Starter Sequences: Generalizations and Applications

by

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Department of Mathematics and Statistics

Memorial University

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Abstract

In this thesis we introduce new types of starter sequences, pseudo-starter sequences, starter-labellings, and generalized (extended) starter sequences. We apply these new sequences to graph labeling. All the necessary conditions for the existence of starter, pseudo-starter, extended, m-fold, excess, and generalized (extended) starter sequences are determined, and some of these conditions are shown to be sufficient. The relationship between starter sequences and graph labellings is introduced. Moreover, the starter-labeling and the minimum hooked starter-labeling of paths, cycles, and k-windmills are investigated. We show that all paths, cycles, and k-windmills can be starter-labelled or minimum starter-labelled.

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First of all, I praise and thank Allah for his many blessings on me.
“ I cannot succeed except with the help of Allah; Upon him I have relied, and to Him I turn for everything ”. (Quran, Surah 11: Verse 88).
Words cannot express how grateful and thankful I am to those whom Allah commands us to be very kind and dutiful to them.
“ Your Lord has decreed that you worship none but him and be good to your parents. Whether one or both of them reach old age in your life, do not say to them a word of annoyance and do not repel them but rather speak to them a noble word. Lower to them the wing of humility for them through mercy and say: My Lord, have mercy upon them as they brought me up when I was small ”. (Quran, Surah 17: Verses 23 and 24).

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To my parents, my wife, my children, and my siblings, with love.

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Chapter 1

Introduction

Starters have been widely used in several combinatorial designs, such as Room squares [47], Howell designs ([3], [12]), Kirkman triple systems [48], and round-robin tournaments [50]. Starters were first used by Stanton and Mullin to construct Room squares [47]. Since then, starters have been widely used in several combinatorial designs. A **starter** in the odd order additive abelian group G is a set of unordered pairs of elements of G such that each non-zero element occurs precisely once in some pair, and also precisely once as the difference of one of the pairs. Thus, $\{x_i : 1 \leq i \leq n\} \bigcup \{y_i : 1 \leq i \leq n\} = G \setminus \{0\}$ and the differences $\{\pm(x_1 - y_1), \pm(x_2 - y_2), \ldots, \pm(x_n - y_n)\}$ are also all the non-zero elements of G, where $|G| = 2n + 1$. For example, the pairs $\{5, 6\}, \{2, 4\}, \{3, 8\}, \{1, 7\}$ form a starter in $\mathbb{Z}_9$.

Similarly, let G be an abelian group of order $2n$ written multiplicatively with identity element e and a unique element g^* of order 2. An **even starter** in G is a set of unordered pairs $S = \{\{a_i, b_i\} : 1 \leq i \leq n - 1\}$ which satisfies the two conditions: $\{a_i : 1 \leq i \leq n - 1\} \bigcup \{b_i : 1 \leq i \leq n - 1\}$ are all the non-identity elements of G except one, denoted m_E, and $\{a_i^{-1}b_i,\, b_i^{-1}a_i : 1 \leq i \leq n - 1\} = G \setminus \{e, g^*\}$.

Now, suppose $S = \{\{x_i, y_i\} : 1 \leq i \leq n\}$ and $T = \{\{u_i, v_i\} : 1 \leq i \leq n\}$ are two starters in the same group G, without loss of generality, assume that $y_i - x_i = v_i - u_i$ for all i. Then S and T are **orthogonal starters** if $u_i - x_i = u_j - x_j$ implies $i = j$, and if $u_i \neq x_i$ for any i. For example, two starters $T = \{\{4,5\},\{1,3\},\{2,6\}\}$ and $V = \{\{2,3\},\{4,6\},\{1,5\}\}$ are orthogonal starters in $\mathbb{Z}_7$.

Perhaps, the most well known applications for starters are in Room squares and one-factorizations of the complete graph. A Room square of side n (or of order $n+1$) is a square array with n cells in each row and each column, such that each cell is either empty or contains an unordered pair of symbols chosen from a set of $n+1$ elements. Each row and each column contains each element precisely once. For example, we can construct a Room square of side 7 from two orthogonal starters $S = \{\{3,4\},\{6,1\},\{5,2\}\}$ and $T = \{\{4,5\},\{1,3\},\{2,6\}\}$, where S and T can be written as starter-sequences $(5,3,1,1,3,5)$ and $(2,4,2,1,1,4)$, respectively.

Table 1.1: A Room square of side 7.

$\infty 0$			52		61	34
45	$\infty 1$			63		02
13	56	$\infty 2$			04	
	24	60	$\infty 3$			15
26		35	01	$\infty 4$		
	30		46	12	$\infty 5$	
		41		50	23	$\infty 6$

It is obvious that the rows of a Room square of side n are all one-factors of the complete graph K_{n+1}, as are the columns. Therefore, a Room square can be interpreted as two orthogonal one-factorizations, a row factorization and a column factorization. For an extensive bibliography of results on Room squares and related designs, the reader is referred to [9] and [14].

A Skolem sequence of order n is an integer sequence of length $2n$, in which each of the integers $1, 2, \ldots, n$ occurs exactly twice in the sequence, and for each $1 \leq i \leq n$, the two occurrences of i are separated by exactly $i-1$ integers. For example, the sequence $(1, 1, 3, 4, 5, 3, 2, 4, 2, 5)$ is equivalent to the partition of the numbers $1, 2, \ldots, 10$ into the pairs $(1, 2), (7, 9), (3, 6), (4, 8), (5, 10)$; this sequence is known as Skolem sequence of order 5. Skolem sequences were introduced in 1957 by Thoralf Skolem [45]; he also determined the necessary and sufficient conditions for their existence. In 1961, Okeefe [29] proved that hooked Skolem sequences exist if and only if $n \equiv 2, 3 \pmod 4$. In 1966, Rosa [37] introduced split Skolem sequences by adding a hook in the middle of the Skolem sequences, and he also introduced hooked split Skolem sequences. Usually in the literature these sequences are called Rosa and hooked Rosa sequences. In 1981, Abraham and Kotziy [2] introduced the extended Skolem sequence where the hook occurs anywhere in the sequence. In 1981, Stanton and Goulden [46] introduced near-Skolem sequences in order to construct cyclic Steiner triple systems. In 1991, Mendelsohn and Shalaby [26] introduced the concept of Skolem-labellings of graphs, and also provided the necessary and sufficient conditions for Skolem-labelling of paths and cycles. In 1992, Shalaby [43] proved the existence theorems for near-Skolem sequences. In 1998, Shalaby and Al-Gwazi [41] introduced generalized extended and near-Skolem sequences. In 1999, Mendelsohn and Shalaby [27] determined the condition for the existence of Skolem-labelling for k-windmills. In 2008, Baker and Manzer [5] obtained the necessary conditions for the Skolem-labeling of generalized k-windmills in which the vanes need not be of the same length and proved that these conditions are sufficient in the case where $k = 3$.

A Langford sequence of defect d and length m is a Skolem sequence where $D = \{d, d+1, \ldots, d+m-1\}$. For example, the sequence $(10, 8, 6, 4, 9, 7, 5, 4, 6, 8, 10, 5, 7, 9)$

is a Langford sequence of defect 4 and length 7. It corresponds to the partition $\{(1,11),(2,10),(3,9),(4,8),(5,14),(6,13),(7,12)\}$ of $P = [1,14]$ into differences in $D = [4,10]$. Throughout this thesis, the notation $[a,b] = \{a, a+1, \ldots, b\}$. Langford sequences were introduced in 1958 by C. Dudly Langford [21]; since that time, Langford sequences have been approached in different ways. In 1959, Priday [35] and Davies [11] solved the problem completely for $d = 2$. In 1966, Gillespia [17] introduced the generalized Langford sequences of order n with multiplicity s; he showed that there are no generalized Langford sequences for $n = 2,3,4,5$, or 6 where $\lambda = 3$. He also showed that there is no generalized Langford sequence if the multiplicity is bigger than the order. In 1968, Levine [22] proved that if the multiplicity (λ) is even, then there is no generalized (λ)-Langford sequence for $n \equiv 1$ or $2 \pmod 4$, and also that if $\lambda = 6t+3$, then there is no generalized λ-Langford sequence for $n \equiv\ 2,3,4,5,6$, or $7 \pmod 9$. In 1971, Roselle and Thomasson [39] determined the necessary conditions for the existence of λ-Langford sequences, where $\lambda = p^e t$, and p is prime, and e and t are positive integers. These conditions are $n \equiv -1, 0, 1, \ldots,$ or $p-2 \pmod{p^{e+1}}$. In 1971, Roselle [40] also introduced generalized Skolem sequences. In 2014, Mata-Montero, Normore, and Shalaby [25] provided new algorithms for generalized Skolem and Langford sequences with a minimum number of hooks.

Influenced by Skolem sequences, Shalaby [42] introduced the concept of starter-sequences which he called pseudo-Skolem sequences. We notice that Skolem sequences are a special case of starter sequences when the number of defects is zero. It is well known that a Skolem sequence of order n can be used to construct a starter in Z_{2n+1}. Moreover, an extended Skolem sequence of order n can be used to construct an even starter in Z_{2n+2}. For more details about Skolem-type sequences and Langford sequences, the reader is referred to [9] and [16].

A starter sequence of order n, is a sequence of $2n$ positive integers $(s_1, s_2, \ldots, s_{2n})$

such that, for every $r \in \{1, 2, \ldots, n\}$, either r or $-r$ appears exactly twice in the sequence, and if $s_i = s_j = r$ or $-r$, then $|j - i| = r$ or $-r$, respectively, where $-r$ is the additive inverse of r in $\mathbb{Z}_{2n+1}$, and $-r$ is referred to as a defect of the sequence. For example, the sequence $(6, 2, 5, 2, 1, 1, 6, 5)$ is a starter sequence of order 4 with two defects: -3 and -4 in the group $\mathbb{Z}_9$. We can also define a starter sequence of order n as a partition of the set $\{1, 2, \ldots, 2n\}$ into n disjoint ordered pairs $\{(a_r, b_r)\}$ such that $a_r < b_r$ and $b_r = a_r + r$ for $r = 1(-1)$, $2(-2)$, $\ldots$, n(−n).

In this thesis, we study starters as sequences. The motivation behind this is that writing Skolem's partition in sequences has produced massively valuable results in combinatorics along with a wide range of applications. For example, Skolem sequences were applied in triple systems, graph labeling, graph factorizations, coding and communication networks. Another motivation is that starter sequences are generalizations of Skolem sequences. In addition, in 2015 Mariusz Meszka and Alexander Rosa [28] determined cubic graphs with at most 22 vertices which are leaves of partial triple systems by using Skolem-type sequences. In 2016, Lianton Lan and others [20] completely determined the sizes of optimal cyclic $(n, d, 3)_3$ codes with minimum distance $1 \leq d \leq 6$ via Skolem-type sequences. Due to the significance of Skolem-type sequences and starters, in this thesis we investigate starter-type sequences as a generalization of Skolem-type sequences.

The thesis consists of six chapters. In chapter 1, we provide a brief historical background and we present some useful concepts and results which are used in this thesis. In chapter 2, we introduce some new starter sequences, we establish the necessary and sufficient conditions for the existence of starter sequences with one defect for all admissible orders and defects. We also determine all the necessary conditions for the existence of starter sequences with two or more defects for all admissible orders. In chapter 3, we introduce the concept of starter-labelled graphs, we show that every path of length m can be starter-labelled with one defect if and only if $m \equiv 3, 5 \pmod 8$. We show that all 3-windmills with $m \equiv 1, 3, 5, 7 \pmod 8$ can be starter-labelled, except for the case $m = 1$. In fact, we investigate necessary and sufficient conditions for the existence of the starter-labelled graph and the minimum hooked starter-labelled graph of paths, cycles, and k-windmils. The results of starter-labelled k-windmills can be found in [31] and the results regarding starter-labelled paths and cycles can be found in [33].

In chapter 4, we introduce the concept of pseudo-starter sequences, we present several types of pseudo-starter sequences, and we determine some of the conditions for the existence of such sequences. The results of this chapter can be found in [33]. In chapter 5, we generalize the concept of starter sequences, we introduce generalized starter sequences and generalized extended starter sequences, and we determine the necessary conditions for their existence. Moreover, we obtain, with few possible exceptions, the minimum number of hooks and their permissible locations. The results of this chapter can be found in [32]. In chapter 6, we give a brief summary of the thesis and open questions for further research.

1.1 Preliminaries

In this section, we provide useful concepts and theorems which are relevant to the results presented in this thesis.

Definition 1.1.1. *A* ***starter*** *for an abelian additive group G of odd order $(2n+1)$ is a set of unordered pairs $S = \{\{a_i, b_i\} : 1 \leq i \leq n\}$ which satisfies the conditions:*

1. *The set S partitions the non-zero elements of G; in other words* $\{a_i : 1 \leq i \leq n\} \bigcup \{b_i : 1 \leq i \leq n\} = G \setminus \{0\}$, *and*

2. *every non-zero element of G occurs as a difference of exactly one pair of the set S; that is $\{\pm(a_i - b_i) : 1 \leq i \leq n\} = G \setminus \{0\}$.*

Example 1.1.1. *The pairs $\{\{2,3\},\{4,6\},\{1,5\}\}$ form a starter in $\mathbb{Z}_7$.*

Definition 1.1.2. *[9] A **Skolem sequence** of order n is a sequence $S_n = (s_1, s_2, \ldots, s_{2n})$ of $2n$ integers satisfying the conditions:*

1. *for every $k \in \{1, 2, \ldots, n\}$, there exist exactly two elements s_i, $s_j \in S_n$ such that $s_i = s_j = k$, and*

2. *if $s_i = s_j = k$ with $i < j$, then $j - i = k$.*

Example 1.1.2. *If $n = 5$, the partition $\{\{8,9\},\{3,5\},\{1,4\},\{6,10\},\{2,7\}\}$, or equivalently, the sequence $(3,5,2,3,2,4,5,1,1,4)$, is a Skolem sequence of order 5.*

Definition 1.1.3. *[43] A **near-Skolem sequence** of order n and defect q is an integer sequence $(s_1, s_2, \ldots, s_{2n-2})$, denoted by q-near S_n, such that for every $k \in \{1, 2, \ldots, q-1, q+1, \ldots, n\}$, there are exactly two elements $s_i, s_j \in q$-near S_n where $s_i = s_j = k$, and if $s_i = s_j = k$, then $|i - j| = k$.*

Definition 1.1.4. *[7] A **k-extended q-near-Skolem sequence** of order n is a sequence $\mathcal{N}_n^q(k) = (s_1, s_2, \ldots, s_{2n-1})$ satisfying the conditions of near-Skolem sequences, with the additional condition that it contains exactly one empty position, which is in position k, denoted by $*$ or 0.*

Example 1.1.3. *$\mathcal{N}_7^3(8) = (4,2,6,2,4,7,5,0,6,1,1,5,7)$, or equivalently, the collection $\{(10,11),(2,4),(1,5),(7,12),(3,9),(6,13)\}$ is an 8-extended 3-near-Skolem sequence of order 7.*

Definition 1.1.5. *[34] Suppose that $\{k, n\} \subset \mathbb{N}$ such that $n \geq 2$ and $1 \leq k \leq 2n-1$. A **k-pseudo-Skolem sequence** of order n, denoted k-pseudo-S_n, is a distribution* *of the elements of the multiset* $\{1, 2, \ldots, 2n-1, k\}$ *into a collection of ordered pairs* $\{(a_i, b_i) : i = 1, 2, \ldots, n\}$ *such that* $a_i < b_i$ *and* $b_i - a_i = i$*, and the pairs that do not contain* k *are mutually disjoint (there are exactly two pairs containing* k*).*

Definition 1.1.6. *[16] A* ***generalized Skolem sequence*** *of order* n *and multiplicity* λ *is a sequence* $GS_n = (s_1, s_2, ..., s_{\lambda n})$ *of integers such that, for each* $i \in \{1, 2, \ldots, n\}$*, there are exactly* λ *positions in the sequence* $GS_n :$ $j_1,$ $j_2 = j_1 + i, \ldots, j_\lambda = j_1 + (\lambda-1)i$ *and* $s_{j_1} = s_{j_2} = \ldots = s_{j_\lambda} = i$*.*

Definition 1.1.7. *[9] A* ***Langford sequence*** *of defect* d *and with* m *differences is an integer sequence* $L_d^m = (l_1, l_2, \ldots, l_{2m})$ *such that the following conditions hold:*

1. *for each* $k \in \{d, d+1, \ldots, d+m-1\}$*, there are exactly two elements* $l_i, l_j \in L_d^m$ *such that* $l_i = l_j = k$*, and*
2. *if* $l_i = l_j = k$*, then* $|j - i| = k$*.*

In this thesis, the largest difference in a Langford sequence, $d + m - 1$, is called the order of the Langford sequence.

Definition 1.1.8. *[9] A* k***-extended Langford sequence*** *of defect* d *with* m *differences is a langford sequence* $L_d^m(k) = (l_1, l_2, \ldots, l_{2m+1})$ *with the added condition that it contains exactly one empty position, which is in position* k*.*

A hooked Langford sequence of defect d with m differences, hL_d^m, is a $2m$-extended Langford sequence of defect d with m differences.

Definition 1.1.9. *[36] A* ***Rosa sequence*** *of order* n *is a sequence* $(r_1, r_2, \ldots, r_{2n+1})$ *satisfying the conditions of Skolem sequences, with the added condition that* $r_{n+1} = 0$*.*

Definition 1.1.10. *[23] A* (p, q)***-extended Rosa sequence*** *is a sequence of length* $2n + 2$ *containing each of the symbols* $\{0, 1, \ldots, n\}$ *exactly twice, and in which two* *occurrences of the integer $j > 0$ are separated by exactly $j-1$ symbols. Such sequences are denoted by $R_n(p, q)$.*

Theorem 1.1.1. *[45] A Skolem sequence of order n exists if and only if $n \equiv 0, 1 \pmod 4$.*

Theorem 1.1.2. *[36] A Rosa sequence of order n exists if and only if $n \equiv 0, 3 \pmod 4$.*

Theorem 1.1.3. *[23] An $\mathcal{R}_n(p, q)$ exists if and only if $p \not\equiv q \pmod 2$ and $n \equiv 0, 1 \pmod 4$, or if $p \equiv q \pmod 2$ and $n \equiv 2, 3 \pmod 4$, with the exception of $\mathcal{R}_1(2, 3)$ and $\mathcal{R}_4(5, 6)$.*

Theorem 1.1.4. *[44] A Langford sequence of defect d and m differences exists if and only if the following conditions hold:*

1. *$m \geq 2d - 1$, and*
2. *$m \equiv 0, 1 \pmod 4$ when d is odd, or $m \equiv 0, 3 \pmod 4$ when d is even.*

Theorem 1.1.5. *[44] A hooked Langford sequence of defect d and m differences exists if and only if the following conditions hold:*

1. *$m(m + 1 - 2d) + 2 \geq 0$, and*
2. *$m \equiv 2, 3 \pmod 4$ when d is odd, or $m \equiv 1, 2 \pmod 4$ when d is even.*

Theorem 1.1.6. *[24] A k-extended Langford sequence of length m and defect 2 exists if and only if $m \equiv 0, 3 \pmod 4$ if k is odd, or $m \equiv 1, 2 \pmod 4$ if k is even.*

Theorem 1.1.7. *[24] A k-extended Langford sequence of length m and defect 3 exists if and only if the following conditions hold:*

1. *$m \geq 3$, and*
2. *$m \equiv 0, 1 \pmod 4$ when k is odd, or $m \equiv 2, 3 \pmod 4$ when k is even, with the exception of $(m, k) = (3, 2), (3, 6), (4, 1), (4, 5), (4, 9)$.*

Theorem 1.1.8. *[43] An m-near-Skolem sequence of order n exists if and only if $n \equiv 0, 1 \pmod 4$ and m is odd, or $n \equiv 2, 3 \pmod 4$ and m is even.*

Theorem 1.1.9. *[7] There exists a k-extended q-near-Skolem sequence of order n for all triples (n, q, k) such that either*

1. *$n \equiv 0, 1 \pmod 4$ and q, k have the same parity, or*

2. *$n \equiv 2, 3 \pmod 4$ and q, k have opposite parity,*

   with the exception of $(n, q, k) = (3, 2, 3), (4, 2, 4)$.

Theorem 1.1.10. *[34] A k-pseudo-Skolem-S_n exists when k is odd and $n \equiv 2$, 3 $\pmod 4$, or when k is even and $n \equiv 0, 1 \pmod 4$.*

Chapter 2

The Existence of Starter Sequences and m–fold Starter Sequences

2.1 Starter Sequences

In this section, we establish the necessary and sufficient conditions for the existence of starter sequences with one defect for all admissible orders and defects. We also determine all necessary conditions for the existence of starter sequences with two or more defects for all admissible orders. We begin this section with the definition of starter sequences.

Definition 2.1.1. *A* ***starter sequence*** *of order n, denoted by SS_n, is a sequence of $2n$ positive integers $(s_1, s_2, \ldots, s_{2n})$ such that, for every positive integer $i \in [1, n]$, either i or $-i$ appears exactly twice in the sequence SS_n, and if $s_a = s_b = i$ or $-i$, then $|b - a| = i$ or $-i$, respectively, where $-i$ is the additive inverse of i in $\mathbb{Z}_{2n+1}$, and $-i$ is refereed to as a defect of the sequence.*

Throughout this thesis, we shall use the notation SS_n^m to denote a starter sequence of order n with m defects.

By Definition 2.1.1, it is obvious that the defect $2n$ does not occur in any starter sequence of order n. Moreover, the defects $2n-1$ and $2n-2$ cannot appear together in any starter sequence of order n.

Theorem 2.1.1. *There exists a starter sequence of order n with one defect (SS_n^1) if and only if $n \equiv 2, 3 \pmod 4$.*

Proof. We first determine the necessary conditions for the existence of the required sequences. Let $(s_1, s_2, ..., s_{2n})$ be a starter sequence of order n with one defect. For each $r \in \{\{1, 2, \ldots, n\} \setminus \{i\}\} \cup \{2n+1-i\}$, we construct an ordered pair (a_r, b_r) such that $s_{a_r} = s_{b_r} = r$. Now consider the sum of all sums and the sum of all differences of these subscripts:

$$\sum_r (b_r + a_r) = \sum_{r=1}^{2n} r = \frac{(2n)(2n+1)}{2} = n(2n+1) \tag{2.1.1}$$

$$\sum_r (b_r - a_r) = \sum_{r=1}^{n} r - i + (2n+1) - i = \frac{n(n+1)}{2} + 2n - 2i + 1, \tag{2.1.2}$$

where $i \in [1, n]$. Solving this system for $\sum_r b_r$, we obtain $\sum_r b_r = \frac{5n^2+7n-4i+2}{4}$. Since the left hand side of the last equation is an integer, this implies that $(5n^2+7n-4i+2)$ must be divisible by 4, which happens only when $n \equiv 2, 3 \pmod 4$.

Now we prove that these necessary conditions are sufficient. For small orders $n = 2$ and $n = 3$, it is straightforward to construct the required sequences. If $n \equiv 2, 3 \pmod 4$ and $n > 3$, then we use k-extended q-near Skolem sequences of order n to construct SS_n^1. By Theorem 1.1.9 we ensure that there exists an $\mathcal{N}_n^q(k)$ for all triples (n, q, k) such that $n \equiv 2, 3 \pmod 4$ and q, k are of opposite parity. As such, we choose a suitable $\mathcal{N}_n^q(k)$ such that we can place the defect $(2n+1-q)$ at the beginning of the sequence and we fill the empty position (k) with the defect $2n+1-q$. ■

For example, to construct a starter sequence of order 6 and one defect $(d = 9)$, we begin with $\mathcal{N}_6^4(9) = (5,1,1,3,6,5,3,2,0,2,6)$, and then we place the defect at the beginning of the sequence and we also fill the hook of the sequence as follows: $(9,5,1,1,3,6,5,3,2,9,2,6) = SS_6^1$.

Theorem 2.1.2. *There exists a starter sequence of order* n *with two defects* (SS_n^2) *only if* $n \equiv 0,1 \pmod 4$.

Proof. Let $(s_1, s_2, ..., s_{2n})$ be a starter sequence with two defects. For each $r \in \{\{1,2,\ldots,n\} \setminus \{i,j\}\} \cup \{2n+1-i, 2n+1-j\}$, we construct an ordered pair (a_r, b_r) such that $s_{a_r} = s_{b_r} = r$. By using the same technique as in the proof of Theorem 2.1.1, we obtain the following equations:

$$\sum_r (a_r + b_r) = \sum_{r=1}^{2n} r = \frac{(2n)(2n+1)}{2} = 2n^2 + n \tag{2.1.3}$$

$$\sum_r (b_r - a_r) = \sum_{r=1}^{n} r - 2i - 2j + 4n + 2 = \frac{n(n+1)}{2} - 2i - 2j + 4n + 2 \tag{2.1.4}$$

Hence,

$$\sum_r b_r = \frac{5n^2 + 11n - 4i - 4j + 4}{4}. \tag{2.1.5}$$

Since the left hand side of the equation (2.1.5) is a non-negative integer, this implies that $(5n^2+11n-4i-4j+4)$ must be divisible by 4, which happens only when $n \equiv 0,1 \pmod 4$.

Moreover, we prove that these necessary conditions are sufficient in some certain cases of defects. Suppose that $n \equiv 0,1 \pmod 4$, where $d_1 = n+1$ and $d_2 = n+2$. Then, according to Theorem 1.1.3, we can use (p,q)-extended Rosa sequences to construct starter sequences for all admissible orders of n with two defects as follows: the defect d_1 is placed in front of the sequence $R_{n-2}(n-3, n+1)$ and the second hook at the position $(n+1)$ is filled with d_1, and the other defect (d_2) is appended to the end of the sequence of $R_{n-2}(n-3, n+1)$ and the first hook at the position $(n-3)$ is filled with the defect d_2. For instance, in order to construct a starter sequence of order 8 with two defects, $d_1 = 9$ and $d_2 = 10$, we begin by constructing $R_6(5,9) = (6,4,1,1,0,4,6,5,0,2,3,2,5,3)$, and then we place d_1 and d_2 in suitable places as follows $(9,6,4,1,1,10,4,6,5,9,2,3,2,5,3,10)$. Similarly, we can use $R_7(6,10) = (7,5,6,1,1,0,5,7,6,0,4,2,3,2,4,3)$ to obtain a starter sequence of order 9 with two defects, $(10,7,5,6,1,1,11,5,7,6,10,4,2,3,2,4,3,11)$, where the set of defects is {10,11}.

In the case that ($d_1 = 2n-1$, $d_2 = 2n-3$), we can use Langford sequences to construct starter sequences for all the admissible orders of n with two defects as follows: for $n \equiv 0 \pmod 4$, let $n = 4s$, and $s \geq 4$; we know that L_5^{4s-4} exists according to Theorem (1.1.4); then the required starter sequence is: $(2n-1, 2n-3, L_5^{4s-4}, 3, 1, 1, 3, 2n-3, 2n-1)$.

For $n \equiv 1 \pmod 4$, let $n = 1+4s$, and $s > 2$; then the required starter sequence is $(2n-1, 2n-3, L_5^{4s-3}, 3, 1, 1, 3, 2n-3, 2n-1)$. To complete this part, we list below the sequences for $n = 4, 5, 8, 9$, and 12, respectively:

$5,6,1,1,2,5,2,6$

$9,7,5,3,1,1,3,5,7,9$

$15,13,7,5,3,8,6,3,5,7,1,1,6,8,13,15$

$17,15,9,7,5,3,6,8,3,5,7,9,6,1,1,8,15,17$

$23,21,10,8,6,3,11,12,3,9,6,8,10,7,1,1,5,11,9,12,7,5,21,23.$

In the case that $d_1 = n+1$, and $d_2 = 2n-1$, we use extended Langford sequences, which exist according to Theorem (1.1.7). For $n \equiv 0, 1 \pmod 4$, and $n > 5$, we first construct an extended Langford sequence with $n-3$ differences, and then the defect d_2 is placed in the first and last positions of the required sequence, and the other defect (d_1) is placed in the second position and the hook of the extended Langford sequence as follows: $SS_n^2 = (2n-1), (n+1)(L_3^{n-3}(n+1)), 1, 1, (2n-1)$. For instance, we use an

extended Langford sequence, $L_3^5(9) = (4, 6, 7, 3, 4, 5, 3, 6, *, 7, 5)$, to construct a starter sequence of order 8 and two defects: $(15, 9, 4, 6, 7, 3, 4, 5, 3, 6, 9, 7, 5, 1, 1, 15)$.
To complete this part, we list the required sequences of orders 4 and 5, respectively: $(7, 5, 3, 1, 1, 3, 5, 7)$ and $(9, 6, 4, 1, 1, 3, 4, 6, 3, 9)$. ■

Starter sequences with more than two defects are natural generalizations of the starter sequences which are represented early in this section. Now we determine the necessary conditions for the existence of starter sequences with more than two defects, and we may conjecture that these necessary conditions are sufficient.

For example, the sequence $(10, 7, 4, 8, 1, 1, 4, 2, 7, 2, 10, 8)$ is a starter sequence of order 6 with three defects, while the sequence $(12, 10, 1, 1, 11, 9, 2, 3, 2, 4, 3, 10, 12, 4, 9, 11)$ is a starter sequence of order 8 with four defects.

