# Classification of conservation laws of shallow-water equations

by

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# Abstract

We carry out the complete classification of zero-order conservation laws of the classes of one- and two-dimensional shallow-water equations with variable bottom topography. We also find the complete equivalence group for the one-dimensional case, using the direct method, and for the two-dimensional case, using the algebraic method. Using conservation-law characteristics, we find all inequivalent cases of bottom topographies (up to the equivalence group), which give different spaces of conservation laws. Analogously, using additionally the method of furcate splitting, we solve the classification problem for conservation laws for the two-dimensional case.

# Acknowledgements

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# Statement of contributions

The general subject of my research was proposed by Dr. Alexander Bihlo and was further specified during my studies. Chapters 3 and 4 are the original parts of the thesis. The main results, which are formulated in Theorems 34 and 39 in Chapters 3 and 4, respectively, were derived by me independently after discussions with Dr. Bihlo and Prof. Roman O. Popovych.

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# List of symbols

| | |
|---|---|
| $\mathrm{d}$ | differential |
| $\partial_i$ | partial derivative |
| $\mathrm{D}$ | total derivative |
| $\mathsf{D}_P$ | Fréchet derivative of differential function $P$ |
| $\mathsf{D}_P^\star$ | adjoint of Fréchet derivative |
| $\mathrm{Div}$ | total divergence |
| $\mathrm{E}$ | Euler operator |
| $\exp$ | exponential map |
| $\exp(\varepsilon \mathbf{v})$ | flow of vector field $\mathbf{v}$ |
| $G$ | Lie group |
| $\mathfrak{g}$ | Lie algebra |
| $\mathrm{CC}(\mathcal{L})$ | conserved currents of system $\mathcal{L}$ |
| $\mathrm{CC}_0(\mathcal{L})$ | trivial conserved currents of system $\mathcal{L}$ |
| $\mathrm{CL}(\mathcal{L})$ | conserved laws of system $\mathcal{L}$ |
| $\mathrm{CL}_0(\mathcal{L})$ | trivial conserved laws of system $\mathcal{L}$ |
| $\mathcal{L}$ | system of differential equations |
| $M$ | open subset of space of independent and dependent variables |
| $\mathrm{mod}$ | modular addition |

# List of symbols

| | |
|---|---|
| $\mathrm{pr}^{(n)}f$ | prolongation of function $f$ |
| $\mathrm{pr}^{(n)}g$ | prolongation of group transformation $g$ |
| $\mathrm{pr}^{(n)}G$ | prolongation of group $G$ |
| $\mathrm{pr}^{(n)}\mathbf{v}$ | prolongation of vector field $\mathbf{v}$ |
| $\mathbb{R}$ | real numbers |
| $u$ | dependent variables |
| $[u]$ | dependence on derivatives |
| $U$ | space of dependent variables |
| $u_i$ | partial derivative of $u$: $\partial u/\partial x_i$ |
| $u^{(k)}$ | partial derivatives of $u$ up to order $k$ |
| $\mathbf{v}$ | vector field |
| $x$ | independent variables |
| $X$ | space of independent variables |
| $[\mathbf{v}, \mathbf{w}]$ | Lie bracket |

# Chapter 1

# Introduction

Conservation laws play a distinguished role in mathematics and physics. They are mathematical versions of the physical laws of conservation, which are principles that state that certain physical properties (such as energy, linear momentum, angular momentum, mass and so on) do not change over time within a physical system. Conservation laws have multiple applications in areas related to differential equations such as geometrical numerical integration, integrability theory, linearization problems, etc. They provide useful tools for the analysis of properties of the solutions of differential equations and their physical investigations, see for example [32,33,40,48,50,54,56] and references therein. Furthermore, one can use them for the construction of new exact solutions. In addition, conservation laws can be used to prove global existence theorems, e.g. in [31] conservation laws are used to prove uniqueness theorems for elastic equilibria. Therefore, the computation and analysis of conservation laws differential equations is an important problem in mathematical physics.

**Symmetries.** The Norwegian mathematician Sophus Lie (1842–1899) is known as the founder of the theory of transformation groups, which is the foundation of the modern theory of Lie groups. This theory comes from Felix Klein's (1849–1925) vision, that the geometry of space is determined by the group of its symmetries. Therefore, the background of Lie theory is geometric. Roughly speaking, a symmetry of a geometrical object is a transformation under the action of which the object does not change. Symmetries are routinely used for classifying geometrical objects.

Since differential equations can be considered as geometrical objects, symmetries map the equations to itself. Moreover, as an equivalent definition, one introduces point transformations of the independent and dependent variables depending on continuous parameters, that map solutions of a system of differential equations to solutions of the same system. These transformations form a Lie symmetry group. The main tools in the theory of Lie groups are the "infinitesimal transformations". This concept necessitates the introduction of vector fields, which loosely speaking, give a tangent vector at each point of the manifold. For each Lie group there are certain vector fields, that form a vector space, the so-called Lie algebra, which is the infinitesimal generator of a given Lie group.

It has to be noticed that finding general point symmetries typically requires one to solve a system of nonlinear partial differential equations, which is often hard to do. Hence, one introduces Lie symmetries for which the determining equations are always linear and can be solved algorithmically, nowadays even with computer algebra systems. Considering all of the above, the main advantage of Lie symmetries is that they transform the invariance condition (with a nonlinear system of partial differential equations) to infinitesimal counterparts of this condition (with a linear system of partial differential equations). Moreover, one can consider generalized symmetries, whose infinitesimal generators besides the independent and dependent variables, additionally depend on the derivatives of the dependent variables. In this way, the corresponding group transformations can be found by integrating an evolutionary system of partial differential equations. The classical textbooks such as, for instance [41, 42] and [29], provide the theoretical background of the application of Lie theory to differential equations.

**Group classification.** Most differential equations include arbitrary parameters (or arbitrary parameter functions). These parameters are specified sometimes experimentally and can often be estimated from data. For instance, if viscosity in the Navier–Stokes equations is zero, then one gets the Euler equations which have different symmetry properties. Thus, to study the symmetries of systems of differential equations with arbitrary elements, one needs to investigate what happens as these parameters take on special values. Namely, for some values of these arbitrary elements the associated system of equations can admit more symmetries than for the general case. Let us specify that the (system of) differential equations with arbitrary elements is called class of (systems of) differential equations. The group classification for a class of systems of differential equations is the classification of Lie symmetry properties of systems from this class, depending on the values of the arbitrary elements, see for example [11, 43]. Depending on the structure of arbitrary elements, group classification can be a hard problem since it requires to solve a complicated overdetermined system of partial differential equations. However, to simplify this problem one can use equivalence transformations, which map each system from the class to another system of the same class and which form the equivalence group, see for example [12, 19]. Equivalence transformations give the opportunity to select the simplest representative among similar systems. The group classification problem is based on the description of all inequivalent (up to the equivalence group) values of arbitrary elements together with the corresponding symmetry groups for each case.

**Conservation laws.** In the beginning of the 20th century Emmy Noether proved in [38] (translated in [39]) that every conservation laws of particular systems of differential equations is associated with a symmetry in the underlying physics (namely, every symmetry of a Lagrangian induces a conservation law of the corresponding Euler–Lagrange equations). Noether's theorem requires a form of a variational structure of the considered system of equations. The extension to systems which do not have a variational principle is provided in [4]. It says, that one can replace symmetries by co-symmetries (adjoint symmetries), the invariance condition is replaced by the adjoint invariance condition on co-symmetries, and thus, conservation laws can be constructed in terms of co-symmetries.

An important part of studying conservation laws is the investigation of trivial conservation laws, which provide no new information on the behavior of solutions and can be applied to any system of differential equations. Using the notion of triviality of conservation laws, one can consider specific functions, so-called characteristic (also refereed to as multipliers), which uniquely characterize each non-trivial conservation law for a given system of equations. The important notion is the order of characteristic, which determines the order of the included derivatives. As soon as we have a connection between conservation laws and their characteristics, we consider the characteristic form of conservation laws instead of computing conservation laws directly.

**Classification problem on conservation laws.** The classification problem on conservation laws is similar to the symmetry group classification of differential equations. One can find, up to the equivalence group, certain systems of the class of equations such as they admit more conservation laws than the most general system from the class. For computing the conservation laws for these certain systems one can use conservation-law characteristics, Noether's theorem, some techniques from the direct method, etc. (see [5, 6, 46, 48]).

**Shallow-water equations.** In this thesis we consider the classes of one- and two-dimensional shallow-water equations with variable bottom topography. The shallow-water equations are derived from Euler's equation under the assumption that the vertical length scale is small in comparison with the horizontal one. The shallow-water equations describe many important physical processes and models. For instance, in ocean dynamics shallow-water equations can be a good model for the propagation of tsunamis across the open ocean, away from shore. This model for tsunami propagation is described in many papers, for instance in [20, 26, 55]. Moreover, the shallow-water equations can be used for modeling flood propagation in urban flooding, floodplains, dam-break computations etc. In these specific areas of the application of shallow-water equations there are many useful results from "pure" mathematics which are described, for example in [2, 8, 25, 51, 53, 59].

**Thesis goal.** The main idea of this thesis is solving the classification problem of zero-order conservation laws of the classes of one- and two-dimensional shallow-water equations with variable bottom topography. The one-dimensional case is considered in [1], where the classification problem for conservation laws was studied. In this thesis we repeat and optimize the computations for the one-dimensional case. We then also solve the conservation law classification problem for the two-dimensional case. Namely, for the full classification of conservation laws of the class of shallow-water equations, we need to compute the equivalence group of this class first. It allows us to compute conservation laws only for the inequivalent cases, since all of the other cases can be calculated directly by applying the equivalence transformations.

**Thesis structure.** The structure of this thesis as follows. In Chapter 2 we provide the theoretical background on symmetry groups (Section 2.1) and conservation laws (Section 2.3) of systems of differential equations as well as on equivalence groups of classes of such systems (Section 2.2), following mostly the classical textbooks [27–29, 39, 41, 42] and papers [9, 10, 14, 18, 22, 30, 32, 50, 57], as well as references therein. We present the main definitions, theorems and provide some methods, which we will use to compute the conservation laws in the next chapter.

Chapter 3 is devoted to the class of one-dimensional shallow-water equations with variable bottom topography. In Section 3.1, we compute the equivalence group $G^{\sim}$ of this class using the direct method. Then in Section 3.2, we classify zero-order conservation laws of one-dimensional shallow-water equations up to $G^{\sim}$-equivalence. This specifies results of the paper [1], where Lie symmetries and zero-order conservation laws of the above equations were described without involving $G^{\sim}$-equivalence. Moreover, the initial objects to be classified in the thesis are conservation-law characteristics whereas the authors of [1] directly classified conserved currents.

In Chapter 4 we extend the study to the class of two-dimensional shallow-water equations with variable bottom topography. This chapter has a similar structure as the previous one. We start with computing the equivalence group of the above class (Section 4.1) by the algebraic method and use the obtained result to solve the classification problem for zero-order conservation laws of systems from this class (Section 4.2), using conservation-law characteristics and the method of furcate splitting, which appeared first in [37,47] in the course of classifying Lie symmetries, see also [44]. In the thesis, this method is applied for the first time for classifying conservation laws.

In the last chapter we summarize the main results of this thesis and provide some possible directions of further work.

# Chapter 2

# Theoretical background

We present the theory of symmetry groups and equivalence groups closely following the classical textbooks such as [28,29,41,42,45] and the relevant works [11,12,19,43,47,49]. All computations and results of this thesis should be interpreted within the local approach, cf. [42].

## 2.1 Symmetry groups

Here and in the following we consider the local group of transformations $G$ acting on an open subset $M \subset X \times U$ of the space of independent and dependent variables for the system $\mathcal{L}\colon\ \Delta^i(x, u^{(p)}) = 0$, $i = 1, \dots, l$, where $x = (x_1, \dots, x_n)$ are the independent variables, $u = (u^1, \dots, u^m)$ are the dependent variables and $u^{(p)}$ denotes the tuple of derivatives of $u$ up to order $p$, including $u$ as the zeroth-order derivative. Let us start with basic notions and definitions (see [42]).

**Definition 1.** A smooth function $F$ depending on $x$, $u$ and a finite number of derivatives of $u$ is called a *differential function* of $u$. Denote this function as $F = F[u]$.

**Definition 2.** The *$j$-th total derivative* of a differential function $F = F[u]$ is

$$\mathrm{D}_j F = \frac{\partial F}{\partial x_j} + \sum_{\alpha=1}^{m} \sum_{J} \frac{\partial u^\alpha_J}{\partial x_j} \frac{\partial F}{\partial u^\alpha_J},$$

where $J = (j_1, \dots, j_k)$, $k \geq 0$, is an unordered $k$-tuple of integers, with entries $1 \leq j_{k'} \leq n$, $k' = 1, \dots, k$, indicating which derivatives are being taken.

**Definition 3.** A $p$-th prolongation of a smooth function $u = f(x)$, $f : X \to U$ is a function $u^{(p)} = \mathrm{pr}^{(p)} f(x)$ defined by the following formula

$$u^\alpha_J = \partial_J f^\alpha(x),$$

where $J = (j_1, \ldots, j_k)$, $k \geq 0$ and $1 \leq j_{k'} \leq n$, $k' = 1, \ldots, k$.

To define of prolongation of group actions, we start with the basic definition of jet spaces.

**Definition 4.** The *p-th order jet space* of the space $X \times U$ is the space $X \times U^{(p)}$, whose coordinates are the independent variables, the dependent variables and their derivatives up to order $p$.

Thus, an induced local action of $G$ on the $p$-th order jet space $M^{(p)}$ is called the *p-th prolongation* of $G$. Analogously to the prolongation of group transformations we can define the prolongation of the corresponding vector fields. Let

$$\mathbf{v} = \sum_{k=1}^{n} \xi^k(x,u)\frac{\partial}{\partial x_k} + \sum_{\alpha=1}^{m} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{2.1}$$

be a vector field on $M$. Then let us define the parametrized maximal integral curve $\exp(\varepsilon\mathbf{v})x$, which is passing through $x \in M$ and called the *flow* generated by $\mathbf{v}$. It has to be mentioned that the flow generated by $\mathbf{v}$ coincides with the local group action of the Lie group on the manifold $M$ (so-called one-parameter group of transformations), and the vector field $\mathbf{v}$ is called the *infinitesimal generator* of the action.

**Definition 5.** The $p$-th *prolongation* of $\mathbf{v}$ is a vector field on the space $M^{(p)}$ such that

$$\mathrm{pr}^{(p)}\mathbf{v}|_{(x,u^{(p)})} = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \mathrm{pr}^{(p)}[\exp(\epsilon\mathbf{v})](x,u^{(p)})$$

for any $(x, u^{(p)}) \in M^{(p)}$.

Suppose that for $g \in G$ the function $g \cdot f$ is defined in a neighborhood of the point

$$(\tilde{x}_0, \tilde{u}_0) = g \cdot (x_0, u_0),$$

where $u_0 = f(x_0)$. Thus, one can determine the action of the prolonged group transformation $\mathrm{pr}^{(p)}g$ on the point $(x_0, u_0^{(p)})$ by the following expression

$$\mathrm{pr}^{(p)}g \cdot (x_0, u_0^{(p)}) = (\tilde{x}_0, \tilde{u}_0^{(p)}),$$

where

$$\tilde{u}_0^{(p)} = \mathrm{pr}^{(p)}(g \cdot f)(\tilde{x}_0).$$

Below we present the basic notions of Lie theory.

**Definition 6.** A *Lie algebra* $\mathfrak{g}$ is a vector space, closed under a bilinear map (so-called Lie multiplication) $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, with $[v, v] = 0$ and $[u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0$ for $u, v, w \in \mathfrak{g}$.

**Definition 7.** An *$r$-parameter Lie group* is a group $G$ which also carries the structure of an $r$-dimensional smooth manifold in such a way that both the group operation $m\colon G \times G \to G$, $m(g, h) = g \cdot h$ for all $g, h \in G$, and the inversion $i\colon G \to G$, $i(g) = g^{-1}$ for all $g \in G$, are smooth maps between manifolds.

**Definition 8.** A *local group of transformations* acting on $M$ is given by a Lie group $G$, an open subset $\mathcal{K}$, with $e \times M \subset \mathcal{K} \subset G \times M$, which is the domain of definition of the group action, and a smooth map $\Psi\colon \mathcal{K} \to M$ with the following properties,

(i) If $(h, x) \in \mathcal{K}$, $(g, \Psi(h, x)) \in \mathcal{K}$ and $(g{\cdot}h, x) \in \mathcal{K}$, then $\Psi(g, \Psi(h, x)) = \Psi(g{\cdot}h, x)$.

(ii) $\Psi(e, x) = x$ for all $x \in M$.

(iii) If $(h, x) \in \mathcal{K}$, then $(g^{-1}, \Psi(g, x)) \in \mathcal{K}$ and $\Psi(g^{-1}, \Psi(g, x)) = x$.

**Definition 9.** A *symmetry group* of the system $\mathcal{L}$ is a local group of transformations $G$ acting on $M$ with the property that whenever $u = f(x)$ is a solution of $\mathcal{L}$, and $g \cdot f$ is defined for $g \in G$, then $u = g \cdot f(x)$ is also a solution of this system.

Namely, a symmetry group of the system $\mathcal{L}$ is a local group of transformations $G$, which transform solutions of the system to solutions of the same system.

**Theorem 10.** *(see [42, Theorem 2.27, p. 100]) Suppose that for every $(x, u^{(p)}) \in \mathcal{J} \subset M^{(p)}$ we have $\mathrm{pr}^{(p)}g \cdot (x, u^{(p)}) \in \mathcal{J}$ for all $g \in G$. Then $G$ is a symmetry group of the system $\mathcal{L}$.*

The following theorem comes directly from Theorem 10, see [42].

**Theorem 11.** *(see [42, Theorem 2.31, p. 104]) If* $\mathrm{pr}^{(p)}\mathbf{v}(\Delta^i(x, u^{(p)})) = 0$, *whenever* $\Delta^i(x, u^{(p)}) = 0$, $i = 1, \ldots, l$ *for every infinitesimal generator* $\mathbf{v}$ *of the local group of transformations* $G$, *then* $G$ *is a symmetry group of the system* $\mathcal{L}$.

For using the infinitesimal criterion, one needs to find a formula for the prolongation of a vector field. Let us provide the general prolongation formula.

**Theorem 12.** *(see [42, Theorem 2.36, p. 110]) For the vector field* $\mathbf{v}$, *its* $p$*-th prolongation is*

$$\mathrm{pr}^{(p)}\mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^{m} \sum_{J} \phi_\alpha^J(x, u^{(p)}) \frac{\partial}{\partial u_J^\alpha}, \tag{2.2}$$

*where*

$$\phi_\alpha^J(x, u^{(p)}) = \mathrm{D}_J \left( \phi_\alpha - \sum_{k=1}^{n} \xi^k u_k^\alpha \right) + \sum_{k=1}^{n} \xi^k u_{J,k}^\alpha, \tag{2.3}$$

*with* $u_k^\alpha = \partial u^\alpha / \partial x_k$ *and* $u_{J,k}^\alpha = \partial u_J^\alpha / \partial x_k$.

