General Volumes in the Orlicz-Brunn-Minkowski Theory and Related Minkowski Problems

by

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Abstract

The Minkowski problem is one of the core problems in convex geometry, which aims to characterize the surface area measures of convex bodies in $\mathbb{R}^n$. Various extensions and their dual have been introduced in recent years, among which the newly proposed dual Minkowski problem for the qth dual curvature measure is one of the most important. This thesis deals with the general (dual) volumes, the general dual Orlicz curvature measure $\widetilde{C}_{G,\psi}$, and related Minkowski type problems.

A typical problem we investigate in this thesis is the following general dual Orlicz-Minkowski problem: *under what conditions on a given measure μ defined on the unit sphere, a two-variable function $G(\cdot,\cdot)$ and a one-variable function $\psi(\cdot)$, does there exist a convex body K such that μ equals to the general dual Orlicz curvature measure of K up to a constant τ, i.e., $\mu = \tau\widetilde{C}_{G,\psi}(K,\cdot)$?* In particular, we will study the existence, continuity, and uniqueness of the solutions to the above general dual Orlicz-Minkowski problem. These will be done in Chapters 3-5, where Chapter 3 deals with the special case of $\widetilde{C}_{G,\psi}$ obtained from $\mathscr{V}_\phi$, Chapter 4 studies the case $\widetilde{C}_{G,\psi}$ with $G(t,\cdot)$ decreasing on t, and Chapter 5 concentrates on the case $\widetilde{C}_{G,\psi}$ with $G(t,\cdot)$ increasing on t. Techniques used in Chapters 3 and 4 are the Blaschke selection theorem and the method of Lagrange multipliers, whereas in Chapter 5 we use the approximation arguments from discrete measures to general measures. In Chapter 6, we investigate the "polar" of the general dual Orlicz-Minkowski problem. This type of problem is a typical extension of many fundamental geometric invariants, such as the L_p/Orlicz geominimal surface areas and the L_p/Orlicz-Petty bodies. The existence, continuity, and uniqueness of the solutions to the general dual-polar Orlicz-Minkowski problem are provided in Chapter 6. Our techniques also follow the approximation arguments from discrete measures to general measures.

To my dear parents and sisters

Lay summary

The classical Brunn-Minkowski theory centers around the relation between Minkowski sum and Lebesgue measure in high dimensional Euclidean space. In recent decades, the classical Brunn-Minkowski theory has been expanded considerably into the L_p Brunn-Minkowski theory, the Orlicz-Brunn-Minkowski theory, and the dual Orlicz-Brunn-Minkowski theory. This thesis seeks to extend the classical Minkowski problem into its most general setting up to now, i.e., characterizing the general dual Orlicz curvature measure $\widetilde{C}_{G,\psi}$ of convex bodies.

For a convex body, the characterization problem for the surface area measure leads to the classical Minkowski problem in the Brunn-Minkowski theory, which in certain circumstance is equivalent to finding the solutions of Monge-Ampère type equations. With the development of the Orlicz-Brunn-Minkowski theory and the dual Brunn-Minkowski theory, a series of Minkowski problems have arisen due to the appearance of various extensions of the surface area measures, including the L_p dual curvature measure, the Orlicz surface area measure, etc. It is our aim to further study the dual Orlicz-Minkowski problem.

Our work addresses several aspects. First, we introduce the background of the dual Orlicz-Minkowski problems by proposing a series of dual Orlicz curvature measures. To characterize these measures, we propose the corresponding dual Orlicz-Minkowski problems. Following the method of Lagrange multipliers and variational formulas with respect to the general dual volumes for convex bodies, the solutions to the series of dual Orlicz-Minkowski problems are provided. This theory complements and enriches the theory of the dual Brunn-Minkowski theory and Minkowski type problems.

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Statement of contributions

Chapters 3-6 consist of four self-contained papers. These papers were completed under the supervision of Professor Deping Ye. Below I will provide more information:

Chapter 3 is based on the paper: S. Xing and D. Ye, *On the general dual Orlicz-Minkowski problem*, Indiana Univ. Math. J., to appear.

Chapter 4 is based on the paper: R. Gardner, D. Hug, W. Weil, S. Xing and D. Ye, *General volumes in the Orlicz-Brunn-Minkowski theory and a related Minkowski Problem I*, Calc. Var. Partial Differential Equations, 58 (2019), 1-35.

Chapter 5 is based on the paper: R. Gardner, D. Hug, S. Xing and D. Ye, *General volumes in the Orlicz-Brunn-Minkowski theory and a related Minkowski Problem II*, arXiv: 1809.09753, submitted.

Chapter 6 is based on the paper: S. Xing, D. Ye and B. Zhu, *The general dual-polar Orlicz-Minkowski problem*, submitted.

In each paper listed above, my contributions are essential including (but are not limited to)

- verifying the feasibility of our ideas by carefully performing the mathematical calculations of (almost all) results in these papers;
- writing the preliminary versions of the statements and their proofs of almost all results in these papers;
- compositing the preliminary versions of the manuscripts of these papers;
- proofreading various versions of our papers by providing feedback and comments to polish the papers.

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List of symbols

h_K	the support function of K
ρ_K	the radial function of K
$\mathscr{K}^n$	the set of nonempty compact convex sets
$\mathscr{K}_o^n$	the set of convex bodies containing o
$\mathscr{K}_{(o)}^n$	the set of convex bodies containing o in their interiors
$\mathscr{S}$	the set of star-shaped sets w.r.t. o with measurable radial functions
$\mathscr{S}^n$	the set of star-shaped sets w.r.t. o with bounded measurable radial functions
$\mathscr{S}_+^n$	the set of $L \in \mathscr{S}^n$ with $\rho_L > 0$
$\mathscr{S}_{c+}^n$	the set of $L \in \mathscr{S}_+^n$ such that ρ_L is continuous on S^{n-1}
$\delta(\cdot,\cdot)$	the Hausdorff metric
$\widetilde{\delta}(\cdot,\cdot)$	the radial metric
$H(K,\cdot)$	the supporting hyperplane of K
$S(K,\cdot)$	the surface area measure of K
$\boldsymbol{\nu}_K$	the spherical image of K
$\boldsymbol{x}_K$	the reverse spherical image of K
r_K	the radial map of K
$\boldsymbol{\alpha}_K$	the radial Gauss image of K
$\boldsymbol{\alpha}_K^*$	the reverse radial Gauss image of K
$N(K,o)$	the normal cone of K w.r.t. o
$N(K,o)^*$	the dual cone of $N(K,o)$
$C^+(E)$	the class of strictly positive continuous functions on E
$[f]$	the Alexsandrov body associated to f
$\langle f\rangle$	the convex hull associated to f
$\widetilde{C}_{G,\psi}$	the general dual Orlicz curvature measure
$\widetilde{C}_{\phi,\mathscr{V}}$	the general dual Orlicz curvature measure of $\widetilde{C}_{G,\psi}$ obtained from $\mathscr{V}_\phi$
$\mathscr{V}_\phi$	the general dual Orlicz quermassintegral
$\widetilde{V}_G$	the general dual volume
$\widehat{V}_G$	the homogeneous general dual volume
V_G	the general volume
$\overline{V}_G$	the homogeneous general volume

Chapter 1

Introduction

The study of Minkowski problems, initiated by Minkowski [55, 56] over a century ago, took on a new life when Lutwak [47] introduced the L_p surface area measure and the L_p mixed volume for $p > 1$. Among those fundamental objects related to the L_p surface area measure and the L_p mixed volume, the L_p Minkowski problem (for $p = 1$ in [55, 56] by Minkowski and for $p \in \mathbb{R}$ in [47] by Lutwak) and the L_p affine surface area (for $p = 1$ in [2] by Blaschke, for $p > 1$ in [48] by Lutwak and for $p < 1$ in [60] by Schütt and Werner) arguably have the greatest influence.

For $p \in \mathbb{R}$, the L_p Minkowski problem asks for the necessary and/or sufficient conditions on a finite nonzero Borel measure μ defined on the unit sphere S^{n-1} to be the L_p surface area measure of a convex body (i.e., a convex compact set in $\mathbb{R}^n$ with nonempty interior) K; that is,

$$d\mu = h_K^{1-p}\, dS(K, \cdot),$$

where h_K is the support function of the convex body K and $S(K, \cdot)$ is the surface area measure of K. The L_p surface area measure can be obtained via a first-order variation of volume with respect to the L_p combination of convex bodies, see e.g. [33, 47, 51]. For example, for $p > 1$ and $K, L \subseteq \mathbb{R}^n$ [47] convex bodies containing the origin o in their interiors, one has

$$\int_{S^{n-1}} h_L^p(u) h_K^{1-p}(u)\, dS(K, u) = p \cdot \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}, \tag{1.1}$$

where $V(K)$ stands for the volume of K and $K+_p\varepsilon\cdot L$ is a convex body determined by the support function $h_{K+_p\varepsilon\cdot L}:S^{n-1}\to(0,\infty)$: for any $\varepsilon>0$, $h^p_{K+_p\varepsilon\cdot L}=h^p_K+\varepsilon h^p_L$. When $p=0$, the L_p combination $K+_p\varepsilon\cdot L$ is called the L_0 addition or logarithmic addition of K and L, and is obtained from, in the sense of Aleksandrov body, the function $h_{K+_0\varepsilon L}=h_K h^{\varepsilon}_L$. When $p=1$, it recovers the classical Minkowski sum $h_{K+\varepsilon L}=h_K+\varepsilon h_L$ and the L_p Minkowski problem reduces to the classical Minkowski problem, i.e., $\mu=S(K,\cdot)$? Let μ be a measure on S^{n-1} such that the Radon-Nikodym derivative of μ with respect to the spherical measure on S^{n-1} exists, i.e., $d\mu=f du$ for a positive function f on S^{n-1}. Hence, the classical Minkowski problem, if the convex bodies involved are smooth enough, can be formulated by the following Monge-Ampère type equation,

$$f(u)=\det(\bar{\nabla}^2h(u)+h(u)I) \text{ on } S^{n-1}, \tag{1.2}$$

where $\det(\bar{\nabla}^2h(u)+h(u)I)$ is the reciprocal Gauss curvature on the boundary point of a convex body whose outer normal vector is u, $\bar{\nabla}^2$ is the Hessian matrix of h with respect to an orthonormal frame on S^{n-1}, and I is the identity matrix. In this case, finding a solution of the classical Minkowski problem requires to solve (1.2).

The L_p Minkowski problem and the L_p affine surface area were apparently developed in completely different approaches, however, they were nicely connected through the L_p geominimal surface area and the L_p Petty bodies [48, 71, 77]. As the bridge to connect several geometries (affine, Minkowski and relative), the L_p geominimal surface area is crucial in convex geometry and, in particular, sharing many properties similar to those for the L_p affine surface area. Let $\mathscr{K}^n_{(o)}$ be the set of convex compact sets in $\mathbb{R}^n$ with the origin o in their interiors. Finding the L_p Petty bodies of $K\in\mathscr{K}^n_{(o)}$ for $p\in\mathbb{R}\setminus\{0,-n\}$ requires to solve the following optimization problem (with μ being the L_p surface area measure of K):

$$\inf/\sup\left\{\int_{S^{n-1}}h^p_{L^*}(u)\,d\mu(u):\quad L\in\mathscr{K}^n_{(o)}\ \text{ and }\ V(L)=V(B^n)\right\}, \tag{1.3}$$

where B^n is the unit Euclidean ball in $\mathbb{R}^n$ and L^* denotes the polar body of $L\in\mathscr{K}^n_{(o)}$. As explained in [44], the L_p Minkowski problem can be viewed as the "polar" of (1.3) (in particular, for μ nice enough such as μ being even) aiming to find convex bodies (ideally in $\mathscr{K}^n_{(o)}$) to solve the optimization problem similar to (1.3), namely with L^* replaced by L. On the other hand, the L_p affine surface area of $K\in\mathscr{K}^n_{(o)}$ can be defined through a formula similar to (1.3) for μ being the L_p surface area measure of

K, but with $L \in \mathscr{K}^n_{(o)}$ and h_{L^*} replaced by L belong to star bodies about the origin and, respectively, ρ_L^{-1} where ρ_L is the radial function of L (see [48, 71, 77] for more details).

The L_p Minkowski problem has attracted tremendous attention in different areas, such as analysis, convex geometry, and partial differential equations (see e.g., [7, 11, 12, 26, 33, 51, 81, 83] among others). In particular, it is closely related to the far-reaching optimal mass transportation problem via the Monge-Ampère type equations. The case $p = 0$ is of particular significance because the L_0 surface area measure, the so-called cone volume measure, is affine invariant. The L_0 or logarithmic Minkowski problem is challenging and was only solved recently for even measures by Böröczky, Lutwak, Yang, and Zhang [7] and discrete planar cases by Stancu [61, 62, 63]. More recent contributions to the logarithmic Minkowski problem are [5, 80] and further references and background on the L_p Minkowski problem may be found in [28, 36, 37, 42, 49, 59, 65, 66].

Various extensions of interest appear and advance towards this theory. For example, Livshyts [41] proposed a surface area measure of K with respect to a measure μ_g, where g, the density function of μ_g with respect to the Lebesgue measure, is continuous on its support. A variational formula for μ_g similar to (1.1) for $p = 1$ was also provided in [41], which gives a variational interpretation of the surface area measure of K with respect to μ_g. The related Minkowski problem was posed and a solution to this problem was given under certain conditions on μ_g (such as, μ_g being a measure with positive degree of concavity and positive degree of homogeneity). An L_p extension of Livshyts' result was obtained by Wu [66], where the L_p surface area measure with respect to μ_g was proposed and related L_p Minkowski problem was solved under certain conditions on μ_g.

With the qth dual volume $\widetilde{V}_q$ (see (4.5)) involved, Huang, Lutwak, Yang, and Zhang in their seminal work [29] brought new ingredients, the qth dual curvature measure $\widetilde{C}_q$ (see (3.15)) into the family of Minkowski problems. These measures were obtained via a first-order variation of the qth dual volume $\widetilde{V}_q$ with respect to the L_0 addition (logarithmic addition) of convex bodies (see [29, Theorem 4.5]), the case $q = n$ being the L_0 surface area measure. The authors of [29] posed a corresponding Minkowski problem—the dual Minkowski problem—of finding necessary and/or sufficient conditions for a measure μ on S^{n-1} to be the qth dual curvature measure

$\widetilde{C}_q$ of some convex body, and they provided a partial solution when μ is even. Note that, the logarithmic Minkowski problem is a special case of this dual Minkowski problem. Naturally, the dual Minkowski problem has become very important for the dual Brunn-Minkowski theory introduced by Lutwak [45, 46]. Since then, progress includes a complete solution for $q < 0$ by Zhao [75], solutions for even μ in [6, 8, 23, 76], and solutions via curvature flows and partial differential equations in [10, 38, 40]. An important extension of the dual Minkowski problem was carried out by Lutwak, Yang, and Zhang [54], who introduced the L_p dual curvature measures $\widetilde{C}_{p,q}$ (see (4.26)) and posed corresponding L_p dual Minkowski problems. In [54], the L_0 addition in [29] is replaced by the L_p addition, while the qth dual volume $\widetilde{V}_q$ remains unchanged. The first contribution to the L_p dual Minkowski problem, by Huang and Zhao [30], proves the existence of solutions for $p, q \in \mathbb{R}$ when $p > 0$ and $q < 0$, and for even μ when $pq > 0$, $p \neq q$. Their results were augmented by Chen, Huang, and Zhao [9], who used curvature flows to show the smoothness of solutions for even μ and $pq \geq 0$. Böröczky and Fodor [3] provide a beautiful solution to the L_p dual Minkowski problem for general μ when $p > 1$ and $q > 0$.

The first Orlicz version of the Minkowski problem appeared in [22], at the inception of the Orlicz-Brunn-Minkowski theory in 2010. Let $\varphi : (0, \infty) \to (0, \infty)$ be a continuous function and μ be a nonzero finite Borel measure defined on the unit sphere S^{n-1}, the Orlicz-Minkowski problem asks whether there exist a convex body K and a constant $\tau > 0$, such that,

$$d\mu = \tau \cdot \varphi(h_K)\, dS(K, \cdot). \tag{1.4}$$

The typical case for $\varphi(t) = t^{1-p}$ recovers the L_p Minkowski problem. The Orlicz-Minkowski problem was first investigated by Haberl, Lutwak, Yang and Zhang in their seminal paper [22] for the even measure μ. Solutions to the Orlicz-Minkowski problem for μ being a discrete and/or general (not necessarily even) measure were provided by Huang and He [27] and Li [39]. The planar Orlicz-Minkowski problem in the L_1-sense was investigated by Sun and Long [64]. The p-capacitary Orlicz-Minkowski problem was posed and studied in [24]. The Orlicz-Minkowski problems are central objects in the recent but rapidly developing Orlicz-Brunn-Minkowski theory for convex bodies [16, 43, 52, 53, 68].

Analogous to the way that the Orlicz-Minkowski problem generalizes L_p Minkowski problem, the dual Minkowski problem was extended to the Orlicz setting and partially solved in [78] by Zhu, Xing and Ye; here, the qth dual volume $\widetilde{V}_q$ in [29] was replaced by certain dual Orlicz quermassintegrals, while L_0 addition is retained. In particular,

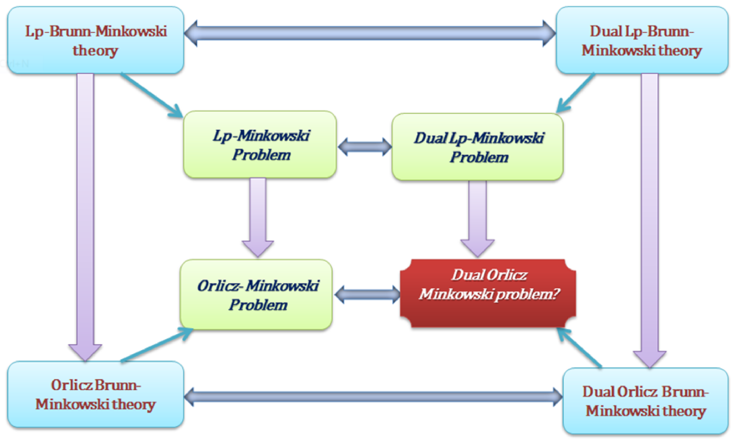

Figure 1.1: Motivation of dual Orlicz-Minkowski problem

the authors in [78] proposed the dual Orlicz-Minkowski problem, which belongs to the recently initiated dual Orlicz-Brunn-Minkowski theory [18, 73, 79]. Chapters 3-5 are motivated by [41, 66, 78] where the general dual Orlicz-Minkowski problems are studied (please see Figure 1.1 for our motivation).

In Chapter 3, the qth dual volume $\widetilde{V}_q$ in [29] is replaced by a general dual Orlicz quermassintegral $\mathscr{V}_\phi$, while L_0 addition is retained. To formulate the general dual Orlicz-Minkowski problem, we propose the definition of the general dual (L_ϕ) Orlicz curvature measure $\widetilde{C}_{\phi,\mathscr{V}}$. For $K \in \mathscr{K}^n_{(o)}$ and a subset $\eta \subseteq S^{n-1}$, denote by ρ_K the radial function of K and $\boldsymbol{\alpha}^*_K(\eta) \subseteq S^{n-1}$ the reverse radial Gauss image of η, respectively. Define $\widetilde{C}_{\phi,\mathscr{V}}$, the general dual ($L_\phi$) Orlicz curvature measure of K with $\phi : \mathbb{R}^n \setminus \{o\} \to (0,\infty)$ a continuous function satisfying condition C1) (following Definition 3.1.1 in Section 3.1), by

$$\widetilde{C}_{\phi,\mathscr{V}}(K,\eta) = \int_{\boldsymbol{\alpha}^*_K(\eta)} \phi(\rho_K(u)u)[\rho_K(u)]^n \, du$$

for any Borel set $\eta \subseteq S^{n-1}$, where du is the spherical measure of S^{n-1}. Convenient formulas to calculate integrals with respect to $\widetilde{C}_{\phi,\mathscr{V}}$ are given in Lemma 3.2.2, and the weak convergence of $\widetilde{C}_{\phi,\mathscr{V}}$ is summarized in Proposition 3.2.3. These properties are crucial in solving the general dual Orlicz-Minkowski problem: *given a nonzero finite Borel measure μ defined on S^{n-1} and a continuous function $\phi : \mathbb{R}^n \setminus \{o\} \to (0,\infty)$, can* *one find a constant $\tau > 0$ and a convex body K (ideally, containing o in its interior), such that,*

$$\mu = \tau \widetilde{C}_{\phi,\mathscr{V}}(K,\cdot)?$$

With the method of Lagrange multipliers and the established variational formula, a solution to the general dual Orlicz-Minkowski problem is provided. In special cases for ϕ, we provide the condition for the uniqueness of the solution to the general dual Orlicz-Minkowski problem.

In Chapter 4, a common generalization of the problems in [54, 69, 78] and Chapter 3 is proposed, in which a very general notion of dual volume denoted by $\widetilde{V}_G$ is introduced and, simultaneously, L_0 addition (in Chapter 3) is replaced by an extension of the L_p addition called the Orlicz addition. The two-variable function G allows $\widetilde{V}_G$ to include not only the qth dual volume $\widetilde{V}_q$, the dual Orlicz quermassintegrals in [78] and the general dual Orlicz quermassintegral $\mathscr{V}_\phi$ in Chapter 3, but several other related notions as well. By combining the general dual volume $\widetilde{V}_G$ with the Orlicz addition, a general dual Orlicz curvature measure denoted by $\widetilde{C}_{G,\psi}$ is defined as

$$\widetilde{C}_{G,\psi}(K,E) = \frac{1}{n}\int_{\boldsymbol{\alpha}^*_K(E)} \frac{\rho_K(u)\, G_t(\rho_K(u),u)}{\psi(h_K(\alpha_K(u)))}\, du \tag{1.5}$$

for each Borel set $E \subset S^{n-1}$, where $K \in \mathscr{K}^n_{(o)}$, $G : (0,\infty) \times S^{n-1} \to (0,\infty)$ and $\psi : (0,\infty) \to (0,\infty)$ are continuous. The related Minkowski problem can be stated as follows: *for which nonzero finite Borel measures μ on S^{n-1} and continuous functions $G : (0,\infty) \times S^{n-1} \to (0,\infty)$ and $\psi : (0,\infty) \to (0,\infty)$ do there exist $\tau \in \mathbb{R}$ and a convex body $K \in \mathscr{K}^n_{(o)}$ such that $\mu = \tau\, \widetilde{C}_{G,\psi}(K,\cdot)$?* The corresponding equivalent Monge-Ampère type equation for this general dual Orlicz-Minkowski problem states that for given G, ψ, and $f : S^{n-1} \to [0,\infty)$, an $h : S^{n-1} \to (0,\infty)$ and $\tau \in \mathbb{R}$,

$$\frac{\tau h}{\psi \circ h}\, P(\bar{\nabla} h + h\iota)\, \det(\bar{\nabla}^2 h + hI) = f, \tag{1.6}$$

where $P(x) = |x|^{1-n} G_t(|x|, \bar{x})$, $\bar{x} = x/|x|$, $\bar{\nabla}$ and $\bar{\nabla}^2$ are the gradient vector and Hessian matrix of h, respectively, with respect to an orthonormal frame on S^{n-1}, ι is the identity map on S^{n-1}, and I is the identity matrix. The problem, which requires solving this Monge-Ampère equation, contains all previously known Minkowski problems as special cases. A solution was presented in Theorem 4.3.3 for general measures μ, assuming that $G_t = \partial G(t,u)/\partial t < 0$, G satisfies some growth conditions, and ψ

satisfies (4.54) below. Some general uniqueness theorems are also demonstrated. In our partial solution, the lack of homogeneity of G and ψ necessitates extra efforts in the variational method we employ. In another contribution, we prove very general Orlicz inequalities of the Minkowski and Brunn-Minkowski type which include others in the literature, such as [54, Theorem 7.4], as special cases.

Chapter 5 aims to complement the results in Chapter 4 by dealing with the case when $G_t > 0$. This requires extending $\widetilde{V}_G$ and $\widetilde{C}_{G,\psi}$ to more general functions $G : [0,\infty) \times S^{n-1} \to [0,\infty)$ and to compact convex sets K containing the origin, but not necessarily in their interiors. It is also necessary to show that $\widetilde{V}_G$ and $\widetilde{C}_{G,\psi}$ are continuous (see Lemma 5.1.2 and Proposition 5.3.2 (iii)), a task necessitating a more delicate treatment of the various maps and cones related to a compact convex set than the one when the origin is contained in the interior.

Unlike the proof of Theorem 4.3.3 in Chapter 4, we approach the Minkowski problem stated above when $G_t > 0$ by first dealing with the case when μ is discrete. This is achieved in Theorem 5.2.4, where we establish, under certain growth conditions on ψ, the existence of a convex polytope P with the origin in its interior, such that μ equals $\widetilde{C}_{G,\psi}(P,\cdot)$ (up to a normalization constant). If $G(t,u) = t^n/n$, then $\widetilde{V}_G(K)$ is the volume of K, so Theorem 5.2.4 recovers the solutions to the Orlicz-Minkowski problem for discrete measures by Huang and He [27] and Li [39]. When $\psi(t) = t^p$ for $p > 1$ and $G(t,u) = t^q\phi(u)$ for $q > 0$ and $\phi \in C^+(S^{n-1})$, Theorem 5.2.4 recovers the solution to the L_p dual Minkowski problem for discrete measures by Böröczky and Fodor [3, Theorem 1.1]. The techniques in these works are similar to and based on those in [33], but some of our arguments differ from and are rather more complicated than those in [3, 27, 39]. In particular, the general volume $\widetilde{V}_G$ prohibits the use of Minkowski's inequality as in [27, 39], and in general the two-variable function G, and the lack of homogeneity of G and ψ, require somewhat more delicate analysis than the special case considered in [3]. On the other hand, we are able to avoid some constructions in [3] by making use of the absolute continuity of $\widetilde{C}_{G,\psi}$ with respect to the surface area measure proved in Proposition 5.3.2 (ii).

With Theorem 5.2.4 in hand, our Minkowski problem for general measures μ can be solved by approximation. This is accomplished in Theorem 5.4.3, where it is shown that under certain conditions on G and ψ, including $G_t > 0$, a finite Borel measure μ on S^{n-1} is not concentrated on any closed hemisphere if and only if there exists a convex body $K \in \mathscr{K}_o^n$, i.e., the set of all nonempty compact convex sets containing o but not necessarily in their interiors, such that

$$(\psi \circ h_K)\mu = \left(\int_{S^{n-1}} \psi(h_K(u))\, d\mu(u) \right) \frac{\widetilde{C}_G(K, \cdot)}{\widetilde{C}_G(K, S^{n-1})}.$$

Here $\widetilde{C}_G(K, \cdot)$ equals to $\widetilde{C}_{G,\psi}(K, \cdot)$ when $\psi \equiv 1$. Again, this result recovers (in a slightly different form) and strengthens the solutions to the Orlicz-Minkowski problem in [27, Theorem 1.2] and the L_p dual Minkowski problem in [3, Theorem 1.2]. In Theorem 5.4.4, we use the same approximation techniques to prove a variant of [17, Theorem 6.4] in the case when $G_t < 0$. When $\psi(t) = t^p$, $p > 0$, and $G(t, u) = t^q$, $q < 0$, Theorem 5.4.4 implies [30, Theorem 3.5]. We end Section 5.4 with Theorem 5.4.5, a uniqueness result related to Theorem 5.4.4 under some additional assumptions on the underlying convex bodies. As far as we know, this is the first uniqueness result for Orlicz-Minkowski problems that applies when $G(t, u) = t^n/n$ and $\widetilde{V}_G(K)$ is the volume of K. A special case of Theorem 5.4.5 contributes to [54, Problem 8.2] by providing a counterpart to [54, Theorem 8.3] for sufficiently smooth convex bodies and generalizing the uniqueness assertion in [30, Theorem 4.1]. The uniqueness problem for the general dual Orlicz curvature measures still remains open.

In Section 5.5, we focus on the case when the measure μ is even, in which case one expects the solution to be an origin-symmetric convex body. Each such body generates a norm on $\mathbb{R}^n$, and every norm on $\mathbb{R}^n$ arises from an origin-symmetric convex body. This lends special significance to Minkowski problems for even measures, particularly in applications to analysis; for example, in proving the L_p affine Sobolev inequality [50] and the affine Moser-Trudinger and Morrey-Sobolev inequalities [13]. Corresponding to Theorems 5.4.3 and 5.4.4, we prove Theorems 5.5.1 and 5.5.2 for the even measure μ, where it is natural to impose weaker conditions on ψ but an extra assumption on G (i.e., that $G_t(t, \cdot)$ is even in t). In our final result of this chapter, Theorem 5.5.3, we solve our Minkowski problem under the assumption that μ is an even measure vanishing on any great subsphere, when $G_t < 0$ and ψ is decreasing. Again, if $G(t, u) = t^n/n$, $\widetilde{V}_G(K)$ is the volume of K and it recovers the solution to the Orlicz-Minkowski problem for even measures by Haberl, Lutwak, Yang, and Zhang [22]. Moreover, when $\psi(t) = t^p$ and $G(t, u) = t^q$, Theorems 5.5.1 and 5.5.3 yield the results of Huang and Zhao [30, Theorem 3.9] for $p, q > 0$ and $p \neq q$ and [30, Theorem 3.11] for $p, q < 0$ and $p \neq q$, respectively. The method we employ both avoids the use of John ellipsoids in the proof of [30, Theorem 3.9] and provides detailed information, not given in [30], on the polytopal solutions to our Minkowski problem when μ is an even discrete measure.

The main purpose of Chapter 6 is to give a systematic study to the general dual-polar Orlicz-Minkowski problem, which extends problem (1.3) in the arguably most general way: with L, the function t^p (from the integrand of the objective functional) and $V(L)$ in problem (1.3) replaced by L^*, a (general nonhomogeneous) continuous function $\varphi:(0,\infty)\rightarrow(0,\infty)$ and $\widetilde{V}_G(L)$, respectively. Namely, we pose the following problem: *under what conditions on a nonzero finite Borel measure μ defined on S^{n-1}, continuous functions $\varphi:(0,\infty)\rightarrow(0,\infty)$ and $G:(0,\infty)\times S^{n-1}\rightarrow(0,\infty)$ can we find a convex body $K\in\mathscr{K}^n_{(o)}$ solving the following optimization problem:*

$$\inf/\sup\left\{\int_{S^{n-1}}\varphi(h_L(u))d\mu(u):L\in\mathscr{K}^n_{(o)}\ \text{ and }\ \widetilde{V}_G(L^*)=\widetilde{V}_G(B^n)\right\}? \tag{1.7}$$

Moreover, problem (1.7) contains as a special case the recent polar Orlicz-Minkowski problem introduced in [44] by Luo, Ye and Zhu, i.e., solving the following optimization problem:

$$\inf/\sup\left\{\int_{S^{n-1}}\varphi(h_{L^*}(u))d\mu(u):L\in\mathscr{K}^n_{(o)}\ \text{ and }\ V(L)=V(B^n)\right\}. \tag{1.8}$$

Note that closely related to (1.8) are the Orlicz affine and geominimal surface areas, which were proposed in [72, 74, 77]. In fact, one can observe that (1.7) not only generalizes (1.8), but also is "dual" to (1.8). This is one of our motivations to study the general dual-polar Orlicz-Minkowski problem.

Another motivation for our general dual-polar Orlicz-Minkowski problem is its close connection with the general dual Orlicz-Minkowski problems in Chapters 4 and 5. Note that, in many circumstances, solving the general dual Orlicz-Minkowski problem requires to find solutions to the following optimization problem:

$$\inf/\sup\left\{\int_{S^{n-1}}\varphi(h_Q(u))d\mu(u):Q\in\mathscr{K}^n_{(o)}\ \text{ and }\ \widetilde{V}_G(Q)=\widetilde{V}_G(B^n)\right\}. \tag{1.9}$$

In particular, if $G(t,u)=t^n/n$, (1.9) recovers the Orlicz-Minkowski problem [22]. In view of (1.7), one sees that (1.7) is "polar" to (1.9). It is our belief that, like the general dual Orlicz-Minkowski problem, the newly proposed general dual-polar

Orlicz-Minkowski problem will constitute one of the core objectives in the rapidly developing dual Orlicz-Brunn-Minkowski theory recently started from the work [18] by Gardner, Hug, Weil and Ye, and independently the work [79] by Zhu, Zhou and Xu. In summary, our motivations of the general dual-polar Orlicz-Minkowski problem

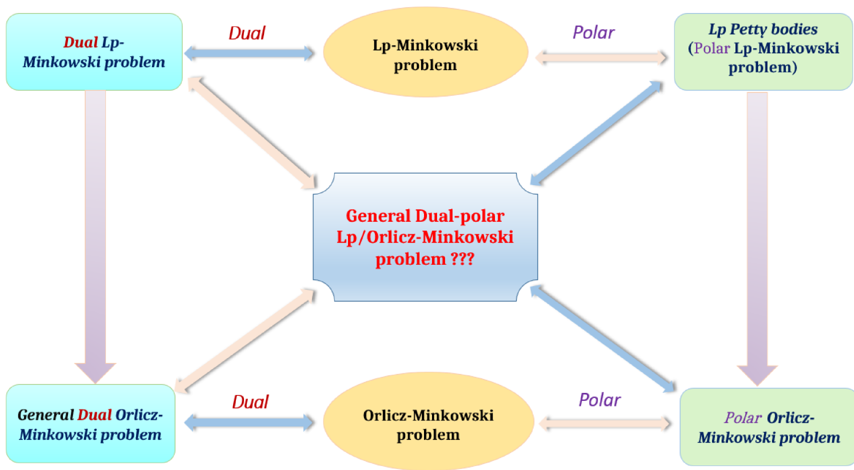

Figure 1.2: Motivation of general dual-polar Orlicz-Minkowski problem

are summarized in Figure 1.2.

Section 6.2 dedicates to establish the continuity, uniqueness, and existence of solutions to the general dual-polar Orlicz-Minkowski problem. In particular, we first obtain the polytopal solutions to the general dual-polar Orlicz-Minkowski problem when the measure μ is discrete under certain conditions such as φ being increasing and the infimum in (1.7) being considered; the detailed statements can be found in Theorem 6.2.3. In Proposition 6.2.4, the nonexistence of solutions to the general dual-polar Orlicz-Minkowski problem for discrete measures are proved by counterexamples if the supremum in (1.7) is considered, or if the infimum is considered with φ being decreasing. As $\widetilde{V}_G$ is not invariant under volume-preserving linear transforms on $\mathbb{R}^n$, our calculations in Proposition 6.2.4 are more delicate than those in [44] where the volume is considered. Our main results are given in Theorem 6.2.7 and Corollary 6.2.8, where the existence, uniqueness and continuity of solutions to the general dual-polar Orlicz-Minkowski problem for general nonzero finite Borel measure μ (instead of discrete measures) are provided.

Section 6.3 aims to investigate several variations of the general dual-polar Orlicz-Minkowski problem, including those leading to the most general definitions extending the L_p Petty bodies. In Section 6.3.1, the objective functional $\int_{S^{n-1}} \varphi(h_L(u))d\mu(u)$ in (1.7) is replaced by the "Orlicz norm" $\|h_L\|_{\mu,\varphi}$. In this case, the continuity, uniqueness, and existence of solutions are rather similar to those in Section 6.2. The second variation, considered in Section 6.3.2, is quite different from the general dual-polar Orlicz-Minkowski problem (1.7). It replaces the general dual volume $\widetilde{V}_G$ by the general volume formulated as follows: for $K \in \mathscr{K}^n_{(o)}$,

$$V_G(K) = \int_{S^{n-1}} G(h_K(u), u)\, dS(K, u).$$

Although the geometric invariant V_G has most properties required to solve the related polar Orlicz-Minkowski problem, it lacks monotonicity in set inclusion, a key ingredient in the proofs of main results in Section 6.2. With the help of the celebrated isoperimetric inequality, we are able to find a substitution of Lemma 6.1.4 for V_G and this will be stated in Lemma 6.3.11. Consequently, the existence of solutions to the related polar Orlicz-Minkowski problem is established in Theorem 6.3.12.

Chapter 2

Preliminaries

This chapter is dedicated to present some terminologies and basic notations in this thesis. Readers are referred to [21, 59] for more detailed information.

2.1 Basic facts about convex geometry

Throughout this thesis, we work on the n-dimensional Euclidean space $\mathbb{R}^n$ with the inner product $\langle \cdot, \cdot \rangle$ and the standard Euclidean norm $|\cdot|$. The origin and canonical orthonormal basis are denoted by o and $\{e_1, \dots, e_n\}$, respectively. Let $B^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ and $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit ball and unit sphere in $\mathbb{R}^n$. The characteristic function of a set X is signified by $\mathbf{1}_X$.

If X is a set, we denote by $\operatorname{conv} X$, $\operatorname{cl} X$, $\operatorname{int} X$, $\operatorname{relint} X$, ∂X, and $\dim X$ the *convex hull*, *closure*, *interior*, *relative interior* (that is, the interior with respect to the affine hull), *boundary*, and *dimension* (that is, the dimension of the affine hull) of X, respectively. If $x \in \mathbb{R}^n \setminus \{o\}$, then $x^\perp$ is the $(n-1)$-dimensional subspace orthogonal to x. We write $\mathscr{H}^k$ for k-dimensional Hausdorff measure in $\mathbb{R}^n$, where $k \in \{1, \dots, n\}$. For compact sets K, we also write $V(K) = \mathscr{H}^n(K)$ for the volume of E. The volume of the unit ball is $\kappa_n = V(B^n)$ and then $\mathscr{H}^{n-1}(S^{n-1}) = n\kappa_n$. The notation dx means $d\mathscr{H}^k(x)$ for the appropriate $k \in \{1, \dots, n\}$, unless stated otherwise. In particular, integration on S^{n-1} is usually denoted by $du = d\mathscr{H}^{n-1}(u)$.

A *convex body* in $\mathbb{R}^n$ is a compact convex subset with nonempty interior. A *convex polytope* is the convex hull of finitely many points. The class of nonempty compact convex sets in $\mathbb{R}^n$ is written as $\mathscr{K}^n$. Let $\mathscr{K}^n_o \subset \mathscr{K}^n$ denote the set of all convex bodies containing o. Let $\mathscr{K}^n_{(o)} \subset \mathscr{K}^n_o$ be the set of all convex bodies containing o in their interiors.

The *support function* of $K \in \mathscr{K}^n$, $h_K(u) : S^{n-1} \to \mathbb{R}$, is defined by

$$h_K(u) = \sup_{x\in K} \langle x, u\rangle \quad \text{for each } u \in S^{n-1}. \tag{2.1}$$

The standard metric on $\mathscr{K}^n$ is the *Hausdorff metric* $\delta(\cdot,\cdot)$, which can be defined by

$$\delta(K, L) = \|h_K - h_L\|_\infty = \sup_{u\in S^{n-1}} |h_K(u) - h_L(u)|$$

for $K, L \in \mathscr{K}^n$. We say that a sequence $K_1, K_2, \cdots, K_i, \cdots \in \mathscr{K}^n$ converges to $K \in \mathscr{K}^n$ in the Hausdorff metric, denoted by $K_i \to K$, if $\lim_{i\to\infty} \delta(K_i, K) = 0$. The *Blaschke selection theorem* provides a powerful machinery to solve Minkowski type problems. It asserts that if $K_i \in \mathscr{K}^n$ and there exists a constant $R > 0$ such that $K_i \subset RB^n$ for all $i \in \mathbb{N}$, then there exists a subsequence $\{K_{i_j}\}_{j\geq 1}$ of $\{K_i\}_{i\geq 1}$ and $K \in \mathscr{K}^n$ such that $K_{i_j} \to K$ as $j \to \infty$ in the Hausdorff metric.

A set $L \subseteq \mathbb{R}^n$ is said to be a *star-shaped set* with respect to o, if $o \in L$ and the line segment $[o, x] \subseteq L$ for all $x \in L$. For a star-shaped set L with respect to o, one can define its *radial function* $\rho_L : S^{n-1} \to [0, \infty]$ by

$$\rho_L(u) = \sup\big\{\lambda > 0 : \ \lambda u \in L\big\} \quad \text{for each } u \in S^{n-1}.$$

The function ρ_L is homogeneous of degree -1, that is, $\rho_L(rx) = r^{-1}\rho_L(x)$ for $x \in \mathbb{R}^n \setminus \{o\}$. It can also be easily checked that $\rho_{sL} = s \cdot \rho_L$ and $h_{sL} = s \cdot h_L$ for $s > 0$ and $L \in \mathscr{K}^n_o$.

Denote by $\mathscr{S}$ the set of all star-shaped sets in $\mathbb{R}^n$ with respect to o whose radial functions are measurable. Let $\mathscr{S}^n$ be the class of star-shaped sets with respect to o in $\mathbb{R}^n$ that are bounded Borel sets and whose radial functions are therefore bounded Borel measurable functions on S^{n-1}. The class of $L \in \mathscr{S}^n$ with $\rho_L > 0$ is denoted by $\mathscr{S}^n_+$ and the class $\mathscr{S}^n_{c+}$ of compact *star bodies* comprises those $L \in \mathscr{S}^n_+$ such that ρ_L is continuous on S^{n-1}. If $L \in \mathscr{S}^n_+$, then $\rho_L(u)u \in \partial L$ and $\rho_L(x) = 1$ for $x \in \partial L$, the boundary of L. The natural metric on $\mathscr{S}^n$ is the *radial metric* $\widetilde{\delta}(\cdot,\cdot)$, which can be defined by

$$\widetilde{\delta}(L_1, L_2) = \|\rho_{L_1} - \rho_{L_2}\|_\infty = \sup_{u \in S^{n-1}} |\rho_{L_1}(u) - \rho_{L_2}(u)|,$$

for $L_1, L_2 \in \mathscr{S}^n$. Consequently, we can define convergence in $\mathscr{S}^n$ by $\lim_{j\to\infty} \widetilde{\delta}(L_j, L) = 0$ for $L, L_1, L_2, \ldots \in \mathscr{S}^n$.

It follows directly from the relations between the metrics δ and $\widetilde{\delta}$ in [20, Lemma 2.3.2] that if $K, K_1, K_2, \ldots \in \mathscr{K}^n_{(o)}$, then $K_i \to K$ in the Hausdorff metric if and only if $K_i \to K$ in the radial metric. That is, for a sequence of convex bodies $\{K_i\}_{i=1}^\infty \subseteq \mathscr{K}^n_{(o)}$ and a convex body $K \in \mathscr{K}^n_{(o)}$,

$$\lim_{i\to\infty} \delta(K_i, K) = \lim_{i\to\infty} \|h_{K_i} - h_K\|_\infty = 0, \tag{2.2}$$

is equivalent to

$$\lim_{i\to\infty} \widetilde{\delta}(K_i, K) = \lim_{i\to\infty} \|\rho_{K_i} - \rho_K\|_\infty = 0. \tag{2.3}$$

For each $K \in \mathscr{K}^n_{(o)}$, one can define K^*, the *polar body* of K, by

$$K^* = \{x \in \mathbb{R}^n : \langle x, y\rangle \le 1 \text{ for all } y \in K\}.$$

Clearly $K^* \in \mathscr{K}^n_{(o)}$. Moreover, the bipolar theorem asserts that $(K^*)^* = K$ (see e.g., [59]) and then

$$\rho_K(x) h_{K^*}(x) = h_K(x)\rho_{K^*}(x) = 1 \quad \text{for } x \in \mathbb{R}^n \setminus \{o\}. \tag{2.4}$$

The *supporting hyperplane* of K in direction $u \in S^{n-1}$ is given by

$$H(K, u) = \big\{x \in \mathbb{R}^n : \langle x, u\rangle = h_K(u)\big\}.$$

The corresponding support set of K in direction u is $F(K, u) = K \cap H(K, u)$.

The *surface area measure* $S(K, \cdot)$ of a convex body K in $\mathbb{R}^n$ is defined for Borel sets $E \subset S^{n-1}$ by

$$S(K, E) = \mathscr{H}^{n-1}(\nu_K^{-1}(E)), \tag{2.5}$$

where $\nu_K^{-1}(E) = \{x \in \partial K : \ \nu_K(x) \in E\}$ is the *inverse Gauss map* of K. Clearly, $S(tK, u) = t^{n-1} S(K, u)$ for any $t > 0$ and $K \in \mathscr{K}^n_{(o)}$.

It is worthwhile to mention that for $K \in \mathscr{K}^n_{(o)}$, its volume takes the following forms:

$$V(K) = \frac{1}{n}\int_{S^{n-1}} h_K(u)\, dS(K,u) = \frac{1}{n}\int_{S^{n-1}} \rho_K(u)^n\, du.$$

Let μ be a nonzero finite Borel measure on S^{n-1}. We say that μ is *not concentrated on any closed hemisphere* if

$$\int_{S^{n-1}} \langle u, v\rangle_+ \, d\mu(u) > 0 \quad \text{for any } v \in S^{n-1}, \tag{2.6}$$

where $a_+ = \max\{a, 0\}$ for $a \in \mathbb{R}$. Condition (2.6) for measure μ is necessary to solve the classical Minkowski problem and its extensions, since it guarantees convex sets to be bounded (and hence compact). See the following picture of examples in two-dimensional discrete case, where the right one has no chances to construct a bounded set.

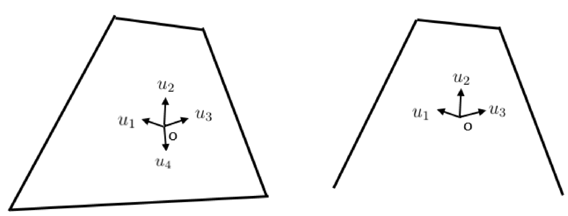

Figure 2.1: Support of μ on the plane

2.2 Maps related to a convex body

2.2.1 The radial Gauss map of a convex body for $K \in \mathscr{K}^n_{(o)}$

In this subsection, we will collect some important maps related to a convex body $K \in \mathscr{K}^n_{(o)}$.

Let $K \in \mathscr{K}^n_{(o)}$. For $E \subset \partial K$, the *spherical image (Gauss map)* of E is defined by

$$\boldsymbol{\nu}_K(E) = \{u \in S^{n-1} : x \in H(K,u) \text{ for some } x \in E\}. \tag{2.7}$$

For $E \subset S^{n-1}$, the *reverse spherical image* of E is defined as

$$\boldsymbol{x}_K(E) = \{x \in \partial K : x \in H(K,u) \text{ for some } u \in E\},$$

and the *radial Gauss image/map* is defined by

$$\boldsymbol{\alpha}_K(E) = \boldsymbol{\nu}_K\big(\{\rho_K(u)u \in \partial K : u \in E\}\big)$$

for $E \subset S^{n-1}$. (Please see Figure 2.2 for the radial Gauss image.) Let $\sigma_K \subset \partial K$,

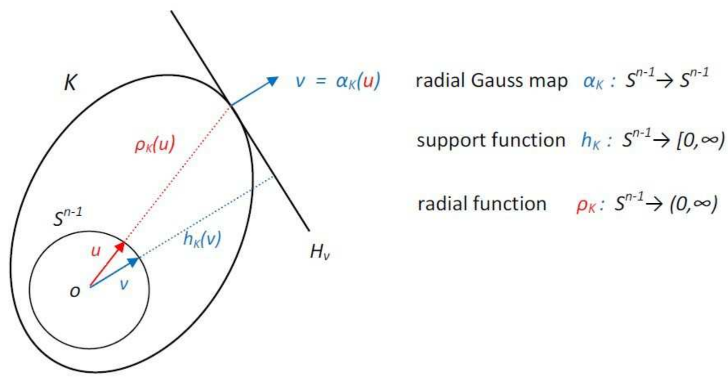

Figure 2.2: The radial Gauss image

$\eta_K \subset S^{n-1}$, and $\omega_K \subset S^{n-1}$ be the sets where $\boldsymbol{\nu}_K(\{x\})$ for $x \in \partial K$, $\boldsymbol{x}_K(\{u\})$ for $u \in \eta_K$, and $\boldsymbol{\alpha}_K(\{u\})$ for $u \in \omega_K$, respectively, have two or more elements. Then

$$\mathscr{H}^{n-1}(\sigma_K) = \mathscr{H}^{n-1}(\eta_K) = \mathscr{H}^{n-1}(\omega_K) = 0. \tag{2.8}$$

Elements of $S^{n-1} \setminus \eta_K$ are called *regular normal vectors* of K and $\operatorname{reg} K = \partial K \setminus \sigma_K$ is the set of *regular boundary points* of K. We write $\nu_K(x)$, $x_K(u)$, and $\alpha_K(u)$ instead of $\boldsymbol{\nu}_K(\{x\})$, $\boldsymbol{x}_K(\{u\})$, and $\boldsymbol{\alpha}_K(\{u\})$ if $x \in \operatorname{reg} K$, $u \in S^{n-1} \setminus \eta_K$, and $u \in S^{n-1} \setminus \omega_K$, respectively.

Next, the *inverse radial Gauss image* is defined by

$$\boldsymbol{\alpha}^*_K(E) = \{\bar{x} : x \in \partial K \cap H(K,u) \text{ for some } u \in E\} = \{\bar{x} : x \in \boldsymbol{x}_K(E)\}$$

for $E \subset S^{n-1}$, where $\bar{x} = x/|x|$. In particular, one can define a continuous map $\alpha^*_K(u) = x_K(u)/|x_K(u)|$ for $u \in S^{n-1} \setminus \eta_K$. For $E \subset S^{n-1}$, we have $\boldsymbol{\alpha}^*_K(E) = \boldsymbol{\alpha}_{K^*}(E)$.

Moreover, for $\mathscr{H}^{n-1}$-almost all $u \in S^{n-1}$,

$$\alpha_K^*(u) = \alpha_{K^*}(u), \tag{2.9}$$

and

$$u \in \boldsymbol{\alpha}_K^*(E) \quad \text{if and only if} \quad \alpha_K(u) \in E. \tag{2.10}$$

2.2.2 Maps and cones related to a convex body $K \in \mathscr{K}_o^n$

Some notations of maps in Section 2.2.1 such as ν_K, ν_K^{-1}, x_K and α_K for $K \in \mathscr{K}_{(o)}^n$ can be carried to $K \in \mathscr{K}_o^n$. However, when the inverse radial function is involved, such as $\boldsymbol{\alpha}_K^*$, things become more complicated. Moreover, additional notions are needed if $K \in \mathscr{K}_o^n$. This section is mainly used in Chapter 5.

For $K \in \mathscr{K}_o^n$, the *normal cone* of K at $z \in K$ is defined by

$$N(K,z) = \{y \in \mathbb{R}^n : \langle y, x - z\rangle \leq 0 \ \text{ for all } x \in K\}.$$

This is a closed convex cone, and $N(K,z) = \{o\}$ if $z \in \operatorname{int} K$. In particular, if $o \in \partial K$, then

$$N(K,o) = \{y \in \mathbb{R}^n : \langle y, x\rangle \leq 0 \ \text{ for all } x \in K\}. \tag{2.11}$$

Let $K \in \mathscr{K}_o^n$. Then the *dual cone* $N(K,o)^*$ of $N(K,o)$ is given by

$$N(K,o)^* = \{x \in \mathbb{R}^n : \langle x, y\rangle \leq 0 \ \text{ for all } y \in N(K,o)\} = \operatorname{cl}\{\lambda x : x \in K \text{ and } \lambda \geq 0\}; \tag{2.12}$$

the set on the right side is called the *support cone* of K at o (see [59, p. 91]). If $o \in \operatorname{int} K$, then $N(K,o)^* = \mathbb{R}^n$, and if $o \in \partial K$, then

$$\mathscr{H}^{n-1}(S^{n-1} \cap \partial N(K,o)^*) = 0. \tag{2.13}$$

For $K \in \mathscr{K}_o^n$, let $r_K : S^{n-1} \to \partial K$ denote the *radial map* of K, defined by $r_K(u) = \rho_K(u)u$. If $o \in \partial K$, then r_K need not be injective, since $\rho_K(u)u = o$ for $u \in S^{n-1} \setminus N(K,o)^*$. The radial map also need not be continuous, but it is upper semicontinuous and hence Borel measurable. However, the restriction of the radial map to $S^{n-1} \cap \operatorname{relint} N(K,o)^*$ is injective and locally bi-Lipschitz. Moreover,

$$\rho_K(u) \begin{cases} = 0 & \text{if } u \in S^{n-1} \setminus N(K,o)^*, \\ > 0 & \text{if } u \in S^{n-1} \cap \operatorname{relint} N(K,o)^*. \end{cases} \tag{2.14}$$

Here $\operatorname{relint} N(K,o)^*$ is the relative interior of the dual cone $N(K,o)^*$. For $u \in S^{n-1} \cap \partial N(K,o)^*$, we only have $\rho_K(u) \geq 0$, but if K is a convex polytope, then

$$\rho_K(u) > 0 \qquad \text{if and only if} \qquad u \in S^{n-1} \cap N(K,o)^*. \tag{2.15}$$

We recall some terminology and facts from [29, Section 2.2] and [59, Section 2.2], presented in a slightly different form (see also [3]). The *radial projection* $\tilde{\pi} : \mathbb{R}^n \setminus \{o\} \to S^{n-1}$ is defined by $\tilde{\pi}(x) = \bar{x} = x/|x|$ and $\tilde{\pi}(A) = \{\bar{x} : x \in A\}$.

For $K \in \mathscr{K}_o^n$ and $E \subset \partial K$, the spherical image of E defined in (2.7) can be expressed in terms of the normal cones, i.e.,

$$\boldsymbol{\nu}_K(E) = S^{n-1} \cap \bigcup_{x \in E} N(K,x).$$

Recall that for a Borel set $E \subset \partial K$, the spherical image $\boldsymbol{\nu}_K(E) \subset S^{n-1}$ is $\mathscr{H}^{n-1}$-measurable (see [59, Lemma 2.2.13]). Following the definition of $\boldsymbol{\alpha}_K(E)$, if $E \subset S^{n-1} \cap \operatorname{relint} N(K,o)^*$ is a Borel set, then so is $r_K(E)$, and $\boldsymbol{\alpha}_K(E)$ is $\mathscr{H}^{n-1}$-measurable. If $\emptyset \neq E \subset S^{n-1} \setminus N(K,o)^*$, then $r_K(E) = \{o\}$ and again $\boldsymbol{\alpha}_K(E) \subset S^{n-1} \cap N(K,o)$ is $\mathscr{H}^{n-1}$-measurable. The situation for a Borel set $E \subset S^{n-1}$ contained in the relative boundary of $N(K,o)^*$ seems to be unclear but will not be needed.

Recall in Section 2.2.1 that the set of $u \in S^{n-1}$ such that $\boldsymbol{\nu}_K^{-1}(\{u\})$ has at least two elements, i.e, the set of singular normal vectors, is a Borel set of $\mathscr{H}^{n-1}$-measure zero. Note that $\boldsymbol{\nu}_K$ and $\boldsymbol{\nu}_K^{-1}$ do not necessarily map disjoint sets to disjoint sets. However, the intersections are sets of singular normal vectors and sets of singular boundary points, respectively, and hence have $\mathscr{H}^{n-1}$-measure zero.

For later use, we also define

$$\Xi_K = \boldsymbol{\nu}_K^{-1}\left(S^{n-1} \cap N(K,o)\right) = K \cap \partial N(K,o)^*. \tag{2.16}$$

Clearly, $\Xi_K = \emptyset$ if $o \in \operatorname{int} K$. Moreover, if $\dim K \leq n-1$, then $\Xi_K = K$.

Following the definition of $\boldsymbol{\alpha}_K^*(E)$, if $E \subset S^{n-1}$ is a Borel set, then $\boldsymbol{\alpha}_K^*(E)$ is $\mathscr{H}^{n-1}$-measurable. This is shown in [29, Lemma 2.1] when $o \in \operatorname{int} K$. To see that it is true in general, first observe that $\boldsymbol{\nu}_K^{-1}(E) \subset \partial K$ is $\mathscr{H}^{n-1}$-measurable. If $A = \boldsymbol{\nu}_K^{-1}(E) \cap \operatorname{relint} N(K,o)^*$, then since r_K is locally bi-Lipschitz on $\operatorname{relint} N(K,o)^*$, it follows that $r_K^{-1}(A)$ is also $\mathscr{H}^{n-1}$-measurable. Let B denote the intersection of $\boldsymbol{\nu}_K^{-1}(E)$ with the relative boundary of $N(K,o)^*$. Then $r_K^{-1}(B) \subset S^{n-1} \cap \partial N(K,o)^*$, which is $\mathscr{H}^{n-1}$-measurable due to (2.13). Therefore $\boldsymbol{\alpha}_K^*(E) = r_K^{-1}(A \cup B)$ is $\mathscr{H}^{n-1}$-measurable.

For $u \in S^{n-1} \cap \operatorname{int} N(K,o)^* \setminus \omega_K$, and hence for $\mathscr{H}^{n-1}$-almost all $u \in S^{n-1} \cap \operatorname{int} N(K,o)^*$, we have

$$u \in \boldsymbol{\alpha}_K^*(E) \quad \text{if and only if} \quad \alpha_K(u) \in E. \tag{2.17}$$

Finally, we remark that

$$\boldsymbol{\alpha}_K^*(E) \cap \operatorname{relint} N(K,o)^* \subset \boldsymbol{\alpha}_K^*(E \setminus N(K,o)) \subset \boldsymbol{\alpha}_K^*(E) \cap N(K,o)^*.$$

Examples show that both inclusions can be strict, but in view of (2.13), we have

$$\boldsymbol{\alpha}_K^*(E) \cap \operatorname{relint} N(K,o)^* = \boldsymbol{\alpha}_K^*(E \setminus N(K,o)) = \boldsymbol{\alpha}_K^*(E) \cap N(K,o)^* \tag{2.18}$$

up to sets of $\mathscr{H}^{n-1}$-measure zero.

2.3 Background on functions and Orlicz linear combinations

As usual, $C(E)$ denotes the class of continuous functions on a topological space E, and we shall write $C^+(E)$ for the (strictly) positive functions in $C(E)$. Let $\Omega \subset S^{n-1}$ be a closed set not contained in any closed hemisphere of S^{n-1}. For each $f \in C^+(\Omega)$, one can define a convex body $[f]$, the *Aleksandrov body* (or *Wulff shape*) associated to it, by setting

$$[f] = \bigcap_{u \in \Omega} \left\{x \in \mathbb{R}^n : \langle x, u\rangle \le f(u)\right\}. \tag{2.19}$$

In particular, when $\Omega = S^{n-1}$ and $f = h_K$ for $K \in \mathscr{K}^n_{(o)}$, one has

$$K = [h_K] = \bigcap_{u \in S^{n-1}} \big\{x \in \mathbb{R}^n : \langle x, u\rangle \le h_K(u)\big\}.$$

When $\Omega = S^{n-1}$ and $f \in C^+(S^{n-1})$, the set of all positive continuous functions defined on S^{n-1}, it is obvious that $h_{[f]}(u) \le f(u)$ for all $u \in S^{n-1}$. A less trivial fact is that $h_{[f]}(u) = f(u)$ for almost all $u \in S^{n-1}$ with respect to the surface area measure $S([f], \cdot)$ ([69]).

If $\Omega \subset S^{n-1}$ is a closed set not contained in any closed hemisphere of S^{n-1}, and $f \in C^+(\Omega)$, define $\langle f\rangle \in \mathscr{K}^n_{(o)}$, the *convex hull* of f, by

$$\langle f\rangle = \mathrm{conv}\,\{f(u)u : u \in \Omega\}.$$

The properties of $\langle f\rangle$ are similar to those of the Aleksandrov body. In particular, taking $\Omega = S^{n-1}$, we have $\langle \rho_K\rangle = K$ for each $K \in \mathscr{K}^n_{(o)}$. It can be checked (see [29, Lemma 2.8]) that

$$[f]^* = \langle 1/f\rangle. \tag{2.20}$$

For $\varphi : (0,\infty) \to (0,\infty)$, denote

$$\begin{aligned} \mathscr{I} &= \{\varphi \text{ strictly increasing with } \varphi(1) = 1, \varphi(0) = 0, \text{ and } \varphi(\infty) = \infty\},\\ \mathscr{D} &= \{\varphi \text{ strictly decreasing with } \varphi(1) = 1, \varphi(0) = \infty, \text{ and } \varphi(\infty) = 0\}, \end{aligned}$$

where $\varphi(0)$ and $\varphi(\infty)$ are considered as limits, $\varphi(0) = \lim_{t\to 0^+} \varphi(t)$ and $\varphi(\infty) = \lim_{t\to\infty} \varphi(t)$. Note that the values of φ at $t = 0, 1, \infty$ are chosen for technical reasons; results may still hold for other values of φ at $t = 0, 1, \infty$. For $a \in \mathbb{R} \cup \{-\infty\}$, we also require the following class of functions $\varphi : (0,\infty) \to (a,\infty)$:

$$\mathscr{J}_a = \{\varphi \text{ is continuous and strictly monotonic, } \inf_{t>0} \varphi(t) = a, \text{ and } \sup_{t>0} \varphi(t) = \infty\}. \tag{2.21}$$

Note that the log function belongs to $\mathscr{J}_{-\infty}$ and $\mathscr{I} \cup \mathscr{D} \subset \mathscr{J}_0$.

Let $f_0 \in C^+(S^{n-1})$, $g \in C(S^{n-1})$, and $\varphi \in \mathscr{J}_a$ for some $a \in \mathbb{R} \cup \{-\infty\}$. Then $\varphi^{-1} : (a,\infty) \to (0,\infty)$, and since S^{n-1} is compact, we have $0 < c \le f_0 \le C$ for some $0 < c \le C$. It is then easy to check that for $\varepsilon \in \mathbb{R}$ close to 0, one can define

$f_\varepsilon = f_\varepsilon(f_0, g, \varphi) \in C^+(S^{n-1})$ by

$$f_\varepsilon(u) = \varphi^{-1}\left(\varphi(f_0(u)) + \varepsilon g(u)\right). \tag{2.22}$$

Note that we can apply (2.22) when $f_0 = h_K$ for some $K \in \mathscr{K}^n_{(o)}$ or when $f_0 = \rho_K$ for some $K \in \mathscr{S}^n_{c+}$. Sometimes we will use this definition when S^{n-1} is replaced by a closed set $\Omega \subset S^{n-1}$ not contained in any closed hemisphere of S^{n-1}. The *left derivative* and *right derivative* of a real-valued function f are denoted by f'_l and f'_r, respectively.

Let $K, L \in \mathscr{K}^n_{(o)}$. For $\varepsilon > 0$, and either $\varphi_1, \varphi_2 \in \mathscr{I}$ or $\varphi_1, \varphi_2 \in \mathscr{D}$, define $h_\varepsilon \in C^+(S^{n-1})$ (implicitly and uniquely) by

$$\varphi_1\left(\frac{h_K(u)}{h_\varepsilon(u)}\right) + \varepsilon\varphi_2\left(\frac{h_L(u)}{h_\varepsilon(u)}\right) = 1 \qquad \text{for } u \in S^{n-1}. \tag{2.23}$$

Note that $h_\varepsilon = h_\varepsilon(K, L, \varphi_1, \varphi_2)$ may not be a support function of a convex body unless $\varphi_1, \varphi_2 \in \mathscr{I}$ are convex, in which case $h_\varepsilon = h_{K+_{\varphi,\varepsilon}L}$, where $K +_{\varphi,\varepsilon} L \in \mathscr{S}^n_{c+}$ is an *Orlicz linear combination of* K *and* L (see [16, p. 463]). However, the Aleksandrov body $[h_\varepsilon]$ of h_ε belongs to $\mathscr{K}^n_{(o)}$.

An alternative approach to define Orlicz linear combinations is as follows. Let $K \in \mathscr{K}^n_{(o)}$, $g \in C(S^{n-1})$, $\varphi \in \mathscr{J}_a$ for some $a \in \mathbb{R} \cup \{-\infty\}$, and let $\widehat{h}_\varepsilon$ be defined by (2.22) with $f_0 = h_K$. This approach goes back to Aleksandrov [1] in the case when $\varphi(t) = t$. Again, the Aleksandrov body $[\widehat{h}_\varepsilon]$ of $\widehat{h}_\varepsilon$ belongs to $\mathscr{K}^n_{(o)}$. When $g = \varphi \circ h_L$ and $\varphi \in \mathscr{I} \subset \mathscr{J}_0$ is convex, $[\widehat{h}_\varepsilon] = K \widehat{+}_\varphi \varepsilon \cdot L$, as defined in [16, (10.4), p. 471].

Suppose that $K, L \in \mathscr{K}^n_{(o)}$, $\varphi \in \mathscr{I}$ is convex, and $K +_{\varphi,\varepsilon} L$ is defined by (2.23) with $\varphi_1 = \varphi_2 = \varphi$. Then, both $K +_{\varphi,\varepsilon} L$ and $K \widehat{+}_\varphi \varepsilon \cdot L$ belong to $\mathscr{K}^n_{(o)}$ and coincide when $\varphi(t) = t^p$ for some $p \geq 1$, but they differ in general (to see this, compare the corresponding different variational formulas given by [16, (8.11) and (8.12), p. 466] and [16, p. 471]).

It is known (see [16, Lemma 8.2], [24, p. 18], and [68, Lemma 3.2]) that $h_\varepsilon \to h_K$ and $\widehat{h}_\varepsilon \to h_K$ uniformly on S^{n-1} as $\varepsilon \to 0$ and hence, by [59, Lemma 7.5.2], both $[h_\varepsilon]$ and $[\widehat{h}_\varepsilon]$ converge to $K \in \mathscr{K}^n_{(o)}$ as $\varepsilon \to 0$. Part (ii) of the following lemma is proved in [24, (5.38)] for the case when $\varphi \in \mathscr{I} \cup \mathscr{D}$, but the same proof applies to the more general result stated.

