im}\left\langle h, \frac{e^{it\omega}-it\omega-1}{-\omega^2}h\right\rangle. \tag{3.68}$$

Finally,

$$\mathrm{Im}\langle h, \frac{e^{it\omega}-it\omega-1}{-\omega^2}h\rangle = \mathrm{Im}\int_{\mathbb{R}^3}|h(k)|^2\frac{\cos t\omega + i\sin t\omega - it\omega - 1}{-\omega^2}d^3k = 2S(t), \tag{3.69}$$

see (3.21).

Combining (3.56), 3.63, (3.64), (3.66) and (3.21) yields formula (3.37). This completes the proof of Lemma 1 and hence that of Theorem 1. ■

3.2.2 Dynamics of states

We now assume that $\langle Q\rangle = 0$.

Example. Consider S to be a spin, with $Q = tfrac12\sigma_z$ (Pauli matrix, see also [15])

$$Q = \tfrac{1}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \tag{3.70}$$

and let $\rho_{\rm S} = \frac{1}{2}\mathbb{1}$ be the initial particle state (this density matrix is the equilibrium state at very high (infinite) temperature). Then one easily sees that $\langle Q\rangle = 0$.

Then Theorem 1 says that for all observables of the form (3.22), we have

$$\lim_{N\to\infty}\langle\mathcal{O}\rangle_N(t) = \Big(\omega_{{\rm S},t}\otimes\cdots\otimes\omega_{{\rm S},t}\otimes\omega_{{\rm R},t}\Big)(\mathcal{O}), \tag{3.71}$$

where the state on the right side is an n fold tensor product of the single particle state $\omega_{{\rm S},t}$ tensored with the field state $\omega_{{\rm R},t}$, given by

$$\begin{aligned}\omega_{{\rm S},t}(\mathcal{O}_{\rm S}) &= {\rm Tr}_{\rm S}(e^{-itA}\widetilde{\rho}e^{itA}\ \mathcal{O}_{\rm S}), &(3.72)\\ \omega_{{\rm R},t}(W(f)) &= e^{-\frac12\lambda^2{\rm var}(Q)({\rm Im}\langle h,\frac{e^{i\omega t}-1}{i\omega}f\rangle)^2}\ {\rm Tr}_{\rm R}\Big(\rho_{\rm R}W(e^{i\omega t}f)\Big) &(3.73)\end{aligned}$$

with $\mathcal{O}_{\rm S}\in\mathcal{B}(\mathcal{H}_{\rm S})$ and $f\in L^2(\mathbb{R}^3, d^3k)$. By linearity, relation (3.71) extends to all (finite) linear combinations of observables of the form (3.22). We now introduce the 'local' (n finite) C^*-algebra

$$\mathcal{A}_n = \mathcal{B}(\mathcal{H}_{\rm S})^{\otimes n}\otimes\mathcal{W}, \tag{3.74}$$

where $\mathcal{W}$ is the Weyl algebra (see section 2.4.4). By taking the partial trace of the total state (3.11) over the particle spaces with indices $n+1,\ldots,N$, we obtain the reduced density matrix of the first n particles and the reservoir,

$$\rho_{n,N}(t) = {\rm Tr}_{[n+1,\ldots,N]}\,\rho_N(t). \tag{3.75}$$

Of course, $\rho_{n,N}(0) = \rho_\mathrm{S} \otimes \cdots \otimes \rho_\mathrm{S} \otimes \rho_\mathrm{R}$ is of product form, but for $t \neq 0$, $\rho_{n,N}(t)$ is not. However in the limit $N \to \infty$ the product structure is reinstated, albeit with a more complicated dynamics.

Consider the state $\omega_{n,N}^t$ on $\mathcal{A}_n$ associated to $\rho_{n,N}(t)$, *i.e.*,

$$\omega_{n,N}^t(\mathcal{O}) = \mathrm{Tr}(\rho_{n,N}(t)\mathcal{O}), \qquad \mathcal{O} \in \mathcal{A}_n. \tag{3.76}$$

Theorem 2 (Dynamics of the state) *We have for all $t \in \mathbb{R}$,*

$$\lim_{N\to\infty} \omega_{n,N}^t = \omega_{\mathrm{S},t}^{\otimes n} \otimes \omega_{\mathrm{R},t}, \tag{3.77}$$

where the limit is understood in the weak $*$ topology.

