Group Gradings on the Classical Lie Superalgebras $Q(n)$, $P(n)$ and $B(m, n)$

by

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Abstract

Let G be an abelian group. We classify, up to isomorphism, the G-gradings on the classical Lie superalgebras B, P and Q, as well as the fine gradings up to equivalence. Also, we revisit the problem for the associative matrix superalgebras. Everything is done over an algebraically closed field of characteristic zero. In summary, this work completes the classification of the group gradings for some of the non-exceptional classical Lie superalgebras. Part of this work is published in [1] and [9].

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To my parents

Lay Summary

The simple finite-dimensional Lie superalgebras have been classified by V. G. Kac, see [16], [19]. They are widely used in applications, such as Quantum Mechanics. Joris Van der Jeugt wrote in [8]

> *Gradings for Lie (super)algebras are important because they give rise to preferred bases of the algebra which admit 'additive quantum numbers'.*

Another motivation for this work is that the classification problem for color Lie superalgebras can be partially reduced to the classification of the gradings on Lie superalgebras, see [3].

The classification of gradings on the exceptional Lie superalgebras is known, see [10]. A particular case of $\mathbb{Z}$-gradings on classical Lie superalgebras is also known, see [15]. Finally, we should also mention that the classification of gradings on simple Lie algebras is essentially complete, see [12].

We classify, up to isomorphism and up to equivalence, the (abelian) group gradings on the classical Lie superalgebras of the types B, P and Q and we also make a contribution to the same problem in the case of associative matrix superalgebras. These results are presented, respectively, in Sections 4.4, 4.6, 5.4, 5.5, 6.4, 3.3 and 3.5.

Some examples are given in Sections 4.5, 5.6 and 6.6.

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— Helen Samara Dos Santos

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Chapter 1

Introduction

Throughout this thesis, all algebras and vector spaces are finite dimensional over a fixed algebraically closed field $\mathbb{F}$ which we assume to be of characteristic zero, unless stated otherwise.

Our purpose is to classify the group gradings on some of the non-exceptional classical Lie superalgebras and provide some examples.

Let G be a group. A G-grading on an algebra A is a vector space decomposition $A = \bigoplus_{g \in G} A_g$ satisfying $A_{g_1} A_{g_2} \subseteq A_{g_1 g_2}$ for all $g_1, g_2 \in G$ where $g_1 g_2$ denotes the product in G. We will denote by e the identity element of the group G.

Definition 1.0.1 (Superalgebra)**.** A $\mathbb{Z}_2$-graded algebra is called a *superalgebra*.

Analogously, a $\mathbb{Z}_2$-graded vector space is said to be a *superspace*. The components of the $\mathbb{Z}_2$-grading that is a part of the definition of a superspace or superalgebra will be labeled by superscripts, reserving subscripts for the components of other gradings. The elements of the $\bar{0}$-component are said to be *even* and those of the $\bar{1}$-component *odd*.

As we have seen, a *superalgebra* is simply a $\mathbb{Z}_2$-graded algebra; however, the concept of superalgebra of certain varieties (e.g., associative, Lie, Jordan) is less straightforward: the "super" analog of a class of algebras can be defined using its Grassman envelope. We start by recalling that if V is a vector space then, by definition, the Grassmann algebra of V is $\mathcal{G}(V) = \sum_{k=0}^{+\infty} \bigwedge^k V$ with product given by the "$\wedge$" operation. It has a natural $\mathbb{Z}_2$-grading, given by $\mathcal{G}(V)^{\bar{0}} = \sum_{i=0}^{+\infty} \bigwedge^{2i} V$ and $\mathcal{G}(V)^{\bar{1}} = \sum_{i=0}^{+\infty} \bigwedge^{2i+1} V$, so we can consider it as a superalgebra.

If V is a fixed vector space with a countably infinite basis, we denote its Grassmann algebra simply by $\mathcal{G}$. If $A = A^{\bar{0}} \oplus A^{\bar{1}}$ is a superalgebra, we define its *Grassmann envelope* by $A^{\bar{0}} \otimes \mathcal{G}^{\bar{0}} + A^{\bar{1}} \otimes \mathcal{G}^{\bar{1}} \subseteq A \otimes \mathcal{G}$, considered as an algebra. In this way, if $\mathfrak{R}$ is a class of algebras (e.g., associative, Lie, Jordan), we define the class of $\mathfrak{R}$-superalgebras, denoted by $\mathfrak{R}_S$, as the class of all superalgebras A such that the Grassmann envelope of A is in $\mathfrak{R}$. In particular, if $\mathfrak{R}$ is the class of Lie algebras, we call the elements of $\mathfrak{R}_S$ Lie superalgebras.

Following this procedure, one can see that an *associative superalgebra* is just a $\mathbb{Z}_2$-graded associative algebra but a *Lie superalgebra* is not (in general) a $\mathbb{Z}_2$-graded Lie algebra. The notion can be axiomatized as follows (with product denoted by brackets).

Definition 1.0.2. A *Lie superalgebra* $(L = L^{\bar{0}} \oplus L^{\bar{1}}, [,])$ is a superalgebra that satisfies, for all homogeneous elements $a \in L^\alpha$, $b \in L^\beta$ and $c \in L$, the following two axioms:

- $[a, b] = -(-1)^{\alpha\beta}[b, a]$ (super anticommutativity)
- $[a, [b, c]] = [[a, b], c] + (-1)^{\alpha\beta}[b, [a, c]]$ (super Jacobi identity)

If $0 \neq a \in A^\alpha$ for $\alpha \in \mathbb{Z}_2$ then α may be denoted by $|a|$ and called the *parity* of a.

Note that if $L = L^{\bar{0}} \oplus L^{\bar{1}}$ is a Lie superalgebra then $L^{\bar{0}}$ is a Lie algebra and $L^{\bar{1}}$ is an $L^{\bar{0}}$-module.

A *super ideal* (also called a graded ideal) of a superalgebra $A = A^{\bar{0}} \oplus A^{\bar{1}}$ is an ideal I of A such that $I = (I \cap A^{\bar{0}}) \oplus (I \cap A^{\bar{1}})$.

A Lie superalgebra L is said to be *simple* if it contains no non-trivial super ideals (i.e. no super ideals aside from 0 and L itself) and if $[L, L] \neq 0$. The condition $[L, L] \neq 0$ is to eliminate $L = 0$ or L being one-dimensional.

Recall that every one-dimensional Lie superalgebra is abelian, so we have two one-dimensional Lie superalgebras (one with $L = L^{\bar{0}} \oplus 0$ and one with $L = 0 \oplus L^{\bar{1}}$) that we are eliminating from the definition of simplicity by the condition $[L, L] \neq 0$. Note that if L is a Lie superalgebra then the derived algebra $[L, L]$ is a super ideal of L.

The simple Lie superalgebras have been classified in [16] (see also [19]). They are of two types, *classical* and *Cartan*. Our interest in this work is the classical Lie superalgebras. They can be divided in two classes: the ones for which the odd component is a simple module over the even component and the ones for which the odd component is the sum of two simple modules. In this second case, when $L^{\bar{1}}$ is not simple, we will say that $L = L^{\bar{0}} \oplus L^{\bar{1}}$ is a *Lie* $\mathbb{Z}$*-superalgebra* or a $\mathbb{Z}$*-graded Lie superalgebra*, as it is possible to write $L^{\bar{1}} = L^{-1} \oplus L^{1}$, where L^{-1} and L^{1} are simple $L^{\bar{0}}$-modules, and $L = L^{-1} \oplus L^{0} \oplus L^{1}$ is a $\mathbb{Z}$-grading with $L^{0} = L^{\bar{0}}$. Note that we also label its components by superscripts.

Definition 1.0.3. We say that a superalgebra $A = A^{\bar{0}} \oplus A^{\bar{1}}$ is a G-graded superalgebra if $\Gamma\colon A = \bigoplus_{g \in G} A_g$ is a grading on A such that $A_g = (A_g \cap A^{\bar{0}}) \oplus (A_g \cap A^{\bar{1}})$, for all

$g \in G$, or, equivalently, each of $A^{\bar{0}}$ and $A^{\bar{1}}$ is G-graded: $A^i = \bigoplus_{g\in G}(A_g \cap A^i)$, $i \in \mathbb{Z}_2$.

If a grading Γ is fixed on a superalgebra A, A is referred to as a *G-graded superalgebra*. The nonzero elements $x \in A_g$ are said to be *homogeneous of degree g*.

The *support* of Γ is the set $\operatorname{supp}(\Gamma) = \{g \in G \mid A_g \neq 0\}$. We have $\operatorname{supp}(\Gamma) = \operatorname{supp}^{\bar{0}}(\Gamma) \cup \operatorname{supp}^{\bar{1}}(\Gamma)$ where $\operatorname{supp}^i(\Gamma) = \{g \in G \mid A_g^i \neq 0\}$.

We will denote by $\Gamma = \Gamma^{\bar{0}} \oplus \Gamma^{\bar{1}}$ the G-grading Γ on the $\mathbb{Z}_2$-superalgebra $L = L^{\bar{0}} \oplus L^{\bar{1}}$, where L^i is equipped with the G-grading Γ^i, $i \in \mathbb{Z}_2$; analogously, we will denote by $\Gamma = \Gamma^{-1} \oplus \Gamma^0 \oplus \Gamma^1$ the G-grading Γ on the $\mathbb{Z}$-superalgebra $L = L^{-1} \oplus L^0 \oplus L^1$, where L^i is equipped with the G-grading Γ^i, $i \in \{-1, 0, 1\}$.

The next proposition is the "super" analog of Proposition 2.16 of [12]; the proof presented in [12], with some obvious alterations, works.

Proposition 1.0.4. *Let L be a simple Lie superalgebra over any field. If G is a semi-group and $L = \bigoplus_{g\in G} L_g$ is a G-grading on L with support T where T generates G, then G is an abelian group.* □

Due to this proposition, throughout this work, we will only consider abelian group gradings unless stated otherwise.

The classical Lie superalgebras are:

- 6 series $A(m, n)$, $B(m, n)$, $C(n)$, $D(m, n)$, $P(n)$ and $Q(n)$;

- 3 exceptional cases: $F(4)$, $G(3)$ and the family $D(2, 1, \alpha)$, $\alpha \in \mathbb{F}$.

Following [16] and [17], we have the following tables:

Table 1.1: Even components of the classical Lie superalgebras

Type of L	$L^{\bar{0}}$
$A(m,n)$, $m > n$, $n \geq 0$	$\mathfrak{sl}_{m+1} \oplus \mathfrak{sl}_{n+1} \oplus \mathbb{F} = A_m \oplus A_n \oplus Z$
$A(n,n)$, $n \geq 1$	$\mathfrak{sl}_{n+1} \oplus \mathfrak{sl}_{n+1} = A_n \oplus A_n$
$B(m,n)$, $m \geq 0$, $n \geq 1$	$\mathfrak{so}_{2m+1} \oplus \mathfrak{sp}_{2n} = B_m \oplus C_n$
$D(m,n)$, $m \geq 2$, $n \geq 1$	$\mathfrak{so}_{2m} \oplus \mathfrak{sp}_{2n} = D_m \oplus C_n$
$C(n)$, $n \geq 2$	$\mathfrak{sp}_{2n-2} \oplus \mathbb{F} = C_{n-1} \oplus Z$
$P(n)$, $n \geq 2$	$\mathfrak{sl}_{n+1} = A_n$
$Q(n)$, $n \geq 2$	$\mathfrak{sl}_{n+1} = A_n$
$D(2,1,\alpha)$, $\alpha \in \mathbb{F} \setminus \{0,-1\}$	$\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 = A_1 \oplus A_1 \oplus A_1$
$F(4)$	$\mathfrak{so}_7 \oplus \mathfrak{sl}_2 = B_3 \oplus A_1$
$G(3)$	$G_2 \oplus A_1$

Table 1.2: Odd components of the classical Lie superalgebras

Type of L	$L^{\bar{1}}$	L^{-1}	L^{1}
$A(m,n)$, $m>n$, $n\geq 0$		$\mathbb{F}^{m+1}\otimes\mathbb{F}^{n+1}\otimes\mathbb{F}$	$(L^{-1})^*$
$A(n,n)$, $n\geq 1$		$\mathbb{F}^{n+1}\otimes\mathbb{F}^{n+1}$	$(L^{-1})^*$
$B(m,n)$, $m\geq 0$, $n\geq 1$	$\mathbb{F}^{2m+1}\otimes\mathbb{F}^{2n}$		
$D(m,n)$, $m\geq 2$, $n\geq 1$	$\mathbb{F}^{2m}\otimes\mathbb{F}^{2n}$		
$C(n)$, $n\geq 2$		$\mathbb{F}^{2n-2}\otimes\mathbb{F}$	$(L^{-1})^*$
$P(n)$, $n\geq 2$		$(\bigwedge^2\mathbb{F}^{n+1})^*$	$\mathrm{S}^2\mathbb{F}^{n+1}$
$Q(n)$, $n\geq 2$	adjoint		
$D(2,1,\alpha)$, $\alpha\in\mathbb{F}\setminus\{0,-1\}$	$\mathbb{F}^2\otimes\mathbb{F}^2\otimes\mathbb{F}^2$		
$F(4)$	$\mathrm{spin}_7\otimes\mathbb{F}^2$		
$G(3)$	$\{\text{traceless octonions}\}\otimes\mathbb{F}^2$		

The superalgebras $A(m,n)$, $B(m,n)$, $C(n)$ and $D(m,n)$ are extensions of the corresponding series of simple Lie algebras, $Q(n)$ and $P(n)$ are known as *strange*, and $D(2,1,\alpha)$, $F(4)$ and $G(3)$ are called the *exceptional classical Lie superalgebras*. For the latter, the classification of (fine) gradings is already known (see [10]). More details on the relevant superalgebras will be provided in the corresponding chapters.

All infinite series of classical Lie superalgebras are closely related to the general linear superalgebras. As an example, we will describe a model for $A(m,n)$, $m,n \geq 0$.

Let $U = U^{\bar{0}} \oplus U^{\bar{1}}$ be a superspace. The algebra of endomorphisms of U has an induced $\mathbb{Z}_2$-grading, so it can be regarded as a superalgebra. It is convenient to write it in matrix form:

$$\operatorname{End}(U) = \begin{pmatrix} \operatorname{End}(U^{\bar{0}}) & \operatorname{Hom}(U^{\bar{1}}, U^{\bar{0}}) \\ \operatorname{Hom}(U^{\bar{0}}, U^{\bar{1}}) & \operatorname{End}(U^{\bar{1}}) \end{pmatrix}. \tag{1.0.1}$$

Choosing bases, we may assume that $U^{\bar{0}} = \mathbb{F}^m$ and $U^{\bar{1}} = \mathbb{F}^n$, so the superalgebra $\operatorname{End}(U)$ can be seen as a matrix superalgebra, which is denoted by $M(m,n)$.

Given a superspace U, the *general linear Lie superalgebra*, denoted by $\mathfrak{gl}(U)$, is the superspace $\operatorname{End}(U)$ with the induced $\mathbb{Z}_2$-grading and with the product given by the *supercommutator*

$$[a,b] = ab - (-1)^{|a||b|} ba.$$

If $U^{\bar{0}} = \mathbb{F}^m$ and $U^{\bar{1}} = \mathbb{F}^n$, then $\mathfrak{gl}(U)$ is also denoted by $\mathfrak{gl}(m,n)$. The *special linear Lie superalgebra*, denoted by $\mathfrak{sl}(U)$, is the derived algebra of $\mathfrak{gl}(U)$. As in the Lie algebra case, we describe it as an algebra of "traceless" operators. The analog of trace in the “super” setting is the so called *supertrace*:

$$\operatorname{str}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \operatorname{tr} a - \operatorname{tr} d,$$

and we have that

$$\mathfrak{sl}(U) = \{T \in \mathfrak{gl}(U) \mid \operatorname{str} T = 0\}.$$

If $U^{\bar{0}} = \mathbb{F}^m$ and $U^{\bar{1}} = \mathbb{F}^n$ then $\mathfrak{sl}(U)$ is also denoted by $\mathfrak{sl}(m,n)$. If one of the parameters m or n is zero, we get a Lie algebra in the ordinary sense, so we assume this is not the case. If $m \neq n$ then $\mathfrak{sl}(m,n)$ is a simple Lie superalgebra. If $m = n$, the identity map $1_U \in \mathfrak{sl}(n,n)$ is a central element and hence $\mathfrak{sl}(n,n)$ is not simple, but the quotient $\mathfrak{sl}(n,n)/\mathbb{F}1_U$ is simple if $n > 1$. For $m,n \geq 0$, the simple Lie superalgebra $A(m,n)$ is $\mathfrak{sl}(m+1,n+1)$ if $m \neq n$, and $\mathfrak{sl}(n+1,n+1)/\mathbb{F}1_U$ if $m = n$.

The definitions of the Lie superalgebras B, C, D and P involve bilinear forms.

Let $U = U^{\bar{0}} \oplus U^{\bar{1}}$ be a superspace with $\dim U^{\bar{0}} = m$ and $\dim U^{\bar{1}} = n$.

Definition 1.0.5. A bilinear form F on a superspace U is said to be

- *even* if $F(U^0, U^1) = F(U^1, U^0) = 0$;
- *odd* if $F(U^0, U^0) = F(U^1, U^1) = 0$.

Definition 1.0.6. A bilinear form F on a superspace U is said to be *supersymmetric* if

$$F(x,y) = (-1)^{\alpha\beta} F(y,x), \forall x \in U^{\alpha}, y \in U^{\beta}.$$

We will come back to this in the appropriate chapters.

Definition 1.0.7. Consider the pair $(L, M) \in \{\big(A(m,n), M(m+1,n+1)\big)$, $\big(B(m,n), M(2m+1,2n)\big)$, $\big(C(n), M(2,2n-2)\big)$, $\big(D(m,n), M(2m,2n)\big)$,

$\big(P(n), M(n{+}1, n{+}1)\big)$, $\big(Q(n), M(n{+}1, n{+}1)\big)\}$ where $M = M^{\bar{0}} \oplus M^{\bar{1}}$ is an associative superalgebra and L is a sub-Lie-superalgebra of M. Let Γ be a G-grading on M.

If $L \subseteq M$ is a graded subspace, then Γ induces a G-grading on L. The gradings on L obtained this way are called *Type I* gradings. The other possible gradings on L are called *Type II* gradings.

The following is a summary of what is done in each chapter, in the order they were written. We started our investigation with the Lie superalgebras of type $Q(n)$. This choice was natural because both even and odd parts of $Q(n)$ are simple. Our approach for classifying the gradings on the superalgebra $Q(n)$ was to consider the corresponding gradings on the Lie algebra $Q^{\bar{0}}$ and on the $Q^{\bar{0}}$-module $Q^{\bar{1}}$. The results regarding this part and examples of gradings on $Q(n)$ are presented in Chapter 6, which is based on [1].

The next step was to work on $P(n)$, in order to complete the classification of the gradings on the "strange" classical Lie superalgebras. As we worked on $P(n)$, it turned out to be useful to classify the group gradings on associative superalgebras that are simple as algebras, therefore we also classified the gradings on the matrix superalgebras $M(m, n)$. The results of this part are presented in Chapters 3 and 5, which are based on [9]. An example of grading on $P(n)$ can also be found in Chapter 5.

In Chapter 4, we give the classification of gradings on the Lie superalgebras of type $B(m, n)$ and an example. The arguments in this chapter are analogous to those in Chapter 3 of [12].

Unlike what was done for $Q(n)$, our approach for classifying abelian group gradings on the Lie superalgebras of type $P(n)$ and $B(m, n)$ was to consider them as sub-Lie-superalgebras of suitable associative matrix superalgebras and use the classification of gradings for the latter. This technique worked well for $P(n)$ and $B(m,n)$ because, as it happens, all gradings on these Lie superalgebras are restrictions from the gradings on the associative superalgebras (i.e. are of Type I in our terminology). The same technique could also be used to classify gradings on the Lie superalgebras of type $C(n)$ and $D(m,n)$. However, in the case of $A(m,n)$ (as well as $Q(n)$), there are gradings that are not restrictions from the associative superalgebra (i.e. are gradings of Type II). A partial result for $A(m,n)$ is presented in [9].

Finally, Chapter 2 gives an exposition of the previously known facts and generalities on gradings that will be useful throughout this thesis.

Chapter 2

Preliminaries

2.1 Graded spaces and modules

First we consider some examples of group gradings on matrix algebras.

Example 2.1.1. Consider the algebra of $2n \times 2n$ matrices over $\mathbb{F}$.

$$M_{2n} = \left\{ \begin{pmatrix} R & M \\ N & S \end{pmatrix} \,\middle|\, R, M, N \text{ and } S \text{ are } n \times n \text{ matrices} \right\}$$

equipped with the $\mathbb{Z}_2$-grading

$$\Gamma : \; M_{2n}^{\bar{0}} \oplus M_{2n}^{\bar{1}}$$

where

$$M_{2n}^{\bar{0}} = \left\{ \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} \,\middle|\, R \text{ and } S \text{ are } n \times n \text{ matrices} \right\}$$

and

$$M_{2n}^{\bar{1}} = \left\{ \begin{pmatrix} 0 & M \\ N & 0 \end{pmatrix} \,\middle|\, M \text{ and } N \text{ are } n \times n \text{ matrices} \right\}.$$

This grading turns M_{2n} into the superalgebra $M(n,n)$ mentioned in Chapter 1.

Analogously, given integers $1 \le n \le m$, we can construct a $\mathbb{Z}_2$-grading on the algebra

$$M_{m+n} = \left\{ \begin{pmatrix} R & u \\ v & S \end{pmatrix} \mid R \in M_{m\times m}(\mathbb{F}),\ S \in M_{n\times n}(\mathbb{F}),\ u \in M_{n\times m}(\mathbb{F}), \text{ and } v \in M_{m\times n}(\mathbb{F}) \right\},$$

which turns it into the superalgebra $M(m,n)$.

The next example is a generalization of the above $\mathbb{Z}_2$-gradings to an arbitrary group G.

Example 2.1.2. Let $(g_1,\dots,g_n)$ be a n-tuple of elements in G. We have the following grading: $M_n = \bigoplus_{g\in G}(M_n)_g$ where $(M_n)_g = \operatorname{span}\{E_{ij} \mid g_i(g_j)^{-1} = g\}$.

The following definition was introduced in [4] (also see [12, Definition 2.19]).

Definition 2.1.3. A G-grading on $M_n = M_n(\mathbb{F})$ is called *elementary* if it is induced from a decomposition of the vector space $\mathbb{F}^n$ in the following way: $\mathbb{F}^n = V_{g_1} \oplus \cdots \oplus V_{g_s}$ and $r \in M_n$ is homogeneous of degree g if and only if $r(V_h) \subset V_{gh}$ for all $h \in G$.

Note that with a suitable choice of basis of matrix units and choosing the n-tuple as follows: $(\underbrace{g_1,\dots,g_1}_{k_1 \text{ times}}, \underbrace{g_2,\dots,g_2}_{k_2 \text{ times}}, \dots, \underbrace{g_s,\dots,g_s}_{k_s \text{ times}})$, where k_i denotes the dimension of V_{g_i}, this grading becomes the one in Example 2.1.2.

The decomposition of $\mathbb{F}^n$ in Definition 2.1.3 fits into the general concept of graded vector spaces and (bi)modules.

Definition 2.1.4. Let G be a group. By a *G-grading* on a vector space V we mean simply a vector space decomposition $\Gamma : V = \bigoplus_{g\in G} V_g$ where the summands are labeled by elements of G. If Γ is fixed, V is referred to as a *G-graded vector space.*

Definition 2.1.5. Given two G-graded vector spaces $V = \bigoplus_{g\in G} V_g$ and $W = \bigoplus_{g\in G} W_g$, we define their tensor product to be the vector space $V \otimes W$ together with the G-grading given by $(V \otimes W)_g = \bigoplus_{ab=g} V_a \otimes W_b$.

The grading on M_n in Definition 2.1.3 is a special case of the following definition:

Definition 2.1.6. If $V = \bigoplus_{g\in G} V_g$ and $W = \bigoplus_{g\in G} W_g$ are two graded vector spaces and $T : V \to W$ is a linear map, we say that T is *homogeneous of degree* t, for some $t \in G$, if $T(V_g) \subseteq W_{tg}$ for all $g \in G$.

Given G-graded vector spaces U, V and W and homogeneous linear maps $S : U \to V$ and $T : V \to W$ of degrees s and t, respectively, we have that $T \circ S$ is homogeneous of degree ts.

We define the *space of graded linear transformations* from V to W to be:

$$\mathrm{Hom}^{\mathrm{gr}}(V, W) = \bigoplus_{g\in G} \mathrm{Hom}(V, W)_g$$

where $\mathrm{Hom}(V, W)_g$ denotes the set of all linear maps from V to W that are homogeneous of degree g.

If we assume V and W to be finite dimensional then we have

$$\mathrm{Hom}(V, W) = \mathrm{Hom}^{\mathrm{gr}}(V, W)$$

and, in particular,

$$\mathrm{End}(V) = \bigoplus_{g\in G} \mathrm{End}(V)_g$$

is a graded algebra.

Definition 2.1.7. Let $A = \bigoplus_{g\in G} A_g$ be a G-graded associative algebra and let $V = \bigoplus_{g\in G} V_g$ be a left A-module that is also a G-graded vector space. We say that V is a *G-graded module over A* if $A_g \cdot V_h \subseteq V_{gh}$ for all $g, h \in G$.

Note that a finite dimensional graded vector space V becomes a left graded module over $\mathrm{End}(V)$.

Similar definitions are given for right modules and for bimodules. Assuming G abelian, we can also define G-graded modules over a G-graded Lie algebra L. These can be regarded as G-graded left $U(L)$-modules if we make the universal enveloping algebra $U(L)$ a G-graded associative algebra by extending the grading from L.

Definition 2.1.8. Let (V, Γ) be a G-graded vector space and $g \in G$. We denote by $\Gamma^{[g]}$ the grading given by relabeling the component V_h by V_{hg}, for all $h \in G$. This is called the (right) *shift of the grading* Γ by g. Also, we denote the space $(V, \Gamma^{[g]})$ by $V^{[g]}$.

We will assume that G is abelian, so there will be no need to distinguish between left and right shift.

Lemma 2.1.9. *If V is a graded (bi)module over a graded associative or Lie algebra, then $V^{[g]}$ is also a graded (bi)module over the same algebra.* $\square$

Proposition 2.1.10. *Let A be a G-graded associative or Lie algebra over an algebraically closed field and let V be a finite-dimensional (ungraded) simple A-module. If Γ and Γ' are two G-gradings that make V a graded module over A then Γ' is a shift of Γ.*

Proof. The Lie case reduces to the associative case by considering the universal enveloping algebra. So, let A be an associative algebra. Since V is finite-dimensional, the Jacobson Density Theorem implies that the representation $\rho\colon A \to \mathrm{End}(V)$ is surjective. It will be convenient to denote V equipped with the gradings Γ and Γ' by

V_Γ and $V_{\Gamma'}$, respectively. Since ρ is a surjective homomorphism of graded algebras, we have $\mathrm{End}(V_\Gamma)_g = \rho(A_g) = \mathrm{End}(V_{\Gamma'})_g$ for all $g \in G$. The result follows from Lemma 2.1.12. $\square$

Note that Proposition 2.1.10 can be applied to (A, B)-bimodules, since we can regard them as left $A \otimes B^{\mathrm{op}}$-modules. Thus we obtain the following:

Corollary 2.1.11. *Let A and B be G-graded associative algebras over an algebraically closed field and let V be a finite-dimensional (ungraded) simple (A, B)-bimodule. If Γ and Γ' are two G-gradings that make V a graded bimodule, then Γ' is a shift of Γ.* $\square$

Lemma 2.1.12. *Let V be a finite-dimensional vector space and let Γ, Γ' be G-gradings on V that induce the same grading on $\mathrm{End}(V)$. Then Γ' is a shift of Γ.*

Proof. Again, let us denote V equipped with Γ and Γ' by V_Γ and $V_{\Gamma'}$, respectively. Let I be a minimal graded left ideal of $R = \mathrm{End}(V)$. By Lemma 2.7 of [12], there exist $g, g' \in G$ such that $V_\Gamma \cong I^{[g]}$ and $V_{\Gamma'} \cong I^{[g']}$. Thus, we have an isomorphism $V_{\Gamma'} \to (V_\Gamma)^{[g^{-1}g']}$ of graded R-modules. Such an isomorphism must be a scalar operator and, therefore, it leaves all subspaces invariant. We conclude that Γ' is a shift of Γ. $\square$

If we have a G-grading on a Lie superalgebra $L = L^{\bar 0} \oplus L^{\bar 1}$ then, in particular, we have a grading on the Lie algebra $L^{\bar 0}$ and a grading on the space $L^{\bar 1}$ that makes it a graded $L^{\bar 0}$-module. If we have a G-grading on an associative superalgebra $S = S^{\bar 0} \oplus S^{\bar 1}$, then $S^{\bar 1}$ becomes a graded bimodule over $S^{\bar 0}$.

In particular, certain shifts of grading may be applied to graded Lie $\mathbb{Z}$ or $\mathbb{Z}_2$-superalgebras. In the case of a $\mathbb{Z}$-superalgebra $L = L^{-1} \oplus L^0 \oplus L^1$, we have the following:

Lemma 2.1.13. *Let $L = L^{-1} \oplus L^0 \oplus L^1$ be a $\mathbb{Z}$-superalgebra such that $L^1 L^{-1} \neq 0$. If we shift the grading on L^1 by $g \in G$ and the grading on L^{-1} by $g' \in G$, then we have a grading on L if and only if $g' = g^{-1}$.* □

We may describe this situation as "shifts in opposite directions".

For our next example of gradings, recall that a *Morita context* is a sextuple $\mathcal{C} = (R, S, M, N, \varphi, \psi)$ where R and S are associative algebras, M is an (R, S)-bimodule, N is a (S, R)-bimodule and $\varphi : M \otimes_S N \to R$ and $\psi : N \otimes_R M \to S$ are linear maps satisfying the necessary and sufficient conditions for

$$C = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

to be an associative algebra, i.e.,

$$\varphi(m_1 \otimes n_1) \cdot m_2 = m_1 \cdot \psi(n_1 \otimes m_2) \text{ and } \psi(n_1 \otimes m_1) \cdot n_2 = n_1 \cdot \varphi(m_1 \otimes n_2)$$

for all $m_1, m_2 \in M$ and $n_1, n_2 \in N$.