Lemma 2.3.1. *Let $K, L \in \mathcal{K}^n_{(o)}$.*

(i) ([16, Lemma 8.4], [68, Lemma 5.2].) *If $\varphi_1, \varphi_2 \in \mathcal{I}$ and $(\varphi_1)'_l(1) > 0$, then*

$$\lim_{\varepsilon\to 0^+} \frac{h_\varepsilon(u) - h_K(u)}{\varepsilon} \;=\; \frac{h_K(u)}{(\varphi_1)'_l(1)}\,\varphi_2\!\left(\frac{h_L(u)}{h_K(u)}\right) \tag{2.24}$$

uniformly on S^{n-1}. For $\varphi_1, \varphi_2 \in \mathcal{D}$, (2.24) holds when $(\varphi_1)'_r(1) < 0$, with $(\varphi_1)'_l(1)$ replaced by $(\varphi_1)'_r(1)$.

(ii) (cf. [24, (5.38)].) *Let $a \in \mathbb{R} \cup \{-\infty\}$. If $\varphi \in \mathcal{J}_a$ and φ' is continuous and nonzero on $(0, \infty)$, then for $g \in C(S^{n-1})$,*

$$\lim_{\varepsilon\to 0} \frac{\widehat{h}_\varepsilon(u) - h_K(u)}{\varepsilon} = \frac{g(u)}{\varphi'\left(h_K(u)\right)}$$

uniformly on S^{n-1}, where $\widehat{h}_\varepsilon$ is defined by (2.22) with $f_0 = h_K$.

Analogous results hold for radial functions of star bodies. Let $K, L \in \mathcal{S}^n_{c+}$. For $\varepsilon > 0$, and either $\varphi_1, \varphi_2 \in \mathcal{I}$ or $\varphi_1, \varphi_2 \in \mathcal{D}$, define $\rho_\varepsilon \in C^+(S^{n-1})$ by

$$\varphi_1\left(\frac{\rho_K(u)}{\rho_\varepsilon(u)}\right) + \varepsilon\varphi_2\left(\frac{\rho_L(u)}{\rho_\varepsilon(u)}\right) = 1 \qquad \text{for } u \in S^{n-1}. \tag{2.25}$$

Then ρ_ε is the radial function of the *radial Orlicz linear combination $K\widetilde{+}_{\varphi,\varepsilon}L$ of K and L* (see [18, (22), p. 822]).

Let $a \in \mathbb{R} \cup \{-\infty\}$. For $\varphi \in \mathcal{J}_a$, $g \in C(S^{n-1})$, and $\varepsilon \in \mathbb{R}$ close to 0, define $\widehat{\rho}_\varepsilon \in C^+(S^{n-1})$ by (2.22) with $f_0 = \rho_K$. The definitions of both ρ_ε and $\widehat{\rho}_\varepsilon$ can be extended to $K, L \in \mathcal{S}^n_+$ (or even $L \in \mathcal{S}^n$), but we shall mainly work with star bodies and hence focus on $\mathcal{S}^n_{c+}$. It is known (see [18, Lemma 5.1], [24, p. 18] (with h replaced by ρ), and [79, Lemma 3.5]) that $\rho_\varepsilon \to \rho_K$ and $\widehat{\rho}_\varepsilon \to \rho_K$ uniformly on S^{n-1} as $\varepsilon \to 0$. From this and the equivalence between convergence in the Hausdorff and radial metrics for sets in $\mathcal{K}^n_{(o)}$, one sees that, for each $K \in \mathcal{K}^n_{(o)}$, both $\langle\rho_\varepsilon\rangle$ and $\langle\widehat{\rho}_\varepsilon\rangle$ converge to K in either metric.

Lemma 2.3.2. *Let $K, L \in \mathcal{S}^n_{c+}$.*

(i) ([18, Lemma 5.3]; see also [79, Lemma 4.1].) *If* $\varphi_1, \varphi_2 \in \mathscr{I}$ *and* $(\varphi_1)'_l(1) > 0$*, then*

$$\lim_{\varepsilon \to 0^+} \frac{\rho_\varepsilon(u) - \rho_K(u)}{\varepsilon} \quad = \quad \frac{\rho_K(u)}{(\varphi_1)'_l(1)} \varphi_2\left(\frac{\rho_L(u)}{\rho_K(u)}\right) \tag{2.26}$$

uniformly on S^{n-1}*. For* $\varphi_1, \varphi_2 \in \mathscr{D}$*,* (2.26) *holds when* $(\varphi_1)'_r(1) > 0$*, with* $(\varphi_1)'_l(1)$ *replaced by* $(\varphi_1)'_r(1)$*.*

(ii) (cf. [24, (5.38)].) *Let* $a \in \mathbb{R} \cup \{-\infty\}$*. If* $\varphi \in \mathscr{J}_a$ *and* φ' *is continuous and nonzero on* $(0, \infty)$*, then for* $g \in C(S^{n-1})$*,*

$$\lim_{\varepsilon \to 0} \frac{\widehat{\rho}_\varepsilon(u) - \rho_K(u)}{\varepsilon} = \frac{g(u)}{\varphi'\left(\rho_K(u)\right)} \tag{2.27}$$

uniformly on S^{n-1}*, where* $\widehat{\rho}_\varepsilon$ *is defined by* (2.22) *with* $f_0 = \rho_K$*.*

The following simple observation will be frequently used in later context.

Lemma 2.3.3. *Let* μ, μ_i *for each* $i \in \mathbb{N}$ *be nonzero finite Borel measures on* S^{n-1} *such that* $\mu_i \to \mu$ *weakly. Let* f, f_i *for each* $i \in \mathbb{N}$ *be continuous functions on* S^{n-1} *such that* $f_i \to f$ *uniformly on* S^{n-1}*. Then,*

$$\lim_{i \to \infty} \int_{S^{n-1}} f_i \, d\mu_i = \int_{S^{n-1}} f \, d\mu.$$

Chapter 3

On the general dual Orlicz-Minkowski problem

This chapter is based on our paper [69]. In this chapter, we investigate the general dual Orlicz-Minkowski problem with respect to $\widetilde{C}_{\phi,\mathscr{V}}$: *given a nonzero finite Borel measure μ defined on S^{n-1} and a continuous function $\phi : \mathbb{R}^n \setminus \{o\} \to (0, \infty)$, can one find a constant $\tau > 0$ and a convex body $K \in \mathscr{K}^n_{(o)}$ such that $\mu = \tau \widetilde{C}_{\phi,\mathscr{V}}(K, \cdot)$?* Here $\widetilde{C}_{\phi,\mathscr{V}}$ is obtained from a variational formula of $\mathscr{V}_\phi$ based on L_0 addition of convex bodies, where $\mathscr{V}_\phi$ is given by $\mathscr{V}_\phi(K) = \int_{\mathbb{R}^n \setminus K} \phi(x)dx$.

3.1 The general dual Orlicz quermassintegral

First, we consider the general dual Orlicz quermassintegral $\mathscr{V}_\phi$, whose density function with respect to the Lebesgue measure dx is a continuous function $\phi : \mathbb{R}^n \setminus \{o\} \to (0, \infty)$.

Definition 3.1.1. *For a measurable subset $E \subseteq \mathbb{R}^n$ with $o \in \mathrm{int}E$, define $\mathscr{V}_\phi(E)$ by*

$$\mathscr{V}_\phi(E) = \int_{\mathbb{R}^n \setminus E} \phi(x)\, dx. \tag{3.1}$$

Clearly, $\mathscr{V}_\phi(\cdot)$ is monotone decreasing, that is, if $E \subseteq F$ with $o \in \mathrm{int}E$, then

$\mathbb{R}^n \setminus E \supseteq \mathbb{R}^n \setminus F$ and hence

$$\mathscr{V}_\phi(E) = \int_{\mathbb{R}^n \setminus E} \phi(x)\, dx \geq \int_{\mathbb{R}^n \setminus F} \phi(x)\, dx = \mathscr{V}_\phi(F),$$

due to the positivity of ϕ.

When E is a star-shaped set in $\mathbb{R}^n$, $\mathscr{V}_\phi(E)$ can be reformulated through the radial function of E and the spherical measure du on S^{n-1}. Namely, for $E \in \mathscr{S}$, $\mathscr{V}_\phi(E)$ can be calculated by

$$\mathscr{V}_\phi(E) = \int_{\mathbb{R}^n \setminus E} \phi(x)\, dx = \int_{S^{n-1}} \left(\int_{\rho_E(u)}^{\infty} \phi(ru) r^{n-1}\, dr \right) du. \tag{3.2}$$

For convenience, let

$$\Phi(t, u) = \int_t^{\infty} \phi(ru) r^{n-1}\, dr,$$

and hence formula (3.2) can be rewritten as

$$\mathscr{V}_\phi(E) = \int_{S^{n-1}} \Phi(\rho_E(u), u)\, du. \tag{3.3}$$

For each $K \in \mathscr{K}^n_{(o)}$, one gets, by formula (3.3)

$$\mathscr{V}_\phi(K) = \int_{S^{n-1}} \Phi(\rho_K(u), u)\, du = \int_{S^{n-1}} \Phi(h_{K^*}(u)^{-1}, u)\, du. \tag{3.4}$$

In later context, for each $K \in \mathscr{K}^n_{(o)}$, $\mathscr{V}_\phi(K)$ will be called the *general dual Orlicz quermassintegral* of K.

Now we list the basic conditions for function ϕ:

C1) $\phi : \mathbb{R}^n \setminus \{o\} \to (0, \infty)$ is a continuous function, such that, for any fixed $t > 0$, the function

$$\Phi(t, u) = \int_t^{\infty} \phi(ru) r^{n-1}\, dr$$

is positive and continuous on S^{n-1};

C2) for any fixed $u_0 \in S^{n-1}$ and any fixed positive constant $b_0 \in (0, 1)$, one has

$$\lim_{a \to 0^+} \mathscr{V}_\phi(\mathbb{R}^n \setminus \mathscr{C}(u_0, a, b_0)) = \infty,$$

where $\mathscr{C}(u_0, a, b_0)$ is defined by

$$\mathscr{C}(u_0, a, b_0) = \Big\{x \in \mathbb{R}^n : \ \langle \bar{x}, u_0 \rangle \geq b_0 \text{ and } |x| \geq a\Big\}.$$

In fact, condition C1) guarantees that $\mathscr{V}_\phi(K) < \infty$ for each $K \in \mathscr{K}^n_{(o)}$. To see this, as $o \in \mathrm{int}K$, there exists a constant $r_0 > 0$ such that $r_0 B^n \subseteq K$. By formula (3.4) and the fact that $\mathscr{V}_\phi(\cdot)$ is monotone decreasing, one has,

$$\mathscr{V}_\phi(K) \leq \mathscr{V}_\phi(r_0 B^n) = \int_{S^{n-1}} \Phi(r_0, u)\, du < \infty.$$

Condition C2) is for the solution of the general dual Orlicz-Minkowski problem.

A typical function satisfying conditions C1) and C2) is a continuous function $\phi : \mathbb{R}^n \setminus \{o\} \to (0, \infty)$ such that

$$\sup_{|x|>r_1} \phi(x)|x|^{n-\alpha_1-1} \leq C_1 \quad \text{and} \quad \inf_{|x|<r_1} \phi(x)|x|^{n-\alpha_2-1} \geq C_2 \tag{3.5}$$

hold for some constants $0 < r_1 < \infty$, $C_1 < \infty$, $C_2 > 0$ and $-\infty < \alpha_1, \alpha_2 < -1$. In particular, if

$$\lim_{|x|\to\infty} \phi(x)|x|^{n-\alpha_1-1} = C_1 \ \text{ and } \ \lim_{|x|\to 0} \phi(x)|x|^{n-\alpha_2-1} = C_2$$

for some constants $0 < C_1, C_2 < \infty$ and $-\infty < \alpha_1, \alpha_2 < -1$, then such ϕ satisfies (3.5) (for different constants). Now let us check that a continuous function ϕ satisfying (3.5) must also satisfy conditions C1) and C2). To this end, let $t > 0$ and $u \in S^{n-1}$ be fixed. It is obvious to have $\Phi(t, u) > 0$. Moreover

$$\begin{aligned}
\Phi(t, u) &= \int_t^{r_1} \phi(ru) r^{n-1}\, dr + \int_{r_1}^{\infty} \phi(ru) r^{n-1}\, dr \\
&\leq \Big| \int_t^{r_1} \phi(ru) r^{n-1}\, dr \Big| + \int_{r_1}^{\infty} \phi(ru) r^{n-1}\, dr \\
&\leq \Big| \int_t^{r_1} \phi(ru) r^{n-1}\, dr \Big| + C_1 \int_{r_1}^{\infty} r^{\alpha_1}\, dr \\
&= \Big| \int_t^{r_1} \phi(ru) r^{n-1}\, dr \Big| - \frac{C_1}{\alpha_1 + 1} \cdot r_1^{\alpha_1+1}.
\end{aligned}$$

Thus $\Phi(t, u) < \infty$ due to the continuity of ϕ, and $\Phi(t, u)$ is well defined. Now we claim that $\Phi(t,\cdot)$ is continuous on S^{n-1}. For fixed t and for an arbitrary sequence $u_i \to u$ with $u_i, u \in S^{n-1}$, one has, for all $r \geq t$, $\phi(ru_i)r^{n-1} \to \phi(ru)r^{n-1}$ and

$$\phi(ru_i)r^{n-1} \leq C_1 r^{\alpha_1} + M$$

for all $i \geq 1$, where, due to the continuity of ϕ,

$$M = \max\Big\{\phi(x)|x|^{n-1}: \quad |x| \text{ is between } t \text{ and } r_1\Big\} < \infty.$$

It follows from the dominated convergence theorem that

$$\begin{aligned}
\lim_{i\to\infty} \Phi(t,u_i) &= \lim_{i\to\infty} \int_t^\infty \phi(ru_i)r^{n-1}\,dr \\
&= \int_t^\infty \lim_{i\to\infty} \phi(ru_i)r^{n-1}\,dr \\
&= \int_t^\infty \phi(ru)r^{n-1}\,dr \\
&= \Phi(t,u).
\end{aligned}$$

Hence $\Phi(t,u)$ is continuous on S^{n-1} and C1) is verified. Now let us verify C2) as follows: for any $b_0 \in (0,1)$,

$$\begin{aligned}
\lim_{a\to 0^+} \mathscr{V}_\phi(\mathbb{R}^n \setminus \mathscr{C}(u_0,a,b_0)) &= \lim_{a\to 0^+} \int_{\{u\in S^{n-1}:\langle u,u_0\rangle \geq b_0\}} \int_a^\infty \phi(ru)r^{n-1}\,dr\,du \\
&\geq \limsup_{a\to 0^+} \int_{\{u\in S^{n-1}:\langle u,u_0\rangle \geq b_0\}} \int_a^{r_1} \phi(ru)r^{n-1}\,dr\,du \\
&\geq C_2 \cdot \limsup_{a\to 0^+} \int_{\{u\in S^{n-1}:\langle u,u_0\rangle \geq b_0\}} \int_a^{r_1} r^{\alpha_2}\,dr\,du \\
&= C_2 \cdot \left(\int_{\{u\in S^{n-1}:\langle u,u_0\rangle \geq b_0\}} du\right) \cdot \limsup_{a\to 0^+} \frac{r_1^{1+\alpha_2} - a^{1+\alpha_2}}{1+\alpha_2} \\
&= \infty,
\end{aligned}$$

where we have used (3.2), (3.5) and $\alpha_2 < -1$.

Now let us provide several special cases of functions satisfying conditions C1) and C2).

Case 1: $\phi(x) = \psi(|x|)$ for all $x \in \mathbb{R}^n \setminus \{o\}$ with $\psi : (0,\infty) \to (0,\infty)$ a continuous function. In this case,

$$\Phi(t,u)=\int_t^\infty \phi(ru)r^{n-1}\,dr=\int_t^\infty \psi(r)r^{n-1}\,dr:=\frac{1}{n}\cdot\hat{\phi}(t). \tag{3.6}$$

Equivalently,

$$\psi(t)=-\hat{\phi}'(t)t^{1-n}/n. \tag{3.7}$$

By formula (3.4), one has, for $K\in\mathscr{K}^n_{(o)}$,

$$\mathscr{V}_\phi(K)=\int_{S^{n-1}}\Phi(\rho_K(u),u)\,du=\frac{1}{n}\int_{S^{n-1}}\hat{\phi}(\rho_K(u))\,du=\widetilde{V}_{\hat{\phi}}(K),$$

where $\widetilde{V}_\varphi$ is the dual (L_φ) Orlicz quermassintegral proposed in [78], namely

$$\widetilde{V}_\varphi(K)=\frac{1}{n}\int_{S^{n-1}}\varphi(\rho_K(u))du.$$

In [78], the dual Orlicz-Minkowski problem is solved under the following conditions:

A1) $\hat{\phi}:(0,\infty)\to(0,\infty)$ is a strictly decreasing continuous function with

$$\lim_{t\to 0^+}\hat{\phi}(t)=\infty \quad \text{and} \quad \lim_{t\to\infty}\hat{\phi}(t)=0;$$

A2) $\hat{\phi}'$, the derivative of $\hat{\phi}$, exists and is strictly negative on $(0,\infty)$;

A3) $\hat{\varphi}(t)=-\hat{\phi}'(t)t:(0,\infty)\to(0,\infty)$ is continuous; hence

$$\hat{\phi}(t)=\int_t^\infty \frac{\hat{\varphi}(s)}{s}\,ds.$$

In Case 1, it is obvious that $\hat{\varphi}(t)=n\psi(t)t^n$. Now let us check that if $\hat{\phi}$ and its companion function $\hat{\varphi}$ satisfy conditions A1)-A3), then $\phi(x)=\psi(|x|)$ with ψ given by (3.7) satisfies conditions C1) and C2). In fact, condition C1) can be easily checked by (3.6) and A1). Let us verify condition C2) as follows: for any $b_0\in(0,1)$,

$$\begin{aligned}\lim_{a\to 0^+}\mathscr{V}_\phi\big(\mathbb{R}^n\setminus\mathscr{C}(u_0,a,b_0)\big) &= \lim_{a\to 0^+}\int_{\{u\in S^{n-1}:\langle u,u_0\rangle\geq b_0\}}\int_a^\infty \phi(ru)r^{n-1}\,dr\,du\\ &= \frac{1}{n}\cdot\lim_{a\to 0^+}\hat{\phi}(a)\cdot\bigg(\int_{\{u\in S^{n-1}:\langle u,u_0\rangle\geq b_0\}}du\bigg)\\ &= \infty,\end{aligned}$$

where we have used (3.2), (3.6), and condition A1).

Case 2: $\phi(x) = \psi(|x|)\phi_2(\bar{x})$ where $\bar{x} = x/|x|$, $\psi : (0,\infty) \to (0,\infty)$ is a continuous function on $(0,\infty)$, and $\phi_2 : S^{n-1} \to (0,\infty)$ is a continuous function on S^{n-1}. In this case, the general dual (L_ϕ) Orlicz quermassintegral of $K \in \mathscr{K}^n_{(o)}$ has the following form:

$$\begin{aligned}
\mathscr{V}_\phi(K) &= \int_{S^{n-1}} \int_{\rho_K(u)}^{\infty} \phi(ru) r^{n-1}\, dr\, du \\
&= \int_{S^{n-1}} \left(\int_{\rho_K(u)}^{\infty} \psi(r) r^{n-1}\, dr \right) \phi_2(u)\, du \\
&= \frac{1}{n} \int_{S^{n-1}} \hat{\phi}(\rho_K(u)) \phi_2(u)\, du, \qquad (3.8)
\end{aligned}$$

where $\hat{\phi}$ is given by (3.6). Again, if $\hat{\phi}$ and its companion function $\hat{\varphi}$ satisfy conditions A1)-A3), then $\phi(x) = \psi(|x|)\phi_2(\bar{x})$ with ψ give by (3.7) satisfies conditions C1) and C2); this follows from an argument similar to the one as in Case 1. A typical example in this case is

$$\phi(x) = \|x\|^{q-n} = |x|^{q-n} \cdot \|\bar{x}\|^{q-n}$$

where $q < 0$ is a constant and $\|\cdot\| : \mathbb{R}^n \to [0,\infty)$ is any norm on $\mathbb{R}^n$. (Note that $\phi_2(\bar{x}) = \|\bar{x}\|^{q-n}$ is always positive, due to the equivalence between the two norms $\|\cdot\|$ and $|\cdot|$.) Indeed, when $\phi(x) = \|x\|^{q-n} = |x|^{q-n} \cdot \|\bar{x}\|^{q-n}$, then $\psi(|x|) = |x|^{q-n}$. Hence

$$\hat{\phi}(t) = n \int_t^{\infty} \psi(r) r^{n-1} dr = n \int_t^{\infty} r^{q-1}\, dr = -\frac{n}{q} \cdot t^q,$$

and $\hat{\varphi} = nt^q$, which satisfy conditions A1)-A3).

Lemma 3.1.2. *Assume that ϕ is a function satisfying condition C1). If the sequence $\{K_i\}_{i=1}^{\infty} \subseteq \mathscr{K}^n_{(o)}$ converges to $K \in \mathscr{K}^n_{(o)}$ in the sense of Hausdorff metric, then*

$$\lim_{i\to\infty} \mathscr{V}_\phi(K_i) = \mathscr{V}_\phi(K).$$

Proof. Let $\phi : \mathbb{R}^n \setminus \{o\} \to (0,\infty)$ be a continuous function satisfying C1). It can be checked that, for any fixed $u \in S^{n-1}$ and for any fixed constant $t_0 > 0$,

$$\lim_{t\to t_0} \Phi(t,u) = \Phi(t_0,u). \qquad (3.9)$$

In fact, for any fixed $u \in S^{n-1}$, $\Phi(t,u)$ is a decreasing function on $t \in (0,\infty)$. Let $t \to t_0$, and without loss of generality assume that $t > t_0/2$. By condition C1) and the fact that $\Phi(t,u)$ is decreasing on t, one has,

$$\Phi(t,u) = \int_t^{\infty} \phi(ru)r^{n-1}\,dr \le \int_{t_0/2}^{\infty} \phi(ru)r^{n-1}dr = \Phi(t_0/2,u) < \infty.$$

It follows from the dominated convergence theorem that

$$\lim_{t\to t_0} \Phi(t,u) = \lim_{t\to t_0} \int_t^{\infty} \phi(ru)r^{n-1}\,dr = \int_{t_0}^{\infty} \phi(ru)r^{n-1}\,dr = \Phi(t_0,u).$$

Let $\{K_i\}_{i=1}^{\infty} \subseteq \mathscr{K}_{(o)}^n$ be a sequence of convex bodies converging to $K \in \mathscr{K}_{(o)}^n$ in the Hausdorff metric. Based on (2.3), ρ_{K_i} converges to ρ_K uniformly on S^{n-1}. Moreover, as $K \in \mathscr{K}_{(o)}^n$, one can find a constant $R_1 > 0$, such that, for all $u \in S^{n-1}$ and for all $i = 1, 2, \cdots$,

$$R_1 \;\le\; \rho_{K_i}(u) \quad \text{and} \quad R_1 \le \rho_K(u).$$

Together with the fact that $\Phi(t,u)$ is a decreasing function on $t \in (0,\infty)$, one has

$$\Phi(\rho_{K_i}(u),u) \le \Phi(R_1,u) \;\text{ and }\; \Phi(\rho_K(u),u) \le \Phi(R_1,u) \quad \text{for all } u \in S^{n-1}.$$

By condition C1), $\Phi(R_1,u)$ is positive and continuous on S^{n-1}. Hence,

$$\int_{S^{n-1}} \Phi(R_1,u)\,du < \infty.$$

It follows from (3.4), (3.9) and the dominated convergence theorem that

$$\begin{aligned}
\lim_{i\to\infty} \mathscr{V}_\phi(K_i) &= \lim_{i\to\infty} \int_{S^{n-1}} \Phi(\rho_{K_i}(u),u)\,du \\
&= \int_{S^{n-1}} \lim_{i\to\infty} \Phi(\rho_{K_i}(u),u)\,du \\
&= \int_{S^{n-1}} \Phi(\rho_K(u),u)\,du \\
&= \mathscr{V}_\phi(K).
\end{aligned}$$

This concludes the proof of Lemma 3.1.2. $\square$

3.2 The general dual Orlicz curvature measure $\widetilde{C}_{\phi,\mathscr{V}}$

We are ready to give the definition of the general dual Orlicz curvature measure $\widetilde{C}_{\phi,\mathscr{V}}$. For $K \in \mathscr{K}^n_{(o)}$, let $\Psi_K(u) = \phi(\rho_K(u)u)[\rho_K(u)]^n$ for $u \in S^{n-1}$. In fact, for any $x \in \partial K$, one has $\Psi_K(\bar{x}) = \phi(x)|x|^n$.

Definition 3.2.1. *For any $K \in \mathscr{K}^n_{(o)}$ and for any function ϕ satisfying condition C1), the general dual (L_ϕ) Orlicz curvature measure of K, denoted by $\widetilde{C}_{\phi,\mathscr{V}}(K,\cdot)$, is given by*

$$\widetilde{C}_{\phi,\mathscr{V}}(K,\eta) = \int_{\boldsymbol{\alpha}^*_K(\eta)} \Psi_K(u)\,du$$

for any Borel set $\eta \subseteq S^{n-1}$.

Indeed, for each $K \in \mathscr{K}^n_{(o)}$, $\widetilde{C}_{\phi,\mathscr{V}}(K,\cdot)$ does define a Borel measure on S^{n-1}. To this end, we only need to show that $\widetilde{C}_{\phi,\mathscr{V}}(K,\cdot)$ satisfies the countable additivity, as $\widetilde{C}_{\phi,\mathscr{V}}(K,\emptyset) = 0$ holds trivially. That is, we need to prove

$$\widetilde{C}_{\phi,\mathscr{V}}(K,\cup_{i=1}^\infty \eta_i) = \sum_{i=1}^\infty \widetilde{C}_{\phi,\mathscr{V}}(K,\eta_i)$$

for any sequence of pairwise disjoint Borel sets $\eta_1,\eta_2,\cdots \subseteq S^{n-1}$. Recall that $\boldsymbol{\alpha}^*_K\big(\cup_{i=1}^\infty \eta_i\big) = \cup_{i=1}^\infty \boldsymbol{\alpha}^*_K(\eta_i)$ by [29, Lemma 2.3], and

$$\boldsymbol{\alpha}^*_K(\eta_i) = \tilde{\pi}(\boldsymbol{x}_K(\eta_i)) = \{\bar{x}:\ x \in \boldsymbol{x}_K(\eta_i)\} \subseteq S^{n-1}$$

is spherical measurable for each $i \geq 1$ by [29, Lemma 2.1], where $\boldsymbol{x}_K(\eta_i)$ is the reverse spherical image of $\eta_i \subseteq S^{n-1}$ given by

$$\boldsymbol{x}_K(\eta_i) = \big\{x \in \partial K : x \in H(K,u) \ \text{ for some } \ u \in \eta_i\big\} \subseteq \partial K.$$

Therefore,

$$\widetilde{C}_{\phi,\mathscr{V}}(K,\cup_{i=1}^\infty \eta_i) = \int_{\boldsymbol{\alpha}^*_K(\cup_{i=1}^\infty \eta_i)} \Psi_K(u)\,du = \int_{\cup_{i=1}^\infty \boldsymbol{\alpha}^*_K(\eta_i)} \Psi_K(u)\,du. \tag{3.10}$$

The countable additivity will follow immediately if $\cup_{i=1}^\infty \boldsymbol{\alpha}^*_K(\eta_i)$ is pairwise disjoint. However, by [29, Lemma 2.4], one gets that $\{\boldsymbol{\alpha}^*_K(\eta_j) \setminus \omega_K\}_{j=1}^\infty$ is pairwise disjoint. Since measure of ω_K turns out to be zero [29, p.339-340], then by (3.10), one can obtain

$$\begin{aligned}
\widetilde{C}_{\phi,\mathscr{V}}(K,\cup_{i=1}^{\infty}\eta_i) &= \int_{\cup_{i=1}^{\infty}(\boldsymbol{\alpha}_K^*(\eta_i)\setminus\omega_K)} \Psi_K(u)\,du \\
&= \sum_{i=1}^{\infty}\int_{\boldsymbol{\alpha}_K^*(\eta_i)\setminus\omega_K} \Psi_K(u)du \\
&= \sum_{i=1}^{\infty}\int_{\boldsymbol{\alpha}_K^*(\eta_i)} \Psi_K(u)du \\
&= \sum_{i=1}^{\infty}\widetilde{C}_{\phi,\mathscr{V}}(K,\eta_i).
\end{aligned}$$

This concludes that $\widetilde{C}_{\phi,\mathscr{V}}$ is a Borel measure.

The following lemma provides convenient formulas to calculate integrals with respect to the measure $\widetilde{C}_{\phi,\mathscr{V}}(K,\cdot)$. Recall that $\Psi_K(u) = \phi(\rho_K(u)u)[\rho_K(u)]^n$ for all $u \in S^{n-1}$.

Lemma 3.2.2. *Let ϕ be a function satisfying condition $C1$). For each $K \in \mathscr{K}_{(o)}^n$, the following formulas*

$$\begin{aligned}
\int_{S^{n-1}} g(v)d\widetilde{C}_{\phi,\mathscr{V}}(K,v) &= \int_{S^{n-1}} g(\alpha_K(u))\Psi_K(u)du && (3.11)\\
&= \int_{\operatorname{reg} K} \langle x,\nu_K(x)\rangle g(\nu_K(x))\phi(x)\,d\mathscr{H}^{n-1}(x) && (3.12)
\end{aligned}$$

hold for any bounded Borel function $g : S^{n-1} \to \mathbb{R}$.

Proof. First, we prove (3.11). Let $\gamma(v) = \sum_{i=1}^m a_i\mathbf{1}_{\eta_i}(v)$ for any $v \in S^{n-1}$ be an arbitrary simple function, where $\eta_i \subseteq S^{n-1}$ are Borel sets and $\mathbf{1}_A$ denotes the indicator function of the set A. By [29, (2.21)], one has $u \in \boldsymbol{\alpha}_K^*(\eta)$ if and only if $\alpha_K(u) \in \eta$, and this further yields that

$$\begin{aligned}
\int_{S^{n-1}} \gamma(\alpha_K(u))\Psi_K(u)\,du &= \int_{S^{n-1}} \sum_{i=1}^m a_i\mathbf{1}_{\eta_i}(\alpha_K(u))\Psi_K(u)\,du \\
&= \int_{S^{n-1}} \sum_{i=1}^m a_i\mathbf{1}_{\boldsymbol{\alpha}_K^*(\eta_i)}(u)\Psi_K(u)\,du \\
&= \sum_{i=1}^m a_i\int_{S^{n-1}} \mathbf{1}_{\boldsymbol{\alpha}_K^*(\eta_i)}(u)\Psi_K(u)\,du.
\end{aligned}$$

Together with Definition 3.2.1, one has

$$\begin{aligned}
\int_{S^{n-1}} \gamma(\alpha_K(u))\Psi_K(u)\,du &= \sum_{i=1}^{m} a_i \int_{S^{n-1}} \mathbf{1}_{\boldsymbol{\alpha}^*_K(\eta_i)}(u)\Psi_K(u)\,du \\
&= \sum_{i=1}^{m} a_i \widetilde{C}_{\phi,\mathscr{V}}(K,\eta_i) \\
&= \sum_{i=1}^{m} a_i \int_{S^{n-1}} \mathbf{1}_{\eta_i}(v)\,d\widetilde{C}_{\phi,\mathscr{V}}(K,v) \\
&= \int_{S^{n-1}} \gamma(v)\,d\widetilde{C}_{\phi,\mathscr{V}}(K,v).
\end{aligned}$$

That is, (3.11) holds true for simple functions. Following from a standard limit approach by simple functions, one can prove formula (3.11) for general bounded Borel functions $g : S^{n-1} \to \mathbb{R}$.

Next we prove (3.12). According to [29, (2.31)], for each bounded integrable function $f : S^{n-1} \to \mathbb{R}$, one has

$$\begin{aligned}
\int_{S^{n-1}} f(u)\phi(\rho_K(u)u)\,du &= \int_{\operatorname{reg} K} \langle x, \nu_K(x)\rangle f(\bar{x}) \frac{\phi(\rho_K(\bar{x})\bar{x})}{\rho^n_K(\bar{x})}\,d\mathscr{H}^{n-1}(x) \\
&= \int_{\operatorname{reg} K} \langle x, \nu_K(x)\rangle f(\bar{x}) \frac{\phi(x)}{|x|^n}\,d\mathscr{H}^{n-1}(x),
\end{aligned}$$

where $\bar{x} = x/|x|$, $\rho_K(\bar{x})\bar{x} = x$, and $\rho_K(\bar{x}) = |x|$. Together with (3.11) and the fact that $f = g \circ \alpha_K$ is bounded integrable on S^{n-1}, one has

$$\begin{aligned}
\int_{S^{n-1}} g(v)\,d\widetilde{C}_{\phi,\mathscr{V}}(K,v) &= \int_{S^{n-1}} g(\alpha_K(u))\Psi_K(u)\,du \\
&= \int_{\operatorname{reg} K} \langle x, \nu_K(x)\rangle g(\nu_K(x))\phi(x)\,d\mathscr{H}^{n-1}(x).
\end{aligned}$$

Hence, (3.12) holds true. □

The weak convergence of the general dual Orlicz curvature measure is proved in the following proposition.

Proposition 3.2.3. *Let ϕ be a function satisfying condition $C1$). If the sequence $\{K_i\}_{i=1}^{\infty} \subseteq \mathscr{K}^n_{(o)}$ converges to $K \in \mathscr{K}^n_{(o)}$ in the Hausdorff metric, then $\widetilde{C}_{\phi,\mathscr{V}}(K_i,\cdot)$ converges to $\widetilde{C}_{\phi,\mathscr{V}}(K,\cdot)$ weakly.*

Proof. As $\{K_i\}_{i=1}^{\infty} \subseteq \mathscr{K}_{(o)}^n$ converges to $K \in \mathscr{K}_{(o)}^n$, then ρ_{K_i} converges to ρ_K uniformly (see (2.3)) and hence one can find constants $R_1, R_2 > 0$, such that, for all $u \in S^{n-1}$ and for all $i \geq 1$,

$$R_1 \leq \rho_{K_i}(u) \leq R_2 \quad \text{and} \quad R_1 \leq \rho_K(u) \leq R_2.$$

For any fixed $u \in S^{n-1}$ and for any function ϕ satisfying condition C1), it can be checked that

$$\Psi_{K_i}(u) = \phi(\rho_{K_i}(u)u)[\rho_{K_i}(u)]^n \to \phi(\rho_K(u)u)[\rho_K(u)]^n = \Psi_K(u) \quad \text{uniformly on} \quad S^{n-1}. \tag{3.13}$$

Note that $\alpha_{K_i} \to \alpha_K$ almost everywhere on S^{n-1} (see [29, Lemma 2.2]). For any continuous function $g : S^{n-1} \to \mathbb{R}$, by (3.13), there exists a constant $M > 0$, such that, for all $u \in S^{n-1}$ and for all $i = 1, 2, \cdots$,

$$|g(\alpha_{K_i}(u))\Psi_{K_i}(u)| \leq M \quad \text{and} \quad |g(\alpha_K(u))\Psi_K(u)| \leq M.$$

It follows from the dominated convergence theorem that

$$\begin{aligned} \lim_{i\to\infty} \int_{S^{n-1}} g(\alpha_{K_i}(u))\Psi_{K_i}(u)\, du &= \int_{S^{n-1}} \lim_{i\to\infty} g(\alpha_{K_i}(u))\Psi_{K_i}(u)\, du \\ &= \int_{S^{n-1}} g(\alpha_K(u))\Psi_K(u)\, du. \end{aligned}$$

Together with (3.11), then

$$\begin{aligned} \lim_{i\to\infty} \int_{S^{n-1}} g(v)\, d\widetilde{C}_{\phi,\mathscr{V}}(K_i, v) &= \lim_{i\to\infty} \int_{S^{n-1}} g(\alpha_{K_i}(u))\Psi_{K_i}(u)\, du \\ &= \int_{S^{n-1}} g(v)\, d\widetilde{C}_{\phi,\mathscr{V}}(K, v), \end{aligned}$$

hold for any continuous function $g : S^{n-1} \to \mathbb{R}$. In conclusion, $\widetilde{C}_{\phi,\mathscr{V}}(K_i, \cdot)$ converges weakly to $\widetilde{C}_{\phi,\mathscr{V}}(K, \cdot)$ as desired. □

Proposition 3.2.4. *Let $K \in \mathscr{K}_{(o)}^n$ and ϕ be a function satisfying condition C1). Then, $\widetilde{C}_{\phi,\mathscr{V}}(K, \cdot)$ is absolutely continuous with respect to the surface area measure $S(K, \cdot)$.*

Proof. Let $\eta \subseteq S^{n-1}$ be a Borel set and $g = \mathbf{1}_\eta$ in (3.12). Then

$$\widetilde{C}_{\phi,\mathscr{V}}(K,\eta) = \int_{\nu_K^{-1}(\eta)} \langle x, \nu_K(x)\rangle \phi(x) d\mathscr{H}^{n-1}(x).$$

Since $K \in \mathscr{K}_{(o)}^n$ and ϕ is a function satisfying condition C1), there exists a constant $T < \infty$, such that, $\langle x, \nu_K(x)\rangle \phi(x) \le T$ for all $x \in \partial K$. Then

$$\int_{\nu_K^{-1}(\eta)} \langle x, \nu_K(x)\rangle \phi(x) d\mathscr{H}^{n-1}(x) \le T \int_{\nu_K^{-1}(\eta)} d\mathscr{H}^{n-1}(x).$$

If $\eta \subseteq S^{n-1}$ is a Borel set such that $S(K,\eta) = 0$, then $\mathscr{H}^{n-1}(\nu_K^{-1}(\eta)) = 0$ and thus

$$\widetilde{C}_{\phi,\mathscr{V}}(K,\eta) \le T \cdot \mathscr{H}^{n-1}(\nu_K^{-1}(\eta)) = 0.$$

As a result, $\widetilde{C}_{\phi,\mathscr{V}}(K,\cdot)$ is absolutely continuous with respect to $S(K,\cdot)$. $\square$

Let us discuss the measure $\widetilde{C}_{\phi,\mathscr{V}}(K,\cdot)$ for $K \in \mathscr{K}_{(o)}^n$ under Case 1 and Case 2 given in Section 3.1. In Case 1, i.e., $\phi(x) = \psi(|x|)$, it follows from Definition 3.2.1 that for any Borel set $\eta \subseteq S^{n-1}$,

$$\begin{aligned}
\widetilde{C}_{\phi,\mathscr{V}}(K,\eta) &= \int_{\boldsymbol{\alpha}_K^*(\eta)} \phi(\rho_K(u)u)[\rho_K(u)]^n \, du \\
&= \int_{\boldsymbol{\alpha}_K^*(\eta)} \psi(\rho_K(u))[\rho_K(u)]^n \, du \\
&= \frac{1}{n}\int_{\boldsymbol{\alpha}_K^*(\eta)} \hat{\varphi}(\rho_K(u)) \, du, \qquad (3.14)
\end{aligned}$$

where $\hat{\varphi}(t) = n\psi(t)t^n$. Recall that for $K \in \mathscr{K}_{(o)}^n$ and $\varphi : (0,\infty) \to (0,\infty)$ a continuous function, the dual L_φ Orlicz curvature measure of K, denoted by $\widetilde{C}_\varphi(K,\cdot)$, is defined in [78] as follows: for each Borel set $\eta \subseteq S^{n-1}$,

$$\widetilde{C}_\varphi(K,\eta) = \frac{1}{n}\int_{\boldsymbol{\alpha}_K^*(\eta)} \varphi(\rho_K(u)) du.$$

Hence, (3.14) asserts that $\widetilde{C}_{\phi,\mathscr{V}}(K,\cdot) = \widetilde{C}_{\hat{\varphi}}(K,\cdot)$. In particular, if $\phi(x) = \frac{|x|^{q-n}}{n}$ which leads to $\hat{\varphi}(t) = t^q$, then $\widetilde{C}_{\phi,\mathscr{V}}(K,\cdot)$ is just the qth dual curvature measure of K [29];

that is, for any Borel set $\eta \subseteq S^{n-1}$,

$$\begin{aligned}\widetilde{C}_{\phi,\mathscr{V}}(K,\eta) &= \widetilde{C}_{\hat{\varphi}}(K,\eta)\\ &= \frac{1}{n}\int_{\boldsymbol{\alpha}_K^*(\eta)} [\rho_K(u)]^q\,du\\ &= \widetilde{C}_q(K,\eta). \end{aligned} \tag{3.15}$$

In Case 2, i.e., $\phi(x) = \psi(|x|)\phi_2(\bar{x})$, one has, for any Borel set $\eta \subseteq S^{n-1}$,

$$\begin{aligned}\widetilde{C}_{\phi,\mathscr{V}}(K,\eta) &= \int_{\boldsymbol{\alpha}_K^*(\eta)} \phi(\rho_K(u)u)[\rho_K(u)]^n\,du\\ &= \int_{\boldsymbol{\alpha}_K^*(\eta)} \psi(\rho_K(u))[\rho_K(u)]^n\phi_2(u)\,du\\ &= \frac{1}{n}\int_{\boldsymbol{\alpha}_K^*(\eta)} \hat{\varphi}(\rho_K(u))\phi_2(u)\,du. \end{aligned} \tag{3.16}$$

In this case, Lemma 3.2.2 can be rewritten as follows.

Corollary 3.2.5. *Let $\phi(x) = \psi(|x|)\phi_2(\bar{x})$ satisfy condition C1). For $K \in \mathscr{K}_{(o)}^n$, then*

$$\begin{aligned}\int_{S^{n-1}} g(v)d\widetilde{C}_{\phi,\mathscr{V}}(K,v) &= \frac{1}{n}\int_{S^{n-1}} g(\alpha_K(u))\hat{\varphi}(\rho_K(u))\phi_2(u)du\\ &= \frac{1}{n}\int_{\operatorname{reg} K} \langle x, \nu_K(x)\rangle \cdot g(\nu_K(x))\frac{\hat{\varphi}(|x|)\phi_2(\bar{x})}{|x|^n}\,d\mathscr{H}^{n-1}(x)\\ &= \int_{\operatorname{reg} K} \langle x, \nu_K(x)\rangle \cdot g(\nu_K(x))\psi(|x|)\phi_2(\bar{x})\,d\mathscr{H}^{n-1}(x), \end{aligned}$$

hold for each bounded Borel function $g : S^{n-1} \to \mathbb{R}$.

3.3 A variational interpretation for the general dual Orlicz curvature measure

The variational interpretation of L_0 addition (logarithmic addition) for the general dual Orlicz curvature measure is stated as follows.

Let Ω be a closed set of S^{n-1} such that Ω is not contained in any closed hemisphere of S^{n-1}. Recall the definition of general dual Orlicz quermassintegral in Section 3.1

that if $\Phi(t,u)=\int_t^\infty \phi(ru)r^{n-1}\,dr$, then $\mathscr{V}_\phi(E)=\int_{S^{n-1}}\Phi(\rho_E(u),u)\,du$.

Theorem 3.3.1. *Let $h_0:\Omega\to(0,\infty)$ and $g:\Omega\to\mathbb{R}$ be two continuous functions. Define h_t by*

$$\log(h_t(u))=\log(h_0(u))+tg(u)+o(t,u)\quad\text{for all } u\in\Omega, \tag{3.17}$$

where $o(t,\cdot):\Omega\to\mathbb{R}$ is continuous and $o(t,u)/t\to 0$ uniformly on Ω as $t\to 0$. Let ϕ be a function satisfying condition C1). Then

$$\frac{d}{dt}\mathscr{V}_\phi([h_t])\Big|_{t=0}=-\int_\Omega g(u)\,d\widetilde{C}_{\phi,\mathscr{V}}([h_0],u). \tag{3.18}$$

Remark. An immediate consequence of (3.18) and the chain rule for derivative is the following formula, which will be used in solving the general dual Orlicz-Minkowski problem:

$$\frac{d}{dt}\log\mathscr{V}_\phi([h_t])\Big|_{t=0}=-\frac{1}{\mathscr{V}_\phi([h_0])}\int_\Omega g(u)\,d\widetilde{C}_{\phi,\mathscr{V}}([h_0],u). \tag{3.19}$$

Proof. Let $\rho_0:\Omega\to(0,\infty)$ be a continuous function. For $\delta>0$ and $t\in(-\delta,\delta)$, let

$$\log(\rho_t(u))=\log(\rho_0(u))+tg(u)+o(t,u)\quad\text{for all } u\in\Omega,$$

where $o(t,\cdot):\Omega\to\mathbb{R}$ is continuous and $o(t,u)/t\to 0$ uniformly on Ω as $t\to 0$.

First of all, let us prove the following formula: for almost every $u\in S^{n-1}$ (with respect to the spherical measure), since $\Psi_K(u)=\phi(\rho_K(u)u)[\rho_K(u)]^n$ for $u\in S^{n-1}$, one has

$$\frac{d}{dt}\Phi(\rho_{\langle\rho_t\rangle^*}(u),u)\Big|_{t=0}=\frac{d}{dt}\int_{\rho_{\langle\rho_t\rangle^*}(u)}^\infty \phi(ru)r^{n-1}\,dr\Big|_{t=0}=\Psi_{\langle\rho_0\rangle^*}(u)g(\alpha^*_{\langle\rho_0\rangle}(u)). \tag{3.20}$$

In fact, it follows from the chain rule and $\rho_{\langle\rho_t\rangle^*}(u)=h^{-1}_{\langle\rho_t\rangle}(u)$ for all $u\in S^{n-1}$ that

$$\begin{aligned}\frac{d}{dt}\Phi(\rho_{\langle\rho_t\rangle^*}(u),u)\Big|_{t=0} &= \frac{d}{dt}\int_{e^{-\log h_{\langle\rho_t\rangle}(u)}}^\infty \phi(ru)r^{n-1}\,dr\Big|_{t=0}\\ &= \phi(h^{-1}_{\langle\rho_0\rangle}(u)u)h^{-n}_{\langle\rho_0\rangle}(u)\cdot\frac{d}{dt}\log h_{\langle\rho_t\rangle}(u)\Big|_{t=0}\\ &= \Psi_{\langle\rho_0\rangle^*}(u)\cdot g(\alpha^*_{\langle\rho_0\rangle}(u)),\end{aligned}$$

where the last equality follows from [29, (4.4)], i.e.,

$$\lim_{t\to 0}\frac{\log h_{\langle\rho_t\rangle}(v)-\log h_{\langle\rho_0\rangle}(v)}{t}=g(\alpha^*_{\langle\rho_0\rangle}(v))$$

holds for any $v\in S^{n-1}\setminus\eta_0$, with $\eta_0=\eta_{\langle\rho_0\rangle}$ the complement of the set of the regular normal vectors of $\langle\rho_0\rangle$. Note that the spherical measure of η_0 is zero.

We shall need the following argument in order to use the dominated convergence theorem: there exist two constants $\delta>0$ and $M>0$, such that, for all $t\in(-\delta,\delta)$ and for all $u\in S^{n-1}$,

$$\left|\Phi(\rho_{\langle\rho_t\rangle^*}(u),u)-\Phi(\rho_{\langle\rho_0\rangle^*}(u),u)\right|\le M|t|. \tag{3.21}$$

Note that $\langle\rho_t\rangle\to\langle\rho_0\rangle$ in the Hausdorff metric; this is a direct consequence of the Aleksandrov's convergence lemma [1] and formula (2.3). Therefore, $\rho_{\langle\rho_t\rangle^*}\to\rho_{\langle\rho_0\rangle^*}$ uniformly on S^{n-1}. As $\langle\rho_0\rangle^*\in\mathscr{K}^n_{(o)}$, one can find constants $l_1,l_2,\delta_1>0$, such that, $l_1<\rho_{\langle\rho_t\rangle^*}(u)<l_2$ holds for all $u\in S^{n-1}$ and for all $t\in(-\delta_1,\delta_1)$. It follows from condition C1) and the continuity of ϕ that

$$\left|\big[\log\Phi(e^{-s},u)\big]'\right|=\left|\phi(e^{-s}u)e^{-sn}/\Phi(e^{-s},u)\right|\le L_2 \tag{3.22}$$

holds for some finite constant L_2 independent of $u\in S^{n-1}$ and for $s\in(-\log l_2,-\log l_1)$. Note that $\log h_{\langle\rho_t\rangle}(u)\in(-\log l_2,-\log l_1)$ and $\log h_{\langle\rho_0\rangle}(u)\in(-\log l_2,-\log l_1)$ for all $u\in S^{n-1}$ and for all $t\in(-\delta_1,\delta_1)$. By (3.22) and the mean value theorem, one has, for all $u\in S^{n-1}$ and for all $t\in(-\delta,\delta)$ (without loss of generality, we can assume that $0<\delta<\delta_1$),

$$\begin{aligned}\left|\log\Phi(h^{-1}_{\langle\rho_t\rangle}(u),u)-\log\Phi(h^{-1}_{\langle\rho_0\rangle}(u),u)\right| &\le L_2\left|\log h_{\langle\rho_t\rangle}(u)-\log h_{\langle\rho_0\rangle}(u)\right|\\ &\le L_2M_1|t|,\end{aligned} \tag{3.23}$$

where the last inequality follows from [29, Lemma 4.1], i.e, there exist constants $0<\delta,M_1<\infty$ such that, for all $u\in S^{n-1}$ and for all $t\in(-\delta,\delta)$,

$$\left|\log h_{\langle\rho_t\rangle}(u)-\log h_{\langle\rho_0\rangle}(u)\right|\le M_1|t|.$$

It follows from condition C1) that there is a constant L_1 (independent of $u\in S^{n-1}$), such that, for all $u \in S^{n-1}$ and for all $t \in (-\delta, \delta)$,

$$0 < s_t = \frac{\Phi(\rho_{\langle \rho_t \rangle^*}(u), u)}{\Phi(\rho_{\langle \rho_0 \rangle^*}(u), u)} = \frac{\Phi(h^{-1}_{\langle \rho_t \rangle}(u), u)}{\Phi(h^{-1}_{\langle \rho_0 \rangle}(u), u)} < L_1.$$

Hence $|s_t - 1| \le L_1 \cdot |\log s_t|$ (see e.g., [29, p.362]). Together with inequality (3.23), one gets, for all $u \in S^{n-1}$ and for all $t \in (-\delta, \delta)$,

$$\begin{aligned}
&\Big|\Phi(\rho_{\langle \rho_t \rangle^*}(u), u) - \Phi(\rho_{\langle \rho_0 \rangle^*}(u), u)\Big| \\
&\quad = \Big|\Phi(h^{-1}_{\langle \rho_t \rangle}(u), u) - \Phi(h^{-1}_{\langle \rho_0 \rangle}(u), u)\Big| \\
&\quad \le \Phi(h^{-1}_{\langle \rho_0 \rangle}(u), u) \cdot L_1 \cdot \Big|\log \Phi(h^{-1}_{\langle \rho_t \rangle}(u), u) - \log \Phi(h^{-1}_{\langle \rho_0 \rangle}(u), u)\Big| \\
&\quad \le \Phi(h^{-1}_{\langle \rho_0 \rangle}(u), u) \cdot L_1 L_2 M_1 \cdot |t| \\
&\quad \le \Phi(l_1, u) \cdot L_1 L_2 M_1 \cdot |t|.
\end{aligned}$$

That is, inequality (3.21) holds by letting $M = L_1 L_2 M_1 \cdot \max_{u \in S^{n-1}} \Phi(l_1, u) < \infty$.

Now we are ready to prove formula (3.18). To this end, let $[h_t]$ be the Wulff shape associated to h_t with h_t given by (3.17). Consider $\kappa_t = 1/h_t$ and then

$$\log \kappa_t = -\log h_t = -\log h_0 - tg - o(t, \cdot) = \log \kappa_0 - tg - o(t, \cdot).$$

Moreover, $[h_t] = \langle 1/h_t \rangle^* = \langle \kappa_t \rangle^*$ due to the bipolar theorem and (2.20). It follows from (3.4), (3.20), (3.21) and the dominated convergence theorem that

$$\begin{aligned}
\frac{d}{dt} \mathscr{V}_\phi([h_t])\Big|_{t=0} &= \frac{d}{dt} \mathscr{V}_\phi(\langle \kappa_t \rangle^*)\Big|_{t=0} \\
&= \frac{d}{dt} \int_{S^{n-1}} \Phi(\rho_{\langle \kappa_t \rangle^*}(u), u)\, du\Big|_{t=0} \\
&= \int_{S^{n-1}} \frac{d}{dt} \Phi(\rho_{\langle \kappa_t \rangle^*}(u), u)\Big|_{t=0} du \\
&= -\int_{S^{n-1}} \Psi_{\langle \kappa_0 \rangle^*}(u) \cdot g(\alpha^*_{\langle \kappa_0 \rangle}(u))\, du.
\end{aligned}$$

Together with (3.11) and the fact that the spherical measure of η_0 is zero, one can prove formula (3.18) as follows:

$$\begin{aligned}\frac{d}{dt}\mathscr{V}_\phi([h_t])\Big|_{t=0} &= -\int_{S^{n-1}\setminus\eta_0}\Psi_{\langle\kappa_0\rangle^*}(u)\cdot g(\alpha^*_{\langle\kappa_0\rangle}(u))\,du\\ &= -\int_{S^{n-1}}(\hat{g}\mathbf{1}_\Omega)(\alpha_{\langle\kappa_0\rangle^*}(u))\Psi_{\langle\kappa_0\rangle^*}(u)\,du\\ &= -\int_{S^{n-1}}(\hat{g}\mathbf{1}_\Omega)(u)\,d\widetilde{C}_{\phi,\mathscr{V}}(\langle\kappa_0\rangle^*,u)\\ &= -\int_\Omega g(u)\,d\widetilde{C}_{\phi,\mathscr{V}}([h_0],u),\end{aligned}$$

where $\hat{g}:S^{n-1}\to\mathbb{R}$ is a continuous function, such that, for all $v\in S^{n-1}\setminus\eta_0$,

$$g(\alpha_{\langle\rho_0\rangle^*}(v))=(\hat{g}\mathbf{1}_\Omega)(\alpha_{\langle\rho_0\rangle^*}(v)).$$

The existence of such $\hat{g}$ was proved in [29, p.364]. □

3.4 A solution to the general dual Orlicz-Minkowski problem

In this section, we provide a solution to the following general dual Orlicz-Minkowski problem.

The general dual Orlicz-Minkowski problem: *given a nonzero finite Borel measure μ defined on S^{n-1} and a continuous function $\phi:\mathbb{R}^n\setminus\{o\}\to(0,\infty)$, can one find a constant $\tau>0$ and a convex body K (ideally $K\in\mathscr{K}^n_{(o)}$), such that, $\mu=\tau\widetilde{C}_{\phi,\mathscr{V}}(K,\cdot)$?*

Clearly, if the general dual Orlicz-Minkowski problem has solutions, the constant τ can be calculated by

$$|\mu|=\int_{S^{n-1}}d\mu(v)=\tau\int_{S^{n-1}}d\widetilde{C}_{\phi,\mathscr{V}}(K,v)=\tau\cdot\widetilde{C}_{\phi,\mathscr{V}}(K,S^{n-1}),$$

and equivalently

$$\tau=\frac{|\mu|}{\widetilde{C}_{\phi,\mathscr{V}}(K,S^{n-1})}. \tag{3.24}$$

It is well known that, to have the various Minkowski problems solvable, the given measure μ must satisfy that μ is not concentrated in any closed hemisphere (2.6), i.e.,

$$\int_{S^{n-1}} \langle u, v\rangle_+ \, d\mu(u) > 0 \quad \text{for all} \quad v \in S^{n-1}.$$

In fact, (2.6) is also a necessary condition in our setting. That is, if there exists a convex body $K \in \mathscr{K}^n_{(o)}$, such that,

$$\frac{\mu}{|\mu|} = \frac{\widetilde{C}_{\phi,\mathscr{V}}(K,\cdot)}{\widetilde{C}_{\phi,\mathscr{V}}(K,S^{n-1})},$$

then μ satisfies (2.6).

To this end, let $v \in S^{n-1}$ be given. Then

$$\int_{S^{n-1}} \langle u, v\rangle_+ \, d\mu(u) = \frac{|\mu|}{\widetilde{C}_{\phi,\mathscr{V}}(K,S^{n-1})} \int_{S^{n-1}} \langle u, v\rangle_+ \, d\widetilde{C}_{\phi,\mathscr{V}}(K,u). \tag{3.25}$$

Hence, in order to show that μ satisfies (2.6), it is enough to show that

$$\int_{S^{n-1}} \langle u, v\rangle_+ \, d\widetilde{C}_{\phi,\mathscr{V}}(K,u) > 0.$$

In fact, it follows from (3.12) that

$$\int_{S^{n-1}} \langle u, v\rangle_+ \, d\widetilde{C}_{\phi,\mathscr{V}}(K,u) \;\; = \;\; \int_{\operatorname{reg} K} \langle \nu_K(x), v\rangle_+ \cdot \langle x, \nu_K(x)\rangle \phi(x) \, d\mathscr{H}^{n-1}(x).$$

As $K \in \mathscr{K}^n_{(o)}$, one can find a constant M such that $\langle x, \nu_K(x)\rangle\phi(x) \geq M$ for all $x \in \operatorname{reg} K$. Consequently,

$$\int_{S^{n-1}} \langle u, v\rangle_+ \, d\widetilde{C}_{\phi,\mathscr{V}}(K,u) \geq M \int_{\operatorname{reg} K} \langle \nu_K(x), v\rangle_+ \, d\mathscr{H}^{n-1}(x) > 0, \tag{3.26}$$

as the surface area measure $S(K,\cdot)$ satisfies

$$\int_{\operatorname{reg} K} \langle \nu_K(x), v\rangle_+ \, d\mathscr{H}^{n-1}(x) = \int_{\partial K} \langle \nu_K(x), v\rangle_+ \, d\mathscr{H}^{n-1}(x) = \int_{S^{n-1}} \langle u, v\rangle_+ \, dS(K,u) > 0.$$

The following theorem also shows that (2.6) is a sufficient condition for the general dual Orlicz-Minkowski problem.

Theorem 3.4.1. *Let μ be a nonzero finite Borel measure on S^{n-1} satisfying (2.6) and let ϕ be a function satisfying conditions C1) and C2). Then there exists a convex body $K \in \mathscr{K}^n_{(o)}$, such that,*

$$\frac{\mu}{|\mu|} = \frac{\widetilde{C}_{\phi,\mathscr{V}}(K,\cdot)}{\widetilde{C}_{\phi,\mathscr{V}}(K,S^{n-1})}.$$

In order to prove Theorem 3.4.1, we need the following lemma.

Lemma 3.4.2. *Let μ be a nonzero finite Borel measure on S^{n-1} satisfying (2.6) and let ϕ be a function satisfying conditions C1) and C2). Then there exists a convex body $Q_0 \in \mathscr{K}^n_{(o)}$ such that $\mathscr{V}_\phi(Q_0) = |\mu|$ and*

$$\mathscr{F}(Q_0) = \sup\left\{\mathscr{F}(K) : \mathscr{V}_\phi(K) = |\mu| \text{ and } K \in \mathscr{K}^n_{(o)}\right\}, \tag{3.27}$$

where $\mathscr{F} : \mathscr{K}^n_{(o)} \to \mathbb{R}$ is defined by

$$\mathscr{F}(K) = -\frac{1}{|\mu|}\int_{S^{n-1}} \log h_K(v)\,d\mu(v). \tag{3.28}$$

Proof. Let $\{Q_i\}_{i=1}^\infty \subseteq \mathscr{K}^n_{(o)}$ be such that $\mathscr{V}_\phi(Q_i) = |\mu|$, and

$$\lim_{i\to\infty} \mathscr{F}(Q_i) = \sup\Big\{\mathscr{F}(K) : \mathscr{V}_\phi(K) = |\mu| \text{ and } K \in \mathscr{K}^n_{(o)}\Big\}. \tag{3.29}$$

First of all, we claim that the sequence $\{Q_i^*\}_{i=1}^\infty$ is uniformly bounded. That is, we need to prove that there exists a constant $R > 0$ such that $Q_i^* \subseteq RB^n$ for all $i = 1, 2, \cdots$

Assume not, i.e., there are no finite constants R such that $Q_i^* \subseteq RB^n$ for all $i = 1, 2, \cdots$. Let $v_i \in S^{n-1}$ be such that $\rho_{Q_i^*}(v_i) = \max_{u\in S^{n-1}} \rho_{Q_i^*}(u)$ and $R_{Q_i^*} = \rho_{Q_i^*}(v_i)$. Without loss of generality, we can assume that $R_{Q_i^*} \to \infty$ (otherwise, the sequence $\{Q_i^*\}_{i=1}^\infty$ is uniformly bounded) and $v_i \to v_0$ (due to the compactness of S^{n-1}) as $i \to \infty$. Consequently, for any $M > 0$, there exists $i_M > 0$ such that $R_{Q_i^*} \geq M$ for all $i > i_M$. Clearly, for all $i > i_M$, $h_{Q_i^*}(u) \geq \langle u, v_i\rangle_+ \, R_{Q_i^*} \geq M\langle u, v_i\rangle_+$. Recall that $\Phi(t,u) = \int_t^\infty \phi(ru)r^{n-1}dr$ is decreasing on t. Then for all $i > i_M$ and for all $u \in S^{n-1}$,

$$\Phi(\rho_{Q_i}(u), u) = \Phi(h^{-1}_{Q_i^*}(u), u) \geq \Phi([M\langle u, v_i\rangle_+]^{-1}, u), \tag{3.30}$$

where we let $\Phi([M\langle u, v_i\rangle_+]^{-1}, u) = 0$ if $\langle u, v_i\rangle_+ = 0$.

Fatou's lemma implies that

$$\begin{aligned}
&\liminf_{i\to\infty}\int_{S^{n-1}}\Phi([M\langle u,v_i\rangle_+]^{-1},u)\,du\\
&\quad=\ \liminf_{i\to\infty}\int_{S^{n-1}}\int_{[M\langle u,v_i\rangle_+]^{-1}}^{\infty}\phi(ru)r^{n-1}\,dr\,du\\
&\quad\ge\ \int_{S^{n-1}}\liminf_{i\to\infty}\int_0^{\infty}\mathbf{1}_{([M\langle u,v_i\rangle_+]^{-1},\infty)}\phi(ru)r^{n-1}\,dr\,du\\
&\quad\ge\ \int_{S^{n-1}}\int_0^{\infty}\liminf_{i\to\infty}\mathbf{1}_{([M\langle u,v_i\rangle_+]^{-1},\infty)}\phi(ru)r^{n-1}\,dr\,du\\
&\quad=\ \int_{S^{n-1}}\int_{[M\langle u,v_0\rangle_+]^{-1}}^{\infty}\phi(ru)r^{n-1}\,dr\,du\\
&\quad=\ \int_{S^{n-1}}\Phi([M\langle u,v_0\rangle_+]^{-1},u)\,du.
\end{aligned}$$

Together with (3.4) and (3.30), one has

$$\begin{aligned}
|\mu| &= \lim_{i\to\infty}\mathscr{V}_\phi(Q_i)\\
&= \lim_{i\to\infty}\int_{S^{n-1}}\Phi(\rho_{Q_i}(u),u)\,du\\
&\ge \liminf_{i\to\infty}\int_{S^{n-1}}\Phi([M\langle u,v_i\rangle_+]^{-1},u)\,du\\
&\ge \int_{S^{n-1}}\Phi([M\langle u,v_0\rangle_+]^{-1},u)\,du.
\end{aligned}\tag{3.31}$$

For all $j\ge 2$, let

$$\Sigma_j(v_0):=\Big\{u\in S^{n-1}:\langle u,v_0\rangle_+>1/j\Big\}.$$

It follows from the monotone convergence theorem and the fact $\Sigma_j(v_0)\subseteq\Sigma_{j+1}(v_0)\subseteq\cup_{j=1}^{\infty}\Sigma_j(v_0)=S^{n-1}\setminus\{u\in S^{n-1}:\ \langle u,v_0\rangle=0\}$ that

$$\lim_{j\to\infty}\int_{\Sigma_j(v_0)}\langle u,v_0\rangle_+\,du=\int_{\cup_{j=1}^{\infty}\Sigma_j(v_0)}\langle u,v_0\rangle_+\,du=\int_{S^{n-1}}\langle u,v_0\rangle_+\,du>0,$$

where the last inequality is due to the fact that the spherical measure is not concentrated on any closed hemisphere. Hence, there exists $j_0\ge 2$, such that,

$$\int_{\Sigma_{j_0}(v_0)}du\ge\int_{\Sigma_{j_0}(v_0)}\langle u,v_0\rangle_+\,du\ge\frac{1}{2}\int_{S^{n-1}}\langle u,v_0\rangle_+\,du>0.$$

It can be checked that $[M\langle u, v_0\rangle_+]^{-1} \le j_0/M$ for all $u \in \Sigma_{j_0}(v_0)$. By (3.3) and (3.31), one gets

$$\begin{aligned} |\mu| &\ge \int_{S^{n-1}} \Phi([M\langle u, v_0\rangle_+]^{-1}, u)\, du \\ &\ge \int_{\Sigma_{j_0}(v_0)} \Phi(j_0/M, u)\, du \\ &= \mathscr{V}_\phi\big(\mathbb{R}^n \setminus \mathscr{C}(v_0, j_0/M, 1/j_0)\big), \end{aligned}$$

where for any fixed $u_0 \in S^{n-1}$, $a > 0$ and $b_0 \in (0,1)$,

$$\mathscr{C}(u_0, a, b_0) = \Big\{x \in \mathbb{R}^n : \ \langle \bar{x}, u_0\rangle \ge b_0 \text{ and } |x| \ge a\Big\}.$$

As ϕ satisfies condition C2), one gets a contradiction as follows:

$$\infty > |\mu| \ge \lim_{M\to\infty} \mathscr{V}_\phi(\mathbb{R}^n \setminus \mathscr{C}\big(v_0, j_0/M, 1/j_0)\big) = \infty.$$

Therefore, the sequence $\{Q_i^*\}_{i=1}^\infty$ is uniformly bounded.