Theorem 2.1.3. *Assuming that D is a set of defects for starter sequences of order n such that $|D| > 2$, then the necessary conditions for the existence of these sequences with the set D of defects are either $n \equiv 0, 1 \pmod 4$ and $|D|$ is even, or $n \equiv 2, 3 \pmod 4$ and $|D|$ is odd.*

Proof. Assume that $D = \{d_1, d_2, \ldots, d_m\}$, and that $(s_1, s_2, ..., s_{2n})$ is a starter sequence of order n with m defects, such that $d_j = 2n + 1 - i_j$. By using the same technique in Theorem (2.1.1), we construct ordered pairs $\{(a_r, b_r)\}$ such that $s_{a_r} = s_{b_r} = r$, and we obtain the following equations:

$$\sum_r (a_r + b_r) = \sum_{r=1}^{2n} r = \frac{(2n)(2n+1)}{2} = 2n^2 + n \qquad (2.1.6)$$

$$\sum_r (b_r - a_r) = \sum_{r=1}^{n} r - 2(i_1 + i_2 + ... + i_m) + m(2n+1) \qquad (2.1.7)$$

By Solving this system for $\sum_r b_r$, we obtain: $\sum_{r=1}^n b_r = \frac{5n^2+2n+4mn+2m-4(i_1+i_2+...+i_m)}{4}$.
Since $\sum_{r=1}^n b_r$ is an integer, this implies that $5n^2+2n+4mn+2m-4(i_1+i_2+...+i_m)$

must be divisible by 4. This implies that either $n \equiv 0, 1 \pmod 4$ and m is even, or $n \equiv 2, 3 \pmod 4$ and m is odd. ■

2.2 Hooked Starter Sequences

In this section, we introduce hooked starter sequences, establish the necessary and sufficient conditions for the existence of hooked starter sequences with one defect for all admissible orders and defects, and determine some of the conditions for the existence of hooked starter sequences with two defects.

Definition 2.2.1. *A* ***hooked starter sequence*** *of order n, denoted by hSS_n, is a sequence of $2n+1$ integers $(s_1, s_2, ..., s_{2n+1})$ satisfying conditions (1) and (2) for starter sequences, with the added condition that $s_{2n} = 0$. This element is called the hook or the null element, and it is denoted by 0, or $*$.*

For example, $(8, 3, 4, 2, 3, 2, 4, 0, 8)$ is a hooked starter sequence of order 4 with one defect, and $(10, 1, 1, 4, 5, 7, 2, 4, 2, 5, 10, 0, 7)$ is a hooked starter sequence of order 6 with two defects.

Theorem 2.2.1. *There exists a hooked starter sequence of order n with one defect (hSS_n^1) if and only if $n \equiv 0, 1 \pmod 4$.*

Proof. We determine the necessary conditions by using the same technique as in Theorem 2.1.1. Hence,

$$\sum_r (a_r + b_r) = \sum_{r=1}^{2n+1} r - 2n = \frac{(2n+1)(2n+2)}{2} - 2n \tag{2.2.1}$$

$$\sum_r (b_r - a_r) = \sum_{r=1}^{n} r - 2k + 2n + 1 = \frac{n(n+1)}{2} - 2k + 2n + 1 \tag{2.2.2}$$

By solving the system for $\sum b_r$, we obtain: $\sum_r b_r = \frac{5n^2+7n-4k+4}{4}$. Since the values of b_r are positive integers, this implies that $\frac{5n^2+7n-4k+4}{4}$ must be a positive integer, which happens only when $n \equiv 0, 1 \pmod 4$.

Concerning the sufficiency, suppose that $n \equiv 0, 1 \pmod 4$. For the minimum value of the defects ($d = n+1$), we can use a Rosa sequence of order n, where $n \equiv 0,\ 3 \pmod 4$, which exists according to Theorem (1.1.2). Then we fill the middle hook with $(n+1)$, and we put the second copy of $(n+1)$ at the end of the sequence. For instance, we can use a Rosa sequence of order three, $(1, 1, 3, 0, 2, 3, 2)$, to construct a hooked starter sequence of order 4 and one defect: $(1, 1, 3, 5, 2, 3, 2, 0, 5)$. For all the remaining possible cases of the defect ($n + 1 < d \leq 2n$), we can use extended near-Skolem sequences to construct the required hooked starter sequences of all the permissible orders. Therefore, if $n \equiv 0, 1 \pmod 4$, and $d = 2n + 1 - m$, then an m-extended m-near Skolem sequence of order n exists according to Theorem (1.1.9). Now we can construct a hooked starter sequence of order n with one defect d by appending $0, m$ to the end of $\mathcal{N}_n^q(k)$. ∎

For instance, to build a hooked starter sequence of order 4 and one defect ($d = 6$), we can use a 3-extended 3-near-Skolem sequence of order four $(4, 2, 0, 2, 4, 1, 1)$, fill the hook with the defect $d = 6$, and put the second copy of d in the position S_9 in order to obtain a hooked starter sequence of order 4 with one defect such as the following: $(4, 2, 6, 2, 4, 1, 1, 0, 6)$.

Theorem 2.2.2. *There exists a hooked starter sequence of order n with two defects (hSS_n^2) only if $n \equiv 2, 3 \pmod 4$ and $n > 2$.*

Proof. By using a similar technique as in the proof of Theorem (2.2.1), we determine the necessary conditions of hSS_n^2. Moreover, we prove that these necessary conditions are sufficient when the pair of defects are $\{n + 1, 2n\}$ or $\{n + 1, n + 2\}$.

In the case that the set of defects $D = \{n+1, 2n\}$, we can construct the required sequences by using extended Langford sequences with ($d = 2$), which exist by Theorem (1.1.6). To begin, the defect $2n$ is placed at the beginning and the end of the required sequence, and the defect $n+1$ is placed in the second position, and the positions from 3 to $2n-1$, inclusive, are occupied by $L_2^{n-1}(n+1)$ such that the hook of the Langford sequence is filled by the defect $n+1$. Therefore, the required sequences are given by: $2n$, $n+1(L_2^{n-1}(n+1))$, 0, $2n$. For example, using $L_2^4(7) = (2,4,2,5,3,4,0,3,5)$, we can build a hooked starter sequence of order 6 with two defects $hSS_6^2 = (12,7,2,4,2,5,3,4,7,3,5,*,12)$.

In the case that $D = \{n+1, n+2\}$, to construct starter sequences of order n with two defects, we use (p,q)-extended Rosa sequences, which exist according to Theorem (1.1.3). First, we place the defect $n+1$ in the first position of the required sequence and $R_{n-2}(p,q)$ in the positions from 2 to $2n-1$, inclusive, such that we can replace the two zeros p and q by $n+2$ and $n+1$, respectively; then we place the second copy of $n+2$ in the last position $(2n+1)$ of the required sequence, and thus the position $2n$ will be occupied by zero. For instance, we use $R_4(4,7) = (2,3,2,*,3,4,*,1,1,4)$ to construct $hSS_6^2 = (7,2,3,2,8,3,4,7,1,1,4,*,8)$. Similarly, we use $R_5(5,8) = (1,1,3,5,*,3,4,*,5,2,4,2)$ to construct a hooked starter sequence of order 7 with two defects: $hSS_7^2 = (8,1,1,3,5,9,3,4,8,5,2,4,2,*,9)$. ■

2.3 Extended Starter Sequences

Definition 2.3.1. *A k-**extended starter sequence** of order n, denoted by $SS_n(k)$ or k-ext SS_n, is a sequence $(s_1, s_2, \ldots, s_{2n+1})$ of $2n+1$ non-negative integers that satisfy the conditions of a starter sequence with the added condition that $s_k = 0$ for some $k \in \{1, 2, \ldots, 2n+1\}$.*

We notice that an extended starter sequence of order n exists for all values of n such that the defect is either $n+1$ or $n+2$. This is not difficult to prove by using the same technique used in [2].

Theorem 2.3.1. *An extended starter sequence of order n with one defect $(SS_n^1(k))$ exists for each positive integer n where d is either $(n+1)$ or $(n+2)$.*

Proof. Given a positive integer (n), we can construct an extended starter sequence of order n with one defect of either $(n+1)$ or $(n+2)$ as follows:
$(k, k-2, \ldots, 5, 3, 1, 1, 3, 5, \ldots, k-2, k, e, e-2, \ldots, 4, 2, 0, 2, 4, \ldots, e-2, e)$, where k and e are the largest odd and even numbers, respectively, in the set $E = \{1, 2, 3, \ldots, n-1, n+1\}$, when $d = (n+1)$, and if $d = (n+2)$ then $E = \{1, 2, \ldots, n-2, n, n+2\}$. ■

Theorem 2.3.2. *There exists a k-extended starter sequence of order n with one defect $(SS_n^1(k))$ only if either*

1. *k is even and $n \equiv 0, 1 \pmod 4$, or*

2. *k is odd and $n \equiv 2, 3 \pmod 4$.*

Proof. Consider the set of subscripts $\{(a_r, b_r) : r = 1, 2, ..., n\}$; then

$$\sum_r (a_r + b_r) = \sum_{r=1}^{2n+1} r - k = (2n+1)(n+1) - k, \tag{2.3.1}$$

$$\sum_r (b_r - a_r) = \sum_{r=1}^{n} r - m + (2n+1-m) = \frac{n(n+1)}{2} + 2n - 2m + 1. \tag{2.3.2}$$

By adding (2.3.1) and (2.3.2) we obtain $\sum_r b_r = \frac{5n^2+11n-2k+4}{4} - m$. This implies that $\frac{5n^2+11n-2k+4}{4}$ must be an integer, which happens only if either k is even and $n \equiv 0, 1 \pmod 4$ or k is odd and $n \equiv 2, 3 \pmod 4$.

For small orders of n, it is straightforward to construct a k-extended starter sequence of order n with one defect, where k is odd. For large orders, we prove that these necessary conditions are sufficient in the case $d = n+1$ or $d = 2n$. If $n \equiv 2, 3 \pmod 4$, $n > 3$, and k is odd, then we can construct extended starter sequences of order n, with the lowest possible defect ($d = n + 1$) by using extended near-Skolem sequences. In the case that $n \equiv 2, \pmod 4$, we use extended near-Skolem sequences of order 3 (mod 4) and $q = n$, which exist by Theorem (1.1.9). For example, a 5-extended 6-near Skolem sequence of order seven $\mathcal{N}_7^6(5) = (7, 4, 1, 1, 0, 4, 5, 7, 2, 3, 2, 5, 3)$ is equivalent to a 5-extended starter sequence of order 6 with one defect provided that ($d = n+1$). In the case that $n \equiv 3 \pmod 4$, we use extended near-Skolem sequences of order $0 \pmod 4$ and $q = n$, which also exist according to Theorem (1.1.9). For instance, using an 11-extended 7-near Skolem sequence of order eight, $\mathcal{N}_8^7(11) = (8, 6, 4, 2, 5, 2, 4, 6, 8, 5, 0, 3, 1, 1, 3)$, is equivalent to using an 11-extended starter sequence of order 7 with one defect.

Similarly, in the case that $n \equiv 0, 1 \pmod 4$, k is even, and $d = n + 1$, we use a k-extended near-Skolem sequence of order $1, 2 \pmod 4$ respectively where $q = n$. In the case that $n \equiv 0, 1 \pmod 4$ and k is even, and $d = 2n$, we can construct a k-extended starter sequence of order n with one defect by using an extended Langford sequence with the defect $d = 2$. Therefore, $SS_n^1(k) = 2n, L_2^{n-1}(k), 2n$. For example, by using $L_2^4(3) = (3, 5, 0, 3, 4, 2, 5, 2, 4)$, we can construct a 4-extended Starter sequence of order 5 with one defect: $SS_5^1(4) = (10, 3, 5, 0, 3, 4, 2, 5, 2, 4, 10)$. Similarly, in case $n \equiv 2, 3 \pmod 4$, $d = 2n$, and k is odd, then $SS_n^1(k) = 2n, L_2^{n-1}(k), 2n$. In the case that $k = 2$, $d = 2n - 1$, and $n \equiv 0, 1 \pmod 4$, we use a hooked Langford sequence to construct the required sequence :$SS_n^1(2) = (2n - 1, 0, 1, 1, hL_3^{n-2})$. ■

By a similar argument, we can determine the necessary conditions for the existence of k-starter sequences with r defects, where $(1 < r \leq n - 1)$.

Theorem 2.3.3. *Let D be a set of defects for starter sequences of order n, where $D = \{d_1, d_2, \ldots, d_r\}$, $1 < r \leq n - 1$, and $n + 1 \leq |d_i| \leq 2n - 2$; then there exists a*

k-extended starter sequence of order n such that $1 \leq k \leq 2n+1$ with r defects only if either

1. *r and k have different parities, and $n \equiv 0,\ 1 \pmod 4$, or*

2. *r and k have the same parity, and $n \equiv 2,\ 3 \pmod 4$.*

2.4 m–fold starter sequences

The concept of m–fold starter sequences was introduced in [1]. In this section, we provide more details about $m$fold starter sequences. Throughout this thesis, for every integer a the notation $a(-a)$ means that either a or $-a$ appears in the sequence.

Definition 2.4.1. *An m–**fold starter sequence** $(s_1, s_2, \ldots, s_{2nm})$ of order n is a sequence of $2nm$ integers such that, for every $k \in \{1(-1), 2(-2), \ldots, n(-n)\}$, there exists m disjoint pairs $(i, i+k)$, $i \in \{1, 2, \ldots, 2nm\}$ such that $s_i = s_{i+k} = k$.*

For example, $(5, 6, 1, 1, 2, 5, 2, 6, 5, 6, 1, 1, 2, 5, 2, 6, 5, 6, 1, 1, 2, 5, 2, 6)$ is a 3–fold starter sequence of order 4 with two defects.

Similarly, an m–**fold extended starter sequence** of order n is a sequence $(s_1, s_2, \ldots, s_{2nm+1})$ of $2nm+1$ integers satisfying the conditions of the m–fold starter sequences with the added condition that there is exactly one subscript y, such that $s_y = 0$. If $s_{2nm} = 0$, then the extended sequence is called a **hooked m–fold starter sequence**. For example, $(4, 5, 1, 1, 4, 5, 5, 4, 1, 1, 5, 4, 4, 5, 1, 1, 4, 0, 5)$ is a hooked 3–fold starter sequence of order 3 with two defects. Note that a 1–fold starter sequence is a starter sequence. Hence, throughout the rest of this chapter, we assume that $m \geq 2$.

2.4.1 The existence of m–fold starter sequences

Now we are ready to establish the existence of m–fold starter sequences with one defect, and we generalize this result to m–fold starter sequences with more than one defect.

Theorem 2.4.1. *An m–fold starter sequence of order n with one defect exists if and only if $n \equiv 2,\,3 \pmod 4$, or $n \equiv 0,\,1 \pmod 4$ and m is even.*

Proof. Suppose that $(s_1, s_2, \ldots, s_{2mn})$ is an m–fold starter sequence of order n with one defect. Consider the sum of all sums, and the sum of all differences of the subscripts i and j:

$$\sum_{i=1}^{n}\sum_{j=1}^{m}(b_{ij}+a_{ij}) = \frac{2mn(2mn+1)}{2}, \tag{2.4.1}$$

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m}(b_{ij}-a_{ij}) &= \frac{mn(n+1)}{2} - mk + m(2n+1-k)\\ &= \frac{mn(n+1)-4m(k-n)+2m}{2}.\end{aligned} \tag{2.4.2}$$

By adding (2.4.1) and (2.4.2) and then dividing by 2, we obtain

$$\sum_{i=1}^{n}\sum_{j=1}^{m} b_{ij} = \frac{mn(4mn+n+7)-4mk+2m}{4} \tag{2.4.3}$$

The left side of (2.4.3) is always an integer. Hence $n \equiv 2,\,3 \pmod 4$, or $n \equiv 0, 1$ (mod 4) and m is even.

Now we prove that these necessary conditions are also sufficient. If $n \equiv 2,\,3$ (mod 4), then we construct an m–fold starter sequence of order n by placing m copies of the starter sequences of order n with one defect side by side.

If $n \equiv 0, 1 \pmod 4$ and m is even, then any hooked starter sequence of order n with one defect can be hooked together with its reverse to build a 2-fold starter sequence with one defect. By putting the first term of the reverse sequence in the hook of the sequence and fitting the last term of the sequence in the hook of the reverse, then we place $\frac{m}{2}$ of these 2-fold starter sequences with one defect together side by side to form an m–fold starter sequence with one defect. For example, the hooked starter sequence of order 4 with one defect $(8, 3, 4, 2, 3, 2, 4, *, 8)$, and its reverse $(8, *, 4, 2, 3, 2, 4, 3, 8)$ form a 2-fold starter sequence of order 4 with one defect $(8, 3, 4, 2, 3, 2, 4, 8, 8, 4, 2, 3, 2, 4, 3, 8)$. Therefore, we can construct any m–fold starter sequence of order 4 with one defect where m is even. For instance, $(8, 3, 4, 2, 3, 2, 4, 8, 8, 4, 2, 3, 2, 4, 3, 8, 8, 3, 4, 2, 3, 2, 4, 8, 8, 4, 2, 3, 2, 4, 3, 8)$ is 4-fold starter sequence of order 4 with one defect. ■

Theorem 2.4.2. *An m–fold starter sequence of order n with two defects exists if and only if*

1. $n \equiv 0,\ 1 \pmod 4$, *or*

2. $n \equiv 2,\ 3 \pmod 4$ *and m is even.*

Proof. Suppose that $(s_1, s_2, \dots, s_{2mn})$ is an m–fold starter sequence of order n with two defects. By using the same technique as in the proof of Theorem (2.4.1), we obtain:

$$\sum_{i=1}^{n}\sum_{j=1}^{m} b_{ij} = \frac{mn(4mn + n + 11) - 4m(k_1 + k_2 - 1)}{4} \tag{2.4.4}$$

The left side of (2.4.4) is always an integer; hence, $n \equiv 0,\ 1 \pmod 4$, or $n \equiv 2,\ 3 \pmod 4$ and m is even. We prove that the necessary conditions are also sufficient. If $n \equiv 0,\ 1 \pmod 4$, then we construct an m–fold starter sequence of order n with two defects by placing m copies of the starter sequence of order n with two defects side by side.

If $n \equiv 2,\ 3 \pmod 4$ and m is even, then any hooked starter sequence of order n with two defects can be hooked together with its reverse to build a 2-fold starter sequence of order n with two defects. Then, $\frac{m}{2}$ of these 2-fold starter sequence with two defects are placed together side by side to form an m–fold starter sequence of order n with two defects. For example, the hooked starter sequence of order 6 with two defects $(12, 9, 5, 6, 2, 3, 2, 5, 3, 6, 9, *, 12)$, and its reverse $(12, *, 9, 6, 3, 5, 2, 3, 2, 6, 5, 9, 12)$, form any m–fold starter sequence of order 6 with two defects, where m is even. For instance, the sequence: (12, 9, 5, 6, 2, 3, 2,5, 3, 6, 9, 12, 12, 9, 6,3, 5,2, 3, 2, 6, 5, 9,12, 12, 9, 5, 6, 2, 3, 2, 5, 3, 6, 9, 12, 12, 9,6, 3, 5, 2, 3, 2, 6, 5, 9, 12) is a 4–fold starter sequence of order 6 with two defects. ∎

Similarly, we establish the necessary conditions for m–fold starter sequences of order n with r defects, where $r > 2$, and we may conjecture that these necessary conditions are sufficient.

Theorem 2.4.3. *Let n, m, and r be positive integers, $(3 \leq r \leq n-1)$, and let $(s_1, s_2, s_3, \ldots, s_{2mn})$ be an m–fold starter sequence of order n with r defects $(d_1, d_2, \ldots, d_r)$. Then one of the following must hold:*

1. $n \equiv 0,\ 1 \pmod 4$ *and r is even,*
2. $n \equiv 0,\ 1 \pmod 4$, *and r is odd and m is even,*
3. $n \equiv 2,\ 3 \pmod 4$ *and r is odd, or*
4. $n \equiv 2,\ 3 \pmod 4$, *and r and m are even.*

2.4.2 Hooked m–fold starter sequences

In this section, we introduce hooked m–fold starter sequences, and we investigate the conditions for their existence.

Theorem 2.4.4. *A hooked m–fold starter sequence of order n with one defect exists if and only if $n \equiv 0$ or $1 \pmod 4$ and m is odd.*

Proof. Suppose that $(s_1, s_2, \ldots, s_{2mn+1})$ is a hooked m–fold starter sequence of order n with one defect. By using a similar argument as in the proof of Theorem (2.4.1), we obtain:

$$\sum_{i=1}^{n}\sum_{j=1}^{m}(b_{ij} + a_{ij}) = \frac{(2mn+1)(2mn+2)}{2} - 2mn \tag{2.4.5}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m}(b_{ij} - a_{ij}) = \frac{mn(n+5) - 4mk + 2m}{2} \tag{2.4.6}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} b_{ij} = \frac{mn(4mn+n+7) - 4mk + 2m + 2}{4} \tag{2.4.7}$$

The left side of (2.4.7) is always an integer. Hence, $n \equiv 0$ or $1 \pmod 4$ and m is odd. Concerning the sufficiency, suppose that $n \equiv 0,\ 1 \pmod 4$, and m is odd. Then, a hooked m–fold starter sequence of order n with one defect can be presented by using $\frac{m-1}{2}$ combinations of a hooked starter sequence of order n with one defect and its reverse, followed by a copy of the hooked starter sequence with one defect. For example, the hooked starter sequence of order 5 with one defect $(0, 3, 4, 5, 3, 2, 4, 2, 5, *, 10)$, and its reverse $(10, *, 5, 2, 4, 2, 3, 5, 4, 3, 10)$, form a 2-fold starter sequence of order 5 with one defect $(10, 3, 4, 5, 3, 2, 4, 2, 5, 10, 10, 5, 2, 4, 2, 3, 5, 4, 3, 10)$. We can easily construct any hooked m–fold starter sequence of order 5 with one defect, where m is odd. For instance, $(10, 3, 4, 5, 3, 2, 4, 2, 5, 10, 10, 5, 2, 4, 2, 3, 5, 4, 3, 10, 10, 3, 4, 5, 3, 2, 4, 2, 5, *, 10)$ is a hooked 3-fold starter sequence of order 5 with one defect. ■

Theorem 2.4.5. *A hooked m–fold starter sequence of order n with two defects exists if and only if $n \equiv 2$ or $3 \pmod 4$ and m is odd.*

Proof. Suppose that $(s_1, s_2, \ldots, s_{2mn+1})$ is a hooked m–fold starter sequence of order n with two defects. Similarly, we obtain:

$$\sum_{i=1}^{n}\sum_{j=1}^{m} b_{ij} = \frac{mn(4mn+n+11) - 4m(k_1+k_2) + 4m + 2}{4}. \tag{2.4.8}$$

The left side of (2.4.8) is always an integer; hence, $n \equiv 2$ or $3 \pmod 4$ and m is odd. Sufficiency: if $n \equiv 2$ or $3 \pmod 4$, and m is odd, then a hooked m–fold starter sequence of order n with two defects can be presented by using $\frac{m-1}{2}$ combinations of a hooked starter sequence of order n with two defects and its reverse, and adding the hooked starter sequence with two defects at the end. This completes the proof. ■

By a similar approach, we establish the necessary conditions for hooked starter sequences with more than two defects.

Theorem 2.4.6. *Let n, m, r, and d_r be positive integers, where $(3 \leq r \leq n-1)$, and $n+1 \leq d_i \leq (2n)$, and $(s_1, s_2, \dots, s_{2mn+1})$ is a hooked m–fold starter sequence of order n with r defects $(d_1, d_2, \dots, d_r)$. Then one of the following must hold:*

1. *$n \equiv 0$ or $1 \pmod 4$, and r and m are odd; or*

2. *$n \equiv 2$ or $3 \pmod 4$, and r is even and m is odd.*

2.4.3 Extended m–fold starter sequences

In this section, we introduce extended m–fold starter sequences, and we determine the necessary conditions and some of the sufficient conditions for the existence of extended m–fold starter sequences of order n.

Theorem 2.4.7. *Let $(s_1, s_2, \dots, s_{2nm+1})$ be a k-extended m–fold starter sequence of order n with one defect. Then one of the following must hold:*

1. *$n \equiv 2, 3 \pmod 4$, and k is odd; or*

2. *$n \equiv 0, 1 \pmod 4$, and m and k are of opposite parity.*

Proof. Suppose that $(s_1, s_2, \dots, s_{2mn+1})$ is a k-extended m–fold starter sequence of order n with one defect, and $s_k = 0$. By using the same technique as in the proof of

Theorem (2.4.1), we obtain:

$$\sum_{i=1}^{n}\sum_{j=1}^{m}(b_{ij}+a_{ij})=\frac{(2mn+1)(2mn+2)}{2}-k, \tag{2.4.9}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m}(b_{ij}-a_{ij})=\frac{mn^2+mn-4mr+4mn+2m}{2}. \tag{2.4.10}$$

By adding (2.4.9) and (2.4.10), and then dividing by 2, we obtain

$$\sum_{i=1}^{n}\sum_{j=1}^{m}b_{ij}=\frac{mn(4mn+n+11)-4mr+2m+2-2k}{4}. \tag{2.4.11}$$

The left side of (2.4.11) is always an integer, hence

1. $n \equiv 2, 3 \pmod 4$, and k is odd; or

2. $n \equiv 0, 1 \pmod 4$, and m and k are of opposite parity.

We investigate the sufficiency: first, suppose that $n \equiv 2, 3 \pmod 4$ and k is odd. It will be sufficient to build the required sequences with zeros in the odd positions from 1 to $2n+1$ inclusive. We first exhibit the two smallest cases:

In the case that $n = 2$:

$0, 3, 1, 1, 3, 3, 1, 1, 3$

$3, 3, 0, 3, 3, 1, 1, 1, 1$

$1, 1, 3, 3, 0, 3, 3, 1, 1$

$0, 4, 2, 4, 2, 4, 2, 4, 2$

$2, 4, 2, 4, 0, 4, 2, 4, 2.$

In the case that $n = 3$:

$0, 4, 4, 1, 1, 4, 4, 2, 2, 2, 2, 1, 1$

$1, 1, 0, 4, 4, 1, 1, 4, 4, 2, 2, 2, 2$

$2, 2, 2, 2, 0, 4, 4, 1, 1, 4, 4, 1, 1$

$2, 2, 2, 2, 1, 1, 0, 4, 4, 1, 1, 4, 4$

$0, 5, 5, 1, 1, 3, 5, 5, 3, 3, 1, 1, 3$

$5, 3, 0, 5, 3, 5, 1, 1, 5, 3, 1, 1, 3$

$5, 1, 1, 5, 0, 5, 3, 3, 5, 3, 3, 1, 1$

$5, 3, 1, 1, 3, 5, 0, 5, 3, 1, 1, 3, 5$

$0, 6, 2, 3, 2, 6, 3, 6, 2, 3, 2, 6, 3$

$6, 2, 0, 2, 3, 6, 6, 3, 2, 3, 2, 6, 3$

$2, 6, 2, 3, 0, 6, 3, 6, 2, 3, 2, 6, 3$

$3, 6, 2, 3, 2, 6, 0, 6, 2, 3, 2, 6, 3.$

If $n \equiv 2, 3 \pmod 4$ and k is odd, and $n > 3$, then according to Theorem (2.1.1), we know that there exists a starter sequence of order (n) with one defect. We also know that there exists a $SS_n^1(k)$, and that k is odd, so we can use $m-1$ copies of SS_n^1 and one copy for the extended starter sequence (k-ext-SS_n). If $n \equiv 0$ or $1 \pmod 4$, m is even, and k is odd, then according to Theorem (2.2.1), we know that a hooked starter sequence of order n with one defect exists (hSS_n^1), and since m is even, we hook hSS_n^1 with its reverse and we concatenate hSS_n^1 with SS_n^1. In particular, if $n \equiv 2, 3 \pmod 4$, $n > 3$, k is odd, and $(d = n+1)$, then we can construct the required extended m–fold starter sequence by using $m-1$ copies of n-near Skolem sequence of order $n+1$ and a k-extended n-near Skolem sequence of order $n+1$ in the appropriate position. For example, a 6-near Skolem sequences of order seven, $(7, 2, 3, 2, 4, 3, 5, 7, 4, 1, 1, 5)$, and the 5-extended 6-near Skolem sequence of order seven, $(7, 4, 1, 1, 0, 4, 5, 7, 2, 3, 2, 5, 3)$, can be used to obtain an extended m–fold starter sequence of order 6 with one defect.