Now, we need to define graded Morita contexts.

Definition 2.1.14. A Morita context $(R, S, M, N, \varphi, \psi)$ is said to be G-*graded* if the algebras R, S are graded, the bimodules M, N are graded and the maps φ and ψ are homogeneous of degree e.

Definition 2.1.15. A *Morita algebra* is a pair (C, ϵ) where C is an associative algebra and $\epsilon \in C$ is an idempotent. A *homomorphism* of Morita algebras $(C_1, \epsilon_1) \to (C_2, \epsilon_2)$ is a homomorphism of algebras $C_1 \to C_2$ that sends $\epsilon_1 \mapsto \epsilon_2$. We will say that (C, ϵ) is G-graded if C is graded and ϵ is a homogeneous element (necessarily of degree e).

Example 2.1.16. Finally, note that a Morita algebra (C, ϵ) has a natural $\mathbb{Z}$-grading given by $C^{-1} = \epsilon C(1 - \epsilon)$, $C^0 = \epsilon C \epsilon \oplus (1 - \epsilon)C(1 - \epsilon)$ and $C^1 = (1 - \epsilon)C\epsilon$.

2.2 Equivalence and isomorphism of gradings

There is a concept of grading not involving groups. This is a decomposition $\Gamma : A = \bigoplus_{s\in S} A_s$ as a direct sum of nonzero subspaces indexed by a set S and having the property that, for any $s_1, s_2 \in S$ with $A_{s_1}A_{s_2} \neq 0$, there exists (unique) $s_3 \in S$ such that $A_{s_1}A_{s_2} \subseteq A_{s_3}$. For such a decomposition Γ, there may or may not exist a group G containing S that makes Γ a G-grading. If such a group exists, Γ is said to be a *group grading*. However, G is usually not unique even if we require that it should be generated by S.

The *universal grading group* is a group G generated by S and has the defining relations $s_1s_2 = s_3$ for all $s_1, s_2, s_3 \in S$ such that $0 \neq A_{s_1}A_{s_2} \subseteq A_{s_3}$. This group is universal among all groups that realize the grading Γ, in other words, if Γ admits a realization as a G_0-grading by some group G_0 then there exists a unique group homomorphism $G \to G_0$ that restricts to identity on the support of Γ.

In this thesis we only consider group gradings, more specifically, abelian group gradings. The universal abelian grading group is defined similarly to the above. In view of Proposition 1.0.4, there is no difference in the case of Lie (super)algebras.

Let $\Gamma : A = \bigoplus_{g\in G} A_g$ and $\Delta : B = \bigoplus_{h\in H} B_h$ be two group gradings on the (super)algebras A and B, with supports S and T, respectively. We say that Γ and Δ are *equivalent* if there exists an isomorphism of (super)algebras $\varphi : A \to B$ and a bijection $\alpha : S \to T$ such that $\varphi(A_s) = B_{\alpha(s)}$ for all $s \in S$. If G and H are universal grading groups then α extends to an isomorphism $G \to H$.

In the case $G = H$, the G-gradings Γ and Δ are said to be *isomorphic* if A and B are isomorphic as G-graded (super)algebras, i.e., if there exists an isomorphism of (super)algebras $\varphi : A \to B$ such that $\varphi(A_g) = B_g$ for all $g \in G$.

If $\Gamma : A = \bigoplus_{g\in G} A_g$ and $\Gamma' : A = \bigoplus_{h\in H} A'_h$ are two gradings on the same superalgebra A, with supports S and T, respectively, then we say that Γ' is a *refinement* of Γ (or Γ is a *coarsening* of Γ') if, for any $t \in T$, there exists (unique) $s \in S$ such that $A'_t \subseteq A_s$. If, moreover, $A'_t \neq A_s$, for at least one $t \in T$, then the refinement is said to be *proper*. A grading Γ is said to be *fine* if it does not admit any proper refinements. Note that $A = \bigoplus_{(g,i)\in G\times\mathbb{Z}_2} A^i_g$ is a refinement of Γ. It follows that if Γ is fine then the sets $\mathrm{supp}^{\bar{0}}(\Gamma)$ and $\mathrm{supp}^{\bar{1}}(\Gamma)$ are disjoint (where $\mathrm{supp}(\Gamma) = \mathrm{supp}^{\bar{0}}(\Gamma) \cup \mathrm{supp}^{\bar{1}}(\Gamma)$ with $\mathrm{supp}^i(\Gamma) = \{g \in G,\ A^i_g \neq 0\}$).

Definition 2.2.1. Let A be a (super)algebra, G and H be groups, $\alpha : G \to H$ be a homomorphism of groups and $\Gamma : A = \bigoplus_{g\in G} A_g$ be a G-grading on A. We define the *coarsening induced by* α to be the H-grading ${}^{\alpha}\Gamma : A = \bigoplus_{h\in H} A_h$ where $A_h = \bigoplus_{g\in\alpha^{-1}(h)} A_g$.

Lemma 2.2.2. *Let $\mathcal{F} = \{\Gamma_i\}_{i\in I}$, be a family of pairwise nonequivalent fine (abelian) group gradings on a (super)algebra A, where Γ_i is a G_i-grading and G_i is generated by* $\mathrm{supp}(\Gamma_i)$*. Suppose that $\mathcal{F}$ has the following property: for any grading Γ on A by an (abelian) group H, there exists $i \in I$ and a homomorphism $\alpha : G_i \to H$ such that Γ is isomorphic to ${}^{\alpha}\Gamma_i$. Then*

(i) every fine (abelian) group grading on A is equivalent to a unique Γ_i;

(ii) for all i, G_i is the universal (abelian) grading group of Γ_i.

Proof. Let Γ be a fine grading on A, realized over its universal group H. Then there is $i \in I$ and $\alpha : G_i \to H$ such that ${}^{\alpha}\Gamma_i \simeq \Gamma$. Writing $\Gamma_i : A = \bigoplus_{g\in G_i} A_g$ and

$\Gamma : A = \bigoplus_{h \in H} B_h$, we then have $\varphi \in \mathrm{Aut}(A)$ such that

$$\varphi\Big(\bigoplus_{g \in \alpha^{-1}(h)} A_g \Big) = B_h$$

for all $h \in H$. Since Γ is fine, we must have $B_h \neq 0$ if, and only if, there is a unique $g \in G_i$ such that $\alpha(g) = h$, $A_g \neq 0$ and $\varphi(A_g) = B_h$. Equivalently, α restricts to a bijection $\mathrm{supp}(\Gamma_i) \to \mathrm{supp}(\Gamma)$ and $\varphi(A_g) = B_{\alpha(g)}$ for all $g \in S_i := \mathrm{supp}(\Gamma_i)$. This proves assertion (i).

Let G be the universal group of Γ_i. It follows that, for all $s_1, s_2, s_3 \in S_i$,

$$\begin{aligned} & s_1 s_2 = s_3 \text{ is a defining relation of } G \\ \iff & 0 \neq A_{s_1} A_{s_2} \subseteq A_{s_3} \\ \iff & 0 \neq B_{\alpha(s_1)} B_{\alpha(s_2)} \subseteq B_{\alpha(s_3)} \\ \iff & \alpha(s_1)\alpha(s_2) = \alpha(s_3) \text{ is a defining relation of } H. \end{aligned}$$

Therefore, the bijection $\alpha|_{S_i}$ extends uniquely to an isomorphism $\beta : G \to H$.

By the universal property of G, there is a unique homomorphism $\gamma : G \to G_i$ that restricts to the identity on S_i.

Since β is an isomorphism, γ must be injective. But γ is also surjective since S_i generates G_i. Hence G_i is isomorphic to G. Since Γ was an arbitrary fine grading, for each given $j \in I$, we can take $\Gamma = \Gamma_j$ (hence, $i = j$ and $H = G$). This concludes the proof of (ii). $\square$

2.3 Gradings and actions

Definition 2.3.1. Let $R = R^{\bar{0}} \oplus R^{\bar{1}}$ be a $\mathbb{Z}_2$-graded algebra and let υ be the linear operator such that $\upsilon(x) = x$ if x is even and $\upsilon(x) = -x$ if x is odd. Then υ is an automorphism of R. It is called the *parity automorphism*.

Note that the $\mathbb{Z}_2$-grading can be recovered as the eigenspace of υ.

Proposition 2.3.2. *Let $R = R^{\bar{0}} \oplus R^{\bar{1}}$ be a $\mathbb{Z}_2$-graded algebra and let υ be the parity automorphism. A linear map $\phi\colon R \to R$ preserves the $\mathbb{Z}_2$-grading if and only if ϕ commutes with υ.*

Proof. Suppose ϕ preserves the $\mathbb{Z}_2$-grading. If $x \in R^{\bar{0}}$ then $\phi(x) \in R^{\bar{0}}$ and $(\phi \circ \upsilon)(x) = \phi(\upsilon(x)) = \phi(x) = \upsilon(\phi(x)) = (\upsilon \circ \phi)(x)$. If $x \in R^{\bar{1}}$ then $\upsilon(x) = -x$ and $(\phi \circ \upsilon)(x) = \phi(\upsilon(x)) = \phi(-x) = -\phi(x) = \upsilon(\phi(x)) = (\upsilon \circ \phi)(x)$.

Conversely, let $x \in R^{\bar{0}}$, if $(\phi \circ \upsilon)(x) = \phi(x) = (\upsilon \circ \phi)(x)$ then, in particular, $\upsilon(\phi(x)) = \phi(x)$ and hence $\phi(x) \in R^{\bar{0}}$. Let $x \in R^{\bar{1}}$, if $\phi(\upsilon(x)) = -\phi(x) = \upsilon(\phi(x))$, then $\phi(x) \in R^{\bar{1}}$. □

More generally, assuming $\mathbb{F}$ is algebraically closed of characteristic zero, all gradings by $\mathbb{Z}_n$ on an algebra R are associated with the algebra automorphisms of order n. In fact, let R be an algebra and let $\alpha\colon R \to R$ be an automorphism of order $n \in \mathbb{N}$. Denote by ω a n-th primitive root of unity. Then α is diagonalizable with eigenvalues $\{1, \omega, \omega^2, \ldots, \omega^{n-1}\}$. For each $0 \leq j \leq n-1$, we declare the eigenspace associated with the eigenvalue ω^j to be the component of degree $\bar{j} \in \mathbb{Z}_n$.

We will now show an example of a grading on $L = A(m,n)$ that is not a restriction of a grading on $M(m,n)$. The following definition will be used in Chapters 4 and 5.

Definition 2.3.3. The *supertranspose* of a matrix in $M(m,n)$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{s\top} = \begin{pmatrix} a^\top & c^\top \\ -b^\top & d^\top \end{pmatrix}. \tag{2.3.1}$$

The map $\tau : L \to L$ given by $\tau(X) = -X^{s\top}$ is, then, an automorphism of order 4 in $\mathrm{Aut}(A(n,n))$.

Remark 2.3.4. By [20], the group of automorphisms of $A(m,n)$ is generated by the group $\mathcal{E}$ of automorphisms of $\mathrm{End}(U)$ and τ. In other words, if $m \neq n$, $\mathrm{Aut}(A(m,n))$ is generated by $\mathcal{E} \cup \{\tau\}$ and, if $m = n$, by $\mathcal{E} \cup \{\pi\, ,\, \tau\}$, where π is defined in Section 3.2. In both cases, $\mathcal{E}$ is a normal subgroup of $\mathrm{Aut}(A(m,n))$. Note that $\pi^2 = \mathrm{id}$, $\tau^2 = \upsilon$ (the parity automorphism – see Definition 2.3.1) and $\pi\tau = \upsilon\tau\pi$. Hence $\mathrm{Aut}(A)/\mathcal{E}$ is isomorphic to $\mathbb{Z}_2$ if $m \neq n$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ if $m = n$.

Now let us construct a grading Γ_τ on $A(m,n)$ associated to τ. From now on, i denotes a square root of -1 (i.e., a primitive 4-th root of unity).

Following the construction above, we have the following $\mathbb{Z}_4$-grading on $L = A(n,n)$:

$$L_{\bar{1}} = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \in L \mid c^\top = ib \right\};$$

$$L_{\bar{2}} = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in L \mid a = a^\top,\, d = d^\top \right\};$$

$$L_{\bar{3}} = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \in L \mid -c^\top = ib \right\};$$

$$L_{\bar{0}} = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in L \mid a = -a^\top,\, d = -d^\top \right\}.$$

Note that this grading is not compatible with the $\mathbb{Z}$-superalgebra structure. Indeed, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A(n,n)_{\bar{1}}$ with $b \neq 0$ then $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ is not Γ_τ-homogeneous.

Another important tool for dealing with gradings on (super)algebras is to translate a G-grading into a $\widehat{G}$-action, where $\widehat{G}$ is the group of characters of G, i.e., group homomorphisms $G \to \mathbb{F}^\times$.

The group $\widehat{G}$ acts on any G-graded (super)algebra $A = \bigoplus_{g\in G} A_g$ by $\chi \cdot a = \chi(g)a$ for all $a \in A_g$. The map given by the action of a character $\chi \in \widehat{G}$ is an automorphism of A.

In our case, where $\mathbb{F}$ is algebraically closed of characteristic zero, $A_g = \{a \in A \mid \chi \cdot a = \chi(g)a\}$, so the grading can be recovered from the action.

For example, if $A = A^{\bar{0}} \oplus A^{\bar{1}}$ is a superalgebra, the action of the non-trivial character of $\widehat{\mathbb{Z}_2}$ yields the *parity automorphism* υ (recall Definition 2.3.1), which acts as identity (id) on $A^{\bar{0}}$ and as negative identity on $A^{\bar{1}}$. If A is a $\mathbb{Z}$-graded algebra, we get the representation $\widehat{\mathbb{Z}} = \mathbb{F}^\times \to \mathrm{Aut}(A)$ given by $\lambda \mapsto \upsilon_\lambda$, where υ_λ acts as $\lambda^i \,\mathrm{id}$ on A^i.

Another result we will make use of is the following:

Proposition 2.3.5. *Let V be a G-graded vector space and let $\eta\colon \widehat{G} \to \mathrm{GL}(V)$ be the corresponding action. A subspace $W \subseteq V$ is graded if and only if W is invariant under $\eta(\widehat{G})$. Moreover, the restriction of the G-grading ($W_g = V_g \cap W$) corresponds to the restriction of the $\widehat{G}$-action.*

Proof. Assume that $W \subseteq V$ is graded. Take $w \in W$ and write $w = w_{g_1} + \cdots + w_{g_l}$ with l a natural number and $w_{g_i} \in V_{g_i}$ for all $1 \leq i \leq l$. For every $\chi \in \widehat{G}$, $\eta(\chi)(w) = \chi(g_1)w_{g_1} + \cdots + \chi(g_l)w_{g_l}$, hence if, for all $1 \leq i \leq l$, $w_{g_i} \in W$ then $\eta(\chi)(w) \in W$. Conversely, assume W invariant under $\eta(\widehat{G})$. We want to prove that if $w = w_1 + w_2 + \cdots + w_m$ where $w_i \in V_{g_i}$, $g_i \neq g_j$ for $i \neq j$ then all $w_i \in W$. Proceed by induction on m, with trivial basis $m = 1$. Let $m > 1$. By induction, we may assume that all w_i are nonzero. Since $\widehat{G}$ separates the elements of G, there is $\chi \in \widehat{G}$ such that

$\chi(g_1) \neq \chi(g_2)$. The elements $\chi(g_1)w$ and χw are in W. Their difference w' has the form $w' = (\chi(g_1) - \chi(g_2))w_2 + \cdots + (\chi(g_1) - \chi(g_m))w_m$ is still in W and one can apply induction. Then all summands of w' are in W. Since $(\chi(g_1) - \chi(g_2)) \neq 0$, we have $w_2 \in W$. Now $w'' = w - w_2 \in W$ and $w'' = w_1 + w_3 + \cdots + w_m \in W$. By induction, $w_1, w_3, ..., w_m \in W$. Thus W is a graded subspace. $\square$

2.4 Gradings on matrix (super)algebras

In this section we will recall the classification of gradings on matrix algebras. We will follow Chapter 2 from [12] but use a slightly different notation, which we will extend to superalgebras in Chapter 3.

The following is the graded version of a classical result (see e.g. [12, Theorem 2.6]). We recall that a *graded division algebra* is a graded unital associative algebra such that every nonzero homogeneous element is invertible.

Theorem 2.4.1. *Let G be a group and let R be a G-graded associative algebra that has no nontrivial graded ideals and satisfies the descending chain condition on graded left ideals. Then there is a G-graded division algebra $\mathcal{D}$ and a graded (right) $\mathcal{D}$-module $\mathcal{V}$ such that $R \simeq \operatorname{End}_{\mathcal{D}}(\mathcal{V})$ as graded algebras.* $\square$

We apply this result to the algebra $R = M_n(\mathbb{F})$ equipped with a grading by an abelian group G. We will now introduce the parameters that determine $\mathcal{D}$ and $\mathcal{V}$, and give an explicit isomorphism $\operatorname{End}_{\mathcal{D}}(\mathcal{V}) \simeq M_n(\mathbb{F})$ (see Definition 2.4.4). It should be mentioned that a classification of G-gradings on R is known for non-abelian G (see [12, Corollary 2.22]); since our final goal here is to use the classification of the group gradings in the associative case as a tool to classify the gradings on some classical Lie superalgebras, we restrict ourselves to abelian groups.

Let $\mathcal{D}$ be a finite-dimensional G-graded division algebra. It is easy to see that $T = \operatorname{supp} \mathcal{D}$ is a finite subgroup of G. Also, since we are over an algebraically closed field, each homogeneous component $\mathcal{D}_t$, for $t \in T$, is one-dimensional. We can choose a generator X_t for each $\mathcal{D}_t$. It follows that, for every $u, v \in T$, there is a unique nonzero scalar $\beta(u, v)$ such that $X_u X_v = \beta(u, v) X_v X_u$. Clearly, $\beta(u, v)$ does not depend on the choice of X_u and X_v. The map $\beta : T \times T \to \mathbb{F}^\times$ is a *bicharacter*, i.e., both maps $\beta(t, \cdot)$ and $\beta(\cdot, t)$ are characters for every $t \in T$. It is also *alternating* in the sense that $\beta(t, t) = 1$ for all $t \in T$. We define the *radical* of β as the set $\operatorname{rad} \beta = \{t \in T \mid \beta(t, T) = 1\}$. In the case we are interested in, where $\mathcal{D}$ is simple as an algebra, the bicharacter β is *nondegenerate*, i.e., $\operatorname{rad} \beta = \{e\}$. The isomorphism classes of G-graded division algebras that are finite-dimensional and simple as algebras are in one-to-one correspondence with the pairs (T, β) where T is a finite subgroup of G and β is an alternating nondegenerate bicharacter on T (see e.g. [12, Section 2.2] for a proof).

Using the fact that the bicharacter β is nondegenerate, we can decompose the group T as $A \times B$, where the restrictions of β to each of the subgroups A and B is trivial, and hence A and B are in duality by β. We can choose the elements $X_t \in \mathcal{D}_t$ in a convenient way (see [12, Remark 2.16] and [13, Remark 18]) so that $X_{ab} = X_a X_b$ for all $a \in A$ and $b \in B$. Using this choice, we can define an action of $\mathcal{D}$ on the vector space underlying the group algebra $\mathbb{F}B$, by declaring $X_a \cdot e_{b'} = \beta(a, b') e_{b'}$ and $X_b \cdot e_{b'} = e_{bb'}$. This action allows us to identify $\mathcal{D}$ with $\operatorname{End}(\mathbb{F}B)$. Using the basis

$\{e_b \mid b \in B\}$ in $\mathbb{F}B$, we can view $\operatorname{End}(\mathbb{F}B)$ as a matrix algebra, where

$$X_{ab} = \sum_{b' \in B} \beta(a, bb') E_{bb',b'}$$

and $E_{b'',b'}$ with b', $b'' \in B$, is a matrix unit, namely, the matrix of the operator that sends $e_{b'}$ to $e_{b''}$ and sends all other basis elements to zero.

Definition 2.4.2. We will refer to these matrix models of $\mathcal{D}$ as its *standard realizations.*

Remark 2.4.3. The matrix transposition is always an involution of the algebra structure. As to the grading, we have

$$X_{ab}^{\top} = \sum_{b' \in B} \beta(a, bb') E_{b',bb'} = \beta(a, b) \sum_{b'' \in B} \beta(a, b^{-1}b'') E_{b^{-1}b'',b''} = \beta(a, b) X_{ab^{-1}}.$$

It follows that if T is an elementary 2-group, then the transposition preserves the degree. In this case, we will use the transposition to fix an identification between the graded algebras $\mathcal{D}$ and $\mathcal{D}^{\mathrm{op}}$.

Graded modules over a graded division algebra $\mathcal{D}$ behave similarly to vector spaces. The usual proof that every vector space has a basis, with obvious modifications, shows that every graded $\mathcal{D}$-module has a *homogeneous basis*, i.e., a basis formed by homogeneous elements. Let $\mathcal{V}$ be such a module of finite rank k, fix a homogeneous basis $\mathcal{B} = \{v_1, \dots, v_k\}$ and let $g_i := \deg v_i$. We then have $\mathcal{V} \simeq \mathcal{D}^{[g_1]} \oplus \cdots \oplus \mathcal{D}^{[g_k]}$, so, the graded $\mathcal{D}$-module $\mathcal{V}$ is determined by the k-tuple $\gamma = (g_1, \dots, g_k)$. The tuple γ is not unique. To capture the precise information that determines the isomorphism class of $\mathcal{V}$, we use the concept of *multiset*, i.e., a set together with a map from it to the set of positive integers. If $\gamma = (g_1, \dots, g_k)$ and $T = \operatorname{supp} \mathcal{D}$, we denote by $\Xi(\gamma)$

the multiset whose underlying set is $\{g_1T, \ldots, g_kT\} \subseteq G/T$ and the multiplicity of g_iT, for $1 \leq i \leq k$, is the number of entries of γ that are congruent to g_i modulo T.

Using $\mathcal{B}$ to represent the linear maps by matrices in $M_k(\mathcal{D}) = M_k(\mathbb{F}) \otimes \mathcal{D}$, we now construct an explicit matrix model for $\mathrm{End}_{\mathcal{D}}(\mathcal{V})$.

Definition 2.4.4. Let $T \subseteq G$ be a finite subgroup, β a nondegenerate alternating bicharacter on T, and $\gamma = (g_1, \ldots, g_k)$ a k-tuple of elements of G. Let $\mathcal{D}$ be a standard realization of a graded division algebra associated to (T, β). Identify $M_k(\mathbb{F}) \otimes \mathcal{D} \simeq M_n(\mathbb{F})$ by means of the Kronecker product, where $n = k\sqrt{|T|}$. We will denote by $\Gamma(T, \beta, \gamma)$ the grading on $M_n(\mathbb{F})$ given by $\deg(E_{ij} \otimes d) := g_i(\deg d)g_j^{-1}$ for $i, j \in \{1, \ldots, k\}$ and homogeneous $d \in \mathcal{D}$, where E_{ij} is the (i, j)-th matrix unit.

If $\mathrm{End}(V)$, equipped with a grading, is isomorphic to $M_n(\mathbb{F})$ with $\Gamma(T, \beta, \gamma)$, we may abuse notation and also denote the grading on $\mathrm{End}(V)$ by $\Gamma(T, \beta, \gamma)$. We restate [12, Theorem 2.27] (see also [2, Theorem 2.6]) using our notation:

Theorem 2.4.5. *Every G-grading on the algebra $M_n(\mathbb{F})$ is isomorphic to some $\Gamma(T, \beta, \gamma)$ as in Definition 2.4.4. Two gradings, $\Gamma(T, \beta, \gamma)$ and $\Gamma(T', \beta', \gamma')$, on the algebra $M_n(\mathbb{F})$ are isomorphic if, and only if, $T = T'$, $\beta = \beta'$ and there is an element $g \in G$ such that $g\Xi(\gamma) = \Xi(\gamma')$.* □

The proof of this theorem is based on the following result (see Theorem 2.10 and Proposition 2.18 from [12]), which will also be needed:

Proposition 2.4.6. *If $\phi : \mathrm{End}_{\mathcal{D}}(\mathcal{V}) \to \mathrm{End}_{\mathcal{D}}(\mathcal{V}')$ is an isomorphism of graded algebras, then there is a homogeneous invertible $\mathcal{D}$-linear map $\psi : \mathcal{V} \to \mathcal{V}'$ such that $\phi(r) = \psi \circ r \circ \psi^{-1}$, for all $r \in \mathrm{End}_{\mathcal{D}}(\mathcal{V})$.* □

We can study matrix superalgebras in the same manner using the following trick: if A is an associative superalgebra with a grading by an abelian group G, we may consider A graded by the group $G^{\#} = G \times \mathbb{Z}_2$. Then the parity $|a|$ of a homogeneous element $a \in A$ is determined by $\deg a \in G^{\#}$.

2.5 Superdual of a graded module

Let $\mathcal{U}$ be a $G^{\#}$-graded module over $\mathcal{D}$. The *superdual module* of $\mathcal{U}$ is $\mathcal{U}^\star = \operatorname{Hom}_{\mathcal{D}}(\mathcal{U}, \mathcal{D})$, with its natural $G^{\#}$-grading and the $\mathcal{D}$-action defined on the *left*: if $d \in \mathcal{D}$ and $f \in \mathcal{U}^\star$, then $(df)(u) = d\,f(u)$ for all $u \in \mathcal{U}$.

We define the *opposite superalgebra* of $\mathcal{D}$, denoted by $\mathcal{D}^{\mathrm{sop}}$, to be the same graded superspace $\mathcal{D}$, but with a new product $a * b = (-1)^{|a||b|} ba$ for every pair of $\mathbb{Z}_2$-homogeneous elements $a, b \in \mathcal{D}$. The left $\mathcal{D}$-module $\mathcal{U}^\star$ can be considered as a right $\mathcal{D}^{\mathrm{sop}}$-module by means of the action defined by $f \cdot d := (-1)^{|d||f|} df$, for every $\mathbb{Z}_2$-homogeneous $d \in \mathcal{D}$ and $f \in \mathcal{U}^\star$.

Lemma 2.5.1. *If $\mathcal{D}$ is a graded division superalgebra associated to the pair (T, β), then $\mathcal{D}^{\mathrm{sop}}$ is associated to the pair (T, β^{-1}).* $\square$

If $\mathcal{U}$ has a homogeneous $\mathcal{D}$-basis $\mathcal{B} = \{e_1, \ldots, e_k\}$, we can consider its *superdual basis* $\mathcal{B}^\star = \{e_1^\star, \ldots, e_k^\star\}$ in $\mathcal{U}^\star$, where $e_i^\star : \mathcal{U} \to \mathcal{D}$ is defined by $e_i^\star(e_j) = (-1)^{|e_i||e_j|}\delta_{ij}$.

Remark 2.5.2. The superdual basis is a homogeneous basis of $\mathcal{U}^\star$, with $\deg e_i^\star = (\deg e_i)^{-1}$. So, if $\gamma = (g_1, \ldots, g_k)$ is the k-tuple of degrees of $\mathcal{B}$, then $\gamma^{-1} = (g_1^{-1}, \ldots, g_k^{-1})$ is the k-tuple of degrees of $\mathcal{B}^\star$.

For graded right $\mathcal{D}$-modules $\mathcal{U}$ and $\mathcal{V}$, we consider $\mathcal{U}^\star$ and $\mathcal{V}^\star$ as right $\mathcal{D}^{\mathrm{sop}}$-modules as defined above. If $L : \mathcal{U} \to \mathcal{V}$ is a $\mathbb{Z}_2$-homogeneous $\mathcal{D}$-linear map, then the *superadjoint* of L is the $\mathcal{D}^{\mathrm{sop}}$-linear map $L^\star : \mathcal{V}^\star \to \mathcal{U}^\star$ defined by $L^\star(f) = (-1)^{|L||f|} f \circ L$. We extend the definition of superadjoint to any map in $\mathrm{Hom}_{\mathcal{D}}(\mathcal{U}, \mathcal{V})$ by linearity.

Remark 2.5.3. In the case $\mathcal{D} = \mathbb{F}$, if we denote by $[L]$ the matrix of L with respect to the homogeneous bases $\mathcal{B}$ of $\mathcal{U}$ and $\mathcal{C}$ of $\mathcal{V}$, then the supertranspose $[L]^{s\top}$ is the matrix corresponding to $L^\star$ with respect to the superdual bases $\mathcal{C}^\star$ and $\mathcal{B}^\star$.

We denote by $\varphi : \mathrm{End}_{\mathcal{D}}(\mathcal{U}) \to \mathrm{End}_{\mathcal{D}^{\mathrm{sop}}}(\mathcal{U}^\star)$ the map $L \mapsto L^\star$. It is clearly bijective and homogeneous of degree e, but it is not a superalgebra isomorphism. Instead, it is a *super-anti-isomorphism*, i.e., it is $\mathbb{F}$-linear and

$$\varphi(L_1 L_2) = (-1)^{|L_1||L_2|} \varphi(L_2)\varphi(L_1) \, .$$

It follows that, if we consider the Lie superalgebras $\mathrm{End}_{\mathcal{D}}(U)^{(-)}$ and $\mathrm{End}_{\mathcal{D}^{\mathrm{sop}}}(U^\star)^{(-)}$, the map $-\varphi$ is an isomorphism.

