Time Delayed Models in Population Biology and Epidemiology

by

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Abstract

In this dissertation, we focus on the development and analysis of time-delayed mathematical models to represent real world applications in biology and epidemiology, especially, population growth and disease spread. Throughout five projects, we establish then analyze the models using various theorems and methods in the literature, such as, the comparison principle and the method of fluctuations, to study qualitative features of the models including existence and uniqueness of solutions, boundedness, steady states, persistence, local, and global stability, with respect to the adult/basic reproduction number $\mathcal{R}_A/\mathcal{R}_0$, which is a key threshold parameter.

Firstly, we discuss ecological models in Chapters 2–4. In Chapter 2, we derive a single species–fish model with three stages: juveniles, small adults and large adults with two harvesting strategies depending on the size and maturity. We study the population extinction and persistence with respect to $\mathcal{R}_A$ and find that the over-harvesting of large matured fish after a certain age can lead to population extinction under certain circumstances. Numerically, we investigate the influence of harvesting functions and discuss the optimal harvesting rates. In Chapter 3, we develop a model for the growth of sea lice with three stages such that the development age for non-infectious larvae to develop into infectious larvae relates to the size of adult population size. As a beginning, we describe the nonlinear dynamics by a system of partial differential equations, then, we transformed it into a system of delay differential equations with constant delay by using the method of characteristics and an appropriate change of variables. We address the system threshold dynamics for the established model with respect to the adult reproduction number, including the global stability of the trivial steady state, persistence, and global attractivity of a coexistence unique positive steady state. As a case study, we provide some numerical simulation results using *Lepeophtheirus salmonis* growth parameters. To explore the biological control of sea lice using one of their predators, "cleaner fish", we propose a model with predator-prey interaction at the adult level of sea lice in Chapter 4. Mathematically, we address threshold dynamics with respect to the adult reproduction number for sea lice $\mathcal{R}_s$ and the net reproductive number of cleaner fish $\mathcal{R}_f$, including the global stability of the trivial steady state when $\mathcal{R}_s < 1$, global attractivity of the predator-free equilibrium point when $\mathcal{R}_s > 1$ and $\mathcal{R}_f < 1$, persistence and coexistence of a unique positive steady state when $\mathcal{R}_s > 1$ and $\mathcal{R}_f > 1$. Furthermore, we discuss the local stability

of the positive equilibrium point and investigate the Hopf bifurcation. Numerically, we compare between two cleaner fish species, goldsinny and ballan wrasse, as a case study.

For epidemiological models, in Chapter 5, we propose an SEIRD model for Ebola disease transmission that incorporates both the transmission of infection between the living humans and from the infected corpses to the living individuals, with a constant latent period. Through mathematical analysis, we prove the globally stability of the disease-free and a unique endemic equilibria with respect to $\mathcal{R}_0$. Moreover, we find that the long latent period or low transmission rate from infectious corpses may reduce the spread of Ebola. In Chapters 6, we consider the influence of seasonal fluctuations on disease transmission and develop a periodic infectious disease model where asymptomatic carriers are potential sources for disease transmission. We consider a general nonlinear incidence rate function with the asymptomatic carriage and latent periods. We implement a case study regarding the meningococcal meningitis disease transmission in Dori, Burkina Faso. Our numerical simulation indicates an irregular pattern of epidemics varying in size and duration, which is consistent with the reported data in Burkina Faso from 1940 to 2014.

In summery, in population growth models, we find that the basic reproduction ratio depends on maturation time, indicating that this key parameter can play an important role in population extinction and persistence. In disease transmission model, we understand that latent period can play a positive role in eliminating or slowing a disease spread.

To my parents Mattar and Nahla

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Statement of contributions

This dissertation is my joint work under the supervision of Professor Yuan Yuan. It combines the following five papers:

i. I. Al-Darabsah and Y. Yuan, *A Stage-Structured Model For Fish Stock with Harvesting*, SIAM Journal on Applied Mathematics, , 78 (2018), pp. 145-170, doi:10.1137/16M1097092.

ii. Y. Tian, I. Al-Darabsah[1] and Y. Yuan, *Nonlinear Dynamics in Modeling Sea Lice with Stage Structure*, Nonlinear Analysis: Real World Applications, 44 (2018), pp. 283-304, doi:10.1016/j.nonrwa.2018.05.007.

iii. I. Al-Darabsah and Y. Yuan, *A Stage-Structured Model for the Biocontrol of Sea Lice Using Cleaner Fish*, under review.

iv. I. Al-Darabsah and Y. Yuan, *A Time-Delayed Epidemic Model for Ebola Disease Transmission*, Applied Mathematics and Computation, 290 (2016), pp. 307-325, doi:10.1016/j.amc.2016.05.043.

v. I. Al-Darabsah and Y. Yuan, *A Periodic Disease Transmission Model with Asymptomatic Carriage and Latency Periods*, Journal of Mathematical Biology (2017), doi:10.1007/s00285-017-1199-1.

[1]The first two authors are co-first authors and contributed equally to this work.

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Chapter 1

Introduction and Preliminaries

Mathematical models are widely used in the natural sciences (such as physics and biology), engineering disciplines, and social sciences (such as economics, sociology and political science). Eykhoff (1974) defined a mathematical model as "a representation of the essential aspects of an existing system (or a system to be constructed) which presents knowledge of that system in usable form" [195]. Mathematical modelling provides an explanation and prediction of the behavior of ecosystems and helps to obtain new theoretical knowledge of nature, and can play an important role in helping to quantify possible disease control strategies by focusing on the important aspects of a disease. Mathematical models can take many forms, including differential equations, statistical models, game theory models, and so on.

Delay differential equations (DDEs) have become popular in ecological and epidemiological models, such as the study of age-structured populations growth and epidemics. In nature, individuals in a population differ from each other in age, life cycle stage, and other physiological characteristics. Sometimes, these differences play critical roles in population dynamics [70]. In the literature, there has been a fair amount of work on modeling age-structured populations using DDEs in various settings [7–9, 11, 26, 56, 71, 102, 122, 137, 154, 164, 178]. For example, in [7–9, 71, 178], the authors considered a fixed time delays in their models. Distributed maturation delays were assumed in [11, 26, 56]. In [122, 154], the authors used time-dependent delays in the proposed models. The authors in [102, 137, 164] have posed models in which maturation is determined by the consumption of a threshold amount of resource.

Mathematical epidemiology has a long history, going back to the smallpox model of Daniel Bernoulli in 1760. Much of the basic theory was developed between 1900 and

1935, and there has been steady progress since that time. More recently, models have been used to assist in the evaluation of the effect of control measures and formulation of policy decisions. Mathematical models for the spread of an infectious disease in a population are depending on many factors: patterns of contacts among infectious and susceptible individuals; the latency period from being infected to becoming infectious; the duration of infectiousness; and the immunity period [15, 89]. Usually, DDEs are used to model such periods in disease transmission mechanisms. For example, in [120, 159, 200, 201], the authors studied Malaria disease transmission. The spread of Lyme disease has bean discussed in [57, 199]. Further, in the literature, many authors proposed general compartmental models with time-delays. For instance, in [212], a disease transmission model of SEIRS type with distributed delays in latent and temporary immune periods is discussed. The authors found that distributed delays can change the dynamics and destabilize the endemic steady state.

In the following, we introduce some terminologies and theories related to delay differential equations. Then we present results for basic reproduction ratios for periodic compartmental models with time delay.

1.1 Brief review of delay differential equations

For a given $t_0 \in J \subset \mathbb{R}$ and $\tau > 0$, let $C = C([t_0 - \tau, t_0], \mathbb{R}^n)$ and $C^+ = C([t_0 - \tau, t_0], \mathbb{R}^n_+)$. Then (C, C^+) is an ordered Banach space equipped with the maximum norm $\|\phi\| = \sum_{i=1}^n \|\phi_i\|_\infty$ where

$$\|\phi_i\|_\infty = \max_{\theta \in [t_0 - \tau, t_0]} |\phi_i(\theta)|$$

for any $\phi = (\phi_1, \ldots, \phi_n) \in C$. For any given continuous function $u(t) \in C([t_0 - \tau, \zeta], \mathbb{R}^n)$ with $\zeta > t_0$, we define $u_t \in C$ for each $t \geq t_0$ by $u_t(\theta) = u(t+\theta)$, $\theta \in [t_0 - \tau, t_0]$. For $D \subset \mathbb{R}^n$ let $C_D = C([t_0 - \tau, t_0], D)$ denote the set of continuous functions mapping $[t_0 - \tau, t_0]$ into D.

A general type of DDE is in the following form

$$x'(t) = f(t, x_t). \tag{1.1}$$

With n discrete delays, $\tau_1, \ldots, \tau_n$, (1.1) can be written as

$$x'(t) = f(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_n)). \tag{1.2}$$

In this case, the above τ is $\max\{\tau_1, \ldots, \tau_n\}$.

For a given $\phi_0 := \phi_0(t) \in C_D$, the initial value problem associated with (1.1) is

$$\begin{aligned} x'(t) &= f(t, x_t), \quad t > t_0 \\ x_{t_0} &= \phi_0, \end{aligned} \tag{1.3}$$

Definition 1.1.1. *A function $x(t)$ is said to be a solution of (1.3) on $[t_0 - \tau, \beta]$ if there are t_0 and β such that $x \in C([t_0 - \tau, \beta], D)$, $[t_0 - \tau, \beta] \subset \mathbb{R}$, and $x(t)$ satisfies (1.3) on $[t_0 - \tau, \beta]$.*

Theorem 1.1.1. *[74, Lemma 2.1.1] Finding a solution of the IVP (1.3) is equivalent to solving the integral equation*

$$\begin{aligned} x(t) &= \phi_0(t_0) + \int_{t_0}^{t} f(s, x_s)ds, \quad t_0 \le t \le \beta, \\ x_{t_0} &= \phi_0. \end{aligned}$$

Definition 1.1.2. *Let $f : J \times C_D \to \mathbb{R}^n$ and let $S \subset J \times C_D$. Then f is Lipschitz on S if there exists a constant $L \ge 0$ such that*

$$|f(t, \phi) - f(t, \psi)| \le L\|\phi - \psi\|$$

whenever (t, ϕ), $(t, \psi) \in S$.

Theorem 1.1.2. *(Local Existence) [74, Theorem 2.2.1] Assume $J \times C_D$ is an open set in $\mathbb{R} \times C$, and $f : J \times C_D \to \mathbb{R}^n$ is continuous on its domain. If $(t_0, \phi_0) \in J \times C_D$, then there exists a solution of the initial value problem (1.3) passing through (t_0, ϕ_0) on $[t_0 - \tau, t_0 + \delta)$ for some $\delta > 0$.*

Theorem 1.1.3. *(Uniqueness) [74, Theorem 2.2.2] Suppose $J \times C_D$ is an open set in $\mathbb{R} \times C$, and $f : J \times C_D \to \mathbb{R}^n$ is continuous and Lipschitz on each compact set of $J \times C_D$. Then there exists a unique solution of the initial value problem (1.3) passing through (t_0, ϕ_0) for any $(t_0, \phi_0) \in J \times C_D$.*

The interval $[t_0-\tau, t_0+\delta)$ described in Theorem 1.1.2 is called the maximal interval of existence of the solution of (1.3).

Theorem 1.1.4. *[169, Theorem 5.2.1] Let C_D is open. Assume that whenever $\phi \in C_D$ satisfies $\phi \geq 0$, $\phi_i(0) = 0$ for some i and $t \in \mathbb{R}$, then $f_i(t, \phi) \geq 0$. If $\phi \in C_D$ satisfies $\phi \geq 0$ and $t_0 \in \mathbb{R}$, then the solution $x(t) \geq 0$, through (t_0, ϕ), for all $t > t_0$ in its maximal interval of existence.*

Definition 1.1.3. *f is said to be quasimonotone if for any $\phi \leq \psi$ with $\phi_i(t_)) = \psi_i(t_))$ for some i, then $f_i(t, \phi) \leq f_i(t, \psi)$ for each t.*

Theorem 1.1.5. *[169, Theorem 5.1.1] Let $f, g : J \times C_D \to \mathbb{R}^n$ be continuous and Lipschitz on each compact set of $J \times C_D$, and assume that either f or g is quasimonotone. Assume also that $f(t, \phi) \leq g(t, \phi)$ for all $(t, \phi) \in J \times C_D$. If (t_0, ϕ), $(t_0, \psi) \in J \times C_D$ satisfy $\phi \leq \psi$, then*

$$x(t, t_0, \phi, f) \leq x(t, t_0, \psi, g)$$

holds for all $t \geq t_0$ for which both are defined.

Assume (1.1) is autonomous system, that is,

$$x'(t) = f(x_t). \tag{1.4}$$

Then an equilibrium point of (1.4) is a constant solution $x(t) = x^*$ that satisfies $f(x^*) = 0$.

Definition 1.1.4. *Assume x^* is an equilibrium point of (1.4). Then*

i. x^* is said to be stable if for any $\epsilon > 0$, there is a $\delta := \delta(\epsilon)$ such that $\|\phi_0 - x^*\| < \delta$ for $t \in [t_0 - \tau, t_0]$ implies $|x(t, \phi_0) - x^*| < \epsilon$ for all $t \geq t_0$. Otherwise, it is called unstable.

ii. x^* is said to be asymptotically stable if it is stable and x^* is attractive, that is, there is η such that $\|\phi_0 - x^*\| < \eta$ for $t \in [t_0 - \tau, t_0]$ implies $\lim_{t\to\infty} |x(t, \phi_0) - x^*| = 0$.

iii. x^* is said globally asymptotically stable if it is stable and $\lim_{t\to\infty} |x(t, \phi_0) - x^*| = 0$ for all $\phi_0 \in C_D$

The general form of a linear autonomous system is

$$x'(t) = L(x_t) \tag{1.5}$$

where $L : C \to \mathbb{R}^n$ is continuous and linear. By the Riesz Representation Theorem, the linear operator L can be expressed in an integral form, in the sense that, there exists an $n \times n$ matrix-valued function

$$\eta(\cdot) : (-\infty, 0] \to \mathbb{R}^n$$

whose components are bounded variation such that

$$L(\phi) = \int_{-\infty}^{0} \phi(\theta)\, d\eta(\theta) \quad \text{for any} \quad \phi \in C.$$

The characteristic equation corresponding to the linear autonomous system (1.5) is

$$\Delta(\lambda) = \det\left(\lambda I_n - \int_{-\infty}^{0} e^{\lambda\theta} d\eta(\theta)\right) = 0 \tag{1.6}$$

where I_n is the $n \times n$ identity matrix [176].

The local stability of the equilibrium point x^* of (1.4) can be determined by the stability of zero equilibrium of the corresponding linearized system, by setting $u = x - x^*$.

For linear autonomous DDEs system with discrete delays, $\tau_1, \ldots, \tau_n$,

$$x'(t) = Ax(t) + \sum_{i=1}^{n} B_i x(t - \tau_i),$$

the characteristic equation (1.6) becomes

$$\Delta(\lambda) = \det\left(\lambda I_n - A - \sum_{i=1}^{n} B_i e^{-\lambda\tau i}\right) = 0.$$

1.2 Basic reproduction ratios

In infectious disease models, a very important threshold quantity for disease survival is the basic reproduction ratio (number), denoted by $\mathcal{R}_0$. In epidemiology, $\mathcal{R}_0$ is the expected number of secondary cases produced by one infected individual introduced into a population of susceptible individuals [51, 193, 217]. In general, when $\mathcal{R}_0$ is introduced for an epidemic model, the disease cannot invade if $\mathcal{R}_0 < 1$ and can invade if $\mathcal{R}_0 > 1$. In this section, we present the theory of basic reproduction ratios for periodic compartmental models with time delay developed in [217].

Let $t_0 = 0$ and consider a linear and periodic functional differential system

$$u'(t) = \mathcal{F}(t)u_t - \mathcal{V}(t)u(t), \qquad t \geq 0, \tag{1.7}$$

where $\mathcal{F}(t) : C \to \mathbb{R}^n$ is a map and $\mathcal{V}(t)$ is a continuous $n \times n$ matrix function on $\mathbb{R}$, and both $\mathcal{F}(t)$ and $\mathcal{V}(t)$ are $T-$periodic in t for some real number $T > 0$. In (1.7), $\mathcal{F}(t)u_t$ describes the newly infected individuals at time t evolving over the time interval $[t - \tau, t]$ and the internal evolution of individuals in the infectious compartments (e.g., natural and disease-induced deaths, and movements among compartments) is governed by the linear ordinary differential system:

$$\frac{du}{dt} = -\mathcal{V}(t)u(t). \tag{1.8}$$

Let $Z(t, s)$ be the evolution operator associated with the system (1.8), that is, $Z(t, s)$ satisfies

$$\frac{\partial}{\partial t}Z(t, s) = -\mathcal{V}(t)Z(t, s), \ \forall t \geq s, \text{ and } Z(s, s) = I, \ \forall s \in \mathbb{R}$$

and $\omega(Z)$ be the exponential growth bound of $Z(t, s)$, that is,

$$\omega(Z) = \inf\left\{\tilde{\omega} : \exists M \geq 1 \text{ such that } \|Z(t + s, s)\| \leq Me^{\tilde{\omega}t}, \forall s \in \mathbb{R}, t \geq 0\right\}.$$

It is clear that, for any $t \geq s$ and $s \in \mathbb{R}$

$$Z(t, s) = e^{-\int_s^t \mathcal{V}(\eta)d\eta}.$$

Assume that

(H1) Each operator $\mathcal{F}(t)$ is positive in the sense that $\mathcal{F}(t)C^+ \subseteq \mathbb{R}^n_+$.

(H2) Each matrix $-\mathcal{V}(t)$ is cooperative, and $\omega(Z) < 0$.

Let C_T be the ordered Banach space of all $T-$periodic functions from $\mathbb{R}$ to $\mathbb{R}^n$, which is equipped with the maximum norm and the positive cone $C_T^+ = \{v \in C_T : v(t) \geq 0, \forall t \in \mathbb{R}\}$. Then the basic reproduction ratio $\mathcal{R}_0$ is the spectral radius of the linear operator on C_T

$$[Lv](t) = \int_0^\infty Z(t, t-s)\mathcal{F}(t-s)v(t-s+\cdot)ds, \ v \in C_T,$$

that is, $\mathcal{R}_0 = \rho(L)$.

Let $P(t)$ be the solution maps of the linear system (1.7) on C, that is, $P(t)\phi = \nu(t, \phi)$, $t \geq 0$, where $\nu(t, \phi)$ is the unique solution of (1.7) satisfying $\nu_0 = \phi \in C$. Then $P := P(T)$ is the Poincaré (period) map associated with the system (1.7). The following result indicates the instability and local stability of the zero solution for periodic system (1.7).

Theorem 1.2.1. *[217, Theorem 2.1] $\mathcal{R}_0 - 1$ has the same sign as $\rho(P) - 1$. Thus, the zero solution in (1.7) is locally asymptotically stable if $\mathcal{R}_0 < 1$, and unstable if $\mathcal{R}_0 > 1$.*

When (1.7) is autonomous system

$$u'(t) = \hat{\mathcal{F}}u_t - \mathcal{V}u(t), \tag{1.9}$$

the basic reproduction ratio $\mathcal{R}_0$ is the spectral radius of the matrix $\hat{\mathcal{F}}\mathcal{V}^{-1}$ [217, Corollary 2.1].

1.3 Uniform persistence

In this section, we present a general persistence theory for infinite dimensional systems due to Hale and Waltman [75] and Smith and Zhao [171].

1.3.1 Hale and Waltman theory

Let X be a complete metric space with metric d and suppose that $T(t) : X \to X$, $t \geq 0$, satisfies

$$T(0) = I \text{ and } T(t+s) = T(t)T(s) \text{ for } t, s \geq 0,$$

and $T(t)x$ is continuous in t and x.

The distance $d(x, Y)$ of a point $x \in X$ from a subset Y of X is defined by

$$d(x, Y) = \inf_{y \in Y} d(x, y).$$

Recall that the positive orbit $\gamma^+(x)$ through x is defined as $\gamma^+(x) = \bigcup_{t \geq 0} \{T(t)x\}$, and its omega limit set is $\omega(x) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} \{T(t)x\}}$. Define $W^s(M)$ the stable set of a compact invariant set M as

$$W^s(M) = \{x : x \in X, \omega(x) \neq \emptyset, \omega(x) \subset M\}$$

Suppose that X^0 is an open and dense set in X with $X^0 \cup X_0 = X$ and $X^0 \cap X_0 = \emptyset$ and $T(t)$ satisfies

$$T(t) : X^0 \longrightarrow X^0 \text{ and } T(t) : X_0 \longrightarrow X_0. \tag{1.10}$$

Theorem 1.3.1. [75, Theorem 4.1] *Suppose $T(t)$ satisfies (1.10) and we have the following:*

(i) *There is a t_0 such that $T(t)$ is compact for $t > t_0$;*

(ii) *$T(t)$ is point dissipative in X;*

(iii) *$A = \bigcup_{x \in A_\partial} \omega(x)$ is isolated and has an acyclic covering $M = \bigcup_{i=1}^{k} M_k$, where A_∂ is the global attractor of $T(t)$ restricted to X_0 and $\omega(x)$ is the omega-1limit set;*

(iv) *For each $M_i \in M$, $W^s(M_i) \cap X^0 = \emptyset$ where W^s is the stable manifold of M_i.*

Then $T(t)$ is uniformly persistent with respect to X^0, that is, there is an $\eta > 0$ such that for any $x \in X^0$,

$$\liminf_{t \to \infty} d\left(T(t)x, X_0\right) \geq \eta.$$

For the case in which $T(t)$ is only asymptotically smooth, the following theorem can be viewed as a corollary of the preceding theorem.

Theorem 1.3.2. [75, Theorem 4.2] *Suppose $T(t)$ satisfies (1.10) and we have the following:*

(i) *$T(t)$ is point dissipative in X;*

(ii) *$\gamma^+(U)$ is bounded in X if U is bounded in X where $\gamma^+(x)$ is the positive orbit through x;*

(iii) *$T(t)$ is asymptotically smooth;*

(iv) *$A = \bigcup_{x\in A_\partial} \omega(x)$ is isolated and has an acyclic covering $M = \bigcup_{i=1}^{k} M_k$, where A_∂ is the global attractor of $T(t)$ restricted to X_0 and $\omega(x)$ is the omega-1limit set;*

(v) *For each $M_i \in M$, $W^s(M_i) \cap X^0 = \emptyset$ where W^s is the stable manifold of M_i.*

Then the conclusion of Theorem 1.3.1 is valid.

1.3.2 Smith and Zhao theory

Suppose X be a complete metric space with metric d and $X_0 \subset X$ an open set.. Let $f : X \longrightarrow X$ be a continuous map. Define $\partial X_0 = X\backslash X_0$ and $M_\partial = \{x \in \partial X_0 : f^n(x) \in \partial X_0,\ \forall n \geq 0\}$, which may be empty.

Let Φ is a semiflow on X. A generalized distance function p for Φ is a continuous function $p : X \to [0, \infty)$ satisfying:

$$p(\Phi_t(x)) > 0 \text{ for } t > 0 \text{ if either } p(x) = 0 \text{ and } x \in X_0 \text{ or if } p(x) > 0.$$

Φ is said to be uniformly persistent with respect to $(X_0, \partial X_0, p)$ if there exists $\eta > 0$ such that

$$\inf_{t\to\infty} \lim p\left(\Phi_t(x)\right) \geq \eta$$

for all $x \in X_0$.

Theorem 1.3.3. *[171, Theorem 3] Let p be a generalized distance function for the semiflow $\Phi(t) : X \to X$ with $\Phi(t)X_0 \subset X_0$ for all $t \geq 0$. Assume that*

(i) $\Phi(t) : X \to X$ has a global attractor A;

(ii) There exists a finite sequence $M = \{M_1, \ldots, M_k\}$ of disjoint, compact, and isolated invariant sets in ∂X_0 with the following properties:

(a) $\bigcup_{x\in M_\partial} \omega(x) \subset \bigcup_{i=1}^{k} M_i$;

(b) No subset of M forms a cycle in ∂X_0;

(c) Each M_i is isolated in X;

(d) $W^s(M_i) \cap p^{-1}(0,\infty) = \emptyset$ for each $1 \leq i \leq k$.

Then there exists $\eta > 0$ such that

$$\inf_{t\to\infty} \lim p\left(\Phi_t(x)\right) \geq \eta$$

for all $x \in X_0$.

The aim of this thesis is to develop and analyze time-delayed mathematical models arising from real-world-applications related to the natural time-lags on the population growth and disease spread. More specifically, modeling age-structured population, and infectious disease at the population level, to understand population dynamics. The rest of this dissertation is organized as follows. In Chapter 2, we develop a model for a single species-fish with three stages: juveniles, small adults and large adults, and two harvesting strategies depend on size and maturity. Chapter 3 is devoted to the study the growth process and the dynamical behavior of sea lice population. In Chapter 4, we study the growth of sea lice while one of their predators, "cleaner fish", exists by proposing a model with predator-prey interaction at the adult level of sea lice. In Chapter 5, we propose a model that incorporates both the transmission of infection between the living humans and from the infected corpses to the living individuals, with a constant latent period of Ebola. In Chapter 6, we analyze an infectious disease model under the influence of seasonal fluctuations (all parameters are time-dependent) with a general nonlinear incidence rate function and consideration of the asymptomatic carriage and latent periods. A brief summary and some future works are presented in Chapter 7

Chapter 2

A Stage-Structured Mathematical Model for Fish Stock with Harvesting

2.1 Introduction

Humans have harvested fish from the ocean for thousands of years, and there has been a dramatic increase in the amount of fish taken from the sea since 1960. In 2006, a team of researchers in a study based on an analysis of regional and global fisheries data over the past 50 years, foresaw that by the year 2048 commercial fishers will have almost nothing left to catch [97, 129]. Keeping a sustainable fisheries and sustainable flow are the fundamental goals of most fisheries management. One of the key factors to manage and preserve genetic diversity of fish stocks, where a group of fish of the same species live in the same geographic area and mix enough to breed with each other when mature [97, 109], is optimal harvesting of fish resources which maximizes the average catch of fish and does not reduce stock's abundance over time. Harvesting, one of the most important parts of fish farming, is often underequipped and poorly planned, and hence leads to the destruction of the resource. To sustain catches and abundance levels, management of fish populations can be based on several alternative means of strategic catch regulation.

Stock harvesting strategies are the plans for adjusting management options in relation to the status of the fish stock. The two most common harvest strategies are maturity selectivity and size preference because catching juveniles is not allowed per

fishing regulations [49, 50]. Many of the world's fisheries use size-selective harvesting, because of gear design, targeting incentives of fishers, or management regulations. For instance, size-selectivity has been implemented to harvest British Columbia pink salmon (Oncorhynchus gorbuscha) since 1950 [127]. Also gillnets were used for catching striped bass in Maryland during the 1950s [124]. On the other hand, maturity selectivity occurs when fisheries target a stock spawning grounds since juveniles and adults are spatially segregated during spawning [107, 145]. Spawning grounds are the areas of water where female fish lay eggs and male fish spread sperm over the eggs. Using maturity selectivity, it is possible to target both small and large matured fish and avoid juveniles due to the segregation. An illustration of maturity selectivity is given in Figure 2.1. For example, in the last 50 years the northeast Arctic cod have been caught around spawning grounds at the Lofoten archipelago and northwards [62, 141]. Maturity-selective harvesting had been used to catch the Norwegian spring-spawning herring (Clupea harengus) at their spawning grounds on the west coast of Norway during the 20th century [54, 55]. In general, harvesting based on maturity is less common than size selectivity in world fisheries.

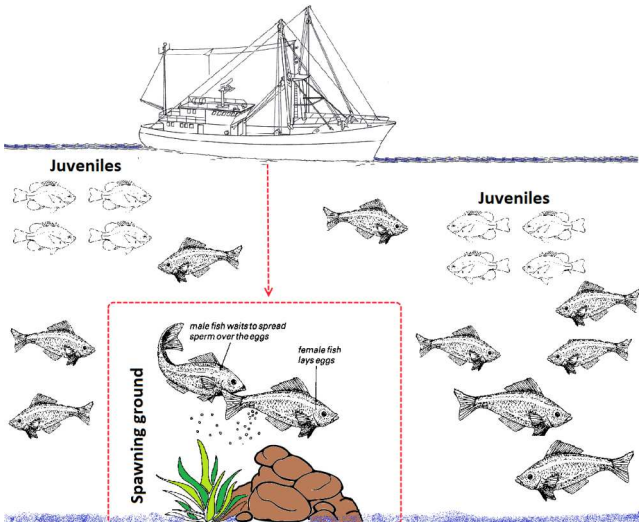


Figure 2.1 Illustration of maturity-selective harvesting.

There has been a fair amount of work on modeling populations with various life history stages [7, 8, 10, 23, 31–35, 58, 67, 126, 211]. For instance, Aiello and Freedman proposed a stage-structured model of single species with two life stages, immature and mature and with a constant time from birth to maturity in [7]. Yuan [211] considered a prey-predation model concerning the interactions between phytoplankton

and herbivorous zooplankton with a maturation time in predators and additional effect of zooplankton predation by a constant fish population, studied the stability and bifurcation with or without the maturation time delay. In [67], Gourley and Lou formulated a predator-prey model at the larval rather than adult level to control the Asian longhorned beetle Anoplophora glabripennis by one of its natural predators and discussed the stability of equilibria and persistence of the predator. Fang et al. [58] proposed a system of distributed delay equations for a system population of larvae and adults and established its positivity, boundedness, stability and uniform persistence. In [35], Brauer and Soudack studied the harvesting of predators in a predator–prey system as an optimal control problem with to maximize the long–term yield. In [33], Brauer discussed a nonlinear age–dependent population growth model with constant-Rate Harvesting. A three-stage structured SEIR epidemic model is discussed in [23].

As pointed out in [59], in marine habitats, preferential removal of larger individuals of a species has been shown to have a negative effect on its demography, life history and ecology. Moreover, size selectivity has shown a reduction in both age and size at puberty in several exploited fish populations, indicating the phenotypic plasticity or genetic changes in the organisms (population). Overusing the size selectivity for large fish will drive the population to become so small and lose its genetic variability because there are not enough individuals in the gene pool to carry the variety of traits that were once found in the population [97, 142]. In the natural world, many species go through two or more life stages as they proceed from birth to death. For instance, mammalian populations exhibit two distinct stages: immature and mature, while mosquito species go through four distinct stages during their life cycle. Unlike birds and mammals, most of fish can continue growing in length and weight after they have reached sexual maturity [206]. In this chapter we consider a fish stock with spawning region and classify the single species fish into three-stages (juveniles, small adults and large adults). We assume harvesting based on two harvesting strategies, maturity selectivity and size selectivity, where the difference is that under maturity selectivity small fish are allowed to be caught on the spawning ground.

The purpose of the work is to investigate the dynamical behavior of the system and discuss the effect of harvesting. As a starting point, we use the McKendrick-von-Foerster equation to construct a simple equation for each class, then by using the technique of integration along characteristics, we reduce each equation to a delay

differential equation for the number of fish in each category. With two growth periods (from juvenile to small adult and from small to large adult), mathematically, the proposed model has several time delay terms and the equations are coupled with each other which is different from the other models those in the literature e.g., [23, 58, 67]. The complexity of the model does not allow us to reduce the equations with respect to the matured fish to a single equation, which enhances the difficulties of theoretical analysis although we can derive the adult reproduction number $\mathcal{R}_A$ which is a threshold parameter that determines the persistence and extinction of the population. We further study the local and global stability of $E_0 = (0, 0, 0)$ when $\mathcal{R}_A < 1$ and prove population persistence when $\mathcal{R}_A > 1$ indicating the survivability of some or all species in an ecosystem. In addition, we provide a condition for the existence of a unique positive equilibrium, implying that the three groups of fish (juvenile, small and large matured fish) can exist in certain ideal environment, from biological point of view. Numerically, we investigate the influence of harvesting functions, discuss the optimal harvesting rates by addressing a relative maximum region and explore the effect of periodic coefficients on the dynamical system.

This chapter is organized as follows: in Section 2.2, we present the model system by deriving a delay differential equation for each population category. In Section 2.3, we discuss the well-posedness property by verifying the non-negativity and boundedness of the solutions with reasonable initial data. The adult reproduction number $\mathcal{R}_A$ is calculated in Section 2.4. In Section 2.5, we study the local and global stability of the trivial equilibrium point when $\mathcal{R}_A < 1$; prove the population persistence and explore the existence of a unique positive equilibrium point when $\mathcal{R}_A > 1$; and address the relation between $\mathcal{R}_A$ and either the harvesting functions or the time delay periods τ and ω. In Section 2.6, we investigate the influence of harvesting functions on the positive equilibrium point, discuss the optimal harvesting rates and explore the effect of periodic coefficients on the dynamical system. The summary and discussion are given in Section 2.7.

2.2 Model derivation

In fish life history, juveniles mature to small adults first and then keep growing to a large size [79, 145]. We consider a fish stock in which individuals are classified into three size classes: immature $I(t)$, small adults $S(t)$, and large adults $L(t)$. Since

it is possible for fishers to target only the matured fish, regardless of size, during spawning on stock spawning grounds [107, 145], we assume harvesting for both the small and large mature classes and there is no competition between different groups for two reasons: (i) the growth of immature and mature fish depends on different levels of food resources, for instance, in bluegill and Pacific salmon species, adults do not compete directly with juveniles [4, 53]; (ii) the harvesting of small and large matured fish can reduce their population sizes, and hence, more resources/spaces become available to the reaming matured fish. Let τ be the maturation time and ω be the time to reach large size after maturation, that is, juveniles become small size adults at age τ and reach a large size at age $\tau + \omega$.

Denote $u(t, a)$ as the density of fish of age a at time t. The total number of juveniles, small and large matured fish is, at time t, respectively,

$$I(t) = \int_0^{\tau} u(t,a)da, \quad S(t) = \int_{\tau}^{\tau+\omega} u(t,a)da \quad \text{and} \quad L(t) = \int_{\tau+\omega}^{+\infty} u(t,a)da.$$

First, we derive a functional differential equation to address variation of the number of juveniles $I(t)$ at time t. Using the McKendrick-von-Foerster age-structured equation, we have

$$\frac{\partial u(t,a)}{\partial t} + \frac{\partial u(t,a)}{\partial a} = -\mu_I u(t,a) - \alpha I(t) u(t,a) \text{ for } a < \tau, \tag{2.1}$$

to describe the loss of juveniles either through natural death with per-capita mortality rate μ_I or through crowding and competition between juveniles which depends on the total population size of the immature individuals, as presented by $\alpha I(t)$. When spawner numbers are high, juveniles mortality rate increases sharply due to severe competition among juveniles for space and food, e.g., coho salmon species [196].
Then the total number of juveniles satisfies

$$\begin{aligned} \frac{dI(t)}{dt} = \int_0^{\tau} \frac{\partial u(t,a)}{\partial t} da &= \int_0^{\tau} \left(-\mu_I u(t,a) - \alpha I(t) u(t,a) - \frac{\partial u(t,a)}{\partial a} \right) da \\ &= -\mu_I I(t) - \alpha I^2(t) + u(t,0) - u(t,\tau), \end{aligned}$$

where $u(t, 0)$ is the egg laying rate of matured fish which is taken to be a function $b(S, L)$ of the total number of mature population. Thus

$$u(t,0) = b\left(S(t), L(t)\right). \tag{2.2}$$

A simple choice of $b\left(S(t), L(t)\right)$ is $b_1\left(S(t)\right) + b_2\left(L(t)\right)$ where $b_1(S(t))$ $(b_2(L(t)))$ are egg laying rates of small (large) matured fish. For example, $b_1\left(S(t)\right) = r_1 S(t)$ and $b_2\left(L(t)\right) = r_2 L(t)$ where r_1, r_2 are the fecundity rates of maturing small, large fish respectively [107, 145].

Next, to calculate $u(t,\tau)$, define $V^{\xi}(a) = u(\xi + a, a)$. Then,

$$\frac{dV^{\xi}(a)}{da} = \left[\frac{\partial u(t,a)}{\partial t} + \frac{\partial u(t,a)}{\partial a}\right]\Bigg|_{t=\xi+a} = -\left(\mu_I + \alpha I(\xi + a)\right) V^{\xi}(a).$$

Thus, $V^{\xi}(a) = V^{\xi}(0) \exp\left\{-\int\limits_0^a \left(\mu_I + \alpha I(\xi + \eta)\right) d\eta\right\}$. When $a = \tau$, $\xi = t - \tau$ and $t \geq \tau$, we obtain

$$u(t,\tau) = u(t-\tau, 0) \exp\left\{-\int\limits_0^{\tau} \left(\mu_I + \alpha I(t - \tau + \eta)\right) d\eta\right\}.$$

By using (2.2), for $t \geq \tau$, we have

$$u(t,\tau) = b\left(S(t-\tau), L(t-\tau)\right) \exp\left\{-\int\limits_0^{\tau} \left(\mu_I + \alpha I(t - \tau + \eta)\right) d\eta\right\}. \tag{2.3}$$

The right-hand side in (2.3) is the rate at which juveniles mature into small adults. Therefore, we have the following equation with respect to the juveniles population,

$$\begin{aligned} \frac{dI(t)}{dt} &= b\left(S(t), L(t)\right) - \mu_I I(t) - \alpha I^2(t) \\ &\quad - b\left(S(t-\tau), L(t-\tau)\right) \exp\left\{-\int\limits_{t-\tau}^{t} \left(\mu_I + \alpha I(\eta)\right) d\eta\right\}, \end{aligned} \tag{2.4}$$

where the integral $\int\limits_{t-\tau}^{t} \alpha I(\eta) d\eta$ represents the accumulated death of juveniles due to competition between $t - \tau$ and t. Thus, the exponential term in (2.4) corrects the birth rate at the earlier time $t - \tau$. Equation (2.4) can also be written in the integral

equation form

$$I(t) = \int_{t-\tau}^{t} b\left(S(\xi), L(\xi)\right) \exp\left\{ -\int_{\xi}^{t} \left(\mu_I + \alpha I(\eta)\right) d\eta \right\} d\xi. \tag{2.5}$$

Similarly for the groups of small and large matured fish, we have

$$\frac{\partial u(t,a)}{\partial t} + \frac{\partial u(t,a)}{\partial a} = -\mu_S u(t,a) - H_1(S(t))u(t,a) \text{ for } \tau \le a < \tau + \omega, \tag{2.6}$$

$$\frac{\partial u(t,a)}{\partial t} + \frac{\partial u(t,a)}{\partial a} = -\mu_L u(t,a) - H_2(L(t))u(t,a) \text{ for } \tau + \omega \le a, \tag{2.7}$$

respectively, where corresponding to the small/large matured fish, μ_S/μ_L is natural death rate with the per-capita mortality and the function $H_i(\cdot)$ $(i = 1, 2)$ represents the harvesting which depends on the total population size of $S(t)/L(t)$. For instance, $H_i(x) = k_i x + d_i$ with constants k_i and d_i $(i = 1, 2)$ [208].

Assuming that no individual dies at the very moment when it becomes adult or reaches large size, using a similar procedure, we can obtain

$$\begin{aligned} \frac{dS(t)}{dt} &= b\left(S(t-\tau), L(t-\tau)\right) \exp\left\{ -\int_{t-\tau}^{t} \left(\mu_I + \alpha I(\eta)\right) d\eta \right\} \\ &\quad -H_1(S(t))S(t) - \mu_S S(t) - u(t, \tau+\omega), \end{aligned}$$

and, for $t \ge \tau + \omega$,

$$u(t, \tau+\omega) = u(t-\omega, \tau) \exp\left\{ -\int_{\tau}^{\tau+\omega} \left(\mu_S + H_1\left(S(t-\omega-\tau+\eta)\right)\right) d\eta \right\}.$$

By using (2.3), for $t \ge \tau + \omega$, we have

$$\begin{aligned} u(t, \tau+\omega) = b\left(S(t-(\tau+\omega)), L(t-(\tau+\omega))\right) &\exp\left\{ -\int_{t-(\tau+\omega)}^{t-\omega} \left(\mu_I + \alpha I(\eta)\right) d\eta \right\} \\ &\exp\left\{ -\int_{t-\omega}^{t} \left(\mu_S + H_1\left(S(\eta)\right)\right) d\eta \right\}. \end{aligned} \tag{2.8}$$

Biologically, the integral $\int_{t-\omega}^{t} H_1(S(\eta))d\eta$ represents accumulated removal of small mature individual due to harvesting between $t-\omega$ and t. The rate of reaching a large size mature individual at time t, $u(t,\tau+\omega)$ depends on a previous generation over the earlier time interval $[t-(\tau+\omega),t]$ including four factors: $b(S(t-(\tau+\omega)),L(t-(\tau+\omega)))$, the egg laying rate at time $t-(\tau+\omega)$; $e^{-\mu_I\tau}\ e^{-\mu_S\omega}$, the probability of surviving natural death; $\exp\left\{-\int_{t-(\tau+\omega)}^{t-\omega}\alpha I(\eta)d\eta\right\}$, the survival probability due to competition among juveniles in $[t-(\tau+\omega),t-\omega]$; and $\exp\left\{\int_{t-\omega}^{t}-H_1(S(\eta))d\eta\right\}$, the probability of surviving due to harvesting of small matured fish in $[t-\omega,t]$.

Thus, the equation with respect to the small matured fish becomes

$$\begin{aligned}\frac{dS(t)}{dt} &= b(S(t-\tau),L(t-\tau))\exp\left\{-\int_{t-\tau}^{t}(\mu_I+\alpha I(\eta))\,d\eta\right\}-\mu_S S(t)-H_1(S(t))S(t)\\ &- b(S(t-(\tau+\omega)),L(t-(\tau+\omega)))\exp\left\{-\int_{t-(\tau+\omega)}^{t-\omega}(\mu_I+\alpha I(\eta))\,d\eta\right\}\\ &\qquad\exp\left\{-\int_{t-\omega}^{t}(\mu_S+H_1(S(\eta)))\,d\eta\right\}.\end{aligned}\tag{2.9}$$

Alternatively (2.9) can be rewritten as

$$S(t) = \int_{t-(\tau+\omega)}^{t-\tau} b(S(\xi),L(\xi))\exp\left\{-\int_{\xi}^{\xi+\tau}(\mu_I+\alpha I(\eta)d\eta)\right\}\exp\left\{-\int_{\xi+\tau}^{t}(\mu_S+H_1(S(\eta))d\eta)\right\}d\xi.\tag{2.10}$$

Parallelly,

$$\begin{aligned}\frac{dL(t)}{dt} &= -\mu_L L(t)-H_2(L(t))L(t)\\ &+b(S(t-(\tau+\omega)),L(t-(\tau+\omega)))\exp\left\{-\int_{t-(\tau+\omega)}^{t-\omega}(\mu_I+\alpha I(\eta))\,d\eta\right\}\\ &\qquad\exp\left\{-\int_{t-\omega}^{t}(\mu_S+H_1(S(\eta)))\,d\eta\right\}.\end{aligned}\tag{2.11}$$

Up to now, the age-structured system of equations (2.1),(2.6) and (2.7) can be reduced

to the following system:

$$\begin{aligned}
\frac{dI(t)}{dt} &= b\left(S(t),L(t)\right)-\mu_I I(t)-\alpha I^2(t)\\
&\quad -b\left(S(t-\tau),L(t-\tau)\right)\exp\left\{-\int_{t-\tau}^{t}\left(\mu_I+\alpha I(\eta)\right)d\eta\right\}, \qquad (2.12)\\
\frac{dS(t)}{dt} &= b\left(S(t-\tau),L(t-\tau)\right)\exp\left\{-\int_{t-\tau}^{t}\left(\mu_I+\alpha I(\eta)\right)d\eta\right\}-\mu_S S(t)-H_1(S(t))S(t)\\
&\quad -b\left(S(t-(\tau+\omega)),L(t-(\tau+\omega))\right)\exp\left\{-\int_{t-(\tau+\omega)}^{t-\omega}\left(\mu_I+\alpha I(\eta)\right)d\eta\right\}\\
&\quad \exp\left\{-\int_{t-\omega}^{t}\left(\mu_S+H_1\left(S(\eta)\right)\right)d\eta\right\},\\
\frac{dL(t)}{dt} &= -\mu_L L(t)-H_2(L(t))L(t)\\
&\quad +b\left(S(t-(\tau+\omega)),L(t-(\tau+\omega))\right)\exp\left\{-\int_{t-(\tau+\omega)}^{t-\omega}\left(\mu_I+\alpha I(\eta)\right)d\eta\right\}\\
&\quad \exp\left\{-\int_{t-\omega}^{t}\left(\mu_S+H_1\left(S(\eta)\right)\right)d\eta\right\}.
\end{aligned}$$

An architecture of the model (2.12) is given in Fig. 2.2.

2.3 Well-posedness property

Denote $C := C([-\tau-\omega,0),\mathbb{R}^3)$. For $\phi=(\phi_1,\phi_2,\phi_3)\in C$, define $\|\phi\| = \sum_{i=1}^{3}\|\phi_i\|_\infty$ where

$$\|\phi_i\|_\infty = \max_{\theta\in[-\tau-\omega,0]}\left|\phi_i(\theta)\right|.$$

Then C is a Banach space and $C^+=\{\phi\in C:\phi_i(\theta)\geq 0,\forall i\in\{1,2,3\},\theta\in[-\tau-\omega,0]\}$ is a normal cone of C with nonempty interior in C. For any given continuous function $u=(u_1,u_2,u_3):[-\tau-\omega,\zeta)\to\mathbb{R}^3$ with $\zeta>0$, we define $u_t\in C$ for each $t\geq 0$ by $u_t(\theta)=(u_1(t+\theta),u_2(t+\theta),u_3(t+\theta))$ for all $\theta\in[-\tau-\omega,0]$.

There are two versions of our model, system (2.12) and the one consisting of (2.5), (2.10) and (2.11). The two systems are equivalent for the class of initial data of the

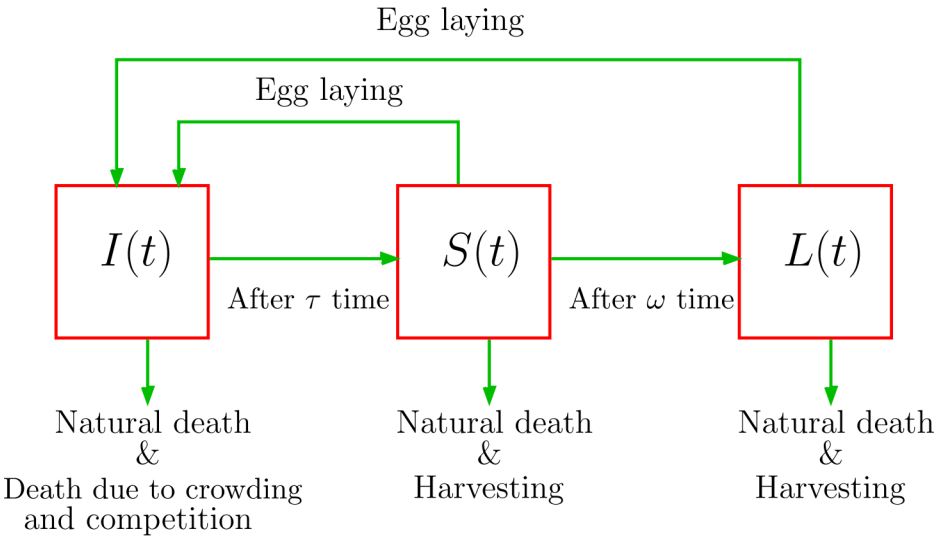


Figure 2.2 Model diagram.

form

$$\mathcal{X} = \left\{ \phi \in C^+ : \phi_1(0) = \int_{-\tau}^{0} b(\phi_2(\theta), \phi_3(\theta)) \exp\left\{ -\int_{\theta}^{0} (\mu_I + \alpha\phi_1(\eta))\, d\eta \right\} d\theta \right.$$

$$\text{and} \quad \phi_2(0) = \int_{-(\tau+\omega)}^{-\tau} b(\phi_2(\theta), \phi_3(\theta)) \exp\left\{ -\int_{\theta}^{\theta+\tau} (\mu_I + \alpha\phi_1(\eta) d\eta) \right\}$$

$$\left. \exp\left\{ -\int_{\theta+\tau}^{0} (\mu_S + H_1(\phi_2(\eta)) d\eta) \right\} d\theta \right\}.$$

From the biological point of view, we assume all the functions in the model system (2.12) are continuous and differentiable over $\mathbb{R}^+$ and satisfy the following hypotheses

(A_1) The harvesting function $H_i(x) > 0$ for $x > 0$, $H_i(0) \geq 0$ $(i = 1, 2)$ and satisfies one of the following:
(a) nondecreasing for all $x > 0$; or (b) strictly increasing for all $x > 0$.

(A_2) The egg laying rate function $b(S, L) \geq 0$ and $\frac{\partial b}{\partial S} \geq 0$, $\frac{\partial b}{\partial L} \geq 0$ for all $(S, L) \in \mathbb{R}_+ \times \mathbb{R}_+ \cup \{(0, 0)\}$, $b(0, 0) = 0$, $b(S, 0) > 0$ for $S > 0$, $b(0, L) > 0$ for $L > 0$ and there exists an increasing function B such that $b\,(S(t), L(t)) \leq B\,(S(t) + L(t))$

for $(S, L) \in \mathbb{R}_+ \times \mathbb{R}_+$, $B(0) = 0$ and satisfies one of the following:

(a) B is strictly sub-homogeneous (strictly sublinear), that is, for any $p \in (0, 1)$, $x > 0$, $B(px) > pB(x)$; or

(b) B is sub-homogeneous (sublinear), that is, for any $p \in (0, 1)$, $x \geq 0$, $B(px) \geq pB(x)$.

We can view (A_2) as a mathematical interpretation of the biological factor that the growth of the population is slower than exponential, so we restrict the growth rate function with the sub-homogeneous condition.

The following theorem demonstrates that the solutions of (2.12) are nonnegative and bounded.

Theorem 2.3.1. *Suppose hypotheses (A_1) and (A_2)-(a), or (A_1)-(b) and (A_2)-(b) are satisfied, and $\phi \in \mathcal{X}$ with $\phi_2(0) > 0$. Then system (2.12) has a unique solution $(I(t), S(t), L(t))$ that satisfies $I(t) > 0$, $S(t) > 0$ and $L(t) \geq 0$ for all $t \geq 0$. Furthermore, the solution is ultimately bounded in C.*

Proof. For any $\phi \in \mathcal{X}$, we define $F(\phi) = (F_1(\phi), F_2(\phi), F_3(\phi))^T$, where

$$
\begin{aligned}
F_1(\phi) &= b\left(\phi_2(0), \phi_3(0)\right) - \mu_I \phi_1(0) - \alpha \phi_1(0)^2 \\
&\quad -b\left(\phi_2(-\tau), \phi_3(-\tau)\right) \exp\left\{-\int_{-\tau}^{0} \mu_I + \alpha\phi_1(\eta) d\eta\right\}, \\
F_2(\phi) &= b\left(\phi_2(-\tau), \phi_3(-\tau)\right) \exp\left\{-\int_{-\tau}^{0} \mu_I + \alpha\phi_1(\eta) d\eta\right\} - \mu_S \phi_2(0) - H_1\left(\phi_2(0)\right)\phi_2(0) \\
&\quad -b\left(\phi_2(-\tau-\omega), \phi_3(-\tau-\omega)\right) \exp\left\{-\int_{-\tau-\omega}^{-\omega} \mu_I + \alpha\phi_1(\eta) d\eta\right\} \\
&\qquad \exp\left\{-\int_{-\omega}^{0} \mu_S + H_1\left(\phi_2(\eta)\right) d\eta\right\}, \\
F_3(\phi) &= b\left(\phi_2(-\tau-\omega), \phi_3(-\tau-\omega)\right) \exp\left\{-\int_{-\tau-\omega}^{-\omega} \mu_I + \alpha\phi_1(\eta) d\eta\right\} \\
&\qquad \exp\left\{-\int_{-\omega}^{0} \mu_S + H_1\left(\phi_2(\eta)\right) d\eta\right\} - \mu_L \phi_3(0) - H_2\left(\phi_3(0)\right)\phi_3(0).
\end{aligned}
$$

Note that $\mathcal{X}$ is closed in C and for any $\phi \in \mathcal{X}$, $F(\phi)$ is continuous and Lipschitz in ϕ in each compact set in $\mathbb{R} \times \mathcal{X}$. By [74, Theorem 2.2.3], it follows that (2.12) has a unique solution $u(t, \phi)$ through $(0, \phi)$ on its maximal interval $[0, \sigma)$ of existence.

Since $F_3(\phi) \geq 0$ when $\phi \in \mathcal{X}$ with $\phi_3(0) = 0$, thus $L(t)$ is nonnegative for all $t \in [0, \sigma)$, i.e., $L(t) \geq 0$, see [169, Theorem 5.2.1]. To prove the positivity of $S(t)$,

from $\phi_2(0) > 0$ and the continuity of solutions, suppose there exists $\tilde{t} \in [0, \sigma)$ such that $S(\tilde{t}) = 0$ and $S(t) > 0$ for $t \in (0, \tilde{t})$. Thus, by (A_2), we have $b(S(t), L(t)) > 0$ for $t \in (0, \tilde{t})$. Then, it follows from the integral form (2.10) that $S(\tilde{t}) > 0$, which is a contradiction. Therefore, $S(t) > 0$ for all $t \in [0, \sigma)$. $I(t) > 0$ is straightforward from (2.5) and the positivity of $S(t)$. Therefore, the solution of the system (2.12) satisfies $I(t) > 0$, $S(t) > 0$ and $L(t) \geq 0$ for $t \in [0, \sigma)$.

Now we prove the boundedness. Let $T > 0$ be sufficiently large. First, assume that $L(t-\tau-\omega) \equiv 0$ for $t > T+\tau+\omega$. Then from the second equation in (2.12) and (A_2), when $t > T+\tau+\omega$, we have

$$\begin{aligned} \frac{dS(t)}{dt} &\leq b\left(S(t-\tau), 0\right) \exp\left\{-\int_{t-\tau}^{t} \left(\mu_I + \alpha I(\eta)\right) d\eta\right\} - \mu_S S(t) - H_1(S(t))S(t) \\ &\leq B\left(S(t-\tau)\right) \exp\left\{-\int_{t-\tau}^{t} \left(\mu_I + \alpha I(\eta)\right) d\eta\right\} - \mu_S S(t) - H_1(S(t))S(t). \end{aligned} \tag{2.13}$$

The boundedness of $S(t)$ in (2.13) and $I(t)$ can be proved by arguments similar to those in the proof of [58, Theorem 2.1].

Assume $L(t-\tau-\omega) \not\equiv 0$ for $t > T+\tau+\omega$. If $L(t)$ is unbounded, then for a real number $K > 0$, $L(t) \geq K$ for $t > T_0$ and some $T_0 > T+\tau+\omega$. It then follows from $S(t) > 0$ and the assumption (A_1) that $0 < H_i(\cdot) \leq H_i(S(t)+L(t))$ for $i = 1, 2$ and $t > T_0$, hence; there exist $\vartheta_1 > 0$ and $\vartheta_2 > 0$ such that $\vartheta_1 \leq \frac{H_1(S(t))}{H_1(S(t)+L(t))} \leq 1$ and $\vartheta_2 \leq \frac{H_2(L(t))}{H_2(S(t)+L(t))} \leq 1$. Let $\vartheta = \min\{\vartheta_1, \vartheta_2\}$ and $H(S(t)+L(t)) = \min\{H_1(S(t)+L(t)), H_2(S(t)+L(t))\}$, it then follows from the second and third equations in (2.12) that

$$\begin{aligned} \frac{d}{dt}\left(S(t)+L(t)\right) &\leq b\left(S(t-\tau), L(t-\tau)\right) - \min\left\{\mu_S, \mu_L\right\}\left(S(t)+L(t)\right) \\ &\quad -H_1(S)S - H_2(L)L, \\ &\leq B\left(S(t-\tau)+L(t-\tau)\right) - \min\left\{\mu_S, \mu_L\right\}\left(S(t)+L(t)\right) \\ &\quad -\vartheta H(S(t)+L(t))(S(t)+L(t)), \end{aligned}$$

for $t > T_0$. Consider the equation

$$\frac{du(t)}{dt} = B(u(t-\tau)) - \min\left\{\mu_S, \mu_L\right\} u(t) - \vartheta H(u(t))u(t). \tag{2.14}$$

Let $C_\tau^+ = C([-\tau, 0), [0, \infty))$. Define $g : C_\tau^+ \to \mathbb{R}$ and $\hat{g} : \mathbb{R}_+ \to \mathbb{R}$ by

$$\begin{aligned} g(\phi) &= B(\phi(-\tau)) - \min\{\mu_S, \mu_L\}\phi(0) - \vartheta H(\phi(0))\phi(0), \\ \hat{g}(u) &= B(u) - \min\{\mu_S, \mu_L\} u - \vartheta H(u)u. \end{aligned}$$

Clearly, $g \geq 0$ when $\phi(0) = 0$.

Claim 1. When either (i) (A_1) and (A_2)-(a); or (ii) (A_1)-(b) and (A_2)-(b) is satisfied, both g and $\hat{g}$ are strictly sub-homogeneous.

We prove Claim 1 for $\hat{g}$. A similar proof works for g. Assume (i) ((ii)) is satisfied then,

$$\begin{aligned} \hat{g}(pu) \quad &= \quad B(pu) - p\min\{\mu_S, \mu_S\} u - p\vartheta H(pu)u \\ &> (\geq) \quad pB(u) - p\min\{\mu_S, \mu_S\} u - p\vartheta H(pu)u \\ &\geq (>) \quad pB(u) - p\min\{\mu_S, \mu_S\} u - p\vartheta H(u)u = p\hat{g}(u). \end{aligned}$$

Thus, $\hat{g}$ is strictly sub-homogeneous. By [219, Theorem 3.2], (2.14) admits a globally asymptotically stable equilibrium which attracts all positive solutions. Therefore, any solution of equation (2.14) is bounded, that is, there exist $\tilde{T} > T_0$ and $M > 0$ such that $u(t) \leq M$ for $t > \tilde{T}$. By the comparison principle, $S(t) + L(t)$ must be bounded which is a contradiction. Therefore,

$$\limsup_{t\to+\infty} S(t) \leq M \quad \text{and} \quad \limsup_{t\to+\infty} L(t) \leq M. \tag{2.15}$$

From (2.5) and using the reverse Fatou lemma (see e.g., [64]), we have

$$\begin{aligned} \limsup_{t\to+\infty} I(t) \quad &\leq \quad \limsup_{t\to+\infty} \int_{t-\tau}^{t} b\left(S(\xi), L(\xi)\right) d\xi \\ &\leq \quad \int_0^{\tau} \limsup_{t\to+\infty} \left(b\left(S(t-\xi), L(t-\xi)\right)\right) d\xi \leq \tau b\left(M, M\right) := K. \end{aligned} \tag{2.16}$$

By (2.15) and (2.16), we have $\sigma = +\infty$ (see e.g., [74, Theorem 2.3.1]). Thus, all the solutions exist globally, and are ultimately bounded. □

2.4 The adult reproduction number

From the system (2.12) it is easy to see that the equilibrium $E_0 = (0, 0, 0)$ always exists for all values of the parameters. When one of the classes is zero, the others become zeros. That is, E_0 is the only boundary equilibrium point. The linearization of (2.12) at E_0 is

$$\begin{aligned}
\frac{dI(t)}{dt} &= b_s S(t) + b_l L(t) - \mu_I I(t) - b_s e^{-\mu_I \tau} S(t-\tau) - b_l e^{-\mu_I \tau} L(t-\tau), \\
\frac{dS(t)}{dt} &= b_s e^{-\mu_I \tau} S(t-\tau) + b_l e^{-\mu_I \tau} L(t-\tau) - (\mu_S + H_1(0)) S(t) \\
&\quad - b_s e^{-\mu_I \tau} e^{-(\mu_S + H_1(0))\omega} S(t-\tau-\omega) - b_l e^{-\mu_I \tau} e^{-(\mu_S + H_1(0))\omega} L(t-\tau-\omega), \\
\frac{dL(t)}{dt} &= b_s e^{-\mu_I \tau} e^{-(\mu_S + H_1(0))\omega} S(t-\tau-\omega) + b_l e^{-\mu_I \tau} e^{-(\mu_S + H_1(0))\omega} L(t-\tau-\omega) \\
&\quad - (\mu_L + H_2(0)) L(t),
\end{aligned} \tag{2.17}$$

where $b_s = \frac{\partial b(S,L)}{\partial S}\Big|_{(0,0)}$ and $b_l = \frac{\partial b(S,L)}{\partial L}\Big|_{(0,0)}$. To calculate the average number of adults produced by one adult over its life span, we consider the decoupled equations for mature classes, $S(t)$ and $L(t)$ in (2.17), which can be rewritten as

$$\frac{d}{dt}\boldsymbol{A}(t) = M_1 \boldsymbol{A}(t-\tau) + M_2 \boldsymbol{A}(t-\tau-\omega) - M_3 \boldsymbol{A}(t), \tag{2.18}$$

with

$$\boldsymbol{A}(t) = \begin{bmatrix} S(t) \\ L(t) \end{bmatrix}, \; M_1 = \begin{bmatrix} b_s e^{-\mu_I \tau} & b_l e^{-\mu_I \tau} \\ 0 & 0 \end{bmatrix}, M_3 = \begin{bmatrix} \mu_S + H_1(0) & 0 \\ 0 & \mu_L + H_2(0) \end{bmatrix}$$

and $M_2 = \begin{bmatrix} -b_s e^{-\mu_I \tau} e^{-(\mu_S + H_1(0))\omega} & -b_l e^{-\mu_I \tau} e^{-(\mu_S + H_1(0))\omega} \\ b_s e^{-\mu_I \tau} e^{-(\mu_S + H_1(0))\omega} & b_l e^{-\mu_I \tau} e^{-(\mu_S + H_1(0))\omega} \end{bmatrix}$.

In (2.18), $\boldsymbol{A}(t)$ is the total number in the adult classes, and $M_1 \boldsymbol{A}(t-\tau) + M_2 \boldsymbol{A}(t-\tau-\omega)$ is the newly mature individuals at time t which depends on the adult individuals over the time interval $[t-\tau-\omega, t]$. Further, the internal evolution of mature individuals through natural death is governed by the linear ordinary differential system:

$$\frac{d}{dt}\boldsymbol{A}(t) = -M_3 \boldsymbol{A}(t). \tag{2.19}$$

Let $\mathscr{A}_{\mathbf{0}} = (A_1, A_2)^T$ be the number of classes $S(t)$ and $L(t)$ at $t = 0$, then from (2.19) the distribution of the remaining population at time $t > 0$ is

$$\mathscr{A}(t) = e^{-M_3 t}\mathscr{A}_0.$$

The total number of newly matured individuals is

$$\bar{\mathscr{A}} = \int_{\tau}^{\infty} M_1\mathscr{A}(t-\tau)dt + \int_{\tau+\omega}^{\infty} M_2\mathscr{A}(t-\tau-\omega)dt = \int_{\tau}^{\infty} M_1 e^{-M_3(t-\tau)}\mathscr{A}_0 dt + \int_{\tau+\omega}^{\infty} M_2 e^{-M_3(t-\tau-\omega)}\mathscr{A}_0 dt.$$

Due to the nonsingularity of the matrix M_3, we have

$$\bar{\mathscr{A}} = (M_1 + M_2)M_3^{-1}\mathscr{A}_{\mathbf{0}}.$$

Adopting the concept of the basic reproduction number from literature (see e.g., [51, 105, 207, 217]), the matrix $(M_1 + M_2)$, which tracks new mature individuals, is positive. Then the next generation operator is

$$M_0 = (M_1 + M_2)M_3^{-1} = \begin{bmatrix} \frac{b_s e^{-\mu_I\tau}(1-e^{-(\mu_S+H_1(0))\omega})}{\mu_S+H_1(0)} & \frac{b_l e^{-\mu_I\tau}(1-e^{-(\mu_S+H_1(0))\omega})}{\mu_L+H_2(0)} \\ \frac{b_s e^{-\mu_I\tau}e^{-(\mu_S+H_1(0))\omega}}{\mu_S+H_1(0)} & \frac{b_l e^{-\mu_I\tau}e^{-(\mu_S+H_1(0))\omega}}{\mu_L+H_2(0)} \end{bmatrix},$$

and hence, the spectral radius of the matrix M_0 is called the adult reproduction number for system (2.12) denoted by $\mathcal{R}_A$, that is,

$$\mathcal{R}_A = \frac{\frac{\partial b(S,L)}{\partial S}\Big|_{(0,0)}(e^{-\mu_I\tau} - e^{-\mu_I\tau}e^{-(\mu_S+H_1(0))\omega})}{\mu_S + H_1(0)} + \frac{\frac{\partial b(S,L)}{\partial L}\Big|_{(0,0)} e^{-\mu_I\tau}e^{-(\mu_S+H_1(0))\omega}}{\mu_L + H_2(0)}. \tag{2.20}$$

For simplicity, in the rest of the chapter, we assume that

$$\mu_S = \mu_L := \mu_A, \ H_1(0) = H_2(0) := H_A$$

and the egg laying rate function $b(S, L)$ has the form

$$b(S(t), L(t)) = b_1(S(t)) + b_2(L(t))$$

with $b_1(0) = b_2(0) = 0$, where $b_1(S(t))$ $(b_2(L(t)))$ is egg laying rate of small (large) matured fish. Thus, the adult reproduction number becomes

$$\mathcal{R}_A = \frac{b_1'(0)(e^{-\mu_I\tau} - e^{-\mu_I\tau}e^{-(\mu_A+H_A)\omega}) + b_2'(0)e^{-\mu_I\tau}e^{-(\mu_A+H_A)\omega}}{\mu_A + H_A}. \tag{2.21}$$

Biologically, $b_1'(0)$ ($b_2'(0)$) is the number of eggs produced by one small (large) matured fish per unit time, $\frac{1}{\mu_A+H_A}$ is the life expectancy of matured fish, $e^{-\mu_I\tau-(\mu_A+H_A)\omega}$ is the survival rate of an individual from immature to large mature period and $e^{-\mu_I\tau}(1-e^{-(\mu_A+H_A)\omega})$ is the probability of matured fish to be in the small size class. Therefore, $\frac{b_1'(0)(e^{-\mu_I\tau}-e^{-\mu_I\tau}e^{-(\mu_A+H_A)\omega})}{\mu_A+H_A}$ is the average number of adults produced by one small matured fish over its expected lifetime and $\frac{b_2'(0)e^{-\mu_I\tau}e^{-(\mu_A+H_A)\omega}}{\mu_A+H_A}$ is the average number of adults produced by one large matured fish over its expected lifetime.

As we know, usually the adult reproduction number $\mathcal{R}_A$ is a threshold parameter that determines the persistence and extinction of a population. Obviously, $\mathcal{R}_A$ depends on maturation time (τ), the growth time to certain size of fish (ω) and the initial harvesting value (H_A). The detailed relations will be discussed in the following section.

2.5 Population persistence and extinction

In the real world, mature small fish produce fewer eggs than large fish (see e.g., [80, 94, 157]). Thus, it is natural to assume that

$$(C_1) \qquad b_1'(0) \le b_2'(0).$$

About the stability of the trivial steady state E_0, we have the following result:

Theorem 2.5.1. *When $\mathcal{R}_A < 1$ and (C_1) holds, E_0 in (2.12) is locally asymptotically stable. When $\mathcal{R}_A > 1$, E_0 is unstable.*

Proof. At the point $(0,0,0)$ in (2.12), the corresponding characteristic equation is

$$\Delta(\lambda,\tau,\omega) = (\lambda+\mu_I)(\lambda+\mu_A+H_A)Q(\lambda,\tau,\omega) = 0, \tag{2.22}$$

where

$$\begin{aligned} Q(\lambda,\tau,\omega) &= \lambda+\mu_A+H_A-b_1'(0)e^{-\mu_I\tau}e^{-\lambda\tau} \\ &\quad -(b_2'(0)-b_1'(0))e^{-\mu_I\tau}e^{-(\mu_A+H_A)\omega}e^{-\lambda(\tau+\omega)}. \end{aligned} \tag{2.23}$$

Since $\lambda=-\mu_I$ and $\lambda=-(\mu_A+H_A)$ are negative roots in (2.22), it suffices to consider the roots in $Q(\lambda,\tau,\omega)=0$. First, if $\mathcal{R}_A<1$, then $Q(0,\tau,\omega)>0$, implying $\lambda=0$

is not a root of the characteristic equation. Now, by contradiction, assume there is a root $\lambda = x + iy$ in $Q(\lambda, \tau, \omega) = 0$ with $x \geq 0$. The real part of $Q(\lambda, \tau, \omega) = 0$ for $\lambda = x + iy$ is

$$\begin{aligned} x &= -\mu_A - H_A + b_1'(0)e^{-\mu_I \tau}\cos(\tau y)e^{-\tau x} \\ &\quad + \left(b_2'(0) - b_1'(0)\right) e^{-\mu_I \tau} e^{-(\mu_A + H_A)\omega} \cos((\tau + \omega)y) e^{-(\tau+\omega)x}. \end{aligned}$$

Since $\mathcal{R}_A < 1$, we have

$$b_1'(0)e^{-\mu_I \tau} + \left(b_2'(0) - b_1'(0)\right) e^{-\mu_I \tau} e^{-(\mu_A + H_A)\omega} < \mu_A + H_A.$$

Thus,

$$\begin{aligned} x &< -b_1'(0)e^{-\mu_I \tau}\left(1 - \cos(\tau y)e^{-\tau x}\right) \\ &\quad - \left(b_2'(0) - b_1'(0)\right) e^{-\mu_I \tau} e^{-(\mu_A + H_A)\omega} \left(1 - \cos((\tau + \omega)y) e^{-(\tau+\omega)x}\right). \end{aligned}$$

When (C_1) holds, the right-hand side in the above inequality is always negative because $x \geq 0$ and $1 - \cos(y)e^{-x} \geq 0$ for any $y \in \mathbb{R}$, which leads to a contradiction. Hence all the eigenvalues in (2.22) have negative real parts, implying E_0 is locally asymptotically stable.

When $\mathcal{R}_A > 1$, $Q(0, \tau, \omega) = (\mu_A + H_A)(1 - \mathcal{R}_A) < 0$. Since $Q(\lambda, \tau, \omega)$ is continuous with respect to λ and $\lim_{\lambda \to \infty} Q(\lambda, \tau, \omega) = +\infty$, there exists $\bar{\lambda} > 0$ such that $Q(\bar{\lambda}, \tau, \omega) = 0$. Therefore $(0, 0, 0)$ is unstable in system (2.12) when $\mathcal{R}_A > 1$. □

Remark 2.5.1. *The system (2.17) is not a cooperative system, here we need the condition (C_1) to ensure the local stability of E_0 when $\mathcal{R}_A < 1$.*

Remark 2.5.2. *Theorem 2.5.1 can be obtained from [217, Theorem 2.1 and Corollary 2.1].*

Given additional restriction

$$(C_2) \qquad b_1(S) \leq b_1'(0)S \text{ and } b_2(L) \leq b_2'(0)L,$$

we can discuss the global stability of E_0.