For $n \equiv 2 \pmod 4$, $k = 1$, and $d = n+2$, the required sequence has the form:

$$\frac{0}{1}, \frac{n+2}{2}, \frac{n}{3}, \overset{-2}{\rightarrow}\cdots, \frac{2}{\frac{n}{2}+2}, \frac{n+2}{\frac{n}{2}+3}, \frac{2}{\frac{n}{2}+4}, \overset{+2}{\rightarrow}\cdots, \frac{n}{n+3}, \frac{n+2}{n+4}, \frac{n}{n+5}, \frac{n-2}{n+6}, \overset{-2}{\rightarrow}\cdots, \frac{2}{\frac{3n}{2}+4}, \frac{n+2}{\frac{3n}{2}+5}, \frac{2}{\frac{3n}{2}+6}, \overset{+2}{\rightarrow}\cdots, \frac{n-2}{2n+4}, \frac{n}{2n+5},$$
$$\frac{n-3}{2n+6}, \frac{n-5}{2n+7}, \overset{-2}{\rightarrow}\cdots, \frac{1}{\frac{5n}{2}+4}, \frac{1}{\frac{5n}{2}+5}, \overset{+2}{\rightarrow}\cdots, \frac{n-5}{3n+2}, \frac{n-3}{3n+3}, \frac{n-3}{3n+4}, \frac{n-5}{3n+5}, \overset{-2}{\rightarrow}\cdots, \frac{1}{\frac{7n}{2}+2}, \frac{1}{\frac{7n}{2}+3}, \overset{+2}{\rightarrow}\cdots, \frac{n-5}{4n}, \frac{n-3}{4n+1}.$$

For $k = 3$ and $n > 6$, we can construct the required sequence using the following formula:

$$\frac{4}{1}, \frac{2}{2}, \frac{0}{3}, \frac{2}{4}, \frac{4}{5}, \frac{n+2}{6}, \frac{n}{7}, \overset{\xrightarrow{-2}}{\ldots}, \frac{2}{\frac{n}{2}+6}, \frac{n+2}{\frac{n}{2}+7}, \frac{2}{\frac{n}{2}+8}, \overset{\xrightarrow{+2}}{\ldots}, \frac{n}{n+7}, \frac{n+2}{n+8}, \frac{n}{n+9}, \frac{n-2}{n+10}, \overset{\xrightarrow{-2}}{\ldots}, \frac{6}{\frac{3n}{2}+6}, \frac{1}{\frac{3n}{2}+7}, \frac{1}{\frac{3n}{2}+8}, \frac{n+2}{\frac{3n}{2}+9},$$
$$\frac{n-3}{\frac{3n}{2}+10}, \frac{n-5}{\frac{3n}{2}+11}, \frac{6}{\frac{3n}{2}+12}, \overset{\xrightarrow{+2}}{\ldots}, \frac{n-2}{2n+8}, \frac{n}{2n+9}, \frac{n-7}{2n+10}, \frac{n-9}{2n+11}, \overset{\xrightarrow{-2}}{\ldots}, \frac{3}{\frac{5n}{2}+5}, \frac{n-5}{\frac{5n}{2}+6}, \frac{n-3}{\frac{5n}{2}+7}, \frac{3}{\frac{5n}{2}+8}, \overset{\xrightarrow{+2}}{\ldots}, \frac{n-7}{3n+3},$$
$$\frac{n-3}{3n+4}, \frac{n-5}{3n+5}, \overset{\xrightarrow{-2}}{\ldots}, \frac{1}{\frac{7n}{2}+2}, \frac{1}{\frac{7n}{2}+3}, \overset{\xrightarrow{+2}}{\ldots}, \frac{n-5}{4n}, \frac{n-3}{4n+1}.$$

If $n = 6$, the required sequence is: $(4, 2, 0, 2, 4, 8, 6, 4, 2, 8, 2, 4, 6, 8, 6, 1, 1, 8, 3, 3, 6, 3, 3, 1, 1)$. Where $n \equiv 3 \pmod 4$, $k = 1$, and $d = n + 2$, the required sequence has the form:

$$\frac{0}{1}, \frac{n+2}{2}, \frac{n}{3}, \overset{\xrightarrow{-2}}{\ldots}, \frac{1}{\frac{n+1}{2}+2}, \frac{1}{\frac{n+1}{2}+3}, \overset{\xrightarrow{+2}}{\ldots}, \frac{n}{n+3}, \frac{n+2}{n+4}, \frac{n-3}{n+5}, \frac{n-5}{n+6}, \overset{\xrightarrow{-2}}{\ldots}, \frac{2}{\frac{3n+5}{2}}, \frac{n-3}{\frac{3n+7}{2}}, \frac{2}{\frac{3n+9}{2}}, \overset{\xrightarrow{+2}}{\ldots}, \frac{n-5}{2n+1}, \frac{n-3}{2n+2},$$
$$, \frac{n-5}{2n+3}, \frac{n-7}{2n+4}, \overset{\xrightarrow{-2}}{\ldots}, \frac{2}{\frac{5n-1}{2}}, \frac{n-3}{\frac{5n+1}{2}}, \frac{2}{\frac{5n+3}{2}}, \overset{\xrightarrow{+2}}{\ldots}, \frac{n-7}{3n-3}, \frac{n-5}{3n-2}, \frac{n+2}{3n-1}, \frac{n}{3n}, \overset{\xrightarrow{-2}}{\ldots}, \frac{1}{\frac{7n-1}{2}}, \frac{1}{\frac{7n+1}{2}}, \overset{\xrightarrow{+2}}{\ldots}, \frac{n}{4n}, \frac{n+2}{4n+1}.$$

In the case that $k = 1$ and $d > n + 1$, in order to construct a 1-extended starter sequence of order n and one defect, we put the zero at the first position of the required sequence. Then we use a k extended q-near Skolem sequence of order n and insert the inverse of the missing number at the front of the near-Skolem and the second q^{-1} to fill the hook of the sequence. For example, we use the sequence $(5, 1, 1, 3, 6, 5, 3, 2, *, 2, 6)$ which is an $N_6^4(10)$, to construct a 1-extended starter sequence of order 6 and one defect: $(0, 9, 5, 1, 1, 3, 6, 5, 3, 2, 9, 2, 6)$. ■

Similarly, we can also determine the necessary conditions for the existence of extended m–fold starter sequences of order n with more than one defect.

Theorem 2.4.8. *Let n, m, r, and d_r be positive integers, where $2 \le r \le n - 1$ and $n + 1 \le d_r \le (2n - 1)$, and $(s_1, s_2, s_3, \ldots, s_{2mn+1})$ is a k-extended m–fold starter sequence of order n with r defects $(d_1, d_2, \ldots, d_r)$. Then one of the following must hold:*

1. $n \equiv 0, 1 \pmod 4$, *where r is even and k is odd;*

2. $n \equiv 0, 1 \pmod 4$, *where r and k are odd, and m is even;*

3. $n \equiv 0, 1 \pmod 4$, *where* r *and* m *are odd, and* k *is even; or*

4. $n \equiv 2, 3 \pmod 4$, *where* r *and* k *are odd.*

2.5 Excess starter sequences

In this section, we introduce another new generalization of starter sequences in which one integer occurs in two sets. We begin this section by introducing the concept of excess starter sequences of order n and surplus p.

Definition 2.5.1. *An excess starter sequence of order* n *and surplus* p *is a sequence* $\mathscr{X}_n^p = (s_1, s_2, \ldots, s_{2n+2})$ *of* $2n+2$ *integers satisfying:*

1. *for every* $j \in \{1(-1), 2(-2), \ldots, p-1(-(P-1)), p+1(-(P+1)), \ldots, n(-n)\}$ *there are exactly two elements* s_u, $s_v \in \mathscr{X}_n^p$ *such that* $s_u = s_v = j$ *and* $|v-u| = j$,

2. *there are exactly four elements* $s_a = s_b = s_c = s_d = p$ *and* $|b-a| = |d-c| = p$.

For example, $\mathscr{X}_4^4 = (6, 4, 1, 1, 4, 4, 6, 2, 4, 2)$ is an excess-starter sequence of order 4 with one defect and a surplus $p = 4$.

It is not difficult to determine the necessary conditions for the existence of such sequences. Moreover, the settling of the existence problem for the extended starter sequences with one defect will imply the sufficient conditions for existence of the excess starter sequences with one defect.

Theorem 2.5.1. *An excess starter sequence of order* n *with one defect and surplus* p *exists only if either* $n \equiv 0, 1 \pmod 4$ *and* p *is even, or* $n \equiv 2, 3 \pmod 4$ *and* p *is odd.*

Proof. Let $(s_1, s_2, \ldots, s_{2n+2})$ be an excess starter sequence of order n with one defect and surplus p. Consider the sum of all sums, and the sum of all differences of the subscripts a_r and b_r, such that $b_r - a_r = k$, and k appears at least twice in $\mathscr{X}_n^p$.

$$\sum_r (b_r + a_r) = \sum_{r=1}^{2n+2} r = \frac{(2n+2)(2n+3)}{2}; \tag{2.5.1}$$

$$\sum_r (b_r - a_r) = \sum_{r=1}^{n} r - d + p + (2n+1-d) = \frac{(n)(n+1)}{2} + 2n - 2d + p + 1. \tag{2.5.2}$$

where $1 \leq p \leq 2n$ and $1 \leq d \leq n$.

By adding (2.5.1) and (2.5.2), then dividing by 2, we obtain

$$\sum_r b_r = \frac{n(5n+15) - 4d + 2p + 8}{4}. \tag{2.5.3}$$

The left side of (2.5.3) is always an integer; hence, $n \equiv 0, 1 \pmod 4$ and p is even, or $n \equiv 2, 3 \pmod 4$ and p is odd. ■

Similarly, we determine the necessary conditions for the existence of excess starter sequences for all admissible numbers of defects.

Theorem 2.5.2. *An excess starter sequence of order n with r defects $(r > 1)$ and surplus p exists only if either*

1. $n \equiv 0, 1 \pmod 4$*, and r and p have different parities; or*

2. $n \equiv 2, 3 \pmod 4$*, and r and p have the same parity.*

Proof. Let $(s_1, s_2, \ldots, s_{2n+2})$ be an excess starter sequence of order n with r-defects and surplus p. Consider the sum of all sums, and the sum of all differences of the subscripts (a_r, b_r). Hence,

$$\sum_r b_r = \frac{n(5n+11) - 4(d_1 + d_2 + \ldots + d_r) + r(4n+2) + 2p + 6}{4}. \tag{2.5.4}$$

The left side of (2.5.4) is always an integer; hence, either $n \equiv 0,\, 1 \pmod 4$, and r and p have different parities, or $n \equiv 2,\, 3 \pmod 4$, and r and p have the same parity. ■

For example, $\mathscr{X}_6^2 = (12, 10, 8, 2, 4, 2, 7, 2, 4, 2, 8, 10, 12, 7)$ is an excess-starter sequence of order 6 with four defects and surplus $p = 2$. In addition, we can generalize the excess starter sequence to more than one surplus; for example, ($\mathscr{X}_4^{2,5} = 5, 5, 1, 1, 3, 5, 5, 3, 2, 2, 2, 2)$ is a double excess starter sequence of order 4 with one defect and two surpluses 2 and 5.

Theorem 2.5.3. *If $1 \leq u < v \leq 2n$, then a double excess starter sequence of order n with one defect, and with two surpluses u and v ($\mathscr{X}_n^{u,v}$), exists only if either:*

1. $n \equiv 2,\, 3 \pmod 4$ *and* $u \equiv v \pmod 2$*; or*

2. $n \equiv 0,\, 1 \pmod 4$ *and* $u \not\equiv v \pmod 2$.

Proof. Let $(s_1, s_2, \ldots, s_{2n+4})$ be a double excess starter sequence of order n with one defect and two surpluses u and v. Consider the sum of all sums, and the sum of all differences of the subscripts a_r and b_r:

$$\sum_r (a_r + b_r) = \sum_{r=1}^{2n+4} r = \frac{(2n+4)(2n+5)}{2} \tag{2.5.5}$$

$$\sum_r (b_r - a_r) = \frac{(n)(n+1)}{2} - 2d + 2n + u + v + 1 = \frac{n(n+5) - 4d + 2(u+v+1)}{2} \tag{2.5.6}$$

where $1 \leq m \leq n$ and $1 \leq u,\, v \leq 2n$.

By subtracting (2.5.6) from (2.5.5) we obtain:

$$\sum_r a_r = \frac{n(5n+23) - 4d + 2(u+v) + 18}{4} \tag{2.5.7}$$

The left side of (2.5.7) is always an integer; hence, $n \equiv 2,\, 3 \pmod 4$ and $u \equiv v \pmod 2$ or, $n \equiv 0,\, 1 \pmod 4$ and $u \not\equiv v \pmod 2$. ■

It is interesting to realize that the settling of the existence question for the extended starter sequences will imply the sufficient conditions for the existence of the excess starter sequences.

Example 2.5.1. $\mathscr{X}_5^{3,4} = (10, 3, 9, 3, 3, 4, 3, 6, 4, 4, 10, 9, 4, 6)$ *is a double excess starter sequence of order* 5 *with three defects and two surpluses* 3 *and* 4*.*

Theorem 2.5.4. *An excess starter sequence of order n with r defects $(r > 1)$ and two surpluses u and v exists only if either*

1. $n \equiv 0, 1 \pmod 4$*, and r is even and* $u \equiv v \pmod 2$*;*
2. $n \equiv 0, 1 \pmod 4$*, and r is odd and* $u \not\equiv v \pmod 2$*;*
3. $n \equiv 2, 3 \pmod 4$*, and r is even and* $u \not\equiv v \pmod 2$*; or*
4. $n \equiv 2, 3 \pmod 4$*, and r is odd and* $u \equiv v \pmod 2$*;*

Proof. Let $(s_1, s_2, \ldots, s_{2n+2})$ be an excess starter sequence of order n with r-defects and two surpluses u and v. Consider the sum of all sums, and the sum of all differences of the subscripts. Hence,

$$\sum_r b_r = \frac{n(5n+19) - 4(d_1 + d_2 + \ldots + d_r) + 4nr + 2r + 2u + 2v + 20}{4} \tag{2.5.8}$$

The left side of (2.5.8) is always an integer; this implies the items (1), (2), (3), and (4) in the statement of the theorem. ■

Chapter 3

Starter Labelling of Paths, Cycles, and Windmills

In this chapter, we introduce the concept of starter-labelled graphs and we explore the necessary and sufficient conditions for the existence of starter and minimum hooked starter labelling of paths, cycles, and k-windmils. We begin this chapter by introducing starter-labelled graphs and hooked starter-labelled graphs. Throughout this chapter, $G = (V, E)$ is an undirected graph and $d_G(u, v)$ is the length of a shortest path in G connecting vertices u and v.
The results of starter-labelled k-windmills have appeared in [31].

Definition 3.0.2. *A starter-labelled graph is a pair (G, L), where:*

(a) $G = (V, E)$ *is a graph with* $2n$ *vertices,*

(b) $L : V \longrightarrow \{1(1^{-1}), 2(2)^{-1}, \ldots, n(n)^{-1}\}$ *is a surjection, and* i^{-1} *is the additive inverse in the group* $\mathbb{Z}_{2n+1}$*,*

(c) *for each* $k \in \{1(1^{-1}), 2(2)^{-1}, \ldots, n(n)^{-1}\}$*, there exist exactly two vertices* $u, w \in V$ *such that* $L(u) = L(w) = k$*, and* $d_G(u, v) = k$*,and*

(d) *if* $\hat{G} = (V, \hat{E})$ *and* $\hat{E} \subset E$ *then* $(\hat{G}, L)$ *violates* (c)*.*

Definition 3.0.3. *A hooked starter-labelled graph is a pair* (G, L) *satisfying the conditions of Definition* (1.1) *with* ($\hat{\text{b}}$) *instead of* (b)*:*

($\hat{\text{b}}$) $L : V \longrightarrow \{0\} \cup \{1(1^{-1}), 2(2)^{-1}, \ldots, n(n)^{-1}\}$ *is a surjection.*

3.1 Starter Labelling of Paths

In this section, we will exhibit the starter (hooked starter) labelling of paths with as few hooks as possible.

Theorem 3.1.1. *Every path of length* m *can be:*

1. *Starter-labelled with an odd number of defects* $(\lfloor \frac{m+1}{4} \rfloor)$ *if* $m \equiv 3,\, 5 \pmod 8$*.*
2. *Starter-labelled with an even number of defects* $(\lfloor \frac{m+1}{4} \rfloor)$ *if* $m \equiv 1,\, 7 \pmod 8$*.*
3. *Hooked starter-labelled if* m *is even.*

Proof. 1-(a) For paths of length $m = 3 + 8s$, we label the vertices $1, 2, \ldots, 4 + 8\text{s}$, and then use the construction: $\{(r, 5 + 8s - r),\ 1 \leq r \leq \frac{m+1}{2}\}$.

(b) For paths of length m $= 5 + 8\text{s}$, the vertices are labeled $1, 2, \ldots, 6 + 8s$, and we use the construction: $(r, 7 + 8s - r),\ 1 \leq r \leq \frac{m+1}{2}$.

2-(a) For paths of length $m = 7 + 8s$, the vertices are labeled $1, 2, \ldots, 8\text{s} + 8$, and we can use the construction $(r, 9 + 8s - r),\ 1 \leq r \leq \frac{m+1}{2}$.

(b) For paths of length $m = 1 + 8s$, the vertices are labeled $1, 2, \ldots, 2 + 8\text{s}$, and we can use the construction $(r, 3 + 8s - r),\ 1 \leq r \leq \frac{m+1}{2}$. Moreover, we can use another construction:

$(r+1, 8s+3-r)$, $r = 1, 2, \ldots, \lfloor \frac{m+1}{4} \rfloor$

$(4s+1-r, 4s+1+r)$, $r = 1, 2, \ldots, \lfloor \frac{m+1}{4} \rfloor - 1$

$(1, 4s+1), (6s+1, 6s+2)$.

3- Since m is even, then $m \equiv 0,\ 2,\ 4,\ 6 \pmod 8$. In the case that $m \equiv 0 (\mathrm{mod}\ 8)$, we label the vertices $1, 2, 3, \ldots, 8s+1$, and we can use the construction: $\{(r+1, 8s+1-r),\ \ 0 \leq r \leq \frac{m}{2} - 1\}$ to obtain a hooked starter labelled with one hook in the middle. Similarly, we can prove the remaining cases ($m \equiv 2,\ 4,\ 6 \pmod 8$, and $m > 6$).

■

Theorem 3.1.2. *Every path of length m can be starter-labelled with one defect if and only if $m \equiv 3, 5 \pmod 8$.*

Proof. Let P_m be a path of length m. Its vertices are labeled by a starter sequence of order n with one defect (d); $(S_n^d = s_1, s_2, s_3, ..., s_{2n})$ such that $D = \{\{1, 2, 3, \ldots, n\} \backslash \{q\}\} \cup \{d\}$, where $d = 2n+1-q$. Consider the set of subscripts $\{(a_r, b_r) : b_r - a_r = r\}$, the sum of all sums, and the sum of all differences of the subscripts:

$$\sum_{r=1}^{n} b_r + \sum_{r=1}^{n} a_r = \sum_{r=1}^{n} (a_r + b_r) = \sum_{j=1}^{2n} j = \frac{(2n)(2n+1)}{2} = 2n^2 + n \tag{3.1.1}$$

$$\sum_{r=1}^{n} b_r - \sum_{r=1}^{n} a_r = \sum_{r=1}^{n} r + 2n - 2q + 1 = \frac{n(n+1)}{2} + 2n - 2q + 1. \tag{3.1.2}$$

By subtracting (3.1.2) from (3.1.1), we obtain $\sum_{r=1}^{n} a_r = \frac{3n^2-3n+4q-2}{4}$. Since the left hand side of the equation is an integer, this implies that $3n^2 - 3n + 4q - 2$ must be divisible by 4, which occurs only when $n \equiv 2, 3 \pmod 4$. This implies that $m \equiv 3, 5 \pmod 8$.

Now the following is the proof of the sufficiency. By Theorem (1.1.9), we know that if $n \equiv 2, 3 \pmod 4$, and q and k have opposite parity, then there exists a k-extended q-near-Skolem sequence $\mathcal{N}_n^q(k)$ except $(n, q, k) = (3, 2, 3)$. Therefore, we can use $\mathcal{N}_n^q(k)$ to construct the required labeling. Then we append the defect d at the end of $\mathcal{N}_n^q(k)$, and fill the hole of $\mathcal{N}_n^q(k)$ by putting the defect at the position k. For instance, we can use $\mathcal{N}_6^5(4)$, which is a 4-extended 5-near-Skolem sequence with one defect to construct a starter-labelled sequence with one defect to label the path P_{12}. We begin with $\mathcal{N}_6^5(4)$, $(6, 4, 2, *, 2, 4, 6, 3, 1, 1, 3)$, and then we insert the defect 8 at the positions $(4, 12)$. Hence, we will obtain a starter labelling of order 6 with one defect of the path, as shown in Figure 3.1.

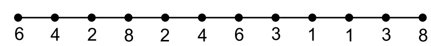

Figure 3.1: A starter labelling with one defect for P_{12}.

In the case that $(n, q, k) = (3, 2, 3)$, we can construct only two starter sequences of order 3 with one defect, $S_3^5 = 5 - 3 - 1 - 1 - 3 - 5$, and $S_3^4 = 4 - 1 - 1 - 2 - 4 - 2$. Clearly, the existence of starter and hooked starter sequences is equivalent to starter and hooked starter labelings of paths. Now we provide additional constructions to obtain starter-labelled paths with one defect, where $d = n + 1, n + 2$. In each case, the solution is given in the form of a table. The first column indicates the difference $i = b_i - a_i$, the second and third columns of the tables give the two subscripts of a_i and b_i, respectively. The difference i will be placed in the sequence at two positions: a_i and b_i.

Case 1: Paths of length $m \equiv 3 \pmod 8$ and $d = n+1$. Let the vertices be $1, 2, \ldots, 4+8s$.

Table 3.1: The construction for $m = 3 + 8s$ and $d = n + 1$.

i	a_i	b_i	r
1	$5s + 2$	$5s + 3$	
2r	$2s + 1 - r$	$2s + 1 + r$	$1 \leq r \leq 2s$
$4s + 3$	$2s + 1$	$6s + 4$	
$2s + 1$	$6s + 3$	$8s + 4$	
$4s + 3 - 2r$	$4s + 1 + r$	$8s + 4 - r$	$1 \leq r \leq s$
$2s + 1 - 2r$	$5s + 3 + r$	$7s + 4 - r$	$1 \leq r \leq (s - 1)$

Similarly, we can label the paths of length $m \equiv 3 \pmod 8$ when $d = n + 2$.

Table 3.2: The construction for $m = 3 + 8s$ and $d = n + 2$.

i	a_i	b_i	r
1	$5s + 4$	$5s + 5$	-
2r	$2s + 2 - r$	$2s + 2 + r$	$1 \leq r \leq 2s + 1$
$4s + 4$	$2s + 2$	$6s + 6$	-
$2r + 1$	$6s + 5 - r$	$6s + 6 + r$	-
$4s + 3 - 2r$	$4s + 1 + r$	$8s + 4 - r$	$1 \leq r \leq s$
$2s + 2r + 1$	$5s + 4 - r$	$7s + 5 + r$	$1 \leq r \leq (s - 1)$

Case 2: Paths of length $m \equiv 5 \pmod 8$ and $d = n + 1$. Let the vertices be $1, 2, \ldots, 6 + 8s$.

Table 3.3: The construction for $m = 5 + 8s$ and $d = n + 1$.

i	a_i	b_i	r
$2r$	$2s + 3 - r$	$2s + 3 + r$	$1 \leq r \leq 2m + 2$
$4s + 1$	$2s + 3$	$6s + 4$	
$4s - 1 - 2r$	$4s + 6 + r$	$8s + 5 - r$	$0 \leq r \leq s - 2$
$2s - 1 - 2r$	$5s + 5 + r$	$7s + 4 - r$	$0 \leq r \leq s - 2$
1	$7s + 5$	$7s + 6$	
$2s + 1$	$6s + 5$	$8s + 6$	

If the defect is $n + 2$, then we can produce a starter-labeling with one defect for a path of length $m \equiv 5 \pmod 8$. Let the vertices be $1, 2, 3, \ldots, 6 + 8s$, where $s \in \mathbb{N}$. The following construction is valid for all $s \geq 1$.

Table 3.4: The construction for $m = 5 + 8s$ and $d = n + 2$.

i	a_i	b_i	r
$4s - 2r + 5$	r	$4s - r + 5$	$1 \leq r \leq 2s + 1$
$4s + 5$	$2s + 2$	$6s + 7$	–
1	$5s + 5$	$5s + 6$	–
$2s + 2r + 4$	$5s - r + 4$	$7s + r + 8$	$0 \leq r \leq s - 2$
$2s + 2$	$2s + 3$	$4s + 5$	–
$2r$	$6s + 7 - r$	$6s + 7 + r$	$0 \leq r \leq s$

■

Theorem 3.1.3. *Every path of length m can be starter-labelled with two defects only if $m \equiv 1, 7 \pmod 8$.*

Proof. Similar to Theorem 2.1.2. ■

In a similar vein, we determine the necessary conditions for the existence of starter sequences with n defects.

Theorem 3.1.4. *Every path of length m can be starter-labelled with n defects only if n is even and $m \equiv 1, 7 \pmod 8$, or n is odd and $m \equiv 3, 5 \pmod 8$.*

3.2 Starter Labelling of Cycles

In this section, we investigate a starter (hooked starter) labelling of cycles with as few hooks as possible. Although cycles cannot be starter-labelled, they can be hooked starter-labelled.

Theorem 3.2.1. *Every cycle of length $m \geq 5$ can be:*

1. *Starter-labelled with two hooks and one defect when m is even.*

2. *Starter-labelled with three hooks and one defect when m is odd.*

Proof. We divide the proof into four subcases as shown in Table(5):

Table 3.5: The construction for $C_m,\ m \geq 5$.

The length	Label	a_i	b_i	r
$4s$	$2r$	$s-r+1$	$s+r+1$	$1 \leq r \leq s$
	$2r+1$	$3s-r$	$3s+r+1$	$0 \leq r \leq s-2$
$2+4s$	$2r$	$s+2-r$	$s+r+2$	$1 \leq r \leq s-1$
	$2r+1$	$3s+2-r$	$3s+3+r$	$0 \leq r \leq s-1$
$1+4s$	$2r$	$s+1-r$	$s+r+1$	$1 \leq r \leq s$
	$2r+1$	$3s+2-r$	$3s+1+r$	$0 \leq r \leq s-1$
$3+4s$	$2r$	$s+2-r$	$s+2+1$	$1 \leq r \leq s-1$
	$2r+1$	$3s+3-r$	$3s+4-r$	$0 \leq r \leq s-1$

■

3.3 Starter-Labelling of Windmills

In this section, we investigate starter-labelled k-windmills. The results of this section have been published in [31].

Definition 3.3.1. *A k-windmill is a tree containing k paths of equal positive lengths, called vanes, which share a center vertex called the pivot or the center.*

Example 3.3.1. *Figure* 3.2 *illustrates a hooked starter-labelled* 4*-windmill.*

According to definition (3.0.2), a hooked starter-labeled graph can have some vertices labelled zero, but every edge is still essential. This leads us to the definition of the strong (weak) starter-labelled graph.

Definition 3.3.2. *A graph on* $2n$ *vertices can be strongly starter-labelled if the removal of any edge destroys the starter-labelled.*

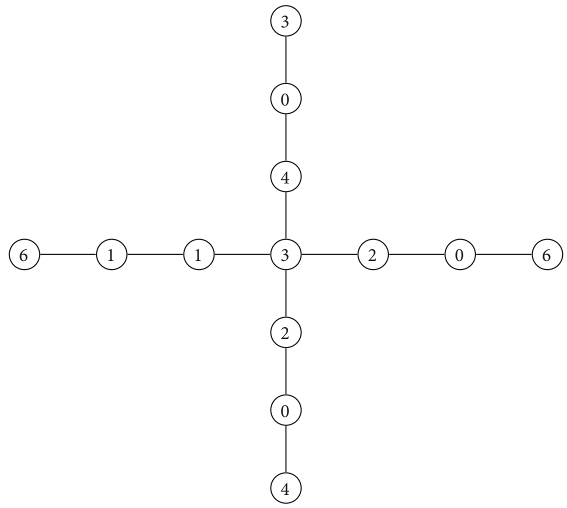

Figure 3.2: A hooked starter-labelled 4-windmills

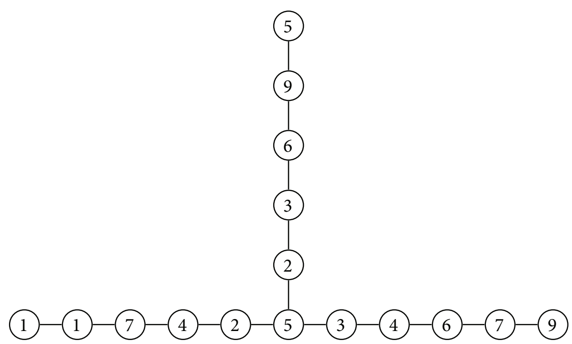

Figure 3.3: A weak starter-labelled 3-windmill

Definition 3.3.3. *A graph on* $2n$ *vertices can be weakly starter-labelled if there exists at least one edge in the graph such that the removal of that edge does not destroy the starter-labelled graph.*

Example 3.3.2. *Figures* 3.3 *and* 3.4 *show a weak starter-labelled* 3*-windmill and a strong starter-labelled* 3*-windmill, respectively.*

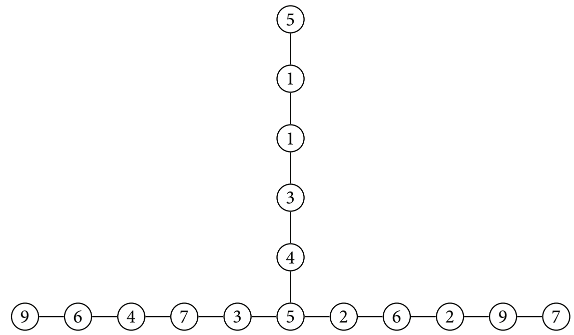

Figure 3.4: A strong starter-labelled 3-windmill

3.3.1 Necessity

We notice that a tree $T = (V, E)$ can only be a starter-labelled graph if the number of vertices is even ($|V| = 2n$). This implies that the length of the vane must be odd and that all k-windmills where k is even cannot be starter-labelled. In addition, an obvious degeneracy condition for a starter-labeling (a hooked starter-labeling) of a tree T is that the tree must have a path of a length of at least $(n + 1)$. Thus, only 3-windmills can be starter-labelled.

3.3.2 Starter Parity

Shalaby and Mendelsohn [27] defined Skolem parity and proved that it was necessary for the existence of any Skolem-labelled tree. Similarly, we establish the parity conditions for starter-labelled k-windmills.

Definition 3.3.4. *The starter parity of a vertex u of a tree $T = (V, E)$ is the sum of the lengths of the paths from u to all the verices of the tree* (T). *Thus,* $P_u = \sum_{v \in V} d(u, v) \pmod 2$.