We summarize these considerations in the following result:

Lemma 2.5.4. *If $\Gamma = \Gamma(T, \beta, \gamma)$ and $\Gamma' = \Gamma(T, \beta^{-1}, \gamma^{-1})$ are G-gradings (considered as $G^\#$-gradings) on the associative superalgebra $M(m, n)$, then, as gradings on the Lie superalgebra $M(m, n)^{(-)}$, Γ and Γ' are isomorphic via an automorphism of $M(m, n)^{(-)}$ that is the negative of a super-anti-automorphism of $M(m, n)$.*

Proof. Let $\mathcal{D}$ be a graded division superalgebra associated to (T, β) and let $\mathcal{U}$ be the graded right $\mathcal{D}$-module associated to γ. The grading Γ is obtained by an identification $\psi : M(m, n) \xrightarrow{\sim} \mathrm{End}_{\mathcal{D}}(\mathcal{U})$. By Lemma 2.5.1 and Remark 2.5.2, Γ' is obtained by an identification $\psi' : M(m, n) \xrightarrow{\sim} \mathrm{End}_{\mathcal{D}^{\mathrm{sop}}}(\mathcal{U}^\star)$. Hence we have that the composition

$(\psi')^{-1}\,(-\varphi)\,\psi$ is an automorphism of the Lie superalgebra $M(m,n)^{(-)}$ sending Γ to Γ'. Note that, it is not an automorphism of the the associative superalgebra $M(m,n)$ since it does not fix the identity element. $\square$

Chapter 3

Matrix superalgebras $M(m,n)$

3.1 The associative superalgebra $M(m,n)$

In this chapter we assume $\mathbb{F}$ to be algebraically closed but not necessarily of characteristic zero.

Let $U = U^{\bar{0}} \oplus U^{\bar{1}}$ be a superspace. Recall that the algebra of endomorphisms of U has an induced $\mathbb{Z}_2$-grading, so it can be regarded as a superalgebra and we can write it in the form (1.0.1). In addition, recall that the superalgebra $\mathrm{End}(U)$ can be seen as a matrix superalgebra, which is denoted by $M(m,n)$ if $\dim U^{\bar{0}} = m$ and $\dim U^{\bar{1}} = n$.

We may also regard U as a $\mathbb{Z}$-graded vector space, putting $U^0 = U^{\bar{0}}$ and $U^1 = U^{\bar{1}}$. By doing so, we obtain an induced $\mathbb{Z}$-grading on $M(m,n) = \mathrm{End}(U)$ such that

$$(\mathrm{End}\ U)^{\bar{0}} = (\mathrm{End}\ U)^{0} = \begin{pmatrix} \mathrm{End}(U^{\bar{0}}) & 0 \\ 0 & \mathrm{End}(U^{\bar{1}}) \end{pmatrix}$$

and $(\text{End } U)^{\bar{1}} = (\text{End } U)^{-1} \oplus (\text{End } U)^{1}$ where

$$(\operatorname{End} U)^1 = \begin{pmatrix} 0 & 0 \\ \operatorname{Hom}(U^{\bar{0}}, U^{\bar{1}}) & 0 \end{pmatrix} \text{ and } (\operatorname{End} U)^{-1} = \begin{pmatrix} 0 & \operatorname{Hom}(U^{\bar{1}}, U^{\bar{0}}) \\ 0 & 0 \end{pmatrix}.$$

This grading will be called the *canonical $\mathbb{Z}$-grading* on $M(m, n)$.

3.2 Automorphisms of $M(m, n)$

It is known that the automorphisms of the superalgebra $\text{End}(U)$ are conjugations by invertible homogeneous operators. (This follows, for example, from Proposition 2.4.6.) The invertible even operators are of the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ where $a \in \text{GL}(m)$ and $d \in \text{GL}(n)$. The corresponding inner automorphisms of $M(m, n)$ will be called *even automorphisms*. They form a normal subgroup of $\text{Aut}(M(m, n))$, which we denote by $\mathcal{E}$.

The inner automorphisms given by odd operators will be called *odd automorphisms*. Note that an invertible odd operator must be of the form $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ where both b and c are invertible, and this forces $m = n$. In this case, the set of odd automorphisms is a coset of $\mathcal{E}$, namely, $\pi\mathcal{E}$, where π is the conjugation by the matrix $\begin{pmatrix} 0_n & I_n \\ I_n & 0_n \end{pmatrix}$. This automorphism is called the *parity transpose* and is usually denoted by superscript:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\pi} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}.$$

Thus, $\text{Aut}(M(m, n)) = \mathcal{E}$ if $m \neq n$, and $\text{Aut}(M(n, n)) = \mathcal{E} \rtimes \langle \pi \rangle$.

Remark 3.2.1. It is worth noting that $\mathcal{E}$ is the automorphism group of the $\mathbb{Z}$-superalgebra structure of $M(m,n)$, regardless of the values m and n. Indeed, the elements of this group are conjugations by homogeneous matrices with respect to the canonical $\mathbb{Z}$-grading, but all the matrices of degree -1 or 1 are degenerate.

3.3 Gradings on matrix superalgebras

We are now going to generalize the results of Section 2.4 to the superalgebra $M(m,n)$. It is clear that a G-graded associative superalgebra is equivalent to a $(G \times \mathbb{Z}_2)$-graded associative algebra. Hence one could think that no new problem arises. But the description of gradings on matrix algebras presented in Section 2.4 does not allow us to readily see the gradings on the even and odd components of the superalgebra, so we are going to refine that description. We will denote the group $G \times \mathbb{Z}_2$ by $G^\#$ and the projection on the second factor by $p\colon G^\# \to \mathbb{Z}_2$. Also, we will abuse the notation and identify G with $G \times \{\bar{0}\} \subseteq G^\#$.

Remark 3.3.1. If the canonical $\mathbb{Z}_2$-grading is a coarsening of the G-grading by means of a homomorphism $p\colon G \to \mathbb{Z}_2$ (referred to as the *parity homomorphism*), then we have another isomorphic copy of G in $G^\#$, namely, the image of the embedding $g \mapsto (g, p(g))$, which contains the support of the $G^\#$-grading. In this case, we do not need $G^\#$ and can work with the original G-grading.

A G-graded superalgebra $\mathcal{D}$ is called a *graded division superalgebra* if every nonzero homogeneous element in $\mathcal{D}^{\bar{0}} \cup \mathcal{D}^{\bar{1}}$ is invertible — in other words, $\mathcal{D}$ is a $G^\#$-graded division algebra.

We split all the gradings on $M(m,n)$ in two classes depending on the superalgebra structure on $\mathcal{D}$: if $\mathcal{D}^{\bar{1}} = 0$, we say that we have an *even grading* and, if $\mathcal{D}^{\bar{1}} \neq 0$, we have an *odd grading*.

To see the difference between even and odd gradings, consider the $G^{\#}$-graded algebra $E = \mathrm{End}_{\mathcal{D}}(\mathcal{U})$, where $\mathcal{D}$ is a $G^{\#}$-graded division algebra and $\mathcal{U}$ is a graded module over $\mathcal{D}$. Define

$$\mathcal{U}^{\bar{0}} = \bigoplus_{g\in G^{\#}} \{u \in \mathcal{U}_g \mid p(g) = \bar{0}\} \text{ and } \mathcal{U}^{\bar{1}} = \bigoplus_{g\in G^{\#}} \{u \in \mathcal{U}_g \mid p(g) = \bar{1}\}.$$

Then $\mathcal{U}^{\bar{0}}$ and $\mathcal{U}^{\bar{1}}$ are $\mathcal{D}^{\bar{0}}$-modules, but they are $\mathcal{D}$-modules if and only if $\mathcal{D}^{\bar{1}} = 0$. So, in the case of an even grading, $\mathcal{U}$ is as a direct sum of $\mathcal{D}$-modules, and all the information related to the canonical $\mathbb{Z}_2$-grading on $\mathrm{End}_{\mathcal{D}}(\mathcal{U})$ comes from the decomposition $\mathcal{U} = \mathcal{U}^{\bar{0}} \oplus \mathcal{U}^{\bar{1}}$.

Definition 3.3.2. Similarly to Definition 2.4.4, we will parametrize the even gradings on $M(m,n)$ as $\Gamma(T,\beta,\gamma_0,\gamma_1)$, where the pair (T,β) characterizes $\mathcal{D}$ and γ_0 and γ_1 are tuples of elements of G corresponding to the degrees of homogeneous bases for $\mathcal{U}^{\bar{0}}$ and $\mathcal{U}^{\bar{1}}$, respectively. Here γ_0 is a k_0-tuple and γ_1 is a k_1-tuple, with $k_0\sqrt{|T|} = m$ and $k_1\sqrt{|T|} = n$.

On the other hand, in the case of an odd grading, the information about the canonical $\mathbb{Z}_2$-grading is encoded in $\mathcal{D}$. To see that, take a homogeneous $\mathcal{D}$-basis of $\mathcal{U}$ and multiply all the odd elements by some nonzero homogeneous element in $\mathcal{D}^{\bar{1}}$. This way we get a homogeneous $\mathcal{D}$-basis of $\mathcal{U}$ such that the degrees are all in the subgroup G of $G^{\#}$. If we denote the $\mathbb{F}$-span of this new basis by $\widetilde{U}$, then $E \simeq \mathrm{End}(\widetilde{U}) \otimes \mathcal{D}$ where the first factor has the trivial $\mathbb{Z}_2$-grading.

Definition 3.3.3. We parametrize the odd gradings by $\Gamma(T, \beta, \gamma)$ where $T \subseteq G^{\#}$ but $T \nsubseteq G$, the pair (T, β) characterizes $\mathcal{D}$, and γ is a tuple of elements of $G = G \times \{\bar{0}\}$ corresponding to the degrees of a homogeneous basis of $\mathcal{U}$ with only even elements.

Remark 3.3.4. Note that, for odd gradings, the homomorphism $p\colon T \to \mathbb{Z}_2$ such that $p(g, i) = i$ is not constant. We can define a character $\chi\colon T \to \mathbb{F}^{\times}$ such that $\chi(t) = (-1)^{p(t)}$ and, since β is nondegenerate, there is an element of order two, $t_0 \in T$, such that $\chi = \beta(t_0, \cdot)$. Hence with this element t_0, one could recover the parity of the elements of $\mathcal{D}$.

Clearly, it is impossible for an even grading to be isomorphic to an odd grading. The classification of even gradings is the following:

Theorem 3.3.5. *Every even G-grading on the superalgebra $M(m, n)$ is isomorphic to some $\Gamma(T, \beta, \gamma_0, \gamma_1)$ as in Definition 3.3.2. Two even gradings, $\Gamma = \Gamma(T, \beta, \gamma_0, \gamma_1)$ and $\Gamma' = \Gamma(T', \beta', \gamma_0', \gamma_1')$, are isomorphic if, and only if, $T = T'$, $\beta = \beta'$, and there is $g \in G$ such that*

(i) for $m \neq n$: $g\Xi(\gamma_0) = \Xi(\gamma_0')$ and $g\Xi(\gamma_1) = \Xi(\gamma_1')$;

(ii) for $m = n$: either $g\Xi(\gamma_0) = \Xi(\gamma_0')$ and $g\Xi(\gamma_1) = \Xi(\gamma_1')$ or $g\Xi(\gamma_0) = \Xi(\gamma_1')$ and $g\Xi(\gamma_1) = \Xi(\gamma_0')$.

Proof. We have already proved the first assertion. For the second assertion, we consider Γ and Γ' as $G^{\#}$-gradings on the algebra $M(m+n)$ and use Theorem 2.4.5 to conclude that they are isomorphic if, and only if, $T = T'$, $\beta = \beta'$ and there is $(g, s) \in G^{\#}$ such that $(g, s)\Xi(\gamma) = \Xi(\gamma')$, where γ is the concatenation of γ_0 and γ_1, where we regard the entries as elements of $G^{\#} = G \times \mathbb{Z}_2$ appending $\bar{0}$ in the second coordinate of the entries of γ_0 and $\bar{1}$ in the second coordinates of the entries of γ_1.

If $m \neq n$, the condition $(g,s)\Xi(\gamma) = \Xi(\gamma)$ must have $s = \bar{0}$, since the size of γ_0 is different from the size of γ_1.

If $m = n$, the condition $(g,s)\Xi(\gamma) = \Xi(\gamma')$ becomes $g\Xi(\gamma_1) = \Xi(\gamma_1')$ if $s = \bar{0}$ and $g\Xi(\gamma_1) = \Xi(\gamma_0')$ if $s = \bar{1}$. $\square$

We now turn to the classification of odd gradings. Recall that here we choose the tuple γ to consist of elements of $G = G \times \{\bar{0}\} \subseteq G^{\#}$ (see Definition 3.3.3). The corresponding multiset $\Xi(\gamma)$ is contained in $\frac{G^{\#}}{T} \simeq \frac{G}{T \cap G}$.

Theorem 3.3.6. *Every odd G-grading on the superalgebra $M(m,n)$ is isomorphic to some $\Gamma(T,\beta,\gamma)$ as in Definition 3.3.3. Two odd gradings, $\Gamma = \Gamma(T,\beta,\gamma)$ and $\Gamma' = \Gamma(T',\beta',\gamma')$, are isomorphic if, and only if, $T = T'$, $\beta = \beta'$, and there is $g \in G$ such that $g\Xi(\gamma) = \Xi(\gamma')$.*

Proof. We have already proved the first assertion. For the second assertion, we again consider Γ and Γ' as $G^{\#}$-gradings and use Theorem 2.4.5: they are isomorphic if, and only if, $T = T'$, $\beta = \beta'$ and there is $(g,s) \in G^{\#}$ such that $(g,s)\Xi(\gamma) = \Xi(\gamma')$. Since T contains an element t_1 with $p(t_1) = \bar{1}$, we may assume $s = \bar{0}$. $\square$

3.4 Even gradings and Morita context

First we observe that every grading on $M(m,n)$ compatible with the $\mathbb{Z}$-superalgebra structure is an even grading. This follows from the fact that $T = \operatorname{supp} \mathcal{D}$ is a finite group, and if a finite group is contained in $G \times \mathbb{Z}$, then it must be contained in $G \times \{0\}$.

Hence, when we look at the corresponding $(G \times \mathbb{Z}_2)$-grading, we have that $T \subseteq G$, so no element of $\mathcal{D}$ has an odd degree.

The converse is also true. Actually, we can prove a stronger assertion: if we write $M(m,n)$ as in Equation (1.0.1), the subspaces given by each of the four blocks are graded. To capture this information, it is convenient to use the concepts of Morita context and Morita algebra.

Recall that a *Morita context* is a sextuple $\mathcal{C} = (R, S, M, N, \varphi, \psi)$ where R and S are unital associative algebras, M is an (R,S)-bimodule, N is a (S,R)-bimodule and $\varphi : M \otimes_S N \to R$ and $\psi : N \otimes_R M \to S$ are linear maps satisfying the necessary and sufficient conditions for

$$C = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$$

to be an associative algebra (see Section 2.1).

We can associate a Morita context to a superspace $U = U^{\bar{0}} \oplus U^{\bar{1}}$ by taking $R = \mathrm{End}(U^{\bar{0}})$, $S = \mathrm{End}(U^{\bar{1}})$, $M = \mathrm{Hom}(U^{\bar{1}}, U^{\bar{0}})$, $N = \mathrm{Hom}(U^{\bar{0}}, U^{\bar{1}})$, with φ and ψ given by the composition of operators.

Given an algebra C as above and the idempotent $\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, we can recover all the data of the Morita context (up to isomorphism), as follows: $R \simeq \epsilon C \epsilon$, $S \simeq (1-\epsilon)C(1-\epsilon)$, $M \simeq \epsilon C(1-\epsilon)$, $N \simeq (1-\epsilon)C\epsilon$ and ϕ and ψ are given by the multiplication in C. In other words, the concept of Morita context is equivalent to the concept of *Morita algebra*, which is a pair (C, ϵ) where C is a unital associative algebra and $\epsilon \in C$ is an idempotent. For example, we may consider $M(m,n)$ as a Morita algebra by fixing the idempotent $\epsilon = \begin{pmatrix} I_m & 0_{m\times n} \\ 0_{n\times m} & 0_n \end{pmatrix}$, i.e., $M(m,n)$ is the

Morita algebra corresponding to the Morita context associated to the superspace $U = \mathbb{F}^m \oplus \mathbb{F}^n$.

Definition 3.4.1. [7] A Morita context $(R, S, M, N, \varphi, \psi)$ is said to be *G-graded* if the algebras R and S are graded, the bimodules M and N are graded, and the maps φ and ψ are homogeneous of degree e. A Morita algebra (C, ϵ) is said to be *G-graded* if C is G-graded and ϵ is a homogeneous element (necessarily of degree e).

Clearly, a Morita context is graded if, and only if, the corresponding Morita algebra is graded.

Remark 3.4.2. For every graded Morita algebra (C, ϵ), we can define a $\mathbb{Z}$-grading by taking $C^{-1} = \epsilon C(1-\epsilon)$, $C^0 = \epsilon C \epsilon \oplus (1-\epsilon)C(1-\epsilon)$ and $C^1 = (1-\epsilon)C\epsilon$. In the case of $M(m, n)$, this is precisely the canonical $\mathbb{Z}$-grading.

Proposition 3.4.3. *Let Γ be a G-grading on the superalgebra $M(m, n)$. The following are equivalent:*

(i) Γ is compatible with the canonical $\mathbb{Z}$-grading;

(ii) Γ is even;

(iii) $M(m, n)$ equipped with Γ is a graded Morita algebra.

Further, if we assume $\operatorname{char} \mathbb{F} = 0$*, the above statements are also equivalent to:*

(iv) Γ corresponds to a $\widehat{G}$-action by even automorphisms.

Proof. (i) $\Rightarrow$ (ii): See the beginning of this section.

(ii) $\Rightarrow$ (iii): Regard Γ as a $G^\#$-grading. By Theorem 2.4.1, there is a graded division algebra $\mathcal{D}$ and a graded right $\mathcal{D}$-module $\mathcal{U}$ such that $\operatorname{End}_{\mathcal{D}}(\mathcal{U}) \simeq M(m, n)$.

Take an isomorphism of graded algebras $\phi : \mathrm{End}_{\mathcal{D}}(\mathcal{U}) \to M(m,n)$. Since Γ is even, $\mathcal{U}^{\bar{0}}$ and $\mathcal{U}^{\bar{1}}$ are graded $\mathcal{D}$-submodules. Take $\epsilon' \in \mathrm{End}_{\mathcal{D}}(\mathcal{U})$ to be the projection onto $\mathcal{U}^{\bar{0}}$ associated to the decomposition $\mathcal{U} = \mathcal{U}^{\bar{0}} \oplus \mathcal{U}^{\bar{1}}$. Clearly, ϵ' is a central idempotent of $\mathrm{End}_{\mathcal{D}}(\mathcal{U})^{\bar{0}}$, hence $\phi(\epsilon')$ is a central idempotent of $M(m,n)^{\bar{0}}$, so either $\phi(\epsilon') = \epsilon$ or $\phi(\epsilon') = 1 - \epsilon$. Either way, $\phi^{-1}(\epsilon)$ is homogeneous, hence so is ϵ.

(iii) $\Rightarrow$ (i): Follows from Remark 3.4.2.

(i) $\Leftrightarrow$ (iv): This follows from the fact that the group of even automorphisms is precisely the group of automorphisms of the $\mathbb{Z}$-superalgebra structure on $M(m,n)$ (see Remark 3.2.1). $\square$

Corollary 3.4.4. *If* $\operatorname{char} \mathbb{F} = 0$*, odd gradings exist only if* $m = n$. $\square$

We now know that the gradings on the $\mathbb{Z}$-superalgebra $M(m,n)$ are precisely the even gradings, but since the automorphism group is different from the $\mathbb{Z}_2$-superalgebra case, the classification of gradings up to isomorphism is also different. The proof of the next result is similar to the proof of Theorem 3.3.5.

Theorem 3.4.5. *Let* $\Gamma(T, \beta, \gamma_0, \gamma_1)$ *and* $\Gamma'(T', \beta', \gamma_0', \gamma_1')$ *be G-gradings on the $\mathbb{Z}$-superalgebra $M(m,n)$. Then Γ and Γ' are isomorphic if, and only if, $T = T'$, $\beta = \beta'$, and there is $g \in G$ such that $g\Xi(\gamma_i) = \Xi(\gamma_i')$ for $i = 0, 1$.* $\square$

Recall that we can always shift the grading on a graded (bi)module and still have a graded (bi)module. In a graded Morita context, as in the case of a graded superalgebra (see Lemma 2.1.13), we have more structure to preserve: if we shift one of the bimodules by an element $g \in G$ and at least one of the bilinear maps is nonzero, then we are forced to shift the other bimodule by g^{-1}. As in the superalgebra case, we will refer to this situation as *shift in opposite directions*.

Theorem 3.4.6. *Let $\mathcal{C} = (R, S, M, N, \varphi, \psi)$ be the Morita context associated with a superspace U and fix G-gradings on R and S making them graded algebras. The bimodules M and N admit G-gradings so that $\mathcal{C}$ becomes a graded Morita context if, and only if, there exists a graded division algebra $\mathcal{D}$ and graded right $\mathcal{D}$-modules $\mathcal{V}$ and $\mathcal{W}$ such that $R \simeq \operatorname{End}_{\mathcal{D}}(\mathcal{V})$ and $S \simeq \operatorname{End}_{\mathcal{D}}(\mathcal{W})$ as graded algebras. Moreover, all such gradings on M and N have the form $M \simeq \operatorname{Hom}_{\mathcal{D}}(\mathcal{W}, \mathcal{V})^{[g]}$ and $N \simeq \operatorname{Hom}_{\mathcal{D}}(\mathcal{V}, \mathcal{W})^{[g^{-1}]}$ as graded bimodules, where $g \in G$ is arbitrary.*

Proof. Suppose M and N admit G-gradings so that the Morita algebra (C, ϵ) associated to $\mathcal{C}$ becomes G-graded. By Theorem 2.4.1 there exists a graded division algebra $\mathcal{D}$ and a graded $\mathcal{D}$-module $\mathcal{U}$ such that $C \simeq \operatorname{End}_{\mathcal{D}}(\mathcal{U})$. Denote the image of ϵ under this isomorphism by ϵ' and let $\mathcal{V} = \epsilon'(\mathcal{U})$ and $\mathcal{W} = (1 - \epsilon')(\mathcal{U})$. Since ϵ is homogeneous, so is ϵ', hence $\mathcal{V}$ and $\mathcal{W}$ are graded $\mathcal{D}$-modules. It follows that $R \simeq \epsilon M(m, n)\epsilon \simeq \epsilon' \operatorname{End}_{\mathcal{D}}(\mathcal{U})\epsilon' \simeq \operatorname{End}_{\mathcal{D}}(\mathcal{V})$ and, analogously, $S \simeq \operatorname{End}_{\mathcal{D}}(\mathcal{W})$.

For the converse, write C in the matrix form by fixing a basis in U and identify $\operatorname{End}_{\mathcal{D}}(\mathcal{V})$ and $\operatorname{End}_{\mathcal{D}}(\mathcal{W})$ with matrix algebras as in Definition 2.4.4. Suppose there exist isomorphisms of graded algebras $\theta_1 \colon R \to \operatorname{End}_{\mathcal{D}}(\mathcal{V})$ and $\theta_2 \colon S \to \operatorname{End}_{\mathcal{D}}(\mathcal{W})$. Then there are $x \in \mathrm{GL}(m)$ and $y \in \mathrm{GL}(n)$ such that θ_1 is the conjugation by x and θ_2 is the conjugation by y. It follows that the conjugation by $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ is an isomorphism of algebras between $C \simeq M(m, n)$ and

$$\operatorname{End}_{\mathcal{D}}(\mathcal{V} \oplus \mathcal{W}) = \begin{pmatrix} \operatorname{End}_{\mathcal{D}}(\mathcal{V}) & \operatorname{Hom}_{\mathcal{D}}(\mathcal{W}, \mathcal{V}) \\ \operatorname{Hom}_{\mathcal{D}}(\mathcal{V}, \mathcal{W}) & \operatorname{End}_{\mathcal{D}}(\mathcal{W}) \end{pmatrix}.$$

Hence we transport the gradings on $\operatorname{Hom}_{\mathcal{D}}(\mathcal{W}, \mathcal{V})$ and $\operatorname{Hom}_{\mathcal{D}}(\mathcal{V}, \mathcal{W})$ to M and N, respectively.

It remains to prove that the gradings on M and N are determined up to shift in opposite directions. Since in our case the Morita algebra C is simple, M and N are simple bimodules. By Corollary 2.1.11, the gradings on M and N are determined up to shifts, and the shifts have to be in opposite directions in order for φ and ψ to be degree-preserving. $\square$

3.5 Fine gradings up to equivalence

We start by investigating the gradings on the superalgebra $M(m,n)$ that are fine among even gradings. By Proposition 3.4.3, this is the same as fine gradings on $M(m,n)$ as a $\mathbb{Z}$-superalgebra.

We will use the following notation. Let H be a finite abelian group whose order is not divisible by $\operatorname{char}\mathbb{F}$. Set $T_H = H \times \widehat{H}$ and define $\beta_H : T_H \times T_H \to \mathbb{F}^\times$ by

$$\beta_H((h_1,\chi_1),(h_2,\chi_2)) = \chi_1(h_2)\,\chi_2(h_1)^{-1}.$$

Then β_H is a nondegenerate alternating bicharacter on T_H.

Definition 3.5.1. Let $\ell \mid \gcd(m,n)$ be a natural number such that $\operatorname{char}\mathbb{F} \nmid \ell$ and put $k_0 := \frac{m}{\ell}$ and $k_1 := \frac{n}{\ell}$. Let Θ_ℓ be a set of representatives of the isomorphism classes of abelian groups of order ℓ. For every H in Θ_ℓ, we define $\Gamma(H,k_0,k_1)$ to be the even $T_H \times \mathbb{Z}^{k_0+k_1}$-grading $\Gamma(T_H,\beta_H,(e_1,\ldots,e_{k_0}),(e_{k_0+1},\ldots,e_{k_0+k_1}))$ on $M(m,n)$, where $\{e_1,\ldots,e_{k_0+k_1}\}$ is the standard basis of $\mathbb{Z}^{k_0+k_1}$. If m and n are clear from the context, we will simply write $\Gamma(H)$.

Let G_H be the subgroup of $T_H \times \mathbb{Z}^{k_0+k_1}$ generated by the support of $\Gamma(H,k_0,k_1)$, i.e., $G_H = T_H \times \mathbb{Z}_0^{k_0+k_1}$, where $\mathbb{Z}_0^k := \{(x_1,\ldots,x_k) \in \mathbb{Z}^k \mid x_1 + \cdots + x_k = 0\}$.

Theorem 3.5.2. *The fine gradings on $M(m,n)$ as a $\mathbb{Z}$-superalgebra are precisely the even fine gradings. Every such grading is equivalent to a unique $\Gamma(H)$ as in Definition 3.5.1. Moreover, every grading $\Gamma(H)$ is fine, and G_H is its universal abelian group.*

Proof. By [12, Proposition 2.35], if we consider $\Gamma(H)$ as a grading on the algebra $M_{n+m}(\mathbb{F})$, it is a fine grading and G_H is its universal group. It follows that the same is true of $\Gamma(H)$ as a grading on the superalgebra $M(m,n)$.

Let $\Gamma = \Gamma(T, \beta, \gamma_0, \gamma_1)$ be any even G-grading on $M(m,n)$. We can write $T = A \times B$ where the restrictions of β to the subgroups A and B are trivial and, hence, there is an isomorphism $\alpha : T_A \to T$ such that $\beta_A = \beta \circ (\alpha \times \alpha)$. We can extend α to a homomorphism $G_A \to G$ (also denoted by α) by sending the elements $e_1, \ldots, e_{k_0}$ to the entries of γ_0 and the elements $e_{k_0+1}, \ldots, e_{k_0+k_1}$ to the entries of γ_1. It follows that ${}^{\alpha}\Gamma(A) \simeq \Gamma$. Since all $\Gamma(H)$ are fine and pairwise nonequivalent (because their universal groups are pairwise nonisomorphic), we can apply Lemma 2.2.2, concluding that every fine grading on $M(m,n)$ as a $\mathbb{Z}$-superalgebra is equivalent to a unique $\Gamma(H)$. $\square$

We now consider odd fine gradings on $M(n,n)$, so $\operatorname{char} \mathbb{F} \neq 2$. We first define some gradings on the algebra $M_{2n}(\mathbb{F})$ and then impose a superalgebra structure.

Definition 3.5.3. Let $\ell \mid n$ be a natural number such that $\operatorname{char} \mathbb{F} \nmid \ell$ and put $k := \frac{n}{\ell}$. Let $\Theta_{2\ell}$ be a set of representatives of the isomorphism classes of abelian groups of order 2ℓ. For every H in $\Theta_{2\ell}$, we consider the $T_H \times \mathbb{Z}^k$-grading $\Gamma = \Gamma(T_H, \beta_H, (e_1, \ldots, e_k))$ on $M_{2n}(\mathbb{F})$, where $\{e_1, \ldots, e_k\}$ is the standard basis of $\mathbb{Z}^k$. Then we choose an element

$t_0 \in T_H$ of order 2 and define a group homomorphism $p : T_H \times \mathbb{Z}^k \to \mathbb{Z}_2$ by

$$p(t, x_1, \dots, x_k) = \begin{cases} \bar{0} & \text{if } \beta_H(t_0, t) = 1, \\ \bar{1} & \text{if } \beta_H(t_0, t) = -1. \end{cases}$$

This defines a superalgebra structure on $M_{2n}(\mathbb{F})$. By construction, Γ is odd as a grading on this superalgebra $(M_{2n}(\mathbb{F}), p)$, and this forces the superalgebra to be isomorphic to $M(n, n)$. We denote by $\Gamma(H, t_0, k)$ the grading Γ considered as a grading on $M(n, n)$. If n is clear from the context, we will simply write $\Gamma(H, t_0)$.

Note that the parameter t_0 of $\Gamma(H, t_0, k)$ does not affect the grading on the algebra $M_{2n}(\mathbb{F})$, but, as we will see in Proposition 3.5.6, different choices of t_0 can yield nonequivalent gradings on the superalgebra $M(n, n)$.

Proposition 3.5.4. *Each grading $\Gamma(H, t_0)$ on $M(n, n)$ is fine, and its universal abelian group is $G_H = T_H \times \mathbb{Z}_0^k$. Every odd fine grading on $M(n, n)$ is equivalent to at least one $\Gamma(H, t_0)$.*

Proof. As in the proof of Theorem 3.5.2, the first assertion follows from [12, Proposition 2.35].

Let $\Gamma(T, \beta, \gamma)$ be an odd G-grading on $M(n, n)$ and let t_0 be its parity element. Then we can find subgroups A and B such that $T = A \times B$ and there exists an isomorphism $\alpha : T_A \to T$ such that $\beta_A = \beta \circ (\alpha \times \alpha)$. We define $t_0' := \alpha^{-1}(t_0)$ and extend α to a homomorphism $G_A \to G$ (also denoted by α) by sending the elements $e_1, \dots, e_k$ to the entries of γ. Then ${}^{\alpha}\Gamma(A, t_0') \simeq \Gamma$.