Without loss of generality, we assume that $Q_i^* \to Q$ (more precisely, a subsequence of $\{Q_i^*\}_{i=1}^\infty$) in the Hausdorff metric for some compact convex set $Q \subseteq \mathbb{R}^n$, due to the Blaschke selection theorem (see e.g., [59]). Note that Q may not be a convex body, however, the support function of Q can be defined as in (2.1) and $Q_i^* \to Q$ in the Hausdorff metric is defined as in (2.2).

We now show $Q \in \mathscr{K}_{(o)}^n$ and the proof can be obtained by an argument almost identical to those in [75, 78]. In fact, assume that $Q \notin \mathscr{K}_{(o)}^n$ and $o \in \partial Q$. Then, there exists $u_0 \in S^{n-1}$ such that $\lim_{i\to\infty} h_{Q_i^*}(u_0) = h_Q(u_0) = 0$. Let

$$\Sigma_{\delta_0}(u_0) = \{v \in S^{n-1} : \langle v, u_0\rangle > \delta_0\}.$$

By (3.28), $\mathscr{V}_\phi(Q_i) = |\mu|$ and $Q_i^* \subseteq RB^n$ (without loss of generality, let $R > 1$) for all i, one has

$$\begin{aligned} \mathscr{F}(Q_i) &= -\frac{1}{|\mu|}\int_{S^{n-1}} \log h_{Q_i}(v)\, d\mu(v) \\ &= \frac{1}{|\mu|}\int_{\Sigma_{\delta_0}(u_0)} \log \rho_{Q_i^*}(v)\, d\mu(v) + \frac{1}{|\mu|}\int_{S^{n-1}\setminus\Sigma_{\delta_0}(u_0)} \log \rho_{Q_i^*}(v)\, d\mu(v) \\ &\le \frac{1}{|\mu|}\int_{\Sigma_{\delta_0}(u_0)} \log \rho_{Q_i^*}(v)\, d\mu(v) + \log R. \end{aligned}$$

It follows from $\mu(\Sigma_{\delta_0}(u_0)) > 0$ and $\rho_{Q_i^*} \to 0$ on $\Sigma_{\delta_0}(u_0)$ uniformly for some $\delta_0 > 0$ that

$$\lim_{i\to\infty} \mathscr{F}(Q_i) = -\infty,$$

which is impossible. Hence, $o \in \mathrm{int}Q$ and then $Q \in \mathscr{K}^n_{(o)}$.

Finally, let us check that $Q_0 = Q^* \in \mathscr{K}^n_{(o)}$ satisfies $\mathscr{V}_\phi(Q_0) = |\mu|$ and (3.27). In fact, as $Q_i^* \to Q$, one has $Q_i \to Q^* = Q_0$ due to the bipolar theorem. Then

$$\mathscr{V}_\phi(Q_0) = \lim_{i\to\infty} \mathscr{V}_\phi(Q_i) = |\mu|$$

is an immediate consequence of Lemma 3.1.2. On the other hand, $h_{Q_i} \to h_{Q_0}$ uniformly on S^{n-1} due to $Q_i \to Q_0 \in \mathscr{K}^n_{(o)}$ and (2.2). Moreover, there exist constants $R_1, R_2 \in (0,\infty)$, such that, for all $u \in S^{n-1}$ and for all $i \geq 1$,

$$R_1 \leq h_{Q_i}(u) \leq R_2 \quad \text{and} \quad R_1 \leq h_{Q_0}(u) \leq R_2.$$

These further imply that, for all $u \in S^{n-1}$ and for all $i \geq 1$,

$$|\log h_{Q_i}(u)| \leq \max\{|\log R_1|, |\log R_2|\} < \infty.$$

It follows from the dominated convergence theorem that

$$\begin{aligned}
\lim_{i\to\infty} \mathscr{F}(Q_i) &= \lim_{i\to\infty} -\frac{1}{|\mu|}\int_{S^{n-1}} \log h_{Q_i}(v)\,d\mu(v) \\
&= -\frac{1}{|\mu|}\int_{S^{n-1}} \lim_{i\to\infty} \log h_{Q_i}(v)\,d\mu(v) \\
&= -\frac{1}{|\mu|}\int_{S^{n-1}} \log h_{Q_0}(v)\,d\mu(v) \\
&= \mathscr{F}(Q_0).
\end{aligned}$$

Together with (3.29), one can easily get the desired formula (3.27). □

Proof of Theorem 3.4.1. Recall that each $K \in \mathscr{K}^n_{(o)}$ can be uniquely determined by its support function and vice versa. Thus we can let $\mathscr{V}_\phi(h_{[f]}) = \mathscr{V}_\phi([f])$ for all $f \in C^+(S^{n-1})$. On the other hand, as $f \geq h_{[f]}$ for all $f \in C^+(S^{n-1})$, then

$$\mathscr{F}(f) := -\frac{1}{|\mu|}\int_{S^{n-1}} \log f(v)\,d\mu(v) \leq \mathscr{F}(h_{[f]}). \tag{3.32}$$

Consider the following optimization problem:

$$\sup\Big\{\mathscr{F}(f):\mathscr{V}_\phi([f])=|\mu| \text{ for } f\in C^+(S^{n-1})\Big\}. \tag{3.33}$$

According to (3.32) and Lemma 3.4.2, the support function of convex body $Q_0\in\mathscr{K}^n_{(o)}$ found in Lemma 3.4.2 is an optimizer for the optimization problem (3.33).

On the other hand, the method of Lagrange multipliers can be used to find the necessary conditions for the optimizers for the optimization problem (3.33). In fact, for $\delta>0$ small enough, let $h_t(v)=h_{Q_0}(v)e^{tg(v)}$ for $t\in(-\delta,\delta)$ and for $v\in S^{n-1}$, where $g:S^{n-1}\to\mathbb{R}$ is an arbitrary continuous function. Let

$$\mathscr{L}(t,\tau)=\mathscr{F}(h_t)-\tau\big(\log\mathscr{V}_\phi([h_t])-\log|\mu|\big).$$

As h_{Q_0} is an optimizer to (3.33), the following equation holds:

$$\frac{\partial}{\partial t}\mathscr{L}(t,\tau)\Big|_{t=0}=0. \tag{3.34}$$

It is easily checked that

$$\frac{\partial}{\partial t}\mathscr{F}(h_t)\Big|_{t=0}=\frac{\partial}{\partial t}\bigg(-\frac{1}{|\mu|}\int_{S^{n-1}}[\log h_{Q_0}(v)+tg(v)]\,d\mu(v)\bigg)\bigg|_{t=0}=-\frac{1}{|\mu|}\int_{S^{n-1}}g(v)\,d\mu(v).$$

It follows from (3.19) that

$$\frac{\partial}{\partial t}\log\mathscr{V}_\phi([h_t])\Big|_{t=0}=-\frac{1}{\mathscr{V}_\phi(Q_0)}\int_{S^{n-1}}g(v)\,d\widetilde{C}_{\phi,\mathscr{V}}(Q_0,v).$$

Due to $\mathscr{V}_\phi(Q_0)=|\mu|$, one can rewrite (3.34) as follows:

$$\int_{S^{n-1}}g(v)\,d\mu(v)=\tau\int_{S^{n-1}}g(v)\,d\widetilde{C}_{\phi,\mathscr{V}}(Q_0,v)$$

holding for arbitrary continuous function $g:S^{n-1}\to\mathbb{R}$. Consequently, $\mu=\tau\widetilde{C}_{\phi,\mathscr{V}}(Q_0,\cdot)$ with the constant τ given by (3.24), that is,

$$\tau=\frac{|\mu|}{\widetilde{C}_{\phi,\mathscr{V}}(Q_0,S^{n-1})}.$$

In summary, a solution to the general dual Orlicz-Minkowski problem has been found.

□

The following corollary provides a solution to the general dual Orlicz-Minkowski problem under the Case 2 in Section 3.1, i.e., $\phi(x) = \psi(|x|)\phi_2(\bar{x})$ with $\psi : (0, \infty) \to (0, \infty)$ and $\phi_2 : S^{n-1} \to (0, \infty)$ continuous functions. Again, let $\hat{\phi}$ and ψ be given as in (3.6) or (3.7), and $\hat{\varphi}(t) = n\psi(t)t^n$.

Corollary 3.4.3. *Let $\phi(x) = \psi(|x|)\phi_2(\bar{x})$ be a continuous function such that the continuous function ϕ_2 is positive on S^{n-1}, and the functions $\hat{\phi}$ and $\hat{\varphi}$ satisfy conditions A1)-A3). Then the following are equivalent:*

i) μ is a nonzero finite Borel measure on S^{n-1} satisfying (2.6);

ii) there exists a convex body $K \in \mathscr{K}^n_{(o)}$ such that

$$\frac{\int_{S^{n-1}} g(v)\, d\mu(v)}{|\mu|} = \frac{\int_{S^{n-1}} g(v)\, d\widetilde{C}_{\phi,\mathscr{V}}(K, v)}{\int_{S^{n-1}}\, d\widetilde{C}_{\phi,\mathscr{V}}(K, v)} = \frac{\int_{S^{n-1}} g(\alpha_K(u))\hat{\varphi}(\rho_K(u))\phi_2(u)\, du}{\int_{S^{n-1}} \hat{\varphi}(\rho_K(u))\phi_2(u)\, du}$$

hold for each bounded Borel function $g : S^{n-1} \to \mathbb{R}$.

Proof. As explained in Section 3.1, under the conditions given in Corollary 3.4.3, $\phi(x) = \psi(|x|)\phi_2(\bar{x})$ satisfies conditions C1) and C2). The argument in ii) is equivalent to

$$\frac{\mu}{|\mu|} = \frac{\widetilde{C}_{\phi,\mathscr{V}}(K, \cdot)}{\widetilde{C}_{\phi,\mathscr{V}}(K, S^{n-1})}.$$

The equivalence between i) and ii) is an immediate consequence from (3.25), (3.26), Corollary 3.2.5 and Theorem 3.4.1. □

3.5 Uniqueness of solutions to the general dual Orlicz-Minkowski problem

It seems very difficult and maybe even impossible to obtain the uniqueness of solutions of the general dual Orlicz-Minkowski problem for general ϕ, due to the lack of homogeneity. In this section, the uniqueness will be proved in special cases. In order to get this done, we need the following theorem.

Theorem 3.5.1. *Let ϕ be a function satisfying condition C1) and that $\phi(x)|x|^n$ is strictly radially decreasing on $\mathbb{R}^n \setminus \{o\}$. If $K, L \in \mathscr{K}^n_{(o)}$ satisfy that $\widetilde{C}_{\phi,\mathscr{V}}(K,\cdot) = \widetilde{C}_{\phi,\mathscr{V}}(L,\cdot)$, then $K = L$.*

The proof of Theorem 3.5.1 follows an argument similar to those in [75, 78], and heavily relies on [75, Lemma 5.1]. For readers' convenience, we list [75, Lemma 5.1] below as Lemma 3.5.2 and provide a brief sketch of the proof of Theorem 3.5.1.

Lemma 3.5.2. *Suppose that $K', L \in \mathscr{K}^n_{(o)}$. If the following sets*

$$\begin{aligned}
\eta_1 &= \big\{v \in S^{n-1}: \quad h_{K'}(v) > h_L(v)\big\},\\
\eta_2 &= \big\{v \in S^{n-1}: \quad h_{K'}(v) < h_L(v)\big\},\\
\eta_3 &= \big\{v \in S^{n-1}: \quad h_{K'}(v) = h_L(v)\big\}
\end{aligned}$$

are nonempty, then the following statements are true:

(a) if $u \in \boldsymbol{\alpha}^*_{K'}(\eta_1)$, then $\rho_{K'}(u) > \rho_L(u)$;

(b) if $u \in \boldsymbol{\alpha}^*_{L}(\eta_2 \cup \eta_3)$, then $\rho_L(u) \geq \rho_{K'}(u)$;

(c) $\boldsymbol{\alpha}^*_{K'}(\eta_1) \subset \boldsymbol{\alpha}^*_L(\eta_1)$;

(d) $\mathscr{H}^{n-1}(\boldsymbol{\alpha}^*_L(\eta_1)) > 0$ and $\mathscr{H}^{n-1}(\boldsymbol{\alpha}^*_{K'}(\eta_2)) > 0$.

Proof of Theorem 3.5.1. Assume that $K, L \in \mathscr{K}^n_{(o)}$ with $\widetilde{C}_{\phi,\mathscr{V}}(K,\cdot) = \widetilde{C}_{\phi,\mathscr{V}}(L,\cdot)$ are not dilates of each other, namely, $K \neq tL$ for any $t > 0$. Hence, there exists some constant $t_0 > 0$ such that $K' = t_0 K$ is a convex body with η_1, η_2, η_3 defined in Lemma 3.5.2 being nonempty.

Recall that $\Psi_K(u) = \phi(\rho_K(u)u)[\rho_K(u)]^n$ for $u \in S^{n-1}$. Due to Lemma 3.5.2 and the fact that $\phi(x)|x|^n$ is strictly radially decreasing on $\mathbb{R}^n \setminus \{o\}$, one has, for all $u \in \boldsymbol{\alpha}^*_{K'}(\eta_1)$,

$$0 < \Psi_{K'}(u) = \phi(\rho_{K'}(u)u)[\rho_{K'}(u)]^n < \phi(\rho_L(u)u)[\rho_L(u)]^n = \Psi_L(u). \tag{3.35}$$

Now we claim that the spherical measure of $\boldsymbol{\alpha}^*_{K'}(\eta_1)$ is positive. In fact, this claim follows from Definition 3.2.1 and Lemma 3.5.2 as follows:

$$\int_{\boldsymbol{\alpha}^*_K(\eta_1)} \Psi_K(u)\,du = \widetilde{C}_{\phi,\mathscr{V}}(K,\eta_1) = \widetilde{C}_{\phi,\mathscr{V}}(L,\eta_1) = \int_{\boldsymbol{\alpha}^*_L(\eta_1)} \Psi_L(u)\,du > 0.$$

Moreover, by (3.35) and Lemma 3.5.2, one has

$$\begin{aligned}\widetilde{C}_{\phi,\mathscr{V}}(K,\eta_1) &= \int_{\boldsymbol{\alpha}^*_L(\eta_1)} \Psi_L(u)\,du \\ &\geq \int_{\boldsymbol{\alpha}^*_{K'}(\eta_1)} \Psi_L(u)\,du \\ &> \int_{\boldsymbol{\alpha}^*_{K'}(\eta_1)} \Psi_{K'}(u)\,du \\ &> 0.\end{aligned}$$

Due to the easily checked fact $\boldsymbol{\alpha}^*_{K'}(\eta_1) = \boldsymbol{\alpha}^*_K(\eta_1)$ and Definition 3.2.1, one gets

$$\begin{aligned}\widetilde{C}_{\phi,\mathscr{V}}(K,\eta_1) &= \int_{\boldsymbol{\alpha}^*_K(\eta_1)} \Psi_K(u)\,du \\ &= \int_{\boldsymbol{\alpha}^*_{K'}(\eta_1)} \phi(\rho_K(u)u)[\rho_K(u)]^n\,du \\ &> \int_{\boldsymbol{\alpha}^*_{K'}(\eta_1)} \Psi_{K'}(u)\,du \\ &= \int_{\boldsymbol{\alpha}^*_{K'}(\eta_1)} \phi(t_0\rho_K(u)u)[t_0\rho_K(u)]^n\,du > 0.\end{aligned}$$

Together with the fact that $\phi(x)|x|^n$ is strictly radially decreasing on $\mathbb{R}^n \setminus \{o\}$, one has $t_0 > 1$ and moreover

$$\phi(\rho_K(u)u)[\rho_K(u)]^n > \phi(t_0\rho_K(u)u)[t_0\rho_K(u)]^n \tag{3.36}$$

holds for all $u \in S^{n-1}$.

Similarly, one can check that the spherical measure of $\boldsymbol{\alpha}^*_L(\eta_2)$ is positive. It follows from Lemma 3.5.2 that $\boldsymbol{\alpha}^*_L(\eta_2) \subseteq \boldsymbol{\alpha}^*_{K'}(\eta_2)$ and

$$0 < \widetilde{C}_{\phi,\mathscr{V}}(K,\eta_2) = \widetilde{C}_{\phi,\mathscr{V}}(L,\eta_2) = \int_{\boldsymbol{\alpha}^*_L(\eta_2)} \Psi_L(u)\,du \leq \int_{\boldsymbol{\alpha}^*_{K'}(\eta_2)} \Psi_{K'}(u)\,du = \widetilde{C}_{\phi,\mathscr{V}}(K',\eta_2).$$

Together with (3.36), Definition 3.2.1, and $\boldsymbol{\alpha}^*_{K'}(\eta_2) = \boldsymbol{\alpha}^*_K(\eta_2)$, one has

$$\widetilde{C}_{\phi,\mathscr{V}}(K,\eta_2) \leq \widetilde{C}_{\phi,\mathscr{V}}(K',\eta_2) < \widetilde{C}_{\phi,\mathscr{V}}(K,\eta_2).$$

This is impossible, and hence K and L are dilates of each other.

Now we claim that $K = L$. Assume not, i.e., there exists a constant $t \neq 1$ such that $K = tL$. Let $t > 1$ and hence $\phi(\rho_L(u)u)[\rho_L(u)]^n > \phi(\rho_K(u)u)[\rho_K(u)]^n$ for all $u \in S^{n-1}$. We can get a contradiction as follows:

$$\begin{aligned}\widetilde{C}_{\phi,\mathscr{V}}(K, S^{n-1}) &= \widetilde{C}_{\phi,\mathscr{V}}(L, S^{n-1}) \\ &= \int_{S^{n-1}} \phi(\rho_L(u)u)[\rho_L(u)]^n \, du \\ &> \int_{S^{n-1}} \phi(\rho_K(u)u)[\rho_K(u)]^n \, du \\ &= \widetilde{C}_{\phi,\mathscr{V}}(K, S^{n-1}),\end{aligned}$$

where we have used the assumption that $\widetilde{C}_{\phi,\mathscr{V}}(K, \cdot) = \widetilde{C}_{\phi,\mathscr{V}}(L, \cdot)$.

Similarly, one can show that $t < 1$ is not possible, and hence $K = L$ as desired. $\square$

Remark. When $\phi(x) = \psi(|x|)\phi_2(\bar{x})$ as stated in Case 2 in Section 3.1, $\phi(x)|x|^n$ is a strictly radially decreasing function if $\hat{\varphi}(t) = n\psi(t)t^n$ is a strictly decreasing function on $t \in (0, \infty)$. For instance, if $\phi(x) = \|x\|^{q-n}$ for $q < 0$, then

$$\phi(x)|x|^n = \|x\|^{q-n}|x|^n = |x|^q \|\bar{x}\|^{q-n}$$

is a strictly radially decreasing function on $\mathbb{R}^n \setminus \{o\}$. On the other hand, if ϕ is smooth enough, say the gradient of ϕ (denoted by $\nabla\phi$) exists on $\mathbb{R}^n \setminus \{o\}$, a typical condition to make $\phi(x)|x|^n$ strictly radially decreasing is $\big\langle \nabla\big(\phi(x)|x|^n\big), x \big\rangle < 0$ or equivalently $\langle \nabla\phi(x), x\rangle + n\phi(x) < 0$ for all $x \in \mathbb{R}^n \setminus \{o\}$.

We are now ready to state our result regarding the uniqueness of solutions to the general dual Orlicz-Minkowski problem. If $\phi_2(u) = 1$ for all $u \in S^{n-1}$, it goes back to the case proved by Zhao [75].

Corollary 3.5.3. *Let $\phi(x) = |x|^{q-n}\phi_2(\bar{x})$ with $q < 0$ and $\phi_2 : S^{n-1} \to (0, \infty)$ a positive continuous function. Then the following statements are equivalent:*

i) μ is a nonzero finite Borel measure on S^{n-1} satisfying (2.6);

ii) there exists a unique convex body $K \in \mathscr{K}^n_{(o)}$, such that, $\mu = \widetilde{C}_{\phi,\mathscr{V}}(K, \cdot)$.

Proof. The argument from ii) to i) follows along the same lines as the arguments for

(3.25) and (3.26). On the other hand, it follows from Theorem 3.4.1 that, if μ is a nonzero finite Borel measure on S^{n-1} satisfying (2.6), then there is a convex body $\widetilde{K} \in \mathscr{K}^n_{(o)}$ such that

$$\frac{\mu}{|\mu|} = \frac{\widetilde{C}_{\phi,\mathscr{V}}(\widetilde{K},\cdot)}{\widetilde{C}_{\phi,\mathscr{V}}(\widetilde{K},S^{n-1})}.$$

By Corollary 3.2.5, $\boldsymbol{\alpha}^*_{\lambda K}(\eta) = \boldsymbol{\alpha}^*_K(\eta)$ and $\rho_{\lambda K} = \lambda\rho_K$ for any constant $\lambda > 0$, and the fact that $u \in \boldsymbol{\alpha}^*_K(\eta)$ if and only if $\alpha_K(u) \in \eta$ (see [29, (2.21)]), one has, for any $\lambda > 0$ and for any Borel set $\eta \subseteq S^{n-1}$,

$$\begin{aligned}
\widetilde{C}_{\phi,\mathscr{V}}(\lambda K,\eta) &= \int_{\boldsymbol{\alpha}^*_{\lambda K}(\eta)} \big[\rho_{\lambda K}(u)\big]^q \phi_2(u)du \\
&= \lambda^q \int_{\boldsymbol{\alpha}^*_K(\eta)} \big[\rho_K(u)\big]^q \phi_2(u)du \\
&= \lambda^q \widetilde{C}_{\phi,\mathscr{V}}(K,\eta). \qquad (3.37)
\end{aligned}$$

Hence, $\widetilde{C}_{\phi,\mathscr{V}}(\lambda K,\cdot) = \lambda^q \widetilde{C}_{\phi,\mathscr{V}}(K,\cdot)$ and

$$\mu = \frac{|\mu|}{\widetilde{C}_{\phi,\mathscr{V}}(\widetilde{K},S^{n-1})}\widetilde{C}_{\phi,\mathscr{V}}(\widetilde{K},\cdot) = \widetilde{C}_{\phi,\mathscr{V}}(K,\cdot),$$

where

$$K = \left(\frac{|\mu|}{\widetilde{C}_{\phi,\mathscr{V}}(\widetilde{K},S^{n-1})}\right)^{\frac{1}{q}} \widetilde{K}.$$

Hence, $K \in \mathscr{K}^n_{(o)}$ is a convex body such that $\mu = \widetilde{C}_{\phi,\mathscr{V}}(K,\cdot)$, if μ is a nonzero finite Borel measure on S^{n-1} satisfying (2.6). The uniqueness of K is an immediate consequence of Theorem 3.5.1 and the remark after its proof. □

The solution for μ being a discrete measure is stated in the following proposition.

Proposition 3.5.4. *Let $\phi(x) = |x|^{q-n}\phi_2(\bar{x})$ with $q < 0$ and $\phi_2 : S^{n-1} \to (0,\infty)$ a positive continuous function. Suppose that $\mu = \sum_{i=1}^m \lambda_i \delta_{u_i}$ with all $\lambda_i > 0$ is a discrete measure not concentrated in any closed hemisphere (i.e., satisfying (2.6)). Then, there exists a unique polytope $P \in \mathscr{K}^n_{(o)}$, such that, $\mu = \widetilde{C}_{\phi,\mathscr{V}}(P,\cdot)$ and $u_1, u_2, \cdots, u_m$ are the unit normal vectors of the faces of P.*

Proof. It follows from Corollary 3.5.3 that there exists a unique convex body $K_0 \in \mathscr{K}^n_{(o)}$, such that, $\mu = \widetilde{C}_{\phi,\mathscr{V}}(K_0,\cdot)$. The desired argument in this proposition follows if we can prove that K_0 is a polytope with $u_1, u_2, \cdots, u_m$ being the unit normal vectors of its faces. To this end, let $M \in \mathscr{K}^n_{(o)}$ be a polytope circumscribed about K_0 whose faces have the unit normal vectors being exactly $u_1, u_2, \cdots, u_m$. Hence $K_0 \subseteq M$ and $h_M(u_i) = h_{K_0}(u_i)$ for all $i = 1, 2, \cdots, m$.

Suppose that $K_0 \neq M$ (as otherwise, nothing to prove). In this case, there exists a set $\eta_M \subseteq S^{n-1}$, such that, the spherical measure of η_M is positive and $\rho_M(u) > \rho_{K_0}(u)$ on η_M. It follows from (3.8) and (3.16) that

$$\mathscr{V}_\phi(M) < \mathscr{V}_\phi(K_0) \quad \text{and} \quad \widetilde{C}_{\phi,\mathscr{V}}(L, S^{n-1}) = -q\mathscr{V}_\phi(L)$$

for all $L \in \mathscr{K}^n_{(o)}$. Hence, $\widetilde{C}_{\phi,\mathscr{V}}(M, S^{n-1}) < \widetilde{C}_{\phi,\mathscr{V}}(K_0, S^{n-1}) = |\mu|$. By (3.37), there exists a constant $0 < c < 1$, such that

$$\widetilde{C}_{\phi,\mathscr{V}}(cM, S^{n-1}) = \widetilde{C}_{\phi,\mathscr{V}}(K_0, S^{n-1}) = |\mu|.$$

On the other hand, from Corollary 3.5.3 and the proof of Theorem 3.4.1, the convex body $(-q)^{1/q}K_0 \in \mathscr{K}^n_{(o)}$ is the unique convex body such that $\mathscr{V}_\phi\big((-q)^{1/q}K_0\big) = |\mu|$ and

$$\mathscr{F}\big((-q)^{1/q}K_0\big) = \sup\big\{\mathscr{F}(K) : \mathscr{V}_\phi(K) = |\mu| \text{ and } K \in \mathscr{K}^n_{(o)}\big\}.$$

However, this is impossible because $\mathscr{V}_\phi\big((-q)^{1/q}cM\big) = |\mu|$ and

$$\begin{aligned}
\mathscr{F}\big((-q)^{1/q}cM\big) &= -\frac{1}{|\mu|}\int_{S^{n-1}} \big[\log h_M(v) + \log c + \log(-q)/q\big]\, d\mu(v) \\
&> -\frac{1}{|\mu|}\int_{S^{n-1}} \big[\log h_M(v) + \log(-q)/q\big]\, d\mu(v) \\
&= -\frac{1}{|\mu|} \cdot \sum_{i=1}^{m} \lambda_i \big[\log h_M(u_i) + \log(-q)/q\big] \\
&= -\frac{1}{|\mu|} \cdot \sum_{i=1}^{m} \lambda_i \big[\log h_{K_0}(u_i) + \log(-q)/q\big] \\
&= \mathscr{F}((-q)^{1/q}K_0),
\end{aligned}$$

where the inequality is due to $0 < c < 1$. Hence $M = K_0$ is a polytope. Moreover, it is easy to get the relation between λ_i and the polytope K_0. In fact,

$$\begin{aligned}
\lambda_i &= \int_{\{u_i\}} d\mu \\
&= \int_{\{u_i\}} d\widetilde{C}_{\phi,\mathscr{V}}(K_0, v) \\
&= \int_{\boldsymbol{\alpha}^*_{K_0}(u_i)} \big[\rho_{K_0}(v)\big]^q \phi_2(v)\, dv \\
&= \int_{\nu^{-1}_{K_0}(\{u_i\})} \langle x, \nu_{K_0}(x)\rangle |x|^{q-n} \phi_2(\bar{x})\, d\mathscr{H}^{n-1}(x) \\
&= \int_{\nu^{-1}_{K_0}(\{u_i\})} \langle x, \nu_{K_0}(x)\rangle \phi(x)\, d\mathscr{H}^{n-1}(x) \\
&> 0,
\end{aligned}$$

where the third equality follows from (3.37) and the fourth equality follows from Corollary 3.2.5. Let $P = K_0$, and then P is the desired polytope, such that, $\mu = \widetilde{C}_{\phi,\mathscr{V}}(P, \cdot)$ and $u_1, u_2, \cdots, u_m$ are the unit normal vectors of the faces of P. $\square$

Chapter 4

General volumes and Minkowski problem for $G(t,\cdot)$ decreasing

This chapter is based on our paper [17]. In this chapter, we investigate the following general dual Orlicz-Minkowski problem: *under what conditions on a given measure μ defined on the unit sphere, a two-variable function $G(\cdot,\cdot)$ and one variable function $\psi(\cdot)$, does there exist a convex body $K \in \mathscr{K}_{(o)}^n$ such that μ equals to the general dual Orlicz curvature measure of $K \in \mathscr{K}_{(o)}^n$ up to a constant τ, i.e., $\mu = \tau\widetilde{C}_{G,\psi}(K,\cdot)$?* A solution to this problem will be provided for the case that $G(t,\cdot)$ is decreasing on t. Moreover, we also investigate some important inequalities with respect to the general dual volume, including the dual Orlicz-Brunn-Minkowski inequalities and dual Orlicz-Minkowski inequalities.

4.1 General dual curvature measures for $\mathscr{K}_{(o)}^n$

First, to give a variational interpretation for the general dual Orlicz curvature measure $\widetilde{C}_{G,\psi}$, we introduce a general dual volume $\widetilde{V}_G$ which is a generalization of the general dual Orlicz quermassintegral $\mathscr{V}_\phi$ in Chapter 3.

Definition 4.1.1. *Let $G:(0,\infty)\times S^{n-1}\to(0,\infty)$ be continuous. For $K\in\mathscr{S}_+^n$, define the* general dual volume $\widetilde{V}_G(K)$ *of K by*

$$\widetilde{V}_G(K) = \int_{S^{n-1}} G(\rho_K(u),u)\,du. \tag{4.1}$$

Our approach will be to obtain results for this rather general set function that yield geometrically interesting consequences for particular functions G. (Remark 4.2.6 addresses the possibility of allowing $G:(0,\infty)\times S^{n-1}\to\mathbb{R}$.) Let $\phi:\mathbb{R}^n\setminus\{o\}\to(0,\infty)$ be a continuous function. One special case of interest is when $G=\overline{\Phi}$, where

$$\overline{\Phi}(t,u)=\int_t^\infty \phi(ru)r^{n-1}\,dr \tag{4.2}$$

for $t>0$ and $u\in S^{n-1}$. Then we define $\overline{V}_\phi(K)=\widetilde{V}_{\overline{\Phi}}(K)$, so that

$$\begin{aligned}\overline{V}_\phi(K) &= \int_{S^{n-1}}\overline{\Phi}(\rho_K(u),u)\,du\\ &= \int_{S^{n-1}}\int_{\rho_K(u)}^\infty \phi(ru)r^{n-1}\,dr\,du\\ &= \int_{\mathbb{R}^n\setminus K}\phi(x)\,dx\\ &= \mathscr{V}_\phi(K),\end{aligned} \tag{4.3}$$

where the integral may be infinite. It recovers the definition of the general dual Orlicz quermassintegral in (3.1).

Similarly, taking $G=\underline{\Phi}$, where

$$\underline{\Phi}(t,u)=\int_0^t \phi(ru)r^{n-1}\,dr$$

for $t>0$ and $u\in S^{n-1}$, we define $\underline{V}_\phi(K)=\widetilde{V}_{\underline{\Phi}}(K)$, whence

$$\underline{V}_\phi(K)=\int_{S^{n-1}}\underline{\Phi}(\rho_K(u),u)\,du=\int_K\phi(x)\,dx, \tag{4.4}$$

where again the integral may be infinite. We refer to both $\underline{V}_\phi(K)$ and $\overline{V}_\phi(K)$ as a *general dual Orlicz volume* of $K\in\mathscr{S}^n$. Indeed, if $q\neq 0$ and $\phi(x)=(|q|/n)|x|^{q-n}$, then

$$\begin{aligned}\widetilde{V}_q(K) &= \frac{1}{n}\int_{S^{n-1}}\rho_K(u)^q\,du \\ &= \begin{cases}\overline{V}_\phi(K), & \text{if } q<0,\\ \underline{V}_\phi(K), & \text{if } q>0,\end{cases}\end{aligned} \tag{4.5}$$

is the *qth dual volume of* K; see [15, p. 410]. In particular, when $q = n$, we have $\underline{V}_\phi(K) = V(K)$, the volume of K. More generally, if $\phi(x) = (|q|/n)|x|^{q-n}\rho_Q(x/|x|)^{n-q}$, where $q \neq 0$ and $Q \in \mathscr{S}^n$, then

$$\widetilde{V}_q(K,Q) = \frac{1}{n}\int_{S^{n-1}} \rho_K(u)^q \rho_Q(u)^{n-q}\,du = \begin{cases} \overline{V}_\phi(K), & \text{if } q < 0, \\ \underline{V}_\phi(K), & \text{if } q > 0, \end{cases} \tag{4.6}$$

is the *qth dual mixed volume of* K *and* Q; see [15, p. 410].

Other special cases of $\widetilde{V}_G(K)$ of interest, the general Orlicz dual mixed volumes $\widetilde{V}_{\phi,\varphi}(K,L)$ and $\breve{V}_{\phi,\varphi}(K,g)$, are given in (4.35) and (4.36).

Lemma 4.1.2. *Let* $G : (0,\infty) \times S^{n-1} \to (0,\infty)$ *be continuous. If* $K_i \in \mathscr{K}^n_{(o)}$, $i \in \mathbb{N}$, *and* $K_i \to K \in \mathscr{K}^n_{(o)}$ *as* $i \to \infty$, *then* $\lim_{i\to\infty} \widetilde{V}_G(K_i) = \widetilde{V}_G(K)$.

Proof. Since $K_i \to K \in \mathscr{K}^n_{(o)}$, $\rho_{K_i} \to \rho_K$ uniformly on S^{n-1}. By the continuity of G, we have $\lim_{i\to\infty} G(\rho_{K_i}(u),u) = G(\rho_K(u),u)$ and $\sup\{G(\rho_{K_i}(u),u) : i \in \mathbb{N}, u \in S^{n-1}\} < \infty$. It follows from the dominated convergence theorem that

$$\lim_{i\to\infty} \widetilde{V}_G(K_i) = \lim_{i\to\infty} \int_{S^{n-1}} G(\rho_{K_i}(u),u)\,du = \int_{S^{n-1}} \lim_{i\to\infty} G(\rho_{K_i}(u),u)\,du = \widetilde{V}_G(K).$$

□

Next we will give the generalized definition of general dual Orlicz curvature measure for $K \in \mathscr{K}^n_{(o)}$.

Definition 4.1.3. Let $K \in \mathscr{K}^n_{(o)}$, $\psi : (0,\infty) \to (0,\infty)$ be continuous, and $G_t(t,u) = \partial G(t,u)/\partial t$ be such that $u \mapsto G_t(\rho_K(u),u)$ is integrable on S^{n-1}. Define the finite *signed* Borel measure $\widetilde{C}_{G,\psi}(K,\cdot)$ on S^{n-1} by

$$\widetilde{C}_{G,\psi}(K,E) = \frac{1}{n}\int_{\boldsymbol{\alpha}^*_K(E)} \frac{\rho_K(u)\,G_t(\rho_K(u),u)}{\psi(h_K(\alpha_K(u)))}\,du \tag{4.7}$$

for each Borel set $E \subset S^{n-1}$. If $\psi \equiv 1$, we often write $\widetilde{C}_G(K,\cdot)$ instead of $\widetilde{C}_{G,\psi}(K,\cdot)$.

To see that $\widetilde{C}_{G,\psi}(K,\cdot)$ is indeed a finite signed Borel measure on S^{n-1}, note firstly that $\widetilde{C}_{G,\psi}(K,\emptyset) = 0$. Since $K \in \mathscr{K}^n_{(o)}$ and $u \mapsto G_t(\rho_K(u),u)$ is integrable, $\widetilde{C}_{G,\psi}(K,\cdot)$ is finite. Let $E_i \subset S^{n-1}$, $i \in \mathbb{N}$, be disjoint Borel sets. By [29, Lemmas 2.3 and 2.4],

$\boldsymbol{\alpha}_K^*(\cup_i E_i) = \cup_i \boldsymbol{\alpha}_K^*(E_i)$ and the intersection of any two of these sets has $\mathscr{H}^{n-1}$-measure zero. The dominated convergence theorem then implies that

$$\begin{aligned}\widetilde{C}_{G,\psi}(K, \cup_i E_i) &= \frac{1}{n}\int_{\cup_i \boldsymbol{\alpha}_K^*(E_i)} \frac{\rho_K(u)\, G_t(\rho_K(u), u)}{\psi(h_K(\alpha_K(u)))}\, du \\ &= \frac{1}{n}\sum_{i=1}^{\infty}\int_{\boldsymbol{\alpha}_K^*(E_i)} \frac{\rho_K(u)\, G_t(\rho_K(u), u)}{\psi(h_K(\alpha_K(u)))}\, du \\ &= \sum_{i=1}^{\infty} \widetilde{C}_{G,\psi}(K, E_i),\end{aligned}$$

so $\widetilde{C}_{G,\psi}(K, \cdot)$ is countably additive.

Integrals with respect to $\widetilde{C}_{G,\psi}(K, \cdot)$ can be calculated as follows. For any bounded Borel function $g : S^{n-1} \to \mathbb{R}$, we have

$$\begin{aligned}\int_{S^{n-1}} g(u)\, d\widetilde{C}_{G,\psi}(K, u) &= \frac{1}{n}\int_{S^{n-1}} \frac{g(\alpha_K(u))\rho_K(u)\, G_t(\rho_K(u), u)}{\psi(h_K(\alpha_K(u)))}\, du \qquad (4.8)\\ &= \frac{1}{n}\int_{\partial K} \frac{g(\nu_K(x))\langle x, \nu_K(x)\rangle}{\psi(\langle x, \nu_K(x)\rangle)}\, |x|^{1-n}\, G_t(|x|, \bar{x})\; dx, \qquad (4.9)\end{aligned}$$

where $\bar{x} = x/|x|$. Relation (4.8) follows immediately from (2.10), and (4.9) follows from the fact that the bi-Lipschitz radial projection $\tilde{\pi} : \partial K \to S^{n-1}$, given by $\tilde{\pi}(x) = x/|x|$, has Jacobian $J\tilde{\pi}(x) = \langle x, \nu_K(x)\rangle |x|^{-n}$ for all regular boundary points, and hence for $\mathscr{H}^{n-1}$-almost all $x \in \partial K$.

If K is strictly convex, then the gradient $\nabla h_K(u)$ of h_K at $u \in S^{n-1}$ equals the unique $x_K(u) \in \partial K$ with outer unit normal vector u, and $\nabla h_K(\nu_K(x)) = x$ for $\mathscr{H}^{n-1}$-almost all $x \in \partial K$. Using this and [54, Lemma 2.10], (4.9) yields

$$\begin{aligned}&\int_{S^{n-1}} g(u)\, d\widetilde{C}_{G,\psi}(K, u)\\ &\quad = \frac{1}{n}\int_{S^{n-1}} \frac{g(u) h_K(u)}{\psi(h_K(u))}\, |\nabla h_K(u)|^{1-n}\, G_t\left(|\nabla h_K(u)|, \frac{\nabla h_K(u)}{|\nabla h_K(u)|}\right) dS(K, u). \qquad (4.10)\end{aligned}$$

The following result could be proved in the same way as [54, Lemma 5.5], using Weil's Approximation Lemma. Here we provide an argument which avoids the use of this lemma.

Theorem 4.1.4. *Let $K \in \mathscr{K}^n_{(o)}$, and G, ψ be as in Definition 4.1.3. Then the measure-valued map $K \mapsto \widetilde{C}_{G,\psi}(K, \cdot)$ is a valuation on $\mathscr{K}^n_{(o)}$.*

Proof. Let $K, L \in \mathscr{K}^n_{(o)}$ be such that $K \cup L \in \mathscr{K}^n_{(o)}$. It suffices to show that for any bounded Borel function $g : S^{n-1} \to \mathbb{R}$, we have

$$I(K \cap L) + I(K \cup L) = I(K) + I(L), \tag{4.11}$$

where $I(M) = \int_{S^{n-1}} g(u)\, d\widetilde{C}_{G,\psi}(M, u)$ for $M \in \mathscr{K}^n_{(o)}$. The sets $K \cap L$, $K \cup L$, K, and L can each be partitioned into three disjoint sets, as follows:

$$\partial(K \cap L) = (\partial K \cap \operatorname{int} L) \cup (\partial L \cap \operatorname{int} K) \cup (\partial K \cap \partial L), \tag{4.12}$$
$$\partial(K \cup L) = (\partial K \setminus L) \cup (\partial L \setminus K) \cup (\partial K \cap \partial L), \tag{4.13}$$
$$\partial K = (\partial K \cap \operatorname{int} L) \cup (\partial K \setminus L) \cup (\partial K \cap \partial L), \tag{4.14}$$
$$\partial L = (\partial L \cap \operatorname{int} K) \cup (\partial L \setminus K) \cup (\partial K \cap \partial L). \tag{4.15}$$

Let $\bar{x} = x/|x|$. For $\mathscr{H}^{n-1}$-almost all $x \in \partial(K \cap L)$, we have

$$x \in \partial K \cap \operatorname{int} L \quad \Rightarrow \quad \nu_{K\cap L}(x) = \nu_K(x) \text{ and } \rho_{K\cap L}(\bar{x}) = \rho_K(\bar{x}), \tag{4.16}$$
$$x \in \partial L \cap \operatorname{int} K \quad \Rightarrow \quad \nu_{K\cap L}(x) = \nu_L(x) \text{ and } \rho_{K\cap L}(\bar{x}) = \rho_L(\bar{x}), \tag{4.17}$$
$$x \in \partial K \cap \partial L \quad \Rightarrow \quad \nu_{K\cap L}(x) = \nu_K(x) = \nu_L(x) \text{ and } \rho_{K\cap L}(\bar{x}) = \rho_K(\bar{x}) = \rho_L(\bar{x}), \tag{4.18}$$

where the first set of equations in (4.18) hold for $x \in \operatorname{reg}(K \cap L) \cap \operatorname{reg} K \cap \operatorname{reg} L$ since $K \cap L \subset K, L$. Also, for $\mathscr{H}^{n-1}$-almost all $x \in \partial(K \cup L)$, we have

$$x \in \partial K \setminus L \quad \Rightarrow \quad \nu_{K\cup L}(x) = \nu_K(x) \text{ and } \rho_{K\cup L}(\bar{x}) = \rho_K(\bar{x}), \tag{4.19}$$
$$x \in \partial L \setminus K \quad \Rightarrow \quad \nu_{K\cup L}(x) = \nu_L(x) \text{ and } \rho_{K\cup L}(\bar{x}) = \rho_L(\bar{x}), \tag{4.20}$$
$$x \in \partial K \cap \partial L \quad \Rightarrow \quad \nu_{K\cup L}(x) = \nu_K(x) = \nu_L(x) \text{ and } \rho_{K\cup L}(\bar{x}) = \rho_K(\bar{x}) = \rho_L(\bar{x}), \tag{4.21}$$

where the first set of equations in (4.21) hold for $x \in \operatorname{reg}(K \cup L) \cap \operatorname{reg} K \cap \operatorname{reg} L$ since $K, L \subset K \cup L$. Now (4.11) follows easily from (4.9), by first decomposing the integrations over $\partial(K \cap L)$ and $\partial(K \cup L)$ into six contributions via (4.12) and (4.13), using (4.16–4.21), and then recombining these contributions via (4.14) and (4.15). $\square$

Some particular cases of (4.7) are worthy of mention. Firstly, with $G = \underline{\Phi}$ and general ψ, we prefer to write $\widetilde{C}_{\phi,\psi}(K,E)$ instead of $\widetilde{C}_{\underline{\Phi},\psi}(K,E)$. Then we have

$$\widetilde{C}_{\phi,\psi}(K,E) = \frac{1}{n}\int_{\boldsymbol{\alpha}^*_K(E)} \frac{\phi(\rho_K(u)u)\rho_K(u)^n}{\psi(h_K(\alpha_K(u)))}\,du, \tag{4.22}$$

and by specializing (4.8) and (4.9) we get

$$\begin{aligned}\int_{S^{n-1}} g(u)\,d\widetilde{C}_{\phi,\psi}(K,u) &= \frac{1}{n}\int_{S^{n-1}} g(\nu_K(\rho_K(u)u))\frac{\phi(\rho_K(u)u)\rho_K(u)^n}{\psi(h_K(\alpha_K(u)))}\,du \\ &= \frac{1}{n}\int_{\partial K} g(\nu_K(x))\frac{\langle x,\nu_K(x)\rangle}{\psi(\langle x,\nu_K(x)\rangle)}\phi(x)\,dx\end{aligned}$$

for any bounded Borel function $g : S^{n-1} \to \mathbb{R}$. Here we used

$$G_t(\rho_K(u),u) = \phi(\rho_K(u)u)\rho_K(u)^{n-1}. \tag{4.23}$$

If we also choose $\psi = 1$ and write $\widetilde{C}_{\phi,\mathscr{V}}(K,E)$ instead of $\widetilde{C}_{\underline{\Phi}}(K,E)$, we obtain

$$\widetilde{C}_{\phi,\mathscr{V}}(K,E) = \frac{1}{n}\int_{\boldsymbol{\alpha}^*_K(E)} \phi(\rho_K(u)u)\rho_K(u)^n\,du,$$

the general dual Orlicz curvature measure $\widetilde{C}_{\phi,\mathscr{V}}$ introduced in Chapter 3, and in particular we see that

$$\begin{aligned}\int_{S^{n-1}} g(u)\,d\widetilde{C}_{\phi,\mathscr{V}}(K,u) &= \frac{1}{n}\int_{S^{n-1}} g(\alpha_K(u))\phi(\rho_K(u)u)\rho_K(u)^n\,du \\ &= \frac{1}{n}\int_{\operatorname{reg} K} g(\nu_K(x))\,\phi(x)\,\langle x,\nu_K(x)\rangle\,dx,\end{aligned} \tag{4.24}$$

as in (3.12).

Note that when $G = \overline{\Phi}$ is given by (4.2), we have $\widetilde{V}_G(K) = \overline{V}_\phi(K)$ as in (4.3), in which case $G_t(\rho_K(u),u) = -\phi(\rho_K(u)u)\rho_K(u)^{n-1}$ and hence $\widetilde{C}_{\overline{\Phi},\psi}(K,E) = -\widetilde{C}_{\phi,\psi}(K,E)$. Comparing (4.7) and (4.8), and using (4.23), we see that

$$\int_{S^{n-1}} \frac{g(u)}{\psi(h_K(u))}\,d\widetilde{C}_{\phi,\mathscr{V}}(K,u) = \begin{cases} -\displaystyle\int_{S^{n-1}} g(u)\,d\widetilde{C}_{\overline{\Phi},\psi}(K,u) & \text{(4.25a)} \\ \displaystyle\int_{S^{n-1}} g(u)\,d\widetilde{C}_{\underline{\Phi},\psi}(K,u). & \text{(4.25b)}\end{cases}$$

Taking $\phi(x) = |x|^{q-n}\rho_Q(x/|x|)^{n-q}$, for some $Q \in \mathscr{S}^n_{c+}$ and $q \in \mathbb{R}$, and $\psi(t) = t^p$, $p \in \mathbb{R}$, from (4.22) we get $\widetilde{C}_{\phi,\psi}(K,E) = \widetilde{C}_{p,q}(K,Q,E)$, where

$$\widetilde{C}_{p,q}(K,Q,E) = \frac{1}{n}\int_{\boldsymbol{\alpha}^*_K(E)} h_K(\alpha_K(u))^{-p}\,\rho_K(u)^q\,\rho_Q(u)^{n-q}\,du \tag{4.26}$$

is the (p,q)*-dual curvature measure of* K *relative to* Q introduced in [54, Definition 4.2]. The formula [54, (5.1), p. 114] or the preceding discussion shows that for any bounded Borel function $g : S^{n-1} \to \mathbb{R}$, we have

$$\int_{S^{n-1}} g(u)\,d\widetilde{C}_{p,q}(K,Q,u) = \frac{1}{n}\int_{S^{n-1}} g(\alpha_K(u))\,h_K(\alpha_K(u))^{-p}\,\rho_K(u)^q\,\rho_Q(u)^{n-q}\,du. \tag{4.27}$$

4.2 Variational formulas for the general dual volume

In this section, variational interpretation in terms of Orlicz addition—variational formulas for the general dual volume $\widetilde{V}_G$ of convex bodies are provided.

4.2.1 General variational formulas for radial Orlicz linear combinations

Our main result in this section is the following variational formula for $\widetilde{V}_G$, where $G_t(t,u) = \partial G(t,u)/\partial t$. Recall the Orlicz combinations introduced in Section 2.3 that for functions $f_0, g, h_K, h_L, \rho_K, \rho_L$, one has

$$f_\varepsilon(u) = \varphi^{-1}\left(\varphi(f_0(u)) + \varepsilon g(u)\right) \qquad \text{for } u \in S^{n-1}, \tag{4.28}$$

$$\varphi_1\left(\frac{h_K(u)}{h_\varepsilon(u)}\right) + \varepsilon\varphi_2\left(\frac{h_L(u)}{h_\varepsilon(u)}\right) = 1 \qquad \text{for } u \in S^{n-1}, \tag{4.29}$$

and

$$\varphi_1\left(\frac{\rho_K(u)}{\rho_\varepsilon(u)}\right) + \varepsilon\varphi_2\left(\frac{\rho_L(u)}{\rho_\varepsilon(u)}\right) = 1 \qquad \text{for } u \in S^{n-1}. \tag{4.30}$$

Note that we can apply (4.28) and (4.30) when $f_0 = h_K$ for some $K \in \mathscr{K}^n_{(o)}$ or when $f_0 = \rho_K$ for some $K \in \mathscr{S}^n_{c+}$.

Theorem 4.2.1. *Let G and G_t be continuous on $(0,\infty)\times S^{n-1}$ and let $K, L \in \mathscr{S}^n_{c+}$.*

(i) *If $\varphi_1, \varphi_2 \in \mathscr{I}$ and $(\varphi_1)'_l(1) > 0$, then*

$$\lim_{\varepsilon\to 0^+} \frac{\widetilde{V}_G(K_\varepsilon) - \widetilde{V}_G(K)}{\varepsilon} = \frac{1}{(\varphi_1)'_l(1)} \int_{S^{n-1}} \varphi_2\left(\frac{\rho_L(u)}{\rho_K(u)}\right) \rho_K(u)\, G_t(\rho_K(u), u)\, du, \quad (4.31)$$

where $K_\varepsilon = K \widetilde{+}_{\varphi,\varepsilon} L \in \mathscr{S}^n_{c+}$ has radial function ρ_ε given by (4.30). For $\varphi_1, \varphi_2 \in \mathscr{D}$, (4.31) holds when $(\varphi_1)'_r(1) < 0$, with $(\varphi_1)'_l(1)$ replaced by $(\varphi_1)'_r(1)$.

(ii) *Let $a \in \mathbb{R}\cup\{-\infty\}$. If $\varphi \in \mathscr{J}_a$ and φ' is continuous and nonzero on $(0,\infty)$, then for $g \in C(S^{n-1})$,*

$$\lim_{\varepsilon\to 0} \frac{\widetilde{V}_G(\widehat{K}_\varepsilon) - \widetilde{V}_G(K)}{\varepsilon} = \int_{S^{n-1}} \frac{g(u)\, G_t(\rho_K(u), u)}{\varphi'\left(\rho_K(u)\right)}\, du,$$

where $\widehat{K}_\varepsilon \in \mathscr{S}^n_{c+}$ has radial function $\widehat{\rho}_\varepsilon$ given by (4.28) with $f_0 = \rho_K$.

Proof. (i) By (4.1),

$$\lim_{\varepsilon\to 0^+} \frac{\widetilde{V}_G(K_\varepsilon) - \widetilde{V}_G(K)}{\varepsilon} = \lim_{\varepsilon\to 0^+} \int_{S^{n-1}} \frac{G(\rho_\varepsilon(u), u) - G(\rho_K(u), u)}{\varepsilon}\, du. \quad (4.32)$$

Also, by (2.26),

$$\begin{aligned} \lim_{\varepsilon\to 0^+} \frac{G(\rho_\varepsilon(u), u) - G(\rho_K(u), u)}{\varepsilon} &= G_t(\rho_K(u), u) \lim_{\varepsilon\to 0^+} \frac{\rho_\varepsilon(u) - \rho_K(u)}{\varepsilon} \\ &= \frac{1}{(\varphi_1)'_l(1)} \varphi_2\left(\frac{\rho_L(u)}{\rho_K(u)}\right) \rho_K(u) G_t(\rho_K(u), u), \end{aligned}$$

where the previous limit is uniform on S^{n-1}. Therefore (4.31) will follow if we show that the limit and integral in (4.32) can be interchanged. To this end, assume that $\varphi_1, \varphi_2 \in \mathscr{I}$ and $(\varphi_1)'_l(1) > 0$; the proof when $\varphi_1, \varphi_2 \in \mathscr{D}$ and $(\varphi_1)'_r(1) < 0$ is similar. If $\rho_1(u) = \rho_\varepsilon(u)\big|_{\varepsilon=1}$, it is easy to see from (4.30) that $\rho_K \le \rho_\varepsilon \le \rho_1$ on S^{n-1} when $\varepsilon \in (0,1)$. Since G_t is continuous on $(0,\infty)\times S^{n-1}$,

$$\sup\{|G_t(t,u)| : \rho_K(u) \le t \le \rho_1(u),\ u \in S^{n-1}\} = m_1 < \infty.$$

By the mean value theorem and Lemma 2.3.2 (i),

$$\left| \frac{G(\rho_\varepsilon(u), u) - G(\rho_K(u), u)}{\varepsilon} \right| \leq m_2$$

for $0 < \varepsilon < 1$. Thus we may apply the dominated convergence theorem in (4.32) to complete the proof.

(ii) The argument is very similar to that for (i) above. Since

$$\lim_{\varepsilon \to 0} \frac{\widetilde{V}_G(\widehat{K}_\varepsilon) - \widetilde{V}_G(K)}{\varepsilon} = \lim_{\varepsilon \to 0} \int_{S^{n-1}} \frac{G(\widehat{\rho}_\varepsilon(u), u) - G(\rho_K(u), u)}{\varepsilon} \, du, \tag{4.33}$$

we can use (2.27) instead of (2.26) and need only justify interchanging the limit and integral in (4.33). To see that this is valid, suppose that $\varphi \in \mathscr{J}_a$ is strictly increasing; the proof is similar when φ is strictly decreasing. Then there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and $u \in S^{n-1}$, we have

$$0 < b_1(u) = \varphi^{-1}\left(\varphi\left(\rho_K(u)\right) - \varepsilon_0 m_3\right) \leq \widehat{\rho}_\varepsilon(u) \leq \varphi^{-1}\left(\varphi\left(\rho_K(u)\right) + \varepsilon_0 m_3\right) = b_2(u) < \infty,$$

where $m_3 = \sup_{u \in S^{n-1}} |g(u)| < \infty$ due to $g \in C(S^{n-1})$. Since G_t is continuous on $(0, \infty) \times S^{n-1}$, then $\sup\{|G_t(t, u)| : b_1(u) \leq t \leq b_2(u),\ u \in S^{n-1}\} = m_4 < \infty$. By the mean value theorem and Lemma 2.3.2 (ii),

$$\left| \frac{G(\widehat{\rho}_\varepsilon(u), u) - G(\rho_K(u), u)}{\varepsilon} \right| \leq m_5$$

for $-\varepsilon_0 < \varepsilon < \varepsilon_0$. Thus we may apply the dominated convergence theorem in (4.33) to complete the proof. $\square$

Recall that $\overline{V}_\phi$ and $\underline{V}_\phi$ are defined by (4.3) and (4.4), respectively. Note that when $G = \overline{\Phi}$ or $\underline{\Phi}$, $G_t(t, u) = \pm\phi(tu)t^{n-1}$ is continuous on $(0, \infty) \times S^{n-1}$ because ϕ is assumed to be continuous. The following result is then a direct consequence of the previous theorem.

Corollary 4.2.2. *Let $\phi : \mathbb{R}^n \setminus \{o\} \to (0, \infty)$ be a continuous function and let $K, L \in \mathscr{S}^n_{c+}$.*

(i) *If* $\varphi_1, \varphi_2 \in \mathscr{I}$ *and* $(\varphi_1)'_l(1) > 0$*, then*

$$\frac{\int_{S^{n-1}} \phi(\rho_K(u)u)\, \varphi_2\Big(\frac{\rho_L(u)}{\rho_K(u)}\Big)\, \rho_K(u)^n\, du}{(\varphi_1)'_l(1)} = \begin{cases} \lim\limits_{\varepsilon \to 0^+} \dfrac{\overline{V}_\phi(K) - \overline{V}_\phi(\rho_\varepsilon)}{\varepsilon} & \text{(4.34a)} \\ \lim\limits_{\varepsilon \to 0^+} \dfrac{\underline{V}_\phi(\rho_\varepsilon) - \underline{V}_\phi(K)}{\varepsilon}, & \text{(4.34b)} \end{cases}$$

where ρ_ε *is given by* (4.30)*, provided* $\overline{\Phi}$ *(or* $\underline{\Phi}$*, respectively) is continuous. For* $\varphi_1, \varphi_2 \in \mathscr{D}$*,* (4.34a) *and* (4.34b) *hold when* $(\varphi_1)'_r(1) < 0$*, with* $(\varphi_1)'_l(1)$ *replaced by* $(\varphi_1)'_r(1)$*.*
(ii) *Let* $a \in \mathbb{R} \cup \{-\infty\}$*. If* $\varphi \in \mathscr{J}_a$ *and* φ' *is continuous and nonzero on* $(0, \infty)$*, then for all* $g \in C(S^{n-1})$*,*

$$\int_{S^{n-1}} \frac{\phi(\rho_K(u)u)\, \rho_K(u)^{n-1}}{\varphi'\left(\rho_K(u)\right)} g(u)\, du = \begin{cases} \lim\limits_{\varepsilon \to 0} \dfrac{\overline{V}_\phi(K) - \overline{V}_\phi(\widehat{\rho}_\varepsilon)}{\varepsilon} \\ \lim\limits_{\varepsilon \to 0} \dfrac{\underline{V}_\phi(\widehat{\rho}_\varepsilon) - \underline{V}_\phi(K)}{\varepsilon}, \end{cases}$$

where $\widehat{\rho}_\varepsilon$ *is given by* (4.28) *with* $f_0 = \rho_K$*.*

Formulas (4.34a) and (4.34b) motivate the following definition of the *general dual Orlicz mixed volume* $\widetilde{V}_{\phi,\varphi}(K, L)$. For $K, L \in \mathscr{S}^n_{c+}$, continuous $\phi : \mathbb{R}^n \setminus \{o\} \to (0, \infty)$, and continuous $\varphi : (0, \infty) \to (0, \infty)$, let

$$\widetilde{V}_{\phi,\varphi}(K, L) = \frac{1}{n} \int_{S^{n-1}} \phi(\rho_K(u)u)\, \varphi\left(\frac{\rho_L(u)}{\rho_K(u)}\right) \rho_K(u)^n\, du. \tag{4.35}$$

Then (4.34a) and (4.34b) become

$$\widetilde{V}_{\phi,\varphi}(K, L) = \begin{cases} \dfrac{(\varphi_1)'_l(1)}{n} \lim\limits_{\varepsilon \to 0^+} \dfrac{\overline{V}_\phi(K) - \overline{V}_\phi(\rho_\varepsilon)}{\varepsilon} \\ \dfrac{(\varphi_1)'_l(1)}{n} \lim\limits_{\varepsilon \to 0^+} \dfrac{\underline{V}_\phi(\rho_\varepsilon) - \underline{V}_\phi(K)}{\varepsilon}. \end{cases}$$

The special case of (4.34a) and (4.34b) when $\phi \equiv 1$ was proved in [18, Theorem 5.4] (see also [79, Theorem 4.1]) and the corresponding quantity $\widetilde{V}_{\phi,\varphi}(K, L)$ was called the *Orlicz dual mixed volume*.

On the other hand, Corollary 4.2.2 (ii) suggests an alternative definition of the general dual mixed volume. For all $K \in \mathscr{S}^n_{c+}$, $g \in C(S^{n-1})$, continuous $\phi : \mathbb{R}^n \setminus \{o\} \to$

$(0,\infty)$, and continuous $\varphi:(0,\infty)\to(0,\infty)$, define

$$\breve{V}_{\phi,\varphi}(K,g)=\frac{1}{n}\int_{S^{n-1}}\phi(\rho_K(u)u)\,\varphi(\rho_K(u))\,g(u)\,du. \tag{4.36}$$

Then the formulas in Corollary 4.2.2 (ii) can be rewritten as

$$\breve{V}_{\phi,\varphi_0}(K,g)=\begin{cases}\lim\limits_{\varepsilon\to 0}\dfrac{\overline{V}_\phi(K)-\overline{V}_\phi(\widehat{\rho}_\varepsilon)}{\varepsilon}\\ \lim\limits_{\varepsilon\to 0}\dfrac{\underline{V}_\phi(\widehat{\rho}_\varepsilon)-\underline{V}_\phi(K)}{\varepsilon},\end{cases}$$

where $\varphi_0(t)=nt^{n-1}/\varphi'(t)$. In particular, one can define a dual Orlicz mixed volume of K and L by letting $g=\psi(\rho_L)$, where $\psi:(0,\infty)\to(0,\infty)$ is continuous and $L\in\mathscr{S}^n_{c+}$, namely

$$\breve{V}_{\phi,\varphi,\psi}(K,L)=\frac{1}{n}\int_{S^{n-1}}\phi(\rho_K(u)u)\,\varphi(\rho_K(u))\;\psi(\rho_L(u))\,du.$$

Note that both $\widetilde{V}_{\phi,\varphi}(K,L)$ and $\breve{V}_{\phi,\varphi}(K,g)$ are special cases of $\widetilde{V}_G(K)$, corresponding to setting $G(t,u)=\frac{1}{n}\phi(tu)\,\varphi\left(\frac{\rho_L(u)}{t}\right)\,t^n$ and $G(t,u)=\frac{1}{n}\phi(tu)\,\varphi(t)\,g(u)$, respectively.

4.2.2 General variational formulas for Orlicz linear combinations

We shall assume throughout the section that $\Omega\subset S^{n-1}$ is a closed set not contained in any closed hemisphere of S^{n-1}.

Let $h_0,\rho_0\in C^+(\Omega)$ and let h_ε and ρ_ε be defined by (4.28) with $f_0=h_0$ and $f_0=\rho_0$, respectively. In Lemma 2.3.2 (ii), we may replace ρ_K by h_0 or ρ_0 to conclude that $h_\varepsilon\to h_0$ and $\rho_\varepsilon\to\rho_0$ uniformly on Ω. Hence $[h_\varepsilon]\to[h_0]$ and $\langle\rho_\varepsilon\rangle\to\langle\rho_0\rangle$ as $\varepsilon\to 0$. However, in order to get a variational formula for the general dual Orlicz volume $\widetilde{V}_G$, we shall need the following lemma. It was proved for $\varphi(t)=\log t$ in [29, Lemmas 4.1 and 4.2] and was noted for t^p, $p\neq 0$, in the proof of [54, Theorem 6.5]. Recall from Section 2.2 that $S^{n-1}\setminus\eta_{\langle\rho_0\rangle}$ is the set of regular normal vectors of $\langle\rho_0\rangle\in\mathscr{K}^n_{(o)}$.

Lemma 4.2.3. *Let $g\in C(\Omega)$, $\rho_0\in C^+(\Omega)$, and $a\in\mathbb{R}\cup\{-\infty\}$. Suppose that $\varphi\in\mathscr{J}_a$ is continuously differentiable and such that φ' is nonzero on $(0,\infty)$. For*

$v \in S^{n-1} \setminus \eta_{\langle\rho_0\rangle}$,

$$\lim_{\varepsilon\to 0} \frac{\log h_{\langle\rho_\varepsilon\rangle}(v) - \log h_{\langle\rho_0\rangle}(v)}{\varepsilon} = \frac{g(\alpha_{\langle\rho_0\rangle^*}(v))}{\rho_0(\alpha_{\langle\rho_0\rangle^*}(v))\,\varphi'(\rho_0(\alpha_{\langle\rho_0\rangle^*}(v)))}, \tag{4.38}$$

where ρ_ε is defined by (4.28) with $f_0 = \rho_0$. Moreover, there exist $\delta, m_0 > 0$ such that

$$|\log h_{\langle\rho_\varepsilon\rangle}(v) - \log h_{\langle\rho_0\rangle}(v)| \le m_0|\varepsilon| \tag{4.39}$$

for $\varepsilon \in (-\delta, \delta)$ and $v \in S^{n-1}$.