Lemma 3.3.1. *[27] If T is a tree with $2n$ vertices, then the starter parity of T is independent of $u \in V$.*

Lemma 3.3.2. *If G is a starter-labelled k-windmill with $2n$ vertices and k vanes, then either:*

(1) $n \equiv 0, 2 \pmod 4$, *and the starter parity of G is odd, or*

(2) $n \equiv 1, 3 \pmod 4$, *and the starter parity of G is even.*

Proof. Assume that G is starter-labelled k-windmill with $2n$ vertices and k vanes of length m. Using the center point c to calculate the starter parity, we obtain:

$$\begin{aligned} P_c = \sum_{v \in V} d(c, v) &= \sum_{i=1}^{k} m(m+1)/2 \\ &= \frac{km^2 - 1}{2} + n. \end{aligned}$$

Since G is starter-labelled graph, $k = 3$ and m must be odd ($m \equiv 1, 3 \pmod 4$). We notice that if $m \equiv 1 \pmod 4 \Rightarrow 3m^2 \equiv 3 \pmod 4 \Rightarrow 3m^2 - 1 \equiv 2 \pmod 4$. Similarly, if $m \equiv 3 \pmod 4 \Rightarrow 3m^2 \equiv 27 \pmod 4$ and since $27 \equiv 3 \pmod 4$, then $3m^2 - 1 \equiv 2 \pmod 4$ (by the transitivity). Now we consider all the cases of n:

(1) If $n \equiv 0 \pmod 4$, then $P_c = (1 + 2j) + (4r) \Rightarrow$ the starter parity is odd;

(2) If $n \equiv 1 \pmod 4$, then $P_c = (1 + 2j) + (1 + 4r) \Rightarrow$ the starter parity is even;

(3) If $n \equiv 2 \pmod 4$, then $P_c = (1 + 2j) + (2 + 4r) \Rightarrow$ the starter parity is odd;

(4) If $n \equiv 3 \pmod 4$, then $P_c = (1 + 2j) + (3 + 4r) \Rightarrow$ the starter parity is even.

■

3.3.3 The Degeneracy Condition

We saw that a graph with $2n$ vertices must have at least a path of length $(n + 1)$ in order to be starter-labelled. Therefore, all windmills with more than 3 vanes cannot be labelled by a starter-sequence. For a (possibly hooked) starter-labelled k-windmill with equal vanes of length m, the largest label is $2m$, the maximum number of edges in the corresponding path not used in any other path is $2m$, and covering all edges of 2 vanes. Also, labels that are bigger than m must cover parts of 2 vanes. The label m may cover the complete vane. Thus, for all labels $\mathrm{m_i}$ with $m \leq m_i \leq 2\mathrm{m}$, the maximum number of edges covered is no more than:

$$2m + (2m-1) + ... + m = \frac{3(m^2+m)}{2}. \tag{3.3.1}$$

Moreover, the labels $n_i < m$ must cover at least one edge covered by another label, so the total number of edges for these labels is at most

$$1 + 2 + \ldots + (m-1) = \frac{m^2-m}{2}. \tag{3.3.2}$$

Therefore, the maximum number of edges is $\leq (3.3.1) + (3.3.2) = 2m^2 + m$. Since the total number of edges in a k-windmill is km, hence $k \leq 2m+1$.

3.3.4 Sufficiency

In this section, we provide and prove the sufficient conditions for obtaining the starter-label (minimum hooked starter label) for all k-windmills, where k is the number of the vanes; we count them arbitrarily (say counter clockwise) from 1 to k. Let m indicate the length of the vane of the windmill; then each vertex v can be represented by a pair (i, j) where i is the number of the vane and j is its distance from the center, and the center point is denoted by $(0, 0)$.

3.3.5 3–Windmills

Lemma 3.3.3. *All* 3*–windmills with* $m \equiv 1, 3, 5, 7 \pmod 8$ *can be starter-labelled, except for the case* $m = 1$.

Proof. The following construction gives us the pairs $\mathrm{a_{i,j}}$ and $\mathrm{b_{i,j}}$, where the number of defects is $\lfloor \frac{m}{4} \rfloor$ in the case $m \equiv 1, 5 \pmod 8$, and $\lceil \frac{m}{4} \rceil$ in the the case $m \equiv 3, 7 \pmod 8$.

$b_{i,j}$	$a_{i,j}$	$\leq r \leq$	label
$(2, \frac{m-1}{2} + r + 1)$	$(2, \frac{m-1}{2} - r)$	$0 \leq r \leq \frac{m-1}{2}$	$2r + 1$
$(3, r)$	$(1, r)$	$1 \leq r \leq m$	$2r$

■

Lemma 3.3.4. *All* 3*–windmills with vane length* $m \equiv 0, 2, 4, 6 \pmod 8$ *can be minimum hooked starter-labelled with exactly one-hook.*

Proof. The solution is given by the following table, where the number of the defects is $\lfloor \frac{m}{4} \rfloor$.

$a_{i,j}$	$b_{i,j}$	$\leq r \leq$	label
$(2, \frac{m}{2} - r)$	$(2, \frac{m}{2} + r + 1)$	$0 \leq r \leq \frac{m}{2} - 1$	$2r + 1$
$(3, m)$	$(0, 0)$	$-$	m
$(3, r)$	$(1, r)$	$1 \leq r \leq \frac{m}{2} - 1$	$2r$
$(3, r)$	$(1, r + 1)$	$\frac{m}{2} \leq r \leq m - 1$	2r+1

■

3.3.6 4−Windmills

All 4-windmills have an odd number of vertices, so there is no starter labelling. The minimum hooked starter labelling in this case has at least three hooks.

Lemma 3.3.5. *All 4–windmills with $m \geq 2$ can be minimum hooked starter-labelled with exactly three hooks.*

Proof. Case $1 : m$ is odd. The solution is given by the following table.

$a_{i,j}$	$b_{i,j}$	$\leq r \leq$	label
$(3,r)$	$(1,r)$	$1 \leq r \leq m$	$2r$
$(2,m)$	$(0,0)$	$-$	m
$(4,r+1)$	$(2,r)$	$1 \leq r < m - (\frac{m+1}{2})$	$2r+1$
$(4, m - \frac{m+1}{2} + 1)$	$(4, m - \frac{m+1}{2} + 2)$	$-$	1
$(4,r+2)$	$(2,r-1)$	$m - \frac{m+1}{2} < r \leq m-2$	$2r+1$

Case $2 :$ m is even. The solution is given by the following table.

$a_{i,j}$	$b_{i,j}$	$\leq r \leq$	label
$(3,r)$	$(1,r)$	$1 \leq r \leq m$	$2r$
$(4,1)$	$(2,m)$	$-$	$m+1$
$(4,r+1)$	$(2,r)$	$1 \leq r < \frac{m}{2}$	$2r+1$
$(4, \frac{m}{2}+2)$	$(4, \frac{m}{2}+1)$	$-$	1
$(4,r+2)$	$(2,r-1)$	$\frac{m}{2} < r \leq m-2$	$2r+1$

The following table provides the construction of the pairs $a_{i,j}$ and $b_{i,j}$ for a weak starter-labelling of 4–windmills.

$a_{i,j}$	$b_{i,j}$	$\leq r \leq$	label
$(3,r)$	$(1,r)$	$1 \leq r \leq m$	$2r$
$(4,r+1)$	$(2,r)$	$0 \leq r \leq m-2$	$2r+1$

Remark: We can construct a hooked starter-labelling with zero defects (skolem labelling) and one hook for all 4–windmills; the following tables provide such a required construction:

Case $1: m \equiv 0 \pmod 2$

$a_{i,j}$	$b_{i,j}$	$\leq r \leq$	label
$(2,r)$	$(1,r)$	$1 \leq r \leq m$	$2r$
$(3,1)$	$(3,m)$	$-$	$m-1$
$(4,\frac{m}{2})$	$(4,\frac{m}{2}-1)$	$-$	1
$(3,r)$	$(4,r+1)$	$\frac{m}{2} \leq r \leq m-1$	$2r+1$
$(3,r+1)$	$(4,r)$	$1 \leq r \leq \frac{m}{2}-2$	$2r+1$

Case $2: m \equiv 1 \pmod 2$

$a_{i,j}$	$b_{i,j}$	$\leq r \leq$	label
$(2,r)$	$(1,r)$	$1 \leq r \leq m$	$2r$
$(0,0)$	$(3,m)$	$-$	m
$(3,r)$	$(4,r+1)$	$\frac{m+1}{2} \leq r \leq m-1$	$2r+1$
$(3,r+1)$	$(4,r)$	$1 \leq r \leq \frac{m-1}{2}-1$	$2r+1$
$(4,\frac{m-1}{2})$	$(4,\frac{m-1}{2}+1)$	$-$	1

3.3.7 k–Windmills, k > 4

In this case there is no starter-labelling. Thus, the only possibility is minimum hooked starter labelling.

Lemma 3.3.6. *For any* k*–windmill, the condition* $\mathrm{k}+1 < 2\mathrm{m}$ *is sufficient for minimum hooked starter labelling.*

Proof. Fix m and consider separate cases for k.

Case (1) The number of vanes is even ($\mathrm{k} = 2t$). Label the vanes $L_1, L_k, L_2, L_{k-1}, \ldots, L_t, L_{t+1}$, and the solution is given in Table 3.3.7.

$a_{i,j}$	$b_{i,j}$	$\leq r \leq$	label
(k, m)	$(1, m)$	$-$	$2m$
$(k-r+1, m-r)$	(r, m)	$2 \leq r \leq t$	$2m-r$
$(\mathrm{k}-\mathrm{r}+2, \mathrm{m}-\mathrm{r})$	$(\mathrm{k}-\mathrm{r}+2, \mathrm{m})$	$3 \leq r \leq t+1$	r
$(k, r-1)$	$(1, r+1)$	$t+2 \leq 2r \leq 2m-t-1$	$2r$
$(k-1, r-1)$	$(2, r+2)$	$t+2 \leq 2r+1 \leq 2m-t-1$	$2r+1$
$(3, 2)$	$(3, 1)$	$-$	1
$(4, 2)$	$(4, 0)$	$-$	2

Case (2) $\mathrm{k} = 2\mathrm{t}+1$, $t > 2$. Label the vanes $L_1, L_k, L_2, L_{k-1}, \ldots, L_t, L_{k+1-t}, L_{k-t}$.

$a_{i,j}$	$b_{i,j}$	$\leq r \leq$	label
(k, m)	$(1, m)$	$-$	$2m$
$(k-r+1, m-r)$	(r, m)	$2 \leq r \leq t$	$2m-r$
$(3, \mathrm{m}-\mathrm{t}-1)$	$(\mathrm{k}-\mathrm{t}, \mathrm{m})$	$-$	$2m-t-1$
$(\mathrm{k}-\mathrm{r}+2, \mathrm{m}-\mathrm{r})$	$(\mathrm{k}-\mathrm{r}+2, \mathrm{m})$	$3 \leq r \leq t+1$	r
$(k, r-1)$	$(1, r+1)$	$t+2 \leq 2r < 2m-t-1$	$2r$
$(k-1, r-1)$	$(2, r+2)$	$t+2 \leq 2r+1 < 2m-t-1$	$2r+1$
$(4, 1)$	$(0, 0)$	$-$	1
$(4, 4)$	$(4, 2)$	$-$	2

Case (3) k $= 5,.$ Label the vanes $L_1, L_2, \ldots, L_5$.

$a_{i,j}$	$b_{i,j}$	$\leq r \leq$	label
$(5, m)$	$(1, m)$	$-$	$2m$
$(4, m-2)$	$(2, m)$	$-$	$2m-2$
$(1, m-2)$	$(3, m-1)$	$-$	$2m-3$
$(4, m-3)$	$(4, m)$	$-$	3
$(5, r)$	$(1, r)$	$4 \leq 2r < 2m-4$	$2r$
$(4, r-1)$	$(2, r+2)$	$4 \leq 2r+1 < 2m-4$	$2r+1$
$(3, m-3)$	$(1, m-1)$	$-$	$2m-4$
$(5, m-2)$	$(5, m-1)$	$-$	1
$(3, m-2)$	$(3, m)$	$-$	2

■

Chapter 4

Pseudo-Starter Sequences

In this chapter, we introduce new sequences called pseudo-starter sequences, present several types of pseudo-starter sequences, and determine some of the conditions for their existence. In a starter sequence, every term (*position*) is occupied by only one number. If we allow some of the terms to be occupied by more than one number, then we construct a pseudo-starter sequence. In fact, pseudo-starter sequences are a generalization of pseudo-Skolem sequences. The results of this chapter are to appear in [33].

Definition 4.0.5. *Suppose that k and n are positive integers, such that $n \geq 3$ and $1 \leq k \leq 2n-1$. A k-pseudo starter sequence of order n (k-pseudo-starter-S_n) is a sequence $(s_1, s_2, \ldots, s_{k-1}, \overset{\hat{s_k}}{s_k}, s_{k+1}, \ldots, s_{2n-1})$ with the property that, for each $i \in \{1(1^{-1}), 2(2^{-1}), \ldots, n(n^{-1})\}$, either i or i^{-1} occurs in the sequence twice; they are separated by either $i-1$ elements or $i^{-1}-1$ elements, respectively, where i^{-1} is the additive inverse of i in the group $\mathbb{Z}_{2n+1}$.*

Example 4.0.3. *The sequence* $(\overset{6}{4}, 1, 1, 2, 4, 2, 6)$ *or* $\{(1,5), (1,7), (2,3), (4,6)\}$ *is a 1-pseudo starter sequence-S_4 with one defect, where the pocket is located in the first term of the sequence as shown in Figure 4.1. For $n = 5$, the sequence* $(2, \overset{7}{6}, 2, 3, 1, 1,$

$,3,6,7)$ *or* $\{(5,6),\,(1,3),\,(4,7),\,(2,8),\,(2,9)\}$ *is a* 2*-pseudo starter sequence-*S_5 *with one defect, and the pocket is located in the second term of the sequence, as shown in Figure* 4.2*. Similarly,* $(7,\,3,\,2,\,5,\,\overset{3}{2},\,1,\,1,\,7,\,5)$ *or* $\{(6,7),\,(3,5),(2,5),(4,9),$ $(1,8)\}$ *is a* 5*-pseudo starter-*S_5 *with one defect and is shown in Figure* 4.3*.*

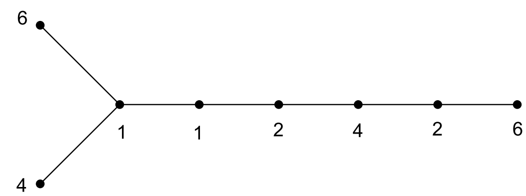

Figure 4.1: A starter-labelled graph corresponding to $(\overset{6}{4},\,1,\,1,\,2,\,4,\,2,\,6)$.

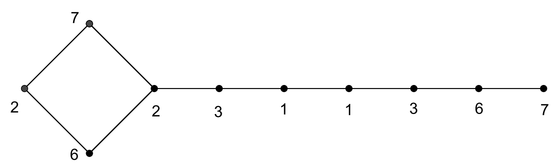

Figure 4.2: A starter-labelled graph corresponding to $(2,\,\overset{7}{6}\,2,\,3,\,1,\,1,\,3,\,6,\,7)$.

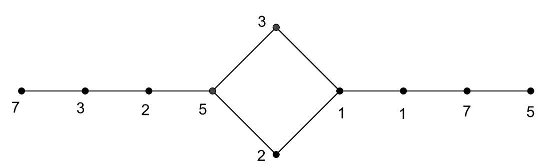

Figure 4.3: A starter-labelled graph corresponding to $(7,\,3,\,2,\,5,\,\overset{3}{2},\,1,\,1,\,7,\,5)$.

According to Definition (4.0.5), a k-pseudo-starter-S_n has exactly one pocket, which is in position k. Similarly, we can also define pseudo-starter sequences with more than one pocket.

Definition 4.0.6. *Suppose that* $k_1,\,k_2,\,\ldots,k_m$ *and* n *are positive integers, such that* $n \geq 2$ *and* $1 \leq k_\jmath \leq 2n-m$ *, for each* $1 \leq \jmath \leq m$*. A* $\{k_1,\,k_2,\,\ldots,k_m\}$*-pseudo starter* *sequence of order* n, *denoted by* $\{k_1, k_2, \ldots, k_m\}$*-pseudo starter-*$S_n$, *is a sequence* $(s_1, s_2, \ldots, \overset{s_{\hat{k}_1}}{s_{k_1}}, s_{k_1+1}, \ldots, \overset{s_{\hat{k}_2}}{s_{k_2}}, s_{k_2+1}, \ldots, \overset{s_{\hat{k}_m}}{s_{k_m}}, s_{k_m+1}, \ldots, s_{2n-m})$ *of positive integers such that, for each* $i \in \{1(1^{-1}), 2(2^{-1}), \ldots, n(n^{-1})\}$, *either* i *or* i^{-1} *occurs in the sequence twice; they are separated by either* $i-1$ *or* $i^{-1}-1$ *elements, respectively.*

Example 4.0.4. *The sequence* $(\overset{9}{4}, 10, 3, 5, 4, 3, 6, 2, 5, \overset{9}{2}, 1, \overset{10}{1}, 6)$ *is a* $\{1, 10, 12\}$*-pseudo starter-*S_8 *with two defects; the pockets are located at the first, tenth, and twelfth positions with* 1, 10, *and* 12 *being the pockets of the sequence as shown in Figure* 4.4.

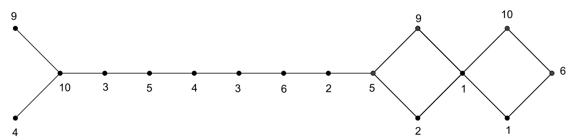

Figure 4.4: A starter-labelled graph corresponding to $(\overset{9}{4}, 10, 3, 5, 4, 3, 6, 2, 5, \overset{9}{2}, 1, \overset{10}{1}, 6)$.

In this chapter, we are interested in pseudo-starter sequences with two elements in each pocket. However, the definitions above can be easily generalized to pseudo-starter sequences with more than two elements in each pocket. We can also define sequences such that some of the positions are filled by null elements. By using known Skolem-type sequences, we can obtain pseudo-starter sequences, and consequently starter label classes of hexagonal chains or rail-siding graphs. For example, using the 6-near Skolem sequence of order seven S_7 $(7, 2, 3, 2, 4, 3, 5, 7, 4, 1, 1, 5)$, we can build pseudo-starter sequences with one defect:$(9, 7, 2, 3, 2, 4, 3, 5, 7, \overset{9}{4}, 1, 1, 5)$, $(\overset{9}{7}, 2, 3, 2, 4, 3, 5, 7, 4, \overset{9}{1}, 1, 5)$,$(7, \overset{9}{2}, 3, 2, 4, 3, 5, 7, 4, 1, \overset{9}{1}, 5)$, $(7, 2, \overset{9}{3}, 2, 4, 3, 5, 7, 4, 1, 1, \overset{9}{5})$, and $(7, 2, 3, \overset{9}{2}, 4, 3, 5, 7, 4, 1, 1, 5, 9)$. Hence, we obtain starter-labellings for the graphs. Similarly, we can obtain infinite families of pseudo-starter sequences by using known Skolem-type sequences. For example, using the Skolem sequence S_5 $(2, 4, 2, 3, 5, 4, 3, 1, 1, 5)$, and assigning labels 6 and 8 to suitable positions of S_5, we can build $(8, 2, \overset{6}{4}, 2, 3, 5, 4, 3, \overset{8}{\overset{6}{1}}$

$, 1, 5)$, $(8, \overset{6}{2}, 4, 2, 3, 5, 4, \overset{6}{3}, \overset{8}{1}, 1, 5)$, and $(\overset{8}{2}, 4, 2, \overset{6}{3}, 5, 4, 3, 1, \overset{8}{1}, \overset{6}{5})$, as shown in Figures 4.5, 4.6, and 4.7, respectively.

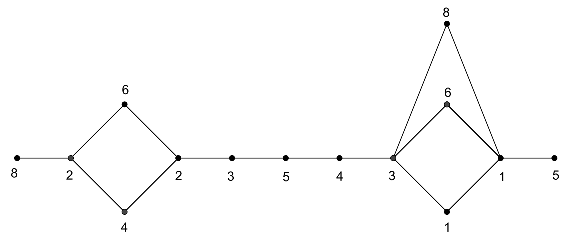

Figure 4.5: A starter-labelled graph corresponding to $(8, 2, \overset{6}{4}, 2, 3, 5, 4, 3, \overset{8}{\overset{6}{1}}, 1, 5)$.

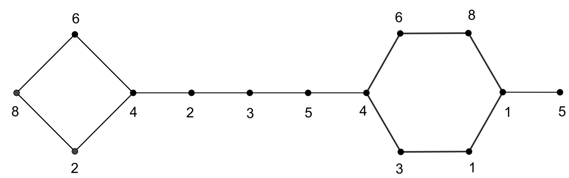

Figure 4.6: A starter-labelled graph corresponding to $(8, \overset{6}{2}, 4, 2, 3, 5, 4, \overset{6}{3}, \overset{8}{1}, 1, 5)$.

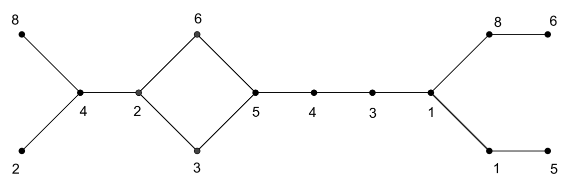

Figure 4.7: A starter-labelled graph corresponding to $(\overset{8}{2}, 4, 2, \overset{6}{3}, 5, 4, 3, 1, \overset{8}{1}, \overset{6}{5})$.

4.0.8 Pseudo-Starter Sequences with One Pocket

In this section, we obtain the necessary and sufficient conditions for the existence of a k-pseudo starter sequence-S_n.

Theorem 4.0.1. *Let $\{k, n\} \subset \mathbb{N}$ such that $n \geq 2$ and $1 \leq k \leq 2n-1$. If a k-pseudo starter sequence-S_n of order n with one defect exists, then either k is odd and $n \equiv 0, 1 \pmod 4$, or k is even and $n \equiv 2, 3 \pmod 4$.*

Proof. Suppose that $\{k, n\} \subset \mathbb{N}$ and there exists a k-pseudo starter-S_n with one defect. We will first find the necessary conditions for such sequences:

$$\begin{aligned}\sum_{i=1}^{n}(a_i + b_i) &= \sum_{i=1}^{2n-1} i + k \\ &= n(2n-1) + k, \qquad (4.0.1)\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{n}(b_i - a_i) &= \sum_{i=1}^{n} i - r + (2n+1-r), \qquad 1 \le r \le n \\ &= \frac{n(n+5)}{2} - 2r + 1. \qquad (4.0.2)\end{aligned}$$

By adding (4.0.1) to (4.0.2), we obtain $\sum_{i=1}^{n} b_i = \frac{n(5n+3)}{4} + \frac{k}{2} + \frac{1}{2} - r$. Since $\sum_{i=1}^{n} b_i \in \mathbb{N}$, we can conclude that either k is odd and $n \equiv 0, 1 \pmod 4$, or k is even and $n \equiv 2, 3 \pmod 4$. ■

Now we can prove that the necessary conditions for having a k-pseudo-starter sequence S_n are also sufficient by building such sequences for any suitable pair of k and n. In order to explain the idea of the construction, we can give constructions for $k \in \{1, 2\}$ before presenting the theorem and its proof.

Case 1: When $k = 1$, we know that if $n \equiv 0, 1 \pmod 4$, then a 3-near Skolem sequence of order n exists, which implies the existence of 1-pseudo starter sequence of order n with one defect. We obtain such a sequence by putting $2n - 2$ in both the first and the $2n - 1$ positions; thus, we obtain a 1-pseudo starter sequence with one pocket and one defect. For example, the sequence $(4, 5, 1, 1, 4, 2, 5, 2)$ is a 3-near Skolem sequence of order 5. By putting label 8 in the first and ninth positions, we obtain a 1-pseudo starter sequence of order 5 with one defect and one pocket: $(\overset{8}{4}, 5, 1,$

$1, 4, 2, 5, 2, 8)$. Similarly, the sequence $(4, 1, 1, 2, 4, 2)$ is a 3-near Skolem sequence of order 4. By putting 6 in the first and the seventh positions, we obtain a 1-pseudo starter sequence of order 4 with one defect: $(\overset{6}{4}, 1, 1, 2, 4, 2, 6)$. We notice that we can find a 1-pseudo starter sequence of order n with one defect by using a $(2n-3)$-extended near-Skolem sequence or a hooked near-Skolem sequence. If we use a $(2n-3)$-extended near-Skolem sequence, then we can acquire a 1-pseudo starter sequence of order n with one defect by filling the hook with $(2n-4)$ and by placing the pocket at the first position. For example, given $(7, 3, 6, 2, 3, 2, 8, 7, 6, 4, 1, 1, 0, 4, 8)$, which is a 13-extended 5-near-Skolem sequence of order 8, we can obtain $(\overset{12}{7}, 3, 6, 2, 3, 2, 8, 7, 6, 4, 1, 1, 12, 4, 8)$, which is a 1-pseudo starter sequence of order 8 with one defect.

Similarly, we can obtain a 1-pseudo starter sequence of order n from a hooked near-Skolem sequence by filling the hook with $(2n-3)$ and placing the pocket at the first position. For example, $(7, 5, 3, 9, 6, 3, 5, 7, 8, 2, 6, 2, 9, 1, 1, 0, 8)$ is a hooked 4-near Skolem sequence of order 9. By putting label 15 into the hook and by having the pocket at the first position, we can obtain a 1-pseudo starter sequence of order 9 with one defect: $(\overset{15}{7}, 5, 3\ , 9, 6, 3, 5, 7\ , 8, 2, 6, 2, 9, 1, 1, 15, 8)$.

Case 2: When $k = 2$, if a 2-pseudo starter-S_n with one defect exists, then $n \equiv 2, 3 \pmod 4$ and $n > 3$. If we have a $4-$near Skolem sequence of order n, then we will have a $2-$pseudo starter sequence of order n with one defect by assigning the label $(2n-3)$ to the second and $(2n-1)$ positions. For example, given the 4-near Skolem S_{10} $(3, 9, 2, 3, 2, 7, 5, 10, 8, 6, 9, 5, 7, 1, 1, 6, 8, 10)$, we can obtain $(3, \overset{17}{9}$ $, 2, 3, 2, 7, 5, 10, 8, 6, 9, 5, 7, 1, 1, 6, 8, 10, 17)$, which is a 2-pseudo starter sequence of order 10 with one defect.

Theorem 4.0.2. *A k-pseudo starter sequence-S_n of order n with one defect exists when k is odd and $n \equiv 0, 1 \pmod 4$, or when k is even and $n \equiv 2, 3 \pmod 4$.*

Proof. The cases $k = 1, 2$ have already been established. By symmetry, the cases $k = 2n-1, 2n-2$ are also established. We discuss the rest of the cases in two parts: $3 \leq k < (n-1)$ and $k = n-1, n$. For $3 \leq k < (n-1)$, we prove that there exists a k-pseudo starter sequence. Assuming that k is odd and $n \equiv 0, 1 \pmod 4$, we notice that $k+2$ is odd; hence, a $(k+2)$-near Skolem sequence of order n exists when $n \equiv 0, 1 \pmod 4$ by Theorem (1.1.8). Therefore, we can obtain a pseudo-starter sequence of order n with one defect from the near-Skolem sequence by assigning the label $(2n-k-1)$ to the positions k and $(2n-1)$. Assuming that k is even, and $n \equiv 2, 3 \pmod 4$, then a $(k+2)$-near Skolem sequence of order n exists by Theorem (1.1.8). Hence, we can obtain a pseudo-starter sequence of order n with one defect from the near-Skolem sequence by assigning the label $(2n+1-(k+2))$ to the positions k and $(2n-1)$.

If $n \equiv 0 \pmod 4$, $n \geq 12$ and $k = n-1$, then we can build a k-pseudo-starter sequence-S_n with one defect as follows:

$$\frac{n-2}{1}, \frac{n-4}{2}, \overset{-2}{\rightarrow}\cdots, \frac{4}{\frac{n}{2}-2}, \frac{2}{\frac{n}{2}-1}, \frac{n+1}{\frac{n}{2}}, \frac{2}{\frac{n}{2}+1}, \frac{4}{\frac{n}{2}+2}, \overset{+2}{\rightarrow}\cdots, \frac{n-4}{n-2}, \overset{\frac{n}{2}+1}{\frac{n-2}{n-1}}, \frac{n-1}{n}, \frac{n-3}{n+1}, \overset{-2}{\rightarrow}\cdots, \frac{\frac{n}{2}+3}{\frac{5n}{4}-2}, \frac{1}{\frac{5n}{4}-1}, \frac{1}{\frac{5n}{4}}, \frac{\frac{n}{2}-1}{\frac{5n}{4}+1}, \overset{-2}{\rightarrow}\cdots, \frac{3}{\frac{3n}{2}-1}, \frac{\frac{n}{2}+1}{\frac{3n}{2}}, \frac{n+1}{\frac{3n}{2}+1}, \frac{3}{\frac{3n}{2}+2}, \overset{+2}{\rightarrow}\cdots, \frac{\frac{n}{2}-1}{\frac{7n}{4}}, \frac{\frac{n}{2}+3}{\frac{7n}{4}+1}, \overset{+2}{\rightarrow}\cdots, \frac{n-3}{2n-2}, \frac{n-1}{2n-1}.$$

For $n = 4: \ (2, 5, \overset{3}{2}, 1, 1, 3, 5)$.

For $n = 8: \ (6, 4, 2, 9, 2, 4, \overset{6}{5}, 7, 1, 1, 3, 5, 9, 3, 7)$.