Theorem 2.5.2. *When $\mathcal{R}_A < 1$ and (C_1)-(C_2) hold, E_0 is globally asymptotically stable.*

Proof. Let $t_0 > 0$ be sufficiently large. Since (C_2) holds, from (2.10) and the third equation in (2.12), for $t \geq t_0 + \tau + \omega$, we have

$$\begin{aligned} S(t) &\leq \int_{t-(\tau+\omega)}^{t-\tau} e^{-\mu_I \tau}\left(b_1{}'(0)S(\xi) + b_2{}'(0)L(\xi)\right)e^{-(\mu_A+H_A)(t-\tau-\xi)}d\xi, \\ \frac{dL(t)}{dt} &\leq e^{-\mu_I\tau}e^{-(\mu_A+H_A)\omega}\left(b_1{}'(0)S(t-\tau-\omega) + b_2{}'(0)L(t-\tau-\omega)\right) - (\mu_A+H_A)L(t). \end{aligned}$$

Adopting the idea of comparison theory from [67, 169, 197], we can claim that, any solution $(S(t), L(t))$ is bounded above by the solution of the linear system

$$\begin{aligned} U_1(t) &= \int_{t-(\tau+\omega)}^{t-\tau} e^{-\mu_I \tau}\left((b_1{}'(0))U_1(\xi) + (b_2{}'(0))U_2(\xi)\right)e^{-(\mu_A+H_A(0))(t-\tau-\xi)}d\xi, \\ \frac{dU_2(t)}{dt} &= e^{-\mu_I\tau}e^{-(\mu_A+H_A(0))\omega}\left((b_1{}'(0))U_1(t-\tau-\omega) + (b_2{}'(0))U_2(t-\tau-\omega)\right) \\ &\quad -(\mu_A+H_A(0))U_2(t) \end{aligned} \tag{2.24}$$

such that $(S(\theta), L(\theta)) \leq (U_1(\theta), U_2(\theta))$ for all $\theta \in [t_0, t_0+\tau+\omega]$. Notice that system (2.24) has a unique solution when

$$U_1(t_0+\tau+\omega) = \int_{t_0}^{t_0+\omega} e^{-\mu_I\tau}\left((b_1{}'(0))U_1(\xi) + (b_2{}'(0))U_2(\xi)\right)e^{-(\mu_A+H_A(0))(-\tau-\xi)}d\xi.$$

To prove the claim, let $\varepsilon > 0$ and consider the linear system

$$\begin{aligned} U_{1\varepsilon}(t) &= \int_{t-(\tau+\omega)}^{t-\tau} e^{-\mu_I \tau}\left((b_1{}'(0)+\varepsilon)U_{1\varepsilon}(\xi) + (b_2{}'(0)+\varepsilon)U_{2\varepsilon}(\xi)\right)e^{-(\mu_A+H_A(0))(t-\tau-\xi)}d\xi, \\ \frac{dU_{2\varepsilon}(t)}{dt} &= e^{-\mu_I\tau}e^{-(\mu_A+H_A(0))\omega}\left((b_1{}'(0)+\varepsilon)U_{1\varepsilon}(t-\tau-\omega) + (b_2{}'(0)+\varepsilon)U_{2\varepsilon}(t-\tau-\omega)\right) \\ &\quad -(\mu_A+H_A(0))U_{2\varepsilon}(t) \end{aligned} \tag{2.25}$$

with $(U_{1\varepsilon}(\theta), U_{2\varepsilon}(\theta)) := (U_1(\theta)+\varepsilon, U_2(\theta)+\varepsilon)$ for all $\theta \in [t_0, t_0+\tau+\omega]$. It is enough to prove that for any sufficiently small positive ε, $S(t) < U_{1\varepsilon}(t)$ and $L(t) < U_{2\varepsilon}(t)$ for all $t \geq t_0+\tau+\omega$ with $(S(\theta), L(\theta)) < (U_{1\varepsilon}(\theta), U_{2\varepsilon}(\theta))$ for all $\theta \in [t_0, t_0+\tau+\omega]$. If this is false for some ε, then there exist a $T_1 \in (t_0+\tau+\omega, \infty)$ such that $S(t) < U_{1\varepsilon}(t)$ and $L(t) < U_{2\varepsilon}(t)$ for all $t \in [t_0+\tau+\omega, T_1)$, and $S(T_1) = U_{1\varepsilon}(T_1)$ or $L(T_1) = U_{2\varepsilon}(T_1)$.

Assume $S(T_1) = U_{1\varepsilon}(T_1)$ holds. From the first equation in (2.25), we have

$$\begin{aligned} U_{1\varepsilon}(T_1) &= \int_{T_1-(\tau+\omega)}^{T_1-\tau} e^{-\mu_I\tau}\left((b_1{}'(0)+\varepsilon)U_{1\varepsilon}(\xi)+(b_2{}'(0)+\varepsilon)U_{2\varepsilon}(\xi)\right)e^{-(\mu_A+H_A(0))(t-\tau-\xi)}d\xi \\ &> \int_{T_1-(\tau+\omega)}^{T_1-\tau} e^{-\mu_I\tau}\left((b_1{}'(0)+\varepsilon)S(\xi)+(b_2{}'(0)+\varepsilon)S(\xi)\right)e^{-(\mu_A+H_A(0))(t-\tau-\xi)}d\xi \geq S(T_1), \end{aligned}$$

which is a contradiction. When $L(T_1) = U_{2\varepsilon}(T_1)$, then $\frac{dL}{dt}\big|_{t=T_1} \geq \frac{dU_{2\varepsilon}}{dt}\big|_{t=T_1}$ must hold. But from the second equation in (2.25), we have

$$\begin{aligned} \frac{dU_{2\varepsilon}}{dt}\Big|_{t=T_1} &= e^{-\mu_I\tau}e^{-(\mu_A+H_A(0))\omega}\left((b_1{}'(0)+\varepsilon)U_{1\varepsilon}(T_1-\tau-\omega)+(b_2{}'(0)+\varepsilon)U_{2\varepsilon}(T_1-\tau-\omega)\right) \\ &\quad -(\mu_A+H_A(0))U_{2\varepsilon}(T_1) \\ &> e^{-\mu_I\tau}e^{-(\mu_A+H_A(0))\omega}\left((b_1{}'(0)+\varepsilon)S(T_1-\tau-\omega)+(b_2{}'(0)+\varepsilon)L(T_1-\tau-\omega)\right) \\ &\quad -(\mu_A+H_A(0))L(T_1) \geq \frac{dL}{dt}\Big|_{t=T_1}. \end{aligned}$$

These contradictions imply that no such T_1 can exist for any sufficiently small $\varepsilon > 0$. Letting $\varepsilon \to 0$, we have $S(t) \leq U_1(t)$ and $L(t) \leq U_2(t)$ for all $t \geq t_0 + \tau + \omega$ with $(S(\theta), L(\theta)) \leq (U_1(\theta), U_2(\theta))$ for all $\theta \in [t_0, t_0 + \tau + \omega]$.

Solutions of system (2.24) are in the form $(U_1(t), U_2(t)) = (c_1, c_2)e^{\lambda t}$ (see e.g. [58]) where λ satisfies $(\lambda + \mu_A + H_A)Q(\lambda, \tau, \omega) = 0$ where Q is given in (2.23). We know from the proof of Theorem 2.5.1 that all roots have negative real parts if $\mathcal{R}_A < 1$ and (C_1) holds. Hence, $(U_1(t), U_2(t)) \to (0, 0)$ as $t \to \infty$. Thus, $(S(t), L(t)) \to (0, 0)$ as $t \to \infty$.

From (2.5) and using the reverse Fatou lemma (see, e.g., [64]), we have

$$\limsup_{t\to+\infty} I(t) \leq \int_0^{\tau} \limsup_{t\to+\infty} \left(b_1(S(t-\xi)) + b_2(L(t-\xi))\right)d\xi = 0.$$

Thus, $\lim_{t\to\infty} I(t) = 0$. Hence, E_0 is globally attractive, which, together with the local stability of E_0 established in Theorem 2.5.1, confirms the global asymptotic stability of E_0 when $\mathcal{R}_A < 1$ and both (C_1) and (C_2) hold. This completes the proof. □

Uniform persistence is an important concept in population dynamics which describes the survival of some or all species in an ecosystem. Now, we study the system persistence and discuss the existence of a unique positive equilibrium point.

Theorem 2.5.3. *If $\mathcal{R}_A > 1$, (2.12) is uniformly persistent, in the sense that, there is a positive number $\eta > 0$ such that every solution in system (2.12) with $\phi \in \mathcal{X}$, $\phi_2(\theta) > 0$ or $\phi_3(\theta) > 0$ for some $\theta \in [-\tau - \omega, 0]$, satisfies*

$$\liminf_{t\to\infty}(I(t), S(t), L(t)) \geq (\eta, \eta, \eta).$$

Proof. Define

$$X^0 = \{\phi \in \mathcal{X} : \phi_2(\theta) > 0 \text{ or } \phi_3(\theta) > 0 \text{ for some } \theta \in [-\tau - \omega, 0]\}$$

and

$$X_0 = \mathcal{X} \setminus X_0 = \{\phi \in \mathcal{X} : \phi_2(\theta) = \phi_3(\theta) = 0 \text{ for all } \theta \in [-\tau - \omega, 0]\}.$$

In the following, we verify the conditions in [75, Theorem 4.2]. Let $\Phi(t)$, $t > 0$, be the solution semiflow of model system (2.12). Notice that X^0 is an open and dense set in $\mathcal{X}$ with $X^0 \cup X_0 = \mathcal{X}$ and $X^0 \cap X_0 = \emptyset$. From (2.10) and (2.11), we have $\Phi(t)X^0 \subset X^0$ and $\Phi(t)X_0 \subset X_0$. (i) has been confirmed in Theorem 2.3.1. Noticing that the bounds in (2.15) and (2.16) are all independent of initial functions, hence, condition (ii) is verified [207]. (iii) It follows from [104, Theorem 2.2.8] that $\Phi(t)$ is asymptotically smooth. For condition (iv) it is clear that $A = \{E_0\}$, and it is isolated. Thus, the covering M is $\{E_0\}$, which is acyclic because there is no orbit connecting E_0 to itself in X_0.

Finally, to verify (v), we prove the following claim.

Claim 2. $W^s(E_0) \cap X^0 = \emptyset$.

By contradiction, suppose that there exists a solution in X^0 such that

$$\lim_{t\to\infty} I(t) = 0, \quad \lim_{t\to\infty} S(t) = 0, \quad \lim_{t\to\infty} L(t) = 0.$$

Thus, for sufficiently small $\delta > 0$, there exists $t_1 > 0$ such that $0 \leq I(t) < \delta$, $0 \leq S(t) < \delta$ and $0 \leq L(t) < \delta$ for $t > t_1 + \tau + \omega$.

For $i = 1, 2$, fix a small $\epsilon_i > 0$. Since $\lim_{x\to 0} \frac{b_i(x)}{x} = b_i{}'(0)$, in a neighborhood of E_0, we have

$$\left|\frac{b_1(S(t))}{S(t)} - b_1'(0)\right| < \epsilon_1 \quad \text{and} \quad \left|\frac{b_2(L(t))}{L(t)} - b_2'(0)\right| < \epsilon_2.$$

Set $\epsilon = \max\{\epsilon_1, \epsilon_2\}$.

Since $\mathcal{R}_A > 1$, then under hypothesis (A_1), we have

$$\mathcal{R}_{A\delta\epsilon} = \frac{(b_1'(0)-\epsilon)e^{-(\mu_I+\delta\alpha)\tau}\left(1-e^{-(\mu_A+H_A(\delta))\omega}\right)+(b_2'(0)-\epsilon)e^{-(\mu_I+\delta\alpha)\tau}e^{-(\mu_A+H_A(\delta))\omega}}{\mu_A+H_A(\delta)} > 1$$

where $H_A(\delta) = \max\{H_1(\delta), H_2(\delta)\}$.
For $t \geq t_1+\tau+\omega$, we obtain, from (2.10) and the third equation in (2.12), that

$$\begin{aligned} S(t) &> \int_{t-(\tau+\omega)}^{t-\tau} e^{-(\mu_I+\delta\alpha)\tau}\left((b_1{}'(0)-\epsilon)S(\xi)+(b_2{}'(0)-\epsilon)L(\xi)\right)e^{-(\mu_A+H_A(\delta))(t-\tau-\xi)}d\xi, \\ \frac{dL(t)}{dt} &> e^{-(\mu_I+\delta\alpha)\tau}e^{-(\mu_A+H_A(\delta))\omega}\left((b_1{}'(0)-\epsilon)S(t-\tau-\omega)+(b_2{}'(0)-\epsilon)L(t-\tau-\omega)\right) \\ &\quad -(\mu_A+H_A(\delta))L(t). \end{aligned} \tag{2.26}$$

Consider the following linear system

$$\begin{aligned} V_1(t) &= \int_{t-(\tau+\omega)}^{t-\tau} e^{-(\mu_I+\delta\alpha)\tau}\left((b_1{}'(0)-\epsilon)V_1(\xi)+(b_2{}'(0)-\epsilon)V_2(\xi)\right)e^{-(\mu_A+H_A(\delta))(t-\tau-\xi)}d\xi, \\ \frac{dV_2(t)}{dt} &= e^{-(\mu_I+\delta\alpha)\tau}e^{-(\mu_A+H_A(\delta))\omega}\left((b_1{}'(0)-\epsilon)V_1(t-\tau-\omega)+(b_2{}'(0)-\epsilon)V_2(t-\tau-\omega)\right) \\ &\quad -(\mu_A+H_A(\delta))V_2(t). \end{aligned} \tag{2.27}$$

Since $\mathcal{R}_{A\delta\epsilon} > 1$, the characteristic equation of system (2.27)

$$\begin{aligned} &(\lambda+\mu_A+H_A(\delta))\left(\lambda+\mu_A+H_A(\delta)-(b_1'-\epsilon)(0)e^{-(\mu_I+\delta\alpha)\tau}e^{-\lambda\tau}\right. \\ &\left.-(b_2'(0)-b_1'(0))e^{-(\mu_I+\delta\alpha)\tau}e^{-(\mu_A+H_A(\delta))\omega}e^{-\lambda(\tau+\omega)}\right) = 0, \end{aligned}$$

has a positive real eigenvalue, which is a contradiction to $\lim_{t\to\infty}(S(t), L(t)) = (0,0)$. Thus, we have $W^s(E_0)\cap X^0 = \emptyset$, confirming condition (vii).

Now, by [75, Theorem 4.2], there exists a $\eta_1 > 0$ such that $\liminf_{t\to\infty} d\left(\Phi(t)\phi, X_0\right) \geq \eta_1$ for any $\phi \in X^0$ implying that the S and L components of the solution with initial function $\phi \in X^0$ satisfy

$$\liminf_{t\to\infty}(S(t), L(t)) \geq (\eta_1, \eta_1).$$

From (2.5) and (2.16), we have

$$\liminf_{t\to+\infty} I(t) \geq \liminf_{t\to+\infty}\int_{t-\tau}^{t} b\left(S(\xi), L(\xi)\right)e^{-(\mu_I+\alpha K)(t-\xi)}d\xi$$

$$= \liminf_{t\to+\infty} \int_0^\tau b\left(S(t-\xi), L(t-\xi)\right) e^{-(\mu_I+\alpha K)\xi} d\xi$$

$$\geq \liminf_{t\to+\infty} \int_0^\tau b\left(\eta_1, \eta_1\right) e^{-(\mu_I+\alpha K)\xi} d\xi = \frac{b\left(\eta_1, \eta_1\right)}{\mu_I + \alpha K}\left(1 - e^{-(\mu_I+\alpha K)\tau}\right) := \eta_2.$$

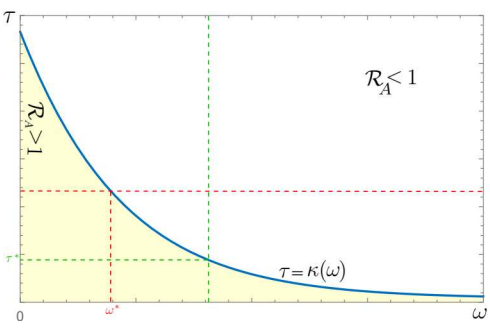
Choose $\eta = \min\{\eta_1, \eta_2\}$. This completes the proof. □

Now, we study the relation of $\mathcal{R}_A$ with respect to τ, ω and H_A. It is clear that the longer maturation period (τ) decreases $\mathcal{R}_A$ and can drive the population to extinction because $\frac{\partial \mathcal{R}_A}{\partial \tau} = -\mu_I \mathcal{R}_A < 0$. Furthermore, if we denote

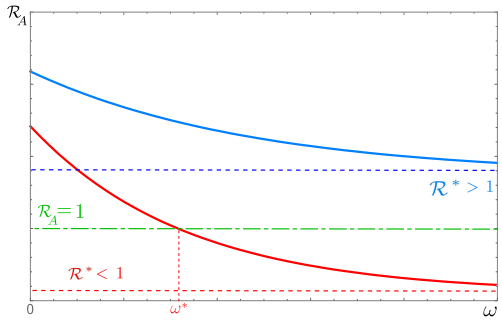
$$\kappa(\omega) = \frac{1}{\mu_I} \ln\left(\frac{b_1'(0) + (b_2'(0) - b_1'(0))e^{-(\mu_A+H_A)\omega}}{\mu_A + H_A}\right), \tag{2.28}$$

which is valid with (C_1), then $\mathcal{R}_A > 1$ (the population persists) if and only if $\tau < \kappa(\omega)$. The delay related domain of population persistence is $\{(\omega, \tau) \in \mathbb{R}_+ \times \mathbb{R}_+ : \tau < \kappa(\omega)\}$. When ω is fixed, $\mathcal{R}_A > 1$ when $\tau < \tau^* = \kappa(\omega)$. By fixing the value of τ, $\mathcal{R}_A > 1$ when $\omega < \omega^* = \frac{1}{\mu_A+H_A} \ln\left(\frac{e^{-\mu_I\tau}\left(b_2'(0)-b_1'(0)\right)}{\mu_A+H_A-b_1'(0)e^{-\mu_I\tau}}\right)$ if ω^* is valid. $\kappa(\omega)$, τ^* and ω^* are shown in Fig. 2.3a. With respect to ω, when (C_1) holds, $\frac{\partial \mathcal{R}_A}{\partial \omega} = -(b_2'(0) - b_1'(0))e^{-\mu_I\tau} e^{-(\mu_A+H_A(0))\omega} \leq 0$ and $\lim_{\omega\to\infty} \mathcal{R}_A = \frac{b_1'(0)e^{-\mu_I\tau}}{\mu_A+H_A(0)} := \mathcal{R}^*$. Thus, longer time period (ω) decreases $\mathcal{R}_A$ when $b_2'(0) > b_1'(0)$ and $\mathcal{R}_A = \mathcal{R}^*$ completely loses its dependence on ω when $b_2'(0) = b_1'(0)$. Actually, when $b_2'(0) = b_1'(0)$, both mature classes (small and large) are treated as one group (mature class), $\mathcal{R}_A$ is related only to the maturation time (τ) and ω becomes redundant. Consequently, the influence of ω on the population persistence is determined in two cases: (i) when $\mathcal{R}^* > 1$, the population persists for all $\omega > 0$ (the blue line in Fig. 2.3b); (ii) when $\mathcal{R}^* < 1$, ω^* exists such that $\mathcal{R}_A < 1$ for $\omega > \omega^*$, that is, large ω leads to extinction (the red line in Fig. 2.3b). Notice that $H_A \geq 0$. If $H_A = 0$, $\mathcal{R}_A$ is independent of the harvesting functions H_i ($i = 1, 2$) in (2.21). Thus, under the assumption (A_i) ($i = 1, 2$), the choice of the harvesting functions will not drive the population to extinction but can change the structure of the fishery resource for sure. While when $H_A > 0$, it is easy to check that $\frac{\partial \mathcal{R}_A}{\partial H_A} = \frac{-(b_2'(0)-b_1'(0))\omega e^{-\mu_I\tau} e^{-(\mu_A+H_A(0))\omega}}{(\mu_A+H_A)^2} - \frac{\mathcal{R}_A}{\mu_A+H_A} < 0$, hence, the population is at risk of dying out when H_A is large enough (see Fig. 2.4). Biologically, the maturation age of fish (τ) depends on environmental factors and genetic changes [142] while most of fish can continue growing in length and weight after they have reached sexual

maturity [206]. Normally, large ω leads to a large size fish. Therefore, through the theoretical analysis, in size-selective harvesting, when fish are left to grow until they reach a preferred specific (large) size, we find that the value of ω should be lower than ω^* (if it exists) for population persistence. In real life, ω^* is out of our control because it depends on natural parameters, such as the per-capita mortality rate μ_S/μ_L for small/large matured fish. On the other hand, we can control the initial harvesting value H_A, by keeping it small enough which has important consequences for population persistence.



(a) The delay related domain of population persistence $\kappa(\omega)$ in (2.28).



(b) $\mathcal{R}_A$ vs ω (blue line $\mathcal{R}^*>1$, red line $\mathcal{R}^*<1$).

Figure 2.3 The relationship between $\mathcal{R}_A$, τ and ω.

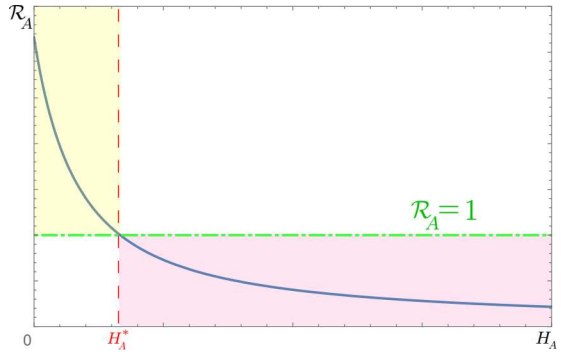


Figure 2.4 $\mathcal{R}_A$ vs H_A.

In view of the uniform persistence result, Theorem 2.5.3, we cannot guarantee the existence of a positive steady state because of the uncertainty about the convexity of $\mathcal{X}$. However, with an additional condition, we have the following result for the existence of a unique positive equilibrium point.

Theorem 2.5.4. *Let* $\mathcal{R}_A > 1$ *and*

$$(C_3) \qquad \mu_I \mu_A e^{\mu_A \omega} > (e^{\mu_A \omega} - 1)(e^{\mu_I \tau} - 1).$$

Then, a positive equilibrium point $E^* = (I^*, S^*, L^*)$ *exists and is unique.*

Proof. Let (I^*, S^*, L^*) be a positive steady state. Define

$$f(x) = \frac{x(\mu_I + \alpha x)}{\exp\{(\mu_I + \alpha x)\tau\} - 1} \text{ and } g(y) = \frac{y(\mu_A + H_1(y))}{1 - \exp\{-(\mu_A + H_1(y))\omega\}}.$$

Then it follows from the first and second equations in (2.12) that

$$f(I^*) = \exp\{-(\mu_I + \alpha I^*)\tau\}(b_1(S^*) + b_2(L^*)) = g(S^*). \tag{2.29}$$

Now, in the (x, y) plane, we prove that functions $f(x)$ and $g(y)$ intersect at a unique point. First, we rewrite g as

$$g(y) = \frac{y}{\omega\ k((\mu_A + H_1(y))\omega)},$$

where $k(y) = (1 - e^{-y})/y$. Since $k(y)$ is decreasing and $H_1(y)$ is nondecreasing or increasing for $y > 0$, $g(y)$ is increasing for $y > 0$ and $g(0) = 0$ while $f(0) = 0$ and $\lim_{x \to +\infty} f(x) = 0$.

Claim 3. There exists a unique $\hat{x} > 0$ such that $f(x) \leq f(\hat{x})$ for all $x \in (0, \infty)$.

To prove Claim 3, it is enough to show that $\frac{df}{dx} = 0$ has exactly one positive root since f is smooth enough, $f(0) = 0$, $f(x) > 0$ and $\lim_{x \to +\infty} f(x) = 0$.

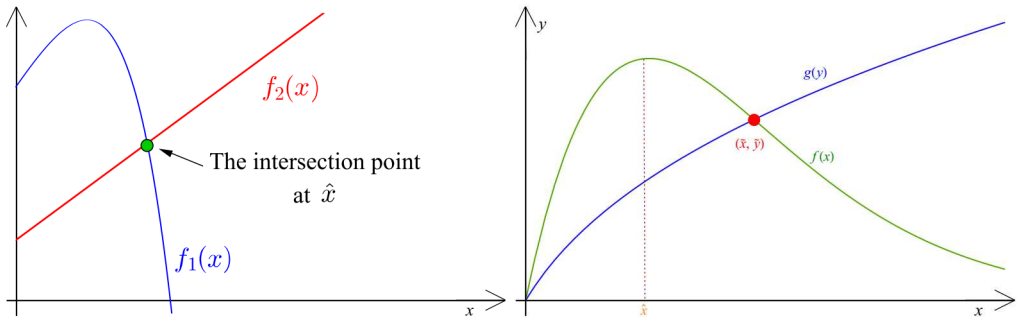
$$\frac{df(x)}{dx} = \frac{f_1(x) - f_2(x)}{(\exp\{(\mu_I + \alpha x)\tau\} - 1)^2},$$

where

$$f_1(x) = \left(\mu_I + \alpha(2 - \mu_I \tau)x - \alpha^2 \tau x^2\right) \exp\{(\mu_I + \alpha x)\tau\} \quad \text{and} \quad f_2(x) = \mu_I + 2\alpha x.$$

$\frac{df(x)}{dx} = 0$ if and only if $f_1(x) = f_2(x)$. For $f_1(x)$, it is easy to see that $f_1(0) = \mu_I e^{\mu_I \tau}$ and $\lim_{x \to +\infty} f_1(x) = -\infty$. We can check that f_1 is concave down for $x > 0$ and there is a unique intersection point, $\hat{x}$, between f_1 and f_2, since f_2 is increasing for all $x > 0$ and $f_2(0) = \mu_I \leq f_1(0)$ (see Fig. 2.5a).

Notice that $f(0) = 0$, $f'(0) = \frac{\mu_I}{e^{\mu_I \tau} - 1} > 0$, f is positive for $x > 0$ and $\lim_{x \to +\infty} f(x) = 0$.



(a) $f_1(x)$ and $f_2(x)$. (b) $f(x)$ and $g(y)$.

Figure 2.5 Intersection points.

Thus, f has exactly one extreme value at $\hat{x} > 0$ satisfying $f(x) \leq f(\hat{x})$ for all $x \in (0,\text{-}\infty)$. This proves the claim.
From Claim 3, we know that f is increasing in $(0, \hat{x})$ and decreasing in $(\hat{x}, +\infty)$.

$$\left.\frac{df}{dx}\right|_{x=0} = \frac{\mu_I}{e^{\mu_I \tau} - 1} \quad \text{and} \quad \left.\frac{dg}{dy}\right|_{y=0} = \frac{\mu_A e^{\mu_A \omega}}{e^{\mu_A \omega} - 1}.$$

Thus, if

$$\frac{\mu_I}{e^{\mu_I \tau} - 1} > \frac{e^{\mu_A \omega} - 1}{\mu_A e^{\mu_A \omega}},$$

there exists exactly one point $(\tilde{x}, \tilde{y})$ satisfying $f(\tilde{x}) = g(\tilde{y})$ because $g(y)$ is increasing for $y > 0$, $f(0) = g(0) = 0$ and $\lim_{y \to +\infty} g(y) = +\infty$ (see Fig. 2.5b), which implies that there exists a unique (I^*, S^*) satisfying $f(I^*) = g(S^*)$. Now we prove the uniqueness of L^*. From the third equation in (2.12), we have

$$(b_1(S^*) + b_2(L^*)) \exp\{-(\mu_I + \alpha I^*)\tau\} \exp\{-(\mu_S + H_1(S^*))\omega\} = L^*(\mu_A + H_2(L^*)).$$

Hence, from (2.29), we obtain

$$g(S^*) \exp\{-(\mu_S + H_1(S^*))\omega\} = L^*(\mu_A + H_2(L^*)). \tag{2.30}$$

Since the left-hand side of the latter equation is positive and the right-hand side function $h(z) = z(\mu_A + H_2(z))$ has the properties $h(0) = 0$, $\frac{dh}{dz} > 0$ and $\lim_{z \to +\infty} h(z) = +\infty$, there exists exactly one positive solution in L^* satisfying (2.30). This completes the proof. □

The condition (C_3) is equivalent to

$$e^{-\mu_I \tau} > \sqrt{\left(\frac{1-e^{-\mu_I \tau}}{\mu_I}\right)\left(\frac{e^{-\mu_I \tau}-e^{-\mu_I \tau-\mu_A \omega}}{\mu_A}\right)}.$$

Biologically, $1-e^{-\mu_I \tau}$ is the probability of being in the immature period and $\frac{1}{\mu_I}$ is the average lifespan of juveniles. Thus, $\frac{1-e^{-\mu_I \tau}}{\mu_I}$ is the time period of the immature class. Parallely, $\frac{e^{-\mu_I \tau}-e^{-\mu_I \tau-\mu_A \omega}}{\mu_A}$ is the time period of the small mature class. The term $e^{-\mu_I \tau}$ is the survival rate of individuals from the immature to small mature periods. Therefore, the condition (C_3) means that the survival rate of juveniles becoming small matured fish is higher than the geometric mean of the time period of juveniles and that of the small mature class.

2.6 Numerical simulations

In this section, we explore the influence of various harvesting function forms on the positive steady state, discuss the optimal harvesting rates and address the effect of periodic coefficients on the dynamical system.

2.6.1 Effect of harvesting functions

First, we investigate the influence of different harvesting functions on the positive equilibrium point. Without loss of generality, we choose the egg laying rate function to be linear, $b_i(x) = r_i x$ $(i = 1, 2)$. Hypothesis (A_2)-(b) requires the harvesting functions to be strictly increasing. Thus, we test harvesting function forms in linear $H_i^{(1)}(x) = k_i x + d$ and nonlinear $H_i^{(2)}(x) = \frac{a_i x}{1+c_i x}$ $(i = 1, 2)$ [208]. We adopt the parameter values μ_I, μ_A, r_1, r_2, τ, ω from [107] and α from [76] which are given in Table 2.1, and choose the other parameters as in Table 2.2.

Parameter	μ_I	μ_A	r_1	r_2
Value	0.4 year^{-1}	$(0.2, 0.3)$ year^{-1}	0.8 year^{-1}	1 year^{-1}
Parameter	τ	ω	α	
Value	1 year	1.25 year	0.05	

Table 2.1 Parameter values for the model (2.12).

When we fix $\mu_A = 0.25$, the adult reproduction number $\mathcal{R}_A = 2.44 > 1$ ($\mathcal{R}_A = 2.54 > 1$) with $H_i^{(1)}$ ($H_i^{(2)}$) and (C_3) holds. Then the positive equilibrium point is

Parameter	k_1	k_2	d	a_1	a_2	c_1	c_2
Value	0.05	0.1	0.01	0.2	0.3	0.5	0.4

Table 2.2 Parameter values for the model (2.12).

$(4.87, 3.28,\ 3.91)$ and $(2.53, 1.74, 1.76)$ with $H_i^{(1)}$ and $H_i^{(2)}$, respectively. Although theoretically, we could not analyze the local stability of the positive equilibrium point E^* due to the complexity of the linearization system of (2.12) at E^*, numerically, the local stability of E^* with the harvesting functions $H_i^{(1)}$ and $H_i^{(2)}$ is shown in Fig. 2.6. On the other hand, when there is no harvesting (the harvesting terms in (2.12) are zeros), the positive equilibrium point is $(18.62, 9.56, 25.94)$. We can notice that harvesting affects the sustainability of the population and different harvesting functions have different impacts on the population.

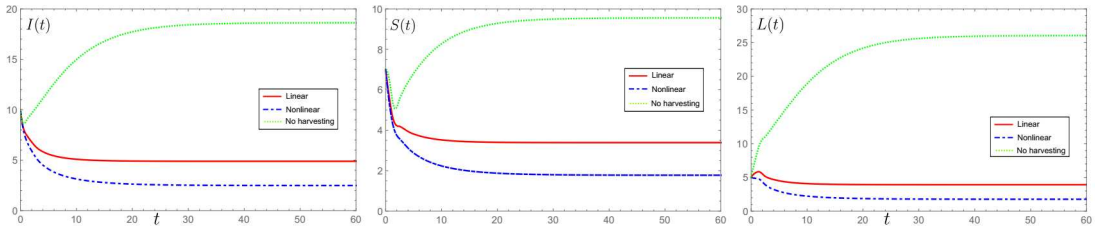


Figure 2.6 Time series with linear egg laying rate functions and linear (solid red lines)/nonlinear (dashed blue lines) harvesting functions/no harvesting (dotted green lines).

In addition, we can conjecture that when $\mathcal{R}_A > 1$ and (C_3) holds, the unique positive equilibrium is not only locally asymptotically stable but also globally asymptotically stable (Fig. 2.7).

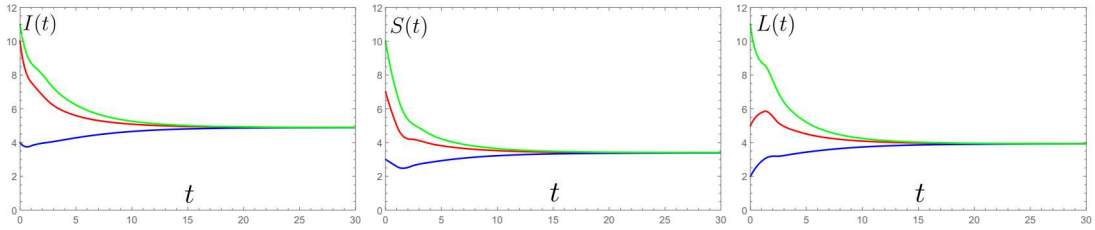


Figure 2.7 Global stability of $E^* = (4.87, 3.28, 3.91)$ under different initial conditions.

2.6.2 Optimal harvesting rates

In fisheries management, the maximum sustainable yield is used to describe the highest average catch that does not reduce a stock's abundance over time. In general, the harvesting function depends on the harvesting rate, and the optimal harvesting rate maximizes the sustainable yield. Mathematically, the harvest is sustainable when the population eventually reaches a positive steady state. So to find the optimal harvesting rates it suffices to maximize the harvesting terms in the mathematical model at a positive equilibrium point (see e.g., [77,145,174,208]). Let h_1, h_2 be the harvesting rate of small, large matured fish, respectively, and denote $H_1(S(t), h_1) := H_1(S(t))$ and $H_2(L(t), h_2) := H_2(L(t))$ to emphasize the dependence on h_i $(i = 1, 2)$. It is clear from (2.29) and (2.30) that the value of the positive equilibrium point $E^* = (I^*, S^*, L^*)$ depends on h_1 and h_2 as well. Then, the sustainable yield in (2.12) is

$$\Gamma(h_1, h_2) = H_1(S^*(h_1, h_2), h_1)S^*(h_1, h_2) + H_2(L^*(h_1, h_2), h_2)L^*(h_1, h_2). \tag{2.31}$$

To calculate the optimal harvesting rates, we need to find values of h_1 and h_2 that maximize $\Gamma(h_1, h_2)$. From

$$\begin{aligned}
\frac{\partial \Gamma}{\partial h_1} &= \left(\frac{\partial H_1}{\partial h_1} + \frac{\partial H_1}{\partial S^*}\frac{\partial S^*}{\partial h_1}\right) S^* + H_1(S^*, h_1)\frac{\partial S^*}{\partial h_1} + \frac{\partial H_2}{\partial L^*}\frac{\partial L^*}{\partial h_1}L^* + H_2(L^*, h_2)\frac{\partial L^*}{\partial h_1} = 0,\\
\frac{\partial \Gamma}{\partial h_2} &= \left(\frac{\partial H_2}{\partial h_2} + \frac{\partial H_2}{\partial L^*}\frac{\partial L^*}{\partial h_2}\right) L^* + H_2(L^*, h_2)\frac{\partial L^*}{\partial h_2} + \frac{\partial H_1}{\partial S^*}\frac{\partial S^*}{\partial h_2}S^* + H_1(S^*, h_1)\frac{\partial S^*}{\partial h_2} = 0,
\end{aligned}$$

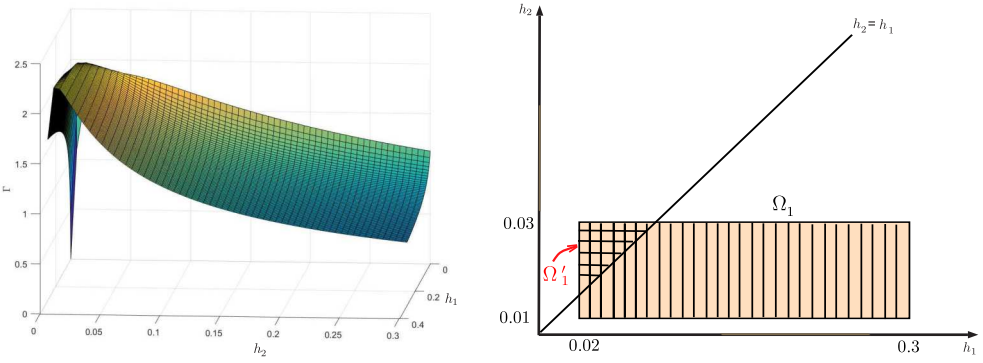
and $D(h_1, h_2) = \frac{\partial^2 \Gamma}{\partial h_1^2}\frac{\partial^2 \Gamma}{\partial h_2^2} - \left(\frac{\partial^2 \Gamma}{\partial h_1 \partial h_2}\right)^2$, we know that the critical point (h_1^*, h_2^*) is a relative (local) maximum if $\frac{\partial \Gamma}{\partial h_1}\Big|_{(h_1^*, h_2^*)} = \frac{\partial \Gamma}{\partial h_2}\Big|_{(h_1^*, h_2^*)} = 0$, $D(h_1^*, h_2^*) > 0$ and $\frac{\partial^2 \Gamma}{\partial h_1^2}\Big|_{(h_1^*, h_2^*)} < 0$. Theoretically, it is impossible to find an explicit form of $S^*(h_1, h_2)$ and $L^*(h_1, h_2)$ because of the exponential terms, $e^{-\int_{t-\tau}^{t} \alpha I(\eta)d\eta}$ and $e^{-\int_{t-\omega}^{t} H_1(S(\eta))d\eta}$ in (2.12), even when we choose linear egg laying rate function $b_i(x) = r_i x$ and linear harvesting function $H_i(x) = h_i x + d$. In the following, we use numerical methods to find an optimal harvesting rates h_1^* and h_2^*. Fixing all the parameters and choosing $H_i^{(1)}(x)$ with a released value h_i, $i = 1, 2$, given in Table 2.2, we see that (2.31) becomes

$$\Gamma_1(h_1, h_2) = h_1 S^{*^2}(h_1, h_2) + h_2 L^{*^2}(h_1, h_2) + d(S^*(h_1, h_2) + L^*(h_1, h_2)).$$

We can see that, in Fig. 2.8a, the peak of the surface $\Gamma_1(h_1, h_2)$ occurs in the region $\Omega_1 = \{(h_1, h_2)|0.02 \leq h_1 \leq 0.3,\ 0.01 \leq h_2 \leq 0.03\}$ which is shown in Fig. 2.8b (the orange shaded region). Similarly, with $H_i^{(2)}(x)$, the sustainable yield is

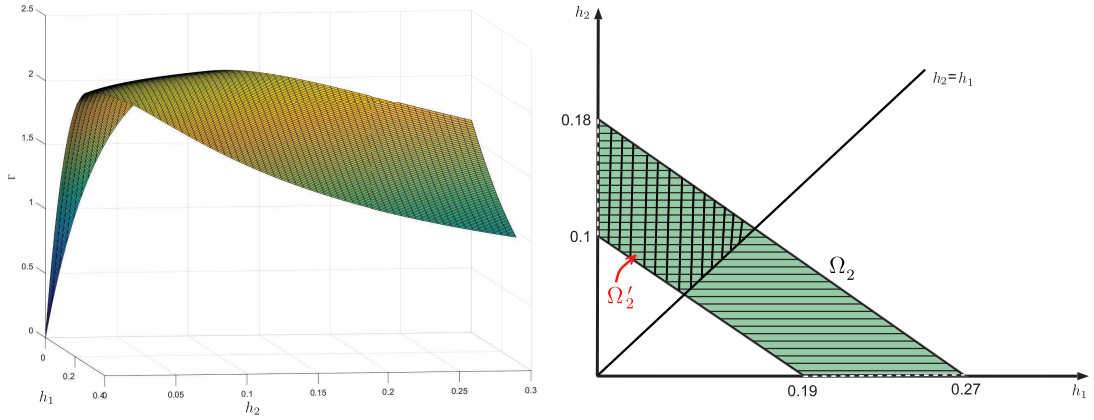
$$\Gamma_2(h_1, h_2) \quad = \quad \frac{h_1 S^{*2}(h_1, h_2)}{1 + 0.5S^*(h_1, h_2)} + \frac{h_2 L^{*2}(h_1, h_2)}{1 + 0.4L^*(h_1, h_2)}.$$

Consequently, the optimal harvesting rates lie in the region $\Omega_2 = \{(h_1, h_2)|h_1 > 0, h_2 > 0, -0.526h_1 + 0.01 \leq h_2 \leq -0.677h_1 + 0.18\}$ (the green shaded region in Fig. 2.9b). Although targeting large fish (size-selective harvesting) is preferable for many of the world's fisheries, our model predicts that a maximum sustainable yield happens when there is harvesting for both large and small matured fish since in this maximum region either h_1 or h_2 is nonzero. Therefore, we can foresee that an appropriate variety portion in harvesting the two classes of matured fish can produce a maximum sustainable yield, in the sense that, targeting large fish in the whole stock while catching small matured fish is also reasonable in the stock spawning grounds. This can decrease the rate of catching large fish and lead to a balance in the size, structure, and distribution of the population with a maximum sustainable yield. Due to the fact that, catching large fish is allowed in the whole stock while catching small matured fish only happens in the stock spawning grounds, in general $h_1 < h_2$. Therefore, the optimal harvesting rates should be chosen from the mesh shaded regions Ω_1' or Ω_2' in Fig. 2.8b and 2.9b, with respect to the harvesting functions $H_i^{(1)}(x)$ or $H_i^{(2)}(x)$, respectively.



(a) $\Gamma_1(h_1, h_2)$ (b) Relative maximum region

Figure 2.8 The sustainable yield with linear harvesting functions.



(a) $\Gamma_2(h_1, h_2)$ (b) Relative maximum region

Figure 2.9 The sustainable yield with nonlinear harvesting functions.

2.6.3 Periodic coefficients

Spawning is one of the most significant reproductive phases in fish life cycle which has a direct impact on the population size, continuation and survival of the species. The term "spawning season" is referred to the period of ovulation or spermiation in a population. A majority of fishes all over the world are seasonal spawner. This is related to environmental factors, such as, temperature and rainfall. For example, in the Indian sub-continent, a vast majority of the freshwater fishes spawn at the time of heavy rainfall [6]. Harvesting, as a method of exploitation and management of fishery resources, often has a direct relationship to demographic variation and population fluctuations. To discuss the effect of seasonal fluctuations on the dynamical system, we assume that the egg laying rate function for small and large matured fish is $b_1 = r_1(t)S(t)$, $b_2 = r_2(t)L(t)$ and the harvesting functions are $H_1 = k_1(t)S(t) + d$, $H_2 = k_2(t)L(t) + d$ with annually periodic time-dependent fecundity and harvesting rates, respectively, where

$$\begin{aligned} r_1(t) &= 0.2(\cos(2\pi t) + 4)\ \text{year}^{-1}, \quad r_2(t) = 0.25(\sin(2\pi t) + 4)\ \text{year}^{-1}, \\ k_1(t) &= 0.025(\cos(2\pi t) + 2)\ \text{year}^{-1}, \quad k_2(t) = 0.05(\sin(2\pi t) + 2)\ \text{year}^{-1}. \end{aligned} \tag{2.32}$$

Notice that the average value for $t \geq 0$ is, respectively, $r_1 = 0.8$, $r_2 = 1$, $k_1 = 0.05$ and $k_2 = 0.1$, which are the same as in Fig. 2.6, then the average adult reproduction number $\mathcal{R}_A = 2.44 > 1$. The upper row of Fig. 2.10 shows the existence of a positive

periodic solution and the three classes become oscillatory around the values of E^* arising from constant parameters. When we increase τ from 1 to 3.5 and ω from 1.25 to 2.5, the average adult reproduction number $\mathcal{R}_A$ decreases to $0.85 < 1$. The lower row of Fig. 2.10 shows the extinction of all classes, which is consistent with our analysis when the average adult reproduction number $\mathcal{R}_A < 1$ and all parameters are constants. We would like to mention that the adult reproduction ratio of time-delayed models with periodic coefficients can be obtained theoretically by using the method developed by Zhao in [217], although the explicit value is only possible by numerical approximation, using the algorithms developed by Bacaër in [19] for instance. In general, the adult reproduction number of the time-averaged autonomous system may coincide with the adult reproduction ratio of the periodic model or underestimate/overestimate population persistence [198].

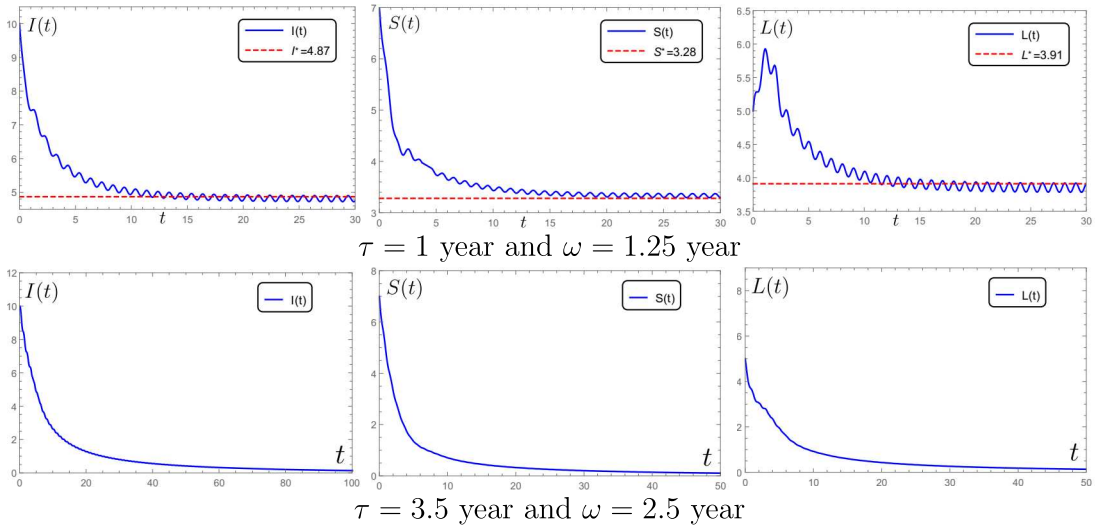


Figure 2.10 Time series for $I(t)$, $S(t)$ and $L(t)$ with periodic functions in (2.32).

2.7 Discussion

Mathematical models can provide an important approach to understanding the risk of human exploitation on fish resources. In this chapter, we have proposed a growth model of a fish stock with three stages: juveniles, small adults and large adults, and introduced two harvesting strategies: maturity and size selectivity. The maturity selectivity is based on targeting matured fish in the stock spawning grounds, while in

the whole stock, size selectivity is used to catch large fish. From the dynamical point of view, we have studied the qualitative features of the system, such as existence and uniqueness of solutions, boundedness, equilibrium points, persistence, and stability, with respect to the adult reproduction number $\mathcal{R}_A$. We also have addressed some important ecological features, such as the adult reproduction number and the optimal harvesting rates.

In many of the world's fisheries, size-selective harvesting is preferable, where the large individuals of fish are preferentially taken. However, this action has a negative effect on the population demography, life history and genetics. This is consistent with what we have explored in this chapter that overharvesting large matured fish after a certain age can lead to population extinction under certain circumstances (e.g. $\mathcal{R}^* < 1$). Through our theoretical model analysis, we have found the adult basic reproduction number $\mathcal{R}_A$ decreases when either maturation period τ or initial harvesting value H_A increases, and hence, large value of either τ or H_A leads to population extinction. $\mathcal{R}_A$ completely loses its dependence on ω when $b_2{}'(0) = b_1{}'(0)$ and it decreases for long time period ω when $b_2{}'(0) > b_1{}'(0)$. In addition, we have found that when $\mathcal{R}^* > 1$, the population persists for any value of ω since $\mathcal{R}_A \geq \mathcal{R}^* > 1$. In this case, targeting large matured fish does not affect the population persistence. While when $\mathcal{R}^* < 1$, large ω leads to population extinction. Hence, there exists ω^* such that $\mathcal{R}_A > 1$ when $\omega < \omega^*$. Although theoretically, harvesting large matured fish before age $\tau + \omega^*$ can help the population to persist, in the real life, we cannot control ω because it depends on natural parameters-the per-capita mortality rates for small and large matured fish.

In the numerical study, we have explored the sustainable yield $\Gamma(h_1, h_2)$ and found that its maximum occurs when there is harvesting for both large and small matured fish. We have addressed a relative maximum region for Γ which contains the optimal harvesting rates h_1^*, h_2^*. In real-life application, we suggest using two harvesting strategies, in the sense of targeting large fish in the whole stock while allowing small matured fish to be caught in the stock spawning grounds. Also, we should choose the harvesting rate of large fish to be bigger than the harvesting rate of small matured fish due to the fishing region size. In this scenario, we can guarantee a balance in the size, structure, and distribution of the population with a maximum sustainable yield.

In [107, 145], the authors used ordinary differential equations to study a stage-structured (size-structured) aquatic ecosystem population model with the assumption

that the immature individuals can mature early or late and then continue growing to become large matured individuals. However, there is no indication about the effect of two important criteria in fishery regulations on the population: age at puberty and age at large size. In general, in a structured population model based on age, the state of the population depends on its time history. A realistic approach is to use delays to incorporate the history of the population into a mathematical model [27]. In this chapter, we have considered the ages τ, the maturation age, and ω, the time to reach large size after maturation ($\tau + \omega$ is the age at large size), and we have built a time lagged model and studied their influence on the population dynamics. In addition, it is shown in [145] that the basic (adult) reproduction number depends on the average fecundity rate of an individual in different size mature classes and the average time spent on each class. Meanwhile in our model, we have found that the basic reproduction ratio depends on both time periods τ, ω and the initial harvesting value H_A (see (2.20) and (2.21)), indicating that these key parameters can play an important role in population extinction and persistence.

Recently, an ecosystem-based fisheries management strategy "balanced harvesting" has been suggested in several studies to increase yields, and is calling for harvesting all species and sizes in an ecosystem in proportion to their productivity [37,61,108]. However, in [37], Burgess et. al mentioned that some aspects of balanced harvesting are controversial, like its call for extensive harvesting of juveniles, and clarified that implementation steps towards balanced harvesting are still premature until it has a clear definition in both conceptually and practice. Also they addressed some important questions to evaluate balanced harvesting. For instance, what sizes should be off limits to fishing? What sizes should be fished to maximize yields? What are the ecological benefits of balanced harvesting and how do we value these? We believe our model may help to answer some of these questions and create a better understating of balanced harvesting. More precisely, we have studied balanced harvesting in the sense of allowing the catching of matured fish regardless of size and avoiding juveniles. Also based on the importance of knowing fish age for stock assessments and by taking into consideration the relation between fish age and size [81], we have allowed fishing at age greater than τ and identified an age $\tau + \omega^*$ for population persistence with an optimal harvesting rates for the maximum sustainable yield.

Chapter 3

Sea Lice Model with Stage Structure

3.1 Introduction

Since the 1970s, salmon production in farms has increased exponentially throughout Canada, Chile, Ireland, Norway and Scotland. The number of farmed salmon was growing from a few thousand tonnes in 1980 to about 2.5 million tonnes in 2014. Sea lice infection became one of the major threats on farmed salmon during the past 40 years [3, 151]. Sea lice are marine ectoparasites that feed on the mucus and tissue of salmonids. In salmon-producing countries, salmon lice are responsible for many outbreaks of disease in salmonid aquaculture, causing enormous economic losses in the salmon aquaculture industry and costing millions of dollars annually [3, 162]. According to the Atlantic Salmon Federation, the costs of fighting sea lice is growing. For instance, in Norway, there was an increase of 32.5% in the cost of sea lice mitigation. Therefore, it is essential to develop mathematical models to predict the variations in sea lice abundance.

Since the first age-structured population model designed by McKendrick in 1926 [128], there has been a fair amount of work on modeling populations with various stages of life history [5, 7–9, 102, 118, 122, 154, 156, 164, 166–168, 211]. For example, the authors proposed a stage-structured model of single species with two life stages, immature and mature and with a constant time from birth to maturity in [7]. In [5], the authors formulated and analyzed a single-species growth model with stage-structure consisting of immature and mature stages for the effects of toxicants with

constant maturation time delay. In [211], the author considered a prey-predation model with a maturation time in predator, studied the stability and bifurcation with or without the maturation time delay. In [122, 154], parasite life stage models were derived with periodic delays due to the seasonally varying temperature on parasite maturation. The basic reproduction ratio is introduced and the long-term behavior of solutions is investigated in [122]. In a series of papers [164, 166–168], Smith has discussed a single species population model with two classes, immature and mature, where the immature individuals grow at a non-constant rate. In [164], he assumed a competition between adults and juveniles; and the maturation rate of the juveniles depends on the density of adults, in the sense that, as the adult population increases, the rate of maturation of juveniles decreases, which causes juveniles to remain in the juvenile stage longer and therefore be exposed to increased mortality. In [166], he assumed the maturation rate of the larvae depends on the food density in which the maturation rate increases as the food density increases. Recently, in [102], the authors used Smith's techniques to derive a nutrient-phytoplankton-zooplankton model where the maturation rate of the juvenile zooplankton (predator) depends on the quantity of phytoplankton (prey).

The sea louse exhibits several distinct life stages in two separate phases: the free-living phase and the parasitic phase. In general, during the life cycle of a sea louse, eggs hatch into a non-infectious free-living nauplius, then after 2 to 14 days this nauplius moults into an infective copepodid and starts searching for a host. Once attached to a host, the copepodid feeds on the mucous and skin of the host and begins to develop into a parasitic sea louse, i.e., the stage of sea lice attached to the fish. In this chapter, we assume the growth of sea lice through three stages: non-infectious larvae and infectious larvae in the free-living phase, adults in the parasitic phase, and the development age for non-infectious larvae to develop into infectious larvae depends on the size of adult population size, in the sense that, a larger mature population is more favorable than a smaller one for facilitating development. Biologically, this is reasonable, since the growth of immature and matured sea lice depends on different levels of food resources. In nature, nauplii and copepodids are non-feeding and live on their energy reserves [29], while adult sea lice feed on the mucus, skin tissue, and the blood of fish [60]. The main goal of this work is to model the growth process and study the dynamical behavior of sea lice population. As a starting point, we adopt the idea from Smith's work [164, 167, 168] to construct a system of partial differential

equations (PDE) to describe the dynamics, then by using the technique of integration along characteristics, we reduce the system to threshold delayed differential equations system (TDE). By the changing variables suggested in [164, 167, 168], we remove the state-dependent delay in (TDE) and transform it into a standard time-delayed differential equations (DDE). Based on the proposed delay mathematical model DDE, we study the nonlinear dynamics in the system including the threshold dynamics with respect to the adult reproduction number $\mathcal{R}_s$ under biologically reasonable conditions.

The rest of the chapter is organized as follows: in Section 3.2, we propose a PDE system to describe the population dynamics of sea lice, solve the PDE model to construct a system of threshold type delay TDE, and transform the obtained TDE system into a DDE with constant delay.. In Section 3.3, we discuss the well-posedness property by verifying the non-negativity and boundedness of the solutions with reasonable initial data, calculate the adult reproduction number $\mathcal{R}_s$ and address the local stability of the trivial equilibrium point. In Section 3.4, we establish the threshold dynamics for the system in terms of $\mathcal{R}_s$ by proving the global stability of the trivial equilibrium point when $\mathcal{R}_s < 1$, sea lice persistence, coexistence and global attractivity of positive steady state when $\mathcal{R}_s > 1$, and discuss the sensitivity of $\mathcal{R}_s$ with respect to the related parameters. In Section 3.5, we present some numerical simulations using *Lepeophtheirus salmonis* growth as a case study. Finally, conclusion and remarks are drawn in Section 3.6.

3.2 The mathematical model

We model the growth of sea lice on salmon farms by using the idea of age structure. As we know, the sea louse life cycle has two distinct phases, the free-living (immature) phase and the parasitic phase, we assume the immature population in the free-living phase is split into two classes: non-infectious larvae "nauplii" and infectious larvae "copepodids", and the parasitic phase includes "adult sea lice" live over the surface of the fish and can move among the host fish [44, 147]. We include the copepodid stage in the immature phase due to the crucial role in sea louse life cycle. According to [112, 143], the copepodid larva is the invasive stage which transforms, on reaching the fish skin, to chalimus larvae, which are attached to the fish skin by an invasive frontal filament. It perhaps the most important stage in the life cycle of sea louse since it leaves the free-living phase and becomes parasitic after crossing this stage.

Let $A(t)$ and $C(t)$ denote the adult parasites and copepodids population sizes at time t, respectively. We assume that the nauplii are not identical but differ in their "level of development" (x) which represents the age. The development level varies between $x = 0$ (newborn non-infectious larvae) and $x = m$ (infectious larvae); an nauplius attaining infective level (age) m immediately enters the copepodid population and becoming infectious individual. In general, the free-living nauplii take from 2 to 14 days to molt into the infective copepodid stage [42, 95]. Let $u(t, x)$ denote the density of nauplii at development level x and time t. Thus, at time t, when $x \in [x_1, x_2] \subset [0, m]$, the number of nauplii is $\int_{x_1}^{x_2} u(t, x)dx$. Adopting the idea from [164, 167, 168], we assume the rate $\frac{dx}{dt}$ depends on the current adult parasites population size:

$$\frac{dx}{dt} = P\left(A(t)\right) \tag{3.1}$$

where $P : [0, \infty) \to [P_0, \infty)$, for some $P_0 > 0$, is continuously differentiable function and satisfies the following property:

(Q_1) $P(0) = P_0 > 0$, $\lim_{s\to\infty} P(s) = P_\infty < \infty$ and $\frac{dP(s)}{ds} \geq 0$.

Biologically, this means a larger mature population is more favorable than a smaller one for facilitating development [164].

Parasitic sea lice is disrupting salmon farms around the world and inescapable part of salmon aquaculture. Generally speaking in a salmon farm, the salmon eggs are fertilized and raised on hatcheries. Then, the smolt (juvenile salmon) are transferred to floating sea cages or net pens (salmon farms) until they are harvested [2, 99]. In other words, salmon can be introduced to sea cages or pens at most times of the year [154]. To avoid the waste of productivity and to make more profits, the population of farmed salmons would be kept around the best suitable number. Therefore, we assume the number of salmon (host) in the salmon farm is constant H.

Based on the above assumptions, we propose the following equations to model the interaction among the three species for $t \geq 0$ and $0 \leq x \leq m$.

$$\frac{\partial u(t, x)}{\partial t} + P\left(A(t)\right)\frac{\partial u(t, x)}{\partial x} = -\mu_N u(t, x), \tag{3.2a}$$

$$\frac{dC}{dt} = P\left(A(t)\right)u(t, m) - \mu_C C(t) - \beta H C(t), \tag{3.2b}$$

$$\frac{dA}{dt} = \beta H C(t) - \mu_A A(t), \tag{3.2c}$$

$$P\left(A(t)\right)u(t, 0) = b(A(t)), \tag{3.2d}$$

where μ_z, $z \in \{N, C, A\}$, is the mortality rate of nauplii, copepodids and adult parasites, respectively; β is the infection rate; and $b(\cdot)$ is the egg laying rate function of adult sea lice. An architecture of the model (3.2) is given in Figure 3.1. We will refer to system (3.2) as the PDE model.

Appropriate initial conditions for the PDE model are:

$$C(0) = C_0, \quad A(0) = A_0, \quad u(0, x) = u_0(x),$$

where C_0, A_0 are nonnegative real numbers and u_0 is a nonnegative continuous function on the interval $[0, m]$.

From the view points in biology and analysis, we assume that the function b is continuous and differentiable, and satisfies the following property:

(Q_2) $b(0) = 0$, $b(s) > 0$ for $s > 0$, $\frac{db(s)}{ds} \geq 0$ and there exists an increasing function B such that $b(s) \leq B(s)$ for $s \geq 0$, $B(0) = 0$, and for any $\vartheta \in (0, 1)$, $s > 0$, $B(\vartheta s) > \vartheta B(s)$ (i.e. B is strictly sub-homogeneous).

We can view (Q_2) as a mathematical interpretation of the biological factor that the growth of the population is slower than exponential, so we restrict the growth rate function with sub-homogeneously condition.

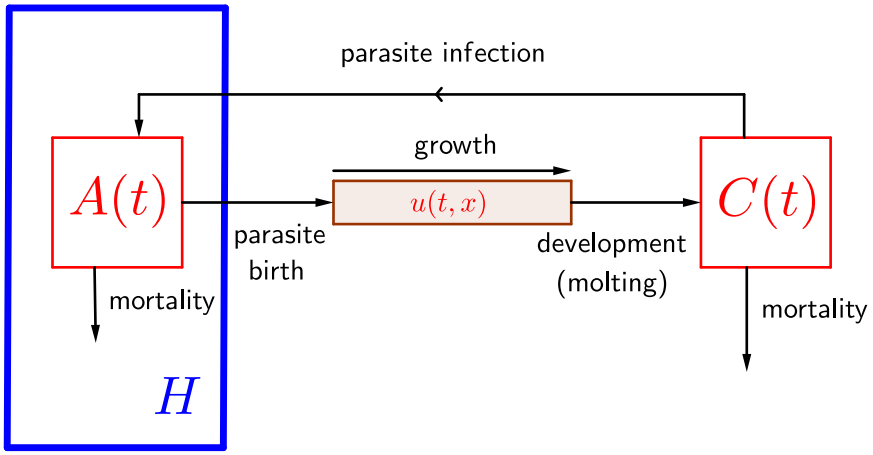


Figure 3.1 Schematic chart of the PDE model.

Next, we reduce the system (3.2) into a threshold delay system with a state-dependent delay. From (3.1), we have

$$x = \int_0^t P\left(A(\eta)\right) d\eta := r(t).$$

Let $t_0 > 0$ be the first time after which the non-infectious larvae population (nauplii) present at $t = 0$ has become infectious (copepodids), that is,

$$r(t_0) = \int_0^{t_0} P\left(A(\eta)\right) d\eta = m.$$

Then the characteristic curve in (3.2a) is

$$C = \left\{(x,t) : 0 \le t \le t_0 \text{ and } x = r(t)\right\}.$$

We divide the strip $S = [0,\infty) \times [0,m]$ into two regions,

$$S_1 = \left\{(t,x) : 0 \le x \le m, x > r(t)\right\},$$
$$S_2 = \left\{(t,x) : 0 \le x \le m, x \le r(t)\right\}.$$

When $(t,x) \in S_1$, the development level of the nauplius at $t = 0$ is $x - r(t)$. When $(t,x) \in S_2$, the nauplius has a development level 0 at time $t - \tau(t,x)$ where $\tau(t,x)$ satisfies

$$\int_{t-\tau(t,x)}^{t} P\left(A(\eta)\right) d\eta = x.$$

For $t \ge t_0$, let $\tau(t) := \tau(t,m)$. Clearly, $\tau(t)$ satisfies

$$\int_{t-\tau(t)}^{t} P\left(A(\eta)\right) d\eta = m \tag{3.3}$$

and $\tau(t_0) = t_0$ since $r(t_0) = m$. Then $\tau(t)$ is the duration for the nauplii cohort becoming infectious larvae ($x = m$) at time t. See Figure 3.2.

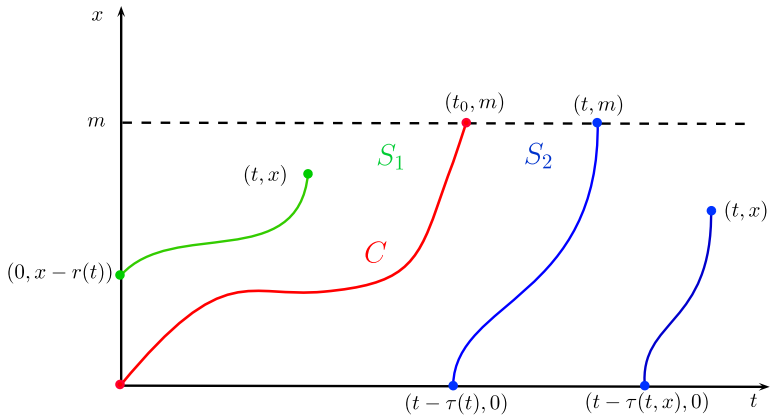


Figure 3.2 Illustration of $\tau(t, x)$ and $\tau(t)$.

Following [164], we use the method of characteristics to obtain

$$u(t,x) = \begin{cases} u_0(x - r(t))e^{-\mu_N t} & \text{for } (t,x) \in S_1, \\ \frac{b(A(t-\tau(t,x)))}{P(A(t-\tau(t,x)))} e^{-\mu_N \tau(t,x)} & \text{for } (t,x) \in S_2. \end{cases}$$

In particular, when $x = m$, we have

$$u(t,m) = \begin{cases} u_0(m - r(t))e^{-\mu_N t} & \text{for } 0 \le t \le t_0, \\ \frac{b(A(t-\tau(t)))}{P(A(t-\tau(t)))} e^{-\mu_N \tau(t)} & \text{for } t > t_0. \end{cases}$$

Thus, for $0 \le t \le t_0$, the system (3.2) becomes

$$\begin{aligned} \frac{dC}{dt} &= P\left(A(t)\right) u_0(m - r(t))e^{-\mu_N t} - \mu_C C(t) - \beta H C(t), \\ \frac{dA}{dt} &= \beta H C(t) - \mu_A A(t), \\ \frac{dr}{dt} &= P\left(A(t)\right), \end{aligned} \tag{3.4}$$

with $C(0) = C_0$, $A(0) = A_0$ and $r(0) = 0$. Notice that t_0 is determined as the solution of $r(t_0) = m$.

For $t > t_0$, the system (3.2) becomes

$$\frac{dC}{dt} = P\left(A(t)\right) \frac{b\left(A(t - \tau(t))\right)}{P\left(A(t - \tau(t))\right)} e^{-\mu_N \tau(t)} - \mu_C C(t) - \beta H C(t), \tag{3.5a}$$

$$\frac{dA}{dt} = \beta H C(t) - \mu_A A(t), \tag{3.5b}$$

$$\int_{t-\tau(t)}^{t} P\left(A(\eta)\right) d\eta = m, \tag{3.5c}$$

with initial data for $t \in [0, t_0]$ given by the solution of (3.4). It is clear that (3.5) is a threshold delay differential system (TDE) because of the threshold condition and that $\tau(t)$ is a state-dependent delay due to the dependence of $A(\cdot)$ in (3.5c). The biological interpretation is that the time spent by the nauplii cohort becoming copepodids at time t ($\tau(t)$) depends on the past history of adult population size ($A(\cdot)$). We will refer to system (3.5) as the TDE model.

To remove the state-dependent delay in (3.5) and transform it into a standard time-delayed model, for $t \geq 0$, let

$$T = \int_{0}^{t} P\left(A(\eta)\right) d\eta, \tag{3.6a}$$

$$\mathcal{C}(T) = C(t), \qquad \mathcal{A}(T) = A(t). \tag{3.6b}$$

Then $0 \leq t \leq t_0$ corresponds to $0 \leq T \leq m$, and system (3.4) becomes

$$\begin{aligned} \frac{d\mathcal{C}}{dT} &= u_0 \left(m - T\right) \exp\left\{ -\mu_N \int_{0}^{T} \frac{1}{P\left(\mathcal{A}(\eta)\right)} d\eta \right\} - \left(\mu_C + \beta H\right) \frac{\mathcal{C}(T)}{P\left(\mathcal{A}(T)\right)}, \\ \frac{d\mathcal{A}}{dT} &= \frac{1}{P\left(\mathcal{A}(T)\right)} \left[\beta H \mathcal{C}(T) - \mu_A \mathcal{A}(T)\right], \end{aligned} \tag{3.7}$$

with $\mathcal{C}(0) = C_0$ and $\mathcal{A}(0) = A_0$.

When $t > t_0$, i.e., $T > m$, it follows from (3.3) and (3.6a) that

$$T - m = \int_{0}^{t-\tau(t)} P\left(A(\eta)\right) d\eta,$$

and hence,

$$\tau(t) = t - \left(t - \tau(t)\right) = \int_{T-m}^{T} \frac{1}{P\left(\mathcal{A}(\eta)\right)} d\eta = \int_{-m}^{0} \frac{1}{P\left(\mathcal{A}_T(\theta)\right)} d\theta$$

where $\mathcal{A}_T(\theta) = \mathcal{A}(T+\theta)$, $-m \leq \theta \leq 0$. This implies that $\tau(t)$ turns into a state-independent delay. Therefore, for $T > m$, (3.5) becomes a functional differential system in the form

$$\frac{d\mathcal{C}}{dT} = \frac{b\left(\mathcal{A}(T-m)\right)}{P\left(\mathcal{A}(T-m)\right)} e^{-\mu_N \hat{\tau}(\mathcal{A}_T)} - \left(\mu_C + \beta H\right) \frac{\mathcal{C}(T)}{P\left(\mathcal{A}(T)\right)}, \tag{3.8a}$$

$$\frac{d\mathcal{A}}{dT} = \frac{1}{P\left(\mathcal{A}(T)\right)} \left[\beta H \mathcal{C}(T) - \mu_A \mathcal{A}(T)\right], \tag{3.8b}$$

where the initial data for $T \in [0, m]$ is given by the solution of (3.7) and $\hat{\tau}$ is defined on $C^+ = C([-m, 0], \mathbb{R}_+)$ by

$$\hat{\tau}(\phi) = \int_{-m}^{0} \frac{1}{P\left(\phi(\theta)\right)} d\theta.$$

We will refer to system (3.8) as the DDE model.