Proof. We shall assume that $\varphi \in \mathscr{J}_a$ is strictly increasing, since the case when it is strictly decreasing is similar. Since $g \in C(\Omega)$, we have $m_1 = \sup_{u\in\Omega}|g(u)| < \infty$. Then there exists $\delta_0 > 0$ such that for $\varepsilon \in [-\delta_0, \delta_0]$ and $u \in \Omega$,

$$0 < \varphi^{-1}\left(\varphi\left(\rho_0(u)\right) - \delta_0 m_1\right) \le \rho_\varepsilon(u) \le \varphi^{-1}\left(\varphi\left(\rho_0(u)\right) + \delta_0 m_1\right),$$

and $\inf_{u\in\Omega}|\varphi'(\rho_\varepsilon(u))| > 0$. For $u \in \Omega$ and $\varepsilon \in (-\delta_0, \delta_0)$, let

$$H_u(\varepsilon) = \log \rho_\varepsilon(u) = \log\left(\varphi^{-1}(\varphi(\rho_0(u)) + \varepsilon\, g(u))\right),$$

from which we obtain

$$H_u'(\varepsilon) = \frac{g(u)}{\rho_\varepsilon(u)\varphi'(\rho_\varepsilon(u))}.$$

By the mean value theorem, for all $u \in \Omega$ and $\varepsilon \in (-\delta_0, \delta_0)$, we get

$$H_u(\varepsilon) - H_u(0) = \varepsilon\, H_u'(\theta\,\varepsilon),$$

where $\theta = \theta(u, \varepsilon) \in (0, 1)$. In other words,

$$\log \rho_\varepsilon(u) - \log \rho_0(u) = \varepsilon\, \frac{g(u)}{\rho_{\theta(u,\varepsilon)\varepsilon}(u)\,\varphi'(\rho_{\theta(u,\varepsilon)\varepsilon}(u))} \tag{4.40}$$

for $u \in \Omega$ and $\varepsilon \in (-\delta_0, \delta_0)$.

Let $v \in S^{n-1} \setminus \eta_{\langle\rho_0\rangle}$. If $\varepsilon \in (-\delta_0, \delta_0)$, there is a $u_\varepsilon \in \Omega$ such that for $u \in \Omega$,

$$\begin{aligned} h_{\langle\rho_\varepsilon\rangle}(v) &= \langle u_\varepsilon, v\rangle\rho_\varepsilon(u_\varepsilon), \quad h_{\langle\rho_\varepsilon\rangle}(v) \ge \langle u, v\rangle\rho_\varepsilon(u), \\ h_{\langle\rho_0\rangle}(v) &\ge \langle u_\varepsilon, v\rangle\rho_{\langle\rho_0\rangle}(u_\varepsilon), \quad \text{and} \quad \rho_{\langle\rho_0\rangle}(u_\varepsilon) \ge \rho_0(u_\varepsilon). \end{aligned} \tag{4.41}$$

Moreover, $\langle u_\varepsilon, v\rangle > 0$ for $\varepsilon \in (-\delta_0, \delta_0)$. Hence, using the equation in (4.41), the inequality in (4.41) with $u = u_\varepsilon$, and (4.40) for $u = u_\varepsilon$, we get

$$\begin{aligned} \log h_{\langle\rho_\varepsilon\rangle}(v) - \log h_{\langle\rho_0\rangle}(v) &\le \log \rho_\varepsilon(u_\varepsilon) - \log \rho_0(u_\varepsilon) \\ &= \varepsilon \frac{g(u_\varepsilon)}{\rho_{\theta(u_\varepsilon,\varepsilon)\varepsilon}(u_\varepsilon)\,\varphi'(\rho_{\theta(u_\varepsilon,\varepsilon)\varepsilon}(u_\varepsilon))}. \end{aligned} \tag{4.42}$$

From the equation in (4.41) with $\varepsilon = 0$, the inequality in (4.41) with $u = u_0$, and from (4.40) with $u = u_0$, we obtain

$$\begin{aligned} \log h_{\langle\rho_\varepsilon\rangle}(v) - \log h_{\langle\rho_0\rangle}(v) &= \log h_{\langle\rho_\varepsilon\rangle}(v) - \log \rho_0(u_0) - \log\langle u_0, v\rangle \\ &\ge \log \rho_\varepsilon(u_0) - \log \rho_0(u_0) \\ &= \varepsilon \frac{g(u_0)}{\rho_{\theta(u_0,\varepsilon)\varepsilon}(u_0)\,\varphi'(\rho_{\theta(u_0,\varepsilon)\varepsilon}(u_0))}. \end{aligned} \tag{4.43}$$

Exactly as in [29, (4.7), (4.8)], we have $u_0 = \alpha^*_{\langle\rho_0\rangle}(v) = \alpha_{\langle\rho_0\rangle^*}(v)$ and $\lim_{\varepsilon\to 0} u_\varepsilon = u_0$. Since g is continuous and $u_\varepsilon \to u_0$, we get $g(u_\varepsilon) \to g(u_0)$ as $\varepsilon \to 0$. From $\theta(\cdot) \in (0,1)$ it follows that $\theta(\cdot)\varepsilon \to 0$ as $\varepsilon \to 0$. Moreover, $\rho_{\theta(\cdot)\varepsilon}(u_\varepsilon) = \varphi^{-1}(\varphi(\rho_0(u_\varepsilon)) + \theta(\cdot)\varepsilon g(u_\varepsilon)) \to \rho_0(u_0)$ and, similarly, $\rho_{\theta(\cdot)\varepsilon}(u_0) \to \rho_0(u_0)$ as $\varepsilon \to 0$. Thus we conclude that

$$\lim_{\varepsilon\to 0} \frac{\log h_{\langle\rho_\varepsilon\rangle}(v) - \log h_{\langle\rho_0\rangle}(v)}{\varepsilon} = \frac{g(u_0)}{\rho_0(u_0)\,\varphi'(\rho_0(u_0))}.$$

Substituting $u_0 = \alpha^*_{\langle\rho_0\rangle}(v)$, we obtain (4.38).

If δ_0 is sufficiently small, then (4.42) and (4.43) imply that if $v \in S^{n-1} \setminus \eta_{\langle\rho_0\rangle}$ then

$$\left|\log h_{\langle\rho_\varepsilon\rangle}(v) - \log h_{\langle\rho_0\rangle}(v)\right| \le |\varepsilon| \sup_{u\in\Omega,\,\theta\in[0,1]} \left|\frac{g(u)}{\rho_{\theta\varepsilon}(u)\,\varphi'(\rho_{\theta\varepsilon}(u))}\right| = m_2|\varepsilon|,$$

say, for some $m_2 < \infty$. From this, we see that (4.39) holds for $v \in S^{n-1} \setminus \eta_{\langle\rho_0\rangle}$ and hence, by (2.8) and the continuity of support functions, for $v \in S^{n-1}$. □

Lemma 4.2.4. *Let $g \in C(\Omega)$, $h_0 \in C^+(\Omega)$, and $a \in \mathbb{R}\cup\{-\infty\}$. Suppose that $\varphi \in \mathscr{J}_a$ is continuously differentiable and such that φ' is nonzero on $(0,\infty)$. If G and G_t are continuous on $(0,\infty)\times S^{n-1}$, then*

$$\lim_{\varepsilon\to 0} \frac{\widetilde{V}_G([h_\varepsilon]) - \widetilde{V}_G([h_0])}{\varepsilon} = \int_{S^{n-1}\setminus\eta_{\langle\kappa_0\rangle}} J(0,u) \frac{\kappa_0(\alpha_{\langle\kappa_0\rangle^*}(u))\, g(\alpha_{\langle\kappa_0\rangle^*}(u))}{\varphi'(\kappa_0(\alpha_{\langle\kappa_0\rangle^*}(u))^{-1})}\, du, \tag{4.44}$$

where h_ε is given by (4.28) with $f_0 = h_0$, and for ε sufficiently close to 0, $\kappa_\varepsilon = 1/h_\varepsilon$ and

$$J(\varepsilon, u) = \rho_{\langle\kappa_\varepsilon\rangle^*}(u)\, G_t(\rho_{\langle\kappa_\varepsilon\rangle^*}(u), u). \tag{4.45}$$

Proof. Let $\overline{\varphi}(t) = \varphi(1/t)$ for all $t \in (0, \infty)$. Clearly $\overline{\varphi} \in \mathscr{J}_a$. Also, for $t \in (0, \infty)$, we have $\overline{\varphi}'(t) = -t^{-2}\varphi'(1/t)$. Hence $\overline{\varphi}$ satisfies the conditions for φ in Lemma 4.2.3. It is easy to check that $\kappa_\varepsilon(u) = \overline{\varphi}^{-1}\left(\overline{\varphi}\left(\kappa_0(u)\right) + \varepsilon g(u)\right)$, that is, κ_ε is given by (4.28) when φ and f_0 are replaced by $\overline{\varphi}$ and κ_0. By (4.38), with ρ_ε and φ replaced by κ_ε and $\overline{\varphi}$, respectively, for sufficiently small $|\varepsilon|$, we obtain, for $u \in S^{n-1} \setminus \eta_{\langle\kappa_0\rangle}$,

$$\begin{aligned}
\lim_{\varepsilon\to 0} \frac{\log \rho_{\langle\kappa_\varepsilon\rangle^*}(u) - \log \rho_{\langle\kappa_0\rangle^*}(u)}{\varepsilon} &= -\lim_{\varepsilon\to 0} \frac{\log h_{\langle\kappa_\varepsilon\rangle}(u) - \log h_{\langle\kappa_0\rangle}(u)}{\varepsilon} \\
&= -\frac{g(\alpha_{\langle\kappa_0\rangle^*}(u))}{\kappa_0(\alpha_{\langle\kappa_0\rangle^*}(u))\,\overline{\varphi}'(\kappa_0(\alpha_{\langle\kappa_0\rangle^*}(u)))} \\
&= \frac{\kappa_0(\alpha_{\langle\kappa_0\rangle^*}(u))\, g(\alpha_{\langle\kappa_0\rangle^*}(u))}{\varphi'(\kappa_0(\alpha_{\langle\kappa_0\rangle^*}(u))^{-1})}.
\end{aligned} \tag{4.46}$$

Moreover, comparing (4.39), there exist $\delta, m_0 > 0$ such that for $\varepsilon \in (-\delta, \delta)$ and $u \in S^{n-1}$,

$$|\log h_{\langle\kappa_\varepsilon\rangle}(u) - \log h_{\langle\kappa_0\rangle}(u)| \le m_0|\varepsilon|. \tag{4.47}$$

Note that

$$\begin{aligned}
\frac{dG(\rho_{\langle\kappa_\varepsilon\rangle^*}(u), u)}{d\varepsilon} &= G_t(\rho_{\langle\kappa_\varepsilon\rangle^*}(u), u)\, \frac{d}{d\varepsilon}\rho_{\langle\kappa_\varepsilon\rangle^*}(u) \\
&= J(\varepsilon, u)\, \frac{d}{d\varepsilon} \log \rho_{\langle\kappa_\varepsilon\rangle^*}(u).
\end{aligned} \tag{4.48}$$

By our assumptions, there exists $0 < \delta_1 \le \delta$ and $m_1 > 0$ such that $|J(\varepsilon, u)| < m_1$ for $\varepsilon \in (-\delta_1, \delta_1)$ and $u \in S^{n-1}$. It follows from (4.47), (4.48), and the mean value theorem that, for $\varepsilon \in (-\delta_1, \delta_1)$ and $u \in S^{n-1}$,

$$\left|\frac{G(\rho_{\langle\kappa_\varepsilon\rangle^*}(u), u) - G(\rho_{\langle\kappa_0\rangle^*}(u), u)}{\varepsilon}\right| < m_0 m_1.$$

From (2.20), we know that $[h_\varepsilon] = \langle\kappa_\varepsilon\rangle^*$, so $\langle\kappa_\varepsilon\rangle^* \to \langle\kappa_0\rangle^*$ as $\varepsilon \to 0$. By the dominated convergence theorem, (4.46), and (4.48), we obtain

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\frac{\widetilde{V}_G([h_\varepsilon])-\widetilde{V}_G([h_0])}{\varepsilon} &= \lim_{\varepsilon\to 0}\int_{S^{n-1}}\frac{G(\rho_{\langle\kappa_\varepsilon\rangle^*}(u),u)-G(\rho_{\langle\kappa_0\rangle^*}(u),u)}{\varepsilon}\,du\\
&= \int_{S^{n-1}}\lim_{\varepsilon\to 0}\frac{G(\rho_{\langle\kappa_\varepsilon\rangle^*}(u),u)-G(\rho_{\langle\kappa_0\rangle^*}(u),u)}{\varepsilon}\,du\\
&= \int_{S^{n-1}\setminus\eta_{\langle\kappa_0\rangle}} J(0,u)\frac{\kappa_0(\alpha_{\langle\kappa_0\rangle^*}(u))\,g(\alpha_{\langle\kappa_0\rangle^*}(u))}{\varphi'(\kappa_0(\alpha_{\langle\kappa_0\rangle^*}(u))^{-1})}\,du,
\end{aligned}
$$

where we have used the fact that $\mathscr{H}^{n-1}(\eta_{\langle\kappa_0\rangle})=0$ by (2.8). □

The next theorem will be used in the proof of Theorem 4.3.3. It generalizes previous results of this type, which originated from [29, Theorem 4.5]; see the discussion after Corollary 4.2.7.

Theorem 4.2.5. *Let $g\in C(\Omega)$, $h_0\in C^+(\Omega)$, and $a\in\mathbb{R}\cup\{-\infty\}$. Suppose that $\varphi\in\mathscr{J}_a$ is continuously differentiable and such that φ' is nonzero on $(0,\infty)$. If G and G_t are continuous on $(0,\infty)\times S^{n-1}$, then*

$$
\lim_{\varepsilon\to 0}\frac{\widetilde{V}_G([h_\varepsilon])-\widetilde{V}_G([h_0])}{\varepsilon}=n\int_\Omega g(u)\,d\widetilde{C}_{G,\psi}([h_0],u), \tag{4.49}
$$

where h_ε is given by (4.28) with $f_0=h_0$, and $\psi(t)=t\varphi'(t)$.

Proof. It follows from [29, p. 364] that there exists a continuous function $\overline{g}:S^{n-1}\to\mathbb{R}$, such that, for $u\in S^{n-1}\setminus\eta_{\langle\kappa_0\rangle}$,

$$
g(\alpha_{\langle\kappa_0\rangle^*}(u))=(\overline{g}\mathbf{1}_\Omega)(\alpha_{\langle\kappa_0\rangle^*}(u)).
$$

Using this, $\kappa_0=1/h_0$, the relation $\langle\kappa_0\rangle^*=[h_0]$ given by (2.9), (2.20), (4.45) with $\varepsilon=0$, $\mathscr{H}^{n-1}(\eta_{\langle\kappa_0\rangle})=0$ from (2.8), and (4.8), the formula (4.44) becomes

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\frac{\widetilde{V}_G([h_\varepsilon])-\widetilde{V}_G([h_0])}{\varepsilon} &= \int_{S^{n-1}\setminus\eta_{\langle\kappa_0\rangle}}\frac{(\overline{g}\mathbf{1}_\Omega)(\alpha_{[h_0]}(u))\,\rho_{[h_0]}(u)\,G_t(\rho_{[h_0]}(u),u)}{h_0(\alpha_{[h_0]}(u))\,\varphi'(h_0(\alpha_{[h_0]}(u)))}\,du\\
&= \int_{S^{n-1}}\frac{(\overline{g}\mathbf{1}_\Omega)(\alpha_{[h_0]}(u))\,\rho_{[h_0]}(u)\,G_t(\rho_{[h_0]}(u),u)}{\psi(h_0(\alpha_{[h_0]}(u)))}\,du\\
&= n\int_\Omega g(u)\,d\widetilde{C}_{G,\psi}([h_0],u),
\end{aligned}
$$

where we also used the fact that

$$h_{[h_0]}(\alpha_{[h_0]}(u)) = h_0(\alpha_{[h_0]}(u)) \quad \text{for } \mathscr{H}^{n-1}\text{-almost all } u \in S^{n-1}.$$

To see this, note that for $\mathscr{H}^{n-1}$-almost all $u \in S^{n-1}$, we have $\alpha_{[h_0]}(u) = \nu_{[h_0]}(\rho_{[h_0]}(u)u)$ and $\rho_{[h_0]}(u)u$ is a regular boundary point of $[h_0]$. The rest is done by the proof of Lemma 7.5.1 in [59, p. 411], which shows that if $x \in \partial[h_0]$ is a regular boundary point, then $h_{[h_0]}(\nu_{[h_0]}(x)) = h_0(\nu_{[h_0]}(x))$. □

Remark 4.2.6. It is possible to extend the definition (4.1) of the general dual volume $\widetilde{V}_G$ by allowing continuous functions $G : (0,\infty) \times S^{n-1} \to \mathbb{R}$. In this case, of course, $\widetilde{V}_G$ may be negative, but the extended definition has the advantage of including fundamental concepts such as the dual entropy $\widetilde{E}(K)$ of K. This is defined by

$$\widetilde{E}(K) = \frac{1}{n}\int_{S^{n-1}} \log \rho_K(u)\, du,$$

corresponding to taking $G(t,u) = (1/n)\log t$ in (4.1). Definition 4.1.3 of the measure $\widetilde{C}_{G,\psi}$ and the integral formulas (4.8) and (4.9) remain valid for continuous functions $G : (0,\infty)\times S^{n-1} \to \mathbb{R}$, as do Theorems 4.1.4, 4.2.1, and 4.2.5, as well as Theorem 4.2.8 below.

Theorem 4.2.5 and its extended form indicated in Remark 4.2.6 may be used to retrieve the formulas in [54, Theorem 6.5], which in turn generalize those in [29, Corollary 4.8]. To see this, let $K, L \in \mathscr{K}^n_{(o)}$ and let $\varphi(t) = t^p$, $p \neq 0$. Setting $h_0 = h_K$ and $g = h_L^p$, we see from (4.28) with $f_0 = h_0$ that $[h_\varepsilon] = K \widehat{+}_p \varepsilon \cdot L$, the L_p linear combination of K and L. Taking $G(t,u) = (1/n)t^q \rho_Q(u)^{n-q}$, for some $Q \in \mathscr{S}^n_{c+}$ and $q \neq 0$, where $t > 0$ and $u \in S^{n-1}$, we have $\widetilde{V}_G(K) = \widetilde{V}_q(K,Q)$ as in (4.6). With $\Omega = S^{n-1}$ and $\psi(t) = t\varphi'(t) = pt^p$, and using (4.8) and (4.27), we obtain

$$\begin{aligned} n\int_\Omega g(u)\, d\widetilde{C}_{G,\psi}([h_0],u) &= n\int_{S^{n-1}} h_L(u)^p\, d\widetilde{C}_{G,\psi}(K,u) \\ &= \frac{q}{np}\int_{S^{n-1}} \left(\frac{h_L(\alpha_K(u))}{h_K(\alpha_K(u))}\right)^p \rho_K(u)^q \rho_Q(u)^{n-q}\, du \\ &= \frac{q}{p}\int_{S^{n-1}} h_L(u)^p\, d\widetilde{C}_{p,q}(K,Q,u). \end{aligned}$$

Thus (4.49) becomes

$$\lim_{\varepsilon\to 0} \frac{\widetilde{V}_q(K \widehat{+}_p \varepsilon \cdot L, Q) - \widetilde{V}_q(K,Q)}{\varepsilon} = \frac{q}{p}\int_{S^{n-1}} h_L(u)^p\, d\widetilde{C}_{p,q}(K,Q,u),$$

the formula in [54, (6.3), Theorem 6.5] (where $\widehat{+}_p$ is denoted by $+_p$; in our usage, the two are equivalent for $p \geq 1$, when h_ε above is a support function). Next, we take instead $\varphi(t) = \log t$ and $g = \log h_L$, noting from (4.28) with $f_0 = h_0$ that $[h_\varepsilon] = K \widehat{+}_0 \, \varepsilon \cdot L$, the logarithmic linear combination of K and L. Then, again with $\Omega = S^{n-1}$ and $\psi(t) = t\varphi'(t) = 1$, an argument similar to that above shows that (4.49) becomes

$$\lim_{\varepsilon \to 0} \frac{\widetilde{V}_q(K \widehat{+}_0 \, \varepsilon \cdot L, Q) - \widetilde{V}_q(K, Q)}{\varepsilon} = q \int_{S^{n-1}} \log h_L(u) \, d\widetilde{C}_{0,q}(K, Q, u),$$

the formula in [54, (6.4), Theorem 6.5] (where $\widehat{+}_0$ is denoted by $+_0$).

If instead we take $G(t, u) = (1/n) \log(t/\rho_Q(u)) \, \rho_Q(u)^n$, for some $Q \in \mathscr{S}^n_{c+}$, where $t > 0$ and $u \in S^{n-1}$, we have

$$\widetilde{V}_G(K) = \frac{1}{n} \int_{S^{n-1}} \log \left(\frac{\rho_K(u)}{\rho_Q(u)} \right) \rho_Q(u)^n \, du = \widetilde{E}(K, Q),$$

the dual mixed entropy of K and Q. Then similar computations to those above show that (4.49) (now justified via Remark 4.2.6) yield the variational formulas [54, (6.5) and (6.6), Theorem 6.5] for $\widetilde{E}(K, Q)$.

The following corollary is a direct consequence of the previous theorem with $G = \overline{\Phi}$ or $\underline{\Phi}$, and (4.25a) and (4.25b) with $\psi(t) = t\varphi'(t)$. When $\varphi(t) = \log t$, it was proved in [69, Theorem 4.1].

Corollary 4.2.7. *Let $g \in C(\Omega)$, $h_0 \in C^+(\Omega)$, and $a \in \mathbb{R} \cup \{-\infty\}$. Suppose that $\varphi \in \mathscr{J}_a$ is continuously differentiable and such that φ' is nonzero on $(0, \infty)$. If $\phi : \mathbb{R}^n \setminus \{o\} \to (0, \infty)$ and $\overline{\Phi}$ (or $\underline{\Phi}$, as appropriate) are continuous, then*

$$n \int_\Omega \frac{g(u)}{h_0(u) \, \varphi'(h_0(u))} \, d\widetilde{C}_{\phi,\mathscr{V}}([h_0], u) = \begin{cases} \lim\limits_{\varepsilon \to 0} \dfrac{\overline{V}_\phi([h_0]) - \overline{V}_\phi([h_\varepsilon])}{\varepsilon} \\ \lim\limits_{\varepsilon \to 0} \dfrac{\underline{V}_\phi([h_\varepsilon]) - \underline{V}_\phi([h_0])}{\varepsilon} \end{cases}, \tag{4.50}$$

where h_ε is given by (4.28) with $f_0 = h_0$.

The following version of Theorem 4.2.5 for Orlicz linear combination of the form (4.29) can be proved in a similar fashion. We omit the proof. Recall that $\widetilde{C}_G([h_1], \cdot) = \widetilde{C}_{G,\psi}([h_1], \cdot)$ when $\psi \equiv 1$, as in Definition 4.1.3.

Theorem 4.2.8. *Let $h_1, h_2 \in C^+(\Omega)$ and let $\varphi_1, \varphi_2 \in \mathscr{I}$ or $\varphi_1, \varphi_2 \in \mathscr{D}$. Suppose that for $i = 1, 2$, φ_i is continuously differentiable and such that φ_i' is nonzero on $(0,\infty)$. If G and G_t are continuous on $(0,\infty) \times S^{n-1}$, then*

$$\lim_{\varepsilon \to 0^+} \frac{\widetilde{V}_G([h_\varepsilon]) - \widetilde{V}_G([h_1])}{\varepsilon} = \frac{n}{\varphi_1'(1)} \int_\Omega \varphi_2\left(\frac{h_2(u)}{h_1(u)}\right) \, d\widetilde{C}_G([h_1], u),$$

where h_ε is given by (4.29) with h_K and h_L replaced by h_1 and h_2, respectively.

Again, the following corollary is a direct consequence of the previous theorem with $G = \overline{\Phi}$ or $\underline{\Phi}$.

Corollary 4.2.9. *Let $h_1, h_2 \in C^+(\Omega)$ and let $\varphi_1, \varphi_2 \in \mathscr{I}$ or $\varphi_1, \varphi_2 \in \mathscr{D}$. Suppose that for $i = 1, 2$, φ_i is continuously differentiable and such that φ_i' is nonzero on $(0,\infty)$. If $\phi : \mathbb{R}^n \setminus \{o\} \to (0,\infty)$ and $\overline{\Phi}$ (or $\underline{\Phi}$, as appropriate) are continuous, then*

$$\frac{n}{\varphi_1'(1)} \int_\Omega \varphi_2\left(\frac{h_2(u)}{h_1(u)}\right) \, d\widetilde{C}_{\phi,\mathscr{V}}([h_1], u) = \begin{cases} \displaystyle\lim_{\varepsilon \to 0^+} \frac{\overline{V}_\phi([h_1]) - \overline{V}_\phi([h_\varepsilon])}{\varepsilon} \\ \displaystyle\lim_{\varepsilon \to 0^+} \frac{\underline{V}_\phi([h_\varepsilon]) - \underline{V}_\phi([h_1])}{\varepsilon}, \end{cases}$$

where h_ε is given by (4.29) with h_K and h_L replaced by h_1 and h_2, respectively.

4.3 Minkowski-type problems

This section is dedicated to providing a partial solution ($G_t < 0$) to the Orlicz-Minkowski problem for the measure $\widetilde{C}_{G,\psi}(K, \cdot)$.

Proposition 4.3.1. *Let G and G_t be continuous on $(0,\infty) \times S^{n-1}$, let $\psi : (0,\infty) \to (0,\infty)$ be continuous, and let $K \in \mathscr{K}^n_{(o)}$. The following statements hold.*

(i) *The signed measure $\widetilde{C}_{G,\psi}(K, \cdot)$ is absolutely continuous with respect to $S(K, \cdot)$.*

(ii) *If $K_i \in \mathscr{K}^n_{(o)}$, $i \in \mathbb{N}$, and $K_i \to K \in \mathscr{K}^n_{(o)}$ as $i \to \infty$, then $\widetilde{C}_{G,\psi}(K_i, \cdot) \to \widetilde{C}_{G,\psi}(K, \cdot)$ weakly.*

(iii) *If $G_t > 0$ on $(0,\infty) \times S^{n-1}$ (or $G_t < 0$ on $(0,\infty) \times S^{n-1}$), then $\widetilde{C}_{G,\psi}(K, \cdot)$ (or $-\widetilde{C}_{G,\psi}(K, \cdot)$, respectively) is a nonzero finite Borel measure not concentrated on any closed hemisphere.*

Proof. (i) Let $E \subset S^{n-1}$ be a Borel set such that $S(K, E) = 0$. If $g = \mathbf{1}_E$, the left-hand side of (4.8) is $\widetilde{C}_{G,\psi}(K, E)$. This equals the expression in (4.9), in which we observe that since $K \in \mathscr{K}^n_{(o)}$, for $x \in \partial K$ both $|x|$ and $\langle x, \nu_K(x)\rangle = h_K(\nu_K(x))$ are bounded away from zero and bounded above, and hence our assumptions imply that

$$\sup_{x\in\partial K}\left|\frac{\rho_K(\bar{x})\, G_t(\rho_K(\bar{x}), \bar{x})\, \langle x, \nu_K(x)\rangle}{\psi(\langle x, \nu_K(x)\rangle)\, |x|^n}\right| = c < \infty,$$

where $\bar{x} = x/|x|$. Then from (4.8) and (4.9) we conclude, using (2.5), that

$$\left|\widetilde{C}_{G,\psi}(K, E)\right| \le c \int_{\partial K} \mathbf{1}_E(\nu_K(x))\, dx = c\, \mathscr{H}^{n-1}(\nu_K^{-1}(E)) = c\, S(K, E) = 0.$$

(ii) Let $g : S^{n-1} \to \mathbb{R}$ be continuous and let

$$I_K(u) = g(\alpha_K(u))\frac{\rho_K(u)\, G_t(\rho_K(u), u)}{\psi(h_K(\alpha_K(u)))}$$

be the integrand of the right-hand side of (4.8). Suppose that $K_i \in \mathscr{K}^n_{(o)}$, $i \in \mathbb{N}$, and $K_i \to K \in \mathscr{K}^n_{(o)}$. By [29, Lemma 2.2], $\alpha_{K_i} \to \alpha_K$ and hence, by the continuity of G_t and the continuity of the map $(K, u) \mapsto h_K(u)$ (see [59, Lemma 1.8.12]), $I_{K_i} \to I_K$, $\mathscr{H}^{n-1}$-almost everywhere on S^{n-1}. Moreover, our assumptions clearly yield $\sup\{I_{K_i}(u) : i \in \mathbb{N}, u \in S^{n-1}\} < \infty$. It follows from (4.8) and the dominated convergence theorem that

$$\int_{S^{n-1}} g(u)\, d\widetilde{C}_{G,\psi}(K_i, u) \to \int_{S^{n-1}} g(u)\, d\widetilde{C}_{G,\psi}(K, u)$$

as $i \to \infty$, as required.

(iii) Suppose that $G_t > 0$ on $(0, \infty) \times S^{n-1}$; the case when $G_t < 0$ on $(0, \infty) \times S^{n-1}$ is similar. Let $m = \min_{x\in\partial K} J_K(x)$, where

$$J_K(x) = \frac{\rho_K(\bar{x})\, G_t(\rho_K(\bar{x}), \bar{x})\, \langle x, \nu_K(x)\rangle}{\psi(\langle x, \nu_K(x)\rangle)\, |x|^n}, \quad x \in \partial K,$$

and $\bar{x} = x/|x|$. Since $K \in \mathscr{K}^n_{(o)}$, our assumptions imply that $m > 0$. By (4.8) and (4.9),

$$\begin{aligned}
\int_{S^{n-1}} \langle u, v\rangle_+ \, d\widetilde{C}_{G,\psi}(K, v) &= \int_{\partial K} \langle u, \nu_K(x)\rangle_+ \, J_K(x)\, dx \\
&\geq m \int_{\partial K} \langle u, \nu_K(x)\rangle_+ \, dx \\
&= m \int_{S^{n-1}} \langle u, v\rangle_+ \, dS(K, v) \\
&> 0.
\end{aligned}$$

This shows that $\widetilde{C}_{G,\psi}(K, \cdot)$ also satisfies (2.6). □

In view of Proposition 4.3.1 (iii), one can ask the following Minkowski-type problem for the signed measure $\widetilde{C}_{G,\psi}(\cdot, \cdot)$.

Problem 4.3.2. *For which nonzero finite Borel measures μ on S^{n-1} and continuous functions $G : (0, \infty) \times S^{n-1} \to (0, \infty)$ and $\psi : (0, \infty) \to (0, \infty)$ do there exist $\tau \in \mathbb{R}$ and $K \in \mathscr{K}^n_{(o)}$ such that $\mu = \tau\, \widetilde{C}_{G,\psi}(K, \cdot)$?*

It follows immediately from (4.10), on using [54, (2.2), p. 93 and (3.28), p. 106], that solving Problem 4.3.2 requires finding an $h : S^{n-1} \to (0, \infty)$ and $\tau \in \mathbb{R}$ that solve (in the weak sense) the Monge-Ampère equation

$$\frac{\tau h}{\psi \circ h}\, P(\bar{\nabla} h + h\iota)\, \det(\bar{\nabla}^2 h + hI) = f, \tag{4.51}$$

where $P(x) = |x|^{1-n} G_t(|x|, \bar{x})$ for $x \in \mathbb{R}^n$. Here f plays the role of the density function of the measure μ in Problem 4.3.2 if μ is absolutely continuous with respect to spherical Lebesgue measure. Formally, then, Problem 4.3.2 is more difficult, since it calls for h in (4.51) to be the support function of a convex body and also a solution for measures that may not have a density function f. The Minkowski problem in [54, Problem 8.1] requires finding, for given $p, q \in \mathbb{R}$, n-dimensional Banach norm $\|\cdot\|$, and $f : S^{n-1} \to [0, \infty)$, an $h : S^{n-1} \to (0, \infty)$ that solves the Monge-Ampère equation

$$h^{1-p}\, \|\bar{\nabla} h + h\iota\|^{q-n} \det(\bar{\nabla}^2 h + hI) = f \tag{4.52}$$

on the unit sphere S^{n-1}, where $\bar{\nabla}$ and $\bar{\nabla}^2$ are the gradient vector and Hessian matrix of h, respectively, with respect to an orthonormal frame on S^{n-1}, ι is the identity map on S^{n-1}, and I is the identity matrix.

To see that (4.51) is more general than (4.52), note firstly that the homogeneity of the left-hand side of (4.52) allows us to set $\tau = 1$, without loss of generality (if $p \neq q$, which is true in the case $p > 0$, $q < 0$ of particular interest in the present chapter). Let $p, q \in \mathbb{R}$ and let $Q \in \mathscr{S}^n_{c+}$. For $t > 0$ and $u \in S^{n-1}$, we set $\psi(t) = t^p$ and $G(t,u) = (1/q)t^q \rho_Q(u)^{n-q}$, if $q \neq 0$, and $G(t,u) = (\log t)\rho_Q(u)^n$, otherwise. (When $q \leq 0$, we have $G : (0,\infty) \times S^{n-1} \to \mathbb{R}$ and Remark 4.2.6 applies.) Then, using the fact that ρ_Q is homogeneous of degree -1, we have $P(x) = \rho_Q(x)^{n-q}$, for $q \in \mathbb{R}$ and $x \in \mathbb{R}^n \setminus \{o\}$. Therefore (4.51) becomes

$$h^{1-p}\, \|\bar{\nabla} h + h\iota\|_Q^{q-n} \det(\bar{\nabla}^2 h + hI) = f,$$

where $\|\cdot\|_Q = 1/\rho_Q$ is the gauge function of Q. Note that $\|\cdot\|_Q$ is an n-dimensional Banach norm if Q is convex and origin symmetric.

Our contribution to Problem 4.3.2 is as follows. Recall that $\Sigma_\varepsilon(v) = \{u \in S^{n-1} : \langle u, v\rangle \geq \varepsilon\}$ for $v \in S^{n-1}$ and $\varepsilon \in (0,1)$.

Theorem 4.3.3. *Let μ be a nonzero finite Borel measure on S^{n-1} not concentrated on any closed hemisphere. Let G and G_t be continuous on $(0,\infty) \times S^{n-1}$ and let $G_t < 0$ on $(0,\infty) \times S^{n-1}$. Let $0 < \varepsilon_0 < 1$ and suppose that for $v \in S^{n-1}$,*

$$\lim_{t \to 0+} \int_{\Sigma_{\varepsilon_0}(v)} G(t,u)\, du = \infty \qquad \textit{and} \qquad \lim_{t\to\infty} \int_{S^{n-1}} G(t,u)\, du = 0. \tag{4.53}$$

Let $\psi : (0,\infty) \to (0,\infty)$ be continuous and satisfy

$$\int_1^\infty \frac{\psi(s)}{s}\, ds = \infty. \tag{4.54}$$

Then there exists $K \in \mathscr{K}^n_{(o)}$ such that

$$\frac{\mu}{|\mu|} = \frac{\widetilde{C}_{G,\psi}(K,\cdot)}{\widetilde{C}_{G,\psi}(K,S^{n-1})}. \tag{4.55}$$

Proof. Note that the limits in (4.53) exist, since $t \mapsto G(t,u)$ is decreasing. Define

$$\varphi(t) = \int_1^t \frac{\psi(s)}{s}\, ds, \quad t > 0, \tag{4.56}$$

and

$$a = -\int_0^1 \frac{\psi(s)}{s}\, ds \in \mathbb{R} \cup \{-\infty\}. \tag{4.57}$$

Then, by (4.54), (4.56), and (4.57), $\varphi \in \mathscr{J}_a$ is strictly increasing and continuously differentiable with $t\varphi'(t) = \psi(t)$ for $t > 0$; the latter equality implies that φ' is nonzero on $(0, \infty)$. For $f \in C^+(S^{n-1})$, let

$$F(f) = \frac{1}{|\mu|} \int_{S^{n-1}} \varphi(f(u))\, d\mu(u), \tag{4.58}$$

and for $K \in \mathscr{K}^n_{(o)}$, define $F(K) = F(h_K)$. We claim that

$$\alpha = \inf \left\{ F(K) : \widetilde{V}_G(K) = |\mu| \text{ and } K \in \mathscr{K}^n_{(o)} \right\} \tag{4.59}$$

is well defined with $\alpha \in \mathbb{R} \cup \{-\infty\}$ because there is a $K \in \mathscr{K}^n_{(o)}$ with $\widetilde{V}_G(K) = |\mu|$. To see this, note that

$$\widetilde{V}_G(rB^n) = \int_{S^{n-1}} G(r, u)\, du \geq \int_{\Sigma_{\varepsilon_0}(v)} G(r, u)\, du$$

for any $v \in S^{n-1}$. Then (4.53) yields $\widetilde{V}_G(rB^n) \to \infty$ as $r \to 0$, and $\widetilde{V}_G(rB^n) \to 0$ as $r \to \infty$. Since $r \mapsto \widetilde{V}_G(rB^n)$ is continuous, there is an $r_0 > 0$ such that $\widetilde{V}_G(r_0 B^n) = |\mu|$. It follows from (4.59) that $\alpha \in \mathbb{R} \cup \{-\infty\}$.

By (4.59), there are $K_i \in \mathscr{K}^n_{(o)}$, $i \in \mathbb{N}$, such that $\widetilde{V}_G(K_i) = |\mu|$ and

$$\lim_{i \to \infty} F(K_i) = \alpha. \tag{4.60}$$

We aim to show that there is a $K_0 \in \mathscr{K}^n_{(o)}$ with $\widetilde{V}_G(K_0) = |\mu|$ and $F(K_0) = \alpha$.

To this end, we first claim that there is an $R > 0$ such that $K_i^* \subset RB^n$, $i \in \mathbb{N}$. Suppose on the contrary that $\sup_{i \in \mathbb{N}} R_i = \infty$, where $R_i = \max_{u \in S^{n-1}} \rho_{K_i^*}(u) = \rho_{K_i^*}(v_i)$. By taking a subsequence, if necessary, we may suppose that $v_i \to v_0 \in S^{n-1}$ and $\lim_{i \to \infty} R_i = \infty$. There exists $i_0 \in \mathbb{N}$ such that $|v_i - v_0| < \varepsilon_0/2$ whenever $i \geq i_0$. Hence, if $u \in \Sigma_{\varepsilon_0}(v_0)$ and $i \geq i_0$, then $\langle u, v_i \rangle \geq \varepsilon_0/2$. It follows that for $u \in \Sigma_{\varepsilon_0}(v_0)$ and $i \geq i_0$, we have

$$\begin{aligned} h_{K_i^*}(u) &\geq \rho_{K_i^*}(v_i)\langle u, v_i \rangle \\ &= R_i \langle u, v_i \rangle \\ &\geq R_i \varepsilon_0/2, \end{aligned}$$

and therefore

$$\begin{aligned}
|\mu| &= \int_{S^{n-1}} G(\rho_{K_i}(u), u)\, du \\
&= \int_{S^{n-1}} G(h_{K_i^*}(u)^{-1}, u)\, du \\
&\geq \int_{\Sigma_{\varepsilon_0}(v_0)} G(h_{K_i^*}(u)^{-1}, u)\, du \\
&\geq \int_{\Sigma_{\varepsilon_0}(v_0)} G(2/(R_i\varepsilon_0), u)\, du \\
&\to \infty
\end{aligned}$$

as $i \to \infty$. This contradiction proves our claim.

By the Blaschke selection theorem, we may assume that $K_i^* \to L$ for some $L \in \mathscr{K}^n$. Suppose that $L \notin \mathscr{K}^n_{(o)}$. Then $o \in \partial L$, so there exists $w_0 \in S^{n-1}$ such that $\lim_{i\to\infty} h_{K_i^*}(w_0) = h_L(w_0) = 0$. Since $|\mu| > 0$ and μ is not concentrated on any closed hemisphere, there is an $\varepsilon \in (0,1)$ such that $\mu(\Sigma_\varepsilon(w_0)) > 0$. Let $v \in \Sigma_\varepsilon(w_0)$. Since

$$0 \leq \rho_{K_i^*}(v) \leq \frac{1}{\langle v, w_0\rangle} h_{K_i^*}(w_0) \leq \frac{1}{\varepsilon} h_{K_i^*}(w_0) \to 0$$

as $i \to \infty$, it follows that $\rho_{K_i^*} \to 0$ uniformly on $\Sigma_\varepsilon(w_0)$. As $\widetilde{V}_G(K_i) = |\mu|$ and $K_i^* \subset RB^n$, using (2.4), (4.58), (4.59), and (4.60), we obtain

$$\begin{aligned}
\alpha &= \lim_{i\to\infty} F(K_i) = \lim_{i\to\infty} \frac{1}{|\mu|} \int_{S^{n-1}} \varphi\left(\rho_{K_i^*}(u)^{-1}\right)\, d\mu(u) \\
&\geq \liminf_{i\to\infty} \frac{1}{|\mu|} \int_{\Sigma_\varepsilon(w_0)} \varphi\left(\rho_{K_i^*}(u)^{-1}\right)\, d\mu(u) + \frac{1}{|\mu|} \int_{S^{n-1}\setminus\Sigma_\varepsilon(w_0)} \varphi\left(1/R\right)\, d\mu(u) \\
&\geq \frac{\mu(\Sigma_\varepsilon(w_0))}{|\mu|} \liminf_{i\to\infty} \min\left\{\varphi\left(\rho_{K_i^*}(u)^{-1}\right) : u \in \Sigma_\varepsilon(w_0)\right\} + \frac{\mu(S^{n-1}\setminus\Sigma_\varepsilon(w_0))}{|\mu|}\varphi\left(1/R\right) \\
&= \infty.
\end{aligned}$$

This is not possible, so $L \in \mathscr{K}^n_{(o)}$.

Let $K_0 = L^* \in \mathscr{K}^n_{(o)}$. Then $K_i \to K_0$ as $i \to \infty$ in $\mathscr{K}^n_{(o)}$. Hence, $h_{K_i} \to h_{K_0} > 0$ uniformly on S^{n-1}. The continuity of φ ensures that

$$\sup\{|\varphi(h_{K_i}(u))| : i \in \mathbb{N}, u \in S^{n-1}\} < \infty.$$

Now it follows from (4.58), (4.60), and the dominated convergence theorem that

$$\alpha = \lim_{i\to\infty} F(K_i) = \frac{1}{|\mu|}\int_{S^{n-1}} \lim_{i\to\infty} \varphi(h_{K_i}(u))\,d\mu(u) = \frac{1}{|\mu|}\int_{S^{n-1}} \varphi(h_{K_0}(u))\,d\mu(u) = F(K_0). \tag{4.61}$$

Also, by Lemma 4.1.2, we have $\widetilde{V}_G(K_0) = |\mu|$, so the aim stated earlier has been achieved. It also follows from (4.61) that $\alpha \in \mathbb{R}$.

We now show that K_0 satisfies (4.55) with K replaced by K_0. Due to $\varphi \in \mathscr{J}_a$ and $f \geq h_{[f]}$, one has $F(f) \geq F(h_{[f]}) = F([f])$ for $f \in C^+(S^{n-1})$. By (4.61),

$$F(h_{K_0}) = F(K_0) = \alpha = \inf\{F(f) : \widetilde{V}_G([f]) = |\mu| \text{ and } f \in C^+(S^{n-1})\}. \tag{4.62}$$

Let $g \in C(S^{n-1})$. For $u \in S^{n-1}$ and sufficiently small $\varepsilon_1, \varepsilon_2 \geq 0$, let $h_{\varepsilon_1,\varepsilon_2}$ be defined by (4.28) with f_0 and εg replaced by h_{K_0} and $\varepsilon_1 g + \varepsilon_2$, respectively, i.e.,

$$h_{\varepsilon_1,\varepsilon_2}(u) = \varphi^{-1}\left(\varphi(h_{K_0}(u)) + \varepsilon_1 g(u) + \varepsilon_2\right). \tag{4.63}$$

Then for sufficiently small ε, we have

$$h_{\varepsilon_1+\varepsilon,\varepsilon_2}(u) = \varphi^{-1}\left(\varphi(h_{\varepsilon_1,\varepsilon_2}(u)) + \varepsilon g(u)\right),$$

and

$$h_{\varepsilon_1,\varepsilon_2+\varepsilon}(u) = \varphi^{-1}\left(\varphi(h_{\varepsilon_1,\varepsilon_2}(u)) + \varepsilon\right).$$

The properties of φ listed after (4.57) allow us to apply (4.49), with $\Omega = S^{n-1}$ and with h_0 and h_ε replaced by $h_{\varepsilon_1,\varepsilon_2}$ and $h_{\varepsilon_1+\varepsilon,\varepsilon_2}$, respectively, to obtain

$$\frac{\partial}{\partial \varepsilon_1}\widetilde{V}_G([h_{\varepsilon_1,\varepsilon_2}]) = \lim_{\varepsilon\to 0}\frac{\widetilde{V}_G([h_{\varepsilon_1+\varepsilon,\varepsilon_2}]) - \widetilde{V}_G([h_{\varepsilon_1,\varepsilon_2}])}{\varepsilon} = n\int_{S^{n-1}} g(u)\,d\widetilde{C}_{G,\psi}([h_{\varepsilon_1,\varepsilon_2}], u), \tag{4.64}$$

and with g, h_0, and h_ε replaced by 1, $h_{\varepsilon_1,\varepsilon_2}$ and $h_{\varepsilon_1,\varepsilon_2+\varepsilon}$, respectively, to yield

$$\frac{\partial}{\partial \varepsilon_2}\widetilde{V}_G([h_{\varepsilon_1,\varepsilon_2}]) = n\int_{S^{n-1}} 1\,d\widetilde{C}_{G,\psi}([h_{\varepsilon_1,\varepsilon_2}], u) = n\,\widetilde{C}_{G,\psi}([h_{\varepsilon_1,\varepsilon_2}], S^{n-1}) \neq 0. \tag{4.65}$$

Since $[h_{\varepsilon_1,\varepsilon_2}]$ depends continuously on $\varepsilon_1, \varepsilon_2$ and in view of Proposition 4.3.1 (ii), (4.64) and (4.65) show that the gradient of the map $(\varepsilon_1, \varepsilon_2) \mapsto \widetilde{V}_G([h_{\varepsilon_1,\varepsilon_2}])$ has rank 1 and depends continuously on $(\varepsilon_1, \varepsilon_2)$, implying that this map is continuously differentiable.

Hence we may apply the method of Lagrange multipliers to conclude from (4.62) that there is a constant $\tau = \tau(g)$ such that

$$\frac{\partial}{\partial \varepsilon_1}\Big(F(h_{\varepsilon_1,\varepsilon_2}) + \tau(\log \widetilde{V}_G([h_{\varepsilon_1,\varepsilon_2}]) - \log|\mu|)\Big)\Big|_{\varepsilon_1=\varepsilon_2=0} = 0, \tag{4.66}$$

and

$$\frac{\partial}{\partial \varepsilon_2}\Big(F(h_{\varepsilon_1,\varepsilon_2}) + \tau(\log \widetilde{V}_G([h_{\varepsilon_1,\varepsilon_2}]) - \log|\mu|)\Big)\Big|_{\varepsilon_1=\varepsilon_2=0} = 0. \tag{4.67}$$

By (4.58) and (4.63), we have

$$\begin{aligned}\frac{\partial}{\partial \varepsilon_1}F(h_{\varepsilon_1,\varepsilon_2})\Big|_{\varepsilon_1=\varepsilon_2=0} &= \frac{1}{|\mu|}\left(\frac{\partial}{\partial \varepsilon_1}\int_{S^{n-1}}(\varphi(h_0(u)) + \varepsilon_1 g(u) + \varepsilon_2)\,d\mu(u)\right)\Big|_{\varepsilon_1=\varepsilon_2=0}\\ &= \frac{1}{|\mu|}\int_{S^{n-1}} g(u)\,d\mu(u), \end{aligned}\tag{4.68}$$

and

$$\frac{\partial}{\partial \varepsilon_2}F(h_{\varepsilon_1,\varepsilon_2})\Big|_{\varepsilon_1=\varepsilon_2=0} = \frac{1}{|\mu|}\int_{S^{n-1}} 1\,d\mu(u) = 1. \tag{4.69}$$

Since $\widetilde{V}_G(K_0) = |\mu|$ and (4.63) gives $h_{0,0} = h_{K_0}$, (4.64) and (4.65) imply that

$$\frac{\partial}{\partial \varepsilon_1}\log \widetilde{V}_G([h_{\varepsilon_1,\varepsilon_2}])\Big|_{\varepsilon_1=\varepsilon_2=0} = \frac{n}{|\mu|}\int_{S^{n-1}} g(u)\,d\widetilde{C}_{G,\psi}(K_0,u), \tag{4.70}$$

and

$$\frac{\partial}{\partial \varepsilon_2}\log \widetilde{V}_G([h_{\varepsilon_1,\varepsilon_2}])\Big|_{\varepsilon_1=\varepsilon_2=0} = \frac{n}{|\mu|}\widetilde{C}_{G,\psi}(K_0,S^{n-1}). \tag{4.71}$$

It follows from (4.66), (4.68), and (4.70) that

$$\int_{S^{n-1}} g(u)\,d\mu(u) = -n\tau\int_{S^{n-1}} g(u)\,d\widetilde{C}_{G,\psi}(K_0,u), \tag{4.72}$$

and from (4.67), (4.69), and (4.71) that

$$\tau = -\frac{|\mu|}{n\,\widetilde{C}_{G,\psi}(K_0,S^{n-1})}. \tag{4.73}$$

In particular, we see from (4.73) that τ is independent of g. Finally, (4.72) and (4.73) show that (4.55) holds with K replaced by K_0. $\square$

We remark that $-\widetilde{C}_{G,\psi}(K,\cdot)$ is a nonnegative measure since $G_t < 0$. Note that

(4.53) holds if $\lim_{t\to 0+} G(t,u) = \infty$ for $u \in S^{n-1}$ and $\lim_{t\to\infty} G(t,u) = 0$ for $u \in \Sigma_\varepsilon(v)$. This follows from the monotone convergence theorem, since $t \mapsto G(t,u)$ is decreasing. In order to solve Problem 4.3.2 when $t \mapsto G(t,u)$ is increasing, one needs to use different techniques and we leave it for future work in Chapter 5.

When $\psi \equiv 1$ (and hence $\varphi(t) = \log t \in \mathscr{J}_{-\infty}$), the following result was proved in [69, Theorem 5.1].

Corollary 4.3.4. *Let μ be a nonzero finite Borel measure on S^{n-1} not concentrated on any closed hemisphere. Let $\phi : \mathbb{R}^n \setminus \{o\} \to (0,\infty)$ be continuous and such that $\overline{\Phi}$ is continuous on $(0,\infty) \times S^{n-1}$, where $\overline{\Phi}$ is defined by (4.2). Let $0 < c < 1$ and suppose that for $v \in S^{n-1}$,*

$$\lim_{b\to 0+} \overline{V}_\phi(C(v,b,c)) = \infty, \tag{4.74}$$

where $C(v,b,c) = \{x \in \mathbb{R}^n : |x| \ge b \text{ and } \langle x/|x|, v\rangle \ge c\}$ and $\overline{V}_\phi(\cdot)$ is defined by (4.3). Let $\psi : (0,\infty) \to (0,\infty)$ be continuous and satisfy (4.54). Then there exists $K \in \mathscr{K}^n_{(o)}$ such that

$$\frac{\mu}{|\mu|} = \frac{\widetilde{C}_{\phi,\psi}(K,\cdot)}{\widetilde{C}_{\phi,\psi}(K,S^{n-1})}.$$

Proof. By assumption, $\overline{\Phi}$ is continuous on $(0,\infty)\times S^{n-1}$, and $\lim_{t\to\infty} \overline{\Phi}(t,u) = 0$ for $u \in S^{n-1}$. Hence the second condition in (4.53) holds with G replaced by $\overline{\Phi}$. Clearly, $\partial\overline{\Phi}(t,u)/\partial t = -\phi(tu)t^{n-1} < 0$. By (4.74),

$$\infty = \lim_{b\to 0+} \overline{V}_\phi(C(v,b,c)) = \lim_{b\to 0+} \int_{\Sigma_c(v)} \int_b^\infty \phi(ru) r^{n-1} dr du = \lim_{b\to 0+} \int_{\Sigma_c(v)} \overline{\Phi}(b,u) du.$$

Therefore the first condition in (4.53) also holds with G replaced by $\overline{\Phi}$. Due to the fact $\widetilde{C}_{\overline{\Phi},\psi}(K,\cdot) = -\widetilde{C}_{\phi,\psi}(K,\cdot)$, Theorem 4.3.3 yields the result. $\square$

Another special case arises if μ is a discrete measure on S^{n-1}, that is, $\mu = \sum_{i=1}^m c_i \delta_{v_i}$, where $c_i > 0$ for $i = 1,\dots,m$, and $v_1,\dots,v_m \in S^{n-1}$ are not contained in any closed hemisphere. Let G and ψ be as in Theorem 4.3.3. Then there exists a polytope $P \in \mathscr{K}^n_{(o)}$ such that

$$\frac{\mu}{|\mu|} = \frac{\widetilde{C}_{G,\psi}(P,\cdot)}{\widetilde{C}_{G,\psi}(P,S^{n-1})}.$$

To see this, note that Theorem 4.3.3 ensures the existence of a $K \in \mathscr{K}^n_{(o)}$ such that (4.55) holds. Since μ is discrete, we obtain

$$\widetilde{C}_{G,\psi}(K,\cdot) = \sum_{i=1}^{m} \bar{c}_i \delta_{v_i},$$

where $\bar{c}_i = \widetilde{C}_{G,\psi}(K, S^{n-1})c_i/|\mu| < 0$ for $i = 1, \dots, m$. Proposition 4.3.1 (i) shows that there is a measurable function $g : S^{n-1} \to (-\infty, 0]$ such that

$$\sum_{i=1}^{m} \bar{c}_i \delta_{v_i}(E) = \int_E g(u)\, dS(K,u)$$

for Borel sets $E \subset S^{n-1}$. Hence $S(K,\cdot)$ is a discrete measure and [59, Theorem 4.5.4] implies that K is a polytope.

4.4 Inequalities for the general dual volume

In this section, we investigate some important inequalities with respect to the general dual volume $\widetilde{V}_G$, including the dual Orlicz-Brunn-Minkowski inequalities and dual Orlicz-Minkowski inequalities.

4.4.1 Dual Orlicz-Brunn-Minkowski inequalities

Let $\overline{\Phi}_m$ be the set of continuous functions $\varphi : [0,\infty)^m \to [0,\infty)$ that are strictly increasing in each component and such that $\varphi(o) = 0$, $\varphi(e_j) = 1$ for $1 \le j \le m$, and $\lim_{t\to\infty} \varphi(tx) = \infty$ for $x \in [0,\infty)^m \setminus \{o\}$. By Ψ_m we mean the set of continuous functions $\varphi : (0,\infty)^m \to (0,\infty)$, such that for $x = (x_1, \dots, x_m) \in (0,\infty)^m$,

$$\varphi(x) = \varphi_0(1/x_1, \dots, 1/x_m) \tag{4.75}$$

for some $\varphi_0 \in \overline{\Phi}_m$. It is easy to see that if $\varphi \in \Psi_m$, then φ is strictly decreasing in each component and such that $\lim_{t\to 0} \varphi(tx) = \infty$ and $\lim_{t\to\infty} \varphi(tx) = 0$ for $x \in (0,\infty)^m$.

Let $K_1, \dots, K_m \in \mathscr{S}^n_{c+}$ and let $\varphi \in \overline{\Phi}_m \cup \Psi_m$. Define $\widetilde{+}_\varphi(K_1, \dots, K_m) \in \mathscr{S}^n_{c+}$, the *radial Orlicz sum* of $K_1, \ldots, K_m$, to be the star body whose radial function satisfies

$$\varphi\left(\frac{\rho_{K_1}(u)}{\rho_{\widetilde{+}_\varphi(K_1,\ldots,K_m)}(u)}, \ldots, \frac{\rho_{K_m}(u)}{\rho_{\widetilde{+}_\varphi(K_1,\ldots,K_m)}(u)}\right) = 1 \tag{4.76}$$

for $u \in S^{n-1}$. It was proved in [18, Theorem 3.2(v) and (vi)] that if $\varphi \in \overline{\Phi}_m$, then

$$\rho_{\widetilde{+}_\varphi(K_1,\ldots,K_m)}(u) > \rho_{K_j}(u) \quad \text{for } u \in S^{n-1}. \tag{4.77}$$

Together with (4.75) and (4.76), this implies that if $\varphi \in \Psi_m$, then

$$\rho_{\widetilde{+}_\varphi(K_1,\ldots,K_m)}(u) < \rho_{K_j}(u) \quad \text{for } u \in S^{n-1}. \tag{4.78}$$

For each $0 \neq q \in \mathbb{R}$ and $\varphi \in \overline{\Phi}_m \cup \Psi_m$, let

$$\varphi_q(x) = \varphi\left(x_1^{1/q}, x_2^{1/q}, \ldots, x_m^{1/q}\right) \qquad \text{for } x = (x_1, \ldots, x_m) \in (0,\infty)^m. \tag{4.79}$$

Then (4.76) is equivalent to

$$\varphi_q\left(\left(\frac{\rho_{K_1}(u)}{\rho_{\widetilde{+}_\varphi(K_1,\ldots,K_m)}(u)}\right)^q, \ldots, \left(\frac{\rho_{K_m}(u)}{\rho_{\widetilde{+}_\varphi(K_1,\ldots,K_m)}(u)}\right)^q\right) = 1. \tag{4.80}$$

For $t \in (0,\infty)$ and $u \in S^{n-1}$, let

$$G_q(t,u) = \frac{G(t,u)}{t^q}. \tag{4.81}$$

The proof of the following result closely follows that of [18, Theorem 4.1].

Theorem 4.4.1. *Let $m, n \geq 2$, let $\varphi \in \overline{\Phi}_m \cup \Psi_m$, let $K_1, \ldots, K_m \in \mathscr{S}^n_{c+}$, let $G : (0,\infty) \times S^{n-1} \to (0,\infty)$ be continuous, and let φ_q and G_q be defined by (4.79) and (4.81). Suppose that φ_q is convex and either $q > 0$ and $G_q(t,\cdot)$ is increasing, or $q < 0$ and $G_q(t,\cdot)$ is decreasing. Then*

$$1 \geq \varphi\left(\left(\frac{\widetilde{V}_G(K_1)}{\widetilde{V}_G(\widetilde{+}_\varphi(K_1,\ldots,K_m))}\right)^{1/q}, \ldots, \left(\frac{\widetilde{V}_G(K_m)}{\widetilde{V}_G(\widetilde{+}_\varphi(K_1,\ldots,K_m))}\right)^{1/q}\right). \tag{4.82}$$

The reverse inequality holds if instead φ_q is concave and either $q > 0$ and $G_q(t,\cdot)$ is decreasing, or $q < 0$ and $G_q(t,\cdot)$ is increasing.

If in addition φ_q is strictly convex (or concave, as appropriate) and equality holds in (4.82)*, then $K_1, \dots, K_m$ are dilatates of each other.*

Proof. Let $\varphi \in \overline{\Phi}_m \cup \Psi_m$ and let $K_1, \dots, K_m \in \mathscr{S}^n_{c+}$. It follows from (4.76) that $\rho_{\widetilde{+}_\varphi(K_1,\dots,K_m)}(u) > 0$ for $u \in S^{n-1}$. By (4.1), one can define a probability measure μ on S^{n-1} by

$$d\mu(u) = \frac{G(\rho_{\widetilde{+}_\varphi(K_1,\dots,K_m)}(u), u)}{\widetilde{V}_G(\widetilde{+}_\varphi(K_1,\dots,K_m))}\, du. \tag{4.83}$$

Suppose that $\varphi \in \overline{\Phi}_m$, $q > 0$, and $G_q(t, \cdot)$ is increasing. By (4.80) and Jensen's inequality [18, Proposition 2.2] applied to the convex function φ_q, similarly to the proof of [18, Theorem 4.1], we have

$$\begin{aligned} 1 &= \int_{S^{n-1}} \varphi_q\left(\left(\frac{\rho_{K_1}(u)}{\rho_{\widetilde{+}_\varphi(K_1,\dots,K_m)}(u)}\right)^q, \dots, \left(\frac{\rho_{K_m}(u)}{\rho_{\widetilde{+}_\varphi(K_1,\dots,K_m)}(u)}\right)^q\right) d\mu(u) \\ &\geq \varphi_q\left(\int_{S^{n-1}} \frac{\rho_{K_1}(u)^q}{\rho_{\widetilde{+}_\varphi(K_1,\dots,K_m)}(u)^q}\, d\mu(u), \dots, \int_{S^{n-1}} \frac{\rho_{K_m}(u)^q}{\rho_{\widetilde{+}_\varphi(K_1,\dots,K_m)}(u)^q}\, d\mu(u)\right). \end{aligned} \tag{4.84}$$

Since $\varphi \in \overline{\Phi}_m$ and $q > 0$, φ_q is strictly increasing in each component. According to (4.77) and the fact that $G_q(t, \cdot)$ is increasing, we have

$$\frac{\rho_{K_j}(u)^q}{\rho_{\widetilde{+}_\varphi(K_1,\dots,K_m)}(u)^q} G(\rho_{\widetilde{+}_\varphi(K_1,\dots,K_m)}(u), u) \geq G(\rho_{K_j}(u), u) \tag{4.85}$$

for $j = 1, \dots, m$. Using (4.83), we obtain for $j = 1, \dots, m$,

$$\begin{aligned} &\frac{\widetilde{V}_G(K_j)}{\widetilde{V}_G(\widetilde{+}_\varphi(K_1,\dots,K_m))} \\ &\quad= \frac{1}{\widetilde{V}_G(\widetilde{+}_\varphi(K_1,\dots,K_m))} \int_{S^{n-1}} G(\rho_{K_j}(u), u)\, du \\ &\quad\leq \frac{1}{\widetilde{V}_G(\widetilde{+}_\varphi(K_1,\dots,K_m))} \int_{S^{n-1}} \frac{\rho_{K_j}(u)^q\, G(\rho_{\widetilde{+}_\varphi(K_1,\dots,K_m)}(u), u)}{\rho_{\widetilde{+}_\varphi(K_1,\dots,K_m)}(u)^q}\, du \\ &\quad= \int_{S^{n-1}} \frac{\rho_{K_j}(u)^q}{\rho_{\widetilde{+}_\varphi(K_1,\dots,K_m)}(u)^q}\, d\mu(u). \end{aligned}$$

Since φ_q is strictly increasing in each component and (4.84) holds, we get

$$\begin{aligned}
1 &\geq \varphi_q\left(\int_{S^{n-1}} \frac{\rho_{K_1}(u)^q}{\rho_{\widetilde{+}_\varphi(K_1,\dots,K_m)}(u)^q}\, d\mu(u), \dots, \int_{S^{n-1}} \frac{\rho_{K_m}(u)^q}{\rho_{\widetilde{+}_\varphi(K_1,\dots,K_m)}(u)^q}\, d\mu(u)\right) \\
&\geq \varphi_q\left(\frac{\widetilde{V}_G(K_1)}{\widetilde{V}_G(\widetilde{+}_\varphi(K_1,\dots,K_m))}, \dots, \frac{\widetilde{V}_G(K_m)}{\widetilde{V}_G(\widetilde{+}_\varphi(K_1,\dots,K_m))}\right) \\
&= \varphi\left(\left(\frac{\widetilde{V}_G(K_1)}{\widetilde{V}_G(\widetilde{+}_\varphi(K_1,\dots,K_m))}\right)^{1/q}, \dots, \left(\frac{\widetilde{V}_G(K_m)}{\widetilde{V}_G(\widetilde{+}_\varphi(K_1,\dots,K_m))}\right)^{1/q}\right), \quad (4.86)
\end{aligned}$$

which yields (4.82).

Suppose in addition that φ_q is strictly convex and equality holds in (4.82). Then equality holds throughout (4.86) and hence in (4.84). Therefore equality holds in Jensen's inequality as used above. Since $G > 0$, the definition (4.83) of μ shows that its support is the whole of S^{n-1}. Then, exactly as in the proof of [18, Theorem 4.1], we can conclude that $K_1, \dots, K_m$ are dilatates of each other.

This proves (4.82) and the implication in case of equality when $\varphi \in \overline{\Phi}_m$, $q > 0$, and $G_q(t, \cdot)$ is increasing. The other cases are similar, noting that if $\varphi \in \Psi_m$, we can use (4.78) instead of (4.77), and if φ_q is concave, Jensen's inequality [18, Proposition 2.2] yields the reverse of inequality (4.84). $\square$

It is possible to state more general versions of Theorem 4.4.1 that hold when $K_1, \dots, K_m \in \mathscr{S}^n$. Indeed, the definition (4.76) of the radial Orlicz sum can be modified, as in [18, p. 817], so that it applies when $K_1, \dots, K_m \in \mathscr{S}^n$. Then extra assumptions would have to be made in Theorem 4.4.1, analogous to the one in [18, Theorem 4.1] that $V(K_j) > 0$ for some j, but now also involving the function G. Note that the stronger assumption that $K_1, \dots, K_m \in \mathscr{S}^n_{c+}$ is still required for the implication in case of equality, as it is in [18, Theorem 4.1].

Under certain cases, equality holds in Theorem 4.4.1 if and only if $K_1, \dots, K_m$ are dilatates of each other. One such is given in Corollary 4.4.2, and it is easy to see that this is true more generally if G is of the form $G(t, u) = t^q H(u)$, where $t > 0$ and $u \in S^{n-1}$, for some $q \neq 0$ and suitable function H, since equality then holds in (4.85). However, it does not seem straightforward to formulate a precise condition and we do not pursue the matter here.