**Example 13.** As an example of computations within the framework of the infinitesimal approach, we find the Lie symmetry algebra of an equation equivalent to the system of the Prandtl equations

$$\mathcal{L}: \quad u_x + v_y = 0, \qquad u_{yy} = u u_x + v u_y. \tag{2.4}$$

The dependent variable $v$ can be expressed in terms of $u$ and its derivatives

$$v = \frac{u_{yy} - u u_x}{u_y}.$$

Then we can also differentiate this expression with respect to $y$ and from the equations (2.4) obtain

$$u u_x u_{yy} - u u_y u_{xy} - u_{yy}^2 + u_y u_{yyy} = 0. \tag{2.5}$$

Let

$$\mathbf{v} = \xi(x,y,u)\frac{\partial}{\partial x} + \eta(x,y,u)\frac{\partial}{\partial y} + \phi(x,y,u)\frac{\partial}{\partial u} \tag{2.6}$$

be the infinitesimal generator of a one-parameter Lie symmetry group of equation (2.5). Then according to Theorems 11 and 12 we act with the third prolongation

$$\begin{aligned}\mathrm{pr}^{(3)}\mathbf{v} = \mathbf{v} + \phi^x\frac{\partial}{\partial u_x} + \phi^y\frac{\partial}{\partial u_y} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xy}\frac{\partial}{\partial u_{xy}} + \phi^{yy}\frac{\partial}{\partial u_{yy}}\\ +\phi^{xxx}\frac{\partial}{\partial u_{xxx}} + \phi^{xxy}\frac{\partial}{\partial u_{xxy}} + \phi^{xyy}\frac{\partial}{\partial u_{xyy}} + \phi^{yyy}\frac{\partial}{\partial u_{yyy}}\end{aligned}$$

to the equation (2.5) and obtain

$$\begin{aligned}&u_x u_{yy}\phi - u_y u_{xy}\phi + u u_{yy}\phi^x - u u_{xy}\phi^y + u_{yyy}\phi^y + u u_x\phi^{yy}\\ &\quad - 2u_{yy}\phi^{yy} - u u_y\phi^{xy} + u_y\phi^{yyy} = 0,\end{aligned}$$

whenever equation (2.5) holds, and $\phi^x$, $\phi^y$, $\phi^{yy}$, $\phi^{xy}$, $\phi^{yyy}$ can be computed by the formula (2.3). The terms with derivative $u_{xyy}$ are contained only in $\phi^{yyy}$, and after collecting coefficients with this monomial by $u_y^2 u_{xyy}$ and $u_y u_{xyy}$ we get $-3\xi_u = 0$, $-3\xi_y = 0$. Thus, we conclude that $\xi = \xi(x)$. The coefficient of $u_y u_{yy}^2$ gives us the condition $\eta = \eta(x,y)$. Continuing to split with respect to the remaining derivatives we obtain

$$\eta_{xy} = 0, \quad \eta_{yy} = 0, \quad \phi_{uu} = 0, \quad \phi_x = 0, \quad \phi_y = 0, \quad \phi - u\xi_x + 2u\eta_y = 0.$$

Thus the infinitesimal generator of the equation (2.5) has coefficient functions of the form

$$\xi = 2c_1 x + c_2, \quad \eta = c_1 y + F(x), \quad \phi = c_3 u,$$

where $c_1, c_2, c_3$ are arbitrary constants and $F(x)$ is an arbitrary function. Hence, the Lie symmetry algebra of the Prandtl equations (2.5) is spanned by four vector fields

$$\mathbf{v}_1 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad \mathbf{v}_2 = \frac{\partial}{\partial x}, \quad \mathbf{v}_3 = u\frac{\partial}{\partial u}, \quad \mathbf{v}_4 = F(x)\frac{\partial}{\partial y}.$$

The complete point symmetry group (including both continuous and discrete symmetries) of a system of differential equations can be computed in different ways. The direct method involves the main tools of solving of system of partial differential equations and can be very voluminous. At the same time, there is the algebraic method to compute the complete point symmetry group of the system, see for example [12]. Start with the fact that each symmetry transformation $\mathcal{T}$ of a system of differential equations generates an automorphism of the maximal Lie invariance algebra via push-forwarding of vector fields in the space of variables of the system. Namely, we can fix a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of this algebra and compute the structure constants $c_{ij}^k$ such that $[\mathbf{v}_i, \mathbf{v}_j] = c_{ij}^k \mathbf{v}_k$. One can obtain the general form $(a_j^i)$ of automorphism matrices by solving the system with structure constants

$$c_{i'j'}^{k'} a_i^{i'} a_j^{j'} = c_{ij}^k a_k^{k'}. \tag{2.7}$$

As a next step we define a system of differential equations for components of a transformation $\mathcal{T}$ by solving the equations $\mathcal{T}_\star \mathbf{v}_i = a_i^j \mathbf{v}_j$, $i = 1, \ldots, n$, where $\mathcal{T}_\star$ is the push-forward of vector fields induced by $\mathcal{T}$. By integrating this system we obtain the final form of $\mathcal{T}$, using the framework of the direct method. The found set of transformations constitute the complete point symmetry group of the system of equations.

## 2.2 Equivalence groups

We consider a system of differential equations $\mathcal{L}_\Theta\colon \Delta^i(x, u^{(p)}, \Theta^{(q)}(x, u^{(p)})) = 0$, $i = 1, \ldots, l$, where $\Theta(x, u^{(p)}) = (\Theta^1(x, u^{(p)}), \ldots, \Theta^k(x, u^{(p)}))$ and $\Theta^{(q)}(x, u^{(p)})$ denote the tuple of derivatives of $\Theta$ with respect to $x$ and $u^{(p)}$ up to order $q$. Consider the system of auxiliary differential equations $S^{i'}(x, u^{(p)}, \Theta^{(q')}(x, u^{(p)})) = 0$, $i' = 1, \ldots, l'$ and the inequalities $\Sigma^{i''}(x, u^{(p)}, \Theta^{(q')}(x, u^{(p)})) \neq 0$, $i'' = 1, \ldots, l''$, thus the arbitrary elements $\Theta$ are constrained by the solution set, denoted by $\mathcal{S}$, of both the auxiliary equations $S^{i'} = 0$ and inequalities $\Sigma^{i''} \neq 0$.

**Definition 14.** [12, Definition 1, p. 4] The set $\{\mathcal{L}_\Theta | \Theta \in \mathcal{S}\}$ denoted by $\mathcal{L}|_\mathcal{S}$ is called a *class of differential equations* defined by the parametrized form of systems $\mathcal{L}_\Theta$ and the set $\mathcal{S}$ of the arbitrary elements $\Theta$.

**Definition 15.** [12, Definition 2, p. 4] The equivalence group $G^\sim = G^\sim(\mathcal{L}|_\mathcal{S})$ of the class $\mathcal{L}|_\mathcal{S}$ is the group of point transformations in the space of $(x, u^{(p)}, \Theta)$ which are projectable to the space of $(x, u^{(p')})$ for any $0 \le p' \le p$, that are consistent with the contact structure on the space of $(x, u^{(p)})$, preserve the set $\mathcal{S}$ of arbitrary elements and preserve the parametrized form of systems $\mathcal{L}_\Theta$. Elements of $G^\sim$ are called *equivalence transformations*.

If $\Theta$ does not depend on derivatives of the dependent variables, i.e. $\Theta = \Theta(x, u)$, then one can introduce the *generalized equivalence group* $G^\sim_{gen}$ of the class $\mathcal{L}|_\mathcal{S}$.

**Definition 16.** The Lie algebra $\mathfrak{g}^\sim$ of vector fields in the space of $(x, u^{(p)}, \Theta)$, which for any $0 \le p' \le p$ are projectable to the space of $(x, u^{(p)})$ with the property that their projections to the space of $(x, u^{(p')})$ are the $p'$-th order prolongations of their projections to the space of $(x, u)$, is called the *equivalence algebra* of the class $\mathcal{L}|_\mathcal{S}$.

Analogously to the generalized equivalence group, one can define the *generalized equivalence algebra* of the class $\mathcal{L}|_\mathcal{S}$.

In the next chapters we consider two methods for the computation of equivalence transformations of systems of differential equations: the direct method (Section 3.1) and the algebraic one (Section 4.1). The direct method is exactly the application of the definition of the equivalence group, and it usually requires the solution of nonlinear systems of differential equations.

The algebraic method was proposed by Hydon in [27, 29], see also [12] and [11]. Let us notice that the algebraic method of constructing a symmetry group, which is described in Section 2.1, can be extended to the framework of equivalence transformations by the following theorem.

**Theorem 17.** *(see [12, Theorem 1, p. 6]) Let $\mathcal{L}|_\mathcal{S}$ be a class of systems of differential equations, $G^\sim$ and $\mathfrak{g}^\sim$ the equivalence group and the equivalence algebra of this class, respectively. Any transformation $\mathcal{T}$ from $G^\sim$ induces an automorphism of $\mathfrak{g}^\sim$ via push-forwarding of vector fields in the relevant space of independent variables, derivatives of unknown functions and arbitrary elements of the class.*

## 2.3 Conservation laws

In this section we provide the theory of conservation laws closely following [5, 6, 39, 42, 46, 50] and [58]. Here we consider the same system of differential equations as in

Section 2.1. Let us start with some basic notions and definitions, which we will use below.

**Definition 18.** The order of the differential function $f[u]$ is the highest order of derivatives involved in $f$, which is denoted by $\operatorname{ord} f$. If $f$ does not depend on derivatives of $u$, then $\operatorname{ord} f = -\infty$.

**Definition 19.** An $n$-tuple of differential functions $P = (P^1[u], \dots, P^n[u])$ is called a *conserved current* of the system $\mathcal{L}$ if

$$(\operatorname{Div} P)|_{\mathcal{L}} = 0, \tag{2.8}$$

where Div is the total divergence, $\operatorname{Div} P = \sum\limits_{i=1}^{n} \mathrm{D}_i P^i$, and $\mathrm{D}_i$ is the operator of total differentiation with respect to $x_i$.

**Definition 20.** A conserved current $P$ of the system $\mathcal{L}$ is called *trivial*

- of the first type if $P$ vanishes on the solutions of $\mathcal{L}$;
- of the second type if $P$ is a null divergence, which means that $\operatorname{Div} P = 0$ regardless of the system $\mathcal{L}$.

A *general trivial conserved current* is a linear combination of trivial conserved currents of the above two types.

**Definition 21.** Conserved currents $P$ and $\tilde{P}$ of the system $\mathcal{L}$ are called *equivalent* if $P - \tilde{P}$ is a trivial conserved current.

**Definition 22.** The *space of conservation laws* of the system $\mathcal{L}$ is the quotient space $\mathrm{CL}(\mathcal{L}) = \mathrm{CC}(\mathcal{L})/\mathrm{CC}_0(\mathcal{L})$. Its elements are called the *conservation laws* (CLs) of the system $\mathcal{L}$.

The order of conserved current $P$ is the maximal order of derivative explicitly appearing in $P$, the *order of the conservation law* is defined as minimum of the set of orders of the corresponding conserved currents. Conserved currents of the system $\mathcal{L}$ generate the linear space $\mathrm{CC}(\mathcal{L})$, the trivial conserved currents of this system span its subspace $\mathrm{CC}_0(\mathcal{L})$.

By Hadamard's lemma the equality (2.8) holds if there exist differential functions $K^{i,J}$ such that

$$\operatorname{Div} P = \sum_{i=1}^{l} \sum_{J} K^{i,J} \mathrm{D}_J \Delta^i. \tag{2.9}$$

The right-hand side of the equality (2.9) can be rewritten as

$$\sum_{i=1}^{l}\sum_{J}\mathrm{D}_J(K^{i,J}\Delta^i)-\mathrm{D}_J(K^{i,J})\Delta^i=\dots=\mathrm{Div}\,\tilde{P}+\sum_{i}\Lambda^i\Delta^i,$$

applying repeated integration by parts, where $P$ and $\tilde{P}$ are equivalent conserved currents.

Thus, the equality $\mathrm{Div}\,P=0$ vanishes on the solutions of this system if and only if there exists an $l$-tuple of differential functions $\Lambda=(\Lambda^1,\dots,\Lambda^l)$ such that

$$\mathrm{Div}\,P=\sum_{i=1}^{l}\Lambda^i\Delta^i. \tag{2.10}$$

It has to be mentioned that this statement is only correct up to equivalent conserved currents.

The equality (2.10) is called a *characteristic form* of the conservation law associated with the conserved current $P$.

**Definition 23.** If the equality (2.10) holds, then the $l$-tuple of differential functions

$$\Lambda=(\Lambda^1[u],\dots,\Lambda^l[u])$$

is called a *characteristic* of the conservation law, which is associated with the conserved current $P$ of the system $\mathcal{L}$.

Analogously to the triviality of conserved currents, let us provide the definition of trivial CL-characteristics.

**Definition 24.** A CL-characteristic $\Lambda$ of the system $\mathcal{L}$ is called *trivial* if it vanishes for all solutions of this system. Two CL-characteristics $\Lambda$ and $\tilde{\Lambda}$ of the system $\mathcal{L}$ are *equivalent* if $\Lambda-\tilde{\Lambda}$ is a trivial CL-characteristic of $\mathcal{L}$.

We denote the linear space of CL-characteristics of the system $\mathcal{L}$ by $\mathrm{Ch}(\mathcal{L})$, the trivial CL-characteristics of this system constitute its subspace $\mathrm{Ch}_0(\mathcal{L})$.

**Definition 25.** The *space of CL-characteristics* of the system $\mathcal{L}$ is the quotient space $\mathrm{Ch}_{\mathrm{q}}(\mathcal{L})=\mathrm{Ch}(\mathcal{L})/\mathrm{Ch}_0(\mathcal{L})$.

**Example 26.** The system of the Prandtl equations (2.4),

$$\mathcal{L}\colon\quad u_x + v_y = 0, \qquad u_{yy} = uu_x + vu_y,$$

can be represented in the extended Kovalevskaya form under assuming $u_{yy}$ and $v_y$ as the leading derivatives. The space of reduced conservation-law characteristics of order not greater than four for the system (2.4) with the above choice of leading derivatives was presented in [36] without showing the details of computations. This space is in fact spanned by two linearly independent zero-order conservation-law characteristics, $(1, 0)$ and $(u, -1)$. The corresponding conserved currents are $(u, v)$ and $(u^2, uv - u_y)$. Indeed,

$$\begin{aligned}
\mathrm{Div}\,(u, v) &= \mathrm{D}_x(u) + \mathrm{D}_y(v) = 1\cdot(u_x + v_y) + 0\cdot(u_{yy} - uu_x - vu_y) = 0,\\
\mathrm{Div}\,(u^2, uv - u_y) &= \mathrm{D}_x(u^2) + \mathrm{D}_y(uv - u_y)\\
&= u\cdot(u_x + v_y) - 1\cdot(u_{yy} - uu_x - vu_y) = 0.
\end{aligned}$$

In Example 29 below, we present the computation of the above zero-order conservation-law characteristics.

In Sections 3.2 and 4.2 we use CL-characteristics to compute the conservation laws by applying the Euler operator to the characteristic form of the conservation law.

**Definition 27.** A differential operator

$$\mathrm{E}_\alpha = \sum_J (-\mathrm{D})_J \frac{\partial}{\partial u^\alpha_J} \tag{2.11}$$

is called the $\alpha$-th Euler operator, where $1 \le \alpha \le m$, $J = (j_1, \dots, j_k)$, $1 \le\ j_k \le n$ and $k \ge 0$, $(-\mathrm{D})_J = (-\mathrm{D}_{j_1})\cdots(-\mathrm{D}_{j_n})$, where $\mathrm{D}_{j_k}$ is the total derivative with respect to the variable $x_{j_k}$.

For the differential function $f$ there is an $l$-tuple of differential functions $F$ such that $f = \mathrm{Div}\,F$ if and only if $\mathrm{E}_\alpha f = 0$. After applying the Euler operator to the characteristic form of the conservation law (2.10), one obtains

$$\mathrm{E}_\alpha\left(\sum_{i=1}^{l} \Lambda^i \Delta^i\right) = 0, \qquad \alpha = 1, \dots, m. \tag{2.12}$$

An $l$-tuple $\Lambda$ is a CL-characteristic of the system $\mathcal{L}$ if and only if the equality (2.12) holds.

**Definition 28.** (see [42, Definition 5.24, p. 307]) Let $F[u] = F(x, u^{(p)}) \in \mathcal{A}^r$ be an $r$-tuple of differential functions. Here $\mathcal{A}^r$ denotes the space of $r$-tuples of differential functions. The *Fréchet derivative* of $F$ is the differential operator $\mathsf{D}_F\colon \mathcal{A}^q \to \mathcal{A}^r$ such that

$$\mathsf{D}_F(Q) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} F[u + \varepsilon Q[u]]$$

for any $Q \in \mathcal{A}^q$.

Let us note that the adjoint of the Fréchet derivative has entries

$$(\mathsf{D}^*_{\mathcal{L}})_{ki} = \sum_J (-\mathrm{D})_J \frac{\partial \Delta^i}{\partial u^k_J}, \qquad k = 1, \dots, m,\ i = 1, \dots, l.$$

The left-hand side of the equality (2.12) gives

$$\mathsf{D}^*_{\Lambda}(\mathcal{L}) + \mathsf{D}^*_{\mathcal{L}}(\Lambda) = 0. \tag{2.13}$$

The differential operators $\mathsf{D}^*_{\Lambda}$ and $\mathsf{D}^*_{\mathcal{L}}$ are the adjoints of the Fréchet derivatives $\mathsf{D}_{\Lambda}$ and $\mathsf{D}_{\mathcal{L}}$. The term $\mathsf{D}^*_{\Lambda}(\mathcal{L})$ vanishes on solutions of $\mathcal{L}$. Therefore, the necessary condition for a CL-characteristic of the system $\mathcal{L}$ is

$$\mathsf{D}^*_{\mathcal{L}}(\Lambda)\big|_{\mathcal{L}} = 0. \tag{2.14}$$

A tuple of differential functions satisfying (2.14) is called a *co-symmetry* of the system $\mathcal{L}$.

**Example 29.** Now we show how to obtain the conservation laws from Example 26. Let $\Lambda = \Lambda(x, y, u, v)$ and $\mathrm{M} = \mathrm{M}(x, y, u, v)$ be the components of a zero-order conservation-law characteristic of the system (2.4). The entries of the adjoint of the Fréchet derivative of the left-hand side of the system (2.4) are

$$(\mathsf{D}^*_{\mathcal{L}})_{11} = -\mathrm{D}_x, \quad (\mathsf{D}^*_{\mathcal{L}})_{12} = \mathrm{D}_y^2 + u\mathrm{D}_x + v\mathrm{D}_y - u_x,$$
$$(\mathsf{D}^*_{\mathcal{L}})_{21} = -\mathrm{D}_y, \quad (\mathsf{D}^*_{\mathcal{L}})_{22} = -u_y.$$

Since $\Lambda$ and $\mathrm{M}$ do not depend on nonzero-order derivatives of $u$ and $v$, we can split, after substituting expressions for principal derivatives in view of $\mathcal{L}$, the condition (2.14) for the corresponding characteristic with respect to parametric first- and second-order derivatives and obtain the following system of equations

$$\Lambda_x = \Lambda_y = \Lambda_v = 0, \quad \mathrm{M}_x = \mathrm{M}_y = \mathrm{M}_u = \mathrm{M}_v = 0, \quad \mathrm{M} + \Lambda_u = 0.$$

Thus, $\Lambda = c_1 u + c_2$ and $\mathrm{M} = -c_1$, where $c_1, c_2 \in \mathbb{R}$. Therefore, for the system (2.4), the space of its CL-characteristics of order not greater than zero is two-dimensional and spanned by the tuples $(1, 0)$ and $(u, -1)$.

Conservation laws classification problem of classes of systems of differential equations is similar to group classification problem. Using equivalence transformations, one can find, up to equivalence, such systems of the class that admit more conservation laws than the most general system from the class.

The following result was provided first in [48], see also [13, 17, 50].

**Theorem 30.** *Any point transformation* $\mathcal{T}$: $\tilde{x} = \mathcal{T}^x(x, u)$, $\tilde{u} = \mathcal{T}^u(x, u)$ *prolonged to the jet space* $J^{(p+1)}$ *transforms the equation* $\mathrm{D_i} F^i = 0$ *to the equation* $\tilde{\mathrm{D}}_\mathrm{i} \tilde{F}^i = 0$, *where the transformed conserved vector* $\tilde{F} = \mathcal{T}^F(x, u^{(p)}, F)$ *is determined as follows*

$$\tilde{F}^i(\tilde{x}, \tilde{u}^{(p)}) = \frac{1}{|\mathrm{D}_x \tilde{x}|} (\mathrm{D}_x \tilde{x}) F(x, u^{(p)}), \tag{2.15}$$

*where* $|\mathrm{D}_x \tilde{x}|$ *is the determinant of the matrix* $\mathrm{D}_x \tilde{x} = (\mathrm{D}_{x_j} \tilde{x}_i)_{i,j=1}^n$.

One can use the characteristics to compute the general conservation laws of a system of equations. However, during application of the necessary condition (2.14) we cannot split it with respect to the arbitrary elements and their derivatives. Namely, we need to solve the system of determining equations with parameters, considering such values of the arbitrary elements, which have an influence on the solutions. This will be illustrated for the shallow-water equations below.

# Chapter 3

# One-dimensional shallow-water equations

In the next two sections we consider the class of one-dimensional shallow-water equations, assuming the bottom topography as the arbitrary element of the class. More specifically, we define the class of systems of differential equations

$$
\begin{aligned}
u_t + uu_x + h_x &= b_x, \\
h_t + uh_x + u_x h &= 0,
\end{aligned}
\tag{3.1}
$$

where

- $u = u(t, x)$ is the horizontal fluid velocity averaged over the height of the fluid column,
- $h = h(t, x)$ is is the thickness of a fluid column, and
- $b = b(x)$ is the bottom topography measured downward with respect to a fixed reference level and considered as the arbitrary element of the class (3.1).