Selecting a representative from each equivalence class of gradings of the form $\Gamma(H, t_0)$, we can apply Lemma 2.2.2, which proves the second assertion. □

It remains to determine which of the gradings $\Gamma(H, t_0)$ are equivalent to each other. For the next proposition, we recall the concept of the *Weyl group* of a grading (introduced in [18], also see Definition 1.16 of [12]).

Definition 3.5.5. Let Γ be a grading on an algebra A. The *automorphism group* of Γ, denoted by $\mathrm{Aut}(\Gamma)$, is the group of all automorphisms of A that permute the components of Γ. Each automorphism $\theta \in \mathrm{Aut}(\Gamma)$ determines a self-bijection, $\alpha(\theta)$, of the support S, an element of the group $\mathrm{Sym}(S)$. The kernel of the homomorphism $\theta \mapsto \alpha(\theta)$ is denoted by $\mathrm{Stab}(\Gamma)$ and called the *stabilizer* of Γ. The *Weyl group* of Γ, denoted by $W(\Gamma)$, is the quotient group $W(\Gamma) := \mathrm{Aut}(\Gamma)/\mathrm{Stab}(\Gamma)$ and is a subgroup of $\mathrm{Sym}(S)$.

If Γ is an (abelian) group grading, then $W(\Gamma)$ can be regarded as a subgroup of $\mathrm{Aut}(G)$ where G is the universal (abelian) group of Γ.

Proposition 3.5.6. *The gradings $\Gamma = \Gamma(H, t_0)$ and $\Gamma' = \Gamma(H, t_0')$ on $M(n, n)$ are equivalent if, and only if, there is $\alpha \in \mathrm{Aut}(T_H, \beta_H)$ such that $\alpha(t_0) = t_0'$.*

Proof. We will denote by $p : G_H \to \mathbb{Z}_2$ the parity homomorphism associated to the grading Γ and by $p' : G_H \to \mathbb{Z}_2$ the one associated to Γ'.

If Γ is equivalent to Γ', there is an isomorphism $\varphi : (M_{2n}(\mathbb{F}), p) \to (M_{2n}(\mathbb{F}), p')$ of superalgebras that is a self-equivalence of the grading on $M_{2n}(\mathbb{F})$. Hence, we have the corresponding group automorphism $\alpha : G_H \to G_H$ in the Weyl group of the grading, and $p' \circ \alpha = p$.

The automorphism α must send the torsion subgroup of G_H to itself, so we can consider the restriction $\alpha|_{T_H}$. By [12, Corollary 2.45], this restriction is in

$\mathrm{Aut}(T_H, \beta_H)$. Hence, by the definition of p and p', the condition $p' \circ \alpha = p$ is equivalent to $\alpha(t_0) = \alpha(t_0')$.

For the converse, we use the same [12, Corollary 2.45] to extend α to an automorphism $G_H \to G_H$ in the Weyl group. Hence, there is an automorphism φ of the algebra $M_{2n}(\mathbb{F})$ that permutes the components of the grading according to α. The condition $\alpha(t_0) = \alpha(t_0')$ is equivalent to $p' \circ \alpha = p$, which shows that $\varphi : (M_{2n}(\mathbb{F}), p) \to (M_{2n}(\mathbb{F}), p')$ is an isomorphism of superalgebras. □

Combining Propositions 3.5.4 and 3.5.6, we obtain:

Theorem 3.5.7. *Every odd fine grading on $M(n,n)$ is equivalent to some $\Gamma(H, t_0)$ as in Definition 3.5.3. Every grading $\Gamma(H, t_0)$ is fine, and G_H is its universal abelian group. Two gradings, $\Gamma(H, t_0)$ and $\Gamma(H', t_0')$, are equivalent if, and only if, $H = H'$ and t_0' lies in the orbit of t_0 under the natural action of* $\mathrm{Aut}(T_H, \beta_H)$. □

For a matrix description of the group $\mathrm{Aut}(T_H, \beta_H)$, we refer the reader to [12, Remark 2.46].

Chapter 4

Lie Superalgebras of Type $B(m, n)$

4.1 Preliminaries and definitions

Let $U = U^{\bar{0}} \oplus U^{\bar{1}}$ be a superspace and let $\langle\, ,\rangle : U \times U \to \mathbb{F}$ be a bilinear form that is homogeneous with respect to the $\mathbb{Z}_2$-grading, i.e., has parity as a linear map $U \otimes U \to \mathbb{F}$. We say that $\langle\, ,\rangle$ is *supersymmetric* if $\langle x, y\rangle = (-1)^{|x||y|}\langle y, x\rangle$ for all homogeneous elements $x, y \in U$.

Let $\langle\, ,\rangle : U \times U \to \mathbb{F}$ be a bilinear form that is supersymmetric, nondegenerate, and even. We define the Lie superalgebra $\mathfrak{osp}(U) = \mathfrak{osp}(U)^{\bar{0}} \oplus \mathfrak{osp}(U)^{\bar{1}} \subset \mathrm{End}(U)^{(-)}$ by

$$\mathfrak{osp}(U)^i := \{T \in \mathrm{End}(U)^i \mid \langle T(x), y\rangle = -(-1)^{i|x|}\langle x, T(y)\rangle, \forall\, x,\ y\,\},\ i \in \mathbb{Z}_2.$$

In the case $U^{\bar{0}} = \mathbb{F}^m$ and $U^{\bar{1}} = \mathbb{F}^n$, we denote $\mathfrak{osp}(U)$ by $\mathfrak{osp}(m, n)$. It is convenient to separate the following cases according to the type of the even component:

- $B(m, n) = \mathfrak{osp}(2m + 1, 2n), m \geq 0, n \geq 1$: the even part is $\mathfrak{so}(2m + 1) \oplus \mathfrak{sp}(2n)$,

- $D(m,n) = \mathfrak{osp}(2m,2n), m \geq 2, n \geq 1$: the even part is $\mathfrak{so}(2m) \oplus \mathfrak{sp}(2n)$, and

- $C(n) = \mathfrak{osp}(2, 2n-2), n \geq 2$: the even part is $\mathfrak{sp}(2n-2) \oplus \mathbb{F}$.

We will now rewrite this definition in a slightly different way. Since the form $\langle\, , \rangle$ is nondegenerate, the following is well defined:

Definition 4.1.1. Let $T : U \to U$ be a homogeneous linear map. The *superadjoint* of T with respect to $\langle\, , \rangle$ is the linear map $\varphi(T) : U \to U$ such that

$$\langle T(x), y\rangle = (-1)^{|T||x|}\langle x, \varphi(T)(y)\rangle$$

for all homogeneous $x, y \in U$. We extend this definition to all operators by linearity, giving us an even linear map $\varphi : \mathrm{End}(U) \to \mathrm{End}(U)$.

Remark 4.1.2. Note that $|\varphi(T)| = |T|$.

Recall the definition of superadjoint, without reference to a bilinear form, given in Section 2.5. It is connected to Definition 4.1.1 by the fact that $\langle\, , \rangle$ allows us to identify U with its superdual (compare Remark 2.5.3 and Equation (4.1.1)).

Definition 4.1.3. Let $L = L^{\bar{0}} \oplus L^{\bar{1}}$ be a superalgebra with product denoted by $*$. An invertible linear map $\varphi \colon L \to L$ is said to be a *super-anti-automorphism* if $\varphi(x * y) = (-1)^{|x||y|}\, \varphi(y) * \varphi(x)$, for all homogeneous elements $x, y \in L$ (with respect to the $\mathbb{Z}_2$-grading). A super-anti-automorphism φ is said to be a *superinvolution* if $\varphi^2 = \mathrm{id}$.

Remark 4.1.4. Note that, in general, if $x_1, \ldots, x_n$ are homogeneous elements in L then $\varphi(x_1 * \cdots * x_n) = (-1)^{\sum_{i<j} |x_i||x_j|}\varphi(x_n) * \cdots * \varphi(x_1)$.

It is well known that superadjoint is a superinvolution. Indeed, letting T,$S \in \mathrm{End}(U)$, x,$y \in U$, we have

$$\langle TS(x), y\rangle = \langle x, (-1)^{|TS||x|}\varphi(TS)y\rangle = \langle x, (-1)^{|x|(|T|+|S|)}\varphi(TS)y\rangle.$$

On the other hand,

$$\langle TS(x), y\rangle = \langle S(x), (-1)^{|T||S(x)|}\varphi(T)y\rangle = \langle x, (-1)^{|x||S|+|T|(|S|+|x|))}\varphi(S)\varphi(T)y\rangle.$$

Hence $\varphi(TS) = (-1)^{|T||S|}\varphi(S)\varphi(T)$.

Now we will show that $\varphi^2 = \mathrm{id}$. Since the form $\langle\,,\,\rangle$ is supersymmetric, we have

$$\langle T(x), y\rangle = (-1)^{|T||x|}\langle x, \varphi(T)y\rangle = (-1)^{|T||x|+|x||\varphi(T)(y)|}\langle \varphi(T)(y), x\rangle.$$

Since $|x||\varphi(T)(y)| = |x|(|\varphi(T)| + |y|) = |x||T| + |x||y|$, we obtain

$$\langle T(x), y\rangle = (-1)^{|x||y|}\langle \varphi(T)(y), x\rangle = (-1)^{|x||y|+|T||y|}\langle y, \varphi^2(T)x\rangle.$$

Due to supersymmetry again, we have

$$\langle T(x), y\rangle = (-1)^{|x||y|+|T||y|+|\varphi^2(T)(x)||y|}\langle \varphi^2(T)(x), y\rangle.$$

Finally, since $|x||y| + |T||y| + |\varphi^2(T)(x)||y| = |x||y| + |T||y| + (|T| + |x|)|y| = 0$, we conclude that $\varphi^2(T) = T$. Hence, φ is a superinvolution as claimed.

Thus, we can define $\mathfrak{osp}(U)$ as the space of skew elements in the superalgebra $R = \mathrm{End}(U)$ with respect to the superinvolution φ :

$$\mathfrak{osp}(U) = \{r \in R \mid \varphi(r) = -r\} =: K(R, \varphi).$$

Fixing bases $\{v_1, \ldots, v_m\}$ in $U^{\bar{0}}$ and $\{v_{m+1}, \ldots, v_{m+n}\}$ in $U^{\bar{1}}$, we can identify R with $M(m, n)$. Then, we get $\mathfrak{osp}(m, n) = \{X \in M(m, n) |\, X = -\varphi(X)\}$. Let us express φ in the matrix form.

Let Φ be the matrix representing the form $\langle\, ,\, \rangle$, i.e., $\Phi = (\lambda_{ij})$ where $\lambda_{ij} = \langle v_i, v_j \rangle$. Suppose $r \in R$ is represented by matrix $X = (x_{ij})$ and $\varphi(r)$ has matrix $Y = (y_{ij})$. Let $k = m + n$ and $j, l \in \{1, \ldots, k\}$. Then, we have the following two equations:

$$\langle rv_j, v_l \rangle = \sum_i \langle x_{ij} v_i, v_l \rangle = \sum_i x_{ij} \lambda_{il}$$

and

$$\langle v_j, \varphi(r) v_l \rangle = \sum_i \langle v_j, y_{il} v_i \rangle = \sum_i y_{il} \lambda_{ji}.$$

Since for all $j, l \in \{1, \ldots, k\}$, $\langle rv_j, v_l \rangle = (-1)^{|v_j||r|} \langle v_j, \varphi(r) v_l \rangle$, we have

$$(-1)^{|v_j||r|} \sum_i x_{ij} \lambda_{il} = \sum_i \lambda_{ji} y_{il},$$

which means

$$\varphi(X) = \Phi^{-1}\, X^{s\top}\, \Phi, \tag{4.1.1}$$

where $s\top$ denotes the supertranspose matrix from Definition 2.3.3.

Lemma 4.1.5. *Let φ be a super-anti-automorphism of $M(m,n)$ and let ϕ_S denote the automorphism given by the conjugation by an even or odd matrix $S \in M(m,n)$. Then ϕ_S commutes with φ if and only if there exists $\lambda \in \mathbb{F}^\times$ such that $\varphi(S) = \lambda S^{-1}$.*

Proof. Let $X \in M(m,n)^{\bar{0}} \cup M(m,n)^{\bar{1}}$. Then we have

$$\phi_S \varphi(X) = S \varphi(X) S^{-1}, \tag{4.1.2}$$

and since $|X||S| + |X||S^{-1}| + |S||S^{-1}| = |S|$, Remark 4.1.4 gives

$$\varphi\phi_S(X) = \varphi(SXS^{-1}) = (-1)^{|S|} \varphi(S^{-1}) \varphi(X) \varphi(S). \tag{4.1.3}$$

Comparing Equations (4.1.2) and (4.1.3) and taking into account that $\varphi(S^{-1}) = (-1)^{|S|}\varphi(S)^{-1}$, we conclude that $\phi_S\varphi = \varphi\phi_S$ if and only if $\varphi(S)S$ is a scalar multiple of the identity matrix. $\square$

Suppose $S \in M(m,n)$ satisfies $\varphi(S) = \lambda S^{-1}$ with $\lambda \in \mathbb{F}^\times$ and let u,$v \in U^{\bar{1}}$. If S is odd, then $\langle Su, Sv\rangle = (-1)^{|S||u|}\langle u, v\rangle\lambda = -\langle u, v\rangle\lambda = \langle v, u\rangle\lambda$; on the other hand, Su, Sv are even elements and $\langle Su, Sv\rangle = \langle Sv, Su\rangle = (-1)^{|S||v|}\langle v, u\rangle\lambda = -\langle v, u\rangle\lambda$. Therefore $-\langle v, u\rangle = \langle v, u\rangle$ for all u, $v \in U^{\bar{1}}$, which is a contradiction; indeed, since $\langle\,,\,\rangle$ is even, $\langle U^{\bar{1}}, U^{\bar{1}}\rangle \neq 0$. Hence, S is even.

Let $R = M(m,n)$ and denote by $\mathrm{Aut}(R,\varphi)$ the group of automorphisms of R that commute with the superinvolution φ. Therefore, Lemma 4.1.5 gives us the following characterization:

$$\mathrm{Aut}(R,\varphi) = \{\phi_S \in \mathrm{Aut}(R) \mid S = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, a \in \mathrm{O}(m), d \in \mathrm{Sp}(n)\}. \tag{4.1.4}$$

The following is well known, but we include a proof for completeness.

Lemma 4.1.6. *The Lie superalgebra $\mathfrak{osp}(m,n)$ generates $M(m,n)$ as an associative superalgebra.*

Proof. Let $U = U^{\bar{0}} \oplus U^{\bar{1}}$ be the natural $\mathfrak{osp}(m,n)$-module: $U^{\bar{0}} = \mathbb{F}^m$, $U^{\bar{1}} = \mathbb{F}^n$.

First, we claim that if $V \subset U$ is a nonzero proper $\mathfrak{osp}(m,n)^{\bar{0}}$-invariant subspace, then either $V = U^{\bar{0}}$ or $V = U^{\bar{1}}$. Indeed, for $i \in \mathbb{Z}_2$, let π_i denote the projection map $U^{\bar{0}} \oplus U^{\bar{1}} \to U^i$. If $\pi_{\bar{0}}(V) = 0$ then $V \subseteq U^{\bar{1}}$ and hence $V = U^{\bar{1}}$ since $U^{\bar{1}}$ is a simple $\mathfrak{osp}(m,n)^{\bar{0}}$-module. Analogously, if $\pi_{\bar{1}}(V) = 0$ then $V = U^{\bar{0}}$. Hence we can assume that $\pi_i(V) \neq 0$, for all $i \in \mathbb{Z}_2$, and therefore $\pi_{\bar{0}}|_V$ and $\pi_{\bar{1}}|_V$ are surjective maps by simplicity of $U^{\bar{0}}$ and $U^{\bar{1}}$. If $\pi_{\bar{0}}|_V$ and $\pi_{\bar{1}}|_V$ are both injective maps, then we have

$U^{\bar{0}} \simeq V \simeq U^{\bar{1}}$, which is a contradiction because $U^{\bar{0}}$ is not isomorphic to $U^{\bar{1}}$. Indeed, $\mathfrak{osp}(m,n)^{\bar{0}} \simeq \mathfrak{so}(m) \oplus \mathfrak{sp}(n)$ and, for all $l_1 \in \mathfrak{so}(m)$, $l_2 \in \mathfrak{sp}(n)$, the action on $U^{\bar{0}}$ is given by $(l_1, l_2).u := l_1 u$, and on $U^{\bar{1}}$ by $(l_1, l_2).u := l_2 u$.

Without loss of generality, suppose $\pi_{\bar{0}}|_V$ is not injective, i.e. , $V \cap U^{\bar{1}} \neq 0$. Then $U^{\bar{1}} \subseteq V$, since $U^{\bar{1}}$ is simple. Hence $V = (U^{\bar{0}} \cap V) \oplus U^{\bar{1}}$, which implies $V = U^{\bar{1}}$ by simplicity of $U^{\bar{0}}$, completing the proof of the claim.

Let A be the associative algebra generated by $\mathfrak{osp}(m,n)$. We claim that U is a simple module over A. Indeed, if V is a nonzero submodule of U then, in particular, V is $\mathfrak{osp}(m,n)^{\bar{0}}$-invariant and, by the claim, we have that $V \in \{U^{\bar{0}}, U^{\bar{1}}, U\}$. Without loss of generality, suppose $U^{\bar{0}} \subseteq V$ and let $u \in U^{\bar{0}}$ and $x \in \mathfrak{osp}(m,n)^{\bar{1}}$ such that $xu \neq 0$. Since $xu \in U^{\bar{1}}$, we conclude that $U^{\bar{1}} \subseteq V$. Hence $V = U$.

It follows that $D := \mathrm{End}_A(U)$ is a division algebra, which must be $\mathbb{F}$ because $\mathbb{F}$ is algebraically closed. By [14, Corollary 1.16] (or by the Jacobson Density Theorem), we conclude that $A = \mathrm{End}(U)$. □

Note that if $\psi \in \mathrm{Aut}(R, \varphi)$, then $K(R, \varphi)$ is invariant under ψ, so we can consider the restriction of ψ to $K(R, \varphi)$.

Proposition 4.1.7. *Let $R = \mathrm{End}(U)$ and $L = \mathfrak{osp}(U)$. Every automorphism of L is a conjugation by an even element of* $\mathrm{End}(U)$. *Also, the restriction map $\theta\colon \mathrm{Aut}(R, \varphi) \to \mathrm{Aut}(L)$ is an isomorphism of (algebraic) groups.*

Proof. Every inner automorphism of L is a conjugation by an even matrix. Due to [20], every automorphism of $B(m,n)$ is inner. Also due to [20], the groups of outer automorphisms of both $C(n)$ and $D(m,n)$ are isomorphic to $\mathbb{Z}_2$ and generated by an improper isometry of $U^{\bar{0}}$. Thus, every $\phi \in \mathrm{Aut}(L)$ is the restriction of some

$\phi_S \in \mathrm{Aut}(R)$. In fact, S can be chosen to be an isometry of $\langle\, , \rangle$. For example, this follows from Equation (4.1.4). Indeed, let $l \in L$, i.e., $\varphi(l) = -l$. Since $\phi_S(l) \in L$, we also have $\varphi(\phi_S(l)) = -\phi_S(l)$. Hence, $\varphi(\phi_S(l)) = \phi_S(\varphi(l))$, for all $l \in L$. In other words, $(\varphi \circ \phi_S)|_L = (\phi_S \circ \varphi)|_L$. Since L generates R as an associative algebra (Lemma 4.1.6), we conclude that $\varphi \circ \phi_S = \phi_S \circ \varphi$, i.e., $\phi_S \in \mathrm{Aut}(R, \varphi)$.

The injectivity of θ also follows from the fact that L generates R. $\square$

4.2 From gradings on $\mathfrak{osp}(m,n)$ to φ-gradings on $M(m,n)$

Definition 4.2.1. Let $L = L^{\bar{0}} \oplus L^{\bar{1}}$ be a G-graded superalgebra. A super-anti-automorphism φ is said to be *G-graded* if $\varphi(L_g) \subseteq L_g$ for all $g \in G$, that is, φ is a homogeneous map of degree $e \in G$.

Note that, for a super anti-automorphism φ, we actually have $\varphi(L_g) = L_g$ for all $g \in G$. Following [11, 12], we will also use the following terminology.

Definition 4.2.2. Let R be an associative (super)algebra and φ be a (super)involution on R. We say that Γ is a *φ-grading* on R if Γ is a grading on R such that φ is a homogeneous map of degree e.

Definition 4.2.3. Let R be an associative (super)algebra and φ be a (super)involution on R. Let (Γ, φ) and (Γ', φ') be gradings on R. We say that (Γ, φ) is isomorphic to (Γ', φ') if there exists an automorphism, $\psi\colon R \to R$, such that $\psi(\Gamma) = \Gamma'$ and $\varphi' = \psi\varphi\psi^{-1}$.

The next theorem uses Proposition 4.1.7 to transfer the problem of classifying the gradings on $\mathfrak{osp}(m,n)$ to the problem of classifying the φ-gradings on the associative superalgebra $R = M(m,n)$ equipped with a superinvolution φ. We will make use of the following two lemmas. The second one is a particular case of Proposition 2.3.5.

Lemma 4.2.4. *Let Γ be a grading on R. A superinvolution φ is a homogeneous map of degree e if and only if the image of $\eta_\Gamma \colon \widehat{G} \to \mathrm{Aut}(R)$ is contained in $\mathrm{Aut}(R,\varphi)$.*

Proof. Let $\chi \in \widehat{G}$ and consider $\psi = \eta_\Gamma(\chi)$. Suppose φ is homogeneous of degree e. We want to show $\varphi\psi = \psi\varphi$. Let $r \in R_g$. We have $\varphi(\psi(r)) = \varphi(\chi(g)r) = \chi(g)\varphi(r)$. Since φ has degree e, $\varphi(r) \in R_g$. Hence $\chi(g)\varphi(r) = \psi(\varphi(r))$.

Conversely, suppose $\varphi\psi = \psi\varphi$ and let $r \in R_g$. We have $\psi(\varphi(r)) = \varphi(\psi(r)) = \varphi(\chi(g)r)$, so $\eta_\Gamma(\chi)(\varphi(r)) = \chi(g)\varphi(r)$, for all $\chi \in \widehat{G}$. Therefore, $\varphi(r) \in R_g$. $\square$

Lemma 4.2.5. *Let Γ be a φ-grading on R. Consider the associated $\widehat{G}$-action $\eta_\Gamma \colon \widehat{G} \to \mathrm{Aut}(R,\varphi)$ and the map θ defined in Proposition 4.1.7. Then $\theta \circ \eta_\Gamma$ corresponds to the restriction of Γ to $\mathfrak{osp}(m,n)$.* $\square$

Theorem 4.2.6. *Let $R = M(m,n)$ and $L = \mathfrak{osp}(m,n) = K(R,\varphi)$. Fix an abelian group G and consider G-gradings on R and on L. Then the mapping that sends a φ-grading on R to its restriction on L is a bijection between φ-gradings on R and all gradings on L. Moreover, this mapping yields a bijection between the isomorphism classes of the said gradings.*

Proof. Let Δ be a grading on L and $\eta_\Delta \colon \widehat{G} \to \mathrm{Aut}(L)$ be its corresponding $\widehat{G}$-action.

Since $\theta\colon \mathrm{Aut}(R,\varphi) \to \mathrm{Aut}(L)$ is an isomorphism by Proposition 4.1.7, we can consider $\theta^{-1} \circ \eta_\Delta \colon \widehat{G} \to \mathrm{Aut}(R,\varphi)$. By Lemma 4.2.4, this homomorphism corresponds to a unique φ -grading Γ on R. By construction, $\eta_\Delta = \theta \circ \eta_\Gamma$, hence Δ is the restriction of Γ to L by Lemma 4.2.5. The uniqueness of Γ is clear.

For the moreover part, let $\Gamma\colon R = \bigoplus_{g\in G} R_g$ and $\Gamma'\colon R = \bigoplus_{g\in G} R'_g$ be isomorphic φ-gradings on R. Then there exists an automorphism $\psi\colon R \to R$ such that $\varphi\psi = \psi\varphi$ and ψ sends Γ to Γ'. We have $\psi \in \mathrm{Aut}(R, \varphi)$. Hence $\theta(\psi) = \psi|_L \in \mathrm{Aut}(L)$. Consider the restrictions of Γ and Γ' to L: $L = \bigoplus_{g\in G} L_g$ and $L = \bigoplus_{g\in G} L'_g$ where $L_g = R_g \cap L$ and $L'_g = R'_g \cap L$. Since $\phi|_L$ sends L_g to L'_g, for all $g \in G$, we conclude that $\Gamma|_L$ and $\Gamma'|_L$ are isomorphic. The proof of the converse is similar (using the invertibility of θ). $\square$

4.3 Generalities on φ-gradings

The following proposition is a super analog of Theorem 1 in [6]. A similar result is proven in [5] for when the algebra admits a super-anti-automorphism of order two.

Proposition 4.3.1. *Let G be a group and let $R = R^{\bar{0}} \oplus R^{\bar{1}}$ be a G-graded unital associative superalgebra such that the support of the G-grading generates G. If R is graded simple and admits a super-anti-automorphism φ (as a G-graded superalgebra) then G is abelian.*

Proof. We have $\varphi(R^i_g R^j_h) = (-1)^{ij}\varphi(R^j_h)\varphi(R^i_g) = R^j_h R^i_g$, with i, $j \in \mathbb{Z}_2$ and g,$h \in G$. Hence $gh = hg$ or $R^i_g R^j_h = 0$.

Now, let g,h be elements in the support of the G-grading. Since R is unital, $I = RR_gR$ is a nonzero two-sided ideal. By a standard argument, I is graded. Since R is graded simple, we have $I = R$. In particular, $0 \neq R_h \subseteq RR_gR$. Let $0 \neq r_h \in R_h$, then there exist elements $y_a \in R_a$ $(a \in G)$ and $z_b \in R_b$ $(b \in G)$, only finitely many of which are nonzero, such that

$$r_h = \sum_{\substack{a,b \in G \\ agb=h}} y_a x_g z_b.$$

Therefore, there are a, $b \in G$ such that $agb = h$ and $y_a x_g z_b \neq 0$. Hence $0 \neq R_a R_g R_b \subseteq R_h$, which implies that $R_a R_g \neq 0$ and $R_g R_b \neq 0$. From the first paragraph of this proof, we conclude that $ag = ga$ and $gb = bg$. Hence, $gh = hg$. Since G is generated by the support, G is abelian. □

We now return to the case $R = M(m, n)$ and assume G abelian. We can write $R \simeq \mathrm{End}_{\mathcal{D}}(\mathcal{U})$ where $\mathcal{D}$ is a graded division superalgebra and $\mathcal{U}$ is a graded (right) $\mathcal{D}$-module – see Section 2.4.

We are now going to obtain some restrictions on $\mathcal{D}$. Recall from Section 2.5 that $\mathcal{U}^\star = \mathrm{Hom}_{\mathcal{D}}(\mathcal{U}, \mathcal{D})$ can be regarded as a right $\mathcal{D}^{\mathrm{sop}}$-module. Also, since $\mathcal{U}$ is a graded simple left R-module, we can consider $\mathcal{U}^\star$ as a left R^{sop}-module (also simple). By using φ, we can identify R with R^{sop}, so $\mathcal{U}^\star$ is a graded simple left R-module. Since R has only one graded simple module up to isomorphism and shift (see Lemma 2.7 in [12]), we have $g_0 \in G$ and a graded R-module isomorphism $\varphi_1 : \mathcal{U}^{[g_0]} \to \mathcal{U}^\star$. Using right $\mathcal{D}$-action on $\mathcal{U}$ and the right $\mathcal{D}^{\mathrm{sop}}$-action on $\mathcal{U}^\star$, we can identify $\mathcal{D}$ with $\mathrm{End}_R(\mathcal{U})$ and $\mathcal{D}^{\mathrm{sop}}$ with $\mathrm{End}_R(\mathcal{U}^\star)$. It follows that $\varphi_0 : \mathcal{D} \to \mathcal{D}^{\mathrm{sop}}$ given by $\varphi_0(d) = \varphi_1\, d\, \varphi_1^{-1}$ is an isomorphism of graded superalgebras. Thus, φ_0 can be seen as a super-anti-automorphism of $\mathcal{D}$ as a graded superalgebra. Based on the following lemma, which is the super analog of Lemma 2.50 in [12], we conclude that $T := \operatorname{supp} \mathcal{D}$ is an elementary (abelian) 2-group.

Lemma 4.3.2. *Let $\mathcal{D}$ be a matrix superalgebra and suppose $\mathcal{D}$ is equipped with a division G-grading with support T. If $\mathcal{D}$ admits a super-anti-automorphism (as a G-graded superalgebra), then T is an elementary (abelian) 2-group.*

Proof. Let β be the alternating nondegenerate bicharacter on T associated to $\mathcal{D}$. Suppose T is not an elementary 2-group. Then there exists an element $a \in T$ of order $l > 2$. Since β is nondegenerate, there also exists $b \in T$ such that $\beta(a, b) = \zeta$ where ζ is a primitive l-th root of unity. Indeed, for all $b \in T$, $\beta(a, b)^l = \beta(a^l, b) = \beta(e, b) = 1$, so the image of the homomorphism $\beta(a, \cdot)\colon T \to \mathbb{F}^\times$ is a subgroup of the group of l-th roots of unity; if it were a proper subgroup, there would exist $l' < l$ such that $\beta(a, b)^{l'} = 1$, for all $b \in T$, which would imply $a^{l'} = e$, a contradiction.