Dual Orlicz-Brunn-Minkowski inequalities for $\overline{V}_\phi(\cdot)$, $\underline{V}_\phi(\cdot)$ and $\breve{V}_{\phi,\varphi}(\cdot,\cdot)$ follow directly from Theorem 4.4.1, once the corresponding assumptions are verified. We shall only state the special case when $G(t,u) = t^q \rho_Q(u)^{n-q}/n$ for some $Q \in \mathscr{S}^n_{c+}$. Then, for $q \neq 0$, we have

$$\widetilde{V}_G(K) = \int_{S^{n-1}} G(\rho_K(u),u)\,du = \frac{1}{n}\int_{S^{n-1}} \rho_K(u)^q\,\rho_Q(u)^{n-q}\,du = \widetilde{V}_q(K,Q), \tag{4.87}$$

the qth dual mixed volume of K and Q, as in (4.6).

The following result was proved for $q = n$ and $Q = B^n$ in [18, Theorem 4.1].

Corollary 4.4.2. *Let $m, n \geq 2$, let $q \neq 0$, let $\varphi \in \overline{\Phi}_m \cup \Psi_m$, and let $Q, K_1, \dots, K_m \in \mathscr{S}^n_{c+}$. If φ_q is convex, then*

$$1 \geq \varphi\left(\left(\frac{\widetilde{V}_q(K_1,Q)}{\widetilde{V}_q(\widetilde{+}_\varphi(K_1,\dots,K_m),Q)}\right)^{1/q}, \dots, \left(\frac{\widetilde{V}_q(K_m,Q)}{\widetilde{V}_q(\widetilde{+}_\varphi(K_1,\dots,K_m),Q)}\right)^{1/q}\right). \tag{4.88}$$

If φ_q is concave, the inequality is reversed. If instead φ_q is strictly convex or strictly concave, respectively, then equality holds in (4.82) *if and only if $K_1, \dots, K_m$ are dilatates of each other.*

Proof. The required inequalities and the necessity of the equality condition follow immediately from Theorem 4.4.1 on noting that $G_q(t,u) = \rho_Q(u)^{n-q}/n$ is a constant function of t. Suppose that $K_1, \dots, K_m$ are dilatates of each other, so $K_i = c_i K$ and hence $\rho_{K_i} = c_i \rho_K$ for some $K \in \mathscr{S}^n_{c+}$ and $c_i > 0$, $i = 1, \dots, m$. Let $d > 0$ be the unique solution of

$$\varphi\left(\frac{c_1}{d}, \dots, \frac{c_m}{d}\right) = 1. \tag{4.89}$$

Comparing (4.76), we obtain $\rho_{\widetilde{+}_\varphi(K_1,\dots,K_m)}(u) = d\rho_K(u)$ for $u \in S^{n-1}$ and hence we have $\widetilde{+}_\varphi(K_1,\dots,K_m) = dK$. From (4.87), we get $\widetilde{V}_q(K_i,Q) = c_i^q\,\widetilde{V}_q(K,Q)$, $i = 1, \dots, m$, and $\widetilde{V}_q(\widetilde{+}_\varphi(K_1,\dots,K_m),Q) = d^q\,\widetilde{V}_q(K,Q)$. Substituting for c_i, $i = 1, \dots, m$, and d from the latter two equations into (4.89), we obtain (4.88) with equality. $\square$

4.4.2 Dual Orlicz-Minkowski inequalities and uniqueness results

Let $K, L, Q \in \mathscr{S}^n_{c+}$, let $q \neq 0$, and let $\varphi : (0,\infty) \to (0,\infty)$ be continuous. It will be convenient to define

$$\widetilde{V}_{q,\varphi}(K,L,Q) = \frac{1}{n}\int_{S^{n-1}} \varphi\left(\frac{\rho_L(u)}{\rho_K(u)}\right) \rho_K(u)^q\, \rho_Q(u)^{n-q}\, du. \tag{4.90}$$

Note that this is a special case of the general dual Orlicz mixed volume $\widetilde{V}_{\phi,\varphi}(K,L)$ defined in (4.35), obtained by setting $\phi(x) = |x|^{q-n}\rho_Q(x/|x|)^{n-q}$. When $q = n$, (4.90) becomes the dual Orlicz mixed volume introduced in [18, 79], and when $q = n$ and $Q = B^n$, the following result yields the dual Orlicz-Minkowski inequality established in [18, Theorem 6.1] and [79, Theorem 5.1].

Theorem 4.4.3. *Let $K, L, Q \in \mathscr{S}^n_{c+}$, let $q \neq 0$, let $\varphi : (0,\infty) \to (0,\infty)$ be continuous, and let $\varphi_q(t) = \varphi(t^{1/q})$ for $t \in (0,\infty)$. If φ_q is convex, then*

$$\widetilde{V}_{q,\varphi}(K,L,Q) \geq \widetilde{V}_q(K,Q)\, \varphi\left(\left(\frac{\widetilde{V}_q(L,Q)}{\widetilde{V}_q(K,Q)}\right)^{1/q}\right). \tag{4.91}$$

The reverse inequality holds if φ_q is concave. If φ_q is strictly convex or strictly concave, respectively, equality holds in the above inequalities if and only if K and L are dilatates of each other.

Proof. Let $q \neq 0$ and let φ_q be convex. By (4.87), one can define a probability measure $\tilde{\mu}$ by $d\tilde{\mu}(u) = \frac{\rho_K(u)^q\, \rho_Q(u)^{n-q}}{n\widetilde{V}_q(K,Q)}\, du$. Jensen's inequality [18, Proposition 2.2] implies that

$$\begin{aligned}
\widetilde{V}_{q,\varphi}(K,L,Q) &= \frac{1}{n}\int_{S^{n-1}} \varphi\left(\frac{\rho_L(u)}{\rho_K(u)}\right) \rho_K(u)^q\, \rho_Q(u)^{n-q}\, du \\
&= \widetilde{V}_q(K,Q) \int_{S^{n-1}} \varphi_q\left(\left(\frac{\rho_L(u)}{\rho_K(u)}\right)^q\right) d\tilde{\mu}(u) \\
&\geq \widetilde{V}_q(K,Q)\, \varphi_q\left(\int_{S^{n-1}} \left(\frac{\rho_L(u)}{\rho_K(u)}\right)^q d\tilde{\mu}(u)\right) \\
&= \widetilde{V}_q(K,Q)\, \varphi_q\left(\int_{S^{n-1}} \frac{\rho_L(u)^q\, \rho_Q(u)^{n-q}}{n\widetilde{V}_q(K,Q)}\, du\right) \\
&= \widetilde{V}_q(K,Q)\, \varphi\left(\left(\frac{\widetilde{V}_q(L,Q)}{\widetilde{V}_q(K,Q)}\right)^{1/q}\right),
\end{aligned}$$

where the first and the last equalities are due to (4.90) and (4.87), respectively.

Suppose that φ_q is strictly convex and equality holds in (4.91). Then the above proof and the equality condition for Jensen's equality show that $\rho_L(u)/\rho_K(u)$ is a constant for $\tilde{\mu}$-almost all $u \in S^{n-1}$ and hence for $\mathscr{H}^{n-1}$-almost all $u \in S^{n-1}$. Since ρ_K and ρ_L are continuous, $\rho_L(u)/\rho_K(u)$ is a constant for $u \in S^{n-1}$ and so K and L are dilatates of each other.

If instead φ_q is concave, the proof is similar since Jensen's inequality [18, Proposition 2.2] also reverses. $\square$

Corollary 4.4.4. *Let $K, L, Q \in \mathscr{S}^n_{c+}$, let $q \neq 0$, let $\varphi : (0,\infty) \to (0,\infty)$, and let $\varphi_q(t) = \varphi(t^{1/q})$ for $t \in (0,\infty)$. Suppose that φ is either increasing or decreasing, and that φ_q is either strictly convex or strictly concave. Then $K = L$ if either*

$$\frac{\widetilde{V}_{q,\varphi}(K,M,Q)}{\widetilde{V}_q(K,Q)} = \frac{\widetilde{V}_{q,\varphi}(L,M,Q)}{\widetilde{V}_q(L,Q)} \tag{4.92}$$

holds for all $M \in \mathscr{S}^n_{c+}$, or

$$\widetilde{V}_{q,\varphi}(M,K,Q) = \widetilde{V}_{q,\varphi}(M,L,Q) \tag{4.93}$$

holds for all $M \in \mathscr{S}^n_{c+}$.

Proof. Let $q \neq 0$ and suppose that (4.92) holds for all $M \in \mathscr{S}^n_{c+}$. Assume that φ is increasing and φ_q is strictly convex; the other three cases can be dealt with similarly. Taking $M = K$ in (4.92), it follows from (4.6), (4.90) with $L = K$, and (4.91) with K and L interchanged, that

$$\varphi(1) = \frac{\widetilde{V}_{q,\varphi}(K,K,Q)}{\widetilde{V}_q(K,Q)} = \frac{\widetilde{V}_{q,\varphi}(L,K,Q)}{\widetilde{V}_q(L,Q)} \geq \varphi\left(\left(\frac{\widetilde{V}_q(K,Q)}{\widetilde{V}_q(L,Q)}\right)^{1/q}\right). \tag{4.94}$$

Since φ is increasing, we get

$$1 \geq \left(\frac{\widetilde{V}_q(K,Q)}{\widetilde{V}_q(L,Q)}\right)^{1/q}. \tag{4.95}$$

Repeating the argument with K and L interchanged yields the reverse inequality. Hence we get $\widetilde{V}_q(K,Q) = \widetilde{V}_q(L,Q)$, from which we obtain equality in (4.94). The equality condition for (4.91) implies that $L = rK$ for some $r > 0$. This together with $\widetilde{V}_q(K,Q) = \widetilde{V}_q(L,Q)$ easily yields $K = L$.

Now suppose that (4.93) holds for all $M \in \mathscr{S}^n_{c+}$. Taking $M = K$ and arguing as above, we get

$$\varphi(1)\,\widetilde{V}_q(K,Q) = \widetilde{V}_{q,\varphi}(K,K,Q) = \widetilde{V}_{q,\varphi}(K,L,Q) \geq \widetilde{V}_q(K,Q)\,\varphi\left(\left(\frac{\widetilde{V}_q(L,Q)}{\widetilde{V}_q(K,Q)}\right)^{1/q}\right). \tag{4.96}$$

Therefore (4.95) holds. Interchanging K and L yields the reverse inequality and hence we have $\widetilde{V}_q(K,Q) = \widetilde{V}_q(L,Q)$, giving equality in (4.96). Exactly as above, we conclude that $K = L$. $\square$

Corollary 4.4.5. *Let $K, L, Q \in \mathscr{S}^n_{c+}$, let $q \neq 0$, let $\varphi : (0,\infty) \to (0,\infty)$ be continuous, and let $\varphi_q(t) = \varphi(t^{1/q})$ for $t \in (0,\infty)$. If φ_q is strictly convex or strictly concave and*

$$\widetilde{V}_{q,\varphi}(K,M,Q) = \widetilde{V}_{q,\varphi}(L,M,Q) \tag{4.97}$$

for all $M \in \mathscr{S}^n_{c+}$, then $K = L$.

Proof. Let $q \neq 0$ and let $\alpha > 0$. Replacing K and L by L and αL, respectively, in (4.90), and taking (4.87) into account, we obtain

$$\widetilde{V}_{q,\varphi}(L,\alpha L,Q) = \frac{\varphi(\alpha)}{\varphi(1)}\widetilde{V}_{q,\varphi}(L,L,Q) = \varphi(\alpha)\widetilde{V}_q(L,Q).$$

Suppose that φ_q is strictly convex; the case when φ_q is strictly concave is similar. Using (4.97) with $M = \alpha L$, (4.91) implies that

$$\varphi(\alpha)\widetilde{V}_q(L,Q) = \widetilde{V}_{q,\varphi}(L,\alpha L,Q) = \widetilde{V}_{q,\varphi}(K,\alpha L,Q) \geq \widetilde{V}_q(K,Q)\,\varphi\left(\alpha\left(\frac{\widetilde{V}_q(L,Q)}{\widetilde{V}_q(K,Q)}\right)^{1/q}\right). \tag{4.98}$$

Let

$$c = \left(\frac{\widetilde{V}_q(L,Q)}{\widetilde{V}_q(K,Q)}\right)^{1/q}.$$

Then (4.98) reads $c^q\varphi(\alpha) \geq \varphi(\alpha c)$. When $\alpha = 1$, we obtain

$$c^q\varphi(1) \geq \varphi(c). \tag{4.99}$$

Repeating the argument with K and L interchanged yields $c^{-q}\varphi(\alpha) \geq \varphi(\alpha c^{-1})$. Setting $\alpha = c$, we get $c^{-q}\varphi(c) \geq \varphi(1)$ and hence

$$c^q\varphi(1) \leq \varphi(c). \tag{4.100}$$

By (4.99) and (4.100), $\varphi(c) = c^q\varphi(1)$, which means that

$$\varphi\left(\left(\frac{\widetilde{V}_q(L,Q)}{\widetilde{V}_q(K,Q)}\right)^{1/q}\right) = \frac{\widetilde{V}_q(L,Q)}{\widetilde{V}_q(K,Q)}\,\varphi(1).$$

Thus equality holds in (4.98) when $\alpha = 1$. By the equality condition for (4.91), we conclude that $L = rK$ for some $r > 0$. That is, K and L are dilatates of each other.

Suppose that $L = rK$, where $r > 0$ and $r \neq 1$. Let $\alpha > 0$. Then (4.87), (4.90), and (4.97) with $M = \alpha K$ yield

$$\varphi(\alpha)\widetilde{V}_q(K,Q) = \widetilde{V}_{q,\varphi}(K,\alpha K,Q) = \widetilde{V}_{q,\varphi}(rK,\alpha K,Q) = \varphi(\alpha/r)r^q\,\widetilde{V}_q(K,Q).$$

Consequently, $\varphi(rs) = r^q\varphi(s)$ for $s > 0$. Equivalently, setting $\beta = r^q$ and $t = s^q$, we obtain $\varphi_q(\beta t) = \beta\varphi_q(t)$ for $t > 0$, where $\beta \neq 1$. But then the points $(\beta^m, \varphi_q(\beta^m))$, $m \in \mathbb{N}$, all lie on the line $y = \varphi(1)x$ in $\mathbb{R}^2$, so φ_q cannot be strictly convex. This contradiction proves that $r = 1$ and hence $K = L$. $\square$

Let $K, L \in \mathscr{K}^n_{(o)}$. We recall from [16, 68] that for $\varphi \in (0,\infty) \to (0,\infty)$, the *Orlicz mixed volume* $V_\varphi(K,L)$ is defined by

$$V_\varphi(K,L) = \frac{1}{n}\int_{S^{n-1}} \varphi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u)\,dS(K,u). \tag{4.101}$$

The *Orlicz-Minkowski inequality* [16, Theorem 9.2] (see also [68, Theorem 2]) states that if $\varphi \in \mathscr{I}$ is convex, then

$$V_\varphi(K,L) \geq V(K)\,\varphi\left(\left(\frac{V(L)}{V(K)}\right)^{1/n}\right). \tag{4.102}$$

If φ is strictly convex, equality in (4.102) holds if and only if K and L are dilatates of each other. When $\varphi(t) = t$, we write $V_\varphi(K, L) = V_1(K, L)$ and retrieve from (4.102) *Minkowski's first inequality*

$$V_1(K, L) \geq V(K)^{(n-1)/n} V(L)^{1/n}. \tag{4.103}$$

Note that (4.103) actually holds for all $K, L \in \mathscr{K}^n$, with equality if and only if K and L lie in parallel hyperplanes or are homothetic; see [15, Theorem B.2.1] or [59, Theorem 6.2.1].

Let $\varphi \in \mathscr{I} \cup \mathscr{D}$ and let $n \in \mathbb{N}$, $n \geq 2$. We say that φ behaves like t^n if there is $r > 0$, $r \neq 1$, such that $\varphi(rt) = r^n \varphi(t)$ for $t > 0$. Of course, if $\varphi(t) = t^n$, then φ behaves like t^n, but there is a $\varphi \in \mathscr{I} \cup \mathscr{D}$ that behaves like t^n such that $\varphi(t) \neq t^n$ for some $t > 0$. To see this, let $f(t) = t^n$ and define $\varphi(t)$ on $[1, 2]$, such that (i) φ is increasing and strictly convex, (ii) $\varphi(t) = f(t)$ at $t = 1$ and $t = 2$, (iii) $\varphi'_r(1) = f'(1)$ and $\varphi'_l(2) = f'(2)$, (iv) $\varphi(t) < f(t)$ on $(1, 2)$. Then define φ on $[1/2, 1]$ by $\varphi(t) = \varphi(2t)/2^n$ and on $[2, 4]$ by $\varphi(t) = 2^n \varphi(t/2)$. It follows that φ is increasing and strictly convex on $[1/2, 1]$ and on $[2, 4]$, $\varphi(t) = f(t)$ at $t = 1/2$ and $t = 4$, $\varphi'_r(1/2) = \varphi'_r(1)/2^{n-1} = f'(1/2)$, $\varphi'_l(4) = 2^{n-1}\varphi'_l(2) = f'(4)$ and $\varphi(t) < f(t)$ on $(1/2, 1) \cup (2, 4)$. Moreover, $\varphi'_l(t) = \varphi'_r(t)$ at $t = 1$ and $t = 2$, so φ is increasing and strictly convex on $[1/2, 4]$. Continuing inductively, we define φ on $[1/2^m, 2^{m+1}]$, $m \in \mathbb{N}$, and hence on $(0, \infty)$, so that it is increasing and strictly convex, $\varphi(t) = t^n$ for $t = 1/2^m$ and $t = 2^m$, $m \in \mathbb{N}$, and $\varphi(t/2) = 2^{-n}\varphi(t)$ for $t > 0$, but φ is not identically equal to t^n. This construction for $r = 1/2$ (or, equivalently, $r = 2$) can be easily modified for other values of $r > 0$, $r \neq 1$.

The following result can be obtained from (4.102) and the argument in the proof of Corollary 4.4.5.

Corollary 4.4.6. *Let $K, L \in \mathscr{K}^n_{(o)}$. Suppose that $\varphi \in \mathscr{I}$ is strictly convex and $V_\varphi(K, M) = V_\varphi(L, M)$ for all $M \in \mathscr{K}^n_{(o)}$. Then K and L are dilatates of each other. Moreover, $K = L$ unless φ behaves like t^n.*

Note that the restriction in the second statement of the previous theorem is necessary, since it is evident from (4.101) that if φ behaves like t^n, then for the corresponding $r \neq 1$, we have $V(K, M) = V(rK, M)$ for all $M \in \mathscr{K}^n_{(o)}$.

Let $K, L \in \mathscr{K}^n_{(o)}$, let $Q \in \mathscr{S}^n_{c+}$, and let $p, q \in \mathbb{R}$. In [54, (1.13), p. 91], the

(p, q)-*mixed volume* $\widetilde{V}_{p,q}(K, L, Q)$ was defined by setting $g = h_L^p$ in (4.27):

$$\begin{aligned}\widetilde{V}_{p,q}(K, L, Q) &= \int_{S^{n-1}} h_L(u)^p \, d\widetilde{C}_{p,q}(K, Q, u) \\ &= \frac{1}{n}\int_{S^{n-1}} h_L(\alpha_K(u))^p \, h_K(\alpha_K(u))^{-p} \, \rho_K(u)^q \, \rho_Q(u)^{n-q} \, du. \\ &= \frac{1}{n}\int_{S^{n-1}} \left(\frac{h_L(\alpha_K(u))}{h_K(\alpha_K(u))}\right)^p \left(\frac{\rho_K(u)}{\rho_Q(u)}\right)^q \rho_Q(u)^n \, du. \qquad (4.104)\end{aligned}$$

Inspired by (4.104), we can consider the nonlinear Orlicz dual curvature functionals defined by

$$\frac{1}{n}\int_{S^{n-1}} \varphi\left(\psi\left(\frac{f(\alpha_K(u))}{h_K(\alpha_K(u))}\right)\left(\frac{\rho_K(u)}{\rho_Q(u)}\right)^n\right) \rho_Q(u)^n \, du,$$

where $\varphi, \psi : (0, \infty) \to (0, \infty)$ are continuous functions and $f \in C^+(S^{n-1})$. We can then take $f = h_L$ to define the (φ, ψ)-*mixed volume*

$$\widetilde{V}_{\varphi,\psi}(K, L, Q) = \frac{1}{n}\int_{S^{n-1}} \varphi\left(\psi\left(\frac{h_L(\alpha_K(u))}{h_K(\alpha_K(u))}\right)\left(\frac{\rho_K(u)}{\rho_Q(u)}\right)^n\right) \rho_Q(u)^n \, du.$$

This is a natural generalization of (4.104) when $q \neq 0$, corresponding to taking $\varphi(t) = t^{q/n}$ and $\psi(t) = t^{np/q}$.

When $L \in \mathscr{K}^n_{(o)}$, the following result provides a common generalization of [16, Theorem 9.2], [18, Theorem 6.1] (see also [79, Theorem 2]), and [54, Theorem 7.4]. The first corresponds to taking $K = Q$ when φ there is replaced by $\varphi \circ \psi$, the second corresponds to taking $K = L$, and the third is obtained by the choices of φ and ψ given in the previous paragraph. Note that in the latter case, for the convexity of φ and ψ we then require that $1 \leq q/n \leq p$, which is precisely the assumption made in [54].

Theorem 4.4.7. *Let $K, L \in \mathscr{K}^n_{(o)}$ and let $Q \in \mathscr{S}^n_{c+}$. If $\varphi, \psi : (0, \infty) \to (0, \infty)$ are increasing and convex, then*

$$\widetilde{V}_{\varphi,\psi}(K, L, Q) \geq \varphi\left(\frac{V(K)}{V(Q)}\, \psi\left(\left(\frac{V(L)}{V(K)}\right)^{1/n}\right)\right) V(Q). \qquad (4.105)$$

If φ and ψ are strictly convex, equality holds if and only if K, L, and Q are dilatates of each other.

Proof. Setting $Q = K$ and $p = 1$ in [54, (7.6), Proposition 7.2], (4.104), and (4.27), we have, for any $q \neq 0$,

$$\begin{aligned} V_1(K,L) &= \widetilde{V}_{1,q}(K,L,K) \\ &= \int_{S^{n-1}} h_L(u)\, d\widetilde{C}_{1,q}(K,K,u) \\ &= \frac{1}{n}\int_{S^{n-1}} \frac{h_L(\alpha_K(u))}{h_K(\alpha_K(u))}\rho_K(u)^n\, du. \end{aligned} \tag{4.106}$$

We use Jensen's inequality [18, Proposition 2.2] twice, once with φ and once with ψ, Minkowski's first inequality (4.103), and (4.106) to obtain

$$\begin{aligned} \frac{\widetilde{V}_{\varphi,\psi}(K,L,Q)}{V(Q)} &= \frac{1}{nV(Q)}\int_{S^{n-1}} \varphi\left(\psi\left(\frac{h_L(\alpha_K(u))}{h_K(\alpha_K(u))}\right)\left(\frac{\rho_K(u)}{\rho_Q(u)}\right)^n\right)\rho_Q(u)^n\, du \\ &\geq \varphi\left(\frac{1}{nV(Q)}\int_{S^{n-1}} \psi\left(\frac{h_L(\alpha_K(u))}{h_K(\alpha_K(u))}\right)\left(\frac{\rho_K(u)}{\rho_Q(u)}\right)^n \rho_Q(u)^n\, du\right) \\ &= \varphi\left(\frac{V(K)}{V(Q)}\cdot\frac{1}{nV(K)}\int_{S^{n-1}} \psi\left(\frac{h_L(\alpha_K(u))}{h_K(\alpha_K(u))}\right)\rho_K(u)^n\, du\right) \\ &\geq \varphi\left(\frac{V(K)}{V(Q)}\,\psi\left(\frac{1}{nV(K)}\int_{S^{n-1}} \frac{h_L(\alpha_K(u))}{h_K(\alpha_K(u))}\rho_K(u)^n\, du\right)\right) \\ &= \varphi\left(\frac{V(K)}{V(Q)}\,\psi\left(\frac{V_1(K,L)}{V(K)}\right)\right) \\ &\geq \varphi\left(\frac{V(K)}{V(Q)}\,\psi\left(\left(\frac{V(L)}{V(K)}\right)^{1/n}\right)\right), \end{aligned}$$

as required.

Suppose that φ and ψ are strictly convex and that equality holds in (4.105). Then equality holds throughout the previous display. As in the proof of [16, Lemma 9.1], equalities in Minkowski's first inequality and in Jensen's inequality with ψ implies that K and L are dilatates of each other. Then equality in Jensen's inequality with φ implies that K and Q are dilatates of each other. □

We omit the proof of the following corollary, which is again similar to that of Corollary 4.4.5.

Corollary 4.4.8. *Let $K, L \in \mathcal{K}^n_{(o)}$, and suppose that $\varphi, \psi : (0,\infty) \to (0,\infty)$ are increasing and strictly convex. If $\widetilde{V}_{\varphi,\psi}(K,M,Q) = \widetilde{V}_{\varphi,\psi}(L,M,Q)$ for $M = \alpha K$, $\alpha > 0$, $Q = K$ and for $M = \alpha L$, $\alpha > 0$, $Q = L$, then K and L are dilatates of each other.*

Moreover, $K = L$ unless ψ behaves like t^n. If ψ behaves like t^n with $\psi(rt) = r^n\psi(t)$, $t > 0$, for some $r > 0$, then $\widetilde{V}_{\varphi,\psi}(K, M, Q) = \widetilde{V}_{\varphi,\psi}(rK, M, Q)$ for all $K, M, Q \in \mathscr{K}^n_{(o)}$.

Chapter 5

General volumes and Minkowski problem for $G(t,\cdot)$ increasing

This chapter is based on our paper [19]. In this chapter, we extend the general dual volume $\widetilde{V}_G$ and the general dual Orlicz curvature measure $\widetilde{C}_{G,\psi}$ defined in Chapter 4 for $K \in \mathscr{K}^n_{(o)}$ to more general functions $G : [0,\infty) \times S^{n-1} \to [0,\infty)$ and to compact convex sets $K \in \mathscr{K}^n_o$ containing the origin (but not necessarily in their interiors). Again we investigate the general dual Orlicz-Minkowski problem with respect to $\widetilde{C}_{G,\psi}$ for $G(t,\cdot)$ increasing. Methods used in this chapter are the approximation arguments from discrete measures to general measures.

5.1 The general dual volume on compact convex sets

First, we extend the $G : (0,\infty) \times S^{n-1} \to (0,\infty)$ to $G : [0,\infty) \times S^{n-1} \to [0,\infty)$ and obtain the generalized dual volume.

Definition 5.1.1. *Let $G : [0,\infty) \times S^{n-1} \to [0,\infty)$ be such that $u \mapsto G(\rho_K(u),u)$ is integrable on S^{n-1} for $K \in \mathscr{K}^n_o$. Define the* general dual volume $\widetilde{V}_G(K)$ *of $K \in \mathscr{K}^n_o$ by*

$$\widetilde{V}_G(K) = \int_{S^{n-1}} G(\rho_K(u),u)\,du. \tag{5.1}$$

If $K \in \mathscr{K}^n_o$ has empty interior, then $\rho_K = 0$ outside a great subsphere of S^{n-1}.

Since $\mathscr{H}^{n-1}$ vanishes on such great subspheres, we then have

$$\widetilde{V}_G(K) = \int_{S^{n-1}} G(0,u)\,du. \tag{5.2}$$

In particular, if $\operatorname{int} K = \emptyset$ and $G(0,u) = 0$ for $u \in S^{n-1}$, then $\widetilde{V}_G(K) = 0$.

The general dual volume was introduced for $K \in \mathscr{K}^n_{(o)}$ via (4.1) is clearly subsumed under Definition 5.1.1, since such a G can be extended to $[0,\infty) \times S^{n-1}$ by setting $G(0,u) = 0$ for $u \in S^{n-1}$.

In Lemma 4.1.2, $\widetilde{V}_G$ was shown to be continuous on $\mathscr{K}^n_{(o)}$ in the Hausdorff metric when $G : (0,\infty)\times S^{n-1} \to (0,\infty)$ is continuous. We next prove a corresponding result for $\widetilde{V}_G$ on $\mathscr{K}_o^n$.

Lemma 5.1.2. *Let $G : [0,\infty) \times S^{n-1} \to [0,\infty)$ be continuous. If $K_i \in \mathscr{K}_o^n$, $i \in \mathbb{N}$, and $K_i \to K \in \mathscr{K}_o^n$ as $i \to \infty$, then* $\lim_{i\to\infty} \widetilde{V}_G(K_i) = \widetilde{V}_G(K)$.

Proof. Let $K_i \in \mathscr{K}_o^n$, $i \in \mathbb{N}$, and $K_i \to K \in \mathscr{K}_o^n$ as $i \to \infty$. If $K \in \mathscr{K}^n_{(o)}$, we can assume without loss of generality that $K_i \in \mathscr{K}^n_{(o)}$ for all $i \in \mathbb{N}$, and the result then follows from Lemma 4.1.2. It therefore suffices to prove the lemma when $o \notin \operatorname{int} K$.

To this end, suppose first that $\operatorname{int} K = \emptyset$, so that $K \subset v^{\perp}$ for some $v \in S^{n-1}$. First, we show that $\rho_{K_i}(u) \to \rho_K(u)$ as $i \to \infty$ for $\mathscr{H}^{n-1}$-almost all $u \in S^{n-1}$. Since $\mathscr{H}^{n-1}(S^{n-1}\cap v^{\perp}) = 0$, it suffices to consider a fixed $u \in S^{n-1}\setminus v^{\perp}$. Let $\varepsilon \in (0, |\langle u, v\rangle|)$. There exists $i_\varepsilon \in \mathbb{N}$ so that $K_i \subset K + \varepsilon^2 B^n \subset \{x \in \mathbb{R}^n : |\langle x, v\rangle| \le \varepsilon^2\}$ for $i > i_\varepsilon$. Hence, for $i > i_\varepsilon$ we get

$$0 \le \rho_{K_i}(u)\varepsilon < \rho_{K_i}(u)|\langle u, v\rangle| = |\langle \rho_{K_i}(u)u, v\rangle| \le \varepsilon^2,$$

and therefore $0 \le \rho_{K_i}(u) < \varepsilon$ for $i > i_\varepsilon$. Thus $\rho_{K_i}(u) \to 0 = \rho_K(u)$ as $i \to \infty$, as required.

Since $K_i \to K$, there exists $R > 0$ such that $\rho_{K_i} \le R$ for $i \in \mathbb{N}$. The continuity of G on $[0,\infty)\times S^{n-1}$ implies that $M_0 = \max\{G(t,u) : (t,u) \in [0,R]\times S^{n-1}\} < \infty$. Hence, since G is continuous, $G(\rho_{K_i}(u), u) \to G(\rho_K(u), u)$ for $\mathscr{H}^{n-1}$-almost all $u \in S^{n-1}$ and the dominated convergence theorem applies. This yields

$$\lim_{i\to\infty} \widetilde{V}_G(K_i) = \lim_{i\to\infty} \int_{S^{n-1}} G(\rho_{K_i}(u), u)\,du = \int_{S^{n-1}} G(\rho_K(u), u)\,du = \widetilde{V}_G(K),$$

as required.

It remains to consider the case when $\operatorname{int} K \neq \emptyset$ and $o \in \partial K$. It is shown in the proof of [3, Lemma 2.2] that $\lim_{i\to\infty} \rho_{K_i}(u) = \rho_K(u)$ for $u \in S^{n-1} \setminus \partial N(K,o)^*$. Using this and (2.13), we obtain

$$\begin{aligned}\lim_{i\to\infty} \widetilde{V}_G(K_i) &= \lim_{i\to\infty} \int_{S^{n-1}\setminus \partial N(K,o)^*} G(\rho_{K_i}(u),u)\,du \\ &= \int_{S^{n-1}\setminus \partial N(K,o)^*} G(\rho_K(u),u)\,du \\ &= \widetilde{V}_G(K),\end{aligned}$$

where the second equality follows again from the dominated convergence theorem and the fact that $\rho_{K_i} \le R$ for $i \in \mathbb{N}$. □

We shall also need the following lemma. For $v \in S^{n-1}$ and $\varepsilon \in (0,1)$, let

$$\Sigma_\varepsilon(v) = \{u \in S^{n-1} : \langle u, v\rangle \ge \varepsilon\}. \tag{5.3}$$

Lemma 5.1.3. *Let $G(t,u)$ be continuous on $(0,\infty)\times S^{n-1}$ and decreasing in t on $(0,\infty)$. Let $0<\varepsilon_0<1$ and suppose that for $v \in S^{n-1}$,*

$$\lim_{t\to 0+} \int_{\Sigma_{\varepsilon_0}(v)} G(t,u)\,du = \infty. \tag{5.4}$$

If $K_i \in \mathscr{K}^n_{(o)}$, $i \in \mathbb{N}$, and $K_i \to K \in \mathscr{K}^n_o$ as $i\to\infty$ with $o \in \partial K$, then

$$\lim_{i\to\infty} \widetilde{V}_G(K_i) = \infty.$$

Proof. Let $K_i \in \mathscr{K}^n_{(o)}$, $i \in \mathbb{N}$, and $K_i \to K \in \mathscr{K}^n_o$ as $i\to\infty$ with $o \in \partial K$. Choose $v \in S^{n-1} \cap N(K,o)$. Let $t \in (0,\varepsilon_0)$. Then there is an $i_t \in \mathbb{N}$ such that for $i > i_t$, we have $K_i \subset \{z \in \mathbb{R}^n : \langle z, v\rangle \le t^2\}$. If $u \in \Sigma_{\varepsilon_0}(v)$ and $i > i_t$, then

$$t < \varepsilon_0 \le \langle u, v\rangle = \frac{\langle \rho_{K_i}(u)u, v\rangle}{\rho_{K_i}(u)} \le \frac{t^2}{\rho_{K_i}(u)},$$

and therefore $0 \le \rho_{K_i}(u) < t$ for $i > i_t$ and $u \in \Sigma_{\varepsilon_0}(v)$. Hence, for $i > i_t$ we get

$$\begin{aligned} \widetilde{V}_G(K_i) &\geq \int_{\Sigma_{\varepsilon_0}(v)} G(\rho_{K_i}(u), u)\, du \\ &\geq \int_{\Sigma_{\varepsilon_0}(v)} G(t, u)\, du, \end{aligned} \tag{5.5}$$

since $G(t, u)$ is decreasing for t. In (5.5), we let $i \to \infty$ and then $t \to 0+$. This and (5.4) yield the assertion. $\square$

5.2 The general dual Orlicz-Minkowski problem for discrete measures

Recall the definition of the general dual Orlicz curvature measure in (4.7). Let $K \in \mathscr{K}^n_{(o)}$ and let $G : (0, \infty) \times S^{n-1} \to (0, \infty)$ and $\psi : (0, \infty) \to (0, \infty)$ be continuous. Suppose that $G_t(t, u) = \partial G(t, u)/\partial t$ is such that $u \mapsto G_t(\rho_K(u), u)$ is integrable on S^{n-1},

$$\widetilde{C}_{G,\psi}(K, E) = \frac{1}{n} \int_{\boldsymbol{\alpha}^*_K(E)} \frac{\rho_K(u)\, G_t(\rho_K(u), u)}{\psi(h_K(\alpha_K(u)))}\, du \tag{5.6}$$

for each Borel set $E \subset S^{n-1}$.

Problem 5.2.1. *For which nonzero finite Borel measures μ on S^{n-1} and continuous functions $G : (0, \infty) \times S^{n-1} \to (0, \infty)$ and $\psi : (0, \infty) \to (0, \infty)$ do there exist $\tau \in \mathbb{R}$ and $K \in \mathscr{K}^n_{(o)}$ such that $\mu = \tau\, \widetilde{C}_{G,\psi}(K, \cdot)$?*

A solution to Problem 5.2.1 was presented in Section 4.3, assuming that $G_t < 0$, G satisfies some growth conditions, and

$$\int_1^\infty \frac{\psi(s)}{s}\, ds = \infty. \tag{5.7}$$

Our aim here is to provide a solution to Problem 5.2.1 when $G_t > 0$. To do this, we first deal with the case when μ is discrete, i.e., when $\mu = \sum_{i=1}^m \lambda_i \delta_{u_i}$, where δ_{u_i} denotes the Dirac measure at $u_i \in S^{n-1}$, $\lambda_i > 0$ for each i, and $\{u_1, \ldots, u_m\}$ is not contained in a closed hemisphere. In this case we seek a solution for which $K \in \mathscr{K}^n_{(o)}$ is a convex polytope. This discrete Minkowski-type problem has been solved in several special cases. Indeed, when $G(t, u) = t^n/n$, then $\widetilde{V}_G(K)$ is the volume of K and the corresponding Orlicz-Minkowski problem for discrete measures was solved in [27, 33, 39]. When $\psi(t) = t^p$ for $p > 1$ and $G(t,u) = t^q\phi_1(u)$ for $q > 0$ and $\phi_1 \in C^+(S^{n-1})$, the problem becomes the L_p dual Minkowski problem for discrete measures proposed in [54] and solved in [3]. Our solution to Problem 5.2.1 in Theorem 5.4.3 below significantly extends those in [3, 27, 33, 39].

We utilize the techniques in [33] which were also found effective in other works, such as [3, 24, 27]. Let $m > n$ be an integer and suppose that $\{u_1, \dots, u_m\}$ is not contained in a closed hemisphere. For each $z = (z_1, \dots, z_m) \in [0,\infty)^m$, let

$$P(z) = \{x \in \mathbb{R}^n : \langle x, u_i\rangle \le z_i, \text{ for } i = 1, \dots, m\}. \tag{5.8}$$

Then $P(z) \in \mathscr{K}_o^n$ is a convex polytope. We point out that the facets of $P(z)$ are among the support sets $F(P(z), u_i)$, $i \in \{1, \dots, m\}$, of $P(z)$, but not all of these necessarily are facets. We have $h(P(z), u_i) \le z_i$ for $i \in \{1, \dots, m\}$, with equality if $F(P(z), u_i)$ is a facet of $P(z)$.

Lemma 5.2.2. *Let $P \in \mathscr{K}_{(o)}^n$ be a convex polytope with facets $F(P, u_1), \dots, F(P, u_m)$. Then*

$$\widetilde{C}_{G,\psi}(P, \cdot) = \sum_{i=1}^m \gamma_i \delta_{u_i}, \tag{5.9}$$

where

$$\gamma_i = \frac{\widetilde{C}_G(P, \{u_i\})}{\psi(h_P(u_i))} \tag{5.10}$$

for $i = 1, \dots, m$. If $G_t > 0$ (or $G_t < 0$) on $(0,\infty) \times S^{n-1}$, then $\gamma_i > 0$ (or $\gamma_i < 0$, respectively) for $i = 1, \dots, m$.

Proof. That $\widetilde{C}_{G,\psi}(P, \cdot)$ is of the form (5.9) follows immediately from the absolute continuity of $\widetilde{C}_{G,\psi}(P, \cdot)$ with respect to $S(P, \cdot)$ in Proposition 4.3.1 (i), since the latter measure is concentrated on $\{u_1, \dots, u_m\}$. Using (5.6) and the fact that $\boldsymbol{\alpha}_K^*(\{u_i\}) = \tilde{\pi}(F(P, u_i))$, we obtain

$$\begin{aligned}
\gamma_i &= \widetilde{C}_{G,\psi}(P, \{u_i\}) \\
&= \frac{1}{n} \int_{\tilde{\pi}(F(P,u_i))} \frac{\rho_P(u)\, G_t(\rho_P(u), u)}{\psi(h_P(u_i))}\, du \\
&= \frac{1}{n\psi(h_P(u_i))} \int_{\tilde{\pi}(F(P,u_i))} \rho_P(u)\, G_t(\rho_P(u), u)\, du \\
&= \frac{\widetilde{C}_G(P, \{u_i\})}{\psi(h_P(u_i))},
\end{aligned} \tag{5.11}$$

proving (5.10). (Recall that $\widetilde{C}_G(P,\cdot)$ denotes $\widetilde{C}_{G,\psi}(P,\cdot)$ with $\psi \equiv 1$ and note that $h_P(u_i) > 0$, so $\psi(h_P(u_i)) > 0$ is defined.)

Suppose that $G_t > 0$ (or $G_t < 0$) on $(0,\infty)\times S^{n-1}$. Since $o \in \operatorname{int} P$, we have $\rho_P > 0$, so the integrand in (5.11) is positive on S^{n-1} (or negative on S^{n-1}, respectively). It follows that $\gamma_i > 0$ (or $\gamma_i < 0$, respectively) for $i = 1,\dots,m$. $\square$

Lemma 5.2.3. *Let $f : (0,\infty) \to (0,\infty)$ be continuously differentiable and let $\alpha_2 > 2\alpha_1 > 0$. There exists $c_0 = c_0(\alpha_1,\alpha_2) > 0$ such that*

$$f(\alpha - s) \geq f(\alpha) - c_0 s \tag{5.12}$$

for $\alpha \in [2\alpha_1,\alpha_2]$ and $s \in [0,\alpha_1]$.

Proof. If $\alpha \in [2\alpha_1,\alpha_2]$ and $s \in [0,\alpha_1]$, then $\alpha - s \in [\alpha_1,\alpha_2]$. Let $c_0 = \max\{|f'(s)| : s \in [\alpha_1,\alpha_2]\}$. Define

$$g(s) = f(\alpha - s) - f(\alpha) + c_0 s$$

for $s \geq 0$. Then $g(0) = 0$ and $g'(s) = c_0 - f'(\alpha - s) \geq 0$ for $s \in [0,\alpha_1]$. Therefore on $[0,\alpha_1]$, g is increasing and hence $g(s) \geq g(0) = 0$, which proves (5.12). $\square$

For $K \in \mathscr{K}^n_{(o)}$ and $\varphi \in \mathscr{I}$, we let

$$\|h_K\|_{\mu,\varphi} = \inf\left\{\lambda > 0 : \frac{1}{\varphi(1)\,\mu(S^{n-1})}\int_{S^{n-1}} \varphi\left(\frac{h_K(u)}{\lambda}\right) d\mu(u) \leq 1\right\}. \tag{5.13}$$

Under appropriate assumptions, $\|\cdot\|_{\mu,\varphi}$ is a norm, the Orlicz or Luxemburg norm. For example, in [16, Section 4], and elsewehere, the triangle inequality is proved for suitable convex φ. We do not need this restriction on φ, since we merely use (5.13) for normalization purposes.

Suppose that $\psi : (0,\infty) \to (0,\infty)$ is continuous. Define

$$\varphi(t) = \int_0^t \frac{\psi(s)}{s}\,ds \quad \text{for } t > 0 \quad \text{and} \quad \varphi(0) = 0. \tag{5.14}$$

(A similar, but slightly different, function was employed in (4.56).) If $\varphi < \infty$ on $(0,\infty)$, then it is continuous (by the dominated convergence theorem) and strictly increasing on $[0,\infty)$, and $\varphi'(t) = \psi(t)/t$ for $t > 0$. Note that this assumption on φ imposes a weak growth condition on $\psi(t)$ as $t \downarrow 0$.

The hypotheses of the next theorem allow $\psi(t) = t^p$ for $p > 1$ and $G(t,u) = t^q$ for $q > 0$, for example.

Theorem 5.2.4. *Let* $\mu = \sum_{i=1}^m \lambda_i \delta_{u_i}$*, where* $\lambda_i > 0$*,* $i = 1, \dots, m$*, and* $\{u_1, \dots, u_m\} \subset S^{n-1}$ *is not contained in a closed hemisphere. Let* $G : [0,\infty) \times S^{n-1} \to [0,\infty)$ *be continuous and such that* G_t *is continuous and positive on* $(0,\infty) \times S^{n-1}$*. Suppose that* $\psi : (0,\infty) \to (0,\infty)$ *is continuous and such that* $\lim_{t\to 0+} \psi(t)/t = 0$ *and* (5.7) *holds. Then there exist a convex polytope* $P \in \mathscr{K}^n_{(o)}$ *and* $\tau > 0$ *such that*

$$\mu = \tau\, \widetilde{C}_{G,\psi}(P,\cdot) \quad \text{and} \quad \|h_P\|_{\mu,\varphi} = 1, \tag{5.15}$$

where φ *and* τ *are given by* (5.14) *and* (5.25)*, respectively.*

Proof. Define φ by (5.14). The assumptions on ψ imply that φ is finite, so $\varphi : [0,\infty) \to [0,\infty)$ is continuous and strictly increasing, and that $\varphi(t) \to \infty$ as $t \to \infty$. It follows that the set

$$M = \left\{ (z_1, \dots, z_m) \in [0,\infty)^m : \sum_{i=1}^m \lambda_i \varphi(z_i) = \sum_{i=1}^m \lambda_i \varphi(1) \right\} \tag{5.16}$$

is compact and nonempty as $(1, \dots, 1) \in M$. By Lemma 5.1.2 and since $z \mapsto P(z)$, $z \in [0,\infty)^m$, is continuous, there is a $z^0 = (z_1^0, \dots, z_m^0) \in M$ such that

$$\widetilde{V}_G(P(z^0)) = \max\{\widetilde{V}_G(P(z)) : \quad z \in M\}. \tag{5.17}$$

As $G_t > 0$, $G(t,u)$ is strictly increasing in $t \in [0,\infty)$, and then (5.1) implies that $\widetilde{V}_G(\cdot)$ is also increasing, i.e., if $K \subset K'$, then $\widetilde{V}_G(K) \le \widetilde{V}_G(K')$. From (5.8) we see that $B^n \subset P((1, \dots, 1))$ and then (5.17) yields

$$\infty > \widetilde{V}_G(P(z^0)) \ge \widetilde{V}_G(P((1, \dots, 1))) \ge \widetilde{V}_G(B^n) = \int_{S^{n-1}} G(1,u)\, du > \int_{S^{n-1}} G(0,u)\, du. \tag{5.18}$$

In view of (5.2), this implies that $\dim P(z^0) = n$.

Let $h_0 \in C^+(S^{n-1})$, $g \in C(S^{n-1})$, and $\varphi_0 \in \mathscr{I}_a$ for some $a \in \mathbb{R} \cup \{-\infty\}$. Then we have $\varphi_0^{-1} : (a,\infty) \to (0,\infty)$, and since S^{n-1} is compact, $0 < c \le h_0 \le C$ for some $0 < c \le C$. It is then easy to check that for $\varepsilon \in \mathbb{R}$ close to 0, one can define

$h_\varepsilon = h_\varepsilon(h_0, g, \varphi_0) \in C^+(S^{n-1})$ by

$$h_\varepsilon(u) = {\varphi_0}^{-1}\left(\varphi_0(h_0(u)) + \varepsilon g(u)\right). \tag{5.19}$$

In particular, we can apply (5.19) when $h_0 = h_K$ for some $K \in \mathscr{K}^n_{(o)}$.

We shall first prove the theorem assuming that $o \in \operatorname{int} P(z^0)$, in which case $P(z^0) \in \mathscr{K}^n_{(o)}$ and $z_i^0 > 0$ for $i = 1, \dots, m$. Fix $i \in \{1, \dots, m\}$ for the moment. Let $h_0 \in C^+(S^{n-1})$ be such that $h_0(u_j) = z_j^0 > 0$ for $j = 1, \dots, m$. Further, let $g_i \in C(S^{n-1})$ be such that $g_i(u_j) = \delta_{ij}$ for $j = 1, \dots, m$. If $|\varepsilon|$ is small enough, we may define h_ε via (5.19) with $\varphi_0(t) = t$, so that $h_\varepsilon = h_0 + \varepsilon g_i \in C^+(S^{n-1})$ and $\psi_0(t) = t\varphi_0'(t) = t$. Moreover, if the Alexandrov body $[h_\varepsilon]$ of h_ε is taken with respect to the set $\Omega = \{u_1, \dots, u_m\}$, we have $[h_\varepsilon] = P(z^0 + \varepsilon e_i)$. Using this, (4.49), and (5.11), we obtain

$$\begin{aligned}
\left.\frac{\partial \widetilde{V}_G(P(z))}{\partial z_i}\right|_{z=z^0} &= \lim_{\varepsilon \to 0} \frac{\widetilde{V}_G([h_\varepsilon]) - \widetilde{V}_G([h_0])}{\varepsilon} \\
&= n \sum_{j=1}^m g_i(u_j) \widetilde{C}_{G,\psi_0}(P(z^0), \{u_j\}) \\
&= n \widetilde{C}_{G,\psi_0}(P(z^0), \{u_i\}) \\
&= n \frac{\widetilde{C}_G(P(z^0), \{u_i\})}{h_{P(z^0)}(u_i)}.
\end{aligned} \tag{5.20}$$

The argument does not depend on the choice of $z \in \mathbb{R}^m$ with positive coordinates z_i, so the calculation shows that the map $z \mapsto \widetilde{V}_G(P(z))$ has continuous partial derivatives and therefore is continuously differentiable. Moreover, since $\lambda_i > 0$ and $\varphi' > 0$, the rank condition in the Lagrange multipliers theorem is satisfied. In view of (5.16) and (5.17), that theorem provides a $\tau \in \mathbb{R}$ such that

$$\left.\frac{\tau}{n} \frac{\partial \widetilde{V}_G(P(z))}{\partial z_i}\right|_{z=z^0} = \left.\frac{\partial}{\partial z_i} \sum_{i=1}^m \lambda_i \varphi(z_i)\right|_{z=z^0} \tag{5.21}$$

for $i = 1, \dots, m$. As $o \in \operatorname{int} P(z^0)$, (5.8) implies that $z_i^0 > 0$ for each i. This, (5.20), (5.21), and $\varphi'(t) = \psi(t)/t$ for $t > 0$ yield

$$\tau \frac{\widetilde{C}_G(P(z^0), \{u_i\})}{h_{P(z^0)}(u_i)} = \lambda_i \varphi'(z_i^0) = \lambda_i \frac{\psi(z_i^0)}{z_i^0} \qquad \text{for} \quad i = 1, \dots, m, \tag{5.22}$$

while $z^0 \in M$ implies that

$$\sum_{i=1}^{m} \lambda_i \varphi(z_i^0) = \varphi(1) \sum_{i=1}^{m} \lambda_i. \tag{5.23}$$

For each i, we have $\lambda_i > 0$ and hence $\widetilde{C}_G(P(z^0), \{u_i\}) > 0$, by (5.22). The absolute continuity of $\widetilde{C}_{G,\psi}(P, \cdot)$ with respect to $S(P, \cdot)$ (see Proposition 4.3.1) implies that the face $F(P(z^0), u_i)$ is actually a facet, hence we have $z_i^0 = h_{P(z^0)}(u_i)$. From (5.22), we conclude that

$$\lambda_i = \tau \frac{\widetilde{C}_G(P(z^0), \{u_i\})}{\psi(h_{P(z^0)}(u_i))} = \tau\, \widetilde{C}_{G,\psi}(P(z^0), \{u_i\}) \tag{5.24}$$

for $i = 1, \ldots, m$. This proves that $\mu = \tau\, \widetilde{C}_{G,\psi}(P(z^0), \cdot)$ because both measures are concentrated on $\{u_1, \ldots, u_m\}$. Summing (5.24) over i, we obtain

$$\tau = \frac{\mu(S^{n-1})}{\widetilde{C}_{G,\psi}(P(z^0), S^{n-1})} = \frac{1}{\widetilde{C}_G(P(z^0), S^{n-1})} \int_{S^{n-1}} \psi(h_{P(z^0)}(u))\, d\mu(u). \tag{5.25}$$

Moreover, in view of (5.13), (5.23) is equivalent to $\|h_{P(z^0)}\|_{\mu,\varphi} = 1$.

This proves the theorem under the assumption that $o \in \operatorname{int} P(z^0)$, which we now claim is true. Suppose that $o \in \partial P(z^0)$. To obtain a contradiction, we use an argument similar to that in the proof of [3, Lemma 3.2]. By relabeling, if necessary, we may suppose that for some $1 \le k < m$, $z_j^0 = 0$ for $j = 1, \ldots, k$ and $z_j^0 > 0$ for $j = k+1, \ldots, m$. Note that $k < m$ because otherwise, $z_j^0 = 0$ for $j = 1, \ldots, m$ implies $P(z^0) = \{o\}$, which is impossible. Let

$$\lambda = \frac{\lambda_1 + \cdots + \lambda_k}{\lambda_{k+1} + \cdots + \lambda_m} > 0, \tag{5.26}$$

and choose $t_0 > 0$ small enough that $\varphi(z_i^0) - \lambda\varphi(t_0) > 0$ for $i = k+1, \ldots, m$. For $t \in (0, t_0)$, let

$$a^t = \left(0, \ldots, 0, \varphi^{-1}(\varphi(z_{k+1}^0) - \lambda\varphi(t)), \ldots, \varphi^{-1}(\varphi(z_m^0) - \lambda\varphi(t))\right), \tag{5.27}$$

where the first k components of a^t are equal to 0, and let

$$b^t = a^t + t(e_1 + \cdots + e_k), \tag{5.28}$$

so that b^t is obtained from a^t by setting the first k components equal to t. By (5.14), φ, and hence φ^{-1}, is increasing on $[0,\infty)$. Therefore $a_i^t = \varphi^{-1}(\varphi(z_i^0) - \lambda\varphi(t)) \leq z_i^0$ for $i = k+1,\ldots,m$. This yields

$$P(a^t) \subset P(z^0) \text{ and hence } \widetilde{V}_G(P(a^t)) \leq \widetilde{V}_G(P(z^0)). \tag{5.29}$$

For $t \in (0,t_0)$, we have $o \in \operatorname{int} P(b^t)$ and from (5.27) and (5.28),

$$P(a^t) \subset P(b^t) \text{ and hence } \widetilde{V}_G(P(a^t)) \leq \widetilde{V}_G(P(b^t)). \tag{5.30}$$

Using (5.26), (5.27), $\varphi(z_1^0) = \cdots = \varphi(z_k^0) = \varphi(0) = 0$, and $z^0 \in M$, we obtain

$$\begin{aligned}
\sum_{i=1}^m \lambda_i \varphi(b_i^t) &= \sum_{i=1}^k \lambda_i\varphi(t) + \sum_{i=k+1}^m \lambda_i(\varphi(z_i^0) - \lambda\varphi(t)) \\
&= \varphi(t)\left(\sum_{i=1}^k \lambda_i - \lambda \sum_{i=k+1}^m \lambda_i\right) + \sum_{i=k+1}^m \lambda_i\varphi(z_i^0) \\
&= \sum_{i=1}^m \lambda_i\varphi(z_i^0) \\
&= \sum_{i=1}^m \lambda_i\varphi(1),
\end{aligned}$$

from which we see via (5.16) that $b^t \in M$.

Let $r_0 = \min\{z_i^0 : i = k+1,\ldots,m\} > 0$ and let $R_0 > \max\{z_i^0 : i = k+1,\ldots,m\}$ be such that $P(z^0) \subset \operatorname{int} R_0B^n$. We apply Lemma 5.2.3 with $f = \varphi^{-1}$, $\alpha = \varphi(z_i^0)$ for $i = k+1,\ldots,m$, $s = \lambda\varphi(t) > 0$, $\alpha_1 = \varphi(r_0)/2$, and $\alpha_2 = \varphi(R_0)$. We conclude that with

$$c_0 = \max\{(\varphi^{-1})'(s) : s \in [\alpha_1,\alpha_2]\} = \max\left\{\frac{1}{\varphi'(\varphi^{-1}(s))} : s \in [\alpha_1,\alpha_2]\right\},$$

and

$$\beta = \min\left\{\varphi^{-1}\left(\frac{\varphi(r_0)}{2\lambda}\right), \varphi^{-1}\left(\frac{r_0}{2\lambda c_0}\right)\right\} > 0,$$

we have

$$\varphi^{-1}(\varphi(z_i^0) - \lambda\varphi(t)) \geq z_i^0 - c_0\lambda\varphi(t) > \frac{r_0}{2} \tag{5.31}$$

for $t \in (0,\beta)$, where the second inequality follows from the definition of β and $z_i^0 > r_0$.

Note that (5.31) and the definition of t_0 ensure that we can choose $t_0 = \beta$.

Recall (see (2.15)) that $\rho_{P(z^0)}(u) > 0$ if and only if $u \in S^{n-1} \cap N(P(z^0), o)^*$. By (5.8), the inclusion in (5.29), and the fact that $a_i^t = 0$ if and only if $z_i^0 = 0$, $\rho_{P(a^t)}(u) > 0$ if and only if $\rho_{P(z^0)}(u) > 0$. In fact, by the definition of r_0 and R_0, (5.27), and (5.31), we have $\rho_{P(z^0)}(u) \in (r_0, R_0)$ and $\rho_{P(a^t)}(u) \in (r_0/2, R_0)$ for $u \in S^{n-1} \cap N(P(z^0), o)^*$. Consequently, in view of the continuity of G_t on $(0, \infty) \times S^{n-1}$, there are constants $c_1 > 0$ and $\beta_1 \in (0, \beta)$ such that

$$G(\rho_{P(z^0)}(u), u) - G(\rho_{P(a^t)}(u), u) \le c_1(\rho_{P(z^0)}(u) - \rho_{P(a^t)}(u)) \tag{5.32}$$

for $u \in S^{n-1} \cap N(P(z^0), o)^*$.

Let $u \in S^{n-1} \cap N(P(z^0), o)^*$ and choose $i_0 \in \{k+1, \dots, m\}$ so that the ray from o in the direction u meets the facet $F(P(a^t), u_{i_0})$, and hence, by the inclusion in (5.29), the facet $F(P(z^0), u_{i_0})$ as well. Then, by (5.31),

$$R_0\langle u, u_{i_0}\rangle > \rho_{P(a^t)}(u)\langle u, u_{i_0}\rangle = a_{i_0}^t = \varphi^{-1}(\varphi(z_{i_0}^0) - \lambda\varphi(t)) > \frac{r_0}{2},$$

and $z_{i_0}^0 = \rho_{P(z^0)}(u)\langle u, u_{i_0}\rangle$.

Using these relations and (5.31) again, we obtain

$$\begin{aligned} \rho_{P(a^t)}(u) &= \frac{\varphi^{-1}(\varphi(z_{i_0}^0) - \lambda\varphi(t))}{\langle u, u_{i_0}\rangle} \\ &\ge \frac{z_{i_0}^0 - c_0\lambda\varphi(t)}{\langle u, u_{i_0}\rangle} \\ &\ge \rho_{P(z^0)}(u) - \frac{2R_0c_0\lambda}{r_0}\varphi(t). \end{aligned} \tag{5.33}$$

From (5.32) and (5.33) we get

$$\begin{aligned} \widetilde{V}_G(P(z^0)) - \widetilde{V}_G(P(a^t)) &= \int_{S^{n-1}\cap N(P(z^0),o)^*} \big(G(\rho_{P(z^0)}(u), u) - G(\rho_{P(a^t)}(u), u)\big)\, du \\ &\le c_1 \int_{S^{n-1}\cap N(P(z^0),o)^*} \big(\rho_{P(z^0)}(u) - \rho_{P(a^t)}(u)\big)\, du \\ &\le \frac{2R_0c_0c_1\lambda}{r_0}\varphi(t) \int_{S^{n-1}\cap N(P(z^0),o)^*} du \\ &\le c_2\varphi(t) \end{aligned} \tag{5.34}$$

for $t \in (0, \beta_1)$, where $c_2 = 2R_0c_0c_1\lambda n\kappa_n/r_0$.

Using $o \in \operatorname{int} P(b^t)$ and the containments in (5.29) and (5.30), one can show that there is a closed set $E_t \subset S^{n-1}$ and constants $r_1 > 0$, $\beta_2 \in (0, \beta_1)$, and $c_3 > 0$, depending only on n, r_0, and R_0, satisfying $\mathscr{H}^{n-1}(E_t) \geq c_3 t$ for $t \in (0, \beta_2)$ and such that $\rho_{P(a^t)}(u) = 0$ and $\rho_{P(b^t)}(u) \geq r_1$ for $u \in E_t$. We omit the details, since these are given in the proof of [3, p. 13]; there, the set E_t is denoted by $\widetilde{G}_t$ and is the radial projection on S^{n-1} of a certain $(n-1)$-dimensional spherical cylinder of height t. For $u \in E_t$, we have $G(\rho_{P(a^t)}(u), u) = G(0, u)$ and $G(\rho_{P(b^t)}(u), u) \geq G(r_1, u)$ as $G_t > 0$. Consequently,

$$\begin{aligned} \widetilde{V}_G(P(b^t)) &= \int_{S^{n-1}\setminus E_t} G(\rho_{P(b^t)}(u), u)\, du + \int_{E_t} G(\rho_{P(b^t)}(u), u)\, du \\ &\geq \widetilde{V}_G(P(a^t)) + \int_{E_t} \left(G(r_1, u) - G(0, u)\right)\, du \\ &\geq \widetilde{V}_G(P(a^t)) + c_4 t \end{aligned} \tag{5.35}$$

for $t \in (0, \beta_2)$, where

$$c_4 = c_3 \min\{G(r_1, u) - G(0, u) : u \in S^{n-1}\} > 0.$$

From (5.34) and (5.35), we obtain

$$\liminf_{t\to 0+} \frac{\widetilde{V}_G(P(b^t)) - \widetilde{V}_G(P(z^0))}{t} \geq \lim_{t\to 0+} \frac{c_4 t - c_2\,\varphi(t)}{t} = c_4 > 0, \tag{5.36}$$

since

$$\lim_{t\to 0+} \frac{\varphi(t)}{t} = \lim_{t\to 0+} \varphi'(t) = \lim_{t\to 0+} \frac{\psi(t)}{t} = 0.$$

By (5.36), there exists $t_1 \in (0, \beta_2)$ such that $\widetilde{V}_G(P(b^{t_1})) > \widetilde{V}_G(P(z^0))$. It was shown above that $b^{t_1} \in M$, so this contradicts (5.17). Thus $o \in \operatorname{int} P(z^0)$ and the proof is complete. □

Recall that for $v \in S^{n-1}$ and $\varepsilon \in (0, 1)$, $\Sigma_\varepsilon(v)$ is defined by (5.3) and that $\|\cdot\|_{\mu,\varphi}$ is defined by (5.13). The hypotheses of the next theorem allow $\psi(t) = t^p$ for $p > 0$ and $G(t, u) = t^q$ for $q < 0$, for example.

Theorem 5.2.5. *Let $\mu = \sum_{i=1}^m \lambda_i \delta_{u_i}$, where $\lambda_i > 0$, $i = 1, \ldots, m$, and $\{u_1, \ldots, u_m\} \subset S^{n-1}$ is not contained in a closed hemisphere. Let $G : (0, \infty) \times S^{n-1} \to (0, \infty)$ be* *continuous and such that* G_t *is continuous and negative on* $(0,\infty)\times S^{n-1}$*. Let* $0<\varepsilon_0<1$ *and suppose that* (5.4) *holds for* $v\in S^{n-1}$*. Suppose that* $\psi:(0,\infty)\to(0,\infty)$ *is continuous,* (5.7) *holds, and that* φ *is finite when defined by* (5.14)*. Then there exist a convex polytope* $P\in\mathscr{K}^n_{(o)}$ *and* $\tau<0$ *such that*

$$\mu=\tau\,\widetilde{C}_{G,\psi}(P,\cdot)\quad\text{ and }\quad\|h_P\|_{\mu,\varphi}=1,$$

where τ *is given by* (5.25)*.*

Proof. Define φ by (5.14). The assumption that φ is finite implies that $\varphi:[0,\infty)\to[0,\infty)$ is continuous and strictly increasing, and from (5.7), we have $\varphi(t)\to\infty$ as $t\to\infty$. The set

$$M'=\left\{(z_1,\dots,z_m)\in(0,\infty)^m:\sum_{i=1}^m\lambda_i\varphi(z_i)=\sum_{i=1}^m\lambda_i\varphi(1)\right\}$$

is bounded and nonempty as $(1,\dots,1)\in M'$. Let

$$\alpha=\inf\{\widetilde{V}_G(P(z)):\quad z\in M'\},\tag{5.37}$$

where $P(z)$ is defined by (5.8). Choose $z^j\in M'$, $j\in\mathbb{N}$, such that

$$\lim_{j\to\infty}\widetilde{V}_G(P(z^j))=\alpha.\tag{5.38}$$

Since M' is bounded, we can assume, by taking a subsequence, if necessary, that $z^j\to z^0\in M$, where M is defined by (5.16). However, we actually have $o\in\operatorname{int}P(z^0)$ and hence $z^0\in M'$. To see this, suppose to the contrary that $o\in\partial P(z^0)$. Since $P(z^j)\in\mathscr{K}^n_{(o)}$ for $j\in\mathbb{N}$ and $P(z^j)\to P(z^0)$ as $j\to\infty$, Lemma 5.1.3 yields

$$\lim_{j\to\infty}\widetilde{V}_G(P(z^j))=\infty.$$

By (5.8), $B^n\subset P((1,\dots,1))$. Also, as $G_t<0$, $G(t,\cdot)$ is decreasing on $(0,\infty)$ and hence $\widetilde{V}_G(\cdot)$ is also decreasing, i.e., if $K\subset K'$, then $\widetilde{V}_G(K)\ge\widetilde{V}_G(K')$. Therefore, using $(1,\dots,1)\in M'$, (5.37), and (5.38), we obtain

$$\infty>\widetilde{V}_G(B^n)\ge\widetilde{V}_G(P((1,\dots,1)))\ge\alpha=\lim_{j\to\infty}\widetilde{V}_G(P(z^j)),$$

a contradiction proving that $z^0 \in M'$ and $P(z^0) \in \mathscr{K}^n_{(o)}$. By Lemma 4.1.2, $\widetilde{V}_G(\cdot)$ is continuous in the Hausdorff metric on $\mathscr{K}^n_{(o)}$, so

$$\infty > \widetilde{V}_G(B^n) \geq \widetilde{V}_G(P(z^0)) = \lim_{j\to\infty} \widetilde{V}_G(P(z^j)) = \alpha > 0. \tag{5.39}$$

The remainder of the proof is precisely the same as the passage from (5.20) to (5.25) in the proof of Theorem 5.2.4. □

Under the conditions on μ, G, and ψ stated in Theorem 5.2.5, but with the assumption that $\varphi < \infty$ replaced by the condition

$$\lim_{t\to\infty} \int_{S^{n-1}} G(t,u)\,du = 0, \tag{5.40}$$

Theorem 4.3.3 proves that there is a $K \in \mathscr{K}^n_{(o)}$ such that

$$\frac{\mu}{\mu(S^{n-1})} = \frac{\widetilde{C}_{G,\psi}(K,\cdot)}{\widetilde{C}_{G,\psi}(K,S^{n-1})}. \tag{5.41}$$

If μ is discrete, Theorem 4.3.3 does not prove that K is a convex polytope, but, as is explained in the discussion after Corollary 4.3.4, this is an easy consequence of (5.41). Thus Theorem 5.2.5 is a variant of Theorem 4.3.3 for discrete μ.