Figure 3.1 represents these values graphically.

We will refer to (3.1) as a class or a system, depending on the context. Below we construct the equivalence group of the class (3.1) by the direct method and then classify the zero-order conservation laws of this system using the method based on conservation-law characteristics. Recall that Lie symmetries and zero-order conservation laws of systems from the class (3.1) were studied recently by Aksenov and Druzhkov in [1] without involving equivalence transformations.

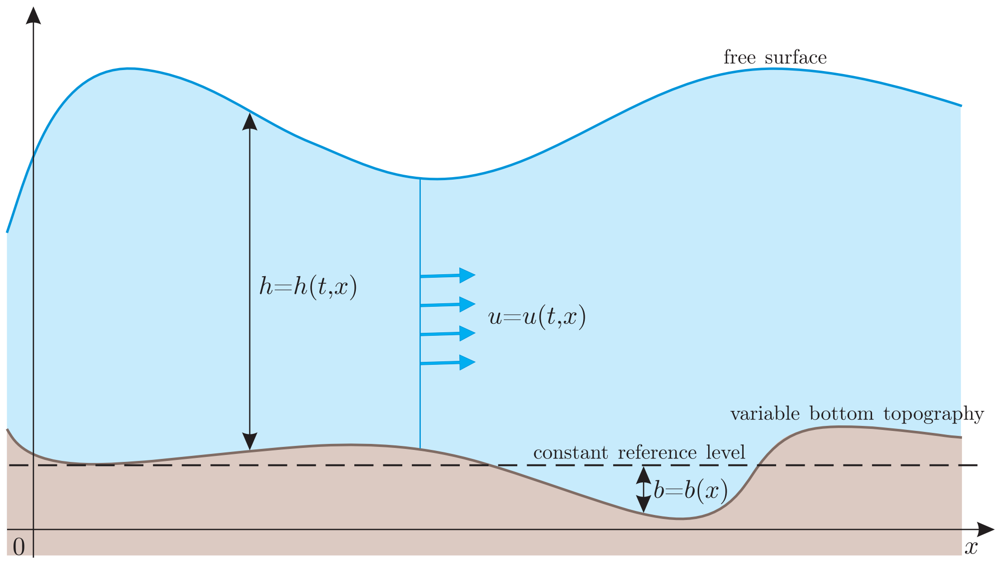


Figure 3.1: The shallow-water model

## 3.1 Equivalence group of class of one-dimensional shallow-water equations

We compute the complete usual equivalence group of the class (3.1) by the direct method. For easier calculation we set $B = b_x$. Thus we fix two systems in the class (3.1),

$$\mathcal{S}: \; u_t + uu_x + h_x = B, \quad h_t + uh_x + u_x h = 0; \tag{3.2}$$

$$\tilde{\mathcal{S}}: \; \tilde u_{\tilde t} + \tilde u\tilde u_{\tilde x} + \tilde h_{\tilde x} = \tilde B, \quad \tilde h_{\tilde t} + \tilde u\tilde h_{\tilde x} + \tilde u_{\tilde x}\tilde h = 0. \tag{3.2$'$}$$

The general form of point transformations that map the system (3.2) to the system (3.2′) is

$$\tilde t = T(t,x,u,h), \quad \tilde x = X(t,x,u,h), \quad \tilde u = U(t,x,u,h), \quad \tilde h = H(t,x,u,h),$$

where $J = |\partial(T,X,U,H)/\partial(t,x,u,h)| \neq 0$. By the chain rule we represent the derivatives of the new dependent variables in terms of the initial ones,

$$\begin{aligned} \tilde u_{\tilde t} &= \frac{1}{K}\left((\mathrm{D}_tU)\mathrm{D}_xX - (\mathrm{D}_xU)\mathrm{D}_tX\right), \quad \tilde u_{\tilde x} = \frac{1}{K}\left((\mathrm{D}_tT)\mathrm{D}_xU - (\mathrm{D}_xT)\mathrm{D}_tU\right), \\ \tilde h_{\tilde t} &= \frac{1}{K}\left(\mathrm{D}_tH)\mathrm{D}_xX - (\mathrm{D}_xH)\mathrm{D}_tX\right), \quad \tilde h_{\tilde x} = \frac{1}{K}\left(\mathrm{D}_tT)\mathrm{D}_xH - (\mathrm{D}_xT)\mathrm{D}_tH\right), \end{aligned} \tag{3.3}$$

where $K = (\mathrm{D}_t T)\mathrm{D}_x X - (\mathrm{D}_x T)\mathrm{D}_t X$. We expand (3.2′) on the solutions of the system (3.2) substituting the expressions (3.3) and the expressions for $u_t$ and $h_t$ in view of (3.2):

$$(\mathrm{D}_t U)\mathrm{D}_x X - (\mathrm{D}_x U)\mathrm{D}_t X + (\mathrm{D}_t T)\mathrm{D}_x(U^2/2 + H) - (\mathrm{D}_x T)\mathrm{D}_t(U^2/2 + H) \\ = \tilde{B}((\mathrm{D}_t T)\mathrm{D}_x X - (\mathrm{D}_x T)\mathrm{D}_t X), \tag{3.4}$$

$$(\mathrm{D}_t H)\mathrm{D}_x X - (\mathrm{D}_x H)\mathrm{D}_t X + (\mathrm{D}_t T)\mathrm{D}_x(UH) - (\mathrm{D}_x T)\mathrm{D}_t(UH) = 0.$$

Given functions $F$ and $G$ of variables $(y, z)$, we abbreviate the notation of the Jacobian

$$\det \frac{\partial(F, G)}{\partial(y, z)} \quad \text{to} \quad \frac{F, G}{y, z}$$

and will use it to make formulas less cumbersome. Collecting the coefficients of the expressions $hu_x^2 - h_x^2$, $u_x$ and $h_x$ in the equations (3.4) we respectively obtain

$$\frac{U, X}{u, h} + \frac{T, U^2/2 + H}{u, h} = \tilde{B}\frac{T, X}{u, h}, \tag{3.5a}$$

$$\frac{U, X}{t, u} - u\frac{U, X}{u, x} - h\frac{U, X}{h, x} \\ + \frac{T, U^2/2 + H}{t, u} - u\frac{T, U^2/2 + H}{u, x} - h\frac{T, U^2/2 + H}{h, x} \\ = \tilde{B}\left(\frac{T, X}{t, u} - u\frac{T, X}{u, x} - h\frac{T, X}{h, x}\right), \tag{3.5b}$$

$$\frac{U, X}{t, h} - \frac{U, X}{u, x} - u\frac{U, X}{h, x} \\ + \frac{T, U^2/2 + H}{t, h} - \frac{T, U^2/2 + H}{u, x} - u\frac{T, U^2/2 + H}{h, x} \\ = \tilde{B}\left(\frac{T, X}{t, h} - \frac{T, X}{u, x} - u\frac{T, X}{h, x}\right), \tag{3.5c}$$

$$\frac{U, X}{t, x} + \frac{U, X}{u, x}B + \frac{T, U^2/2 + H}{t, x} + \frac{T, U^2/2 + H}{u, x}B \\ = \tilde{B}\left(\frac{T, X}{t, x} + \frac{T, X}{u, x}B\right). \tag{3.5d}$$

and

$$\frac{H,X}{u,h}+\frac{T,UH}{u,h}=0, \tag{3.6a}$$

$$\frac{H,X}{t,u}-u\frac{H,X}{u,x}-h\frac{H,X}{h,x}+\frac{T,UH}{t,u}-u\frac{T,UH}{u,x}-h\frac{T,UH}{h,x}=0, \tag{3.6b}$$

$$\begin{aligned}&\frac{H,X}{t,h}+\frac{H,X}{u,h}B-\frac{H,X}{u,x}-u\frac{H,X}{h,x}\\&\qquad+\frac{T,UH}{t,h}+\frac{T,UH}{u,h}B-\frac{T,UH}{u,x}-u\frac{T,UH}{h,x}=0,\end{aligned} \tag{3.6c}$$

$$\frac{H,X}{t,x}+\frac{H,X}{u,x}B+\frac{T,UH}{t,x}+\frac{T,UH}{u,x}B=0. \tag{3.6d}$$

We are looking for the usual equivalence group of the class (3.1), whose elements generate admissible transformations within the class (3.1) for any source value for the arbitrary element $B$. This is why it is possible to split such among equations (3.5) and (3.6) with respect to $\tilde{B}$ (resp. $B$) that do not contain $B$ (resp. $\tilde{B}$). Note that the splitting of equations just means the collecting of coefficients of arbitrary element and its derivatives or coefficients of polynomials etc.. Therefore, collecting the coefficients of $\tilde{B}$ in the equations (3.5a), (3.5b) and (3.5c), we obtain

$$\frac{T,X}{u,h}=0, \tag{3.7}$$
$$\frac{T,X}{t,u}-u\frac{T,X}{u,x}-h\frac{T,X}{h,x}=0, \tag{3.8}$$
$$\frac{T,X}{t,h}-\frac{T,X}{u,x}-u\frac{T,X}{h,x}=0. \tag{3.9}$$

From equation (3.7) we conclude that there exists a smooth function $\Phi=\Phi(t,x,u,h)$ such that $T=T(t,x,\Phi)$, $X=X(t,x,\Phi)$. Substituting these forms into equations (3.8) and (3.9) we get the system, which can be considered as a system of linear algebraic equations with respect to $\dfrac{T,X}{t,\Phi}$ and $\dfrac{T,X}{x,\Phi}$,

$$\begin{aligned}\Phi_u\frac{T,X}{t,\Phi}+(u\Phi_u+h\Phi_h)\frac{T,X}{x,\Phi}&=0,\\ \Phi_h\frac{T,X}{t,\Phi}+(\Phi_u+u\Phi_h)\frac{T,X}{x,\Phi}&=0.\end{aligned} \tag{3.10}$$

Suppose that this system has a nonzero solution. Then its determinant equals zero, i.e.,

$$h\Phi_h^2 - \Phi_u^2 = 0,$$

which implies that $h \geq 0$ and $\Phi_u = \varepsilon\sqrt{h}\Phi_h$, where $\varepsilon = \pm 1$. Substituting this result in the last equation, we solve that with respect to $\Phi$, $\Phi = \Psi(t, x, \omega)$ with $\omega = u + 2\varepsilon\sqrt{h}$ and $\Psi_\omega \neq 0$. Then the system (3.10) reduces to the single equation

$$\frac{T, X}{t, \Phi} + (u + \varepsilon\sqrt{h})\frac{T, X}{x, \Phi} = 0, \quad \text{i.e.,} \quad \frac{T, X}{t, \Phi} + (\omega - \varepsilon\sqrt{h})\frac{T, X}{x, \Phi} = 0.$$

Splitting this equation with respect to $h$, we have $\Phi_u = \Phi_h = 0$, which contradicts the supposition.

Therefore, $\dfrac{T, X}{t, \Phi} = \dfrac{T, X}{x, \Phi} = 0$. Then $T_\Phi = X_\Phi = 0$ since

$$\operatorname{rank}\begin{pmatrix} T_t & T_x & T_\Phi \\ X_t & X_x & X_\Phi \end{pmatrix} = 2.$$

In other words, $T = T(t, x)$, $X = X(t, x)$ and $\dfrac{T, X}{t, x} \neq 0$. By equation (3.5d),

$$\tilde{B} = \frac{1}{K}\left(\frac{U, X}{t, x} + U_u X_x B + T_t(UU_x + H_x) - T_x(UU_t + H_t) - T_x(UU_u + H_u)B\right).$$

We can differentiate both the sides of this equation with respect to $\tilde{t}$, where the operator $\partial_{\tilde{t}}$ has the form

$$\begin{aligned} \partial_{\tilde{t}} = \left(\frac{H, U}{h, u}\frac{T, X}{t, x}\right)^{-1} &\left(\frac{H, U}{h, u}\,(X_x\partial_t - X_t\partial_x)\right. \\ &\left. + \left(H_h\frac{U, X}{x, t} + U_h\frac{H, X}{t, x}\right)\partial_u + \left(X_t\frac{H, U}{x, u} + X_x\frac{H, U}{u, t}\right)\partial_h\right). \end{aligned} \tag{3.11}$$

Since $\partial_{\tilde{t}}\tilde{B} = 0$ and $\dfrac{T, X}{t, x} \neq 0$, collecting of the coefficient of $B_x$ in the resulting equation leads to the equation

$$X_t\big(U_u X_x - T_x(UU_u + H_u)\big) = 0.$$

Suppose that $X_t \neq 0$. Then considering the above equation jointly with an equation derived by splitting of (3.6d) with respect to $B$, we have the system, which can be considered as a system of linear algebraic equations with respect to $(T_x, X_x) \neq (0,0)$,

$$(UU_u + H_u)T_x - U_u X_x = 0, \tag{3.12}$$

$$(UH)_u T_x - H_u X_x = 0. \tag{3.13}$$

This system has a nonzero solution, so its determinant equals zero, which gives $-H_u^2 + U_u^2 H = 0$. Thus,

$$-H_u^2 + U_u^2 H = 0, \quad \text{i.e.,} \quad H_u = \pm\sqrt{H} U_u, \tag{3.14}$$

with the assumption $H \geq 0$. Recalling that $\dfrac{U, H}{u, h} \neq 0$, we have $U_u \neq 0$ and $H_u \neq 0$. After substituting the expression (3.14) for $H$ and $U$ in view of (3.12) and factoring out $U_u$, we get $X_x = T_x(U \pm \sqrt{H})$. We differentiate this equation with respect to $u$ and obtain $T_x U_u = 0$. Since $U_u \neq 0$, we conclude that $T_x = X_x = 0$, which leads to the contradiction with the condition $(T_x, X_x) \neq (0,0)$.

Therefore, $X_t = 0$, and thus $T_t X_x \neq 0$. We consider the system of equations (3.6b)–(3.6d) as a system of linear algebraic equations with respect to $(T_t, T_x, X_x)$. Since this system has a nonzero solution, its determinant equals zero,

$$H \frac{U, H}{u, h} \big( h(UH)_h - u(UH)_u \big) = 0,$$

or equivalently,

$$h(UH)_h - u(UH)_u = 0, \tag{3.15}$$

in view of $H \neq 0$ (non-degeneracy) and $\dfrac{U, H}{u, h} \neq 0$.

The general solution of the equation (3.15) with respect to $(U, H)$ is $UH = G(t, x, \omega)$, where $\omega = uh$ and $G$ is a smooth function of its arguments. Then $U = G/H$. The condition $\dfrac{U, H}{u, h} \neq 0$ implies that $G_\omega \neq 0$.

After substituting these expressions in equation (3.6d) and collecting the coefficient of $B$ we get $H_u X_x - G_u T_x = 0$, so $H = G T_x / X_x + F$, where $F = F(t, x, h)$

is an arbitrary smooth function of its arguments. In view of equation (3.15), equation (3.6c) is a differential consequence of equations (3.6b) and (3.6d). We multiply equation (3.6d) by $2u$ and subtract the result from equation (3.6b), deriving

$$(uH_u - hH_h)X_x + (UH)_u T_t = 0.$$

After substituting the expression for $UH$ and $H$ we obtain

$$G_\omega T_t - X_x F_h = 0. \tag{3.16}$$

The differentiations of (3.16) with respect to $\omega$ and $h$ give, respectively, that $G_{\omega\omega} = 0$ and $F_{hh} = 0$. Thus,

$$G = G^1(t,x)\omega + G^0(t,x), \quad F = F^1(t,x)h + F^0(t,x),$$

where $G^1 = G_\omega \neq 0$ and $F^1 = G^1 T_t / X_x$ by equation (3.16). Now we are ready to substitute all of these expressions into equation (3.5b) and split it with respect to $\omega$, after excluding the common denominator. Then the coefficient of $\omega^3$ equals zero,

$$2(G^1)^4 \frac{T_x^5}{X_x^4} = 0,$$

where we denote $K := T_x/X_x$. Then $T_x = 0$ and $H = F^1(t,x)h + F^0(t,x)$. Thus, the equation (3.5b) reduces to

$$-(G^1F^1\omega h + 2G^1F^0\omega - G^0F^1h)X_x + G^1(G^1\omega + G^0)T_t h = 0,$$

which only implies that $F^0 = G^0 = 0$. Therefore, $H = H^1(t,x)h$ and $U = U^1(t,x)u$, where $H^1$ and $U^1$ are smooth functions of $(t,x)$ with $H^1U^1 \neq 0$. Moreover, $U^1 = X_x/T_t$. Under the derived conditions, the equation (3.5c) is equivalent to $-U^1X_x + H^1T_t = 0$ and implies that $H^1 = (X_x/T_t)^2$. We consider the reduced equation (3.6d),

$$\frac{H, X}{t, x} + \frac{T, UH}{t, x} = 0, \quad \text{i.e.,} \quad H^1_t X_x h + T_t (U^1H^1)_x uh = 0.$$

Then, $H^1_t = (U^1H^1)_x = 0$ and as consequence $T_{tt} = X_{xx} = 0$. Therefore,

$$X = X_1 x + X_0, \quad T = T_1 t + T_0, \quad U = \frac{X_1}{T_1}u, \quad H = \frac{X_1^2}{T_1^2}h.$$

After substituting these expressions into equation (3.5d), we obtain $\tilde{B} = X_1B/T_1^2$. This proves the following theorem.

**Theorem 31.** *The usual equivalence group $G^{\sim}$ of the class (3.2) consists of the transformations*

$$\tilde{t} = T_1t + T_0, \quad \tilde{x} = X_1x + X_0, \quad \tilde{u} = \frac{X_1}{T_1}u, \quad \tilde{h} = \frac{X_1^2}{T_1^2}h, \quad \tilde{B} = \frac{X_1}{T_1^2}B,$$

*where $X_1, X_0, T_1$ and $T_0$ are arbitrary constants with $T_1X_1 \neq 0$.*

**Remark 32.** In the beginning of this section, we reparametrize the class (3.1) by setting $B = b_x$. The $b$-components of equivalence transformations are obviously of the form

$$\tilde{b} = \frac{X_1^2}{T_1^2}b + B^0,$$

where $B^0$ is an arbitrary constant. The equivalence transformations associated with the group parameter $B^0$, $\tilde{t} = t$, $\tilde{x} = x$, $\tilde{u} = u$, $\tilde{h} = h$, $\tilde{b} = b + B^0$, are (trivial) *gauge equivalence transformations* for the class (3.1) and constitute the gauge equivalence group of this class since these and only these equivalence transformations act only on the arbitrary element $b$ and do not change each system of the class (3.1). See [35, 49] for related definitions. The gauge equivalence group is a normal subgroup of the group $G^{\sim}$.

**Remark 33.** The systems from the class (3.1) that are associated with the values of the arbitrary element that are linear functions of $x$, $b(x) = B^1x + B^0$, can be reduced to the systems of the same class with $b(x) = 0$,

$$u_t + uu_x + h_x = 0, \quad h_t + uh_x + u_xh = 0, \tag{3.17}$$

by the change of variables

$$\tilde{t} = t, \quad \tilde{x} = x - \tfrac{1}{2}B^1t^2, \quad \tilde{u} = u - B^1t, \quad \tilde{h} = h,$$

which was presented in [24]. This change of variables helps to simplify the study of symmetry properties and conservation laws of the one-dimensional shallow-water equations with linear bottom topography, see [24].

The system (3.17) can be linearized by the two-dimensional hodograph transformation, which exchanges the roles of dependent and independent variables. We set $y = u$, $z = h$, $p = t$, $q = x$, where $(y, z)$ are new independent variables, $(p, q)$ are new dependent variables, i.e., $p(u, h) = t$, $q(u, h) = x$. We differentiate these expressions with respect to $t$ and $x$ and get the following system of equations

$$p_y u_t + p_z h_t = 1, \quad q_y u_t + q_z h_t = 0,$$
$$p_y u_x + p_z h_x = 0, \quad q_y u_x + q_z h_x = 1.$$

We assume $K := p_y q_z - p_z q_y \neq 0$. Then the solution of this system is

$$u_t = \frac{q_z}{K}, \quad h_t = -\frac{q_y}{K}, \quad u_x = -\frac{p_z}{K}, \quad h_x = \frac{p_y}{K}.$$

Thus, the system (3.17) reduces to the linear system

$$q_z - y p_z + p_y = 0, \quad -q_y - z p_z + y p_y = 0$$

in the new variables. We differentiate the first and the second equation with respect $y$ and $z$, respectively, and subtract the second result from the first one. This leads to the differential consequence $p_{yy} = z p_{zz} + 2 p_z$, which can be mapped to the equation

$$p_{uu} = s^3 p_{ss}$$

by the point transformation $s = 1/z$. The last equation is the Tricomi equation, the symmetry analysis of which can be found in [15, 16].