If $n \equiv 1 \pmod 4$ and $k = n$, then we can build a k-pseudo-starter sequence-S_n with one defect $(d = n+1)$ as follows:

$$\frac{n-1}{1}, \frac{n-3}{2}, \overset{-2}{\rightarrow}\cdots, \frac{4}{(\frac{n-3}{2})}, \frac{2}{(\frac{n-1}{2})}, \frac{n+1}{(\frac{n+1}{2})}, \frac{2}{(\frac{n+3}{2})}, \frac{4}{(\frac{n+5}{2})}, \overset{+2}{\rightarrow}\cdots, \frac{n-3}{n-1}, \overset{(\frac{n+1}{2})}{\frac{n-1}{n}}, \frac{n-2}{n+1}, \frac{n-4}{n+2}, \overset{-2}{\rightarrow}\cdots, \frac{(\frac{n+5}{2})}{(\frac{5(n-1)}{4})}, \frac{1}{(\frac{5n-1}{4})}, \frac{1}{(\frac{5n+3}{4})}, \frac{(\frac{n-3}{2})}{(\frac{5n+7}{4})}, \overset{-2}{\rightarrow}\cdots, \frac{5}{\frac{3(n-1)}{2}}, \frac{3}{(\frac{3n-1}{2})}, \frac{(\frac{n+1}{2})}{(\frac{3n+1}{2})}, \frac{n+1}{\frac{3(n+1)}{2}}, \frac{3}{(\frac{3n+5}{2})}, \frac{5}{(\frac{3n+7}{2})}, \overset{+2}{\rightarrow}\cdots, \frac{(\frac{n-3}{2})}{(\frac{7n+1}{4})}, \frac{(\frac{n+5}{2})}{(\frac{7n+5}{4})}, \overset{+2}{\rightarrow}\cdots, \frac{n-4}{2n-2}, \frac{n-2}{2n-1}.$$

If $n \equiv 2 \pmod 4$, and $k = n$, then we can construct a k-pseudo-starter sequence-S_n with one defect $(d = n+1)$ by the flowing construction:

$$\frac{n-4}{1}, \frac{n-6}{2}, \overset{-2}{\rightarrow}\cdots, \frac{2}{\frac{n}{2}-2}, \frac{n-2}{\frac{n}{2}-1}, \frac{2}{\frac{n}{2}}, \overset{+2}{\rightarrow}\cdots, \frac{n-6}{n-4}, \frac{n-4}{n-3}, \frac{n+1}{n-2}, \frac{n-1}{n-1}, \overset{(\frac{n}{2}-2)}{\frac{n-3}{n}}, \frac{n-5}{n+1}, \overset{-2}{\rightarrow}\cdots, \frac{\frac{n}{2}}{(\frac{5n-6}{4})}, \frac{\frac{n}{2}-4}{(\frac{5n-2}{4})}, \overset{-2}{\rightarrow}\cdots, \frac{3}{\frac{3n}{2}-4},$$

$\frac{n-2}{\frac{3n}{2}-3}, \frac{\frac{n}{2}-2}{\frac{3n}{2}-2}, \frac{3}{\frac{3n}{2}-1}, \overset{+2}{\rightarrow}, \ldots, \frac{1}{(\frac{7n-14}{4})}, \frac{1}{(\frac{7n-10}{4})}, \frac{\frac{n}{2}}{(\frac{7n-6}{4})}, \overset{+2}{\rightarrow}, \ldots, \frac{n-1}{2n-2}, \frac{n+1}{2n-1}.$ ■

Theorem 4.0.3. *Let $\{k, n\} \subset \mathbb{N}$ such that $n > 4$ and $1 \le k \le 2n-1$. A k-pseudo starter sequence-S_n of order n with two defects exists if and only if either k is odd and $n \equiv 2, 3 \pmod 4$, or k is even and $n \equiv 0, 1 \pmod 4$.*

Proof. Suppose that $\{k, n\} \subset \mathbb{N}$ and there exists a k-pseudo starter-S_n with two defects. We will first find the necessary conditions for such sequences as well.

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{2n-1} i + k$$
$$= n(2n-1) + k, \tag{4.0.3}$$

$$\sum_{i=1}^{n}(b_i - a_i) = \sum_{i=1}^{n} i - r_1 - r_2 + (2n+1-r_1) + (2n+1-r_2)$$
$$= \frac{n(n+1)}{2} - 2r_1 - 2r_2 + 4n + 2. \tag{4.0.4}$$

Thus, $\sum_{i=1}^{n} b_i = \frac{n(5n+7)}{4} + \frac{k}{2} - r_1 - r_2 + 1$, where $r_1, r_2 \in \mathbb{N}$, $1 \le r_1 < r_2 \le n$. Since $\sum_{i=1}^{n} b_i \in \mathbb{N}$, we can conclude that either k is odd and $n \equiv 2, 3 \pmod 4$, or k is even and $n \equiv 0, 1 \pmod 4$. This completes the proof of the necessity. Now if $n \equiv 0 \pmod 4$, then we can construct a k-pseudo-starter sequence S_n with two defects for $n = 4m$, where $m \ge 4$:

$\frac{n+2}{1}, \frac{n}{2}, \frac{n-2}{3}, \overset{-2}{\rightarrow}, \ldots, \frac{2}{\frac{n}{2}+1}, \frac{n+4}{\frac{n}{2}+2}, \frac{2}{\frac{n}{2}+3}, \overset{+2}{\rightarrow}, \ldots, \frac{n-2}{n+1}, \frac{n}{n+2}, \frac{n+2}{n+3}, \frac{n-5}{n+4}, \frac{n-7}{n+5}, \overset{-2}{\rightarrow}, \ldots, \frac{13}{\frac{3n}{2}-5}, \frac{\overset{9}{11}}{\frac{3n}{2}-4}, \overset{-2}{\rightarrow}, \ldots, \frac{3}{\frac{3n}{2}-1}, \frac{1}{\frac{3n}{2}},$
$\frac{1}{\frac{3n}{2}+1}, \frac{3}{\frac{3n}{2}+2}, \overset{+2}{\rightarrow}, \ldots, \frac{9}{\frac{3n}{2}+5}, \frac{n+4}{\frac{3n}{2}+6}, \frac{11}{\frac{3n}{2}+7}, \overset{+2}{\rightarrow}, \ldots, \frac{n-5}{2n-1}.$

For $n = 8:$ $(12, 10, 8, 6, 4, 2, 3, 2, 4, \overset{3}{6}, 8, 10, 12, 1, 1)$.

For $n = 12:$ $(16, 14, 12, 10, 8, 6, 4, 2, 7, 2, 4, 6, 8, 10, 12, \overset{14}{7}, 16, 5, 3, 1, 1, 3, 5)$.

Now if $n = 4m + 1$ where $m \ge 3$, then we can construct a k-pseudo-starter sequence S_n with two defects:

$\frac{n+3}{1}, \frac{n+1}{2}, \frac{n-1}{3}, \overset{\xrightarrow{-2}}{\ldots}, \frac{2}{(\frac{n+3}{2})}, \frac{n-4}{(\frac{n+5}{2})}, \frac{2}{(\frac{n+7}{2})}, \overset{\xrightarrow{+2}}{\ldots}, \frac{n-1}{n+2}, \frac{n+1}{n+3}, \frac{n+3}{n+4}, \frac{n-6}{n+5}, \frac{n-8}{n+6}, \overset{\xrightarrow{-2}}{\ldots}, \frac{9}{(\frac{3n-5}{2})}, \overset{7}{\frac{n-4}{(\frac{3n-3}{2})}}, \frac{5}{(\frac{3n-1}{2})},$
$\overset{\xrightarrow{-2}}{\ldots}, \frac{1}{(\frac{3n+3}{2})}, \frac{1}{(\frac{3n+5}{2})}, \overset{\xrightarrow{+2}}{\ldots}, \frac{n-8}{2n-2}, \frac{n-6}{2n-1}.$

For $n = 5: \ (8, 6, 4, 2, 1, \overset{1}{2}, 4, 6, 8)$.

For $n = 9: \ (12, 10, 8, 6, 4, 2, 5, 2, 4, 6, 8, \overset{5}{10}, 12, 3, 1, 1, 3)$. ∎

Theorem 4.0.4. *Let $\{k, n, m\} \subset \mathbb{N}$, such that $n \geq 2$, $1 \leq k \leq 2n-1$ and $1 \leq m < n$. If a k-pseudo starter sequence-S_n of order n with m defects exists, then either $n \equiv 0, 1 \pmod 4$ and k and m have the same parity, or $n \equiv 2, 3 \pmod 4$ and n and m have opposite parity.*

Proof. Suppose that $\{k, n\} \subset \mathbb{N}$ and there exists a k-pseudo starter-S_n with m defects. We will first find the necessary conditions for such sequences:

$$\begin{aligned} \sum_{i=1}^{n}(a_i + b_i) &= \sum_{i=1}^{2n-1} i + k \\ &= n(2n-1) + k, \end{aligned} \tag{4.0.5}$$

$$\begin{aligned} \sum_{i=1}^{n}(b_i - a_i) &= \sum_{i=1}^{n} i - 2(r_1 + r_2 + \ldots + r_m) + 2nm + m \\ &= \frac{n(n+1+4m)}{2} - 2(r_1 + r_2 + \ldots + r_m) + m. \end{aligned} \tag{4.0.6}$$

So, $\sum_{i=1}^{n} b_i = \frac{n(5n+4m-1)}{4} - (r_1 + r_2 + \ldots + r_m) + \frac{k}{2} + \frac{m}{2}$, and $1 \leq r_1 < r_2 < \ldots < r_m \leq n$. Since $\sum_{i=1}^{n} b_i \in \mathbb{N}$, we can conclude that either $n \equiv 0, 1 \pmod 4$ and k and m have the same parity, or $n \equiv 2, 3 \pmod 4$ and k and m have opposite parity. ∎

4.0.9 Pseudo-Starter Sequences with Two Pockets

Theorem 4.0.5. *Let $\{k_1, k_2, n\} \subset \mathbb{N}$ such that $n \geq 2$ and $1 \leq k_1 < k_2 \leq 2n-2$. A $\{k_1, k_2\}$-pseudo starter-S_n of order n with one defect exists only if one of the following* *two conditions holds:*

1. $n \equiv 0, 1 \pmod 4$ *and* k_1 *and* k_2 *have the same parity;*

2. $n \equiv 2, 3 \pmod 4$ *and* k_1 *and* k_2 *have opposite parity.*

Proof. If there exists a $\{k_1, k_2\}$-pseudo starter sequence-S_n with one defect, then:

$$\begin{aligned}\sum_{i=1}^{n}(a_i + b_i) &= \sum_{i=1}^{2n-2} i + k_1 + k_2 \\ &= (n-1)(2n-1) + k_1 + k_2, \end{aligned} \tag{4.0.7}$$

$$\begin{aligned}\sum_{i=1}^{n}(b_i - a_i) &= \sum_{i=1}^{n} i - r + (2n+1-r) \\ &= \frac{n(n+5)}{2} - 2r + 1. \end{aligned} \tag{4.0.8}$$

Thus, $\sum_{i=1}^{n} b_i = \frac{n(5n-1)}{4} + \frac{k_1}{2} + \frac{k_2}{2} - r + 1$. Since $\sum_{i=1}^{n} b_i \in \mathbb{N}$, we can conclude that, if a pseudo-starter sequence with two pockets and one defect exists, then $n \equiv 0, 1 \pmod 4$ and k_1 and k_2 have the same parity, or $n \equiv 2, 3 \pmod 4$ and k_1 and k_2 have different parities. This completes the proof of the necessity. We notice that these necessity conditions are sufficient when $2n - 3 > k_2 - k_1 > n$.

Now if $n \equiv 2, 3 \pmod 4$ and k_1 and k_2 have opposite parity, then $k_2 - k_1$ is odd. Assuming that $m = k_2 - k_1$, then m^{-1} is even. As a result of this, we know that an m^{-1}-near-Skolem S_n exists. Hence, we can construct a $\{k_1, k_2\}$-pseudo starter S_n with one defect by putting the label m in positions k_1 and k_2 of the sequence. Similarly, if $n \equiv 0, 1 \pmod 4$ and k_1 and k_2 have the same parity, then m^{-1} is odd. Therefore, an m^{-1}-near-Skolem S_n exists according to Theorem (1.1.8). Thus, we can construct a $\{k_1, k_2\}$-pseudo starter S_n with one defect by assigning the label m to positions k_1 and k_2 of the sequence. ∎

Theorem 4.0.6. *Let* $\{k_1, k_2, n, m\} \subset \mathbb{N}$ *such that* $n \geq 2,\ 1 \leq k_1 < k_2 \leq 2n-2$ *and* $1 \leq m < n$. *A* $\{k_1, k_2\}$*-pseudo starter-*S_n *of order* n *with* m *defects exists only if one of the following two conditions hold:*

1. $n \equiv 0, 1 \pmod 4$ *and one of* $\{m,\ k_1,\ k_2\}$ *is odd and the remaining of them have the same parity;*

2. $n \equiv 2, 3 \pmod 4$ *and one of* $\{m,\ k_1,\ k_2\}$ *is even and the remaining of them have the same parity.*

Proof. Assume that a $\{k_1, k_2\}$-pseudo starter-S_n of order n with m defects exists. Hence,

$$\sum_{i=1}^{n} b_i = \frac{n(5n+4m-5)+2}{4} - (r_1 + r_2 + \ldots + r_m) + \frac{k_1}{2} + \frac{k_2}{2} + \frac{m}{2}.$$

Therefore, we conclude that if $n \equiv 0, 1 \pmod 4$, then one of $\{m,\ k_1,\ k_2\}$ is odd and the remaining of them have the same parity. If $n \equiv 2, 3 \pmod 4$, then one of $\{m,\ k_1,\ k_2\}$ is even and the remaining of them have the same parity. ■

4.0.10 Pseudo-Starter Sequences with Three Pockets

Theorem 4.0.7. *Let* $\{k_1, k_2, k_3, n\} \subset \mathbb{N}$ *such that* $n \geq 3$ *and* $1 \leq k_1 < k_2 < k_3 \leq 2n-3$. *A* $\{k_1, k_2, k_3\}$*-pseudo starter sequence-*S_n *of order* n *with one defect exists only if one of the following two conditions holds:*

1. $n \equiv 0, 1 \pmod 4$ *and either only one of* $\{k_1, k_2, k_3\}$ *is even or all three of them are even;*

2. $n \equiv 2, 3 \pmod 4$ *and either only one of* $\{k_1, k_2, k_3\}$ *is odd or all three of them are odd.*

Proof. Assuming that a $\{k_1, k_2, k_3\}$-pseudo starter sequence-S_n of order n with one defect exists, then

$$\begin{aligned}\sum_{i=1}^{n}(a_i + b_i) &= \sum_{i=1}^{2n-3} i + k_1 + k_2 + k_3 \\ &= (n-1)(2n-3) + k_1 + k_2 + k_3, \qquad (4.0.9)\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{n}(b_i - a_i) &= \sum_{i=1}^{n} i - r + (2n+1-r) \\ &= \frac{n(n+1)}{2} + 2n - 2r + 1. \qquad (4.0.10)\end{aligned}$$

Thus, $\sum_{i=1}^{n} b_i = \frac{5n(n-1)}{4} + \frac{k_1}{2} + \frac{k_2}{2} + \frac{k_3}{2} + 2 - r$. Since $\sum_{i=1}^{n} b_i \in \mathbb{N}$, we conclude that $n \equiv 0, 1 \pmod 4$, and either only one or each k_i is even for $i \in \{1, 2, 3\}$, or $n \equiv 2, 3 \pmod 4$ and either only one or each k_i is odd for $i \in \{1, 2, 3\}$. This completes the proof of the necessity. ■

Now we will present some constructions for prescribed pseudo-starter sequences with three pockets and one defect.

(i) Pockets in positions i, $n-2$, and $n+1+i$, where $1 \leq i \leq n-4$:

We are looking to find a $\{i, n-2, n+1+i\}$-pseudo-starter-S_n with one defect. By theorem (4.0.7), $\{i, n-2, n+1+i\}$-pseudo-starter-S_n with one defect exists only if $n \equiv 2, 3 \pmod 4$. Theses conditions are sufficient. If we have a $\{n-2\}$-pseudo-Skolem-S_{n-1}, then we can construct a $\{i,\ n-2,\ n+1+i\}$-pseudo-starter sequence S_n with one defect. By assigning the label $(n+1)$ to positions i and $n+1+i$, by theorem (1.1.10) we know that a $\{n-2\}$-pseudo Skolem sequence-S_{n-1} exists when $n \equiv 2, 3 \pmod 4$.

Example 4.0.5. *Consider* $(5, 3, 1, \overset{4}{1}, 3, 5, 2, 4, 2)$ *as a* 4*-pseudo Skolem sequence* *of order five. We can obtain* $(\overset{7}{5}, 3, 1, \overset{4}{1}, 3, 5, 2\ \overset{7}{4}, 2)$ *and* $(5, \overset{7}{3}, 1, \overset{4}{1}, 3, 5, 2, 4, \overset{7}{2})$ *as a* $\{1, 4, 8\}$*-and a* $\{2, 4, 9\}$*-pseudo starter sequences of order six with one defect.*

(ii) Pockets are in the first, middle, and last positions:

We are looking to construct a $\{1, n-1, 2n-3\}$-pseudo starter-S_n with one defect. By theorem (4.0.7), such a sequence exists only if $n \equiv 1, \text{or}\, 2 \pmod 4$ (which implies that only one of k_i is even or each k_i is odd).

For case $n \equiv 1 \pmod 4$, let $n = 1 + 4s$. For $s \geq 3$, the solution is given by the following construction:

$$\overset{n+1}{\frac{n-1}{1}}, \frac{n-3}{2}, \frac{n-5}{3}, \overset{-2}{\overset{\to}{\dots}}, \frac{4}{(\frac{n-3}{2})}, \frac{2}{(\frac{n-1}{2})}, \frac{n-2}{(\frac{n+1}{2})}, \frac{2}{(\frac{n+3}{2})}, \frac{4}{(\frac{n+5}{2})}, \overset{+2}{\overset{\to}{\dots}}, \frac{n-5}{n-2}, \overset{(\frac{n+1}{2})}{\frac{n-3}{n-1}}, \frac{n-1}{n}, \frac{n-4}{n+1}, \frac{n+1}{n+2}, \frac{n-6}{n+3}, \frac{n-8}{n+4},$$
$$\overset{-2}{\overset{\to}{\dots}}, \frac{(\frac{n+5}{2})}{(\frac{5n-5}{4})}, \frac{(\frac{n-3}{2})}{(\frac{5n-1}{4})}, \overset{-2}{\overset{\to}{\dots}}, \frac{3}{(\frac{3n-5}{2})}, \frac{n-2}{(\frac{3n-3}{2})}, \frac{(\frac{n+1}{2})}{(\frac{3n-1}{2})}, \frac{3}{(\frac{3n+1}{2})}, \overset{+2}{\overset{\to}{\dots}}, \frac{(\frac{n-3}{2})}{(\frac{7n-7}{4})}, \frac{1}{(\frac{7n-3}{4})}, \frac{1}{(\frac{7n+1}{4})}, \frac{(\frac{n+5}{2})}{(\frac{7n+5}{4})}, \overset{-2}{\overset{\to}{\dots}}, \frac{n-8}{2n-4},$$
$$\overset{n-4}{\frac{n-6}{2n-3}}.$$

To complete the proof, we list the sequences for $n = 5$ and 9 :

$n = 5 : (\overset{2}{6}, 4, 2, \overset{1}{3}, 1, 4, \overset{3}{6})$.

$n = 9 : (\overset{8}{10}, 6, 4, 2, 5, 2, 4, \overset{7}{6}, 8, 5, 10, 3, 1, 1, \overset{3}{7})$.

For $n \equiv 2 \pmod 4$, the general construction for $\{1, n-1, 2n-3\}$-S_n with one defect can be written as follows:

$$\overset{n-1}{\frac{n-3}{1}}, \frac{n-5}{2}, \frac{n-7}{3}, \overset{-2}{\overset{\to}{\dots}}, \frac{(\frac{n}{2}+2)}{(\frac{n-6}{4})}, \frac{(\frac{n}{2}-2)}{(\frac{n-2}{4})}, \overset{-2}{\overset{\to}{\dots}}, \frac{3}{(\frac{n}{2}-3)}, \frac{n+1}{(\frac{n}{2}-2)}, \frac{\frac{n}{2}}{(\frac{n}{2}-1)}, \frac{3}{(\frac{n}{2})}, \overset{+2}{\overset{\to}{\dots}}, \frac{\frac{n}{2}-2}{(\frac{3n-10}{4})}, \frac{1}{(\frac{3n-6}{4})}, \frac{1}{(\frac{3n-2}{4})}, \frac{\frac{n}{2}+2}{(\frac{3n+2}{4})},$$
$$\overset{+2}{\overset{\to}{\dots}}, \frac{n-7}{n-4}, \frac{n-5}{n-3}, \frac{n-3}{n-2}, \overset{\frac{n}{2}}{\frac{n-2}{n-1}}, \frac{n-1}{n}, \frac{n-4}{n+1}, \frac{n-6}{n+2}, \frac{n-8}{n+3}, \overset{-2}{\overset{\to}{\dots}}, \frac{4}{(\frac{3n}{2}-3)}, \frac{2}{(\frac{3n}{2}-2)}, \frac{n+1}{\frac{3n}{2}-1}, \frac{2}{(\frac{3n}{2})}, \frac{4}{\frac{3n}{2}+1}, \overset{+2}{\overset{\to}{\dots}}, \frac{n-8}{2n-5}, \frac{n-6}{2n-4},$$
$$\overset{n-4}{\frac{n-2}{2n-3}}.$$

(iii) Pockets are in the second, middle, and penultimate positions: we are looking to construct $\{2, n-1, 2n-4\}$-pseudo starter-S_n with one defect. By the necessary conditions obtained above, such sequences exists only if $n \equiv 1, 2 \pmod 4$. The general construction for $n \equiv 1 \pmod 4$ for the case $\{2, n-1, 2n-4\} - S_n$ with one defect $(d = n+1)$ can be written as follows:

$$\frac{n-1}{1}, \overset{n-3}{\frac{n-5}{2}}, \frac{n-7}{3}, \frac{n-9}{4}, \overset{\overset{-2}{\to}}{\cdots}, \frac{2}{(\frac{n-3}{2})}, \frac{n+1}{(\frac{n-1}{2})}, \frac{2}{(\frac{n+1}{2})}, \frac{4}{(\frac{n+3}{2})}, \overset{\overset{+2}{\to}}{\cdots}, \frac{n-5}{n-3}, \frac{n-2}{n-2}, \overset{(\frac{n+1}{2})}{\frac{n-3}{n-1}}, \frac{n-1}{n}, \frac{n-4}{n+1}, \frac{n-6}{n+2}, \overset{\overset{-2}{\to}}{\cdots},$$
$$\frac{(\frac{n+5}{2})}{(\frac{5n-9}{4})}, \frac{1}{(\frac{5n-5}{4})}, \frac{1}{(\frac{5n-1}{4})}, \frac{(\frac{n-3}{2})}{(\frac{5n+3}{4})}, \overset{\overset{-2}{\to}}{\cdots}, \frac{3}{(\frac{3n-3}{2})}, \frac{(\frac{n+1}{2})}{(\frac{3n-1}{2})}, \frac{n+1}{(\frac{3n+1}{2})}, \frac{3}{(\frac{3n+3}{2})}, \frac{5}{(\frac{3n+5}{2})}, \overset{\overset{+2}{\to}}{\cdots}, \frac{(\frac{n-3}{2})}{(\frac{7n-3}{4})}, \frac{(\frac{n+5}{2})}{(\frac{7n+1}{4})}, \overset{\overset{+2}{\to}}{\cdots},$$
$$\frac{n-8}{2n-5}, \overset{n-6}{\frac{n-4}{2n-4}}, \frac{n-4}{2n-3}.$$

The construction for $n \equiv 1 \pmod 4$ for the case $\{2, n-1, 2n-4\}$-S_n with one defect ($d = n+2$) can be written as follows:

$$\frac{n-5}{1}, \overset{n-9}{\frac{n-7}{2}}, \frac{n-11}{3}, \overset{\overset{-2}{\to}}{\cdots}, \frac{2}{(\frac{n-7}{2})}, \frac{n-3}{(\frac{n-5}{2})}, \frac{2}{(\frac{n-3}{2})}, \overset{\overset{+2}{\to}}{\cdots}, \frac{n-9}{n-7}, \frac{n+2}{n-6}, \frac{n-7}{n-5}, \frac{n-5}{n-4}, \frac{n}{n-3}, \frac{n-2}{n-2}, \overset{n-4}{\frac{n-6}{n-1}}, \frac{n-8}{n}, \overset{\overset{-2}{\to}}{\cdots},$$
$$\frac{(\frac{n+1}{2})}{(\frac{5n-17}{4})}, \frac{(\frac{n-7}{2})}{(\frac{5n-13}{4})}, \overset{\overset{+2}{\to}}{\cdots}, \frac{3}{(\frac{3n-13}{2})}, \frac{n-3}{(\frac{3n-11}{2})}, \frac{(\frac{n-3}{2})}{(\frac{3n-9}{2})}, \frac{3}{(\frac{3n-7}{2})}, \overset{\overset{+2}{\to}}{\cdots}, \frac{(\frac{n-7}{2})}{(\frac{7n-27}{4})}, \frac{1}{(\frac{7n-23}{4})}, \frac{1}{(\frac{7n-19}{4})}, \frac{(\frac{n+1}{2})}{(\frac{7n-15}{4})},$$
$$\overset{\overset{+2}{\to}}{\cdots}, \frac{n-6}{2n-7}, \frac{(\frac{n-3}{2})}{2n-6}, \frac{n-4}{2n-5}, \overset{n+2}{\frac{n-2}{2n-4}}, \frac{n}{2n-3}.$$

The general construction for $n \equiv 2 \pmod 4$ for the case $\{2, n-1, 2n-4\} - S_n$ with one defect $(d = n+1)$ as follows:

$$\frac{n-1}{1}, \overset{n-3}{\frac{n+1}{2}}, \frac{n-5}{3}, \frac{n-7}{4}, \overset{\overset{-2}{\to}}{\cdots}, \frac{\frac{n}{2}}{(\frac{n+2}{4})}, \frac{1}{(\frac{n+6}{4})}, \frac{1}{(\frac{n+10}{4})}, \frac{(\frac{n}{2}-4)}{(\frac{n+14}{4})}, \overset{\overset{-2}{\to}}{\cdots}, \frac{3}{(\frac{n}{2})}, \frac{(\frac{n}{2}-2)}{(\frac{n}{2}-2)}, \frac{n-2}{(\frac{n}{2}+2)}, \frac{3}{(\frac{n}{2}+3)}, \overset{\overset{+2}{\to}}{\cdots}, \frac{(\frac{n}{2}-4)}{(\frac{3n-2}{4})},$$
$$\frac{\frac{n}{2}}{(\frac{3n+2}{4})}, \overset{\overset{+2}{\to}}{\cdots}, \frac{n-5}{n-2}, \overset{(\frac{n}{2}-2)}{\frac{n-3}{n-1}}, \frac{n-1}{n}, \frac{n-4}{n+1}, \frac{n-6}{n+2}, \frac{n+1}{n+3}, \frac{n-8}{n+4}, \frac{n-10}{n+5}, \overset{\overset{-2}{\to}}{\ldots}, \frac{2}{(\frac{3n}{2}-1)}, \frac{n-2}{(\frac{3n}{2})}, \frac{2}{(\frac{3n}{2}+1)}, \frac{4}{(\frac{3n}{2}+2)}, \overset{\overset{+2}{\to}}{\cdots},$$
$$\frac{n-10}{2n-5}, \overset{n-8}{\frac{n-6}{2n-4}}, \frac{n-4}{2n-3}.$$

We can also obtain another construction for $n \equiv 2 \pmod 4$ such that $n \geq 18$:

$$\frac{n-3}{1}, \overset{n-1}{\frac{n-5}{2}}, \frac{n-7}{3}, \frac{n-9}{4}, \overset{\overset{-2}{\to}}{\cdots}, \frac{(\frac{n}{2}+2)}{(\frac{n-6}{4})}, \frac{(\frac{n}{2}-2)}{(\frac{n-2}{4})}, \overset{\overset{-2}{\to}}{\cdots}, \frac{3}{(\frac{n}{2}-3)}, \frac{n+1}{(\frac{n}{2}-2)}, \frac{\frac{n}{2}}{(\frac{n}{2}-1)}, \frac{3}{\frac{n}{2}}, \overset{\overset{+2}{\to}}{\cdots}, \frac{\frac{n}{2}-2}{(\frac{3n-10}{4})}, \frac{1}{(\frac{3n-6}{4})}, \frac{1}{(\frac{3n-2}{4})},$$
$$\frac{\frac{n}{2}+2}{(\frac{3n+2}{4})}, \overset{\overset{+2}{\to}}{\cdots}, \frac{n-3}{n-2}, \overset{\frac{n}{2}}{\frac{n-2}{n-1}}, \frac{n-4}{n}, \frac{n-1}{n+1}, \frac{n-6}{n+2}, \frac{n-8}{n+3}, \frac{n-10}{n+4}, \overset{\overset{-2}{\to}}{\cdots}, \frac{4}{(\frac{3n}{2}-3)}, \frac{2}{(\frac{3n}{2}-2)}, \frac{n+1}{\frac{3n}{2}-1}, \frac{2}{\frac{3n}{2}}, \frac{4}{\frac{3n}{2}+1}, \overset{\overset{+2}{\to}}{\cdots},$$
$$\frac{n-10}{2n-6}, \frac{n-8}{2n-5}, \overset{n-6}{\frac{n-4}{2n-4}}, \frac{n-2}{2n-3}.$$

Theorem 4.0.8. *Let $\{k_1, k_2, k_3, n, m\} \subset \mathbb{N}$ such that $n \geq 3,\ 1 \leq k_1 < k_2 < k_3 \leq 2n-3$ and $1 \leq m < n$. A $\{k_1, k_2, k_3\}$-pseudo starter-S_n of order n with m defects exists only if one of the following two conditions holds:*

1. $n \equiv 0, 1 \pmod 4$ *and either only one of $\{m,\ k_1,\ k_2, k_3\}$ is even or only one of them is odd;*

2. $n \equiv 2, 3 \pmod 4$ *and either all four of $\{m,\ k_1,\ k_2,\ k_3\}$ have the same parity or precisely two of them are even.*

Proof. Assume that a $\{k_1, k_2, k_3\}$-pseudo starter-S_n of order n with m defects exists. Hence,

$$\sum_{i=1}^{n} b_i = \frac{n(5n+4m-9)+6}{4} - (r_1 + r_2 + \ldots + r_m) + \frac{k_1}{2} + \frac{k_2}{2} + \frac{k_3}{2} + \frac{m}{2}.$$

Since $\sum_{i=1}^{n} b_i$ is an integer, then $\frac{n(5n+4m-9)+6}{4} + \frac{k_1}{2} + \frac{k_2}{2} + \frac{k_3}{2} + \frac{m}{2}$ must be an integer. Therefore, we conclude that if $n \equiv 0, 1 \pmod 4$, then either one of $\{m, k_1, k_2, k_3\}$ is even or one of them is odd, and if $n \equiv 2, 3 \pmod 4$ then, either all the elements of $\{m, k_1, k_2, k_3\}$ have the same parity or only two of them are even. ■

4.1 Starter Labelling of Classes of Hexagonal Chains

In this section, we determine the necessary conditions for labelling classes of hexagonal chains by using (hooked) starter sequences with one defect. We use h to denote the number of hexagonal subgraphs of a hexagonal chain.