5.3 General dual Orlicz curvature measures for $\mathscr{K}^n_o$

The general dual Orlicz curvature measure $\widetilde{C}_{G,\psi}(K,\cdot)$ was defined by (5.6) for $K \in \mathscr{K}^n_{(o)}$. In this section, we extend the definition to $K \in \mathscr{K}^n_o$.

Let $K \in \mathscr{K}^n_o$. Recall that $N(K,o)$ and $N(K,o)^*$ are defined by (2.11) and (2.12).

Definition 5.3.1. *Define the* general Orlicz curvature measure $\widetilde{C}_{G,\psi}(K,\cdot)$ *by*

$$\widetilde{C}_{G,\psi}(K,E) = \frac{1}{n}\int_{\boldsymbol{\alpha}^*_K(E\setminus N(K,o))} \frac{\rho_K(u)\,G_t(\rho_K(u),u)}{\psi(h_K(\alpha_K(u)))}\,du \tag{5.42}$$

for each Borel set $E \subset S^{n-1}$, whenever $G : [0,\infty)\times S^{n-1} \to [0,\infty)$ and $\psi : [0,\infty) \to [0,\infty)$ with $\psi(t) > 0$ for $t > 0$ are such that the integral in (5.42) exists for all $K \in \mathscr{K}^n_o$ and Borel sets $E \subset S^{n-1}$.

Note that if $\dim K < n$, then $\widetilde{C}_{G,\psi}(K,E) = 0$, since $S^{n-1} \cap N(K,o)^*$ is then at most $(n-2)$-dimensional. Furthermore, if $\dim K = n$, then in view of (2.13) and (2.18), the integral in (5.42) may equivalently be taken over $\boldsymbol{\alpha}^*_K(E) \cap N(K,o)^*$ or over $\boldsymbol{\alpha}^*_K(E) \cap \operatorname{int} N(K,o)^*$. For $\mathscr{H}^{n-1}$-almost all $u \in \boldsymbol{\alpha}^*_K(E \setminus N(K,o))$, the vector $\alpha_K(u)$ is well defined and $\alpha_K(u) \notin N(K,o)$, hence $h_K(\alpha_K(u)) > 0$ and $\psi(h_K(\alpha_K(u))) > 0$.

As before, if $\psi \equiv 1$, we often write $\widetilde{C}_G(K,\cdot)$ instead of $\widetilde{C}_{G,\psi}(K,\cdot)$. The integral in (5.42) should be considered as 0 if $\boldsymbol{\alpha}^*_K(E \setminus N(K,o)) = \emptyset$, in particular for $E \subset S^{n-1} \cap N(K,o)$. In other words, $\widetilde{C}_{G,\psi}(K,E) = 0$ for each Borel set $E \subset S^{n-1} \cap N(K,o)$.

When $o \in \operatorname{int} K$, we have $N(K,o) = \{o\}$ and hence $E \setminus N(K,o) = E$, so (5.42) agrees with (5.6). Moreover, if $G : (0,\infty) \times S^{n-1} \to (0,\infty)$ and $\psi : (0,\infty) \to (0,\infty)$ are continuous and $G_t(t,u) = \partial G(t,u)/\partial t$ is such that $u \mapsto G_t(\rho_K(u),u)$ is integrable on S^{n-1}, then we can extend G and ψ by setting $G(0,u) = 0$ for $u \in S^{n-1}$ and $\psi(0) = 0$ and the integral in (5.42) will exist for all $K \in \mathscr{K}^n_{(o)}$ and Borel sets $E \subset S^{n-1}$. Thus Definition 4.1.3 is subsumed under Definition 5.3.1.

Suppose that G and ψ are such that $\widetilde{C}_{G,\psi}(K,\cdot)$ is indeed a finite signed Borel measure on S^{n-1}. Then integrals with respect to $\widetilde{C}_{G,\psi}(K,\cdot)$ can be calculated as follows. For any bounded Borel function $g : S^{n-1} \to \mathbb{R}$, we have

$$\begin{aligned}
&\int_{S^{n-1}} g(u)\, d\widetilde{C}_{G,\psi}(K,u) \\
&\quad = \frac{1}{n} \int_{S^{n-1} \cap \operatorname{int} N(K,o)^*} g(\alpha_K(u))\, \frac{\rho_K(u)\, G_t(\rho_K(u),u)}{\psi(h_K(\alpha_K(u)))}\, du \qquad (5.43) \\
&\quad = \frac{1}{n} \int_{\partial K \setminus \Xi_K} g(\nu_K(x)) \frac{\langle x, \nu_K(x)\rangle}{\psi(\langle x, \nu_K(x)\rangle)}\, |x|^{1-n} G_t(|x|, \bar{x})\, dx. \qquad (5.44)
\end{aligned}$$

Indeed, it suffices to prove (5.43) for $g = \mathbf{1}_E$, where $E \subset S^{n-1}$ is a Borel set. If $\dim K \le n-1$, then all integrals are zero, so we can assume that $\dim K = n$. Then, using (2.17) and (2.18), we obtain

$$\begin{aligned}
\int_{S^{n-1}} \mathbf{1}_E(u)\, d\widetilde{C}_{G,\psi}(K,u) &= \widetilde{C}_{G,\psi}(K,E) \\
&= \frac{1}{n} \int_{\boldsymbol{\alpha}^*_K(E) \cap \operatorname{int} N(K,o)^*} \frac{\rho_K(u)\, G_t(\rho_K(u),u)}{\psi(h_K(\alpha_K(u)))}\, du \\
&= \frac{1}{n} \int_{S^{n-1} \cap \operatorname{int} N(K,o)^*} \mathbf{1}_E(\alpha_K(u))\, \frac{\rho_K(u)\, G_t(\rho_K(u),u)}{\psi(h_K(\alpha_K(u)))}\, du,
\end{aligned}$$

as required, thus proving (5.43). Now (2.16) and a standard change of variables (see, e.g., [3, (21)], (4.9), or [29, (2.30)]) gives (5.44). When ψ does not vanish or when $\mathscr{H}^{n-1}(\Xi_K)=0$, (5.44) becomes

$$\int_{S^{n-1}} g(u)\, d\widetilde{C}_{G,\psi}(K,u) = \frac{1}{n}\int_{\partial K} g(\nu_K(x))\frac{\langle x,\nu_K(x)\rangle}{\psi(\langle x,\nu_K(x)\rangle)}\,|x|^{1-n}G_t(|x|,\bar{x})\,dx, \qquad (5.45)$$

since it is easy to see that $\langle \nu_K(x),x\rangle=0$ for $x\in\Xi_K\cap\partial K$. We emphasize that these observations are made under the assumption that the integrals exist.

In Proposition 5.3.2 (iii) below, we will find use for a simplified version of (5.43) that holds when $\psi\equiv 1$ and $tG_t(t,u)=0$ at $t=0$ for $u\in S^{n-1}$. The latter of these two conditions simply means that $tG_t(t,u)\to 0$ as $t\to 0+$ for $u\in S^{n-1}$. The apparently weaker condition that $\lim_{t\to 0+} tG_t(t,u)$ exists for $u\in S^{n-1}$ is in fact equivalent. Indeed, suppose that $u\in S^{n-1}$ and $tG_t(t,u)\to c\neq 0$ as $t\to 0+$. If $c>0$, there exist $0<c_1\le c_2<\infty$ and $t_0>0$ such that $0<c_1\le tG_t(t,u)\le c_2$ for $t\in(0,t_0]$. If $s\in(0,t_0]$, we can divide by t and integrate from s to t_0 to obtain

$$G(t_0,u)-c_2\ln t_0+c_2\ln s\le G(s,u)\le G(t_0,u)-c_1\ln t_0+c_1\ln s.$$

But then $G(s,u)\to-\infty$ as $s\to 0+$, a contradiction. If $c<0$, there exist $c_1\le c_2<0$ and t_0 as above and a similar argument leads to $G(s,u)\to\infty$ as $s\to 0+$, again a contradiction.

The following proposition focuses on the case when $\psi\equiv 1$. In this case, provided $tG_t(t,u)=0$ at $t=0$ for $u\in S^{n-1}$, (5.43) simplifies to

$$\int_{S^{n-1}} g(u)\, d\widetilde{C}_G(K,u)=\frac{1}{n}\int_{S^{n-1}} g(\alpha_K(u))\,\rho_K(u)\,G_t(\rho_K(u),u)\,du \qquad (5.46)$$

for any bounded Borel function $g:S^{n-1}\to\mathbb{R}$, since the integral may be restricted to $\operatorname{int} N(K,o)^*$ due to (2.13) and (2.14).

Proposition 5.3.2. *Let $G:[0,\infty)\times S^{n-1}\to[0,\infty)$ and let $K\in\mathscr{K}_o^n$. The following statements hold.*

(i) *$\widetilde{C}_G(K,\cdot)$ is a finite signed measure on S^{n-1}.*

(ii) *Suppose that $t^{1-n}G_t(t,u)$ is continuous on $[0,\infty)\times S^{n-1}$, where the value of $t^{1-n}G_t(t,u)$ for each $u\in S^{n-1}$ at $t=0$ is taken to be the value of the limit as*

$t \to 0+$. Then $\widetilde{C}_G(K,\cdot)$ is absolutely continuous with respect to $S(K,\cdot)$.

(iii) *Suppose that $tG_t(t,u)$ is continuous on $[0,\infty)\times S^{n-1}$, where $tG_t(t,u)=0$ at $t=0$ for $u\in S^{n-1}$. If $K_i\in\mathscr{K}_o^n$ and $K_i\to K\in\mathscr{K}_o^n$ as $i\to\infty$, then $\widetilde{C}_G(K_i,\cdot)\to\widetilde{C}_G(K,\cdot)$ weakly as $i\to\infty$.*

Proof. (i) As was pointed out before, $\widetilde{C}_G(K,\cdot)$ is the zero measure if $\dim K\le n-1$. Hence let $\dim K=n$. The assumption on G_t ensures that with $\psi\equiv 1$, the integral in (5.42) exists. For the σ-additivity, it suffices to show that

$$\widetilde{C}_G(K,\cup_{i=1}^{\infty}E_i)=\sum_{i=1}^{\infty}\widetilde{C}_G(K,E_i), \tag{5.47}$$

for disjoint Borel sets $E_i\subset S^{n-1}$, $i\in\mathbb{N}$. To this end, observe that it has $\boldsymbol{\alpha}_K^*((\cup_{i=1}^{\infty}E_i)\setminus N(K,o))=\cup_{i=1}^{\infty}\boldsymbol{\alpha}_K^*(E_i\setminus N(K,o))$. From (5.42), we see that (5.47) will be proved if we can show that

$$\mathscr{H}^{n-1}\big(\boldsymbol{\alpha}_K^*(E_i\setminus N(K,o))\cap\boldsymbol{\alpha}_K^*(E_j\setminus N(K,o))\big)=0 \tag{5.48}$$

for $i\ne j$. To see this, note that since r_K is locally bi-Lipschitz on $S^{n-1}\cap\operatorname{int}N(K,o)^*$, we have

$$\mathscr{H}^{n-1}(r_K^{-1}(\partial K\setminus\operatorname{reg}K)\cap\operatorname{int}N(K,o)^*)=0.$$

Using this and (2.18), we get

$$\begin{aligned}&\mathscr{H}^{n-1}\big(\boldsymbol{\alpha}_K^*(E_i\setminus N(K,o))\cap\boldsymbol{\alpha}_K^*(E_j\setminus N(K,o))\big)\\ =\;&\mathscr{H}^{n-1}\big(\boldsymbol{\alpha}_K^*(E_i)\cap\boldsymbol{\alpha}_K^*(E_j)\cap\operatorname{int}N(K,o)^*\cap r_K^{-1}(\operatorname{reg}K)\big).\end{aligned}$$

But the latter set is empty, because if it contained a point u, we would have

$$r_K(u)\in\boldsymbol{\nu}_K^{-1}(E_i)\cap\boldsymbol{\nu}_K^{-1}(E_j)\cap\operatorname{reg}K=\emptyset,$$

as $E_i\cap E_j=\emptyset$. This proves (5.48) and hence (5.47).

(ii) If $\operatorname{int}K=\emptyset$, then $\widetilde{C}_G(K,\cdot)=0$ and there is nothing to prove. Suppose that $\operatorname{int}K\ne\emptyset$. Let $E\subset S^{n-1}$ be a Borel set such that $S(K,E)=0$, let $g=\mathbf{1}_E$, and choose $R<\infty$ such that $K\subset RB^n$. By (5.45) with $\psi\equiv 1$, the continuity of $t^{1-n}G_t(t,u)$, and the fact that $\langle x, \nu_K(x)\rangle \le R$ for $x \in \partial K$, we obtain

$$\begin{aligned}
&\widetilde{C}_G(K,E)\\
&\quad= \frac{1}{n}\int_{\partial K} \mathbf{1}_E(\nu_K(x))\,\langle x, \nu_K(x)\rangle\,|x|^{1-n} G_t(|x|,\bar{x})\,dx\\
&\quad\le \frac{R}{n}\max\left\{t^{1-n}G_t(t,u) : (t,u)\in[0,R]\times S^{n-1}\right\}\mathscr{H}^{n-1}\left(\{x\in\partial K : \nu_K(x)\in E\}\right)\\
&\quad= 0,
\end{aligned}$$

as required.

(iii) The case when $K \in \mathscr{K}^n_{(o)}$ was proved in Proposition 4.3.1 (ii). First assume that $o \in \partial K$ and $\operatorname{int} K \neq \emptyset$. Let $g \in C(S^{n-1})$ and let

$$I_K(u) = g(\alpha_K(u))\,\rho_K(u)\,G_t(\rho_K(u),u)$$

be the integrand of the right-hand side of (5.46). If $u \in \operatorname{int} N(K,o)^*$, then $u \in \operatorname{int} N(K_i,o)^*$ for $i \ge i_u$ and $\rho_{K_i}(u) \to \rho_K(u)$ as $i \to \infty$. Let Z be the set consisting of those $u \in S^{n-1} \cap \operatorname{int} N(K,o)^*$ for which $\rho_K(u)u \notin \operatorname{reg} K$ and those $u \in S^{n-1} \cap \operatorname{int} N(K_i,o)^*$ for which $\rho_{K_i}(u)u \notin \operatorname{reg} K_i$ for some $i \in \mathbb{N}$. Then (2.8) yields $\mathscr{H}^{n-1}(Z) = 0$. Also, since $\alpha_{K_i}(u) \to \alpha_K(u)$ as $i \to \infty$ for $u \in \operatorname{int} N(K,o)^* \backslash Z$ (cf. [29, Lemma 2.2]), we have $I_{K_i}(u) \to I_K(u)$ as $i \to \infty$ for $u \in \operatorname{int} N(K,o)^* \setminus Z$.

On the other hand, if $u \in S^{n-1} \backslash N(K,o)^*$, then $\rho_K(u) = 0$ by (2.15) and $\rho_{K_i}(u) \to 0$ as $i \to \infty$ (as can be seen by a separation argument), and hence, using the assumption that $tG_t(t,u) = 0$ at $t = 0$ for $u \in S^{n-1}$, we have $I_{K_i}(u) \to 0$ as $i \to \infty$. Thus we have shown that $I_{K_i}(u) \to I_K(u)$ as $i \to \infty$ for $\mathscr{H}^{n-1}$-almost all $u \in S^{n-1}$. We also have $\sup\{|I_K(u)| : u \in S^{n-1}\}| < \infty$, by the continuity of $tG_t(t,u)$ on $[0,\infty) \times S^{n-1}$.

Using these facts, (5.46), and the dominated convergence theorem, we obtain

$$\begin{aligned}
\int_{S^{n-1}} g(u)\,d\widetilde{C}_G(K,u) &= \frac{1}{n}\int_{S^{n-1}} I_K(u)\,du\\
&= \lim_{i\to\infty}\frac{1}{n}\int_{S^{n-1}} I_{K_i}(u)\,du\\
&= \lim_{i\to\infty}\int_{S^{n-1}} g(u)\,d\widetilde{C}_G(K_i,u),
\end{aligned}$$

proving the result when $\operatorname{int} K \neq \emptyset$.

Now assume that $\operatorname{int} K = \emptyset$. Since g is continuous on S^{n-1}, it is bounded, so there is a $c > 0$ such that $|g(\alpha_{K_i}(u))| \le c$ for $u \in S^{n-1}$ and $i \in \mathbb{N}$. We apply Lemma 5.1.2 with $G(t,u)$ replaced by $t|G_t(t,u)|$ to obtain

$$\begin{aligned}
\limsup_{i\to\infty} \int_{S^{n-1}} & |g(\alpha_{K_i}(u))|\, \rho_{K_i}(u)\, |G_t(\rho_{K_i}(u),u)|\, du \\
&\le c \limsup_{i\to\infty} \int_{S^{n-1}} \rho_{K_i}(u)\, |G_t(\rho_{K_i}(u),u)|\, du \\
&\le c \int_{S^{n-1}} \rho_K(u)\, |G_t(\rho_K(u),u)|\, du = 0,
\end{aligned}$$

where we have used again the assumption that $tG_t(t,u) = 0$ at $t = 0$ for $u \in S^{n-1}$. This and (5.46) yield

$$\begin{aligned}
\lim_{i\to\infty} \int_{S^{n-1}} g(u)\, d\widetilde{C}_G(K_i,u) &= \int_{S^{n-1}} g(u)\, d\widetilde{C}_G(K,u) \\
&= 0,
\end{aligned}$$

completing the proof. □

Finally, we provide a generalization of uniqueness results for $\widetilde{C}_G(K,\cdot)$ in [69, Theorem 6.1], [75, Theorem 5.2], and [78, Theorem 3.1], with a simpler proof.

Theorem 5.3.3. *Let $G : (0,\infty) \times S^{n-1} \to (0,\infty)$ be continuous and such that G_t is continuous and negative on $(0,\infty) \times S^{n-1}$. Suppose that $tG_t(t,u)$ is strictly increasing on t for $u \in S^{n-1}$. If $K, L \in \mathscr{K}^n_{(o)}$ satisfy*

$$\widetilde{C}_G(K,\cdot) = \widetilde{C}_G(L,\cdot),$$

then $K = L$.

Proof. Suppose that $K \neq L$. Then we may assume that $L \not\subset K$. Let $E = \{v \in S^{n-1} : h_L(v) > h_K(v)\} \neq \emptyset$. We apply Lemma 3.5.2 with $K' = L$ and $L = K$. Using (5.46) and the fact that $G_t < 0$, Lemma 3.5.2 (a), the assumption that $tG_t(t,u)$ is strictly increasing on t for $u \in S^{n-1}$, and Lemma 3.5.2 (c) together with (2.17), we obtain

$$
\begin{aligned}
\widetilde{C}_G(K,E) = \widetilde{C}_G(L,E) & \\
&= \int_{S^{n-1}} \mathbf{1}_E(\alpha_L(u))\, \rho_L(u)\, G_t(\rho_L(u),u)\, du \\
&\geq \int_{S^{n-1}} \mathbf{1}_E(\alpha_L(u))\, \rho_K(u)\, G_t(\rho_K(u),u)\, du \\
&\geq \int_{S^{n-1}} \mathbf{1}_{\boldsymbol{\alpha}_K^*(E)}(u)\, \rho_K(u)\, G_t(\rho_K(u),u)\, du \\
&= \widetilde{C}_G(K,E).
\end{aligned}
$$

If $\mathscr{H}^{n-1}(\boldsymbol{\alpha}_K^*(E) \setminus \boldsymbol{\alpha}_L^*(E)) > 0$, then the second inequality is strict. If $\mathscr{H}^{n-1}(\boldsymbol{\alpha}_K^*(E) \setminus \boldsymbol{\alpha}_L^*(E)) = 0$, then Lemma 3.5.2 (d) implies that $\mathscr{H}^{n-1}(\boldsymbol{\alpha}_L^*(E)) > 0$ and therefore the first inequality is strict. Thus, in any case we arrive at a contradiction. $\square$

In the following, we provide solutions to the Minkowski type problems in terms of general measures and even measures.

5.4 Minkowski problems for general measures

In view of (5.42), one sees that

$$\frac{d\widetilde{C}_{G,\psi}(K,u)}{d\widetilde{C}_G(K,u)} = \frac{1}{\psi(h_K(u))}.$$

We consider the following Minkowski-type problem.

Problem 5.4.1. *For which nonzero finite Borel measures μ on S^{n-1} and continuous functions $G : (0,\infty) \times S^{n-1} \to (0,\infty)$ and $\psi : (0,\infty) \to (0,\infty)$ do there exist $\tau \in \mathbb{R}$ and $K \in \mathscr{K}_o^n$ with* $\operatorname{int} K \neq \emptyset$ *such that*

$$\mu = \tau\, \widetilde{C}_{G,\psi}(K,\cdot) \quad \text{and/or} \quad (\psi \circ h_K)\mu = \tau\, \widetilde{C}_G(K,\cdot)?$$

For our contribution to this problem, we need the following lemma. It is essentially known (see e.g., [24, 27, 44, 77]), but we provide an explicit dependence of R on μ that will be needed in the proof of Theorems 5.4.3 and 5.5.3.

Lemma 5.4.2. *Let μ be a finite Borel measure on S^{n-1} not concentrated on any closed hemisphere and let $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ be continuous and* *strictly increasing. Suppose that $K \in \mathscr{K}^n_{(o)}$ satisfies $\|h_K\|_{\mu,\varphi} = 1$ (see (5.13)). Then there is an $R = R(\mu, \varphi) > 0$ such that $K \subset RB^n$.*

Proof. There is a $\delta = \delta(\mu) > 0$ such that

$$\int_{S^{n-1}} \langle u, v\rangle_+ \, d\mu(u) \geq \delta \, \mu(S^{n-1}) \tag{5.49}$$

for $v \in S^{n-1}$, since the integral on the left is continuous in v on S^{n-1} and μ is not concentrated on a closed hemisphere. Let $K \in \mathscr{K}^n_{(o)}$ satisfy $\|h_K\|_{\mu,\varphi} = 1$ and let $rv_0 \in K$, where $r \geq 0$ and $v_0 \in S^{n-1}$. Then $[o, rv_0] \subset K$ implies that $h_K(u) \geq r\langle u, v_0\rangle_+$ for $u \in S^{n-1}$, so using (5.3), (5.13), and our assumptions on φ, we obtain

$$\varphi(1)\, \mu(S^{n-1}) = \int_{S^{n-1}} \varphi(h_K(u))\, d\mu(u) \geq \int_{\Sigma_{\delta/2}(v_0)} \varphi(r\delta/2)\, d\mu(u) = \varphi(r\delta/2)\mu\left(\Sigma_{\delta/2}(v_0)\right). \tag{5.50}$$

Splitting the integral in (5.49) with $v = v_0$ into one over $\Sigma_{\delta/2}(v_0)$ and one over $S^{n-1} \setminus \Sigma_{\delta/2}(v_0)$, and using the obvious bounds for the integrand in these cases, we get

$$\delta\, \mu(S^{n-1}) \leq \mu\left(\Sigma_{\delta/2}(v_0)\right) + (\delta/2)\mu(S^{n-1}),$$

and therefore $\mu\left(\Sigma_{\delta/2}(v_0)\right) \geq (\delta/2)\mu(S^{n-1})$. Substituting this into (5.50), we see that $r \leq R$, where

$$R = (2/\delta)\varphi^{-1}(2\varphi(1)/\delta), \tag{5.51}$$

proving that $K \subset RB^n$. □

We can now state the first main theorem of this section, whose hypotheses allow $\psi(t) = t^p$ for $p > 1$ and $G(t, u) = t^q$ for $q > 0$, for example.

Theorem 5.4.3. *Let $G : [0, \infty) \times S^{n-1} \to [0, \infty)$ be continuous and such that $G_t > 0$ on $(0, \infty) \times S^{n-1}$ and $tG_t(t, u)$ is continuous on $[0, \infty) \times S^{n-1}$, where $tG_t(t, u) = 0$ at $t = 0$ for $u \in S^{n-1}$. Suppose that $\psi : (0, \infty) \to (0, \infty)$ is continuous and such that $\lim_{t\to 0+} \psi(t)/t = 0$ and (5.7) holds. Then the following statements are equivalent:*

(i) *The finite Borel measure μ on S^{n-1} is not concentrated on any closed hemisphere.*

(ii) *There exist* $K \in \mathscr{K}_o^n$ *with* $\operatorname{int} K \neq \emptyset$ *and* $\tau > 0$ *such that* $\mathscr{H}^{n-1}(\Xi_K) = 0$ *and*

$$(\psi \circ h_K)\mu = \tau\, \widetilde{C}_G(K, \cdot), \tag{5.52}$$

where

$$\tau = \frac{1}{\widetilde{C}_G(K, S^{n-1})} \int_{S^{n-1}} \psi(h_K(u))\, d\mu(u). \tag{5.53}$$

Proof. Assume that (i) is true. Following the proof of [59, Theorem 8.2.2], we can construct nonzero finite discrete Borel measures μ_j, $j \in \mathbb{N}$, such that $\mu_j \to \mu$ weakly as $j \to \infty$ and such that there is a $\delta > 0$ so that (5.49) holds for μ and also with μ replaced by μ_j, $j \in \mathbb{N}$. In particular, μ_j, $j \in \mathbb{N}$, is not concentrated on any closed hemisphere. By Theorem 5.2.4, for each j, there exists a convex polytope $P_j \in \mathscr{K}_{(o)}^n$ such that $\mu_j = \tau_j\, \widetilde{C}_{G,\psi}(P_j, \cdot)$, where

$$\tau_j = \frac{\mu_j(S^{n-1})}{\widetilde{C}_{G,\psi}(P_j, S^{n-1})} = \frac{1}{\widetilde{C}_G(P_j, S^{n-1})} \int_{S^{n-1}} \psi(h_{P_j}(u))\, d\mu_j(u). \tag{5.54}$$

Moreover, from (5.18), we have

$$\widetilde{V}_G(P_j) \geq \widetilde{V}_G(B^n) > \int_{S^{n-1}} G(0, u)\, du \tag{5.55}$$

for $j \in \mathbb{N}$. Theorem 5.2.4 also gives $\|h_{P_j}\|_{\mu_j,\varphi} = 1$, where φ is defined by (5.14). By Lemma 5.4.2, we have $P_j \subset RB^n$ for $j \in \mathbb{N}$, where R is given by (5.51). Then Blaschke selection theorem implies that $P_j \to K$ for some $K \in \mathscr{K}_o^n$, as $j \to \infty$, in the Hausdorff metric. By Lemma 5.1.2, $\lim_{j\to\infty} \widetilde{V}_G(P_j) = \widetilde{V}_G(K)$. This and (5.55) imply that

$$\widetilde{V}_G(K) \geq \widetilde{V}_G(B^n) > \int_{S^{n-1}} G(0, u)\, du.$$

In view of (5.2), this shows that $\operatorname{int} K \neq \emptyset$.

By Proposition 5.3.2 (iii) and the fact that $\operatorname{int} K \neq \emptyset$, we have $\widetilde{C}_G(P_j, \cdot) \to \widetilde{C}_G(K, \cdot)$ weakly as $j \to \infty$ and hence

$$\widetilde{C}_G(P_j, S^{n-1}) \to \widetilde{C}_G(K, S^{n-1}) > 0$$

as $j \to \infty$. Our assumption that $\lim_{t\to 0+} \psi(t)/t = 0$ shows that $\psi(0) = 0$ provides a continuous extension of ψ to $[0, \infty)$. This and the uniform convergence of h_{P_j} to h_K

imply that $\psi(h_{P_j}) \to \psi(h_K)$ uniformly as $j \to \infty$. Now from the weak convergence of μ_j to μ and of $\widetilde{C}_G(P_j, \cdot)$ to $\widetilde{C}_G(K, \cdot)$, along with $\mu_j = \tau_j \, \widetilde{C}_{G,\psi}(P_j, \cdot)$, which can be expressed in the form

$$(\psi \circ h_{P_j})\mu_j = \tau_j \, \widetilde{C}_G(P_j, \cdot),$$

with τ_j as in (5.54), we conclude that (5.52) holds, with τ given by (5.53). That τ is finite is a direct consequence of the continuity of ψ.

Since $G_t > 0$, we have $\tau \geq 0$. We claim that $\tau > 0$. To see this, use $\operatorname{int} K \neq \emptyset$ to choose $v \in S^{n-1}$ such that $\rho_K(v) > 0$. As μ is not concentrated on any closed hemisphere, the monotone convergence theorem yields

$$\lim_{j\to\infty} \int_{\Sigma_{1/j}(v)} \langle u, v\rangle \, d\mu(u) = \int_{\{u \in S^{n-1} : \langle u,v\rangle > 0\}} \langle u, v\rangle \, d\mu(u) > 0,$$

where $\Sigma_\varepsilon(v)$ is defined for $\varepsilon \in (0,1)$ by (5.3). Hence a $j_0 \geq 2$ exists such that

$$\mu\left(\Sigma_{1/j_0}(v)\right) \geq \int_{\Sigma_{1/j_0}(v)} \langle u, v\rangle \, d\mu(u) > 0.$$

We use this, (5.53), and the fact that

$$h_K(u) \geq \rho_K(v)\langle u, v\rangle \geq \rho_K(v)/j_0$$

for $u \in \Sigma_{1/j_0}(v)$ to obtain

$$\begin{aligned} \tau &\geq \frac{1}{\widetilde{C}_G(K, S^{n-1})} \int_{\Sigma_{1/j_0}(v)} \psi(h_K(u)) \, d\mu(u) \\ &\geq \min\{\psi(t) : t \in [\rho_K(v)/j_0, R]\} \, \frac{\mu\left(\Sigma_{1/j_0}(v)\right)}{\widetilde{C}_G(K, S^{n-1})} > 0, \end{aligned}$$

proving our claim and (5.53).

It remains to be shown that $\mathscr{H}^{n-1}(\Xi_K) = 0$. To see this, suppose to the contrary that $\mathscr{H}^{n-1}(\Xi_K) \neq 0$. Then (see (2.16)) we have $o \in \partial K$. Since $\tau > 0$, we can, in view of (5.54) and the fact that $\mu_j \to \mu$ and $\tau_j \to \tau$ as $j \to \infty$, assume without loss of generality that

$$\widetilde{C}_{G,\psi}(P_j, S^{n-1}) \leq \frac{2\mu(S^{n-1})}{\tau} < \infty \tag{5.56}$$

for $j \in \mathbb{N}$, where P_j is as above. Let $z \in \operatorname{int} K$ be fixed. For $E \subset \partial K$, define

$$\sigma(E) = \{z + \lambda(x - z) : \quad x \in E \ \text{ and } \ \lambda > 0\}.$$

Let $a, b > 0$, and let $\varepsilon > 0$. From statements (a'), (b'), and (c') in the proof of [3, Lemma 4.4], we know (recall that all $P_j \in \mathscr{K}^n_{(o)}$) that there exist $U \subset \partial K$ and $j_\varepsilon \in \mathbb{N}$ such that for $j \geq j_\varepsilon$, one has $a \leq |x| \leq R$ for all $x \in \sigma(U) \cap \partial P_j$, $\mathscr{H}^{n-1}(\sigma(U) \cap \partial P_j) \geq b/2$, and $h_{P_j}(u) \leq 2\varepsilon$ if $u \in S^{n-1}$ is an outer normal vector at $x \in \sigma(U) \cap \partial P_j$. Using these facts, (5.45) for P_j with $o \in \operatorname{int} P_j$, and the continuity of G_t on $(0, \infty) \times S^{n-1}$, we obtain, for $j \geq j_\varepsilon$,

$$\begin{aligned} \widetilde{C}_{G,\psi}(P_j, S^{n-1}) &= \frac{1}{n} \int_{\partial P_j} \frac{\langle x, \nu_{P_j}(x) \rangle}{\psi(\langle x, \nu_{P_j}(x) \rangle)} \, |x|^{1-n} G_t(|x|, \bar{x}) \, dx \\ &\geq \frac{1}{n} \int_{\sigma(U) \cap \partial P_j} \frac{\langle x, \nu_{P_j}(x) \rangle}{\psi(\langle x, \nu_{P_j}(x) \rangle)} \, |x|^{1-n} G_t(|x|, \bar{x}) \, dx \\ &\geq \frac{bc\, d_\varepsilon}{2n}, \end{aligned} \tag{5.57}$$

where $c = \min\{t^{1-n} G_t(t, u) : (t, u) \in [a, R] \times S^{n-1}\} > 0$ and $d_\varepsilon = \inf\{t/\psi(t) : t \in (0, 2\varepsilon]\}$. Since $\lim_{t \to 0+} \psi(t)/t = 0$, (5.56) and (5.57) yield

$$\infty > \frac{2\mu(S^{n-1})}{\tau} \geq \widetilde{C}_{G,\psi}(P_j, S^{n-1}) \geq \lim_{\varepsilon \to 0+} \frac{bc\, d_\varepsilon}{2n} = \infty.$$

This contradiction proves that $\mathscr{H}^{n-1}(\Xi_K) = 0$. Therefore (ii) holds.

Now assume that (ii) is true. We claim that $\widetilde{C}_G(K, \cdot)$ is not concentrated on any closed hemisphere; by (5.52), this will yield (i). To prove the claim, we must show that (2.6) holds when μ there is replaced by $\widetilde{C}_G(K, \cdot)$. If this is not true, there is a $v_0 \in S^{n-1}$ such that

$$\int_{S^{n-1}} \langle u, v_0 \rangle_+ \, d\widetilde{C}_G(K, u) = \frac{1}{n} \int_{S^{n-1} \cap \operatorname{int} N(K,o)^*} \langle \alpha_K(u), v_0 \rangle_+ \, \rho_K(u) \, G_t(\rho_K(u), u) \, du = 0, \tag{5.58}$$

where the first equality is due to (5.43) with $\psi \equiv 1$. By (2.14), we have $\rho_K(u) > 0$ if $u \in S^{n-1} \cap \operatorname{int} N(K, o)^*$. It follows from (5.58) that $\langle \alpha_K(u), v_0 \rangle_+ = 0$ for $\mathscr{H}^{n-1}$-almost all $u \in S^{n-1} \cap \operatorname{int} N(K, o)^*$. Define $X = \Xi_K \cup \sigma_K \cup Y \subset \partial K$, where

$$Y = \left\{ r_K(u) = \rho_K(u) u : u \in S^{n-1} \cap \operatorname{int} N(K, o)^* \text{ and } \langle \alpha_K(u), v_0 \rangle_+ \neq 0 \right\}.$$

Then the observations just made imply that $\mathscr{H}^{n-1}(Y) = 0$, and since $\mathscr{H}^{n-1}(\Xi_K) = 0$ by assumption and (2.8) holds, it follows that $\mathscr{H}^{n-1}(X) = 0$. Moreover, for $x = r_K(u) \in \partial K \setminus X$, we have $\langle \alpha_K(u), v_0\rangle_+ = \langle \nu_K(r_K(u)), v_0\rangle_+ = 0$ and hence

$$\langle \nu_K(x), v_0\rangle \leq 0. \tag{5.59}$$

Next, note that if $A \subset \operatorname{reg} K = \partial K \setminus \sigma_K$ and $\mathscr{H}^{n-1}(\partial K \setminus A) = 0$, then

$$K = \bigcap_{x \in A} H^-(K, x), \tag{5.60}$$

where $H^-(K, x)$ is the unique supporting halfspace of K containing K whose bounding hyperplane $H(K, x)$ passes through x. Indeed, K is contained in the set on the right-hand side of (5.60). For the reverse inclusion, let $z \in \mathbb{R}^n \setminus K$. Choose a ball $B \subset \operatorname{int} K$. Then $\operatorname{conv}(\{z\} \cup B) \cap \partial K$ is open relative to ∂K and since it has positive $\mathscr{H}^{n-1}$-measure, it must contain an $x \in A$. Then $B \subset \operatorname{int} H^-(K, x)$ and therefore $z \notin H^-(K, x)$. This proves (5.60).

The representation (5.60) immediately implies that the positive hull of $\{\nu_K(x) : x \in A\}$ is $\mathbb{R}^n$. Noting that $\partial K \setminus X \subset \operatorname{reg} K$ by the definition of X, we see that when $A = \partial K \setminus X$, this contradicts (5.59) and completes the proof. $\square$

It is not true in general that the set K in Theorem 5.4.3 (ii) satisfies $K \in \mathscr{K}^n_{(o)}$. In fact this is already the situation for the L_p Minkowski problem, corresponding to $G(t, u) = t^n$ and $\psi(t) = t^p$ for $p > 1$; see [33, Example 4.1]. However, additional assumptions can be imposed ensuring that we can find a solution K of (5.52) with $K \in \mathscr{K}^n_{(o)}$. For example, suppose that $t/\psi(t)$ is decreasing on $(0, 1]$ and there exists $c_0 > 0$ such that

$$\inf\left\{\frac{t\, G_t(t, u)}{\psi(t)} : (t, u) \in (0, 1] \times S^{n-1}\right\} > nc_0. \tag{5.61}$$

We show that it is not possible to have $o \in \partial K$ and $\operatorname{int} K \neq \emptyset$. Using (5.57), $\langle x, \nu_{P_j}(x)\rangle \leq |x|$ for $j \in \mathbb{N}$ and $x \in \partial P_j$, the fact that $\psi(t)/t$ is increasing on $(0, 1]$, and (5.61), we obtain

$$\begin{aligned}\frac{2\mu(S^{n-1})}{\tau} &\geq \frac{1}{n}\int_{B^n\cap\partial P_j}\frac{\langle x,\nu_{P_j}(x)\rangle}{\psi(\langle x,\nu_{P_j}(x)\rangle)}\,|x|^{1-n}G_t(|x|,\bar{x})\,dx\\ &\geq \frac{1}{n}\int_{B^n\cap\partial P_j}\frac{|x|}{\psi(|x|)}\,|x|^{1-n}G_t(|x|,\bar{x})\,dx\\ &\geq c_0\int_{B^n\cap\partial P_j}|x|^{1-n}\,dx.\end{aligned}$$

The argument then follows directly from [3, (55)-(57)]. In particular, we can find $v\in S^{n-1}$, $c_1>0$, and $0<r_0<r_1<1$ such that

$$\frac{2\mu(S^{n-1})}{\tau}\geq c_0\int_{B^n\cap\partial P_j}|x|^{1-n}\,dx\geq c_1\int_{B(r_1)\setminus B(r_0)}|x|^{1-n}\,dx>\frac{2\mu(S^{n-1})}{\tau},$$

where $B(r)=rB^n\cap v^{\perp}$. This contradiction proves that $K\in\mathscr{K}^n_{(o)}$.

Instead of assuming the monotonicity of $\psi(t)/t$, one can assume that there exists an $\alpha\geq n-1$ such that

$$\inf\left\{t^{1-n}\,G_t(t,u):(t,u)\in(0,1]\times S^{n-1}\right\}>0\quad\text{and}\quad\inf_{t\in(0,1]}\frac{t^{1+\alpha}}{\psi(t)}>0.$$

Indeed, by (5.57), we then have

$$\begin{aligned}\frac{2\mu(S^{n-1})}{\tau} &\geq \frac{1}{n}\int_{B^n\cap\partial P_j}\frac{\langle x,\nu_{P_j}(x)\rangle}{\psi(\langle x,\nu_{P_j}(x)\rangle)}\,|x|^{1-n}G_t(|x|,\bar{x})\,dx\\ &\geq c_2\int_{B^n\cap\partial P_j}\langle x,\nu_{P_j}(x)\rangle^{-\alpha}\,dx,\end{aligned}$$

for some $c_2>0$. It then follows directly from the arguments on [33, p. 713] that $o\in\operatorname{int}K$ and hence $K\in\mathscr{K}^n_{(o)}$.

The following result provides a variant of Theorem 4.3.3, not requiring the condition (5.40) but with a weak additional growth condition at 0 on ψ (see the discussion after Theorem 5.2.5). The hypotheses allow $\psi(t)=t^p$ for $p>0$ and $G(t,u)=t^q$ for $q<0$, for example.

Theorem 5.4.4. *Let $G:(0,\infty)\times S^{n-1}\to(0,\infty)$ be continuous and such that G_t is continuous and negative on $(0,\infty)\times S^{n-1}$. Let $0<\varepsilon_0<1$ and suppose that* (5.4) *holds for $v\in S^{n-1}$. Suppose that $\psi:(0,\infty)\to(0,\infty)$ is continuous,* (5.7) *holds, and that φ is finite when defined by* (5.14)*. Then the following statements are equivalent:*

(i) *The finite Borel measure μ on S^{n-1} is not concentrated on any closed hemisphere.*

(ii) *There exist $K \in \mathscr{K}^n_{(o)}$ and $\tau < 0$ such that*

$$\mu = \tau\, \widetilde{C}_{G,\psi}(K, \cdot), \tag{5.62}$$

where

$$\tau = \frac{\mu(S^{n-1})}{\widetilde{C}_{G,\psi}(K, S^{n-1})}. \tag{5.63}$$

Proof. Assume that (i) is true. Define φ as in (5.14). As at the beginning of the proof of Theorem 5.4.3, but using Theorem 5.2.5 instead of Theorem 5.2.4, we can find nonzero finite discrete Borel measures μ_j, $j \in \mathbb{N}$, not concentrated on any closed hemisphere, such that $\mu_j \to \mu$ weakly as $j \to \infty$, and convex polytopes $P_j \in \mathscr{K}^n_{(o)}$ such that $\mu_j = \tau_j\, \widetilde{C}_{G,\psi}(P_j, \cdot)$, where (in view of (5.25)) τ_j satisfies (5.54) and $\|h_{P_j}\|_{\mu_j,\varphi} = 1$ for $j \in \mathbb{N}$. From the latter property and Lemma 5.4.2, it follows as in the proof of Theorem 5.4.3 that $(P_j)_{j\in\mathbb{N}}$ is bounded. Hence, we can extract a subsequence that converges to $K \in \mathscr{K}^n_o$.

Next, we show that $o \in \operatorname{int} K$. In fact, if $o \in \partial K$, we can apply Lemma 5.1.3 to get $\lim_{j\to\infty} \widetilde{V}_G(P_j) = \infty$. However, since P_j corresponds to $P(z^0)$ in Theorem 5.2.5, (5.39) implies that

$$\widetilde{V}_G(P_j) \le \widetilde{V}_G(B^n) < \infty$$

for all $j \in \mathbb{N}$, a contradiction.

Then (5.62) and (5.63) follow from the weak convergence of μ_j to μ and of $\widetilde{C}_{G,\psi}(P_j, \cdot)$ to $\widetilde{C}_{G,\psi}(K, \cdot)$, the latter a consequence of Proposition 4.3.1 (ii). In particular, we use that $\widetilde{C}_{G,\psi}(P_j, S^{n-1}) \to \widetilde{C}_{G,\psi}(K, S^{n-1}) \in (0, \infty)$ to ensure the convergence of $(\tau_j)_{j\in\mathbb{N}}$. Suppose that (ii) holds. By Proposition 4.3.1 (iii), $\widetilde{C}_{G,\psi}(K, \cdot)$ is not concentrated on any closed hemisphere, so by (5.62), this is also the case for μ. □

The final result in this section addresses the uniqueness problem related to Theorem 5.4.4 and generalizes and extends [54, Theorem 8.3]. It can be applied, for example, when $G(t, u) = t^q$, $q \neq 0$, and $\psi(s) = s^p$ with $q < p$. Note that when $\psi \equiv 1$ and $G_t < 0$, the result holds for general $K, K' \in \mathscr{K}^n_{(o)}$ by Theorem 5.3.3, since the assumption there that $tG_t(t, u)$ is strictly increasing on t for $u \in S^{n-1}$ implies the second inequality in (5.64). We do not know if the result holds for general ψ and general $K, K' \in \mathscr{K}^n_{(o)}$.

Theorem 5.4.5. *Let $G : (0,\infty) \times S^{n-1} \to (0,\infty)$ and $\psi : (0,\infty) \to (0,\infty)$ be continuous. Suppose that $G_t > 0$ (or $G_t < 0$) on $(0,\infty) \times S^{n-1}$ and that if*

$$\frac{G_t(t,u)}{\psi(s)} \geq \frac{\lambda G_t(\lambda t, u)}{\psi(\lambda s)} \quad \textit{(or} \quad \frac{G_t(t,u)}{\psi(s)} \leq \frac{\lambda G_t(\lambda t, u)}{\psi(\lambda s)}, \quad \textit{respectively)} \tag{5.64}$$

for some $\lambda, s, t > 0$ and $u \in S^{n-1}$, then $\lambda \geq 1$. If $K, K' \in \mathscr{K}^n_{(o)}$ are both polytopes or both have support functions in C^2 and $\widetilde{C}_{G,\psi}(K,\cdot) = \widetilde{C}_{G,\psi}(K',\cdot)$, then $K = K'$.

Proof. Suppose that $K, K' \in \mathscr{K}^n_{(o)}$ are such that $\widetilde{C}_{G,\psi}(K,\cdot) = \widetilde{C}_{G,\psi}(K',\cdot)$ and $K \neq K'$. Then we can assume without loss of generality that $K \not\subset K'$, so there is a maximal $\lambda < 1$ such that $\lambda K \subset K'$.

Consider first the case when K and K' are polytopes. By Lemma 5.2.2, the facets of K and K' have the same outer unit normal vectors, $u_1, \dots, u_m$, say, and from (5.9) and (5.11), we have

$$\widetilde{C}_{G,\psi}(K,\cdot) = \widetilde{C}_{G,\psi}(K',\cdot) = \sum_{i=1}^m \gamma_i \delta_{u_i},$$

where

$$\gamma_i = \int_{\tilde{\pi}(F(K,u_i))} \frac{\rho_K(u)\, G_t(\rho_K(u),u)}{n\psi(h_K(u_i))}\, du = \int_{\tilde{\pi}(F(K',u_i))} \frac{\rho_{K'}(u)\, G_t(\rho_{K'}(u),u)}{n\psi(h_{K'}(u_i))}\, du. \tag{5.65}$$

Since the facets of λK and K' also have the same outer unit normal vectors and λ is maximal, at least one facet of λK is contained in a facet of K'. If this facet has outer unit normal vector u_i, then

$$h_{\lambda K}(u_i) = h_{K'}(u_i), \qquad \tilde{\pi}(F(K,u_i)) = \tilde{\pi}(F(\lambda K,u_i)) \subset \tilde{\pi}(F(K',u_i)), \tag{5.66}$$

and

$$\rho_{\lambda K}(u) = \rho_{K'}(u) \quad \text{for} \quad u \in \tilde{\pi}(F(K,u_i)). \tag{5.67}$$

If $G_t > 0$ (the argument when $G_t < 0$ is similar), we conclude from (5.65), (5.66), and (5.67) that

$$\int_{\tilde{\pi}(F(K,u_i))} \frac{\rho_K(u)\, G_t(\rho_K(u),u)}{n\psi(h_K(u_i))}\, du \geq \int_{\tilde{\pi}(F(K,u_i))} \frac{\rho_{\lambda K}(u)\, G_t(\rho_{\lambda K}(u),u)}{n\psi(h_{\lambda K}(u_i))}\, du.$$

Since $\mathscr{H}^{n-1}(\tilde{\pi}(F(K,u_i)) > 0$, there is a $u \in \tilde{\pi}(F(K,u_i))$ such that

$$\frac{\rho_K(u)\,G_t(\rho_K(u),u)}{n\psi(h_K(u_i))} \geq \frac{\rho_{\lambda K}(u)\,G_t(\rho_{\lambda K}(u),u)}{n\psi(h_{\lambda K}(u_i))},$$

that is,

$$\frac{G_t(\rho_K(u),u)}{\psi(h_K(u_i))} \geq \frac{\lambda\,G_t(\lambda\rho_K(u),u)}{\psi(\lambda h_K(u_i))}.$$

Now the first inequality in (5.64) with $s = h_K(u_i)$ and $t = \rho_K(u)$ yields $\lambda \geq 1$, a contradiction. This completes the proof for when K and K' are polytopes.

For the other case, note firstly that if $L \in \mathscr{K}^n_{(o)}$ and $h_L \in C^2$, then $S(L,\cdot)$ is absolutely continuous with respect to $\mathscr{H}^{n-1}$ with continuous density $R(L,\cdot)$, where $R(L,u)$ is the product of the principal radii of curvature of L at $u \in S^{n-1}$. (This is well known when L is of class C^2_+; see, for example, [59, (4.26), p. 217]. When $h_L \in C^2$, one can observe that [4, Lemma 5.1] implies that [32, Theorem 3.7(c)] holds, and then [32, Theorem 3.7(a)] yields the absolute continuity of $S(L,\cdot)$. The form of the density is then given by [31, Theorem 3.5].) Let $K, K' \in \mathscr{K}^n_{(o)}$ and $h_K, h_{K'} \in C^2$. Using (4.10), we obtain

$$\begin{aligned}&\frac{h_K(u)\,|\nabla h_K(u)|^{1-n}\,G_t\left(|\nabla h_K(u)|, \nabla h_K(u)/|\nabla h_K(u)|\right)R(K,u)}{\psi(h_K(u))}\\ =\;&\frac{h_{K'}(u)\,|\nabla h_{K'}(u)|^{1-n}\,G_t\left(|\nabla h_{K'}(u)|, \nabla h_{K'}(u)/|\nabla h_{K'}(u)|\right)R(K',u)}{\psi(h_{K'}(u))}\end{aligned} \tag{5.68}$$

for all $u \in S^{n-1}$, since both sides of (5.68) are continuous functions. Since $\lambda K \subset K'$ and $\lambda < 1$ is maximal, there exists $u_0 \in S^{n-1}$ such that $h_{\lambda K}(u_0) = h_{K'}(u_0)$ and $\nabla h_{\lambda K}(u_0) = \nabla h_{K'}(u_0)$, i.e., λK and K' have a common boundary point with common outer unit normal vector u_0.

We claim that

$$R(K',u_0) \geq R(\lambda K,u_0) = \lambda^{n-1}R(K,u_0). \tag{5.69}$$

It suffices to prove the inequality, since the equality follows by homogeneity. Let $u = u_0 + av$, where $a > 0$ and $v \in S^{n-1}$. For $L \in \mathscr{K}^n_{(o)}$ with $h_L \in C^2$, and $u \in S^{n-1}$, let $d^2h_L[u]$ denote the second differential of h_L at u, considered as a bilinear form on $\mathbb{R}^n$. Since $h_{\lambda K} \leq h_{K'}$, $h_{\lambda K}(u_0) = h_{K'}(u_0)$, and $\nabla h_{\lambda K}(u_0) = \nabla h_{K'}(u_0)$, we may apply the first displayed equation in [59, p. 31, Note 3] (with $f = h_L$, $Af(x) = d^2h_L[x]$,

$x = u_0$, and $y = u$, for $L = \lambda K$ and $L = K'$), to obtain

$$\frac{1}{2}d^2h_{\lambda K}[u_0](av,av) + r_{\lambda K}(u_0,a)a^2 \le \frac{1}{2}d^2h_{K'}[u_0](av,av) + r_{K'}(u_0,a)a^2,$$

where $r_{\lambda K}(u_0,a), r_{K'}(u_0,a) \to 0$ as $a \to 0+$. Dividing by a^2 letting $a \to 0+$, we get $d^2h_{\lambda K}[u_0](v,v) \le d^2h_{K'}[u_0](v,v)$ for $v \in S^{n-1}$. We write $d^2h_{\lambda K}[u_0]|u_0^\perp$ and $d^2h_{K'}[u_0]|u_0^\perp$ for the symmetric, positive semidefinite linear maps from $u_0^\perp$ to itself, associated with the restrictions of the bilinear forms to $u_0^\perp \times u_0^\perp$. By [59, Corollary 2.5.2], which in particular guarantees that for both maps u_0 is an eigenvector with eigenvalue zero, [59, p. 124, l. -3], and with the help of [25, Corollary 7.7.4(e)], we conclude that

$$R(\lambda K, u_0) = \det\left(d^2h_{\lambda K}[u_0]|u_0^\perp\right) \le \det\left(d^2h_{K'}[u_0]|u_0^\perp\right) = R(K', u_0),$$

proving the claim.

Suppose that $G_t > 0$ on $(0,\infty) \times S^{n-1}$; a similar argument applies when $G_t < 0$ instead. By (5.68) with $u = u_0$, and (5.69), we have

$$\begin{aligned}
& \frac{h_K(u_0)}{\psi(h_K(u_0))}\,|\nabla h_K(u_0)|^{1-n}\,G_t\left(|\nabla h_K(u_0)|, \nabla h_K(u_0)/|\nabla h_K(u_0)|\right)R(K,u_0)\\
= \;& \frac{h_{\lambda K}(u_0)}{\psi(h_{\lambda K}(u_0))}\,|\nabla h_{\lambda K}(u_0)|^{1-n}\,G_t\left(|\nabla h_{\lambda K}(u_0)|, \nabla h_{\lambda K}(u_0)/|\nabla h_{\lambda K}(u_0)|\right)R(K',u_0)\\
\ge \;& \lambda\frac{h_K(u_0)}{\psi(\lambda h_K(u_0))}\,\lambda^{1-n}|\nabla h_K(u_0)|^{1-n}\,G_t\left(\lambda|\nabla h_K(u_0)|, \nabla h_K(u_0)|/\nabla h_K(u_0)|\right)\lambda^{n-1}R(K,u_0).
\end{aligned}$$

Therefore

$$\frac{G_t\left(|\nabla h_K(u_0)|, \nabla h_K(u_0)/|\nabla h_K(u_0)|\right)}{\psi(h_K(u_0))} \ge \frac{\lambda G_t\left(\lambda|\nabla h_K(u_0)|, \nabla h_K(u_0)/|\nabla h_K(u_0)|\right)}{\psi(\lambda h_K(u_0))}.$$

But then the first inequality in (5.64) implies that $\lambda \ge 1$, a contradiction proving that $K = K'$. $\square$

5.5 Minkowski problems for even measures

In this section we revisit the Minkowski problems considered in earlier sections, focusing on the case of even measures and attempting to keep the discussion as brief as possible. We say that a set K is origin symmetric if it is centrally symmetric with the center o, i.e, $K = -K$. Let $\mathscr{K}^n_{os}$ (or $\mathscr{K}^n_{(o)s}$) denote the class of origin-symmetric compact convex sets containing the origin (or containing the origin in their interiors, respectively). Also, note that an even measure is not concentrated on any closed hemisphere if and only if it is not concentrated on a great subsphere.

The hypotheses of the next theorem allow $\psi(t) = t^p$ for $p > 0$ and $G(t,u) = t^q$ for $q > 0$, for example.

Theorem 5.5.1. *Let $G : [0,\infty) \times S^{n-1} \to [0,\infty)$ be continuous and such that G_t is continuous and positive on $(0,\infty)\times S^{n-1}$. Assume that $G_t(t,u) = G_t(t,-u)$ for $(t,u) \in (0,\infty) \times S^{n-1}$. Suppose that $\psi : (0,\infty) \to (0,\infty)$ is continuous,* (5.7) *holds, and that φ is finite when defined by* (5.14). *Then the following statements are equivalent:*

(i) *The finite even Borel measure μ on S^{n-1} is not concentrated on any closed hemisphere.*

(ii) *There is a $K \in \mathscr{K}^n_{(o)s}$ such that $\mu = \tau\widetilde{C}_{G,\psi}(K,\cdot)$, with $\tau > 0$ as in* (5.53).

Proof. We first observe that under our extra assumption that $G_t(t,u) = G_t(t,-u)$ for $(t,u) \in (0,\infty) \times S^{n-1}$, Theorem 5.2.4 holds for even discrete measures. Specifically, if

$$\mu = \sum_{i=1}^{m} \lambda_i(\delta_{u_i} + \delta_{-u_i}),$$

where $\lambda_i > 0$ for $i = 1,\ldots,m$ and $\{\pm u_1, \pm u_2, \ldots, \pm u_m\} \subset S^{n-1}$, there is a convex polytope $P \in \mathscr{K}^n_{(o)s}$ satisfying (5.15). Indeed, the proof of Theorem 5.2.4 can be easily adapted, as follows. For each $z = (z_1,\ldots,z_m) \in [0,\infty)^m$, let

$$P_e(z) = \{x \in \mathbb{R}^n : |\langle x, u_i\rangle| \le z_i, \text{ for } i = 1,\ldots,m\},$$

so that $P_e(z)$ is a convex polytope in $\mathscr{K}^n_{os}$. As in the proof of Theorem 5.2.4, one can find $z^0 = (z^0_1,\ldots,z^0_m) \in M_+$ such that

$$\widetilde{V}_G(P_e(z^0)) = \max\left\{\widetilde{V}_G(P_e(z)) : z \in M_+\right\}.$$

Moreover, (5.18) holds with $P(z^0)$ replaced by $P_e(z^0)$. From this, we see that $P_e(z^0) \in \mathscr{K}^n_{(o)s}$ and $z^0_i > 0$ for $i = 1,\ldots,m$. One can adjust the argument used to prove (5.20) to obtain

$$\left.\frac{\partial \widetilde{V}_G(P_e(z))}{\partial z_i}\right|_{z=z^0} = 2n\frac{\widetilde{C}_G(P_e(z^0),\{u_i\})}{h_{P_e(z^0)}(u_i)} \tag{5.70}$$

for $i = 1,\ldots,m$. The method of Lagrange multipliers provides $\tau \in \mathbb{R}$ such that

$$\left.\frac{\tau}{2n}\frac{\partial \widetilde{V}_G(P_e(z))}{\partial z_i}\right|_{z=z^0} = \left.\frac{\partial \sum_{i=1}^m \lambda_i \varphi(z_i)}{\partial z_i}\right|_{z=z^0} \tag{5.71}$$

for $i = 1,\ldots,m$. Then (5.70) and (5.71) can be used instead of (5.20) and (5.21), respectively, and the rest of the proof of Theorem 5.2.4 can be followed up to (5.25) to conclude the proof in the case of an even discrete measure.

With Theorem 5.2.4 for even discrete measures in hand, the proof of (i)⇒(ii) in Theorem 5.4.3 can be followed without difficulty to obtain the same implication for even measures, where K is origin symmetric. In particular, we can take advantage of the fact that it easily follows that $P_j \to K \in \mathscr{K}^n_{os}$ and $\operatorname{int} K \neq \emptyset$, hence $K \in \mathscr{K}^n_{(o)s}$. But then h_K is bounded away from zero and no continuous extension of ψ at 0 is needed.

The implication (ii)⇒(i) follows from the proof of the same implication in Theorem 5.4.3 together with the evenness of $\widetilde{C}_G(K,\cdot)$ when K is origin symmetric and our extra assumption on G holds. (Recall that $\Xi_K = \emptyset$ if $K \in \mathscr{K}^n_{(o)}$.) □

We omit the proof of the following result, which provides the even analogue of Theorem 4.3.3, since it follows without difficulty from the argument given in the proof of Theorem 5.4.4. The hypotheses allow $\psi(t) = t^p$ for $p > 0$ and $G(t,u) = t^q$ for $q < 0$, for example.

Theorem 5.5.2. *Let $G : (0,\infty)\times S^{n-1} \to (0,\infty)$ and $\psi : (0,\infty) \to (0,\infty)$ satisfy the assumptions of Theorem 5.4.4 and suppose also that $G_t(t,u) = G_t(t,-u)$ for $(t,u) \in (0,\infty)\times S^{n-1}$. Then Theorem 5.4.4 holds when in (i) μ is an even measure and in (ii) K is origin symmetric.*

Our final result addresses Problem 5.2.1 when $G_t < 0$ and ψ is decreasing. The hypotheses allow $\psi(t) = t^p$ for $p < 0$ and $G(t,u) = t^q$ for $q < 0$, for example.

If $\psi : (0,\infty) \to (0,\infty)$ is continuous, define

$$\overline{\varphi}(t) = \int_t^\infty \frac{\psi(s)}{s}\,ds \tag{5.72}$$

for $t > 0$.

Theorem 5.5.3. *Let μ be a nonzero finite even Borel measure vanishing on great subspheres. Let G and G_t be continuous on $(0,\infty) \times S^{n-1}$, where $G_t < 0$ and where $G_t(t,u) = G_t(t,-u)$ for $(t,u) \in (0,\infty) \times S^{n-1}$. Suppose that there is some $0 < \varepsilon_0 < 1$ such that (5.4) holds for $v \in S^{n-1}$. Let $\psi : (0,\infty) \to (0,\infty)$ be continuous and suppose that $\overline{\varphi}$ is finite when defined by (5.72). Then there exists a $K \in \mathscr{K}^n_{(o)s}$ such that*

$$\frac{\mu}{\mu(S^{n-1})} = \frac{\widetilde{C}_{G,\psi}(K,\cdot)}{\widetilde{C}_{G,\psi}(K,S^{n-1})}. \tag{5.73}$$

Proof. Since $G_t < 0$, we may define $a_0 \in [0,\infty)$ by $a_0 = \lim_{t\to\infty} \int_{S^{n-1}} G(t,u)\,du$. Define the functional $F : C^+(S^{n-1}) \to \mathbb{R}$ by

$$F(f) = \frac{1}{\mu(S^{n-1}) + a_0} \int_{S^{n-1}} \overline{\varphi}(f(u))\,d\mu(u)$$

for $f \in C^+(S^{n-1})$, and define $F(K) = F(h_K)$ for $K \in \mathscr{K}^n_{(o)s}$. Let

$$\alpha = \sup\left\{F(K) : K \in \mathscr{K}_{(o)s} \text{ and } \widetilde{V}_G(K) = \mu(S^{n-1}) + a_0\right\}. \tag{5.74}$$

As in the proof of Theorem 4.3.3, there is an $r_0 > 0$ such that $\widetilde{V}_G(r_0 B^n) = \mu(S^{n-1}) + a_0$, so the supremum in (5.74) is taken over a nonempty set. (Note that our assumptions on the even measure μ imply in particular that it is not concentrated on any closed hemisphere, as is assumed in Theorem 4.3.3.) Choose $K_j \in \mathscr{K}^n_{(o)s}$, $j \in \mathbb{N}$, such that $\widetilde{V}_G(K_j) = \mu(S^{n-1}) + a_0$ and $\lim_{j\to\infty} F(K_j) = \alpha$. The proof of Theorem 4.3.3 shows that there is an $R > 0$ such that the polar bodies satisfy $K_j^* \subset RB^n$ for $j \in \mathbb{N}$. By relabeling, if necessary, using Blaschke selection theorem, and noting that K_j^* is also origin symmetric for $j \in \mathbb{N}$, a $Q_0 \in \mathscr{K}^n_{os}$ can be found such that $K_j^* \to Q_0$ as $j \to \infty$.

Define $\widetilde{\varphi}$ by $\widetilde{\varphi}(t) = \overline{\varphi}(1/t)$ for $t > 0$. The dominated convergence theorem shows that $\overline{\varphi}(t) \to 0$ as $t \to \infty$. Then our assumption on $\overline{\varphi}$ implies that

$$\widetilde{\varphi}(0) = \lim_{t\to 0+} \widetilde{\varphi}(t) = \lim_{t\to 0+} \overline{\varphi}(1/t) = 0$$

defines a continuous extension of $\widetilde{\varphi}$ at 0. By (2.4), we have

$$F(h_{K_j}) = \frac{1}{\mu(S^{n-1}) + a_0} \int_{S^{n-1}} \overline{\varphi}(h_{K_j}(u))\,d\mu(u) = \frac{1}{\mu(S^{n-1}) + a_0} \int_{S^{n-1}} \widetilde{\varphi}(\rho_{K_j^*}(u))\,d\mu(u).$$

We claim that $Q_0 \in \mathscr{K}^n_{(o)s}$. In fact, assume that $\operatorname{int} Q_0 = \emptyset$, so that $Q_0 \subset v^\perp$ for some $v \subset S^{n-1}$. Then, as shown in the proof of Lemma 5.1.2, $\rho_{K_j^*}(u) \to 0$ as $j \to \infty$ for $u \in S^{n-1} \setminus v^\perp$. Since $\widetilde{\varphi} : [0,\infty) \to [0,\infty)$ is continuous, it follows that $\widetilde{\varphi}(\rho_{K_j^*}(u)) \to \widetilde{\varphi}(0) = 0$ as $j \to \infty$ for $u \in S^{n-1} \setminus v^\perp$, and hence for μ-almost all $u \in S^{n-1}$, as μ vanishes on the great subsphere $S^{n-1} \cap v^\perp$. The continuity of $\widetilde{\varphi}$ also implies that $M_1 = \max\{\widetilde{\varphi}(t) : t \in [0,R]\} < \infty$. Hence the dominated convergence theorem can be applied and yields

$$\alpha = \lim_{j\to\infty} F(h_{K_j}) = \lim_{j\to\infty} \frac{1}{\mu(S^{n-1}) + a_0} \int_{S^{n-1}} \widetilde{\varphi}(\rho_{K_j^*}(u))\, d\mu(u) = 0.$$

But this is impossible because $\alpha \geq F(r_0 B^n) = \overline{\varphi}(r_0) > 0$. This proves the claim.