## 3.2 Zero-order conservation laws of one-dimensional shallow-water equations

Consider a system $\mathcal{L}$ of one-dimensional shallow-water equations from the class (3.1),

$$u_t + u u_x + h_x = b_x,$$
$$h_t + u h_x + u_x h = 0.$$

The entries of the adjoint of Fréchet derivative of the left hand-side of this system are

$$(\mathsf{D}^*_{\mathcal{L}})_{11} = -\mathrm{D}_t - u\mathrm{D}_x, \quad (\mathsf{D}^*_{\mathcal{L}})_{12} = -h\mathrm{D}_x,$$
$$(\mathsf{D}^*_{\mathcal{L}})_{21} = -\mathrm{D}_x, \quad (\mathsf{D}^*_{\mathcal{L}})_{22} = -\mathrm{D}_t - u\mathrm{D}_x.$$

Each zero-order conservation law of the system $\mathcal{L}$ possesses a characteristic with components $\Lambda = \Lambda(t, x, u, h)$ and $\mathrm{M} = \mathrm{M}(t, x, u, h)$. By applying the necessary condition (2.14) to the characteristic $(\Lambda, \mathrm{M})$ and replacing $u_t$ and $h_t$ by $-uu_x - h_x + b_x$ and $-(uh)_x$ respectively, we derive the equations

$$-u\Lambda_x - \Lambda_t + \Lambda_u(h_x - b_x) + \Lambda_h u_x h - h(\mathrm{M}_x + \mathrm{M}_u u_x + \mathrm{M}_h h_x) = 0,$$
$$-(\Lambda_x + \Lambda_u u_x + \Lambda_h h_x) - \mathrm{M}_t - \mathrm{M}_x u + \mathrm{M}_u(h_x - b_x) + \mathrm{M}_h u_x h = 0.$$

We can then split these equations with respect to the derivatives $u_x$ and $h_x$ because $\Lambda$ and M do not depend on them, which respectively gives

$$u_x : \quad h\Lambda_h - h\mathrm{M}_u = 0,$$
$$h_x : \quad \Lambda_u - h\mathrm{M}_h = 0,$$
$$1 : \quad \Lambda_x u + M_x h + \Lambda_t + \Lambda_u b_x = 0$$

and

$$u_x : \quad -\Lambda_u + h\mathrm{M}_h = 0,$$
$$h_x : \quad -\Lambda_h + \mathrm{M}_u = 0,$$
$$1 : \quad \Lambda_x + u\mathrm{M}_x + \mathrm{M}_t + \mathrm{M}_u b_x = 0.$$

We rewrite this collection of equations as

$$R^1 := \Lambda_u - h\mathrm{M}_h = 0, \tag{3.18a}$$
$$R^2 := \Lambda_h - \mathrm{M}_u = 0, \tag{3.18b}$$
$$R^3 := \Lambda_t + u\Lambda_x + b_x\Lambda_u + h\mathrm{M}_x = 0, \tag{3.18c}$$
$$R^4 := \mathrm{M}_t + u\mathrm{M}_x + b_x\mathrm{M}_u + \Lambda_x = 0, \tag{3.18d}$$

which represents the complete system of determining equations for characteristics of zero-order conservation laws of systems from the class (3.1).

First of all, we derive some short nontrivial differential consequences of the system (3.18). In particular, we obtain the differential consequence

$$R^3_u - hR^4_h - (R^1_t + uR^1_x + b_xR^1_u) + hR^2_x = \Lambda_x = 0.$$

Taking it into account, we rewrite the system (3.18) in the form

$$\Lambda_x = 0, \quad \Lambda_u = h\mathrm{M}_h, \quad \Lambda_h = \mathrm{M}_u, \quad \Lambda_t = -h(\mathrm{M}_x + b_x\mathrm{M}_h), \tag{3.19a}$$

$$\tilde{R}^4 := \mathrm{M}_t + u\mathrm{M}_x + b_x\mathrm{M}_u = 0, \tag{3.19b}$$

Excluding $\Lambda$ by cross differentiation of equations (3.19a), we derive more equations for the single function M that are not differential consequences of (3.19c),

$$\mathrm{M}_{hx} = 0, \quad \mathrm{M}_{ux} = 0, \quad \mathrm{M}_{xx} + b_{xx}\mathrm{M}_h = 0, \quad h\mathrm{M}_{hh} + \mathrm{M}_h = \mathrm{M}_{uu}. \tag{3.19c}$$

Differentiating the second equation in (3.19c) separately with respect to $u$ and $h$ and combining the results with other equations in (3.19c) give the differential consequences

$$b_{xx}\mathrm{M}_{hh} = 0, \quad b_{xx}\mathrm{M}_{hu} = 0, \quad b_{xx}(\mathrm{M}_h - \mathrm{M}_{uu}) = 0. \tag{3.19d}$$

It is easy to find zero-order conservation-law characteristics that are common for all systems from the class (3.1). Looking for such characteristics, we can split the equations (3.18c) and (3.18d), which in particular gives $\Lambda_u = \mathrm{M}_u = 0$. In view of the equations (3.18a) and (3.18b), we also have $\Lambda_h = \mathrm{M}_h = 0$. Then the equations (3.18c) and (3.18d) can be further split with respect to $u$ and $h$ to

$$\Lambda_x = \Lambda_t = \mathrm{M}_x = \mathrm{M}_t = 0.$$

Therefore, a zero-order conservation-law characteristic is common for all systems from the class (3.1) if and only if its components are constants. The space of such characteristics is two-dimensional and is spanned by the characteristics $(\Lambda, \mathrm{M}) = (1, 0)$ and $(\Lambda, \mathrm{M}) = (0, 1)$.

Now we find families of zero-order conservation-law characteristics that are pointwise parametrized by the arbitrary element $b$, i.e., the components of these characteristics are of the form $\Lambda = \check{\Lambda}(t, x, u, b(x))$ and $\mathrm{M} = \check{\mathrm{M}}(t, x, u, b(x))$. It is obvious that such characteristics include the above common ones. Since $b$ is an arbitrary function of $x$ here, the substitution of such characteristics into the system (3.18) and its differential consequences should not lead to equations in $b$. This is why after expanding derivatives of $\Lambda$ and $\mathrm{M}$ with respect to $x$,

$$\Lambda_x = \check\Lambda_x + \check\Lambda_b b_x,$$
$$\mathrm{M}_x = \check{\mathrm{M}}_x + \check{\mathrm{M}}_b b_x, \quad \mathrm{M}_{xx} = \check{\mathrm{M}}_{xx} + 2\check{\mathrm{M}}_{xb} b_x + \check{\mathrm{M}}_{bb} b_x^2 + \check{\mathrm{M}}_b b_{xx},$$

we can split the resulting equations with respect to $b_x$ and $b_{xx}$. Thus, the equations (3.19b), (3.19c) and (3.19d) respectively imply

$$\check{\mathrm{M}}_t + u\check{\mathrm{M}}_x = 0, \quad u\check{\mathrm{M}}_b + \check{\mathrm{M}}_u = 0,$$
$$\check{\mathrm{M}}_{hx} = \check{\mathrm{M}}_{hb} = 0, \quad \check{\mathrm{M}}_{ux} = \check{\mathrm{M}}_{ub} = 0, \quad \check{\mathrm{M}}_{xx} = \check{\mathrm{M}}_{xb} = \check{\mathrm{M}}_{bb} = \check{\mathrm{M}}_b + \check{\mathrm{M}}_h = 0,$$
$$\check{\mathrm{M}}_{hh} = 0, \quad \check{\mathrm{M}}_{hu} = 0, \quad \check{\mathrm{M}}_h = \check{\mathrm{M}}_{uu}.$$

Then we get $(\check{\mathrm{M}}_t + u\check{\mathrm{M}}_x)_u = \check{\mathrm{M}}_x = 0$, and thus also $\check{\mathrm{M}}_t = 0$. The equations for $\check\Lambda$ follow from (3.19a), $\check\Lambda_x = \check\Lambda_b = 0$, $\check\Lambda_u = h\check{\mathrm{M}}_h$, $\check\Lambda_h = \check{\mathrm{M}}_u$, $\check\Lambda_t = 0$. The general solution of the derived system of determining equations for $(\check\Lambda, \check{\mathrm{M}})$ is

$$\Lambda = c_1 uh + c_2, \quad \mathrm{M} = c_1\left(\frac{u^2}{2} + h - b\right) + c_3, \tag{3.20}$$

where $c_1$, $c_2$ and $c_3$ are arbitrary constants. Therefore, any system from the class (3.1) possesses the conservation-law characteristics

$$(1, 0), \quad (0, 1), \quad \left(uh, \frac{u^2}{2} + h - b\right)$$

with the corresponding value of the arbitrary element $b$. The associated conserved currents are respectively

$$\left(u,\, \frac{u^2}{2} + h - b\right), \quad (h,\, uh), \quad \left(\frac{u^2 h}{2},\, \frac{u^3 h}{2} + uh^2 - uhb\right).$$

Let us classify possible extensions of the space of zero-order conservation laws for the system $\mathcal{L}$ depending on specific forms of $b$. The equations (3.19d) show that there are two cases that should be considered separately, $b_{xx} \neq 0$ and $b_{xx} = 0$.

$\boldsymbol{b_{xx} \neq 0}$**.** Then the equations (3.19d) imply $\mathrm{M}_{hh} = \mathrm{M}_{hu} = \mathrm{M}_h - \mathrm{M}_{uu} = 0$ and hence $\tilde{R}^4_h = \mathrm{M}_{th} = 0$. Integrating the joint system of these equations with the equations $\mathrm{M}_{hx} = 0$, $\mathrm{M}_{ux} = 0$ and $\mathrm{M}_{xx} + b_{xx}\mathrm{M}_h = 0$ from (3.19c), we obtain a preliminary expression for M,

$$\mathrm{M} = c_1\left(\frac{u^2}{2} + h - b\right) + \mu^2(t)u + \mu^1(t)x + \mu^0(t),$$

where $c_1$ is a constant and $\mu^0$, $\mu^1$ and $\mu^2$ are sufficiently smooth functions of $t$. The substitution of this expression into (3.19b) leads to the equation

$$\mu^2_t u + \mu^1_t x + \mu^0_t + \mu^1 u + \mu^2 b_x = 0, \tag{3.21}$$

which splits with respect to $u$ into $\mu^2_t + \mu^1 = 0$ and $\mu^1_t x + \mu^0_t + \mu^2 b_x = 0$. The system (3.19a) reduces to $\Lambda_x = 0$, $\Lambda_u = c_1 h$, $\Lambda_h = c_1 u + \mu^2$, $\Lambda_t = -\mu^1 h$ and thus integrates to

$$\Lambda = c_1 uh + \mu^2(t)h + c_2.$$

After differentiating the equation $\mu^1_t x + \mu^0_t + \mu^2 b_x = 0$ twice with respect to $x$, we get that $b_{xxx}\mu^2 = 0$.

If $b_{xxx} \neq 0$, then $\mu^2 = 0$, and the equation (3.21) splits with respect to $x$ and $u$ into $\mu^1 = 0$ and $\mu^0_t = 0$. As a result, there is no extension of the space of zero-order conservation laws in this case.

Therefore, $b_{xxx} = 0$, i.e., $b = \frac{1}{2}d_2x^2 + d_1x + d_0$ for some constants $d_2$, $d_1$ and $d_0$. Up to equivalence transformations of the class (3.1) (shifts of $x$ and $b$ and scalings), we can set $d_1 = d_0 = 0$ and $d_2 = \pm 1$. Then the equation (3.21) splits with respect to $x$ and $u$ into the system

$$\mu^2_t + \mu^1 = 0, \quad \mu^1_t + d_2\mu^2 = 0, \quad \mu^0_t = 0.$$

The general solution of this system is $\mu^0 = c_3$ and

$$\mu^2 = c_4e^t + c_5e^{-t}, \quad \mu^1 = -c_4e^t + c_5e^{-t} \quad \text{if} \quad d_2 = 1,$$
$$\mu^2 = c_4\cos t - c_5\sin t, \quad \mu^1 = c_4\sin t + c_5\cos t \quad \text{if} \quad d_2 = -1.$$

This leads to the following expressions for conservation-law characteristics:

$$\Lambda = c_1uh + (c_4e^t + c_5e^{-t})h + c_2,$$
$$\mathrm{M} = c_1\left(\frac{u^2}{2} + h - b\right) + (c_4e^t + c_5e^{-t})u - (c_4e^t - c_5e^{-t})x + c_3$$

if $d_2 = 1$ and

$$\Lambda = c_1uh + (c_4\cos t + c_5\sin t)h + c_2,$$
$$\mathrm{M} = c_1\left(\frac{u^2}{2} + h - b\right) + (c_4\cos t + c_5\sin t)u - (c_4\sin t - c_5\cos t)x + c_3$$

if $d_2 = -1$. Therefore, in comparison with the general value of $b$, the system with $b = \pm\frac{1}{2}x^2$ admits two more linearly independent zero-order conservation-law characteristics, which are associated with the constants $c_4$ and $c_5$ in the above expressions. The corresponding conserved currents are

$$\left(e^t(u-x)h,\ e^tu(u-x)h + e^t\frac{h^2}{2}\right), \quad \left(e^{-t}(u+x)h,\ e^{-t}u(u+x)h + e^{-t}\frac{h^2}{2}\right)$$

if $b(x) = x^2/2$ and

$$\left((u\cos t + x\sin t)h,\ u(u\cos t + x\sin t)h + \frac{h^2}{2}\cos t\right),$$
$$\left((u\sin t - x\cos t)h,\ u(u\sin t - x\cos t)h + \frac{h^2}{2}\sin t\right)$$

if $b(x) = -x^2/2$.

$\boldsymbol{b_{xx} = 0}$**.** In other words, $b = d_1x + d_0$ with constants $d_1$ and $d_0$. Here the constant $d_0$ can be set to 0 by gauge equivalence transformation of shifts with respect to $b$, and $d_1 \in \{0, 1\}$ mod $G^{\sim}$. Then we produce the following system of equations

$$\begin{gathered}\mathrm{M}_{xx} = 0, \quad \mathrm{M}_{hx} = 0, \quad \mathrm{M}_{ux} = 0, \quad \mathrm{M}_t + d_1\Lambda_h + u\mathrm{M}_x = 0,\\ \Lambda_x = 0, \quad \Lambda_t + h(d_1\mathrm{M}_h + \mathrm{M}_x) = 0, \quad \Lambda_h = \mathrm{M}_u, \quad \Lambda_u = h\mathrm{M}_h.\end{gathered} \tag{3.22}$$

The system (3.22) holds and has infinite-dimensional space of solutions, which means that the space of zero-order characteristics and the space of zero-order conservation laws are infinite-dimensional. By Remark 33, we can set $d_1 = 0$ by a point transformation, and the system (3.22) reduces to

$$\mathrm{M}_{xx} = 0, \quad \mathrm{M}_{hx} = 0, \quad \mathrm{M}_{ux} = 0, \quad \mathrm{M}_t + u\mathrm{M}_x = 0,$$
$$\Lambda_x = 0, \quad \Lambda_t + h\mathrm{M}_x = 0, \quad \Lambda_h = \mathrm{M}_u, \quad \Lambda_u = h\mathrm{M}_h.$$

We can solve this system and obtain $\Lambda = c_1 th + F(u,h)$, $\mathrm{M} = c_1(tu - x) + G(u,h)$, where $c_1$ is an arbitrary constant, $F$ and $G$ are smooth functions of $(u,h)$ that satisfy the system

$$F_h = G_u, \quad F_u = hG_h, \tag{3.23}$$

whose differential consequence is $G_{uu} - hG_{hh} - G_h = 0$. Therefore, if $b = d_1 x$, then the components of the corresponding conserved currents are of the form

$$\begin{aligned} Q &= c\big(tsh - xh + \tfrac{1}{2}d_1 t^2 h\big) + Q^1, \\ P &= c\big(ts^2 h + \tfrac{3}{2}d_1 t^2 hs + \tfrac{1}{2}th^2 - xhs - d_1 txh + \tfrac{1}{2}d_1^2 t^3 h\big) + d_1 tQ^1 + P^1, \end{aligned} \tag{3.24}$$

where $c$ is an arbitrary constant, $s := u - d_1 t$ and $(Q^1, P^1) = (Q^1, P^1)(s,h)$ is an arbitrary solution of the system

$$P^1_s = sQ^1_s + hQ^1_h,$$
$$P^1_h = Q^1_s + sQ^1_h.$$

Summing up, we prove the following theorem.

**Theorem 34.** *Each system of the class (3.1) for a fixed choice of the function $b$ admits the three-dimensional space of zero-order conservation laws that is spanned by*

$$\mathrm{D}_t(u) + \mathrm{D}_x(u^2/2 + h - b) = 0, \quad \mathrm{D}_t(h) + \mathrm{D}_x(uh) = 0,$$
$$\mathrm{D}_t\left(\frac{u^2 h}{2}\right) + \mathrm{D}_x\left(\frac{u^3 h}{2} + uh^2 - uhb\right) = 0,$$

*which are the conservation laws of momentum, mass and energy, respectively.*

*All $G^\sim$-inequivalent extensions of the space of zero-order conservation laws in the class (3.1) are exhausted by the following cases:*

*$b(x) = x^2/2$:*

$$\mathrm{D}_t\big(e^t(u-x)h\big) + \mathrm{D}_x\left(e^t u(u-x)h + e^t\frac{h^2}{2}\right) = 0,$$

$$\mathrm{D}_t\big(e^{-t}(u+x)h\big) + \mathrm{D}_x\left(e^{-t} u(u+x)h + e^{-t}\frac{h^2}{2}\right) = 0;$$

*$b(x) = -x^2/2$:*

$$\mathrm{D}_t\big((u\cos t + x\sin t)h\big) + \mathrm{D}_x\left(u(u\cos t + x\sin t)h + \frac{h^2}{2}\cos t\right) = 0,$$

$$\mathrm{D}_t\big((u\sin t - x\cos t)h\big) + \mathrm{D}_x\left(u(u\sin t - x\cos t)h + \frac{h^2}{2}\sin t\right) = 0;$$

*$b(x) = x$:*

$$\mathrm{D}_t\big(tsh - xh + \tfrac{1}{2}t^2h\big) + \mathrm{D}_x\big(ts^2h + \tfrac{3}{2}t^2sh + \tfrac{1}{2}th^2 - xsh - txh + \tfrac{1}{2}^2t^3h\big) = 0,$$

$$\mathrm{D}_t\big(Q^1\big) + \mathrm{D}_x\big(tQ^1 + P^1\big) = 0,$$

*where $s := u - t$ and $(Q^1, P^1) = (Q^1, P^1)(s, h)$ runs through the solution set of the system*

$$P^1_s = sQ^1_s + hQ^1_h, \quad P^1_h = Q^1_s + sQ^1_h;$$

*$b(x) = 0$:*

$$\mathrm{D}_t\big(tuh - xh\big) + \mathrm{D}_x\big(tu^2h + \tfrac{1}{2}th^2 - xuh\big) = 0,$$

$$\mathrm{D}_t\big(Q^1\big) + \mathrm{D}_x\big(P^1\big) = 0,$$

*where $(Q^1, P^1) = (Q^1, P^1)(u, h)$ runs through the solution set of the system*

$$P^1_u = uQ^1_u + hQ^1_h, \quad P^1_h = Q^1_u + uQ^1_h.$$

**Remark 35.** The case $b(x) = x$ is reduced to the case $b(x) = 0$ by the point transformation $\tilde{t} = t$, $\tilde{x} = x - \frac{1}{2}t^2$, $\tilde{u} = u - t$, $\tilde{h} = h$, which is an admissible transformation in the class (3.1), see Remark 33.

**Remark 36.** The systems from the class (3.1) with linear bottom topographies can be linearized using a hodograph transformation, see Remark 33. Exact solutions of system (3.23) were presented in [24]. The system (3.23) determines the range of zero-order conservation laws and, additionally, the set of non-degenerate exact solutions of the shallow-water equations (see [24]). All the degenerate solutions of the shallow-water equations are found in [24].