Pick nonzero homogeneous X_a and X_b of degrees a and b, respectively. Then we have $X_a X_b = \zeta X_b X_a$, and applying φ to both sides, we obtain $(-1)^{|a||b|}\varphi(X_b)\varphi(X_a) = (-1)^{|b||a|}\zeta\varphi(X_a)\varphi(X_b)$ hence $\varphi(X_b)\varphi(X_a) = \zeta\varphi(X_a)\varphi(X_b)$. On the other hand, there are nonzero scalars λ_a, λ_b such that $\varphi(X_b) = \lambda_b X_b$ and $\varphi(X_a) = \lambda_a X_a$. Hence $X_b X_a = \zeta X_a X_b$ and therefore $X_b X_a = \zeta^2 X_b X_a$. We conclude $\zeta^2 = 1$, a contradiction with $l > 2$. $\square$

From now on, we will focus on the Lie superalgebra $B(m, n) \subseteq M(2m+1, 2n)^{(-)}$. Recall what elementary grading is – see Definition 2.1.3.

Proposition 4.3.3. *Every φ-grading on $M(2m+1, 2n)$ is elementary.*

Proof. Due to Lemma 4.3.2, T is an elementary 2-group.

Suppose T is nontrivial. Then $|T|$ is even. On the other hand, as graded algebras, $R \simeq \operatorname{End}_{\mathcal{D}}(\mathcal{U})$ and hence $\dim R = (\dim_{\mathcal{D}} \mathcal{U})^2 \dim \mathcal{D}$. Since every homogeneous component of $\mathcal{D}$ is one-dimensional, $\dim \mathcal{D} = |T|$. This implies that $\dim R = (2m+1+2n)^2$ is even, a contradiction. $\square$

Corollary 4.3.4. *Every grading on $B(m,n)$ is a restriction of a unique elementary grading.*

Proof. Theorem 4.2.6 and Proposition 4.3.3 give the result. $\square$

Our goal now will be to investigate the elementary gradings on $M(2m+1, 2n)$ that are also φ-gradings and go towards a classification of gradings on $B(m,n)$.

From now on, we will assume $\mathcal{D} = \mathbb{F}$, that is, we will consider only elementary gradings. Thus, let U be a finite-dimensional G-graded superspace, $\langle\,,\,\rangle$ be a supersymmetric even bilinear form, and φ be the corresponding superinvolution on $R = \mathrm{End}(U)$, that is, the superadjoint with respect to $\langle\,,\,\rangle$ (Definition 4.1.1). We claim that φ is homogeneous of degree e if and only if there exists $g_0 \in G$ such that $\langle\,,\,\rangle\colon U \otimes U \to \mathbb{F}$ is homogeneous of degree g_0. Indeed, let $\langle\,,\,\rangle\colon U \otimes U \to \mathbb{F}$ be homogeneous of degree g_0 and let $r \in \mathrm{End}(U)_k^i$ be a nonzero element, then for all $g, h \in G$, $v \in U_g^j$ and $w \in U_h$, we have $\langle rv, w\rangle = (-1)^{ij}\langle v, \varphi(r)w\rangle$, which is homogeneous of degree g_0kgh. Hence, for every $v \in U_g$, $r \in \mathrm{End}(U)_k$ and $w \in U_h$, we have $\langle v, \varphi(r)w\rangle = 0$ unless $g_0gkh = e$. Since the form is nondegenerate, $\varphi(r)w \in U_{kh}$ and therefore φ is homogeneous of degree e.

The converse follows from the next two results.

Proposition 4.3.5. *Let U be a finite dimensional graded superspace and let φ be a superinvolution on $\mathrm{End}(U)$ which is homogeneous of degree e. Then there is a nondegenerate bilinear form $\langle\,,\,\rangle$ on U such that $\langle rv, w\rangle = (-1)^{|r||v|}\langle v, \varphi(r)w\rangle$ for all v, $w \in U$, $r \in \mathrm{End}(U)$, and $\langle\,,\,\rangle\colon U \otimes U \to \mathbb{F}$ is a homogeneous map.*

Proof. Consider U as a graded left module over $R = \mathrm{End}(U)$. Then U^* is a graded right R-module with the action, for all $f \in U^*$, $r \in R$, given by

$$f.r := f \circ r. \tag{4.3.1}$$

Note that U and U^* are simple R-modules.

We can equip U^* with a left R-module structure by setting, for all $f \in U^*$ and $r \in R$,

$$r.f := (-1)^{|f||r|} f \circ \varphi(r). \tag{4.3.2}$$

(Note that Equations (4.3.1) and (4.3.2) are special cases of what was discussed in the paragraph preceding Lemma 4.3.2 – also see Section 2.5.)

Using Lemma 2.7 of [12], there is $g_0 \in G$ and an isomorphism of graded modules $\theta\colon U^{[g_0]} \to U^*$. We define a bilinear map $\langle\, ,\, \rangle\colon U \times U \to \mathbb{F}$ by setting $\langle v, w\rangle = \theta(v)(w)$. If v has degree g in U and w has degree h in U, then v has degree $g_0 g$ in $U^{[g_0]}$ and $\langle v, w\rangle$ has degree $g_0 g h$, hence the degree of $\langle\, ,\, \rangle$ is g_0.

Moreover, $\langle rv, w\rangle = \theta(rv)(w) = (r.\theta(v))(w) = (-1)^{|r||\theta(v)|}\theta(v)(\varphi(r)(w))$
$= (-1)^{|r||v|}\langle v, \varphi(r)w\rangle$, as desired. $\square$

Lemma 4.3.6. *Let U be a finite dimensional graded superspace and let φ be a superinvolution on $\mathrm{End}(U)$. Let $\langle\, ,\, \rangle_1\colon U \times U \to \mathbb{F}$ and $\langle\, ,\, \rangle_2\colon U \times U \to \mathbb{F}$ be two nondegenerate bilinear forms on U such that φ is the superadjoint with respect to both $\langle\, ,\, \rangle_1$ and $\langle\, ,\, \rangle_2$. Then there is $0 \neq \lambda \in \mathbb{F}$ such that $\langle v, w\rangle_1 = \lambda\langle v, w\rangle_2$, for all v, $w \in U$.*

Proof. We define $\theta_i\colon U \to U^*$ by sending $v \mapsto \langle v, \cdot\rangle_i$, for $1 \leq i \leq 2$. We claim that θ_i is an isomorphism of left R-modules. Let $0 \neq r \in \mathrm{End}(U)^j_k$ and v, $w \in U$. Then

$\theta_i(rv)(w) = \langle rv, w\rangle_i = (-1)^{j|v|}\langle v, \varphi(r)w\rangle_i = (-1)^{j|v|}\theta_i(v)(\varphi(r)w) = (-1)^{j|v|}(\theta_i(v) \circ \varphi(r))(w) = (r\theta_i(v))(w)$. The latter equality is due to Equation (4.3.2).

Consider the composition $\theta_2^{-1}\theta_1 \colon U \to U$. We have $\theta_2^{-1}\theta_1 \in \mathrm{End}_R(U) = \mathbb{F}$, so there is a nonzero $\lambda \in \mathbb{F}$ such that $\theta_1 = \lambda\theta_2$, that is, $\theta_1(v)(w) = \lambda\theta_2(v)(w)$ for all v, $w \in U$. The result follows by the definition of θ_i. □

4.4 Gradings on $B(m, n)$ up to isomorphism

Due to Theorem 4.2.6, the classification of group gradings on the Lie superalgebra $B(m, n)$ is equivalent to the classification of φ-gradings on the associative superalgebra $M(2m + 1, 2n)$. Moreover, all such φ-gradings are elementary by Proposition 4.3.3. Therefore we need to classify the elementary φ-gradings on $M(2m + 1, 2n)$ up to isomorphism.

Consider $M(2m+1, 2n) \simeq \mathrm{End}(U)$ where $U^{\bar 0} = \mathbb{F}^{2m+1}$, $U^{\bar 1} = \mathbb{F}^{2n}$, and $U = U^{\bar 0} \oplus U^{\bar 1}$ is equipped with a supersymmetric nondegenerate even bilinear form $\langle\, ,\, \rangle$. Moreover, the superspace U is G-graded and $\langle\, ,\, \rangle$ is homogeneous of degree g_0. Let $a \in G$ be such that $U^{\bar 0}_a \neq \{0\}$. Since $\langle\, ,\, \rangle$ is even and nondegenerate, there is $b \in G$ with $\langle U^{\bar 0}_a, U^{\bar 0}_b\rangle \neq 0$, which implies, $abg_0 = e$, hence $b = g_0^{-1}a^{-1}$.

We have two cases:

i If $a = b$ then $\langle\, ,\, \rangle|_{U^{\bar 0}_a}$ is nondegenerate. Therefore, in this case, $U^{\bar 0}$ is self-dual with respect to $\langle\, ,\, \rangle$.

ii If $a \neq b$ then $\langle , \rangle|_{U^{\bar 0}_a} = 0$, but $U^{\bar 0}_a$ and $U^{\bar 0}_b$ are dual to each other. In particular, $\dim U^{\bar 0}_a = \dim U^{\bar 0}_b$.

Note that the first case occurs if and only if $a^2 = g_0^{-1}$.

Therefore, we can write

$$U^{\bar{0}} = U^{\bar{0}}_{g_1} \oplus \cdots \oplus U^{\bar{0}}_{g_{l_0}} \oplus (U^{\bar{0}}_{g'_{l_0+1}} \oplus U^{\bar{0}}_{g''_{l_0+1}}) \oplus \cdots \oplus (U^{\bar{0}}_{g'_{k_0}} \oplus U^{\bar{0}}_{g''_{k_0}}) \tag{4.4.1}$$

where $U^{\bar{0}}_{g_i}$ is self-dual for $1 \le i \le l_0$ and $U^{\bar{0}}_{g'_i}$ is dual to $U^{\bar{0}}_{g''_i}$ via $\langle\, , \rangle$ for $l_0 + 1 \le i \le k_0$, and set

$$\gamma_{\bar{0}} = (g_1^{(\kappa_1^0)}, \ldots, g_{l_0}^{(\kappa_{l_0}^0)}, g'^{\,(\kappa_{l_0+1}^0)}_{l_0+1}, g''^{\,(\kappa_{l_0+1}^0)}_{l_0+1}, \ldots, g'^{\,(\kappa_{k_0}^0)}_{k_0}, g''^{\,(\kappa_{k_0}^0)}_{k_0}) \tag{4.4.2}$$

where, for $1 \le i \le l_0$, κ_i^0 denotes $\dim U^{\bar{0}}_{g_i}$ for $l_0 + 1 \le i \le k_0$, κ_i^0 denotes $\dim U^{\bar{0}}_{g'_i} = \dim U^{\bar{0}}_{g''_i}$ and $g^{(s)}$ is short for $\underbrace{g, \ldots, g}_{s \text{ times}}$. Observe that, with this notation, $\gamma_{\bar{0}}$ is a $(2m+1)$-tuple, and it will sometimes be more convenient to relabel its entries consecutively:

$$g_1^2 = \cdots = g_{q_0}^2 = g'_{q_0+1} g''_{q_0+1} = \cdots = g'_{q_0+s_0} g''_{q_0+s_0} = g_0^{-1}, \tag{4.4.3}$$

where $q_0 = \kappa_1^0 + \cdots + \kappa_{l_0}^0$ and $s_0 = \kappa_{l_0+1}^0 + \cdots + \kappa_{k_0}^0$.

Thus, $\gamma_{\bar{0}}$ satisfies the following:

Definition 4.4.1. Let $\gamma = (g_1, \ldots, g_k)$ be a k-tuple of elements of G. We call γ and the corresponding multiset $\Xi(\gamma)$ g_0-*balanced* if, for all $g \in G$, the number of times that g occurs in γ is equal to the number of times $g_0^{-1} g^{-1}$ occurs in γ.

Conversely, any g_0-balanced tuple can be relabelled as in Equation (4.4.2) and 4.4.3.

Analogously, for $U^{\bar{1}}$, we write

$$U^{\bar{1}} = U^{\bar{1}}_{h_1} \oplus \cdots \oplus U^{\bar{1}}_{h_{l_1}} \oplus (U^{\bar{1}}_{h'_{l_1+1}} \oplus U^{\bar{1}}_{h''_{l_1+1}}) \oplus \cdots \oplus (U^{\bar{1}}_{h_{k'_1}} \oplus U^{\bar{1}}_{h_{k''_1}}) \tag{4.4.4}$$

and set

$$\gamma_{\bar{1}} = (h_1^{(\kappa_1^1)}, \ldots, h_{l_1}^{(\kappa_{l_1}^1)}, h'_{l_1+1}{}^{(\kappa_{l_1+1}^1)}, h''_{l_1+1}{}^{(\kappa_{l_1+1}^1)}, \ldots, h'_{k_1}{}^{(\kappa_{k_1}^1)}, h''_{k_1}{}^{(\kappa_{k_1}^1)}). \tag{4.4.5}$$

Analogously, observe that, with this notation, $\gamma_{\bar{1}}$ is a $2n$-tuple; and it will sometimes be more convenient to relabel its entries consecutively:

$$h_1^2 = \cdots = h_{q_1}^2 = h'_{q_1+1}h''_{q_1+1} = \cdots = h'_{q_1+s_1}h''_{q_1+s_1} = g_0^{-1}, \tag{4.4.6}$$

where $q_1 = \kappa_1^1 + \cdots + \kappa_{l_1}^1$ and $s_1 = \kappa_{l_1+1}^0 + \cdots + \kappa_{k_1}^0$. Note that $\gamma_{\bar{1}}$ is also g_0-balanced.

Moreover, for $1 \leq i \leq l_1$, the dimension of $U_{h_i}^{\bar{1}}$ is even. Indeed, $\langle\, , \rangle|_{U^{\bar{1}}}$ is skew-symmetric and $\langle\, , \rangle|_{W_{h_i}}$ is nondegenerate for all $1 \leq i \leq l_1$.

Definition 4.4.2. A pair $(\gamma_{\bar{0}}, \gamma_{\bar{1}})$ is said to be g_0-*admissible* if both $\gamma_{\bar{0}}$ and $\gamma_{\bar{1}}$ are g_0-balanced and each $g \in G$ satisfying $g^2 = g_0^{-1}$ occurs in γ_1 an even (possibly zero) number of times.

Since $U^{\bar{0}}$ has odd dimension, we have $l_0 > 0$. It follows by equation (4.4.3) that g_0 is a square. Since $R = \mathrm{End}(U) = \mathrm{End}(U^{[g]})$, as graded algebras, for any $g \in G$, we will replace U with $U^{[g]}$ where $g^2 = g_0$. Therefore, we may assume $g_0 = e$. In this case, we will say "admissible" instead of "e-admissible".

Definition 4.4.3. Given $\gamma_{\bar{0}}$ and $\gamma_{\bar{1}}$, we will denote the corresponding elementary grading on $R = \mathrm{End}(U)$ by $\Gamma(\gamma_{\bar{0}}, \gamma_{\bar{1}})$. Moreover, let a pair $(\gamma_{\bar{0}}, \gamma_{\bar{1}})$ be admissible and write $\gamma_{\bar{0}}$ and $\gamma_{\bar{1}}$ as in Equations (4.4.2) and (4.4.5) with $g_0 = e$. We will consider $\Gamma(\gamma_{\bar{0}}, \gamma_{\bar{1}})$ as a φ-grading where φ maps every $r \in \mathrm{End}(U)$ to its adjoint with respect to the bilinear form $\langle\, , \rangle$ on $U = \mathbb{F}^{2m+1} \oplus \mathbb{F}^{2n}$ represented by the matrix:

$$\Phi = \left(\begin{array}{c|c} \Phi_0 & 0 \\ \hline 0 & \Phi_1 \end{array}\right)$$

where

$$\Phi_0 = \left(\begin{array}{c|ccccc} I_{\kappa_1^0+\cdots+\kappa_{l_0}^0} & & & 0 & & \\ \hline & 0 & I_{\kappa_{l_0+1}^0} & 0 & \cdots & 0 \\ & I_{\kappa_{l_0+1}^0} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ & \vdots & \vdots & 0 & 0 & I_{\kappa_{k_0}^0} \\ & 0 & 0 & 0 & I_{\kappa_{k_0}^0} & 0 \end{array}\right);$$

$$\Phi_1 = \left(\begin{array}{ccccc|ccccc} 0 & I_{\kappa_1^1/2} & 0 & \cdots & 0 & & & & & \\ -I_{\kappa_1^1/2} & 0 & 0 & \cdots & 0 & & & & & \\ 0 & 0 & \ddots & 0 & 0 & & & 0 & & \\ \vdots & \vdots & 0 & 0 & I_{\kappa_{l_1}^1/2} & & & & & \\ 0 & 0 & 0 & -I_{\kappa_{l_1}^1/2} & 0 & & & & & \\ \hline & & & & & 0 & I_{\kappa_{l_1+1}^1} & 0 & \cdots & 0 \\ & & & & & -I_{\kappa_{l_1+1}^1} & 0 & 0 & \cdots & 0 \\ & & 0 & & & 0 & 0 & \ddots & 0 & 0 \\ & & & & & \vdots & \vdots & 0 & 0 & I_{\kappa_{k_1}^1} \\ & & & & & 0 & 0 & 0 & -I_{\kappa_{k_1}^1} & 0 \end{array}\right)$$

Note that the form $\langle\, ,\, \rangle$ in Definition 4.4.3 is even, supersymmetric and nondegenerate. Moreover, it is homogenous of degree e with respect to the G-grading. We summarize the discussion above:

Theorem 4.4.4. *Let G be an abelian group and let $R = M(2m+1, 2n)$ be a matrix superalgebra over $\mathbb{F}$ equipped with a superinvolution φ that arises from a supersymmetric even bilinear form $\langle\,,\,\rangle$. Up to isomorphism, the φ-gradings on R are precisely $\Gamma(\gamma_{\bar{0}}, \gamma_{\bar{1}})$, where $(\gamma_{\bar{0}}, \gamma_{\bar{1}})$ is admissible and φ is given by $\varphi(X) = \Phi^{-1}X^{s\top}\Phi$, for all $X \in R$, where Φ is given in Definition 4.4.3.* $\square$

Finally, we classify these φ-gradings up to isomorphism.

Lemma 4.4.5. *Let $\theta\colon U \to U'$ be an isomorphism of superspaces. Let $\langle\,,\,\rangle$ and $\langle\,,\,\rangle'$ be nondegenerate bilinear forms on U and U', respectively, and let φ and φ' be their corresponding superadjoints. Define $\psi\colon \operatorname{End}(U) \to \operatorname{End}(U')$ by $\psi(T) = \theta \circ T \circ \theta^{-1}$. Then $\varphi' = \psi \circ \varphi \circ \psi^{-1}$ if and only if there exists $\lambda \in \mathbb{F}^\times$ such that $\langle \theta(x), \theta(y)\rangle' = \lambda\langle x, y\rangle$ for all x,$y \in U$.*

Proof. Define the following bilinear form on U: $\langle x, y\rangle'' := \langle\theta(x), \theta(y)\rangle'$, for all x,$y \in U$. Suppose $\varphi' = \psi \circ \varphi \circ \psi^{-1}$, we claim that φ is the superadjoint with respect to this form. Indeed, for every $T \in \operatorname{End}(U)$, we have

$$\begin{aligned}
\langle T(x), y\rangle'' &= \langle\theta(T(x)), \theta(y)\rangle' = \langle\psi(T)(\theta(x)), \theta(y)\rangle' \\
&= (-1)^{|\theta(x)||\psi(T)|}\langle\theta(x), \varphi'(\psi(T))(\theta(y)\rangle') \\
&= (-1)^{|x||T|}\langle\theta(x), (\psi \circ \varphi \circ \psi^{-1})(\psi(T))(\theta(y))\rangle' \\
&= (-1)^{|x||T|}\langle\theta(x), \psi(\varphi(T))(\theta(y))\rangle' = (-1)^{|x||T|}\langle\theta(x), (\theta\varphi(T)\theta^{-1})(\theta(y))\rangle' \\
&= (-1)^{|x||T|}\langle\theta(x), \theta(\varphi(T)(y))\rangle' = (-1)^{|x||T|}\langle x, \varphi(T)(y)\rangle''.
\end{aligned}$$

By Lemma 4.3.6, there exists $\lambda' \in \mathbb{F}^\times$ such that $\langle x, y\rangle = \lambda'\langle x, y\rangle'' = \lambda'\langle\theta(x), \theta(y)\rangle'$, and by taking $\lambda = 1/\lambda'$, we have that $\langle\theta(x), \theta(y)\rangle' = \lambda\langle x, y\rangle$ for all x,$y \in U$.

Conversely, if $\langle\theta(x),\theta(y)\rangle' = \lambda\langle x,y\rangle$ for all $x, y \in U$, then φ is the superadjoint with respect to $\langle\,,\,\rangle''$. On the other hand, for all $T \in \mathrm{End}(U)$, we have

$$\begin{aligned}\langle Tx,y\rangle'' &= \langle\psi(T)(\theta(x)),\theta(y)\rangle' = (-1)^{|\psi(T)||\theta(x)|}\langle\theta(x),\varphi'(\psi(T))(\theta(y))\rangle' \\ &= (-1)^{|T||x|}\langle\theta(x),\theta\big((\theta^{-1}\circ(\varphi'(\psi(T)))\circ\theta)(y)\big)\rangle' \\ &= (-1)^{|T||x|}\langle x,(\psi^{-1}\circ\varphi'\circ\psi)(T)(y)\rangle''.\end{aligned}$$

Therefore, $\varphi = \psi^{-1}\circ\varphi'\circ\psi$. ☐

Theorem 4.4.6. *Let G and R be as in Theorem 4.4.4. Two φ-gradings $\Gamma = \Gamma(\gamma_{\bar{0}},\gamma_{\bar{1}})$ and $\Gamma' = \Gamma'(\gamma'_{\bar{0}},\gamma'_{\bar{1}})$ on R are isomorphic if and only if there is $g \in G$ such that $\Xi(\gamma'_{\bar{0}}) = g\Xi(\gamma_{\bar{0}})$, $\Xi(\gamma'_{\bar{1}}) = g\Xi(\gamma_{\bar{1}})$ and $g^2 = e$.*

Proof. Let R_Γ and $R_{\Gamma'}$ denote R equipped with the grading Γ and Γ', respectively, and let φ and φ' denote the superinvolutions on R_Γ and on $R_{\Gamma'}$, respectively. Also, let U_Γ and $U_{\Gamma'}$ denote the superspace U equipped with a grading given by $(\gamma_{\bar{0}},\gamma_{\bar{1}})$ and $(\gamma'_{\bar{0}},\gamma'_{\bar{1}})$, respectively. Let $\langle\,,\,\rangle$ and $\langle\,,\,\rangle'$ denote bilinear forms on U and U' such that φ and φ' are the superadjoint with respect to them.

Suppose there is an isomorphism of graded superalgebras $\psi\colon R_\Gamma \to R_{\Gamma'}$ such that $\varphi' = \psi\varphi\psi^{-1}$. By [12, Theorem 2.10], there is a $G\times\mathbb{Z}_2$-homogeneous linear bijection $\theta\colon U_\Gamma \to U_{\Gamma'}$ such that $\psi(T) = \theta\circ T\circ\theta^{-1}$ for all $T \in R_\Gamma$. Note that θ is even because of the dimensions of the even and odd subsuperspaces of U_Γ and $U_{\Gamma'}$.

Due to Lemma 4.4.5, for all x,$y \in U_\Gamma$, we have $\langle x,y\rangle = \lambda\langle\theta(x),\theta(y)\rangle'$. Hence $g^2 = e$ where $g \in G$ is the degree of θ. Moreover, since $\theta\colon U_\Gamma^{[g]} \to U_{\Gamma'}$ is an isomorphism of graded superspaces, it follows that $\Xi(\gamma_{\bar{0}}) = g\Xi(\gamma'_{\bar{0}})$ and $\Xi(\gamma_{\bar{1}}) = g\Xi(\gamma'_{\bar{1}})$.

Conversely, permuting the entries, if necessary, from $\Xi(\gamma'_{\bar{0}}) = g\Xi(\gamma_{\bar{0}})$ and $\Xi(\gamma'_{\bar{1}}) = g\Xi(\gamma_{\bar{1}})$, we have $\gamma'_i = g\gamma_i$, $i \in \mathbb{Z}_2$. Let $\{e_i\}$ and $\{e'_i\}$ be the (homogeneous) standard bases of U_Γ and $U_{\Gamma'}$, respectively. Then, the map $\theta\colon U_\Gamma \to U_{\Gamma'}$ defined by $\theta(e_i) = e'_i$ for all $1 \leq i \leq 2m+1+2n$ is a homogeneous bijection of degree g.

Let Φ and Φ' be the matrices representing the bilinear forms $\langle\, ,\, \rangle$ and $\langle\, ,\, \rangle'$, respectively. From $\gamma'_i = g\gamma_i$, we have $\Phi = \Phi'$. Hence $\langle \theta(x), \theta(y)\rangle' = \langle x, y\rangle$. By Lemma 4.4.5, the map $\psi\colon \operatorname{End}(U_\Gamma) \to \operatorname{End}(U_{\Gamma'})$ given by $\psi(T) = \theta \circ T \circ \theta^{-1}$ is an isomorphism of graded superalgebras with involution. $\square$

Our final goal is to classify the fine gradings on $B(m,n)$ up to equivalence. Before that, we will consider an example that will turn out to be a fine grading.

4.5 An example of grading on $B(0,1)$

Consider $B(0,1) = \mathfrak{osp}(1,2) \subseteq M(1,2)$. Take $U^{\bar{0}} = \mathbb{F}$, $U^{\bar{1}} = \mathbb{F}^2$ and $\{e_1, e_2, e_3\}$ to be the standard basis of $U = U^{\bar{0}} \oplus U^{\bar{1}}$.

Let $G = \mathbb{Z}_2 \times \mathbb{Z}$, $\gamma_{\bar{0}} = ((\bar{1}, 0))$ and $\gamma_{\bar{1}} = ((\bar{0}, -1), (\bar{0}, 1))$.

Set

$$\Phi = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & -1 & 0 \end{array}\right).$$

Then we have the following grading on $M(1,2) \simeq \operatorname{End}(U)$:

$$\begin{cases} \mathrm{End}(U)_{(\bar{0},0)} = \langle E_{11}, E_{22}, E_{33} \rangle \\ \mathrm{End}(U)_{(\bar{1},-1)} = \langle E_{21}, E_{13} \rangle \\ \mathrm{End}(U)_{(\bar{1},1)} = \langle E_{12}, E_{31} \rangle \\ \mathrm{End}(U)_{(\bar{0},-2)} = \langle E_{23} \rangle \\ \mathrm{End}(U)_{(\bar{0},2)} = \langle E_{32} \rangle \end{cases}$$

Note that $\varphi(E_{12}) = \Phi^{-1} E_{12}^{s\top} \Phi = -E_{31}$. Since φ is involutive, we have $\varphi(E_{31}) = -E_{12}$. Analogously, $\varphi(E_{13}) = E_{21}$, $\varphi(E_{21}) = E_{13}$, $\varphi(E_{23}) = -E_{23}$, $\varphi(E_{32}) = -E_{32}$, $\varphi(E_{11}) = E_{11}$, $\varphi(E_{22}) = E_{33}$, and $\varphi(E_{33}) = E_{22}$, which gives us

$$B(0,1) = \left\{ \begin{pmatrix} 0 & d & e \\ -e & a & b \\ d & c & -a \end{pmatrix} \mid a, b, c, d, e \in \mathbb{F} \right\}.$$

Therefore, we have the following grading on $B(0,1)$:

$$\Gamma\colon \bigoplus_{(i,j) \in \mathbb{Z}_2 \times \mathbb{Z}} B(0,1)_{(i,j)}$$

where $B(0,1)_{(\bar{0},0)} = \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$, $B(0,1)_{(\bar{1},1)} = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle$, $B(0,1)_{(\bar{1},-1)} = \left\langle \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle$, $B(0,1)_{(\bar{0},-2)} = \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle$, $B(0,1)_{(\bar{0},2)} = \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$ and $B(0,1)_{(i,j)} = 0$ otherwise.

Note that the support of the grading on $M(1,2)$ generates G, however, the support of the grading on $B(0,1)$ does not. Indeed, $(\bar{1},0)$ is not in the subgroup generated by the support of the grading on $B(0,1)$. This subgroup is generated by $(\bar{1},1)$ and isomorphic to $\mathbb{Z}$.

The grading we have just constructed is a fine grading because its nonzero components have dimension 1. We will generalize this construction and classify all fine gradings in the next section.

4.6 Fine gradings on $B(m,n)$

As before, consider $B(m,n) \subseteq \mathfrak{osp}(2m+1,2n) \subseteq M(2m+1,2n) \simeq \mathrm{End}(U)$ where $U = U^{\bar{0}} \oplus U^{\bar{1}}$, $U^{\bar{0}} = \mathbb{F}^{2m+1}$ and $U^{\bar{1}} = \mathbb{F}^{2n}$.

We will now define a φ-grading on $\mathrm{End}(U)$ and will later prove that it is fine. All components of U will be 1-dimensional. The number of self-dual components of $U^{\bar{0}}$ will be denoted by q_0 and the number of pairs of the components in duality of U^i will be denoted by s_i, $i \in \mathbb{Z}_2$. (There will be no self-dual components of $U^{\bar{1}}$.) Note that, according to Equations (4.4.1) and (4.4.4), we have $q_0 = l_0$, $q_0 + 2s_0 = 2m+1$, $l_1 = 0$ and $s_1 = n$. Let the standard bases of $U^{\bar{0}}$ and $U^{\bar{1}}$ be $\{e^{\bar{0}}_1, \ldots, e^{\bar{0}}_{q_0+2s_0}\}$ and $\{e^{\bar{1}}_{q_0+2s_0+1}, \ldots, e^{\bar{1}}_{q_0+2s_0+2s_1}\}$, respectively.

Definition 4.6.1. Let $\widetilde{G}(q_0,s_0,s_1) = \mathbb{Z}_2^{q_0} \times \mathbb{Z}^{s_0+s_1}$. Denote the standard basis of $\mathbb{Z}_2^{q_0}$ by $\{\varepsilon_1, \ldots, \varepsilon_{q_0}\}$ and of $\mathbb{Z}^{s_0+s_1}$ by $\{\delta_1, \ldots, \delta_{s_0}, \delta_{s_0+1}, \ldots, \delta_{s_0+s_1}\}$.