Remark. Convergence in the weak $*$ topology simply means that $\lim_{N\to\infty} \omega_{n,N}^t(\mathcal{O}) = \omega_{\mathrm{S},t}^{\otimes n} \otimes \omega_{\mathrm{R},t}(\mathcal{O})$, for all $\mathcal{O} \in \mathcal{A}_n$.

Proof Theorem 2. Define $\mathcal{P}_n$ to be the set of all finite linear combinations of observables of the form $\mathcal{O}_1 \otimes \ldots \mathcal{O}_n \otimes W(f)$. The completion of $\mathcal{P}_n$ in the operator norm topology is simply $\mathcal{A}_n$. Since the linear functionals $\omega_{n,N}^t$ and $\omega_{\mathrm{S},t}^{\otimes n} \otimes \omega_{\mathrm{R},t}$ are bounded and have norm one, they can be extended by continuity to $\mathcal{A}_n$ and have norm one, for all $t \in \mathbb{R}$ and $N \geq 1$.

Given any $A \in \mathcal{A}_n$ and any $\epsilon > 0$, we can find an element $P_\epsilon \in \mathcal{P}_n$ such that $\|A - P_\epsilon\| \leq \frac{\epsilon}{3}$. We have $\forall t \in \mathbb{R}$ and $\forall A \in \mathcal{A}_n$

$$\begin{aligned} &\|\omega_{n,N}^t(A) - \omega_{\mathrm{S},t}^{\otimes n} \otimes \omega_{\mathrm{R},t}(A)\| \\ &\quad \leq \ \|\omega_{n,N}^t(A) - \omega_{n,N}^t(P_\epsilon)\| + \|\omega_{n,N}^t(P_\epsilon) - \omega_{\mathrm{S},t}^{\otimes n} \otimes \omega_{\mathrm{R},t}(P_\epsilon)\| \\ &\qquad + \|\omega_{\mathrm{S},t}^{\otimes n} \otimes \omega_{\mathrm{R},t}(P_\epsilon) - \omega_{\mathrm{S},t}^{\otimes n} \otimes \omega_{\mathrm{R},t}(A)\|. \end{aligned} \tag{3.78}$$

The first and the third terms on the right side of (3.78) are bounded above each by $\epsilon/3$, due to $\|A - P_\epsilon\| \leq \epsilon/3$. Next, due to Theorem 1, there exists an N_ϵ s.t. $\forall N \geq N_\epsilon$, we have $\|\omega_{n,N}^t(P_\epsilon) - \omega_{\mathrm{S},t}^{\otimes n} \otimes \omega_{\mathrm{R},t}(P_\epsilon)\| \leq \epsilon/3$. This shows (3.77) and completes the proof of Theorem 2. ■

3.3 Hilbert space representation

Recall the definition of the reservoir state $\omega_{\mathrm{R},t}$, (3.73),

$$\omega_{\mathrm{R},t}(W(f)) = e^{-\frac{1}{2}\lambda^2 \mathrm{var}(Q)(\mathrm{Im}\langle h, \frac{e^{i\omega t}-1}{i\omega} f\rangle)^2}\ \mathrm{Tr}\Big(\rho_{\mathrm{R}} W(e^{i\omega t} f)\Big), \tag{3.79}$$

where $f \in L^2(\mathbb{R}^3, d^3k)$ is arbitrary and h is the form factor in the interaction, see (3.8).

For a fixed $t \in \mathbb{R}$, we denote the Hilbert space (GNS) representation of $\omega_{\mathrm{R},t}$ by $(\mathcal{H}_t, \pi_t, \Omega_t)$, meaning that

$$\omega_{\mathrm{R},t}(W(f)) = \langle \Omega_t, \pi_t(W(f))\Omega_t\rangle, \qquad \forall f. \tag{3.80}$$

The main result of this section is Theorem 3, in which we construct the representation explicitly.