4.1.1 Type (I) Hexagonal Chains

A type (I) hexagonal chain H_h^1 is a graph with P_{4h+2} being its main path, the maximum degree of the graph is 3 ($\Delta(H_h^1) = 3$), $V(H_h^1) = 6h + 2$, and the end vertices are not inflated. The general form of such graphs is shown in Figure 4.8.

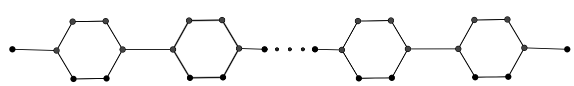

Figure 4.8: Type (I) Hexagonal Chains.

Theorem 4.1.1. *The graph H_h^1 can be starter-labelled with one defect only if $h \equiv 2, 3$ (mod 4).*

Proof. If a_i and b_i are the smallest and largest positions respectively of the label i in the sequence, then

$$\begin{aligned}\sum_{i=1}^{3h+1}(a_i+b_i) &= \sum_{i=1}^{4h+2} i + \sum_{i=0}^{h-1}(4i+3) + \sum_{i=0}^{h-1}(4i+4)\\ &= (2h+1)(4h+3) + 4\sum_{i=0}^{h-1} i + 3h + 4\sum_{i=0}^{h-1} i + 4h\\ &= (2h+1)(4h+3) + 8\sum_{i=1}^{h-1} i + 7h\\ &= 12h^2 + 13h + 3, \end{aligned} \tag{4.1.1}$$

$$\begin{aligned}\sum_{i=1}^{3h+1}(b_i-a_i) &= \sum_{i=1}^{3h+1} i - r + (6h+3-r)\\ &= \frac{(3h+1)(3h+2)}{2} - 2r + 6h + 3. \end{aligned} \tag{4.1.2}$$

By adding (4.1.1) and (4.1.2), we obtain $\sum_{i=1}^{3h+1} b_i = \frac{33h^2+47h+14}{4} - r$. Since $\sum_{i=1}^{3h+1} b_i + r \in \mathbb{N}$, we conclude that $h \equiv 2, 3 \pmod 4$. ■

Similarly, we can prove Theorems (4.1.2) and (4.1.3).

Example 4.1.1. *Figure* 4.9 *illustrates a starter-labelled with one defect of* H_2^1*.*

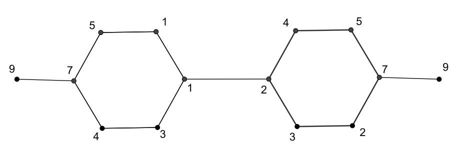

Figure 4.9: A starter-labelled H_2^1 with one defect.

4.1.2 Type (II) Hexagonal Chains

A type (II) hexagonal chain H_h^2 is a graph with P_{4h+1} being its main path, $\Delta(H_h^2) = 3$, $V(H_h^2) = 6h + 1$, and non-inflated end vertices. The general form of such graphs is shown in Figure (4.10).

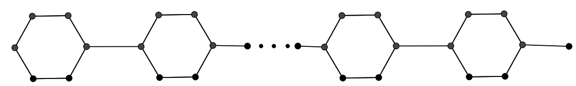

Figure 4.10: Type (II) Hexagonal Chains.

Theorem 4.1.2. *The graph H_h^2 can be starter-labelled with one defect only if $h \equiv 1, 2 \pmod 4$.*

4.1.3 Type (III) Hexagonal Chains

A type (III) hexagonal chain H_h^3 is a graph with P_{4h-1} being its main path, $\Delta = 3$, $V(H_h^3) = 6h$, and non-inflated end vertices. The general form of such graphs is shown in Figure 4.11.

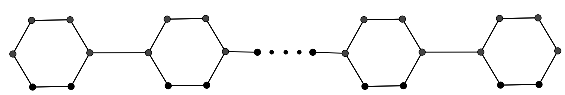

Figure 4.11: Type (III) Hexagonal Chains.

Example 4.1.2. *Figure 4.12 shows a starter-labelled with one defect of H_2^3.*

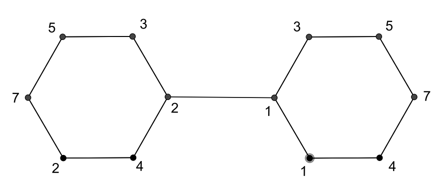

Figure 4.12: A starter labelled with one defect H_2^3.

Theorem 4.1.3. *The graph H_h^3 can be starter-labelled with one defect only if* $h \equiv 1, 2 \pmod 4$.

4.1.4 Type (IV) Hexagonal Chains

A type *IV* hexagonal chain H_h^4 is a graph with P_{3h+1} being its main rail, $\Delta = 4$, $V(H_h^4) = 5h+1$, and the end vertices are not inflated. The general form of such graphs is shown in Figure 4.13.

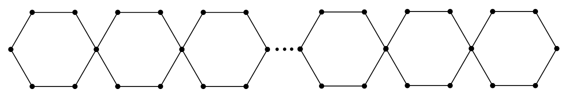

Figure 4.13: Type (IV) Hexagonal Chains.

Theorem 4.1.4. *The graph H_h^4 can be starter-labelled with one defect only if* $h \equiv 1, 3, 5 \pmod 8$.

Proof. If a_i and b_i are the smallest and largest positions respectively of the label i in the sequence, then

$$\begin{aligned}\sum_{i=1}^{\frac{5h+1}{2}}(a_i+b_i) &= \sum_{i=1}^{3h+1} i + \sum_{i=0}^{h-1}(3i+2) + \sum_{i=0}^{h-1}(3i+3)\\ &= \frac{(3h+1)(3h+2)}{2} + 6\sum_{i=1}^{h-1} i + 5h\\ &= \frac{15h^2+13h+2}{2}, \qquad (4.1.3)\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{\frac{5h+1}{2}}(b_i-a_i) &= \sum_{i=1}^{\frac{5h+1}{2}} i - r + (5h+2-r)\\ &= \frac{25h^2+60h+19}{8} - 2r. \qquad (4.1.4)\end{aligned}$$

By adding (4.1.3) and (4.1.4), we obtain $2\sum_{i=1}^{\frac{5h+1}{2}} b_i = \frac{85h^2+112h+27}{8} - 2r$. This implies that $h \equiv 1, 3, 5 \pmod 8$. ■

Chapter 5

Generalized Starter Sequences

In this chapter, we introduce generalized (extended) starter sequences and provide some of the conditions for the existence of generalized (extended) starter sequences. The results of this chapter have been accepted for the publication in Journal of Information and Optimization Sciences [32].

Definition 5.0.1. *A generalized starter sequence of order n and multiplicity λ is a sequence $(s_1,\, s_2, ..., s_{\lambda n})$ of λn integers satisfying the following conditions:*

1. *for every $i \in \{1, 2, \ldots, n\}$, either i or $-i$ appears exactly in λ positions in the sequence, $j_1,\, j_2 = j_1 + i, \ldots, j_\lambda = j_1 + (\lambda - 1)i$ or $j_1,\, j_2 = j_1 + (-i), \ldots, j_\lambda = j_1 + (\lambda - 1)(-i)$;*

2. *$s_{j_1} = s_{j_2} = \ldots = s_{j_\lambda} = i$ or $-i$, respectively. Where $-i$ is the additive inverse of i in $\mathbb{Z}_{2n+1}$, and $-i$ is referred to as a defect of the sequence.*

We shall use the notation (λ, n)-starter sequence to label a starter sequence of order n with multiplicity λ. For example, the sequence $(2, 13, 2, 5, 2, 7, 9, 11, 5, 1, 1, 1, 7, 5, 13, 9, 6, 4,11, 7,3, 4, 6, 3, 9, 4, 3, 13, 6, 11)$ is a $(3, 10)$-starter sequence with two defects.

Definition 5.0.2. *A **generalized extended starter sequence** of order n and multiplicity λ is a sequence $GESS_n = (s_1, s_2, s_3, ..., s_{\lambda n+h})$ of $\lambda n + h$ integers satisfying the conditions of generalized starter sequences, and in addition there are exactly h zeros in the sequence, where h is the minimum number of zeros such that the sequence exists.*

For example, the sequence $(3, 7, 0, 3, 6, 0, 3, 2, 7, 2, 6, 2, 1, 1, 1, 7, 6)$ is a generalized extended starter sequence of order 5 where the multiplicity is $\lambda = 3$, with two defects, and a minimum number of hooks ($h = 2$). The following proposition gives a partial result in the case $\lambda = 3$.

Proposition 5.0.1. *There are no generalized starter sequences for orders $n = 2, 3, 4, 5, 6$, or 7 with multiplicity $\lambda = 3$.*

Proof. The case $n = 2$ is straightforward.
Consider the case $n = 3$. Assume that there is a generalized starter sequence of order $n = 3$ and multiplicity $\lambda = 3$. Hence, the sequence must be of the form: $(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9)$. Neither 5 nor 6 appear in the sequence. So, only one defect ($d = 4$) must be in the desired sequence. Therefore, the entire sequence must be of the form: $(4, s_2, s_3, s_4, 4, s_6, s_7, s_8, 4)$. Clearly, the first occurrence of 2 in the sequence is at s_2 or s_4; hence, either $s_2 = s_4 = s_6 = 2$ or $s_4 = s_6 = s_8 = 2$. Since the spaces are already occupied by the $2^{'}s$ and $4^{'}s$, it is not possible to put the $1^{'}s$ in this sequence. Therefore, there is no generalized starter sequence of order $n = 3$ and multiplicity $\lambda = 3$.

In the case $n = 4$, if the desired sequence is possible, then the sequence must be of the form: $(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12})$. Therefore, the defects $6, 7$, and 8 cannot occur in the sequence. Hence, the defect ($d = 5$) must be in the sequence. Therefore, the sequence either begins or ends with a 5. Consider the case

$(5, s_2, s_3, s_4, s_5, 5, s_7, s_8, s_9, s_{10}, 5, s_{12})$; here $s_2 = s_5 = s_8 = 3$ or $s_4 = s_7 = s_{10} = 3$, but in both cases it is not possible to put the $2's$ in the sequence. A similar argument can be made when the sequence ends with a 5. Therefore, there is no generalized starter sequence of order $n = 4$ and multiplicity $\lambda = 3$.

Now, suppose that the order is $n = 5$. Clearly the defects 8, 9, and 10 cannot occur as elements in the sequence. Hence, either 6 or 7 appears in the sequence as a defect. If $(d = 7)$, then the entire sequence must be of the form: $(7, s_2, s_3, s_4, s_5, s_6, s_7, 7, s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}, 7)$. So, the first occurrence of 5 in the sequence must be at s_2 or s_4. Thus, either $s_2 = s_7 = s_{12} = 5$, or $s_4 = s_9 = s_{14} = 5$. When we consider the case $s_2 = s_7 = s_{12} = 5$, there is only one way to insert the 3's in the sequence: $s_3 = s_6 = s_9 = 3$. Since the spaces are already occupied by the $7's, 5's$,and $3's$, we cannot put the $2's$ in the sequence. A similar argument can be applied when $s_4 = s_9 = s_{14} = 5$. Therefore, there is no generalized starter sequence of order $n = 5$, $\lambda = 3$, and $d = 7$.

In the alternate case $(d = 6)$, we must have the subsequence $(6, s_2, s_3, s_4, s_5, s_6, 6, s_8,$ $s_9, s_{10},\ s_{11},\ s_{12}, 6)$ in the proposed sequence. Hence, without loss of generality, the entire sequence must be of the form: $(6, s_2, s_3, s_4, s_5, s_6, 6, s_8, s_9, s_{10}, s_{11}, s_{12}, 6, s_{14}, s_{15})$, or $(s_1, 6, s_3, s_4, s_5\ , s_6, s_7, 6, s_9,\ s_{10}, s_{11}, s_{12}\ , s_{13}, 6, s_{15})$. But it is not possible to insert all the $4's, 3's, 2's$, and $1's$ in the desired sequence. Therefore, there is no starter sequence of order $n = 5$ and multiplicity $\lambda = 3$. Similar arguments can be made to prove the remaining cases $(n = 6, \text{and}\, 7)$. Hence, there are no starter sequences of orders $n = 2, 3, 4, 5, 6, \text{ or } 7$ with multiplicity $\lambda = 3$. ■

Remark: We have shown by computer search that there are no generalized starter sequences of orders $n = 13$ or $n = 16$ with multiplicity $\lambda = 3$.

Propositions 5.0.2 and 5.0.3 provide some of the necessary conditions for the existence of $(\lambda,\ n)$-generalized starter sequences.

Proposition 5.0.2. *There exists a generalized starter sequence of order n and multiplicity λ only if $\lambda \leq n$.*

Proof. Assume that there exists a generalized starter sequence of order n and multiplicity λ such that $\lambda > n$. Hence, the sequence will be of length λn, and will have at least one defect ($d \geq n+1$). Consider the minimum value of d. Then, the integer $n+1$ appears λ times in the sequence. By the definition of generalized starter sequences, we know that between any two consecutive occurrences of the integer $n+1$ there are exactly n entries; so the total length of the subsequence containing all the integers $n+1$ will be $\lambda + n(\lambda - 1)$. This implies that the length of the subsequence is greater than λn which leads to a contradiction. This completes the proof. ■

By using the same technique as in the proof of proposition 5.0.2, we can obtain another necessary condition for the existence of a generalized starter sequence of order n, multiplicity λ, and in which the largest defect appears in the sequence d, as shown in the following proposition.

Proposition 5.0.3. *There exists a generalized starter sequence of order n and multiplicity λ only if $(\lambda-1)d+1 \leq \lambda n$, where d is the largest defect which appears in the sequence.*

5.1 More Results

In this section we extend the results of generalized Langford sequences ([39], [25]) and generalized Skolem sequences [41].

Theorem 5.1.1. *Let $\lambda = p^e t$, where p is the smallest prime factor of λ and e, t are positive integers. If a generalized starter sequence of order n, one defect d, and multiplicity λ exists, then n must satisfy one of the following congruences:*

$$n \equiv \begin{cases} -p, 1-p, 2-p, \ldots, -1 \pmod{p^{e+1}}, & \textit{if}\, d \equiv 0 \textit{ and } -d \not\equiv 0 \,(\mathrm{mod}\; p); \\ 0, 1, 2, \ldots, p-1 \pmod{p^{e+1}}, & \textit{if both}\, d,\, -d \equiv 0, \textit{ or both}\, d,\, -d \not\equiv 0 \,(\mathrm{mod}\; p); \\ p, 1+p, 2+p, \ldots, 2p-1 \pmod{p^{e+1}}, & \textit{if}\, d \not\equiv 0 \textit{ and } -d \equiv 0 \,(\mathrm{mod}\; p). \end{cases}$$

Such that $d = -r$, *where* $-r$ *is the additive inverse of* r *in the group* $\mathbb{Z}_{2n+1}$.

Proof. Let $GSS_{n,1} = (s_1, s_2, ..., s_{\lambda n})$ be a generalized starter sequence of order n, one defect (d), and multiplicity $\lambda = tp^e$. Arrange the terms of the sequence $GSS_{n,1}$ into the $ntp^{e-1} \times p$ matrix $B = (b_{ij})$ according to the definition: $b_{ij} = s_{(i-1)p+j}$ $(1 \leq i \leq ntp^{e-1};\ 1 \leq j \leq p)$:

$$B = \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,p} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \cdots & \vdots \\ b_{ntp^{e-1},1} & b_{ntp^{e-1},2} & \cdots & b_{ntp^{e-1},p} \end{pmatrix}.$$

Consider the set $E = \{\{1, 2, \ldots, n\} \setminus \{r\}\} \cup \{d\}$, where $1 \leq r \leq n$, $d = -r$, and $-r$ is the additive inverse of r in $\mathbb{Z}_{2n+1}$. Now, let $M = |\{b \in E : b \equiv 0 \pmod{p}\}|$. Hence,

$$M = \begin{cases} \lfloor \frac{n}{p} \rfloor, & \text{if both } d,\, -d \equiv 0, \text{ or both } d,\, -d \not\equiv 0 \,(\mathrm{mod}\; p); \\ \lfloor \frac{n}{p} \rfloor + 1, & \text{if}\, d \equiv 0 \text{ and } -d \not\equiv 0 \,(\mathrm{mod}\; p); \\ \lfloor \frac{n}{p} \rfloor - 1, & \text{if}\, d \not\equiv 0 \text{ and } -d \equiv 0 \,(\mathrm{mod}\; p). \end{cases}$$

Notice that for every b in the set E, two observations are made:

1. if $b \equiv 0 \pmod{p}$, then b will always appear $\lambda = tp^e$ times in a single column of the matrix B;

2. if $b \not\equiv 0 \pmod{p}$, then b appears exactly tp^{e-1} times in every column of the matrix B.

The first condition implies that, in order to have a perfect starter sequence, M must be a multiple of p. Hence, $\frac{M}{tp^{e-1}} \equiv 0 \pmod{p} \Rightarrow M = cs$ for some integer c. Now we will consider all the possible values of M.

Case (1): If $M = \lfloor \frac{n}{p} \rfloor$, then $\lfloor \frac{n}{p} \rfloor \equiv 0 \pmod{s}$. But if $n \equiv i \pmod{p}$, then $n = n_0 p + i$ $\Longrightarrow n_0 = \frac{n-i}{p}$, $0 \leq i \leq p-1$; thus $\lfloor \frac{n_0 p+i}{p} \rfloor \equiv 0 \pmod{s} \Longrightarrow n_0 \equiv 0 \pmod{s}$ $\Longrightarrow \frac{n-i}{p} = c_0 s$, for some integer $c_0 \Longrightarrow n - i = c_0 t p^{e+1} \Longrightarrow n \equiv i \pmod{p^{e+1}}$.

Therefore, in the case of generalized starter sequence of order n, multiplicity $s = tp^e$, and one defect, where the defect and its inverse are both multiples of p or neither of them is a multiple of p, in order for a such sequence to exist, n must satisfy one of the following congruency classes: $n \equiv 0, 1, 2, \ldots, p-1 \pmod{p^{e+1}}$.

Case (2): If $M = \lfloor \frac{n}{p} \rfloor + 1$, then $\lfloor \frac{n}{p} \rfloor + 1 \equiv 0 \pmod{s}$. But if $n \equiv i \pmod{p}$ $\Longrightarrow n_0 = \frac{n-i}{p}$, $0 \leq i \leq p-1$, then $\lfloor \frac{n_0 p+i}{p} \rfloor + 1 \equiv 0 \pmod{s} \Rightarrow n_0 + 1 \equiv 0 \pmod{s}$ $\Longrightarrow \frac{n-i}{p} + 1 = c_0 s$, for some integer $c_0 \Longrightarrow n - i = -p + c_0 t p^{e+1} \Longrightarrow n \equiv i - p$ $\pmod{p^{e+1}}$.

Therefore, if a generalized starter sequence of order n exists with multiplicity $s = tp^e$, and with one defect, where the defect is a multiple of p but its inverse is not a multiple of p, then n must satisfy one of the following congruency classes: $n \equiv -p, 1-p, 2-p, \ldots, -1 \pmod{p^{e+1}}$.

Case (3): If $M = \lfloor \frac{n}{p} \rfloor - 1$, then $\lfloor \frac{n}{p} \rfloor - 1 \equiv 0 \pmod{s}$. But if $n \equiv i \pmod{p} \Rightarrow n_0 = \frac{n-i}{p}$,
$0 \leq i \leq p-1$, then $\lfloor \frac{n_0 p+i}{p} \rfloor - 1 \equiv 0 \pmod{s} \Rightarrow n_0 - 1 \equiv 0 \pmod{s} \Rightarrow \frac{n-i}{p} - 1 = c_0 s$, for some integer c_0. Hence, $n - i = p + c_0 t p^{e+1} \Longrightarrow n \equiv i + p \pmod{p^{e+1}}$. Therefore, n must satisfy one of the congruences: $n \equiv p, 1+p, 2+p, \ldots, 2p-1 \pmod{p^{e+1}}$. ■

Theorem 5.1.2. *Let $\lambda = p^e t$, $A_2 = \{r_1, r_2\}$, and $D_2 = \{d_1, d_2\}$, where p is the smallest prime factor of λ, and e, t, r_1, r_2, d_1, and d_2 are positive integers, $A_2^* = \{x \in A_2 : x \equiv 0 \pmod{p}\}$, and $D_2^* = \{d \in D_2 : d \equiv 0 \pmod{p}\}$. If a generalized* *starter sequence of order n, two defects d_1 and d_2, and multiplicity λ exists, then n must satisfy one of the following congruences:*

$$n \equiv \begin{cases} -2p, 1-2p, 2-2p, \ldots, -1-p \pmod{p^{e+1}}, & if |D_2^*| = 2 \, and |A_2^*| = 0; \\ 0, 1, 2, \ldots, p-1 \pmod{p^{e+1}}, & if |D_2^*| = |A_2^*| = i, \, where \, i \in \{0,1,2\}; \\ -p, 1-p, 2-p, \ldots, -1 \pmod{p^{e+1}}, & if |D_2^*| = 1 \, and |A_2^*| = 0, \, or |D_2^*| = 2 \, and |A_2^*| = 1; \\ p, 1+p, 2+p, \ldots, 2p-1 \pmod{p^{e+1}}, & if |D_2^*| = 0 \, and |A_2^*| = 1, \, or |D_2^*| = 1 \, and |A_2^*| = 2; \\ 2p, 1+2p, 2+2p, \ldots, 3p-1 \pmod{p^{e+1}}, & if |D_2^*| = 0, \, and |A_2^*| = 2. \end{cases}$$

Where $d_1 = -r_1$, $d_2 = -r_2$, and the inverse is the additive inverse in the group $\mathbb{Z}_{2n+1}$.

Proof. Let $GSS_{n,2} = (s_1, s_2, ..., s_{\lambda n})$ be a generalized starter sequence of order (n), multiplicity $\lambda = tp^e$, and two defects: $d_1 = -r_1$, and $d_2 = -r_2$. Arrange the terms of the sequence $GSS_{n,2}$ into the $ntp^{e-1} \times p$ matrix $B = (b_{ij})$ according to the definition: $b_{ij} = s_{(i-1)p+j}$ $(1 \leq i \leq ntp^{e-1};\ 1 \leq j \leq p)$:

$$B = \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,p} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \cdots & \vdots \\ b_{ntp^{e-1},1} & b_{ntp^{e-1},2} & \cdots & b_{ntp^{e-1},p} \end{pmatrix}.$$

Consider the set $E = \{\{1, 2, \ldots, n\} \setminus \{r_1, r_2\}\} \cup \{-r_1, -r_2\}$, where $1 \leq r_1 < r_2 \leq n$ and $-r$ is the additive inverse of r in $\mathbb{Z}_{2n+1}$. Now, let $M = |\{b \in E : b \equiv 0 \pmod p\}|$, $A_2 = \{r_1, r_2\}$, $D_2 = \{d_1, d_2\}$, $A_2^* = \{x \in A_2 : x \equiv 0 \pmod p\}$, and $D_2^* = \{d \in D_2 : d \equiv 0 \pmod p\}$. Hence,

$$M = \begin{cases} \lfloor \frac{n}{p} \rfloor + 2, & \text{if } |D_2^*| = 2 \text{ and } |A_2^*| = 0; \\ \lfloor \frac{n}{p} \rfloor + 1, & \text{if } |D_2^*| = 1 \text{ and } |A_2^*| = 0, \text{ or } |D_2^*| = 2 \text{ and } |A_2^*| = 1; \\ \lfloor \frac{n}{p} \rfloor, & \text{if } |D_2^*| = |A_2^*| = i, \text{ where } i \in \{0,1,2\}; \\ \lfloor \frac{n}{p} \rfloor - 1, & \text{if } |D_2^*| = 0 \text{ and } |A_2^*| = 1, \text{ or } |D_2^*| = 1 \text{ and } |A_2^*| = 2; \\ \lfloor \frac{n}{p} \rfloor - 2, & \text{if } |D_2^*| = 0, \text{ and } |A_2^*| = 2. \end{cases}$$

Notice that for every b in the set E, two observations are made:

1. if $b \equiv 0 \pmod p$, then b appears all $\lambda = tp^e$ times in a single column of the matrix B;
2. if $b \not\equiv 0 \pmod p$, then b appears exactly tp^{e-1} times in every column of the matrix B.

The first condition implies that, to have a perfect starter sequence, M must be a multiple of p. Hence, $\frac{M}{tp^{e-1}} \equiv 0 \,(\text{mod } p) \Rightarrow M = cs$ for some integer c. Now we consider all the possible values of M:

Case (1): If $M = \lfloor \frac{n}{p} \rfloor + 2$, then $\lfloor \frac{n}{p} \rfloor + 2 \equiv 0 \pmod s$.

But if $n \equiv i \pmod p \Longrightarrow n_0 = \frac{n-i}{p}$, $0 \le i \le p-1$, then

$\lfloor \frac{n_0 p + i}{p} \rfloor + 2 \equiv 0 \pmod s \Longrightarrow n_0 + 2 \equiv 0 \pmod s \Longrightarrow \frac{n-i}{p} + 2 = c_0 s$, for some integer c_0

$\Longrightarrow n - i = -2p + c_0 t p^{e+1} \Longrightarrow n \equiv i - 2p \pmod{p^{e+1}}$. Therefore, for a generalized starter sequence of order n, multiplicity $s = tp^e$, and two defects, where the defects are both multiples of p but none of their inverses is multiple of p, to exist, n must satisfy one of the following congruency classes: $n \equiv -2p, 1-2p, 2-2p, \dots, -1-p \pmod{p^{e+1}}$.

By the proof of Theorem (5.1.1), we obtain the results of the following three cases:

Case (2): If $M = \lfloor \frac{n}{p} \rfloor + 1$, then $n \equiv -p, 1-p, 2-p, \dots, -1 \pmod{p^{e+1}}$;

Case (3): If $M = \lfloor \frac{n}{p} \rfloor$, then $n \equiv 0, 1, 2, \ldots, p-1 \pmod{p^{e+1}}$;

Case (4): If $M = \lfloor \frac{n}{p} \rfloor - 1$, then $n \equiv p, 1+p, 2+p, \ldots, 2p-1 \pmod{p^{e+1}}$.

Case (5): If $M = \lfloor \frac{n}{p} \rfloor - 2$, then $\lfloor \frac{n}{p} \rfloor - 2 \equiv 0 \pmod{s}$. But if $n \equiv i \pmod{p}$

$\Longrightarrow n_0 = \frac{n-i}{p}$, $0 \leq i \leq p-1$. Therefore,

$\lfloor \frac{n_0 p + i}{p} \rfloor - 2 \equiv 0 \pmod{s} \Longrightarrow n_0 - 2 \equiv 0 \pmod{s} \Longrightarrow \frac{n-i}{p} - 2 = c_0 s$, for some integer c_0

$\Longrightarrow n - i = 2p + c_0 t p^{e+1} \Longrightarrow n \equiv i + 2p \pmod{p^{e+1}}$.