# Chapter 4

# Two-dimensional shallow-water equations

Let us consider the class of systems of two-dimensional shallow-water equations

$$
\begin{aligned}
&u_t + uu_x + vu_y + h_x = b_x, \\
&v_t + uv_x + vv_y + h_y = b_y, \\
&h_t + (uh)_x + (vh)_y = 0,
\end{aligned}
\tag{4.1}
$$

where $(u(t,x,y), v(t,x,y))$ is the horizontal fluid velocity averaged over the height of the fluid column, $h = h(t,x,y)$ is is the thickness of a fluid column and $b = b(x,y)$ is the bottom topography measured downward with respect to a fixed reference level and considered as the arbitrary element. We will refer to (4.1) as a class or a system depending on the context. Systems of equations from the class (4.1) were also intensively investigated within the framework of symmetry analysis of differential equations, but not for arbitrary values of the bottom topography. Special attention was paid for the flat bottom topography $b(x,y) = 0$ and for $b$ quadratic in $(x,y)$, i.e.,

$$
b(x,y) = Ax^2 + By^2,
$$

although it was explored in both resting and rotating reference frames, see e.g. [23, 34, 52].

## 4.1 Equivalence group of class of two-dimensional shallow-water equations

We compute the complete equivalence group of the class of two-dimensional shallow-water equations (4.1) by the algebraic method. This class is parametrized by the arbitrary element $b$, which satisfies the following system of auxiliary equations

$$b_t = 0, \quad b_u = 0, \quad b_v = 0, \quad b_h = 0, \quad b_{u_x} = 0, \quad b_{u_y} = 0,$$
$$b_{v_x} = 0, \quad b_{v_y} = 0, \quad b_{h_x} = 0, \quad b_{h_y} = 0,$$

meaning that $b$ does not depend on $u, v, h$ and its derivatives. This is why we can replace the observation the study of the class (4.1) by the consideration of the following single system

$$\mathcal{S}\colon \quad \begin{aligned} & u_t + uu_x + vu_y + h_x = b_x, \\ & v_t + uv_x + vv_y + h_y = b_y, \\ & h_t + (uh)_x + (vh)_y = 0, \\ & b_t = 0, \end{aligned} \tag{4.2}$$

where $b$ is assumed one more dependent variable.

The generalized equivalence group $G^{\sim}$ of the class (4.1) coincides with the complete point symmetry group $G$ of the system $\mathcal{S}$. Thus, we also have the similar relation between the infinitesimal counterparts. In other words, the generalized equivalence algebra $\mathfrak{g}^{\sim}$ of the class (4.1) can be identified with the maximal Lie invariance algebra $\mathfrak{g}$ of the system $\mathcal{S}$. Therefore, below we use the following strategy.

The maximal Lie invariance algebra $\mathfrak{g}$ of the system $\mathcal{S}$ consists of the infinitesimal generators of one-parameter Lie symmetry groups of the system (4.2), which are vector fields in the space with coordinates $(t, x, y, u, v, h)$,

$$\mathbf{v} = \tau\partial_t + \xi\partial_x + \eta\partial_y + \Phi\partial_u + \Psi\partial_v + \Xi\partial_h + \Omega\partial_b, \tag{4.3}$$

where the components $\tau$, $\xi$, $\eta$, $\Phi$, $\Psi$, $\Xi$ and $\Omega$ of $\mathbf{v}$ are smooth functions of $t$, $x$, $y$, $u$, $v$, $h$ and $b$.

The infinitesimal invariance criterion formulated in Theorem 11 requires that

$$\mathrm{pr}^{(1)}\mathbf{v}(u_t + uu_x + vu_y + h_x - b_x) = 0,$$
$$\mathrm{pr}^{(1)}\mathbf{v}(v_t + uv_x + vv_y + h_y - b_y) = 0,$$
$$\mathrm{pr}^{(1)}\mathbf{v}(h_t + (uh)_x + (vh)_y) = 0,$$
$$\mathrm{pr}^{(1)}\mathbf{v}(b_t) = 0,$$

whenever the system (4.2) holds. Here the first prolongation $\mathrm{pr}^{(1)}\mathbf{v}$ of $\mathbf{v}$ is defined by Theorem 12. Thus, the infinitesimal invariance criterion for the system (4.2) takes the form

$$\begin{aligned}
&\Phi^t + \Phi u_x + \Phi^x u + \Psi u_y + \Phi^y v + \Xi^x - \Omega^x = 0,\\
&\Psi^t + \Phi v_x + \Psi^x u + \Psi v_y + \Psi^y v + \Xi^y - \Omega^y = 0,\\
&\Xi^t + \Phi h_x + \Xi^x u + \Psi h_y + \Xi^y v + \Xi u_x + \Phi^x h + \Xi v_y + \Psi^y h = 0,\\
&\Omega^t = 0,
\end{aligned} \tag{4.4}$$

which should be satisfied whenever the system (4.2) holds. We substitute the expressions for the derivatives $u_t$, $v_t$, $h_t$ and $b_t$ in view of the system $\mathcal{S}$ into the conditions (4.4) and then split them with respect to the derivatives $u_x$, $v_x$, $h_x$, $u_y$, $v_y$, $h_y$, $b_x$ and $b_y$, which gives the system of determining equations for the components of $\mathbf{v}$. Integrating this system, we get the exact forms for $\tau$, $\xi$, $\eta$, $\Phi$, $\Psi$, $\Xi$ and $\Omega$,

$$\tau = c_1 + (c_5 - c_7)t, \quad \xi = c_2 + c_5 x + c_6 y, \quad \eta = c_3 + c_5 y - c_6 x,$$
$$\Phi = c_6 v + c_7 u, \quad \Psi = -c_6 u + c_7 v, \quad \Xi = 2c_7 h, \quad \Omega = c_4 + 2c_7 b,$$

where $c_1, \ldots, c_7$ are arbitrary real constants. Therefore, we have that the maximal Lie invariance algebra $\mathfrak{g}$ of the system (4.2) is spanned by the vector fields

$$\mathbf{v}_1 = \partial_t, \quad \mathbf{v}_2 = \partial_x, \quad \mathbf{v}_3 = \partial_y, \quad \mathbf{v}_4 = \partial_b,$$
$$\mathbf{v}_5 = t\partial_t + x\partial_x + y\partial_y, \quad \mathbf{v}_6 = -u\partial_v + v\partial_u - x\partial_y + y\partial_x,$$
$$\mathbf{v}_7 = 2b\partial_b + 2h\partial_h + u\partial_u + v\partial_v - t\partial_t.$$

The non-zero commutation relation between the basis elements of the algebra $\mathfrak{g}$ are exhausted (up to anti-commutativity) by the following ones

$$[\mathbf{v}_1,\mathbf{v}_5]=\mathbf{v}_1,\quad [\mathbf{v}_1,\mathbf{v}_7]=-\mathbf{v}_1,\quad [\mathbf{v}_2,\mathbf{v}_5]=\mathbf{v}_2,\quad [\mathbf{v}_2,\mathbf{v}_6]=-\mathbf{v}_3,$$
$$[\mathbf{v}_3,\mathbf{v}_5]=\mathbf{v}_3,\quad [\mathbf{v}_3,\mathbf{v}_6]=\mathbf{v}_2,\quad [\mathbf{v}_4,\mathbf{v}_7]=2\mathbf{v}_4,$$

i.e., the only nonzero structure constants of $\mathfrak{g}$ are, up to permutation of subscripts, $c^1_{15}=1$, $c^1_{17}=-1$, $c^2_{25}=1$, $c^3_{26}=-1$, $c^3_{35}=1$, $c^2_{36}=1$, $c^4_{47}=2$. We fix the basis $\mathcal{B}=(\mathbf{v}_1,\dots,\mathbf{v}_7)$ of $\mathfrak{g}$ and obtain the general form of automorphism matrices for $\mathfrak{g}$ by the formula (2.7). Hence, the automorphism group of $\mathfrak{g}$ consists of the linear operators on $\mathfrak{g}$ whose matrices are, in the basis $\mathcal{B}$, of the form

$$A=\begin{bmatrix} a_{11} & 0 & 0 & 0 & -a_{17} & 0 & a_{17}\\ 0 & a_{22} & -\varepsilon a_{32} & 0 & a_{25} & a_{26} & 0\\ 0 & a_{32} & \varepsilon a_{22} & 0 & \varepsilon a_{26} & -\varepsilon a_{25} & 0\\ 0 & 0 & 0 & a_{44} & 0 & 0 & a_{47}\\ 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \varepsilon & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.5}$$

where $a_{ij}\in\mathbb{R}$, $a_{11}a_{22}a_{44}\neq 0$ and $\varepsilon=\pm1$.

Elements of the complete point symmetry group $G$ of the system $\mathcal{S}$ are point transformations in the extended space with the coordinates $(t,x,y,u,v,h,b)$ and thus are of general form

$$\mathcal{T}\colon\quad (\tilde t,\tilde x,\tilde y,\tilde u,\tilde v,\tilde h,\tilde b)=(T,X,Y,U,V,H,B)(t,x,y,u,v,h,b),$$

where the Jacobian $|\partial(T,X,Y,U,V,H,B)/\partial(t,x,y,u,v,h,b)|$ of the right hand-side does not vanish.

For each point transformation $\mathcal{T}\in G$, the push-forward $\mathcal{T}_\star$ of vector fields in the space with coordinates $(t,x,y,u,v,h,b)$ by $\mathcal{T}$ induces an automorphism of $\mathfrak{g}$, i.e.,

$$\mathcal{T}_*\mathbf{v}_j=\sum_{i=1}^{7}a_{ij}\mathbf{v}_i,\quad j=1,\dots,7, \tag{4.6}$$

where the matrix $A = (a_{ij})_{i,j=1}^7$ is of the form (4.5). Now we write down the equalities (4.6) that correspond to the basis elements $\mathbf{v}_1, \ldots, \mathbf{v}_4$ of $\mathfrak{g}$,

$$T_t\partial_{\tilde t} + X_t\partial_{\tilde x} + Y_t\partial_{\tilde y} + U_t\partial_{\tilde u} + V_t\partial_{\tilde v} + H_t\partial_{\tilde h} + B_t\partial_{\tilde b} = a_{11}\partial_{\tilde t},$$

$$T_x\partial_{\tilde t} + X_x\partial_{\tilde x} + Y_x\partial_{\tilde y} + U_x\partial_{\tilde u} + V_x\partial_{\tilde v} + H_x\partial_{\tilde h} + B_x\partial_{\tilde b} = a_{22}\partial_{\tilde x} + a_{32}\partial_{\tilde y},$$

$$T_y\partial_{\tilde t} + X_y\partial_{\tilde x} + Y_y\partial_{\tilde y} + U_y\partial_{\tilde u} + V_y\partial_{\tilde v} + H_y\partial_{\tilde h} + B_y\partial_{\tilde b} = -\varepsilon a_{32}\partial_{\tilde x} + \varepsilon a_{22}\partial_{\tilde y},$$

$$T_b\partial_{\tilde t} + X_b\partial_{\tilde x} + Y_b\partial_{\tilde y} + U_b\partial_{\tilde u} + V_b\partial_{\tilde v} + H_b\partial_{\tilde h} + B_b\partial_{\tilde b} = a_{44}\partial_{\tilde b}.$$

Splitting these equalities component-wise, we derive that

- $T$ does not depend on $x$, $y$ and $b$, and $T_t = a_{11}$;
- $X$ does not depend on $t$ and $b$, and $X_x = a_{22}$, $X_y = -\varepsilon a_{32}$;
- $Y$ does not depend on $t$ and $b$, and $Y_x = a_{32}$, $Y_y = \varepsilon a_{22}$;
- $B$ does not depend on $t, x$ and $y$, and $B_b = a_{44}$;
- $U$, $V$ and $H$ do not depend on $t$, $x$, $y$ and $b$.

In view of these conditions, the equalities (4.6) corresponding to the basis elements $\mathbf{v}_5$, $\mathbf{v}_6$ and $\mathbf{v}_7$ of $\mathfrak{g}$ respectively take the form

$$\begin{aligned}&tT_t\partial_{\tilde t} + (X_x + yX_y)\partial_{\tilde x} + (xY_x + yY_y)\partial_{\tilde y}\\ &\qquad = -a_{17}\partial_{\tilde t} + a_{25}\partial_{\tilde x} - \varepsilon a_{26}\partial_{\tilde y} + T\partial_{\tilde t} + X\partial_{\tilde x} + Y\partial_{\tilde y},\end{aligned}$$

$$\begin{aligned}&(-uT_v + vT_u)\partial_{\tilde t} + (-uX_v + vX_u - xX_y + yX_x)\partial_{\tilde x}\\ &+ (-uY_v + vY_u - xY_y + yY_x)\partial_{\tilde y} + +(-uU_v + vU_u)\partial_{\tilde u} + (-uV_v + vV_u)\partial_{\tilde v}\\ &+ (-uH_v + vH_u)\partial_{\tilde h} + (-uB_v + vB_u)\partial_{\tilde b}\\ &\qquad = a_{26}\partial_{\tilde x} + \varepsilon a_{25}\partial_{\tilde y} - \varepsilon(-U\partial_{\tilde v} + V\partial_{\tilde u} - X\partial_{\tilde y} + Y\partial_{\tilde x}),\end{aligned}$$

$$\begin{aligned}&(2hT_h + uT_u + vT_v - tT_t)\partial_{\tilde t} + (2hX_h + uX_u + vX_v)\partial_{\tilde x} + (2hY_h + uY_u + vY_v)\partial_{\tilde y}\\ &+ (2hU_h + uU_u + vU_v)\partial_{\tilde u} + (2hV_h + uV_u + vV_v)\partial_{\tilde v} + (2hH_h + uH_u + vH_v)\partial\tilde h\\ &+ (2bB_b + 2hB_h + uB_u + vB_v)\partial_{\tilde b}\\ &\qquad = a_{17}\partial_{\tilde t} + a_{47}\partial_{\tilde b} + 2B\partial_{\tilde b} + 2H\partial_{\tilde h} + U\partial_{\tilde u} + V\partial_{\tilde v} - T\partial_{\tilde t}.\end{aligned}$$

Analogously splitting these equations component-wise, we find the exact form of $T$, $X$ and $Y$,

$$T = a_{11}t + a_{17}, \quad X = a_{22}x - \varepsilon a_{32}y - a_{25}, \quad Y = a_{32}x + \varepsilon a_{22}y - \varepsilon a_{26},$$

where $a_{11} \neq 0$, $a_{22}^2 + a_{32}^2 \neq 0$, and the determining equations for $U$, $V$, $H$ and $B$,

$$\begin{aligned}
&vU_u - uU_v = \varepsilon V, \quad 2hU_h + vU_u + vU_v = U,\\
&vV_u - uV_v = -\varepsilon U, \quad 2hV_h + uV_u + vV_v = V,\\
&vH_u - uH_v = 0, \quad 2hH_h + uH_u + vH_v = 2H,\\
&vB_u - uB_v = 0, \quad 2a_{44}b + 2hB_h + uB_u + vB_v = 2B + a_{47}.
\end{aligned}$$

The last four equations integrate with respect to $H$ and $B$ to

$$H = (u^2 + v^2)F\left(\frac{h}{u^2 + v^2}\right), \quad B = (u^2 + v^2)G\left(\frac{h}{u^2 + v^2}, b\right) + a_{44}b - \frac{a_{47}}{2},$$

where $F$ and $G$ are arbitrary smooth functions of their arguments.

The above forms and equations for the components of equivalence transformations, which have been derived within the algebraic method, are further used to continue computations within the direct method and to get the precise, more strictly form of equivalence transformation components. For this purpose, we compute the operators $\partial_{\tilde{t}}$, $\partial_{\tilde{x}}$ and $\partial_{\tilde{y}}$ in terms of the old partial derivative operators

$$\partial_{\tilde{t}} = \frac{1}{a_{11}}\partial_t, \quad \partial_{\tilde{x}} = \frac{1}{K}(-\varepsilon a_{22}\partial_x + a_{32}\partial_y), \quad \partial_{\tilde{y}} = \frac{1}{K}(-\varepsilon a_{32}\partial_x - a_{22}\partial_y),$$

where $K = \varepsilon(a_{22}^2 + a_{32}^2)$, to express all required transformed derivatives in terms of the initial coordinates.

Then we substitute the obtained expressions into the system $\mathcal{S}$ written in the new variables $(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}, \tilde{h}, \tilde{b})$,

$$\tilde{\mathcal{S}}\colon \quad \begin{aligned}
&\tilde{u}_{\tilde{t}} + \tilde{u}\tilde{u}_{\tilde{x}} + \tilde{v}\tilde{u}_{\tilde{y}} + \tilde{h}_{\tilde{x}} = \tilde{b}_{\tilde{x}},\\
&\tilde{v}_{\tilde{t}} + \tilde{u}\tilde{v}_{\tilde{x}} + \tilde{v}\tilde{v}_{\tilde{y}} + \tilde{h}_{\tilde{y}} = \tilde{b}_{\tilde{y}},\\
&\tilde{h}_{\tilde{t}} + (\tilde{u}\tilde{h})_{\tilde{x}} + (\tilde{v}\tilde{h})_{\tilde{y}} = 0,\\
&\tilde{b}_{\tilde{t}} = 0.
\end{aligned} \tag{4.7}$$

Each solution of system (4.2) should identically satisfy the expanded equations in (4.7). We substitute expressions for derivatives of $u$, $v$ and $h$ with respect to $t$ in view of the system $\mathcal{S}$ into the expanded equations (4.7) and then collect the coefficients of the first-order derivatives of $u, v, h$ and $b$ with respect to $x$ and $y$ in the resulting equations. First of all, we expand the third equation of the system (4.7). Collecting the coefficients of $b_x$, $u_y$ and $v_x$ respectively gives

$$F' = \frac{u^2+v^2}{h}F, \quad a_{32}U_u - a_{22}V_u = 0, \quad a_{22}U_v + a_{32}V_v = 0,$$

where $F'$ denotes the derivative of $F$ with respect to its argument.

We combine these equations for $U$ and $V$ with form for them constructed in the course of applying the algebraic method. As a result, we get

$$U = \frac{1}{a_{11}}\left(a_{22}u - \varepsilon a_{32}v\right), \qquad V = \frac{1}{a_{11}}\left(a_{32}u + \varepsilon a_{22}v\right).$$

Further, expanding the fourth equation of the system (4.7) gives that $G = 0$, and thus $B = a_{44}b - a_{47}/2$. Additionally, expanding of the first equation leads to

$$F = \frac{a_{22}^2 + a_{32}^2}{a_{11}^2}\frac{h}{u^2+v^2}, \quad \text{i.e.,} \quad H = \frac{a_{22}^2 + a_{32}^2}{a_{11}^2}h \quad \text{and} \quad a_{44} = \frac{a_{22}^2 + a_{32}^2}{a_{11}^2}.$$

Summing up, we prove the following theorem.

**Theorem 37.** *The generalized equivalence group $G^{\sim}$ of the class (4.1) consists of the point transformations in the space with the coordinates $(t, x, y, u, v, h, b)$ whose components are of the form*

$$\begin{aligned}
&\tilde t = T_1 t + T_0, \quad \tilde x = X_1 x - \varepsilon Y_1 y + X_0 \quad \tilde y = Y_1 x + \varepsilon X_1 y + Y_0,\\
&\tilde u = \frac{1}{T_1}(X_1 u - \varepsilon Y_1 v), \quad \tilde v = \frac{1}{T_1}(Y_1 u + \varepsilon X_1 v), \quad \tilde h = \frac{1}{T_1^2}(X_1^2 + Y_1^2)h,\\
&\tilde b = \frac{1}{T_1^2}(X_1^2 + Y_1^2)b + B_0, \quad \varepsilon = \pm 1,
\end{aligned}$$

*where $T_1$, $T_0$, $X_1$, $X_0$, $Y_1$, $Y_0$ and $B_0$ are arbitrary constants with $T_1(X_1^2 + Y_1^2) \neq 0$.*

Since the components $T$, $X$, $Y$, $U$, $V$ and $H$ of elements of the generalized equivalence group $G^{\sim}$ do not depend on $b$, these elements are in fact usual equivalence transformations, which implies the following corollary.