We would like to mention that Smith showed that the solution of the DDE model defines a solution of the TDE model for $t > t_0$ by appropriate change of variables [164, 165]. In the rest of the chapter, we work on the DDE models (3.8) by applying the standard theory for delay differential equations, see e.g. [74, 169, 218].

3.3 Well-posedness property and the adult reproduction number

First, we verify the well-posedness property for the system (3.7) when $0 \leq T \leq m$.

Proposition 3.3.1. *Suppose hypotheses (Q_1)-(Q_2) are satisfied. Then system (3.7) has a unique nonnegative bounded solution $(\mathcal{C}(T), \mathcal{A}(T))$ with the initial value $(C_0, A_0) \in \mathbb{R}^2_+$. Furthermore, if $C_0 > 0$ and $A_0 > 0$, then the solution is strictly positive.*

Proof. From the properties of P given in (Q_1), the function is locally Lipschitz for any closed bounded set in $\mathbb{R}$. Thus, it follows from [73, Theorem 1.3.1] that system (3.7) admits a unique solution $(\mathcal{C}(T), \mathcal{A}(T))$ through an initial value $(C_0, A_0) \in \mathbb{R}^2_+$ with the maximal interval of existence $[0, \xi)$ for some $\xi > 0$.

From (3.7) and comparison principle (see, e.g. [170, Theorem B.1]), we have

$$\begin{aligned}\mathcal{C}(T) &\geq C_0 \exp\left\{-\int_0^T \frac{\mu_C + \beta H}{P(\mathcal{A}(\theta))} d\theta\right\} \geq 0,\\ \mathcal{A}(T) &\geq A_0 \exp\left\{-\int_0^T \frac{\mu_A}{P(\mathcal{A}(\theta))} d\theta\right\} \geq 0.\end{aligned}$$

Furthermore, for any $T \in [0, m]$,

$$\mathcal{C}(T) \leq C_0 + \int_0^T u_0(m-\theta)\, d\theta < \infty.$$

Therefore

$$\mathcal{A}(T) \leq A_0 + \beta H \int_0^T \frac{\mathcal{C}(\theta)}{P(\mathcal{A}(\theta))} d\theta < \infty.$$

Thus, all solutions are bounded and exist for any $T \in [0, m]$. The strictly positive solution when $C_0 > 0$ and $A_0 > 0$ is clear. □

To analyze (3.8) mathematically, let $X := C([0, m], \mathbb{R}^2)$ and $X^+ = C([0, m], \mathbb{R}^2_+)$. For $\phi = (\phi_1, \phi_2) \in X$, denote $\|\phi\| = \sum_{i=1}^2 \|\phi_i\|_\infty$ with $\|\phi_i\|_\infty = \max_{-m\leq\theta\leq 0} |\phi_i(\theta)|$. Then, (X, X^+) is an ordered Banach space and X^+ is a normal cone of X with nonempty interior in X. For any given continuous function $z : [0, \sigma_\phi) \to \mathbb{R}^2$ with $\sigma_\phi > 0$, we define $z_t \in X$ for $t \geq m$ by $z_T(\theta) = z(T - \theta)$ for all $\theta \in [0, m]$.

Denote the initial data set for system (3.8) by

$$D_X = \left\{ \begin{array}{c} \phi \in X^+ : (\phi_1(\theta), \phi_2(\theta)) \text{ is a solution of (3.7) for } \theta \in [0, m] \\ \text{with } (\phi_1(0), \phi_2(0)) = (C_0, A_0) \end{array} \right\},$$

then the well-posedness property for the system (3.8) is followed.

Theorem 3.3.1. *Suppose hypotheses (Q_1) and (Q_2) are satisfied. Then, for any $\phi \in D_X$, the system (3.8) has a unique nonnegative solution $z(T, \phi)$ with the initial condition $z_0 = \phi$, and all solutions are ultimately bounded. Furthermore, $z(T, \phi)$ is strictly positive when $\mathcal{C}(m) > 0$ and $\mathcal{A}(m) > 0$.*

Proof. Given $\phi \in D_X$, we define $G(\phi) = (G_1(\phi), G_2(\phi))$, with

$$\begin{aligned} G_1(\phi) =& \frac{b\left(\phi_2(0)\right)}{P\left(\phi_2(0)\right)} e^{-\mu_N \hat{\tau}(\phi_2)} - \left(\mu_C + \beta H\right) \frac{\phi_1(-m)}{P\left(\phi_2(-m)\right)}, \\ G_2(\phi) =& \frac{1}{P\left(\phi_2(-m)\right)} \left[\beta H \phi_1(-m) - \mu_A \phi_2(-m)\right], \end{aligned}$$

where

$$\hat{\tau}(\phi_2) = \int_{-m}^{0} \frac{1}{P\left(\phi_2(m+\theta)\right)} d\theta.$$

It is easy to see that D_X is closed in X, $G(\phi)$ is continuous and is Lipschitz in ϕ in each compact set in D_X. By [74, Theorem 2.2.3] there is a unique solution in (3.8) on its maximal existence interval $[0, \sigma_\phi)$ through ϕ for any $\phi \in D_X$.

Furthermore, for any $\phi \in D_X$ with $\phi_i(m) = 0$, it is obvious that $G_i(\phi) \geq 0$ for $i = 1, 2$. Thus $\phi_i(T) \geq 0$ for all $t \in [0, \sigma_\phi)$, $i = 1, 2$, see [169, Theorem 5.2.1]. Thus all the solutions of (3.8) are nonnegative for any $t \in [0, \sigma_\phi)$. More precisely, it is easy to see that when $\phi \in D_X$,

$$\mathcal{C}(T) \geq \mathcal{C}(m) \exp\left\{ - \int_{m}^{T} \frac{\mu_C + \beta H}{P\left(\mathcal{A}(\theta)\right)} d\theta \right\}.$$

Thus, $\mathcal{C}(T) > 0$ when $\mathcal{C}(m) > 0$. The positivity of $\mathcal{A}(T)$ can be obtained as in Proposition 3.3.1 when $\mathcal{A}(m) > 0$.

Now we prove the boundedness. From (3.8), we have

$$\begin{aligned} \frac{d\mathcal{C}}{dT} &\leq \frac{b\left(\mathcal{A}(T-m)\right)}{P_0} - \frac{\mu_C}{P_\infty} \mathcal{C}(T) - \beta H \frac{\mathcal{C}(T)}{P\left(\mathcal{A}(T)\right)}, \\ \frac{d\mathcal{A}}{dT} &\leq \beta H \frac{\mathcal{C}(T)}{P\left(\mathcal{A}(T)\right)} - \frac{\mu_A}{P_\infty} \mathcal{A}(T). \end{aligned}$$

Thus,

$$\frac{d(\mathcal{C}+\mathcal{A})}{dT} \leq \frac{1}{P_0} b\left(\mathcal{A}(T-m)\right) - \frac{\tilde{\mu}}{P_\infty} \left(\mathcal{C}(T) + \mathcal{A}(T)\right).$$

Since b is an increasing function and $\mathcal{C}(T) \geq 0$, we have

$$\frac{d(\mathcal{C}+\mathcal{A})}{dT} \leq \frac{1}{P_0} b\left(\mathcal{C}(T-m) + \mathcal{A}(T-m)\right) - \frac{\tilde{\mu}}{P_\infty} \left(\mathcal{C}(T) + \mathcal{A}(T)\right),$$

$$\leq \frac{1}{P_0} B\left(\mathcal{C}(T-m)+\mathcal{A}(T-m)\right) - \frac{\tilde{\mu}}{P_\infty}\left(\mathcal{C}(T)+\mathcal{A}(T)\right),$$

where $\tilde{\mu} = \min\{\mu_C, \mu_A\}$.

Consider the delay differential equation

$$\frac{dv}{dT} = \frac{1}{P_0} B\left(v(T-m)\right) - \frac{\tilde{\mu}}{P_\infty} v(T). \tag{3.9}$$

Since the function B is strictly sub-homogeneous and $\frac{dv}{dT}\big|_{v=0} = 0$, it follows from [219, Theorem 3.2] that (3.9) admits a globally asymptotically stable equilibrium which attracts all positive solutions. Therefore, $\mathcal{C}+\mathcal{A}$ is bounded, and hence, any solution of (3.8) is bounded due to the non-negativity of $\mathcal{C}$ and $\mathcal{A}$. Thus, $\sigma_\phi = +\infty$ (see e.g., [74, Theorem 2.3.1]). Hence, all the solutions exist globally, and are ultimately bounded. □

Remark 3.3.1. *Theorem 3.3.1 implies that there exists $T_1 > 0$ and $M > 0$ such that $0 \leq \mathcal{C}(T) \leq K$ and $0 \leq \mathcal{A}(T) \leq K$ for $T > T_1$.*

As we know, the adult reproduction number is an important index to measure the individual expected lifetime reproduction. In the following, we calculate the adult reproduction number for sea lice ($\mathcal{R}_s$) and prove that $\mathcal{R}_s$ is a threshold value for the stability of the zero solution of system (3.8).

Since $b(0) = 0$, it is easy to see that the equilibrium $E_0 = (0,0)$ always exists in (3.8) for all values of the parameters. The linearization of (3.8) at E_0 is

$$\begin{aligned} \frac{d\mathcal{C}}{dT} &= \frac{b'(0)}{P_0} e^{\frac{-\mu_N m}{P_0}} \mathcal{A}(T-m) - \frac{\mu_C + \beta H}{P_0}\mathcal{C}(T), \\ \frac{d\mathcal{A}}{dT} &= \frac{\beta H}{P_0}\mathcal{C}(T) - \frac{\mu_A}{P_0}\mathcal{A}(T). \end{aligned} \tag{3.10}$$

System (3.10) can be rewritten as

$$\frac{d}{dT}\boldsymbol{J}(T) = \mathbb{M}_1\ \boldsymbol{J}(T-m) - \mathbb{M}_2\ \boldsymbol{J}(T), \tag{3.11}$$

with

$$\boldsymbol{J}(T) = \begin{bmatrix} \mathcal{C}(T) \\ \mathcal{A}(T) \end{bmatrix},\ \mathbb{M}_1 = \frac{1}{P_0}\begin{bmatrix} 0 & b'(0)e^{\frac{-\mu_N m}{P_0}} \\ 0 & 0 \end{bmatrix} \text{ and } \mathbb{M}_2 = \frac{1}{P_0}\begin{bmatrix} \mu_C + \beta H & 0 \\ -\beta H & \mu_A \end{bmatrix}.$$

In (3.11), $\boldsymbol{J}(T)$ is the number of the sea lice population, and $\mathbb{M}_1\ \boldsymbol{J}(T-m)$ is the new sea lice individuals at time T evolving over the time interval $[T-m, T]$. Further, for time $T > m$, the internal evolution of sea lice population, through natural deaths and movements between classes, is

$$\mathscr{J}(T) = e^{-\mathbb{M}_2 T}\,\mathscr{J}_m,$$

from $\frac{d}{dT}\boldsymbol{J}(T) = -\mathbb{M}_2\ \boldsymbol{J}(T)$ with initial $\mathscr{J}_{\boldsymbol{m}} = (J_1, J_2)^T$ at $T = m$.

The total number of the new sea lice individuals is

$$\widehat{\mathscr{J}} \;=\; \int_m^\infty \mathbb{M}_1\ \mathscr{J}(T-m)dT = \int_m^\infty \mathbb{M}_1\ e^{-\mathbb{M}_2(T-m)}\,\mathscr{J}_m dT.$$

Due to the nonsingularity of the matrix $\mathbb{M}_2$, we have

$$\widehat{\mathscr{J}} = \mathbb{M}_1\ \mathbb{M}_2^{-1}\mathscr{J}_{\boldsymbol{m}}.$$

Then the next generation operator is

$$\mathbb{M}_0 = \mathbb{M}_1\mathbb{M}_2^{-1} = \frac{1}{P_0^2}\begin{bmatrix} \frac{b'(0)\beta H}{(\mu_C+\beta H)\mu_A}e^{\frac{-\mu_N m}{P_0}} & \frac{b'(0)}{\mu_A}e^{\frac{-\mu_N m}{P_0}} \\ 0 & 0 \end{bmatrix},$$

and hence, the spectral radius of the matrix $\mathbb{M}_0$, denoted by $\mathcal{R}_s$ (see e.g., [217, Corollary 2.1]), is called the adult reproduction number for sea lice, that is,

$$\mathcal{R}_s := \frac{b'(0)\beta H}{\mu_A(\mu_C+\beta H)}e^{\frac{-\mu_N m}{P_0}}. \tag{3.12}$$

Biologically, $\frac{1}{\mu_A}$ gives the average lifespan of adult sea lice and $b'(0)$ is the number of eggs produced by one matured sea louse per unit time which will survive from the free-living phase to the parasite phase with the probability of $\frac{\beta H}{\mu_C+\beta H}e^{\frac{-\mu_N m}{P_0}}$. Note that this probability is the surviving probability in both immature stages, the nauplii stage $\left(e^{\frac{-\mu_N m}{P_0}}\right)$ and copepodid stage $\left(\frac{\beta H}{\mu_C+\beta H}\right)$. Therefore, $\mathcal{R}_s$ is the average number of adult sea lice that a single adult sea louse can produce in its expected lifespan.

About the stability of the trivial steady state $E_0 = (0, 0)$, we have the following result:

Theorem 3.3.2. *When $\mathcal{R}_s < 1$ and (Q_1)-(Q_2) hold, E_0 in (3.8) is locally asymptotically stable. When $\mathcal{R}_s > 1$, $(0,0)$ is unstable.*

Proof. The characteristic equation of (3.10) is

$$\Delta_0(\lambda) = \lambda^2 + \left(\frac{\mu_A + \mu_C + \beta H}{P_0}\right)\lambda + \frac{\mu_A(\mu_C + \beta H)}{P_0^2} - \frac{b'(0)\beta H}{P_0^2} e^{\frac{-\mu_N m}{P_0}} e^{-\lambda m} = 0. \quad (3.13)$$

Thus

$$\left(\lambda + \frac{\mu_A}{P_0}\right)\left(\lambda + \frac{\mu_C + \beta H}{P_0}\right) = \frac{b'(0)\beta H}{P_0^2} e^{\frac{-\mu_N m}{P_0}} e^{-\lambda m}.$$

The latter equation is equivalent to

$$\left(\frac{P_0}{\mu_A}\lambda + 1\right)\left(\frac{P_0}{\mu_C + \beta H}\lambda + 1\right) = \mathcal{R}_s e^{-\lambda m}.$$

Assume there exists a zero in $\Delta_0(\lambda) = 0$ with $Re(\lambda) \geq 0$, then

$$\left|\frac{P_0}{\mu_A}\lambda + 1\right| \geq 1, \quad \left|\frac{P_0}{\mu_C + \beta H}\lambda + 1\right| \geq 1 \quad \text{and} \quad \left|e^{-\lambda m}\right| \leq 1.$$

Therefore, when $\mathcal{R}_s < 1$, we have

$$\left|\left(\frac{P_0}{\mu_A}\lambda + 1\right)\left(\frac{P_0}{\mu_C + \beta H}\lambda + 1\right)\right| > \left|\mathcal{R}_s e^{-\lambda m}\right|$$

which leads to a contradiction. Hence all the eigenvalues in (3.13) have negative real parts, implying $(0,0)$ is locally asymptotically stable.

From (3.13), when $\mathcal{R}_s > 1$, it is clear that

$$\Delta_0(0) = \frac{\mu_A(\mu_C + \beta H)}{P_0^2}(1 - \mathcal{R}_s) < 0.$$

Since $\Delta_0(\lambda)$ is continuous and $\lim_{\lambda \to \infty} \Delta_0(\lambda) = +\infty$, there exists $\bar{\lambda} > 0$ such that $\Delta_0(\bar{\lambda}) = 0$. Therefore $(0,0)$ is unstable in system (3.10) when $\mathcal{R}_s > 1$. □

Remark 3.3.2. *Theorem 3.3.2 can be obtained from [217, Theorem 2.1 and Corollary 2.1].*

In the next section, we establish the threshold dynamics for the system (3.8) in terms of the adult reproduction number for sea lice.

3.4 Threshold dynamics

In order to use the comparison principle to study the global dynamics of system (3.8), we need to make sure the solution semiflow of (3.8) is eventually strongly monotone and strongly order preserving (see [169, Chapter 5]). Therefore, we choose $Z = \mathbb{R} \times C([0,m],\mathbb{R})$, $Z^+ = \mathbb{R}_+ \times C([0,m],\mathbb{R}_+)$, Then (Z, Z^+) is an ordered Banach space. For a continuous function $z : [0,\infty) \times [0,\infty) \to \mathbb{R}^2$ and $T \geq m$, we define the solution semiflow $z_T(\cdot) : Z \to \mathbb{R}^2$ of (3.8) by

$$z_T(\mathcal{C}_m, \theta) = (z_1(T), z_2(T-\theta)) \quad \forall \theta \in [0,m]$$

where $\mathcal{C}_m = \mathcal{C}(m) = z_1(m)$. We choose the initial data set for (3.8) in the following set:

$$D_Z = \left\{ \begin{array}{c} \phi_c = (\mathcal{C}_m, \tilde{\phi}) \in Z^+ : \ \mathcal{C}_m = \psi_1(m) \text{ and } \tilde{\phi}(\theta) = \psi_2(\theta) \ \forall \ \theta \in [0,m] \text{ where} \\ (\psi_1(\theta), \psi_2(\theta)) \text{ is a solution of (3.7) for } \theta \in [0,m] \text{ with } (\psi_1(0), \psi_2(0)) = (C_0, A_0) \end{array} \right\}.$$

The following result can be proved as Theorem 3.3.1.

Lemma 3.4.1. *For any $\phi_c \in D_Z$, system (3.8) admits a unique bounded nonnegative solution $z_T(\phi_c)$ on $[m,\infty)$ with $z_m = \phi_c$.*

In the following, we will discuss the sea lice population dynamics with respect to $\mathcal{R}_s$. Given additional restriction

$$b(s)\, e^{\frac{-\mu_N m}{P(s)}} \leq b'(0)\, e^{\frac{-\mu_N m}{P_0}} s, \tag{K_1}$$

we can discuss the global stability of $(0,0)$.

Theorem 3.4.1. *Assume that $\mathcal{R}_s < 1$, (Q_1), (Q_2) and (K_1) hold. Then*

(i) No positive equilibrium exists in (3.8);

(ii) $E_0 = (0,0)$ of system (3.8) is globally asymptotically stable for any $\phi_c \in D_Z$.

Proof. (i) By contradiction, assume a positive equilibrium $(\mathcal{C}^*, \mathcal{A}^*)$ exists. Then, from (3.8) we have

$$b(\mathcal{A}^*) \exp\left\{\frac{-m\mu_N}{P(\mathcal{A}^*)}\right\} = \frac{(\mu_C + \beta H)\,\mu_A}{\beta H}\mathcal{A}^*. \tag{3.14}$$

When $\mathcal{R}_s < 1$ and (K_1) holds, we have

$$b'(0)\, e^{\frac{-\mu_N m}{P_0}}\mathcal{A}^* \geq b(\mathcal{A}^*) \exp\left\{\frac{-m\mu_N}{P(\mathcal{A}^*)}\right\} = \frac{(\mu_C + \beta H)\,\mu_A}{\beta H}\mathcal{A}^* > b'(0)\, e^{\frac{-\mu_N m}{P_0}}\mathcal{A}^*,$$

which is a contradiction. Therefore, no $\mathcal{A}^* > 0$ exists and satisfies (3.14), hence, the positive equilibrium does not exist.

(ii) We use the method of fluctuations (see e.g. [86, 187, 213]) to prove that the limit supremum of $\mathcal{C}(T)$ and $\mathcal{A}(T)$ is zero when $T \to \infty$. In order to use this method, we let

$$\mathcal{C}^\infty = \limsup_{T\to\infty} \mathcal{C}(T) \text{ and } \mathcal{A}^\infty = \limsup_{T\to\infty} \mathcal{A}(T).$$

By [86, Lemma 4.2], there exist two sequences $\alpha_n \to \infty$ and $\kappa_n \to \infty$ as $n \to \infty$, such that

$$\begin{aligned} &\lim_{n\to\infty} \mathcal{C}(\alpha_n) = \mathcal{C}^\infty, \qquad \left.\frac{d\mathcal{C}}{dT}\right|_{T=\alpha_n} = 0, \quad \forall n \geq 1; \\ &\lim_{n\to\infty} \mathcal{A}(\kappa_n) = \mathcal{A}^\infty, \qquad \left.\frac{d\mathcal{A}}{dT}\right|_{T=\kappa_n} = 0, \quad \forall n \geq 1. \end{aligned} \tag{3.15}$$

Let $n \geq 1$. From (3.8), we have

$$\begin{aligned} 0 &= \left.\frac{d\mathcal{C}}{dT}\right|_{T=\alpha_n} = \frac{b\left(\mathcal{A}(\alpha_n - m)\right)}{P\left(\mathcal{A}(\alpha_n - m)\right)} e^{-\mu_N \hat{\tau}(\mathcal{A}(\alpha_n))} - (\mu_C + \beta H)\frac{\mathcal{C}(\alpha_n)}{P\left(\mathcal{A}(\alpha_n)\right)}, \\ 0 &= \left.\frac{d\mathcal{A}}{dT}\right|_{T=\kappa_n} = \frac{1}{P\left(\mathcal{A}(\kappa_n)\right)}\left[\beta H \mathcal{C}(\kappa_n) - \mu_A \mathcal{A}(\kappa_n)\right]. \end{aligned}$$

Notice that $\lim_{n\to\infty} P\left(\mathcal{A}(\alpha_n - m)\right) = \lim_{n\to\infty} P\left(\mathcal{A}(\alpha_n)\right)$ since $\lim_{n\to\infty} \alpha_n = \infty$. Since $b(s)e^{\frac{-\mu_N m}{P(s)}}$ is increasing function , we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \left(b\left(\mathcal{A}(\alpha_n - m)\right) e^{-\mu_N \hat{\tau}(\mathcal{A}(\alpha_n))}\right) - (\mu_C + \beta H)\mathcal{C}^\infty \leq b\left(\mathcal{A}^\infty\right) e^{\frac{-\mu_N m}{P(\mathcal{A}^\infty)}} - (\mu_C + \beta H)\mathcal{C}^\infty, \\ 0 &= \beta H \lim_{n\to\infty} \mathcal{C}(\kappa_n) - \mu_A \mathcal{A}^\infty \leq \beta H \mathcal{C}^\infty - \mu_A \mathcal{A}^\infty. \end{aligned}$$

Hence

$$\mathcal{C}^\infty \leq \frac{1}{\mu_C + \beta H} b\left(\mathcal{A}^\infty\right) e^{\frac{-\mu_N m}{P(\mathcal{A}^\infty)}} \quad \text{and} \quad \mathcal{A}^\infty \leq \frac{\beta H}{\mu_A}\mathcal{C}^\infty.$$

Since $\mathcal{R}_s < 1$ and (Q_2)-(K_1) hold, we have

$$\mathcal{C}^\infty \leq \frac{1}{\mu_C + \beta H} b\left(\frac{\beta H}{\mu_A}\mathcal{C}^\infty\right) \exp\left\{\frac{-\mu_N m}{P\left(\frac{\beta H}{\mu_A}\mathcal{C}^\infty\right)}\right\} \leq \frac{b'(0)\,\beta H}{(\mu_C + \beta H)\,\mu_A} e^{\frac{-\mu_N m}{P_0}}\mathcal{C}^\infty = \mathcal{R}_s \mathcal{C}^\infty.$$

Thus, $(1 - \mathcal{R}_s)\mathcal{C}^\infty \leq 0$. When $\mathcal{R}_s < 1$, $\mathcal{C}^\infty = 0$ since $0 \leq \mathcal{C}(T)$, and hence, $\mathcal{A}^\infty = 0$. Therefore $\lim_{T\to\infty} (\mathcal{C}(T), \mathcal{A}(T)) = (0, 0)$. That is, $E_0 = (0, 0)$ is globally attractive in (3.8), which, together with the local stability of E_0 established in Theorem 3.3.2, confirms the global asymptotic stability of E_0.

□

Obviously, (K_1) is equivalent to

$$\frac{b(\mathcal{A})\ e^{\frac{-\mu_N m}{P(\mathcal{A})}}}{\mathcal{A}} \leq b'(0)\ e^{\frac{-\mu_N m}{P_0}}.$$

Biologically, $b'(0)\ e^{\frac{-\mu_N m}{P_0}}$ represents the new recruited copepodids after the egg laying at $t = 0$. Thus, (K_1) means that, as time going ($t > 0$) the proportion of the new recruited copepodids $\left(\frac{b(\mathcal{A})e^{\frac{-\mu m}{P(\mathcal{A})}}}{\mathcal{A}}\right)$ to the total number of adult sea lice population is less than initial. Therefore, the sea lice population cannot survive since each adult produces less than one adult sea lice in its expected lifespan ($\mathcal{R}_s < 1$).

Uniform persistence is an important concept in population dynamics which describes the survival of some or all species in an ecosystem. When $\mathcal{R}_s > 1$, we can study the system persistence and discuss the existence and global attractivity of a unique positive equilibrium point.

Theorem 3.4.2. *If $\mathcal{R}_s > 1$ and (Q_1)- (Q_2) hold, (3.8) is uniformly persistent, in the sense that, there is a positive number $\eta > 0$ such that every solution in system (3.8) with $\phi_c \in D_Z$ such that $\mathcal{C}_m > 0$ and $\tilde{\phi}(m) > 0$, satisfies*

$$\liminf_{T\to\infty}(\mathcal{C}(T), \mathcal{A}(T)) \geq (\eta, \eta).$$

Proof. Let

$$Z_0 = \{\phi_c \in D_Z \ : \ \mathcal{C}_m > 0 \text{ and } \tilde{\phi}(m) > 0, \}$$

$$\partial Z_0 = Z \setminus Z_0 = \{\phi_c \in D_Z \ : \ \mathcal{C}_m = 0 \text{ or } \tilde{\phi}(m) = 0.\}$$

and

$$N_\partial = \{\phi_c \in D_Z : z_T(\phi_c) \in \partial Z_0, T \geq m\}.$$

Since $\lim_{\mathcal{A}\to 0} \frac{b(\mathcal{A})}{\mathcal{A}P(\mathcal{A})} = \frac{b'(0)}{P_0}$, in a neighborhood of $\mathcal{A} = 0$, there exists a small $\epsilon > 0$ such that

$$\frac{b'(0)}{P_0} - \epsilon < \frac{b\,(\mathcal{A})}{\mathcal{A}P(A)} < \frac{b'(0)}{P_0} + \epsilon. \tag{3.16}$$

We have the following claim:

Claim 1. There exists a $\delta(\epsilon) > 0$, such that for any $\phi_c \in Z_0$,

$$\limsup_{T\to\infty} \|z_T(\mathcal{C}_m, \tilde{\phi}) - E_0\| \geq \delta(\epsilon).$$

By contradiction, suppose that $\limsup_{T\to\infty} \|z_T(\psi_c) - E_0\| < \delta(\epsilon)$ for some $\psi_c \in Z_0$. Thus, there exists $T_2 > m$ such that $|\mathcal{A}(T)| < \delta(\epsilon)$ for $T > T_2 + m$. Hence, (3.16) is satisfied.

From (3.8), we obtain

$$\begin{aligned}\frac{d\mathcal{C}}{dT} &> \left(\frac{b'(0)}{P_0}-\epsilon\right)e^{\frac{-m\mu_N}{P_0}}A(T-m)-\frac{\mu_C+\beta H}{P_0}C(T),\\ \frac{d\mathcal{A}}{dT} &> \frac{\beta H}{P(\delta(\epsilon))}\mathcal{C}(T)-\frac{\mu_A}{P_0}\mathcal{A}(T).\end{aligned}\tag{3.17}$$

Since the system obtained from (3.17) by replacing $>$ with $=$ is quasimonotone and irreducible, it is sufficient to consider only the real roots of the characteristic equation because any complex roots would have smaller real part than the largest real root (see [169, Theorem 5.5.1]).

The characteristic equation is

$$\Delta_1(\lambda)=\lambda^2+\left(\frac{\mu_A+\mu_C+\beta H}{P_0}\right)\lambda+\frac{\mu_A\left(\mu_C+\beta H\right)}{P_0^2}-\left(\frac{b'(0)}{P_0}-\epsilon\right)\frac{e^{\frac{-\mu_N m}{P_0}}\beta H}{P\left(\delta_0(\epsilon)\right)}e^{-\lambda m}=0.\tag{3.18}$$

Let $\lambda_1(\epsilon)$ be the principle eigenvalue. When $\epsilon=0$, it is clear that $\Delta_1(0)=\frac{\mu_A(\mu_C+\beta H)}{P_0^2}\left(1-\mathcal{R}_s\right)<0$ when $\mathcal{R}_s>1$. Hence, there exists $\tilde{\lambda}>0$ such that $\Delta_1(\tilde{\lambda})=0$ because $\Delta_1(\lambda)$ is continuous and $\lim_{\lambda\to\infty}\Delta_1(\lambda)=+\infty$. Thus $\lambda_1(0)>0$, and $\lambda_1(\epsilon)>0$ for sufficiently small $\epsilon>0$ due the continuity of λ_1.

It follows that there exists a solution $\vec{V}(T)=e^{\lambda_1(\epsilon)T}\zeta$, where ζ is the positive eigenfunction associated with $\lambda_1(\epsilon)$, $\vec{V}$ and ζ are vectors with two components. Since $\mathcal{C}(T)\geq 0$ and $\mathcal{A}(T)\geq 0$ for all $T>m$, the comparison theory (see e.g., [169, Theorem 5.1.1]) implies that there exists a small $\ell>0$ such that $(\mathcal{C}(T),\mathcal{A}(T))^T\geq \ell e^{\lambda_1(\epsilon)T}\zeta$ for all $T\geq T_2+m$. Since $\lambda_1(\epsilon)>0$, we have $\lim_{T\to\infty}(\mathcal{C}(T),\mathcal{A}(T))=+\infty$. This contradicts the boundedness of solutions to system (3.8).

Let $\omega(\phi_c)$ be the omega limit set of the orbit $z_T(\phi_c)$ through $\phi_c\in D_Z$.

Claim 2. $\bigcup\{\omega(\phi_c):\phi_c\in N_\partial\}=(0,0)$.

Let $\phi_c\in N_\partial$. If $\mathcal{C}(T)\equiv 0$, then from (3.8b), we have

$$\frac{d\mathcal{A}}{dT}\leq-\frac{\mu_A}{P_\infty}\mathcal{A}(T).$$

By the comparison theory, we get $\mathcal{A}(T)\to 0$ as $T\to\infty$. Parallelly, when $\mathcal{A}(T)\equiv 0$, it follows from (3.8a) that

$$\frac{d\mathcal{C}}{dT}=-\frac{\mu_C+\beta H}{P_0}\mathcal{C}(T).$$

Thus, $\mathcal{C}(T)\to 0$ as $T\to\infty$. This proves the claim.

Define a continuous function $\rho : D_Z \to \mathbb{R}_+$ by

$$\rho(\phi_c) = \min\{\mathcal{C}_m, \tilde{\phi}(m)\},\ \forall \phi_c \in D_Z.$$

It is clear that $\rho^{-1}(0,\infty) \subset Z_0$ and if $\rho(\phi_c) > 0$ then $\rho(z_T(\phi_c)) > 0$ for all $T > m$. By Claim 2, we get that for any forward orbit of z_T in N_∂ converges to $(0,0)$, by Claim 1, we conclude that $W^s((0,0)) \cap Z_0 = \emptyset$, where $W^s(0,0)$ is the stable manifold of $(0,0)$, and there is no cycle in M_∂ from $(0,0)$ to $(0,0)$. By [171], it then follows that there exists $\eta > 0$ such that $\liminf_{T\to\infty}(\mathcal{C}(T), \mathcal{A}(T)) \geq (\eta, \eta)$ for all $\phi \in Z_0$, which implies the uniform persistence. This completes the proof. □

Obviously, $\mathcal{R}_s > 1$ implies that m must be less than

$$m_{\max} = \frac{P_0}{\mu_N} \ln\left(\frac{b'(0)\beta H}{\mu_A(\mu_C + \beta H)}\right). \tag{3.19}$$

Therefore to guarantee the persistence of sea lice, the infection age m should be appropriately low.

To discuss the survival steady state of sea lice, we assume that the function $b(\mathcal{A})$ is concave with respect to the variable $\mathcal{A}$, i.e.

(K_2) $b''(\mathcal{A}) < 0$ for $\mathcal{A} > 0$ and there exists $\mathcal{A}^{**} > 0$ such that $b(\mathcal{A}) > \mu_A\mathcal{A}$ for $\mathcal{A} \in (0, \mathcal{A}^{**})$ and $b(\mathcal{A}) < \mu_A\mathcal{A}$ for $\mathcal{A} > \mathcal{A}^{**}$.

Biologically, the condition (K_2) may be interpreted as a consequence of saturation effects, in the sense that, the egg laying rate ($b(\mathcal{A})$) starts faster than linearly ($\mu_A\mathcal{A}$) until the adult sea lice population reaches a high level $\mathcal{A}^{**}$. Then, as the number of adult sea lice becomes higher ($\mathcal{A} > \mathcal{A}^{**}$), the egg laying rate will respond more slowly than linearly to the increase in $\mathcal{A}$.

With the additional condition (K_2), we have the following result for the existence of a unique positive equilibrium point.

Proposition 3.4.1. *Assume $\mathcal{R}_s > 1$, (Q_1)-(Q_2) and (K_2) hold. Then system (3.8) has a unique positive equilibrium point $E^* = (\mathcal{C}^*, \mathcal{A}^*)$.*

Proof. If $(\mathcal{C}^*, \mathcal{A}^*)$ is a positive equilibrium point in (3.8). Then, we have $\mathcal{C}^* = \frac{\mu_A}{\beta H}\mathcal{A}^*$ and $\mathcal{A}^*$ satisfies

$$b(\mathcal{A}^*) \exp\left\{\frac{-m\mu_N}{P(\mathcal{A}^*)}\right\} = \frac{(\mu_C + \beta H)\mu_A}{\beta H}\mathcal{A}^*. \tag{3.20}$$

From (3.20), we define

$$g(\mathcal{A}) = b(\mathcal{A}) \exp\left\{\frac{-m\mu_N}{P(\mathcal{A})}\right\} \qquad \text{and} \qquad h(\mathcal{A}) = \frac{(\mu_C + \beta H)\mu_A}{\beta H}\mathcal{A}.$$

When (Q_1) and (Q_2) hold, we get

$$g(0) = h(0) = 0, \quad g'(\mathcal{A}) > 0, \quad h'(\mathcal{A}) > 0$$

and since $\mathcal{R}_s > 1$, we obtain

$$b'(0) \geq g'(0) = b'(0) \exp\left\{\frac{-m\mu_N}{P_0}\right\} > \frac{(\mu_C + \beta H)\,\mu_A}{\beta H} = h'(0).$$

Obviously, $h(\mathcal{A}) > \mu_A \mathcal{A}$ for $\mathcal{A} > 0$. Thus, when (K_2) holds, there exists $\hat{\mathcal{A}}^{**} < \mathcal{A}^{**}$ such that

$$b(\mathcal{A}) > h(\mathcal{A}) \text{ for } \mathcal{A} \in (0, \hat{\mathcal{A}}^{**}) \text{ and } b(\mathcal{A}) < h(\mathcal{A}) \text{ for } \mathcal{A} > \hat{\mathcal{A}}^{**}.$$

It is clear that $b(\mathcal{A}) > g(\mathcal{A})$ for $\mathcal{A} > 0$ when (Q_1) holds. Since $g'(\mathcal{A}) > 0$, g and h intersect at exactly one point at $\mathcal{A}^*$ such that $\mathcal{A}^* < \hat{\mathcal{A}}^{**}$ (see Figure 3.3). Consequently, $\mathcal{C}^* = \frac{\mu_A}{\beta H}\mathcal{A}^*$. □

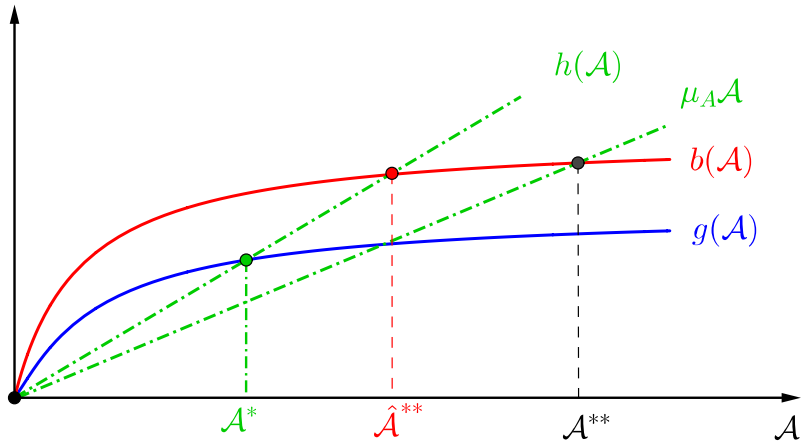


Figure 3.3 Intersection points.

To discuss the global attractivity of the positive steady state $E^* = (\mathcal{C}^*, \mathcal{A}^*)$, we need the following assumption:

(K_3) The function $\frac{b(s)\exp\left\{\frac{-m\mu_N}{P(s)}\right\}}{s}$ is decreasing in $s \in [\eta, K]$ where η is the lower bound given in Theorem 3.4.2 and K is the upper bound given in Remark 3.3.1.

Theorem 3.4.3. *Assume $\mathcal{R}_s > 1$, (Q_1)-(Q_2) and (K_2)-(K_3) hold. Then*

$$\lim_{T\to\infty} (\mathcal{C}(T), \mathcal{A}(T)) = (\mathcal{C}^*, \mathcal{A}^*)$$

for any $\phi_c \in D_Z$ such that $\mathcal{C}_m > 0$ and $\tilde{\phi}(m) > 0$.

Proof. We use the method of fluctuations (see e.g., [86,187,213]) to prove the global attractivity of $(\mathcal{C}^*, \mathcal{A}^*)$ in (3.8). Let

$$\mathcal{C}^\infty = \limsup_{T\to\infty} \mathcal{C}(T),\ \mathcal{C}_\infty = \liminf_{T\to\infty} \mathcal{C}(T),\ \mathcal{A}^\infty = \limsup_{T\to\infty} \mathcal{A}(T) \text{ and } \mathcal{A}_\infty = \liminf_{T\to\infty} \mathcal{A}(T).$$

By [86, Lemma 4.2], there exist four sequences $\alpha_n^i \to \infty$ and $\kappa_n^i \to \infty$ $(i = 1, 2)$ as $n \to \infty$, such that

$$\begin{aligned}
&\lim_{n\to\infty} \mathcal{C}(\alpha_n^1) = \mathcal{C}^\infty,\ \lim_{n\to\infty} \mathcal{C}(\alpha_n^2) = \mathcal{C}_\infty, \quad \left.\frac{d\mathcal{C}}{dT}\right|_{T=\alpha_n^i} = 0 \ \ \forall n \geq 1,\ i = 1, 2;\\
&\lim_{n\to\infty} \mathcal{A}(\kappa_n^1) = \mathcal{A}^\infty,\ \lim_{n\to\infty} \mathcal{A}(\kappa_n^2) = \mathcal{A}_\infty, \quad \left.\frac{d\mathcal{A}}{dT}\right|_{T=\kappa_n^2} = 0 \ \ \forall n \geq 1,\ i = 1, 2.
\end{aligned} \tag{3.21}$$

From the boundedness and persistence of solutions, we know that $0 < \eta \leq \mathcal{C}_\infty \leq \mathcal{C}^\infty \leq K$ and $0 < \eta \leq \mathcal{A}_\infty \leq \mathcal{A}^\infty \leq K$. Let $n \geq 1$. From (3.8a) and (3.21), we have

$$0 \leq b(\mathcal{A}^\infty) \exp\left\{\frac{-m\mu_N}{P(\mathcal{A}^\infty)}\right\} - (\mu_C + \beta H)\mathcal{C}^\infty, \tag{3.22a}$$

$$0 \geq b(\mathcal{A}_\infty) \exp\left\{\frac{-m\mu_N}{P(\mathcal{A}_\infty)}\right\} - (\mu_C + \beta H)\mathcal{C}^\infty, \tag{3.22b}$$

$$0 \leq b(\mathcal{A}^\infty) \exp\left\{\frac{-m\mu_N}{P(\mathcal{A}^\infty)}\right\} - (\mu_C + \beta H)\mathcal{C}_\infty, \tag{3.22c}$$

$$0 \geq b(\mathcal{A}_\infty) \exp\left\{\frac{-m\mu_N}{P(\mathcal{A}_\infty)}\right\} - (\mu_C + \beta H)\mathcal{C}_\infty. \tag{3.22d}$$

In view of (3.8b) and (3.21), we obtain

$$0 \leq \beta H \mathcal{C}^\infty - \mu_A \mathcal{A}^\infty, \tag{3.23a}$$

$$0 \geq \beta H \mathcal{C}_\infty - \mu_A \mathcal{A}^\infty, \tag{3.23b}$$

$$0 \leq \beta H \mathcal{C}^\infty - \mu_A \mathcal{A}_\infty, \tag{3.23c}$$

$$0 \geq \beta H \mathcal{C}_\infty - \mu_A \mathcal{A}_\infty. \tag{3.23d}$$

From (3.23b) and (3.22d), we get

$$\mathcal{A}^\infty \geq \frac{\beta H}{(\mu_C + \beta H)\mu_A} b(\mathcal{A}_\infty) \exp\left\{\frac{-m\mu_N}{P(\mathcal{A}_\infty)}\right\}. \tag{3.24}$$

Parallelly, from (3.23c) and (3.22a) , we have

$$\mathcal{A}_\infty \leq \frac{\beta H}{(\mu_C + \beta H)\mu_A} b(\mathcal{A}^\infty) \exp\left\{\frac{-m\mu_N}{P(\mathcal{A}^\infty)}\right\}. \tag{3.25}$$

It follows from (3.23a), (3.22a) and (3.23d), (3.22d), respectively, that

$$\mathcal{A}^{\infty} \leq \frac{\beta H}{(\mu_C+\beta H)\mu_A} b(\mathcal{A}^{\infty}) \exp\left\{\frac{-m\mu_N}{P(\mathcal{A}^{\infty})}\right\}, \tag{3.26a}$$

$$\mathcal{A}_{\infty} \geq \frac{\beta H}{(\mu_C+\beta H)\mu_A} b(\mathcal{A}_{\infty}) \exp\left\{\frac{-m\mu_N}{P(\mathcal{A}_{\infty})}\right\}. \tag{3.26b}$$

For notational simplicity, we define

$$\mathcal{H}(s) = \frac{\beta H}{(\mu_C+\beta H)\mu_A} b(s) \exp\left\{\frac{-m\mu_N}{P(s)}\right\}, \quad \forall s \in (0, K]. \tag{3.27}$$

Then, (3.25), (3.24) and (3.26) yield

$$\mathcal{A}^{\infty} \geq \mathcal{H}(\mathcal{A}_{\infty}), \quad \mathcal{A}_{\infty} \leq \mathcal{H}(\mathcal{A}^{\infty}) \quad \text{and} \quad \mathcal{A}_{\infty}\mathcal{H}(\mathcal{A}^{\infty}) \geq \mathcal{A}^{\infty}\mathcal{H}(\mathcal{A}_{\infty}). \tag{3.28}$$

It is easy to see that (i) $\frac{\mathcal{H}(s)}{s}$ is decreasing when (K_3) holds; (ii) $\mathcal{H}(x)$ is increasing in $s \in [\eta, K]$ when (Q_1)-(Q_2) hold.

Adopting the idea of the property (P) in [213], we can prove the following claim.

Claim 3. Let $k_1, k_2 \in [\eta, K]$. If $k_1 \geq \mathcal{H}(k_2)$, $k_2 \leq H(k_1)$ and $k_2\mathcal{H}(k_1) \geq k_1\mathcal{H}(k_2)$ then $k_1 = k_2$.

By contrary, suppose $k_1 \neq k_2$. If $k_1 < k_2$, then due to the monotonicity of $\mathcal{H}(s)$, we have $\mathcal{H}(k_1) \leq \mathcal{H}(k_2) \leq k_1$. Since $k_1 < k_2 \leq \mathcal{H}(k_1)$, we get

$$k_1 \geq \mathcal{H}(k_1) > k_1,$$

which is a contradiction. When $k_1 > k_2$, then

$$\frac{\mathcal{H}(k_1)}{k_1} \leq \frac{\mathcal{H}(k_2)}{k_2}$$

yields $k_2\mathcal{H}(k_1) \leq k_1\mathcal{H}(k_2)$. With the condition $k_2\mathcal{H}(k_1) \geq k_1\mathcal{H}(k_2)$, one obtains $k_1 = k_2$, a contradiction. This proves the claim.

Claim 3 and (3.28) imply that $\mathcal{A}^{\infty} = \mathcal{A}_{\infty}$. From (3.22a) and (3.22d), we have

$$\frac{1}{\mu_C+\beta H} b(\mathcal{A}_{\infty}) \exp\left\{\frac{-m\mu_N}{P(\mathcal{A}_{\infty})}\right\} \leq \mathcal{C}_{\infty} \leq \mathcal{C}^{\infty} \leq \frac{1}{\mu_C+\beta H} b(\mathcal{A}^{\infty}) \exp\left\{\frac{-m\mu_N}{P(\mathcal{A}^{\infty})}\right\}.$$

Hence, $\mathcal{C}^{\infty} = \mathcal{C}_{\infty}$. Thus,

$$\lim_{T\to\infty} (\mathcal{C}(T), \mathcal{A}(T)) = (\mathcal{C}^*, \mathcal{A}^*)$$

for any $\phi_c \in D_Z$ with $\mathcal{C}_m > 0$ and $\tilde{\phi}(m) > 0$. □

As we know, $\mathcal{R}_s$ is a key parameter for population persistence and extinction. Next, we discuss the sensitivity of $\mathcal{R}_s$ with respect to the related parameters μ_N, μ_C, μ_A, β, H and m.

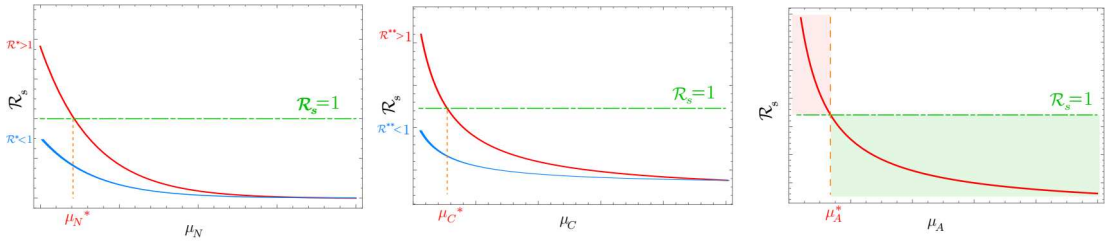
First, we discuss the relationship of $\mathcal{R}_s$ with respect to the mortality rates. For $\mathcal{S} \in \{\mu_N,\ \mu_C,\ \mu_A\}$, it is easy to check that $\frac{\partial \mathcal{R}_s}{\partial \mathcal{S}} < 0$, $\lim\limits_{\mathcal{S}\to\infty} \mathcal{R}_s = 0$, and

$$\lim_{\mu_N\to 0^+} \mathcal{R}_s = \frac{b'(0)\beta H}{\mu_A(\mu_C+\beta H)} := \mathcal{R}^*, \quad \lim_{\mu_C\to 0^+} \mathcal{R}_s = \frac{b'(0)}{\mu_A} e^{\frac{-\mu_N m}{P_0}} := \mathcal{R}^{**}, \quad \lim_{\mu_A\to 0^+} \mathcal{R}_s = +\infty.$$

We understand that, (i) if $\mathcal{R}^* < 1$ ($\mathcal{R}^{**} < 1$), the sea lice population dies out for even small mortality rate $\mu_N > 0$ ($\mu_C > 0$) (the blue line in Figures 3.4a – 3.4b); (ii) if $\mathcal{R}^* > 1$ ($\mathcal{R}^{**} > 1$), there exist the critical values

$$\mu_N^* = \frac{P_0}{m} \ln\left(\frac{b'(0)\beta H}{\mu_A\,(\mu_C+\beta H)}\right) \quad \left(\mu_C^* = \frac{b'(0)e^{\frac{-\mu_N m}{P_0}} - \mu_A}{\mu_A}\beta H\right)$$

such that $\mathcal{R}_s > 1$ for $\mu_N < \mu_N^*$ ($\mu_C < \mu_C^*$). Similarly, there exists $\mu_A^* = \frac{b'(0)\beta H}{\mu_C+\beta H} e^{\frac{-\mu_N m}{P_0}}$ such that $\mathcal{R}_s < 1$ for $\mu_A > \mu_A^*$ (Figure 3.4c). Therefore, any large mortality rate μ_N, μ_C or μ_A leads to the sea lice extinction (Figure 3.4). Biologically, the environmental salinity affects such mortality rates. The increasing of salinity level results the decreasing of μ_N, μ_C and μ_A [36, 39, 95].



(a) $\mathcal{R}_s$ v.s μ_N (b) $\mathcal{R}_s$ v.s μ_C (c) $\mathcal{R}_s$ v.s μ_A

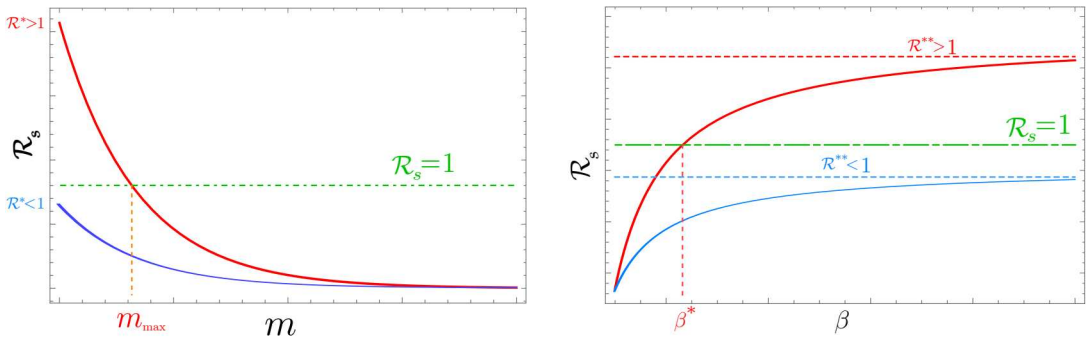
Figure 3.4 The relationship between $\mathcal{R}_s$ and the mortality rates.

To see the influence of the infection development age m, notice that, $\frac{\partial \mathcal{R}_s}{\partial m} = -\frac{\mu_N}{P_0}\mathcal{R}_s < 0$, $\lim\limits_{m\to 0^+} \mathcal{R}_s = \mathcal{R}^*$ and $\lim\limits_{m\to\infty} \mathcal{R}_s = 0$. Therefore, when $\mathcal{R}^* < 1$, the population will be extincted no matter at what age (m) the nauplii are developed (the blue line in Figure 3.5a). While when $\mathcal{R}^* > 1$, $m_{\max}$ exists (given in (3.19)) and the population persists for $m < m_{\max}$ (the red line in Figure 3.5a). In nature, water temperature has an impact on nauplius development age (m). The increasing on the temperature can promote the development age m [177].

The effect of the infection rate β on $\mathcal{R}_s$ is opposite. When $\mathcal{R}^{**} > 1$, there exists the critical

$$\beta^* = \frac{\mu_A \mu_C}{\left(b'(0) e^{\frac{-\mu_N m}{P_0}} - \mu_A \right) H}$$

such that $\mathcal{R}_s < 1$ for $\beta < \beta^*$. Thus, large β leads to population persistence (the red line in Figure 3.5b). If $\mathcal{R}^{**} < 1$, the sea lice population dies out for any $\beta > 0$. Biologically, currents, salinity and light assist copepodids in finding a host, that is, increasing β [42]. Obviously, from the expression of $\mathcal{R}_s$, the relationship of $\mathcal{R}_s$ with respect to H is equivalent to the one with respect to β.



(a) $\mathcal{R}_s$ v.s m (b) $\mathcal{R}_s$ v.s β

Figure 3.5 The relationship between $\mathcal{R}_s$, m and β.

3.5 Numerical simulations

In this section, we provide some numerical simulation results using *Lepeophtheirus salmonis* growth as a case study with appropriate parameters from the literature.

Lepeophtheirus salmonis is a species of sea lice which is a serious problem in salmon aquaculture and causes substantial economic losses on salmon farms [43]. We assume that the egg laying rate function for sea lice is the generalized Michaelis-Menten growth function [28]

$$b(s) = \frac{B_1 s}{B_2 + s^r}, \tag{3.29}$$

with $0 < r \leq 1$, and choose

$$P(s) = \frac{h_1 s + h_2}{h_3 s + h_4}, \tag{3.30}$$

where B_1, B_2 and h_i $(i = 1, 2, 3, 4)$ are positive constants. It is easy to check that $b'(0) = \frac{B_1}{B_2}$. According to [84, 133], the number of eggs per clutch is 592 eggs and the production rate

of egg string is between $0.0476 - 0.0576$ day^{-1}. Therefore, the number of eggs produced by one matured sea louse per day is

the number of eggs per clutch $\times$ the production rate of egg string.

Thus, $\frac{B_1}{B_2} \in (28.1792, 34.0992)$ egg per day. We adopt the parameter values μ_N, μ_C, μ_A, β and m from the literature which are given in Table 3.1. Due to the lack of data in the literature we choose the other parameters as in Table 3.2 for simulation illustration.

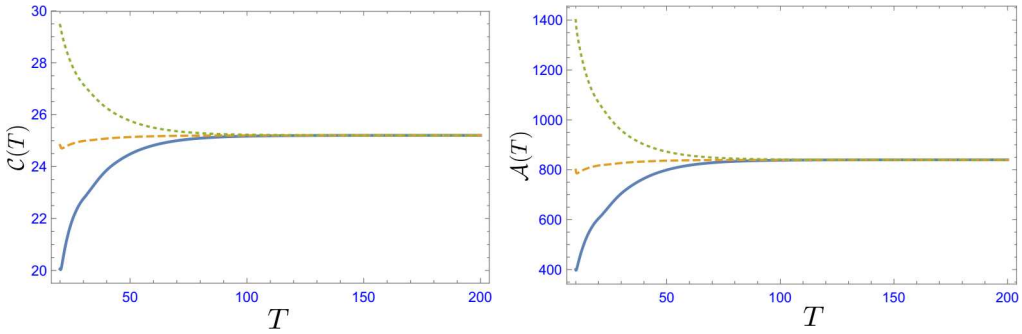
Parameter	$\frac{B_1}{B_2}$	μ_N & μ_C	μ_A	m	β
Dimension	egg per day	day^{-1}	day^{-1}	day	fish per day
Value	$28.1792 - 34.0992$	$0.02 - 0.13$	$0.01 - 0.2$	$2 - 14$	$0.001 - 0.1$
Reference	[84, 133]	[154]	[154]	[42, 95]	[84]

Table 3.1 Parameter values for the model (3.8).

Parameter	h_1	h_2	h_3	h_4
Value	1	0.1	1	0.3

Table 3.2 Parameter values for the model (3.8).

Fixing $B_1 = 28.2$, $B_2 = 1$, $r = 0.7$ (in (3.29)), $\mu_N = 0.05$, $\mu_C = 0.07$, $\mu_A = 0.15$, $\beta = 0.01$ and $H = 500$. Then we can calculate $m_{\max} = 34.82$. Hence, for any m given in Table 3.1, $\mathcal{R}_s > 1$ always holds, so a unique globally attractive positive equilibrium point $E^* = (\mathcal{C}^*, \mathcal{A}^*)$ exists. Figure 3.6 shows that the number of adult sea lice stabilize at a high constant level $\mathcal{A}^* \approx 840.126$ and copepodids stabilize at $\mathcal{C}^* \approx 25$ when $m = 10$.



(a) Time series $\mathcal{C}(T)$ (b) Time series $\mathcal{A}(T)$

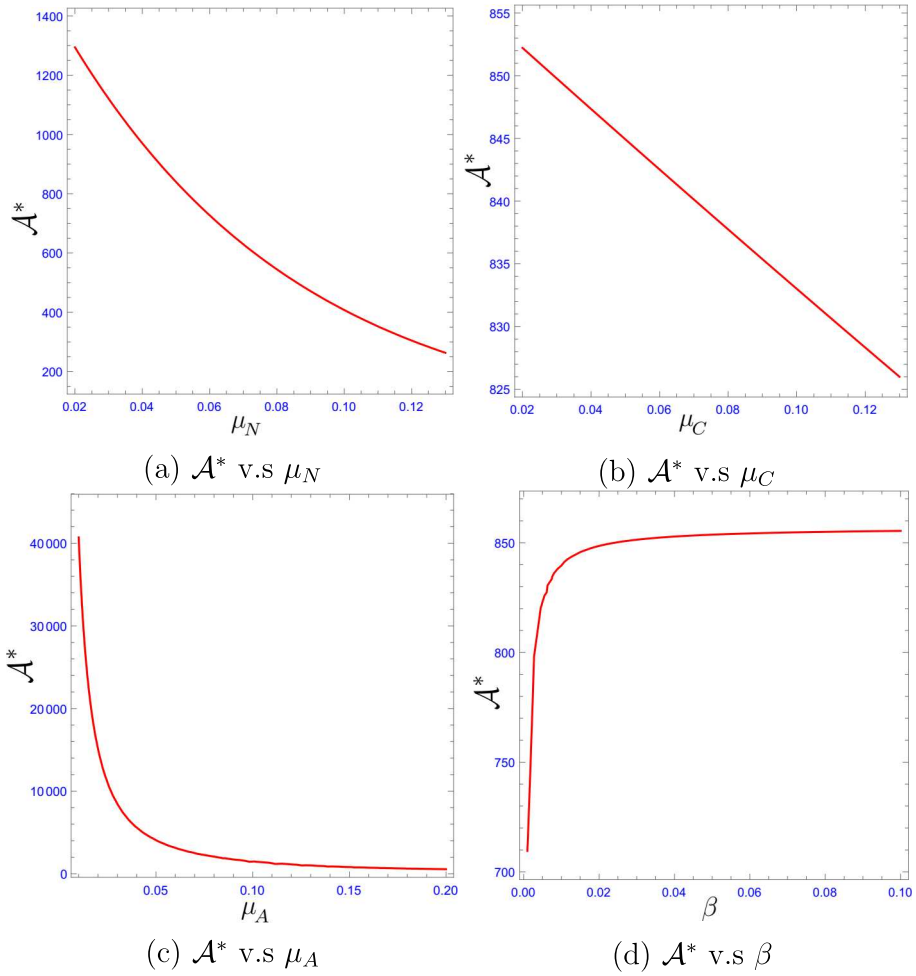
Figure 3.6 The global attractivity of $E^* = (25.2038, 840.126)$ under different initial condition. $m = 10$ and $\mathcal{R}_s = 41.4 > 1$.

From mathematical point of view, the above simulations mean that the suggested biological parameters in the literature may help *Lepeophtheirus salmonis* to persist. This is consistent with the realistic situation of sea lice in the nature. Since $\mathcal{C}^* = \frac{\mu_A}{\beta H}\mathcal{A}^*$, from Proposition 3.4.1, we can explore the effects of parameter variations (μ_N, μ_C, μ_A, β) on the magnitude of $\mathcal{A}^*$ numerically. We allow one of the parameters $\mathcal{S} \in \{\beta, \mu_N, \mu_C, \mu_A\}$ to vary from Table 3.1 while keep the others fixed as in Figure 3.6. We can observe that the increasing of (μ_N, μ_C, μ_A) decreases $\mathcal{A}^*$ in different degrees (See Figures 3.7a – 3.7c). The value of $\mathcal{A}^*$ is highly sensitive with small μ_A. Figure 3.7d indicates that the high infection rate β promotes the magnitude of $\mathcal{A}^*$, and hence, reduces the magnitude of $\mathcal{C}^*$ since $\mathcal{C}^* = \frac{\mu_A}{\beta H}\mathcal{A}^*$. This is biologically consistent, because when the infection rate is high, more copepodids attach to salmon which decreases the population of copepodids and increases the population of adults. The relationship between $\mathcal{A}^*$ and m (H) is equivalent to the one with μ_N (β).

Finally, we would like to mention that solving the TDE model (3.5) numerically can be done by transforming the time series resulting from solving the DDE model (3.8) using (3.6). We got the same time series as in Figure 3.6 by discretizing (3.6a) using a trapezoidal rule.

3.6 Discussion

As marine ectoparasite sea lice are responsible for most disease outbreaks on salmon farms which is causing enormous monetary losses. In this paper, we have proposed a stage-structured mathematical model to study the nonlinear dynamics of sea lice. We have divided the immature phase into two stages: non-infectious larvae (nauplii) and infectious larvae (copepodids), and considered one stage in the parasitic phase (adult sea lice). In the sea lice population, we assume that there is no competition between immatures and adults since each one feeds on different levels of food resources, consequently a a larger adults population is more convenient than a smaller one for facilitating development, that is, the development age for a nauplius to become a copepodid relates on the size of adults population. Based on Smith's work, we have described the dynamics by a system of partial differential equations (PDE), transformed the PDE model into a delay differential equation with a state-dependent threshold-type delay (TDE) and a delay differential equation with a state-independent delay (DDE), by mathematical techniques and appropriate change of variables. Once establishing the mathematical model, we have discussed the well-posedness of the model by proving the existence, uniqueness, non-negativity and boundedness of the solutions theoretically which are important features for ecological models. Then mathematically, we have calculated the adult reproduction number $\mathcal{R}_s$ which is a fundamental concept in growth models. From the



(a) $\mathcal{A}^*$ v.s μ_N (b) $\mathcal{A}^*$ v.s μ_C

(c) $\mathcal{A}^*$ v.s μ_A (d) $\mathcal{A}^*$ v.s β

Figure 3.7 The relationship between $\mathcal{A}^*$ and the related parameters.

dynamical point of view, we have studied the equilibrium points, persistence, local stability and global attractivity with respect to $\mathcal{R}_s$.

Through our theoretical analysis, we have found that the adult reproduction number $\mathcal{R}_s$ depends on deferent parameters related to the natural environment, implying that some parameters $(\mu_N, \mu_C, \mu_A, m, \beta)$ can play important role in the population extinction and persistence since the adult reproduction number $\mathcal{R}_s$ decreases as any mortality rate (μ_N, μ_C, μ_A) or the infection development age (m) increases. While the infection rate β has an opposite effect on $\mathcal{R}_s$. For instance, low salinity density and high water temperature increase the mortality rates and the development age, respectively [36, 39, 95, 177].

In the numerical study, based on biological parameters in the literature we have presented a case study regarding *Lepeophtheirus salmonis* dynamics, and found that the suggested biological parameters in the literature may help *Lepeophtheirus salmonis* to persist, which

is the realistic situation of *Lepeophtheirus salmonis* in the nature. In real-life, *Lepeophtheirus salmonis* have high reproductive capacity allowing them to persist. Recently, the control of sea lice is one of the top priorities in aquaculture research, see e.g., [149, 162, 189], so understand the sea lice dynamics is essential, as a part of the control process of these parasites. Theoretically, controlling the spreading of sea lice equates to decreasing the value of $\mathcal{R}_s$ below to 1. However, in the real world, it is almost inapplicable because of the high dependence of natural parameters. Furthermore, in the numerical simulations, we have noticed that the mortality rate μ_A has a significant effect on the steady state of sea lice population ($\mathcal{A}^*$). Therefore, targeting the adult sea lice can play a major role in controlling sea lice, for instance, the use of the natural predator "cleaner fish" as a biological controller of sea lice [48, 162]. On the other hand, we have predicted that β plays a critical role in decreasing the steady state of copepodids population ($\mathcal{C}^*$). For example, the chemical compound hydrogen peroxide is used to kill the copepodids directly or detach them from the salmon skin surface [149, 189], i.e., decreasing β, and hence, controlling the adult sea lice indirectly.

In the literature, although sea lice growth is studied through field experiments, mathematical analysis and dynamical modeling are lacked. As we know that an important aspect of population dynamics is to study the long-term behavior of modeling systems. Intuition alone is not sufficient to fully grasp the population dynamical behavior. Instead an explicit mathematical description of population growth and its dynamics allows for testing and predicting the behavior theoretically and in computer simulations. That is one of the main contribution in our current work. In real world application, fish scientists may use the theoretical results of the proposed model to study the growth of sea lice and develop strategies or polices to control it.

Chapter 4

A Stage-Structured Model for the Biocontrol of Sea Lice Using Cleaner Fish

4.1 Introduction

In salmon-producing countries, such as Norway and Canada, threats of sea lice are a substantial concern since sea lice can limit the growth of salmon and increase the mortality of them. These parasites attach to salmon and feed on their mucus and tissue, thereby increasing farming costs and reducing the value of the product. Attempting chemical control on sea lice in salmon farms has provoked an often acrimonious debate with environmental organizations. In 1980, a Norwegian fish biologist Asmutid Bjordal did the first field trials on controlling sea lice using cleaner fish rather than chemicals, since then, the use of cleaner fish has became attractive because of its environmental benefits and the economic cost [47,190]. For instance, 30 farm sites in Scotland were using 150,000 wrasse by 1994 [98] and nowadays Norway alone uses around 5 million cleaner fish per year [47]. Economically, the cost of acquisition of cleaning fish is less than chemical treatment due to the relatively high cost of the latter and the large manpower requirements of bath treatments [148,190].

Biological control (biocontrol) is defined as the control of pests and weeds using other living organisms [110]. From the view point of mathematical modeling, biocontrol can be represented as predator-prey model, host–parasite model or competition model, which are central topics in ecology. Various mathematical models have been proposed to study the impact of biological control strategies in different scenarios [13,67,121,140,180]. For instance, in [67], the authors proposed a time-delayed model, with a predator-prey interaction, for

controlling of the Asian longhorned beetle *Anoplophora glabripennis* by its natural predators, the cylindrical bark beetle *Dastarcus longulus*. In [121], an ordinary differential equations model was developed to control malaria by using larvivorous fish. A biological control for schistosomiasis by the introduction of a competitor snail species was investigated in [13]. Using cleaner fish to control of sea lice has bean discussed by ecologists in field experiments [48,90,148,162]. In these experiments, sea lice were counted over randomly collected salmon when no cleaner fish existed, then did the same process with cleaner fish again. For example, in [48], the authors investigated the use of two cleaner fish, corkwing and goldsinny wrasse, in controlling sea lice infestations on two commercial fish farms off the west Irish coast. In [148], the authors discussed the efficiency of lumpfish for sea lice control.

In the literature, although controlling sea lice by cleaner fish is studied through field experiments e.g. [12,68], little mathematical analysis has been carried out to understand the dynamical behaviors of sea lice. In this work, we extend the growth model (3.2) in Chapter 3 to a predator-prey interaction at the adult level of sea lice and we study the dynamical behavior of sea lice population in the present of a control agent. As a starting point, we describe the dynamics of sea lice and cleaner fish by a system of partial differential equations, then by using the technique of integration along characteristics and a particular changing of variables, we transform it into a standard time-delayed model. Based on the proposed delayed mathematical model, we derive the reproduction numbers for the adult sea lice ($\mathcal{R}_s$) and cleaner fish ($\mathcal{R}_f$). Then, we study the threshold dynamics in the system with respect to $\mathcal{R}_s$ and $\mathcal{R}_f$ under biologically reasonable conditions. In addition, we discuss the coexistence of a unique positive equilibrium, implying both sea lice and cleaner fish can coexist, from biological point of view. Then, we study the local stability of the positive equilibrium point, discuss possible oscillations or stability change analytically, we investigate the occurrence of Hopf bifurcations, and explore the direction and stability of the Hopf bifurcation.

The rest of the chapter is organized as follows: in Section 4.2, we formulate the model from the growth of sea lice procedure and the interaction with cleaner fish and discuss the well-posedness property of the model by verifying the non-negativity and boundedness of the solutions with reasonable initial data. In Section 4.3, we calculate the adult reproduction number for sea lice $\mathcal{R}_s$ and the net reproductive number of cleaner fish $\mathcal{R}_f$, establish the threshold dynamics of the model in terms of $\mathcal{R}_s$ and $\mathcal{R}_f$. In Section 4.4, we explore the coexistence and stability of positive steady state, investigate the occurrence of Hopf bifurcations, and identify the direction and stability of the Hopf bifurcation. In Section 4.5, we carry out a case study for comparing two cleaner fish species and investigate the oscillation behavior. Finally, conclusion and remarks are drawn in Section 4.6.

4.2 Model Formulation

The purpose of this section is to formulate a mathematical model for controlling of sea lice by one of its natural predators, "cleaner fish". In nature, cleaner fish ($W(t)$), such as wrasses, have been observed removing parasite sea lice from salmon (see e.g. [47, 98, 144]). Assuming that cleaner fish feed only on parasitic stage of the sea lice, based on the model 3.2, we describe the interaction among the two species (sea lice and cleaner fish) for $t \geq 0$ and $0 \leq x \leq m$, in the following way:

$$\begin{aligned}
&\frac{\partial u(t,x)}{\partial t} + P\left(A(t)\right)\frac{\partial u(t,x)}{\partial x} = -\mu_N u(t,x),\\
&\frac{dC}{dt} = P\left(A(t)\right)u(t,m) - \mu_C C(t) - \beta H C(t),\\
&\frac{dA}{dt} = \beta H C(t) - \mu_A A(t) - W(t)A(t)f(A(t)),\\
&\frac{dW}{dt} = \gamma W(t)A(t)f(A(t)) - \mu_W W(t),\\
&P\left(A(t)\right)u(t,0) = b(A(t)),
\end{aligned} \tag{4.1}$$

where μ_W is the mortality rate cleaner fish; $\gamma \in (0,1]$ denotes cleaner fish food-conversion efficiency (the efficiency of turning food into offspring); and $f(\cdot)$ is the per-prey-per-predator predation rate of cleaner fish on sea lice, so that the product $A(t)f(A(t))$ is the per-predator predation rate (i.e., the 'functional response') [8, 10, 17, 139]. An architecture of the model (4.1) is given in Figure 4.1. We will refer to system (4.1) as the PDE model.

Appropriate initial conditions for the PDE model are:

$$C(0) = C_0, \quad A(0) = A_0, \quad W(0) = W_0, \quad u(0,x) = u_0(x),$$

where C_0, A_0 and W_0 are nonnegative real numbers and u_0 is a nonnegative continuous function on the interval $[0,m]$.