Theorem 3 (Hilbert space representation of the reservoir state)

Denote the GNS representation of $\omega_{\mathrm{R},0}$ by $(\mathcal{H}_0, \pi_0, \Omega_0)$. For any $t \in \mathbb{R}$, we have the following.

(A) The GNS representation of the state $\omega_{\mathrm{R},t}$ is given by

$$\mathcal{H}_t \subset \mathcal{F}(\mathbb{C}) \otimes \mathcal{H}_0, \tag{3.81}$$

$$\pi_t(W(f)) = e^{i\sqrt{2\mathrm{var}(Q)}\lambda \mathrm{Im}\langle h_t, f\rangle\,\varphi} \otimes \pi_0(W(f)), \tag{3.82}$$

$$\Omega_t = \Omega_{\mathrm{HO}} \otimes \Omega_0. \tag{3.83}$$

Here,

$$h_t(k) = h(k)\frac{1-e^{-i\omega t}}{i\omega} \qquad (\omega = \omega(k)) \tag{3.84}$$

and $\varphi = 2^{-1/2}(a^* + a)$ is the field operator of a harmonic oscillator. Ω_{HO} is the ground state vacuum vector in $\mathcal{F}(\mathbb{C})$.

(B) Suppose the reservoir is in the vacuum state at zero temperature, that is, let $\mathcal{H}_0 = \mathcal{F}(L^2(\mathbb{R}^3))$ and let Ω_0 be the vacuum vector in this Fock space, so that $\omega_{\mathrm{R},0}(\cdot) = \langle \Omega_0, \cdot\, \Omega_0\rangle$ is the (Fock) vacuum state. Then, for fixed $t \in \mathbb{R}$, the GNS

Hilbert space is

$$\mathcal{H}_t = \mathcal{F}(\mathbb{C}) \otimes \mathcal{F}(L^2(\mathbb{R}^3, d^3k)) \tag{3.85}$$

if h_t is not the zero function in $L^2(\mathbb{R}^3, d^3k)$. If $h_t \equiv 0$ then $\mathcal{H}_t = \mathcal{F}(L^2(\mathbb{R}^3, d^3k))$.

Discussion.
(1) The GNS Hilbert space is the closure $\mathcal{H}_t = \overline{\pi_t(\mathcal{W})(\Omega_{\mathrm{HO}} \otimes \Omega_0)}$. Relation (3.81) says that it is realized as a subspace of $\mathcal{F}(\mathbb{C}) \otimes \mathcal{H}_0$. Part (B) of the theorem shows that the GNS space of the reservoir, for $t > 0$, is the entire space $\mathcal{F}(\mathbb{C}) \otimes \mathcal{H}_0$ if the reservoir is initially in the vacuum state.

(2) Probably one can carry out the proof of part (B) for any regular representation of the CCR (where the a^* and a exist), or at least for thermal representations (Araki-Woods).

The following result is a basic fact from quantum theory [13] and will be useful for us to characterize the reservoir representations π_t, (3.82).

Theorem III. (Stone von-Neumann uniqueness theorem) *Let $\mathfrak{h}$ be a finite dimensional Hilbert space and let $(\mathcal{H}, \pi)$ be a regular representation of the Weyl CCR algebra $\mathcal{W}(\mathfrak{h})$. Then $(\mathcal{H}, \pi)$ is unitarily equivalent to the direct sum representation*

$$\Big(\oplus_j \, \pi_{\mathrm{F}}, \oplus_j \mathcal{F}(\mathfrak{h}) \Big)$$

(finite or countably infinite) of copies of the Fock representation $(\mathcal{F}(\mathfrak{h}), \pi_{\mathrm{F}})$.

Remarks. (1) $(\mathcal{H}, \pi)$ regular means that $t \mapsto \pi(W(tf))$ is differentiable at $t = 0$, in the strong sense on $\mathcal{H}$. (This guarantees the existence of field and creation and annihilation operators.)