We define the following grading on U:

$$\begin{cases} \deg(e_i^{\bar{0}}) := \varepsilon_i, & \text{for } 1 \le i \le q_0; \\ \deg(e_{q_0+2i}^{\bar{0}}) = -\deg(e_{q_0+2i-1}^{\bar{0}}) := \delta_i, & \text{for } 1 \le i \le s_0; \\ \deg(e_{q_0+2s_0+2i}^{\bar{1}}) = -\deg(e_{q_0+2s_0+2i-1}^{\bar{1}}) := \delta_{s_0+i}, & \text{for } 1 \le i \le s_1. \end{cases}$$

The bilinear form $\langle\, , \,\rangle$ is defined as in Definition 4.4.3, and the degrees of matrix units are determined by the grading on U. Explicitly:

- Case 1) If $i = j$ then $\deg(E_{ij}) = 0$.
- Case 2) If $1 \le i, j \le q_0$, $i \ne j$, then $\deg(E_{ij}) = \varepsilon_i - \varepsilon_j = \varepsilon_j - \varepsilon_i = \deg(E_{ji})$.
- Case 3) If $1 \le i \le q_0$ and $1 \le j \le s_0 + s_1$, then $\deg(E_{i,q_0+2j}) = \varepsilon_i - \delta_j = \deg(E_{q_0+2j-1,i})$ and $\deg(E_{q_0+2j,i}) = \varepsilon_i + \delta_j = \deg(E_{i,q_0+2j-1})$.
- Case 4) If $1 \le i, j \le s_0 + s_1$, $i \ne j$, then

$$\begin{cases} \deg E_{q_0+2i,q_0+2j} = \delta_i - \delta_j = \deg(E_{q_0+2j-1,q_0+2i-1}); \\ \deg(E_{q_0+2i-1,q_0+2j}) = -\delta_i - \delta_j = \deg(E_{q_0+2j-1,q_0+2i}); \\ \deg(E_{q_0+2i,q_0+2j-1}) = \delta_i + \delta_j = \deg(E_{q_0+2j,q_0+2i-1}). \end{cases}$$

- Case 5) If $1 \le i \le s_0 + s_1$,then

$$\begin{cases} \deg(E_{q_0+2i-1,q_0+2i}) = -2\delta_i; \\ \deg(E_{q_0+2i,q_0+2i-1}) = 2\delta_i. \end{cases}$$

This $\widetilde{G}(q_0, s_0, s_1)$-grading can be seen as a $G(q_0, s_0, s_1)$-grading where $G(q_0, s_0, s_1) \simeq \mathbb{Z}_2^{q_0-1} \times \mathbb{Z}^{s_0+s_1}$ is the subgroup generated by the support (note that for Lie superalgebras of type B, $q_0 \ge 1$).

Notation 4.6.2. We denote the $G(q_0, s_0, s_1)$-grading on $(M(2m+1, 2n), \varphi)$ just defined as $\Gamma(q_0, s_0, s_1)$ where $q_0 + 2s_0 = 2m+1$, $s_1 = n$, and $\varphi(X) = \Phi^{-1}X^{s\top}\Phi$, for all $X \in R$, and Φ is given in Definition 4.4.3.

Note that once m and n are fixed, we can abbreviate $G(q_0, s_0, s_1)$ as $G(s_0)$ and $\Gamma(q_0, s_0, s_1)$ as $\Gamma(s_0)$, $0 \leq s_0 \leq m$.

Moreover, note that the component of degree 0 in $M(2m+1, 2n)$ has dimension $2m+1+2n$; it is spanned by $\{E_{ii} \,|\, 1 \leq i \leq 2m+1+2n\}$. The components of dimension 2 are:

- If $1 \leq i < j \leq q_0$, $\mathrm{End}(U)_{\varepsilon_i+\varepsilon_j}$ is spanned by $\{E_{ij}, E_{ji}\}$;
- If $1 \leq i \leq q_0$ and $1 \leq j \leq s_0 + s_1$, then for any $\nu \in \mathbb{Z}_2$,
  $\mathrm{End}(U)_{\varepsilon_i-\delta_j}$ is spanned by $\{E_{i,q_0+2j}, E_{q_0+2j-1,i}\}$;
  $\mathrm{End}(U)_{\varepsilon_i+\delta_j}$ is spanned by $\{E_{q_0+2j,i}, E_{i,q_0+2j-1}\}$.
- If $1 \leq i, j \leq s_0+s_1$, $i \neq j$, then $\mathrm{End}(U)_{-\delta_i-\delta_j}$ is spanned by $\{E_{q_0+2i-1,q_0+2j}, E_{q_0+2j-1,q_0+2i})\}$; $\mathrm{End}(U)_{\delta_i+\delta_j}$ is spanned by $\{E_{q_0+2i,q_0+2j-1}, E_{q_0+2j,q_0+2i-1}\}$; $\mathrm{End}(U)_{\delta_i-\delta_j}$ is spanned by $\{E_{q_0+2i,q_0+2j}, E_{q_0+2j-1,q_0+2i-1})\}$;

There are $2(s_0+s_1)$ components of dimension 1. Indeed each index as in Case 5 gives us two components of dimension 1 each, i.e., if $1 \leq i \leq s_0 + s_1$, then

- $\mathrm{End}(U)_{-2\delta_i}$ is spanned by $\{E_{q_0+2i-1,q_0+2i}\}$;
- $\mathrm{End}(U)_{2\delta_i}$ is spanned by $\{E_{q_0+2i,q_0+2i-1}\}$.

These considerations about the dimensions are important in the proof that the above gradings are, indeed, fine. We will also use the following lemma whose proof is based on the fact that if $X \in R = M(2m+1, 2n)$ is homogeneous in any φ-grading and X is not an eigenvector of φ, then $X + \varphi(X)$ and $X - \varphi(X)$ are in the same component as X and are eigenvectors with eigenvalues 1 and -1, respectively.

Lemma 4.6.3. *Let $\Gamma = \Gamma(q_0, s_0, s_1)$ and let Γ' be a refinement of Γ. Suppose that i, j, k and l are natural numbers such that the span of $\{E_{ij}, E_{kl}\}$ is a 2-dimensional component R_g with respect to Γ with $i \neq k$, $j \neq l$ and that, for some $s \in \{i, j, k, l\}$, E_{ss} is a homogeneous element with respect to Γ'. Then R_g is also a homogeneous component with respect to Γ'.*

Proof. The component R_g is a direct sum of two 1-dimensional eigenspaces for φ: one spanned by $E_{ij} + E_{kl}$ and the other spanned by $E_{ij} - E_{kl}$. Then, both $E_{ij} + E_{kl}$ and $E_{ij} - E_{kl}$ must be homogeneous with respect to Γ'. We conclude the proof by noting that, if $s = i$ or $s = j$ then $E_{ij} = E_{ss}(E_{ij} + E_{kl})$ or $E_{ij} = (E_{ij} + E_{kl})E_{ss}$, respectively. In both cases, since E_{ss} is an idempotent, it must have degree e with respect to Γ' and, hence, E_{ij} is homogeneous of the same degree as $E_{ij} + E_{kl}$, so that E_{kl} is also homogeneous of the same degree as E_{ij} and therefore the component R_g cannot split. Analogously, if $s = k$ or $s = l$ then $E_{kl} = E_{ss}(E_{ij} + E_{kl})$ or $E_{kl} = (E_{ij} + E_{kl})E_{ss}$, respectively. The same argument as above allows us to conclude that, R_g cannot split, so R_g is a homogeneous component of Γ'. □

Proposition 4.6.4. $\Gamma(s_0)$ *is a fine grading on* $(M(2m+1, 2n), \varphi)$.

Proof. Recall that $\Gamma = \Gamma(s_0)$ has one component of dimension $2m + 1 + 2n$, $2(s_0 + s_1)$ components of dimension 1, and all the remaining components have dimension 2.

Let Γ' be a refinement of Γ. We will show that Γ' cannot be proper. Since $s_1 = n > 0$, there is at least one component of Γ of dimension 1, spanned by E_{ij} where i and j corresponds to components of $U^{\bar{1}}$ that are in duality to each other (see Case 5). Since Γ' is a refinement of Γ and the dimension is 1, E_{ij} is also homogeneous with respect to Γ'. Note that E_{ji} also spans a component of Γ of dimension 1 (see Case 5). So, analogously, E_{ji} is Γ'-homogeneous. Therefore, for some (consecutive) $i, j > q_0$, $E_{ii} = E_{ij}E_{ji}$ and $E_{jj} = E_{ji}E_{ij}$ are also Γ'-homogeneous.

Fix $s > q_0$ such that E_{ss} is Γ'-homogeneous. Then, by Lemma 4.6.3, the 2-dimensional components involving index s also do not split (see Cases 3 and 4). Hence, E_{is} and E_{si} are Γ'-homogeneous for all i. It follows that $E_{ii} = E_{is}E_{si}$ is also Γ'-homogeneous for every i, which means that the component of dimension $2m+1+2n$ does not split.

Applying Lemma 4.6.3 again, we conclude that the remaining 2-dimensional components do not split.

$\square$

We will now show that our fine gradings are not equivalent to each other.

Proposition 4.6.5. *Let $\Gamma(s_0)$, $\Gamma(s_0')$ be two fine gradings on $R = M(2m+1, 2n)$. Then $\Gamma(s_0)$ and $\Gamma(s_0')$ are equivalent if and only if $s_0 = s_0'$.*

Proof. If $s_0 = s_0'$, they are the same grading, hence equivalent. If $s_0 \neq s_0'$, then $\Gamma(s_0)$ has $2s_0 + 2n$ components of dimension 1 while $\Gamma(s_0')$ has $2s_0' + n$, hence they are not equivalent. $\square$

Finally, we will show that our list of fine gradings is complete.

Theorem 4.6.6. *The gradings $\Gamma(s_0)$, $0 \le s_0 \le m$, are precisely the fine φ-gradings on $R = M(2m+1, 2n)$ up to equivalence. Moreover, $G(s_0)$ is the universal group of $\Gamma(s_0)$.*

Proof. Let $\Gamma = \Gamma(\gamma_{\bar{0}}, \gamma_{\bar{1}})$ be a φ-grading on R and let $\widetilde{\Gamma}$ be the G-grading on $U = \mathbb{F}^{2m+1} \oplus \mathbb{F}^{2n}$ that induces Γ on $R = \mathrm{End}(U)$. Recall that the entries of $\gamma_{\bar{0}}$ and $\gamma_{\bar{1}}$ can be relabelled so that they satisfy Equations (4.4.3) and (4.4.6) for $g_0 = e \in G$. Let $\widetilde{\Gamma}(s_0)$ denote the $\widetilde{G}(s_0)$-grading on U that induces $\Gamma(s_0)$ on R. By Definitions 4.4.3 and 4.6.1, $\widetilde{\Gamma}(s_0)$ is clearly a refiniment of $\widetilde{\Gamma}$; more precisely, if we define $\alpha\colon \widetilde{G}(s_0) \to G$ by

$$\begin{cases} \varepsilon_i \mapsto g_i, & \text{for } 1 \le i \le q_0, \\ \delta_i \mapsto g_{q_0+2i}, & \text{for } 1 \le i \le s_0, \\ \delta_i \mapsto h_{2i}, & \text{for } s_0 < i \le s_0 + s_1, \end{cases}$$

then ${}^{\alpha}\widetilde{\Gamma}(s_0) = \widetilde{\Gamma}$ and $\langle\, , \,\rangle$ has degree e with respect to $\widetilde{\Gamma}(s_0)$. Restricting α to $G(s_0)$, it follows that ${}^{\alpha}\Gamma(s_0) = \Gamma$, so applying Lemma 2.2.2 and Propositions 4.6.4 and 4.6.5, we have the desired result. $\square$

To finish this chapter, we observe that fine gradings on $(M(2m+1, 2n), \varphi)$ are in one to one correspondence with fine gradings on $B(m, n)$ and equivalent gradings correspond to equivalent gradings. Indeed, it follows from Proposition 4.1.7 and [12, Theorem 1.39] (see also Remark 1.40). For example, $\Gamma(0)$ on $M(1, 2)$ corresponds to the grading on $B(0, 1) = \mathfrak{osp}(1, 2)$ considered in Section 4.5.

Chapter 5

Lie superalgebras of type $P(n)$

5.1 The Lie superalgebra $P(n)$

The definition of the superalgebra $P(n)$ is very similar to the definiton of $B(m,n)$ (see Subsection 4.1).

From now on, we suppose that $\langle\,,\rangle$ is supersymmetric, nondegenerate, and odd. The *periplectic Lie superalgebra* $\mathfrak{p}(U)$ is defined as $\mathfrak{p}(U)^{\bar{0}} \oplus \mathfrak{p}(U)^{\bar{1}}$ where

$$\mathfrak{p}(U)^i = \{L \in \mathfrak{gl}(U)^i \mid \langle L(x), y\rangle = -(-1)^{i|x|}\langle x, L(y)\rangle\}$$

for all $i \in \mathbb{Z}_2$. The superalgebra $\mathfrak{p}(U)$ is not simple, but its derived superalgebra $P(U) = [\mathfrak{p}(U), \mathfrak{p}(U)]$ is simple if $\dim U \geq 6$.

Since $\langle\,,\rangle$ is nondegenerate and odd, it is clear that $U^{\bar{1}}$ is isomorphic to $(U^{\bar{0}})^*$ by $u \mapsto \langle u, \cdot\rangle$. Writing $U^{\bar{0}} = V$, we can identify U with $V \oplus V^*$. Since $\langle\,,\rangle$ is supersymmetric, with this identification we have

$$\langle v_1 + v_1^*, v_2 + v_2^*\rangle = v_1^*(v_2) + v_2^*(v_1)$$

for all $v_1, v_2 \in V$ and $v_1^*, v_2^* \in V^*$. Hence, $P(U)$ is a subsuperspace of

$$\operatorname{End}(U) = \operatorname{End}(V \oplus V^*) = \begin{pmatrix} \operatorname{End}(V) & \operatorname{Hom}(V^*, V) \\ \operatorname{Hom}(V, V^*) & \operatorname{End}(V^*) \end{pmatrix}$$

given by

$$P(U) = \left\{ \begin{pmatrix} a & b \\ c & -a^* \end{pmatrix} \;\middle|\; \operatorname{tr} a = 0,\ b = b^* \text{ and } c = -c^* \right\}.$$

In the case $V = \mathbb{F}^{n+1}$, we write $\mathfrak{p}(n)$ for $\mathfrak{p}(U)$ and define $P(n) = [\mathfrak{p}(n), \mathfrak{p}(n)]$, where $n \geq 2$. Using the standard basis of V, we can identify $P(n)$ with the following subsuperalgebra of $M(n+1, n+1)^{(-)}$:

$$P(n) = \left\{ \begin{pmatrix} a & b \\ c & -a^\top \end{pmatrix} \;\middle|\; \operatorname{tr} a = 0,\ b = b^\top \text{ and } c = -c^\top \right\}. \tag{5.1.1}$$

One can readily check that $P(U)$ is a graded subspace of $\operatorname{End}(U)$ equipped with its canonical $\mathbb{Z}$-grading, so we have $P(U) = P(U)^{-1} \oplus P(U)^0 \oplus P(U)^1$. Also, the map $\iota : \mathfrak{sl}(n+1) \to P(n)^0$ given by $\iota(a) = \begin{pmatrix} a & 0 \\ 0 & -a^\top \end{pmatrix}$ is an isomorphism of Lie algebras. If we identify $\mathfrak{sl}(n+1)$ and $P(n)^0$ via this map, then $P(n)^{-1} \simeq \mathrm{S}^2(U^{\bar{0}}) \simeq V_{2\pi_1}$ and $P(n)^1 \simeq \bigwedge^2(U^{\bar{1}}) \simeq V_{\pi_{n-1}}$ as modules over $P(n)^0$, where π_i denotes the i-th fundamental weight of $\mathfrak{sl}(n+1)$.

5.2 Automorphisms of $P(n)$

The automorphisms of $P(n)$ were originally described by V. Serganova (see [20, Theorem 1]). We give a more explicit description of the automorphism group that we will use for our purposes.

Lemma 5.2.1. *Let U be a finite-dimensional superspace equipped with a supersymmetric nondegenerate odd bilinear form,* $\dim U \geq 4$. *The subset $P(U)$ generates* $\mathrm{End}(U)$ *as an associative superalgebra.*

Proof. Denote by R the associative superalgebra generated by $P(U)$. We claim that U is a simple R-module. Indeed, since $P(U)^{\bar{0}} \simeq \mathfrak{sl}(n+1)$, we have that $U^{\bar{0}} \simeq V_{\pi_1}$ and $U^{\bar{1}} \simeq V_{\pi_n}$ are simple nonisomorphic modules over the Lie algebra $P(U)^{\bar{0}}$. Also, the action of $P(U)^{\bar{1}}$ moves elements from $U^{\bar{0}}$ to $U^{\bar{1}}$ and vice-versa, so U does not have nonzero proper subspaces invariant under $P(U)$.

By the Jacobson Density Theorem, since we are over an algebraically closed field, we conclude that $R = \mathrm{End}(U)$. $\square$

Proposition 5.2.2. *The group of automorphisms of $P(n)$ is* $\frac{\mathrm{GL}(n+1)}{\{-1,+1\}}$ *where* $a \in \mathrm{GL}(n+1)$ *acts as the conjugation by* $\begin{pmatrix} a & 0 \\ 0 & (a^\top)^{-1} \end{pmatrix}$.

Proof. Let $P = P(n)$ and $\varphi : P \to P$ be a Lie superalgebra automorphism. Since it preserves the canonical $\mathbb{Z}_2$-grading, taking its restrictions, we obtain a Lie algebra automorphism $\varphi_{\bar{0}} : P^{\bar{0}} \to P^{\bar{0}}$ and an invertible linear map $\varphi_{\bar{1}} : P^{\bar{1}} \to P^{\bar{1}}$.

Claim 1. The components P^{-1} and P^1 of the canonical $\mathbb{Z}$-grading are invariant under φ.

We denote by $(P^{\bar{1}})^{\varphi_{\bar{0}}}$ the $P^{\bar{0}}$-module $P^{\bar{1}}$ twisted by $\varphi_{\bar{0}}$, i.e., the space $P^{\bar{1}}$ with a new action given by $\ell \cdot x = \varphi_{\bar{0}}(\ell)x$ for all $\ell \in P^{\bar{0}}$ and $x \in P^{\bar{1}}$. Clearly, the map $\varphi_{\bar{1}} : P^{\bar{1}} \to (P^{\bar{1}})^{\varphi_{\bar{0}}}$ is a $P^{\bar{0}}$-module isomorphism. In particular, $(P^{\bar{1}})^{\varphi_{\bar{0}}} = \varphi_{\bar{1}}(P^{-1}) \oplus \varphi_{\bar{1}}(P^1)$, where $\varphi_{\bar{1}}(P^{-1})$ and $\varphi_{\bar{1}}(P^1)$ are simple and nonisomorphic. It follows that either $(P^{-1})^{\varphi_{\bar 0}} = \varphi_{\bar 1}(P^{-1})$ or $(P^{-1})^{\varphi_{\bar 0}} = \varphi_{\bar 1}(P^{1})$. By dimension count, we have $(P^{-1})^{\varphi_{\bar 0}} = \varphi_{\bar 1}(P^{-1})$ and, similarly, $(P^{1})^{\varphi_{\bar 0}} = \varphi_{\bar 1}(P^{1})$.

Claim 2. The automorphism $\varphi_{\bar 0}$ is inner.

If we identify $\mathfrak{sl}(n+1)$ with P^0 via the map ι defined in Subsection 5.1, we have $P^{-1} \simeq V_{2\pi_1}$ as an $\mathfrak{sl}(n+1)$-module. By Claim 1, we know that $\varphi_{\bar 1}|_{P^{-1}} : P^{-1} \to (P^{-1})^{\varphi_{\bar 0}}$ is an isomorphism of modules, but if $\varphi_{\bar 0}$ were an outer automorphism, we would have $(V_{2\pi_1})^{\varphi_{\bar 0}} \simeq V_{2\pi_n}$, which would force $n = 1$, a contradiction.

Claim 3. If $\varphi_{\bar 0} = \mathrm{id}$, then $\varphi = \upsilon_\lambda$ for some $\lambda \in \mathbb{F}^\times$.

The automorphism υ_λ acts as $\lambda^i \,\mathrm{id}$ on P^i. Since $\varphi_{\bar 0} = \mathrm{id}$, $\varphi_{\bar 1} : P^{\bar 1} \to P^{\bar 1}$ is a P^0-module automorphism. By Claim 1 and Schur's Lemma, $\varphi_{\bar 1}|_{P^{-1}}$ and $\varphi_{\bar 1}|_{P^{1}}$ are scalar operators. Due to the superalgebra structure, these two scalars must be inverses of each other, concluding the proof of the claim.

By Claim 2, we know that there is an invertible $a \in \mathrm{End}(U^{\bar 0})$ such that $\varphi_{\bar 0}$ is the conjugation by $A = \begin{pmatrix} a & 0 \\ 0 & (a^\top)^{-1} \end{pmatrix}$. By Claim 3, φ must be equal to the composition of this conjugation with υ_λ for some $\lambda \in \mathbb{F}^\times$. But υ_λ is the conjugation by $\begin{pmatrix} \mu^{-1}\,\mathrm{id} & 0 \\ 0 & \mu\,\mathrm{id} \end{pmatrix}$ where $\mu^2 = \lambda$, so we can adjust a and assume that φ is the conjugation by A. Since P generates $M(n+1, n+1)$ as an associative superalgebra (Lemma 5.2.1), A is determined up to scalar and, clearly, the only possible scalars are -1 and 1. $\square$

Remark 5.2.3. The images of υ_λ, $\lambda \in \mathbb{F}^\times$, cover the group of outer automorphisms of $P(n)$ (see [20, Theorem 1]).

5.3 Restriction of gradings from $M(n+1,n+1)$ to $P(n)$

We start with a consequence of Proposition 5.2.2.

Corollary 5.3.1. *Every automorphism of $P(n)$ is the restriction of a unique even automorphism of $M(n+1,n+1)$ and every grading on $P(n)$ is the restriction of a unique even grading on $M(n+1,n+1)$.*

Proof. Consider the embedding $\mathrm{Aut}(P(n)) \to \mathrm{Aut}(M(n+1,n+1))$ that follows from Proposition 5.2.2. The image consists of even automorphisms, so Proposition 3.4.3(iv) implies that every G-grading on $P(n)$ extends to an even grading on $M(n+1,n+1)$. The uniqueness follows from Lemma 5.2.1. □

Of course, not every even grading on $M(n+1,n+1)$ restricts to $P(n)$. We are going to obtain necessary and sufficient conditions for such restriction to be possible.

Let $\mathcal{D}$ be a finite-dimensional graded division algebra. The concept of *dual of a graded $\mathcal{D}$-module* is a special case of the concept of *superdual* discussed in Subsection 2.5, which arises when the gradings on $\mathcal{D}$ and its graded modules are even. Furthermore, in our situation $T = \operatorname{supp} \mathcal{D}$ must be an elementary 2-group (see Theorem 5.3.3). Let us recall the definitions and specialize them to the case at hand.

Let $\mathcal{V}$ be a *right* graded $\mathcal{D}$-module. Then $\mathcal{V}^\star = \mathrm{Hom}_{\mathcal{D}}(\mathcal{V},\mathcal{D})$ is a *left* $\mathcal{D}$-module with the action given by $(d \cdot f)(v) = df(v)$ for all $d \in \mathcal{D}$, $f \in \mathcal{V}^\star$ and $v \in \mathcal{V}$. If $\mathcal{B} = \{v_1, \ldots, v_k\}$ is a homogeneous basis for $\mathcal{V}$, the dual basis $\mathcal{B}^\star \subseteq \mathcal{V}^\star$ consists of the elements $v_i^\star$, $1 \leq i \leq k$, defined by $v_i^\star(v_j) = \delta_{ij}$. Note that $\deg v_i^\star = (\deg v_i)^{-1}$. Given two right $\mathcal{D}$-modules, $\mathcal{V}$ and $\mathcal{W}$, and a $\mathcal{D}$-linear map $L : \mathcal{V} \to \mathcal{W}$, we have the adjoint $L^\star : \mathcal{W}^\star \to \mathcal{V}^\star$ defined by $L^\star(f) = f \circ L$, for every $f \in \mathcal{W}^\star$.

We now assume that $\mathcal{D}$ is a standard realization associated to a pair (T, β) such that T is an elementary 2-group. With this we can identify $\mathcal{D}$ with $\mathcal{D}^{\mathrm{op}}$ via transposition (see Remark 2.4.3) and, hence, we can regard left $\mathcal{D}$-modules as right $\mathcal{D}$-modules. In particular, if $\mathcal{V}$ is a graded right $\mathcal{D}$-module, then $\mathcal{V}^\star$ is a graded right $\mathcal{D}$-module via $(f \cdot d)(v) = d^\top f(v)$ for all $f \in \mathcal{V}^\star$, $d \in \mathcal{D}$ and $v \in \mathcal{V}$.

Consider the space $\mathrm{Hom}_{\mathcal{D}}(\mathcal{V}, \mathcal{W})$. Fixing homogeneous $\mathcal{D}$-bases $\mathcal{B} = \{v_1, \ldots, v_k\}$ and $\mathcal{C} = \{w_1, \ldots, w_\ell\}$ for $\mathcal{V}$ and $\mathcal{W}$, respectively, we obtain an isomorphism between $\mathrm{Hom}_{\mathcal{D}}(\mathcal{V}, \mathcal{W})$ and $M_{\ell\times k}(\mathcal{D})$. The latter is naturally isomorphic to $M_{\ell\times k}(\mathbb{F}) \otimes \mathcal{D}$, so we will identify them.

Lemma 5.3.2. *Let $L : \mathcal{V} \to \mathcal{W}$ be a $\mathcal{D}$-linear map. We fix homogeneous $\mathcal{D}$-bases on $\mathcal{V}$ and $\mathcal{W}$, respectively, and their dual bases in $\mathcal{V}^\star$ and $\mathcal{W}^\star$. If $A \otimes d \in M_{\ell\times k}(\mathbb{F}) \otimes \mathcal{D}$ represents L, then $A^\top \otimes d^\top$ represents $L^\star$.* $\square$

We can regard the elements of $M_{\ell\times k}(\mathbb{F}) \otimes \mathcal{D}$ as matrices over $\mathbb{F}$ via Kronecker product (as in Definition 2.4.4). Then we have $A^\top \otimes d^\top = (A \otimes d)^\top$.

Theorem 5.3.3. *Let U be a finite-dimensional superspace and let $\Gamma = \Gamma(T, \beta, \gamma_0, \gamma_1)$ be an even G-grading on* $\mathrm{End}(U)$*. The superspace U admits a supersymmetric nondegenerate odd bilinear form such that $P(U)$ is a G-graded subsuperalgebra of* $\mathrm{End}(U)^{(-)}$ *if, and only if, T is an elementary* 2*-group and there is $g_0 \in G$ such that $\Xi(\gamma_1) = g_0\, \Xi(\gamma_0^{-1})$. Moreover, if there are two supersymmetric nondegenerate odd bilinear forms on U such that the corresponding $P_1(U)$ and $P_2(U)$ are G-graded subsuperalgebras, then $P_1(U)$ and $P_2(U)$ are isomorphic up to shift in opposite directions.*

Proof. Assume that, for some form, $P(U)$ is a G-graded subsuperalgebra. Let $V = U^{\bar{0}}$ and consider the identification of $U^{\bar{1}}$ with V^* presented in Subsection 5.1. This way $\Gamma = \Gamma(T, \beta, \gamma_0, \gamma_1)$ is an even grading on

$$\mathrm{End}(U) = \mathrm{End}(V \oplus V^*) = \begin{pmatrix} \mathrm{End}(V) & \mathrm{Hom}(V^*, V) \\ \mathrm{Hom}(V, V^*) & \mathrm{End}(V^*) \end{pmatrix}.$$

In particular, $\mathrm{End}(V)$ and $\mathrm{End}(V^*)$ are graded subspaces of $\mathrm{End}(U)^{\bar{0}}$, with gradings $\Gamma(T, \beta, \gamma_0)$ and $\Gamma(T, \beta, \gamma_1)$, respectively. If

$$x = \begin{pmatrix} a & 0 \\ 0 & -a^* \end{pmatrix}$$

is a homogeneous element in $P(U)^{\bar{0}}$, then both $u(x) := a \in \mathfrak{sl}(V) \subseteq \mathrm{End}(V)$ and $v(x) := -a^* \in \mathfrak{sl}(V^*) \subseteq \mathrm{End}(V^*)$ are homogeneous elements of the same degree. In other words, the maps $u : P(n)^{\bar{0}} \to \mathfrak{sl}(V)$ and $v : P(n)^{\bar{0}} \to \mathfrak{sl}(V^*)$ are homogeneous of degree e. Consider the algebra isomorphism $\varphi : \mathrm{End}(V)^{\mathrm{op}} \to \mathrm{End}(V^*)$ associating to each operator its adjoint. Clearly, $\varphi(a) = -(v \circ u^{-1})(a)$ for all $a \in \mathfrak{sl}(V)$. Since $\mathrm{End}(V) = \mathfrak{sl}(V) \oplus \mathbb{F}\,\mathrm{id}_V$ and $\varphi(\mathrm{id}_V) = \mathrm{id}_{V^*}$, we see that φ is homogeneous of degree e. From Lemma 2.5.1 and Remark 2.5.2, we conclude that $\Gamma(T, \beta^{-1}, \gamma_0^{-1}) \simeq \Gamma(T, \beta, \gamma_0)$, and hence, by Theorem 2.4.5, $\beta^{-1} = \beta$ and there is $g_0 \in G$ such that $g_0\, \Xi(\gamma_0^{-1}) = \Xi(\gamma_1)$. Since β is nondegenerate, $\beta^{-1} = \beta$ if, and only if, T is an elementary 2-group.

Note that the G-graded algebra $P(U)^{\bar{0}}$ is isomorphic (via the map u) to the G-graded subalgebra $\mathfrak{sl}(V)$ of $\mathrm{End}(V)^{(-)}$, where the grading on $\mathrm{End}(V)$ is $\Gamma(T, \beta, \gamma_0)$. Therefore, if we have two forms such that the corresponding $P_1(U)$ and $P_2(U)$ are G-graded subsuperalgebras, then their even parts are isomorphic as G-graded algebras. Using Lemmas 2.1.10 and 2.1.13, we conclude the "moreover" part.