From the view points in biology and analysis, we assume that the function f is continuous and differentiable, and satisfies the property:

(Q_3) $f(s) > 0$ for $s > 0$, $\lim\limits_{s\to 0^+} f(s) \geq 0$, $\lim\limits_{s\to\infty} f(s) = 0$, $\frac{d(sf(s))}{ds} > 0$ and $\lim\limits_{s\to\infty} sf(s) = F_\infty > 0$,

where $sf(s)$, as $s = A$, represents the number of mature sea lice that one cleaner fish consumes, so F_∞ represents cleaner fish satiation (the maximum number of sea lice consumed per one cleaner fish) and $\lim\limits_{s\to\infty} f(s) = 0$ means that when the amount of food available in the ecological system (sea lice density) is very large, the chance of a given sea louse being killed by cleaner fish is very low. The properties of f in (Q_3) allows a unified treatment for many fundamental functional response in ecology, some examples are listed in Table 4.1.

$f(s)$	$sf(s)$	Parameters	Functional response	Ref.
$\frac{a}{1+aqs}$	$\frac{as}{1+aqs}$	a : the attack rate	Holling type II	[87]
$\frac{as^{n-1}}{1+aqs^n}$, $n \geq 2$	$\frac{as^n}{1+aqs^n}$	q : the handling time	Holling type III	[150]
$\frac{\varrho(1-e^{-\alpha s})}{s}$	$\varrho(1-e^{-\alpha s})$	ϱ: hunting success α: the maximal predation rate	Ivlev type	[92]

Table 4.1 Examples of function f.

By using the technique of integration along characteristics, we can reduce the system (4.1) to a system of threshold delayed differential equations (TDE). More specifically, for $t > t_0$, the system (4.1) becomes

$$\begin{aligned}
\frac{dC}{dt} &= P\left(A(t)\right)\frac{b\left(A(t-\tau(t))\right)}{P\left(A(t-\tau(t))\right)}e^{-\mu_N\tau(t)} - \mu_C C(t) - \beta HC(t),\\
\frac{dA}{dt} &= \beta HC(t) - \mu_A A(t) - W(t)A(t)f(A(t)),\\
\frac{dW}{dt} &= \gamma W(t)A(t)f(A(t)) - \mu_W W(t),\\
&\int_{t-\tau(t)}^{t} P\left(A(\eta)\right)d\eta = m,
\end{aligned} \tag{4.2}$$

with initial data for $t \in [0, t_0]$ given by the solution of the system

$$\begin{aligned}
\frac{dC}{dt} &= P\left(A(t)\right)u_0(m - r(t))e^{-\mu_N t} - \mu_C C(t) - \beta HC(t),\\
\frac{dA}{dt} &= \beta HC(t) - \mu_A A(t) - W(t)A(t)f(A(t)),\\
\frac{dW}{dt} &= \gamma W(t)A(t)f(A(t)) - \mu_W W(t),\\
\frac{dr}{dt} &= P\left(A(t)\right),
\end{aligned}$$

with $C(0) = C_0$, $A(0) = A_0$, $W(0) = W_0$ and $r(0) = 0$.

By changing of variables technically we can remove the state-dependent delay in (4.2) and transform it into a standard time-delayed model (4.3), here we omit the procedure. For $t \geq 0$, let

$$T = \int_0^t P\left(A(\eta)\right)d\eta, \qquad \mathcal{C}(T) = C(t), \qquad \mathcal{A}(T) = A(t), \qquad \mathcal{W}(T) = W(t).$$

Then system (4.2) becomes

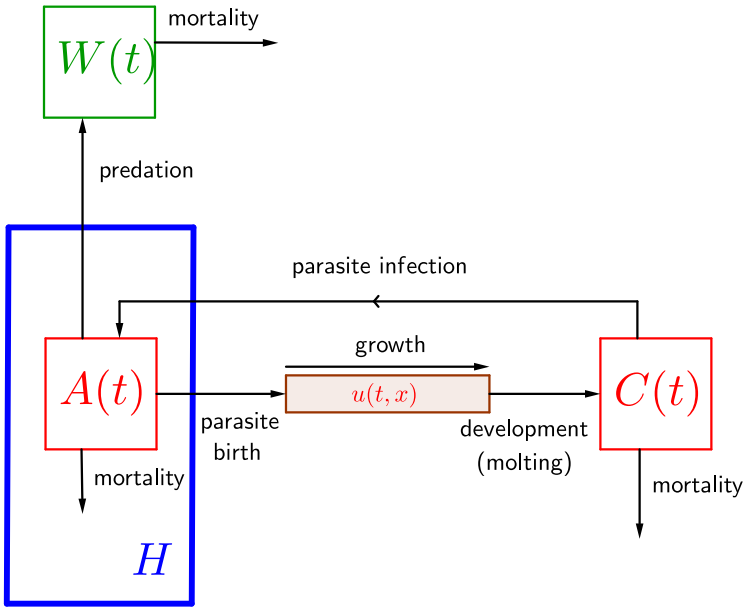


Figure 4.1 Schematic chart of the PDE model.

$$\frac{d\mathcal{C}}{dT} = \frac{b\left(\mathcal{A}(T-m)\right)}{P\left(\mathcal{A}(T-m)\right)} e^{-\mu_N \hat{\tau}(\mathcal{A}_T)} - \left(\mu_C + \beta H\right) \frac{\mathcal{C}(T)}{P\left(\mathcal{A}(T)\right)}, \tag{4.3a}$$

$$\frac{d\mathcal{A}}{dT} = \frac{1}{P\left(\mathcal{A}(T)\right)} \left[\beta H \mathcal{C}(T) - \mu_A \mathcal{A}(T) - \mathcal{W}(T)\mathcal{A}(T) f(\mathcal{A}(T))\right], \tag{4.3b}$$

$$\frac{d\mathcal{W}}{dT} = \frac{1}{P\left(\mathcal{A}(T)\right)} \left[\gamma \mathcal{W}(T)\mathcal{A}(T) f(\mathcal{A}(T)) - \mu_W \mathcal{W}(T)\right], \tag{4.3c}$$

where $\hat{\tau}$ is defined on $C^+ = C([-m, 0], \mathbb{R}_+)$ by

$$\hat{\tau}(\phi) = \int_{-m}^{0} \frac{1}{P\left(\phi(\theta)\right)} d\theta.$$

and the initial data for $T \in [0, m]$ is given by the solution of the system

$$\begin{aligned}
\frac{d\mathcal{C}}{dT} &= u_0\left(m - T\right) \exp\left\{-\mu_N \int_0^T \frac{1}{P\left(A(\eta)\right)} d\eta\right\} - \left(\mu_C + \beta H\right) \frac{\mathcal{C}(T)}{P\left(\mathcal{A}(T)\right)}, \\
\frac{d\mathcal{A}}{dT} &= \frac{1}{P\left(\mathcal{A}(T)\right)} \left[\beta H \mathcal{C}(T) - \mu_A \mathcal{A}(T) - \mathcal{W}(T)\mathcal{A}(T) f(\mathcal{A}(T))\right], \\
\frac{d\mathcal{W}}{dT} &= \frac{1}{P\left(\mathcal{A}(T)\right)} \left[\gamma \mathcal{W}(T)\mathcal{A}(T) f(\mathcal{A}(T)) - \mu_W \mathcal{W}(T)\right],
\end{aligned} \tag{4.4}$$

with $\mathcal{C}(0) = C_0$, $\mathcal{A}(0) = A_0$, $\mathcal{W}(0) = W_0$. We will refer to system (4.3) as the DDE model. In the rest of the manuscript, we mainly work on the model (4.3) by applying the standard theory for delay differential equations, see e.g. [74, 169, 218], based on the conclusions given by Smith in [164, 165].

Next, we demonstrates that the solutions of model (4.4) are nonnegative and bounded when $0 \leq T \leq m$.

Proposition 4.2.1. *Suppose hypotheses (Q_1)-(Q_3) are satisfied. Then system (4.4) has a unique nonnegative bounded solution $(\mathcal{C}(T), \mathcal{A}(T), \mathcal{W}(T))$ with the initial value $(C_0, A_0, W_0) \in \mathbb{R}^3_+$. Furthermore, if $C_0 > 0$, $A_0 > 0$ and $W_0 > 0$, then the solution is strictly positive.*

Proof. From the properties of P and f given in (Q_1) and (Q_3), both functions are locally Lipschitz for any closed bounded set in $\mathbb{R}$. Thus, it follows from [73, Theorem 1.3.1] that system (4.4) admits a unique solution $(\mathcal{C}(T), \mathcal{A}(T), \mathcal{W}(T))$ through an initial value $(C_0, A_0, W_0) \in \mathbb{R}^3_+$ with the maximal interval of existence $[0, \xi)$ for some $\xi > 0$.

Now we prove the non-negativity of the solution. For $T \in [0, m]$, it is clear from the third equation in (4.4) that

$$\mathcal{W}(T) = W_0 \exp\left\{ \int_0^T \frac{\gamma \mathcal{A}(\theta) f\left(\mathcal{A}(\theta)\right) - \mu_W}{P\left(\mathcal{A}(\theta)\right)} d\theta \right\} \geq 0.$$

From the first equation in (4.4) and comparison principle (see, e.g. [170, Theorem B.1]), we have

$$\mathcal{C}(T) \geq C_0 \exp\left\{ -\int_0^T \frac{\mu_C + \beta H}{P\left(\mathcal{A}(\theta)\right)} d\theta \right\} \geq 0.$$

Hence,

$$\mathcal{A}(T) \geq A_0 \exp\left\{ -\int_0^T \frac{\mathcal{W}(\theta) f\left(A(\theta)\right) + \mu_A}{P\left(\mathcal{A}(\theta)\right)} d\theta \right\} \geq 0.$$

Furthermore, for any $T \in [0, m]$,

$$\mathcal{C}(T) \leq C_0 + \int_0^T u_0\left(m - \theta\right) d\theta < \infty \quad \text{and} \quad \mathcal{W}(T) \leq W_0 \int_0^T \frac{\gamma F_\infty}{P\left(\mathcal{A}\left(\theta\right)\right)} d\theta < \infty.$$

Therefore

$$\mathcal{A}(T) \leq A_0 + \beta H \int_0^T \frac{\mathcal{C}\left(\theta\right)}{P\left(\mathcal{A}\left(\theta\right)\right)} d\theta < \infty.$$

Thus, all solutions are bounded and exist for any $T \in [0, m]$. The strictly positive solution when $C_0 > 0$, $A_0 > 0$ and $W_0 > 0$ is clear. □

To analyze (4.3) mathematically, let $X := C([0, m], \mathbb{R}^3)$ and $X^+ = C([0, m], \mathbb{R}^3_+)$. For $\phi = (\phi_1, \phi_2, \phi_3) \in X$, denote $\|\phi\| = \sum_{i=1}^3 \|\phi_i\|_\infty$ with $\|\phi_i\|_\infty = \max_{0\le\theta\le m} |\phi_i(\theta)|$. Then, (X, X^+) is an ordered Banach space and X^+ is a normal cone of X with nonempty interior in X. For any given continuous function $z : [0, \sigma_\phi) \to \mathbb{R}^3$ with $\sigma_\phi > 0$, we define $z_t \in X$ for $t \ge m$ by $z_T(\theta) = z(T - \theta)$ for all $\theta \in [0, m]$.

Denote the initial data set for system (4.3) by

$$D_X = \left\{ \begin{array}{c} \phi \in X^+ : (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \text{ is a solution of (4.4) for } \theta \in [0, m] \\ \text{with } (\phi_1(0), \phi_2(0), \phi_3(0)) = (C_0, A_0, W_0) \end{array} \right\},$$

then the well-posedness property for the system (4.3) is followed.

Theorem 4.2.1. *Suppose hypotheses (Q_1), (Q_2) and (Q_3) are satisfied. Then, for any $\phi \in D_X$, the system (4.3) has a unique nonnegative solution $z(T, \phi)$ with the initial condition $z_0 = \phi$, and all solutions are ultimately bounded. Furthermore, $z(T, \phi)$ is strictly positive when $\mathcal{C}(m) > 0$, $\mathcal{A}(m) > 0$ and $\mathcal{W}(m) > 0$.*

Proof. Given $\phi \in D_X$, we define

$$G(\phi) = (G_1(\phi), G_2(\phi), G_3(\phi)),$$

with

$$\begin{aligned} G_1(\phi) =& \frac{b(\phi_2(0))}{P(\phi_2(0))} e^{-\mu_N \hat{\tau}(\phi_2)} - (\mu_C + \beta H) \frac{\phi_1(-m)}{P(\phi_2(-m))}, \\ G_2(\phi) =& \frac{1}{P(\phi_2(-m))} \left[\beta H \phi_1(-m) - \mu_A \phi_2(-m) - \phi_3(-m)\phi_2(-m) f(\phi_2(-m))\right], \\ G_3(\phi) =& \frac{1}{P(\phi_2(-m))} \left[\gamma \phi_3(-m)\phi_2(-m) f(\phi_2(-m)) - \mu_W \phi_3(-m)\right], \end{aligned}$$

with

$$\hat{\tau}(\phi_2) = \int_{-m}^{0} \frac{1}{P(\phi_2(m + \theta))} d\theta.$$

It is easy to see that D_X is closed in X, $G(\phi)$ is continuous and is Lipschitz in ϕ in each compact set in D_X. By [74, Theorem 2.2.3] there is a unique solution in (4.3) on its maximal existence interval $[0, \sigma_\phi)$ through ϕ for any $\phi \in D_X$.

Furthermore, for any $\phi \in D_X$ with $\phi_i(m) = 0$, it is obvious that $G_i(\phi) \ge 0$ for $i = 1, 2, 3$. Thus $\phi_i(T) \ge 0$ for all $t \in [0, \sigma_\phi)$, $i = 1, 2, 3$, see [169, Theorem 5.2.1]. Thus all the solutions

of (4.3) are nonnegative for any $t \in [0, \sigma_\phi)$. More precisely, it is easy to see that when $\phi \in D_X$,

$$\mathcal{C}(T) \geq \mathcal{C}(m) \exp\left\{ -\int_m^T \frac{\mu_C + \beta H}{P\left(\mathcal{A}(\theta)\right)} d\theta \right\}.$$

Thus, $\mathcal{C}(T) > 0$ when $\mathcal{C}(m) > 0$. The positivity of $\mathcal{A}(T)$ and $\mathcal{W}(T)$ can be obtained as in Proposition 4.2.1 when $\mathcal{A}(m) > 0$ and $\mathcal{W}(m) > 0$.

Now we prove the boundedness. From (4.3), we have

$$\begin{aligned}
\frac{d\mathcal{C}}{dT} &\leq \frac{b\left(\mathcal{A}(T-m)\right)}{P_0} - \frac{\mu_C}{P_\infty}\mathcal{C}(T) - \beta H \frac{\mathcal{C}(T)}{P\left(\mathcal{A}(T)\right)}, \\
\frac{d\mathcal{A}}{dT} &\leq \beta H \frac{\mathcal{C}(T)}{P\left(\mathcal{A}(T)\right)} - \frac{\mu_A}{P_\infty}\mathcal{A}(T) - \frac{\mathcal{W}(T)\mathcal{A}(T) f(\mathcal{A}(T))}{P\left(\mathcal{A}(T)\right)}, \\
\frac{d\mathcal{W}}{dT} &\leq \frac{\mathcal{W}(T)\mathcal{A}(T) f(\mathcal{A}(T))}{P\left(\mathcal{A}(T)\right)} - \frac{\mu_W}{P_\infty}\mathcal{W}(T).
\end{aligned}$$

Thus,

$$\frac{d(\mathcal{C}+\mathcal{A}+\mathcal{W})}{dT} \leq \frac{1}{P_0} b\left(\mathcal{A}(T-m)\right) - \frac{\tilde{\mu}}{P_\infty}\left(\mathcal{C}(T)+\mathcal{A}(T)+\mathcal{W}(T)\right).$$

Since b is an increasing function, $\mathcal{C}(T) \geq 0$ and $\mathcal{W}(T) \geq 0$, we have

$$\begin{aligned}
\frac{d(\mathcal{C}+\mathcal{A}+\mathcal{W})}{dT} &\leq \frac{1}{P_0} b\left(\mathcal{C}(T-m)+\mathcal{A}(T-m)+\mathcal{W}(T-m)\right) - \frac{\tilde{\mu}}{P_\infty}\left(\mathcal{C}(T)+\mathcal{A}(T)+\mathcal{W}(T)\right), \\
&\leq \frac{1}{P_0} B\left(\mathcal{C}(T-m)+\mathcal{A}(T-m)+\mathcal{W}(T-m)\right) - \frac{\tilde{\mu}}{P_\infty}\left(\mathcal{C}(T)+\mathcal{A}(T)+\mathcal{W}(T)\right),
\end{aligned}$$

where $\tilde{\mu} = \min\{\mu_C, \mu_A, \mu_W\}$.

Consider the delay differential equation

$$\frac{dv}{dT} = \frac{1}{P_0} B\left(v(T-m)\right) - \frac{\tilde{\mu}}{P_\infty} v(T). \tag{4.5}$$

Since the function B is strictly sub-homogeneous and $\frac{dv}{dT}\big|_{v=0} = 0$, it follows from [219, Theorem 3.2] that (4.5) admits a globally asymptotically stable equilibrium which attracts all positive solutions. Therefore, $\mathcal{C}+\mathcal{A}+\mathcal{W}$ is bounded, and hence, any solution of (4.3) is bounded due to the non-negativity of $\mathcal{C}$, $\mathcal{A}$ and $\mathcal{W}$. Thus, $\sigma_\phi = +\infty$ (see e.g., [74, Theorem 2.3.1]). Hence, all the solutions exist globally, and are ultimately bounded. □

Remark 4.2.1. *Theorem 4.2.1 implies that there exists $T_1 > 0$ and $M > 0$ such that $0 \leq \mathcal{C}(T) \leq K$, $0 \leq \mathcal{A}(T) \leq K$ and $0 \leq \mathcal{W}(T) \leq K$ for $T > T_1$.*

Comparing with Chapter 3, by introducing the cleaner fish to the model (3.2) we have changed the model from a population growth model to a predator-prey interaction model.

This is more realistic from the biological perspective since each species has enemies in the nature. The model (4.3) with cleaner fish may produce oscillations in the population size of both cleaner fish and sea lice. Mathematically, this interaction influences the dynamics of the cleaner fish and sea lice populations, including the equilibria and their stability. Some of the results, such as well-posedness property and persistence, still hold for systems (4.3) although different mathematical techniques are used in the proof. From the view point of real world application, studying the outcomes of interactions among species is important for the biologists to understand the structure of different communities and their sustained.

4.3 Reproduction numbers and threshold dynamics

In this section, we discuss the adult reproduction number of the prey population (sea lice) and the net reproductive number of the predator population (cleaner fish), and further explore the threshold dynamics for the system (4.3) in terms of the reproduction numbers.

Since $b(0) = 0$, it is easy to see that the equilibrium $E_0 = (0, 0, 0)$ always exists in (4.3) for all values of the parameters.

The linearization of (4.3) at E_0 is

$$\begin{aligned} \frac{d\mathcal{C}}{dT} &= \frac{b'(0)}{P_0} e^{\frac{-\mu_N m}{P_0}} \mathcal{A}(T-m) - \frac{\mu_C + \beta H}{P_0} \mathcal{C}(T), \\ \frac{d\mathcal{A}}{dT} &= \frac{\beta H}{P_0} \mathcal{C}(T) - \frac{\mu_A}{P_0} \mathcal{A}(T), \\ \frac{d\mathcal{W}}{dT} &= -\mu_W \mathcal{W}(T). \end{aligned} \tag{4.6}$$

Since the third equation in (4.6) is decoupled, the adult reproduction number for sea lice, denoted by $\mathcal{R}_s$, is the same as in Chapter 3. Consequently,

$$\mathcal{R}_s := \frac{b'(0)\beta H}{\mu_A(\mu_C + \beta H)} e^{\frac{-\mu_N m}{P_0}}, \tag{4.7}$$

which represents the average number of offspring reaching adulthood from a single adult over its life span, biologically.

With the condition (K_2), we have the following result for the existence of a steady state with $\mathcal{W} = 0$, which we refer to as the predator-free equilibrium (or cleaner fish-free equilibrium).

Proposition 4.3.1. *Assume $\mathcal{R}_s > 1$, (Q_1)-(Q_2) and (K_2) hold. Then system (4.3) has a*

unique equilibrium point $E_1 = (\mathcal{C}_1^*, \mathcal{A}_1^*, 0)$ *where* $\mathcal{C}_1^* = \frac{\mu_A}{\beta H}\mathcal{A}_1^*$ *and* $\mathcal{A}_1^*$ *satisfies*

$$b\left(\mathcal{A}_1^*\right)\exp\left\{\frac{-m\mu_N}{P(\mathcal{A}_1^*)}\right\} = \frac{\left(\mu_C+\beta H\right)\mu_A}{\beta H}\mathcal{A}_1^*. \tag{4.8}$$

Obviously, for the existence of E_1, it is necessary, but is not sufficient, that $\mathcal{R}_s > 1$, that is, the value of m must be relatively small, i.e., $m \le m_{\max}$ where $m_{\max}$ is defined in (3.19).

From the biological point of view, in prey-predator interaction, at a steady prey population (sea lice), the net reproductive number of the predator population (cleaner fish), $\mathcal{R}_f$, is

$$\mathcal{R}_f := \begin{pmatrix}\text{Food-conversion}\\ \text{efficiency}\end{pmatrix} \times \begin{pmatrix}\text{Cleaner fish}\\ \text{predation rate}\end{pmatrix} \times \begin{pmatrix}\text{Life expectancy}\\ \text{of cleaner fish}\end{pmatrix},$$

see e.g., [45,46]. As we know, from the model (4.1), that the food-conversion efficiency is γ, $\mathcal{A}f(\mathcal{A})$ is the number of mature sea lice that one cleaner fish consumes, that is, cleaner fish predation rate and $\frac{1}{\mu_W}$ is the average lifespan of cleaner fish. Therefore, at the steady state of adult sea lice population $\mathcal{A}_1^*$, we have

$$\mathcal{R}_f := \gamma \times \mathcal{A}_1^* f(\mathcal{A}_1^*) \times \frac{1}{\mu_W}. \tag{4.9}$$

In other words, the net reproductive number of cleaner fish $\mathcal{R}_f$ is the expected number of new cleaner fish that can be produced by one cleaner fish over its lifespan when the sea lice population is stable.

Next, we study the stability of the trivial steady state E_0. It is easy to check that the characteristic equation of (4.6) is

$$(\lambda + \mu_W)\Delta_0(\lambda) = 0$$

where $\Delta_0(\lambda)$ is defined in (3.13). Since $\lambda = -\mu_W$ is a negative root, the stability of E_0 is determined by the roots of $\Delta_0(\lambda) = 0$. It then follows from Theorem 3.3.2 that the following result holds.

Theorem 4.3.1. *When* $\mathcal{R}_s < 1$ *and* (Q_1)-(Q_2) *hold,* $E_0 = (0,0,0)$ *in (4.3) is locally asymptotically stable. When* $\mathcal{R}_s > 1$, $E_0 = (0,0,0)$ *is unstable.*

In order to use the comparison principle to study the global dynamics of system (4.3), we need to make sure the solution semiflow of (4.3) is eventually strongly monotone and strongly order preserving (see [169, Chapter 5]). Therefore, we choose $Y = \mathbb{R}\times C([0,m],\mathbb{R})\times\mathbb{R}$, $Y^+ = \mathbb{R}_+\times C([0,m],\mathbb{R}_+)\times\mathbb{R}_+$. Then (Y, Y^+) is an ordered Banach space. For a continuous function $\hat{z} : [0,\infty)\times[0,\infty)\times[0,\infty) \to \mathbb{R}^3$ and $T \ge m$, define the solution semiflow $\hat{z}_T(\cdot) :$

$Y \to \mathbb{R}^3$ of (4.3) by

$$\hat{z}_T(\mathcal{C}_m, \theta, \mathcal{W}_m) = (\hat{z}_1(T), \hat{z}_2(T-\theta), \hat{z}_3(T)) \quad \forall \theta \in [0, m]$$

where $\mathcal{C}_m = \mathcal{C}(m) = \hat{z}_1(m)$ and $\mathcal{W}_m = \mathcal{W}(m) = \hat{z}_3(m)$. We choose the initial data set for (4.3) in the following set:

$$D_Y = \left\{ \begin{array}{c} \phi_{\mathcal{CW}} = (\mathcal{C}_m, \hat{\phi}, \mathcal{W}_m) \in Y^+ : \ \mathcal{C}_m = \hat{\psi}_1(m), \ \hat{\phi}(\theta) = \hat{\psi}_2(\theta) \ \forall \ \theta \in [0, m] \text{ and } \mathcal{W}_m = \hat{\psi}_3(m) \text{ where} \\ (\hat{\psi}_1(\theta), \hat{\psi}_2(\theta), \hat{\psi}_3(\theta)) \text{ is a solution of (4.4) for } \theta \in [0, m] \text{ with } (\hat{\psi}_1(0), \hat{\psi}_2(0), \hat{\psi}_3(0)) = (C_0, A_0, W_0) \end{array} \right\}.$$

The following result can be proved as Theorem 4.2.1.

Lemma 4.3.1. *For any $\phi_{\mathcal{CW}} \in D_Y$, system (4.3) admits a unique bounded nonnegative solution $\hat{z}_T(\phi_{\mathcal{CW}})$ on $[m, \infty)$ with $\hat{z}_m = \phi_{\mathcal{CW}}$.*

With the condition (K_1), we can discuss the global stability of $E_0 = (0, 0, 0)$.

Theorem 4.3.2. *If $\mathcal{R}_s < 1$, (Q_1)-(Q_2) and (K_1) holds, then $E_0 = (0, 0, 0)$ of system (4.3) is globally asymptotically stable for any $\phi_{\mathcal{CW}} \in D_Y$.*

Proof. From (4.3a) and (4.3b), we have

$$\begin{aligned} \frac{d\mathcal{C}}{dT} &= \frac{b\left(\mathcal{A}(T-m)\right)}{P\left(\mathcal{A}(T-m)\right)} e^{-\mu_N \hat{\tau}(\mathcal{A}_T)} - (\mu_C + \beta H) \frac{\mathcal{C}(T)}{P\left(\mathcal{A}(T)\right)}, \\ \frac{d\mathcal{A}}{dT} &\leq \frac{1}{P\left(\mathcal{A}(T)\right)} \left[\beta H \mathcal{C}(T) - \mu_A \mathcal{A}(T)\right]. \end{aligned} \tag{4.10}$$

When $\mathcal{R}_s < 1$, we know that $(0, 0)$ is a globally attractive equilibrium point in the system obtained from (4.10) by replacing $\leq$ with $=$, see Theorem 3.4.1. By the comparison principle (see e.g., [169, Theorem 5.1.1]) and the non-negativity of $\mathcal{C}(T)$ and $\mathcal{A}(T)$, we obtain

$$\lim_{T \to \infty} \mathcal{C}(T) = 0 \qquad \text{and} \qquad \lim_{T \to \infty} \mathcal{A}(T) = 0.$$

Denote $\Phi(T)(\phi_{\mathcal{CW}}) := \hat{z}_T(\phi_{\mathcal{CW}})$ be the solution semiflow associated with (4.3). Let $\omega = \omega(\phi_{\mathcal{CW}})$ be the omega limit set of $\Phi(T)$. By [218, Lemma 1.2.1], ω is an internally chain transitive set for $\Phi(T)$. Thus, $\omega = \{(0, 0)\} \times \bar{\omega}$ for some $\bar{\omega} \subset \mathbb{R}$. It is easy to see that

$$\Phi(0, 0, \mathcal{W}_m) = (0, 0, \hat{\Phi}(\mathcal{W}_m))$$

where $\hat{\Phi}(T)$ is the solution semiflow associated with the equation

$$\frac{d\mathcal{W}}{dT} = \frac{-\mu_W}{P_0} \mathcal{W}(T). \tag{4.11}$$

Since ω is an internally chain transitive set for $\Phi(T)$, it easily follows that $\bar{\omega}$ is an internally chain transitive set for $\hat{\Phi}$.

Obviously $\{0\}$ is globally asymptotically stable for (4.11), we get $\bar{\omega} \cap W^s(0) \neq \emptyset$, where $W^s(0)$ is the stable manifold of 0. Thus, $\bar{\omega} = \{0\}$ (see e.g., [85, Theorem 3.2 and Remark 4.6] or [218, Theorem 1.2.1]). Therefore, we have $\omega = \{(0,0,0)\}$, and hence

$$\lim_{T\to\infty} (\mathcal{C}(T), \mathcal{A}(T), \mathcal{W}(T)) = (0,0,0)\,.$$

Together with the local stability of $(0,0,0)$, we obtain the global asymptotic stability of $(0,0,0)$.

□

From the biological point of view, if $\mathcal{R}_s < 1$, the sea lice population cannot survive since an adult sea lice produces on average less than one secondary adults, and hence, the cleaner fish population may extinct due to the lack of food resource (adult sea lice).

Now we discuss the global attractivity of the predator-free equilibrium $E_1 = (\mathcal{C}^*, \mathcal{A}^*, 0)$.

Theorem 4.3.3. *Assume $\mathcal{R}_s > 1$, $\mathcal{R}_f < 1$, (Q_1)-(Q_3) and (K_2)-(K_3) hold. Then*

$$\lim_{T\to\infty} (\mathcal{C}(T), \mathcal{A}(T), \mathcal{W}(T)) = (\mathcal{C}_1^*, \mathcal{A}_1^*, 0)$$

for any $\phi_{\mathcal{CW}} \in D_Y$ such that $\mathcal{C}_m > 0$, $\hat{\phi}(m) > 0$.

Proof. From Theorem 3.4.3, when $\mathcal{R}_s > 1$, $(\mathcal{C}_1^*, \mathcal{A}_1^*)$ is a globally attractive equilibrium point in the system

$$\frac{d\mathcal{C}}{dT} = \frac{b\left(\mathcal{A}(T-m)\right)}{P\left(\mathcal{A}(T-m)\right)} e^{-\mu_N \hat{\tau}(\mathcal{A}_T)} - (\mu_C + \beta H)\frac{\mathcal{C}(T)}{P\left(\mathcal{A}(T)\right)}, \tag{4.12a}$$

$$\frac{d\mathcal{A}}{dT} = \frac{1}{P\left(\mathcal{A}(T)\right)} \left[\beta H \mathcal{C}(T) - \mu_A \mathcal{A}(T)\right]. \tag{4.12b}$$

By the comparison principle (see e.g., [169, Theorem 5.1.1]), we obtain

$$\limsup_{T\to\infty} \mathcal{C}(T) \leq \mathcal{C}_1^* \qquad \text{and} \qquad \limsup_{T\to\infty} \mathcal{A}(T) \leq \mathcal{A}_1^*.$$

Next, we prove $\mathcal{W}(T) \to 0$ as $T \to \infty$ when $\mathcal{R}_f < 1$. Let T_2 be sufficiently large, from (4.3c), we get

$$\frac{d\mathcal{W}}{dT} \leq \frac{\gamma \mathcal{A}_1^* f(\mathcal{A}_1^*) - \mu_W}{P(A)} \mathcal{W}(T) \leq \frac{\gamma \mathcal{A}_1^* f(\mathcal{A}_1^*) - \mu_W}{P_0} \mathcal{W}(T) = -\frac{1}{\mu_A P_0}\left(1 - \mathcal{R}_f\right) \mathcal{W}(T)$$

for $T \geq T_2$. Therefore

$$\limsup_{T\to\infty} \mathcal{W}(T) \leq 0.$$

Hence,

$$\lim_{T\to\infty} \mathcal{W}(T) = 0$$

due to the non-negativity of $\mathcal{W}$.

Hence, the limiting system of system (4.3) is same as (4.12). Thus, the omega limit set of the solution semiflow $\Phi(T)$ of system (4.3) is $\omega = \bar{\bar{\omega}} \times 0$ for some $\bar{\bar{\omega}} \subset \mathbb{R} \times \mathbb{R}$.

Claim 1. $\bar{\bar{\omega}} \not\subset \{(0,0)\} \cup \{(0,\bar{y}) : \bar{y} \neq 0\} \cup \{(\bar{x},0) : \bar{x} \neq 0\}$.

By contrary, first, assume $\bar{\bar{\omega}} \subset \{(0,\bar{y}) : \bar{y} \neq 0\}$. Then

$$\lim_{T\to\infty} \mathcal{C}(T) = 0.$$

When $\mathcal{R}_s > 1$, the solutions of (4.12) satisfies $\liminf_{T\to\infty}(\mathcal{C}(T), \mathcal{A}(T)) \geq (\eta_1, \eta_1)$ for some positive number $\eta_1 > 0$ from Theorem 3.4.2. Thus,

$$\begin{aligned}\lim_{T\to\infty} \left[\frac{b\left(\mathcal{A}(T-m)\right)}{P\left(\mathcal{A}(T-m)\right)} e^{-\mu_N \hat{\tau}(\mathcal{A}_T)} - \left(\mu_C + \beta H\right) \frac{\mathcal{C}(T)}{P\left(\mathcal{A}(T)\right)}\right] &= \lim_{T\to\infty} \frac{b\left(\mathcal{A}(T-m)\right)}{P\left(\mathcal{A}(T-m)\right)} e^{-\mu_N \hat{\tau}(\mathcal{A}_T)} \\ &> \frac{1}{2} \frac{b\left(\eta_1\right)}{P_\infty} \exp\left\{\frac{-m\mu_N}{P\left(\eta_1\right)}\right\} > 0.\end{aligned}$$

Let $T_3 > 0$ be sufficiently large. Then it follows from (4.12a) that

$$\frac{d\mathcal{C}}{dT} > \frac{1}{2} \frac{b\left(\eta_1\right)}{P_\infty} \exp\left\{\frac{-m\mu_N}{P\left(\eta_1\right)}\right\} > 0, \qquad \forall T \geq T_3.$$

Therefore, $\mathcal{C}(T) \to \infty$ as $T \to \infty$, a contradiction.

Now, assume that $\bar{\bar{\omega}} \subset \{(\bar{x},0) : \bar{x} \neq 0\}$. Then

$$\lim_{T\to\infty} \mathcal{A}(T) = 0.$$

Let $T_4 > 0$ be sufficiently large. Then it follows from (4.12b) that

$$\frac{d\mathcal{A}}{dT} > \frac{1}{2} \frac{\beta H \eta_1}{P_\infty} > 0, \qquad \forall T \geq T_4.$$

Therefore, $\mathcal{A}(T) \to \infty$ as $T \to \infty$, which contradicts the boundedness of solutions. This proves the claim.

For any $\phi_{\mathcal{CW}} \in D_Y$, we have

$$\Phi(\mathcal{C}_m, \hat{\phi}, 0) = (\tilde{\Phi}(\mathcal{C}_m, \hat{\phi}), 0)$$

where $\tilde{\Phi}(T)$ is the solution semiflow associated with system (4.12). By [218, Lemma 1.2.1], ω is an internally chain transitive set, and hence, $\bar{\omega}$ is an internally chain transitive set for $\tilde{\Phi}$. Since $(\mathcal{C}_1^*, \mathcal{A}_1^*)$ is a globally attractive in (4.12) and from Claim 1, we get $\bar{\omega} \cap W^s(\mathcal{C}_1^*, \mathcal{A}_1^*) \neq \emptyset$, where $W^s(\mathcal{C}_1^*, \mathcal{A}_1^*)$ is the stable manifold of $(\mathcal{C}_1^*, \mathcal{A}_1^*)$. Then it follows from [218, Theorem 1.2.1] (or [85, Theorem 3.2 and Remark 4.6]) that $\bar{\omega} = (\mathcal{C}_1^*, \mathcal{A}_1^*)$. This proves that $\omega = (\mathcal{C}_1^*, \mathcal{A}_1^*, 0)$. Consequently,

$$\lim_{T\to\infty} (\mathcal{C}(T), \mathcal{A}(T), \mathcal{W}(T)) = (\mathcal{C}_1^*, \mathcal{A}_1^*, 0)$$

for any $\phi_{\mathcal{CW}} \in D_Y$ with $\mathcal{C}_m > 0$, $\hat{\phi}(m) > 0$, that is, $(\mathcal{C}_1^*, \mathcal{A}_1^*, 0)$ is globally attractive. This completes the proof. □

Uniform persistence is an important concept in population dynamics which describes the survival of both species in an ecosystem. Now, we study the persistence of system (4.3) when $\mathcal{R}_s > 1$ and $\mathcal{R}_f > 1$.

Theorem 4.3.4. *Assume $\mathcal{R}_s > 1$, $\mathcal{R}_f > 1$, (Q_1)-(Q_3) and (K_2)- (K_3) hold. Then there exists a positive number $\eta_2 > 0$ such that every solution in system (4.3) with $\phi_{\mathcal{CW}} \in D_Y$, $\mathcal{C}_m > 0$, $\hat{\phi}(m) > 0$ and $\mathcal{W}_m > 0$, satisfies*

$$\liminf_{T\to\infty} (\mathcal{C}(T), \mathcal{A}(T), \mathcal{W}(T)) \geq (\eta_2, \eta_2, \eta_2).$$

Proof. Let

$$Y_0 = \{\phi_{\mathcal{CW}} \in D_Y \ : \ \mathcal{C}_m > 0, \ \hat{\phi}(m) > 0 \text{ and } \mathcal{W}_m > 0\},$$

$$\begin{aligned} Y_1 &= \{\phi_{\mathcal{CW}} \in D_Y \ : \ \mathcal{C}_m = 0\}, \\ Y_2 &= \{\phi_{\mathcal{CW}} \in D_Y \ : \ \hat{\phi}(m) = 0\}, \\ Y_3 &= \{\phi_{\mathcal{CW}} \in D_Y \ : \ \mathcal{W}_m = 0\}, \end{aligned}$$

and

$$M_\partial = \{\phi_{\mathcal{CW}} \in D_Y \ : \ \hat{z}_T(\phi_{\mathcal{CW}}) \in \partial Y_0, T \geq m\}$$

where $\partial Y_0 = Y \setminus Y_0 = Y_1 \cup Y_2 \cup Y_3$.

For $i = 1, 2$, fix a small $\epsilon_i > 0$. Since $\lim_{\mathcal{A}\to 0} \frac{b(\mathcal{A})}{\mathcal{A}P(\mathcal{A})} = \frac{b'(0)}{P_0}$ and $\lim_{\mathcal{A}\to 0} \mathcal{A}f(\mathcal{A}) = 0$, in a

neighborhood of $\mathcal{A}=0$, we have

$$\left|\frac{b\left(\mathcal{A}\right)}{\mathcal{A}P(\mathcal{A})}-\frac{b'(0)}{P_0}\right|<\epsilon_1 \text{ and } |\mathcal{A}f(\mathcal{A})|<\epsilon_2, \tag{4.14}$$

respectively. Set $\epsilon_0=\max\{\epsilon_1,\epsilon_2\}$. Using the similar arguments as those in Claim 1 in Chapter 3, we have

Claim 2. There exists a $\delta_0(\epsilon_0)>0$, such that for any $\phi_{\mathcal{CW}}\in Y_0$,

$$\limsup_{T\to\infty}\|\hat{z}_T(\phi_{\mathcal{CW}})-E_0\|\geq\delta_0(\epsilon_0).$$

By contradiction, there exists $T_6>m$ such that $|\mathcal{A}(T)|<\delta_0(\epsilon_0)$ for $T>T_6+m$, then, (4.14) is satisfied.

From (4.3), we obtain

$$\begin{aligned}
\frac{d\mathcal{C}}{dT} &> \left(\frac{b'(0)}{P_0}-\epsilon_0\right)e^{\frac{-m\mu_N}{P_0}}A(T-m)-\frac{\mu_C+\beta H}{P_0}C(T),\\
\frac{d\mathcal{A}}{dT} &> \frac{\beta H}{P(\delta_0(\epsilon_0))}\mathcal{C}(T)-\frac{\mu_A}{P_0}\mathcal{A}(T)-\frac{\epsilon_0}{P_0}\mathcal{W},\\
\frac{d\mathcal{A}}{dT} &> -\frac{\mu_W}{P_0}\mathcal{W}.
\end{aligned}\tag{4.15}$$

The characteristic equation of the system obtained from (4.15) by replacing $>$ with $=$ is

$$(\lambda+\frac{\mu_W}{P_0})\Delta_1(\lambda)=0$$

where $\Delta_1(\lambda)$ is defined in (3.18). Therefore, the principle eigenvalue $\lambda_2(\epsilon_0)>0$ for sufficiently small $\epsilon_0>0$. Consequently, there exists a solution $\vec{V_2}(T)=e^{\lambda_2(\epsilon_0)T}\hat{\zeta}$ where $\hat{\zeta}$ is the positive eigenfunction associated with $\lambda_2(\epsilon_0)$, $\vec{V_2}$ and $\hat{\zeta}$ are vectors with three components. By the comparison theory, there exists a small $\ell_2>0$ such that $(\mathcal{C}(T),\mathcal{A}(T),\mathcal{W}(T))\geq\ell_2e^{\lambda_2(\epsilon_0)T}\hat{\zeta}$ for all $T\geq T_6+m$. Thus, $\lim_{T\to\infty}(\mathcal{C}(T),\mathcal{A}(T),\mathcal{W}(T))=\infty$, This contradicts the boundedness of solutions to system (4.3). This proves the claim.

Since $\lim_{\mathcal{A}\to\mathcal{A}_1^*}\mathcal{A}f(\mathcal{A})=\mathcal{A}_1^*f(\mathcal{A}_1^*)$, in a neighborhood of $\mathcal{A}=\mathcal{A}_1^*$, there exists a small $\epsilon_3>0$ such that

$$\mathcal{A}_1^*f(\mathcal{A}_1^*)-\epsilon_4<\mathcal{A}f(\mathcal{A})<\mathcal{A}_1^*f(\mathcal{A}_1^*)+\epsilon_4. \tag{4.16}$$

Claim 3. There exists a $\delta_1(\epsilon_3)=\delta_1>0$, such that for any $\phi_{\mathcal{CW}}\in Y_0$,

$$\limsup_{T\to\infty}\|\hat{z}_T(\phi_{\mathcal{CW}})-E_1\|\geq\delta_1(\epsilon_3).$$

By contradiction, suppose that $\limsup_{T\to\infty} \|\hat{z}_T(\bar{\psi}_{\mathcal{CW}}) - E_1\| < \delta_1(\epsilon_3)$ for some $\bar{\psi}_{\mathcal{CW}} \in Y_0$. Thus, there exists $T_7 > m$ such that $|\mathcal{A}(T) - \mathcal{A}_1^*| < \delta_1(\epsilon_3)$ for $T > T_7 + m$. Hence, (4.16) is satisfied.

From (4.3), we have

$$\begin{aligned}
\frac{d\mathcal{C}}{dT} &> \frac{-(\mu_C + \beta H)}{P(\mathcal{A}_1^* - \delta_1(\epsilon_3))}\mathcal{C}(T), \\
\frac{d\mathcal{A}}{dT} &> \frac{\beta H}{P(\mathcal{A}_1^* + \delta_1(\epsilon_3))}\mathcal{C}(T) - \frac{\mu_A}{P(\mathcal{A}_1^* - \delta_1(\epsilon_3))}\mathcal{A}(T) - \frac{\mathcal{A}_1^* f(\mathcal{A}_1^*) + \epsilon_3}{P(\mathcal{A}_1^* - \delta_1(\epsilon_3))}\mathcal{W}(T), \quad (4.17) \\
\frac{d\mathcal{W}}{dT} &> \left(\gamma\frac{\mathcal{A}_1^* f(\mathcal{A}_1^*) - \epsilon_3}{P(\mathcal{A}_1^* + \delta_1(\epsilon_3))} - \frac{\mu_W}{P(\mathcal{A}_1^* - \delta_1(\epsilon_3))}\right)\mathcal{W}(T).
\end{aligned}$$

The characteristic equation of the system obtained from (4.17) by replacing $>$ with $=$ is

$$\Delta_2(\lambda) = \left(\lambda + \frac{\mu_C + \beta H}{P(\mathcal{A}_1^* - \delta_1(\epsilon_3))}\right)\left(\lambda + \frac{\mu_A}{P(\mathcal{A}_1^* - \delta_1(\epsilon_3))}\right)\left(\lambda - \gamma\frac{\mathcal{A}_1^* f(\mathcal{A}_1^*) - \epsilon_3}{P(\mathcal{A}_1^* + \delta_1(\epsilon_3))} + \frac{\mu_W}{P(\mathcal{A}_1^* - \delta_1(\epsilon_3))}\right) = 0.$$

When $\epsilon_3 = 0$, then $\lambda = \frac{\mathcal{R}_f - 1}{\mu_W P(\mathcal{A}_1^*)}$ is a positive root in $\Delta_2(\lambda) = 0$ when $\mathcal{R}_f > 1$. Hence, the principle eigenvalue is positive. By similar arguments as those in proof of Claim 5 we see that the $(\mathcal{C}(T), \mathcal{A}(T), \mathcal{W}(T)) \to \infty$ as $T \to \infty$, a contradiction.

Let $\hat{\omega}(\phi_{\mathcal{CW}})$ be the omega limit set of the orbit $\hat{z}_T(\phi_{\mathcal{CW}})$ through $v \in D_Y$.

Claim 4. $\bigcup\{\hat{\omega}(\phi_{\mathcal{CW}}) : \phi_{\mathcal{CW}} \in M_\partial\} = E_0 \cup E_1$.

For any $\phi_{\mathcal{CW}} \in M_\partial$, we have $\hat{z}_T(\phi_{\mathcal{CW}}) \in Y_1$ or $\hat{z}_T(\phi_{\mathcal{CW}}) \in Y_2$ or $\hat{z}_T(\phi_{\mathcal{CW}}) \in Y_3$. If $\hat{z}_T(\phi_{\mathcal{CW}}) \in Y_1$, i.e., $\mathcal{C}(T) \equiv 0$, then from (4.3b) we get

$$\frac{d\mathcal{A}}{dT} \leq -\mathcal{A}(T)\left(\frac{\mu_A + \mathcal{W}(T)f(\mathcal{A}(T))}{P_\infty}\right)$$

By the comparison theory, we get $\mathcal{A}(T) \to 0$ as $T \to \infty$. It then follows from (4.3c) that

$$\frac{d\mathcal{W}}{dT} \leq \frac{-\mu_W}{P_\infty}\mathcal{W}(T).$$

Hence, $\mathcal{W}(T) \to 0$ as $T \to \infty$.

Parallelly, when $\hat{z}_T(\phi_{\mathcal{CW}}) \in Y_2$, i.e., $\mathcal{A}(T) \equiv 0$, it follows from (4.3a) and (4.3c) that

$$\frac{d\mathcal{C}}{dT} = -\frac{\mu_C + \beta H}{P_0}\mathcal{C}(T) \qquad \text{and} \qquad \frac{d\mathcal{W}}{dT} = \frac{-\mu_W}{P_0}\mathcal{W}(T),$$

respectively. Thus, $\mathcal{C}(T) \to 0$ and $\mathcal{W}(T) \to 0$ as $T \to \infty$.

If $\hat{z}_T(\phi_{\mathcal{CW}}) \in Y_3$, i.e., $\mathcal{W}(T) \equiv 0$, then (4.3) becomes (3.8). Since $\mathcal{R}_s > 1$, (Q_1)-(Q_3) and

(K_2)-(K_3) hold, it follows from Theorem 3.4.3 that $(\mathcal{C}(T), \mathcal{A}(T)) \to (\mathcal{C}_1^*, \mathcal{A}_1^*)$ as $T \to \infty$. Hence, $\bigcup\{\hat{\omega}(\phi_{c\mathcal{W}}) : \phi_{c\mathcal{W}} \in M_\partial\} = E_0 \cup E_1$. Thus the claim holds.

Define a continuous function $\rho : D_Y \to \mathbb{R}_+$ by

$$\rho(\phi_{c\mathcal{W}}) = \min\{\mathcal{C}_m, \hat{\phi}(m), \mathcal{W}_m\},\ \forall \phi_{c\mathcal{W}} \in D_Y.$$

It is clear that $\rho^{-1}(0,\infty) \subset Y_0$ and if $\rho(\phi_{c\mathcal{W}}) > 0$ then $\rho(\hat{z}_T(\phi_{c\mathcal{W}})) > 0$ for all $T > m$. By Claim 4, we get that for any forward orbit of $\hat{z}_T$ in M_∂ converges to E_0 or E_1, by Claim 2 and 3, we conclude E_0 or E_1 are two isolated invariant in Y, and $(W^s(E_0) \cup W^s(E_1)) \cap Y_0 = \emptyset$ and no subset of $\{E_0, E_1\}$ form a cycle in ∂Y_0. By [171], it then follows that there exists $\eta_2 > 0$ such that $\liminf_{T\to\infty}(\mathcal{C}(T), \mathcal{A}(T), \mathcal{W}(T)) \geq (\eta_2, \eta_2, \eta_2)$ for all $(\mathcal{C}_m, \hat{\phi}, \mathcal{W}_m) \in Y_0$, which implies the uniform persistence. This completes the proof.

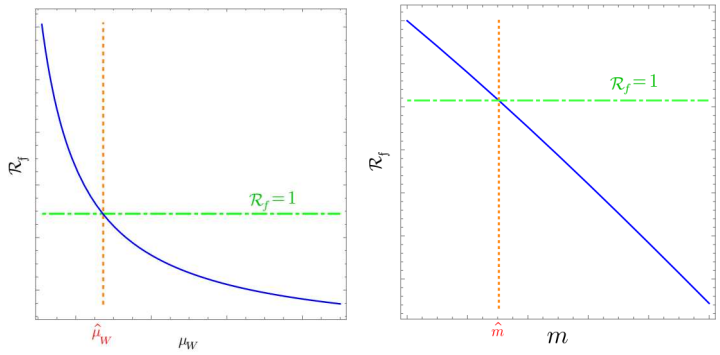
□

In the following, we address the relation of the net reproductive number $\mathcal{R}_f$ with respect to the related parameters (μ_W, γ and m). As we know from (4.9), $\mathcal{R}_f$ is determined by different parameters, such as μ_W, γ and implicitly by m through the dependency of $\mathcal{A}_1^*$. Obviously, $\mathcal{R}_f$ is linear with respect to γ. For the mortality rate μ_W, it is easy to check that,

$$\frac{\partial \mathcal{R}_f}{\partial \mu_W} < 0, \qquad \lim_{\mu_W \to 0^+} \mathcal{R}_f = \infty, \qquad \lim_{\mu_W \to \infty} \mathcal{R}_f = 0.$$

Therefore, there exists $\hat{\mu}_W$ such that $\mathcal{R}_f > 1$ for $\mu_W < \hat{\mu}_W$ and $\mathcal{R}_f < 1$ for $\mu_W > \hat{\mu}_W$ (Figure 4.2a) and large μ_W leads to the cleaner fish extinction. A similar influence by m is shown in Figure 4.2b. Theoretically, $\hat{\mu}_W$ and $\hat{m}$ depends on $\mathcal{A}_1^*$, and hence, they are related to the natural parameters of sea lice. From the biological perspective, the use of cleaner fish with long enough lifespan (i.e. small μ_W) can help to maintain $\mathcal{R}_f$ at a relatively high level. When the development process is too long, food resource (adult sea lice) for cleaner fish decreases which can lead to cleaner fish extinction.

Further, we can investigate the relation between $\mathcal{R}_s$ and $\mathcal{R}_f$ when the development age m varies. Obviously, $\mathcal{R}_s$ depends explicitly on m from (4.7) and $\mathcal{R}_f$ relates to m implicitly via $\mathcal{A}_1^*$ from (4.9). Figure 4.3 illustrates the population dynamics in system 4.3, with respect to $\mathcal{R}_s$, $\mathcal{R}_f$ and m, that is characterized in Theorem 4.3.2, 4.3.3 and 4.3.4. We can view in Figure 4.3 that, as m increases both $\mathcal{R}_f$ and $\mathcal{R}_s$ decreases. More specifically: i) for small m, the adult/net reproduction number $\mathcal{R}_s/\mathcal{R}_f$ is above 1, then both sea lice and cleaner fish can survive (the green segment); ii) with increasing m, $\mathcal{R}_f$ becomes below 1 while $\mathcal{R}_s$ remains above 1, hence, sea lice survive and cleaner fish extinct (the blue segment); iii) for large enough m, both $\mathcal{R}_s$ and $\mathcal{R}_f$ are below 1, then, both species extinct (the red segment). From the biological point of view, when $\mathcal{R}_s$ is larger (> 1), an adult sea lice produces more



(a) $\mathcal{R}_f$ v.s μ_W (b) $\mathcal{R}_f$ v.s m

Figure 4.2 The relationship between $\mathcal{R}_f$ and μ_W, or m.

secondary adults on average, that is, more food resource for the cleaner fish, and hence, $\mathcal{R}_f$ becomes larger.

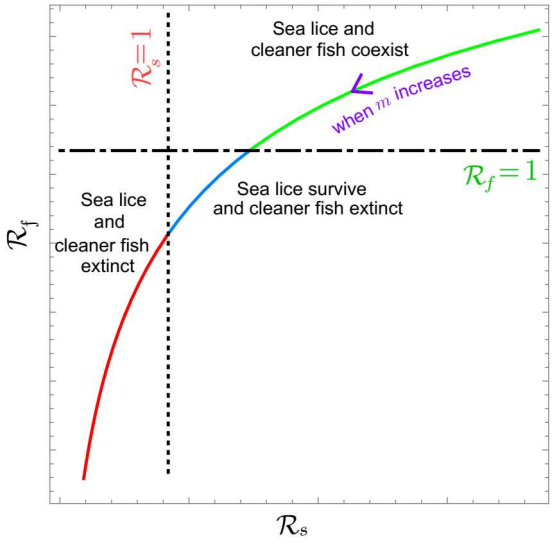


Figure 4.3 R_s v.s R_f.

4.4 Existence and stability of the positive steady state

In this section, we examine the existence and local stability of positive equilibrium point with respect to the age m, investigate the occurrence of Hopf bifurcations and identify the direction and stability of the Hopf bifurcation.

Firstly, the conditions to ensure the coexistence of both species, sea lice and cleaner fish, are provided in the following:

Proposition 4.4.1. *Assume $\mathcal{R}_s > 1$, $\mathcal{R}_f > 1$, (Q_1)-(Q_3) and (K_2) hold. Then there exists a unique positive equilibrium point $E_2 = (\mathcal{C}_2^*, \mathcal{A}_2^*, \mathcal{W}_2^*)$ in system (3.8). Furthermore, $\mathcal{C}_2^* \leq \mathcal{C}_1^*$ and $\mathcal{A}_2^* < \mathcal{A}_1^*$ where $\mathcal{C}_1^*$ and $\mathcal{A}_1^*$ are related to the steady state $E_1 = (\mathcal{C}_1^*, \mathcal{A}_1^*, 0)$.*

Proof. Assume $E_2 = (\mathcal{C}_2^*, \mathcal{A}_2^*, \mathcal{W}_2^*)$ is a positive steady state. Then it follows from (3.8) that E_2 satisfies

$$b\left(\mathcal{A}_2^*\right) \exp\left\{\frac{-m\mu_N}{P(\mathcal{A}_2^*)}\right\} = \left(\mu_C + \beta H\right) \mathcal{C}_2^*, \tag{4.18a}$$

$$\beta H \mathcal{C}_2^* = \mu_A \mathcal{A}_2^* + \mathcal{W}_2^* \mathcal{A}_2^* f(\mathcal{A}_2^*), \tag{4.18b}$$

$$\gamma \mathcal{A}_2^* f(\mathcal{A}_2^*) = \mu_W. \tag{4.18c}$$

For notational simplicity, we define $F(s) = sf(s)$. Then, it follows from (4.18c), (Q_3), $\mathcal{R}_f > 1$, and $F(0) = 0$ that there exist a unique positive value $\mathcal{A}_2^*$ such that

$$0 < \mathcal{A}_2^* = F^{-1}\left(\frac{\mu_W}{\gamma}\right) < \mathcal{A}_1^*.$$

Clearly, $\mathcal{C}_2^*$ is uniquely determined by (4.18a). Thus,

$$\mathcal{W}_2^* = \frac{\gamma\beta H}{\mu_W\left(\mu_C + \beta H\right)}\left(g\left(\mathcal{A}_2^*\right) - h\left(\mathcal{A}_2^*\right)\right)$$

where g and h are defined in the proof of Proposition 3.4.1. Since $g(s) > h(s)$ for $s \in (0, \mathcal{A}_1^*)$, we get $\mathcal{W}_2^* > 0$.

Finally, when (Q_1)-(Q_2) hold, it follows from (4.18a) and (3.20) that

$$\mathcal{C}_2^* = \frac{g(\mathcal{A}_2^*)}{\mu_C + \beta H} \leq \frac{g(\mathcal{A}_1^*)}{\mu_C + \beta H} = \mathcal{C}_1^*$$

□

Proposition 4.4.1 implies that when $\mathcal{R}_f > 1$, $\mathcal{C}_2^* \leq \mathcal{C}_1^*$ and $\mathcal{A}_2^* < \mathcal{A}_1^*$, implying the theoretical prediction matches the ecological factor that the cleaner fish is considered as a control agent for the sea lice. In other words, the introduction of cleaner fish can reduce the population size of sea lice. Thus, the researchers in marine biology may use the model to study the efficiency of cleaner fish as delousing agents or to compare the efficiency of removing sea lice among different cleaner fish species.

To analyze the stability of the positive equilibrium point $E_2 = (\mathcal{C}_2^*, \mathcal{A}_2^*, \mathcal{W}_2^*)$, we linearize (4.3) at E_2, by setting $y_1 = \mathcal{C} - \mathcal{C}_2^*$, $y_2 = \mathcal{A} - \mathcal{A}_2^*$ and $y_3 = \mathcal{W} - \mathcal{W}_2^*$, then the linearization

is:

$$\begin{aligned}
\frac{dy_1}{dT} &= a_{11}y_1(T) + a_{12}y_2(T) + b_{12}y_2(T-m) + c_{12}\int_{-m}^{0} y_2(T+\theta)d\theta, \\
\frac{dy_2}{dT} &= a_{21}y_1(T) + a_{22}y_2(T) + a_{23}y_3(T), \\
\frac{dy_3}{dT} &= a_{32}y_2(T) + a_{33}y_3(T),
\end{aligned} \tag{4.19}$$

where

$$\begin{aligned}
a_{11} &= \frac{-(\mu_C+\beta H)}{P(\mathcal{A}_2^*)} < 0, \quad a_{12} = \frac{(\mu_C+\beta H)\mathcal{C}_2^* P'(\mathcal{A}_2^*)}{P^2(\mathcal{A}_2^*)} > 0, \quad b_{12} = \frac{d}{ds}\left[\frac{b(s)}{P(s)}\right]\Big|_{s=\mathcal{A}_2^*} e^{\frac{-\mu_N m}{P(\mathcal{A}_2^*)}} \\
c_{12} &= \frac{\mu_N b(\mathcal{A}_2^*)P'(\mathcal{A}_2^*)}{P^3(\mathcal{A}_2^*)} e^{\frac{-\mu_N m}{P(\mathcal{A}_2^*)}} > 0, \quad a_{21} = \frac{\beta H}{P(\mathcal{A}_2^*)} > 0, \quad a_{23} = -\frac{\mathcal{A}_2^* f(\mathcal{A}_2^*)}{P(\mathcal{A}_2^*)} < 0, \\
a_{22} &= -\frac{1}{P(\mathcal{A}^*)}\left[\mu_A + \mathcal{W}_2^* \frac{d\left(sf(s)\right)}{ds}\Big|_{s=\mathcal{A}_2^*}\right] < 0, \; a_{32} = \frac{\gamma \mathcal{W}_2^*}{P(\mathcal{A}_2^*)} \frac{d\left(sf(s)\right)}{ds}\Big|_{s=\mathcal{A}_2^*} > 0, \; a_{33} = 0.
\end{aligned}$$

The characteristic equation corresponding to (4.19) is

$$Q(\lambda, m) = \det\left(\lambda I - M_1 - M_2 e^{-m\lambda} - M_3 \int_{-m}^{0} e^{\lambda\theta} d\theta\right) = 0 \tag{4.20}$$

where

$$M_1 = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & 0 \end{pmatrix}, \; M_2 = \begin{pmatrix} 0 & b_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } M_3 = \begin{pmatrix} 0 & c_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check that

$$Q(0, m) = a_{11}a_{23}a_{32} > 0.$$

Thus, $\lambda = 0$ is not a root in (4.20), so (4.20) can be rewritten as

$$Q(\lambda, m) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 + (q_0\lambda + q_1)\, e^{-\lambda m} = 0, \tag{4.21}$$

where

$$\begin{aligned}
&p_1 = -(a_{11} + a_{22}) > 0, \quad p_2 = a_{11}a_{22} - a_{12}a_{21} - a_{23}a_{32}, \\
&p_3 = a_{11}a_{23}a_{32} - a_{12}c_{21}, \quad q_0 = -a_{21}b_{12}, \quad q_1 = a_{21}c_{12} > 0.
\end{aligned} \tag{4.22}$$

Without delay ($m = 0$), in $Q(\lambda, 0)$, the Routh-Hurwitz stability criterion (see e.g., [14]) leads to the following result.

Proposition 4.4.2. *At $m = 0$, the positive equilibrium point $E_2 = (\mathcal{C}_2^*, \mathcal{A}_2^*, \mathcal{W}_2^*)$ is locally asymptotically stable if and only if*

$$p_2 + q_0 > 0, \text{ and } p_1(p_2 + q_0) > p_3 + q_1.$$

When $m > 0$, one notices that the coefficients in $Q(\lambda, m)$ depend on m since m is involved in $\mathcal{C}_2^*$ and $\mathcal{W}_2^*$. We use the methods similar to those in [24, 212] to discuss the existence of the critical values of m where possible oscillations or stability change may occur. Suppose $\lambda = \mathrm{i}\nu$ ($\nu > 0$) is a purely imaginary root of (4.21). Substituting it into (4.21) and separating the real and imaginary parts, we obtain:

$$\begin{aligned} p_3 - p_1\nu^2 &= -q_0\nu \sin m\nu - q_1 \cos m\nu, \\ p_2\nu - \nu^3 &= -q_0\nu \cos m\nu + q_1 \sin m\nu. \end{aligned} \tag{4.23}$$

Squaring and adding both equations of (4.23) lead to

$$\left(p_3 - p_1\nu^2\right)^2 + \left(p_2\nu - \nu^3\right)^2 = \left(q_0\nu\right)^2 + q_1^2. \tag{4.24}$$

Therefore, the solutions of (4.24) are the roots in

$$\mathcal{H}(z) := z^3 + l_1 z^2 + l_2 z + l_3 = 0, \tag{4.25}$$

with $z = \nu^2$ and

$$l_1 = p_1^2 - 2p_2, \quad l_2 = p_2^2 - 2p_1p_3 - q_0^2 \text{ and } l_3 = p_3^2 - q_1^2.$$

To explore the existence of positive real roots in (4.25), we use Descartes' Rule of Signs (see e.g., [173]) and the sign of the discriminant (see e.g., [93])

$$\mathcal{D} := l_1^2 l_2^2 + 18 l_1 l_2 l_3 - 27 l_3^2 - 4 l_1^3 l_3 - 4 l_2^3. \tag{4.26}$$

Regarding the sign of l_i ($i = 1, 2, 3$) and $\mathcal{D}$, there are some different cases which is summarized in Table 4.2.

That no positive real root exists in $\mathcal{H}(z) = 0$ for $m \in (0, m_{\max})$, implies no any root with the form $\lambda = i\nu$ exists in (4.21). Thus, changing the value of the delay cannot move eigenvalues across the imaginary axis, i.e. no stability change may occur.

Case	Sign l_1	Sign l_2	Sign l_3	Number of positive roots: $\mathcal{D} > 0$	$\mathcal{D} = 0$	$\mathcal{D} < 0$
1	$+$	$+$	$+$	0	0	0
2	$-/+/+$	$-/-/+$	$-$	1	1	1
3	$-/+/-$	$+/-/-$	$+$	2 distinct	1 repeated	0
4	$-$	$+$	$-$	3 distinct	2 distinct (one repeated root)	1

Table 4.2 Number of positive real roots in (4.25).

Proposition 4.4.3. *If $E_2 = (\mathcal{C}_2^*, \mathcal{A}_2^*, \mathcal{W}_2^*)$ is stable (unstable) for $m = 0$, then it remains stable (unstable) for any feasible time delay $m > 0$ when one of the following holds for all $m > 0$:*
(i) $l_i > 0$ $(i = 1, 2, 3)$; *(ii) $\mathcal{D} < 0$, and $l_3 > 0$;*

Assume that (4.25) has one simple root $\nu := \nu(m)$. Following [24], $i\nu(m^*) := i\nu^*$ is a root in (4.21) if and only if m^* is a zero of a function S_n for some $n \in \mathbb{N}$ where

$$S_n(m) = m - \frac{\theta(m) + 2n\pi}{\nu(m)}, \quad n \in \mathbb{N}, \tag{4.27}$$

with $\theta(m) \in [0, 2\pi)$ such that

$$\cos\theta(m) = -\frac{q_0\nu\left(p_3\nu - p_1\nu^3\right) + q_1\left(p_3 - p_1\nu^2\right)}{q_0^2\nu^2 + q_1^2},$$
$$\sin\theta(m) = \frac{q_1\left(p_3\nu - p_1\nu^3\right) - q_0\nu\left(p_3 - p_1\nu^2\right)}{q_0^2\nu^2 + q_1^2}.$$

To investigate the occurrences of stability change at the positive equilibrium point E_2 when $m = m^*$, taking derivation with respect to m in (4.21) and substituting $\lambda(m) = \mathrm{i}\nu^*$ leads to

$$\mathrm{Re}\left\{\frac{d\lambda(m)}{dm}\right\}^{-1}\Bigg|_{\substack{m=m^*\\ \lambda=i\nu^*}} = \frac{3p_1\nu^4 + \left(2p_1^2 - p_2p_1 - 3p_3\right)\nu^2 - 2p_1p_3 + p_2p_3 - q_0^2}{q_0^2\nu^2 + q_1^2}\Bigg|_{v=v^*} = \frac{\mathcal{K}(\nu^*)}{q_0^2\nu^{*2} + q_1^2},$$

where

$$\mathcal{K}(\nu) = 3p_1\nu^4 + \left(2p_1^2 - p_2p_1 - 3p_3\right)\nu^2 - 2p_1p_3 + p_2p_3 - q_0^2. \tag{4.29}$$

Since $q_0^2\nu^{*2} + q_1^2 > 0$, we have

$$\operatorname{sign}\{\mathcal{K}(v^*)\} = \operatorname{sign}\left\{ \operatorname{Re}\left\{ \frac{d(\lambda(m))}{dm} \right\}^{-1} \Big|_{\substack{m=m^* \\ \lambda=i\nu^*}} \right\} = \operatorname{sign}\left\{ \frac{d(\operatorname{Re}\{\lambda(m)\})}{dm} \Big|_{m=m^*} \right\}.$$

Therefore, we have the following claim.

Claim 5. Assume ν^* is a root in (4.25). Then

$$\operatorname{sign}\left\{ \frac{d(\operatorname{Re}\{\lambda(m)\})}{dm} \Big|_{m=m^*} \right\} = \operatorname{sign}\{\mathcal{K}(\nu^*)\} \tag{4.30}$$

where $\mathcal{K}$ is given in (4.29).

When (4.25) has simple roots, the eigenvalue $\lambda(m)$ crosses the imaginary axis from left to right (right to left) if $\operatorname{sign}\{\operatorname{Re}\{\lambda(m)\}\} < 0$ (> 0) in a neighborhood of $m = m^*$ and

$$\operatorname{sign}\left\{ \frac{d(\operatorname{Re}\{\lambda(m)\})}{dm} \Big|_{m=m^*} \right\} > 0 \ (< 0),$$

which is called "the transversality\crossing condition" because the eigenvalues cross the imaginary axis with non-zero speed. Therefore, a change in stability may occur at E_2^*.

Summarizing the above analysis, we have the following result.

Proposition 4.4.4. *Assume (4.25) has one simple positive real root ν^*. Then, the following holds:*

i. If E_2^* is stable (unstable) when $m = 0$ and $\mathcal{K}(\nu^*) > 0$ ($\mathcal{K}(\nu^*) < 0$), then it remains stable (unstable) for $m \in (0, m^*)$; and then becomes unstable (stable) for $m > m^*$,

ii. If E_2^* is stable (unstable) when $m = 0$ and $\mathcal{K}(\nu^*) < 0$ ($\mathcal{K}(\nu^*) > 0$), then it remains stable (unstable) for all $m > 0$;

Furthermore, the system (3.8) undergoes a Hopf bifurcation at E_2^* when $m = m^*$.

As we see in Table 4.2, it is possible for (4.25) to have: i) two or three root; ii) one positive real root with double multiplicity, say $\hat{z} = \hat{\nu}^2$, that is, $\mathcal{H}(\hat{z}) = \mathcal{H}'(\hat{z}) = 0$. For case (i), the occurrence of stability switches and a periodic-doubling may occur when m varies. For case (ii), $\lambda = \pm i\hat{\nu}$ is not a simple root in (4.21). In this case, the bifurcation is called a degenerate Hopf bifurcation [66]. These are out of the scope of this work and we leave them for future research.

Next, we investigate the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions by using the normal form and the center manifold theory developed by Hassard et al. [78].

Let

$$\mathcal{I}(t) = \int_{t-m}^{t} \frac{1}{P\left(\mathcal{A}(\eta)\right)} d\eta. \tag{4.31}$$

Then

$$\frac{d\mathcal{I}}{dt} = \frac{1}{P\left(\mathcal{A}(t)\right)} - \frac{1}{P\left(\mathcal{A}(t-m)\right)},$$

with $\mathcal{I}(0) = \frac{m}{P(\mathcal{A}(0))}$. Hence, system (3.8) is equivalent to the system

$$\begin{aligned}
\frac{d\mathcal{C}}{dt} &= \frac{b\left(\mathcal{A}(t-m)\right)}{P\left(\mathcal{A}(t-m)\right)} e^{-\mu_N \mathcal{I}(t)} - \left(\mu_C + \beta H\right) \frac{\mathcal{C}(t)}{P\left(\mathcal{A}(t)\right)}, \\
\frac{d\mathcal{I}}{dt} &= \frac{1}{P\left(\mathcal{A}(t)\right)} - \frac{1}{P\left(\mathcal{A}(t-m)\right)}, \\
\frac{d\mathcal{A}}{dt} &= \frac{1}{P\left(\mathcal{A}(t)\right)} \left[\beta H \mathcal{C}(t) - \mu_A \mathcal{A}(t) - \mathcal{W}(t)\mathcal{A}(t) f(\mathcal{A}(t))\right], \\
\frac{d\mathcal{W}}{dt} &= \frac{1}{P\left(\mathcal{A}(t)\right)} \left[\gamma \mathcal{W}(t)\mathcal{A}(t) f(\mathcal{A}(t)) - \mu_W \mathcal{W}(t)\right].
\end{aligned} \tag{4.32}$$

It is clear that the positive equilibrium point in (4.32) is $E^* = (\mathcal{C}_2^*, \mathcal{I}^*, \mathcal{A}_2^*, \mathcal{W}_2^*)$ where $\mathcal{I}^* = \frac{m}{P(\mathcal{A}_2^*)}$.