**Corollary 38.** *The generalized equivalence group $G^{\sim}$ of the class (4.1) coincides with the usual equivalence group of this class.*

## 4.2 Zero-order conservation laws of two-dimensional shallow-water equations

Consider the class $\mathcal{L}$ of two-dimensional shallow-water equations

$$\begin{aligned} &u_t + uu_x + vu_y + h_x = b_x,\\ &v_t + uv_x + vv_y + h_y = b_y,\\ &h_t + uh_x + vh_y + h(u_x + v_y) = 0. \end{aligned} \tag{4.8}$$

Let $(\Lambda, \mathrm{M}, \mathrm{K})$ be a CL-characteristic of order not greater than zero, whose components depend on $t$, $x$, $y$, $u$, $v$ and $h$. First of all, we write the expressions of the adjoint of the Fréchet derivative for the system (4.8):

$$(\mathsf{D}^*_{\mathcal{L}})_{11} = -\mathrm{D}_t - u\mathrm{D}_x - v\mathrm{D}_y - v_y, \quad (\mathsf{D}^*_{\mathcal{L}})_{12} = v_x, \quad (\mathsf{D}^*_{\mathcal{L}})_{13} = -h\mathrm{D}_x,$$

$$(\mathsf{D}^*_{\mathcal{L}})_{21} = u_y, \quad (\mathsf{D}^*_{\mathcal{L}})_{22} = -\mathrm{D}_t - u\mathrm{D}_x - v\mathrm{D}_y - u_x, \quad (\mathsf{D}^*_{\mathcal{L}})_{23} = -h\mathrm{D}_y,$$

$$(\mathsf{D}^*_{\mathcal{L}})_{31} = -\mathrm{D}_x, \quad (\mathsf{D}^*_{\mathcal{L}})_{32} = -\mathrm{D}_y, \quad (\mathsf{D}^*_{\mathcal{L}})_{33} = -\mathrm{D}_t - u\mathrm{D}_x - v\mathrm{D}_y.$$

Thus, we are ready to apply the necessary condition (2.14) to the tuple $(\Lambda, \mathrm{M}, \mathrm{K})$, which has to vanish on the solutions of the system (4.8). Since $\Lambda$, $\mathrm{M}$, $\mathrm{K}$ do not depend on the derivatives of $u, v$ and $h$, we can split the derived equation with respect to the first-order derivatives of $u, v$ and $h$,

$$\Lambda_v = 0, \quad \mathrm{M}_u = 0, \quad \Lambda_u = \mathrm{M}_v, \quad \Lambda_h = \mathrm{K}_u, \quad \mathrm{M}_h = \mathrm{K}_v, \tag{4.9}$$

$$\Lambda_u = \mathrm{K}_h h, \quad \Lambda = \Lambda_h h, \quad \mathrm{M} = \mathrm{M}_h h, \tag{4.10}$$

$$\Lambda_t + \Lambda_x u + \Lambda_y v + \Lambda_u b_x + \mathrm{K}_x h = 0, \tag{4.11}$$

$$\mathrm{M}_t + \mathrm{M}_x u + \mathrm{M}_y v + \mathrm{M}_v b_y + \mathrm{K}_y h = 0, \tag{4.12}$$

$$\mathrm{K}_t + \mathrm{K}_x u + \mathrm{K}_y v + \mathrm{K}_u b_x + \mathrm{K}_v b_y + \Lambda_x + \mathrm{M}_y = 0. \tag{4.13}$$

One can obtain from the equations (4.9) and (4.10) that

$$\begin{aligned} &\Lambda = \nu(t,x,y)uh + \Lambda^0(t,x,y)h, \\ &\mathrm{M} = \nu(t,x,y)vh + \mathrm{M}^0(t,x,y)h, \\ &\mathrm{K} = \nu(t,x,y)\left(\frac{u^2+v^2}{2} + h\right) + \Lambda^0(t,x,y)u + \mathrm{M}^0(t,x,y)v + \mathrm{K}^0(t,x,y), \end{aligned} \tag{4.14}$$

Substituting the expressions (4.14) for $\Lambda$, M and K into the equations (4.11)–(4.13), we can split the derived equations with respect to $u, v$ and $h$, giving

$$\nu_x = \nu_y = 0, \tag{4.15}$$

$$\nu_t + 2\Lambda^0_x = 0, \Lambda^0_y = -\mathrm{M}^0_x, \quad \Lambda^0_x = \mathrm{M}^0_y, \tag{4.16}$$

$$\mathrm{K}^0_x + \Lambda^0_t + \nu b_x = 0, \quad \mathrm{K}^0_y + \mathrm{M}^0_t + \nu b_y = 0, \quad \mathrm{K}^0_t + \Lambda^0 b_x + \mathrm{M}^0 b_y = 0. \tag{4.17}$$

As differential consequences, we also get the following conditions on $\Lambda^0$ and $\mathrm{M}^0$:

$$\begin{aligned} &\Lambda^0_{ty} = \Lambda^0_{xx} = \Lambda^0_{xy} = \Lambda^0_{yy} = 0, \\ &\mathrm{M}^0_{tx} = \mathrm{M}^0_{xx} = \mathrm{M}^0_{xy} = \mathrm{M}^0_{yy} = 0. \end{aligned}$$

As a result, we obtain that

$$\begin{aligned} &\nu = \nu(t), \\ &\Lambda^0 = A(t)x + By + C(t), \\ &\mathrm{M}^0 = -Bx + A(t)y + D(t), \end{aligned}$$

where $A(t), C(t)$ and $D(t)$ are smooth functions of $t$ and $B$ is an arbitrary real constant. The integration the first two equations of (4.17) with respect to $K^0$ gives

$$\mathrm{K}^0 = -\nu(t)b - A_t\frac{x^2+y^2}{2} - C_t x - D_t y + \mathrm{K}^{00}(t),$$

where $\nu_t = -2A$ in view of equation (4.16). The last equation of (4.17) implies the only classifying equation,

$$\begin{aligned} &(Ax + By + C)b_x + (-Bx + Ay + D)b_y \\ &\quad + 2Ab - A_{tt}\frac{x^2+y^2}{2} - C_{tt}x - D_{tt}y + \mathrm{K}^{00}_t = 0. \end{aligned} \tag{4.18}$$

To solve this classifying condition with respect to both the arbitrary element $b$ and the functions parametrizing CL-characteristics, we use the method of furcate splitting. This method first appeared in the cause of solving the group classification problem for classes of nonlinear Schrödinger equations and of nonlinear diffusion–convection equations in [37,47], see also [44]. The method is based on the construction of template-from equations for the corresponding arbitrary elements via fixing the variables that are not involved in the arbitrary elements and then considering the template-form equations instead of the initial classifying equations. Studying possible cases of the number of independent template-form equations, we solve them with respect to the arbitrary elements and simultaneously find unknown values in the objects under consideration (symmetries, conservation laws, etc).

Here we can fix the various values of the variable $t$ in the classifying equation (4.18). One derives the template form of equation for $b$,

$$\begin{aligned}(a_2x+a_1y+a_3)b_x+(-a_1x+a_2y+a_4)b_y\\ +2a_2b+a_5(x^2+y^2)+a_6x+a_7y+a_8=0,\end{aligned} \tag{4.19}$$

where $a_1,\dots,a_8$ are constants.

Denote by $k$ the number of linearly independent tuples $(a_1,\dots,a_8)$ among those associated with template-form equations for $b$. For an arbitrary $k$ the corresponding template form (4.19) can be rewritten as

$$(a^i_j)^{i=1,\dots,k}_{j=1\dots,4}\cdot S=(a^i_5(x^2+y^2)+a^i_6x+a^i_7y+a^i_8)^k_{i=1}, \tag{4.20}$$

where $S=(xb_x+yb_y+2b,yb_x-xb_y,b_x,b_y)^\mathsf{T}$. Let us note that $0\le k\le 4$.

**Case $\boldsymbol{k=0}$**. The value $k=0$ corresponds to the general value of the arbitrary element $b$ for which the space of conservation laws the common conservation laws of a system of the class (4.8) and to the families of conservation laws of this system parametrized by arbitrary $b$. In other words, after substituting the corresponding CL-characteristics into equation (4.18), we get an identity. That is, the equation (4.18) can be split with respect to $b$ and its derivatives. Thus,

$$A=B=C=D=0,\quad \mathrm{K}^{00}=c_1,$$

moreover, it follows from above that $\nu = c_2 \in \mathbb{R}$. Therefore, $\mathrm{K}^0 = c_1 - c_2 b$. Finally, we get a linearly independent common conservation-law characteristics $W_1$ and a family of CL-characteristics $W_2$ parametrized by $b$,

$$W_1 = (0,0,1), \qquad W_2 = (uh, vh, (u^2+v^2)/2 + h - b). \tag{4.21}$$

**Case $\boldsymbol{k = 1}$**. If $k = 1$, then the following equation identically holds

$$\begin{aligned}
&(Ax + By + C)b_x + (-Bx + Ay + D)b_y \\
&\qquad + 2Ab - A_{tt}\frac{x^2+y^2}{2} - C_{tt}x - D_{tt}y + \mathrm{K}^{00}_t \\
&= \lambda(t)\Big((a_2x + a_1y + a_3)b_x + (-a_1x + a_2y + a_4)b_y \\
&\qquad + 2a_2b + a_5(x^2+y^2) + a_6x + a_7y + a_8\Big),
\end{aligned}$$

where $\lambda$ is a nonvanishing smooth function of $t$. Comparing coefficients of the left- and right-hand sides of this equation, we obtain

$$\begin{aligned}
&a_1\lambda(t) = B, \quad a_2\lambda(t) = A(t), \quad a_3\lambda(t) = C(t), \quad a_4\lambda(t) = D(t), \\
&a_5\lambda(t) = -A_{tt}/2, \quad a_6\lambda(t) = -C_{tt}, \quad a_7\lambda(t) = -D_{tt}, \quad a_8\lambda(t) = \mathrm{K}^{00}_t.
\end{aligned}$$

If $a_1 \neq 0$, then we can choose $a_1 = 1$, then $\lambda(t) = B \in \mathbb{R}$, $A(t) = a_2B$, $C(t) = a_3B$, $D(t) = a_4B$, $\mathrm{K}^{00}(t) = a_8Bt + \mathrm{K}^{000}$ and $a_5 = a_6 = a_7 = 0$.

Here and in the following we will use the point equivalence transformations to obtain $b(x,y)$ in the simplest form. Moreover, for simplicity, we will use the same notion for constants $a_i$, even if they are changed.

For the case $a_1 \neq 0$, using equivalence transformations of shifts with respect to $x$ and $y$, we can rewrite equation (4.19) as

$$(a_2x + y)b_x + (-x + a_2y)b_y + 2a_2b + a_8 = 0.$$

If additionally $a_2 \neq 0$, then

$$b(x,y) = -\frac{a_8}{2a_2} + \frac{1}{x^2+y^2}F\left(\arctan\frac{y}{x} + \frac{\ln\sqrt{x^2+y^2}}{a_2}\right).$$

This case admits only two conservation-law characteristics, which coincide with (4.21). If $a_2 = 0$, then

$$b(x, y) = a_8 \arctan y/x + F(\sqrt{x^2 + y^2}).$$

If $a_1 = 0$, $a_2 \neq 0$, let us choose $a_2 = 1$, then $\lambda(t) = A(t)$, $B = 0$, $C(t) = a_3 A(t)$, $D(t) = a_4 A(t)$, $K^{00}_t = a_8 A(t)$, where $A(t)$ satisfies the equation $A_{tt} + 2a_5 A = 0$.

If $a_5 = 0$, then $A(t) = c_1 t + c_2$, $\mathrm{K}^{00}(t) = a_8(c_1 t^2/2 + c_2 t) + \mathrm{K}^{000}$, where $c_1, c_2, \mathrm{K}^{000} \in \mathbb{R}$ and $a_6 = a_7 = 0$. In this case we have the equation $xb_x + yb_y + 2b + a_8 = 0$, and $b(x, y) = -a_8/2 + F(y/x)/x^2$.

If $a_5 < 0$, then $A(t) = c_1 e^{\sqrt{-2a_5}t} + c_2 e^{-\sqrt{-2a_5}t}$, $\mathrm{K}^{00}(t) = c_1 e^{\sqrt{-2a_5}t} a_8/\sqrt{-2a_5} - c_2 e^{-\sqrt{-2a_5}t} a_8/\sqrt{-2a_5} + \mathrm{K}^{000}$, where $c_1, c_2, \mathrm{K}^{000} \in \mathbb{R}$ and $a_6 = 2a_3 a_5, a_7 = 2a_4 a_5$.

If $a_5 > 0$, then $A(t) = c_1 \cos\sqrt{2a_5}t + c_2 \sin\sqrt{2a_5}t$, $\mathrm{K}^{00}(t) = (c_1 \sin\sqrt{2a_5}t - c_2 \cos\sqrt{2a_5}t) a_8/\sqrt{2a_5} + \mathrm{K}^{000}$, where $c_1, c_2, \mathrm{K}^{000} \in \mathbb{R}$ and $a_6 = 2a_3 a_5, a_7 = 2a_4 a_5$.

In the last two cases the equation (4.19) can be rewritten as $xb_x + yb_y + 2b + a_5(x^2 + y^2) + a_8 - a_5(a_3^2 + a_4^2) = 0$ and $b(x, y) = -a_5(x^2 + y^2)/4 - (a_8 - a_5(a_3^2 + a_4^2))/2 + F(y/x)/x^2$.

If $a_1 = a_2 = 0$ and $(a_3, a_4) \neq (0, 0)$, we can choose $a_3 = 1$, $a_4 = 0$. Thus, $A = B = 0$, $a_5 = 0$, $\lambda(t) = C(t)$, $D(t) = a_4 C(t)$ and $C_{tt} + a_6 C = 0$.

If $a_6 = 0$, then $C(t) = c_1 t + c_2$, $\mathrm{K}^{00}(t) = (c_1 t^2/2 + c_2 t) a_8 + \mathrm{K}^{000}$, where $c_1, c_2, \mathrm{K}^{000} \in \mathbb{R}$ and $a_5 = a_7 = 0$. Hence, $b(x, y) = -a_8 x + b^0(y)$.

If $a_6 < 0$, then $C(t) = c_1 e^{\sqrt{-a_6}t} + c_2 e^{-\sqrt{-a_6}t}$, $\mathrm{K}^{00}(t) = (c_1 e^{\sqrt{-a_6}t} - c_2 e^{-\sqrt{-a_6}t}) \cdot a_8/\sqrt{-a_6} + K^{000}$, where $c_1, c_2, \mathrm{K}^{000} \in \mathbb{R}$ and $a_7 = a_4 a_6$.

If $a_6 > 0$, then $C(t) = c_1 \cos\sqrt{a_6}t + c_2 \sin\sqrt{a_6}t$, $\mathrm{K}^{00}(t) = c_1 \sin\sqrt{a_6}t a_8/\sqrt{a_6} - c_2 \cos\sqrt{a_6}t a_8/\sqrt{a_6} + \mathrm{K}^{000}$, where $c_1, c_2, \mathrm{K}^{000} \in \mathbb{R}$ and $a_7 = a_4 a_6$. For the last two cases $b(x, y) = -a_6 x^2/2 - a_8 x + b^0(y)$.

Finally, we obtain all $G^{\sim}$-inequivalent forms of the arbitrary element $b(x, y)$ for the case $k = 1$ for which the corresponding systems from the class (4.8) admit spaces of CL-characteristics of dimension greater than two. Below we list these values of $b$ jointly with specific linearly independent conservation-law characteristics $Z_i$, $i \geq 1$.

- $b(x, y) = \arctan y/x + F(\sqrt{x^2 + y^2})$:
  $Z_1 = (yh,\ -xh,\ yu - xv + t)$.

- $b(x,y) = d_1(x^2+y^2) + F(y/x)/x^2,\;\; d_1 > 0,\;\; d_1 = 1/2 \bmod G^{\sim}$:

$$Z_1 = (-e^{2t}uh + e^{2t}xh,\; -e^{2t}vh + e^{2t}yh,$$
$$-e^{2t}((u^2+v^2)/2 + h - b) + e^{2t}(xu+yv) - 2e^{2t}(x^2+y^2)),$$
$$Z_2 = e^{-2t}uh + e^{-2t}xh,\; e^{-2t}vh + e^{-2t}yh,$$
$$e^{-2t}((u^2+v^2)/2 + h - b) + e^{-2t}(xu+yv) + 2e^{-2t}(x^2+y^2)).$$

- $b(x,y) = d_1(x^2+y^2) + F(y/x)/x^2,\;\; d_1 = 0$:

$$Z_1 = (-t^2uh+txh,\; -t^2vh+tyh,\; -t^2((u^2+v^2)/2+h-b)+t(xu+yv)-(x^2+y^2)/2),$$
$$Z_2 = (-2tuh + xh,\; -2tvh + yh,\; -2t((u^2+v^2)/2 + h - b) + xu + yv).$$

- $b(x,y) = d_1(x^2+y^2) + F(y/x)/x^2,\;\; d_1 < 0,\;\; d_1 = -1/2 \bmod G^{\sim}$:

$$Z_1 = (-uh\sin 2t + xh\cos 2t,\; -vh\sin 2t + yh\cos 2t,$$
$$-((u^2+v^2)/2 + h - b)\sin 2t + (xu+yv)\cos 2t + 2(x^2+y^2)\sin 2t),$$
$$Z_2 = (uh\cos 2t + xh\sin 2t,\; vh\cos 2t + yh\sin 2t,$$
$$((u^2+v^2)/2 + h - b)\cos 2t + (xu+yv)\sin 2t - 2(x^2+y^2)\cos 2t).$$

- $b(x,y) = d_1x^2 + d_2x + F(y),\;\; d_1 > 0,\;\; d_1 = 1/2 \bmod G^{\sim},\;\; d_2 = 0 \bmod G^{\sim}$:

$$Z_1 = (e^th,\, 0,\, e^t(u-x)),\quad Z_2 = (e^{-t}h,\, 0,\, e^{-t}(u+x)).$$

- $b(x,y) = d_1x^2 + d_2x + F(y),\; d_1 < 0,\;\; d_1 = -1/2 \bmod G^{\sim},\;\; d_2 = 0 \bmod G^{\sim}$:

$$Z_1 = (h\cos t,\, 0,\, u\cos t + x\sin t),\quad Z_2 = (h\sin t,\, 0,\, u\sin t - x\cos t).$$

- $b(x,y) = d_1x^2 + d_2x + F(y),\;\; d_1 = 0,\;\; d_2 = 1 \bmod G^{\sim}$:

$$Z_1 = (th,\, 0,\, tu - x - t^2/2),\quad Z_2 = (h,\, 0,\, u - t),$$

- $b(x,y) = d_1x^2 + d_2x + F(y),\;\; d_1 = d_2 = 0$:

$$Z_1 = (th,\, 0,\, tu - x),\quad Z_2 = (h,\, 0,\, u).$$

**Case $\boldsymbol{k = 2}$.** In the case of $k = 2$ we consider the $2\times 2$ matrix $P = (a^i_j)^2_{i,j=1}$, with $\operatorname{rank} P \le 2$. If $\operatorname{rank} P = 2$, then without loss of generality, this matrix can be considered as the identity matrix. Using shifts of $x, y$ and $b$ we can simplify the system (4.20) to the form

$$xb_x + yb_y + 2b + a^1_5(x^2+y^2) + a^1_6x + a^1_7y = 0,$$
$$(y + a^2_3)b_x + (-x + a^2_4)b_y + a^2_5(x^2+y^2) + a^2_6x + a^2_7y + a^2_8 = 0,$$

where for the simplicity we denote the new coefficients in the same way. Let us consider the two corresponding vector fields

$$x\partial_x + y\partial_y - (2b + a_5^1(x^2+y^2) + a_6^1 x + a_7^1 y)\partial_b,$$
$$(y + a_3^2)\partial_x + (-x + a_4^2)\partial_y - (a_5^2(x^2+y^2) + a_6^2 x + a_7^2 y + a_8^2)\partial_b.$$

Since the commutator of these vector fields has to be equal zero, we get the following conditions

$$a_4^1 = a_4^2 = a_5^2 = a_8^2 = 0, \quad a_7^1 = -3a_6^2, \quad a_6^1 = 3a_7^2.$$

We can find the conditions on $a_6^1$ and $a_6^2$ by solving the equation

$$\begin{aligned} &2Ab - A_{tt}(x^2+y^2)/2 - C_{tt}x - D_{tt}y + \mathrm{K}_t^{00} \\ &\qquad + (Ax + By + C)b_x + (-Bx + Ay + D)b_y \\ &= \sum_{i=1}^{2} \lambda^i(t)\Big(a_1^i(xb_x + yb_y + 2b) + a_2^i(yb_x - xb_y) + a_3^i b_x + a_4^i b_y \\ &\qquad + a_5^i(x^2+y^2) + a_6^i x + a_7^i y + a_8^i\Big), \end{aligned}$$

where $\lambda^1$ and $\lambda^2$ are nonvanishing smooth functions of $t$. We can split this equation with respect to all the powers of $x$ and $y$ and obtain

$$\lambda^1(t) = A(t), \quad \lambda^2(t) = B, \quad C(t) = D(t) = 0, \quad \mathrm{K}^{00} = c,$$
$$A_{tt} + 2a_5^1 A(t) = 0.$$

Thus, system (4.20) can be rewritten as

$$xb_x + yb_y + 2b + a_5^1(x^2+y^2) = 0, \quad yb_x - xb_y = 0.$$

Changing the variables to polar coordinates $(r, \phi)$ gives us the system

$$rb_r + 2b + a_5^1 r^2 = 0, \quad b_\phi = 0,$$

which leads to the result $b(r,\phi) = -a_5^1 r^2/4 + c/r^2, c \in \mathbb{R}$ and

$$b(x,y) = -a_5^1(x^2+y^2) + \frac{c}{x^2+y^2}, \quad c \in \mathbb{R}.$$

If rank $P=1$, then without loss of generality we can choose $a^1_1=1, a^1_2=a^2_1=a^2_2=0$, thus by the certain turn and shifts of $x, y$ and $b$ we can take $(a^2_3, a^2_4)=(1,0)$, $a^1_3=a^1_4=a^1_8=0$, and the system (4.20) has the following form

$$
\begin{aligned}
&xb_x+yb_y+2b+a^1_5(x^2+y^2)+a^1_6x+a^1_7y=0,\\
&b_x+a^2_5(x^2+y^2)+a^2_6x+a^2_7y+a^2_8=0.
\end{aligned}
$$

The commutator of the two corresponding vector fields has to be equal to the vector fields of the second equation multiplied by an arbitrary function $\alpha(t)$. Thus, we get $\alpha(t)=-1$ which means, that the following conditions hold

$$
a^2_5=a^2_7=0, \quad a^1_5=2a^2_6, \quad a^1_6=3a^2_8.
$$

Thus, from the above system we obtain

$$
b(x,y)=\frac{1}{2}(-a^2_6(x^2+y^2))-\frac{a^1_7y}{3}-a^2_8x+\frac{c}{y^2}, \quad c\in\mathbb{R}.
$$

Consider $b(x,y)=d_1(x^2+y^2)+d_2x+d_3y+d_4y^{-2}$, where $d_1, d_2, d_3$ and $d_4$ are arbitrary constants. If $d_4=0$, then this case coincides with the case $k=4$, which is presented below. If $d_4\neq 0$, $d_1>0$, $d_1=1/2 \mod G^{\sim}$, $d_2=0 \mod G^{\sim}$, then we obtain two conservation law characteristics (apart from the characteristics $W_1$ and $W_2$),

$$
Z_1=(e^th,0,e^t(u-x)), \qquad Z_2=(e^{-t}h,0,e^{-t}(u+x)).
$$

This case coincides with the case $b(x,y)=x^2+F(y)$ $(k=1)$. If $d_4\neq 0$, $d_1>0$, $d_1=-1/2 \mod G^{\sim}$, $d_2=0 \mod G^{\sim}$, then

$$
Z_1=(h\cos t,0,u\cos t+x\sin t), \qquad Z_2=(h\sin t,0,u\sin t-x\cos t),
$$

that coincides with the case $b(x,y)=-x^2+F(y)$ $(k=1)$.