Now assume, conversely, that T is an elementary 2-group and $\Xi(\gamma_1) = g_0\,\Xi(\gamma_0^{-1})$. We can adjust γ_1, if necessary, so that $\gamma_1 = g_0\,\gamma_0^{-1}$ and the isomorphism class of Γ does not change.

Let $\mathcal{D}$ be a standard realization of a graded division algebra associated to (T, β) and let $\mathcal{V}$ be a graded right $\mathcal{D}$-module with a homogeneous basis $\mathcal{B}$ whose degrees are given by γ_0. Define $\mathcal{U} = \mathcal{U}^{\bar{0}} \oplus \mathcal{U}^{\bar{1}}$ with $\mathcal{U}^{\bar{0}} = \mathcal{V}$ and $\mathcal{U}^{\bar{1}} = (\mathcal{V}^\star)^{[g_0]}$. The G-grading Γ on $\mathrm{End}(U)$ is defined by means of an isomorphism:

$$\mathrm{End}(U) \simeq \mathrm{End}_{\mathcal{D}}(\mathcal{U}) = \begin{pmatrix} \mathrm{End}_{\mathcal{D}}(\mathcal{V}) & \mathrm{Hom}_{\mathcal{D}}((\mathcal{V}^\star)^{[g_0]}, \mathcal{V}) \\ \mathrm{Hom}_{\mathcal{D}}(\mathcal{V}, (\mathcal{V}^\star)^{[g_0]}) & \mathrm{End}_{\mathcal{D}}((\mathcal{V}^\star)^{[g_0]}) \end{pmatrix}$$
$$= \begin{pmatrix} \mathrm{End}_{\mathcal{D}}(\mathcal{V}) & \mathrm{Hom}_{\mathcal{D}}(\mathcal{V}^\star, \mathcal{V})^{[g_0^{-1}]} \\ \mathrm{Hom}_{\mathcal{D}}(\mathcal{V}, \mathcal{V}^\star)^{[g_0]} & \mathrm{End}_{\mathcal{D}}(\mathcal{V}^\star) \end{pmatrix}.$$

Using the homogeneous $\mathcal{D}$-bases $\mathcal{B}$ for $\mathcal{V}$ and $\mathcal{B}^\star$ for $\mathcal{V}^\star$ to represent $\mathcal{D}$-linear maps by matrices in $M_k(\mathcal{D}) = M_k(\mathbb{F}) \otimes \mathcal{D}$ and using the Kronecker product to identify the latter with $M_{n+1}(\mathbb{F})$, we obtain an isomorphism $\mathrm{End}(U) \xrightarrow{\sim} M(n+1, n+1)$, and $M(n+1, n+1)$ contains $\mathfrak{p}(n)$ and $P(n) = [\mathfrak{p}(n), \mathfrak{p}(n)]$ as in Equation (5.1.1).

The above isomorphism $\mathrm{End}(U) \xrightarrow{\sim} M(n+1, n+1)$ of superalgebras is given by an isomorphism of superspaces $U \xrightarrow{\sim} \mathbb{F}^{n+1} \oplus \mathbb{F}^{n+1}$. Hence, there exists a supersymmetric nondegenerate odd bilinear form on U such that $P(U)$ corresponds to $P(n)$ under the above isomorphism.

Finally, we have to show that $P(U)$ is a G-graded subsuperspace of $\mathrm{End}(U)$. Clearly, it is sufficient to prove the same for $\mathfrak{p}(U)$. But $\mathfrak{p}(U)$ corresponds to

$$\mathfrak{p}(n) = \left\{ \begin{pmatrix} a & b \\ c & -a^\top \end{pmatrix} \,\middle|\, a, b, c \in M_{n+1}(\mathbb{F}),\ b = b^\top \text{ and } c = -c^\top \right\} \subseteq M(n+1, n+1),$$

which, in view of Lemma 5.3.2, corresponds to the subsuperspace

$$\left\{ \begin{pmatrix} a & b \\ c & -a^\star \end{pmatrix} \,\middle|\, a \in \mathrm{End}_{\mathcal{D}}(\mathcal{U}),\, b = b^\star \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{V}^\star, \mathcal{V}),\, c = -c^\star \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{V}, \mathcal{V}^\star) \right\}$$

of $\mathrm{End}_{\mathcal{D}}(\mathcal{U})$, which is clearly a G-graded subsuperspace. □

5.4 G-gradings up to isomorphism

Definition 5.4.1. Let $T \subseteq G$ be a finite elementary 2-subgroup, β be a nondegenerate alternating bicharacter on T, γ be a k-tuple of elements of G, and $g_0 \in G$. We will denote by $\Gamma_P(T, \beta, \gamma, g_0)$ the grading on the superalgebra $P(n)$ obtained by restricting the grading $\Gamma(T, \beta, \gamma, g_0\gamma^{-1})$ on $M(n+1, n+1)$ as in the proof of Theorem 5.3.3. Explicitly, write $\gamma = (g_1, \dots, g_k)$ and take a standard realization of a graded division algebra $\mathcal{D}$ associated to (T, β). Then $M_{n+1}(\mathbb{F}) \simeq M_k(\mathbb{F}) \otimes \mathcal{D}$ by means of Kronecker product, and

$$M(n+1, n+1) \simeq \begin{pmatrix} M_k(\mathbb{F}) \otimes \mathcal{D} & M_k(\mathbb{F}) \otimes \mathcal{D} \\ M_k(\mathbb{F}) \otimes \mathcal{D} & M_k(\mathbb{F}) \otimes \mathcal{D} \end{pmatrix}$$

Denote by E_{ij} the (i, j)-th matrix unit in $M_k(\mathbb{F})$. The grading $\Gamma(T, \beta, \gamma, g_0\gamma^{-1})$ is given by:

- $\deg(E_{ij} \otimes d) = g_i(\deg d)g_j^{-1}$ in the upper left corner;
- $\deg(E_{ij} \otimes d) = g_i(\deg d)g_j\, g_0^{-1}$ in the upper right corner;
- $\deg(E_{ij} \otimes d) = g_i^{-1}(\deg d)g_j^{-1}g_0$ in the lower left corner;
- $\deg(E_{ij} \otimes d) = g_i^{-1}(\deg d)g_j$ in the lower right corner.

Note that the restriction of $\Gamma_P(T, \beta, \gamma, g_0)$ to the even part is the inner grading on $\mathfrak{sl}(n+1)$ with parameters (T, β, γ).

Theorem 5.4.2. *Every G-grading on the Lie superalgebra $P(n)$ is isomorphic to some $\Gamma_P(T,\beta,\gamma,g_0)$ as in Definition 5.4.1. Two gradings, $\Gamma = \Gamma_P(T,\beta,\gamma,g_0)$ and $\Gamma' = \Gamma_P(T',\beta',\gamma',g_0')$, are isomorphic if and only if $T = T'$, $\beta = \beta'$, and there is $g \in G$ such that $g^2 g_0 = g_0'$ and $g\,\Xi(\gamma) = \Xi(\gamma')$.*

Proof. The first assertion follows from Corollary 5.3.1 and Theorem 5.3.3. For the second assertion, recall that Γ and Γ' are, respectively, the restrictions of the gradings $\widetilde{\Gamma} = \Gamma(T,\beta,\gamma,g_0\gamma^{-1})$ and $\widetilde{\Gamma}' = \Gamma(T',\beta',\gamma',g_0'(\gamma')^{-1})$ on $M(n+1,n+1)$.

($\Rightarrow$): Suppose $\Gamma \simeq \Gamma'$. Since every automorphism of $P(n)$ extends to an automorphism of $M(n+1,n+1)$ (Corollary 5.3.1), we have $\widetilde{\Gamma} \simeq \widetilde{\Gamma}'$, which implies $T = T'$ and $\beta = \beta'$ by Theorem 3.3.5.

Let $\mathcal{D}$ be a standard realization associated to (T,β) and let $\mathcal{V}$ be a right $\mathcal{D}$-module with basis $\mathcal{B} = \{v_1,\dots,v_k\}$, which is graded by assigning $\deg v_i = g_i$. The same module, but with $\deg v_i = g_i'$, will be denoted by $\mathcal{W}$. Then $E = \mathrm{End}_{\mathcal{D}}(\mathcal{V} \oplus (\mathcal{V}^\star)^{[g_0]})$ and $E' = \mathrm{End}_{\mathcal{D}}(\mathcal{W} \oplus (\mathcal{W}^\star)^{[g_0']})$ are graded superalgebras. Using the bases $\mathcal{B}$ and $\mathcal{B}^\star$ and the Kronecker product, we can identify them with $M(n+1,n+1)$. The first identification gives the grading $\widetilde{\Gamma}$ on $M(n+1,n+1)$ and the second gives $\widetilde{\Gamma}'$.

Let Φ be an automorphism of $M(n+1,n+1)$ that sends $\widetilde{\Gamma}$ to $\widetilde{\Gamma}'$. By Proposition 5.2.2, Φ is the conjugation by

$$A = \begin{pmatrix} a & 0 \\ 0 & (a^\top)^{-1} \end{pmatrix}$$

for some $a \in \mathrm{GL}(n+1)$. By Lemma 5.3.2, Φ corresponds to the isomorphism $E \to E'$ that is the conjugation by

$$\phi = \begin{pmatrix} \alpha & 0 \\ 0 & (\alpha^\star)^{-1} \end{pmatrix}$$

where $\alpha : \mathcal{V} \to \mathcal{W}$ and $(\alpha^\star)^{-1} : (\mathcal{V}^\star)^{[g_0]} \to (\mathcal{W}^\star)^{[g_0']}$ are $\mathcal{D}$-linear maps. On the other hand, by Proposition 2.4.6, this isomorphism $E \to E'$ is the conjugation by a homogeneous bijective $\mathcal{D}$-linear map

$$\psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}.$$

It follows that there is $\lambda \in \mathbb{F}$ such that $\phi = \lambda\psi$ and, hence, ϕ is homogeneous. Let us denote its degree by g. Then both α and $(\alpha^\star)^{-1}$ must be homogeneous of degree g. Hence, $\alpha : \mathcal{V}^{[g]} \to \mathcal{W}$ is an isomorphism of graded $\mathcal{D}$-modules, so we conclude that $g\Xi(\gamma) = \Xi(\gamma')$. Considered as a map $\mathcal{V}^\star \to \mathcal{W}^\star$, $(\alpha^\star)^{-1}$ would have degree g^{-1}, so taking into account the shifts, it has degree $g^{-1}g_0^{-1}g_0'$, which must be equal to g, so $g_0' = g^2 g_0$.

($\Leftarrow$): We may suppose $\mathcal{D} = \mathcal{D}'$. Since $g\Xi(\gamma) = \Xi(\gamma')$, we have an isomorphism of graded $\mathcal{D}$-modules $\alpha : \mathcal{V}^{[g]} \to \mathcal{W}$. As a map from $\mathcal{V}$ to $\mathcal{W}$, α is homogeneous of degree g, hence $(\alpha^\star)^{-1} : \mathcal{V}^\star \to \mathcal{W}^\star$ has degree g^{-1}. It follows that, as a map from $(\mathcal{V}^\star)^{[g_0]}$ to $(\mathcal{W}^\star)^{[g_0']}$, $(\alpha^\star)^{-1}$ has degree $g^{-1}g_0^{-1}g_0' = g$. The desired automorphism of $P(n)$ that sends Γ to Γ' is the conjugation by the matrix $\psi = \begin{pmatrix} \alpha & 0 \\ 0 & (\alpha^\star)^{-1} \end{pmatrix}$. □

5.5 Fine gradings up to equivalence

For every integer $\ell \geq 0$, we set $T_{(\ell)} = \mathbb{Z}_2^{2\ell}$ and fix a nondegenerate alternating bicharacter $\beta_{(\ell)}$, say,

$$\beta_{(\ell)}(x, y) = (-1)^{\sum_{i=1}^{2\ell} x_i y_{2\ell-i+1}}.$$

Definition 5.5.1. For every ℓ such that 2^ℓ is a divisor of $n+1$, put $k := \frac{n+1}{2^\ell}$ and $\tilde{G}_{(\ell)} = T_{(\ell)} \times \mathbb{Z}^k$. Let $\{e_1, \ldots, e_k\}$ be the standard basis of $\mathbb{Z}^k$ and let $\langle e_0 \rangle$ be the infinite cyclic group generated by a new symbol e_0. We define $\Gamma_P(\ell,k)$ to be the $\tilde{G}_{(\ell)}\times\langle e_0\rangle$-grading $\Gamma_P(T_{(\ell)},\beta_{(\ell)},(e_1,\dots,e_k),e_0)$ on $P(n)$. If n is clear from the context, we will simply write $\Gamma_P(\ell)$.

The subgroup of $\tilde{G}_{(\ell)}\times\langle e_0\rangle$ generated by the support of $\Gamma_P(\ell,k)$ is

$$G_{(\ell)} := (T_{(\ell)}\times\mathbb{Z}_0^k)\oplus\langle 2e_1-e_0\rangle\simeq\mathbb{Z}_2^{2\ell}\times\mathbb{Z}^k.$$

Proposition 5.5.2. *The gradings $\Gamma_P(\ell)$ on $P(n)$ are fine. Moreover, if $\ell\neq\ell'$, then $\Gamma_P(\ell)$ and $\Gamma_P(\ell')$ are not equivalent.*

Proof. We can write $\Gamma_P(\ell)=\Gamma^{-1}\oplus\Gamma^0\oplus\Gamma^1$ where Γ^i is the restriction of $\Gamma_P(\ell)$ to the i-th component of the canonical $\mathbb{Z}$-grading of $P(n)$. We identify $P(n)^0=P(n)^{\bar{0}}$ with $\mathfrak{sl}(n+1)$ via the map ι defined in Subsection 5.1. Then the grading Γ^0 on $P(n)^0$ is the restriction to $\mathfrak{sl}(n+1)$ of a fine grading on $M_{n+1}(\mathbb{F})$ with universal group $T_{(\ell)}\times\mathbb{Z}_0^k$ ([12, Proposition 2.35]), so it has no proper refinements among the inner gradings on $\mathfrak{sl}(n+1)$. Also, $\Gamma_P(\ell)$ and $\Gamma_P(\ell')$ are nonequivalent if $\ell\neq\ell'$, because their restrictions to $P(n)^0$ are nonequivalent.

Note that the supports of Γ^{-1}, Γ^0 and Γ^1 are pairwise disjoint since they project to, respectively, $-e_0$, 0, and e_0 in the direct summand $\langle e_0\rangle$ of $\tilde{G}_{(\ell)}\times\langle e_0\rangle$. Suppose that the grading $\Gamma_P(\ell)$ admits a refinement $\Delta=\Delta^{-1}\oplus\Delta^0\oplus\Delta^1$. Then Δ^0 is an inner grading that is a refinement of Γ^0, hence they are the same grading (up to relabeling). Using Lemma 2.1.10, we conclude that Γ and Δ are the same grading, proving that Γ is fine. □

Theorem 5.5.3. *Every fine grading on $P(n)$ is equivalent to a unique $\Gamma_P(\ell)$ as in Definition 5.5.1. Moreover, every grading $\Gamma_P(\ell)$ is fine, and $G_{(\ell)}$ is its universal group.*

Proof. Let $\Gamma = \Gamma_P(G, T, \beta, \gamma, g_0)$ be any G-grading on $P(n)$. Since T is an elementary 2-group of even rank, we have an isomorphism $\alpha : T_{(\ell)} \to T$, for some ℓ, such that $\beta_{(\ell)} = \beta \circ (\alpha \times \alpha)$. We can extend α to a homomorphism $G_{(\ell)} \to G$ (also denoted by α) by sending the elements $e_1, \dots, e_k$ to the entries of γ, and e_0 to g_0. By construction, ${}^{\alpha}\Gamma_P(\ell) \simeq \Gamma$. It remains to apply Proposition 5.5.2 and Lemma 2.2.2. $\square$

5.6 An example of a grading on $P(3)$

Let $l = 1$, $T_{(1)} = \mathbb{Z}_2 \times \mathbb{Z}_2$, $\beta_{(1)}((x_1, x_2), (y_1, y_2)) = (-1)^{x_1 y_2 + x_2 y_1}$. Consider $n = 3$, then $k = 2$ and $\widetilde{G}_{(1)} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}$. Let $\{e_1 = (1, 0), e_2 = (0, 1)\}$ be the standard basis of $\mathbb{Z} \times \mathbb{Z}$ and let $e_0 = 1$. We will show explicitly the $\widetilde{G}_{(1)} \times \mathbb{Z}$-grading $\Gamma_P(T_{(1)}, \beta_{(1)}, (e_1, e_2), 1)$ on $P(3) \subseteq \mathfrak{sl}(4, 4) \subseteq M(4, 4)$.

First we need to recall how to recover the graded division algebra $\mathcal{D}$ from the pair $(T_{(1)}, \beta_{(1)})$. Note that $T_{(1)} = \mathbb{Z}_2 \times \mathbb{Z}_2 = A \times B$ where $A = B = \mathbb{Z}_2$ and also that $\beta_{(1)}(A, A) = \beta_{(1)}(B, B) = (-1)^0 = 1$. Consider the group algebra $V = \mathbb{F}B = \mathbb{F}\mathbb{Z}_2$ with its basis $\{e_{(\bar{0},\bar{0})}, e_{(\bar{0},\bar{1})}\}$. We define $X_{(\bar{0},\bar{0})}$, $X_{(\bar{0},\bar{1})}$, $X_{(\bar{1},\bar{0})}$ and $X_{(\bar{1},\bar{1})}$ to be such that $X_{(\bar{0},\bar{0})} e_b = e_b$ for all $e_b \in V$, $X_{(\bar{0},\bar{1})} e_{(\bar{0},i)} = e_{(\bar{0},i+\bar{1})}$ for all $i \in \mathbb{Z}_2$, $X_{(\bar{1},\bar{0})} e_{(\bar{0},i)} = \beta_{(1)}((\bar{1},\bar{0}), (\bar{0}, i)) e_{(\bar{0},i)} = (-1)^i e_{(\bar{0},i)}$ for all $i \in \mathbb{Z}_2$ and finally $X_{(\bar{1},\bar{1})} = X_{(\bar{1},\bar{0})+(\bar{0},\bar{1})} = X_{(\bar{1},\bar{0})} X_{(\bar{0},\bar{1})}$. In matrix form, we have $X_{(\bar{0},\bar{0})} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $X_{(\bar{0},\bar{1})} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $X_{(\bar{1},\bar{0})} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $X_{(\bar{1},\bar{1})} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Recall that $\mathcal{D} = \oplus_{(i,j) \in \mathbb{Z}_2} \mathcal{D}_{(i,j)}$ with $\mathcal{D}_{(i,j)} = \mathbb{F} X_{(i,j)}$.

From $\gamma = ((1, 0), (0, 1))$ and $e_0 = 1$ we construct $\gamma_0 = ((1, 0, 0), (0, 1, 0))$ and

$\gamma_1 = \gamma_0^{-1} = ((-1,0,1)(0,-1,1))$. Consider the four dimension vector space generated by the homogeneous basis with symbols $\{e_{(1,0,0)}, e_{(0,1,0)}, e_{(-1,0,1)}, e_{(0,-1,1)}\}$ and consider the matrix algebra $M(4)$ equipped with the induced elementary from that vector space. Denote the basis of $M(4)$ by $\{E_{ij},\ 1 \le i,j \le 4\}$ where E_{ij} denotes the matrix with 1 in the (i,j)-entry and zero elsewhere. We have that

$$\begin{cases} M(4)_{(0,0,0)} = \langle E_{11}, E_{22}, E_{33}, E_{44} \rangle \\ M(4)_{(-1,1,0)} = \langle E_{21}, E_{34} \rangle \\ M(4)_{(-1,-1,1)} = \langle E_{41}, E_{32} \rangle \\ M(4)_{(1,-1,0)} = \langle E_{12}, E_{43} \rangle \\ M(4)_{(1,1,-1)} = \langle E_{23}, E_{14} \rangle \\ M(4)_{(-2,0,1)} = \langle E_{31} \rangle \\ M(4)_{(0,-2,1)} = \langle E_{42} \rangle \\ M(4)_{(2,0,-1)} = \langle E_{13} \rangle \\ M(4)_{(0,2,-1)} = \langle E_{24} \rangle \end{cases}$$

Considering the Kronecker's product we have a grading on $M(8)$ and then we can take the restriction to the following grading on $P(3)$:

- $P(3)_{(0,0,0,\bar{0},\bar{0})} = \langle (E_{11} - E_{33}) \otimes X_{(\bar{0},\bar{0})}, (E_{22} - E_{44}) \otimes X_{(\bar{0},\bar{0})} \rangle$
- $P(3)_{(0,0,0,\bar{0},\bar{1})} = \langle (E_{11} - E_{33}) \otimes X_{(\bar{0},\bar{1})}, (E_{22} - E_{44}) \otimes X_{(\bar{0},\bar{1})} \rangle$
- $P(3)_{(0,0,0,\bar{1},\bar{0})} = \langle (E_{11} - E_{33}) \otimes X_{(\bar{1},\bar{0})}, (E_{22} - E_{44}) \otimes X_{(\bar{1},\bar{0})} \rangle$
- $P(3)_{(0,0,0,\bar{1},\bar{1})} = \langle (E_{11} + E_{33}) \otimes X_{(\bar{1},\bar{1})}, (E_{22} + E_{44}) \otimes X_{(\bar{1},\bar{1})} \rangle$

- $P(3)_{(-1,1,0,\bar{0},\bar{0})} = \langle (E_{21} - E_{34}) \otimes X_{(\bar{0},\bar{0})} \rangle$
- $P(3)_{(-1,1,0,\bar{0},\bar{1})} = \langle (E_{21} - E_{34}) \otimes X_{(\bar{0},\bar{1})} \rangle$
- $P(3)_{(-1,1,0,\bar{1},\bar{0})} = \langle (E_{21} - E_{34}) \otimes X_{(\bar{1},\bar{0})} \rangle$
- $P(3)_{(-1,1,0,\bar{1},\bar{1})} = \langle (E_{21} + E_{34}) \otimes X_{(\bar{1},\bar{1})} \rangle$
- $P(3)_{(-1,-1,1,\bar{0},\bar{0})} = \langle (E_{41} - E_{32}) \otimes X_{(\bar{0},\bar{0})} \rangle$
- $P(3)_{(-1,-1,1,\bar{1},\bar{0})} = \langle (E_{41} - E_{32}) \otimes X_{(\bar{1},\bar{0})} \rangle$
- $P(3)_{(-1,-1,1,\bar{0},\bar{1})} = \langle (E_{41} - E_{32}) \otimes X_{(\bar{0},\bar{1})} \rangle$
- $P(3)_{(-1,-1,1,\bar{1},\bar{1})} = \langle (E_{41} + E_{32}) \otimes X_{(\bar{1},\bar{1})} \rangle$
- $P(3)_{(1,-1,0,\bar{0},\bar{0})} = \langle (E_{12} - E_{43}) \otimes X_{(\bar{0},\bar{0})} \rangle$
- $P(3)_{(1,-1,0,\bar{1},\bar{0})} = \langle (E_{12} - E_{43}) \otimes X_{(\bar{1},\bar{0})} \rangle$
- $P(3)_{(1,-1,0,\bar{0},\bar{1})} = \langle (E_{12} - E_{43}) \otimes X_{(\bar{0},\bar{1})} \rangle$
- $P(3)_{(1,-1,0,\bar{1},\bar{1})} = \langle (E_{12} + E_{43}) \otimes X_{(\bar{1},\bar{1})} \rangle$
- $P(3)_{(1,1,-1,\bar{0},\bar{0})} = \langle (E_{23} + E_{14}) \otimes X_{(\bar{0},\bar{0})} \rangle$
- $P(3)_{(1,1,-1,\bar{1},\bar{0})} = \langle (E_{23} + E_{14}) \otimes X_{(\bar{1},\bar{0})} \rangle$
- $P(3)_{(1,1,-1,\bar{0},\bar{1})} = \langle (E_{23} + E_{14}) \otimes X_{(\bar{0},\bar{1})} \rangle$
- $P(3)_{(1,1,-1,\bar{1},\bar{1})} = \langle (E_{23} - E_{14}) \otimes X_{(\bar{1},\bar{1})} \rangle$
- $P(3)_{(-2,0,1,\bar{1},\bar{1})} = \langle E_{31} \otimes X_{(\bar{1},\bar{1})} \rangle$

- $P(3)_{(0,-2,1,\bar{1},\bar{1})} = \langle E_{42} \otimes X_{(\bar{1},\bar{1})}\rangle$
- $P(3)_{(2,0,-1,\bar{0},\bar{0})} = \langle E_{13} \otimes X_{(\bar{0},\bar{0})}\rangle$
- $P(3)_{(2,0,-1,\bar{1},\bar{0})} = \langle E_{13} \otimes X_{(\bar{1},\bar{0})}\rangle$
- $P(3)_{(2,0,-1,\bar{0},\bar{1})} = \langle E_{13} \otimes X_{(\bar{0},\bar{1})}\rangle$
- $P(3)_{(0,2,-1,\bar{0},\bar{0})} = \langle E_{24} \otimes X_{(\bar{0},\bar{0})}\rangle$
- $P(3)_{(0,2,-1,\bar{1},\bar{0})} = \langle E_{24} \otimes X_{(\bar{1},\bar{0})}\rangle$
- $P(3)_{(0,2,-1,\bar{0},\bar{1})} = \langle E_{24} \otimes X_{(\bar{0},\bar{1})}\rangle$

The signs appear taking into consideration the definition of $P(3)$ and also that $X_{(\bar{0},\bar{0})}^{\top} = X_{(\bar{0},\bar{0})}$, $X_{(\bar{0},\bar{1})}^{\top} = X_{(\bar{0},\bar{1})}$, $X_{(\bar{1},\bar{0})}^{\top} = X_{(\bar{1},\bar{0})}$ and $X_{(\bar{1},\bar{1})}^{\top} = -X_{(\bar{1},\bar{1})}$.

Note that this grading is an example of fine grading and has 28 components, 4 components have dimension 2.

Chapter 6

Lie superalgebras of type $Q(n)$

Just like $P(n)$, the Lie superalgebras of type $Q(n)$ are connected with the associative superalgebra $M(n+1, n+1)$, but in this case, not every automorphism of $Q(n)$ can be realized as restriction of an automorphism of $M(n+1, n+1)$. So we will not use the same technique we used for $P(n)$. Instead, the technique we use here is to look to $Q(n)$ as a graded algebra and a graded module with a graded bilinear map. We will formalize it as the chapter goes.

6.1 Definitions

The special linear Lie superalgebras (series A) are constructed from $M(m, n)^{(-)}$ by taking the quotient of the derived superalgebra modulo its center. The three orthosymplectic series (B, C, D) and the periplectic series (P) are constructed in the same way from the subalgebras of skew-symmetric elements in $M(m, n)^{(-)}$ with respect to appropriate superinvolutions. Series Q is different in that we have to start from an associative superalgebra that is simple as a superalgebra but not as an algebra. Namely, let $A = R \times R$, with component-wise product, where $R = M_{n+1}$, $n \geq 1$. Then $(x, y) \mapsto (y, x)$ is an automorphism of A of order 2 and hence its eigenspace decomposition is a $\mathbb{Z}_2$-grading on A, with $A^{\bar{0}} = \{(x, x) \mid x \in R\}$ and $A^{\bar{1}} = \{(x, -x) \mid x \in R\}$. Note that $A^{\bar{0}}$ is isomorphic to R as an algebra and $A^{\bar{1}} = uA^{\bar{0}}$ where $u = (1, -1)$, so we can write $A = R \oplus uR$ where u is odd, commutes with the elements of R and satisfies $u^2 = 1$. The associative superalgebra A can be identified with a subalgebra of $M(n+1, n+1)^{(-)}$ as follows:

$$\left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in M(n+1, n+1) \;\middle|\; a, b \in M_{n+1} \right\} \xrightarrow{\sim} A, \qquad \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mapsto a + ub.$$

Let $\widetilde{Q}(n)$ be the derived superalgebra of $A^{(-)}$. Then

$$\widetilde{Q}(n) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in M(n+1, n+1) \;\middle|\; a, b \in M_{n+1},\ \mathrm{tr}(b) = 0 \right\}.$$

Set $Q(n)$ to be the quotient of $\widetilde{Q}(n)$ by its center, which is spanned by the identity matrix:

$$Q(n) = \frac{\widetilde{Q}(n)}{\mathbb{F}1}.$$

Thus, the even part of $Q(n)$ can be identified with $\mathfrak{pgl}(n+1) := R/\mathbb{F}1$ and the odd part with $\mathfrak{sl}(n+1)$. If $n \geq 2$ then $Q(n)$ is a simple Lie superalgebra (see [16, 19]). We may denote $Q(n)$ simply by Q when n is clear from the context.

Since the characteristic is 0, the Lie algebras $\mathfrak{pgl}(n+1)$ and $\mathfrak{sl}(n+1)$ are isomorphic by means of the map $a + \mathbb{F}1 \mapsto a^\sharp$ where

$$a^\sharp := a - \frac{1}{n+1}\,\mathrm{tr}(a)1,\ a \in M_{n+1}.$$

We may, therefore, identify both even and odd parts of the Lie superalgebra $Q(n)$ with $\mathfrak{sl}(n+1)$. In this way, we obtain another realization of $Q(n)$, which will be convenient for us:

$$\begin{array}{ccc} Q(n) & \xrightarrow{\sim} & \mathfrak{sl}(n+1)\oplus\mathfrak{sl}(n+1) \\ \begin{pmatrix} a & b \\ b & a \end{pmatrix} + \mathbb{F}1 & \mapsto & (a^\sharp, b). \end{array} \tag{6.1.1}$$

To distinguish between $Q^{\bar{0}}$ and $Q^{\bar{1}}$, we will denote $(x,0)$ by x and $(0,x)$ by $\underline{x}$, for $x \in \mathfrak{sl}(n+1)$. The mapping $x \mapsto \underline{x}$ is an isomorphism $Q^{\bar{0}} \to Q^{\bar{1}}$ as $Q^{\bar{0}}$-modules, which looks as follows in terms of the realization of $Q(n)$ as a subalgebra of $M(n+1,n+1)^{(-)}/\mathbb{F}1$:

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \mathbb{F}1 \mapsto \begin{pmatrix} 0 & a^\sharp \\ a^\sharp & 0 \end{pmatrix} + \mathbb{F}1.$$

It follows that the bracket of the Lie superalgebra $Q(n)$ in the realization (6.1.1) is given by

$$[a,b] = ab - ba,\ \ [a,\underline{b}] = \underline{ab - ba} \text{ and } [\underline{a},\underline{b}] = (ab+ba)^\sharp, \tag{6.1.2}$$

for all $a, b \in \mathfrak{sl}(n+1)$, where juxtaposition denotes multiplication in M_{n+1}.