Let $\mathbf{u} = (u_1, u_2, u_3, u_4)^T := (\mathcal{C}(t), \mathcal{I}(t), \mathcal{A}(t), \mathcal{W}(t)) - E^*$. Then the third order Taylor expansion of the system (4.32) is

$$\frac{d\mathbf{u}}{dt} = \mathbb{J}_1 \mathbf{u}(t) + \mathbb{J}_2 \mathbf{u}(t-m) + \mathbf{F}\left(\mathbf{u}(t), \mathbf{u}(t-m)\right), \tag{4.33}$$

where

$$\mathbb{J}_1 = \begin{pmatrix} a_{11} & -\mu_N b_*^0 & a_{12} & 0 \\ 0 & 0 & P_*^1 & 0 \\ a_{21} & 0 & a_{22} & a_{23} \\ 0 & 0 & a_{32} & 0 \end{pmatrix}, \qquad \mathbb{J}_2 = \begin{pmatrix} 0 & 0 & b_{12} & 0 \\ 0 & 0 & -P_*^1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and $\mathbf{F} = (F_1, F_2, F_3, F_4)^T$ where

$$\begin{aligned}
F_1(\mathbf{u}(t), \mathbf{u}(t-m)) &= d_{22}^1 u_2^2(t) + d_{13}^1 u_1(t) u_3(t) + d_{33}^1 u_3^2(t) + c_{33}^1 u_3^2(t-m) \\
&\quad + c_{23}^1 u_2(t) u_3(t-m) + d_{222}^1 u_2^3(t) + d_{133}^1 u_1(t) u_3^2(t) + d_{333}^1 u_3^3(t) \\
&\quad + c_{333}^1 u_3^3(t-m) + c_{233}^1 u_2(t) u_3^2(t-m) + c_{223}^1 u_2^2(t) u_3(t-m), \\
F_2(\mathbf{u}(t), \mathbf{u}(t-m)) &= d_{33}^2 u_3^2(t) + c_{33}^2 u_3^2(t-m) + d_{333}^2 u_3^3(t) + c_{333}^2 u_3^3(t-m), \\
F_3(\mathbf{u}(t), \mathbf{u}(t-m)) &= d_{13}^3 u_1(t) u_3(t) + d_{33}^3 u_3^2(t) + d_{34}^3 u_3(t) u_4(t) + d_{133}^3 u_1(t) u_3^2(t) \\
&\quad + d_{333}^3 u_3^3(t) + d_{334}^3 u_3^2(t) u_4(t),
\end{aligned}$$

$$F_4(\mathbf{u}(t),\mathbf{u}(t-m)) \;=\; d^4_{33}u_3^2(t)+d^4_{34}u_3(t)u_4(t)+d^4_{333}u_3^3(t)+d^4_{334}u_3^2(t)u_4(t),$$

with

$$\begin{array}{l}
d^1_{22}=\frac{\mu_N^2}{2}b^0_*,\quad d^1_{13}=-(\mu_C+\beta H)P^1_*,\quad d^1_{33}=-\frac{(\mu_C+\beta H)\mathcal{C}^*_2}{2}P^2_*,\quad c^1_{33}=\frac{1}{2}b^2_*,\quad c^1_{23}=\frac{-\mu_N}{2}b^1_*,\\
d^1_{222}=-\frac{\mu_N^3}{6}b^0_*,\quad d^1_{133}=-\frac{\mu_C+\beta H}{6}P^2_*,\quad d^1_{333}=-\frac{(\mu_C+\beta H)\mathcal{C}^*_2}{2}P^3_*,\quad c^1_{333}=\frac{1}{6}b^3_*,\\
c^1_{223}=\frac{\mu_N^2}{6}b^1_*,\quad d^2_{33}=\frac{1}{2}P^2_*,\quad c^2_{33}=-d^2_{33},\quad d^2_{333}=\frac{1}{6}P^3_*,\quad c^2_{333}=-d^2_{333}, d^3_{13}=\beta H P^1_*,\\
d^3_{33}=\frac{-1}{2}P^0_*f^2_*-P^1_*\left(\mu_W+\mathcal{W}^*_2f^1_*\right),\quad d^3_{34}=-g^1_*,\quad d^3_{133}=\beta H P^2_*,\quad d^3_{334}=-g^2_*,\\
d^3_{333}=\frac{-1}{6}P^0_*f^3_*-\frac{1}{2}P^1_*f^2_*-\frac{1}{2}P^2_*\left(\mu_W+\mathcal{W}^*_2f^1_*\right),\quad d^4_{33}=\frac{\gamma\mathcal{W}^*_2}{2}\left(P^0_*f^2_*+2P^1_*f^1_*\right),\quad d^4_{34}=\gamma g^1_*,\\
d^4_{333}=\frac{\gamma\mathcal{W}^*_2}{6}\left(P^0_*f^3_*+3P^1_*f^2_*+3P^2_*f^1_*\right),\quad d^4_{334}=\gamma g^2_*,\quad c^1_{233}=\frac{-\mu_N}{6}b^2_*,
\end{array}$$

where, for $\ell=0,1,2,3$,

$$\begin{array}{rcl}
P^\ell_* &=& \dfrac{d^\ell}{ds^\ell}\left(\dfrac{1}{P(s)}\right)\Big|_{s=\mathcal{A}^*_2}, \qquad b^\ell_*=\dfrac{d^\ell}{ds^\ell}\left(\dfrac{b(s)}{P(s)}\right)\Big|_{s=\mathcal{A}^*_2}e^{-\mu_N\mathcal{I}^*},\\
f^\ell_* &=& \dfrac{d^\ell}{ds^\ell}\left(sf(s)\right)|_{s=\mathcal{A}^*_2}, \qquad g^\ell_*=\dfrac{d^\ell}{ds^\ell}\left(\dfrac{sf(s)}{P(s)}\right)\Big|_{s=\mathcal{A}^*_2}.
\end{array}$$

Since $m>0$, we rescale the time by $t\to\frac{t}{m}$ to normalize the delay in (4.33) and introduce a perturbation parameter $m=m^*+\mu$ with $\mu\in\mathbb{R}$, then (4.33) can be rewritten as a functional differential equation in the phase space $C:=C\left([-1,0],\mathbb{R}^4\right)$, at the Hopf bifurcation point $m=m^*$,

$$\begin{array}{rcl}
\dfrac{d\mathbf{u}}{dt} &=& (\mu+m^*)\mathbb{J}_1\mathbf{u}(t)+(\mu+m^*)\mathbb{J}_2\mathbf{u}(t-1)+(\mu+m^*)\mathbf{F}\left(\mathbf{u}(t),\mathbf{u}(t-1)\right),\\
&=& L_\mu(\mathbf{u}_t)+\mathcal{F}(\mathbf{u}_t,\mu),
\end{array}\tag{4.34}$$

where $L_\mu:C\to\mathbb{R}^4$ is the linear operator

$$L_\mu(\phi)=(m^*+\mu)[\mathbb{J}_1\phi(0)+\mathbb{J}_2\phi(-1)],$$

and the non-linear operator $\mathcal{F}:C\times\mathbb{R}\to\mathbb{R}^4$ is

$$\mathcal{F}(\phi,\mu)=(m^*+\mu)\mathbf{F}\left(\phi(0),\phi(-1)\right).$$

By the Riesz representation theorem, the linear operator L_μ can be expressed in an integral form, in the sense that, there exists an $n\times n$ matrix-valued function

$$\eta(\cdot,\mu):[-1,0]\to\mathbb{R}^4$$

whose components are bounded variation such that

$$L_\mu(\phi) = \int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta) \quad \text{for any} \quad \phi \in C.$$

In fact, we can choose

$$d\eta(\theta,\mu) = (\mu + m^*)[\mathbb{J}_1\delta(\theta) + \mathbb{J}_1\delta(\theta+1)],$$

where δ is the Dirac delta function $\delta(\theta) = \begin{cases} 0, & \theta \neq 0, \\ 1, & \theta = 0. \end{cases}$.

For $\phi \in C^1\left([-1,0],\mathbb{R}^4\right)$, define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 < \theta \leq 0, \\ \int_{-1}^{0} d\eta(r,\mu)\phi(r), & \theta = 0, \end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 0, & -1 < \theta \leq 0, \\ \mathcal{F}(\phi,\mu), & \theta = 0. \end{cases}$$

Thus, (4.34) is equivalent to the operator-differential equation

$$\dot{\mathbf{u}}_t = A(\mu)\mathbf{u}_t + R(\mu)\mathbf{u}_t. \tag{4.35}$$

Following the idea used in Hassard et al. [78] we compute the center manifold C_0 at the Hopf bifurcation value. Let $\mu = 0$ and suppose that the eigenvector q of $A(0)$ corresponding to the eigenvalue iv^*m^* is of the form $q(\theta) = q(0)e^{iv^*m^*\theta}$ and $q(0) := (q_1, q_2, q_3, q_4)^T$ satisfies

$$\left(iv^*I_4 - \mathbb{J}_1 - \mathbb{J}_2 e^{-iv^*m^*}\right) q(0) = (0,0,0,0)^T.$$

The direct computation leads to

$$q_1 = \frac{-v^*a_{22} + i\left(v^{*2} + a_{23}a_{32}\right)}{v^*a_{21}}, \qquad q_2 = \frac{iP_*^1\left(1 - e^{-iv^*m^*}\right)}{v^*}, \qquad q_3 = 1, \qquad q_4 = \frac{-a_{32}i}{v^*}.$$

The eigenvector of the adjoint operator A^* corresponding to eigenvalue $-iv^*m^*$ is of the form $q^*(s) = D\psi e^{iv^*m^*s}$, where $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$ with (see e.g., [175])

$$\psi_1 = \overline{q_4}, \qquad \psi_2 = \overline{q_3}, \qquad \psi_3 = \overline{q_2}, \qquad \psi_4 = \overline{q_1}$$

and

$$D = \left\{ 2\bar{q}_1\bar{q}_4 + 2\bar{q}_2\bar{q}_3 + \left(b_{12}\bar{q}_3\bar{q}_4 - P^1_*\bar{q}^2_3 \right) e^{iv^*m^*} \right\}^{-1}.$$

Next we use the right and left eigenvectors $q(\theta)$ and $q^*(s)$ of operator $A(0)$ to compute coordinates (z, w) describing the center manifold C_0 at $\mu = 0$.

Suppose $\mathbf{u}_t$ is a solution of (4.35) at $\mu = 0$, then (4.35) can be rewritten with a new variables:

$$\begin{aligned} z(t) &= \langle q^*, \mathbf{u}_t \rangle , \\ w(t,\theta) &= \mathbf{u}_t(\theta) - z(\theta)q(\theta) - \bar{z}(\theta)\bar{q}(\theta) = \mathbf{u}_t(\theta) - 2\ \mathrm{Re}\left\{ z(\theta)q(\theta) \right\}. \end{aligned} \tag{4.36}$$

On the center manifold C_0, we have

$$w(t,\theta) = W(z,\bar{z},\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots ,$$

where z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Note that W is real if $\mathbf{u}_t$ is real. We consider only real solutions.

For any real solution $\mathbf{u}_t \in \mathsf{C}_0$ of (4.35), it follows from (4.36), by rigorous and tedious computations, that

$$\begin{aligned} \dot{z} = \langle q^*, \mathbf{u}_t \rangle &= iv^*m^*z + \langle q^*(\theta), \mathcal{F}\left(W(z,\bar{z},\theta) + 2\ Re\left\{ z(\theta)q(\theta) \right\}, 0 \right) \rangle \\ &= iv^*m^*z + \bar{q}^*(0)\mathcal{F}\left(W(z,\bar{z},0) + 2\ Re\left\{ z(0)q(0) \right\}, 0 \right) \\ &\overset{\text{def}}{=} iv^*m^*z + \bar{q}^*(0)\mathcal{F}_0\left(z,\bar{z}\right) \\ &= iv^*m^*z + g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots \end{aligned}$$

with

$$\begin{aligned} g_{20} &= 2m^*D\left(q_4\mathcal{F}^1_{z^2} + q_3\mathcal{F}^2_{z^2} + q_2\mathcal{F}^3_{z^2} + q_1\mathcal{F}^4_{z^2} \right), \\ g_{11} &= m^*D\left(q_4\mathcal{F}^1_{z\bar{z}} + q_3\mathcal{F}^2_{z\bar{z}} + q_2\mathcal{F}^3_{z\bar{z}} + q_1\mathcal{F}^4_{z\bar{z}} \right) \\ g_{02} &= 2m^*D\left(q_4\mathcal{F}^1_{\bar{z}^2} + q_3\mathcal{F}^2_{\bar{z}^2} + q_2\mathcal{F}^3_{\bar{z}^2} + q_1\mathcal{F}^4_{\bar{z}^2} \right), \\ g_{21} &= 2m^*D\left(q_4\mathcal{F}^1_{z^2\bar{z}} + q_3\mathcal{F}^2_{z^2\bar{z}} + q_2\mathcal{F}^3_{z^2\bar{z}} + q_1\mathcal{F}^4_{z^2\bar{z}} \right), \end{aligned} \tag{4.37}$$

where

$$\mathcal{F}^1_{z^2} = d^1_{22}q^2_2 + d^1_{13}q_1q_3 + d^1_{33}q^2_3 + c^1_{33}q^2_3e^{-i2v^*m^*} + c^1_{23}q_2q_3e^{-iv^*m^*},\ \mathcal{F}^2_{z^2} = d^2_{33}q^2_3 + c^2_{33}q^2_3e^{-i2v^*m^*},$$
$$\mathcal{F}^3_{z^2} = d^3_{13}q_1q_3 + d^3_{33}q^2_3 + d^3_{34}q_3q_4, \quad \mathcal{F}^4_{z^2} = d^4_{33}q^2_3 + d^4_{34}q_3q_4,$$

$$
\begin{aligned}
\mathcal{F}^1_{z\bar{z}} &= 2d^1_{22}|q_2|^2 + d^1_{13}\left(q_1\bar{q}_3 + \bar{q}_1 q_3\right) + 2d^1_{33}|q_3|^2 + 2c^1_{33}|q_3|^2 + c^1_{23}\left(q_2\bar{q}_3 e^{iv^*m^*} + \bar{q}_2 q_3 e^{-iv^*m^*}\right),\\
\mathcal{F}^2_{z\bar{z}} &= 2d^2_{33}|q_3|^2 + 2c^2_{33}|q_3|^2 = 0, \quad \mathcal{F}^3_{z\bar{z}} = d^3_{13}\left(q_1\bar{q}_3 + \bar{q}_1 q_3\right) + 2d^3_{33}|q_3|^2 + d^3_{34}\left(q_3\bar{q}_4 + \bar{q}_3 q_4\right),\\
\mathcal{F}^4_{z\bar{z}} &= 2d^4_{33}|q_3|^2 + d^4_{34}\left(q_3\bar{q}_4 + \bar{q}_3 q_4\right), \quad \mathcal{F}^2_{\bar{z}^2} = d^2_{33}\bar{q}^2_3 + c^2_{33}\bar{q}^2_3 e^{i2v^*m^*},\\
\mathcal{F}^1_{\bar{z}^2} &= d^1_{22}\bar{q}^2_2 + d^1_{13}\bar{q}_1\bar{q}_3 + d^1_{33}\bar{q}^2_3 + c^1_{33}\bar{q}^2_3 e^{i2v^*m^*} + c^1_{23}\bar{q}_2\bar{q}_3 e^{iv^*m^*},\\
\mathcal{F}^3_{\bar{z}^2} &= d^3_{13}\bar{q}_1\bar{q}_3 + d^3_{33}\bar{q}^2_3 + d^3_{34}\bar{q}_3\bar{q}_4, \quad \mathcal{F}^4_{\bar{z}^2} = d^4_{33}\bar{q}^2_3 + d^4_{34}\bar{q}_3\bar{q}_4,\\
\mathcal{F}^1_{z^2\bar{z}} &= d^1_{22}\left[2q_2W^{(2)}_{11}(0) + \bar{q}_2W^{(2)}_{20}(0)\right] + d^1_{33}\left[2q_3W^{(3)}_{11}(0) + \bar{q}_3W^{(3)}_{20}(0)\right]\\
&\quad + d^1_{13}\left[\frac{1}{2}\bar{q}_3W^{(1)}_{20}(0) + \frac{1}{2}\bar{q}_1W^{(3)}_{20}(0) + q_3W^{(1)}_{11}(0) + q_1W^{(3)}_{11}(0)\right]\\
&\quad + c^1_{33}\left[2q_3e^{-iv^*m^*}W^{(3)}_{11}(-1) + \bar{q}_3e^{iv^*m^*}W^{(3)}_{20}(-1)\right]\\
&\quad + c^1_{23}\left[\frac{1}{2}\bar{q}_3e^{iv^*m^*}W^{(2)}_{20}(0) + \frac{1}{2}\bar{q}_2W^{(3)}_{20}(-1) + q_3e^{-iv^*m^*}W^{(2)}_{11}(0) + q_2W^{(3)}_{11}(-1)\right]\\
&\quad + 3d^1_{222}q^2_2\bar{q}_2 + d^1_{133}\left[q^2_3\bar{q}_1 + 2q_1|q_3|^2\right] + 3d^1_{333}q^2_3\bar{q}_3\\
&\quad + 3c^1_{333}q^2_3\bar{q}_3e^{-iv^*m^*} + c^1_{233}\left[q^2_3\bar{q}_2e^{-2iv^*m^*} + 2q_2|q_3|^2\right] + c^1_{223}\left[q^2_2\bar{q}_3e^{iv^*m^*} + 2q_3|q_2|^2e^{-iv^*m^*}\right],\\
\mathcal{F}^2_{z^2\bar{z}} &= d^2_{33}\left[2q_3W^{(3)}_{11}(0) + \bar{q}_3W^{(3)}_{20}(0)\right] + c^2_{33}\left[2q_3e^{-iv^*m^*}W^{(3)}_{11}(-1) + \bar{q}_3e^{iv^*m^*}W^{(3)}_{20}(-1)\right]\\
&\quad + 3d^2_{333}q^2_3\bar{q}_3 + 3c^2_{333}q^2_3\bar{q}_3e^{-iv^*m^*},\\
\mathcal{F}^3_{z^2\bar{z}} &= d^3_{13}\left[\frac{1}{2}\bar{q}_3W^{(1)}_{20}(0) + \frac{1}{2}\bar{q}_1W^{(3)}_{20}(0) + q_3W^{(1)}_{11}(0) + q_1W^{(3)}_{11}(0)\right]\\
&\quad + d^3_{33}\left[2q_3W^{(3)}_{11}(0) + \bar{q}_3W^{(3)}_{20}(0)\right] + 3d^3_{333}q^2_3\bar{q}_3 + d^1_{334}\left[q^2_3\bar{q}_4 + 2q_4|q_3|^2\right]\\
&\quad + d^3_{133}\left[q^2_3\bar{q}_1 + 2q_1|q_3|^2\right] + d^3_{34}\left[\frac{1}{2}\bar{q}_3W^{(4)}_{20}(0) + \frac{1}{2}\bar{q}_4W^{(3)}_{20}(0) + q_3W^{(4)}_{11}(0) + q_4W^{(3)}_{11}(0)\right]\\
\mathcal{F}^4_{z^2\bar{z}} &= d^4_{33}\left[2q_3W^{(3)}_{11}(0) + \bar{q}_3W^{(3)}_{20}(0)\right] + d^1_{334}\left[q^2_3\bar{q}_4 + 2q_4|q_3|^2\right]\\
&\quad + d^4_{34}\left[\frac{1}{2}\bar{q}_3W^{(4)}_{20}(0) + \frac{1}{2}\bar{q}_4W^{(3)}_{20}(0) + q_3W^{(4)}_{11}(0) + q_4W^{(3)}_{11}(0)\right] + 3d^4_{333}q^2_3\bar{q}_3.
\end{aligned}
$$

where

$$
\begin{aligned}
W_{11}(\theta) &= \frac{ig_{11}}{v^*m^*}q(0)e^{iv^*m^*\theta} + \frac{i\bar{g}_{11}}{v^*m^*}\bar{q}(0)e^{-iv^*m^*\theta} + K_2,\\
W_{20}(\theta) &= \frac{ig_{20}}{v^*m^*}q(0)e^{iv^*m^*\theta} + \frac{i\bar{g}_{20}}{3v^*m^*}\bar{q}(0)e^{-iv^*m^*\theta} + K_1e^{2iv^*m^*\theta},
\end{aligned}
$$

with the four dimensional vectors K_1 and K_2 satisfy

$$
\begin{aligned}
\left(2iv^*I_4 - \mathbb{J}_1 - \mathbb{J}_2e^{-2iv^*m^*}\right)K_1 &= \frac{1}{m^*}\mathcal{F}_{z^2},\\
\left(\mathbb{J}_1 + \mathbb{J}_2\right)K_2 &= \frac{-1}{m^*}\mathcal{F}_{z\bar{z}},
\end{aligned}
$$

where $\mathcal{F}_{z^2} = (\mathcal{F}^1_{z^2}, \mathcal{F}^2_{z^2}, \mathcal{F}^3_{z^2}, \mathcal{F}^4_{z^2})^T$ and $\mathcal{F}_{z\bar{z}} = (\mathcal{F}^1_{z\bar{z}}, \mathcal{F}^2_{z\bar{z}}, \mathcal{F}^3_{z\bar{z}}, \mathcal{F}^4_{z\bar{z}})^T$.

Finally, we can obtain the most important quantity:

$$C_1(0) = \frac{i}{2v^*m^*}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}.$$

Let

$$\begin{aligned} \mu_2 &= -\frac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\{\lambda'(m^*)\}}, \\ \beta_2 &= 2\mathrm{Re}\{C_1(0)\}, \end{aligned} \tag{4.38}$$

where $\lambda(m) = \lambda_R(m) + i\lambda_I(m)$ is a solution of characteristic equation corresponding to the linear system $\frac{d\mathbf{u}}{dt} = \mathbb{J}_1\mathbf{u}(t) + \mathbb{J}_2\mathbf{u}(t-m)$ satisfying $\lambda_R(m^*) = 0$ and $\lambda_I(m^*) = v^*$.

By using the general theory of Hopf bifurcation, we know that the direction of the Hopf bifurcation is determined by the sign of μ_2 and the stability of the bifurcating periodic solutions is determined by the sign of β_2 [78, 175, 204]. Hence, we have the following result.

Theorem 4.4.1. *Assume that the system (4.32) undergoes a Hopf bifurcation at E^* when $m = m^*$. Then we have the following:*
(i) If $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation occurs as m crosses m^* to the right (to the left);
(ii) If $\beta_2 < 0$ ($\beta_2 > 0$), then the bifurcating periodic solutions are orbitally stable (unstable).

Due to the complexity of the system, we cannot provide the explicit form of μ_2 and β_2 with respect to the system parameters, instead, we will calculate them numerically in the following section.

4.5 Numerical simulations

In this section, we carry out a case study for two cleaner fish species with appropriate parameters from the literature. Then, we explore the stability of E_2 and existence of periodic solution.

4.5.1 A case study.

Aquarium studies have found that some kinds of wrasse species (type of cleaner fish) can remove mobile sea lice (*Lepeophtheirus salmonis*) from salmon in production cages, such as goldsinny (*Ctenolabrus rupestris*), rock cook (*Centrolabrus exoletus*), ballan wrasse (*Labrus bergylta*), cuckoo wrasse (*Labrus mixtus*) and corkwing (*Crenilabrus melops*) [21]. In this

subsection, we study the efficacy of two different species of cleaner wrasses, goldsinny and ballan wrasse, on controlling sea lice in salmon farms based on our proposed model.

First, for the *Lepeophtheirus salmonis* population, we fix the parameters as in Figure 3.6, then the predator-free equilibrium point is $E_1 = (\mathcal{C}_1^*, \mathcal{A}_1^*, 0)$ where $\mathcal{C}_1^* \approx 25$ and $\mathcal{A}_1^* \approx 840.126$ from (3.20).

Now, for the cleaner fish population, we assume that the functional response in the system (3.8) is Holling type II

$$F(s) = sf(s) = \frac{as}{1 + aqs}$$

where a is the attack rate (searching efficiency) and q is the handling time (see e.g., [87]).

In ecology, the attack rate and the handling time can be measured as (see e.g., [22]),

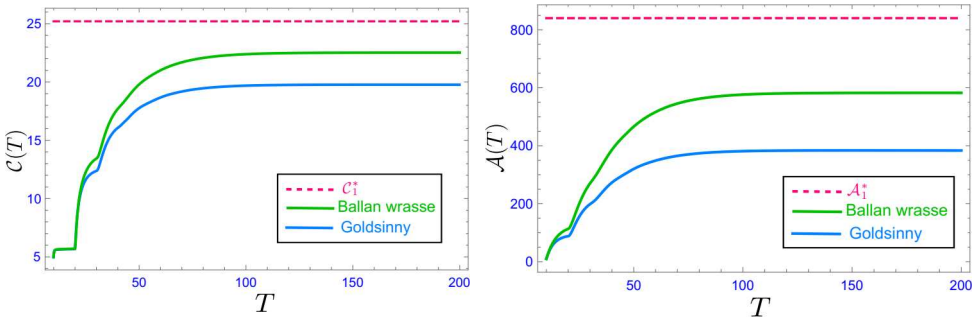
$$\begin{aligned} a &= \frac{\text{The number of prey items consumed by a predator during a period of searching time}}{\text{The searching time} \times \text{The density of prey items}}, \\ q &= \frac{\text{The total time - The searching time}}{\text{The number of prey items consumed by a predator during a period of searching time}}. \end{aligned} \tag{4.39}$$

In the field experiments, goldsinny showed more efficiency in removing adult sea lice from salmon than ballan wrasse [48, 162] although goldsinny is smaller than ballan wrasse in the size. More clearly, goldsinny consumed a mean of 46 lice per wrasse per day in cages of 500 salmon [48] while ballan wrasse consumed 23 lice per wrasse per day in average [162]. We assume the search time is 0.6 day (the total time is one day in these experiments), and to test the efficiency of a cleaner fish for controlling sea lice, we measure the number of sea lice without and with the control agent cleaner fish, separately. We take the density of prey items in (4.39) as density of adult sea lice ($\mathcal{A}_1^* = 840.126$) without predator and use $\mu_W = \frac{1}{\text{The lifespan}}$ to approximate the mortality rate of cleaner fish since the lifespan of goldsinny and ballan wrasse is 17 and 25 years, respectively, [163]. To guarantee the coexistence of E_2, we choose γ such that $F(\mathcal{A}_1^*)\gamma > \mu_W$, that is, $\mathcal{R}_f > 1$. Together with (4.39), we can approximate the parameters values which is listed in Table 4.3. Therefore, the net reproductive number for goldsinny and ballan wrasse is $\mathcal{R}_f^* = 1.713$ and $\mathcal{R}_f^{**} = 1.210$, respectively.

Parameter	a	q	γ	μ_W
Dimension	Sea lice per day	Day	Dimensionless	Day^{-1}
Goldsinny	0.0913	0.0198	0.000009	0.00016
Ballan wrasse	0.0456	0.0415	0.000009	0.00011

Table 4.3 Parameter values for the model (4.3).

For simulations, we choose initial data as $C_0 = 5, A_0 = 10$ and $W_0 = 2$. We can see that in the goldsinny case, $\mathcal{A}_2^* = 300.491$ and $\mathcal{C}_2^* = 18.3387$, that is, it causes 64.23% reduction in the population size of adults sea lice and 27.23% of copepodids, when no clear fish exists ($\mathcal{A}_1^* = 840.126$ and $\mathcal{C}_1^* = 25$). Taking ballan wrasse as the cleaner fish, one can calculate $\mathcal{A}_2^* = 543.919$ and $\mathcal{C}_2^* = 22.0507$, and hence, the reduction is 35.26% in the adults and 12.51% in copepodids population size. The numerical computation indicate that, goldsinny is more aggressive predator than ballan wrasse and causes a dramatically reduction in the adult sea lice (Figure 4.4) which is consistent with the theoretical analysis since $\mathcal{R}_f^* > \mathcal{R}_f^{**}$. The simulation result has good agreement with the field experiments. The reasons might be that cleaner fish can be intimidated by salmon since salmon become irritated and turn over by the presence of adult sea lice [30, 160]. So the smaller size of goldsinny gives more flexibility to approach salmon and remove the adult sea lice.



(a) Time series $\mathcal{C}(T)$ (b) Time series $\mathcal{A}(T)$

Figure 4.4 The sea lice population dynamics after introducing cleaner fish $C_0 = 5, A_0 = 10$ and $W_0 = 2$. Green line: ballan wrasse, blue line: goldsinny.

4.5.2 Stability of E_2.

To capture the oscillation behavior at $E_2 := E_2(m)$ and possible stability switch, we keep the sea lice parameters as those used in Figure 4.4, take

$$\mu_W = 0.4, \qquad \gamma = 0.165, \qquad F(s) = \frac{0.7s}{1 + 0.28s},$$

and let m be varied.

When $m = 0$, we have $\mathcal{R}_s = 187.344 > 1$, $\mathcal{R}_f = 1.02915 > 1$, and

$$p_2 + q_0 = -2.88684 < 0,\ p_1(p_2 + q_0) = -58.5469 < p_3 + q_1 = 0.159243,$$

in (4.22), i.e., the condition in Proposition 4.4.2 does not hold, implying that $E_2(0) = (5.6, 114.3, 39.3)$ is unstable and oscillations occur (see Figure 4.5). From Proposition 4.4.4, when $m > 0$, we can predict that $E_2(m)$ remains unstable and the occurrence of periodic solution may occur when $m < m^*$ for some $m^* > 0$, then becomes stable as $m > m^*$ is increased. To find m^*, we plot $S_n(m)$ defined in (4.27) in Figure 4.6 and obtain $m^* \approx 10.63$. We can further discuss the stability and direction of the bifurcating periodic solutions following Theorem 4.4.1. Numerically, we have $v(m^*) = v^* \approx 0.23$ and $\mathrm{Re}\{\lambda'(m^*)\} < 0$ from (4.25) and (4.30), respectively, and calculate $g_{20} = 0.14036 - 0.161389i$, $g_{11} = -0.278289 - 0.31439i$, $g_{02} = -0.17569 + 0.119498i$ and $g_{21} = -0.88805 - 10.3024i$ in (4.37). Thus, $C_1(0) = -0.444186 - 5.24517i$, implying $\mu_2 < 0$ and $\beta_2 < 0$ in (4.38). Hence, the Hopf bifurcation occurs as m crosses m^* to the left and the bifurcating periodic solutions are orbitally asymptotically stable. To see the changes of the dynamical behavior of orbits with respect to m, first, we choose $m = 2 < m^*$, then $\mathcal{R}_s = 138.788 > 1$ and $\mathcal{R}_f = 1.02883 > 1$. Thus, a positive steady state $E_2(2) = (5.08, 114.28, 34.86)$ exists, unstable, and an oscillations appear (see Figure 4.7). Then, we increase the infection development age time $m = 14 > m^*$, which results the break of the limit cycle, and the system approaches to a stable steady state $E_2(14) = (2.79, 114.28, 15.92)$ (see Figure 4.8). To see the influence of m on the amplitude of the population size of system 3.8, we plot the difference between max and min of the solutions of system 3.8 as m varies. Figure 4.9 shows that the amplitude of the oscillations decreases as m increases until it reaches zero when $m > m^*$. All of these are consistent with our prediction. Furthermore, we can notice in Figure 4.7 that high points in adult sea lice and copepodids numbers occur before peaks in the cleaner fish number. Biologically, abundance of mature sea lice leads to more egg production, and hence, high level of copepodids. On the other hand, when the number of predators (cleaner fish) is low, the prey (adult sea lice) population rises in size. While the food source (adult sea lice) increases, the number of predators increases. When there are enough predators, the prey numbers decline. With a lack of food source, the number of predators crashes and the cycle repeats.

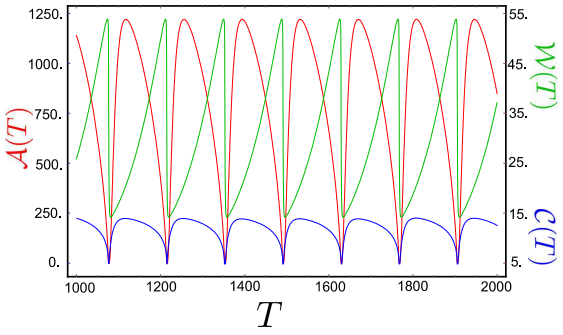


Figure 4.5 Time series of system (3.8) with $m = 0$. $\mathcal{A}(T)$/red, $\mathcal{C}(T)$/blue and $\mathcal{W}(T)$/green.

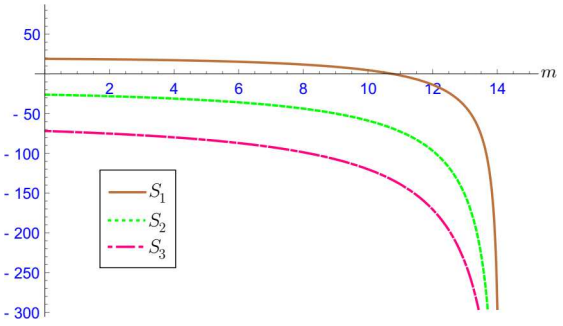
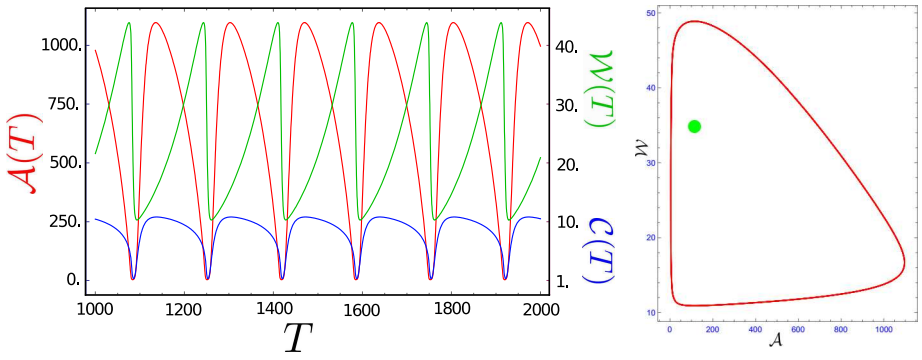


Figure 4.6 Function $S_n(m)$ $(n = 1, 2, 3)$.

4.6 Discussion

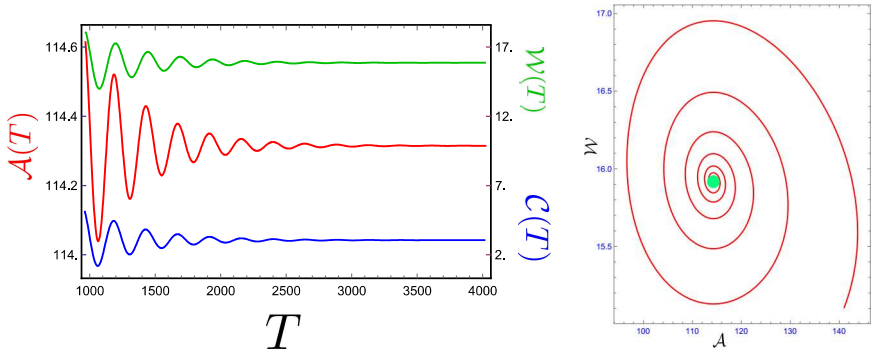
Recently, the control of sea lice becomes one of the top priorities in aquaculture research due to the high reproductive capacity of sea lice. In salmon farms, the biological control of sea lice using cleaner fish is more attractive than using chemicals because of its environmental benefits and cost-effective [47, 148, 190]. Many ecologists studied controlling sea lice by cleaner fish via field experiments without a mathematical modeling. In this paper, we have studied a stage-structured model for biocontrol of sea lice by introducing one of its natural predators cleaner fish. We have considered two stages in the immature phase of sea lice and one stage in mature phase of sea lice, and assumed predator-prey interaction at the adult level of sea lice. From the dynamical point of view, we have studied the qualitative features of the system, such as existence and uniqueness of solutions, boundedness, equilibria, persistence, and stability, with respect to the adult reproduction number for sea lice $\mathcal{R}_s$ and the net reproductive number of cleaner fish $\mathcal{R}_f$.

As a case study, based on our proposed model, we have discussed the efficacy of two different species of cleaner wrasses, goldsinny (*Ctenolabrus rupestris*) and ballan wrasse (*Labrus bergylta*), on controlling sea lice. Our numerical simulation has shown that, goldsinny causes



(a) (b)

Figure 4.7 Time series (a) and phase portrait (b) of system (3.8) with $m = 2$. $\mathcal{A}(T)$/red, $\mathcal{C}(T)$/blue and $\mathcal{W}(T)$/green. The green dot represents $E_2(2) = (5.08, 114.28, 34.86)$.



(a) (b)

Figure 4.8 Time series (a) and phase portrait (b) of system (3.8) with $m = 14$. $\mathcal{A}(T)$/red, $\mathcal{C}(T)$/blue and $\mathcal{W}(T)$/green. The green dot represents $E_2(14) = (2.79, 114.28, 15.92)$.

64.23% reduction in the population size of adults sea lice and 27.23% of copepodids, comparing with those without any cleaner fish, while ballan wrasse causes a reduction of 35.26% in the adults and 12.51% in copepodids population size. This indicates that goldsinny is more aggressive predator than ballan wrasse and causes a dramatically reduction in the adult sea lice, which is consistent with the field experiments.

Through the theoretical analysis of the model, we understand that the net reproductive number of cleaner fish $\mathcal{R}_f$ exceeds 1, implying an rise in the cleaner fish population, hence, larger $\mathcal{R}_f$ is more effective for sea lice control. In real world application, fish biologists in salmon farms may use the theoretical results of the proposed model to develop strategies or polices to improve the efficiency of the control agent, for example, by (i) Use intermittent dietary supplements for cleaner fish to increase their attack rate. In general, if cleaner fish are fed too much, they may not eat sea lice and tend to ignore the salmon [162], (ii)

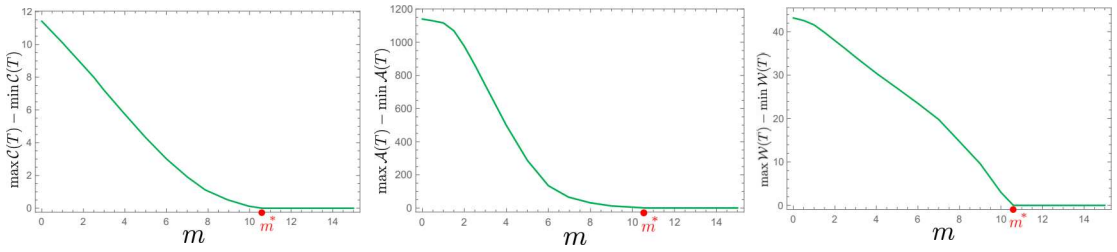


Figure 4.9 The amplitude of the population size when m increases.

Monitor the number of cleaner fish in salmon farms, in the sense that, keep appropriate ratio between the number of salmon and cleaner fish. In nature, when the number of cleaner fish is large enough, the competition between them increases, i.e., many cleaner fish may clean one salmon at the same time, which leads the salmon to resist the cleaner fish, and hence, decreases the number of consumed adult sea lice [162]. From a mathematical point of view, in to our model, (i) and (ii) mean the value $\mathcal{A}f(\mathcal{A})$, in (4.3b), is small. More specifically, in the case study, this implies that the attack rate becomes smaller (see (4.39)). Theoretically, using cleaner fish with long enough lifespan (i.e. larger $\mathcal{R}_f$) is more effective for sea lice control. However, in nature, most cleaner fish have lifespans much longer than a salmon farm production cycle (2 years), thus, cleaner fish mortality during a production cycle is negligible. We hope that this paper would lead to a better understanding of the biological control of sea lice and provide useful conceptual tools for this important subject.

Chapter 5

A Time-Delayed Epidemic Model for Ebola Disease Transmission

5.1 Introduction

Ebola virus disease, or simply Ebola, is a disease of humans and other non-human primates (gorillas, chimpanzees and duikers) caused by Ebola virus (EBOV) which belongs to the family Filoviridae. The virus originates in fruit bats and jumps to humans through an intermediate animal, such as chimpanzees [158, 172]. Ebola first appeared in 1976 in two outbreaks, one in Sudan, and the other in Democratic Republic of Congo. Since then it has resurfaced in Africa several times, for example, in 1994 in Ivory Coasty and Gabon; and in 2000 in Uganda. The outbreak in West Africa (Guinea, Liberia and Sierra Leone) in March 2014, is the largest and most complex Ebola outbreak. During this outbreak, Ebola has infected around 27678 people, roughly 11267 of whom have died [183]. According to Centers for Disease Control and Prevention (CDC) and World Health Organization (WHO), Ebola can be spread through human-to-human transmission via direct contact with the blood, secretions, organs or other bodily fluids (including but not limited to urine, saliva, sweat, feces, vomit, breast milk, and semen) of infected people, and with surfaces and materials (e.g., bedding and clothing) contaminated with these fluids [182, 184].

The mathematical study of infectious disease dynamics is an important aspect of investigating the spread of infections. Compartmental models are the most frequently used to describe the epidemiology of infectious diseases, where the total population is usually divided into a finite number of discrete categories, for example, the classical SEIR model with four stages of susceptible, infected but not yet infectious, infectious and recovered is discussed in [52, 83]. Multiple epidemiological models have been proposed to predict the

spread of Ebola in West Africa [16, 155, 203, 205]. In [16], the author used SEIR model to estimate the basic reproduction numbers of Ebola during the 2014 outbreak in West Africa. The maximum estimation of the basic reproduction number was 1.51 for Guinea, 2.53 for Sierra Leone and 1.59 for Liberia. In [155], the authors divided the population into six compartments: susceptible, exposed, infectious, hospitalized, funeral and removed. They found that increased contact tracing, improved infection control, or a combination of the two can have a substantial impact on the number of Ebola cases. In [203], a model consisting of susceptible, exposed, infectious, contaminated deceased, isolated infectious and removed categories was proposed to indicate that isolating the infectious cases with average time less than three days between the appearance of symptoms and isolation, and the efficiently monitoring of the contact traced incubating infected cases are the most important elements for containment of Ebola within a short time. A SEIRD model (susceptible, exposed, infectious, recovered and dead but still infectious) was studied in [205]. They found robustly that inferences that don't account for post-death transmission tend to underestimate the basic reproductive number, in other words, large amounts of post-death transmission imply larger reproductive numbers.

In the real world, for many diseases such as Ebola, when adequate contact with an infectious individual happen, a susceptible individual becomes infected but is not yet infectious. This individual remains in the exposed class for a certain latent period before becoming infectious. Such period in disease transmission can be modelled by a delay differential equation. For example, in [40] an SEIRS epidemic model with a constant latent and immune periods is presented. In [194], the authors proposed a general mathematical model for a disease with a latent period and relapse. In [212], a disease transmission model of SEIRS type with distributed delays in latent and temporary immune periods is discussed. The authors studied the threshold property of the basic reproduction number R_0 and the dynamical properties of the disease-free/endemic equilibrium points with general and particular probability distributions in both periods.

In West Africa it is common to contact with the bodies and fluids of persons who have died, where family and community members often touch and wash the body of the deceased in preparation for funerals [136, 152]. Since Ebola virus can survive for several days at room temperature in body fluids [181], one of the main infection pathways is through preparation of corpses for burial. In this chapter, we propose a model that incorporates both the transmission of infection between the living humans and from the infected corpses to the living individuals with a constant latent period. From the viewpoint of dynamics, we discuss the existence and stability of equilibrium points and give numerical simulations to show the theoretical results and explore the dynamical behavior of the disease under varied environments. The main difference of this work, from the literature e.g. [16, 155, 203, 205], is

that those models are all given by ordinary differential equations which neglected the effect of latent period. However, the latent period has a profound effect on the generation time, and hence epidemic growth (see e.g, [135]). Moreover, although there are some contain analysis on the basic reproduction number R_0 previously, which offer some interesting insights into Ebola transmission in humans and show some numeric results, in this work, in addition to the mathematical derivation of R_0, we determine the local and global dynamics of the model analytically with respect to the basic reproduction number R_0.

The rest of this chapter is organized as follows. In Section 5.2, we present the model and discuss its well-posedness by verifying the non-negativity and boundedness of the solutions with reasonable initial data. In Section 5.3, we calculate the basic reproduction number R_0; discuss the global stability of the disease-free equilibrium when $R_0 < 1$; explore the existence of a unique endemic equilibrium when $R_0 > 1$ and show the local stability under certain condition. In Section 5.3, we prove the persistence of infection when $R_0 > 1$ and study the global stability of the endemic equilibrium in a special case of the model by considering a limiting system of the model and then using the Lyapunov functional and LaSalle invariance principle. In Section 5.4, numerical simulations are given to demonstrate the theoretical results and explore the disease transmission with the variation of seasonality.

5.2 Mathematical model and the well-posedness property

Motivated by the model in [205], we consider the transmission of infection between the living humans and from the infected corpses to the living individuals in which the latent period of Ebola is incorporated.

We consider the size of the population $N(t)$ is divided into susceptible, exposed (infected but not yet infectious), infectious, recovered individuals and infected corpses who are nonetheless infectious, with class sizes denoted by $S(t)$, $E(t)$, $I(t)$, $R(t)$ and $D(t)$, respectively. Let τ be the latent period, c the probability of transmission of infection from an infectious human to a susceptible individual when a contact occurs, and d the probability of transmission of infection from an infectious corpse to a susceptible individual. Therefore, the rate of new infected individuals at time t is

$$c\frac{S(t)I(t)}{N(t)} + d\frac{S(t)D(t)}{N(t)}.$$

If μ is the natural death rate, then the probability that an individual survives in the latent period $[t-\tau, t]$ is $e^{-\mu\tau}$. Hence, the rate of individuals surviving in the latent period

τ and becoming infectious at time t is

$$c\frac{S(t-\tau)I(t-\tau)}{N(t-\tau)}e^{-\mu\tau} + d\frac{S(t-\tau)D(t-\tau)}{N(t-\tau)}e^{-\mu\tau}.$$

Thus we have the following delayed model:

$$\begin{aligned}
\frac{dS(t)}{dt} &= \Lambda - c\frac{S(t)I(t)}{N(t)} - d\frac{S(t)D(t)}{N(t)} - \mu S(t),\\
\frac{dE(t)}{dt} &= c\frac{S(t)I(t)}{N(t)} + d\frac{S(t)D(t)}{N(t)} - c\frac{S(t-\tau)I(t-\tau)}{N(t-\tau)}e^{-\mu\tau}\\
&\quad -d\frac{S(t-\tau)D(t-\tau)}{N(t-\tau)}e^{-\mu\tau} - \mu E(t),\\
\frac{dI(t)}{dt} &= c\frac{S(t-\tau)I(t-\tau)}{N(t-\tau)}e^{-\mu\tau} + d\frac{S(t-\tau)D(t-\tau)}{N(t-\tau)}e^{-\mu\tau} - (\rho+\mu+\delta)I(t),\\
\frac{dR(t)}{dt} &= \rho I(t) - \mu R(t),\\
\frac{dD(t)}{dt} &= (\mu+\delta)I(t) - \gamma D(t),
\end{aligned} \tag{5.1}$$

where $N(t) = S(t) + E(t) + I(t) + R(t) + D(t)$, Λ is the recruitment rate, δ is an additional death rate due to infection by Ebola, $\frac{1}{\rho}$ is the average duration of the infectious period, with recovery rate ρ, and the average period of infectiousness after death in human corpses is $\frac{1}{\gamma}$. All the parameters are positive constants. An architecture of the model (5.1) is given in Figure 5.1.

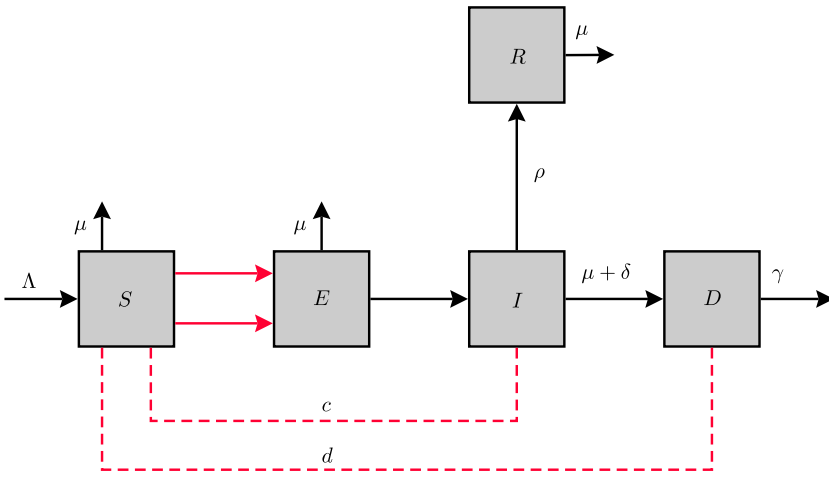


Figure 5.1 Schematic diagram of Ebola transmission.

By choosing a reasonable initial condition, we can find the implicit solution of the differential equation for $E(t)$ in (5.1) in the following result.

Lemma 5.2.1.

$$E(t) = \int_{t-\tau}^{t} \frac{S(\theta)}{N(\theta)}(cI(\theta) + dD(\theta))e^{-\mu(t-\theta)}d\theta$$

is the implicit solution for $E(t)$ in system (5.1) with the initial condition

$$E(0) = \int_{-\tau}^{0} \frac{S(\theta)}{N(\theta)}(cI(\theta) + dD(\theta))e^{\mu\theta}d\theta.$$

Proof. From the second equation in system (5.1), we have

$$e^{\mu t}\left(\frac{dE}{dt} + \mu E(t)\right) = e^{\mu t}\left(\frac{S(t)}{N(t)}(cI(t) + dD(t))\right) - e^{\mu(t-\tau)}\left(\frac{S(t-\tau)}{N(t-\tau)}(cI(t-\tau) + dD(t-\tau))\right).$$

By integrating on both sides of the equation, we get

$$\begin{aligned}
e^{\mu t}E(t) - E(0) &= \int_0^t \left(\frac{S(\theta)}{N(\theta)}(cI(\theta) + dD(\theta))e^{\mu\theta}\right)d\theta \\
&\quad - \int_0^t \left(\frac{S(\theta-\tau)}{N(\theta-\tau)}(cI(\theta-\tau) + dD(\theta-\tau))e^{\mu(\theta-\tau)}\right)d\theta \\
&= \int_0^t \left(\frac{S(\theta)}{N(\theta)}(cI(\theta) + dD(\theta))e^{\mu\theta}\right)d\theta - \int_{-\tau}^{t-\tau} \left(\frac{S(\theta)}{N(\theta)}(cI(\theta) + dD(\theta))e^{\mu\theta}\right)d\theta \\
&= \int_{t-\tau}^t \left(\frac{S(\theta)}{N(\theta)}(cI(\theta) + dD(\theta))e^{\mu\theta}\right)d\theta - \int_{-\tau}^0 \left(\frac{S(\theta)}{N(\theta)}(cI(\theta) + dD(\theta))e^{\mu\theta}\right)d\theta.
\end{aligned}$$

Hence, if $E(0) = \int_{-\tau}^{0} \frac{S(\theta)}{N(\theta)}(cI(\theta) + dD(\theta))e^{\mu\theta}d\theta$, then $E(t) = \int_{t-\tau}^{t} \frac{S(\theta)}{N(\theta)}(cI(\theta) + dD(\theta))e^{-\mu(t-\theta)}d\theta$. □

Denote $C := C([-\tau, 0), \mathbb{R}^5)$. For $\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) \in C$, define $\|\phi\| = \sum_{i=1}^{5} \|\phi_i\|_\infty$ where

$$\|\phi_i\|_\infty = \max_{\theta\in[-\tau,0]} |\phi_i(\theta)|.$$

Then C is a Banach space and $C^+ = \{\phi \in C : \phi_i(\theta) \geq 0, \forall i \in \{1, 2, 3, 4, 5\}, \theta \in [-\tau, 0]\}$ is a normal cone of C with nonempty interior in C. For any given continuous function $u = (u_1, u_2, u_3, u_4, u_5) : [-\tau, \zeta) \to \mathbb{R}^5$ with $\zeta > 0$, we define $u_t \in C$ for each $t \geq 0$ by $u_t(\theta) = (u_1(t + \theta), u_2(t + \theta), u_3(t + \theta), u_4(t + \theta), u_5(t + \theta))$ for all $\theta \in [-\tau, 0]$.

Taking the initial data for system (5.1) as the form

$$\mathcal{X}_\xi = \left\{ \phi \in C^+ : \sum_{i=1}^{5} \phi_i(s) \geq \xi, \forall s \in [-\tau, 0], \phi_2(0) = \int_{-\tau}^{0} \left(\frac{\phi_1(s)(c\phi_3(s) + d\phi_5(s))}{\sum_{i=1}^{5} \phi_i(s)} e^{\mu s} \right) ds \right\}$$

for small $\xi \in (0, \frac{\Lambda}{\max\{\mu,\gamma\}})$, where the form for $\phi_2(0)$ follows from Lemma 5.2.1. Note that the denominators $N(t)$ and $N(t-\tau)$ in the system (5.1) do not approach to zero as $t \to \infty$. It is clear that,

$$\frac{dN(t)}{dt} \geq \Lambda - \max\{\mu, \gamma\} N(t).$$

For the system $\frac{du(t)}{dt} = \Lambda - \max\{\mu, \gamma\} u(t)$, the equilibrium point $\frac{\Lambda}{\max\{\mu,\gamma\}}$ is globally asymptotically stable. This implies that for any $0 < \xi < \frac{\Lambda}{\max\{\mu,\gamma\}}$, $\frac{du}{dt}\big|_{u=\xi} = \Lambda - \max\{\mu, \gamma\}\xi > 0$. So if $u(0) \geq \xi$, then $u(t) \geq \xi$ for all $t \geq 0$.

The following theorem demonstrates that the solutions of model (5.1) are nonnegative and bounded.

Theorem 5.2.1. *For any $\phi \in \mathcal{X}_\xi$, system (5.1) has a unique nonnegative solution $u(t, \phi)$ satisfying $u_0 = \phi$. Furthermore, the solution is ultimately bounded in C. More precisely, $\limsup_{t\to\infty} S(t) \leq \frac{\Lambda}{\mu}$, $\limsup_{t\to\infty} E(t) \leq \frac{(c+d)\Lambda}{\mu^2}$, $\limsup_{t\to\infty} I(t) \leq \frac{\Lambda e^{-\mu\tau}(c+d)}{\mu(\rho+\mu+\delta)} := M$, $\limsup_{t\to\infty} R(t) \leq \frac{\rho M}{\mu}$ and $\limsup_{t\to\infty} D(t) \leq \frac{(\mu+\delta)M}{\gamma}$.*

Proof. For any $\phi \in \mathcal{X}_\xi$, we define $F(\phi) = (F_1(\phi), F_2(\phi), F_3(\phi), F_4(\phi), F_5(\phi))^T$, where

$$\begin{aligned}
F_1(\phi) &= \Lambda - c\frac{\phi_1(0)\phi_3(0)}{\sum_{i=1}^{5} \phi_i(0)} - d\frac{\phi_1(0)\phi_5(0)}{\sum_{i=1}^{5} \phi_i(0)} - \mu\phi_1(0), \\
F_2(\phi) &= c\frac{\phi_1(0)\phi_3(0)}{\sum_{i=1}^{5} \phi_i(0)} + d\frac{\phi_1(0)\phi_5(0)}{\sum_{i=1}^{5} \phi_i(0)} - c\frac{\phi_1(-\tau)\phi_3(-\tau)}{\sum_{i=1}^{5} \phi_i(-\tau)} - d\frac{\phi_1(-\tau)\phi_5(-\tau)}{\sum_{i=1}^{5} \phi_i(-\tau)} - \mu\phi_2(0), \\
F_3(\phi) &= c\frac{\phi_1(-\tau)\phi_3(-\tau)}{\sum_{i=1}^{5} \phi_i(-\tau)} + d\frac{\phi_1(-\tau)\phi_5(-\tau)}{\sum_{i=1}^{5} \phi_i(-\tau)} - (\rho + \mu + \delta)\phi_3(0), \\
F_4(\phi) &= \rho\phi_3(0) - \mu\phi_4(0), \\
F_5(\phi) &= (\mu + \delta)\phi_3(0) - \gamma\phi_5(0).
\end{aligned}$$

Note that $\mathcal{X}_\xi$ is closed in C and for any $\phi \in \mathcal{X}_\xi$, $F(\phi)$ is continuous and Lipschitz in ϕ in each compact set in $\mathbb{R} \times \mathcal{X}_\xi$. By [74, Theorem 2.2.3], it follows that system (5.1) has a unique solution $u(t, \phi)$ through $(0, \phi)$ on its maximal interval $[0, \zeta)$ of existence.

For $i = 1, 3, 4, 5$, obviously $F_i(\phi) \geq 0$ when $\phi \in \mathcal{X}_\xi$ with $\phi_i(0) = 0$, thus the solutions $u_i(t, \phi)$ are nonnegative for all $t \in [0, \zeta)$ see [169, Theorem 5.2.1]. $u_2(t, \phi) \geq 0$ is straightforward from Lemma 5.2.1.

Moreover, from $\frac{dN}{dt} = \Lambda - \mu S(t) - \mu E(t) - \mu I(t) - \mu R(t) - \gamma D(t)$, we have

$$\frac{dN(t)}{dt} \leq \Lambda - \min\{\mu, \gamma\} N(t).$$

By the standard comparison argument and because $y = \frac{\Lambda}{\min\{\mu,\gamma\}}$ is globally asymptotically stable in the system $\frac{dy(t)}{dt} = \Lambda - \min\{\mu, \gamma\} y(t)$, we have

$$\limsup_{t\to\infty} N(t) \leq \frac{\Lambda}{\min\{\mu, \gamma\}}.$$

Thus, $S(t)$, $E(t)$, $I(t)$, $R(t)$ and $D(t)$ are bounded on $t \in [0, \zeta)$. Hence, [74, Theorem 2.3.1] implies that $\zeta = \infty$. That is, all the solutions exist globally, and are ultimately bounded.

More precisely, now in (5.1), from the first equation, we have $\frac{dS(t)}{dt} \leq \Lambda - \mu S(t)$, so $\limsup_{t\to\infty} S(t) \leq \frac{\Lambda}{\mu}$. Using the fact that $\frac{I(t)}{N(t)} \leq 1$, $\frac{D(t)}{N(t)} \leq 1$ for all t in the second and third equations, we get

$$\begin{aligned} \frac{dE(t)}{dt} &\leq \frac{(c+d)\,\Lambda}{\mu} - \mu E(t), \\ \frac{dI(t)}{dt} &\leq \frac{\Lambda e^{-\mu\tau}(c+d)}{\mu} - (\rho + \mu + \delta) I(t). \end{aligned}$$

Hence, $\limsup_{t\to\infty} E(t) \leq \frac{(c+d)\Lambda}{\mu^2}$ and $\limsup_{t\to\infty} I(t) \leq \frac{\Lambda e^{-\mu\tau}(c+d)}{\mu(\rho+\mu+\delta)} := M$. Then it follows that, from the forth and fifth equations

$$\frac{dR(t)}{dt} \leq \rho M - \mu R(t)$$

and

$$\frac{dD(t)}{dt} \leq (\mu + \delta) M - \gamma R(t),$$

respectively. Thus, $\limsup_{t\to\infty} R(t) \leq \frac{\rho M}{\mu}$ and $\limsup_{t\to\infty} D(t) \leq \frac{(\mu+\delta)M}{\gamma}$. □

5.3 Disease-related equilibrium points and basic reproduction number

From the system (5.1) it is easy to see that the class E, I and D are disease-related, when $E(t) = I(t) = D(t) = 0$, $S(t) = \frac{\Lambda}{\mu}$ and $R(t) = 0$, that is, the disease-free equilibrium (DFE)

$E_1 = (\frac{\Lambda}{\mu}, 0, 0, 0, 0)$ always exists for all values of the parameters.

The linearization of (5.1) at E_1 is

$$\begin{aligned}
\frac{dS(t)}{dt} &= -cI(t) - dD(t) - \mu S(t), \\
\frac{dE(t)}{dt} &= cI(t) + dD(t) - ce^{-\mu\tau} I(t-\tau) - de^{-\mu\tau} D(t-\tau) - \mu E(t), \\
\frac{dI(t)}{dt} &= ce^{-\mu\tau} I(t-\tau) + de^{-\mu\tau} D(t-\tau) - (\rho + \mu + \delta) I(t), \\
\frac{dR(t)}{dt} &= \rho I(t) - \mu R(t), \\
\frac{dD(t)}{dt} &= (\mu + \delta) I(t) - \gamma D(t).
\end{aligned} \tag{5.2}$$

To calculate the basic reproduction number, we consider the equations for the diseased classes in system (5.2), which can be rewritten as

$$\frac{d}{dt}\boldsymbol{Y}(t) = A\boldsymbol{Y}(t-\tau) - B\boldsymbol{Y}(t), \tag{5.3}$$

where

$$\boldsymbol{Y}(t) = \begin{bmatrix} E(t) \\ I(t) \\ D(t) \end{bmatrix}, \; A = \begin{bmatrix} 0 & -ce^{-\mu\tau} & -de^{-\mu\tau} \\ 0 & ce^{-\mu\tau} & de^{-\mu\tau} \\ 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} \mu & -c & -d \\ 0 & \rho+\mu+\delta & 0 \\ 0 & -(\mu+\delta) & \gamma \end{bmatrix}.$$

Let $\boldsymbol{\mathcal{Y}}_0 = (y_1, y_2, y_3)^T$ be the number of classes $E(t)$, $I(t)$ and $D(t)$ at $t = 0$, then from (5.3) the distribution of the remaining population of classes $E(t)$, $I(t)$ and $D(t)$ at time $t > 0$ is

$$\boldsymbol{\mathcal{Y}}(t) = e^{-Bt}\boldsymbol{\mathcal{Y}}_0.$$

The total number of newly infected individuals is

$$\bar{\boldsymbol{\mathcal{Y}}} = \int_{\tau}^{\infty} A\boldsymbol{\mathcal{Y}}(t-\tau)dt = \int_{\tau}^{\infty} Ae^{-B(t-\tau)}\boldsymbol{\mathcal{Y}}_0 dt = AB^{-1}\boldsymbol{\mathcal{Y}}_0$$

due to the nonsingularity of the matrix B. Then it follows that, the next infection operator is

$$M_0 = AB^{-1} = \begin{bmatrix} 0 & \frac{-(\gamma c + d(\mu+\delta))e^{-\mu\tau}}{\gamma(\rho+\mu+\delta)} & \frac{-de^{-\mu\tau}}{\gamma} \\ 0 & \frac{(\gamma c + d(\mu+\delta))e^{-\mu\tau}}{\gamma(\rho+\mu+\delta)} & \frac{de^{-\mu\tau}}{\gamma} \\ 0 & 0 & 0 \end{bmatrix}.$$

In literature (see e.g., [105]), the basic reproduction number R_0 for system (5.1) is the

spectral radius of the matrix M_0, which is

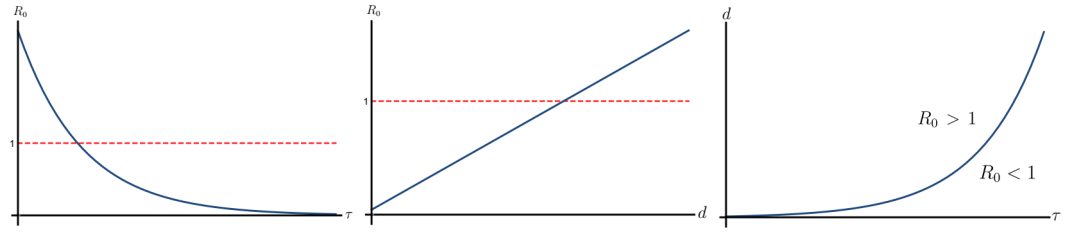
$$R_0 = \frac{(c\gamma + d\,(\mu+\delta))\,e^{-\mu\tau}}{\gamma\,(\rho+\mu+\delta)} = \frac{ce^{-\mu\tau}}{\rho+\mu+\delta} + \frac{d(\mu+\delta)e^{-\mu\tau}}{\gamma(\rho+\mu+\delta)}. \tag{5.4}$$

Biologically, $e^{-\mu\tau}$ is the survival rate of infected individual in latent period, $\frac{\mu+\delta}{\gamma}$ is the number of the death from one infected individual and $\frac{1}{\rho+\mu+\delta}$ is the average period of infected individual to recover/survive or die. Therefore, $\frac{ce^{-\mu\tau}}{\rho+\mu+\delta}$ is the number of newly infected individuals that arise from one infectious individual and $\frac{de^{-\mu\tau}(\mu+\delta)}{\gamma(\rho+\mu+\delta)}$ is the number of newly infected individuals that arise from one infectious corpse. From the point view of epidemiology, if the basic reproduction number $R_0 > 1$, then each individual is causing more than one infection, so the disease will take hold and when $R_0 < 1$ the disease will die out.

Therefore, in order to reduce the spread of the disease, we need to restrict the value of R_0 below 1 by controlling the related parameters. For example, for the fixed parameters c, μ, δ, ρ and γ, first, notice that $\frac{\partial R_0}{\partial \tau} = -\mu R_0 < 0$ for all $\tau \geq 0$, implying the reproduction number decreases as the latent period increases. Hence, we may try to extend the duration of latent to slow Ebola spread by prescription drug or control measures. On the other side, $\frac{\partial R_0}{\partial d} = \frac{e^{-\mu\tau}(\mu+\delta)}{\gamma(\rho+\mu+\delta)} > 0$, R_0 increases as the value of d increases. Therefore, we need to keep the transmission rate d small to reduce the spread of Ebola, one of the control manner is dispose of human remains either by cremation or burial. Mathematically, it is straightforward to obtain:

Lemma 5.3.1. *$R_0 < 1$ when $\tau > \frac{1}{\mu}\ln\left(\frac{\gamma c + d(\mu+\delta)}{\gamma(\rho+\mu+\delta)}\right)$ or $d < \frac{\gamma((\rho+\mu+\delta)e^{\mu\tau} - c)}{\mu+\delta}$.*

Figure 5.2 provides us the relation of the basic reproduction R_0 with respect to the latent period τ and the transmission rate d.



R_0 vs τ $\qquad$ R_0 vs d $\qquad$ d vs τ

Figure 5.2 The relationship between R_0, τ and d.

The following mathematical result is consistent with the biological interpretation.

Theorem 5.3.1. *The disease-free equilibrium E_1 of system (5.1) is locally asymptotically stable if $R_0 < 1$ and it is unstable if $R_0 > 1$.*

Proof. At the DFE point $(\frac{\Lambda}{\mu}, 0, 0, 0, 0)$ in (5.1), the corresponding characteristic equation is

$$\Delta(\lambda, \tau) = (\lambda + \mu)^3 h(\lambda, \tau) = 0 \tag{5.5}$$

where

$$h(\lambda, \tau) = (\lambda + \rho + \mu + \delta)(\lambda + \gamma) - \left(c\lambda e^{-\mu\tau} + (c\gamma + d(\mu + \delta))\, e^{-\mu\tau}\right) e^{-\lambda\tau}. \tag{5.6}$$

Since $\lambda = -\mu$ is a negative root in (5.5), it suffices to consider the roots in $h(\lambda, \tau) = 0$. From

$$h(0, \tau) = \gamma(\rho + \mu + \delta) - (c\gamma + d(\mu + \delta))\, e^{-\mu\tau} = \gamma(\rho + \mu + \delta)(1 - R_0),$$

it is apparent that if $R_0 > 1$, then $h(0, \tau) < 0$. Since $h(\lambda, \tau)$ is continuous with respect to λ and $\lim_{\lambda \to \infty} h(\lambda, \tau) = +\infty$, there exists $\bar{\lambda} > 0$ such that $h(\bar{\lambda}, \tau) = 0$. Therefore $(\frac{\Lambda}{\mu}, 0, 0, 0, 0)$ is unstable in system (5.1) when $R_0 > 1$.

Now, when $R_0 < 1$, we need to show that all the eigenvalues in $h(\lambda, \tau) = 0$ have negative real parts. First, note that any eigenvalue in $h(\lambda, \tau) = 0$ satisfies

$$(\lambda + \rho + \mu + \delta)(\lambda + \gamma) = \left(c\lambda e^{-\mu\tau} + (c\gamma + d(\mu + \delta))\, e^{-\mu\tau}\right) e^{-\lambda\tau},$$

which is equivalent to

$$\left(\frac{\lambda}{\rho + \delta + \mu} + 1\right)\left(\frac{\lambda}{\gamma} + 1\right) = \left(\frac{ce^{-\mu\tau}}{\rho + \delta + \mu}\frac{\lambda}{\gamma} + R_0\right) e^{-\lambda\tau}.$$

That is,

$$\left(\frac{\lambda}{\rho + \delta + \mu} + 1\right)\left(\frac{\lambda}{\gamma} + 1\right) = R_0\left(\frac{ce^{-\mu\tau}}{R_0(\rho + \delta + \mu)}\frac{\lambda}{\gamma} + 1\right) e^{-\lambda\tau}.$$

Assume there exists a zero in $h(\lambda, \tau) = 0$ with $Re(\lambda) \geq 0$, then

$$\left|\frac{\lambda}{\rho + \delta + \mu} + 1\right| \geq 1, \quad \left|e^{-\lambda\tau}\right| \leq 1$$

and

$$\left|\frac{\lambda}{\gamma} + 1\right| > \left|\frac{ce^{-\mu\tau}}{R_0(\rho + \delta + \mu)}\frac{\lambda}{\gamma} + 1\right|$$

due to $R_0 > \frac{ce^{-\mu\tau}}{(\rho + \delta + \mu)}$. Therefore, when $R_0 < 1$, we have

$$\left|\left(\frac{\lambda}{\rho + \delta + \mu} + 1\right)\left(\frac{\lambda}{\gamma} + 1\right)\right| > \left|R_0 e^{-\lambda\tau}\left(\frac{ce^{-\mu\tau}}{R_0(\rho + \delta + \mu)}\frac{\lambda}{\gamma} + 1\right)\right|,$$

which leads to a contradiction. Hence all the eigenvalues in (5.5) have negative real parts,

implying E_1 is locally asymptotically stable. □

Remark 5.3.1. *Theorem 5.3.1 can be obtained from [217, Theorem 2.1 and Corollary 2.1].*

In addition, we can prove that,

Theorem 5.3.2. *If $R_0 < 1$, the disease-free equilibrium E_1 of system (5.1) is globally asymptotically stable.*

Proof. Let $t_0 > 0$ be sufficiently large, since $N(t-\tau) > 0$ and $S(t-\tau) \leq N(t-\tau)$ for any $t > t_0 + \tau$, it then follows from the third and fifth equations in system (5.1) that

$$\begin{aligned} \frac{dI(t)}{dt} &\leq c(1+\sigma_1)e^{-\mu\tau}I(t-\tau) + d(1+\sigma_1)e^{-\mu\tau}D(t-\tau) - (\rho+\mu+\delta)I(t), \\ \frac{dD(t)}{dt} &= (\mu+\delta)I(t) - \gamma D(t), \end{aligned} \tag{5.7}$$

for sufficiently small $\sigma_1 > 0$.
Consider the following linear system

$$\begin{aligned} \frac{dV_1(t)}{dt} &= c(1+\sigma_1)e^{-\mu\tau}V_1(t-\tau) + d(1+\sigma_1)e^{-\mu\tau}V_2(t-\tau) - (\rho+\mu+\delta)V_1(t), \\ \frac{dV_2(t)}{dt} &= (\mu+\delta)V_1(t) - \gamma V_2(t). \end{aligned} \tag{5.8}$$

Let $\lambda_1(\sigma_1)$ be the principle eigenvalue of system (5.8). To prove that $\lambda_1(\sigma_1) < 0$, first, we show that system (5.8) is quasimonotone and irreducible. Then Theorem 5.5.1 in [169] guarantees us that it is sufficient to consider only the real roots of the characteristic equation of (5.8) because any complex roots would have smaller real part than the largest real root.