Let $K_0 = Q_0^*$. Then $K_0 \in \mathscr{K}^n_{(o)s}$. Also, $K_j \to K_0$ as $j \to \infty$, so $\widetilde{V}_G(K_j) \to \widetilde{V}_G(K_0)$ as $j \to \infty$ by the continuity of G and Lemma 4.1.2, yielding $\widetilde{V}_G(K_0) = \mu(S^{n-1}) + a_0$. If $f \in C^+(S^{n-1})$, the support function $h_{[f]}$ of the Wulff shape $[f]$ of f, defined by (2.19), satisfies $h_{[f]} \leq f$. As $\overline{\varphi}$ is decreasing, we have $\overline{\varphi}(h_{[f]}) \geq \overline{\varphi}(f)$. Consequently,

$$F(h_{K_0}) = \alpha = \sup\left\{F(f) : \widetilde{V}_G([f]) = \mu(S^{n-1}) + a_0 \text{ and } f \in C^+(S^{n-1}) \text{ is even}\right\}. \tag{5.75}$$

Let $g \in C(S^{n-1})$ be even. We apply the method of Lagrange multipliers, following the argument in the proof of Theorem 4.3.3 from (4.62) onwards, where $\overline{\varphi}$ play the role of φ. (Note that from (5.72), we have $\psi(t) = -t\overline{\varphi}'(t)$.) The extra constant a_0 in (5.75) has no effect on the conclusion, which is

$$\int_{S^{n-1}} g(u)\, d\mu(u) = -n\tau \int_{S^{n-1}} g(u)\, d\widetilde{C}_{G,\psi}(K_0, u),$$

where

$$\tau = -\frac{\mu(S^{n-1})}{n\, \widetilde{C}_{G,\psi}(K_0, S^{n-1})}.$$

As g is an arbitrary even function in $C(S^{n-1})$, we can use our assumption that $G_t(t,u) = G_t(t,-u)$ for $(t,u) \in (0,\infty) \times S^{n-1}$ to obtain (5.73) with K replaced by K_0. $\square$

Chapter 6

The general dual-polar Orlicz-Minkowski problem

This chapter is based on our paper [70]. In this chapter, we give a systematic study to the general dual-polar Orlicz-Minkowski problem. This problem involves the general dual volume $\widetilde{V}_G$ introduced in Chapter 4. Therefore, the general dual-polar Orlicz-Minkowski problem is "polar" to the general dual Orlicz-Minkowski problem in Chapter 4 and "dual" to the newly proposed polar Orlicz-Minkowski problem in [44]. In particular, we establish the existence, continuity and uniqueness for the solutions to the general dual-polar Orlicz-Minkowski problem. Again our main techniques are the approximation from discrete measures to general measures. Moreover, polytopal solutions and/or counterexamples to the general dual-polar Orlicz-Minkowski problem for discrete measures are also provided. Several variations of the general dual-polar Orlicz-Minkowski problem are discussed as well.

6.1 The homogeneous general dual volumes and properties

In the following, we will define the homogeneous general dual volume and discuss related properties. For simplicity, let

$$\begin{aligned}
\mathscr{G}_I &= \Big\{G:\ G(t,\cdot) \text{ is continuous, strictly increasing on } t,\ \lim_{t\to 0^+} G(t,\cdot)=0,\ \lim_{t\to\infty} G(t,\cdot)=\infty\Big\},\\
\mathscr{G}_d &= \Big\{G:\ G(t,\cdot) \text{ is continuous, strictly decreasing on } t,\ \lim_{t\to 0^+} G(t,\cdot)=\infty,\ \lim_{t\to\infty} G(t,\cdot)=0\Big\}.
\end{aligned}$$

The homogeneous general dual volume of $K \in \mathscr{K}^n_{(o)}$, denoted by $\widehat{V}_G(K)$, can be formulated by

$$\widehat{V}_G(K) = \inf\left\{\eta > 0 : \int_{S^{n-1}} G\left(\frac{\rho_K(u)}{\eta}, u\right) du \le 1\right\}, \quad \text{if } G \in \mathscr{G}_I, \tag{6.1}$$

$$\widehat{V}_G(K) = \inf\left\{\eta > 0 : \int_{S^{n-1}} G\left(\frac{\rho_K(u)}{\eta}, u\right) du \ge 1\right\}, \quad \text{if } G \in \mathscr{G}_d. \tag{6.2}$$

The following proposition provides a more convenient formula for $\widehat{V}_G$.

Proposition 6.1.1. *Let $K \in \mathscr{K}^n_{(o)}$. For any $G \in \mathscr{G}_I \cup \mathscr{G}_d$, there exists a unique $\eta_0 > 0$ such that*

$$\int_{S^{n-1}} G\left(\frac{\rho_K(u)}{\eta_0}, u\right) du = 1. \tag{6.3}$$

Moreover, $\eta_0 = \widehat{V}_G(K)$.

Proof. For $\eta \in (0, \infty)$ and $K \in \mathscr{K}^n_{(o)}$, let $G \in \mathscr{G}_I$ and

$$H_K(\eta) = \int_{S^{n-1}} G\left(\frac{\rho_K(u)}{\eta}, u\right) du.$$

As $K \in \mathscr{K}^n_{(o)}$, there exist positive constants r and R such that $r \le \rho_K \le R$. Thus for any $u \in S^{n-1}$,

$$\int_{S^{n-1}} G\left(\frac{r}{\eta}, u\right) du \ \le\ H_K(\eta) \ \le\ \int_{S^{n-1}} G\left(\frac{R}{\eta}, u\right) du. \tag{6.4}$$

This, together with $G \in \mathscr{G}_I$ and Fatou's lemma, implies that

$$\liminf_{\eta\to 0^+} H_K(\eta) \ \ge\ \liminf_{\eta\to 0^+} \int_{S^{n-1}} G\left(\frac{r}{\eta}, u\right) du \ \ge\ \int_{S^{n-1}} \liminf_{\eta\to 0^+} G\left(\frac{r}{\eta}, u\right) du = \infty.$$

On the other hand, the dominated convergence theorem yields, by (6.4), that

$$\lim_{\eta\to\infty} H_K(\eta) \ \le\ \lim_{\eta\to\infty} \int_{S^{n-1}} G\left(\frac{R}{\eta}, u\right) du = \int_{S^{n-1}} \lim_{\eta\to\infty} G\left(\frac{R}{\eta}, u\right) du = 0.$$

Thus, $\lim_{\eta\to 0^+} H_K(\eta) = \infty$ and $\lim_{\eta\to\infty} H_K(\eta) = 0$. As $G \in \mathscr{G}_I$ is continuous and strictly increasing, $H_K(\eta)$ is clearly continuous and strictly decreasing on $\eta \in (0, \infty)$. Hence, there exists a unique $\eta_0 > 0$ such that $H_K(\eta_0) = 1$, which proves (6.3). Clearly $\eta_0 = \widehat{V}_G(K)$ by (6.1).

The case for $G \in \mathscr{G}_d$ follows along similar lines as above, and its proof will be omitted. $\square$

Clearly, if $G(t,u) = t^q/n$ with $q \neq 0$ for all $(t,u) \in (0,\infty) \times S^{n-1}$, then

$$\widehat{V}_G(K) = \left(\frac{1}{n}\int_{S^{n-1}} \rho_K^q(u)\, du\right)^{1/q} = \big(\widetilde{V}_q(K)\big)^{1/q}.$$

Properties for $\widehat{V}_G$ are summarized in the following proposition.

Proposition 6.1.2. *Let $G \in \mathscr{G}_I \cup \mathscr{G}_d$. Then $\widehat{V}_G(\cdot)$ has the following properties.*

(i) $\widehat{V}_G(\cdot)$ is homogeneous, that is, $\widehat{V}_G(sK) = s\widehat{V}_G(K)$ holds for all $s > 0$ and all $K \in \mathscr{K}^n_{(o)}$.

(ii) $\widehat{V}_G(\cdot)$ is continuous on $\mathscr{K}^n_{(o)}$ in terms of the Hausdorff metric, that is, for any sequence $\{K_i\}_{i\geq 1}$ such that $K_i \in \mathscr{K}^n_{(o)}$ for all $i \in \mathbb{N}$ and $K_i \to K \in \mathscr{K}^n_{(o)}$, then $\widehat{V}_G(K_i) \to \widehat{V}_G(K)$.

(iii) $\widehat{V}_G(\cdot)$ is strictly increasing, that is, for any $K, L \in \mathscr{K}^n_{(o)}$ such that $K \subsetneq L$, then $\widehat{V}_G(K) < \widehat{V}_G(L)$.

Proof. (i) The desired argument follows trivially from Proposition 6.1.1, and $\rho_{sK} = s\rho_K$ for all $s > 0$.

(ii) Let $K_i \in \mathscr{K}^n_{(o)}$ for all $i \in \mathbb{N}$ and $K_i \to K \in \mathscr{K}^n_{(o)}$. Then $\rho_{K_i} \to \rho_K$ uniformly on S^{n-1}. Moreover, there exist two positive constants $r_K < R_K$ such that $r_K \leq \rho_K \leq R_K$ and $r_K \leq \rho_{K_i} \leq R_K$ for all $i \in \mathbb{N}$. For $G \in \mathscr{G}_I$, it follows from Proposition 6.1.1 and (6.4) that for each $i \in \mathbb{N}$,

$$\begin{aligned} \int_{S^{n-1}} G\left(\frac{r_K}{\widehat{V}_G(K_i)}, u\right) du \quad &\leq \quad 1 \\ &= \quad \int_{S^{n-1}} G\left(\frac{\rho_{K_i}(u)}{\widehat{V}_G(K_i)}, u\right) du \\ &\leq \quad \int_{S^{n-1}} G\left(\frac{R_K}{\widehat{V}_G(K_i)}, u\right) du. \end{aligned}$$

Suppose that $\inf_{i\in\mathbb{N}} \widehat{V}_G(K_i) = 0$, and without loss of generality, we assume that $\lim_{i\to\infty} \widehat{V}_G(K_i) = 0$. Then for any $\varepsilon > 0$, there exists $i_\varepsilon \in \mathbb{N}$ such that $\widehat{V}_G(K_i) < \varepsilon$ for all $i > i_\varepsilon$. Thus, for $i > i_\varepsilon$,

$$\begin{aligned} \int_{S^{n-1}} G\Big(\frac{r_K}{\varepsilon}, u\Big)\, du &\leq \int_{S^{n-1}} G\Big(\frac{r_K}{\widehat{V}_G(K_i)}, u\Big)\, du \\ &\leq 1. \end{aligned}$$

Fatou's lemma and the fact that $\lim_{t\to\infty} G(t,\cdot) = \infty$ yield

$$\infty = \int_{S^{n-1}} \liminf_{\varepsilon\to 0^+} G\Big(\frac{r_K}{\varepsilon}, u\Big)\, du \ \leq\ \liminf_{\varepsilon\to 0^+} \int_{S^{n-1}} G\Big(\frac{r_K}{\varepsilon}, u\Big)\, du \ \leq\ 1,$$

a contradiction. Hence,

$$A_1 = \inf_{i\in\mathbb{N}} \widehat{V}_G(K_i) > 0.$$

Moreover, for all $u \in S^{n-1}$ and all $i \in \mathbb{N}$,

$$G\Big(\frac{\rho_{K_i}(u)}{\widehat{V}_G(K_i)}, u\Big) \leq G\Big(\frac{R_K}{A_1}, u\Big).$$

Assume that $\limsup_{i\to\infty} \widehat{V}_G(K_i) > \widehat{V}_G(K)$. There exists a subsequence $\{K_{i_j}\}$ of $\{K_i\}$ such that $\lim_{j\to\infty} \widehat{V}_G(K_{i_j}) > \widehat{V}_G(K)$. Together with Proposition 6.1.1 and the dominated convergence theorem, one has

$$\begin{aligned} 1 &= \lim_{j\to\infty} \int_{S^{n-1}} G\Big(\frac{\rho_{K_{i_j}}(u)}{\widehat{V}_G(K_{i_j})}, u\Big)\, du \\ &= \int_{S^{n-1}} \lim_{j\to\infty} G\Big(\frac{\rho_{K_{i_j}}(u)}{\widehat{V}_G(K_{i_j})}, u\Big)\, du \\ &= \int_{S^{n-1}} G\Big(\frac{\rho_K(u)}{\lim_{j\to\infty} \widehat{V}_G(K_{i_j})}, u\Big)\, du \\ &< \int_{S^{n-1}} G\Big(\frac{\rho_K(u)}{\widehat{V}_G(K)}, u\Big)\, du \\ &= 1. \end{aligned}$$

This is a contradiction and hence $\limsup_{i\to\infty} \widehat{V}_G(K_i) \leq \widehat{V}_G(K)$. Similarly, one can obtain $\liminf_{i\to\infty} \widehat{V}_G(K_i) \geq \widehat{V}_G(K)$, which leads to $\lim_{i\to\infty} \widehat{V}_G(K_i) = \widehat{V}_G(K)$ as desired.

The case for $G \in \mathscr{G}_d$ follows along the same lines, and its proof will be omitted.

(iii) Let $G \in \mathscr{G}_I$ and let $K, L \in \mathscr{K}^n_{(o)}$ such that $K \subsetneq L$. Then, the spherical measure of the set $E = \{u \in S^{n-1} : \rho_K(u) < \rho_L(u)\}$ is positive. By Proposition 6.1.1, one has

$$
\begin{aligned}
1 &= \int_{S^{n-1}} G\bigg(\frac{\rho_L(u)}{\widehat{V}_G(L)}, u\bigg)\, du \\
&= \int_{S^{n-1}} G\bigg(\frac{\rho_K(u)}{\widehat{V}_G(K)}, u\bigg)\, du \\
&= \int_E G\bigg(\frac{\rho_K(u)}{\widehat{V}_G(K)}, u\bigg)\, du + \int_{S^{n-1}\setminus E} G\bigg(\frac{\rho_K(u)}{\widehat{V}_G(K)}, u\bigg)\, du \\
&< \int_E G\bigg(\frac{\rho_L(u)}{\widehat{V}_G(K)}, u\bigg)\, du + \int_{S^{n-1}\setminus E} G\bigg(\frac{\rho_L(u)}{\widehat{V}_G(K)}, u\bigg)\, du \\
&= \int_{S^{n-1}} G\bigg(\frac{\rho_L(u)}{\widehat{V}_G(K)}, u\bigg)\, du.
\end{aligned}
$$

Then $\widehat{V}_G(K) < \widehat{V}_G(L)$ follows from the fact that $G(t, \cdot)$ is strictly increasing on $t \in (0, \infty)$.

The case for $G \in \mathscr{G}_d$ follows along the same lines, and its proof will be omitted. □

The following property may be useful in later context. Denote by $O(n)$ the set of all orthogonal matrices on $\mathbb{R}^n$, that is, for any $T \in O(n)$, one has $TT^t = T^tT = \mathbb{I}_n$, where T^t denotes the transpose of T and $\mathbb{I}_n$ is the identity map on $\mathbb{R}^n$.

Proposition 6.1.3. *Let $K \in \mathscr{K}^n_{(o)}$. If $G(t, u) = \phi(t)$ for all $(t, u) \in (0, \infty) \times S^{n-1}$ with $\phi : (0, \infty) \to (0, \infty)$ being a continuous function, then $\widetilde{V}_G(TK) = \widetilde{V}_G(K)$.*

Proof. Let $G(t, u) = \phi(t)$ for all $t > 0$ and $u \in S^{n-1}$. For $K \in \mathscr{K}^n_{(o)}$ and $T \in O(n)$, then the determinant of T is ± 1 and

$$
\widetilde{V}_G(TK) = \int_{S^{n-1}} \phi(\rho_{TK}(u))\, du = \int_{S^{n-1}} \phi(\rho_K(T^t u))\, du = \int_{S^{n-1}} \phi(\rho_K(v))\, dv = \widetilde{V}_G(K),
$$

if letting $T^t u = v$. This completes the proof. □

In later context, we will employ Proposition 6.1.3 to $G(t, u) = \frac{1}{n} t^q$ for $0 \neq q \in \mathbb{R}$, which implies $\widetilde{V}_q(TK) = \widetilde{V}_q(K)$ for all $T \in O(n)$ and all $K \in \mathscr{K}^n_{(o)}$.

For $G : (0, \infty) \times S^{n-1} \to (0, \infty)$, define two families of convex bodies as follows:

$$
\begin{aligned}
\widetilde{\mathscr{B}} &= \big\{Q \in \mathscr{K}^n_{(o)} : \ \widetilde{V}_G(Q^*) = \widetilde{V}_G(B^n)\big\}; \\
\widehat{\mathscr{B}} &= \big\{Q \in \mathscr{K}^n_{(o)} : \ \widehat{V}_G(Q^*) = \widehat{V}_G(B^n)\big\}, \quad \text{if} \ \ G \in \mathscr{G}_I \cup \mathscr{G}_d.
\end{aligned}
$$

It is obvious that both $\widetilde{\mathscr{B}}$ and $\widehat{\mathscr{B}}$ are nonempty as they all contain the unit Euclidean ball B^n. The following lemma plays essential roles in later context.

Lemma 6.1.4. *Let $G : (0,\infty)\times S^{n-1} \to (0,\infty)$ be a continuous function. For $q \in \mathbb{R}$, let $G_q(t,u) = \frac{G(t,u)}{t^q}$. Suppose that there exists a constant $q \geq n-1$ such that*

$$\inf\Big\{G_q(t,u): \quad t\geq 1 \ \text{ and } \ u\in S^{n-1}\Big\} > 0. \tag{6.5}$$

Then the following statements hold.

(i) If $\{Q_i\}_{i\geq 1}$ with $Q_i \in \widetilde{\mathscr{B}}$ for all $i\in\mathbb{N}$ is a bounded sequence, then there exist a subsequence $\{Q_{i_j}\}_{j\geq 1}$ of $\{Q_i\}_{i\geq 1}$ and a convex body $Q_0 \in \widetilde{\mathscr{B}}$ such that $Q_{i_j}\to Q_0$.

(ii) If in addition $G \in \mathscr{G}_I$, the statement in (i) also holds if $\widetilde{\mathscr{B}}$ is replaced by $\widehat{\mathscr{B}}$.

Remark. Clearly $G(t,u) = t^q$ for some $q \geq n-1$ satisfies (6.5). In particular $G(t,u) = t^n/n$ satisfies (6.5) and hence Lemma 6.1.4 recovers [48, Lemma 3.2]. It can be easily checked that formula (6.5) is equivalent to: there exist constants $c, C > 0$, such that

$$\inf\Big\{G_q(t,u): \quad t\geq c \ \text{ and } \ u\in S^{n-1}\Big\} > C. \tag{6.6}$$

Moreover, if $G\in\mathscr{G}_d$, then G does not satisfy (6.5). In fact, for all $q\geq n-1$ and for all $u\in S^{n-1}$,

$$\lim_{t\to\infty} G_q(t,u) = \lim_{t\to\infty} G(t,u)\times \lim_{t\to\infty} t^{-q} = 0.$$

Proof. Let $\{Q_i\}_{i\geq 1}$ be a bounded sequence with $Q_i\in\widetilde{\mathscr{B}}$ (or, respectively, $Q_i\in\widehat{\mathscr{B}}$) for all $i\in\mathbb{N}$. It follows from the Blaschke selection theorem that there exist a subsequence of $\{Q_i\}_{i\geq 1}$, say $\{Q_{i_j}\}_{j\geq 1}$, and a compact convex set $Q_0\in\mathscr{K}^n$, such that $Q_{i_j}\to Q_0$ in the Hausdorff metric. As $o\in \mathrm{int}Q_{i_j}$ for all $j\in\mathbb{N}$, one has, $o\in Q_0$. In order to show $Q_0\in\widetilde{\mathscr{B}}$ (or, respectively, $Q_0\in\widehat{\mathscr{B}}$), we first need to show $o\in\mathrm{int}Q_0$.

(i) To this end, we assume that $o\in\partial Q_0$ and seek for contradictions. As $\{Q_i\}_{i\geq 1}$ is a bounded sequence, there exists a constant $R>0$ such that $Q_i\subset RB^n$ for each $i\in\mathbb{N}$. For each $j\in\mathbb{N}$, one can find $u_{i_j}\in S^{n-1}$ such that $r_{i_j} = h_{Q_{i_j}}(u_{i_j}) = \min_{u\in S^{n-1}} h_{Q_{i_j}}(u)$. As $o\in\partial Q_0$, one sees that $\lim_{j\to\infty} r_{i_j} = 0$. The fact that $Q_{i_j}\subset RB^n$ implies that $\frac{1}{R}B^n\subset Q_{i_j}^*$, and in particular, $\rho_{Q_{i_j}^*}(u)\geq\frac{1}{R}$ for any $u\in S^{n-1}$.

Let the constant c in (6.6) be $\frac{1}{R}$. For some fixed constants $q \geq n-1$ and $C > 0$,

$$\widetilde{V}_G(Q_{i_j}^*) = \int_{S^{n-1}} G\big(\rho_{Q_{i_j}^*}(u), u\big)\, du \geq C \int_{S^{n-1}} \big(\rho_{Q_{i_j}^*}(u)\big)^q\, du = Cn\widetilde{V}_q(Q_{i_j}^*). \qquad (6.7)$$

For any $T \in O(n)$, $(TQ_{i_j})^* = (T^t)^{-1}Q_{i_j}^* = TQ_{i_j}^*$ as $T^tT = \mathbb{I}_n$ where T^{-1} denotes the inverse map of T. It follows from Proposition 6.1.3 that $\widetilde{V}_q(Q_{i_j}^*)$ is $O(n)$-invariant. Hence, for convenience, one can assume that $u_{i_j} = e_n$. The radial function $\rho_{Q_{i_j}^*}$ can be bounded from below by the radial function of $\mathbf{C}_j = Cone\Big(o, \frac{1}{R}, \frac{e_n}{r_{i_j}}\Big)$, the cone with base $\frac{B^{n-1}}{R}$ and the apex $\frac{e_n}{r_{i_j}}$. Note that

$$\rho_{\mathbf{C}_j}(u) = \begin{cases} \frac{1}{R\sin\theta + r_{i_j}\cos\theta}, & \text{if } u \in S^{n-1} \text{ such that } \langle e_n, u\rangle \geq 0; \\ 0, & \text{if } u \in S^{n-1} \text{ such that } \langle e_n, u\rangle < 0, \end{cases}$$

where $\theta \in [0, \pi/2]$ is the angle between u and e_n (see Figure 6.1).

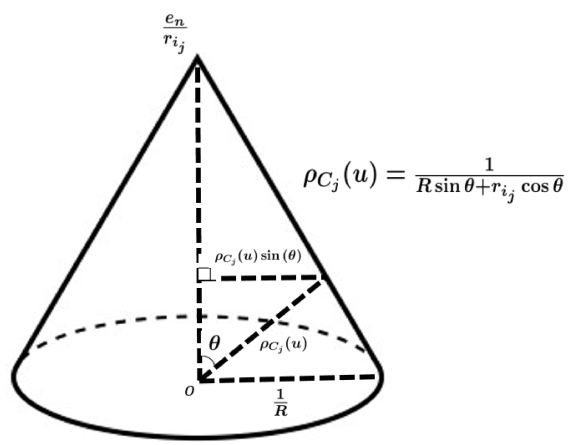

Figure 6.1: The cone $\mathbf{C}_j$

Indeed, from Figure 6.1, for $u \in S^{n-1}$ such that $\langle u, e_n\rangle \geq 0$, one has

$$\frac{\rho_{\mathbf{C}_j}(u)\cdot\sin\theta}{R^{-1}} = \frac{r_{i_j}^{-1} - \rho_{\mathbf{C}_j}(u)\cdot\cos\theta}{r_{i_j}^{-1}} \Longrightarrow \rho_{\mathbf{C}_j}(u) = \frac{1}{R\sin\theta + r_{i_j}\cos\theta}.$$

Using the general spherical coordinate (see, e.g., [8, Page 14]) by letting

$$u = (v\sin\theta, \cos\theta) \in S^{n-1},\ v \in S^{n-2} \text{ and } \theta \in [0, \pi],$$

we have $du = (\sin\theta)^{n-2}\,d\theta\,dv$, where dv denotes the spherical measure of S^{n-2}. Thus

$$\begin{aligned} n\widetilde{V}_q(\mathbf{C}_j) &= \int_{S^{n-2}} \left(\int_0^{\frac{\pi}{2}} \left(\frac{1}{R\sin\theta + r_{i_j}\cos\theta} \right)^q (\sin\theta)^{n-2}\,d\theta \right) dv \\ &= (n-1)V(B^{n-1}) \int_0^{\frac{\pi}{2}} \left(\frac{1}{R\sin\theta + r_{i_j}\cos\theta} \right)^q (\sin\theta)^{n-2}\,d\theta. \end{aligned} \tag{6.8}$$

We will not need the precise value of $\widetilde{V}_q(\mathbf{C}_j)$, however if $q = n$, formula (6.8) does lead to

$$\widetilde{V}_n(\mathbf{C}_j) = V(\mathbf{C}_j) = \frac{V(B^{n-1})}{nR^{n-1}r_{i_j}},$$

which coincides with the calculation provided in [48, Lemma 3.2].

Together with (6.7), $\rho_{Q^*_{i_j}} \geq \rho_{\mathbf{C}_j}$, Fatou's lemma, and $\lim_{j\to\infty} r_{i_j} = 0$, one has, if $q \geq n-1$, then $n-2-q \leq -1$ and

$$\begin{aligned} &\liminf_{j\to\infty} \widetilde{V}_G(Q^*_{i_j}) \\ &\quad\geq \liminf_{j\to\infty} Cn\widetilde{V}_q(\mathbf{C}_j) \\ &\quad= C\cdot(n-1)V(B^{n-1})\cdot \liminf_{j\to\infty} \int_0^{\frac{\pi}{2}} \left(\frac{1}{R\sin\theta + r_{i_j}\cos\theta} \right)^q (\sin\theta)^{n-2} d\theta \\ &\quad\geq C\cdot(n-1)V(B^{n-1})\cdot \int_0^{\frac{\pi}{2}} \liminf_{j\to\infty} \left(\frac{1}{R\sin\theta + r_{i_j}\cos\theta} \right)^q (\sin\theta)^{n-2}\,d\theta \\ &\quad= \frac{C\cdot(n-1)V(B^{n-1})}{R^q} \int_0^{\frac{\pi}{2}} (\sin\theta)^{n-2-q}\,d\theta \\ &\quad\geq \frac{C\cdot(n-1)V(B^{n-1})}{R^q} \int_0^{\frac{\pi}{2}} \frac{1}{\sin\theta}\,d\theta \\ &\quad= \frac{C\cdot(n-1)V(B^{n-1})}{R^q} \cdot \ln\tan(\theta/2)\Big|_{\theta=0}^{\theta=\pi/2} = \infty. \end{aligned} \tag{6.9}$$

On the other hand, as $Q_{i_j} \in \widetilde{\mathscr{B}}$ for each $j \in \mathbb{N}$, then

$$\widetilde{V}_G(Q^*_{i_j}) = \widetilde{V}_G(B^n) = \int_{S^{n-1}} G(1,u)\,du < \infty.$$

This is a contradiction and thus $o \in \mathrm{int}Q_0$.

As $Q_{i_j} \in \mathscr{K}^n_{(o)}$ for each $j \in \mathbb{N}$ and $Q_0 \in \mathscr{K}^n_{(o)}$, $Q_{i_j} \to Q_0$ yields $Q^*_{i_j} \to Q^*_0$. Together with the continuity of $\widetilde{V}_G(\cdot)$ (see Lemma 4.1.2) and the fact that $\widetilde{V}_G(Q^*_{i_j}) = \widetilde{V}_G(B^n)$

for each $j \in \mathbb{N}$, one gets $\widetilde{V}_G(Q_0^*) = \lim_{j\to\infty} \widetilde{V}_G(Q_{i_j}^*) = \widetilde{V}_G(B^n)$. This concludes that $Q_0 \in \widetilde{\mathscr{B}}$ as desired.

(ii) Again, we assume that $o \in \partial Q_0$ and seek for contradictions. It follows from Proposition 6.1.1 that $\widehat{V}_G(B^n) > 0$ is a finite constant. Following notations in (i), Proposition 6.1.1 and $\widehat{V}_G(Q_{i_j}^*) = \widehat{V}_G(B^n)$ for each $j \in \mathbb{N}$ yield that

$$\int_{S^{n-1}} G\bigg(\frac{\rho_{Q_{i_j}^*}(u)}{\widehat{V}_G(B^n)}, u\bigg)\, du = 1. \tag{6.10}$$

As $\frac{1}{R}B^n \subset Q_{i_j}^*$ for each $j \in \mathbb{N}$, one can take the constant c in (6.6) to be $\frac{1}{R\cdot \widehat{V}_G(B^n)}$ and there exists a constant $C > 0$ such that, for all $u \in S^{n-1}$ and some $q \geq n-1$,

$$G\bigg(\frac{\rho_{Q_{i_j}^*}(u)}{\widehat{V}_G(B^n)}, u\bigg) \geq C \cdot \bigg(\frac{\rho_{Q_{i_j}^*}(u)}{\widehat{V}_G(B^n)}\bigg)^q.$$

Together with (6.10), one has,

$$\int_{S^{n-1}} C \cdot \bigg(\frac{\rho_{Q_{i_j}^*}(u)}{\widehat{V}_G(B^n)}\bigg)^q du \leq 1 \Longrightarrow C \cdot \int_{S^{n-1}} \big(\rho_{Q_{i_j}^*}(u)\big)^q\, du \leq \big(\widehat{V}_G(B^n)\big)^q.$$

Similar to (6.9), one gets

$$\begin{aligned} \infty &= \liminf_{j\to\infty} C \cdot \int_{S^{n-1}} \big(\rho_{Q_{i_j}^*}(u)\big)^q\, du \\ &\leq \big(\widehat{V}_G(B^n)\big)^q, \end{aligned}$$

a contradiction and hence $o \in \mathrm{int} Q_0$. The rest of the proof follows along the lines in (i), where the continuity of $\widehat{V}_G(\cdot)$ (see Proposition 6.1.2) shall be used. $\square$

6.2 The general dual-polar Orlicz-Minkowski problem

Motivated by the polar Orlicz-Minkowski problem proposed in [44] and by the general dual Orlicz-Minkowski problem proposed in Chapters 4 and 5, we propose the following general dual-polar Orlicz-Minkowski problem:

Problem 6.2.1. *Under what conditions on a nonzero finite Borel measure μ defined on S^{n-1}, continuous functions $\varphi : (0,\infty) \to (0,\infty)$ and $G \in \mathscr{G}_I \cup \mathscr{G}_d$ can we find a convex body $K \in \mathscr{K}^n_{(o)}$ solving the following optimization problems:*

$$\inf / \sup \left\{ \int_{S^{n-1}} \varphi(h_Q(u)) d\mu(u) : Q \in \widetilde{\mathscr{B}} \right\}; \tag{6.11}$$

$$\inf / \sup \left\{ \int_{S^{n-1}} \varphi(h_Q(u)) d\mu(u) : Q \in \widehat{\mathscr{B}} \right\}. \tag{6.12}$$

Although the function G in the optimization problem (6.11) can be any continuous function $G : (0,\infty) \times S^{n-1} \to (0,\infty)$, to find its solutions, only those $G \in \mathscr{G}_I \cup \mathscr{G}_d$ with monotonicity will be considered. One reason is that most G of interest (such as $G(t,u) = t^q/n$ for $0 \neq q \in \mathbb{R}$) are monotone. More importantly, without the monotonicity of G, the set $\widetilde{\mathscr{B}}$ may contain only one convex body B^n (for instance, if $G(1,u) < G(t,u)$ for all $(t,u) \in (0,\infty) \times S^{n-1}$ such that $t \neq 1$). In this case, the optimization problem (6.11) becomes trivial. Note that when $G(t,u) = t^n/n$, both $\widetilde{V}_G$ and (essentially) $\widehat{V}_G$ are volume, then Problem 6.2.1 becomes the polar Orlicz-Minkowski problem posed in [44].

In the following subsection, we will solve the general dual-polar Orlicz-Minkowski problem for discrete measures first.

6.2.1 The general dual-polar Orlicz-Minkowski problem for discrete measures

Throughout this subsection, let μ be a discrete measure of the following form:

$$\mu = \sum_{i=1}^{m} \lambda_i \delta_{u_i}, \tag{6.13}$$

where $\lambda_i > 0$, δ_{u_i} denotes the Dirac measure at u_i, and $\{u_1, \cdots, u_m\}$ is a subset of S^{n-1} which is not concentrated on any closed hemisphere (clearly $m \geq n+1$). It has been proved in [44, Propositions 3.1 and 3.3] that the solutions to the polar Orlicz-Minkowski problem for discrete measures must be polytopes, the convex hulls of finite points in $\mathbb{R}^n$. It is well known that all convex bodies can be approximated by polytopes, and hence to study the Minkowski type problems for discrete measures is very important and receives extensive attention, see e.g., [3, 5, 14, 19, 24, 27, 33, 35, 39, 67, 81, 82, 83].

The following lemma shows that if, when the infimum is considered, Problem 6.2.1 for discrete measures has solutions, then the solutions must be polytopes.

Lemma 6.2.2. *Let $\varphi \in \mathscr{I}$ and μ be as in (6.13) whose support $\{u_1, \cdots, u_m\}$ is not concentrated on any closed hemisphere. Let $G \in \mathscr{G}_I$.*

(i) If $\widetilde{M} \in \widetilde{\mathscr{B}}$ is a solution to the optimization problem (6.11) when the infimum is considered, then $\widetilde{M}$ is a polytope, and $u_1, \cdots, u_m$ are the corresponding unit normal vectors of its faces.

(ii) If $\widehat{M} \in \widehat{\mathscr{B}}$ is a solution to the optimization problem (6.12) when the infimum is considered, then $\widehat{M}$ is a polytope, and $u_1, \cdots, u_m$ are the corresponding unit normal vectors of its faces.

Proof. Let $G \in \mathscr{G}_I$. For discrete measure μ and $Q \in \mathscr{K}^n_{(o)}$, one has

$$\int_{S^{n-1}} \varphi(h_Q(u))d\mu(u) = \sum_{i=1}^m \varphi(h_Q(u_i))\mu(\{u_i\}) = \sum_{i=1}^m \lambda_i \varphi(h_Q(u_i)).$$

(i) Let $\widetilde{M} \in \widetilde{\mathscr{B}}$ be a solution to the optimization problem (6.11). Define the polytope P as follows: $\widetilde{M} \subseteq P$, $h_P(u_i) = h_{\widetilde{M}}(u_i)$ for $1 \le i \le m$, and $u_1, \cdots, u_m$ are the corresponding unit normal vectors of the faces of P. As $\widetilde{M} \in \widetilde{\mathscr{B}}$, one has $\widetilde{V}_G(\widetilde{M}^*) = \widetilde{V}_G(B^n)$ and $o \in \mathrm{int}\widetilde{M}$. Hence $P \in \mathscr{K}^n_{(o)}$ and $P^* \subseteq \widetilde{M}^*$. Similar to the proof of Proposition 6.1.2 (iii), one can obtain that $\widetilde{V}_G(\cdot)$ for $G \in \mathscr{G}_I$ is strictly increasing in terms of set inclusion. In particular, $\widetilde{V}_G(P^*) \le \widetilde{V}_G(\widetilde{M}^*) = \widetilde{V}_G(B^n)$. As $\lim_{t\to\infty} G(t, \cdot) = \infty$, there exists $t_0 \ge 1$ such that $\widetilde{V}_G(t_0 P^*) = \widetilde{V}_G(B^n)$. That is, $P/t_0 \in \widetilde{\mathscr{B}}$. Due to the minimality of $\widetilde{M}$ and the fact that $\varphi \in \mathscr{I}$ is strictly increasing, one has

$$\sum_{i=1}^m \lambda_i \varphi(h_P(u_i)) = \sum_{i=1}^m \lambda_i \varphi(h_{\widetilde{M}}(u_i)) \le \sum_{i=1}^m \lambda_i \varphi(h_{P/t_0}(u_i)) \le \sum_{i=1}^m \lambda_i \varphi(h_P(u_i)),$$

which yields $t_0 = 1$. Then, $\widetilde{V}_G(P^*) = \widetilde{V}_G(B^n) = \widetilde{V}_G(\widetilde{M}^*)$ and hence $P = \widetilde{M}$ following from $\widetilde{M} \subseteq P$.

(ii) Proposition 6.1.2 (iii) asserts that, if $G \in \mathscr{G}_I$, $\widehat{V}_G(K) < \widehat{V}_G(L)$ for all $K, L \in \mathscr{K}^n_{(o)}$

such that $K \subsetneq L$. The proof in this case then follows along the same lines as in (i), and will be omitted. $\square$

The following result is for the existence of solutions to Problem 6.2.1 for discrete measures if the infimum is considered.

Theorem 6.2.3. *Let $\varphi \in \mathscr{I}$ and μ be as in (6.13) whose support $\{u_1, \cdots, u_m\}$ is not concentrated on any closed hemisphere. Let $G \in \mathscr{G}_I$ be a continuous function such that (6.5) holds for some $q \geq n-1$. Then the following statements hold.*

(i) There exists a polytope $\widetilde{P} \in \widetilde{\mathscr{B}}$ with $u_1, \cdots, u_m$ being the corresponding unit normal vectors of its faces, such that,

$$\sum_{i=1}^{m} \lambda_i \varphi(h_{\widetilde{P}}(u_i)) = \inf \Big\{ \sum_{i=1}^{m} \lambda_i \varphi(h_Q(u_i)) : \quad Q \in \widetilde{\mathscr{B}} \Big\}. \tag{6.14}$$

(ii) There exists a polytope $\widehat{P} \in \widehat{\mathscr{B}}$ with $u_1, \cdots, u_m$ being the corresponding unit normal vectors of its faces, such that,

$$\sum_{i=1}^{m} \lambda_i \varphi(h_{\widehat{P}}(u_i)) = \inf \Big\{ \sum_{i=1}^{m} \lambda_i \varphi(h_Q(u_i)) : \quad Q \in \widehat{\mathscr{B}} \Big\}.$$

Proof. By Lemma 6.2.2, to solve (6.14), it will be enough to find a solution for the following problem:

$$\widetilde{\alpha} = \inf \Big\{ \sum_{i=1}^{m} \lambda_i \varphi(z_i) : \quad z \in \mathbb{R}^m_+ \text{ such that } P(z) \in \widetilde{\mathscr{B}} \Big\}, \tag{6.15}$$

where $z = (z_1, \cdots, z_m) \in \mathbb{R}^m_+$ means that each $z_i > 0$ and

$$P(z) = \bigcap_{i=1}^{m} \Big\{ x \in \mathbb{R}^n : \langle x, u_i \rangle \leq z_i \Big\} \subset \mathscr{K}^n_{(o)}.$$

Clearly $h_{P(z)}(u_i) \leq z_i$ for all $i = 1, 2, \cdots, m$.

Let $P_1 = P(1, \cdots, 1)$. Then $B^n \subsetneq P_1$ and hence $P_1^* \subsetneq B^n$. As $G \in \mathscr{G}_I$ one has $\widetilde{V}_G(P_1^*) < \widetilde{V}_G(B^n)$. The facts that $G(t, \cdot)$ is strictly increasing on t and $\lim_{t\to\infty} G(t, \cdot) = \infty$ imply the existence of $t_1 > 1$ such that $\widetilde{V}_G(t_1 P_1^*) = \widetilde{V}_G(B^n)$. In other words, $P_1/t_1 \in \widetilde{\mathscr{B}}$ and then the infimum in (6.15) is not taken over an empty set. Moreover, due to $\varphi \in \mathscr{I}$ (in particular, φ is strictly increasing and $\varphi(1) = 1$) and $1/t_1 < 1$, one has,

$$\widetilde{\alpha} \leq \varphi(1/t_1) \sum_{i=1}^{m} \lambda_i \leq \sum_{i=1}^{m} \lambda_i.$$

This in turn implies that $z \in \mathbb{R}^m_+$ in (6.15) can be restricted in a bounded set, for instance,

$$z_i \leq \varphi^{-1}\Big(\frac{\lambda_1 + \cdots + \lambda_m}{\min_{1 \leq i \leq m} \lambda_i}\Big), \quad \text{for all } i = 1, 2, \cdots, m. \tag{6.16}$$

Let $z^1, \cdots, z^j \cdots \in \mathbb{R}^m_+$ be the limiting sequence of (6.15), that is,

$$\widetilde{\alpha} = \lim_{j \to \infty} \sum_{i=1}^{m} \lambda_i \varphi(z_i^j) \ \text{ and } \ \widetilde{V}_G(P^*(z^j)) = \widetilde{V}_G(B^n) \quad \text{for all} \quad j \in \mathbb{N}.$$

Due to (6.16), without loss of generality, we can assume that $z^j \to z^0$ for some $z^0 \in \mathbb{R}^m$ and hence $P(z^j) \to P(z^0)$ in the Hausdorff metric (see e.g., [59]). Lemma 6.1.4 yields that $P(z^0) \in \widetilde{\mathscr{B}}$, i.e., $\widetilde{V}_G(P^*(z^0)) = \widetilde{V}_G(B^n)$ and $o \in \mathrm{int}P(z^0)$. In particular, $z_i^0 > 0$ for all $i = 1, 2, \cdots, m$.

On the other hand, we claim that $h_{P(z^0)}(u_i) = z_i^0$ for all $i = 1, 2, \cdots, m$. To this end, assume not, then there exists $i_0 \in \{1, 2, \cdots, m\}$ such that $h_{P(z^0)}(u_{i_0}) < z_{i_0}^0$. As $\varphi \in \mathscr{I}$ is strictly increasing and $\lambda_{i_0} > 0$, one clearly has

$$\widetilde{\alpha} = \sum_{i=1}^{m} \lambda_i \varphi(z_i^0) > \sum_{i \in \{1,2,\cdots,m\} \setminus \{i_0\}} \lambda_i \varphi(z_i^0) + \lambda_{i_0} \varphi(h_{P(z^0)}(u_{i_0})).$$

This contradicts with the minimality of $\widetilde{\alpha}$.

Let $\widetilde{P} = P(z^0)$. Then $\widetilde{P} \in \widetilde{\mathscr{B}}$ solves (6.15) and hence (6.14). This concludes the proof of (i).

(ii) The proof is almost identical to the one for (i), and will be omitted. $\square$

It has been proved in [44] that the existence of solutions to Problem 6.2.1 for discrete measures in general is invalid when $G(t, u) = t^n/n$, if the supremum is considered for $\varphi \in \mathscr{I} \cup \mathscr{D}$, or the infimum is considered for $\varphi \in \mathscr{D}$. One can also prove similar arguments for Problem 6.2.1 for discrete measures with more general $G \in \mathscr{G}_I$, but more delicate calculations are required. We only state the following result as an example.

Proposition 6.2.4. *Let μ be as in (6.13) whose support $\{u_1, \cdots, u_m\}$ is not concentrated on any closed hemisphere. Let $G \in \mathscr{G}_I$ be such that (6.5) holds for some $q \geq n-1$.*

(i) If $\varphi \in \mathscr{D}$ and the first coordinates of $u_1, u_2, \cdots, u_m$ are all nonzero, then

$$\inf_{Q \in \widehat{\mathscr{B}}} \sum_{i=1}^{m} \lambda_i \varphi(h_Q(u_i)) = 0.$$

(ii) If $\varphi \in \mathscr{I} \cup \mathscr{D}$, then

$$\sup_{Q \in \widehat{\mathscr{B}}} \sum_{i=1}^{m} \lambda_i \varphi(h_Q(u_i)) = \infty.$$

Proof. (i) For $0 < \epsilon < 1$, let $T_\epsilon = \mathrm{diag}(1, 1, \cdots, 1, \epsilon)$ and $L_\epsilon = T_\epsilon B^n$. It can be checked that

$$\rho_{L_\epsilon}(w) = \big(w_1^2 + w_2^2 + \cdots + w_{n-1}^2 + w_n^2/\epsilon^2\big)^{-1/2}$$

for all $w = (w_1, \cdots, w_n) \in S^{n-1}$. Thus $\rho_{L_\epsilon}(w)$ is increasing on $\epsilon > 0$ for each $w \in S^{n-1}$ and then L_ϵ is increasing in the sense of set inclusion on $\epsilon > 0$. In particular, $L_\epsilon \subset B^n$ and $B^n \subset L_\epsilon^* = T_\epsilon^{-1} B^n$. Moreover, $L_\epsilon^* = T_\epsilon^{-1} B^n$ is decreasing in the sense of set inclusion on $\epsilon > 0$, and so is $\widehat{V}_G(L_\epsilon^*)$ due to $G \in \mathscr{G}_I$. By the homogeneity of $\widehat{V}_G(\cdot)$, one has

$$\widehat{V}_G(f(\epsilon) L_\epsilon^*) = \widehat{V}_G(B^n),$$

if

$$f(\epsilon) = \frac{\widehat{V}_G(B^n)}{\widehat{V}_G(L_\epsilon^*)}.$$

We now claim that $f(\epsilon) \to 0$, which is equivalent to prove $\widehat{V}_G(L_\epsilon^*) \to \infty$ as $\epsilon \to 0^+$.

To this end, it is enough to prove that $\sup_{0<\epsilon<1} \widehat{V}_G(L_\epsilon^*) = \infty$. Let us assume that $\sup_{0<\epsilon<1} \widehat{V}_G(L_\epsilon^*) = A_0 < \infty$. By $B^n \subset L_\epsilon^*$ and (6.6) with $c = 1/A_0$, there exists a constant $C_A > 0$ such that

$$\int_{S^{n-1}} G\bigg(\frac{\rho_{L_\epsilon^*}(u)}{A_0}, u\bigg)\, du \geq C_A \int_{S^{n-1}} \bigg(\frac{\rho_{L_\epsilon^*}(u)}{A_0}\bigg)^q du \geq C_A \int_{S^{n-1}} \bigg(\frac{\rho_{\mathbf{C}_\epsilon}(u)}{A_0}\bigg)^q du,$$

where $\mathbf{C}_\epsilon \subset L_\epsilon^*$ is the cone with the base B^{n-1} and the apex $\epsilon^{-1} e_n$. It follows from

Proposition 6.1.1, (6.8) and (6.9) that

$$\begin{aligned}
1 &= \liminf_{\epsilon\to 0^+}\int_{S^{n-1}} G\left(\frac{\rho_{L_\epsilon^*}(u)}{\widehat{V}_G(L_\epsilon^*)}, u\right) du \\
&\geq \liminf_{\epsilon\to 0^+}\int_{S^{n-1}} G\left(\frac{\rho_{L_\epsilon^*}(u)}{A_0}, u\right) du \\
&\geq \frac{C_A}{A_0^q}\cdot \liminf_{\epsilon\to 0^+}\int_{S^{n-1}} \left(\rho_{\mathbf{C}_\epsilon}(u)\right)^q du \\
&\geq \frac{C_A(n-1)V(B^{n-1})}{A_0^q}\cdot \liminf_{\epsilon\to 0^+}\int_0^{\frac{\pi}{2}} \left(\frac{1}{\sin\theta+\epsilon\cos\theta}\right)^q (\sin\theta)^{n-2}\, d\theta \\
&= \infty.
\end{aligned}$$

This is a contradiction, which yields $\sup_{0<\epsilon<1}\widehat{V}_G(L_\epsilon^*) = \infty$ and then $f(\epsilon)\to 0$ as $\epsilon\to 0^+$.

Recall that $\widehat{V}_G(f(\epsilon)L_\epsilon^*) = \widehat{V}_G(B^n)$ and then $L_\epsilon/f(\epsilon) = T_\epsilon B^n/f(\epsilon)\in\widehat{\mathscr{B}}$. It is assumed that $\alpha = \min_{1\leq i\leq m}\{|(u_i)_1|\} > 0$, and hence for all $1\leq i\leq m$ (by letting $v_2 = u_i$),

$$\begin{aligned}
h_{L_\epsilon/f(\epsilon)}(u_i) &= \max_{v_1\in L_\epsilon/f(\epsilon)}\langle v_1, u_i\rangle \\
&= \max_{v_2\in B^n}\langle T_\epsilon v_2, u_i\rangle/f(\epsilon) \\
&\geq \alpha^2/f(\epsilon).
\end{aligned}$$

The fact that $\varphi\in\mathscr{D}$ is strictly decreasing yields

$$\begin{aligned}
\inf_{Q\in\widehat{\mathscr{B}}}\sum_{i=1}^m \lambda_i\varphi(h_Q(u_i)) &\leq \sum_{i=1}^m \varphi\left(h_{L_\epsilon/f(\epsilon)}(u_i)\right)\cdot\mu(\{u_i\}) \\
&\leq \varphi\left(\alpha^2/f(\epsilon)\right)\cdot\mu(S^{n-1}) \\
&\to 0,
\end{aligned}$$

where we have used $\lim_{\epsilon\to 0^+} f(\epsilon) = 0$ and $\lim_{t\to\infty}\varphi(t) = 0$. This concludes the proof of (i).

(ii) Note that $\mu(\{u_1\}) > 0$. For any $0<\epsilon<1$, let $\widetilde{L}_\epsilon = TT_\epsilon B^n$, where $T\in O(n)$ is an orthogonal matrix such that $T^t u_1 = e_1$ (indeed, this can always be done by the Gram-Schmidt process). Again $\widetilde{L}_\epsilon\subset B^n$ and hence $B^n\subset\widetilde{L}_\epsilon^*$. As in (i), one can prove that

$$f(\epsilon)=\frac{\widehat{V}_G(B^n)}{\widehat{V}_G(\widetilde{L}_\epsilon^*)}\to 0 \ \text{ as } \ \epsilon\to 0^+.$$

Moreover, $\widehat{V}_G(f(\epsilon)\widetilde{L}_\epsilon^*)=\widehat{V}_G(B^n)$ and thus $\widetilde{L}_\epsilon/f(\epsilon)\in\widehat{\mathscr{B}}$. One can check (by letting $v_2=e_1$) that

$$h_{\widetilde{L}_\epsilon/f(\epsilon)}(u_1)=f(\epsilon)^{-1}\max_{v_2\in B^n}\langle TT_\epsilon v_2,u_1\rangle=f(\epsilon)^{-1}\langle T^t u_1,\mathrm{diag}(1,1,\cdots,1,\epsilon)\cdot e_1\rangle=f(\epsilon)^{-1}.$$

Together with $\varphi\in\mathscr{I}$ (in particular, $\lim_{t\to\infty}\varphi(t)=\infty$), one has

$$\begin{aligned}\sup_{Q\in\widehat{\mathscr{B}}}\sum_{i=1}^m\lambda_i\varphi(h_Q(u_i)) &\geq \sum_{i=1}^m\varphi\left(h_{\widetilde{L}_\epsilon/f(\epsilon)}(u_i)\right)\cdot\mu(\{u_i\})\\ &\geq \varphi\left(h_{\widetilde{L}_\epsilon/f(\epsilon)}(u_1)\right)\cdot\mu(\{u_1\})\\ &= \varphi\left(f(\epsilon)^{-1}\right)\cdot\mu(\{u_1\})\\ &\to \infty,\end{aligned}$$

as $\epsilon\to 0^+$, which follows from the fact that $\lim_{\epsilon\to 0^+}f(\epsilon)=0$.

When $\varphi\in\mathscr{D}$, let $\overline{L}_\epsilon=\widetilde{L}_\epsilon^*=\widetilde{L}_{1/\epsilon}$. Hence $\overline{L}_\epsilon^*\subset B^n$ for all $\epsilon\in(0,1)$. We claim that $\widehat{V}_G(\overline{L}_\epsilon^*)\to 0$ as $\epsilon\to 0^+$. To this end, it can be checked that

$$\rho_{\overline{L}_\epsilon^*}(u)=\frac{\epsilon}{\sqrt{[(T^t u)_n]^2+\epsilon^2(1-[(T^t u)_n]^2)}},$$

where $(T^t u)_n$ denotes the n-th coordinate of $T^t u$. Clearly $\rho_{\overline{L}_\epsilon^*}(u)\leq 1$ for all $u\in S^{n-1}$ and $\rho_{\overline{L}_\epsilon^*}(u)\to 0$ as $\epsilon\to 0^+$ for all $u\in\eta$, where $\eta=\{u\in S^{n-1}:(T^t u)_n\neq 0\}$. Also note that the spherical measure of $S^{n-1}\setminus\eta$ is 0.

On the other hand, $\overline{L}_\epsilon^*$ is increasing (in the sense of set inclusion) and hence $\widehat{V}_G(\overline{L}_\epsilon^*)$ is strictly increasing on ϵ due to Proposition 6.1.2. To show that $\widehat{V}_G(\overline{L}_\epsilon^*)\to 0$ as $\epsilon\to 0^+$, we assume that $\inf_{\epsilon>0}\widehat{V}_G(\overline{L}_\epsilon^*)=\beta>0$ and seek for contradictions. By Proposition 6.1.1, one has, for all $\epsilon\in(0,1)$,

$$\int_{S^{n-1}}G\bigg(\frac{\rho_{\overline{L}_\epsilon^*}(u)}{\beta},u\bigg)\,du\geq\int_{S^{n-1}}G\bigg(\frac{\rho_{\overline{L}_\epsilon^*}(u)}{\widehat{V}_G(\overline{L}_\epsilon^*)},u\bigg)\,du=1. \tag{6.17}$$

Moreover, as $\rho_{\overline{L}_\epsilon^*}(u)\leq 1$ for all $u\in S^{n-1}$, one has, for all $u\in S^{n-1}$,

$$G\left(\frac{\rho_{\overline{L}_\epsilon^*}(u)}{\beta}, u\right) \leq G\left(\frac{1}{\beta}, u\right).$$

Together with (6.17) and the dominated convergence theorem, one gets that

$$1 \leq \lim_{\epsilon\to 0^+} \int_{S^{n-1}} G\left(\frac{\rho_{\overline{L}_\epsilon^*}(u)}{\beta}, u\right) du = \int_{S^{n-1}} \lim_{\epsilon\to 0^+} G\left(\frac{\rho_{\overline{L}_\epsilon^*}(u)}{\beta}, u\right) du = 0.$$

This implies $\widehat{V}_G(\overline{L}_\epsilon^*) \to 0$ as $\epsilon \to 0^+$. Again, $\overline{L}_\epsilon/f(\epsilon) \in \widehat{\mathscr{B}}$ and $h_{\overline{L}_\epsilon/f(\epsilon)}(u_1) = f(\epsilon)^{-1}$, where

$$f(\epsilon) = \frac{\widehat{V}_G(B^n)}{\widehat{V}_G(\overline{L}_\epsilon^*)} \to \infty \quad \text{as} \quad \epsilon \to 0^+.$$

Together with $\varphi \in \mathscr{D}$ (in particular, $\lim_{t\to 0^+} \varphi(t) = \infty$), one has

$$\sup_{Q\in\widehat{\mathscr{B}}} \sum_{i=1}^m \lambda_i \varphi(h_Q(u_i)) \geq \varphi\left(h_{\overline{L}_\epsilon/f(\epsilon)}(u_1)\right) \cdot \mu(\{u_1\}) = \varphi\left(f(\epsilon)^{-1}\right) \cdot \mu(\{u_1\}) \to \infty,$$

as $\epsilon \to 0^+$. This concludes the proof of (ii). □

It is worth to mention that the argument in Proposition 6.2.4 (ii) for the case $\varphi \in \mathscr{D}$ indeed works for all $G \in \mathscr{G}_I$ without assuming (6.5) for some $q \geq n-1$. Moreover, the proof of Proposition 6.2.4 can be slightly modified to show similar results for the case $\widetilde{\mathscr{B}}$ and the details are omitted.

6.2.2 The general dual-polar Orlicz-Minkowski problem

In view of Proposition 6.2.4, in this subsection, we will provide the continuity, uniqueness, and existence of solutions to Problem 6.2.1 for $\varphi \in \mathscr{I}$ and with the infimum considered.

The following lemma is very useful in later context. Its proof can be found in, e.g., the proof of [44, Theorem 3.2] (slight modification is needed) and hence is omitted.

Lemma 6.2.5. *Let $\varphi \in \mathscr{I}$. Let μ_i, μ for $i \in \mathbb{N}$ be nonzero finite Borel measures on S^{n-1} which are not concentrated on any closed hemisphere and $\mu_i \to \mu$ weakly. Suppose that $\{Q_i\}_{i\geq 1}$ is a sequence of convex bodies such that $Q_i \in \mathscr{K}_{(o)}^n$ for each $i \in \mathbb{N}$ and*

$$\sup_{i\geq 1} \left\{ \int_{S^{n-1}} \varphi(h_{Q_i}(u)) d\mu_i(u) \right\} < \infty.$$

Then $\{Q_i\}_{i\geq 1}$ *is a bounded sequence in* $\mathscr{K}^n_{(o)}$.

The continuity of the extreme values for Problem 6.2.1 is given below.

Theorem 6.2.6. *Let* μ_i, μ *for* $i \in \mathbb{N}$ *be finite Borel measures on* S^{n-1} *which are not concentrated on any closed hemisphere and* $\mu_i \to \mu$ *weakly. Let* $G \in \mathscr{G}_I$ *be a continuous function such that (6.5) holds for some* $q \geq n-1$ *and* $\varphi \in \mathscr{I}$. *The following statements hold true.*

(i) If for each $i \in \mathbb{N}$, *there exists* $\widetilde{M_i} \in \widetilde{\mathscr{B}}$ *such that*

$$\int_{S^{n-1}} \varphi\big(h_{\widetilde{M_i}}(u)\big) d\mu_i(u) = \inf\left\{\int_{S^{n-1}} \varphi(h_Q(u)) d\mu_i(u) : Q \in \widetilde{\mathscr{B}}\right\}, \tag{6.18}$$

then there exists $\widetilde{M} \in \widetilde{\mathscr{B}}$ *such that*

$$\int_{S^{n-1}} \varphi\big(h_{\widetilde{M}}(u)\big) d\mu(u) = \inf\left\{\int_{S^{n-1}} \varphi(h_Q(u)) d\mu(u) : Q \in \widetilde{\mathscr{B}}\right\}. \tag{6.19}$$

Moreover,

$$\lim_{i\to\infty} \int_{S^{n-1}} \varphi\big(h_{\widetilde{M_i}}(u)\big) d\mu_i(u) = \int_{S^{n-1}} \varphi\big(h_{\widetilde{M}}(u)\big) d\mu(u). \tag{6.20}$$

(ii) If for each $i \in \mathbb{N}$, *there exists* $\widehat{M_i} \in \widehat{\mathscr{B}}$ *such that*

$$\int_{S^{n-1}} \varphi\big(h_{\widehat{M_i}}(u)\big) d\mu_i(u) = \inf\left\{\int_{S^{n-1}} \varphi(h_Q(u)) d\mu_i(u) : Q \in \widehat{\mathscr{B}}\right\},$$

then there exists $\widehat{M} \in \widehat{\mathscr{B}}$ *such that*

$$\int_{S^{n-1}} \varphi\big(h_{\widehat{M}}(u)\big) d\mu(u) = \inf\left\{\int_{S^{n-1}} \varphi(h_Q(u)) d\mu(u) : Q \in \widehat{\mathscr{B}}\right\}.$$

Moreover,

$$\lim_{i\to\infty} \int_{S^{n-1}} \varphi\big(h_{\widehat{M_i}}(u)\big) d\mu_i(u) = \int_{S^{n-1}} \varphi\big(h_{\widehat{M}}(u)\big) d\mu(u).$$

Proof. For each $i \in \mathbb{N}$, let

$$\mu_i(S^{n-1}) = \int_{S^{n-1}} d\mu_i \quad \text{and} \quad \int_{S^{n-1}} d\mu = \mu(S^{n-1}).$$

(i) It can be easily checked from (6.18) and $B^n \in \widetilde{\mathscr{B}}$ that for each $i \in \mathbb{N}$,

$$\int_{S^{n-1}} \varphi\big(h_{\widetilde{M}_i}(u)\big)d\mu_i(u) \le \varphi(1)\mu_i(S^{n-1}).$$

Moreover, the weak convergence of $\mu_i \to \mu$ yields $\mu_i(S^{n-1}) \to \mu(S^{n-1})$. Hence,

$$\sup_{i\ge 1}\bigg\{\int_{S^{n-1}} \varphi(h_{\widetilde{M}_i}(u))d\mu_i(u)\bigg\} < \infty.$$

By Lemma 6.2.5, one sees that $\{\widetilde{M}_i\}_{i\ge1}$ is a bounded sequence in $\mathscr{K}^n_{(o)}$. As $\widetilde{M}_i \in \widetilde{\mathscr{B}}$ for each $i \in \mathbb{N}$, Lemma 6.1.4 implies that there exist a subsequence $\{\widetilde{M}_{i_j}\}_{j\ge1}$ of $\{\widetilde{M}_i\}_{i\ge1}$ and a convex body $\widetilde{M} \in \widetilde{\mathscr{B}}$ such that $\widetilde{M}_{i_j} \to \widetilde{M}$.

Now we verify that $\widetilde{M}$ satisfies the desired properties. First of all, for any given $Q \in \widetilde{\mathscr{B}}$, one has, for each $j \in \mathbb{N}$,

$$\int_{S^{n-1}} \varphi\big(h_{\widetilde{M}_{i_j}}(u)\big)d\mu_{i_j}(u) \le \int_{S^{n-1}} \varphi\big(h_Q(u)\big)d\mu_{i_j}(u).$$

Together with the weak convergence of $\mu_i \to \mu$, Lemma 2.3.3, $\varphi \in \mathscr{I}$, and $\widetilde{M}_{i_j} \to \widetilde{M}$, one obtains that $\varphi\big(h_{\widetilde{M}_{i_j}}\big) \to \varphi(h_{\widetilde{M}})$ uniformly on S^{n-1} and for each given $Q \in \widetilde{\mathscr{B}}$,

$$\begin{aligned}
\int_{S^{n-1}} \varphi\big(h_{\widetilde{M}}(u)\big)d\mu(u) &= \lim_{j\to\infty}\int_{S^{n-1}} \varphi\big(h_{\widetilde{M}_{i_j}}(u)\big)d\mu_{i_j}(u)\\
&\le \lim_{j\to\infty}\int_{S^{n-1}} \varphi\big(h_Q(u)\big)d\mu_{i_j}(u)\\
&= \int_{S^{n-1}} \varphi\big(h_Q(u)\big)d\mu(u).
\end{aligned}$$

Taking the infimum over $Q \in \widetilde{\mathscr{B}}$ and together with $\widetilde{M} \in \widetilde{\mathscr{B}}$, one gets that

$$\int_{S^{n-1}} \varphi\big(h_{\widetilde{M}}(u)\big)d\mu(u) \le \inf_{Q\in\widetilde{\mathscr{B}}}\bigg\{\int_{S^{n-1}} \varphi\big(h_Q(u)\big)d\mu(u)\bigg\} \le \int_{S^{n-1}} \varphi\big(h_{\widetilde{M}}(u)\big)d\mu(u).$$

Hence, $\widetilde{M} \in \widetilde{\mathscr{B}}$ verifies (6.19).

Now let us verify (6.20). To this end, let $\{\mu_{i_k}\}_{k\ge1}$ be an arbitrary subsequence of $\{\mu_i\}_{i\ge1}$. Repeating the arguments above for μ_{i_k} and $\widetilde{M}_{i_k}$ (replacing μ_i and $\widetilde{M}_i$, respectively), one gets a subsequence $\{\widetilde{M}_{i_{k_j}}\}_{j\ge1}$ of $\{\widetilde{M}_{i_k}\}_{k\ge1}$ such that $\widetilde{M}_{i_{k_j}} \to \widetilde{M}_0 \in$

$\widetilde{\mathscr{B}}$ and $\widetilde{M_0}$ satisfies (6.19). Thus,

$$\begin{aligned}\lim_{j\to\infty}\int_{S^{n-1}}\varphi\Big(h_{\widetilde{M}_{i_{k_j}}}(u)\Big)d\mu_{i_{k_j}}(u) &= \int_{S^{n-1}}\varphi\big(h_{\widetilde{M_0}}(u)\big)d\mu(u)\\ &= \inf\left\{\int_{S^{n-1}}\varphi(h_Q(u))d\mu(u) : Q\in\widetilde{\mathscr{B}}\right\}\\ &= \int_{S^{n-1}}\varphi\big(h_{\widetilde{M}}(u)\big)d\mu(u),\end{aligned}$$

where the first equality follows from Lemma 2.3.3 and the last two equalities follow from (6.19). This concludes the proof of (6.20), i.e.,

$$\lim_{i\to\infty}\int_{S^{n-1}}\varphi\big(h_{\widetilde{M}_i}(u)\big)d\mu_i(u) = \int_{S^{n-1}}\varphi\big(h_{\widetilde{M}}(u)\big)d\mu(u).$$

(ii) The proof of this case is almost identical to the one in (i), and will be omitted. □

The following theorem provides the existence and uniqueness of solutions to Problem 6.2.1 for $\varphi\in\mathscr{I}$ and with the infimum considered.

Theorem 6.2.7. *Let $\varphi\in\mathscr{I}$ and μ be a nonzero finite Borel measure defined on S^{n-1} which is not concentrated on any closed hemisphere. Let $G\in\mathscr{G}_I$ be a continuous function such that (6.5) holds for some $q\geq n-1$. Then the following statements hold.*

(i) There exists a convex body $\widetilde{M}\in\widetilde{\mathscr{B}}$ such that

$$\int_{S^{n-1}}\varphi\big(h_{\widetilde{M}}(u)\big)d\mu(u) = \inf\left\{\int_{S^{n-1}}\varphi(h_Q(u))d\mu(u) : Q\in\widetilde{\mathscr{B}}\right\}. \tag{6.21}$$

If, in addition, both $\varphi(t)$ and $G(t,\cdot)$ are convex on $t\in(0,\infty)$, then the solution is unique.