Therefore, for a generalized starter sequence of order n, multiplicity $s = tp^e$, and two defects, where neither of the them is a multiple of p but their inverses are both multiples of p, to exist, n must satisfy one of the following congruency classes: $n \equiv 2p, 1+2p, \ldots, 3p-1 \pmod{p^{e+1}}$. ■

Theorem 5.1.3. *Let $\lambda = p^e t$, $A_3 = \{r_1, r_2, r_3\}$, and $D_3 = \{d_1, d_2, d_3\}$, where p is the smallest prime factor of λ, the elements $\{e,\, t,\, d_1,\, d_2,\, d_3,\, r_1,\, r_2,\, r_3\}$ are positive integers, $A_3^* = \{x \in A_3 : x \equiv 0 \pmod{p}\}$, and $D_3^* = \{d \in D_2 : d \equiv 0 \pmod{p}\}$. If a generalized starter sequence of order n, three defects ($d_1 = -r_1, d_2 = -r_2$, and $d_3 = -r_3$), and multiplicity λ exists, then n must satisfy one of the following congruences:*

$$n \equiv \begin{cases} -3p, 1-3p, 2-3p, \ldots, -1-2p \pmod{p^{e+1}}, & if |A_3^*| = 0 \; and |D_3^*| = 3; \\ -2p, 1-2p, 2-2p, \ldots, -1-p \pmod{p^{e+1}}, & if |A_3^*| = 0 \; and |D_3^*| = 2, \; or |A_3^*| = 1 \; and |D_3^*| = 3; \\ 0, 1, 2, \ldots, p-1 \pmod{p^{e+1}}, & if |A_3^*| = |D_3^*| = i, \; where\, i \in \{0,1,2,3\}; \\ -p, 1-p, \ldots, -1 \pmod{p^{e+1}}, & if |A_3^*| = 0, \, |D_3^*| = 1, \; or |A_3^*| = 1, \, |D_3^*| = 2, \; or |A_3^*| = 3, \, |D_3^*| = 2; \\ p, 1+p, \ldots, 2p-1 \pmod{p^{e+1}}, & if |A_3^*| = 1, \, |D_3^*| = 0, \; or |A_3^*| = 2, \, |D_3^*| = 1, or |A_3^*| = 3, \, |D_3^*| = 2; \\ 2p, 1+2p, 2+2p, \ldots, 3p-1 \pmod{p^{e+1}}, & if |A_3^*| = 2 \; and |D_3^*| = 0, \; or \; |A_3^*| = 3 \; and |D_3^*| = 1; \\ 3p, 1+3p, 2+3p, \ldots, 4p-1 \pmod{p^{e+1}}, & if |A_3^*| = 3 \; and |D_3^*| = 0. \end{cases}$$

Proof. Let $GSS_{n,3} = (s_1, s_2, ..., s_{\lambda n})$ be a generalized starter sequence of order (n), multiplicity $\lambda = tp^e$, and three defects d_1, d_2, and d_3. Arrange the terms of the sequence $GSS_{n,3}$ into the $ntp^{e-1} \times p$ matrix, $B = (b_{ij})$ according to the following rule: $b_{ij} = s_{(i-1)p+j}$ $(1 \leq i \leq ntp^{e-1};\; 1 \leq j \leq p)$:

$$B = \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,p} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \cdots & \vdots \\ b_{ntp^{e-1},1} & b_{ntp^{e-1},2} & \cdots & b_{ntp^{e-1},p} \end{pmatrix}.$$

Consider the set $E = \{\{1, 2, \ldots, n\} \setminus \{r_1, r_2, r_3\}\} \cup \{-r_1, -r_2, -r_3\}$, where $1 \leq r_1 < r_2 < r_3 \leq n$ and $-r$ is the additive inverse of r in $\mathbb{Z}_{2n+1}$. Now, let $M = |\{b \in E : b \equiv 0 \pmod p)\}|$,

$A_3 = \{r_1, r_2, r_3\}$, $D_3 = \{d_1, d_2, d_3\}$, $A_3^* = \{x \in A_3 : x \equiv 0 \pmod p\}$, and $D_3^* = \{d \in D_3 : d \equiv 0 \pmod p\}$. Hence,

$$M = \begin{cases} \lfloor \frac{n}{p} \rfloor + 3, & \text{if } |A_3^*| = 0 \text{ and } |D_3^*| = 3; \\ \lfloor \frac{n}{p} \rfloor + 2, & \text{if} |A_3^*| = 0 \text{ and } |D_3^*| = 2, \text{ or } |A_3^*| = 1 \text{ and } |D_3^*| = 3; \\ \lfloor \frac{n}{p} \rfloor + 1, & \text{if} |A_3^*| = 0 \text{ and } |D_3^*| = 1, \text{ or } |A_3^*| = 1 \text{ and } |D_3^*| = 2, \text{ or } |A_3^*| = 3 \text{ and } |D_3^*| = 2; \\ \lfloor \frac{n}{p} \rfloor, & \text{if } |A_3^*| = |D_3^*| = i, \text{ where } i \in \{0, 1, 2, 3\}; \\ \lfloor \frac{n}{p} \rfloor - 1, & \text{if} |A_3^*| = 1 \text{ and } |D_3^*| = 0, \text{ or } |A_3^*| = 2 \text{ and } |D_3^*| = 1, \text{ or } |A_3^*| = 3 \text{ and } |D_3^*| = 2; \\ \lfloor \frac{n}{p} \rfloor - 2 & \text{if } |A_3^*| = 2 \text{ and } |D_3^*| = 0, \text{ or } |A_3^*| = 3 \text{ and } |D_3^*| = 1; \\ \lfloor \frac{n}{p} \rfloor - 3, & |A_3^*| = 3 \text{ and } |D_3^*| = 0. \end{cases}$$

Notice that, for every b in the set E, two observations are made:

1. if $b \equiv 0 \pmod p$, then b appears all $\lambda = tp^e$ times in a single column of the array B;

2. if $b \not\equiv 0 \pmod p$, then b appears exactly tp^{e-1} times in every column of the array B.

The first condition implies that, to have a perfect starter sequence, M must be a multiple of p. Hence, $\frac{M}{tp^{e-1}} \equiv 0 \,(\text{mod } p) \Rightarrow M = cs$ for some integer c. Now we will consider all the possible values of M.

Case (1): If $M = \lfloor \frac{n}{p} \rfloor + 3$, then $\lfloor \frac{n}{p} \rfloor + 3 \equiv 0 \pmod{s}$.

But if $n \equiv i \pmod{p} \Longrightarrow n_0 = \frac{n-i}{p}$, $0 \leq i \leq p-1$, then

$\lfloor \frac{n_0 p + i}{p} \rfloor + 3 \equiv 0 \pmod{s} \Longrightarrow n_0 + 3 \equiv 0 \pmod{s} \Longrightarrow \frac{n-i}{p} + 3 = c_0 s$, for some integer c_0

$\Longrightarrow n - i = -3p + c_0 t p^{e+1} \Longrightarrow n \equiv i - 3p \pmod{p^{e+1}}$. Therefore, if a generalized starter sequence of order n exists, with multiplicity $s = tp^e$ and three defects, where the defects are multiple of p but none of their inverses is a multiple of p, then n must satisfy one of the following congruency classes: $n \equiv -3p, 1-3p, 2-3p, \ldots, -1-2p \pmod{p^{e+1}}$.

By the proof of Theorem (5.1.2), we obtain the result of the following case:

Case (2): If $M = \lfloor \frac{n}{p} \rfloor + 2$, then $n \equiv -2p, 1-2p, 2-2p, \ldots, -1-p \pmod{p^{e+1}}$.

Now by the proof of Theorem (5.1.1), we obtain the results of the following three cases:

Case (3): If $M = \lfloor \frac{n}{p} \rfloor + 1$, then $n \equiv -p, 1-p, 2-p, \ldots, -1 \pmod{p^{e+1}}$.

Case (4): If $M = \lfloor \frac{n}{p} \rfloor$, then $n \equiv 0, 1, 2, \ldots, p-1 \pmod{p^{e+1}}$.

Case (5): If $M = \lfloor \frac{n}{p} \rfloor - 1$, then $n \equiv p, 1+p, 2+p, \ldots, 2p-1 \pmod{p^{e+1}}$.

By the proof of Theorem (5.1.2) we obtain the results of the following cases:

Case (6): If $M = \lfloor \frac{n}{p} \rfloor - 2$, then $n \equiv 2p, 1+2p, 2+2p, \ldots, 3p-1 \pmod{p^{e+1}}$.

Case (7): If $M = \lfloor \frac{n}{p} \rfloor - 3$, then $\lfloor \frac{n}{p} \rfloor - 3 \equiv 0 \pmod{s}$.

But if $n \equiv i \pmod{p} \Longrightarrow n_0 = \frac{n-i}{p}$, $0 \leq i \leq p-1$, then

$\lfloor \frac{n_0 p + i}{p} \rfloor - 3 \equiv 0 \pmod{s} \Longrightarrow n_0 - 3 \equiv 0 \pmod{s} \Longrightarrow \frac{n-i}{p} - 3 = c_0 s$, for some integer c_0

$\Longrightarrow n - i = 3p + c_0 t p^{e+1} \Longrightarrow n \equiv i + 3p \pmod{p^{e+1}}$.

Therefore, for a generalized starter sequence of order n exists, with multiplicity $s = tp^e$, and three defects, where none of them is a multiple of p, but their inverses are multiples of p, then n must satisfy one of the following congruency classes: $n \equiv$

$3p, 1+3p, \ldots, 4p-1 \pmod{p^{e+1}}$.

■

Theorem 5.1.4. *Let $\lambda = p^e t$, where p is the smallest prime factor of λ, and e and t are positive integers. If a generalized extended starter sequence of order n, one defect d, and multiplicity λ exists, then n must satisfy one of the following congruences:*

$$n \equiv \begin{cases} kp, kp+1, \ldots, kp+p-1 \pmod{p^{e+1}}, & \textit{if both}\, d,\ -d \equiv 0,\ \textit{or}\ d,\ -d \not\equiv 0 \,(\mathrm{mod}\ p); \\ kp-p, kp+(1-p), \ldots, kp-1 \pmod{p^{e+1}}, & \textit{if}\ d \not\equiv 0,\ \textit{and}\ -d \equiv 0 \,(\mathrm{mod}\ p); \\ (k+1)p, (k+1)p+1, \ldots, p(k+1)+(p-1) \pmod{p^{e+1}}, & \textit{if}\ d \equiv 0,\ \textit{and}\ -d \not\equiv 0 \,(\mathrm{mod}\ p). \end{cases}$$

Such that $d = -r$, where $-r$ is the additive inverse of r in the group $\mathbb{Z}_{2n+1}$

Proof. Let $GES_{n,1} = (s_1, s_2, ..., s_{\lambda n+h})$ be a generalized extended starter sequence of order n, one defect d, multiplicity $\lambda = tp^e$, and h zeros. Arrange the terms of the sequence $GESS_{n,1}$ into the $(ntp^{e-1} + \lceil \frac{h}{p} \rceil) \times p$ matrix, $B = (b_{ij})$ according to the following rule: $b_{ij} = s_{(i-1)p+j}$ $(1 \le i \le ntp^{e-1} + \lceil \frac{h}{p} \rceil;\ 1 \le j \le p)$:

$$B = \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,k} & 0 & \cdots & b_{1,p} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,k} & b_{2,k+1} & \cdots & b_{2,n} \\ \cdots & \cdots & \cdots & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ b_{ntp^{e-1}+\lceil \frac{h}{p} \rceil,1} & b_{ntp^{e-1}+\lceil \frac{h}{p} \rceil,2} & \cdots & b_{ntp^{e-1}+\lceil \frac{h}{p} \rceil,k} & * & * & * \end{pmatrix}.$$

Consider the set $E = \{\{1, 2, \ldots, n\} \setminus \{r\}\} \cup \{-r\}$, where $1 \le r \le n$ and $-r$ is the additive inverse of r in the group $\mathbb{Z}_{2n+1}$. Now, let $M = |\{b \in E : b \equiv 0 \pmod{p}\}|$.

Hence,

$$M = \begin{cases} \lfloor \frac{n}{p} \rfloor, & \text{if both}\, d,\ -d \equiv 0, \text{ or } d\,\text{and } -d \not\equiv 0 \pmod{p}; \\ \lfloor \frac{n}{p} \rfloor + 1, & \text{if } d \equiv 0, \text{ and } -d \not\equiv 0 \pmod{p}; \\ \lfloor \frac{n}{p} \rfloor - 1, & \text{if } -d \equiv 0, \text{ and } d \not\equiv 0 \pmod{p}. \end{cases}$$

Notice that, for every b in the set E, two observations are made:

1. if $b \equiv 0 \pmod{p}$, then b appears all $\lambda = tp^e$ times in a single column of the matrix B;

2. if $b \not\equiv 0 \pmod{p}$, then b appears exactly tp^{e-1} times in every column of the matrix B.

The first condition implies that, to have a perfect starter sequence, M must be a multiple of p. Now, if M is not a multiple of p, then we say that

$M \equiv k (\text{mod } p) \Rightarrow \frac{M}{tp^{e-1}} \equiv \frac{k}{tp^{e-1}} \pmod{p}$. Hence, $M = k + c\lambda$ for some integer c. Therefore, $M \equiv k (\text{mod } \lambda)$. Now, we will consider all the possible values of M.

Case (1): If $M = \lfloor \frac{n}{p} \rfloor$, then $\lfloor \frac{n}{p} \rfloor \equiv k \pmod{\lambda}$.

But if $n \equiv i \pmod{p}$, then $n_0 = \frac{n-i}{p}$, $0 \le i \le p-1$; thus

$\lfloor \frac{n_0 p + i}{p} \rfloor \equiv k \pmod{\lambda} \Longrightarrow n_0 \equiv k \pmod{\lambda} \Longrightarrow \frac{n-i}{p} = k + c_0\lambda$, for some integer c_0

$\Longrightarrow n - i = kp + c_0 tp^{e+1} \Longrightarrow n \equiv kp + i \pmod{p^{e+1}}$.

Therefore, if a generalized extended starter sequence of order n exists, with multiplicity $\lambda = tp^e$ and one defect, where the defect and its inverse are both multiples of p or neither of them is a multiple of p, then n must satisfy one of the congruences:

$n \equiv kp, kp+1, \ldots, kp+p-1 \pmod{p^{e+1}}$.

Case (2): If $M = \lfloor \frac{n}{p} \rfloor + 1$, then $\lfloor \frac{n}{p} \rfloor + 1 \equiv k \pmod{\lambda}$.

But if $n \equiv i \pmod{p} \Longrightarrow n_0 = \frac{n-i}{p}$, $0 \le i \le p-1$, then

$\lfloor \frac{n_0 p+i}{p} \rfloor + 1 \equiv k \pmod{\lambda} \Longrightarrow n_0 + 1 \equiv k \pmod{\lambda} \Longrightarrow \frac{n-i}{p} + 1 = k + c_0\lambda$, for some integer c_0

$\Longrightarrow n - i = kp - p + c_0 tp^{e+1} \Longrightarrow n \equiv kp + (i-p) \pmod{p^{e+1}}$.

Therefore, if a generalized extended starter sequence of order n exists, with multiplicity $\lambda = tp^e$ and one defect, where the defect is a multiple of p but its inverse is not a multiple of p, then n must satisfy one of the congruences:

$n \equiv kp - p, kp + (1-p), kp + (2-p), \ldots, kp - 1 \pmod{p^{e+1}}$.

Case (3): If $M = \lfloor \frac{n}{p} \rfloor - 1$, then $\lfloor \frac{n}{p} \rfloor - 1 \equiv k \pmod{\lambda}$.

But if $n \equiv i \pmod{p} \Longrightarrow n_0 = \frac{n-i}{p}$, $0 \leq i \leq p-1$, then

$\lfloor \frac{n_0 p+i}{p} \rfloor - 1 \equiv k \pmod{\lambda} \Longrightarrow n_0 - 1 \equiv k \pmod{\lambda} \Longrightarrow \frac{n-i}{p} - 1 = k + c_0\lambda$, for some integer c_0

$\Longrightarrow n - i = kp + p + c_0 tp^{e+1} \Longrightarrow n \equiv i + (k+1)p \pmod{p^{e+1}}$.

Therefore, n must satisfy one of the congruences:

$n \equiv (k+1)p, 1 + (k+1)p, \ldots, (p-1) + (k+1)p \pmod{p^{e+1}}$. ■

Theorem 5.1.5. *Let $\lambda = p^e t$, $A_2 = \{r_1, r_2\}$, $D_2 = \{d_1, d_2\}$, $A_2^* = \{x \in A_2 : x \equiv 0 \pmod{p}\}$, and $D_2^* = \{d \in D_2 : d \equiv 0 \pmod{p}\}$, where $d_i = -r_i$, $1 \leq i \leq 2$, and p is the smallest prime factor of λ and e, t, r_1, r_2 are positive integers, and $-r_i$ is the additive inverse of r_i in the group $\mathbb{Z}_{2n+1}$. If a generalized extended starter sequence of order n, two defects d_1, d_2, and multiplicity λ exists, then n must satisfy one of the following congruences:*

$$n \equiv \begin{cases} (k-2)p, 1+(k-2)p, \ldots, (k-1)p-1 \pmod{p^{e+1}}, & if |D_2^*| = 2\, and |A_2^*| = 0; \\ kp, kp+1, \ldots, kp+p-1 \pmod{p^{e+1}}, & if |D_2^*| = |A_2^*| = i,\ where\, i \in \{0,1,2\}; \\ kp-p, kp+(1-p), \ldots, kp-1 \pmod{p^{e+1}}, & if |D_2^*| = 1\, and |A_2^*| = 0,\ or |D_2^*| = 2\, and |A_2^*| = 1; \\ (k+1)p, 1+(k+1)p, \ldots, (2+k)p-1 \pmod{p^{e+1}}, & if |D_2^*| = 0\, and |A_2^*| = 1,\ or |D_2^*| = 1\, and |A_2^*| = 2; \\ (2+k)p, 1+(2+k)p, \ldots, (3+k)p-1 \pmod{p^{e+1}}, & if |D_2^*| = 0,\ and |A_2^*| = 2. \end{cases}$$

Proof. Let $GES_{n,2} = (s_1, s_2, ..., s_{\lambda n+h})$ be a generalized extended starter sequence of order (n), two defects d_1 and d_2, multiplicity $\lambda = tp^e$, and h zeros. Arrange the terms of the sequence $GESS_{n,2}$ into the $(ntp^{e-1} + \lceil \frac{h}{p} \rceil) \times p$ matrix, $B = (b_{ij})$, according to the following rule: $b_{ij} = s_{(i-1)p+j}$ $(1 \le i \le ntp^{e-1} + \lceil \frac{h}{p} \rceil;\ 1 \le j \le p)$.

$$B = \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,k} & 0 & \cdots & b_{1,p} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,k} & b_{2,k+1} & \cdots & b_{2,n} \\ \cdots & \cdots & \cdots & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ b_{ntp^{e-1}+\lceil \frac{h}{p} \rceil,1} & b_{ntp^{e-1}+\lceil \frac{h}{p} \rceil,2} & \cdots & b_{ntp^{e-1}+\lceil \frac{h}{p} \rceil,k} & * & * & * \end{pmatrix}.$$

Consider the set $E = \{\{1, 2, \ldots, n\} \setminus \{r_1, r_2\}\} \cup \{d_1, d_2\}$, where $1 \le r_1 < r_2 \le n$, $d_1 = -r_1$, $d_2 = -r_2$, and $-r$ is the additive inverse of r in the group $\mathbb{Z}_{2n+1}$. Now, let $M = |\{b \in E : b \equiv 0 \pmod{p}\}|$, $A_2 = \{r_1, r_2\}$, $D_2 = \{d_1, d_2\}$, $A_2^* = \{x \in A_2 : x \equiv 0 \pmod{p}\}$, and $D_2^* = \{d \in D_2 : d \equiv 0 \pmod{p}\}$. Hence,

$$M = \begin{cases} \lfloor \frac{n}{p} \rfloor + 2, & \text{if}\, |D_2^*| = 2 \,\text{and}\, |A_2^*| = 0; \\ \lfloor \frac{n}{p} \rfloor + 1, & \text{if}\, |D_2^*| = 1 \,\text{and}\, |A_2^*| = 0, \text{ or}\, |D_2^*| = 2 \,\text{and}\, |A_2^*| = 1; \\ \lfloor \frac{n}{p} \rfloor, & \text{if}\, |D_2^*| = |A_2^*| = i, \text{ where}\, i \in \{0, 1, 2\}; \\ \lfloor \frac{n}{p} \rfloor - 1, & \text{if}\, |D_2^*| = 0 \,\text{and}\, |A_2^*| = 1, \text{ or}\, |D_2^*| = 1 \,\text{and}\, |A_2^*| = 2; \\ \lfloor \frac{n}{p} \rfloor - 2, & \text{if}\, |D_2^*| = 0, \text{ and}\, |A_2^*| = 2. \end{cases}$$

Notice that, for every b in the set E, two observations are made:

1. if $b \equiv 0 \pmod{p}$, then b appears $\lambda = tp^e$ times in a single column of the matrix B;

2. if $b \not\equiv 0 \pmod{p}$, then b appears exactly tp^{e-1} times in every column of the matrix B.

The first condition implies that, to have a perfect starter sequence, M must be a multiple of p. Now, if M is not a multiple of p, then we say that

$M \equiv k \pmod{p} \Rightarrow \frac{M}{tp^{e-1}} \equiv \frac{k}{tp^{e-1}} \pmod{p}$. Hence, $M = k + c\lambda$ for some integer c.

Therefore, $M \equiv k (\bmod \lambda)$. Now, we consider all the possible values of M. We only need to investigate the first case and the last case; the remaining cases are already considered in Theorem (5.1.4).

Case (1): If $M = \lfloor \frac{n}{p} \rfloor + 2$, then $\lfloor \frac{n}{p} \rfloor + 2 \equiv k \pmod{\lambda}$.

But if $n \equiv i \pmod{p} \Longrightarrow n_0 = \frac{n-i}{p}$, $0 \leq i \leq p-1$, then

$\lfloor \frac{n_0 p + i}{p} \rfloor + 2 \equiv k \pmod{s} \Longrightarrow n_0 + 2 \equiv k \pmod{\lambda} \Longrightarrow \frac{n-i}{p} + 2 = k + c_0\lambda$, for some integer c_0

$\Longrightarrow n - i = (k-2)p + c_0 t p^{e+1} \Longrightarrow n \equiv i + (k-2)p \pmod{p^{e+1}}$.

Therefore, n must satisfy one of the following congruency classes:

$n \equiv (k-2)p, 1 + (k-2)p, 2 + (k-2)p, \ldots, (k-1)p - 1 \pmod{p^{e+1}}$.

Case (2): If $M = \lfloor \frac{n}{p} \rfloor$, then $n \equiv kp, kp+1, kp+2, \ldots, kp+p-1 \pmod{p^{e+1}}$.

Case (3): If $M = \lfloor \frac{n}{p} \rfloor + 1$, then $n \equiv kp - p, kp + (1-p), kp + (2-p), \ldots, kp - 1 \pmod{p^{e+1}}$.

Case (4): If $M = \lfloor \frac{n}{p} \rfloor - 1$, then $n \equiv (k+1)p, 1 + (k+1)p, \ldots, (p-1) + (k+1)p \pmod{p^{e+1}}$.

Case (5): If $M = \lfloor \frac{n}{p} \rfloor - 2$, then $\lfloor \frac{n}{p} \rfloor - 2 \equiv k \pmod{\lambda}$.

But if $n \equiv i \pmod{p} \Longrightarrow n_0 = \frac{n-i}{p}$, $0 \leq i \leq p-1$, then

$\lfloor \frac{n_0 p + i}{p} \rfloor - 2 \equiv k \pmod{\lambda} \Longrightarrow n_0 - 2 \equiv k \pmod{\lambda} \Longrightarrow \frac{n-i}{p} - 2 = k + c_0\lambda$, for some integer c_0

$\Longrightarrow n - i = (2+k)p + c_0 t p^{e+1} \Longrightarrow n \equiv i + (2+k)p \pmod{p^{e+1}}$.

Therefore, n must satisfy one of the following congruences:

$n \equiv (2+k)p, 1 + (2+k)p, 2 + (2+k)p, \ldots, (3+k)p - 1 \pmod{p^{e+1}}$. ■

Theorem 5.1.6. *Let* $\lambda = p^e t$, $A_3 = \{r_1, r_2, r_3\}$, *and* $D_3 = \{d_1, d_2, d_3\}$, *where* p *is the smallest prime factor of* λ, *the elements* $\{e,\ t,\ d_1,\ d_2,\ d_3,\ r_1,\ r_2,\ r_3\}$ *are positive integers,* $A_3 = \{r_1, r_2, r_3\}$, $D_3 = \{d_1, d_2, d_3\}$, $A_3^* = \{x \in A_3 : x \equiv 0 \pmod{p}\}$, *and* $D_3^* = \{d \in D_3 : d \equiv 0 \pmod{p}\}$. *If a generalized extended starter sequence of order* n, *with three defects* $(d_i = -r_i, i \in \{1,2,3\})$, *and multiplicity* λ *exists, then* n *must* *satisfy one of the following congruences modulo p^{e+1}:*

$$n \equiv \begin{cases} (k-3)p, 1+(k-3)p, \ldots, (k-2)p-1, & \text{if } |A_3^*| = 0 \text{ and } |D_3^*| = 3; \\ (k-2)p, 1+(k-2)p, \ldots, (k-1)p-1, & \text{if } |A_3^*| = 0 \text{ and } |D_3^*| = 2, \text{ or } |A_3^*| = 1 \text{ and } |D_3^*| = 3; \\ kp, kp+1, \ldots, kp+p-1, & \text{if} |A_3^*| = |D_3^*| = i, \text{ where } i \in \{0,1,2,3\}; \\ kp-p, kp+(1-p), \ldots, kp-1, & \text{if } |A_3^*| = 0, |D_3^*| = 1, \text{ or} |A_3^*| = 1, |D_3^*| = 2, \text{ or} |A_3^*| = 3, |D_3^*| = 2; \\ (k+1)p, 1+(k+1)p, \ldots, (p-1)+(k+1)p, & \text{if} |A_3^*| = 1, |D_3^*| = 0, \text{ or} |A_3^*| = 2, |D_3^*| = 1, \text{ or} |A_3^*| = 3, |D_3^*| = 2; \\ (2+k)p, 1+(2+k)p, \ldots, (3+k)p-1, & \text{if } |A_3^*| = 2 \text{ and } |D_3^*| = 0, \text{ or } |A_3^*| = 3 \text{ and } |D_3^*| = 1; \\ (k+3)p, 1+(k+3)p, \ldots, (k+4)p-1, & \text{if } |A_3^*| = 3 \text{ and } |D_3^*| = 0. \end{cases}$$

Proof. Let $GES_{n,3} = (s_1, s_2, ..., s_{\lambda n+h})$ be a generalized extended starter sequence of order n, multiplicity $\lambda = tp^e$, three defects d_1, d_2, and d_3, and h zeros. Arrange the terms of the sequence $GESS_{n,3}$ into the $(ntp^{e-1} + \lceil \frac{h}{p} \rceil) \times p$ matrix, $B = (b_{ij})$, according to the following rule: $b_{ij} = s_{(i-1)p+j}$ $(1 \leq i \leq ntp^{e-1} + \lceil \frac{h}{p} \rceil;\ 1 \leq j \leq p)$.

$$B = \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,k} & 0 & \cdots & b_{1,p} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,k} & b_{2,k+1} & \cdots & b_{2,n} \\ \cdots & \cdots & \cdots & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ b_{ntp^{e-1}+\lceil \frac{h}{p} \rceil,1} & b_{ntp^{e-1}+\lceil \frac{h}{p} \rceil,2} & \cdots & b_{ntp^{e-1}+\lceil \frac{h}{p} \rceil,k} & * & * & * \end{pmatrix}.$$

Consider the set $E = \{\{1, 2, \ldots, n\} \setminus \{r_1, r_2, r_3\}\} \cup \{-r_1, -r_2, -r_3\}$, where $1 \leq r_1 < r_2 < r_3 \leq n$ and $-r$ is the additive inverse of r in $\mathbb{Z}_{2n+1}$. Now, let $M = |\{b \in E : b \equiv 0 \pmod{p}\}|$,

$A_3 = \{r_1, r_2, r_3\}$, $D_3 = \{d_1, d_2, d_3\}$, $A_3^* = \{x \in A_3 : x \equiv 0 \pmod{p}\}$, and $D_3^* = \{d \in D_3 : d \equiv 0 \pmod{p}\}$. Hence,

$$M = \begin{cases} \lfloor \frac{n}{p} \rfloor + 3, & \text{if}\, |A_3^*| = 0 \,\text{and}\, |D_3^*| = 3; \\ \lfloor \frac{n}{p} \rfloor + 2, & \text{if} |A_3^*| = 0 \,\text{and}\, |D_3^*| = 2, \text{ or } |A_3^*| = 1 \,\text{and}\, |D_3^*| = 3; \\ \lfloor \frac{n}{p} \rfloor + 1, & \text{if} |A_3^*| = 0 \,\text{and}\, |D_3^*| = 1, \text{ or } |A_3^*| = 1 \,\text{and}\, |D_3^*| = 2, \text{ or } |A_3^*| = 3 \,\text{and}\, |D_3^*| = 2; \\ \lfloor \frac{n}{p} \rfloor, & \text{if}\, |A_3^*| = |D_3^*| = i, \text{ where}\, i \in \{0,1,2,3\}; \\ \lfloor \frac{n}{p} \rfloor - 1, & \text{if} |A_3^*| = 1 \,\text{and}\, |D_3^*| = 0, \text{ or } |A_3^*| = 2 \,\text{and}\, |D_3^*| = 1, \text{ or } |A_3^*| = 3 \,\text{and}\, |D_3^*| = 2; \\ \lfloor \frac{n}{p} \rfloor - 2, & \text{if}\, |A_3^*| = 2 \,\text{and}\, |D_3^*| = 0, \text{ or } |A_3^*| = 3 \,\text{and}\, |D_3^*| = 1; \\ \lfloor \frac{n}{p} \rfloor - 3, & |A_3^*| = 3 \,\text{and}\, |D_3^*| = 0. \end{cases}$$

Notice that, for every b in the set E, two observations are made:

1. if $b \equiv 0 \pmod p$, then b appears $\lambda = tp^e$ times in a single column of the array B.

2. if $b \not\equiv 0 \pmod p$, then b appears exactly tp^{e-1} times in every column of the array B.

The first condition implies that, to have a perfect starter sequence, M must be a multiple of p. Hence, $\frac{M}{tp^{e-1}} \equiv k \,(\text{mod } p) \Rightarrow M = c\lambda$ for some integer c. Now we consider all the possible values of M.

Case (1): If $M = \lfloor \frac{n}{p} \rfloor + 3$, then $\lfloor \frac{n}{p} \rfloor + 3 \equiv k \pmod \lambda$.