In fact in (5.8), $c(1+\sigma_1)e^{-\mu\tau}V_1(t-\tau)+d(1+\sigma_1)e^{-\mu\tau}V_2(t-\tau)-(\rho+\mu+\delta)V_1(t)$ is increasing as a function of the delayed variable $V_i(t-\tau)$ $(i=1,2)$ and $(\mu+\delta)V_1(t)-\gamma V_2(t)$ is increasing with respect to $V_1(t)$, thus, from the results in [169], system (5.8) is quasimonotone. To prove that system (5.8) is irreducible, it is enough to show that the matrix

$$A = \begin{pmatrix} c(1+\sigma_1)e^{-\mu\tau} - \rho - \mu - \delta & d(1+\sigma_1)e^{-\mu\tau} \\ \mu+\delta & -\gamma \end{pmatrix}$$

is irreducible, which is straightforward since the directed graph associated with the matrix A is strongly connected (see e.g. [106]).

The characteristic equation of (5.8) is

$$\Delta_1(\lambda) = (\lambda+\rho+\mu+\delta)(\lambda+\gamma) - (1+\sigma_1)\left(c\lambda e^{-\mu\tau} + (c\gamma + d(\mu+\delta))e^{-\mu\tau}\right)e^{-\lambda\tau} = 0.$$

When $\sigma_1 = 0$, it is clear that $\Delta_1(0) > 0$ when $R_0 < 1$. Hence, all real roots are negative because $\lim_{\lambda\to\infty} \Delta_1(\lambda) \to \infty$ and

$$\Delta_1'(\lambda) = 2\lambda + (c\tau\lambda + \tau\,(c\gamma + d(\mu+\delta)))\, e^{-(\lambda+\mu)\tau} + (\rho+\mu+\delta+\gamma) - ce^{-(\lambda+\mu)\tau} \geq 0$$

for $\lambda \geq 0$ because $\frac{ce^{-(\lambda+\mu)\tau}}{\rho+\mu+\delta} \leq \frac{ce^{-\mu\tau}}{\rho+\mu+\delta} < R_0 < 1$. Thus, $\lambda_1(0) < 0$. Due to the continuity of λ_1, $\lambda_1(\sigma_1) < 0$ for sufficiently small $\sigma_1 > 0$.

It follows that there exists a solution $\vec{U_1}(t) = e^{\lambda_1(\sigma_1)t}\phi_0$ of (5.8) , where ϕ_0 is the positive eigenfunction associated with $\lambda_1(\sigma_1)$, $\vec{U_1}$ and ϕ_0 are vectors with two components. Since $I(t) \geq 0$ and $D(t) \geq 0$ for all $t > 0$ in (5.7), the comparison theory implies that there exists a small $\ell_1 > 0$ such that $(I(t), D(t))^T \leq \ell_1 e^{\lambda_1(\sigma_1)t}\phi_0$ for all $t \geq t_0$. From $\lambda_1(\sigma_1) < 0$, we have $\lim_{t\to\infty}(I(t), D(t)) = (0,0)$. By the theory of asymptotically autonomous semiflows (see e.g., [185]), it follows that

$$\lim_{t\to\infty}(S(t), E(t), I(t), R(t), D(t)) = \left(\frac{\Lambda}{\mu}, 0, 0, 0, 0\right).$$

That is, E_1 is globally attractive if $R_0 < 1$, which, together with the local stability of E_1 established in Theorem 5.3.1, confirms the global asymptotic stability of the disease-free equilibrium E_1 when $R_0 < 1$. This completes the proof. □

To discuss the dynamical behavior of the system (5.1) when $R_0 > 1$, first, we have the result about the existence of endemic (disease-present) equilibrium.

Theorem 5.3.3. *If $R_0 > 1$, then system (5.1) has a unique endemic equilibrium point $E_2 = (S^*, E^*, I^*, R^*, D^*)$ with $0 < S^* < \frac{\Lambda}{\mu}$, $E^* > 0$, $I^* > 0$, $R^* > 0$ and , $D^* > 0$.*

Proof. If the positive equilibrium (S^*,E^*,I^*,R^*,D^*) in system (5.1) exists, then S^*, E^*, I^*, R^* and D^* must satisfy

$$\begin{aligned}
\Lambda - \frac{cS^*I^*}{S^*+E^*+I^*+R^*+D^*} - \frac{dS^*D^*}{S^*+E^*+I^*+R^*+D^*} - \mu S^* &= 0,\\
\frac{(1-e^{-\mu\tau})cS^*I^*}{S^*+E^*+I^*+R^*+D^*} + \frac{(1-e^{-\mu\tau})dS^*D^*}{S^*+E^*+I^*+R^*+D^*} - \mu E^* &= 0,\\
\frac{ce^{-\mu\tau}S^*I^*}{S^*+E^*+I^*+R^*+D^*} + \frac{de^{-\mu\tau}S^*D^*}{S^*+E^*+I^*+R^*+D^*} - (\rho+\mu+\delta)I^* &= 0,\\
\rho I^* - \mu R^* &= 0,\\
(\mu+\delta)I^* - \gamma D^* &= 0.
\end{aligned} \tag{5.9}$$

In (5.9) from the forth and fifth equations, we get

$$R^* = \frac{\rho}{\mu} I^* \quad \text{and} \quad D^* = \frac{\mu + \delta}{\gamma} I^*. \tag{5.10}$$

Hence, the remaining three equations become

$$\begin{aligned} \Lambda - g(S^*, E^*, I^*) - \mu S^* &= 0, \\ (1 - e^{-\mu\tau}) g(S^*, E^*, I^*) - \mu E^* &= 0, \\ e^{-\mu\tau} g(S^*, E^*, I^*) - (\rho + \mu + \delta) I^* &= 0, \end{aligned} \tag{5.11}$$

where

$$g(S^*, E^*, I^*) = \frac{cS^*I^* + dS^*D^*}{S^* + E^* + I^* + R^* + D^*}.$$

By (5.10), we have

$$g(S^*, E^*, I^*) = \frac{\mu\left(\gamma c + d(\mu + \delta)\right) S^* I^*}{\mu\gamma S^* + \mu\gamma E^* + \left(\mu^2 + \mu\gamma + \gamma\rho + \mu\delta\right) I^*}.$$

From the first and third equations of (5.11), we obtain

$$I^* = \frac{\Lambda - \mu S^*}{(\rho + \mu + \delta) e^{\mu\tau}}.$$

Clearly, if $S^* \geq \frac{\Lambda}{\mu}$ then $I^* \leq 0$, that is, the positive equilibrium does not exist. Therefore, if the positive equilibrium E_2 exists, then $0 < S^* < \frac{\Lambda}{\mu}$. Notice that $g(S^*, E^*, I^*) = (\rho + \mu + \delta) e^{\mu\tau} I^*$, it follows from the second equation of (5.11) that

$$E^* = \frac{\left(1 - e^{-\mu\tau}\right) g(S^*, E^*, I^*)}{\mu} = \frac{\left(1 - e^{-\mu\tau}\right) e^{\mu\tau} (\rho + \mu + \delta) I^*}{\mu} = \frac{\left(1 - e^{-\mu\tau}\right) (\Lambda - \mu S^*)}{\mu}.$$

Hence, both sides in the third equation in (5.11) must be functions of S^*. Define

$$\begin{aligned} G(S^*) &= g(S^*, E^*, I^*) - (\rho + \mu + \delta) e^{\mu\tau} I^* \\ &= \frac{\mu S^* \left(\Lambda - \mu S^*\right) \left(\gamma c + d(\mu + \delta)\right)}{\mu\gamma \left(\rho + \mu + \delta\right) e^{\mu\tau} S^* + \left(\gamma \left(e^{\mu\tau} - 1\right) \left(\rho + \mu + \delta\right) + \mu^2 + \mu\gamma + \rho\gamma + \mu\delta\right) \left(\Lambda - \mu S^*\right)} - 1 \\ &= \left(\Lambda - \mu S^*\right) \left(\frac{AS^*}{\left(B - C\right) S^* + D} - 1 \right) = 0 \end{aligned}$$

where

$$\begin{aligned} A &= \mu\left(\gamma c + d(\mu + \delta)\right) > 0, \quad B = \mu\gamma \left(\rho + \mu + \delta\right) e^{\mu\tau} > 0, \\ C &= \mu\left(\gamma(e^{\mu\tau} - 1)(\rho + \mu + \delta) + \mu^2 + \mu\gamma + \rho\gamma + \mu\delta\right) > 0, \end{aligned}$$

$$D = \Lambda\left(\gamma(e^{\mu\tau}-1)(\rho+\mu+\delta)+\mu^2+\mu\gamma+\rho\gamma+\mu\delta\right) > 0.$$

From

$$\Lambda - \mu S^* > 0, \quad \frac{AS^*}{(B-C)\,S^*+D} = 1,$$

it is easy to see that $S^* = \frac{D}{A-B+C}$ is unique and positive because $R_0 > 1$ and

$$A - B + C = \mu\gamma\,(\rho+\mu+\delta)\,(e^{\mu\tau}R_0 - 1) + \mu^2 + \mu\gamma + \rho\gamma + \mu\delta > 0.$$

Therefore we can obtain the unique endemic equilibrium, in terms of R_0,

$$\begin{aligned}
S^* &= \frac{\Lambda\left(\gamma(\rho+\mu+\delta)\,(e^{\mu\tau}-1)+\mu(\rho+\mu+\delta)+\gamma\delta\right)}{\mu\left(\gamma(\rho+\mu+\delta)\,(R_0e^{\mu\tau}-1)+\mu(\rho+\mu+\delta)+\gamma\delta\right)},\\
E^* &= \frac{\Lambda\gamma(\rho+\mu+\delta)\,(e^{\mu\tau}-1)\,(R_0-1)}{\mu\left(\gamma(\rho+\mu+\delta)\,(R_0e^{\mu\tau}-1)+\mu(\rho+\mu+\delta)+\gamma\delta\right)},\\
I^* &= \frac{\Lambda\gamma\,(R_0-1)}{\gamma(\rho+\mu+\delta)\,(R_0e^{\mu\tau}-1)+\mu(\rho+\mu+\delta)+\gamma\delta},\\
R^* &= \frac{\Lambda\gamma\rho\,(R_0-1)}{\mu\left(\gamma(\rho+\mu+\delta)\,(R_0e^{\mu\tau}-1)+\mu(\rho+\mu+\delta)+\gamma\delta\right)},\\
D^* &= \frac{\Lambda(\mu+\delta)\,(R_0-1)}{\gamma(\rho+\mu+\delta)\,(R_0e^{\mu\tau}-1)+\mu(\rho+\mu+\delta)+\gamma\delta}.
\end{aligned} \tag{5.12}$$

Furthermore, we have $N^* = S^* + E^* + I^* + R^* + D^* = R_0S^*$. □

Therefore, from Theorems 5.3.1-5.3.3, we understand that system (5.1) has at most two equilibria. More precisely, there is only disease-free equilibrium E_1 which is globally asymptotically stable when $R_0 < 1$ and an additional unique endemic equilibrium E_2 when $R_0 > 1$. When $R_0 = 1$, E_1 and E_2 coalesces into one point.

Next we discuss the local stability of the endemic equilibrium E_2. The linearized system of (5.1) at E_2 is

$$\begin{aligned}
\frac{dS(t)}{dt} &= a_{11}S(t) + a_{12}E(t) + a_{13}I(t) + a_{12}R(t) + a_{15}D(t),\\
\frac{dE(t)}{dt} &= -(a_{11}+\mu)S(t) - (a_{12}+\mu)E(t) - a_{13}I(t) - a_{12}R(t) - a_{15}D(t),\\
&\quad +e^{-\mu\tau}(a_{11}+\mu)S(t-\tau) + e^{-\mu\tau}a_{12}E(t-\tau) + e^{-\mu\tau}a_{13}I(t-\tau),\\
&\quad +e^{-\mu\tau}a_{12}R(t-\tau) + e^{-\mu\tau}a_{15}D(t-\tau),\\
\frac{dI(t)}{dt} &= -(\rho+\mu+\delta)I(t) - e^{-\mu\tau}(a_{11}+\mu)S(t-\tau) - e^{-\mu\tau}a_{12}E(t-\tau),\\
&\quad -e^{-\mu\tau}a_{13}I(t-\tau) - e^{-\mu\tau}a_{12}R(t-\tau) - e^{-\mu\tau}a_{15}D(t-\tau),\\
\frac{dR(t)}{dt} &= \rho I(t) - \mu R(t),
\end{aligned} \tag{5.13}$$

$$\frac{dD(t)}{dt} = (\mu+\delta)I(t) - \gamma D(t),$$

where

$$a_{11} = \frac{\Lambda(1-R_0)}{R_0 S^*} - \frac{\mu}{R_0},\ a_{12} = \frac{\Lambda-\mu S^*}{R_0 S^*},\ a_{13} = \frac{\Lambda-\mu S^*}{R_0 S^*} - \frac{c}{R_0},\ a_{15} = \frac{\Lambda-\mu S^*}{R_0 S^*} - \frac{d}{R_0}.$$

The characteristic equation of (5.13) is

$$\Theta(\lambda,\tau) = (\lambda+\mu)^2\left(P(\lambda,\tau) + Q(\lambda,\tau)e^{-\lambda\tau}\right) = 0, \tag{5.14}$$

where

$$P(\lambda,\tau) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3,$$
$$Q(\lambda,\tau) = B_1\lambda^2 + B_2\lambda + B_3,$$

with

$$A_1 = \rho+\mu+\delta+\gamma+\tfrac{\Lambda}{S^*},\quad A_2 = (\rho+\mu+\delta)\left(\gamma+\tfrac{\Lambda}{S^*}\right)+\tfrac{\gamma\Lambda}{S^*},\quad A_3 = \tfrac{\gamma(\rho+\mu+\delta)\Lambda}{S^*},$$
$$B_1 = \tfrac{-ce^{-\mu\tau}}{R_0},\quad B_2 = \tfrac{\mu e^{-\mu\tau}}{R_0}\left(\tfrac{\Lambda}{S^*}-\mu\right) - \left(\tfrac{c\mu e^{-\mu\tau}}{R_0} + \gamma(\rho+\mu+\delta)\right),$$
$$B_3 = \tfrac{\mu(\mu+\delta)e^{-\mu\tau}}{R_0}\left(\tfrac{\Lambda}{S^*}-\mu\right) - \tfrac{\gamma\delta e^{-\mu\tau}}{R_0}\left(\tfrac{\Lambda}{S^*}-\mu\right) - \gamma\mu(\rho+\mu+\delta).$$

Obviously, it suffices to consider the roots in

$$P(\lambda,\tau) + Q(\lambda,\tau)e^{-\lambda\tau} = 0. \tag{5.15}$$

Note that

$$\begin{aligned} P(0,\tau)+Q(0,\tau) &= A_3+B_3 \\ &= \left(\frac{\Lambda}{S^*}-\mu\right)\left(\gamma(\rho+\mu)+\gamma\delta - \frac{\gamma\delta e^{-\mu\tau}}{R_0} + \frac{\mu(\mu+\delta)e^{-\mu\tau}}{R_0}\right) > 0 \end{aligned} \tag{5.16}$$

since $R_0 > 1$, $e^{-\mu\tau} < 1$ and $0 < S^* < \frac{\Lambda}{\mu}$. Thus, $\lambda = 0$ is not a root in (5.15).

When $\tau = 0$

$$P(\lambda,0) + Q(\lambda,0) = \lambda^3 + (A_1+B_1)\lambda^2 + (A_2+B_2)\lambda + A_3 + B_3 = 0. \tag{5.17}$$

Due to $R_0 > \frac{c}{\rho+\mu+\delta}$ and $R_0 > 1$, we have

$$A_1+B_1 \geq \gamma + \frac{\Lambda}{S^*} > 0,\quad A_2+B_2 \geq \left(\frac{\Lambda}{S^*}-\mu\right)\left(\rho+\mu+\delta+\frac{\mu}{R_0}\right) > 0 \tag{5.18}$$

and $A_3+B_4 > 0$ from (5.16).

In addition,

$$\begin{aligned}(A_1+B_1)(A_2+B_2) &\geq \left(\frac{\Lambda}{S^*}-\mu\right)\left(\gamma+\frac{\Lambda}{S^*}\right)(\rho+\mu+\delta)\\ &\geq \left(\frac{\Lambda}{S^*}-\mu\right)\left(\gamma(\rho+\mu+\delta)+\frac{\mu(\mu+\delta)}{R_0}\right)\\ &\geq \left(\frac{\Lambda}{S^*}-\mu\right)\left(\gamma(\rho+\mu+\delta)+\frac{\mu(\mu+\delta)-\gamma\delta}{R_0}\right)=A_3+B_3.\end{aligned}$$

Thus, E_2 is locally asymptotically stable when $\tau=0$ by Routh-Hurwitz stability criterion. When $\tau>0$, we assume that $\lambda=i\omega$ $(\omega>0)$ is a root in (5.15), then by a straightforward calculation, ω must satisfy

$$\omega^6+\left(A_1^2-2A_2-B_1^2\right)\omega^4+\left(A_2^2-2A_1A_3-B_2^2+2B_1B_3\right)\omega^2+\left(A_3^2-B_3^2\right)=0. \tag{5.19}$$

Through the tedious computation, we have

$$A_1^2-2A_2-B_1^2>\gamma^2+\left(\frac{\Lambda}{S^*}\right)^2>0$$

and

$$\begin{aligned}A_2^2-2A_1A_3+2B_1B_3-B_2^2 &> 2\frac{(\rho+\mu+\delta)(\mu+\delta)(\gamma-\mu)e^{-\mu\tau}}{R_0}\left(\frac{\Lambda}{S^*}-\mu\right)\\ &\quad+2\frac{\mu ce^{-2\mu\tau}}{R_0^2}\left(\frac{\Lambda}{S^*}-\mu\right)+(\rho+\mu+\delta)^2\left(\left(\frac{\Lambda}{S^*}\right)^2-\mu^2\right)\\ &\quad+\left(\frac{\Lambda}{S^*}\right)^2(\gamma^2-\mu^2)+2\frac{c\gamma\delta e^{-2\mu\tau}}{R_0^2}\left(\frac{\Lambda}{S^*}-\mu\right)\end{aligned}$$

and

$$\begin{aligned}A_3^2-B_3^2 &= (A_3+B_3)(A_3-B_3)\\ &> (A_3+B_3)\left(\gamma\mu(\rho+\mu+\delta)+\frac{\gamma\delta e^{-\mu\tau}}{R_0}\left(\frac{\Lambda}{S^*}-\mu\right)+\frac{(\gamma-\mu)(\rho+\mu+\delta)\Lambda}{S^*}\right).\end{aligned}$$

By Descartes' Rule of Signs (see e.g., [173]), equation (5.19) has no positive roots ω^2 if all the coefficients are positive. Therefore, if

$$\begin{aligned}&2\frac{(\rho+\mu+\delta)(\mu+\delta)(\gamma-\mu)e^{-\mu\tau}}{R_0}\left(\frac{\Lambda}{S^*}-\mu\right)\\ &+2\frac{c\gamma\delta e^{-2\mu\tau}}{R_0^2}\left(\frac{\Lambda}{S^*}-\mu\right)+2\frac{\mu ce^{-2\mu\tau}}{R_0^2}\left(\frac{\Lambda}{S^*}-\mu\right)\end{aligned}$$

$$+(\rho+\mu+\delta)^2\left(\left(\frac{\Lambda}{S^*}\right)^2-\mu^2\right)+\left(\frac{\Lambda}{S^*}\right)^2(\gamma^2-\mu^2)>0 \tag{5.20}$$

and

$$\gamma\mu\left(\rho+\mu+\delta\right)+\frac{\gamma\delta e^{-\mu\tau}}{R_0}\left(\frac{\Lambda}{S^*}-\mu\right)+\frac{(\gamma-\mu)\left(\rho+\mu+\delta\right)\Lambda}{S^*}>0, \tag{5.21}$$

no any root with the form $\lambda = i\omega$ exists in (5.19). In other words, all the eigenvalues in (5.15) have negative real parts when $\tau > 0$. Therefore, E_2 is locally asymptotically stable and the variation of the time delay τ cannot destroy this stability.
Summarizing the above analysis, we have proven the following theorem.

Theorem 5.3.4. *When $R_0 > 1$, the endemic equilibrium E_2 of system (5.1) is locally asymptotically stable if (5.20) and (5.21) hold. A obviously local stability condition is $\gamma \geq \mu$.*

Remark 5.3.2. *Biologically, the condition $\gamma \geq \mu$ means the infectiousness rate after death in human corpses is higher than the natural death rate. In nature, this condition always holds because the Ebola virus survived period in body fluids of the infectious corpses ($\frac{1}{\gamma}$) is just for a few days [181], which is obviously shorter than the life expectancy.*

5.4 System persistence

Persistence is a significant property in population dynamics. In this section, we study the persistence of the system and discuss the global stability of the endemic equilibrium.

Theorem 5.4.1. *If $R_0 > 1$, the disease is uniformly persistent in (5.1), in the sense that, there is a positive number $\eta_0 > 0$ such that every solution in system (5.1) with $\phi \in \mathcal{X}_\xi$, $\phi_3(\theta) > 0$ or $\phi_5(\theta) > 0$ for some $\theta \in [-\tau, 0]$, satisfies*

$$\liminf_{t\to\infty}(I(t), D(t)) \geq (\eta_0, \eta_0).$$

Proof. Define

$$X^0 = \{\phi \in \mathcal{X}_\xi : \phi_3(\theta) > 0 \text{ or } \phi_5(\theta) > 0 \text{ for some } \theta \in [-\tau, 0]\}$$

and

$$X_0 = \mathcal{X}_\xi \setminus X_0 = \{\phi \in \mathcal{X}_\xi : \phi_3(\theta) = \phi_5(\theta) = 0 \text{ for all } \theta \in [-\tau, 0]\}.$$

In the following, we verify the conditions in [75, Theorem 4.2]. Let $\Phi(t)$, $t > 0$, be the solution semiflow of model system (5.1). Notice that X^0 is an open and dense set in $\mathcal{X}_\xi$

with $X^0 \cup X_0 = \mathcal{X}_\xi$ and $X^0 \cap X_0 = \emptyset$. It follows from Theorem 5.2.1 that $\Phi(t)X^0 \subset X^0$. Further, for any $\phi \in X_0$, it follows from the third and fifth equations in (5.1) that $I(t,\phi) \equiv 0$ and $D(t,\phi) \equiv 0$ for all $t \geq 0$. (i) has been confirmed in Theorem 5.2.1. Noticing that the bounds in Theorem 5.2.1 are all independent of initial functions, hence, condition (ii) is verified [207]. (iii) It follows from [104, Theorem 2.2.8] that $\Phi(t)$ is asymptotically smooth. For condition (iv) it is clear that $A = \{E_1\}$, and it is isolated. Thus, the covering M is $\{E_1\}$, which is acyclic because there is no orbit connecting E_1 to itself in X_0.

Finally, to verify (v), we prove the following claim.

Claim 1. $W^s(E_1) \cap X^0 = \emptyset$.

By contradiction, suppose that there exists a solution in X^0 such that

$$\lim_{t\to\infty} S(t) = \frac{\Lambda}{\mu}, \quad \lim_{t\to\infty} E(t) = 0,, \quad \lim_{t\to\infty} I(t) = 0, \quad \lim_{t\to\infty} R(t) = 0, \quad \lim_{t\to\infty} D(t) = 0.$$

Thus, for sufficiently small $\delta > 0$, there exists $t_1 > 0$ such that $\left|S(t) - \frac{\Lambda}{\mu}\right| < \delta$, $0 \leq E(t) < \delta$, $0 \leq I(t) < \delta$, $0 \leq R(t) < \delta$ and $0 \leq D(t) < \delta$ for $t > \tau + t_1$.

Fix a small $\epsilon > 0$. Since $\lim_{(S,E,I,R,D)\to E_1} \frac{S}{S+E+I+R+D} = 1$, in a neighborhood of E_1, we have

$$\left|\frac{S}{S+E+I+R+D} - 1\right| < \epsilon,$$

i.e.,

$$1 - \epsilon < \frac{S}{S+E+I+R+D} < 1 + \epsilon. \tag{5.22}$$

From the third and fifth equations in system (5.1) we obtain

$$\begin{aligned} \frac{dI(t)}{dt} &\geq c(1-\epsilon)e^{-\mu\tau}I(t-\tau) + d(1-\epsilon)e^{-\mu\tau}D(t-\tau) - (\rho+\mu+\delta)I(t), \\ \frac{dD(t)}{dt} &= (\mu+\delta)I(t) - \gamma D(t), \end{aligned} \tag{5.23}$$

for $t > \tau + t_1$.

Consider the following linear system

$$\begin{aligned} \frac{d\hat{I}}{dt} &= c(1-\epsilon)e^{-\mu\tau}\hat{I}(t-\tau) + d(1-\epsilon)e^{-\mu\tau}\hat{D}(t-\tau) - (\rho+\mu+\delta)\hat{I}(t) \\ \frac{d\hat{D}}{dt} &= (\mu+\delta)\hat{I}(t) - \gamma\hat{D}(t), \end{aligned} \tag{5.24}$$

for sufficiently small $\epsilon > 0$. Similar to the process in Theorem 5.3.2, we know the system (5.24) is quasimonotone and irreducible. Let $\lambda_2(\epsilon)$ be the principle eigenvalue of system

(5.24). First, we consider the following system of ordinary differential equations

$$\begin{aligned}\frac{dy_1}{dt} &= \left(c(1-\epsilon)e^{-\mu\tau} - (\rho+\mu+\delta)\right) y_1(t) + d(1-\epsilon)e^{-\mu\tau} y_2(t)\\ \frac{dy_2}{dt} &= (\mu+\delta)y_1(t) - \gamma y_2(t),\end{aligned} \tag{5.25}$$

The characteristic equation of (5.25) is

$$\Delta_2(\lambda) = \lambda^2 + \left(\rho+\mu+\delta+\gamma - ce^{-\mu\tau}\right)\lambda + \gamma(\rho+\mu+\delta) - (1-\epsilon)\left(c\gamma + d(\mu+\delta)\right)e^{-\mu\tau} = 0.$$

When $\epsilon = 0$, it is clear that $\Delta_2(0) < 0$ when $R_0 > 1$. Thus the principle eigenvalue is positive in (5.25). By [169, Corollary 5.5.2], we have $\lambda_2(0) > 0$. Thus, $\lambda_2(\epsilon) > 0$ for sufficiently small $\epsilon > 0$ due the continuity of λ_2.

Thus, there exists a solution $\vec{U_2}(t) = e^{\lambda_2(\epsilon)t}\hat{\phi}_0$ of (5.24), where $\hat{\phi}_0$ is the positive eigenfunction associated with $\lambda_2(\epsilon)$. Since $I(t) \gg 0$ and $D(t)) \gg 0$ for all $t > 0$ in (5.23), the comparison theory implies that there exists a small $\ell_2 > 0$ such that $(I(t), D(t))^T \geq \ell_2 e^{\lambda_2(\epsilon)t}\hat{\phi}_0$ for all $t \geq t_1$. Since $\lambda_2(\epsilon) > 0$, we have $\lim\limits_{t\to\infty}(I(t), D(t)) = +\infty$, This contradicts $\lim\limits_{t\to\infty}(I(t), D(t)) = (0,0)$.

Now, by [75, Theorem 4.2], there exists a $\eta_0 > 0$ such that $\liminf\limits_{t\to\infty} d\left(\Phi(t)\phi, X_0\right) \geq \eta_0$ for any $\phi \in X^0$ implying that the I and D components of the solution with initial function $\phi \in X^0$ satisfy

$$\liminf_{t\to\infty}(S(t), L(t)) \geq (\eta_0, \eta_0).$$

□

When $R_0 > 1$, we know E_2 exists and is at least locally asymptotic stable when $\gamma \geq \mu$. Moreover, we have the following result about its global stability.

Theorem 5.4.2. *Assume that $\mu = \gamma$ in system (5.1). If $R_0 > 1$, then for any $\phi \in \mathcal{X}_\xi$ with $\phi_3(\theta) > 0$ or $\phi_5(\theta) > 0$ for some $\theta \in [-\tau, 0]$ the endemic equilibrium $E_2 = (S^*, E^*, I^*, R^*, D^*)$ is globally asymptotically stable.*

Proof. When $\mu = \gamma$, we have

$$\frac{dN(t)}{dt} = \Lambda - \mu N(t). \tag{5.26}$$

Since all solutions of system (5.1) are bounded (see Theorem 5.2.1) and $N^* = \frac{\Lambda}{\mu}$ is globally asymptotically stable for system (5.26), we consider the following decoupled equations in the limiting system (see, e.g., [131, 186, 202])

$$\frac{dW_1(t)}{dt} = \Lambda - c'W_1(t)W_3(t) - d'W_1(t)W_5(t) - \mu W_1(t),$$

$$\begin{aligned}\frac{dW_3(t)}{dt} &= c'e^{-\mu\tau}W_1(t-\tau)W_3(t-\tau)+d'e^{-\mu\tau}W_1(t-\tau)W_5(t-\tau)\\ &\quad -(\rho+\mu+\delta)W_3(t),\\ \frac{dW_5(t)}{dt} &= (\mu+\delta)W_3(t)-\mu W_5(t),\end{aligned} \tag{5.27}$$

where $c'=\frac{c\mu}{\Lambda}$, $d'=\frac{d\mu}{\Lambda}$ and W_1, W_3, W_5 are related to S, I, D, respectively.

Let $C_1^+ = C\left([-\tau,0],\mathbb{R}_+^3\right)$. By similar arguments as in Theorem 5.2.1, it follows that for any $\psi\in C_1^+$, all the solutions of (5.27) exist globally, and are unique, nonnative and ultimately bounded, thus $\limsup_{t\to\infty} W_1(t)\le\frac{\Lambda}{\mu}$, $\limsup_{t\to\infty} W_3(t)\le M$ and $\limsup_{t\to\infty} W_5(t)\le\frac{(\mu+\delta)M}{\gamma}$. That is, C_1^+ is a positively invariant for system (5.27). Further, the system (5.27) is uniformly persistent by the same arguments in Theorem 5.4.1. That is, there is a positive number $\eta_1>0$ such that

$$\liminf_{t\to\infty}\left(W_3(t,\psi),W_5(t,\psi)\right)\ge(\eta_1,\eta_1)$$

for any $\psi=(\psi_1,\psi_2,\psi_3)\in C_1^+$ with $\psi_2(\theta)>0$ or $\psi_3(\theta)>0$ for some $\theta\in[-\tau,0]$.

To lift the dynamics of the limiting system (5.27) to the main system (5.1), define

$$\begin{aligned}\kappa=\{(\psi_1,\psi_2,\psi_3)\in C_1^+ \;:\; &\lim_{n\to\infty}\left(W_1(t_n+\theta),W_3(t_n+\theta),W_5(t_n+\theta)\right)\\ =\;&(\psi_1,\psi_2,\psi_3)\ \text{ for some } t_n\to\infty,\theta\in[-\tau,0]\}.\end{aligned}$$

It is easy to see that κ is a nonempty and compact subset of C_1^+. By the continuous-time version of [218, Lemma 1.2.2], we know that κ is an internally chain transitive set for the solution semiflow of (5.27) on C_1^+.

Let $(W_1(t,\psi),W_3(t,\psi),W_5(t,\psi))=(W_1(t),W_3(t),W_5(t))$ for any $\psi\in C_1^+$ with $\psi_2(\theta)>0$ or $\psi_3(\theta)>0$ for some $\theta\in[-\tau,0]$ and $E_3=(W_1^*,W_3^*,W_5^*)$ be the positive equilibrium point in (5.27). We construct a Lyapunov function

$$V(t)=V_1(t)+c'W_1^*W_3^*V_2(t)+d'W_1^*W_5^*V_3(t),$$

where

$$\begin{aligned}V_1(t) &= W_1(t)-W_1^*\ln\left(\frac{W_1(t)}{W_1^*}\right)+e^{\mu\tau}\left(W_3(t)-W_3^*\ln\left(\frac{W_3(t)}{W_3^*}\right)\right)\\ &\quad +\frac{d'W_1^*W_5^*}{(\mu+\delta)W_3^*}\left(W_5(t)-W_5^*\ln\left(\frac{W_5(t)}{W_5^*}\right)\right),\\ V_2(t) &= \int_{t-\tau}^{t}H\left(\frac{W_1(\theta)W_3(\theta)}{W_1^*W_3^*}\right)ds,\end{aligned}$$

$$V_3(t) = \int_{t-\tau}^{t} H\left(\frac{W_1(\theta)W_5(\theta)}{W_1^*W_5^*}\right)ds,$$

with the function $H(x) = x - 1 - \ln(x)$. It is easy to check that $H(x) \geq 0$ for all $x \in (0, \infty)$ and $H(x) = 0$ if and only if $x = 1$. By the boundedness and persistence of solutions, we know that $V(t)$ is well defined for large t.

From (5.27), it is easy to see that E_3 satisfies the following relations

$$\Lambda - \mu W_1^* = c'W_3^*W_1^* + d'W_1^*W_5^*, \quad e^{\mu\tau}(\rho + \mu + \delta) = c'W_1^* + \frac{d'W_1^*W_5^*}{W_3^*}, \quad \mu = \frac{(\mu+\delta)W_3^*}{W_5^*}.$$

To calculate $\frac{dV_1}{dt}$ along (5.27), we have, $\frac{dV_1}{dt} = \frac{\partial V_1}{\partial W_1}\frac{dW_1}{dt} + \frac{\partial V_1}{\partial W_3}\frac{dW_3}{dt} + \frac{\partial V_1}{\partial W_5}\frac{dW_5}{dt}$ with

$$\begin{aligned}
\frac{\partial V_1}{\partial W_1}\frac{dW_1}{dt} &= \left(1 - \frac{W_1^*}{W_1(t)}\right)\left(\Lambda - \mu W_1(t) - c'W_1(t)W_3(t) - d'W_1(t)W_5(t)\right) \\
&= \mu W_1^*\left(1 - \frac{W_1^*}{W_1(t)}\right)\left(1 - \frac{W_1(t)}{W_1^*}\right) \\
&\quad + c'W_1^*W_3^*\left(1 - \frac{W_1(t)W_3(t)}{W_1^*W_3^*} - \frac{W_1^*}{W_1(t)} + \frac{W_3(t)}{W_3^*}\right) \\
&\quad + d'W_1^*W_5^*\left(1 - \frac{W_1(t)W_5(t)}{W_1^*W_5^*} - \frac{W_1^*}{W_1(t)} + \frac{W_5(t)}{W_5^*}\right), \\
\frac{\partial V_1}{\partial W_3}\frac{dW_3}{dt} &= \left(1 - \frac{W_3^*}{W_3(t)}\right)\left(c'W_1(t-\tau)W_3(t-\tau) + d'W_1(t-\tau)W_5(t-\tau)\right. \\
&\qquad \left. - e^{\mu\tau}(\rho + \mu + \delta)W_3(t)\right) \\
&= c'W_1^*W_3^*\left(1 - \frac{W_3(t)}{W_3^*} + \frac{W_1(t-\tau)W_3(t-\tau)}{W_1^*W_3^*}\right. \\
&\qquad \left. - \frac{W_1(t-\tau)W_3(t-\tau)}{W_1^*W_3(t)}\right) \\
&\quad + d'W_1^*W_5^*\left(1 - \frac{W_3(t)}{W_3^*} + \frac{W_1(t-\tau)W_5(t-\tau)}{W_1^*W_3^*}\right. \\
&\qquad \left. - \frac{W_1(t-\tau)W_5(t-\tau)W_3^*}{W_1^*W_5^*W_3(t)}\right), \\
\frac{\partial V_1}{\partial W_5}\frac{dW_5}{dt} &= \left(1 - \frac{W_5^*}{W_5(t)}\right)\left((\mu+\delta)W_3(t) - \gamma W_5(t)\right) \\
&= d'W_1^*W_5^*\left(1 + \frac{W_3(t)}{W_3^*} - \frac{W_5(t)}{W_5^*} - \frac{W_5^*W_3(t)}{W_3^*W_5(t)}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{dV_2}{dt} &= \frac{W_1(t)W_3(t)}{W_1^*W_3^*} - \frac{W_1(t-\tau)W_3(t-\tau)}{W_1^*W_3^*} + \ln\left(\frac{W_1(t-\tau)W_3(t-\tau)}{W_1(t)W_3(t)}\right),\\ \frac{dV_3}{dt} &= \frac{W_1(t)W_5(t)}{W_1^*W_5^*} - \frac{W_1(t-\tau)W_5(t-\tau)}{W_1^*W_5^*} + \ln\left(\frac{W_1(t-\tau)W_5(t-\tau)}{W_1(t)W_5(t)}\right).\end{aligned}$$

Thus,

$$\begin{aligned}\left.\frac{dV}{dt}\right|_{(5.27)} &= \mu W_1^*\left(1-\frac{W_1^*}{W_1(t)}\right)\left(1-\frac{W_1(t)}{W_1^*}\right)\\ &\quad + c'W_1^*W_3^*\left(2-\frac{W_1^*}{W_1(t)}-\frac{W_1(t-\tau)W_3(t-\tau)}{W_1^*W_3(t)}+\ln\left(\frac{W_1(t-\tau)W_3(t-\tau)}{W_1(t)W_3(t)}\right)\right)\\ &\quad + d'W_1^*W_5^*\left(3-\frac{W_1^*}{W_1(t)}-\frac{W_5^*W_3(t)}{W_3^*W_5(t)}-\frac{W_1(t-\tau)W_5(t-\tau)W_3^*}{W_1^*W_5^*W_3(t)}\right.\\ &\quad \left.+\ln\left(\frac{W_1(t-\tau)W_5(t-\tau)}{W_1(t)W_5(t)}\right)\right).\end{aligned}$$

Notice that

$$\left(1-\frac{W_1^*}{W_1(t)}\right)\left(1-\frac{W_1(t)}{W_1^*}\right)\le 0, \tag{5.28}$$

$$\begin{aligned}&2-\frac{W_1^*}{W_1(t)}-\frac{W_1(t-\tau)W_3(t-\tau)}{W_1^*W_3(t)}+\ln\left(\frac{W_1(t-\tau)W_3(t-\tau)}{W_1(t)W_3(t)}\right)\\ &\quad = -H\left(\frac{W_1^*}{W_1(t)}\right)-H\left(\frac{W_1(t-\tau)W_3(t-\tau)}{W_1^*W_3(t)}\right)\le 0\end{aligned} \tag{5.29}$$

and

$$\begin{aligned}&3-\frac{W_1^*}{W_1(t)}-\frac{W_5^*W_3(t)}{W_3^*W_5(t)}-\frac{W_1(t-\tau)W_5(t-\tau)W_3^*}{W_1^*W_5^*W_3(t)}+\ln\left(\frac{W_1(t-\tau)W_5(t-\tau)}{W_1(t)W_5(t)}\right)\\ &\quad = -H\left(\frac{W_1^*}{W_1(t)}\right)-H\left(\frac{W_5^*W_3(t)}{W_3^*W_5(t)}\right)-H\left(\frac{W_1(t-\tau)W_5(t-\tau)W_3^*}{W_1^*W_5^*W_3(t)}\right)\le 0.\end{aligned} \tag{5.30}$$

Therefore, $\left.\frac{dV}{dt}\right|_{(5.27)} \le 0$.
Let $G=\{(W_1(t),W_3(t),W_5(t)) : \left.\frac{dV}{dt}\right|_{(5.27)}=0\}$ and $\tilde{G}$ be the largest invariant set in G.

Claim 2. $\tilde{G}=\{E_3\}$.

From (5.28-5.30), it is easy to see that $\left.\frac{dV}{dt}\right|_{(5.27)}=0$ if and only if

$$\begin{aligned}&W_1(t)=W_1^*, && W_1(t-\tau)W_3(t-\tau)=W_1^*W_3(t),\\ &W_5^*W_3(t)=W_3^*W_5(t), && W_1(t-\tau)W_5(t-\tau)W_3^*=W_1^*W_5^*W_3(t).\end{aligned}$$

From $W_5^* W_3(t) = W_3^* W_5(t)$ and the third equation in (5.27), we have $\frac{dW_5(t)}{dt} = 0$, implying that $W_5(t)$ is a constant, so is $W_3(t)$. Since $W_1(t) = W_1^*$, by the uniqueness of the positive equilibrium, we then conclude that $W_3(t) = W_3^*$ and $W_5(t) = W_5^*$. Therefore, $\tilde{G} = \{E_3\}$. Thus the claim holds.

By the LaSalle invariance principle (see e.g. [74, Theorem 5.3.1]), E_3 is globally attractive, that is,

$$\lim_{t\to\infty} (W_1(t,\psi), W_3(t,\psi), W_5(t,\psi)) = (W_1^*, W_3^*, W_5^*)$$

for any $\psi \in C_1^+$ with $\psi_2(\theta) > 0$ or $\psi_3(\theta) > 0$ for some $\theta \in [-\tau, 0]$. By the theory of chain transitive sets (see e.g. [218]) and the internally chain transitive set κ for the solution semiflow of (5.27) on C_1^+, we can lift the global attractivity for system (5.27) to the system (5.1). Thus,

$$\lim_{t\to\infty} (S(t,\phi), I(t,\phi), D(t,\phi)) = (S^*, I^*, D^*)$$

for any $\phi \in \mathcal{X}_\xi$ with $\phi_2(\theta) > 0$ or $\phi_3(\theta) > 0$. Using the theory of asymptotically autonomous semiflows (see e.g., [185]), it follows that

$$\lim_{t\to\infty} u(t,\phi) = (S^*, E^*, I^*, R^*, D^*).$$

That is, $(S^*, E^*, I^*, R^*, D^*)$ is globally attractive, which, together with the local stability of E_2 established in Theorem 5.3.4, confirms the global asymptotic stability of the endemic equilibrium E_2. This completes the proof. □

5.5 Numerical simulations

To illustrate our theoretical results obtained in the previous sections and to explore the disease transmission with the variation of seasonality, we provide numerical simulations of model (5.1) with parameters values taken from the literature.

Note that the value of Λ and μ can be calculated by using the life expectancy, the total human population and applying

$$\mu = \frac{1}{\text{The life expectancy} \times 365}$$

and

$$\Lambda = \mu \times \text{The total human population.}$$

First, when we choose the latent period $\tau = 5$, $\Lambda = 554.80$, $\mu = 0.000047$, $c = 0.05$, $d = 0.20$, $\delta = 0.130$, $\gamma = 0.5$ and $\rho = 0.05$, then the basic reproduction number $R_0 = 0.567 < 1$,

Parameter	Dimension	Values	Source
Λ	Humans $\times$ Day^{-1}	$189 - 555$	-
c	Day^{-1}	$0.05 - 0.25$	[205]
d	Day^{-1}	$0.20 - 0.71$	[205]
μ	Day^{-1}	$0.000044 - 0.000060$	-
δ	Day^{-1}	$0.084 - 0.135$	[16]
γ	Day^{-1}	$0.17 - 0.50$	[205]
ρ	Day^{-1}	$0.045 - 0.095$	[16]
τ	Day	$2 - 21$	[184]

Table 5.1 Parameters value for the model (5.1).

implying that the disease-free equilibrium $E_1 = (1.18 \times 10^7, 0, 0, 0, 0)$ is globally asymptotically stable which is shown in Figure 5.3. To see the influence of the transition rate d, we only increase the value of d to 0.5007 and 0.75 and all the other parameters are the same as those used in Figure 5.3, which produces $R_0 = 1$ and $R_0 = 1.3608 > 1$, respectively. When $R_0 = 1$, theoretically we may not provide the stability without additional information, while numerically, Figure 5.4a shows that it is possible for the disease-free equilibrium to be locally asymptotically stable. When $R_0 = 1.3608 > 1$, from Figure 5.4b, we can observe the long-term dynamics behavior displaying the persistence of the infectious and the local stability of the endemic equilibrium point $E_2 = (5.12 \times 10^6, 1566, 1740, 1.9 \times 10^6, 452)$. To see the influence of the latent period, we keep all the parameters same (except τ) as in Figure 5.4b and increase τ from 5 to 12 and 21 consequently, the basic reproductive number R_0 decreases from 1.3608 to 1.3604 and 1.3598, respectively. Figure 5.5 shows the decreasing of infectious population in the first epidemic peak in Figure 5.4b with the same initial condition, showing the effect of raising the latent period.

Next, to demonstrate that the endemic equilibrium is globally asymptotically stable when $\mu = \gamma$ and $R_0 > 1$ obtained in Theorem 5.4.2, we take $\mu = \gamma = 0.125$ as the average of $\mu = 0.00006$ and $\gamma = 0.25$, and choose $\Lambda = 362.84$, $\tau = 10$, $c = 0.25, d = 0.71, \delta = 0.084$ and $\rho = 0.090$, then $R_0 = 1.377 > 1$, and choose different initial history functions as $\phi^I(\theta) = (10^6 \sin\theta + 6 \times 10^6, -6.3\theta + 150, 5\theta + 80, 12 - \theta, 2e^\theta + 12)$ and $\phi^{II}(\theta) = (10^6 e^\theta + 5.5 \times 10^6, \theta + 50, 2\theta \sin\theta + 20, -0.3\theta + 10, -7\sin\theta + 8)$, $\theta \in [-10, 0]$ (see Figure 5.6), we can observe, from Figure 5.7, that the endemic equilibrium $E_2 = (2171.5, 521.72, 87.58, 63.06, 146.43)$ is globally asymptotically stable even the initial conditions are not constants.

Now, we compare our model prediction with real data of Ebola infectious/death cases in Guinea. In Figure 5.8, the blue line shows the model prediction and the red circles shows the real data, the data fitting is based on the data from Jan 26, 2015 to December 29, 2015, see [183]. For the period from January 2015 to June 2015, the cumulative number of Ebola

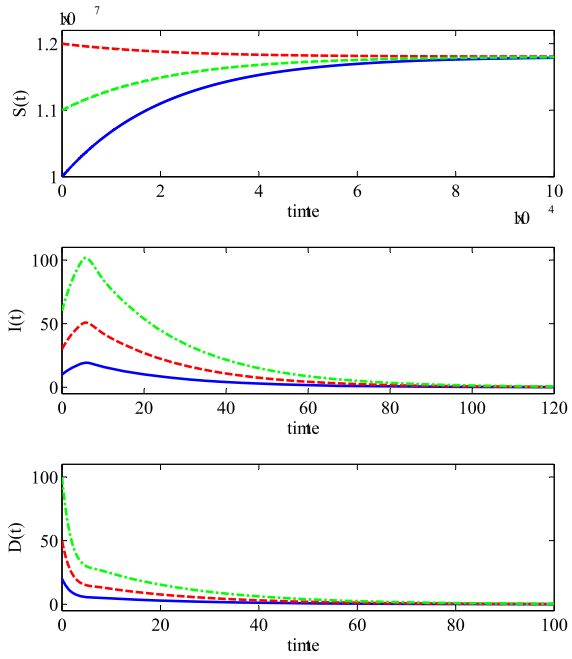
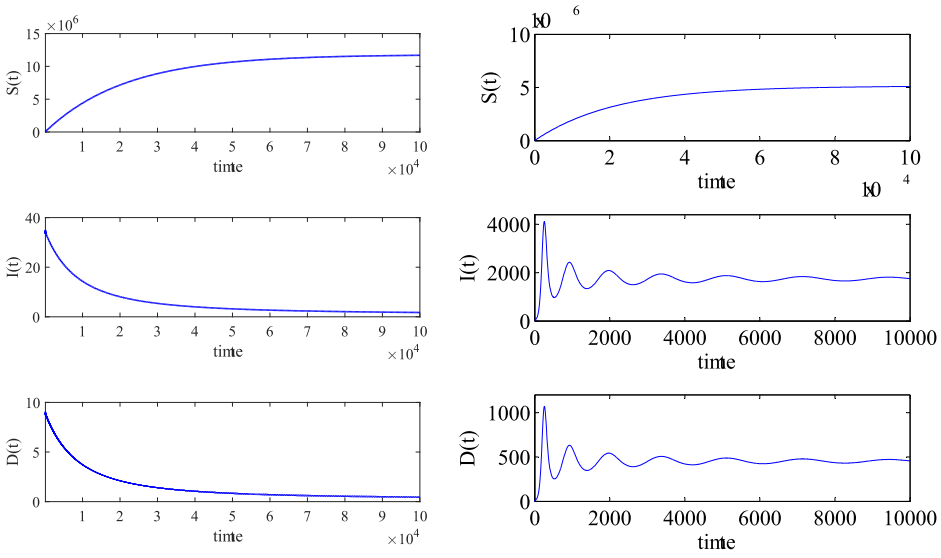


Figure 5.3 The disease-free equilibrium is globally asymptotically stable under different initial condition. $d = 0.20$, $R_0 = 0.567 < 1$.

infectious/death cases is increasing, then it becomes steady until December 2015, due to the interruption of Ebola transmission by WHO, by reducing the risk of human-to-human transmission from direct or close contact with people with Ebola symptoms, particularly with their bodily fluids (reducing c); and applying outbreak containment measures, including prompt and safe burial of the dead (reducing d), identifying people who may have been in contact with someone infected with Ebola and monitoring their health for 21 days (extending τ), see [184]. In January 2016 WHO declared the end of the outbreak of Ebola virus disease. We can see that the model is predicting the stabilization of the number of Ebola infectious/death cases. The simulation result has good agreement with the real data.

Finally, to discuss the effect of climate change (e.g. seasonality) on Ebola transmission. First, we choose $\Lambda = 189.75$, $\mu = 0.000044$, $\tau = 6$, $\delta = 0.125, \gamma = 0.17$, $\rho = 0.049$, $c = 0.15$ and $d = 0.45$, the basic reproduction number $R_0 = 2.763 > 1$, implying that the endemic equilibrium point $(S^*, E^*, I^*, R^*, D^*) = (595000\ , 981.28,\ 939.57,\ 1046300,\ 691.1)$ is locally asymptotically stable (Theorem 5.3.4), Figure 5.9 gives an good agreement. Now, instead of constant transmission rates c and d, we take two periodic time-dependent transmission rates $c_1(t) = 0.15(1 + \sin 2\pi t)$ and $d_1(t) = 0.45(1 + \cos 2\pi t)$. Notice that the average value of $c_1(t)$ and $d_1(t)$ over the interval $[0, \infty)$ are $c = 0.15$ and $d = 0.45$ (same as in Figure 5.9),



(b) The disease-free equilibrium is locally asymptotically stable. $d = 0.5007$, $R_0 = 1$.

(c) The endemic equilibrium is locally asymptotically stable. $d = 0.75$, $R_0 = 1.361 > 1$.

Figure 5.4 Time series for $S(t)$, $I(t)$ and $D(t)$.

respectively, then we can observe in Figure 5.10 that the infectious population (green line) becomes oscillatory around the infectious values I^* and D^* (red line). Biologically, as the temperature changes seasonly, the number of infectious population is varied. Actually, if we keep all the parameters same (except c and d) as in Figure 5.4c and change c and d by a periodic time-dependent transmission rates $c_2(t) = 0.05(1 + \sin 2\pi t)$ and $d_2(t) = 0.75(1 + \cos 2\pi t)$ with the average value over the interval $[0, \infty)$ are $c = 0.05$ and $d = 0.75$ (same as in Figure 5.4), respectively, the dynamical behavior is given in Figure 5.11 which is similar to Figure 5.10, i.e. the infectious population becomes oscillatory around the infectious values that arise from constant transmission rates. We can see that the behavior of the model with constant transmission rates or periodic time-dependent transmission rates is similar, in the sense that, when the endemic equilibrium point exists with constant transmission rates, a positive periodic solution exists with periodic time-dependent transmission rates.

5.6 Discussion

Ebola virus disease is a disease of humans and other non-human primates (gorillas, chimpanzees and duikers) caused by Ebola virus (EBOV) which is belong to the family Filoviridae. The virus is originate in fruit bats and jump to humans through an intermediate

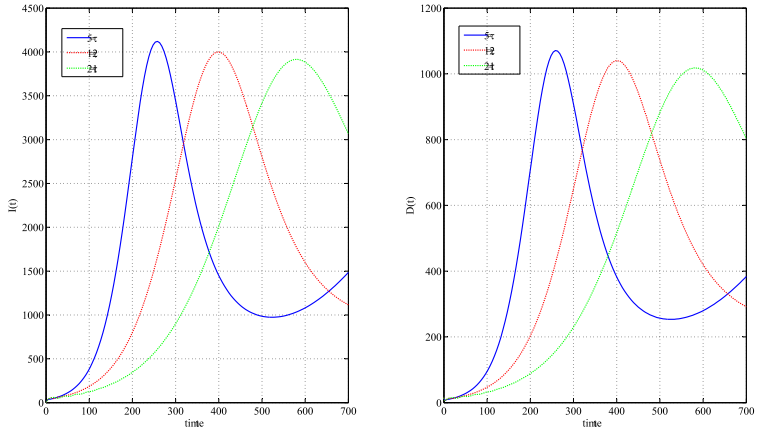
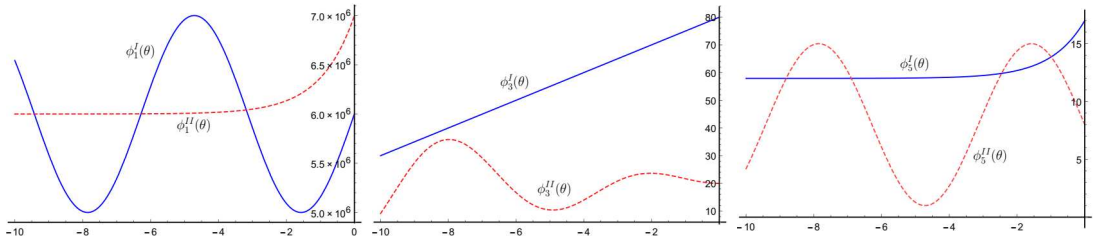


Figure 5.5 The effect of raising the latent period.



$\phi_1^I(\theta)$ and $\phi_1^{II}(\theta)$ $\phi_3^I(\theta)$ and $\phi_3^{II}(\theta)$ $\phi_5^I(\theta)$ and $\phi_5^{II}(\theta)$

Figure 5.6 The initial history functions $\phi^I(\theta)$ and $\phi^{II}(\theta)$.

animal, such as gorillas. Ebola first appeared in 1976 in two outbreaks, one in Sudan, and the other in Democratic Republic of Congo, since then it has resurfaced in Africa several times. The last outbreak in 2014 in West Africa (Guinea, Liberia and Sierra Leone) was the largest and most complex outbreak. In this chapter, we have considered an SEIRD model where the population of the humans is described by a system of susceptible, exposed, infectious, recovered individuals and infected corpses who are nonetheless infectious.

In epidemiology, the basic reproduction number the basic reproduction number R_0 is fundamental concept. Mathematically, we have calculated R_0 and discussed the global dynamics of the model with respect to R_0. Furthermore, we have investigated the relation between the basic reproduction number R_0 and the parameters τ and d. We have shown that R_0 is a decreasing function for τ and increasing for d. From disease control point of view, to reduce the spread of the Ebola, we need to extend the duration of latent and/or dispose of human remains by cremation or burial.

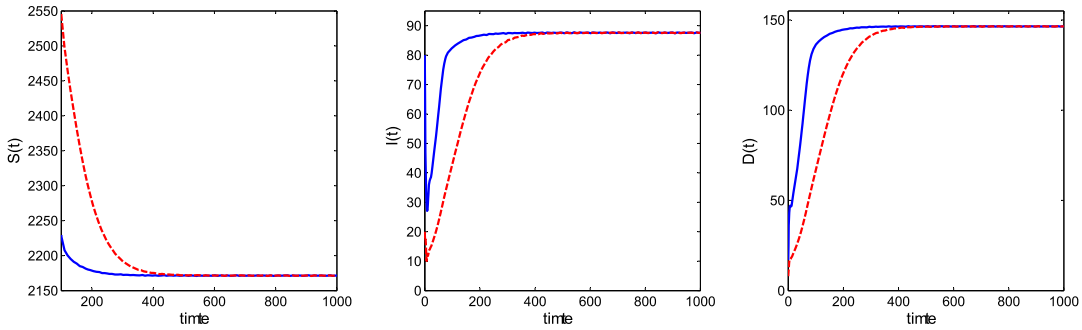


Figure 5.7 The endemic equilibrium is globally asymptotically stable when $\mu = \gamma$ under different initial history function. $R_0 = 1.377 > 1$.

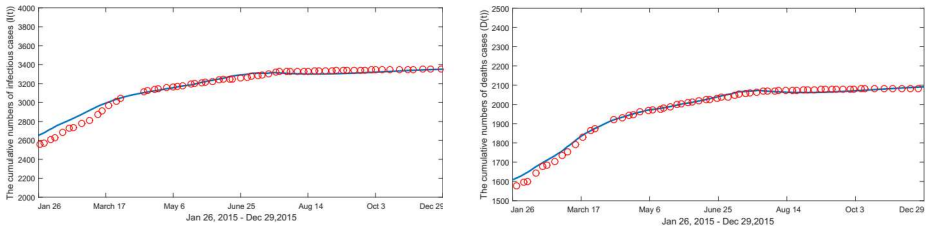


Figure 5.8 Predictions (blue line) and data fitting of the cumulative number of Ebola infectious/death cases in Guinea (red circles). Parameters: $c = 0.23, d = 0.28$, $\mu = 0.000048, \Lambda = 554.80$, $\delta = 0.12$, $\tau = 3.5$, $\gamma = 0.25$ and $\rho = 0.065$.

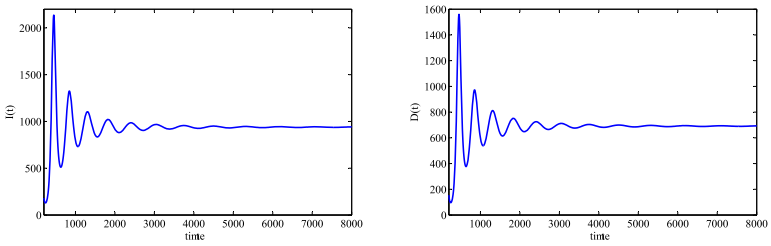


Figure 5.9 Time series for $I(t)$ and $D(t)$ when $c = 0.15$ and $d = 0.45$. The endemic equilibrium is locally asymptotically stable. $R_0 = 2.763 > 1$.

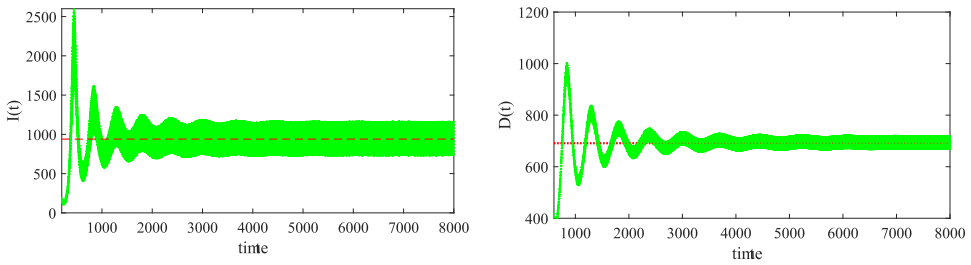


Figure 5.10 Time series for $I(t)$ and $D(t)$ with periodic functions $c_1(t) = 0.15(1 + \sin 2\pi t)$ and $d_1(t) = 0.45(1 + \cos 2\pi t)$.

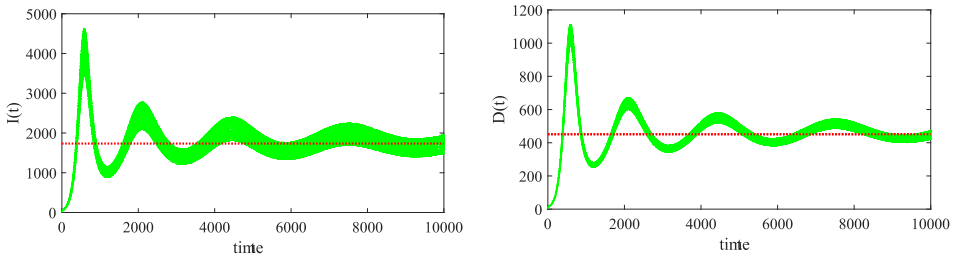


Figure 5.11 Time series for $I(t)$ and $D(t)$ with periodic functions $c_2(t) = 0.05(1 + \sin 2\pi t)$ and $d_2(t) = 0.75(1 + \cos 2\pi t)$.

Chapter 6

A Periodic Disease Transmission Model with Asymptomatic Carriage and Latency Periods

6.1 Introduction

For certain infectious diseases, there are individuals who have been infected and are able to transmit their illness but do not display any symptoms. These individuals are called "asymptomatic carriers". They are a potential source for transmission of some diseases including Typhoid Fever, HIV, Epstein-Barr Virus (EBV), Clostridium difficile and Chlamydia, etc. Since the asymptomatic carriers are common and invisible, they can have serious long term health consequences. For example, in sexually transmitted diseases (STD), Chlamydia is the most common treatable STD, but three-quarters of all women, and half of all men, with chlamydia have no STD symptoms. Some scientists call STDs as "the hidden epidemic" [38]. According to World Health Organization (WHO) report, up to $5-10\%$ of population may be asymptomatic carriers of Meningococcal disease, which is spread by person-to-person contact through respiratory droplets of infected people. Although asymptomatic carriage of some diseases is common in patients staying in health care facilities, the importance of asymptomatic carriers with regard to disease transmission is unclear. In [153], the authors suggest that asymptomatic carriers of epidemic and non-epidemic C. difficile strains have the potential to contribute significantly to disease transmission in long-term care facilities, some clinical factors, may be predictive of asymptomatic carriage. In [101], asymptomatic carriers of Plasmodium falciparum serve as a reservoir of parasites for malaria transmission. Identification and treatment of asymptomatic carriers within a region may reduce the

parasite reservoir and influence malaria transmission in that area.

Over the last decades, mathematical models have been extensively used to study the dynamics of infectious diseases transmission. After 1990, because of the public health significance, the influence of carriers on the dynamics of infectious diseases transmission has received research attention in the mathematical modeling literature [192]. Carriers have been incorporated in a variety of epidemic models, e.g., hepatitis B virus with carriers was discussed in [130, 215], meningococcal meningitis was investigated in [91, 191], a general mathematical models that incorporates disease carriers were studied in [63, 96, 100, 134]. Time delays have been included in a variety of epidemic models such as SEIR, SEIRS and SIR, etc [10, 23, 25, 40, 41, 69, 82, 88, 159, 212]. For instance, in [23], the authors presented a two delays SEIR epidemic model, one delay is the latent period and the second delay is the time taken by an infectious individual to be removed from the infection. In [159], the authors studied a delayed Ross-Macdonald model for Malaria Transmission. A disease transmission model of SEIRS type with distributed delays in latent and temporary immune periods was discussed in [212].

In the real world, periodicity and other oscillatory behaviors have been observed in the incidence of many infectious diseases, including measles, influenza and chickenpox, etc. The appearance of such oscillatory behaviors is mostly due to seasonal changes in environmental factors such as temperature and humidity [111, 161]. Many researchers in the field have shown great interest in seasonal fluctuations in epidemic models with and without delay [117, 119, 120, 214]. For instance, a periodic SIS epidemic model with maturation delay was investigated in [119]. In [120], a malaria transmission model with periodic birth rate and age structure for the vector population was proposed. In [214], the authors studied a nonautonomous SEIRS epidemic model.

To the best of our knowledge, all the mathematical models study involving disease carriers e.g., [63, 91, 96, 100, 130, 134, 191], are all given by autonomous or nonautonomous ordinary differential equations which neglected the effect of latent period. However, the latent period has a profound effect on the generation time, and hence epidemic growth [135]. In the present work, we propose a periodic disease transmission model with asymptomatic carriage and latency periods. In addition to normal latent period in exposed class (see e.g., [88, 125, 212]), we introduce another time delay to represent the time-lag that asymptomatic carriers take to develop the disease symptoms (the asymptomatic carriage latency period).

The main goal of this work is to analyze a seasonal fluctuation infectious disease model with a general nonlinear incidence rate function and with consideration of the asymptomatic carriage and latent periods. Based on the proposed mathematical model, we derive the basic reproduction ratio $\mathcal{R}_0$ mathematically and study the threshold dynamics with respect to $\mathcal{R}_0$ under biologically reasonable hypotheses. The rest of this chapter is organized as follows: in

Section 6.2, we propose a periodic compartmental epidemic model with nonlinear incidence rate function and two constant delays by stage-structure derivation. In Section 6.3, we discuss the well-posedness property by verifying the non-negativity and boundedness of the solutions with reasonable initial data. The basic reproduction ratio $\mathcal{R}_0$ is calculated in Section 6.4. In Section 6.5, we establish the threshold dynamics for the system in terms of the basic reproduction ratio by proving the global attractivity of the disease-free state when $\mathcal{R}_0 < 1$, coexistence of endemic state and disease persistence when $\mathcal{R}_0 > 1$. In Section 6.6, we explore the uniqueness of the endemic state when all coefficients are constants. In Section 6.7, we present numerical algorithm to calculate the basic reproduction ratio $\mathcal{R}_0$; do a case study regarding the meningococcal meningitis disease transmission; and discuss the sensitivity of $\mathcal{R}_0$ (the time-average basic reproduction ratio $[\mathcal{R}_0]$) with respect to the latent periods and carriers related parameters.

6.2 Model derivation

We consider a compartmental model with two latent periods where the total population of size $N(t)$ is divided into six categories: (i) Two disease-free classes: susceptible ($S(t)$) and recovered ($R(t)$); (ii) Four disease-related classes:

- Exposed class ($E(t)$): individuals are infected but not yet infectious. This class is related to the latent period of the infectious diseases [125, 212];
- Asymptomatic carrier class ($C(t)$): individuals are infectious but not showing any disease symptoms [91, 125]. In this class, individuals are unaffected by the disease themselves;
- Carrier-latent class ($E^c(t)$): adopting the idea behind the stage E, we introduce this stage to represent individuals who are developing the disease symptoms. This class is related to the time period that asymptomatic carriers take to show symptoms;
- Ill class ($I(t)$): individuals are infectious and showing disease signs and symptoms [91, 125].

We assume that the new infected susceptible individual remains in an exposed class for a certain latent period τ_1, then part of them, with the proportion p, move into asymptomatic carrier class and others (proportion of $1-p$) become ill, with the infection rate (incidence rate) function $f(t, S(t), C(t), I(t))$, which depends on time t and variables $S(t)$, $C(t)$ and $I(t)$. Usually, asymptomatic carriers take time to show symptoms. We called this time-lag as the asymptomatic carriage latency period τ_2. Biologically $\tau_2 \geq \tau_1$. $q(t)$ is the

transmission rate of the asymptomatic carrier $C(t)$ to ill class $I(t)$ (through $E^c(t)$) at which asymptomatic carriers take time τ_2 to show symptoms, $\delta(t)$ is disease-related death rate and the recover rate is $r(t)$. Asymptomatic carriers can loss the carriage and return to the susceptible class at a rate $\gamma(t)$. We further assume that there is a natural death rate $\mu(t)$ for all the compartments, $\Lambda(t)$ is the recruitment rate into the population. The interaction and corresponding transmission diagram among the six classes is given in Figure 6.1.

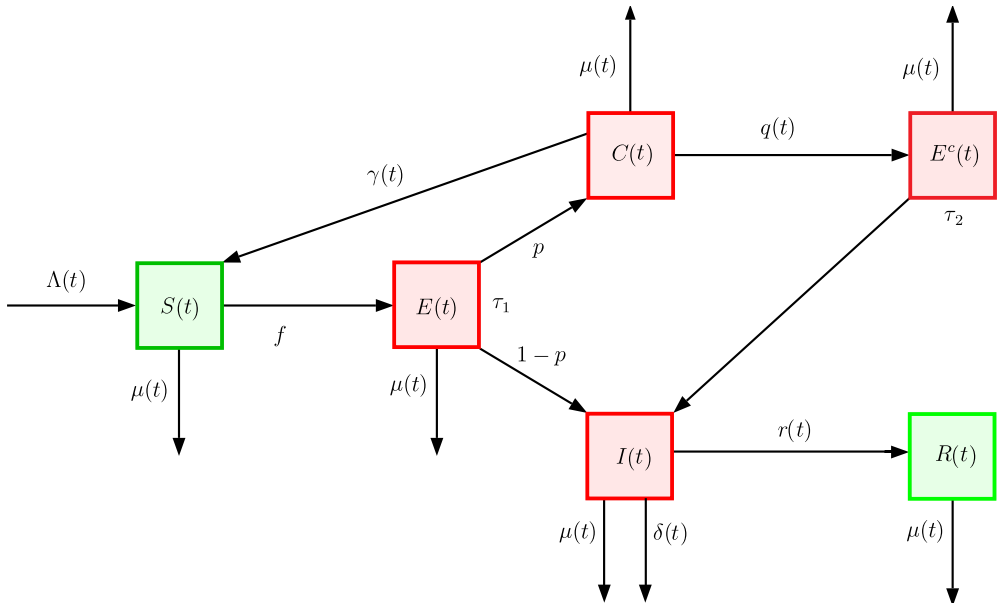


Figure 6.1 Disease Transmission Diagram.