(ii) There exists a convex body $\widehat{M}\in\widehat{\mathscr{B}}$ such that

$$\int_{S^{n-1}}\varphi\big(h_{\widehat{M}}(u)\big)d\mu(u) = \inf\left\{\int_{S^{n-1}}\varphi(h_Q(u))d\mu(u) : Q\in\widehat{\mathscr{B}}\right\}.$$

If, in addition, both $\varphi(t)$ and $G(t,\cdot)$ are convex on $t\in(0,\infty)$, then the solution is unique.

Proof. Let μ be a nonzero finite Borel measure defined on S^{n-1} which is not concentrated on any closed hemisphere. Let μ_i for all $i\in\mathbb{N}$ be nonzero finite discrete Borel measures defined on S^{n-1}, which are not concentrated on any closed hemisphere, such that, $\mu_i\to\mu$ weakly (see e.g., [59]).

(i) By Theorem 6.2.3, for each $i\in\mathbb{N}$, there exists a polytope $\widetilde{P}_i\in\widetilde{\mathscr{B}}$ solving (6.21) with μ replaced by μ_i. It follows from Theorem 6.2.6 that there exists a $\widetilde{M}\in\widetilde{\mathscr{B}}$ such that (6.21) holds.

Now let us prove the uniqueness. Assume that $\widetilde{M}\in\widetilde{\mathscr{B}}$ and $\widetilde{M}_0\in\widetilde{\mathscr{B}}$, such that

$$\int_{S^{n-1}}\varphi\big(h_{\widetilde{M}}(u)\big)d\mu(u)=\int_{S^{n-1}}\varphi\big(h_{\widetilde{M}_0}(u)\big)d\mu(u)=\inf\left\{\int_{S^{n-1}}\varphi(h_Q(u))d\mu(u):Q\in\widetilde{\mathscr{B}}\right\}.$$

Note that both $\widetilde{M}\in\mathscr{K}^n_{(o)}$ and $\widetilde{M}_0\in\mathscr{K}^n_{(o)}$. Let $K_0=\frac{\widetilde{M}+\widetilde{M}_0}{2}\in\mathscr{K}^n_{(o)}$. Then,

$$h_{K_0}=\frac{h_{\widetilde{M}}+h_{\widetilde{M}_0}}{2}\Longrightarrow\ \rho_{K_0^*}=2\cdot\frac{\rho_{\widetilde{M}^*}\cdot\rho_{\widetilde{M}_0^*}}{\rho_{\widetilde{M}^*}+\rho_{\widetilde{M}_0^*}},$$

following from $h_K\cdot\rho_{K^*}=1$ for all $K\in\mathscr{K}^n_{(o)}$. The facts that $G(t,\cdot)$ is convex and $G\in\mathscr{G}_I$ is strictly increasing, together with $\widetilde{M}\in\widetilde{\mathscr{B}}$ and $\widetilde{M}_0\in\widetilde{\mathscr{B}}$, yield that

$$\begin{aligned}
\widetilde{V}_G(K_0^*)&=\int_{S^{n-1}}G\big(\rho_{K_0^*}(u),u\big)\,du\\
&\le\int_{S^{n-1}}G\left(2\cdot\frac{\rho_{\widetilde{M}^*}(u)\cdot\rho_{\widetilde{M}_0^*}(u)}{\rho_{\widetilde{M}^*}(u)+\rho_{\widetilde{M}_0^*}(u)},\ u\right)du\\
&\le\int_{S^{n-1}}G\left(\frac{\rho_{\widetilde{M}^*}(u)+\rho_{\widetilde{M}_0^*}(u)}{2},u\right)du\\
&\le\int_{S^{n-1}}\frac{G\big(\rho_{\widetilde{M}^*}(u),u\big)+G\big(\rho_{\widetilde{M}_0^*}(u),u\big)}{2}\,du\\
&=\frac{\widetilde{V}_G(\widetilde{M}^*)+\widetilde{V}_G(\widetilde{M}_0^*)}{2}\\
&=\widetilde{V}_G(B^n).
\end{aligned}\tag{6.22}$$

Again, as $G\in\mathscr{G}_I$, one can find a constant $t_2\ge1$ such that $\widetilde{V}_G(t_2K_0^*)=\widetilde{V}_G(B^n)$ and $K_0/t_2\in\widetilde{\mathscr{B}}$. Due to $t_2\ge1$ and the facts that $\varphi\in\mathscr{I}$ is convex and strictly increasing, one has

$$\begin{aligned}\int_{S^{n-1}} \varphi(h_{K_0/t_2}(u))\, d\mu(u) &\geq \inf\left\{\int_{S^{n-1}} \varphi(h_Q(u)) d\mu(u) : Q \in \widetilde{\mathscr{B}}\right\} \\ &= \frac{1}{2}\bigg(\int_{S^{n-1}} \varphi\big(h_{\widetilde{M}}(u)\big) d\mu(u) + \int_{S^{n-1}} \varphi\big(h_{\widetilde{M}_0}(u)\big) d\mu(u)\bigg) \\ &\geq \int_{S^{n-1}} \varphi\bigg(\frac{h_{\widetilde{M}}(u) + h_{\widetilde{M}_0}(u)}{2}\bigg) d\mu(u) \\ &= \int_{S^{n-1}} \varphi(h_{K_0}(u))\, d\mu(u) \\ &\geq \int_{S^{n-1}} \varphi(h_{K_0/t_2}(u))\, d\mu(u). \end{aligned} \tag{6.23}$$

Hence all "$\geq$" in (6.23) become "$=$"; and this can happen if and only if $t_2 = 1$ as φ is strictly increasing. This in turn yields that all "$\geq$" in (6.22) become "$=$" as well. In particular, as $G(t, \cdot)$ is strictly increasing, for all $u \in S^{n-1}$,

$$2 \cdot \frac{\rho_{\widetilde{M}^*}(u) \cdot \rho_{\widetilde{M}_0^*}(u)}{\rho_{\widetilde{M}^*}(u) + \rho_{\widetilde{M}_0^*}(u)} = \frac{\rho_{\widetilde{M}^*}(u) + \rho_{\widetilde{M}_0^*}(u)}{2},$$

and hence $\rho_{\widetilde{M}^*}(u) = \rho_{\widetilde{M}_0^*}(u)$ for all $u \in S^{n-1}$. That is, $\widetilde{M} = \widetilde{M}_0$ and the uniqueness follows.

(ii) The proof of this case is almost identical to the one in (i), and will be omitted. $\square$

The following result states that the continuity of solutions to Problem 6.2.1 for $\varphi \in \mathscr{I}$ and with the infimum considered.

Corollary 6.2.8. *Let μ_i, μ for $i \in \mathbb{N}$ be nonzero finite Borel measures on S^{n-1} which are not concentrated on any closed hemisphere and $\mu_i \to \mu$ weakly. Let $G \in \mathscr{G}_I$ be a continuous function such that $G(t, \cdot)$ is convex on $t \in (0, \infty)$ and (6.5) holds for some $q \geq n-1$. Let $\varphi \in \mathscr{I}$ be convex. The following statements hold true.*

(i) Let $\widetilde{M}_i \in \widetilde{\mathscr{B}}$ for each $i \in \mathbb{N}$ and $\widetilde{M} \in \widetilde{\mathscr{B}}$ be the solutions to the optimization problem (6.11) with the infimum considered for measures μ_i and μ, respectively. Then $\widetilde{M}_i \to \widetilde{M}$ as $i \to \infty$.

(ii) Let $\widehat{M}_i \in \widehat{\mathscr{B}}$ for each $i \in \mathbb{N}$ and $\widehat{M} \in \widehat{\mathscr{B}}$ be the solutions to the optimization problem (6.12) with the infimum considered for measures μ_i and μ, respectively. Then

$\widehat{M}_i \to \widehat{M}$ *as* $i \to \infty$.

Proof. (i) The proof of this result follows from the combination of the proof of Theorem 6.2.6 and the uniqueness in Theorem 6.2.7. Indeed, let $\{\widetilde{M}_{i_k}\}_{k\geq 1}$ be an arbitrary subsequence of $\{\widetilde{M}_i\}_{i\geq 1}$. Like in the proof of Theorem 6.2.6, one can check that there exist a subsequence $\{\widetilde{M}_{i_{k_j}}\}_{j\geq 1}$ of $\{\widetilde{M}_{i_k}\}_{k\geq 1}$ and a convex body $\widetilde{M}_0 \in \widetilde{\mathscr{B}}$ such that $\widetilde{M}_{i_{k_j}} \to \widetilde{M}_0$. Moreover, $\widetilde{M}_0$ satisfies that

$$\int_{S^{n-1}} \varphi\big(h_{\widetilde{M}_0}(u)\big) d\mu(u) = \inf \left\{ \int_{S^{n-1}} \varphi(h_Q(u)) d\mu(u) : Q \in \widetilde{\mathscr{B}} \right\}.$$

The uniqueness in Theorem 6.2.7 yields $\widetilde{M}_0 = \widetilde{M}$.

In other words, one shows that every subsequence $\{M_{i_k}\}_{k\geq 1}$ of $\{M_i\}_{i\geq 1}$ must have a subsequence $\widetilde{M}_{i_{k_j}}$ convergent to $\widetilde{M}$. This concludes that $\widetilde{M}_i \to \widetilde{M}$.

(ii) The proof of this case is almost identical to the one in (i), and will be omitted. □

Problem 6.2.1 discussed in Section 6.2 is only typical example of the polar Orlicz-Minkowski type problems. In the following section, several variations of Problem 6.2.1 will be provided.

6.3 Variations of dual-polar Orlicz-Minkowski problem

First, we investigate the general dual-polar Orlicz-Minkowski problem associated with the Orlicz norms.

6.3.1 The general dual-polar Orlicz-Minkowski problem associated with the Orlicz norms

Let μ be a given nonzero finite Borel measure defined on S^{n-1}. For $\varphi \in \mathscr{I} \cup \mathscr{D}$ and for $Q \in \mathscr{K}^n_{(o)}$, the functional $\int_{S^{n-1}} \varphi(h_Q)\, d\mu$ is in general not homogeneous. Similar to the definition for $\widehat{V}_G$, based on (5.13), we give a systematic definition of the "Orlicz norm" with respect to $\varphi \in \mathscr{I} \cup \mathscr{D}$ and $Q \in \mathscr{K}^n_{(o)}$ as follows:

$$\begin{aligned} \|h_Q\|_{\mu,\varphi} &= \inf\left\{\lambda > 0 : \frac{1}{\mu(S^{n-1})}\int_{S^{n-1}} \varphi\left(\frac{h_Q(u)}{\lambda}\right) d\mu(u) \leq 1\right\} \quad \text{if } \varphi \in \mathscr{I}; \\ \|h_Q\|_{\mu,\varphi} &= \inf\left\{\lambda > 0 : \frac{1}{\mu(S^{n-1})}\int_{S^{n-1}} \varphi\left(\frac{h_Q(u)}{\lambda}\right) d\mu(u) \geq 1\right\} \quad \text{if } \varphi \in \mathscr{D}. \end{aligned}$$

Following the proof of Proposition 6.1.1, it can be checked that, for any $Q \in \mathscr{K}^n_{(o)}$ and $\varphi \in \mathscr{I} \cup \mathscr{D}$, $\|h_Q\|_{\mu,\varphi} > 0$ satisfies

$$\frac{1}{\mu(S^{n-1})}\int_{S^{n-1}} \varphi\left(\frac{h_Q(u)}{\|h_Q\|_{\mu,\varphi}}\right) d\mu = 1. \tag{6.24}$$

Moreover, $\|1\|_{\mu,\varphi} = 1$, $\|ch_Q\|_{\mu,\varphi} = c\|h_Q\|_{\mu,\varphi}$ for any constant $c > 0$ and any $Q \in \mathscr{K}^n_{(o)}$, and $\|h_Q\|_{\mu,\varphi} \leq \|h_L\|_{\mu,\varphi}$ for $Q, L \in \mathscr{K}^n_{(o)}$ such that $Q \subseteq L$.

The following lemma for $\varphi \in \mathscr{I} \cup \mathscr{D}$ can be proved similar to the proof of Proposition 6.1.2 (ii). For completeness, we provide a brief proof here. See e.g., [22, Lemma 4] and [27, Lemma 3.4 and Corollary 3.5] for similar results.

Lemma 6.3.1. *Let $Q_i, Q \in \mathscr{K}^n_{(o)}$ for each $i \in \mathbb{N}$, and μ_i, μ for each $i \in \mathbb{N}$ be nonzero finite Borel measures on S^{n-1}. If $Q_i \to Q$ and $\mu_i \to \mu$ weakly, then for all $\varphi \in \mathscr{I} \cup \mathscr{D}$,*

$$\lim_{i\to\infty} \|h_{Q_i}\|_{\mu_i,\varphi} = \|h_Q\|_{\mu,\varphi}.$$

Proof. We only prove the case for $\varphi \in \mathscr{I}$ (and the case for $\varphi \in \mathscr{D}$ follows along the same lines). Let $Q_i \in \mathscr{K}^n_{(o)}$ for all $i \in \mathbb{N}$ and $Q_i \to Q \in \mathscr{K}^n_{(o)}$. Let the constants $0 < r_Q < R_Q < \infty$ be such that $r_Q \leq h_Q \leq R_Q$ and $r_Q \leq h_{Q_i} \leq R_Q$ for all $i \in \mathbb{N}$. It can be checked that

$$r_Q \leq \inf_{i\geq 1} \|h_{Q_i}\|_{\mu_i,\varphi} \leq \sup_{i\geq 1} \|h_{Q_i}\|_{\mu_i,\varphi} \leq R_Q.$$

Assume that $\limsup_{i\to\infty} \|h_{Q_i}\|_{\mu_i,\varphi} > \|h_Q\|_{\mu,\varphi}$. There exists a subsequence $\{Q_{i_j}\}$ of $\{Q_i\}$ such that $\lim_{j\to\infty} \|h_{Q_{i_j}}\|_{\mu_{i_j},\varphi} > \|h_Q\|_{\mu,\varphi}$. Together with (6.24), Lemma 2.3.3, the uniform convergence of $h_{Q_i} \to h_Q$ on S^{n-1}, and the weak convergence of $\mu_i \to \mu$, one has

$$
\begin{aligned}
1 &= \lim_{j\to\infty} \frac{1}{\mu_{i_j}(S^{n-1})} \int_{S^{n-1}} \varphi\left(\frac{h_{Q_{i_j}}(u)}{\|h_{Q_{i_j}}\|_{\mu_{i_j},\varphi}}\right) d\mu_{i_j} \\
&= \frac{1}{\mu(S^{n-1})} \int_{S^{n-1}} \varphi\left(\frac{h_Q(u)}{\lim_{j\to\infty}\|h_{Q_{i_j}}\|_{\mu_{i_j},\varphi}}\right) d\mu \\
&< \frac{1}{\mu(S^{n-1})} \int_{S^{n-1}} \varphi\left(\frac{h_Q(u)}{\|h_Q\|_{\mu,\varphi}}\right) d\mu \\
&= 1.
\end{aligned}
$$

This is a contradiction and hence $\limsup_{i\to\infty}\|h_{Q_i}\|_{\mu_i,\varphi} \le \|h_Q\|_{\mu,\varphi}$. Similarly, one can obtain $\liminf_{i\to\infty}\|h_{Q_i}\|_{\mu_i,\varphi} \ge \|h_Q\|_{\mu,\varphi}$, which leads to $\lim_{i\to\infty}\|h_{Q_i}\|_{\mu_i,\varphi} = \|h_Q\|_{\mu,\varphi}$ as desired. □

For the convenience of later citation, the following lemma is given, whose proof for polytopes and discrete measures has appeared in e.g., [19, 24, 27] and is similar to the proof of Lemma 6.2.5. A brief sketch of the proof is provided for completeness and for future reference.

Lemma 6.3.2. *Let $\varphi \in \mathscr{I}$. Let μ_i, μ for $i \in \mathbb{N}$ be nonzero finite Borel measures on S^{n-1} which are not concentrated on any closed hemisphere and $\mu_i \to \mu$ weakly. Suppose that $\{Q_i\}_{i\ge1}$ is a sequence of convex bodies such that $Q_i \in \mathscr{K}^n_{(o)}$ for each $i \in \mathbb{N}$ and $\sup_{i\ge1}\|h_{Q_i}\|_{\mu_i,\varphi} < \infty$. Then $\{Q_i\}_{i\ge1}$ is a bounded sequence in $\mathscr{K}^n_{(o)}$.*

Proof. Let $a_+ = \max\{a, 0\}$ for all $a \in \mathbb{R}$. For each $i \in \mathbb{N}$, let $u_i \in S^{n-1}$ be such that $\rho_{Q_i}(u_i) = \max_{u\in S^{n-1}} \rho_{Q_i}(u)$, and hence $h_{Q_i}(u) \ge \rho_{Q_i}(u_i)\langle u, u_i\rangle_+$ for any $u \in S^{n-1}$. Assume that $\{Q_i\}_{i\ge1}$ is not bounded in $\mathscr{K}^n_{(o)}$, i.e., $\sup_{i\ge1}\rho_{Q_i}(u_i) = \infty$. Without loss of generality, let $u_i \to v \in S^{n-1}$ and $\lim_{i\to\infty}\rho_{Q_i}(u_i) = \infty$. By formula (6.24) and $\varphi \in \mathscr{I}$, one has for any given $C > 0$, there exists $i_C \in \mathbb{N}$ such that for all $i > i_C$,

$$
\begin{aligned}
1 &= \frac{1}{\mu_i(S^{n-1})} \int_{S^{n-1}} \varphi\left(\frac{h_{Q_i}(u)}{\|h_{Q_i}\|_{\mu_i,\varphi}}\right) d\mu_i(u) \\
&\ge \frac{1}{\mu_i(S^{n-1})} \int_{S^{n-1}} \varphi\left(\frac{\rho_{Q_i}(u_i)\langle u, u_i\rangle_+}{\sup_{i\ge1}\|h_{Q_i}\|_{\mu_i,\varphi}}\right) d\mu_i(u) \\
&\ge \frac{1}{\mu_i(S^{n-1})} \int_{S^{n-1}} \varphi\left(\frac{C\cdot\langle u, u_i\rangle_+}{\sup_{i\ge1}\|h_{Q_i}\|_{\mu_i,\varphi}}\right) d\mu_i(u).
\end{aligned}
$$

By Lemma 2.3.3, the uniform convergence of $\langle u, u_i\rangle_+ \to \langle u, v\rangle_+$ on S^{n-1} as $u_i \to v$, the weak convergence of $\mu_i \to \mu$, and $\varphi \in \mathscr{I}$, one gets

$$
\begin{aligned}
1 &\geq \lim_{i\to\infty} \frac{1}{\mu_i(S^{n-1})} \int_{S^{n-1}} \varphi\left(\frac{C\cdot \langle u, u_i\rangle_+}{\sup_{i\geq 1}\|h_{Q_i}\|_{\mu_i,\varphi}}\right) d\mu_i(u) \\
&= \frac{1}{\mu(S^{n-1})} \int_{S^{n-1}} \varphi\left(\frac{C\cdot \langle u, v\rangle_+}{\sup_{i\geq 1}\|h_{Q_i}\|_{\mu_i,\varphi}}\right) d\mu(u) \\
&\geq \frac{1}{\mu(S^{n-1})}\cdot \varphi\left(\frac{C\cdot c_0}{\sup_{i\geq 1}\|h_{Q_i}\|_{\mu_i,\varphi}}\right)\cdot \int_{\{u\in S^{n-1}:\langle u,v\rangle\geq c_0\}} d\mu(u),
\end{aligned}
$$

where $c_0 > 0$ is a finite constant (which always exists due to the monotone convergence theorem and the assumption that μ is not concentrated on any closed hemisphere) such that $\int_{\{u\in S^{n-1}:\langle u,v\rangle\geq c_0\}} d\mu(u) > 0$. Taking $C \to \infty$, the fact that $\lim_{t\to\infty}\varphi(t) = \infty$ then yields a contradiction to $1 < \infty$. This concludes that $\{Q_i\}_{i\geq 1}$ is a bounded sequence in $\mathscr{K}^n_{(o)}$. $\square$

Our first variation of Problem 6.2.1 is the following general dual-polar Orlicz-Minkowski problem associated with the Orlicz norms:

Problem 6.3.3. *Under what conditions on a nonzero finite Borel measure μ defined on S^{n-1}, continuous functions $\varphi : (0,\infty) \to (0,\infty)$ and $G \in \mathscr{G}_I \cup \mathscr{G}_d$ can we find a convex body $K \in \mathscr{K}^n_{(o)}$ solving the following optimization problems:*

$$\inf/\sup\left\{\|h_Q\|_{\mu,\varphi} : \quad Q \in \widetilde{\mathscr{B}}\right\}; \tag{6.25}$$

$$\inf/\sup\left\{\|h_Q\|_{\mu,\varphi} : \quad Q \in \widehat{\mathscr{B}}\right\}. \tag{6.26}$$

Due to the high similarity of properties of $\int_{S^{n-1}} \varphi(h_Q)\, d\mu$ and $\|h_Q\|_{\mu,\varphi}$, results and their proofs in Section 6.2 can be extended and adopted to Problem 6.3.3. For instance, the existence of solutions to Problem 6.3.3, if the infimum is considered, can be obtained.

Theorem 6.3.4. *Let $\varphi \in \mathscr{I}$ and μ be a nonzero finite Borel measure defined on S^{n-1} which is not concentrated on any closed hemisphere. Let $G \in \mathscr{G}_I$ be a continuous function such that (6.5) holds for some $q \geq n-1$. Then the following statements hold.*

(i) There exists $\widetilde{M} \in \widetilde{\mathscr{B}}$ *such that*

$$\|h_{\widetilde{M}}\|_{\mu,\varphi} = \inf\left\{\|h_Q\|_{\mu,\varphi} :\ Q \in \widetilde{\mathscr{B}}\right\}. \tag{6.27}$$

If, in addition, both $\varphi(t)$ *and* $G(t,\cdot)$ *are convex on* $t \in (0,\infty)$*, then the solution is unique.*

(ii) There exists $\widehat{M} \in \widehat{\mathscr{B}}$ *such that*

$$\|h_{\widehat{M}}\|_{\mu,\varphi} = \inf\left\{\|h_Q\|_{\mu,\varphi} :\ Q \in \widehat{\mathscr{B}}\right\}.$$

If, in addition, both $\varphi(t)$ *and* $G(t,\cdot)$ *are convex on* $t \in (0,\infty)$*, then the solution is unique.*

Proof. Only the brief proof for (i) is provided and the proof for (ii) follows along the same lines. First of all, $B^n \in \widetilde{\mathscr{B}}$, and the optimization problem (6.27) is well defined. In particular, there exists a sequence $\{Q_i\}_{i\geq 1}$ such that each $Q_i \in \widetilde{\mathscr{B}}$ and

$$\lim_{i\to\infty} \|h_{Q_i}\|_{\mu,\varphi} = \inf\left\{\|h_Q\|_{\mu,\varphi} :\ Q \in \widetilde{\mathscr{B}}\right\} < \infty.$$

This further implies that $\sup_{i\geq 1} \|h_{Q_i}\|_{\mu,\varphi} < \infty$, which in turn yields the existence of a subsequence $\{Q_{i_j}\}_{j\geq 1}$ of $\{Q_i\}_{i\geq 1}$ and $\widetilde{M} \in \widetilde{\mathscr{B}}$, such that $Q_{i_j} \to \widetilde{M}$, by Lemmas 6.1.4 and 6.3.2. It then follows from Lemma 6.3.1 that $\lim_{i\to\infty} \|h_{Q_i}\|_{\mu,\varphi} = \lim_{j\to\infty} \|h_{Q_{i_j}}\|_{\mu,\varphi} = \|h_{\widetilde{M}}\|_{\mu,\varphi}$. This concludes the proof, if one notices $\widetilde{M} \in \widetilde{\mathscr{B}}$, for the existence of solutions to the optimization problem (6.27).

For the uniqueness, assume that $\widetilde{M} \in \widetilde{\mathscr{B}}$ and $\widetilde{M_0} \in \widetilde{\mathscr{B}}$, such that

$$\|h_{\widetilde{M}}\|_{\mu,\varphi} = \|h_{\widetilde{M_0}}\|_{\mu,\varphi} = \inf\left\{\|h_Q\|_{\mu,\varphi} :\ Q \in \widetilde{\mathscr{B}}\right\}. \tag{6.28}$$

Note that $G(t,\cdot)$ is convex and $G \in \mathscr{G}_I$ is strictly increasing. Let $K_0 = \frac{\widetilde{M}+\widetilde{M_0}}{2}$. By (6.22), there is a constant $t_2 \geq 1$ such that $\widetilde{V}_G(t_2 K_0^*) = \widetilde{V}_G(B^n)$ and hence $K_0/t_2 \in \widetilde{\mathscr{B}}$. It follows from (6.24), (6.28), $t_2 \geq 1$ and $\varphi \in \mathscr{I}$ being convex and strictly increasing that

$$\begin{aligned}\mu(S^{n-1}) &= \int_{S^{n-1}} \varphi\bigg(\frac{h_{K_0}(u)}{\|h_{K_0}\|_{\mu,\varphi}}\bigg)\, d\mu \\ &= \frac{1}{2}\bigg[\int_{S^{n-1}} \varphi\bigg(\frac{h_{\widetilde{M}}}{\|h_{\widetilde{M}}\|_{\mu,\varphi}}\bigg)\, d\mu + \int_{S^{n-1}} \varphi\bigg(\frac{h_{\widetilde{M}_0}(u)}{\|h_{\widetilde{M}_0}\|_{\mu,\varphi}}\bigg)\, d\mu\bigg] \\ &\geq \int_{S^{n-1}} \varphi\bigg(\frac{h_{K_0}(u)}{\|h_{\widetilde{M}_0}\|_{\mu,\varphi}}\bigg)\, d\mu,\end{aligned}$$

and hence $\|h_{\widetilde{M}_0}\|_{\mu,\varphi} \geq \|h_{K_0}\|_{\mu,\varphi} \geq \|h_{K_0/t_2}\|_{\mu,\varphi} \geq \|h_{\widetilde{M}_0}\|_{\mu,\varphi}$. Thus, all "$\geq$" become "$=$"; and this can happen if and only if $t_2 = 1$. This in turn yields that all "$\geq$" in (6.22) become "$=$" as well. In particular, $\widetilde{M} = \widetilde{M}_0$ and the uniqueness follows. $\square$

Our second example is the continuity for Problem 6.3.3 and its solutions.

Theorem 6.3.5. *Let μ_i, μ for $i \in \mathbb{N}$ be finite Borel measures on S^{n-1} which are not concentrated on any closed hemisphere and $\mu_i \to \mu$ weakly. Let $G \in \mathscr{G}_I$ be a continuous function such that (6.5) holds for some $q \geq n-1$ and $\varphi \in \mathscr{I}$. The following statements hold.*

(i) Let $\widetilde{M}_i, \widetilde{M} \in \widetilde{\mathscr{B}}$, for all $i \in \mathbb{N}$, be solutions to the optimization problem (6.25), with the infimum considered, for measures μ_i and μ, respectively. Then, $\lim_{i\to\infty} \|h_{\widetilde{M}_i}\|_{\mu_i,\varphi} = \|h_{\widetilde{M}}\|_{\mu,\varphi}$. If, in addition, both $\varphi(t)$ and $G(t,\cdot)$ are convex on $t \in (0,\infty)$, then $\widetilde{M}_i \to \widetilde{M}$ as $i \to \infty$.

(ii) Let $\widehat{M}_i, \widehat{M} \in \widehat{\mathscr{B}}$, for all $i \in \mathbb{N}$, be solutions to the optimization problem (6.26), with the infimum considered, for measures μ_i and μ, respectively. Then, $\lim_{i\to\infty} \|h_{\widehat{M}_i}\|_{\mu_i,\varphi} = \|h_{\widehat{M}}\|_{\mu,\varphi}$. If, in addition, both $\varphi(t)$ and $G(t,\cdot)$ are convex on $t \in (0,\infty)$, then $\widehat{M}_i \to \widehat{M}$ as $i \to \infty$.

Proof. Only the brief proof for (i) is provided and the proof for (ii) follows along the same lines. It follows from $B^n \in \widetilde{\mathscr{B}}$, (6.24), and $\varphi \in \mathscr{I}$, in particular $\varphi(1) = 1$ that

$$\sup_{i\geq 1} \|h_{\widetilde{M}_i}\|_{\mu_i,\varphi} \leq \sup_{i\geq 1} \|h_{B^n}\|_{\mu_i,\varphi} = 1.$$

Lemma 6.3.2 yields that $\{\widetilde{M}_i\}_{i\geq 1}$ is a bounded sequence.

Let $\{\widetilde{M}_{i_k}\}_{k\geq 1}$ be an arbitrary subsequence of $\{\widetilde{M}_i\}_{i\geq 1}$. Lemma 6.1.4 yields the existence of a subsequence $\{\widetilde{M}_{i_{k_j}}\}_{j\geq 1}$ of $\{\widetilde{M}_{i_k}\}_{k\geq 1}$ and $\widetilde{M}_0 \in \widetilde{\mathscr{B}}$ such that $\widetilde{M}_{i_{k_j}} \to \widetilde{M}_0$.

Together with the minimality of $\|h_{\widetilde{M}_{i_{k_j}}}\|_{\mu_{i_{k_j}},\varphi}$, Lemma 6.3.1 and the weak convergence of $\mu_i \to \mu$ imply that

$$\|h_{\widetilde{M}_0}\|_{\mu,\varphi} = \lim_{j\to\infty} \|h_{\widetilde{M}_{i_{k_j}}}\|_{\mu_{i_{k_j}},\varphi} \leq \lim_{j\to\infty} \|h_Q\|_{\mu_{i_{k_j}},\varphi} = \|h_Q\|_{\mu,\varphi},$$

for all $Q \in \widetilde{\mathscr{B}}$. Taking the infimum over $Q \in \widetilde{\mathscr{B}}$ and together with $\widetilde{M}_0 \in \widetilde{\mathscr{B}}$, one gets that

$$\|h_{\widetilde{M}_0}\|_{\mu,\varphi} \leq \inf_{Q\in\widetilde{\mathscr{B}}} \|h_Q\|_{\mu,\varphi} = \|h_{\widetilde{M}}\|_{\mu,\varphi} \leq \|h_{\widetilde{M}_0}\|_{\mu,\varphi}. \tag{6.29}$$

In conclusion, every subsequence $\{\widetilde{M}_{i_k}\}_{k\geq 1}$ of $\{\widetilde{M}_i\}_{i\geq 1}$ has a subsequence $\{\widetilde{M}_{i_{k_j}}\}_{j\geq 1}$ such that

$$\|h_{\widetilde{M}}\|_{\mu,\varphi} = \lim_{j\to\infty} \|h_{\widetilde{M}_{i_{k_j}}}\|_{\mu_{i_{k_j}},\varphi},$$

which implies $\lim_{i\to\infty} \|h_{\widetilde{M}_i}\|_{\mu_i,\varphi} = \|h_{\widetilde{M}}\|_{\mu,\varphi}$.

Formula (6.29) asserts that $\widetilde{M}_0 \in \widetilde{\mathscr{B}}$ solves the optimization problem (6.25) with the infimum considered. If, in addition, both $\varphi(t)$ and $G(t,\cdot)$ are convex on $t \in (0,\infty)$, the uniqueness in Theorem 6.3.4 implies $\widetilde{M}_0 = \widetilde{M}$. In conclusion, every subsequence $\{\widetilde{M}_{i_k}\}_{k\geq 1}$ of $\{\widetilde{M}_i\}_{i\geq 1}$ has a subsequence $\{\widetilde{M}_{i_{k_j}}\}_{j\geq 1}$ such that $\widetilde{M}_{i_{k_j}} \to \widetilde{M}$. Hence $\widetilde{M}_i \to \widetilde{M}$ as $i \to \infty$. $\square$

An argument almost identical to Lemma 6.2.2 shows that, if $\varphi \in \mathscr{I}$ and $G \in \mathscr{G}_I$ satisfying (6.5) for some $q \geq n-1$, the solutions to Problem 6.3.3 with the infimum considered for μ being a discrete measure defined in (6.13) (whose support $\{u_1, \cdots, u_m\}$ is not concentrated on any closed hemisphere) must be polytopes with $\{u_1, \cdots, u_m\}$ being the corresponding unit normal vectors of their faces. Counterexamples in Proposition 6.2.4 can be used to prove that the solutions to Problem 6.3.3 may not exist if $\varphi \in \mathscr{I} \cup \mathscr{D}$ and the supremum is considered or if $\varphi \in \mathscr{D}$ and the infimum is considered. We leave the details for readers.

6.3.2 The polar Orlicz-Minkowski problem associated with the general volume

First, we propose the definitions of the general nonhomogeneous and homogeneous volume with respect to a convex body $K \in \mathscr{K}^n_{(o)}$.

Definition 6.3.6. *Let $G : (0,\infty) \times S^{n-1} \to (0,\infty)$ be a continuous function. The general volume of a convex body $K \in \mathscr{K}^n_{(o)}$, denoted by $V_G(K)$, is proposed to be*

$$V_G(K) = \int_{S^{n-1}} G(h_K(u), u)\, dS(K,u).$$

Note that $V_G(K) = V(K)$ if $G(t,u) = t/n$ for any $(t,u) \in (0,\infty) \times S^{n-1}$.

Proposition 6.3.7. *Let $G : (0,\infty) \times S^{n-1} \to (0,\infty)$ be a continuous function. The general volume $V_G(\cdot)$ has the following properties.*

(i) $V_G(\cdot)$ is continuous on $\mathscr{K}^n_{(o)}$ in terms of the Hausdorff metric, that is, for any sequence $\{K_i\}_{i\ge1}$ such that $K_i \in \mathscr{K}^n_{(o)}$ for all $i \in \mathbb{N}$ and $K_i \to K \in \mathscr{K}^n_{(o)}$, then $V_G(K_i) \to V_G(K)$.

(ii) Let $K \in \mathscr{K}^n_{(o)}$. If $\overline{G}(t,\cdot) = t^{n-1}G(t,\cdot) \in \mathscr{G}_I$, then $V_G(tK)$ is strictly increasing on $t \in (0,\infty)$ and

$$\lim_{t\to0^+} V_G(tK) = 0 \quad \text{and} \quad \lim_{t\to\infty} V_G(tK) = \infty;$$

while if $\overline{G} \in \mathscr{G}_d$, then $V_G(tK)$ is strictly decreasing on $t \in (0,\infty)$ and

$$\lim_{t\to0^+} V_G(tK) = \infty \quad \text{and} \quad \lim_{t\to\infty} V_G(tK) = 0.$$

Proof. The fact that $K_i \to K \in \mathscr{K}^n_{(o)}$ with $K_i \in \mathscr{K}^n_{(o)}$ for each $i \in \mathbb{N}$ implies that $h_{K_i} \to h_K$ uniformly on S^{n-1} and $S(K_i) \to S(K)$. Moreover, there exist two positive constants $r_K < R_K$ such that

$$r_K \le h_K \le R_K \quad \text{and} \quad r_K \le h_{K_i} \le R_K \quad \text{for all } \ i \in \mathbb{N}.$$

(i) As $h_{K_i} \to h_K$ uniformly on S^{n-1}, one has $G(h_{K_i}(u), u) \to G(h_K(u), u)$ also uniformly on S^{n-1}. Lemma 2.3.3 and the well known fact that $S(K_i,\cdot) \to S(K,\cdot)$ weakly yield that $V_G(K_i) \to V_G(K)$ as $i \to \infty$.

(ii) Let $K \in \mathscr{K}^n_{(o)}$. For all $t > s > 0$ and all $u \in S^{n-1}$, if $\overline{G} \in \mathscr{G}_I$ (and hence $\overline{G}(t,\cdot)$ is strictly increasing on $t > 0$), then $V_G(tK)$ is strictly increasing on $t > 0$ as follows:

$$\begin{aligned} V_G(tK) &= \int_{S^{n-1}} G(h_{tK}(u),u)\,dS(tK,u) \\ &= \int_{S^{n-1}} t^{n-1}G(t\cdot h_K(u),u)\,dS(K,u) \\ &= \int_{S^{n-1}} \overline{G}(t\cdot h_K(u),u)h_K^{1-n}(u)\,dS(K,u) \\ &> \int_{S^{n-1}} \overline{G}(s\cdot h_K(u),u)h_K^{1-n}(u)\,dS(K,u) \\ &= V_G(sK). \end{aligned}$$

As $r_K \le h_K(u) \le R_K$ for all $u \in S^{n-1}$,

$$\begin{aligned} \lim_{t\to 0^+} V_G(tK) &= \lim_{t\to 0^+} \int_{S^{n-1}} \overline{G}(t\cdot h_K(u),u)h_K^{1-n}(u)\,dS(K,u) \\ &\le \lim_{t\to 0^+} \int_{S^{n-1}} r_K^{1-n}\overline{G}(t\cdot R_K,u)\,dS(K,u) \\ &= \int_{S^{n-1}} \lim_{t\to 0^+} r_K^{1-n}\overline{G}(t\cdot R_K,u)\,dS(K,u) \\ &= 0, \end{aligned}$$

where we have used the dominated convergence theorem and $\lim_{t\to 0^+}\overline{G}(t,\cdot) = 0$. This proves that $\lim_{t\to 0^+} V_G(tK) = 0$. Similarly, $\lim_{t\to\infty} V_G(tK) = \infty$ can be proved as follows:

$$\begin{aligned} \lim_{t\to\infty} V_G(tK) &\ge \liminf_{t\to\infty} \int_{S^{n-1}} \overline{G}(t\cdot r_K,u)R_K^{1-n}\,dS(K,u) \\ &\ge \int_{S^{n-1}} \liminf_{t\to\infty} \overline{G}(t\cdot r_K,u)R_K^{1-n}\,dS(K,u) \\ &= \infty, \end{aligned}$$

where we have used Fatou's lemma and the fact that $\lim_{t\to\infty}\overline{G}(t,\cdot) = \infty$. The desired result for the case $\overline{G} \in \mathscr{G}_d$ follows along the same lines. $\square$

For each $K \in \mathscr{K}^n_{(o)}$, denote by $S(K)$ the surface area of K. A fundamental inequality for $S(K)$ is the celebrated classical isoperimetric inequality (see e.g., [59]):

$$S(K) \ge n\big[V(B^n)\big]^{1/n} V(K)^{\frac{n-1}{n}}. \tag{6.30}$$

Definition 6.3.8. *The homogeneous general volume of $K \in \mathscr{K}^n_{(o)}$, denoted by $\overline{V}_G(K)$ is defined as follows: for $G \in \mathscr{G}_I \cup \mathscr{G}_d$,*

$$\frac{1}{S(K)} \int_{S^{n-1}} G\bigg(\frac{S(K) \cdot h_K(u)}{\overline{V}_G(K)}, u\bigg)\, dS(K,u) = 1. \tag{6.31}$$

Indeed, $\overline{V}_G(K)$ for $K \in \mathscr{K}^n_{(o)}$ exists and is uniquely defined following from similar arguments to Proposition 6.1.1. In particular, $\overline{V}_G(K) = V(K)$ if $G(t,u) = t/n$. Note that $\overline{V}_G(K)$ has equivalent formulas similar to (6.1) and (6.2).

Proposition 6.3.9. *Let $G \in \mathscr{G}_I \cup \mathscr{G}_d$. The homogeneous general volume $\overline{V}_G(\cdot)$ has the following properties.*

(i) $\overline{V}_G(\cdot)$ is homogeneous, that is, $\overline{V}_G(tK) = t^n \overline{V}_G(K)$ holds for all $t > 0$ and all $K \in \mathscr{K}^n_{(o)}$.

(ii) $\overline{V}_G(\cdot)$ is continuous on $\mathscr{K}^n_{(o)}$ in terms of the Hausdorff metric, that is, for any sequence $\{K_i\}_{i \geq 1}$ such that $K_i \in \mathscr{K}^n_{(o)}$ for all $i \in \mathbb{N}$ and $K_i \to K \in \mathscr{K}^n_{(o)}$, then $\overline{V}_G(K_i) \to \overline{V}_G(K)$.

Proof. (i) The desired argument follows trivially from (6.31), the strict monotonicity of G, and the facts that $S(tK) = t^{n-1} S(K)$ and $h_{tK} = t \cdot h_K$ for all $t > 0$.

(ii) Following the notations as in Proposition 6.3.7, we will prove the continuity for $\overline{V}_G(\cdot)$ if $G \in \mathscr{G}_I$ (and the proof for the case $G \in \mathscr{G}_d$ is omitted). It follows from (6.31) that

$$\begin{aligned} \int_{S^{n-1}} G\bigg(\frac{S(K_i) \cdot r_K}{\overline{V}_G(K_i)}, u\bigg)\, dS(K_i,u) &\leq S(K_i) \\ &\leq \int_{S^{n-1}} G\bigg(\frac{S(K_i) \cdot R_K}{\overline{V}_G(K_i)}, u\bigg)\, dS(K_i,u). \end{aligned}$$

Suppose that $\inf_{i \in \mathbb{N}} \overline{V}_G(K_i) = 0$, and without loss of generality, let $\lim_{i \to \infty} \overline{V}_G(K_i) = 0$. Then for any $\varepsilon > 0$, there exists $i_\varepsilon \in \mathbb{N}$ such that $\overline{V}_G(K_i) < \varepsilon$ for all $i > i_\varepsilon$. Hence, for $i \geq i_\varepsilon$,

$$\begin{aligned} \int_{S^{n-1}} G\bigg(\frac{S(K_i) \cdot r_K}{\varepsilon}, u\bigg)\, dS(K_i,u) &\leq \int_{S^{n-1}} G\bigg(\frac{S(K_i) \cdot r_K}{\overline{V}_G(K_i)}, u\bigg)\, dS(K_i,u) \\ &\leq S(K_i). \end{aligned}$$

A contradiction can be obtained from Lemma 2.3.3, the weak convergence of $S(K_i, \cdot) \to S(K, \cdot)$, the facts that $\lim_{t\to\infty} G(t, \cdot) = \infty$ and $S(K_i) \to S(K)$, and Fatou's lemma as follows:

$$\begin{aligned} S(K) &\geq \liminf_{\varepsilon\to 0^+} \left[\lim_{i\to\infty} \int_{S^{n-1}} G\left(\frac{S(K_i)\cdot r_K}{\varepsilon}, u\right) dS(K_i, u) \right] \\ &= \liminf_{\varepsilon\to 0^+} \int_{S^{n-1}} G\left(\frac{S(K)\cdot r_K}{\varepsilon}, u\right) dS(K, u) \\ &\geq \int_{S^{n-1}} \liminf_{\varepsilon\to 0^+} G\left(\frac{S(K)\cdot r_K}{\varepsilon}, u\right) dS(K, u) \\ &= \infty. \end{aligned}$$

This is impossible and hence $\inf_{i\in\mathbb{N}} \overline{V}_G(K_i) > 0$. Similarly, $\sup_{i\in\mathbb{N}} \overline{V}_G(K_i) < \infty$.

Now let us prove $\lim_{i\to\infty} \overline{V}_G(K_i) = \overline{V}_G(K)$. Assume $\overline{V}_G(K) < \limsup_{i\to\infty} \overline{V}_G(K_i)$. There exists a subsequence $\{K_{i_j}\}$ of $\{K_i\}$ such that

$$\overline{V}_G(K) < \lim_{j\to\infty} \overline{V}_G(K_{i_j}) \leq \sup_{i\in\mathbb{N}} \overline{V}_G(K_i) < \infty.$$

By $G \in \mathscr{G}_I$, (6.31), Lemma 2.3.3, $S(K_{i_j}, u) \to S(K, \cdot)$ weakly and $h_{K_{i_j}} \to h_K > 0$ uniformly on S^{n-1}, one gets

$$\begin{aligned} S(K) &= \lim_{j\to\infty} \int_{S^{n-1}} G\left(\frac{S(K_{i_j})\cdot h_{K_{i_j}}(u)}{\overline{V}_G(K_{i_j})}, u\right) dS(K_{i_j}, u) \\ &= \int_{S^{n-1}} G\left(\frac{S(K)\cdot h_K(u)}{\lim_{j\to\infty} \overline{V}_G(K_{i_j})}, u\right) dS(K, u) \\ &< \int_{S^{n-1}} G\left(\frac{S(K)\cdot h_K(u)}{\overline{V}_G(K)}, u\right) dS(K, u) \\ &= S(K). \end{aligned}$$

This is a contradiction and hence $\limsup_{i\to\infty} \overline{V}_G(K_i) \leq \overline{V}_G(K)$. Similarly, one can obtain $\liminf_{i\to\infty} \overline{V}_G(K_i) \geq \overline{V}_G(K)$ and then the desired equality $\lim_{i\to\infty} \overline{V}_G(K_i) = \overline{V}_G(K)$ holds. $\square$

Problems 6.2.1 and 6.3.3 can be asked for V_G and $\overline{V}_G$, respectively.

Problem 6.3.10. *Under what conditions on a nonzero finite Borel measure μ defined on S^{n-1}, continuous functions $\varphi : (0,\infty) \to (0,\infty)$ and $G : (0,\infty) \times S^{n-1} \to (0,\infty)$*

can we find a convex body $K \in \mathscr{K}^n_{(o)}$ solving the following optimization problems:

$$\inf/\sup\left\{\|h_Q\|_{\mu,\varphi}:\ Q\in\mathscr{B}\right\}\quad\text{or}\quad \inf/\sup\left\{\int_{S^{n-1}}\varphi(h_Q(u))\,d\mu(u):\ Q\in\mathscr{B}\right\};$$
$$\inf/\sup\left\{\|h_Q\|_{\mu,\varphi}:\ Q\in\overline{\mathscr{B}}\right\}\quad\text{or}\quad \inf/\sup\left\{\int_{S^{n-1}}\varphi(h_Q(u))\,d\mu(u):\ Q\in\overline{\mathscr{B}}\right\},$$

where $\mathscr{B}$ and $\overline{\mathscr{B}}$ are given by

$$\begin{aligned}\mathscr{B} &= \left\{Q\in\mathscr{K}^n_{(o)}:\ V_G(Q^*)=V_G(B^n)\right\};\\ \overline{\mathscr{B}} &= \left\{Q\in\mathscr{K}^n_{(o)}:\ \overline{V}_G(Q^*)=\overline{V}_G(B^n)\right\},\quad \text{if}\quad G\in\mathscr{G}_I\cup\mathscr{G}_d.\end{aligned}$$

Again, when $G = t/n$, Problem 6.3.10 becomes the polar Orlicz-Minkowski problem [44]. From Sections 6.2 and 6.3.1, one sees that the existence and continuity of solutions to Problems 6.2.1 and 6.3.3 are similar, and their proofs heavily depend on Lemmas 6.1.4, 6.2.5, 6.3.1 and 6.3.2. In particular, if alternative arguments of Lemma 6.1.4 for $V_G(\cdot)$ and $\overline{V}_G(\cdot)$ can be established, the desired existence and continuity of solutions, if applicable, to Problem 6.3.10 will follow. The following lemma is a replacement of Lemma 6.1.4. Note that the monotonicity of $V_G(\cdot)$ and $\overline{V}_G(\cdot)$ in terms of set inclusion, in general, may be invalid. Therefore, our proof for Lemma 6.3.11 is quite different from the one for Lemma 6.1.4.

Lemma 6.3.11. *Let $G:(0,\infty)\times S^{n-1}\to(0,\infty)$ be a continuous function and $G_q(t,u)=\frac{G(t,u)}{t^q}$ for $q\in\mathbb{R}$.*

(i) Suppose that there exists a constant $q\in(1-n,0)$, such that,

$$\inf\left\{G_q(t,u):\ t\geq 1\ \text{ and }\ u\in S^{n-1}\right\}>0. \tag{6.32}$$

If $\{Q_i\}_{i\geq1}$ with $Q_i\in\mathscr{B}$ for all $i\in\mathbb{N}$ is a bounded sequence, then there exist a subsequence $\{Q_{i_j}\}_{j\geq1}$ of $\{Q_i\}_{i\geq1}$ and $Q_0\in\mathscr{B}$ such that $Q_{i_j}\to Q_0$.

(ii) Let $G\in\mathscr{G}_I$ satisfy (6.32) for some $q\geq1$. If $\{Q_i\}_{i\geq1}$ with $Q_i\in\overline{\mathscr{B}}$ for all $i\in\mathbb{N}$ is a bounded sequence, then there exist a subsequence $\{Q_{i_j}\}_{j\geq1}$ of $\{Q_i\}_{i\geq1}$ and $Q_0\in\overline{\mathscr{B}}$ such that $Q_{i_j}\to Q_0$.

Proof. Let $\{Q_i\}_{i\geq1}$ with $Q_i\in\mathscr{K}^n_{(o)}$ for each $i\in\mathbb{N}$ be bounded. There exists a finite constant $R>0$ such that $Q_i\subset RB^n$ for all $i\in\mathbb{N}$, which in turn implies $Q_i^*\supset\frac{1}{R}B^n$. In particular, $h_{Q_i^*} \geq 1/R$ for each $i \in \mathbb{N}$ and $S(Q_i^*) \geq R^{1-n} S(B^n)$ due to the monotonicity of surface area for convex bodies.

(i) Again (6.32) is equivalent to: there exist finite constants $c_0, C_0 > 0$ such that for $q \in (1-n, 0)$,

$$\inf \Big\{ G_q(t,u) : \ t \geq c_0 \ \text{ and } \ u \in S^{n-1} \Big\} > C_0. \tag{6.33}$$

Let $c_0 = 1/R$. Then $G(t,u) \geq C_0 t^q$ for $q \in (1-n, 0)$ and for all $(t,u) \in [1/R, \infty) \times S^{n-1}$. Thus,

$$\begin{aligned}
V_G(Q_i^*) &= \int_{S^{n-1}} G(h_{Q_i^*}(u), u)\, dS(Q_i^*, u) \\
&\geq C_0 \cdot S(Q_i^*) \int_{S^{n-1}} h^q_{Q_i^*}(u) \ \frac{1}{S(Q_i^*)}\, dS(Q_i^*, u) \\
&\geq C_0 \cdot S(Q_i^*) \bigg(\int_{S^{n-1}} h_{Q_i^*}(u) \ \frac{1}{S(Q_i^*)}\, dS(Q_i^*, u) \bigg)^q \\
&= C_0 \cdot S(Q_i^*) \left(\frac{nV(Q_i^*)}{S(Q_i^*)} \right)^q \\
&\geq C_0 \cdot n \big(V(B^n)\big)^{\frac{1}{n}} \big(V(Q_i^*)\big)^{1-\frac{1}{n}} \left(\frac{V(Q_i^*)}{V(B^n)} \right)^{\frac{q}{n}} \\
&= C_0 \cdot n \big(V(B^n)\big)^{\frac{1-q}{n}} \big(V(Q_i^*)\big)^{\frac{n-1+q}{n}},
\end{aligned}$$

where we have used Jensen's inequality and the classical isoperimetric inequality (6.30). Recall that $V_G(Q_i^*) = V_G(B^n)$ for all $i \in \mathbb{N}$ and $1-n < q < 0$, one has

$$\sup_{i \geq 1} \big\{ V(Q_i^*) \big\} \leq \left(\frac{V_G(B^n)}{C_0 \cdot n \big(V(B^n)\big)^{\frac{1-q}{n}}} \right)^{\frac{n}{n-1+q}} < \infty.$$

Note that t^n/n satisfies (6.5). The proof of Lemma 6.1.4 (in particular, (6.9)) can be used to get a subsequence $\{Q_{i_j}\}_{j \geq 1}$ of $\{Q_i\}_{i \geq 1}$ and $Q_0 \in \mathscr{K}^n_{(o)}$ such that $Q_{i_j} \to Q_0$ (see also [48, Lemma 3.2]). Consequently $Q^*_{i_j} \to Q^*_0$, and the continuity of $V_G(\cdot)$ in Proposition 6.3.7 further yields that $Q_0 \in \mathscr{B}$ following from $Q_i \in \mathscr{B}$ for all $i \in \mathbb{N}$.

(ii) Recall that $Q_i^* \supset \frac{1}{R} B^n$ for each $i \in \mathbb{N}$. As $Q_i \in \overline{\mathscr{B}}$ for each $i \in \mathbb{N}$, one has

$$c_0 = \frac{R^{-n} S(B^n)}{\overline{V}_G(B^n)} \leq \frac{S(Q_i^*) \cdot h_{Q_i^*}(u)}{\overline{V}_G(Q_i^*)}.$$

It follows from (6.31), (6.33) and Jensen's inequality for $q \geq 1$ that

$$\begin{aligned} 1 &= \frac{1}{S(Q_i^*)} \int_{S^{n-1}} G\bigg(\frac{S(Q_i^*) \cdot h_{Q_i^*}(u)}{\overline{V}_G(Q_i^*)}, u\bigg)\, dS(Q_i^*, u) \\ &\geq \frac{C_0}{S(Q_i^*)} \int_{S^{n-1}} \bigg(\frac{S(Q_i^*) \cdot h_{Q_i^*}(u)}{\overline{V}_G(B^n)}\bigg)^q dS(Q_i^*, u) \\ &\geq C_0 \bigg(\int_{S^{n-1}} \frac{h_{Q_i^*}(u)}{\overline{V}_G(B^n)}\, dS(Q_i^*, u)\bigg)^q \\ &= C_0 \bigg(\frac{nV(Q_i^*)}{\overline{V}_G(B^n)}\bigg)^q. \end{aligned}$$

This further implies that

$$V(Q_i^*) \leq n^{-1} C_0^{-1/q} \overline{V}_G(B^n)$$

for each $i \in \mathbb{N}$. As in (i) (the last paragraph), one gets a subsequence $\{Q_{i_j}\}_{j\geq 1}$ of $\{Q_i\}_{i\geq 1}$ and $Q_0 \in \mathscr{K}_{(o)}^n$, such that, $Q_{i_j}^* \to Q_0^*$. The continuity of $\overline{V}_G(\cdot)$ in Proposition 6.3.9 further yields that $Q_0 \in \overline{\mathscr{B}}$ following from $Q_i \in \overline{\mathscr{B}}$ for all $i \in \mathbb{N}$. $\square$

Remark. It can be easily checked that if (6.32) holds for some $q \geq 0$, Part (i) of Lemma 6.3.11 also holds. To this end, if (6.32) holds for $q \geq 0$, one can verify that $2q + n - 1 > 0$ and

$$\begin{aligned} & \inf\Big\{G_{\frac{1-n}{2}}(t,u):\ (t,u) \in [1,\infty) \times S^{n-1}\Big\} \\ = & \inf\Big\{G_q(t,u) \cdot t^{\frac{2q+n-1}{2}}:\ (t,u) \in [1,\infty) \times S^{n-1}\Big\} \\ \geq & \inf\Big\{G_q(t,u):\ (t,u) \in [1,\infty) \times S^{n-1}\Big\} \\ > & \ 0. \end{aligned}$$

Hence, (6.32) holds for $\frac{1-n}{2} \in (1-n, 0)$ and then Part (i) of Lemma 6.3.11 also follows. In particular, Part (i) of Lemma 6.3.11 works for $G = t/n$ and $G = 1$ which correspond to the volume and the surface area, respectively. Similar to the remark of Lemma 6.1.4, if $G \in \mathscr{G}_d$, G does not satisfy (6.32) for some $q \geq 1$.

The existence of solutions and the continuity of the extreme values to Problem 6.3.10 for V_G are stated below.

Theorem 6.3.12. *Let $\varphi \in \mathscr{I}$ and let $G : (0,\infty) \times S^{n-1} \to (0,\infty)$ satisfying (6.32)* *for some* $q \in (1-n, 0)$.

(i) Let μ *be a nonzero finite Borel measure on* S^{n-1} *whose support is not concentrated on any great hemisphere. Then there exist* $M_1, M_2 \in \mathscr{B}$ *such that*

$$\begin{aligned} \int_{S^{n-1}} \varphi(h_{M_1}(u)) d\mu(u) &= \inf_{Q\in\mathscr{B}} \int_{S^{n-1}} \varphi(h_Q(u))\, d\mu(u); && (6.34)\\ \|h_{M_2}\|_{\mu,\varphi} &= \inf_{Q\in\mathscr{B}} \|h_Q\|_{\mu,\varphi}. && (6.35) \end{aligned}$$

(ii) Let $\{\mu_i\}_{i=1}^{\infty}$ *and* μ *be nonzero finite Borel measures on* S^{n-1} *whose supports are not concentrated on any closed hemisphere, such that,* $\mu_i \to \mu$ *weakly as* $i \to \infty$*. Then*

$$\begin{aligned} \lim_{i\to\infty} \left(\inf_{Q\in\mathscr{B}} \int_{S^{n-1}} \varphi(h_Q(u))\, d\mu_i(u) \right) &= \inf_{Q\in\mathscr{B}} \int_{S^{n-1}} \varphi(h_Q(u))\, d\mu(u); \\ \lim_{i\to\infty} \left(\inf_{Q\in\mathscr{B}} \|h_Q\|_{\mu_i,\varphi} \right) &= \inf_{Q\in\mathscr{B}} \|h_Q\|_{\mu,\varphi}. \end{aligned}$$

Proof. (i) Note that $B^n \in \mathscr{B}$ and hence the optimization problem in (6.34) is well defined. Let $\{Q_i\}_{i\geq 1}$ be the limiting sequences such that $Q_i \in \mathscr{B}$ for each $i \in \mathbb{N}$ and

$$\mu(S^{n-1}) \geq \inf_{Q\in\mathscr{B}} \int_{S^{n-1}} \varphi(h_Q(u))\, d\mu(u) = \lim_{i\to\infty} \int_{S^{n-1}} \varphi(h_{Q_i}(u))\, d\mu(u).$$

It follows from Lemma 6.2.5 that $\{Q_i\}_{i\geq 1}$ is a bounded sequence in $\mathscr{K}_{(o)}^n$. Together with Lemma 6.3.11, there exist a subsequence $\{Q_{i_j}\}_{j\geq 1}$ of $\{Q_i\}_{i\geq 1}$ and $M_1 \in \mathscr{B}$ such that $Q_{i_j} \to M_1$. Lemma 2.3.3 and $\varphi \in \mathscr{I}$ then yield

$$\int_{S^{n-1}} \varphi(h_{M_1}(u))\, d\mu(u) = \lim_{j\to\infty} \int_{S^{n-1}} \varphi(h_{Q_{i_j}}(u))\, d\mu(u) = \inf_{Q\in\mathscr{B}} \int_{S^{n-1}} \varphi(h_Q(u))\, d\mu(u).$$

The existence of $M_2 \in \mathscr{B}$ that verifies (6.35) can be obtained similarly, with Lemma 6.2.5 and Lemma 2.3.3 replaced by Lemma 6.3.2 and Lemma 6.3.1, respectively, if one notices that

$$1 \geq \inf_{Q\in\mathscr{B}} \|h_Q\|_{\mu,\varphi} = \lim_{i\to\infty} \|h_{Q_i}\|_{\mu,\varphi}.$$

(ii) First, note that from Part (i), the optimization problems (6.34) and (6.35) for μ and μ_i for each $i \in \mathbb{N}$ have solutions. The rest of the proof is almost identical to those for Theorems 6.2.6 and 6.3.5, with Lemma 6.1.4 replaced by Lemma 6.3.11. $\square$

Similarly, one can prove the existence of solutions and the continuity of the extreme values to Problem 6.3.10 for $\overline{V}_G(\cdot)$. The proof will be omitted due to the high similarity to those in e.g., Theorem 6.3.12.

Theorem 6.3.13. *Let $\varphi \in \mathscr{I}$ and let $G \in \mathscr{G}_I$ satisfy (6.32) for some constant $q \geq 1$.*

(i) Let μ be a nonzero finite Borel measure on S^{n-1} whose support is not concentrated on any great hemisphere. There exist $\overline{M}_1, \overline{M}_2 \in \overline{\mathscr{B}}$ such that

$$\int_{S^{n-1}} \varphi(h_{\overline{M}_1}(u))d\mu(u) = \inf_{Q\in\overline{\mathscr{B}}} \int_{S^{n-1}} \varphi(h_Q(u))\,d\mu(u) \quad \text{and} \quad \|h_{\overline{M}_2}\|_{\mu,\varphi} = \inf_{Q\in\overline{\mathscr{B}}} \|h_Q\|_{\mu,\varphi}.$$

(ii) Let $\{\mu_i\}_{i=1}^{\infty}$ and μ be nonzero finite Borel measures on S^{n-1} whose supports are not concentrated on any closed hemisphere, such that, $\mu_i \to \mu$ weakly as $i \to \infty$. Then

$$\begin{aligned} \lim_{i\to\infty}\left(\inf_{Q\in\overline{\mathscr{B}}} \int_{S^{n-1}} \varphi(h_Q(u))\,d\mu_i(u)\right) &= \inf_{Q\in\overline{\mathscr{B}}} \int_{S^{n-1}} \varphi(h_Q(u))\,d\mu(u)); \\ \lim_{i\to\infty}\left(\inf_{Q\in\overline{\mathscr{B}}} \|h_Q\|_{\mu_i,\varphi}\right) &= \inf_{Q\in\overline{\mathscr{B}}} \|h_Q\|_{\mu,\varphi}. \end{aligned}$$

6.3.3 The general Orlicz-Petty bodies

The classical geominimal surface area [57, 58] and its L_p or Orlicz extensions (see e.g., [48, 71, 72, 74, 77]) are central objects in convex geometry. When studying the properties of various geominimal surface areas, the Petty body or its generalizations play fundamental roles. In short, the Orlicz-Petty bodies are the solutions to the following optimization problems [74, 77]:

$$\inf\Big\{nV_\varphi(K,L) : L \in \mathscr{K}^n_{(o)} \quad \text{with} \quad V(L^*) = V(B^n)\Big\}; \tag{6.36}$$

$$\inf\Big\{\widehat{V}_\varphi(K,L) : L \in \mathscr{K}^n_{(o)} \quad \text{with} \quad V(L^*) = V(B^n)\Big\}, \tag{6.37}$$

where $\varphi \in \mathscr{I}$, and $V_\varphi(K,L)$ and $\widehat{V}_\varphi(K,L)$ are the Orlicz L_φ mixed volumes of $K, L \in \mathscr{K}^n_{(o)}$ defined by (see e.g., [16, 68, 77]):

$$V_\varphi(K,L) = \frac{1}{n}\int_{S^{n-1}} \varphi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u)dS(K,u) \quad \text{and} \quad \widehat{V}_\varphi(K,L) = \left\|\frac{h_L}{h_K}\right\|_{S(K,\cdot),\varphi}.$$

The surface area measure $S(K,\cdot)$ may be replaced by other measures; for instance, Luo, Ye and Zhu in [44] obtained the p-capacitary Orlicz-Petty bodies where the surface area measure is replaced by the p-capacitary measure (see e.g., [14, 34]). As explained in [44], the polar Orlicz-Minkowski problem (i.e., Problems 6.2.1 and 6.3.3 with $G = t^n/n$) and the optimization problems (6.36) and (6.37) are quite different in their general forms; however these two problems are also very closely related. In view of their relations, we can ask the following problem aiming to find the general Orlicz-Petty bodies.

Problem 6.3.14. *Let $K \in \mathscr{K}^n_{(o)}$ be a fixed convex body. Let μ_K be a nonzero finite Borel measure associated with K defined on S^{n-1}, which is not concentrated on any closed hemisphere. Under what conditions on continuous functions $\varphi : (0,\infty) \to (0,\infty)$ and $G : (0,\infty) \times S^{n-1} \to (0,\infty)$ can we find a convex body $M \in \mathscr{K}^n_{(o)}$ solving the following optimization problems:*

$$\inf / \sup \left\{ \left\| \frac{h_Q}{h_K} \right\|_{\mu_K,\varphi} : Q \in \mathscr{A} \right\} \text{ or } \inf / \sup \left\{ \int_{S^{n-1}} \varphi\left(\frac{h_Q(u)}{h_K(u)} \right) h_K(u) d\mu_K(u) : Q \in \mathscr{A} \right\}, \tag{6.38}$$

where $\mathscr{A}$ is selected from the following sets: $\widetilde{\mathscr{B}}, \widehat{\mathscr{B}}, \mathscr{B}$ and $\overline{\mathscr{B}}$.

Note that the measure μ_K assumed in Problem 6.3.14 includes many interesting measures, such as, the surface area measure $S(K,\cdot)$, the p-capacitary measure [14, 34], the Orlicz p-capacitary measure [24], the L_p dual curvature measures [29, 54], the general dual Orlicz curvature measures [17, 19, 69, 78], and many more.

Definition 6.3.15. *Let $K \in \mathscr{K}^n_{(o)}$ be a fixed convex body. Let μ_K be a nonzero finite Borel measure associated with K defined on S^{n-1}, which is not concentrated on any closed hemisphere. If $M \in \mathscr{A}$ solving the optimization problem (6.3.14), then M is called a general Orlicz-Petty body of K with respect to μ_K.*

Recall that if $K \in \mathscr{K}^n_{(o)}$, there are two constants $0 < r_K < R_K$ such that $r_K B^n \subset K \subset R_K B^n$. In view of this, the existence, continuity and uniqueness, if applicable, of the general Orlicz-Petty bodies with respect to μ_K can be obtained (almost identically) as in Sections 6.2, 6.3.1 and 6.3.2. Polytopal solutions and counterexamples as in Proposition 6.2.4, when K is a polytope, can be also established accordingly.

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