(2) Two representations $(\mathcal{H}_1, \pi_1)$, $(\mathcal{H}_2, \pi_2)$ of a C^*-algebra $\mathfrak{A}$ are called unitarily equivalent if there is a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\pi_1(A) = U^* \pi_2(A) U$ for all $A \in \mathfrak{A}$. We write simply

$$\pi_1 \simeq \pi_2.$$

In particular, the conclusion of the Stone von Neumann uniqueness theorem reads $\pi \simeq \oplus_j \pi_{\mathrm{F}}$.

(3) If π in Theorem III is an irreducible representation (the only subspaces left invariant by $\pi(\mathcal{W}(\mathfrak{h}))$ are the whole Hilbert space and $\{0\}$), then the direct sum in

Theorem 3.3 has a single copy. In other words, any irreducible representation π of $\mathcal{W}(\mathfrak{h})$ with $\dim \mathfrak{h} < \infty$ is unitarily equivalent to the Fock representation.

For the following result, denote by $\pi_{\rm HO}$ the Fock representation of $\mathcal{F}(\mathbb{C})$ ('the harmonic oscillator').

Theorem 4 *Set for short $L^2 \equiv L^2(\mathbb{R}^3, d^3k)$ and let $\pi_{\rm F}$ be the Fock representation of $\mathcal{W}(L^2)$ on the Hilbert space $\mathcal{F}(L^2)$. For fixed $g \in L^2$ define a representation of $\mathcal{W}(L^2)$ on the Hilbert space $\mathcal{F}(\mathbb{C}) \otimes \mathcal{F}(L^2)$ by*

$$\pi^{(g)}(W(f)) = e^{i\,{\rm Im}\langle g,f\rangle\varphi} \otimes \pi_{\rm F}(W(f)), \qquad f \in L^2. \tag{3.86}$$

Then

$$\pi^{(g)} \simeq \Big(\oplus_j \pi_{\rm HO} \Big) \otimes \pi_\perp, \tag{3.87}$$

where $\pi_{\rm HO}$ is the Fock representation of $\mathcal{F}(\mathbb{C})$ ('the harmonic oscillator') and $\pi_\perp$ is the Fock representation of $\mathcal{W}((\mathbb{C}g)^\perp)$, the orthogonal complement being the one in the space L^2.

The point of Theorem 4 is that in (3.87), the right side does not depend on g, except possibly in the multiplicity of the direct sum. More precisely, we have the following result (which is a corollary to the proof of Theorem 4)

Corollary 1 *Let g and h be fixed elements of L^2. Then we have*

$$\pi^{(X)} \simeq \Big(\oplus_{j=1}^{n_X} \pi_{\rm HO} \Big) \otimes \pi, \qquad X = g, h, \tag{3.88}$$

where $n_X \in \mathbb{N} \cup \{\infty\}$ and where π is independent of the value of X, it is the Fock representation of $\mathcal{W}(M^\perp)$, with $M = {\rm span}\{g,h\} \subset L^2$. This shows that given any $g, h \in L^2$, the representations $\pi^{(g)}$ and $\pi^{(h)}$ are unitarily equivalent, up to multiplicity.

Discussion. In view of Theorem 3, the result of Corollary 1 says that for any two times $t, t' \in \mathbb{R}$, the reservoir representations π_t and $\pi_{t'}$ (see (3.82)) are unitarily equivalent, up to multiplicity.

Proof Theorem 4. Let $g \in L^2(\mathbb{R}^3, d^3k)$ be fixed. We decompose

$$L^2(\mathbb{R}^3, d^3k) \equiv M \oplus M^\perp, \tag{3.89}$$

where

$$M = \mathbb{C}g. \tag{3.90}$$

For an element $f \in L^2(\mathbb{R}^3, d^3k)$, we write the decomposition as

$$f = f_{||} + f_{\perp}. \tag{3.91}$$

According to the decomposition (3.89), the Fock space splits into a tensor product,

$$\mathcal{F}(M \oplus M^{\perp}) = \mathcal{F}(M) \otimes \mathcal{F}(M^{\perp}) \tag{3.92}$$