We use the McKendrick-von Foerster equation (see e.g., [125]) to derive functional differential equations for the *disease-related categories* (E, C, E^c and I), by assuming that each infected individual in these categories has an "infection age" which represents the length of time that the individual has infected. In other words, we consider an epidemiological model with age-structure for the disease. More precisely for the branch $E \to C \to E^c \to I$, once a susceptible individual is infected, the infection age (a) is $a > 0$. The individual becomes asymptomatic carrier at infection age $a = \tau_1$, then starts to develop symptoms at $a = \hat{a}$ for some $\hat{a} > \tau_1$, and finally becomes ill (infectious with symptoms) after time τ_2, that is, at infection age $a = \hat{a} + \tau_2$. An illustration is given in Figure 6.2.

Let $e(t, a)$ be the density of infected individuals at time t with infection age a in the exposed class. Thus, at time t, when $a \in [0, \tau_1]$, the number of individuals in the exposed class is

$$E(t) = \int_0^{\tau_1} e(t, a)da. \tag{6.1}$$

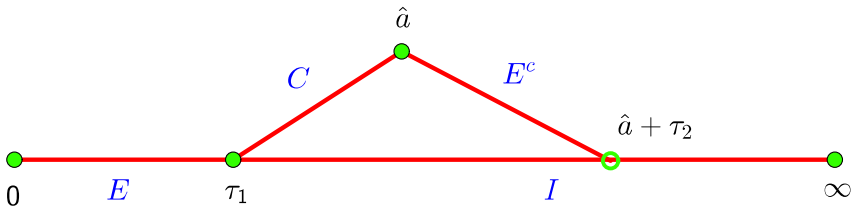


Figure 6.2 The disease-related categories with their corresponding infection age τ_1, $\hat{a}$ and $\hat{a}+\tau_2$.

To derive a functional differential equation to address the variation of the number of individuals in the exposed class at time t, we use the McKendrick von-Foerster equation

$$\frac{\partial e(t,a)}{\partial t}+\frac{\partial e(t,a)}{\partial a}=-\mu(t)e(t,a), \tag{6.2}$$

to describe the evolution of the density $e(t,a)$, where $\mu(t)$ is the loss of infected individuals through natural death.

The population with zero infection age in the exposed class should be the population of new infected at time t, therefore

$$e(t,0)=f(t,S(t),C(t),I(t)).$$

Differentiation of (6.1) and using (6.2) yield

$$\frac{dE(t)}{dt}=-\mu(t)E(t)-e(t,\tau_1)+e(t,0)=f(t,S(t),C(t),I(t))-\mu(t)E(t)-e(t,\tau_1).$$

To evaluate the value of $e(t,\tau_1)$, denote $e_\xi(a)=e(a+\xi,a)$, then

$$\frac{de_\xi(a)}{da}=\frac{\partial e}{\partial t}+\frac{\partial e}{\partial a}=-\mu(a+\xi)e_\xi(a),$$

implying

$$e_\xi(a)=e_\xi(0)e^{-\int_0^a\mu(s+\xi)ds}.$$

Let $a=\tau_1$, then $\xi=t-\tau_1$. From

$$e_{t-\tau_1}(a)=e(t-\tau_1+a,a)\quad\text{and}\quad e_{t-\tau_1}(0)=e(t-\tau_1,0),$$

we have

$$e(t,\tau_1)=f(t-\tau_1,S(t-\tau_1),C(t-\tau_1),I(t-\tau_1))e^{-\int_{t-\tau_1}^t\mu(\eta)d\eta}. \tag{6.3}$$

Finally we obtain the differential equation for $E(t)$,

$$\frac{dE(t)}{dt} = f(t, S(t), C(t), I(t)) - f(t-\tau_1, S(t-\tau_1), C(t-\tau_1), I(t-\tau_1))e^{-\int_{t-\tau_1}^{t}\mu(\eta)d\eta} - \mu(t)E(t). \tag{6.4}$$

Equation (6.4) can also be written in the integral equation form

$$E(t) = \int_{t-\tau_1}^{t} f\left(s, S(s), C(s), I(s)\right) e^{-\int_{s}^{t}\mu(\eta)d\eta} ds. \tag{6.5}$$

Parallelly, let $\hat{e}(t,a)$ be the density of individuals in the asymptomatic carrier class. Then, for $a \in [\tau_1, \hat{a}]$ for some $\hat{a} > \tau_1$, the equation

$$\frac{\partial \hat{e}(t,a)}{\partial t} + \frac{\partial \hat{e}(t,a)}{\partial a} = -(\mu(t) + \gamma(t))\hat{e}(t,a). \tag{6.6}$$

describes the loss of infected individuals through natural death $\mu(t)$ and returning to the susceptible class $\gamma(t)$. The population of the asymptomatic carrier can be expressed by

$$C(t) = \int_{\tau_1}^{\hat{a}} \hat{e}(t,a)da. \tag{6.7}$$

Therefore,

$$\begin{aligned}
\frac{dC(t)}{dt} &= \int_{\tau_1}^{\hat{a}} [-(\mu(t+\gamma(t))\hat{e}(t,a) - \frac{\partial \hat{e}(t,a)}{\partial a}]da \\
&= -(\mu(t)+\gamma(t)) \int_{\tau_1}^{\hat{a}} \hat{e}(t,a)da - \int_{\tau_1}^{\hat{a}} \frac{\partial \hat{e}(t,a)}{\partial a} da \\
&= -(\mu(t)+\gamma(t))C(t) - \hat{e}(t,\hat{a}) + \hat{e}(t,\tau_1).
\end{aligned}$$

As we know, the population in $C(t)$ comes from the exposed class with the proportion p, hence $\hat{e}(t,\tau_1) = pe(t,\tau_1)$. We assume the number of individuals transfer from C to E^c, at time t and age $\hat{a}$, is a proportion $q(t)$ of C and independent on the infection age, that is

$$\hat{e}(t,\hat{a}) = q(t)C(t),$$

although it might overestimate the value of $\hat{e}(t,\hat{a})$.

It then follows from (6.3) that

$$\frac{dC(t)}{dt} = -(\mu(t)+\gamma(t))C(t) + pe(t,\tau_1) - q(t)C(t)$$

$$= pf(t-\tau_1, S(t-\tau_1), C(t-\tau_1), I(t-\tau_1))e^{-\int_{t-\tau_1}^{t}\mu(\eta)d\eta} - (\mu(t)+q(t)+\gamma(t))C(t). \tag{6.8}$$

Analogously, we denote $e^c(t,a)$ as the density of individuals in the carrier-latent class. Then, when $a \in [\hat{a}, \hat{a}+\tau_2]$, the number of individuals in the carrier-latent class can be expressed by

$$E^c(t) = \int_{\hat{a}}^{\hat{a}+\tau_2} e^c(t,a)da. \tag{6.9}$$

Similarly, we have

$$\frac{\partial e^c(t,a)}{\partial t} + \frac{\partial e^c(t,a)}{\partial a} = -\mu(t)e^c(t,a). \tag{6.10}$$

The number of individuals in the carrier-latent class with infection age $a = \hat{a}$ is the new individuals transmitted from asymptomatic carrier class at time t, thus

$$e^c(t,\hat{a}) = \hat{e}(t,\hat{a}) = q(t)C(t).$$

Equation (6.9) can be converted into

$$\frac{dE^c(t)}{dt} = -\mu(t)E^c(t) - e^c(t,\hat{a}+\tau_2) + e^c(t,\hat{a}),$$

by computing the derivative with respect to time t and using (6.10). Using the same procedure to evaluate the value of $e^c(t,\hat{a}+\tau_2)$ yields

$$e^c(t,\hat{a}+\tau_2) = q(t-\tau_2)C(t-\tau_2)e^{-\int_{t-\tau_2}^{t}\mu(\eta)d\eta}.$$

Therefore,

$$\frac{dE^c(t)}{dt} = q(t)C(t) - q(t-\tau_2)C(t-\tau_2)e^{-\int_{t-\tau_2}^{t}\mu(\eta)d\eta} - \mu(t)E^c(t). \tag{6.11}$$

Alternatively (6.11) can be rewritten as

$$E^c(t) = \int_{t-\tau_2}^{t} q(s)C(s)e^{-\int_{s}^{t}\mu(\eta)d\eta}ds. \tag{6.12}$$

Let $\tilde{e}(t,a)$ be the density of individuals in the ill class. Then the population of ill individuals can be expressed by

$$I(t) = \int_{\tau_1}^{\infty} \tilde{e}(t,a)da, \tag{6.13}$$

and $\tilde{e}(t,a)$ satisfies

$$\frac{\partial \tilde{e}(t,a)}{\partial t}+\frac{\partial \tilde{e}(t,a)}{\partial a}=-(\mu(t)+r(t)+\delta(t))\tilde{e}(t,a) \tag{6.14}$$

where $(r(t)+\delta(t))\tilde{e}(t,a)$ represents the loss of infected individuals through recovering and disease-related death. Then,

$$\frac{dI}{dt}=-\left(\mu(t)+r(t)+\delta(t)\right)I(t)-\int_{\tau_1}^{\infty}\frac{\partial \tilde{e}(t,a)}{\partial a}da.$$

From the definitions of $e(t,a)$ and $e^c(t,a)$, the population in the infectious class includes two parts, one from "direct" exposed infective class and another from "indirect" infective transmitted from asymptomatic carrier latent class, thus

$$\tilde{e}(t,a)=\begin{cases}(1-p)e(t,a) & \tau_1<a\leq \hat{a}+\tau_2\\ (1-p)e(t,a)+e^c(t,a) & \hat{a}+\tau_2<a.\end{cases}$$

Since $\mu(t)$ is bounded, we have $e(t,\infty)=e^c(t,\infty)=0$. Consequently, we obtain

$$\begin{aligned}\frac{dI}{dt} &= -\left(\mu(t)+r(t)+\delta(t)\right)I(t)-(1-p)\int_{\tau_1}^{\infty}\frac{\partial e(t,a)}{\partial a}da-\int_{\hat{a}+\tau_2}^{\infty}\frac{\partial e^c(t,a)}{\partial a}da\\ &= (1-p)f(t-\tau_1,S(t-\tau_1),C(t-\tau_1),I(t-\tau_1))e^{-\int_{t-\tau_1}^{t}\mu(\eta)d\eta}\\ &\quad +q(t-\tau_2)C(t-\tau_2)e^{-\int_{t-\tau_2}^{t}\mu(\eta)d\eta}-\left(\mu(t)+r(t)+\delta(t)\right)I(t).\end{aligned} \tag{6.15}$$

Adopting the simplest demographic structure of the population under consideration [113, 207], we have the following equations for the susceptible and recovered classes:

$$\begin{aligned}\frac{dS(t)}{dt} &= \Lambda(t)-\mu(t)S(t)-f(t,S(t),C(t),I(t))+\gamma(t)C(t),\\ \frac{dR(t)}{dt} &= r(t)I(t)-\mu(t)R(t).\end{aligned} \tag{6.16}$$

Combining the equations (6.4), (6.8), (6.11), (6.15) and (6.16), the disease transmission dynamics with asymptomatic carrier and two latent periods can be described by the following nonautonomous delay differential equations

$$\begin{aligned}\frac{dS(t)}{dt} =&\Lambda(t)-\mu(t)S(t)-f(t,S(t),C(t),I(t))+\gamma(t)C(t),\\ \frac{dE(t)}{dt} =&f(t,S(t),C(t),I(t))-f(t-\tau_1,S(t-\tau_1),C(t-\tau_1),I(t-\tau_1))e^{-\int_{t-\tau_1}^{t}\mu(\eta)d\eta}\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad -\mu(t)E(t),\\
\frac{dC(t)}{dt} =& pf(t-\tau_1, S(t-\tau_1), C(t-\tau_1), I(t-\tau_1))e^{-\int_{t-\tau_1}^{t}\mu(\eta)d\eta} - (\mu(t)+q(t)+\gamma(t))C(t),\\
\frac{dE^c(t)}{dt} =& q(t)C(t) - q(t-\tau_2)C(t-\tau_2)e^{-\int_{t-\tau_2}^{t}\mu(\eta)d\eta} - \mu(t)E^c(t),\\
\frac{dI(t)}{dt} =& (1-p)f(t-\tau_1, S(t-\tau_1), C(t-\tau_1), I(t-\tau_1))e^{-\int_{t-\tau_1}^{t}\mu(\eta)d\eta}\\
&\qquad + q(t-\tau_2)C(t-\tau_2)e^{-\int_{t-\tau_2}^{t}\mu(\eta)d\eta} - (\mu(t)+r(t)+\delta(t))I(t),\\
\frac{dR(t)}{dt} =& r(t)I(t) - \mu(t)R(t).
\end{aligned}
\tag{6.17}
$$

As we know, more realistic epidemic model should involve the fluctuation of seasonality [117,119,120,214]. In the following, we focus our attention on the model (6.17) with periodic coefficients. For any positive, T-periodic continuous function $g(t)$, we set

$$g^u = \max_{t\in[0,T]} g(t) \quad \text{and} \quad g^l = \min_{t\in[0,T]} g(t),$$

and assume that all time-dependent coefficients satisfy

(H_1) $\Lambda(t)$, $\mu(t)$, $\delta(t)$, $r(t)$, $q(t)$ and $\gamma(t)$ are all continuous periodic and positive functions with period T.

Although the incidence rate is usually chosen as a standard bilinear form, there is a variety of reasons to incorporate a nonlinear incidence rate, e.g., saturation effect and heterogeneous mixing environment [103]. In the model, we prefer to take a general nonlinear $T-$periodic (in t) incidence form $f(t, S(t), C(t), I(t))$, for some real number $T > 0$, satisfying the following biological feasible condition:

(H_2) $f(t, S, C, I)$ is a nonnegative C^1-function with the following properties:

(i) $f(t, 0, C, I) = 0$ and $f(t, S, 0, 0) = 0$ for all $t \in \mathbb{R}$, $S > 0, C > 0$ and $I > 0$;

(ii) $\frac{\partial f(t,S,C,I)}{\partial S} > 0$, $\frac{\partial f(t,S,C,I)}{\partial C} > 0$, $\frac{\partial f(t,S,C,I)}{\partial I} > 0$ for all $(t, S, C, I) \in \mathbb{R} \times \mathbb{R}^3_+$,

which allows a unified treatment for all important biological cases. The commonly used forms of f include $f(t, S(t), C(t), I(t)) = (\beta(t)C(t) + \gamma(t)I(t))S(t)$, a bilinear incidence rate [96, 188]. For some disease models without the consideration of asymptomatic carrier,

$f(t, S(t), C(t), I(t)) = f(t, S(t), I(t))$ have the following forms

$$\begin{aligned} f(t, S(t), I(t)) &= \beta(t) I(t)^m S(t)^n \ (m, n \text{ are positive parameters [115, 116]}); \\ f(t, S(t), I(t)) &= \frac{\beta(t) I(t) S(t)}{1 + \alpha(t) I(t)} \ \text{(incorporate the saturation of infective [210])}; \\ f(t, S(t), I(t)) &= \frac{\beta(t) I(t)^m S(t)^n}{1 + \alpha(t) I(t)^m} \ \text{(Combination of the above two forms [116])} \end{aligned} \tag{6.18}$$

Further, it is easy to see that the function $h(t, \tau) := e^{-\int_{t-\tau}^{t} \mu(\eta) d\eta}$ in the model (6.17) is also $T-$periodic in t. Therefore the model (6.17) is an $T-$periodic and time-delayed system. Given the hypotheses (H_1) and (H_2), in the following, we will mainly work on the system (6.17).

6.3 Well-posedness property

To analyze (6.17) mathematically, let $\tau = \max\{\tau_1, \tau_2\} = \tau_2$, $X := C([-\tau, 0], \mathbb{R}^6)$ and $X^+ = C([-\tau, 0], \mathbb{R}^6_+)$. For $\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6) \in X$, denote $\|\phi\| = \sum_{i=1}^{6} \|\phi_i\|_\infty$ with $\|\phi_i\|_\infty = \max_{-\tau \le \theta \le 0} |\phi_i(\theta)|$. Then, (X, X^+) is an ordered Banach space and X^+ is a normal cone of X with nonempty interior in X. For any given continuous function $u : [-\tau, \sigma_\phi) \to \mathbb{R}^6$ with $\sigma_\phi > 0$, we define $u_t \in X$ for $t \ge 0$ by $u_t(\theta) = u(t + \theta)$ for all $\theta \in [-\tau, 0]$.

We choose the initial data in the following set:

$$D_X = \Big\{ \phi \in X^+ : \phi_2(0) = \int_{-\tau_1}^{0} f(\vartheta, \phi_1(\vartheta), \phi_3(\vartheta), \phi_4(\vartheta)) e^{-\int_{\vartheta}^{0} \mu(\eta) d\eta} d\vartheta, \\ \phi_4(0) = \int_{-\tau_2}^{0} q(\vartheta) \phi_3(\vartheta) e^{-\int_{\vartheta}^{0} \mu(\eta) d\eta} d\vartheta \Big\}.$$

Then we have the following result to demonstrate the nonnegativity and boundedness of the solution in (6.17).

Theorem 6.3.1. *For any $\phi \in D_X$, under the hypotheses (H_1) and (H_2), the system (6.17) has a unique nonnegative solution $u(t, \phi)$ with the initial condition $u_0 = \phi$, and all solutions are ultimately bounded and uniformly bounded. In addition, the solution semiflow $\Phi(t) = u_t(\cdot) : D_X \to \mathbb{R}^6$ has a compact global attractor and*

$$\Gamma = \left\{ (S, E, C, E^c, I, R) \in \mathbb{R}^6_+ : 0 \le S + E + C + E^c + I + R \le \frac{\Lambda^u}{\mu^l} \right\}$$

is positively invariant for (6.17).

Proof. Given $\phi \in D_X$, we define

$$G(t,\phi) = (G_1(t,\phi), G_2(t,\phi), G_3(t,\phi), G_4(t,\phi), G_5(t,\phi), G_6(t,\phi)),$$

with

$$\begin{aligned}
G_1(t,\phi) =& \Lambda(t) - \mu(t)\phi_1(0) - f(t,\phi_1(0),\phi_3(0),\phi_5(0)) + \gamma(t)\phi_3(0)),\\
G_2(t,\phi) =& f(t,\phi_1(0),\phi_3(0),\phi_5(0)) - f(t-\tau_1,\phi_1(-\tau_1),\phi_3(-\tau_1),\phi_5(-\tau_1))e^{-\int_{t-\tau_1}^{t}\mu(\eta)d\eta}\\
&- \mu(t)\phi_2(0),\\
G_3(t,\phi) =& pf(t-\tau_1,\phi_1(-\tau_1),\phi_3(-\tau_1),\phi_5(-\tau_1))e^{-\int_{t-\tau_1}^{t}\mu(\eta)d\eta}\\
&- q(t)\phi_3(0) - (\gamma(t)+\mu(t))\phi_3(0),\\
G_4(t,\phi) =& q(t)\phi_3(0) - q(t-\tau_2)\phi_3(-\tau_2)e^{-\int_{t-\tau_2}^{t}\mu(\eta)d\eta} - \mu(t)\phi_4(0),\\
G_5(t,\phi) =& (1-p)f(t-\tau_1,\phi_1(-\tau_1),\phi_3(-\tau_1),\phi_5(-\tau_1))e^{-\int_{t-\tau_1}^{t}\mu(\eta)d\eta}\\
&+ q(t-\tau_2)\phi_3(-\tau_2)e^{-\int_{t-\tau_2}^{t}\mu(\eta)d\eta} - (r(t)+\delta(t+\mu(t))\phi_5(0),\\
G_6(t,\phi) =& r(t)\phi_5(0) - \mu(t)\phi_6(0).
\end{aligned}$$

It is easy to see that D_X is closed in X, $G(t,\phi)$ is continuous and is Lipschitz in ϕ in each compact set in $\mathbb{R}\times D_X$. Therefore, there is a unique solution in (6.17) on its maximal existence interval $[0,\sigma_\phi)$ through $(0,\phi)$ for any $\phi \in D_X$.

Furthermore, for any $\phi \in D_X$ with $\phi_i(0) = 0$, it is obvious that $G_i(t,\phi) \geq 0$ for $i = 1,3,5,6$. Thus $\phi_i(t) \geq 0$ for all $t \in [0,\sigma_\phi)$, $i = 1,3,5,6$, see [169, Theorem 5.2.1]. $\phi_2(t) \geq 0$ and $\phi_4(t) \geq 0$ are straightforward from (6.1) and (6.12), respectively. Thus all the solutions of (6.17) are nonnegative for any $t \in [0,\sigma_\phi)$.

Adding the six equations in (6.17) yields that the total population $N(t) = S(t) + E(t) + C(t) + E(t)^c + I(t) + R(t)$ satisfies

$$\frac{dN(t)}{dt} = \Lambda(t) - \mu(t)N(t) - \delta(t)I(t) \leq \Lambda(t) - \mu(t)N(t). \tag{6.19}$$

Under hypothesis (H_1), the equation

$$\frac{d\hat{N}(t)}{dt} = \Lambda(t) - \mu(t)\hat{N}(t). \tag{6.20}$$

has a unique globally asymptotically stable positive $T-$periodic solution (see e.g. [217])

$$\hat{N}^*(t) = e^{-\int_0^t \mu(\eta)d\eta}\left(\hat{N}(0) + \int_0^t e^{\int_0^s \mu(\eta)d\eta}\Lambda(s)ds\right) \tag{6.21}$$

where

$$\hat{N}(0) = \frac{e^{-\int_0^T \mu(\eta)d\eta}}{1 - e^{-\int_0^T \mu(\eta)d\eta}} \int_0^T e^{\int_0^s \mu(\eta)d\eta}\Lambda(s)ds. \tag{6.22}$$

Thus,

$$\lim_{t\to\infty}(\hat{N}(t) - \hat{N}^*(t)) = 0.$$

Therefore all the solutions are ultimately bounded and exist globally ($\sigma_\phi = \infty$) (see e.g., [74, Theorem 2.3.1]). Moreover, from (6.19) and (6.20), we obtain

$$\frac{dN(t)}{dt} \leq \Lambda^u - \mu^l N(t).$$

For the system $\frac{dy(t)}{dt} = \Lambda^u - \mu^l y(t)$, the equilibrium $\frac{\Lambda^u}{\mu^l}$ is globally asymptotically stable. Thus, if $N(t) > \frac{\Lambda^u}{\mu^l}$, $\frac{dN}{dt} < 0$. Hence, all the solutions are uniformly bounded and the solution semi-flow $\Phi(t)$ is point dissipative on D_X. By [72, Theorem 3.4.8], it follows that $\Phi(t)$ admits a compact global attractor in D_X. Furthermore,

$$\Gamma = \left\{(S, E, C, E^c, I, R) \in \mathbb{R}^6_+ : 0 \leq S + E + C + E^c + I + R \leq \frac{\Lambda^u}{\mu^l}\right\}$$

is positively invariant set for (6.17). □

6.4 The basic reproduction ratio

The disease-related classes include either exposed, ill, asymptomatic carrier or carrier-latent i.e., E, I, C and E^c. To find the disease-free state, letting $E = I = C = E^c = 0$, we then get $R = 0$ and

$$\frac{dS(t)}{dt} = \Lambda(t) - \mu(t)S(t). \tag{6.23}$$

Hence, there is only one disease-free $T-$ periodic state $E_1(t) = (S^*(t), 0, 0, 0, 0, 0)$ where $S^*(t) = N^*(t)$ is given in (6.21) which is the positive $T-$periodic solution of (6.20).

Now, we introduce the basic reproduction ratio for system (6.17) according to the theory

developed by Zhao in [217]. In the linearized system of (6.17) at $E_1(t)$, the following disease-related subsystem is decoupled from others:

$$\begin{aligned}\frac{dC(t)}{dt} &= pa_1(t)C(t-\tau_1) + pa_2(t)I(t-\tau_1) - b_1(t)C(t) \\ \frac{dI(t)}{dt} &= (1-p)a_1(t)C(t-\tau_1) + (1-p)a_2(t)I(t-\tau_1) + a_3(t)C(t-\tau_2) - b_2(t)I(t)\end{aligned} \tag{6.24}$$

where

$$a_1(t) = h(t,\tau_1)\frac{\partial f(t-\tau_1, S^*(t-\tau_1), 0, 0)}{\partial C},\ a_2(t) = h(t,\tau_1)\frac{\partial f(t-\tau_1, S^*(t-\tau_1), 0, 0)}{\partial I},$$
$$a_3(t) = h(t,\tau_2)q(t-\tau_2),\ b_1(t) = \mu(t) + q(t) + \gamma(t) \text{ and } b_2(t) = \mu(t) + r(t) + \delta(t).$$

Let

$$\mathcal{F}_1(t) = \begin{bmatrix} pa_1(t) & pa_2(t) \\ (1-p)a_1(t) & (1-p)a_2(t) \end{bmatrix}, \mathcal{F}_2(t) = \begin{bmatrix} 0 & 0 \\ a_3(t) & 0 \end{bmatrix}, \mathcal{V}(t) = \begin{bmatrix} b_1(t) & 0 \\ 0 & b_2(t) \end{bmatrix}.$$

Then we can rewrite (6.24) as

$$\frac{du(t)}{dt} = \mathcal{F}_1(t)u(t-\tau_1) + \mathcal{F}_2(t)u(t-\tau_2) - \mathcal{V}(t)u(t),$$

where $u(t) = (C(t), I(t))^T$.

Let $X_2 = C([-\tau, 0], \mathbb{R}^2)$ and $Z(t,s)$ be the evolution operator associated with the system $\frac{du}{dt} = -\mathcal{V}(t)u(t)$, that is, $Z(t,s)$ satisfies

$$\frac{\partial}{\partial t}Z(t,s) = -\mathcal{V}(t)Z(t,s),\ \forall t \geq s, \text{ and } Z(s,s) = I,\ \forall s \in \mathbb{R}.$$

Thus, for any $t \geq s$, any $s \in \mathbb{R}$

$$Z(t,s) = e^{-\int_s^t \mathcal{V}(\eta)d\eta} = \begin{bmatrix} e^{-\int_s^t b_1(\eta)d\eta} & 0 \\ 0 & e^{-\int_s^t b_2(\eta)d\eta} \end{bmatrix}$$

since $\mathcal{V}(t)$ is a diagonal matrix.
We define $\mathcal{F}(t) : X_2 \to \mathbb{R}^2$ by

$$\mathcal{F}(t)\begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{pmatrix} = \begin{pmatrix} p\ a_1(t)\tilde{\phi}_1(-\tau_1) + p\ a_2(t)\tilde{\phi}_2(-\tau_1) \\ (1-p)\ a_1(t)\tilde{\phi}_1(-\tau_1) + (1-p)\ a_2(t)\tilde{\phi}_2(-\tau_1) + a_3(t)\tilde{\phi}_1(-\tau_2) \end{pmatrix}.$$

Clearly, $-\mathcal{V}$ is cooperative and it follows from the hypotheses (H_1)-(H_2) that $\mathcal{F}(t)$ is positive in the sense that $\mathcal{F}(t)X_2^+ \subseteq \mathbb{R}^2_+$ where $X_2^+ = C([-\tau, 0], \mathbb{R}^2_+)$. Let C_T be the ordered Banach space of all $T-$periodic functions from $\mathbb{R}$ to $\mathbb{R}^2$, which is equipped with the maximum norm and the positive cone $C_T^+ = \{v \in C_T : v(t) \geq 0, \forall t \in \mathbb{R}\}$. Then we can define a linear operator on C_T by

$$[Lv](t) = \int_0^\infty Z(t, t-s)\mathcal{F}(t-s)v(t-s+\cdot)ds, \ v \in C_T.$$

It then follows from [217] that the basic reproduction ratio $\mathcal{R}_0 = \rho(L)$, the spectral radius of L. In periodic environments, the definition of $\mathcal{R}_0$ can be biologically interpreted as the asymptotic per generation growth rate [20].
Clearly, we can write the linear operator L as

$$[Lv](t) = \begin{pmatrix} \ell_{11} + \ell_{12} \\ \ell_{21} + \ell_{22} \end{pmatrix}$$

where

$$\ell_{11} = \int_0^\infty e^{-\int_{t-s}^{t} b_1(\eta)d\eta} pa_1(t-s)v_1(t-s-\tau_1)ds,$$

$$\ell_{12} = \int_0^\infty e^{-\int_{t-s}^{t} b_1(\eta)d\eta} pa_2(t-s)v_2(t-s-\tau_1)ds,$$

$$\ell_{21} = \int_0^\infty e^{-\int_{t-s}^{t} b_2(\eta)d\eta} \left((1-p)a_1(t-s)v_1(t-s-\tau_1) + a_3(t-s)v_1(t-s-\tau_2)\right) ds,$$

$$\ell_{22} = \int_0^\infty e^{-\int_{t-s}^{t} b_2(\eta)d\eta} (1-p)a_2(t-s)v_2(t-s-\tau_1)ds.$$

Further, we can simplify the notation as

$$\ell_{11} = \int_{\tau_1}^\infty e^{-\int_{t-s+\tau_1}^{t} b_1(\eta)d\eta} pa_1(t-s+\tau_1)v_1(t-s)ds := \int_0^\infty K_{11}(t,s)v_1(t-s)ds.$$

Similarly

$$\ell_{12} = \int_0^\infty K_{12}(t,s)v_2(t-s)ds \qquad \text{and} \qquad \ell_{22} = \int_0^\infty K_{22}(t,s)v_2(t-s)ds$$

with, for $s \geq \tau_1$,

$$K_{11}(t,s) = pa_1(t-s+\tau_1)e^{-\int\limits_{t-s+\tau_1}^{t} b_1(\eta)d\eta},$$

$$K_{12}(t,s) = pa_2(t-s+\tau_1)e^{-\int\limits_{t-s+\tau_1}^{t} b_1(\eta)d\eta},$$

$$K_{22}(t,s) = (1-p)\,a_2(t-s+\tau_1)e^{-\int\limits_{t-s+\tau_1}^{t} b_2(\eta)d\eta},$$

and $K_{ij}(t,s) = 0$ for $s < \tau_1$, $i,j \in \{1,2\}$.
The integration ℓ_{21} depends on both time delays τ_1 and τ_2. Since $\tau_2 \geq \tau_1$, we have

$$\begin{aligned}
\ell_{21} &= \int\limits_0^\infty e^{-\int\limits_{t-s}^{t} b_2(\eta)d\eta}(1-p)a_1(t-s)v_1(t-s-\tau_1)ds \\
&\quad + \int\limits_0^\infty e^{-\int\limits_{t-s}^{t} b_2(\eta)d\eta}a_3(t-s)v_1(t-s-\tau_2)ds \\
&= \int\limits_{\tau_1}^\infty e^{-\int\limits_{t-s+\tau_1}^{t} b_2(\eta)d\eta}(1-p)a_1(t-s+\tau_1)v_1(t-s)ds \\
&\quad + \int\limits_{\tau_2}^\infty e^{-\int\limits_{t-s+\tau_2}^{t} b_2(\eta)d\eta}a_3(t-s+\tau_2)v_1(t-s)ds = \int\limits_0^\infty K_{21}(t,s)v_1(t-s)ds
\end{aligned}$$

with

$$K_{21}(t,s) = \begin{cases} (1-p)\,a_1(t-s+\tau_1)e^{-\int\limits_{t-s+\tau_1}^{t} b_2(\eta)d\eta} + a_3(t-s+\tau_2)e^{-\int\limits_{t-s+\tau_2}^{t} b_2(\eta)d\eta} & \text{if} \quad s \geq \tau_2, \\ (1-p)\,a_1(t-s+\tau_1)e^{-\int\limits_{t-s+\tau_1}^{t} b_2(\eta)d\eta} & \text{if} \quad \tau_1 \leq s < \tau_2, \\ 0 & \text{if} \quad s < \tau_1. \end{cases}$$

Therefore, the specific form of the linear operator L becomes

$$[Lv](t) = \begin{pmatrix} \int\limits_0^\infty K_{11}(t,s)v_1(t-s)ds + \int\limits_0^\infty K_{12}(t,s)v_2(t-s)ds \\ \int\limits_0^\infty K_{21}(t,s)v_1(t-s)ds + \int\limits_0^\infty K_{22}(t,s)v_2(t-s)ds \end{pmatrix} := \int\limits_0^\infty \mathcal{K}(t,s)v(t-s)ds \tag{6.25}$$

with

$$\mathcal{K}(t,s) = \begin{pmatrix} K_{11}(t,s) & K_{12}(t,s) \\ K_{21}(t,s) & K_{22}(t,s) \end{pmatrix}.$$

We would mention that, even though we cannot find the explicit form of $\mathcal{R}_0 = \rho(L)$ in general, we will provide a numerical algorithm to obtain an approximation value of $\mathcal{R}_0$ in Section 6.7.

To demonstrate that $\mathcal{R}_0 - 1$ is a threshold value for the stability of the zero solution of system (6.24), let $P(t)$ be the solution maps of the linear system (6.24) on X_2, that is, $P(t)\tilde{\phi} = \nu(t, \tilde{\phi})$, $t \geq 0$, where $\nu(t, \tilde{\phi})$ is the unique solution of (6.24) satisfying $\nu_0 = \tilde{\phi} \in X_2$. Then $P := P(T)$ is the Poincaré map associated with the system (6.24). By [217, Theorem 2.1] or [199, Lemma 1], we have the following result which indicates the instability and local stability of $(0, 0)$ in (6.24).

Theorem 6.4.1. *$\mathcal{R}_0 - 1$ has the same sign as $\rho(P) - 1$. Thus, $(0, 0)$ in (6.24) is locally asymptotically stable if $\mathcal{R}_0 < 1$, and unstable if $\mathcal{R}_0 > 1$.*

Now, we discuss the solution corresponding to $\rho(P)$. Since $\mathcal{F}(t) > 0$, it follows from [74, Theorem 3.6.1] and [169, Lemma 5.3.2] that for each $t \geq 2\tau$, the linear operator $P(t)$ is compact and strongly positive on X_2. Choose an integer $n_0 > 0$ such that $n_0 T \geq 2\tau$. Since $P^{n_0} = P(n_0 T)$, [114, Lemma 3.1] implies that $\rho(P)$ is a simple eigenvalue of P having a strongly positive eigenvector and the modulus of any other eigenvalue is less than $\rho(P)$. Summarizing the above analysis and by argument similar to those in the proof of [209, Proposition 2.1], we have the following result which illustrates a practical form of a positive solution that related to $\rho(P)$.

Lemma 6.4.1. *Let $\rho(P)$ be the spectral radius of P. Then $\rho(P)$ is a positive eigenvalue of P with a positive eigenfunction. Moreover, $\nu(t) = V(t)e^{\lambda t}$ is a solution of (6.24), where $\lambda = \frac{\ln(\rho(P))}{T}$ and $V(t)$ is positive $T-$periodic function $\forall t \geq 0$.*

6.5 Threshold dynamics

In this section, we establish the threshold dynamics for the system in terms of the basic reproduction ratio.

Since the S, C and I equations are decoupled in (6.17), it suffices to study the following $T-$periodic system:

$$\begin{aligned}
\frac{dS(t)}{dt} =& \Lambda(t) - \mu(t)S(t) - f(t, S(t), C(t), I(t)) + \gamma(t)C(t), \\
\frac{dC(t)}{dt} =& pf(t - \tau_1, S(t - \tau_1), C(t - \tau_1), I(t - \tau_1))e^{-\int_{t-\tau_1}^{t} \mu(\eta)d\eta} - (\mu(t) + q(t) + \gamma(t))C(t),
\end{aligned} \tag{6.26}$$

$$\frac{dI(t)}{dt} =(1-p)f(t-\tau_1,S(t-\tau_1),C(t-\tau_1),I(t-\tau_1))e^{-\int\limits_{t-\tau_1}^{t}\mu(\eta)d\eta}$$
$$+q(t-\tau_2)C(t-\tau_2)e^{-\int\limits_{t-\tau_2}^{t}\mu(\eta)d\eta}-(\mu(t)+r(t)+\delta(t))I(t).$$

When a time-delayed system admits the comparison principle, the powerful theory of monotone semiflows can be applied to study the global dynamics, see, e.g., [169, 213]. However, if the system is not monotone, the global stability is a challenging research topic. Therefore, to make sure the solution semiflow of (6.26) is eventually strongly monotone and strongly order preserving (see [169, Chapter 5]), we choose $Y_1 = C([-\tau_1,0],\mathbb{R})\times C([-\tau_2,0],\mathbb{R})\times C([-\tau_1,0],\mathbb{R})$, $Y_1^+ = C([-\tau_1,0],\mathbb{R}_+)\times C([-\tau_2,0],\mathbb{R}_+)\times C([-\tau_1,0],\mathbb{R}_+)$, $Y_0=\{\psi=(\psi_1,\psi_2,\psi_3)\in Y_1^+ \ : \ \psi_2(0)>0 \text{ and } \psi_3(0)>0\}$ and $\partial Y_0 = Y_1\setminus Y_0 = \{\psi\in Y_1^+ : \psi_2(0)=0 \text{ or } \psi_3(0)=0\}$. Then (Y_1,Y_1^+) is an ordered Banach space and Y_0 is an open set relative to Y_1. For a continuous function $\tilde{u}:[-\tau_1,\infty)\times[-\tau_2,\infty)\times[-\tau_1,\infty)\to\mathbb{R}^3$ and $t\geq 0$, define the solution semiflow $\Psi(t)=\tilde{u}_t(\cdot):Y_1\to\mathbb{R}^3$ of (6.26) by

$$\tilde{u}(\theta_1,\theta_2,\theta_3)=(\tilde{u}_1(t+\theta_1),\tilde{u}_2(t+\theta_2),\tilde{u}_3(t+\theta_3)) \quad \forall(\theta_1,\theta_2,\theta_3)\in[-\tau_1,0]\times[-\tau_2,0]\times[-\tau_1,0]\,.$$

The following result can be proved as Theorem 6.3.1.

Lemma 6.5.1. *For any $\psi\in Y_1^+$, system (6.26) admits a bounded unique nonnegative solution $\tilde{u}(t,\psi)$ on $[0,\infty)$ with $\tilde{u}_0=\psi$.*

To see the consistency between Theorem 6.3.1 and Lemma 6.5.1, let $\hat{u}(t,\hat{\psi})$ be the unique solution of (6.26) satisfying $\hat{u}_0=\hat{\psi}\in X_1^+=C([-\tau,0],\mathbb{R}_+^3)$ where $\tau=\max\{\tau_1,\tau_2\}=\tau_2$. For any $\hat{\psi}\in X_1^+$ and $\psi\in Y_1^+$ with $\hat{\psi}_1(\theta_1)=\psi_1(\theta_1)$ $(\forall\theta_1\in[-\tau_1,0])$, $\hat{\psi}_2(\theta_2)=\psi_2(\theta_2)$ $(\forall\theta_2\in[-\tau_2,0])$ and $\hat{\psi}_3(\theta_3)=\psi_3(\theta_3)$ $(\forall\theta_3\in[-\tau_1,0])$, we have $\hat{u}_i(t,\hat{\psi})=\tilde{u}_i(t,\psi)$, where $\hat{u}_i$ and $\tilde{u}_i$ are solutions of system (6.26) satisfying $(\hat{u}_0)_i=\hat{\psi}_i$ and $(\tilde{u}_0)_i=\psi_i$ $(i\in\{1,2,3\})$, respectively, because of the uniqueness of solution and the structure of the model, i.e. the history functions for $S(t)$, $C(t)$ and $I(t)$ are defined on $[-\tau_1,0]$, $[-\tau_2,0]$ and $[-\tau_1,0]$, respectively.

Now, we show that a threshold value for the stability of $(0,0)$ in (6.24) on X_2 is the same as that on $Y_2=C([-\tau_2,0],\mathbb{R})\times C([-\tau_1,0],\mathbb{R})$. For any given $t\geq 0$, let $G(t)$ be the time map of the linear periodic system (6.24) on Y_2, that is, $G(t)=z(t,\hat{\phi})$, where $z(t,\hat{\phi})$ is the unique solution of (6.24) with $z_0=\hat{\phi}\in Y_2$. By similar arguments as in the proof of [122, Lemma 3.8], we have the following result which shows the stability equivalence.

Lemma 6.5.2. *The two Poincare maps $P(T):X_2\to X_2$ and $G(T):Y_2\to Y_2$ have the same spectral radius, that is, $\rho(P(T))=\rho(G(T))$.*

In general, the global stability analysis of the endemic state in the epidemic models is often very difficult, and even impossible sometimes, because the dynamics are highly

nonlinear. A dynamical behavior "permanence" implies that the disease will be maintained globally, irrespective of the initial composition. Even if the endemic equilibrium is unstable, the disease will last forever, possibly with perpetual oscillation or chaotic fluctuation [179]. Since the solutions in (6.26) are bounded, then "permanence" and "persistent" are equivalent (see, e.g. [214]). With the additional assumption:

$$(H_3)\quad \frac{\partial^2 f(t,S,C,I)}{\partial C^2}\leq 0,\ \frac{\partial^2 f(t,S,C,I)}{\partial C\partial I}\leq 0,\ \frac{\partial^2 f(t,S,C,I)}{\partial I^2}\leq 0,\ \forall t\in\mathbb{R},\ S>0,\ C>0\ ,\ I>0,$$

we can obtain the following results about the disease persistence in the system (6.26) and the global attractivity of $(S^*(t),0,0)$. A biological interpretation of (H_3) can be found in [103].

Theorem 6.5.1. *When $\mathcal{R}_0>1$ and $(H_1)-(H_3)$ hold, then (6.26) admits at least one positive periodic solution $E_2(t)=(S^*(t),C^*(t),I^*(t))$, and there exists a positive constant $\eta_1>0$ such that any solution $(S(t,\psi),C(t,\psi),I(t,\psi))$ in (6.26) satisfies*

$$\lim_{t\longrightarrow\infty}\inf(C(t,\psi),I(t,\psi))\geq(\eta_1,\eta_1)$$

for any $\psi\in Y_0$.

Proof. Let $Q(t)\psi=\Psi_t(\psi)$ and $Q=Q(T)$. Hence, $\{Q(t)\}_{t\geq 0}$ is an $T-$periodic semiflow on Y_1 and $Q^n=Q(nT)$ for all $n\geq 0$. Denote $\tilde{E}_1=(S_0^*,0,0)$, where $S_0^*(\theta)=S^*(\theta)$, $\forall\theta\in[-\tau_1,0]$, then $Q(\tilde{E}_1)=\tilde{E}_1$.

Since $\mathcal{R}_0>1$, we have $\rho(G)>1$. Based on (6.24), let G_ϵ be the Poincaré map of the following perturbed linear periodic system

$$\begin{aligned}\frac{dU_1(t)}{dt}&=pa_{1\epsilon}(t)U_1(t-\tau_1)+pa_{2\epsilon}(t)U_2(t-\tau_1)-b_1(t)U_1(t),\\ \frac{dU_2(t)}{dt}&=(1-p)a_{1\epsilon}(t)U_1(t-\tau_1)+(1-p)a_{2\epsilon}(t)U_2(t-\tau_1)+a_3(t)U_1(t-\tau_2)\\ &\quad-b_2(t)U_2(t).\end{aligned}\tag{6.27}$$

where

$$a_{1\epsilon}(t)=h(t,\tau_1)\tfrac{\partial f(t,S^*(t-\tau_1)-\epsilon,\epsilon,\epsilon)}{\partial C}\text{ and }a_{2\epsilon}(t)=h(t,\tau_1)\tfrac{\partial f(t,S^*(t-\tau_1)-\epsilon,\epsilon,\epsilon)}{\partial I}.$$

Since $\lim_{\epsilon\to 0}\rho(G_\epsilon)=\rho(G)>1$, we can fix a small number $\epsilon>0$ such that $\rho(G_\epsilon)>1$. We know $\lim_{\psi\to\tilde{E}_1}\left\|\Psi_t(\psi)-\Psi_t(\tilde{E}_1)\right\|=0$ uniformly for $t\in[0,T]$, then there exists $\varrho<0$ such that

$$S_0^*-\epsilon<S(t,\psi)<S_0^*+\epsilon,\ 0<C(t,\psi)<\epsilon,\ 0<I(t,\psi)<\epsilon\text{ whenever }\left\|\psi-\tilde{E}_1\right\|<\varrho$$

for $t \in [0,T]$. Then, the mean value theorem for two variables and (H_2) imply that

$$f(t,S(t),C(t),I(t)) > f(t,S_0^*-\epsilon,C(t),I(t)) = \frac{\partial f}{\partial C}(t,S_0^*-\epsilon,\hat{C},\hat{I})C(t)+\frac{\partial f}{\partial I}(t,S_0^*-\epsilon,\hat{C},\hat{I})I(t)$$

with some $\hat{C} \in (0,\epsilon)$, $\hat{I} \in (0,\epsilon)$. From (H_3) we have

$$f(t,S(t),C(t),I(t)) > \frac{\partial f}{\partial C}(t,S_0^*-\epsilon,\epsilon,\epsilon)C(t) + \frac{\partial f}{\partial I}(t,S_0^*-\epsilon,\epsilon,\epsilon)I(t) \tag{6.28}$$

for $t \in [0,T]$.

Claim 1. $\limsup_{n\to\infty} \|Q^n(\tilde{E}_1) - \tilde{E}_1\| \geq \epsilon$ for any $\psi \in Y_0$.
By contradiction, suppose that $\limsup_{n\to\infty} \|Q^n(\tilde{E}_1) - \tilde{E}_1\| < \epsilon$ for some $\tilde{\psi} \in Y_0$. Thus, there exists $M_1 > 0$ such that $\|Q^n(\tilde{E}_1) - \tilde{E}_1\| < \epsilon$ for all $n \geq M_1$, implying that, for any $t - \tau_2 \geq M_1 T$, $t = t' + nT$ with $n \geq M_1$ and $t' \in [0,T]$,

$$\begin{aligned} \left\|\Psi_t(\tilde{\psi}) - \Psi_t(\tilde{E}_1)\right\| &= \left\|\Psi_{t'}\left(\Psi_{nT}(\tilde{\psi})\right) - \Psi_{t'}\left(\Psi_{nT}(\tilde{E}_1)\right)\right\| \\ &= \left\|\Psi_{t'}\left(\Psi_{nT}(\tilde{\psi})\right) - \Psi_{t'}\left(\tilde{E}_1\right)\right\| < \epsilon. \end{aligned}$$

Therefore, for $t - \tau_2 \geq M_1 T$, in (6.26), we have, from (6.28)

$$\begin{aligned} \frac{dC(t)}{dt} &> pa_{1\epsilon}(t)C(t-\tau_1) + pa_{2\epsilon}(t)I(t-\tau_1) - b_1(t)C(t), \\ \frac{dI(t)}{dt} &>(1-p)a_{1\epsilon}(t)C(t-\tau_1) + (1-p)a_{2\epsilon}(t)I(t-\tau_1) + a_3(t)C(t-\tau_2) - b_2(t)I(t). \end{aligned}$$

For (6.27), there exists a positive $T-$periodic function $V_\epsilon(t) = (V_{1\epsilon}(t), V_{2\epsilon}(t))$ such that $U_\epsilon(t) = V_\epsilon(t)e^{\lambda_\epsilon t}$ is a solution with $\lambda_\epsilon = \frac{\ln(\rho(G_\epsilon))}{T}$. By the comparison principle ([169, Theorem 5.1.1]) there exists K_1 such that $(C(t),I(t)) \geq K_1 U_\epsilon(t)$. Let $t = nT$, $n \geq 1$, then, $U_\epsilon(nT) = V_\epsilon(nT)e^{\lambda_\epsilon nT}$ and $(C(nT),I(nT)) \geq K_1 U_\epsilon(nT)$. Thus $\lim_{n\to\infty}(C(nT),I(nT)) \to \infty$ due to $\lambda_\epsilon > 0$, a contradiction.

Claim 2. $M_\partial = \{\tilde{E}_1\}$ with $M_\partial = \{\psi \in \partial Y_0 : Q^n(\psi) \in \partial Y_0,\ t \geq 0\}$.
Clearly, $\{\tilde{E}_1\} \subset M_\partial$. To prove $M_\partial \subset \{\tilde{E}_1\}$, it suffices to prove $M_\partial \backslash \{\tilde{E}_1\} = \emptyset$. First, notice that from the second and third equations in (6.26), we have

$$\frac{dC(t)}{dt} \geq -(\mu(t)+q(t)+\gamma(t))C(t) \quad \text{and} \quad \frac{dI(t)}{dt} \geq -(\mu(t)+r(t)+\delta(t))I(t).$$

Consequently

$$C(t) \geq \psi_2(0)e^{-\int_0^t (\mu(\eta)+q(\eta)+\gamma(\eta))d\eta} \geq 0 \quad \text{and} \quad I(t) \geq \psi_3(0)e^{-\int_0^t (\mu(\eta)+r(\eta)+\delta(\eta))d\eta} \geq 0. \quad (6.29)$$

Let $\psi \in \partial Y_0 \backslash \{\tilde{E}_1\}$, for the case, $\psi_2(0) = 0$ and $\psi_3(0) > 0$, we have $I(t, \psi) > 0$ for $t \geq 0$ from (6.29). From the first equation in (6.26) and (H_2), it follows that $\frac{dS}{dt} > -(\mu(t)S(t) + f(t, S(t), \frac{\Lambda^u}{\mu_l}, \frac{\Lambda^u}{\mu_l}))$. Consequently, by Theorem 6.3.1 and the mean value theorem, we have $\frac{dS}{dt} > -(\mu(t) + \frac{\partial f}{\partial S}(t, \hat{s}, \frac{\Lambda^u}{\mu_l}, \frac{\Lambda^u}{\mu_l}))S(t)$ where $\hat{s} \in (0, \frac{\Lambda^u}{\mu_l})$, and hence, there exists t_0 such that $S(t, \psi) > 0$ for all $t \geq t_0$. Choose an integer $n_1 \geq 1$ such that $n_1 T > t_0 + \tau_1$, then $\frac{dC}{dt}\big|_{t=n_1 T} > 0$. Therefore, there exists $t_1 > n_1 T$ such that $C(t, \psi) > 0$ for $t \in [n_1 T, t_1]$. Thus $(S(t, \psi), C(t, \psi), I(t, \psi)) \notin \partial Y_0$. Same proof valid when $\psi_2(0) > 0$ and $\psi_3(0) = 0$. Hence, $M_\partial = \{\tilde{E}_1\}$. Moreover, $\bigcup_{\psi \in M_\partial} \omega(\psi) = \tilde{E}_1$. Therefore, any forward orbit of $\Psi(t)$ in M_∂ converges to $\tilde{E}_1$.

By the acyclicity theorem on uniform persistence for maps [218, Theorem 1.3.1 and Remark 1.3.1], it follows that $Q : Y_1 \to Y_1$ is uniformly persistent with respect to $(Y_0, \partial Y_0)$. Thus, [218, Theorem 3.1.1] implies that the periodic semiflow $\Psi : Y_1 \to Y_1$ is also uniformly persistent with respect to $(Y_0, \partial Y_0)$. It then follows from [216, Theorem 3.1] that system (6.26) admits at least one $T-$periodic solution $E_2(t) = (S(t, \psi^*), C(t, \psi^*), I(t, \psi^*))$ with $\psi^* \in Y_0$. From [123, Theorem 4.5] we know that $Q : Y_1 \to Y_1$ has a compact global attractor Γ_0. Therefore, $\Gamma_0 = Q(\Gamma_0) = \Psi_T(\Gamma_0)$ and $\psi_2(0) > 0$ and $\psi_3(0) > 0$ for any $\psi \in \Gamma_0$. Thus, $\psi_1(0) > 0$ due to the invariance of Γ_0. Let $\Gamma_1 = \bigcup_{t \in [0,T]} \Psi_t(\Gamma_0)$, then $\psi_i(0) > 0$ for all $\psi \in \Gamma_1$ $(i = 1, 2, 3)$. Moreover, $\Gamma_1 \subset Y_0$ and $\lim_{t \to \infty} d(\Psi_t(\psi), \Gamma_1) = 0$ for all $\psi \in Y_0$. Define a continuous function $\kappa : Y_1 \to \mathbb{R}_+$ by

$$\kappa(\psi) = \min\{\psi_2(0), \psi_3(0)\}, \ \forall \psi \in Y_1.$$

Since Γ_1 is a compact subset of Y_0, we have

$$\inf_{\psi \in \Gamma_1} \kappa(\psi) = \min_{\psi \in \Gamma_1} \kappa(\psi) > 0.$$

Consequently, there exists $\eta_1 > 0$ such that

$$\liminf_{t \to \infty} \min(S(t, \psi), C(t, \psi), I(t, \psi)) = \liminf_{t \to \infty} \kappa(\Psi_t(\psi)) \geq \eta_1, \ \forall \psi \in Y_0.$$

In particular, $\liminf_{t \to \infty} \min E_2(t) \geq \eta_1$, and hence, $E_2(t) > 0$ for all $t \geq 0$. This implies that $E_2(t)$ is a positive $T-$periodic solution. □

Theorem 6.5.2. *When $\mathcal{R}_0 < 1$ and $(H_1)-(H_3)$ hold, $(S^*(t), 0, 0)$ is globally attractive for system (6.26) in Y_1^+.*

Proof. Since $\lim_{\zeta\to 0}\rho(G_\zeta) = \rho(G) < 1$, we can fix a small number $\zeta > 0$ such that $\rho(G_\zeta) < 1$. From the global stability of $S^*(t)$ in (6.23), there exists $M_2 > 0$ such that

$$S^*(t) - \zeta < S(t) < S^*(t) + \zeta$$

for all $t - \tau_1 \geq M_2 T$. The mean value theorem for two variables and (H_2)-(H_3) imply that

$$f(t, S(t), C(t), I(t)) \leq \frac{\partial f}{\partial C}(t, S^*(t)+\zeta, 0, 0)C(t) + \frac{\partial f}{\partial I}(t, S^*(t)+\zeta, 0, 0)I(t).$$

Therefore, for $t - \tau \geq M_2 T$, we have

$$\begin{aligned}
\frac{dC(t)}{dt} &\leq pa_{1\zeta}(t)C(t-\tau_1) + pa_{2\zeta}(t)I(t-\tau_1) - b_1(t)C(t), \qquad (6.30)\\
\frac{dI(t)}{dt} &\leq (1-p)a_{1\zeta}(t)C(t-\tau_1) + (1-p)a_{2\zeta}(t)I(t-\tau_1) + a_3(t)C(t-\tau_2) - b_2(t)I(t).
\end{aligned}$$

where

$$a_{1\zeta}(t) = h(t,\tau_1)\frac{\partial f(t, S^*(t-\tau_1)+\zeta, 0, 0)}{\partial C} \text{ and } a_{2\zeta}(t) = h(t,\tau_1)\frac{\partial f(t, S^*(t-\tau_1)+\zeta, 0, 0)}{\partial I}.$$

In (6.30) (replace $\leq$ by $=$), there exists a positive $T-$periodic function $\tilde{V}_\zeta(t) = (\tilde{V}_{1\zeta}(t), \tilde{V}_{2\zeta}(t))$ such that $W_\zeta(t) = \tilde{V}_\zeta(t)e^{\lambda_\zeta t}$ is a solution with $\lambda_\zeta = \frac{\ln(\rho(G_\zeta))}{T}$ where G_ζ is the Poincaré map of the perturbed linear periodic in (6.30). Choose a sufficiently large number $K_2 > 0$ such that $(C(t), I(t)) \leq K_2 W_\zeta(t)$ for all $t \in [M_2T - \tau, M_2T]$. Thus, the comparison theorem for delay differential equations ([169, Theorem 5.1.1]) implies that $(C(t), I(t)) \leq K_2 W_\zeta(t)$ for all $t \geq M_2 T$, and hence, $\lim_{t\to\infty}(C(t), I(t)) = (0, 0)$.

Therefore, the system (6.26) is asymptotic to the limiting system

$$\frac{dS}{dt} = \Lambda(t) - \mu(t)S(t) \qquad (6.31)$$

when $\lim_{t\to\infty}(C(t), I(t)) = (0, 0)$. Note that $S^*(t)$ is a global attractive solution of (6.31). Next, we use the theory of internally chain transitive sets (see e.g., [218]) to prove

$$\lim_{t\to\infty}((S(t), C(t), I(t)) - (S^*(t), 0, 0)) = 0.$$

Let $\psi \in Y_1$ and $\omega = \omega(\psi)$ be the omega limit set of $Q^n(\psi)$. Since $(C(t), I(t)) \to (0, 0)$ as $t \to \infty$, $\omega = \tilde{\omega} \times (0, 0)$. We first claim that $\tilde{\omega} \neq \{0\}$. If $\tilde{\omega} = \{0\}$, then $\lim_{n\to\infty} S(nT, \psi) = 0$.

Thus, there exists $\tilde{T} > nT$ such that

$$\Lambda(t) - \mu(t)S(t) > \frac{1}{2}\Lambda(t) > 0, \qquad \forall t \geq \tilde{T}.$$

Hence, from (6.31), we have

$$\frac{dS}{dt} > \frac{1}{2}\Lambda^l, \qquad \forall t \geq \tilde{T}.$$

This implies that $S(t) \to \infty$ as $t \to \infty$, a contradiction. It is easy to see that $Q^n(\psi_1, 0, 0) = (\tilde{Q}^n(\psi_1), 0, 0)$ where $\tilde{Q}$ is the periodic solution semiflow of (6.31). By [218, Lemma 1.2.1], ω is an internally chain transitive set for Q, and hence, $\tilde{\omega}$ is an internally chain transitive set for $\tilde{Q}$. Since $\tilde{\omega} \neq \{0\}$ and S_0^* is a globally stable fixed point for $\tilde{Q}$ in $C([-\tau_1, 0], \mathbb{R})$, we have $\tilde{\omega} \cap W^s(S_0^*) \neq \emptyset$, where $W^s(S_0^*)$ is the stable set of S_0^*. By [218, Theorem 1.2.1] we then get $\tilde{\omega} = S_0^*$. This proves $\omega = (S_0^*, 0, 0)$, and hence,

$$\lim_{t\to\infty} ((S(t), C(t), I(t)) - (S^*(t), 0, 0)) = 0.$$

That is, $(S^*(t), 0, 0)$ is globally attractive. This completes the proof.

□

We remark that one of the most important questions in mathematical studies of epidemics is the possibility of the eradication of disease. In general, the global dynamics analysis of the disease-free state is enough to answer the question, because it give us information on the global behavior of this state, irrespective of the initial conditions [179].

After we understand the dynamics of (6.26) which is a subsystem of (6.17), we now deduce the dynamics for the other three variables $E(t)$, $E^c(t)$ and $R(t)$. To discuss the system (6.17), first, notice that the integral form of $E(t)$ and $E^c(t)$ in (6.5) and (6.12) can be written as

$$E(t) = \int_0^{\tau_1} f(t-s, S(t-s), C(t-s), I(t-s)) e^{-\int_{t-s}^{s} \mu(\eta) d\eta} ds,$$

$$E^c(t) = \int_0^{\tau_2} q(t-s) C(t-s) e^{-\int_{t-s}^{s} \mu(\eta) d\eta} ds,$$

respectively.

Clearly when $\mathcal{R}_0 < 1$ and $(H_1) - (H_3)$ hold, we have

$$\lim_{t\to\infty} (E(t), E^c(t), R(t)) = (0, 0, 0)\,.$$

When $\mathcal{R}_0 > 1$ and $(H_1) - (H_3)$, it follows from Theorem 6.5.1 that (6.17) admits at least one positive periodic solution $E_2(t) = (S^*(t), E^*(t), C^*(t), E^{c^*}(t), I^*(t), R^*(t))$ with

$$E^*(t) = \int_0^{\tau_1} f(t-s, S^*(t-s), C^*(t-s), I^*(t-s)) e^{-\int_{t-s}^{s} \mu(\eta) d\eta} ds > 0,$$

$$E^{c*}(t) = \int_0^{\tau_2} q(t-s) C^*(t-s) e^{-\int_{t-s}^{s} \mu(\eta) d\eta} ds > 0,$$

$$R^*(t) = R(0) e^{\int_0^t \mu(s) ds} + \int_0^t e^{\int_t^s \mu(\eta) d\eta} r(s) I^*(s) ds > 0.$$

Let $Y = C([-\tau_1, 0], \mathbb{R}) \times \mathbb{R} \times C([-\tau_2, 0], \mathbb{R}) \times \mathbb{R} \times C([-\tau_1, 0], \mathbb{R}) \times \mathbb{R}$, $Y^+ = C([-\tau_1, 0], \mathbb{R}_+) \times \mathbb{R}_+ \times C([-\tau_2, 0], \mathbb{R}_+) \times \mathbb{R}_+ \times C([-\tau_1, 0], \mathbb{R}_+) \times \mathbb{R}_+$,

$$D_Y = \Big\{ \hat{\psi} \in Y^+ : \hat{\psi}_2(0) = \textstyle\int_{-\tau_1}^0 f(\vartheta, \hat{\psi}_1(\vartheta), \hat{\psi}_3(\vartheta), \hat{\psi}_4(\vartheta)) e^{-\int_\vartheta^0 \mu(\eta) d\eta} d\vartheta, \\ \hat{\psi}_4(0) = \textstyle\int_{-\tau_2}^0 q(\vartheta) \hat{\psi}_3(\vartheta) e^{-\int_\vartheta^0 \mu(\eta) d\eta} d\vartheta \Big\},$$

$\hat{Y} = \{\hat{\psi} \in D_Y \ : \ \hat{\psi}_3(0) > 0 \text{ and } \hat{\psi}_5(0) > 0\}$ and $\partial \hat{Y} = D_Y \setminus \hat{Y} = \{\hat{\psi} \in D_Y : \hat{\psi}_3(0) = 0 \text{ or } \hat{\psi}_5(0) = 0\}$. Then we have the following result for system (6.17).

Theorem 6.5.3. *Assume $(H_1) - (H_3)$ hold*

(i) when $\mathcal{R}_0 < 1$, then

$$\lim_{t \to \infty} [(S(t), E(t), C(t), E^c(t), I(t), R(t)) - (S^*(t), 0, 0, 0, 0, 0)] = 0$$

in D_Y;

(ii) when $\mathcal{R}_0 > 1$, then (6.17) admits at least one positive periodic solution

$$(S^*(t), E^*(t), C^*(t), E^{c^*}(t), I^*(t), R^*(t))$$

and there exists a positive constant $\eta_2 > 0$ such that any solution $(S(t, \hat{\psi}), E(t, \hat{\psi}), C(t, \hat{\psi}), E^c(t, \hat{\psi}), I(t, \hat{\psi}), R(t, \hat{\psi}))$ in (6.17) satisfies

$$\lim_{t \longrightarrow \infty} \inf(C(t, \hat{\psi}), I(t, \hat{\psi})) \geq (\eta_2, \eta_2)$$

for any $\hat{\psi} \in \hat{Y}$.

6.6 Uniqueness of the epidemic state with constants coefficients

It is known that the analysis to prove the uniqueness of the endemic state is a challenging topic in nonautonomous epidemic models. In this section, we show the coexistence of a unique endemic state when (6.17) is an autonomous system with all constants coefficients and $f(t,S,C,I) \equiv f(S,C,I)$, then the basic reproduction ratio $\mathcal{R}_0$ with discrete delays becomes explicit (see [217, Corollary 2.1])

$$[\mathcal{R}_0] = \rho((F_1+F_2)V^{-1}) = \frac{T+\sqrt{T^2+4D}}{2} \tag{6.32}$$

with

$$T = \frac{pe^{-\mu\tau_1}\frac{\partial f}{\partial C}(\frac{\Lambda}{\mu},0,0)}{q+\gamma+\mu} + \frac{(1-p)e^{-\mu\tau_1}\frac{\partial f}{\partial I}(\frac{\Lambda}{\mu},0,0)}{r+\delta+\mu} > 0,\ D = \frac{pq\frac{\partial f}{\partial I}(\frac{\Lambda}{\mu},0,0)e^{-\mu(\tau_1+\tau_2)}}{(q+\gamma+\mu)(r+\delta+\mu)} > 0$$

where $\frac{\Lambda}{\mu}$ is the global asymptotic stable equilibrium of $\frac{dN}{dt} = \Lambda - \mu N(t)$.

Beside the disease-free equilibrium $E_1 = (\frac{\Lambda}{\mu},0,0)$, to examine the existence of the positive equilibrium point $E_2 = (S^*,C^*,I^*)$ in (6.26), we have to solve

$$\begin{aligned} f(S,C,I) &= \Lambda - \mu S + \gamma C, && (6.33a)\\ pe^{-\mu\tau_1}f(S,C,I) &= qC + (\gamma+\mu)C, && (6.33b)\\ (1-p)e^{-\mu\tau_1}f(S,C,I) &= (r+\delta+\mu)I - qe^{-\mu\tau_2}C. && (6.33c) \end{aligned}$$

Motivated by [103], in order to find the conditions to guarantee the existence of positive solutions in (6.33), first of all, we have $I = \frac{A_1+A_2}{A_3}C$ from (6.33b) and (6.33c), where

$$A_1 = \frac{q+\gamma+\mu}{pe^{-\mu\tau_1}},\ A_2 = \frac{qe^{-\mu\tau_2}}{(1-p)e^{-\mu\tau_1}},\ A_3 = \frac{r+\delta+\mu}{(1-p)e^{-\mu\tau_1}}.$$

By denoting $g(S,C) = f(S,C,\frac{A_1+A_2}{A_3}C)$, then

$$\begin{aligned} \frac{\partial g(S,C)}{\partial C} &= \frac{\partial f}{\partial C} + \frac{A_1+A_2}{A_3}\frac{\partial f}{\partial I},\\ \frac{\partial^2 g(S,C)}{\partial C^2} &= \frac{\partial^2 f}{\partial C^2} + \frac{2(A_1+A_2)}{A_3}\frac{\partial^2 f}{\partial C\partial I} + (\frac{A_1+A_2}{A_3})^2\frac{\partial^2 f}{\partial I^2}. \end{aligned}$$

Consequently, (6.33) yields

$$\begin{aligned} g(S,C) &= A_1 C, \\ \Lambda - \mu S + \gamma C &= A_1 C. \end{aligned} \tag{6.34}$$

When (H_3) holds, by contradiction, we can prove that $\frac{\partial g(S,C)}{\partial C} \leq A_1$ since $\frac{\partial^2 g(S,C)}{\partial C^2} \leq 0$. Otherwise, if there is a $C_1 \geq 0$ satisfies $\frac{\partial g(S,C_1)}{\partial C} > A_1$, there must exist a $C_2 \in (0, C_1)$ such that $\frac{\partial g(S,C_2)}{\partial C} = \frac{g(S,C_1)-g(S,0)}{C_1} = A_1$ from $g(S,0) = 0$ and (6.34). Therefore, the following proposition holds.

Proposition 6.6.1. *Under the hypothesis $(H_2)-(H_3)$, if the endemic equilibrium point E_2 exists, we have*

$$\frac{\partial f(S^*,C^*,I^*)}{\partial C} + \frac{A_1+A_2}{A_3}\frac{\partial f(S^*,C^*,I^*)}{\partial I} \leq A_1,$$

where the strict equality holds only if

$$\frac{\partial^2 f(S, \frac{A_1+A_2}{A_3}C, C)}{\partial C^2} = \frac{\partial^2 f(S, \frac{A_1+A_2}{A_3}C, C)}{\partial C \partial I} = \frac{\partial^2 f(S, \frac{A_1+A_2}{A_3}C, C)}{\partial I^2} = 0 \text{ for all } S > 0, C > 0.$$

Similar to the proof given in [103] with more complicated analysis and computation, we know that, under the hypothesis $(H_2)-(H_3)$, the condition

$$\frac{\partial f(\frac{\Lambda}{\mu},0,0)}{\partial C} + \frac{A_1+A_2}{A_3}\frac{\partial f(\frac{\Lambda}{\mu},0,0)}{\partial I} > A_1 \tag{6.35}$$

is sufficient to ensure the existence and uniqueness of such endemic equilibrium point E_2 from (6.33a) by noting that $\gamma < A_1$ since $0 < p < 1$ and $e^{-\mu\tau_1} < 1$.

Furthermore, in the expression of the basic reproduction number $[\mathcal{R}_0]$,

$$\begin{aligned} T + D &= \frac{1}{(q+\gamma+\mu)(r+\delta+\mu)}[(r+\delta+\mu)pe^{-\mu\tau_1}\frac{\partial f(\frac{\Lambda}{\mu},0,0)}{\partial C} \\ &\qquad + ((q+\gamma+\mu)(1-p)e^{-\mu\tau_1} + pqe^{-\mu(\tau_1+\tau_2)})\frac{\partial f(\frac{\Lambda}{\mu},0,0)}{\partial I}] \\ &= \frac{pe^{-\mu\tau_1}}{q+\gamma+\mu}[\frac{\partial f(\frac{\Lambda}{\mu},0,0)}{\partial C} + \frac{A_1+A_2}{A_3}\frac{\partial f(\frac{\Lambda}{\mu},0,0)}{\partial I}] \\ &= \frac{1}{A_1}[\frac{\partial f(\frac{\Lambda}{\mu},0,0)}{\partial C} + \frac{A_1+A_2}{A_3}\frac{\partial f(\frac{\Lambda}{\mu},0,0)}{\partial I}]. \end{aligned} \tag{6.36}$$

The condition (6.35) yields $T + D > 1$ which is sufficient to have $[\mathcal{R}_0] > 1$. Therefore,

Theorem 6.6.1. *Given the assumptions* $(H_2)-(H_3)$. *If* $[\mathcal{R}_0]>1$, *then the positive equilibrium point* $E_2=(S^*,C^*,I^*)$ *exists in (6.26) and is uniquely determined in (6.33). Consequently, a unique endemic equilibrium point* $(S^*,E^*,C^*,E^{c*},I^*,R^*)$ *exists in (6.17) with* $E^*=\frac{1}{\mu}(1-e^{-\mu\tau_1})f(S^*,C^*,I^*)$, $E^{c*}=\frac{q}{\mu}(1-e^{-\mu\tau_2})C^*$ *and* $R^*=\frac{r}{\mu}I^*$.

6.7 Numerical computation and simulation

In this section, firstly, we present numerical algorithm to calculate the basic reproduction ratio $\mathcal{R}_0$. Secondly, we do a case study regarding the meningococcal meningitis disease transmission in Dori, Burkina Faso. Thirdly, we discuss the sensitivity of $\mathcal{R}_0$ ($[\mathcal{R}_0]$) with respect to the latent period τ_1 and some key parameters related to carriers including τ_2, p, $\gamma(t)$ and $q(t)$.