But if $n \equiv i \pmod p \Longrightarrow n_0 = \frac{n-i}{p}$, $0 \leq i \leq p-1$, then

$\lfloor \frac{n_0 p + i}{p} \rfloor + 3 \equiv k \pmod \lambda \Longrightarrow n_0 + 3 \equiv k \pmod \lambda \Longrightarrow \frac{n-i}{p} + 3 = k + c_0\lambda$, for some integer c_0

$\Longrightarrow n - i = (k-3)p + c_0 tp^{e+1} \Longrightarrow n \equiv i + (k-3)p \pmod{p^{e+1}}$.

Therefore, n must satisfy one of the following congruency classes: $n \equiv (k-3)p, 1 + (k-3)p, 2 + (k-3)p, \ldots, (k-2)p - 1 \pmod{p^{e+1}}$.

The following cases are already considered in Theorem 5.1.5

Case (2): If $M = \lfloor \frac{n}{p} \rfloor + 2$, then $n \equiv (k-2)p, 1 + (k-2)p, \ldots, (k-1)p - 1 \pmod{p^{e+1}}$.

Case (3): If $M = \lfloor \frac{n}{p} \rfloor$, then $n \equiv kp, kp+1, kp+2, \ldots, kp+p-1 \pmod{p^{e+1}}$.

Case (4): If $M = \lfloor \frac{n}{p} \rfloor + 1$, then $n \equiv kp-p, kp+(1-p), kp+(2-p), \ldots, kp-1 \pmod{p^{e+1}}$.

Case (5): If $M = \lfloor \frac{n}{p} \rfloor - 1$, then $n \equiv (k+1)p, 1+(k+1)p, \ldots, (p-1)+(k+1)p \pmod{p^{e+1}}$.

Case (6): If $M = \lfloor \frac{n}{p} \rfloor - 2$, then $n \equiv (2+k)p, 1+(2+k)p, \ldots, (3+k)p-1 \pmod{p^{e+1}}$.

Case (7): If $M = \lfloor \frac{n}{p} \rfloor - 3$, then $\lfloor \frac{n}{p} \rfloor - 3 \equiv k \pmod{\lambda}$.

But if $n \equiv i \pmod{p} \Longrightarrow n_0 = \frac{n-i}{p}$, $0 \leq i \leq p-1$, then

$\lfloor \frac{n_0 p+i}{p} \rfloor - 3 \equiv k \pmod{\lambda} \Longrightarrow n_0 - 3 \equiv k \pmod{\lambda} \Longrightarrow \frac{n-i}{p} - 3 = k + c_0\lambda$, for some integer c_0

$\Longrightarrow n - i = (k+3)p + c_0 t p^{e+1} \Longrightarrow n \equiv i + (k+3)p \pmod{p^{e+1}}$.

Therefore, n must satisfy one of the congruences:

$n \equiv (k+3)p, 1+(k+3)p, 2+(k+3)p, \ldots, (k+4)p-1 \pmod{p^{e+1}}$.

■

Corollary 5.1.1. *If a generalized extended starter sequence $S = (s_1, s_2, \ldots, s_{\lambda n+h})$ of order n, and multiplicity $\lambda = tp^e$ exists, then the expected minimum number of zeros is given by:*

$h = (p-k)(\lambda-1)$. The permissible locations of these zeros are calculated as s_i such that $i \equiv k+1, \ldots, p-1, 0 \pmod{p}$. Provided that $k \not\equiv 0 \pmod{p}$, where $M = |\{b \in E : b \equiv 0 \pmod{p}| \equiv k \pmod{p}$.

Proof. In case $|\{b \in E : b \equiv 0 \pmod{p}|$ is not a multiple of p, then the array B must have $(p-k)$ columns each containing λ zeros. Hence, the lower bound is attained when the entries $b_{ntp^{e-1},k+1}, b_{ntp^{e-1},k+2}, \ldots, b_{ntp^{e-1},p}$ are all 0 or $*$, but all these will lie outside the sequence; thus, we attain the lower bound of zeros, which is $(p-k)(s-1)$. Since $b_{ij} = s_{(i-1)p+j}$, $(1 \leq i \leq ntp^{e-1};\ 1 \leq j \leq p)$, the permissible locations for the zeros are given by s_i, such that $i \equiv k+1, \ldots, p-1, 0 \pmod{p}$. ■

5.2 Examples

The following example shows the minimum possible value of n for a perfect starter sequence with multiplicity $\lambda = 3$.

Example 5.2.1. *Consider the perfect starter sequence with two defects for the case* $(n, \lambda, h) = (8, 3, 0)$ *given below in matrix form:*

$$\begin{pmatrix} 9 & 6 & 10 \\ 1 & 1 & 1 \\ 2 & 6 & 2 \\ 9 & 2 & 5 \\ 10 & 6 & 3 \\ 4 & 5 & 3 \\ 9 & 4 & 3 \\ 5 & 10 & 4 \end{pmatrix}.$$

Note $\lambda = p^e t \Longrightarrow p = 3$, $t = e = 1$. $A = \{7, 8\}$, $D = \{9, 10\}$, $A^* = \phi$, *and* $D^* = \{9\}$. *Since* $\lfloor \frac{n}{p} \rfloor + 1 \equiv k \pmod{p} \Longrightarrow 0 \equiv k \pmod{3}$, *then*

$$\begin{aligned} n \equiv -p, 1-p, 2-p, \ldots, -1 \pmod{p^{e+1}} &\Leftrightarrow 8 \equiv -1 \pmod{9} \\ &\Leftrightarrow 8 \equiv 8 \pmod{9}. \end{aligned}$$

Example 5.2.2. *Consider the perfect starter sequence with three defects for the case* $(n, \lambda, h) = (18, 4, 0)$, *given below in (transpose) array form. In fact, this example shows the smallest order for the perfect starter sequences with multiplicity* $(\lambda = 4)$.

7	1	1	21	10	13	14	7	3	10	19	3	11	14	10	15	5	9	3	10	14	5	6	11	21
19	1	1	7	8	20	3	15	8	3	7	13	8	21	12	20	8	11	5	19	12	9	15	5	13

$$\begin{matrix} 6 & 9 & 14 & 6 & 19 & 15 & 6 & 2 & 2 & 2 & 2 \\ 20 & 12 & 4 & 11 & 4 & 9 & 4 & 12 & 4 & 21 & 20 \end{matrix}$$

Note $\lambda = p^e t \implies p = 2 = e,\ t = 1.\ A = \{16, 17, 18\},\ D = \{19, 20, 21\},\ |A^*| = 2,$ *and* $|D^*| = 1.$

Since $\lfloor \frac{n}{p} \rfloor - 1 \equiv k \pmod{p} \implies 0 \equiv k \pmod{8}$, *then*

$$\begin{aligned} n \equiv p, 1+p, 2+p, \ldots, 2p-1 \pmod{p^{e+1}} &\Leftrightarrow 18 \equiv 2 \pmod{8} \\ &\Leftrightarrow 2 \equiv 2 \pmod{8}. \end{aligned}$$

Example 5.2.3. *Consider a hooked starter sequence with one defect for the case* $(n, \lambda, h) = (6, 3, 2)$, *given below in array form.*

$$\begin{pmatrix} 2 & 6 & 2 \\ 8 & 2 & 0 \\ 4 & 6 & 0 \\ 3 & 4 & 8 \\ 3 & 6 & 4 \\ 3 & 1 & 1 \\ 1 & 8 & \star \end{pmatrix}.$$

since $\lfloor \frac{n}{p} \rfloor \equiv k \pmod{p} \implies 2 \equiv k \pmod{3}$, *then*

$$\begin{aligned} n \equiv kp, kp+1, kp+2, \ldots, kp+p-1 \pmod{p^{e+1}} &\Leftrightarrow 6 \equiv 2(3) \pmod{3} \\ &\Leftrightarrow 0 \equiv 0 \pmod{3}. \end{aligned}$$

Now, the expected minimum number of zeros is calculated as $h = (p-k)(s-1) \Rightarrow$

$h = 2$. *This is in fact the case for the given sequence, where we have two zeros in the third column of the array. However, the last zero in the array lies outside the sequence represented by a star. The permissible locations of the zeros are given by* s_i, *such that* $i \equiv k+1, \ldots, p-1, 0 \pmod{p} \Rightarrow i \equiv 0 \pmod{3}$. *This is in fact the case for the given sequence, where* $6, 9 \equiv 0 \pmod{3}$.

Example 5.2.4. *Consider a hooked starter sequence with three defects for the case* $(n, \lambda, h) = (9, 4, 3)$, *given below in (transpose) array form:*

$$\begin{array}{cccccccccccccccccccc} 12 & 4 & 11 & 4 & 10 & 4 & 12 & 4 & 3 & 10 & 5 & 3 & 12 & 11 & 10 & 5 & 1 & 1 & 12 & 10 \\ 0 & 2 & 2 & 2 & 2 & 6 & 3 & 11 & 6 & 3 & 0 & 6 & 5 & 0 & 6 & 1 & 1 & 5 & 11 & \star \end{array}$$

Note $\lambda = p^e t \implies p = 2 = e$, $t = 1$. $A = \{7, 8, 9\}$, $D = \{10, 11, 12\}$, $|A^*| = 1$, *and* $|D^*| = 2$. *since,* $\lfloor \frac{n}{p} \rfloor + 1 \equiv k \pmod{p} \implies 1 \equiv k \pmod{2}$ *then*

$$\begin{aligned} n \equiv kp - p, kp + (1-p), \ldots, kp - 1 \pmod{p^{e+1}} &\Leftrightarrow 9 \equiv 1 \pmod{8} \\ &\Leftrightarrow 1 \equiv 1 \pmod{8}. \end{aligned}$$

Now, the expected minimum number of zeros is calculated as $h = (p-k)(s-1) \Rightarrow h = 3$. *This is in fact the case for the given sequence. The permissible locations of the zeros are given by* s_i *such that* $i \equiv k+1, \ldots, p-1, 0 \pmod{p} \Rightarrow i \equiv 0 \pmod{2}$. *This is in fact the case for the given sequence, where* $2, 22, 28 \equiv 0 \pmod{2}$.

5.3 Computational Results

In this section, we present some computational results for generalized (extended) starter sequences with the minimum number of zeros (hooks) and with multiplicity

($\lambda = 3, 4$) as shown in Tables (1) and (2). These results were obtained by an exhaustive computer search. The original program was written by David Churchill in his Honors project under the supervision of Nabil Shalaby. New results are presented here in this section, an exhaustive search shows that there exists one generalized extended starter sequence of order 17 and multiplicity $\lambda = 4$.

$(4, 17)$-GES:

$7, 19, 1, 1, 1, 1, 21, 7, 10, 8, 13, 20, 14, 3, 7, 15, 3, 8, 10, 3, 19, 7, 3, 13, 11, 8, 14, 21, 10, 12, 15, 20, 5, 8, 9, 11, 13, 5, 10, 19, 14, 12, 5, 9, 6, 15, 11, 5, 21, 13, 6, 20, 9, 12, 14, 4, 6, 11, 19, 4, 15, 9, 6, 4, 2, 12, 2, 4, 2, 21, 2, 20.$

The next result is that there exists one generalized starter sequence $(4, 18)$-GSS, one $(4, 19)$-GSS, and four $(4, 20)$-GSS .

$n = 18$ and $\lambda = 4$

$7, 19, 1, 1, 1, 1, 21, 7, 10, , 8, 13, 20, 14, 3, 7, 15, 3, 8, 10, 3, 19, 7, 3, 13, 11, 8, 14, 21, 10, 12, 15, 20, 5, 8, 9, 11, 13, 5, 10, 19, 14, 12, 5, 9, 6, 15, 11, 5, 21, 13, 6, 20, 9, 12, 14, 4, 6, 11, 19, 4, 15, 9, 6, 4, 2, 12, 2, 4, 2, 21, 2, 20.$

$n = 19$ and $\lambda = 4$

$5, 17, 2, 19, 2, 5, 2, 21, 2, 15, 5, 9, 11, 4, 13, 5, 16, 4, 17, 14, 9, 4, 19, 11, 15, 4, 10, 13, 21, 9, 7, 12, 16, 14, 11, 17, 10, 7, 9, 15, 13, 19, 8, 12, 7, 11, 10, 14, 16, 21, 8, 7, 17, 13, 15, 12, 10, 6, 8, 3, 19, 14, 3, 6, 16, 3, 8, 12, 3, 6, 21, 1, 1, 1, 1, 6.$

$n = 20$ and $\lambda = 4$

$20, 11, 22, 1, 1, 1, 1, 24, 14, 16, 3, 18, 11, 3, 7, 12, 3, 13, 15, 3, 20, 7, 14, 11, 22, 16, 6, 12, 7, 18, 13, 24, 6, 15, 11, 7, 14, 8, 6, 12, 20, 16, 10, 13, 6, 8, 22, 18, 15, 9, 14, 12, 10, 8, 5, 24, 13, 16, 9, 5, 20, 8, 10, 15, 5, 18, 4, 9, 22, 5, 4, 2, 10, 2, 4, 2, 9, 2, 4, 24.$

$20, 24, 4, 2, 9, 2, 4, 2, 10, 2, 4, 5, 22, 9, 4, 18, 5, 15, 10, 8, 20, 5, 9, 16, 13, 24, 5, 8, 10, 12, 14, 9, 15, 18, 22, 5, 6, 13, 10, 16, 20, 12, 6, 8, 14, 7, 11, 15, 6, 24, 13, 18, 7, 12, 6, 16, 22, 11, 14, 7, 20, 3, 15, 13, 3, 12, 7, 3, 11, 18, 3, 16, 14, 24, 1, 1, 1, 1, 22, 11.$

5, 11, 6, 19, 25, 5, 20, 14, 6, 23, 5, 13, 11, 15, 6, 5, 1, 1, 1, 1, 6, 14, 19, 11, 13, 17, 20, 3, 15, 25, 3, 7, 23, 3, 11, 14, 3, 13, 7, 12, 8, 19, 17, 15, 9, 7, 20, 10, 8, 14, 13, 12, 7, 9, 25, 23, 8, 10, 15, 17, 19, 4, 9, 12, 8, 4, 20, 10, 2, 4, 2, 9, 2, 4, 2, 12, 17, 10, 23, 25.

26, 2, 16, 2, 5, 2, 17, 2, 4, 5, 18, 21, 4, 8, 5, 10, 4, 7, 16, 5, 4, 8, 19, 17, 7, 10, 26, 12, 18, 8, 13, 7, 21, 14, 16, 10, 6, 8, 7, 12, 17, 19, 6, 13, 11, 10, 18, 14, 6, 9, 16, 12, 26, 21, 6, 11, 13, 17, 9, 3, 19, 14, 3, 12, 18, 3, 11, 9, 3, 13, 1, 1, 1, 1, 21, 14, 9, 11, 26, 19.

n	Samples of the generalized (extended) starter sequences of order (n), multiplicity ($\lambda = 3$), with the minimum number of zeros (h).	h	Number of the starter sequences including Skolem sequences
2	_	_	0
3	_	_	0
4	5,7,0,3,0,5,3,0,7,3,5,1,1,1,0,7 6,0,1,1,1,5,6,0,0,2,5,2,6,2,0,5 2,5,2,6,2,0,5,0,0,6,0,5,1,1,1,6 7,0,4,6,0,0,4,7,0,6,4,1,1,1,7,6	4	52
5	8,5,1,,1,1,4,5,0,8,4,2,5,2,4,2,0,8 6,7,1,1,1,2,6,2,7,2,3,0,6,3,0,7,3 3,7,0,3,6,0,3,2,7,2,6,2,1,1,1,7,6 7,1,1,1,6,0,3,7,0,3,6,2,3,2,7,2,6	2	14
6	9,1,1,1,2,5,2,6,2,9,5,3,0,6,3,5,0,3,9,6 7,9,2,5,2,0,2,7,5,3,9,0,3,5,7,3,1,1,1,9 9,5,6,1,1,1,5,0,6,9,3,5,0,3,6,2,3,2,9,2 8,9,0,2,7,2,3,2,8,3,9,7,3,1,1,1,8,0,7,9	2	48
7	5,7,0,4,9,5,3,4,7,3,5,4,3,9,2,7,2,0,2,1,1,1,9 7,10,0,3,4,0,3,7,4,3,6,10,4,2,7,2,6,2,1,1,1,10,6 11,9,0,5,1,1,1,7,5,3,9,11,3,5,7,3,2,0,2,9,2,7,11 8,10,4,1,1,1,4,6,8,3,4,10,3,6,0,3,8,0,2,6,2,10,2	2	132
8	9,6,10,1,1,1,2,6,2,9,2,5,10,6,3,4,5,3,9,4,3,5,10,4 4,10,5,3,4,9,3,5,4,3,6,10,5,2,9,2,6,2,1,1,1,,10,6,9	0	2
9	4,11,9,3,4,5,3,7,4,3,5,9,11,6,7,5,1,1,1,6,9,7,2,11,2,6 10,8,12,5,1,1,1,4,5,8,10,4,6,5,12,4,3,8,6,3,10,2,3,2,6,2,12 5,10,12,3,4,5,3,8,4,3,5,10,4,6,12,8,1,1,1,6,2,10,2,8,2,6,12 12,8,1,1,1,10,5,3,6,8,3,5,12,3,6,10,5,8,4,2,6,2,4,2,12,10,4	0	36
10	10,14,6,1,1,1,3,9,6,3,10,8,3,4,6,14,9,4,5,8,10,4,2,5,2,9,2,8,5,14 13,10,3,7,5,3,6,9,3,5,7,10,6,13,5,4,9,7,6,4,2,10,2,4,2,9,13,1,1,1 14,10,4,12,8,2,4,2,5,2,4,10,8,5,14,12,3,6,5,3,8,10,3,6,1,1,1,12,14,6 5,3,10,13,3,5,9,3,6,4,5,7,10,4,6,9,13,4,7,2,6,2,10,2,9,7,1,1,1,13	0	172
11	11,9,13,2,5,2,6,2,8,5,9,11,6,7,5,13,8,4,6,9,7,4,11,3,8,4,3,7,13,3,1,1,1 16,5,13,11,4,2,5,2,4,2,8,5,4,9,11,13,16,6,8,1,1,1,9,6,3,11,8,3,13,6,3,9,16 16,8,3,4,11,3,6,4,3,8,10,4,6,9,5,11,16,8,6,5,10,2,9,2,5,2,11,1,1,1,10,9,16 3,4,13,3,14,4,3,12,2,4,2,5,2,7,8,13,5,6,14,12,7,5,8,6,1,1,1,7,13,6,8,12,14	0	196
12	12,16,1,1,1,10,2,7,2,11,2,8,12,5,7,10,6,16,5,8,11,7,6,5,12,10,4,8,6,3,4,11,3,16,4,3	0	550
13	_	_	0
14	14,20,4,13,11,3,4,12,3,8,4,3,10,5,14,11,13,8,5,12,7,20,10,5,6,8,11,7,14,13,6,12, 10,2,7,2,6,2,1,1,1,20. 14,19,17,13,9,6,1,1,1,3,11,6,3,9,14,3,13,6,8,17,19,11,9,2,7,2,8,2,14,13,5,7,11,4, 8,5,17,4,7,19,5,4.	0	5904
15	15,21,12,14,6,2,13,2,5,2,6,9,11,5,12,15,6,14,5,13,9,3,21,11,3,8,12,3,7,9,15,14,13,8, 11,7,4,1,1,1,4,8,7,21,4. 15,18,12,20,14,1,1,1,2,8,2,10,2,7,12,15,6,8,14,18,7,10,6,20,9,8,12,7,6,5,15,10,14,9, 5,3,4,18,3,5,4, 3,9,20,4.	0	27414

Table 5.1: (3,n)-(extended) starter sequences with the minimum number of zeros.

n	Samples of the generalized (extended) starter sequences of order (n), multiplicity ($\lambda = 4$), with the minimum number of zeros (h).	h	Number of the starter sequences including Skolem sequences
2	2,0,2,0,2,0,2,1,1,1,1 1,1,1,1,2,0,2,0,2,0,2	3	2
3	4,2,0,2,4,2,0,2,4,0,0,0,4,1,1,1,1 4,0,0,2,4,2,0,2,4,2,0,0,4,1,1,1,1 4,0,0,0,4,2,0,2,4,2,0,2,4,1,1,1,1	5	6
4	5,0,6,0,0,5,0,2,6,2,5,2,0,2,6,5,1,1,1,1,6 6,1,1,1,1,5,6,2,0,2,5,2,6,2,0,5,0,0,6,0,5	5	2
5	6,3,7,0,3,0,6,3,0,7,3,2,6,2,0,2,7,2,6,1,1,1,1,7 7,1,1,1,1,6,3,7,0,3,0,6,3,0,7,3,2,6,2,0,2,7,2,6 3,7,0,3,0,6,3,0,7,3,2,6,2,0,2,7,2,6,1,1,1,1,7,6 2,7,2,0,2,6,2,3,7,0,3,6,0,3,0,7,3,6,1,1,1,1,7,6	4	8
6	6,8,1,1,1,1,6,3,0,8,3,4,6,3,0,4,3,8,6,4,2,0,2,4,2,8,2 2,8,2,4,2,0,2,4,6,8,3,4,0,3,6,4,3,8,0,3,6,1,1,1,1,8,6	3	2
7	4,5,8,2,4,2,5,2,4,2,8,5,4,6,3,0,5,3,8,6,3,0,0,3,0,6,8,1,1,1,1,6 2,4,2,8,2,4,2,5,6,4,0,8,5,4,6,3,0,5,3,8,6,3,5,0,3,0,6,8,1,1,1,1 6,1,1,1,1,8,6,0,3,0,5,3,6,8,3,5,0,3,6,4,5,8,0,4,2,5,2,4,2,8,2,4 1,1,1,1,2,4,2,8,2,4,2,5,6,4,0,8,5,4,6,3,0,5,3,8,6,3,5,0,3,0,3,6,8	4	12
8	8,10,1,1,1,1,4,6,8,0,4,10,5,6,4,0,8,5,4,6,3,10,5,3,8,6,3,5,2,3,2,10,2,0,2 2,0,2,10,2,3,2,5,3,6,8,3,5,10,3,6,4,5,8,0,4,6,5,10,4,0,8,6,4,1,1,1,1,10,8	3	6
9	10,11,12,5,1,1,1,1,5,6,10,0,11,5,12,6,3,0,5,3,10,6,3,11,4,3,12,2,4,2,11,2,4,2,12 12,0,4,2,11,2,4,2,10,2,4,6,12,3,4,11,3,6,10,3,5,0,3,6,12,5,11,0,10,6,5,1,1,1,1,5,12,11,10	3	10
10	14,5,1,1,1,1,5,6,9,0,10,5,0,6,14,8,5,9,3,6,10,3,0,8,3,6,9,3,14,4,10,8, 2,4,2,9,2,4,2,8,10,4,14. 11,2,4,2,8,2,4,2,9,7,4,11,8,6,4,3,7,9,3,6,8,3,11,7,3,6,9,5,8,0,7,6,5,11, 0,9,0,5,1,1,1,1,5. 11,2,4,2,0,2,4,2,9,7,4,11,8,6,4,3,7,9,3,6,8,3,11,7,3,6,9,5,8,0, 7,6,5,11,0,9,8,5,1,1,1,1,5.	3	6
11	11,13,14,7,1,1,1,1,5,0,7,11,8,5,13,0,14,7,5,6,8,0,11,5,7,6,3,13,8,3,14,6, 3,11,4,3,8,6,4,2,13,2,4,2,14,2,4. 13,1,1,1,1,6,3,7,8,3,12,6,3,13,7,3,8,6,9,0,5,7,12,6,8,5,13,9,7,4,5,0,8,4,12, 5,9,4,2,13,2,4,2,0,2,9,12.	3	30
12	13,5,10,14,0,0,5,1,1,1,1,5,10,13,7,4,5,14,6,4,9,7,10,4,6,8,13,4,7,9,6,14,10, 8,3,7,6,3,9,13,3,8,2,3,2,14,2,9,2,8	2	2
13	2,6,2,0,2,15,2,6,1,1,1,1,11,6,8,13,5,10,7,6,15,5,,10,7,6,15,5,11,9,7,5,10,13, 0,8,5,7,9,11,15,4,10,8,7,4,13,9,3,4,11,3,10,4,3,15,9,3,0,13. 10,1,1,1,1,12,16,4,0,14,10,4,2,0,2,4,2,12,2,4,10,5,16,14,8,9,5,6,7,12,10,5,8,6, 9,7,5,14,16,6,8,12,7,9,3,6,0,3,8,7,3,14,9,3,16.	3	10
14	17,13,14,6,7,1,1,1,1,6,9,7,10,11,13,6,14,17,7,9,3,6,10,3,11,7,3,13,9,3,14,8,10, 5,17,11,0,9,5,8,13,4,10,5,14,4,11,8,5,4,2,17,2,4,2,8,2. 17,9,14,6,7,1,1,1,1,6,9,7,10,11,13,6,14,17,7,9,3,6,10,3,11,7,3,13,9,3,14,8,10, 5,17,11,0,4,5,8,13,4,10,5,14,4,11,8,5,4,2,17,2,13,2,8,2.	1	4
15	4,10,8,19,4,15,11,6,4,7,8,10,4,6,13,14,7,11,8,6,15,10,19,7,9,6,8,13,11,14,7,10,2,9, 2,15,2,0,2,11,13,19,9,14,5,1,1,1,1,5,15,9,3,13,5,3,0,14,3,5,19,3. 8,10,17,2,18,2,4,2,8,2,4,10,0,16,4,5,8,11,4,17,5,10,18,12,8,5,6,7,11,16,5,10,6,9,7,12,17, 0,6,11,18,7,9,3,6,16,3,12,7,3,11,9,3,17,1,1,1,1,18,12,9,16	2	6
16	23,13,16,17,3,10,5,3,20,0,3,5,0,3,13,10,5,14,16,11,17,5,7,23,9,10,6,13,20,7,11,14,6,9, 16,10,7,17,6,8,13,11,9,7,6,14,23,8,20,4,16,9,11,4,17,8,2,4,2,14,2,4,2,8,1,1,1,1,20,23. 18,2,21,2,12,2,13,2,9,6,1,1,1,1,16,6,12,9,18,13,11,6,15,21,5,8,9,6,12,5,16,11,13,8,5,9, 18,15,10,5,12,8,11,7,21,13,16,4,10,8,7,4,15,11,18,4,0,7,10,4,3,0,16,3,7,21,3,15,10,3.	2	12

Table 5.2: (4,n)-(extended) starter sequences with the minimum number of zeros.

Chapter 6

Conclusion and Further Research

In this thesis, new families of the starter sequences were introduced, which will be useful in combinatorial designs and graph theory. Moreover, all the necessary conditions and some of the sufficient conditions of their existence were determined. We introduced starter-labelled graphs, and we provided the conditions for the existence of the minimum hooked starter-labeling of paths, cycles, and k-windmills. We also introduced pseudo-starter sequences, which is a generalization of pseudo-Skolem sequences. Furthermore, we provided some of the conditions for the existence of several types of pseudo-starter sequences. We also introduced generalized (extended) starter sequences, we determined the necessary conditions for their existence, and we determined, with few possible exceptions, the minimum number of zeros and their permissible locations for the existence of generalized (extended) starter sequences.

In future work, we can introduce another generalization of a starter sequence, called near-starter sequences. It is not hard to determine their necessary conditions by using the same technique that we have used in this thesis.

Definition 6.0.1. *An m-near-starter sequence of order n is a sequence of $2n-2$ positive integers m-near-$SS_n = (s_1, s_2, \ldots, s_{2n-2})$ such that, for every positive integer*

$i \in [1, n] \setminus \{m\}$, either i or i^{-1} appears exactly twice in the sequence m-near-SS_n, and if $s_a = s_b = i$ or i^{-1}, then $|b - a| = i$ or i^{-1}, respectively, such that i^{-1} is the additive inverse of i in $\mathbb{Z}_{2n+1}$, where i^{-1} is referred to as a defect of the sequence, and m is the missing point.

For example, the sequence $(9, 6, 2, 5, 2, 1, 1, 6, 5, 9)$ is a 3-near-starter sequence of order 6 with one defect (4^{-1}) where the missing point $(m = 3)$.

Theorem 6.0.1. *There exists an m-near-starter sequence of order n with one defect only if one of the following condition holds:*

1. $n \equiv 0, 1 \pmod 4$ *and m is even, or*
2. $n \equiv 2, 3 \pmod 4$ *and m is odd.*

Definition 6.0.2. *A k-extended m-near-starter sequence of order n is a sequence m-near-$SS_n(k) = (s_1, s_2, \ldots, s_{2n-1})$ satisfying the conditions of near-starter sequences, with the additional condition that it contains exactly one empty position, which is in position k, denoted by $*$ or 0.*

For example, the sequence $(12, 6, 4, 2, 7, 2, 4, 6, 0, 1, 1, 7, 12)$ is a 9-extended-near-starter sequence of order 7 with one defect (3^{-1}) and $m = 5$.

Theorem 6.0.2. *There exists a k-extended m-near-starter sequence of order n with one defect only if one of the following condition holds:*

1. $n \equiv 0, 1 \pmod 4$, *and m and k are of opposite parity, or*
2. $n \equiv 2, 3 \pmod 4$, *and m and k are of the same parity.*

Some open problems include:

1. Determining the sufficient conditions for the existence of near-starter sequences and extended near-starter sequences.

2. Proving that the necessary conditions are sufficient for the existence of perfect, hooked, and extended starter sequences with m defects for all the remaining admissible defects, where $m > 2$.

3. Determining the sufficient conditions for the existence of excess starter sequences.

4. Determining the sufficient conditions for the existence of pseudo-starter sequences for all admissible defects.

5. Determining the sufficient conditions for labelling classes of hexagonal chains by using starter sequences for all admissible defects.

6. Finding some applications of pseudo-starter sequences.

7. Determining the sufficient conditions for the existence of generalized perfect, extended, and near-starter sequences.

8. Finding some applications of the generalized perfect, extended, and near-starter sequences.

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