If rank $P=0$, then we define the matrix $P_1=(a^i_j)^{i=1,2}_{j=3,4}$ where rank $P_1=2$. Thus, we can consider $P_1$ as the identity matrix and system (4.20) can be rewritten as

$$
b_x+a^1_5(x^2+y^2)+a^1_6x+a^1_7y+a^1_8=0, \tag{4.22}
$$

$$
b_y+a^2_5(x^2+y^2)+a^2_6x+a^2_7y+a^2_8=0. \tag{4.23}
$$

Applying turns, which are equivalent transformations we set $a_7^1 = a_6^2 = 0$, moreover, we obtain $a_5^1 = a_5^2 = 0$. Thus,

$$b(x,y) = -\frac{1}{2}(a_6^1 x^2 + a_7^2 y^2) - a_8^1 x - a_8^2 y + c, \quad c \in \mathbb{R}.$$

We can consider only the following cases:

- if $a_6^1 \neq 0$, $a_7^2 \neq 0$, then $b(x,y) = d_1 x^2 + d_2 y^2$, $(d_1 \neq d_2)$,
- if $a_6^1 = 0, a_7^2 \neq 0$, then $b(x,y) = d_1 x^2 + d_2 x$,
- if $a_6^1 \neq 0, a_7^2 = 0$, then $b(x,y) = d_1 x^2 + d_2 y$,
- if $a_6^1 = a_7^2 \neq 0$, then $b(x,y)$ has the same form as in the case $k = 4$, which is presented below.

Since the equivalence group allows us to classify the parameter $b$, up to the equivalence transformations, we have the following cases ($W_1, W_2, Z_i, i \geq 1$ are the characteristics, which span the space of characteristics; we do not repeat the general characteristics $W_1$ and $W_2$, which are described above).

- $b(x,y) = d_1(x^2+y^2) + d_2(x^2+y^2)^{-1}, \ d_1 > 0, \ d_1 = 1/2 \bmod G^{\sim}$:

  $Z_1 = (yh,\ -xh,\ yu - xv)$,

  $Z_2 = (-e^{2t}uh + e^{2t}xh,\ -e^{2t}vh + e^{2t}yh,$
  $\quad -e^{2t}((u^2+v^2)/2 + h - b) + e^{2t}(xu+yv) - 2e^{2t}(x^2+y^2))$,

  $Z_3 = (e^{-2t}uh + e^{-2t}xh,\ e^{-2t}vh + e^{-2t}yh,$
  $\quad e^{-2t}((u^2+v^2)/2 + h - b) + e^{-2t}(xu+yv) + 2e^{-2t}(x^2+y^2))$.

- $b(x,y) = d_1(x^2+y^2) + d_2(x^2+y^2)^{-1}, \ d_1 < 0, \ d_1 = -1/2 \ \bmod \ G^{\sim}$:

  $Z_1 = (yh,\ -xh,\ yu - xv)$,

  $Z_2 = (-uh\sin 2t + xh\cos 2t,\ -vh\sin 2t + yh\cos 2t,$
  $\quad -((u^2+v^2)/2 + h - b)\sin 2t + (xu+yv)\cos 2t + 2(x^2+y^2)\sin 2t)$,

  $Z_3 = (uh\cos 2t + xh\sin 2t,\ vh\cos 2t + yh\sin 2t,$
  $\quad ((u^2+v^2)/2 + h - b)\cos 2t + (xu+yv)\sin 2t - 2(x^2+y^2)\cos 2t)$.

- $b(x,y) = d_1(x^2+y^2) + d_2(x^2+y^2)^{-1}, \ d_1 = 0$:

  $Z_1 = (yh,\ -xh,\ yu - xv)$,

  $Z_2 = (-t^2uh + txh,\ -t^2vh + tyh,\ -t^2((u^2+v^2)/2 + h - b) + t(xu+yv) - (x^2+y^2)/2)$,

  $Z_3 = (-2tuh + xh,\ -2tvh + yh,\ -2t((u^2+v^2)/2 + h - b) + xu + yv)$.

- $b(x,y) = d_1x^2 + d_2y^2,\ \ d_1 \neq d_2,\ \ d_1 > 0,\ d_2 > 0,\ \ s_1 := \sqrt{2d_1},\ \ s_2 := \sqrt{2d_2}$:

  $Z_1 = (e^{s_1t}h,\ 0,\ e^{s_1t}(u - s_1x)),$

  $Z_2 = (e^{-s_1t}h,\ 0,\ e^{-s_1t}(u + s_1x)),$

  $Z_3 = (0,\ e^{s_2t}h,\ e^{s_2t}(v - s_2y)),$

  $Z_4 = (0,\ e^{-s_2t}h,\ e^{-s_2t}(v + s_2y)).$

- $b(x,y) = d_1x^2 + d_2y^2,\ \ d_1 \neq d_2,\ \ d_1 < 0,\ \ d_2 < 0,\ \ s_1 := \sqrt{-2d_1},\ \ s_2 := \sqrt{-2d_2}$:

  $Z_1 = (h\cos s_1t,\ 0,\ u\cos s_1t + s_1x\sin s_1t),$

  $Z_2 = (h\sin s_1t,\ 0,\ u\sin s_1t - s_1x\cos s_1t),$

  $Z_3 = (0,\ h\cos s_2t,\ v\cos s_2t + s_2y\sin s_2t),$

  $Z_4 = (0,\ h\sin s_2t,\ v\sin s_2t - s_2y\cos s_2t).$

- $b(x,y) = d_1x^2 + d_2y^2,\ \ d_1 > 0,\ \ d_2 < 0,\ \ s_1 := \sqrt{2d_1},\ \ s_2 := \sqrt{-2d_2}$:

  $Z_1 = (e^{s_1t}h,\ 0,\ e^{s_1t}(u - s_1x)),$

  $Z_2 = (e^{-s_1t}h,\ 0,\ e^{-s_1t}(u + s_1x)),$

  $Z_3 = (0,\ h\cos s_2t,\ v\cos s_2t + s_2y\sin s_2t),$

  $Z_4 = (0,\ h\sin s_2t,\ v\sin s_2t - s_2y\cos s_2t).$

- $b(x,y) = d_1x^2 + d_2x,\ \ d_1 > 0,\ \ d_1 = 1/2 \bmod G^{\sim},\ \ d_2 = 0 \bmod G^{\sim}$:

  $Z_1 = (e^th,\ 0,\ e^t(u - x)),\quad Z_2 = (e^{-t}h,\ 0,\ e^{-t}(u + x)),$

  $Z_3 = (0,\ th,\ tv - y),\quad Z_4 = (0,\ h,\ v).$

- $b(x,y) = d_1x^2 + d_2x,\ \ d_1 < 0,\ \ d_1 = -1/2 \bmod G^{\sim},\ \ d_2 = 0 \bmod G^{\sim}$:

  $Z_1 = (h\cos t,\ 0,\ u\cos t + x\sin t),\quad Z_2 = (h\sin t,\ 0,\ u\sin t - x\cos t),$

  $Z_3 = (0,\ th,\ tv - y),\quad Z_4 = (0,\ h,\ v).$

- $b(x,y) = d_1x^2 + d_2y,\ \ d_1 > 0,\ \ d_1 = 1/2 \bmod G^{\sim},\ \ d_2 = 1 \bmod G^{\sim}$:

  $Z_1 = (e^th,\ 0,\ e^t(u - x)),\quad Z_2 = (e^{-t}h,\ 0,\ e^{-t}(u + x)),$

  $Z_3 = (0,\ th,\ tv - y - t^2/2),\quad Z_4 = (0,\ h,\ v - t).$

- $b(x,y) = d_1x^2 + d_2y,\ \ d_1 < 0,\ \ d_1 = -1/2 \bmod G^{\sim},\ \ d_2 = 1 \bmod G^{\sim}$:

  $Z_1 = (h\cos t,\ 0,\ u\cos t + x\sin t),\quad Z_2 = (h\sin t,\ 0,\ u\sin t - x\cos t),$

  $Z_3 = (0,\ th,\ tv - y - t^2/2),\quad Z_4 = (0,\ h,\ v - t).$

**Case $\boldsymbol{k=3}$**. This means that $M := (a^i_j)^{i=1,\dots,3}_{j=1\dots,4}$ is of rank 3. Suppose that rank $M_1 = 2$ for the matrix $M_1 := (a^i_j)^{i=1,\dots,3}_{j=1\dots,2}$. Then up to linearly recombining equations of the system (4.20), we can set $(a^i_j)^2_{i,j=1} = \operatorname{diag}(1,1)$ and $a^3_1 = a^3_2 = 0$. Let $\mathbf{v}^1$, $\mathbf{v}^2$ and $\mathbf{v}^3$ denote the vector fields respectively associated with the equations of the system (4.20). Commuting the vector fields $\mathbf{v}^2$ and $\mathbf{v}^3$ gives us a vector field that is associated with a template-form equation and does not linearly depend on $\mathbf{v}^1$, $\mathbf{v}^2$ and $\mathbf{v}^3$, which leads us exactly to the case $k=4$.

Therefore, rank $M_1 = 1$, and recombining equations of the system (4.20), we reduce the matrix $M$ to the form

$$\begin{bmatrix} a^1_1 & a^1_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.24}$$

We can express the formulas for $b_x$ and $b_y$ from the second and third equations of the system (4.20), since $b_{xy} = b_{yx}$, we get the conditions $a^2_5 = a^3_5 = 0$, $a^2_7 = a^3_6$, and the arbitrary element $b(x,y)$ has the form

$$b(x,y) = -\frac{1}{2}(a^2_6x^2 + a^3_7y^2) - a^2_7xy - a^2_8x - a^3_8y + c, \quad c \in \mathbb{R}.$$

After substituting this equation into the first equation of the system (4.20), we get $a^1_1 = a^1_2 = 0$, which is exactly the case $k=2$, rank $P = 0$.

**Case $\boldsymbol{k=4}$**. Then up to linearly recombining equations of the system (4.20), the matrix $(a^i_j)^4_{i,j=1}$ can be considered as the $4\times 4$ identity matrix. Thus, $b_x = N_1$, $b_y = N_2$, $yb_x - xb_y = N_3$, $xb_x + yb_y + 2b = N_4$, where $N_1,\dots,N_4$ are polynomials of the form $d_1(x^2+y^2) + d_2x + d_3y + d_4$. Therefore, $b(x,y)$ is a polynomial of the same form. We have to consider the following $G^\sim$-inequivalent forms of $b$: $b(x,y) = 0$, $b(x,y) = x$ and $b(x,y) = d_1(x^2+y^2)$ with $d_1 = \pm\frac{1}{2} \bmod G^\sim$ if $d_1 = d_2 = 0$, ($d_1 = 0$, $(d_2,d_3) \neq (0,0)$) and $d_1 \neq 0$, respectively. We substitute each of these forms of $b$ into the equation (4.18) and split the obtained equation with respect to $x$ and $y$, which leads to a system of ordinary differential equations for the functions $A$, $B$, $C$, $D$ and $K^{00}$. Integrating this system, we construct the following linearly independent characteristics, which are additional to the characteristics $W^1$ and $W^2$.

- $b(x,y) = d_1(x^2+y^2)$, $d_1 > 0$, $d_1 = 1/2 \bmod G^{\sim}$:

$$Z_1 = (e^t h,\, 0,\, e^t(u-x)), \quad Z_2 = (e^{-t}h,\, 0,\, e^{-t}(u+x)),$$
$$Z_3 = (0,\, e^t h,\, e^t(v-y)), \quad Z_4 = (0,\, e^{-t}h,\, e^{-t}(v+y)),$$
$$Z_5 = (yh,\, -xh,\, yu - xv),$$
$$Z_6 = (-e^{2t}uh + e^{2t}xh,\, -e^{2t}vh + e^{2t}yh,$$
$$-e^{2t}((u^2+v^2)/2 + h - b) + e^{2t}(xu+yv) - 2e^{2t}(x^2+y^2)),$$
$$Z_7 = e^{-2t}uh + e^{-2t}xh,\, e^{-2t}vh + e^{-2t}yh,$$
$$e^{-2t}((u^2+v^2)/2 + h - b) + e^{-2t}(xu+yv) + 2e^{-2t}(x^2+y^2)/2).$$

- $b(x,y) = d_1(x^2+y^2)$, $d_1 > 0$, $d_1 = -1/2 \bmod G^{\sim}$:

$$Z_1 = (h\cos t,\, 0,\, u\cos t + x\sin t), \quad Z_2 = (h\sin t,\, 0,\, u\sin t - x\cos t),$$
$$Z_3 = (0,\, h\cos t,\, v\cos t + y\sin t), \quad Z_4 = (h\sin t,\, 0,\, v\sin t - y\cos t),$$
$$Z_5 = (yh,\, -xh,\, yu - xv),$$
$$Z_6 = (-uh\sin 2t + xh\cos 2t,\, -vh\sin 2t + yh\cos 2t,$$
$$-((u^2+v^2)/2 + h - b)\sin 2t + (xu+yv)\cos 2t + 2(x^2+y^2)\sin 2t),$$
$$Z_7 = (uh\cos 2t + xh\sin 2t,\, vh\cos 2t + yh\sin 2t,$$
$$((u^2+v^2)/2 + h - b)\cos 2t + (xu+yv)\sin 2t - 2(x^2+y^2)\cos 2t).$$

- $b(x,y) = x$:

$$Z_1 = (0,\, h,\, v), \quad Z_2 = (0,\, th,\, tv - y),$$
$$Z_3 = (h,\, 0,\, u - t), \quad Z_4 = (th,\, 0,\, tu - x - t^2/2),$$
$$Z_5 = (yh,\, -xh + t^2h/2,\, yu - xv + t^2v/2 - ty),$$
$$Z_6 = (-t^2uh + txh + t^3h/2,\, -t^2vh + tyh,$$
$$-t^2((u^2+v^2)/2 + h - b) + t(xu+yv) - (x^2+y^2)/2 + t^3u/2 - 3t^2x/2 - t^4/8),$$
$$Z_7 = (-2tuh + xh + 3t^2h/2,\, -2tvh + yh,$$
$$-2t((u^2+v^2)/2 + h - b) + xu + yv + 3t^2u/2 - 3tx - t^3/2).$$

- $b(x,y) = 0$:

$$Z_1 = (-t^2uh + txh,\, -t^2vh + tyh,\, -t^2((u^2+v^2)/2 + h - b) + t(xu+yv) - (x^2+y^2)/2),$$
$$Z_2 = (-2tuh + xh,\, -2tvh + yh,\, -2t((u^2+v^2)/2 + h - b) + xu + yv),$$
$$Z_3 = (yh,\, -xh,\, yu - xv),$$
$$Z_4 = (th,\, 0,\, tu - x), \quad Z_5 = (h,\, 0,\, u),$$
$$Z_6 = (0,\, th,\, tv - y), \quad Z_7 = (0,\, h,\, v).$$

All the above proves the following theorem.

**Theorem 39.** *Each system of the class (4.8) admits two linearly independent zero-order conservation laws with the conserved currents*

$$C_1^0 = (h, uh, vh), \quad C_2^0 = (\alpha h, \beta uh, \beta vh).$$

*Here and in what follows we denote*

$$\alpha = \frac{1}{2}(u^2+v^2) + h/2 - b,$$
$$\beta = \frac{1}{2}(u^2+v^2) + h - b,$$
$$\gamma = xu + yv.$$

*All $G^{\sim}$-inequivalent cases of extensions of the spaces of zero-order conservation laws for systems from the class (4.8) are exhausted those presented in Table 4.1.*

In Table 4.1 we present only linear independent conserved currents which extend the space of conservation laws, notation for which is given below.