6.7.1 Calculation of $\mathcal{R}_0$

For any $v\in C_T$, we can rewrite the infection operator in (6.25) as

$$\begin{aligned}[Lv](t)=\int_0^\infty \mathcal{K}(t,s)v(t-s)ds&=\sum_{m=0}^{\infty}\int_{mT}^{(m+1)T}\mathcal{K}(t,s)v(t-s)ds\\&=\sum_{m=0}^{\infty}\int_0^T\mathcal{K}(t,s+mT)v(t-s-mT)ds\\&=\sum_{m=0}^{\infty}\int_0^T\mathcal{K}(t,s+mT)v(t-s)ds=\int_0^T G_{\mathcal{K}}(t,s)v(t-s)ds\end{aligned}\tag{6.37}$$

with

$$G_{\mathcal{K}}(t,s)=\sum_{m=0}^{\infty}\mathcal{K}(t,s+mT).\tag{6.38}$$

It is clear that, when $(H_1)-(H_2)$ hold, there exist $K_Y>0$ and $\alpha_Y>0$ such that $\|Y(t,s)\|\leq K_Y e^{-\alpha_Y(t-s)}$ for all $t\geq s$ with $s\in\mathbb{R}$. Hence, we may approximate $G_{\mathcal{K}}$ by a finite sum (See [146])

$$G_{\mathcal{K}}(s,t)\approx\sum_{m=0}^{M_{\mathcal{K}}}\mathcal{K}(t,s+mT)\tag{6.39}$$

for some integer $M_{\mathcal{K}}>0$. We would like to mention that, due to the exponential decay of the terms in the summation in (6.38), the large value of $M_{\mathcal{K}}$ is unnecessary [146]. To apply the numerical algorithm in [146], we partition the interval $[0,T]$ uniformly into n subintervals

$[t_i, t_{i+1}]$ with $t_i = \frac{i\,T}{n}$ for $i = 0, \ldots, n-1$ and use the trapezoidal rule to approximate the integral in (6.37) with $G_{\mathcal{K}}(s,t)$ in (6.39). Then $\mathcal{R}_0 \approx \frac{T}{n}\rho(\mathcal{A})$ where $\mathcal{A}$ is $2n \times 2n$ matrix

$$\mathcal{A} = \begin{bmatrix} \tilde{G}_{\mathcal{K}}(t_0,t_0) & G_{\mathcal{K}}(t_0,t_{n-1}) & \cdots & \cdots & \cdots & G_{\mathcal{K}}(t_0,t_2) & G_{\mathcal{K}}(t_0,t_1) \\ G_{\mathcal{K}}(t_1,t_1) & \tilde{G}_{\mathcal{K}}(t_1,t_0) & \cdots & \cdots & \cdots & G_{\mathcal{K}}(t_1,t_3) & G_{\mathcal{K}}(t_1,t_2) \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ G_{\mathcal{K}}(t_j,t_j) & G_{\mathcal{K}}(t_j,t_{j-1}) & \cdots & \tilde{G}_{\mathcal{K}}(t_j,t_0) & \cdots & G_{\mathcal{K}}(t_j,t_{j+2}) & G_{\mathcal{K}}(t_j,t_{j+1}) \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ G_{\mathcal{K}}(t_{n-2},t_{n-2}) & G_{\mathcal{K}}(t_{n-2},t_{n-3}) & \cdots & \cdots & \cdots & \tilde{G}_{\mathcal{K}}(t_{n-2},t_0) & G_{\mathcal{K}}(t_{n-2},t_{n-1}) \\ G_{\mathcal{K}}(t_{n-1},t_{n-1}) & G_{\mathcal{K}}(t_{n-2},t_{n-2}) & \cdots & \cdots & \cdots & \tilde{G}_{\mathcal{K}}(t_{n-1},t_1) & \tilde{G}_{\mathcal{K}}(t_{n-1},t_0) \end{bmatrix}$$

with $\tilde{G}_{\mathcal{K}}(t_j,t_0) = \frac{1}{2}(G_{\mathcal{K}}(t_j,t_0) + G_{\mathcal{K}}(t_j,t_n))$ and $j = 0, \ldots, n-1$.

6.7.2 A case study

To apply the model to realistic example, we study the transmission of meningococcal meningitis disease, a major public health problem in a large area of sub-Saharan Africa, known as the meningitis belt, the area of Africa that lies south of the Sahara desert and stretches from Senegal to Ethiopia. According to WHO, meningococcal meningitis is a bacterial form of meningitis caused by the bacterium *Neisseria meningitidis*, a serious infection of the thin lining that surrounds the brain and spinal cord. The bacteria are transmitted from person-to-person through droplets of respiratory or throat secretions from carriers such as sneezing or coughing on someone.

In Burkina Faso, a landlocked country in West Africa, the annual number of meningitis cases is shown in Figure 6.3, recorded from $1940 - 2014$, which exhibits an oscillatory behavior, although with irregular patterns of epidemics varying in size and duration [1, 91]. The observed behavior can be related to environmental factors, particularly the Harmattan (a dry and dusty trade wind that blows across the region during the dry season).

For the case study, we choose a town in northeastern Burkina Faso named as Dori. According to the 2012 World Bank report, the life expectancy in Burkina Faso is 55.86 year. So we choose the natural death rate $\mu(t) \equiv \mu = 1/55.86 = 0.018$ year^{-1}. The total population in Dori is 21078 (2006), that is, the recruitment rate $\Lambda(t) \equiv \Lambda = 21078 \times \mu \approx 379$ people per year.

In general, the incidence rate function f can be chosen in different forms with condition (H_2). For simplicity and following the literature, we take $f(t,S,C,I) = f_1(t,S,C) + f_2(t,S,I)$ [91, 125]. Let $\beta(t)$ be the transmission rate. Typically, asymptomatic carriers are

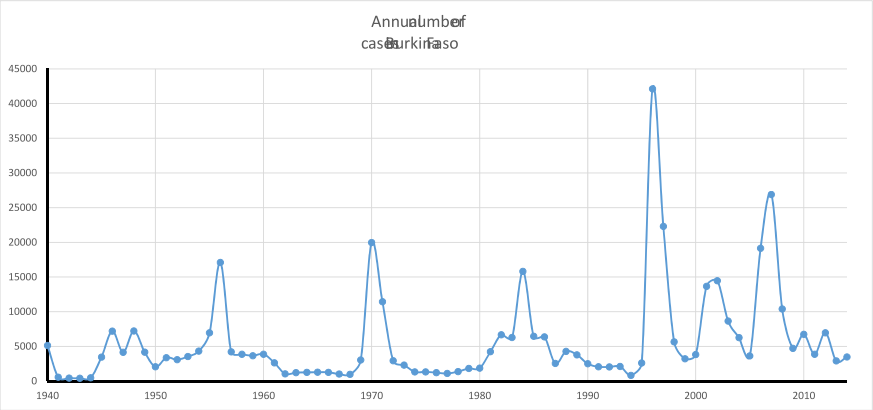


Figure 6.3 Annual number of reported suspected meningitis cases in Burkina Faso, 1940-2014.

infectious at a reduced transmission rate $l\beta(t)$ ($l \in (0,1)$) [125]. We choose

$$f_1(t,S,C) = \frac{l\beta(t)S(t)C(t)}{1+\alpha_1 C(t)} \qquad \text{and} \qquad f_2(t,S,I) = \frac{\beta(t)S(t)I(t)}{1+\alpha_2 I(t)}$$

with $l = 0.8$, $\alpha_1 = 0.07$ and $\alpha_2 = 0.05$ [210]. In general, the transmission rate is between $50-200$ year^{-1} [91]. To estimate a periodic $\beta(t)$, we first notice that usually the meningitis incidence is the lowest during rainfall season and it increases to reach the highest during the dry season in most districts of the meningitis belt due to dust winds, cold nights and upper respiratory tract infections [132, 192]. Now, based on the inverse relation between rainfall and disease transmission, we assume that there is a higher transmission rate in the most dry period and it decreases as the average precipitation increases. From the recorded average precipitation per month in Dori from 2000 to 2012 (Figure 6.4, the data are taken from www.worldweatheronline.com), we see that the rainy season lasts approximately four months from June to September. Hence, we take the lowest disease transmission as 50 year^{-1}, occurring in August and the highest, 200 year^{-1}, in January (the red dots in Figure 6.5 represent the transmission value per month). Consequently, we obtain a one year periodic function $\beta_1(t) = 128.333 + 52.0374\cos(2\pi t) + 50.3707\sin(2\pi t)$ (Figure 6.5a). In [91], the authors assume $\hat{\beta}(t) = \beta_0(1+\beta_1\cos(2\pi t))$ with $\beta_1 \in (0, 0.975)$. Using this form of $\hat{\beta}(t)$, we have found that the best fit function is $\hat{\beta}(t) = 126.667(1+0.384506\cos(2\pi t))$ (Figure 6.5b). Obviously, the error is higher than using $\beta_1(t)$.

According to WHO for meningococcal meningitis disease, the latent (latent) period (τ_1) is $2-10$ days ($0.0055-0.0274$ year). The duration of carriage (τ_2) is $1-3$ months ($0.083-0.25$

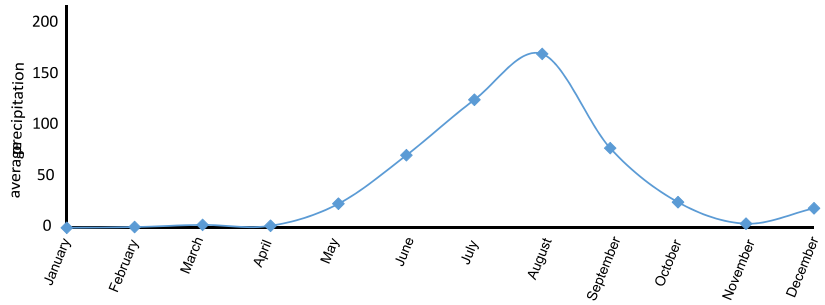
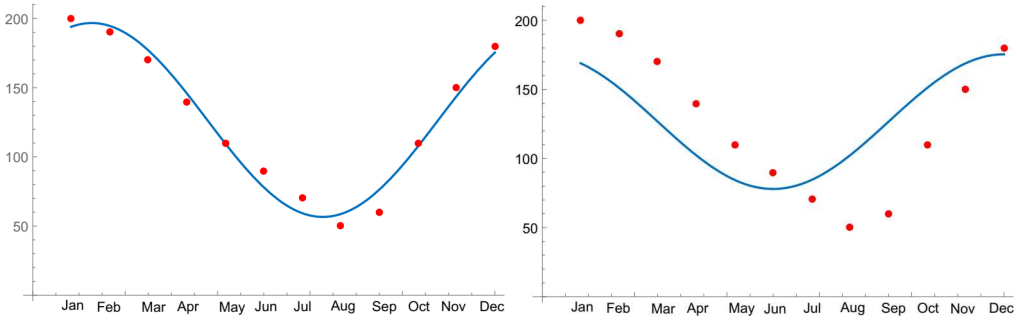


Figure 6.4 Average precipitation per month in Dori from 2000 to 2012.



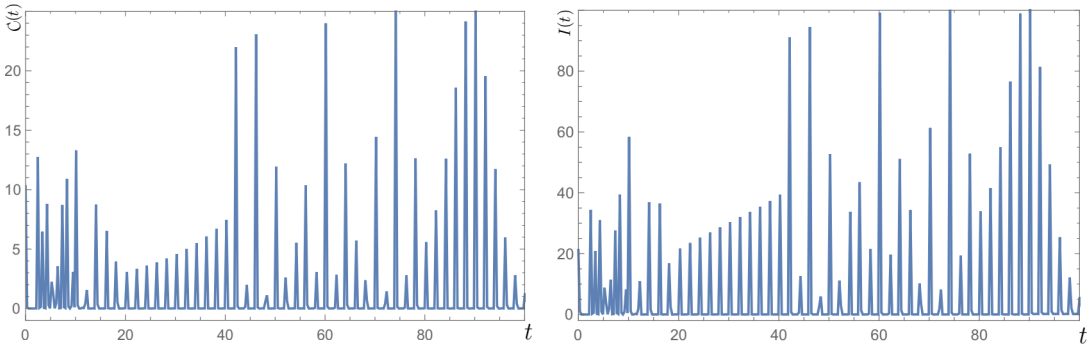
(a) $\beta_1(t) = 128.333 + 52.0374\cos(2\pi t) + 50.3707\sin(2\pi t)$. (b) $\hat{\beta}(t) = 126.667(1 + 0.384506\cos(2\pi t))$

Figure 6.5 Transmission rate function.

year) [91, 192]. The rate at which carriers become ill is between $0.1 - 52$ year^{-1}, we take $q(t) = q_0(1 + q_1\cos(2\pi t))$ with $q_1 \in [0, 0.975]$ [91]. In the following numerical simulation we choose $\tau_1 = 0.008$, $\tau_2 = 0.083$, $q_0 = 30$, $q_1 = 0.5$; and assume that 20% of infected susceptible individuals become carriers (i.e. $p = 0.2$); and all the other parameters are chosen from literature as constants $r(t) \equiv r = 52$ year^{-1}, $\delta(t) \equiv \delta = 5.2$ year^{-1}, and $\gamma(t) \equiv \gamma = 20$ year^{-1} [91]. Thus, $\mathcal{R}_0 \approx 2.5371$ by using the provided algorithm in Section 6.7.1 with $n = 500$ and $M_{\mathcal{K}} = 10$. The numerical simulation result shows the disease persistence and oscillatory behavior with irregular patterns in size and duration (Figure 6.6), which has good agreement with the behavior of the real data (Figure 6.3), although we cannot fully recover Figure 6.3 due to the lack of data for each town there. In fact the pathogenesis of meningococcal meningitis disease is not fully understood [65].

6.7.3 Sensitivity of $\mathcal{R}_0$ and $[\mathcal{R}_0]$

In this subsection, we study the sensitivity of $\mathcal{R}_0$ ($[\mathcal{R}_0]$) with respect to the latent period τ_1 and carriers related parameters τ_2, p, γ and $q(t)$. We keep all parameters value as in Figure 6.6 except $\mu = 0.051$ and $p = 0.4$ to obtain a value for $\mathcal{R}_0$ closing to one to study



(a) $C(t)$ (b) $I(t)$

Figure 6.6 Time series $C(t)$ and $I(t)$. $S(0) = 15000$, $E(0) = 30$, $C(0) = 20$, $E^C(0) = 5$, $I(0) = 20$ and $R(0) = 5$.

the influence of these parameters on controlling the infectious disease.

First, with constant coefficients, from the explicit formula for $[\mathcal{R}_0]$ in (6.32), we know

$$\begin{aligned}
\frac{\partial[\mathcal{R}_0]}{\partial\tau_1} &= -\frac{\mu}{2}\left(T + \frac{T^2+2D}{\sqrt{T^2+4D}}\right) < 0, \qquad \frac{\partial[\mathcal{R}_0]}{\partial\tau_2} = -\frac{-\mu D}{\sqrt{T^2+4D}} < 0, \\
\frac{\partial[\mathcal{R}_0]}{\partial f_C} &= \frac{1}{2}\left(1 + \frac{T}{\sqrt{T^2+4D}}\right)\frac{pe^{-\mu\tau_1}}{q+\gamma+\mu} = \frac{[\mathcal{R}_0]}{\sqrt{T^2+4D}}\frac{pe^{-\mu\tau_1}}{q+\gamma+\mu} > 0,
\end{aligned}$$

and for $s \in \{\gamma, p, q\}$,

$$\frac{\partial[\mathcal{R}_0]}{\partial s} = \frac{1}{\sqrt{T^2+4D}}\left([\mathcal{R}_0]\frac{\partial T}{\partial s} + \frac{\partial D}{\partial s}\right)$$

with

$$\begin{aligned}
\frac{\partial T}{\partial\gamma} &= \frac{-pe^{-\mu\tau_1}f_C}{(q+\gamma+\mu)^2} < 0, \qquad \frac{\partial D}{\partial\gamma} = \frac{-pqe^{-\mu(\tau_1+\tau_2)}f_I}{(r+\delta+\mu)(q+\gamma+\mu)^2} < 0 \\
\frac{\partial T}{\partial p} &= e^{-\mu\tau_1}\left(\frac{f_C}{q+\gamma+\mu} - \frac{f_I}{r+\delta+\mu}\right), \qquad \frac{\partial D}{\partial p} = \frac{qf_Ie^{-\mu(\tau_1+\tau_2)}}{(q+\gamma+\mu)(r+\delta+\mu)} > 0, \\
\frac{\partial T}{\partial q} &= -\frac{pf_Ce^{-\mu\tau_1}}{(q+\gamma+\mu)^2} < 0, \qquad \frac{\partial D}{\partial q} = \frac{pf_Ie^{-\mu(\tau_1+\tau_2)}}{r+\delta+\mu}\frac{\gamma+\mu}{(q+\gamma+\mu)^2} > 0.
\end{aligned}$$

where $\frac{\partial f}{\partial C}\left(\frac{\Lambda}{\mu}, 0, 0\right) = f_C$ and $\frac{\partial f}{\partial I}\left(\frac{\Lambda}{\mu}, 0, 0\right) = f_I$. Based on the above relations, Table 6.1 summarizes the conditions for increasing $[\mathcal{R}_0]$.

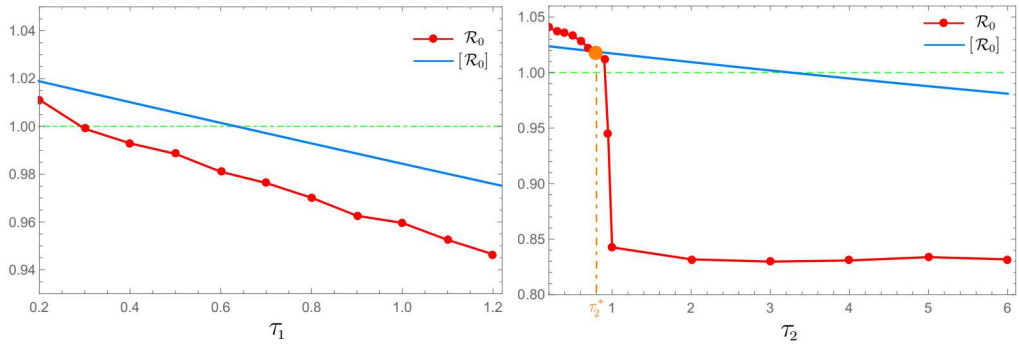
From epidemiological view point, control the outbreak of disease implies to decrease the value of $[\mathcal{R}_0]$ below to 1. Thus it is clear that the longer latency periods (τ_1,τ_2) or the larger portion of return to susceptible class from carrier (γ) decrease $[\mathcal{R}_0]$, while $[\mathcal{R}_0]$ increases under one of the following circumstances: 1) higher transmissibility with respect to the carrier class (f_C), 2) a great portion (or probability) to develop carriage with either

Parameter	Condition
τ_1, τ_2	Reduce τ_1 or τ_2
f_C	Increase f_C
γ	Reduce γ
p	Increase p and one of the following conditions holds (i) $\frac{f_C}{q+\gamma+\mu} > \frac{f_I}{r+\delta+\mu}$ or (ii) $[\mathcal{R}_0] < \frac{qf_Ie^{-\mu\tau_2}}{f_I(q+\gamma+\mu)-f_C(r+\delta+\mu)}$
q	Increase q and the condition (iii) $[\mathcal{R}_0] < e^{-\mu\tau_2}\frac{f_I}{f_C}\frac{\gamma+\mu}{r+\delta+\mu}$ holds

Table 6.1 Conditions for increasing $[\mathcal{R}_0]$.

the number of newly infected individuals that arise from one carrier $\frac{e^{-\mu\tau_1}f_C}{q+\gamma+\mu}$ is larger than the number of newly infected individuals that arise from one infectious individual $\frac{e^{-\mu\tau_1}f_I}{r+\delta+\mu}$ (condition (i)) or when $[\mathcal{R}_0]$ is small enough (condition (ii)), 3) condition (iii) holds and higher transmission rate from asymptomatic carrier latency period to ill class occurs (q).

To compare between $[\mathcal{R}_0]$ in the autonomous system and the basic reproduction number in the periodic system (6.17) $\mathcal{R}_0$, we take the average value of $[\beta_1] = \frac{1}{T}\int_0^T \beta_1(t)dt = 128.333$ and $[q] = 30$ in Figure 6.6 over the interval $[0, T]$ to calculate $[\mathcal{R}_0]$, use the algorithm given in Section 6.7.1 to illustrate the sensitivity of $\mathcal{R}_0$ with respect to τ_1 and τ_1. First, with fixed $\tau_2 = 0.8$, we observe that both $\mathcal{R}_0$ and $[\mathcal{R}_0]$ decrease when τ_1 increases (Figure 6.7a). Similar behavior is shown in Figure 6.7b for fixed $\tau_1 = 0.2$ and variable τ_2. These are consistent with the fact that long latent period leads to less infection spread. The latent period can be extended by, for example, prescription drug or control measures. On the other hand, we observe that the curve of $[\mathcal{R}_0]$ is below the curve of $\mathcal{R}_0$ for $\tau_2 < \tau_2^*$ and above it when $\tau_2 > \tau_2^*$, see Figure 6.7b. This shows that the time-average basic reproduction number $[\mathcal{R}_0]$ underestimates the disease transmission risk when the asymptomatic carriage period is short and overestimates it when asymptomatic carriage duration is long enough. While $[\mathcal{R}_0]$ overestimates $\mathcal{R}_0$ when τ_1 varies (see Figure 6.7a). In general, the time-average basic reproduction ratio $[\mathcal{R}_0]$ may coincide with the basic reproduction ratio $\mathcal{R}_0$ or underestimate/overestimate infection risks [198].



(a) Fixed $\tau_2 = 0.8$ and τ_1 varies. (b) Fixed $\tau_1 = 0.2$ and τ_2 varies.

Figure 6.7 The graph of $\mathcal{R}_0$ and $[\mathcal{R}_0]$ when τ_1, τ_2 varies. $\mu = 0.051$, $p = 0.4$ and the other parameters as in Figure 6.6.

6.8 Discussion

The periodicity in epidemic models reflect the disease dynamics that often occur in the real world, which can be used to explain the biological mechanisms for the emergence of disease outbreaks in a population systems. In this chapter, considering a general nonlinear incidence rate function, we have proposed a stage-structured periodic disease transmission model with asymptomatic carriage and two latency periods, one is the latent period of the disease and another is the time-lag that asymptomatic carriers take to develop the disease symptoms. To derive the model, we have used the McKendrick-von Foerster equation by introducing the infection age in the disease-related classes. Then, by the method of characteristics, we have obtained a system of functional differential equations. After formulating the model, first, we have discussed the well-posedness of the model by proving the existence, uniqueness, non-negativity and boundedness of the solutions theoretically.

In epidemiology, the basic reproduction ratio $\mathcal{R}_0$ is fundamental concept. Using the theory in [217], we have found that $\mathcal{R}_0$ is the spectral radius of a linear operator on the space of continuous periodic functions, and discussed the global dynamics of the model by studying the global attractivity of the disease-free state when $\mathcal{R}_0 < 1$ and disease persistence when $\mathcal{R}_0 > 1$. Furthermore, we have investigated the sensitivity of $\mathcal{R}_0$ and $[\mathcal{R}_0]$ with respect to some key parameters including τ_1, τ_2, p, γ and $q(t)$. Using the explicit formula for $[\mathcal{R}_0]$, we have found that $[\mathcal{R}_0]$ decreases when either τ_1, τ_2 or γ increases. While it increases with respect to p, q under certain conditions. Numerically, we have found similar relationships between $\mathcal{R}_0$ and the latency periods. From disease control point of view, we understand that latent period can play a positive role in eliminating or slowing a disease spread. Further, we have seen that the epidemic might happen when a large proportion of

infected individuals becomes carriers, i.e. hidden infectious individuals, under certain conditions (e.g., condition (i)). On the other hand, in the simplified time-averaged autonomous system, we have shown that $[\mathcal{R}_0]$ underestimates the infection risk for short asymptomatic carriage period and overestimates it when asymptomatic carriage duration is long enough. While it overestimates the infection risk for τ_1.

As a case study, based on the fact that the transmission of meningococcal meningitis in sub-Saharan Africa is greatly influenced by the climatic factor for rainfall, we have estimated the periodic transmission rate function and calculated the basic reproduction ratio $\mathcal{R}_0 \approx 2.5371$ using the monthly average precipitation in Dori. Our numerical simulation has indicated an irregular patterns of epidemics varying in size and duration, which is consistent with the reported data in Burkina Faso from 1940 to 2014. This irregularity might be related to the interactions with the disease carriers (beside the ill individuals) and the seasonal variation in disease transmissibility [91]. One of the most challenging issues for meningococcal meningitis is that the pathogenesis of this disease is not fully understood since its first discovery in 1805 [65]. We hope that with the involvement of asymptomatic carriers the presented research can give some insight in understanding the disease transmission.

Chapter 7

Summary and Future Works

In this chapter, we briefly summarize the main results in this thesis and propose some problems for future investigations.

7.1 Research summary

In this thesis, we study the dynamics of time-delayed mathematical models in ecology and epidemiology.

Mathematical models can provide an important approach to understand the risk of human exploitation on fish resources. In Chapter 2, we proposed a mathematical model for a single species fish stock with three-stage structure: juveniles, small adults and large adults with two harvesting strategies for mature classes: maturity and size selectivity. We investigated the dynamical behavior of the model and discussed the effect of harvesting. After we proved the existence and uniqueness of bounded solutions, we calculated the adult reproduction number $\mathcal{R}_A$ for the model mathematically. Then, we obtained the local and global stability of the trivial equilibrium $(0, 0, 0)$ when $\mathcal{R}_A < 1$, and discussed the population persistence and existence of a unique positive equilibrium (I^*, S^*, L^*) when $\mathcal{R}_A > 1$. Numerically, we investigated the influence of harvesting functions, discussed the optimal harvesting rates, which is important ecological features, and explored the effect of periodic coefficients on the dynamical system.

Sea lice infection is one of the major threats in the marine fishery, especially for farmed salmon. In Chapter 3, we proposed a mathematical model for the growth of sea lice which is one of the major threats in the marine fishery, especially for farmed salmon. We considered three stages in sea louse lifespan: non-infectious immature, infectious immature and adults where the level of non-infectious immature development depends on the size of the adult

population. We described the nonlinear dynamics by a system of partial differential equations, then, we transformed it into a system of delay differential equations with constant delay by mathematical techniques and an appropriate change of variables. We addressed the system threshold dynamics for the established model with respect to the adult reproduction number $\mathcal{R}_s$, including the global stability of the trivial steady state $(0, 0)$ when $\mathcal{R}_s < 1$, persistence and global attractivity of a coexistence unique positive steady state $(\mathcal{C}^*, \mathcal{A}^*)$ when $\mathcal{R}_s > 1$. We provided some numerical simulation results using *Lepeophtheirus salmonis* growth as a case study with appropriate parameters from the literature.

Treatment of sea lice has become one of the top priorities in aquaculture research, due to their responsibility for most disease outbreaks on salmon farms and causing enormous monetary losses. In Chapter 4, we proposed a mathematical model for biological control of sea lice by introduction of one of its natural predators "cleaner fish" as a dynamical systems approach. We classified the growth of sea lice population into three stages: non-invasive immature, invasive immature and adults, and assumed predator prey interaction at the adult level of sea lice. Through mathematical analysis, we addressed threshold dynamics with respect to the adult reproduction number for sea lice $\mathcal{R}_s$ and the net reproductive number of cleaner fish $\mathcal{R}_f$, including the global stability of the trivial steady state when $\mathcal{R}_s < 1$, global attractivity of the predator-free equilibrium point when $\mathcal{R}_s > 1$ and $\mathcal{R}_f < 1$, persistence and coexistence of a unique positive steady state when $\mathcal{R}_s > 1$ and $\mathcal{R}_f > 1$. We discussed the local stability of the positive equilibrium point and investigated the Hopf bifurcation. Numerical simulations were provided to present a case study for comparing between two cleaner fish species, goldsinny and ballan wrasse, and show the oscillation behavior.

In Chapter 5, we considered an SEIRD model where the population of the humans is described by a system of susceptible, exposed, infectious, recovered individuals and infected corpses who are nonetheless infectious. We discussed the non-negativity, boundedness of solutions, equilibria, permanence, the local and global stability. The basic reproduction number R_0 was obtained and it determined the dynamics of the model. We proved that the disease-free equilibrium $(\frac{\Lambda}{\mu}, 0, 0, 0, 0)$ is globally stable and the disease dies out when $R_0 < 1$; and the infection always persists and a unique endemic equilibrium $(S^*, E^*, I^*, R^*, D^*)$ exists when $R_0 > 1$. We also showed that the endemic steady state is locally asymptotically stable under certain condition and globally asymptotically stable in a special case of the model. Moreover, we studied the relation between the basic reproduction number R_0 and the parameters τ and d. We showed that R_0 is a decreasing function for τ and increasing for d, which means, to reduce the spread of the Ebola, we need to extend the duration of latent and/or dispose of human remains by cremation or burial. We gave numerical simulations to demonstrate the theoretical results and show the relation between R_0 and the number of

infectious individuals.

In Chapter 6, we studied the global dynamics of a periodic disease transmission model with two delays in latent and asymptomatic carriage periods. We first derived the model system with a general nonlinear incidence rate function by stage-structure. Then, we identified the basic reproduction ratio $\mathcal{R}_0$ for the model and presented numerical algorithm to calculate it. We obtained the global attractivity of the disease-free state $(S^*(t), 0, 0, 0, 0, 0)$ when $\mathcal{R}_0 < 1$ and discussed the disease persistence when $\mathcal{R}_0 > 1$. We also explored the coexistence of endemic state in the nonautonomous system and proved the uniqueness with constants coefficients. Numerical simulations were provided to present a case study regarding the meningococcal meningitis disease transmission and discussed the influence of carriers on $\mathcal{R}_0$.

7.2 Future works

In proposing the mathematical model in Chapter 2, we assumed that the rate of change of age/size with respect to time is a constant 1. Meanwhile, we understand that such a rate is normally density-dependent in reality, which results in the mathematical system/model as nonlinear ordinary differential equations being coupled to a partial differential equation [102, 167]. Can we transform the system with mixed (age and stage/size) structure into functional differential equations where the delay may be of threshold type or state dependent? And, if so, how? This will be one of the most challenging topics in our future research. Furthermore, it will be interesting to study how seasonal fluctuations in harvesting affect population size and behavior. Spatial structure can also be added to identify spatial patterns and their relationships to ecological behaviors.

In Chapter 3, further work may include consideration of wild salmon. Sea lice from salmon farms have been suggested to be possible threats facing wild salmon. Studies showed that the heaviest infections of wild salmon are limited to the areas with a high concentration of salmon farms. More precisely, besides adult salmons which are natural hosts for the sea louse species, high density of sea lice has been found on juvenile salmons which is rare to occur in areas with no salmon farms [18, 138]. It will be interesting to derive a spatial-temporal infection (by sea lice) model for wild salmon associated in space and time when the wild salmon pass through areas containing salmon farms.

Efficient sea lice control remains one of the most important challenges for the salmon farming industry [148]. Further work in Chapter 4 may include: i) consideration of immigration and emigration in the model: Salmon farms may become infested by sea lice from wild salmon and become point sources for sea lice [44]. In the presence of adult sea lice, salmon become irritated and turn over, hence, cleaner fish can be intimidated by salmon and

leave the salmon farms [30, 160]. Extension of our framework to include immigration of sea lice and emigration of cleaner fish is an interesting future work; ii) modify the model with periodic coefficients: cleaner fish works only during the day [48, 162]. It will be interesting to study how daily fluctuations of functional response affect the population size and behavior of sea lice and cleaner fish.

In Chapters 5 and 6, it will be interesting to consider vaccination, which plays an effective role of preventing infectious diseases and is a powerful tool in the public-health control.

Bibliography

[1] The humanitarian data exchange, total number of suspected meningitis cases reported in burkina faso. November 30, 2016. Available from: https://data.humdata.org/dataset/40c8daaa-a5db-4a3a-941f-07a5bb5975b6.

[2] *Sea Lice and Salmon: Elevating the dialogue on the farmed-wild salmon story.* Watershed Watch Salmon Society, 2004.

[3] J. Abolofia, F. Asche, and J. Wilen. The cost of lice: quantifying the impacts of parasitic sea lice on farmed salmon. *Marine Resource Economics*, 32(3):329 – 349, 2017.

[4] D. Aday, D. Philipp, and D. Wahl. Sex-specific life history patterns in bluegill (lepomis macrochirus): interacting mechanisms influence individual body size. *Oecologia*, 147(1):31–38, 2006.

[5] M. Agarwal and S. Devi. A time-delay model for the effect of toxicant in a single species growth with stage-structure. *Nonlinear Analysis: Real World Applications*, 11(4):2376–2389, 2010.

[6] N. Agarwal. *Fish Reproduction.* APH Publishing Corporation, 2008.

[7] W. Aiello and H. Freedman. A time delay model of single-species growth with stage structure. *Mathematical Biosciences*, 101(2):139–153, 1990.

[8] I. Al-Darabsah, X. Tang, and Y. Yuan. A prey-predator model with migrations and delays. *Discrete and Continuous Dynamical Systems-B*, 21(3):737–761, 2016.

[9] I. Al-Darabsah and Y. Yuan. Dynamics of a general stage structured n parallel food chains. *Proceedings of the International Symposium on Mathematical and Computational Biology, World Scientific*, pages 63–84, May 2014.

[10] I. Al-Darabsah and Y. Yuan. A time-delayed epidemic model for Ebola disease transmission. *Appl. Math. Comput.*, 290:207–325, 2016.

[11] J. Al-Omari and S. Gourley. A nonlocal reaction-diffusion model for a single species with stage structure and distributed maturation delay. *European Journal of Applied Mathematics*, 16(1):37–51, 2005.

[12] M. Aldrin, R. B. Huseby, A. Stien, R. N. Grøntvedt, H. Viljugrein, and P. A. Jansen. A stage-structured bayesian hierarchical model for salmon lice populations at individual salmon farms–estimated from multiple farm data sets. *Ecological Modelling*, 359:333–348, 2017.

[13] E. Allen and H. Victory. Modelling and simulation of a schistosomiasis infection with biological control. *Acta Tropica*, 87(2):251–267, 2003.

[14] L. Allen. *An Introduction to Mathematical Biology*. Pearson, 2006.

[15] L. Allen, C. Bauch, C. Castillo-Chavez, D. Earn, Z. Feng, M. Lewis, J. Li, M. Martcheva, M. Nuno, J. Watmough, et al. *Mathematical Epidemiology (Lecture Notes in Mathematics/Mathematical Biosciences Subseries)*. Springer, 2008.

[16] C. Althaus. Estimating the reproduction number of Ebola virus (EBOV) during the 2014 outbreak in West Africa. *PLOS Currents Outbreaks*, 2014. Sept 2. Edition 1. doi:10.1371/currents.outbreaks.91afb5e0f279e7f29e7056095255b28.

[17] R. Armstrong. Stable model structures for representing biogeochemical diversity and size spectra in plankton communities. *Journal of Plankton research*, 21(3):445–464, 1999.

[18] F. Asche and T. Bjorndal. *The Economics of Salmon Aquaculture*. Wiley-Blackwell, 2nd edition edition, 2011.

[19] N. Bacaër. Approximation of the basic reproduction number r_0 for vector-borne diseases with a periodic vector population. *Bull. Math. Biology*, 69(3):1067–1091, 2007.

[20] N. Bacaër and E. Dads. On the biological interpretation of a definition for the parameter r_0 in periodic population models. *J. Math. Biology*, 65(4):601–621, 2012.

[21] J. Baker, R. Muller, and D. Rollinson, editors. *Advances in Parasitology*, volume 44. Academic Press, 1999.

[22] M. Begon, C. Townsend, and J. Harper. *Ecology: From Individuals to Ecosystems*. Wiley-Blackwell, 4 edition, 2005.

[23] E. Beretta and D. Breda. An SEIR epidemic model with constant latency time and infectious period. *Mathematical Biosciences and Engineering*, 8(4):931–952, 2011.

[24] E. Beretta and Y. Kuang. Geometric stability switch criteria in delay differential systems with delay dependent parameters. *SIAM J Math Anal*, 33(5):1144 – 1165, 2002.

[25] E. Beretta and Y. Takeuchi. Global stability of an SIR epidemic model with time delays. *J. Math. Biology*, 33(3):250–260, 1995.

[26] S. Blythe, R. Nisbet, and W. Gurney. The dynamics of population models with distributed maturation periods. *Theoretical population biology*, 25(3):289–311, 1984.

[27] G. Bocharov and K. Hadeler. Structured population models, conservation laws, and delay equations. *Journal of Differential Equations*, 168(1):212–237, 2000.

[28] B. Bolker. *Ecological Models and Data in R*. Princeton University Press, 2008.

[29] K. Boxaspen. A review of the biology and genetics of sea lice. *ICES Journal of marine Science*, 63(7):1304–1316, 2006.

[30] G. A. Boxshall and D. Defays. *Pathogens of wild and farmed fish: sea lice*. CRC Press, 1993.

[31] F. Brauer. Periodic solutions of some ecological models. *J. theoretical biology*, 69(1):143–152, 1977.

[32] F. Brauer. Stability of some population models with delay. *Mathematical Biosciences*, 33(3-4):345–358, 1977.

[33] F. Brauer. Constant-rate harvesting of age-structured populations. *SIAM J. Math Analysis*, 14(5):947–961, 1983.

[34] F. Brauer and D. A. Sanchez. Constant rate population harvesting: equilibrium and stability. *Theoretical population biology*, 8(1):12–30, 1975.

[35] F. Brauer and A. C. Soudack. Optimal harvesting in predator–prey systems. *International Journal of Control*, 41(1):111–128, 1985.

[36] I. Bricknell, S. Dalesman, B. O'Shea, C. Pert, and A. M. Luntz. Effect of environmental salinity on sea lice lepeophtheirus salmonis settlement success. *Diseases of Aquatic Organisms*, 71(3):201–212, 2006.

[37] M. Burgess, F. Diekert, N. Jacobsen, K. Andersen, and S. Gaines. Remaining questions in the case for balanced harvesting. *Fish and Fisheries*, 2015.

[38] Committee on Prevention and Control of Sexually Transmitted Diseases, Institute of Medicine and National Academy of Sciences. *The hidden epidemic: confronting sexually transmitted diseases*. National Academies Press, 1997.

[39] B. Connors, E. Juarez-Colunga, and L. Dill. Effects of varying salinities on lepeophtheirus salmonis survival on juvenile pink and chum salmon. *Journal of Fish Biology*, 72(7):1825–1830, 2008.

[40] K. Cooke and P. van den Driessche. Analysis of an SEIRS epidemic model with two delays. *J. Math. Bioscience*, 35:240–260, 1996.

[41] K. L. Cooke. Stability analysis for a vector disease model. *The Rocky Mountain J. Math*, 9(1):31–42, 1979.

[42] M. Costello. Ecology of sea lice parasitic on farmed and wild fish. *Trends in Parasitology*, 22(10):475–483, 2006.

[43] M. Costello. The global economic cost of sea lice to the salmonid farming industry. *Journal of Fish Diseases*, 32(1):115–118, 2009.

[44] M. Costello. How sea lice from salmon farms may cause wild salmonid declines in europe and north america and be a threat to fishes elsewhere. *Proc Biol Sci.*, 276(1672):3385–3394, 2009.

[45] J. Cushing. *An Introduction to Structured Population Dynamics.* SIAM, Philadelphia, USA, 1998.

[46] J. Cushing and M. Saleem. A predator prey model with age structure. *J. Math. Biology*, 14(2):231–250, 1982.

[47] J. Davenport, K. Black, G. Burnell, T. Cross, S. Culloty, S. Ekaratne, B. Furness, M. Mulcahy, and H. Thetmeyer. *Aquaculture: The Ecological Issues.* Wiley–Blackwell, 2003.

[48] S. Deady, S. Varian, and J. Fives. The use of cleaner-fish to control sea lice on two irish salmon (salmo salar) farms with particular reference to wrasse behaviour in salmon cages. *Aquaculture*, 131:73 – 90, 1995.

[49] g. o. c. Department of Justice. Newfoundland and labrador fishery regulations (sor/78-443). April 28, 2018. Available at http://laws-lois.justice.gc.ca/eng/regulations/SOR-78-443/page-3.html.

[50] E. DeSombre and J. Barkin. *Fish.* Wiley, 2013.

[51] O. Diekmann, J. Heesterbeek, and J. Metz. On the definition and the computation of the basic reproduction ratio r_0 in models for infectious diseases in heterogeneous populations. *J. math biology*, 28(4):365–382, 1990.

[52] O. Diekmann, J. Heesterbeek, and M. Roberts. The construction of next-generation matrices for compartmental epidemic models. *J. R. Soc. Interface*, 7:873–885, 2010.

[53] U. S. DOC, NOAA, and NMFS. *Final Environmental Impact Statement for American Fisheries Act Amendments 61/61/13/8.* Alaska Region, February 2002.

[54] O. Dragesund, J. Hamre, and Ø. Ulltang. Biology and population dynamics of the norwegian spring- spawning herring. *Rapports et Procs-Verbaux des Runions, Conseil Permanent International pour l'Exploration de la Mer*, 177:43–71, 1980.

[55] G. Engelhard and M. Heino. Maturity changes in norwegian spring-spawning herring clupea harengus: compensatory or evolutionary responses. *Marine Ecology Progress Series*, 272:245–256, 2004.

[56] C. Eurich, A. Thiel, and L. Fahse. Distributed delays stabilize ecological feedback systems. *Physical review letters*, 94(15):158104, 2005.

[57] G. Fan, H. R. Thieme, and H. Zhu. Delay differential systems for tick population dynamics. *Journal of mathematical biology*, 71(5):1017–1048, 2015.

[58] J. Fang, S. Gourley, and Y. Lou. Stage-structured models of intra- and interspecific competition within age classes. *Journal of Differential Equations*, 260(2):1918–1953, 2016.

[59] P. Fenberg and K. Roy. Ecological and evolutionary consequences of size-selective harvesting: How much do we know? *Molecular Ecology*, 17:209–220, 2008.

[60] G. Flik and G. Wiegertjes, editors. *Host-Parasite Interactions*. Taylor & Francis, 2004.

[61] S. Garcia, J. Kolding, J. Rice, M. Rochet, S. Zhou, T. Arimoto, J. Beyer, L. Borges, A. Bundy, D. Dunn, E. Fulton, M. Hall, M. Heino, R. Law, M. Makino, A. Rijnsdorp, F. Simard, and A. Smith. Reconsidering the consequences of selective fisheries. *Science*, 335:1045–1047, 2012.

[62] D. Garrod. Population dynamics of the arcto-norwegian cod. *Journal of the Fisheries Research Board of Canada*, 24:145–190, 1967.

[63] M. Ghosh, P. Chandra, P. Sinha, and J. Shukla. Modelling the spread of carrier-dependent infectious diseases with environmental effect. *Appl. Math. Comput.*, 152(2):385–402, 2004.

[64] M. Giaquinta and G. Modica. *Mathematical Analysis: An Introduction to Functions of Several Variables*. Birkhäuser Basel, 2009.

[65] E. A. Glasper, G. McEwing, and J. Richardson, editors. *Emergencies in Children's and Young People's Nursing*. Oxford University Press, 2011.

[66] M. Golubitsky and D. Schaeffer. *Singularities and Groups in Bifurcation Theory: Volume I (Applied Mathematical Sciences)*. Springer, New York, 1985.

[67] S. Gourley and Y. Lou. A mathematical model for the spatial spread and biocontrol of the Asian longhorned beetle. *SIAM J. Appl. Math.*, 74(3):864–884, 2014.

[68] M. L. Groner, R. Cox, G. Gettinby, and C. W. Revie. Use of agent-based modelling to predict benefits of cleaner fish in controlling sea lice, lepeophtheirus salmonis, infestations on farmed atlantic salmon, salmo salar l. *Journal of Fish Diseases*, 36:195–208, 2013.

[69] Z. Grossman. Oscillatory phenomena in a model of infectious diseases. *Theoretical population biology*, 18(2):204–243, 1980.

[70] W. Gurney and R. Nisbet. *Ecological dynamics*. Oxford University Press, 1998.

[71] W. Gurney, R. Nisbet, and J. Lawton. The systematic formulation of tractable single-species population models incorporating age structure. *The Journal of Animal Ecology*, pages 479–495, 1983.

[72] J. K. Hale. *Asymptotic Behavior of Dissipative Systems*. Math. Surveys Monogr., vol. 25, American Mathematical Society, Providence, RI, 1988.

[73] J. K. Hale. *Ordinary Differential Equations*. Dover Publications, 2009.

[74] J. K. Hale and S. M. Verduyn-Lunel. *Introduction to Functional Differential Equations*. Springer-Verlag, New York, 1993.

[75] J. K. Hale and P. Waltman. Persistence in infinite-dimensional systems. *SIAM J. Math. Anal.*, 20:388–395, 1989.

[76] P. Hancock and H. Godfray. Application of the lumped age-class technique to studying the dynamics of malaria-mosquito-human interactions. *Malaria Journal*, 6(98), 2007.

[77] J. Harte. *Consider A Spherical Cow, A Course in Environmental Problem Solving*. University Science Books, 1988.

[78] B. Hassard, D. Kazarinoff, and Y. Wan. *Theory and Applications of Hopf Bifurcation*. Cambridge University Press, Cambridge, 1981.

[79] M. Heino, B. D. Pauli, and U. Dieckmann. Fisheries-induced evolution. *Annual Review of Ecology, Evolution, and Systematics*, 46(98):461–480, 2015.

[80] G. Helfman. *Fish Conservation: A Guide to Understanding and Restoring Global Aquatic Biodiversity and Fishery Resources*. Island Press, 2007.

[81] G. Helfman, B. Collette, and D. Facey. *The Diversity of Fishes*. Blackwell Science, 1997.

[82] H. Hethcote and P. van den Driessche. Two SIS epidemiologic models with delays. *J. Math. Biology*, 40(1):3–26, 2000.

[83] H. W. Hethcote. The mathematics of infectious diseases. *SIAM review*, 42(4):599–653, 2000.

[84] P. Heuch, J. Nordhagen, and T. Schram. Egg production in the salmon louse [lepeophtheirus salmonis (krøyer)] in relation to origin and water temperature. *Aquaculture Research*, 31(11):805–814, 2000.

[85] M. Hirsch, H. L. Smith, and X.-Q. Zhao. Chain transitivity, attractivity, and strong repellors for semidynamical systems. *Journal of Dynamics and Differential Equations*, 13(1):107–131, 2001.

[86] W. Hirsch, H. Hanisch, and J.-P. Gabriel. Differential equation models of some parasitic infections: Methods for the study of asymptotic behavior. *Communications on Pure and Applied Mathematics*, 38:733–753, 1985.

[87] C. Holling. The functional response of predator to prey density and its role in mimicry and population regulation. *Memoirs of the Entomological Society of Canada*, 97:5–60, 1965.

[88] G. Huang, Y. Takeuchi, W. Ma, and D. Wei. Global stability for delay SIR and SEIR epidemic models with nonlinear incidence rate. *Bull. Math. Biology*, 72(5):1192–1207, 2010.

[89] A. Huppert and G. Katriel. Mathematical modelling and prediction in infectious disease epidemiology. *Clinical Microbiology and Infection*, 19(11):999–1005, 2013.

[90] A. Imsland, P. Reynolds, G. Eliassen, T. Hangstad, , A. Nytrø, A. Foss, E. Vikingstad, and T. Elvegård. Assessment of growth and sea lice infection levels in atlantic salmon stocked in small-scale cages with lumpfish. *Aquaculture*, 433:137–142, 2014.

[91] T. Irving, K. Blyuss, C. Colijn, and C. Trotter. Modelling meningococcal meningitis in the african meningitis belt. *Epidemiol. Infect.*, 140(5):897–905, 2012.

[92] V. Ivlev. *Experimental Ecology of the Feeding of Fishes*. Yale University Press, 1961.

[93] R. James. *Mathematics Dictionary*. Springer, 1992.

[94] K. Johnson, E. Laman, G. Cailliet, and R. Starr. *Fishery Resources of the Monterey Bay National Marine Sanctuary*. University of California, California Sea Grant College System, 1998.

[95] S. Johnson and L. Albright. Development, growth, and survival of lepeophtheirus salmonis (copepoda: Caligidae) under laboratory conditions. *Journal of the Marine Biological Association of the United Kingdom*, 71(2):425–436, 1991.

[96] D. Kalajdzievska and M. Li. Modeling the effects of carriers on transmission dynamics of infectious disease. *Math. Bio. & Eng.*, 8(3):711–722, 2011.

[97] G. Karleskint, R. Turner, and J. Small. *Introduction to Marine Biology*. Brooks Cole, 2012.

[98] G. Kearn. *Leeches, Lice and Lampreys: A Natural History of Skin and Gill Parasites of Fishes*. Springer, 2004.

[99] B. Keller and R. Leslie. *Sea-Silver: Inside British Columbia's Salmon-Farming Industry*. Horsdal & Schubart Publishers, 1996.

[100] J. Kemper. The effects of asymptotic attacks on the spread of infectious disease: A deterministic model. *Bull. Math. Biology*, 40(6):707–718, 1978.

[101] S. Kern, A. Tiono, M. Makanga, A. Gbadoẽ, Z. Premji, O. Gaye, I. Sagara, D. Ubben, M. Cousin, F. Oladiran, O. Sander, and B. Ogutu. Community screening and treatment of asymptomatic carriers of plasmodium falciparum with artemether-lumefantrine to reduce malaria disease burden: a modelling and simulation analysis. *Malaria Journal*, page 10:210, 2011.

[102] M. Kloosterman, S. Campbell, and F. Poulin. An NPZ model with state-dependent delay due to size-structure in juvenile zooplankton. *SIAM J. Appl. Math.*, 76(2):551–577, 2016.

[103] A. Korobeinikov and P. Maini. Non-linear incidence and stability of infectious disease models. *Mathematical Medicine and Biology*, 22:113–128, 2005.

[104] Y. Kuang. *Delay Differential Equation with Application in Population Dynamics*. Academic Press, New York, 1993.

[105] X. Lai and X. Zou. Modeling HIV-1 virus dynamics with both virus-to-cell infection and cell-to-cell transmission. *SIAM J. Appl. Math.*, 74(3):898–917, 2014.

[106] P. Lancaster and M. Tismenetsky. *The Theory of Matrices: With Applications (Computer Science and Scientific Computing)*. Academic Press, second edition, 1985.

[107] P. Landi, C. Hui, and U. Dieckmann. Fisheries-induced disruptive selection. *J. Theoretical Biol.*, 365:204–216, 2015.

[108] R. Law, M. Plank, and J. Kolding. Balanced exploitation and coexistence of interacting, size-structured, fish species. *Fish and Fisheries*, 2014.

[109] J. Lawless, editor. *Statistics in Action: A Canadian Outlook*. Chapman & Hall/CRC, 2014.

[110] E. Lawrence. *Henderson's Dictionary of Biology*. Pearson, 2011.

[111] S. Levin, T. Hallam, and L. Gross, editors. *Applied Mathematical Ecology*. Springer, 1989.

[112] A. Lewis. Life history of the caligid copepod lepeophtheirus dissimulatus wilson, 1905 (crustacea: Caligoida). *Pacif. Sci.*, 17(2):195–242, 1963.

[113] J. Li and X. Zou. Dynamics of an epidemic model with non-local infections for disease with latency over a patch environment. *J. Math. Biology*, 60:645–686, 2010.

[114] X. Liang and X.-Q. Zhao. Asymptotic speeds of spread and traveling waves for monotone semiflows with applications. *Commun. Pure Appl. Math.*, 60:1–40, 2007.

[115] W. Liu, H. Hethcote, and S. Levin. Dynamical behavior of epidemiological models with nonlinear incidence rates. *J. Math. Biology*, 25:359–380, 1987.

[116] W. Liu, S. Levin, and Y. Isawa. Influence of nonlinear incidence rates upon the behaviour of SIRS epidemiological models. *J. Math. Biology*, 23:187–204, 1986.

[117] X. Liu and X.-Q. Zhao. A periodic epidemic model with age structure in a patchy environment. *SIAM J. Appl. Math.*, 71(6):1896–1917, 2011.

[118] Z. Liu and N. Li. Stability and bifurcation in a predator-prey model with age structure and delays. *Journal of Nonlinear Science*, 25(4):937957, 2015.

[119] Y. Lou and X.-Q. Zhao. Threshold dynamics in a time-delayed periodic SIS epidemic model. *Discrete and Continuous Dynamical Systems-B*, 12(1):169–186, 2009.

[120] Y. Lou and X.-Q. Zhao. A climate-based malaria transmission model with structured vector population. *SIAM J. Appl. Math.*, 70(6):2023–2044, 2010.

[121] Y. Lou and X.-Q. Zhao. Modelling malaria control by introduction of larvivorous fish. *Bulletin of mathematical biology*, 74(10):2384–2407, 2011.

[122] Y. Lou and X.-Q. Zhao. A theoretical approach to understanding population dynamics with seasonal developmental durations. *J. Nonlinear Sci.*, 27:573–603, 2017.

[123] P. Magal and X.-Q. Zhao. Global attractors and steady states for uniformly persistent dynamical systems. *SIAM J. Math. Analysis*, 37:251–275, 2005.

[124] R. Mansueti. Age, growth, and movements of the striped bass, roccus saxatilis, taken in size selective fishing gear in maryland. *Chesapeake Science*, 2:9–36, 1961.

[125] M. Martcheva. *An Introduction to Mathematical Epidemiology*. Springer, US, 2015.

[126] A. Martin and S. Ruan. Predator-prey models with delay and prey harvesting. *J. Math Biology*, 43(3):247–267, 2001.

[127] M. McAllister and R. Peterman. Decision analysis of a large-scale fishing experiment designed to test for a genetic effect of size-selective fishing on british columbia pink salmon (oncorhynchus gorbuscha). *Canadian Journal of Fisheries and Aquatic Sciences*, 49:1305–1304, 1992.

[128] L. McKendrick. Applications of mathematics to medical problems. *Proc. Edinburgh Math. Soc.*, 44:98–130, 1926.

[129] P. Meadows and J. Campbell. *An Introduction to Marine Science*. Springer Netherlands, 1988.

[130] G. Medley, N. Lindop, W. Edmunds, and D. Nokes. Hepatitis-b virus edemicity: Heterogeneity, catastrophic dynamics and control. *Nature Medicine*, 7(5):619–624, 2001.

[131] K. Mischaikow, H. L. Smith, and H. R. Thieme. Asymptotically autonomous semiflows: chain recurrence and lyapunov functions. *Trans. Amer. Math. Soc.*, 347:1669–1685, 1995.

[132] J. Mueller and B. Gessner. A hypothetical explanatory model for meningococcal meningitis in the african meningitis belt. *International Journal of Infectious Diseases*, 14:e553–e559, 2009.

[133] A. Mustafa, G. Conboy, and J. Burka. Life-span and reproductive capacity of sea lice, lepeophtheirus salmonis, under laboratory conditions. *Aquaculture Association Of Canada*, 4:113–114, 2001.

[134] R. Naresh, S. Pandey, and A. K. Misra. Analysis of a vaccination model for carrier dependent infectious diseases with environmental effects. *Nonlinear Analysis: Modelling and Control*, 13(3):331–350, 2008.

[135] K. Nelson and C. Williams. *Infectious Disease Epidemiology: Theory and Practice*. Jones & Bartlett Learning, 2014.

[136] C. Nielsen, S. Kidd, A. Sillah, E. Davis, J. Mermin, and P. Kilmarx. Improving burial practices and cemetery management during an Ebola virus disease epidemic-Sierra Leone. *Morbidity and Mortality Weekly Report (*MMWR*)*, 64(1):20–27, 2014.

[137] R. Nisbet and W. Gurney. The systematic formulation of population models for insects with dynamically varying instar duration. *Theoretical Population Biology*, 23(1):114–135, 1983.

[138] ø. Aas, A. Klemetsen, S. Einum, and J. Skurdal, editors. *Atlantic Salmon Ecology*. Wiley-Blackwell, 2010.

[139] A. Oaten and W. Murdoch. Functional response and stability in predator–prey systems. *The American Naturalist*, 109(967):289–298, 1975.

[140] N. Oliveira and F. Hilker. Modelling disease introduction as biological control of invasive predators to preserve endangered prey. *Bulletin of mathematical biology*, 72(2):444–468, 2010.

[141] A. Opdal. Fisheries change spawning ground distribution of northeast arctic cod. *Biology Letters*, 6:261–264, 2010.

[142] T. Pandian. *Genetic Sex Differentiation in Fish*. CRC Press, 2012.

[143] A. Pike. Sea lice–major pathogens of farmed atlantic salmon. *Parasitology Today*, 5(9):291–297, 1989.

[144] A. Pike and S. Wadsworth. Sealice on salmonids: Their biology and control. *Advances in Parasitology*, 44:233–337, 1999.

[145] J. Poos, A. Brannstorm, and U. Dieckmann. Harvest-induced maturation evolution under different life-history trade-offs and harvesting regimes. *J. Theoretical Biol.*, 279:102–112, 2011.

[146] D. Posny and J. Wang. Computing the basic reproductive numbers for epidemiological models in nonhomogeneous environments. *Appl. Math. Comput.*, 242:473–490, 2014.

[147] R. Poulin, editor. *Parasites in Marine Systems*. Cambridge University Press, 2004.

[148] A. Powell, J. Treasurer, C. Pooley, A. Keay, R. Lloyd, A. Imsland, and C. G. de Leaniz. Use of lumpfish for sea-lice control in salmon farming: challenges and opportunities. *Reviews in Aquaculture*, 0:1–20, 2017.

[149] M. Powell, P. Reynolds, and T. Kristensen. Freshwater treatment of amoebic gill disease and sea-lice in seawater salmon production: Considerations of water chemistry and fish welfare in norway. *Aquaculture*, 448:18 – 28, 2015.

[150] L. Real. The kinetics of functional response. *The American Naturalist*, 111(978):289–300, 1977.

[151] C. Revie, C. Robbins, G. Gettinby, L. Kelly, and J. Treasurer. A mathematical model of the growth of sea lice, lepeophtheirus salmonis, populations on farmed atlantic salmon, salmo salar l., in scotland and its use in the assessment of treatment strategies. *Journal of Fish Diseases*, 28:603 – 613, 2005.

[152] P. Richards, J. Amara, M. Ferme, E. Mokuwa, P. Koroma, I. Sheriff, R. Suluku, and M. Voors. Social pathways for Ebola virus disease in rural Sierra Leone, and some implications for containment. *PLOS Neglected Tropical Diseases*, 2015. April 17. doi:10.1371/journal.pntd.0003567.

[153] M. Riggs, A. Sethi, T. Zabarsky, E. Eckstein, R. Jump, and C. Donskey. Asymptomatic carriers are a potential source for transmission of epidemic and nonepidemic clostridium difficile strains among long-term care facility residents. *Clin. Infect. Dis.*, 45(8):992–998, 2007.

[154] M. Rittenhouse, C. Revie, and A. Hurford. A model for sea lice (lepeophtheirus salmonis) dynamics in a seasonally changing environment. *Epidemics*, 16:8 – 16, 2016.

[155] C. Rivers, E. Lofgren, M. Marathe, S. Eubank, and B. Lewis. Modeling the impact of interventions on an epidemic of Ebola in Sierra Leone and Liberia. *PLOS Currents Outbreaks*, 2014. Nov 6. Edition 2. doi: 10.1371/currents.outbreaks.4d41fe5d6c05e9df30ddce33c66d084c.

[156] F. Rizaner and S. Rogovchenko. Dynamics of a single species under periodic habitat fluctuations and allee effect. *Nonlinear Analysis: Real World Applications*, 13(1):141–157, 2012.

[157] C. Roberts. *The Ocean of Life: The Fate of Man and the Sea.* Penguin Books, Reprint edition, 2013.

[158] P. Rouquet, J.-M. Froment, M. Bermejo, A. Kilbourne, W. Karesh, P. Reed, B. Kumulungui, P. Yaba, A. Delicat, P. Rollin, and E. M. Leroy. Wild animal mortality monitoring and human Ebola outbreaks, Gabon and Republic of Congo, 2001-2003. *Emerg. Infect. Dis.*, 11:283–290, 2005.

[159] S. Ruan, D. Xiao, and J. C. Beier. On the delayed ross–macdonald model for malaria transmission. *Bulletin of mathematical biology*, 70(4):1098–1114, 2008.

[160] M. Sayer, J. Treasurer, and M. Costello. *Wrasse: biology and use in aquaculture.* Wiley-Blackwell, 1996.

[161] M. Singer. *Anthropology of Infectious Disease.* Routledge, 2014.

[162] A. Skiftesvik, R. Bjelland, C. Durif, I. Johansen, and H. Browman. Delousing of atlantic salmon (salmo salar) by cultured vs. wild ballan wrasse (labrus bergylta). *Aquaculture*, 402:113–118, 2013.

[163] A. Skiftesvik, G. Blom, A.-L. Agnalt, C. Durif, H. Browman, R. Bjelland, L. Harkestad, E. Farestveit, O. Paulsen, M., T. Havelin, K. Johnsen, and S. Mortensen. Wrasse (labridae) as cleaner fish in salmonid aquaculture the hardangerfjord as a case study. *Marine Biology Research*, 10(3):289–300, 2014.

[164] H. L. Smith. Reduction of structured population models to threshold-type delay equations and functional differential equations: A case study. *Math. Biosci.*, 113:1–23, 1993.

[165] H. L. Smith. Threshold delay differential equations are equivalent to standard FDEs. *in International Conference on Differential Equations (Equadiff-91), World Scientific, River Edge*, pages 899–904, 1993.

[166] H. L. Smith. Existence and uniqueness of global solutions for a size-structured model of an insect population with variable instar duration. *Rocky Mountain J. Math.*, 24:311–334, 1994.

[167] H. L. Smith. A structured population model and a related functional differential equation: Global attractors and uniform persistence. *J. Dynam. Diff. Eq.*, 6(1):71–99, 1994.

[168] H. L. Smith. Equivalent dynamics for a structured population model and a related functional differential equation. *Rocky Mountain J. Math.*, 25(1):491–499, 1995.

[169] H. L. Smith. *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*. American Mathematical Society, 1995.

[170] H. L. Smith and P. Waltman. *The Theory of the Chemostat: Dynamics of Microbial Competition*. Cambridge University Press, 1995.

[171] H. L. Smith and X.-Q. Zhao. Robust persistence for semidynamical systems. *Nonlinear Anal.*, 47:6169–6179, 2001.

[172] T. Smith. E*bola and* M*arburg Virus (Deadly Diseases and Epidemics)*. Chelsea House Pub, second edition, 2010.

[173] C. Smorynski. *History of Mathematics*. Springer-Verlag, New York, 2008.

[174] X. Song and L. Chen. Optimal harvesting and stability for a two-species competitive system with stage structure. *Mathematical Biosciences*, 170(2):173–186, 2001.

[175] Y. Song, J. Wei, and Y. Yuan. Stability switches and hopf bifurcations in a pair of delay-coupled oscillators. *Journal of Nonlinear Science*, 17(2):145–166, 2007.

[176] G. Stépán. *Retarded dynamical systems: stability and characteristic functions*. Longman Scientific & Technical, 1989.

[177] A. Stien, P. Bjørn, P. Heuch, and D. Elston. Population dynamics of salmon lice lepeophtheirus salmonis on atlantic salmon and sea trout. *Marine Ecology Progress Series*, 290:263–275, 2005.

[178] D. Sulsky, R. Vance, and W. Newman. Time delays in age-structured populations. *Journal of Theoretical Biology*, 141(3):403–422, 1989.

[179] Y. Takeuchi, Y. Iwasa, and K. Sato, editors. *Mathematics for Life Science and Medicine*. Springer, 2007.

[180] S. Tang and R. Cheke. Models for integrated pest control and their biological implications. *Mathematical Biosciences*, 215(1):115–125, 2008.

[181] Centers for Disease Control and Prevention. Ebola (Ebola virus disease)-Q&As on transmission. Available at http://www.cdc.gov/vhf/ebola/transmission/qas.html.

[182] Centers for Disease Control and Prevention. Ebola (Ebola virus disease) transmission. Available at http://www.cdc.gov/vhf/ebola/transmission/index.html.

[183] World Health Organization. Ebola situation reports. July 12, 2015. Available at http://apps.who.int/ebola/ebola-situation-reports.

[184] World Health Organization. Ebola virus disease. Available at http://www.who.int/mediacentre/factsheets/fs103/en.

[185] H. R. Thieme. Convergence results and poincare-bendixson trichotomy for asymptotically autonomous differential equations. *J. Math. Biology*, 30:755–763, 1992.

[186] H. R. Thieme. Asymptotically autonomous differential equations in the plane. *Rocky Mountain Journal of Mathematics*, 24(1):351–380, 2003.

[187] H. R. Thieme. *Mathematics in Population Biology*. Princeton University Press, Princeton, NJ, 2003.

[188] Y. Tian and Y. Yuan. Effect of time delays in an hiv virotherapy model with nonlinear incidence. *Proc. R. Soc.*, page A 472:20150626, 2016. http://dx.doi.org/10.1098/rspa.2015.0626.

[189] O. Torrissen, S. Jones, F. Asche, A. Guttormsen, O. Skilbrei, F. Nilsen, T. Horsberg, and D. Jackson. Salmon lice – impact on wild salmonids and salmon aquaculture. *Journal of Fish Diseases*, 36(3):171 – 194, 2013.

[190] J. Treasurer. A review of potential pathogens of sea lice and the application of cleaner fish in biological control. *Pest Management Science*, 58(6):546–558, 2002.

[191] C. L. Trotter, N. Gay, and W. Edmunds. Dynamic models of meningococcal carriage, disease, and the impact of serogroup c conjugate vaccination. *American J. Epidemiology*, 162(1):89–100, 2005.

[192] C. L. Trotter and B. Greenwood. Meningococcal carriage in the african meningitis belt. *Lancet Infect Dis*, 7(12):797–803, 2007.

[193] P. van den Driessche. Reproduction numbers of infectious disease models. *Infectious Disease Modelling*, 2(3):288–303, 2017.

[194] P. van den Driessche, L. Wang, and X. Zou. Modeling diseases with latency and relapse. *Mathematical Biosciences and Engineering*, 4(2):205–219, 2007.

[195] M. Verspoor, K. D. Bot, and W. Lowie. *A dynamic approach to second language development: Methods and techniques*, volume 29. John Benjamins Publishing, 2011.

[196] C. Walters and J. Korman. *Salmon Stocks*. Pacific Fisheries Resource Conservation Council, Vancouver, BC, Canada, June 1999.

[197] W. Wang and X.-Q. Zhao. An epidemic model with population dispersal and infection period. *SIAM J. Appl. Math.*, 66(4):1454–1472., 2006.

[198] W. Wang and X.-Q. Zhao. Threshold dynamics for compartmental epidemic models in periodic environments. *J. Dynamical Differential Equations*, 20(3):699–171, 2008.

[199] X. Wang and X.-Q. Zhao. Dynamics of a time-delayed Lyme disease model with seasonality. *SIAM J. Appl. Math.*, 16(2):853–881, 2017.

[200] X. Wang and X.-Q. Zhao. A malaria transmission model with temperature-dependent incubation period. *Bulletin of mathematical biology*, 79(5):1155–1182, 2017.

[201] X. Wang and X.-Q. Zhao. A periodic vector-bias malaria model with incubation period. *SIAM J. Appl. Math.*, 77(1):181–201, 2017.

[202] Z. Wang and X.-Q. Zhao. Global dynamics of a time-delayed dengue transmission model. *Canadian Applied Mathematics Quarterly*, 20:89–113, 2012.

[203] G. Webb, C. Browne, X.Huo, O. Seydi, M. Seydi, and P. Magal. A model of the 2014 Ebola epidemic in west Africa with contact tracing. *PLOS Currents Outbreaks*, 2015. Jan 30 . Edition 1. doi: 10.1371/currents.outbreaks.846b2a31ef37018b7d1126a9c8adf22a.

[204] J. Wei and M. Li. Hopf bifurcation analysis in a delayed nicholson blowflies equation. *Nonlinear Analysis*, 60:1351–1367, 2003.

[205] J. Weitz and J. Dushoff. Modeling post-death transmission of Ebola: challenges for inference and opportunities for control. *Scientific Reports*, 5, 2015. Article number: 8751. doi:10.1038/srep08751.

[206] R. Wootton. *Fish Ecology*. Springer Science & Business Media, 1991.

[207] Y. Xiao and X. Zou. Transmission dynamics for vector-borne diseases in a patchy environment. *J. Math. Biology*, 69:113–146, 2014.

[208] C. Xu, M. Boyce, and D. Daley. Harvesting in seasonal environments. *J. Math. Biology*, 50:663–668, 2005.

[209] D. Xu and X.-Q. Zhao. Dynamics in a periodic competitive model with stage structure. *J. Math. Anal. Appl.*, 311:417–438, 2005.

[210] R. Xu and Z. Ma. Global stablity of a delayed SEIRS epidemic model with saturation incidence rate. *Nonlinear Dyn.*, 61:229–239, 2010.

[211] Y. Yuan. A coupled plankton system with instantaneous and delayed predation. *Journal of Biological Dynamics*, 6(2):148–165, 2012.

[212] Y. Yuan and J. Bélair. Threshold dynamics in an SEIRS model with latency and temporary immunity. *J. Math. Biology*, 69:875–904, 2014.

[213] Y. Yuan and X.-Q. Zhao. Global stability for non-monotone delay equations (with application to a model of blood cell production). *J. Differential Equations*, 252:2189–2209, 2012.

[214] T. Zhang and Z. Teng. On a nonautonomous SEIRS model in epidemiology. *Bull. Math. Biology*, 69(8):2537–2559, 2007.

[215] S. Zhao, Z. Xu, and Y. Lu. A mathematical model of hepatitis b virus transmission and its application for vaccination strategy in china. *Int. J. Epidemiol.*, 29(4):744–752, 2000.

[216] X.-Q. Zhao. Permanence implies the existence of interior periodic solutions for FDEs. *International Journal of Qualitative Theory of Differential Equations and Applications*, 2:125–137, 2008.

[217] X.-Q. Zhao. Basic reproduction ratios for periodic compartmental models with time delay. *Journal of Dynamics and Differential Equations*, 29(1):67 – 82, 2017.

[218] X.-Q. Zhao. *Dynamical Systems in Population Biology*. Springer-Verlag, New York, second edition, 2017.

[219] X.-Q. Zhao and Z. Jing. Global asymptotic behavior in some cooperative systems of functional differential equations. *Canadian Applied Mathematics Quarterly*, pages 421–444, 1996.