and the representation does as well,

$$\pi_{\mathrm{F}} = \pi_{||} \otimes \pi_{\perp}, \tag{3.93}$$

where $\pi_{||}$ and $\pi_{\perp}$ are the regular representations of $\mathcal{W}(M)$ and $\mathcal{W}(M^{\perp})$, respectively, obtained by restriction of π_{F} to the corresponding subalgebras. The ranges of $\pi_{||}$ and $\pi_{\perp}$ are in the bounded operators acting on $\mathcal{F}(M) \simeq \mathcal{F}(\mathbb{C})$ and $\mathcal{F}(M^{\perp})$, respectively. Then

$$\pi^{(g)}(W(f)) = e^{i\mathrm{Im}\langle g, f_{||}\rangle\varphi} \otimes \pi_{||}(W(f_{||})) \otimes \pi_{\perp}(W(f_{\perp})). \tag{3.94}$$

The latter operator acts on the Hilbert space

$$\mathcal{F}(\mathbb{C}) \otimes \mathcal{F}(M) \otimes \mathcal{F}(M^{\perp}). \tag{3.95}$$

Now we define the representation $\widetilde{\pi}^{(g)}$ of $\mathcal{W}(M)$ on $\mathcal{F}(\mathbb{C}) \otimes \mathcal{F}(M)$ by

$$\widetilde{\pi}^{(g)}(W(f_{||})) = e^{i\mathrm{Im}\langle g, f_{||}\rangle\varphi} \otimes \pi_{||}(W(f_{||})). \tag{3.96}$$

$\widetilde{\pi}^{(g)}$ is a regular representation of $\mathcal{W}(M)$. Since $\dim M < \infty$, the Stone-von Neumann uniqueness theorem implies that $\widetilde{\pi}^{(g)}$ is unitarily equivalent to a multiple of the Fock representation π_{HO} on $\mathcal{W}(M)$ (harmonic oscillator, since $\dim M = 1$). ■

Proof Corollary 1. By redefining the M in the proof of Theorem 4 to be $M = \mathrm{span}\{g, h\}$ and writing $f \in L^2$ as $f = f_M + f_{M^{\perp}}$, we obtain as in (3.94)

$$\pi^{(g)}(W(f)) = e^{i\mathrm{Im}\langle g, f_M\rangle\varphi} \otimes \pi_M(W(f_{M^{\perp}})) \otimes \pi_{M^{\perp}}(W(f_{M^{\perp}})), \tag{3.97}$$

where $\pi_{\mathrm{F}} = \pi_M \otimes \pi_{M^\perp}$ is the splitting analogous to (3.93). Now again, as in (3.96),

$$\widetilde{\pi}^{(X)}(W(f_M)) = e^{i\mathrm{Im}\langle X, f_M\rangle\varphi} \otimes \pi_M(W(f_M))$$

is a regular representation of $\mathcal{W}(M)$ and $\dim M < \infty$. The relation (3.88) follows from the Stone von Neumann uniqueness theorem. ■

3.3.1 Proof of Theorem 3

(A) Let Ω_{HO} be the ground state of a harmonic oscillator with associated Hamiltonian $H_{\mathrm{HO}} = a^*a$. The expectation value of a Weyl operator $e^{i(za^*+\bar{z}a)/\sqrt{2}}$ of the harmonic oscillator, for $z \in \mathbb{C}$, is

$$\langle\Omega_{\mathrm{HO}}, e^{i(za^*+\bar{z}a)/\sqrt{2}}\Omega_{\mathrm{HO}}\rangle = e^{-\frac{1}{4}|z|^2}. \tag{3.98}$$

Choosing

$$z = \sqrt{2\mathrm{var(Q)}}\lambda\mathrm{Im}\langle h_t, f\rangle \tag{3.99}$$

gives $\exp\Big[-\frac{1}{2}\lambda^2\mathrm{var}(Q)[\mathrm{Im}\langle h_t, f\rangle]^2\Big]$. Hence (3.79) can be written as

$$\begin{aligned}\langle\Omega_t, \pi_t(W(f))\Omega_t\rangle &= \langle\Omega_{\mathrm{HO}}, e^{i\sqrt{2\mathrm{var(Q)}}\lambda\mathrm{Im}\langle h_t,f\rangle\varphi}\Omega_{\mathrm{HO}}\rangle\langle\Omega_0, \pi_0(W(e^{i\omega t}f))\Omega_0\rangle \\ &= \omega_{R,t}(W(f)),\end{aligned} \tag{3.100}$$

where

$$\varphi := \frac{1}{\sqrt{2}}(a^* + a) \tag{3.101}$$

is the harmonic oscillator field operator.