6.2 Preliminaries

We will later need the following fact about automorphisms of the superalgebra $Q = Q(n)$, $n \geq 2$. For an automorphism $\varphi\colon Q \to Q$, we will denote by φ_i the restriction of φ to Q^i, $i \in \mathbb{Z}_2$. Let υ be the parity automorphism, i.e., $\upsilon_i = (-1)^i\,\mathrm{id}$.

Proposition 6.2.1. *If* $\sqrt{-1} \in \mathbb{F}$ *then the restriction map* $\mathrm{Aut}(Q) \to \mathrm{Aut}(Q^{\bar{0}})$ *is a surjective homomorphism whose kernel is generated by* υ.

Proof. Suppose $\varphi \in \mathrm{Aut}(Q)$ belongs to the kernel of the restriction map, i.e., $\varphi_{\bar{0}} = \mathrm{id}$. Then $\varphi_{\bar{1}}$ is an automorphism of $Q^{\bar{1}}$ as a $Q^{\bar{0}}$-module, hence $\varphi_{\bar{1}} = \lambda\,\mathrm{id}$ by Schur's

Lemma, where λ is a nonzero scalar. Since the composition $Q^{\bar{1}} \times Q^{\bar{1}} \to Q^{\bar{0}}$ is nonzero and preserved by φ, it follows that $\lambda = \pm 1$. Therefore, $\varphi = \mathrm{id}$ or $\varphi = \upsilon$.

It is well known that the automorphism group of $Q^{\bar{0}}$, which is $\mathfrak{sl}(n+1) \subset M_{n+1}^{(-)}$, is generated by inner automorphisms (i.e., conjugations by invertible elements from the respective matrix algebra) and by an outer automorphism θ, which is $\theta(x) = -x^t$. Thus, to prove the surjectivity of the restriction map, we only need to show that the inner automorphisms of $Q^{\bar{0}}$ and θ can, indeed, be extended to automorphisms of the whole of Q.

In the first case, if $\psi_r(x) = rxr^{-1}$ is an inner automorphism of $Q^{\bar{0}}$ then we set $\varphi(a + \underline{b}) = \psi_r(a) + \underline{\psi_r(b)}$, for all $a, b \in Q^{\bar{0}}$. In the second case, we set $\varphi(a + \underline{b}) = \theta(a) + \sqrt{-1}\,\underline{\theta(b)}$, for all $a, b \in Q^{\bar{0}}$. For both cases, it is straightforward to verify that φ is an automorphism. $\square$

Remark 6.2.2. The above extensions of inner automorphisms of $Q^{\bar{0}}$ are the inner automorphisms of Q. The extension of θ has order 4, as its square equals υ. It generates the group of outer automorphisms of Q. It follows that $\mathrm{Aut}(Q)$ is isomorphic to the semidirect product of the group of inner automorphisms of $Q^{\bar{0}}$, which is $\mathrm{PGL}(n+1)$, and the cyclic group of order 4 – this can be found in [20, Theorem 1].

Corollary 6.2.3 (of the proof). *Let $\varphi\colon Q \to Q$ be an automorphism. Then there is a scalar $\lambda \in \mathbb{F}^\times$ such that $\varphi_{\bar{1}}(\underline{x}) = \lambda \underline{\varphi_{\bar{0}}(x)}$ for all $x \in Q^{\bar{0}}$.*

Proof. For the sake of simplifying notation, let V be $Q^{\bar{1}}$ viewed as a module over $Q^{\bar{0}}$. Note that $\varphi_{\bar{1}}\colon V \to V$ is almost an automorphism of modules: for all $a \in Q^{\bar{0}}$ and all $v \in V$, we have that

$$\varphi_{\bar{1}}(av) = \varphi_{\bar{0}}(a)\varphi_{\bar{1}}(v). \tag{6.2.1}$$

Recall the module V twisted by $\varphi_{\bar{0}}$, i.e. set W to be the same vector space as V but with the following action:

$$a * w = \varphi_{\bar{0}}(a)w, \ \forall a \in Q^{\bar{0}}, \forall w \in W$$

By Equation 6.2.1, we know that $\varphi_{\bar{1}} \colon V \to W$ is a module isomorphism. We can also define

$$\psi(a\underline{b}) = \psi([a, \underline{b}]) = \psi(\underline{[a, b]}) = \underline{\varphi_{\bar{0}}([a, b])} = \underline{[\varphi_{\bar{0}}(a), \varphi_{\bar{0}}(b)]} = [\varphi_{\bar{0}}(a), \underline{\varphi_{\bar{0}}(b)}] = a * \psi(\underline{b}).$$

Now $\psi^{-1} \circ \varphi_{\bar{1}} \colon V \to V$ is an automorphism and, by Schur's Lemma, there is a scalar $\lambda \in \mathbb{F}$ such that $\psi^{-1} \circ \varphi_{\bar{1}} = \lambda\, \mathrm{id}$, concluding the proof. □

6.3 Gradings on $\mathfrak{sl}(n)$

Since we are going to reduce the classification of gradings on Q to the same problem for $Q^{\bar{0}}$, we need to recall some facts about gradings on the Lie algebra $\mathfrak{sl}_n$.

Recall that if a grading Γ on $S = \mathfrak{sl}(n)$ is the restriction of a grading of $R = M_n$, we say that Γ is a *Type I* grading. Otherwise, we say that Γ is a *Type II* grading. If $\mathbb{F}$ is algebraically closed, Type I gradings are characterized by the property that the image of $\eta_\Gamma \colon \widehat{G} \to \mathrm{Aut}(S)$ consists of inner automorphisms of S. Type II gradings are related to Type I gradings in the following way.

Definition 6.3.1 ([2]). If $\Gamma : S = \bigoplus_{g \in G} S_g$ is a G-grading of Type II on S, then there exists a unique element $h \in G$ of order 2 such that the coarsening $\overline{\Gamma}$ induced by the quotient map $G \to \overline{G} = G/\langle h \rangle$ is a $\overline{G}$-grading of Type I (see [12, §3.1]). Moreover, for any $\chi \in \widehat{G}$, the automorphism $\eta_\Gamma(\chi)$ is inner if and only if $\chi(h) = 1$. We call h

the *distinguished element* associated to the grading Γ. For a Type I grading, it is convenient to define $h = e$.

The next lemma will be crucial for describing gradings on Q.

Lemma 6.3.2. *Let Γ be a G-grading on $\mathfrak{sl}_n$ and let h be its distinguished element. If we extend Γ to a grading of $\mathfrak{gl}_n$ by declaring the identity matrix to have degree h then the map*

$$\begin{aligned} J: \quad \mathfrak{gl}_n \otimes \mathfrak{gl}_n &\to \quad \mathfrak{gl}_n \\ x \otimes y \quad &\mapsto \quad xy + yx \end{aligned}$$

is homogeneous of degree h.

Proof. Let Δ be the indicated extension of Γ to the Lie algebra $\mathfrak{gl}_n$. If Γ is a Type I grading then Δ is actually a grading on the associative algebra $R = M_n$ and hence the map J is homogeneous of degree $h = e$.

Now suppose that Γ is a Type II grading and consider its extension $\Delta : R = \bigoplus_{g\in G} R_g$, which is a grading on the Lie algebra $\mathfrak{gl}_n$ but not on the associative algebra M_n. Without loss of generality, we may assume $\mathbb{F}$ is algebraically closed. Let $\overline{G} = G/\langle h \rangle$ and consider the coarsening $\overline{\Delta} : R = \bigoplus_{\bar{g}\in\overline{G}} R_{\bar{g}}$ induced by the quotient map $G \to \overline{G}$, i.e., $R_{\bar{g}} = R_g \oplus R_{gh}$. If $x \in R_a$ and $y \in R_b$ for some $a, b \in G$ then

$$xy \in R_{\overline{ab}} = R_{ab} \oplus R_{abh},$$

since $\overline{\Delta}$ is a grading on the associative algebra R. Hence we can write $xy = z_0 + z_1$ with $z_0 \in R_{ab}$ and $z_1 \in R_{abh}$. Pick a character χ of G such that $\eta_\Gamma(\chi)$ is not an inner automorphism of S, so $\chi(h) \neq 1$. Since h has order 2, $\chi(h) = -1$. Then $-\eta_\Gamma(\chi)$ is the restriction of some anti-automorphism φ of R. We have $\eta_\Delta(\chi)(1_R) = -1_R$ and hence $\varphi = -\eta_\Delta(\chi)$. We compute:

$$\begin{aligned}(-\chi(b)y)(-\chi(a)x) = \varphi(y)\varphi(x) = \varphi(xy) = \varphi(z_0) + \varphi(z_1)\\ = -\chi(ab)(z_0 + \chi(h)z_1).\end{aligned}$$

Hence $yx = -z_0 + z_1$, which implies $xy + yx = 2z_1 \in R_{abh}$, as required. $\square$

6.4 Classification of gradings on $Q(n)$

Now we are going to classify group gradings on the Lie superalgebras $Q = Q(n)$, $n \geq 2$, under the assumption $\operatorname{char}\mathbb{F} \neq 2$ and, in addition, $\operatorname{char}\mathbb{F}$ does not divide $n+1$. It will be convenient to use the following notation.

Let V and W be vector spaces with G-gradings $\Gamma : V = \bigoplus_{g\in G} V_g$ and $\Delta : W = \bigoplus_{g\in G} W_g$. We will denote by $\Gamma \oplus \Delta$ the G-grading on the superspace $V \oplus W$, where V is regarded as the even part and W as the odd part, given by $(V \oplus W)_g = V_g \oplus W_g$.

Recall from Section 6.1 the isomorphism $Q^{\bar{0}} \to Q^{\bar{1}}$ of $Q^{\bar{0}}$-modules, which we denoted by $x \mapsto \underline{x}$. Given a G-grading Γ on the vector space $Q^{\bar{0}}$, we denote by $\underline{\Gamma}$ the image of Γ under this isomorphism, i.e., $\underline{\Gamma}$ is given by $Q^{\bar{1}}_g = \{\underline{x} \mid x \in Q^{\bar{0}}_g\}$ for all $g \in G$.

We are ready to describe all possible G-gradings on Q.

Theorem 6.4.1. *Consider the simple Lie superalgebra $Q = Q(n)$, $n \geq 2$. Let Γ be a G-grading on $Q^{\bar{0}}$, for some abelian group G, and let $h \in G$ be the distinguished element associated to Γ. Then the G-gradings on Q extending Γ are precisely the gradings of the form $\Gamma \oplus \underline{\Gamma}^{[d]}$, where $d \in G$ is such that $d^2 = h$.*

Proof. First we show that if $d^2 = h$ then $\Gamma \oplus \underline{\Gamma}^{[d]}$ is, indeed, a grading on Q, i.e., the product $[,]\colon Q \otimes Q \to Q$ (respectively, $\circ\colon Q \otimes Q \to Q$) is a homogeneous map of degree e with respect to this grading. It is sufficient to consider the restrictions $Q^{\bar 0}\otimes Q^{\bar 0}\to Q^{\bar 0}$, $Q^{\bar 0}\otimes Q^{\bar 1}\to Q^{\bar 1}$ and $Q^{\bar 1}\otimes Q^{\bar 1}\to Q^{\bar 0}$.

The case of $Q^{\bar 0}\otimes Q^{\bar 0}\to Q^{\bar 0}$ is clear since Γ is a grading on the Lie algebra $Q^{\bar 0}$. In the second case, we observe that $Q^{\bar 1}$ equipped with $\underline{\Gamma}$ is a graded $Q^{\bar 0}$-module (by the definition of $\underline{\Gamma}$) and then apply Lemma 2.1.9. For the third case, we will use the realization of Q given by (6.1.1).

According to (6.1.2), the map $[,]\colon Q^{\bar 1}\otimes Q^{\bar 1}\to Q^{\bar 0}$ is the composition of the following four maps:

$$Q^{\bar 1}\otimes Q^{\bar 1}\xrightarrow{\beta} Q^{\bar 0}\otimes Q^{\bar 0}\hookrightarrow \mathfrak{g}\otimes\mathfrak{g}\xrightarrow{J}\mathfrak{g}\xrightarrow{\sharp} Q^{\bar 0},$$

where β is an isomorphism given by $\underline{x}\otimes\underline{y}\mapsto x\otimes y$, $\mathfrak{g}=\mathfrak{gl}(n+1)$ and $J(x\otimes y)=xy+yx$. As in Lemma 6.3.2, we extend the grading Γ to a grading on $\mathfrak{g}$ by declaring the identity matrix to have degree h. Then J is a homogeneous map of degree h. Also, β is homogeneous of degree $d^{-2}=h^{-1}$, while the inclusion $Q^{\bar 0}\hookrightarrow\mathfrak{g}$ and projection $\mathfrak{g}\xrightarrow{\sharp} Q^{\bar 0}$ are homogeneous of degree e. It follows that $[,]\colon Q^{\bar 1}\otimes Q^{\bar 1}\to Q^{\bar 0}$ has degree e, as desired.

It remains to prove that all extensions of the grading Γ on $Q^{\bar 0}$ to the superalgebra Q are of the indicated form. Since $Q^{\bar 1}$ has to be a graded $Q^{\bar 0}$-module, we can apply Proposition 2.1.10 to conclude that the grading on Q must have the form $\Gamma\oplus\underline{\Gamma}^{[d]}$ for some $d\in G$. The above calculation of the degree of the product $Q^{\bar 1}\otimes Q^{\bar 1}\to Q^{\bar 0}$ shows that $d^2=h$. $\square$

Remark 6.4.2. Let $L=L^{\bar 0}\oplus L^{\bar 1}$ be a classical Lie superalgebra over an algebraically closed field of characteristic 0. If L is not isomorphic to $D(2,1,\alpha)$, it is shown in the proof of Proposition 2.1.4 of [16] that, with fixed Lie bracket on $L^{\bar 0}$ and $L^{\bar 0}$-module structure on $L^{\bar{1}}$, the space of symmetric maps $L^{\bar{1}} \otimes L^{\bar{1}} \to L^{\bar{0}}$ that make L a Lie superalgebra has dimension 1. It follows that if $L^{\bar{0}}$ and $L^{\bar{1}}$ are given G-gradings such that $L^{\bar{0}}$ is a graded algebra and $L^{\bar{1}}$ is a graded $L^{\bar{0}}$-module then the product $[,]\colon L^{\bar{1}} \otimes L^{\bar{1}} \to L^{\bar{0}}$ is automatically a homogeneous map of some degree, which we computed to be h in the case of the gradings Γ and $\underline{\Gamma}$ on $Q^{\bar{0}}$ and $Q^{\bar{1}}$, respectively.

Note that if Γ is a Type I grading on $Q^{\bar{0}}$ then we can always extend it to Q: for example, $\Gamma \oplus \underline{\Gamma}$ does the job. But Theorem 6.4.1 also shows that this is not the case for Type II gradings:

Corollary 6.4.3. *If G does not have elements of order 4 then every G-grading on Q restricts to a Type I grading on $Q^{\bar{0}}$.*

Proof. If the grading on $Q^{\bar{0}}$ is of Type II then the distinguished element h has order 2 and hence d must have order 4. □

Now we are going to determine when two gradings on Q are isomorphic.

Theorem 6.4.4. *Consider the simple Lie superalgebra $Q = Q(n)$, $n \geq 2$. Let G be an abelian group, let Γ and Δ be G-gradings on $Q^{\bar{0}}$, and let $c, d \in G$ be such that $\Gamma \oplus \underline{\Gamma}^{[c]}$ and $\Delta \oplus \underline{\Delta}^{[d]}$ are gradings on Q. Then $\Gamma \oplus \underline{\Gamma}^{[c]}$ and $\Delta \oplus \underline{\Delta}^{[d]}$ are isomorphic if and only if Γ and Δ are isomorphic and $c = d$.*

Proof. It will be convenient to give two names to the isomorphism $Q^{\bar{0}} \to Q^{\bar{1}}$, $x \mapsto \underline{x}$, according to what gradings we use. If we consider Γ and $\underline{\Gamma}^{[c]}$ on $Q^{\bar{0}}$ and $Q^{\bar{1}}$, respectively, then we will call it β_c, since it will have degree c. Analogously, if we consider Δ and $\underline{\Delta}^{[d]}$, it will have degree d and we will call it β_d.

Suppose $\varphi\colon Q \to Q$ is an automorphism sending the grading $\Gamma \oplus \underline{\Gamma}^{[c]}$ onto $\Delta \oplus \underline{\Delta}^{[d]}$. By Corollary 6.2.3, we have

$$\varphi_{\bar{1}} \circ \beta_c = \lambda \beta_d \circ \varphi_{\bar{0}} \tag{6.4.1}$$

for some $\lambda \in \mathbb{F}^\times$. By our assumption on φ, both $\varphi_{\bar{0}}$ and $\varphi_{\bar{1}}$ have degree e and, hence, Equation (6.4.1) implies $c = d$.

Conversely, suppose $c = d$ and there exists an automorphism $\psi\colon Q^{\bar{0}} \to Q^{\bar{0}}$ sending Γ onto Δ. By Proposition 6.2.1, there exists an automorphism $\varphi\colon Q \to Q$ such that $\varphi_{\bar{0}} = \psi$. Now Equation (6.4.1) implies that $\varphi_{\bar{1}}$ has degree e, i.e., sends $\underline{\Gamma}^{[c]}$ onto $\underline{\Delta}^{[c]}$. $\square$

For algebraically closed $\mathbb{F}$, the classification of G-gradings on $Q^{\bar{0}} = \mathfrak{sl}(n+1)$ up to isomorphism can be found in [2] or [12, §3.3]. Together with our Theorems 6.4.1 and 6.4.4, this gives a classification of G-gradings on the Lie superalgebra $Q(n)$ up to isomorphism.

6.5 Gradings up to equivalence

The classification of fine gradings up to equivalence on the Lie superalgebra Q is the same as for the Lie algebra $Q^{\bar{0}} = \mathfrak{sl}(n+1)$, which can be found in [11] or [12, §3.3]).

We will need the following notation. Let G be an abelian group and let $h \in G$. We denote by $G[h^{1/2}]$ the abelian group generated by G and a new element d subject only to the relation $d^2 = h$. We will also denote d by $h^{1/2}$.

Theorem 6.5.1. *Consider the simple Lie superalgebra $Q = Q(n)$, $n \geq 2$. If Γ is a fine grading on $Q^{\bar{0}}$ with universal group G then $\tilde{\Gamma} = \Gamma \oplus \underline{\Gamma}^{[h^{1/2}]}$ is a fine grading on*

Q with universal group $G[h^{1/2}]$, and every fine grading on Q has this form (if we use its universal group). Moreover, $\tilde{\Gamma}$ and $\tilde{\Delta}$ are equivalent if and only if Γ and Δ are equivalent.

Proof. If Γ is fine then so is $\tilde{\Gamma}$, because the grading on $Q^{\bar{1}}$ is determined by the grading on $Q^{\bar{0}}$ up to a shift (see Theorem 6.4.1). Note that $\mathrm{supp}(\tilde{\Gamma}) = S_{\bar{0}} \cup S_{\bar{1}}$ (disjoint union) where $S_{\bar{0}} = \mathrm{supp}(\Gamma)$ and $S_{\bar{1}} = S_{\bar{0}} h^{1/2}$.

Let Δ be another fine grading on $Q^{\bar{0}}$. Clearly, if $\tilde{\Gamma}$ and $\tilde{\Delta}$ are equivalent by means of an automorphism $\varphi\colon Q \to Q$ and a bijection $\alpha\colon \mathrm{supp}(\tilde{\Gamma}) \to \mathrm{supp}(\tilde{\Delta})$, then Γ and Δ are equivalent by means of $\varphi_{\bar{0}}$ and $\alpha_{\bar{0}}$ (the restriction of α to $S_{\bar{0}}$). Conversely, suppose Γ and Δ are equivalent. Let G' be the universal group of Δ and let h' be its distinguished element.

Hence there exists an isomorphism $\beta\colon G \to G'$ such that ${}^{\beta}\Gamma$ is isomorphic to Δ. It follows that $\beta(h) = h'$ and, hence, we can extend β to an isomorphism $\tilde{\beta}\colon G[h^{1/2}] \to G'[(h')^{1/2}]$ such that $\tilde{\beta}(h^{1/2}) = (h')^{1/2}$. Then ${}^{\tilde{\beta}}\tilde{\Gamma}$ is isomorphic to $\tilde{\Delta}$ by Theorem 6.4.4, so $\tilde{\Gamma}$ and $\tilde{\Delta}$ are equivalent.

Finally, suppose G' is any abelian group and we have a G'-grading on Q, which, according to Theorem 6.4.1, we can write as $\Gamma' \oplus \underline{\Gamma'}^{[d']}$ for some $d' \in G'$ satisfying $(d')^2 = h'$, where h' is the distinguished element of Γ'. There exists a fine grading Γ on $Q^{\bar{0}}$, with universal group G, and a homomorphism $\beta\colon G \to G'$ such that $\Gamma' = {}^{\beta}\Gamma$. It follows that β sends the distinguished element h of Γ to h' and, hence, we can extend β to a homomorphism $\tilde{\beta}\colon G[h^{1/2}] \to G'$ such that $\tilde{\beta}(h^{1/2}) = d'$. Then we obtain $\Gamma' \oplus \underline{\Gamma'}^{[d']} = {}^{\tilde{\beta}}\tilde{\Gamma}$ by definition of $\tilde{\Gamma}$. Since we started with an arbitrary group grading on Q, it follows that every fine grading on Q has the form $\tilde{\Gamma}$, for some fine grading Γ

on $Q^{\bar{0}}$, and, moreover, $G[h^{1/2}]$ is the universal group of $\tilde{\Gamma}$. $\square$

To determine the structure of the universal groups of fine gradings on Q, note that, since $h^2 = e$, the group $G[h^{1/2}]$ is isomorphic to $G \times \mathbb{Z}_2$ if h is a square in G and $\overline{G} \times \mathbb{Z}_4$ otherwise, where $\overline{G} = G/\langle h \rangle$. If Γ is of Type I, we have $h = e$, so $\overline{G} = G$ and $G[h^{1/2}] \cong G \times \mathbb{Z}_2$. If Γ is of Type II, the group G is computed in [12, §3.3] (for algebraically closed $\mathbb{F}$) in terms of the extension $\langle h \rangle \to G \to \overline{G}$, which splits if and only if h is not a square in G. Since the orders of torsion elements of G are divisors of 4, it follows that $G[h^{1/2}] \cong \overline{G} \times \mathbb{Z}_4$. To summarize,

$$\text{the universal group of } \tilde{\Gamma} \cong \begin{cases} G \times \mathbb{Z}_2 & \text{if } \Gamma \text{ is of Type I;} \\ \overline{G} \times \mathbb{Z}_4 & \text{if } \Gamma \text{ is of Type II.} \end{cases}$$

For Type I, the group G is the universal group of the corresponding fine grading on the associative algebra R, where $R = M_{n+1}$, $n \geq 2$; it is given in [12, §2.3] for all fine gradings. For Type II, $\overline{G}$ is the universal group of the grading on R corresponding to the Type I coarsening induced by the quotient map $G \to \overline{G}$; it is computed in [12, §3.2].

6.6 An example of a grading on $Q(2)$

Example 6.6.1. Let us take $G = \mathbb{Z}_2$ and $n = 2$.

Recall that $Q(2) = \frac{\tilde{Q}(2)}{1\mathbb{F}}$ where $\tilde{Q}(2) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in sl(3,3) \mid \operatorname{tr}(b) = 0 \right\}$.

We know that $Q(2)^{\bar{0}} \simeq \mathfrak{sl}(3)$. The $\mathbb{Z}_2$-gradings on $\mathfrak{sl}(3)$ are the following:

- 1 Type I grading:

$$_{(\mathrm{I})}(\mathfrak{sl}_3)_{\bar{0}} = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in \mathfrak{sl}_3 \,|\, x \in \mathbb{F},\, y \in M_2(\mathbb{F}) \right\}$$

$$_{(\mathrm{I})}(\mathfrak{sl}_3)_{\bar{1}} = \left\{ \begin{pmatrix} 0 & v \\ u & 0 \end{pmatrix} \in \mathfrak{sl}_3 \,|\, v \in M_{1\times 2}(\mathbb{F}),\, u \in M_{2\times 1}(\mathbb{F}) \right\}$$

- 1 Type II grading:

$$_{(\mathrm{II})}(\mathfrak{sl}_3)_{\bar{0}} = \left\{ M \in \mathfrak{sl}_3 \,|\, M = -M^t \right\}$$

$$_{(\mathrm{II})}(\mathfrak{sl}_3)_{\bar{1}} = \left\{ M \in \mathfrak{sl}_3 \,|\, M = M^t \right\}$$

Since G does not have elements of order 4, the Type II grading of $\mathfrak{sl}_3$ does not extend to $Q(2)$. But we have the following two Type I gradings on $Q(2)$:

- $\Gamma^{\bar{0}} \oplus \Gamma^{\bar{1}}$, where $\Gamma^{\bar{0}}$ is the above Type I grading $\mathfrak{sl}_3 = \left({}_{(\mathrm{I})}(\mathfrak{sl}_3)_{\bar{0}}\right) \oplus \left({}_{(\mathrm{I})}(\mathfrak{sl}_3)_{\bar{1}}\right)$ and $\Gamma^{\bar{1}} = \underline{\Gamma^{\bar{0}}}$; and

- $\Delta^{\bar{0}} \oplus \Delta^{\bar{1}}$ where $\Delta^{\bar{0}} = \Gamma^{\bar{0}}$ and $\Delta^{\bar{1}} = {}^{[\bar{1}]}\Gamma^{\bar{1}}$.

Explicitly, the first grading is given by

$$Q^{\bar{0}}_{\bar{0}} = \left\{ \left(\begin{array}{cc|cc} x & 0 & & 0 \\ 0 & y & & \\ \hline & & x & 0 \\ 0 & & 0 & y \end{array}\right) \in M(3,3) \,|\, x \in \mathbb{F},\, y \in M_2(\mathbb{F}) \right\}$$

$$Q^{\bar 1}_{\bar 0} = \left\{ \left(\begin{array}{cc|cc} 0 & & x & 0 \\ & & 0 & y \\ \hline x & 0 & & \\ 0 & y & & 0 \end{array} \right) \in M(3,3) \,|\, x \in \mathbb{F},\, y \in M_2(\mathbb{F}) \right\}$$

$$Q^{\bar 0}_{\bar 1} = \left\{ \left(\begin{array}{cc|cc} 0 & v & & 0 \\ u & 0 & & \\ \hline & & 0 & v \\ 0 & & u & 0 \end{array} \right) \in M(3,3) \,|\, v \in M_{1\times 2}(\mathbb{F}),\, u \in M_{2\times 1}(\mathbb{F}) \right\}$$

$$Q^{\bar 1}_{\bar 1} = \left\{ \left(\begin{array}{cc|cc} 0 & & 0 & v \\ & & u & 0 \\ \hline 0 & v & & \\ u & 0 & & 0 \end{array} \right) \in M(3,3) \,|\, v \in M_{1\times 2}(\mathbb{F}),\, u \in M_{2\times 1}(\mathbb{F}) \right\}$$

and the second grading is the same, but with $Q^{\bar 1}_{\bar 0}$ and $Q^{\bar 1}_{\bar 1}$ interchanged.

There are $2\lfloor \frac{n}{2} \rfloor$ $\mathbb{Z}_2$-gradings on $Q(n)$. Indeed, for each $1 \le k \le \lfloor \frac{n+1}{2} \rfloor$, we have one (elementary) $\mathbb{Z}_2$-grading on $\mathfrak{sl}_{n+1}$, which gives rise to two Type I $\mathbb{Z}_2$-gradings on $Q(n)$.

There are no $\mathbb{Z}_2$-gradings of Type II in $Q(n)$ because $\mathbb{Z}_2$ has no elements of order 4. However, if we change the group to be $\mathbb{Z}_4$, we have examples of Type II gradings on $Q(n)$.

Example 6.6.2.

- $\Gamma^{\bar 0} \oplus \Gamma^{\bar 1}$, where $\Gamma^{\bar 0}$ is the above Type II grading on $\mathfrak{sl}_3$ by the copy of $\mathbb{Z}_2$ contained in $\mathbb{Z}_4$: $Q^{\bar 0}_{\bar 0} = {}_{(\mathrm{II})}(\mathfrak{sl}_3)_{\bar 0}$, $Q^{\bar 0}_{\bar 2} = {}_{(\mathrm{II})}(\mathfrak{sl}_3)_{\bar 1}$ and $\Gamma^{\bar 1} = \underline{\Gamma^{\bar 0}}$; and
- $\Delta^{\bar 0} \oplus \Delta^{\bar 1}$ such that $\Delta^{\bar 0} = \Gamma^{\bar 0}$ and $\Delta^{\bar 1} =^{[2]} \Gamma^{\bar 1} =^{[2]} \underline{\Gamma^{\bar 0}}$.

Explicitly:

$$Q_{0}^{\bar{0}} = \left\{ \left(\begin{array}{c|c} M & 0 \\ \hline 0 & M \end{array} \right) \in M(3,3) \,|\, M = -M^t \right\}$$

$$Q_{1}^{\bar{1}} = \left\{ \left(\begin{array}{c|c} 0 & M \\ \hline M & 0 \end{array} \right) \in M(3,3) \,|\, M = -M^t \right\}$$

$$Q_{2}^{\bar{0}} = \left\{ \left(\begin{array}{c|c} M & 0 \\ \hline 0 & M \end{array} \right) \in M(3,3) \,|\, M = M^t \right\}$$

$$Q_{3}^{\bar{1}} = \left\{ \left(\begin{array}{c|c} 0 & M \\ \hline M & 0 \end{array} \right) \in M(3,3) \,|\, M = M^t \right\}$$

and the second grading is the same, but with $Q_{\bar{1}}^{\bar{1}}$ and $Q_{\bar{3}}^{\bar{1}}$ interchanged.

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