$$C_1 = \Big(h(yu - xv + t),\ uh(yu - xv + t) + \tfrac{1}{2}yh^2,\ vh(yu - xv + t) - \tfrac{1}{2}xh^2\Big),$$

$$C_2 = \Big(-\alpha t^2 h + \gamma th - \tfrac{1}{2}(x^2+y^2)h,\ -\beta t^2 uh + \gamma tuh + \tfrac{1}{2}txh^2 - \tfrac{1}{2}(x^2+y^2)uh,$$
$$-\beta t^2 vh + \gamma tvh + \tfrac{1}{2}tyh^2 - \tfrac{1}{2}(x^2+y^2)vh\Big),$$

$$C_3 = \Big(-2\alpha th + \gamma h,\ -2\beta tuh + \gamma uh + \tfrac{1}{2}xh^2, -2\beta tvh + \gamma vh + \tfrac{1}{2}yh^2\Big),$$

$$C_4 = \Big(h(yu - xv),\ uh(yu - xv) + \tfrac{1}{2}yh^2,\ vh(yu - xv) - \tfrac{1}{2}xh^2\Big),$$

$$C_5 = \Big(h(tu - x),\ uh(tu - x) + \tfrac{1}{2}th^2,\ vh(tu - x)\Big),$$

$$C_6 = \Big(uh,\ u^2 h + \tfrac{1}{2}h^2,\ uvh\Big),$$

$$C_7 = \Big(h(tv - y),\ uh(tv - y),\ vh(tv - y) + \tfrac{1}{2}th^2\Big),$$

$$C_8 = \Big(vh,\ uvh,\ hv^2 + \tfrac{1}{2}h^2\Big),$$

$$C_9 = \Big(h(tu - x - \tfrac{1}{2}t^2),\ uh(tu - x - \tfrac{1}{2}t^2) + \tfrac{1}{2}th^2,\ vh(tu - x - \tfrac{1}{2}t^2)\Big),$$

$$C_{10} = \Big(h(u - t),\ uh(u - t) + \tfrac{1}{2}h^2,\ vh(u - t)\Big),$$

Table 4.1: $G^{\sim}$-inequivalent extensions of space of zero-order CLs for systems of the class (4.8)

| # | $\boldsymbol{b(x,y)}$ | $\boldsymbol{C_i}$ |
|---|---|---|
| 1 | $b(x,y)=\arctan y/x+F(\sqrt{x^2+y^2})$ | $C_1$ |
| 2 | $b(x,y)=F(y/x)/x^2$ | $C_2, C_3$ |
| 2a | $b(x,y)=(x^2+y^2)/2+F(y/x)/x^2$ | $C_{22}, C_{23}$ |
| 2b | $b(x,y)=-(x^2+y^2)/2+F(y/x)/x^2$ | $C_{24}, C_{25}$ |
| 2.1 | $b(x,y)=(x^2+y^2)^{-1}$ | $C_2, C_3, C_4$ |
| 2.1a | $b(x,y)=(x^2+y^2)/2+d_1(x^2+y^2)^{-1}$, $d_1\neq 0$ | $C_4, C_{22}, C_{23}$ |
| 2.1b | $b(x,y)=-(x^2+y^2)/2+d_1(x^2+y^2)^{-1}$, $d_1\neq 0$ | $C_4, C_{24}, C_{25}$ |
| 2.2 | $b(x,y)=0$ | $C_2, C_3, C_4, C_5, C_6, C_7, C_8$ |
| 2.2a | $b(x,y)=(x^2+y^2)/2,\ s_2:=1$ | $C_4, C_{14}, C_{15}, C_{16}, C_{17}, C_{22}, C_{23}$ |
| 2.2b | $b(x,y)=-(x^2+y^2)/2,\ s_2:=1$ | $C_4, C_{18}, C_{19}, C_{20}, C_{21}, C_{24}, C_{25}$ |
| 2.2c | $b(x,y)=x$ | $C_7, C_8, C_9, C_{10}, C_{11}, C_{12}, C_{13}$ |
| 3 | $b(x,y)=F(y)$ | $C_5, C_6$ |
| 3a | $b(x,y)=x+F(y)$ | $C_9, C_{10}$ |
| 4 | $b(x,y)=x^2/2+F(y)$ | $C_{14}, C_{15}$ |
| 4.1 | $b(x,y)=x^2/2+d_2y^2,\ d_2>0$, $d_2\neq 1/2,\ s_2:=\sqrt{2d_2}$ | $C_{14}, C_{15}, C_{16}, C_{17}$ |
| 4.2 | $b(x,y)=x^2/2+d_2y^2,\ d_2<0$, $s_2:=\sqrt{-2d_2}$ | $C_{14}, C_{15}, C_{18}, C_{19}$ |
| 4.3 | $b(x,y)=x^2/2$ | $C_7, C_8, C_{14}, C_{15}$ |
| 4.3a | $b(x,y)=x^2/2+y$ | $C_{26}, C_{27}, C_{14}, C_{15}$ |
| 5 | $b(x,y)=-x^2/2+F(y)$ | $C_{20}, C_{21}$ |
| 5.1 | $b(x,y)=-x^2/2+d_2y^2,\ d_2<0$, $d_2\neq -1/2,\ s_2:=\sqrt{-2d_2}$ | $C_{18}, C_{19}, C_{20}, C_{21}$ |
| 5.2 | $b(x,y)=-x^2/2$ | $C_7, C_8, C_{20}, C_{21}$ |
| 5.2a | $b(x,y)=-x^2/2+y$ | $C_{26}, C_{27}, C_{20}, C_{21}$ |

$$C_{11} = \Big(h(yu - xv - ty + \tfrac{1}{2}t^2 v),\ uh(yu - xv - y + \tfrac{1}{2}t^2 v) + \tfrac{1}{2}yh^2,$$
$$vh(yu - xv - ty + \tfrac{1}{2}t^2 v) - \tfrac{1}{2}xh^2 + \tfrac{1}{4}t^2 h^2\Big),$$

$$C_{12} = \Big( - \alpha t^2 h + \gamma th + \tfrac{1}{2}t^3 uh - (\tfrac{1}{8}t^4 + \tfrac{3}{2}t^2 x + \tfrac{1}{2}(x^2 + y^2))h,$$
$$-\beta t^2 uh + \gamma tuh + \tfrac{1}{2}t^3 uh^2 + \tfrac{1}{2}txh^2 + \tfrac{1}{4}t^3 h^2$$
$$-(\tfrac{1}{8}t^4 + \tfrac{3}{2}t^2 x + \tfrac{1}{2}(x^2 + y^2))uh,$$
$$-\beta t^2 vh + \gamma tvh + \tfrac{1}{2}t^3 uvh + \tfrac{1}{2}tyh^2 - (\tfrac{1}{8}t^4 + \tfrac{3}{2}t^2 x + \tfrac{1}{2}(x^2 + y^2))vh\Big),$$

$$C_{13} = \Big( - 2\alpha th + \gamma h + \tfrac{3}{2}t^2 uh - 3txh - \tfrac{1}{2}t^3 h,$$
$$-2\beta tuh + \gamma uh + \tfrac{1}{2}xh^2 + \tfrac{3}{4}t^2 h^2 + \tfrac{3}{2}t^2 u^2 h - 3txuh - \tfrac{1}{2}t^3 uh,$$
$$-2\beta tvh + \gamma vh + \tfrac{1}{2}yh^2 + \tfrac{3}{2}t^2 uvh - 3txvh - \tfrac{1}{2}t^3 vh\Big),$$

$$C_{14} = \Big(e^t h(u - x),\ e^t uh(u - x) + \tfrac{1}{2}e^t h^2,\ e^t vh(u - x)\Big),$$

$$C_{15} = \Big(e^{-t} h(u + x),\ e^{-t} uh(u + x) + \tfrac{1}{2}e^{-t} h^2,\ e^{-t} vh(u + x)\Big),$$

$$C_{16} = \Big(e^{s_2 t} h(v - s_2 y),\ e^{s_2 t} uh(v - s_2 y),\ e^{s_2 t} vh(v - s_2 y) + \tfrac{1}{2}e^{s_2 t} h^2\Big),$$

$$C_{17} = \Big(e^{-s_2 t} h(v + s_2 y),\ e^{-s_2 t} uh(v + s_2 y) + \tfrac{1}{2}e^{-s_2 t} h^2,\ e^{-s_2 t} vh(v + s_2 y)\Big),$$

$$C_{18} = \Big(h(v\cos s_2 t + s_2 y \sin s_2 t),\ uh(v\cos s_2 t + s_2 y \sin s_2 t),$$
$$vh(v\cos s_2 t + s_2 y \sin s_2 t) + \tfrac{1}{2}h^2 \cos s_2 t\Big),$$

$$C_{19} = \Big(h(v\sin s_2 t - s_2 y \cos s_2 t),\ uh(v\sin s_2 t - s_2 y \cos s_2 t),$$
$$vh(v\sin s_2 t - s_2 y \cos s_2 t) + \tfrac{1}{2}h^2 \sin s_2 t\Big),$$

$$C_{20} = \Big(h(u\cos t + x\sin t),\ uh(u\cos t + x\sin t) + \tfrac{1}{2}h^2 \cos t,$$
$$vh(u\cos t + x\sin t)\Big),$$

$$C_{21} = \Big(h(u\sin t - x\cos t),\ uh(u\sin t - x\cos t) + \tfrac{1}{2}h^2 \sin t,$$
$$vh(u\sin t - x\cos t)\Big),$$

$$C_{22} = \Big( - \alpha e^{2t} + e^{2t} h(\gamma - 2(x^2 + y^2)),\ -\beta e^{2t} + e^{2t} uh(\gamma - 2(x^2 + y^2))$$

$$+\tfrac{1}{2}e^{2t}xh^2,\ -\beta e^{2t}+e^{2t}vh(\gamma-2(x^2+y^2))+\tfrac{1}{2}e^{2t}yh^2\Big),$$

$$C_{23}=\Big(\alpha e^{-2t}+e^{-2t}h(\gamma+2(x^2+y^2)),\ \beta e^{-2t}+e^{-2t}uh(\gamma+2(x^2+y^2))$$
$$+\tfrac{1}{2}e^{-2t}xh^2,\ \beta e^{-2t}+e^{-2t}vh(\gamma+2(x^2+y^2))+\tfrac{1}{2}e^{-2t}yh^2\Big),$$

$$C_{24}=\Big(-\alpha\sin 2t+\gamma h\cos 2t+2(x^2+y^2)h\sin 2t,$$
$$-\beta\sin 2t+\gamma uh\cos 2t+2(x^2+y^2)uh\sin 2t+\tfrac{1}{2}xh^2\tfrac{1}{2}\cos 2t,$$
$$-\beta\sin 2t+\gamma vh\cos 2t+2(x^2+y^2)vh\sin 2t+\tfrac{1}{2}yh^2\tfrac{1}{2}\cos 2t\Big),$$

$$C_{25}=\Big(2\alpha\cos 2t+\gamma h\sin 2t-2(x^2+y^2)h\cos 2t,$$
$$\beta\cos 2t+\gamma uh\sin 2t-2(x^2+y^2)uh\cos 2t+\tfrac{1}{2}xh^2\sin 2t,$$
$$\beta\cos 2t+\gamma vh\sin 2t-2(x^2+y^2)vh\cos 2t+\tfrac{1}{2}yh^2\sin 2t\Big),$$

$$C_{26}=\Big(h(tv-y-\tfrac{1}{2}t^2),\ uh(tv-y-\tfrac{1}{2}t^2),\ vh(tv-y-\tfrac{1}{2}t^2)+\tfrac{1}{2}th^2\Big),$$

$$C_{27}=\Big(h(v-t),\ uh(v-t),\ vh(v-t)+\tfrac{1}{2}h^2\Big).$$

One can notice that within certain groups of classification cases the corresponding spaces of zero-order conservation laws are of the same dimension and respective basis conserved currents are of similar structure. It means that there may exist admissible transformations within the class (4.8) that are not generated by elements of $G^{\sim}$. [1] Such transformations establish additional equivalences within the presented list of $G^{\sim}$-inequivalent systems from the class (4.8) with extended spaces of zero-order conservation laws. In other words, an admissible transformation maps a pair constituted by a system from the class (4.8) and the corresponding space of zero-order conservation laws to a pair of the same kind.

Based on analysis of the forms of the above conserved currents, we make conjectures on additional equivalences among classification cases presented in Table 4.1 and on possible form of associated point transformations in the underlying space with coordinates $(t,x,y,u,v,h)$.

[1]The observation on similarity of spaces of conservation laws had been used in the literature for finding similarity transformations between systems of differential equations. In particular, necessary and sufficient conditions of existence of invertible mappings of a given nonlinear PDE to a linear one in terms of CL-characteristics were derived in [7]. The technique for constructing point transformations realizing such mappings was also proposed therein.

The systems from the class (4.8) where arbitrary element $b(x,y)$ is of the forms 2.2c and 3a are transformed to systems of the same class associated with Cases 2.2 and 3, respectively, by the point transformation

$$\begin{aligned}&\tilde t = t, \quad \tilde x = x - \tfrac{1}{2}t^2, \quad \tilde y = y,\\ &\tilde u = u - t, \quad \tilde v = v, \quad \tilde h = h.\end{aligned}$$

The mapping of the systems induces mappings between the corresponding zero-order conserved currents $C_9, C_{10}, C_{11}, C_{12}$ and $C_{13}$ to $C_5, C_6, C_4, C_2$ and $C_3$, respectively. This follows from Theorem 30, see also an example of computation of such an induced mapping below.

Analogously, Cases 5.2a and 4.3a are mapped to Cases 5.2 and 4.3, respectively, by the point transformation

$$\begin{aligned}&\tilde t = t, \quad \tilde x = x, \quad \tilde y = y - \tfrac{1}{2}t^2,\\ &\tilde u = u, \quad \tilde v = v - t, \quad \tilde h = h.\end{aligned}$$

This transformation coincides with above one up to the permutation of $(x,u)$ and $(y,v)$, which is an equivalent transformation of the class (4.8), see Theorem 37. The induced map relates the conserved currents $C_{26}$ and $C_{27}$ to $C_7$ to the conserved currents $C_8$, respectively.

The similarity of Cases 2a, 2.1a and 2.2a with Cases 2, 2.1 and 2.2, respectively, is established by the point transformation

$$\begin{aligned}&\tilde t = \tfrac{1}{2}e^{2t}, \quad \tilde x = e^t x, \quad \tilde y = e^t y,\\ &\tilde u = e^{-t}(u + x), \quad \tilde v = e^{-t}(v + y), \quad \tilde h = e^{-2t}h,\end{aligned} \tag{4.25}$$

where the conserved currents $C_{14}, C_{15}, C_{16}, C_{17}, C_{22}$ and $C_{23}$ are mapped to the conserved currents $C_5, C_6, C_7, C_8, C_2$ and $C_3$, respectively.

Cases 2b, 2.1b and 2.2b are mapped to Cases 2, 2.1 and 2.2, respectively, by the point transformation

$$\begin{aligned}&\tilde t = \tan t, \quad \tilde x = \frac{x}{\cos t}, \quad \tilde y = \frac{y}{\cos t},\\ &\tilde u = u\cos t + x\sin t, \quad \tilde v = v\cos t + y\sin t, \quad \tilde h = h\cos^2 t.\end{aligned} \tag{4.26}$$

The generalization of this transformation that is also relevant for the rotating reference frame was first found in [23, Theorem 1] for shallow-water equations in cylindrical coordinates. The transformation (4.26) maps the conserved currents $C_{18}, C_{19}, C_{20}, C_{21}, C_{24}$ and $C_{25}$ to the conserved currents $C_5, C_6, C_7, C_8, C_2$ and $C_3$, respectively.

**Example 40.** Consider the mapping $C_{14}$ to $C_5$ by the point transformation (4.25). Applying the formula (2.15), we have to prove the identity

$$\tilde{C}_{14} = \frac{1}{e^{4t}} C_5 \cdot \begin{pmatrix} e^{2t} & e^t x & e^t y \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix}. \tag{4.27}$$

Here the tilde over $C_{14}$ indicates that we use the new coordinates $(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}, \tilde{h})$ for this conserved current. The right hand-side of (4.27) can be rewritten as follows

$$\begin{pmatrix} e^{-t} h(u-x) \\ e^{-2t}(xh(u-x) + uh(u-x) + \frac{1}{2}h^2) \\ e^{-2t}(yh(u-x) + vh(u-x)) \end{pmatrix},$$

where $(t, x, y, u, v, h)$ should be replaced by $(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}, \tilde{h})$ according to the transformation (4.25). As a result, we get exactly the conserved current $\tilde{C}_{14}$.

Unfortunately, we were not able to exhaustively describe the entire equivalence groupoid of the class (4.8). In other words, we do not know all admissible transformations of this class, which would be needed for an accurate consideration of additional equivalences among the $G^{\sim}$-inequivalent classification cases that are presented in Table 4.1. At the same time, the inequivalence of classification cases up to general point equivalence may be proved using other tools for each fixed relevant pair of systems from the class (4.8), although this may require cumbersome computations. Analyzing the form of the listed conserved currents, we suppose that there are no other additional equivalences among the above classification cases. This may be checked using not only conservation laws but, for instance, Lie symmetries, which leads to the following conjecture.

**Conjecture 41.** *A complete list of inequivalent (up to all admissible transformations) extensions of spaces of zero-order conservation laws of systems from the class (4.8) is exhausted by the cases given in Table 4.2.*

**Remark 42.** It has to be noticed that some of the cases in Table 4.2 are the expansions of other cases. Case 2.1 is the specific value of function $F$ from Case 2; Case 2.2 coincides with Case 2 for $F = 0$. Cases 4.1, 4.2 and 4.3 are similarly the sub-cases of Case 4 for specific values of $F$, the same as Case 5.1 and 5.2 are the sub-cases of Case 5 when $F$ is of quadratic or linear form.

Table 4.2: Inequivalent extensions of spaces of zero-order CLs for systems of the class (4.8)

| # | $\boldsymbol{b(x,y)}$ | $\boldsymbol{C_i}$ |
|---|---|---|
| 1 | $b(x,y) = \arctan y/x + F(\sqrt{x^2+y^2})$ | $C_1$ |
| 2 | $b(x,y) = F(y/x)/x^2$ | $C_2, C_3$ |
| 2.1 | $b(x,y) = (x^2+y^2)^{-1}$ | $C_2, C_3, C_4$ |
| 2.2 | $b(x,y) = 0$ | $C_2, C_3, C_4, C_5, C_6, C_7, C_8$ |
| 3 | $b(x,y) = F(y)$ | $C_5, C_6$ |
| 4 | $b(x,y) = x^2/2 + F(y)$ | $C_{14}, C_{15}$ |
| 4.1 | $b(x,y) = x^2/2 + d_2y^2,\ d_2 > 0,$ $d_2 \neq 1/2,\ s_2 := \sqrt{2d_2}$ | $C_{14}, C_{15}, C_{16}, C_{17}$ |
| 4.2 | $b(x,y) = x^2/2 + d_2y^2,\ d_2 < 0,$ $s_2 := \sqrt{-2d_2}$ | $C_{14}, C_{15}, C_{18}, C_{19}$ |
| 4.3 | $b(x,y) = x^2/2$ | $C_7, C_8, C_{14}, C_{15}$ |
| 5 | $b(x,y) = -x^2/2 + F(y)$ | $C_{20}, C_{21}$ |
| 5.1 | $b(x,y) = -x^2/2 + d_2y^2,\ d_2 < 0,$ $d_2 \neq -1/2,\ s_2 := \sqrt{-2d_2}$ | $C_{18}, C_{19}, C_{20}, C_{21}$ |
| 5.2 | $b(x,y) = -x^2/2$ | $C_7, C_8, C_{20}, C_{21}$ |

# Chapter 5

# Conclusion

In this thesis we have considered the classes of one- and two-dimensional shallow-water equations (3.1) and (4.1) with variable bottom topography depending on the space variables. First of all, in Sections 3.1 and 4.1 we have found the equivalence groups for these classes, which are presented in Theorems 31 and 37, respectively. We have used the direct method for computing the equivalence group in the one-dimensional case. This method is based on the definition of the equivalence group, the relation between equivalence and admissible transformations and the chain rule for prolonging transformations to derivatives. Although this method leads to a cumbersome system of nonlinear determining equations for components of equivalence transformations, we were able to solve this system without involving algebraic techniques. Note that we have thus constructed only the usual equivalence group. Since applying the direct method in the two-dimensional case would require much more cumbersome calculations, we have used the algebraic method for this case, which is explained in Section 2.1 and is based on Theorem 17. The fact that the arbitrary element $b$ depends only on the space variables allowed us to replace the construction of the equivalence group for the class (4.1) by finding the point symmetry group for the system (4.2). In contrast to the one-dimensional case, in dimension two we have found the generalized equivalence group and have showed that it coincides with the usual equivalence group.

Further, in Section 3.2 we have solved the problem of classification of zero-order conservation laws for the class (3.1) of one-dimensional shallow-water equations using conservation-law characteristics. To separate cases of extensions of the space of zero-order conservation laws, we integrate the corresponding system of determining equations for their characteristics depending on values of the arbitrary element $b = b(x)$. Since we have computed the equivalence group $G^{\sim}$ of the class (3.1) in Section 3.1, we have classified conservation laws of systems from the class (3.1) up to $G^{\sim}$-equivalence. The result of the classification is summed up in Theorem 34. Moreover, we reduced the classification list using admissible transformation of the class (3.1) for the case of linear $b$.

For the class (4.1) of two-dimensional shallow-water equations, the problem of classification of zero-order conservation laws required involving conservation-law characteristics. However, instead of the direct integration of the system of the determining equations for them up to $G^{\sim}$-equivalence, we applied the more advanced method of furcate splitting. The use of transformations from the corresponding equivalence group $G^{\sim}$, which is found in Section 4.1, is even more important for simplifying all the computations than in the one-dimensional case. After listing $G^{\sim}$-inequivalent extension cases in Theorem 39, we found additional equivalences between them due to the fact that the class (4.1) is not normalized. In other words, there are admissible transformations in this class that are not generated by equivalence transformations.

The results obtained in the thesis can be used practically, for instance, for testing numerical schemes, the construction of new exact solutions and the numerical modeling of shallow-water motion with uneven bottom topography, see for example [3,21,24].

We have found only zero-order conservation laws of the shallow-water equations. The problem is that even for order zero in the two-dimensional case we had to use some modern and powerful techniques to classify all the conservation laws. In case of higher-order conservation laws, the system of determining equations for conservation-law characteristics, which is implied by the necessary condition for conservation-law characteristics in terms of co-symmetries, should be hard to test for consistency. Since the direct test of compatibility cannot be used efficiently, and the method of furcate splitting might give too many cases to observe, these techniques will not be enough. Thus, in a future work we are going to study higher-order conservation laws for the same classes of equations, using conservation-law characteristics, some techniques from the direct method, the method of furcate splitting and some algebraic methods.

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