(B) The Fock representation is given by the Hilbert space $\mathcal{F}(L^2)$, where $L^2 \equiv L^2(\mathbb{R}^3, d^3k)$. Denote by Ω_F the Fock vacuum vector and let $a_F^*(f)$, $a_F(f)$, $\varphi_F(f) = \frac{1}{\sqrt{2}}(a_F^*(f) + a_F(f))$ and $W_F(f) = e^{i\varphi_F(f)}$ be the Fock creation, annihilation, field and Weyl operators.

For $g \in L^2 \equiv L^2(\mathbb{R}^3, d^3k)$, $g \neq 0$ fixed, set

$$\pi_g(W(f)) = e^{i\mathrm{Im}\langle g,f\rangle\,\varphi} \otimes W_F(f), \qquad f \in L^2,$$

which acts on $\mathcal{F}(\mathbb{C}) \otimes \mathcal{F}(L^2)$. Denote by $\mathcal{W}$ the Weyl algebra over the single paticle space L^2. We are going to show that for all $g \neq 0$, the set

$$\mathcal{D} \equiv \pi_g(\mathcal{W})(\Omega_{\rm HO} \otimes \Omega_F) \tag{3.102}$$

is dense in $\mathcal{F}(\mathbb{C}) \otimes \mathcal{F}(L^2)$. For $x \in \mathbb{R}\backslash\{0\}$ and $f \in L^2$, set

$$V(x,f) = \frac{W(xf) - \mathbb{1}}{ix}, \qquad V_F(x,f) = \frac{W_F(xf) - \mathbb{1}}{ix}. \tag{3.103}$$

We have $V_F(x,f) \to \varphi_F(f)$ in the strong sense (on a dense domain), as $x \to 0$. Let $x, x_1, \dots, x_k \in \mathbb{R}$ and $f_1, \dots, f_k \in L^2$ with $f_j \perp g$, $j = 1, \dots, k$. Then

$$\pi_g(V(x_1,f_1) \cdots V(x_k,f_k))(\Omega_{\rm HO} \otimes \Omega_F) = \Omega_{\rm HO} \otimes V_F(x_1,f_1) \cdots V_F(x_k,f_k)\Omega_F. \tag{3.104}$$

By taking in (3.104) the limits $x_j \to 0$, $j = 1, \dots, k$, we see that

$$\Omega_{\rm HO} \otimes \varphi_F(f_1) \cdots \varphi_F(f_k)\Omega_F \in \overline{\mathcal{D}}.$$

Using that $a_F^*(f_j) = \frac{1}{\sqrt{2}}(\varphi_F(f_j) - i\varphi_F(\mathrm{i}f_j))$ and taking linear combinations yields

$$\Omega_{\rm HO} \otimes a_F^*(f_1) \cdots a_F^*(f_k)\Omega_F \in \overline{\mathcal{D}}.$$

We now show that $\Omega_{\rm HO} \otimes a_F^*(g) a_F^*(f_1) \cdots a_F^*(f_k)\Omega_F \in \overline{\mathcal{D}}$ as well. As in (3.104), we have

$$\begin{aligned} &\pi_g(V(x,g)V(x_1,f_1) \cdots V(x_k,f_k))(\Omega_{\rm HO} \otimes \Omega_F) \\ &\quad = \Omega_{\rm HO} \otimes V_F(x,g)V_F(x_1,f_1) \cdots V_F(x_k,f_k)\Omega_F. \end{aligned} \tag{3.105}$$

And by taking the limits of all the x, $x_j \to 0$ gives

$$\Omega_{\rm HO} \otimes \varphi_F(g) a_F^*(f_1) \cdots a_F^*(f_k)\Omega_F \in \overline{\mathcal{D}}. \tag{3.106}$$

Note that we cannot directly take linear combinations to conclude, since $\pi_g(W(ig)) = e^{i\|g\|^2\varphi} \otimes W_F(ig)$ has a nontrivial part on the first factor as well. However, it follows from (3.106) and the fact that $a_F(g)$ commutes with all the $a_F^*(f_j)$, and $a_F(g)\Omega_F = 0$, that

$$\Omega_{\rm HO} \otimes a_F^*(g) a_F^*(f_1) \cdots a_F^*(f_k)\Omega_F \in \overline{\mathcal{D}}. \tag{3.107}$$

Finally, it is clear how to use the above procedure leading to (3.106) to show that

$$\Omega_{\mathrm{HO}} \otimes (\varphi_F(g))^N a_F^*(f_1)\cdots a_F^*(f_k)\Omega_F \in \overline{\mathcal{D}}$$

for all integers N. By writing the field operator $\varphi_F(g)$ as a sum of creators and annihilators, and getting rid of the annihilators $a_F(g)$ by commuting them to hit the vacuum, one sees readily by induction that

$$\Omega_{\mathrm{HO}} \otimes (a_F^*(g))^N a_F^*(f_1)\cdots a_F^*(f_k)\Omega_F \in \overline{\mathcal{D}} \tag{3.108}$$

for all integers N. Since the set

$$\mathrm{span}\Big\{a^*(h_1)\cdots a^*(h_k)\Omega_F \;:\; k\in\mathbb{N}, h_1,\ldots,h_k \in L^2\Big\}$$

is dense in $\mathcal{F}(L^2)$, we conclude from (3.108) that

$$\Omega_{\mathrm{HO}} \otimes \mathcal{F}(L^2) \in \overline{\mathcal{D}}. \tag{3.109}$$

Next, let $\psi \in \mathcal{F}(L^2)$ belong to the finite particle space

$$\mathcal{F}^0(L^2) = \Big\{\psi = \{\psi_n\}_{n\geq 0} \in \mathcal{F}(L^2) \mid \text{all but finitely many } \psi_n \text{ are zero}\Big\}.$$

By the above construction (using the $V(x,f)$), there is a sequence $\mathcal{O}_n \in \mathcal{W}$ such that

$$\pi_g(\mathcal{O}_n)\Omega_{\mathrm{HO}} \otimes \Omega_F \to \Omega_{\mathrm{HO}} \otimes \psi. \tag{3.110}$$

For $y \in \mathbb{R}$,

$$\pi_g(W(\mathrm{i}yg)\mathcal{O}_n)\Omega_{\mathrm{HO}} \otimes \Omega_F = \Big(e^{iy\|g\|^2\varphi} \otimes W_F(iyg)\Big)\pi_g(\mathcal{O}_n)\,\Omega_{\mathrm{HO}} \otimes \Omega_F. \tag{3.111}$$

Taking the derivative w.r.t. y at $y=0$ and the limit $n\to\infty$ shows that

$$\Big(\|g\|^2\varphi \otimes \mathbb{1}_F + \mathbb{1}_{\mathrm{S}} \otimes \varphi(\mathrm{i}g)\Big)\ \Omega_{\mathrm{HO}} \otimes \psi \in \overline{\mathcal{D}}.$$

But the second vector in the sum, $\Omega_{\mathrm{HO}} \otimes \varphi(\mathrm{i}g)\psi$, already belongs to the closure $\overline{\mathcal{D}}$, as shown above, so $a^*\Omega_{\mathrm{HO}} \otimes \psi \in \overline{\mathcal{D}}$. Instead of taking the first derivative, one may take

∂_y^2 which easily implies that $(a^*)^2\Omega_{\mathrm{HO}} \otimes \psi \in \overline{\mathcal{D}}$. Repeating the argument with higher y-derivatives gives that $(a^*)^N\Omega_{\mathrm{HO}} \otimes \psi \in \overline{\mathcal{D}}$ for any N. Consequently $\mathcal{F}(\mathbb{C}) \otimes \psi \in \overline{\mathcal{D}}$ for all ψ in a dense subset of $\mathcal{F}(L^2)$ and hence the closure of $\mathcal{D}$ is all of $\mathcal{F}(\mathbb{C}) \otimes \mathcal{F}(L^2)$.
■

Chapter 4

Conclusion

In this thesis, we consider an open quantum system formed by N particles interacting with an environment, called a *reservoir*. In our main result we find the evolution of any subsystem (plus the reservoir) in the limit $N \to \infty$. A main assumption we make, to be able to carry out the mathematics, is that of an energy conserving interaction. We show that due to high complexity $N \to \infty$, all particles and the reservoir become uncorrelated and evolve independently. We then consider the effect the particle system has on the reservoir and its dynamics. We find the reservoir dynamics in the limit $N \to \infty$ and obtain explicitly its Hilbert space (GNS) representation. By using the Stone von Neumann uniqueness theorem we are able to prove that the representations at any two times unitarily equivalent up to multiplicity.

Bibliography

[1] Blum, K.: *Density matrix theory and applications*, Plenum Press, New York (1996)

[2] Alfio Borzi, Ciaramella, G., and Sprengel, M.: *Formulation and numerical solution of quantum control problems*, SIAM, Philadelphia (2017)

[3] Chruściński, D., Marmo, G.: *Remarks on the GNS representation and the geometry of quantum states*, Open Systems & Information Dynamics, **16** 157-177 (2009)

[4] Gottfried, K., Yan, T.-M.: *Quantum Mechanics: Fundamentals*, Springer Verlag, New York (2005)

[5] Moretti, V.: *Spectral theory and quantum mechanics*, Springer Verlag (2018)

[6] Nielsen, M. A., Chuang, I. L.: *Quantum computation and quantum information*, Cambridge University Press (2000)

[7] Attal, S., Joye, A., C -A Pillet.: *Open quantum systems I: the Hamiltonian approach*, Lecture Notes in Mathematics **1880**, Springer Verlag, Berlin (2006)

[8] Ottesen, J. T.: *Infinite-dimensional groups and algebras in quantum physics*, Springer Verlag, Berlin (1995)

[9] Simon, B.: *Functional integration and quantum physics*, AMS Providence (2005)

[10] Steeb, W.-H., Hardy, Y.: *Problems and solutions in quantum computing and quantum information*, World Scientific Publishing, New Jersey (2018)

[11] Woit, P.: *Quantum theory, groups and representations*, Springer Verlag (2018)

[12] Zhu, K.: *Introduction to operator algebras*, CRC Press, Boca Raton (1993)

[13] Bratteli, O., Robinson, D. W.: *Operator algebras and quantum statistical mechanics I, II*, Springer Verlag, New York (1987)

[14] Hepp, K., Lieb, E. H.: *On the Superradiant Phase Transition for Molecules in a quantized Radiation Field: the Dicke Maser Model*, Ann. Phys. **76**, 360-404 (1973).

[15] Merkli, M., Berman, G. P.: *Mean field evolution of open quantum systems. An exactly solvable model*. Proc. R. Soc. A, **468**, 3398-3412. (2012)

[16] D. Petz: *An Invitation to the Algebra of Canonical Commutation Relations*, Leuven University Press, 1990

[17] Joos, E.: *Decoherence and the appearance of a classical world in quantum theory*, Springer Verlag (2011)

[18] Palma, G. M., Suominen K.-A., Ekert A. K.: *Quantum Computers and Dissipation*, Proc. R. Soc. A, **452**, 567-584 (1996)

[19] Breuer, H.-P., Petruccione, F.: *The theory of open quantum systems*, Oxford University Press (2010)

[20] Gardiner, C. W., Zoller, P.: *Quantum noise*, Springer Verlag, Berlin (2004)

[21] Merkli, M., Rafiyi, A.: *Mean field dynamics of some open quantum systems*, Proc. R. Soc. A, **474**, 20170856 (2018)