Analysis of Scale Effect in Ice Flexural Strength

By

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ABSTRACT

Ice flexural strength is an important parameter in the assessment of ice loads on the hulls of ice-class ships, sloped offshore structures, or sloped bridge piers. While scale effects in compressive ice strength are well known, scale effects in ice flexural strength are not proven. To investigate scale effects during flexural failure of both freshwater and saline ice, a comprehensive up-to-date database of beam flexural strength measurements has been compiled. The database includes 2073 freshwater ice beam tests between 0.00016 to 2.197 m^3 volumes, and 2843 sea-ice beam tests between 0.00048 to 59.87 m^3 volumes. The data show a considerable decrease in flexural strength as the specimen size increases, when examined over a large range of scales. Empirical models of freshwater ice flexural strength as a function of beam volume, and of saline ice as function of beam and brine volumes have been developed using regression analysis. For freshwater ice, the scale-dependent flexural strength is given as: $\sigma_f = 839(V/V_1)^{-0.13}$. For sea ice, the dependence of flexural strength is embedded in: $\sigma = 1324\,(\frac{V}{V_1})^{-0.054} e^{-4.969\sqrt{v_b}}$. To facilitate probabilistic ice load modeling an analysis of the residuals was completed, and probability distributions were fitted to these data. These results have important implications for design, since scale effects can result in significantly lower strength for large-scale interactions as compared to strength values reported for small laboratory specimens.

KEY WORDS

Sea Ice, Freshwater Ice, Flexural Strength; Scale Effects

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NOMENCLATURE

$\sigma_{flexural}$	Flexural strength
F	Peak load at failure
L	Beam length
b	Beam width
h	Beam thickness
V_1	Reference volume
c	Distance from the loading pin to the end support
T_i	Strength of the *i*th element
R	Strength of the whole beam
r	Strength of an element
$F_T(t)$	Distribution function $F_T(t)$ for each element
$F_R(r)$	Failure probability of the beam
n	Number of elements
v	Beam volume
v_0	Volume of each element
$m(r)$	Weibull material function
r_1	Weibull distribution scale parameters
r_0	Lower limit for strength.
α	Weibull distribution shape parameter
x	General three-parameter strength of an element
γ	General three-parameter lower limit for strength (location parameter)
β	General three-parameter Weibull distribution scale parameters
v^*	Reduced volume
ΔVi	Elemental volume
$\sigma(x_i)$	Elemental stress
x_i	Center coordinates of the element
$\phi(x_i)$	Elemental stress as function of position

y	Distance from the neutral axis
M	Moment's magnitude
V	Beam volume
V_1	Reference volume
v_b	Brine volume
$\sigma_{Normalized}$	Normalized Strength
$\sigma_{calculated,v_b}$	Strength calculated at measured brine
$\sigma_{Ref_{v_b,Ref}}$	Strength at reference brine volume
$v_{b,Ref}$	Brine volume at reference ice conditions
r	Residuals
T	Temperature
S	Salinity

1 Introduction

1.1 Overview

For ships and offshore structures operating in ice environments, ice loads are a dominant consideration for design (Gudmestad et al., 2007). Ice loads depend on the failure mode of the ice, which can include crushing, bending, buckling, or mixed mode. Ice flexural strength is an important parameter in the assessment of ice loads on the hulls of ice-class ships, offshore structures with sloped water line geometry, bridge piers, or lighthouses. Moreover, flexural strength is essential in the study of ice ridging and rafting phenomena, and for calculating the bearing capacity of ice cover, which is critical in the design of winter roads, as well as other on-ice operations.

Ships or structures that break the ice in bending typically experience much lower loads than what are associated with other types of failure, such as crushing. Vaudrey (1983) estimated that the flexural strength of ice is around 10% to 50% of its compressive strength. This highlights the opportunity to take advantage of flexural strength in design, and reinforces the need for more investigation of ice flexural strength in general.

1.2 Purpose

Ice, as a geophysical material, contains many flaws and cracks, so it is expected that the probability of encountering such flaws increases with increasing specimen size, which would cause a decrease in strength (Sanderson, 1988; Tozawa and Taguchi, 1986). For example, in the case of compressive ice strength there is strong evidence supporting why such scale effects would be expected (*e.g.* Taylor and Jordaan, 2015; Jordaan et al., 2012; Sanderson, 1988).

While ISO19906 does recommend that full-scale tests be carried out if possible and it does acknowledged that scale effect in flexural strength should be considered, this standard does not provide an equation for flexural strength that accounts for scale. Similarly scale effects for flexural strength are not currently considered in the International Association of Classification Societies (IACS) Polar Class ship rules. The practical implications of accounting for such scale effects are important. For example, Williams and Parsons (1994) suggested that the flexural strength encountered by a specific icebreaker or offshore platform when failing in bending in reality is around 50% of the measured flexural strength from small-scale beam tests. The last extensive study on the subject was carried out by Williams and Parson (1994) and since then a wealth of new data have been collected or made public. All of these factors highlight the need for a more updated investigation of scale effects in ice flexural strength, which is the goal of this thesis.

The main objective of this thesis is to investigate and understand the effects of scale on ice flexural strength for both freshwater ice and sea ice. The effects of other important parameters such as temperature and brine volume on flexural strength are also explored. The scope of this research can be categorized as follows:

- Review theoretical and statistical theories for ice flexural strength, and measurement methodologies used in laboratory and field testing.

- Review relevant literature and previous studies on scale effects in ice mechanics.
- Compile a comprehensive up-to-date database of ice beam flexural strength measurements for freshwater ice and sea ice.
- Examine the effect of beam size on ice flexural strength measurements collected.
- Develop an empirical model for ice flexural strength that can be used in offshore structures and ice-class ship design applications for freshwater ice and sea ice.
- Investigate the influence of other important parameters on flexural strength such as brine volume and temperature.
- Propose conclusions and recommendations for future research.

1.3 Outline of thesis

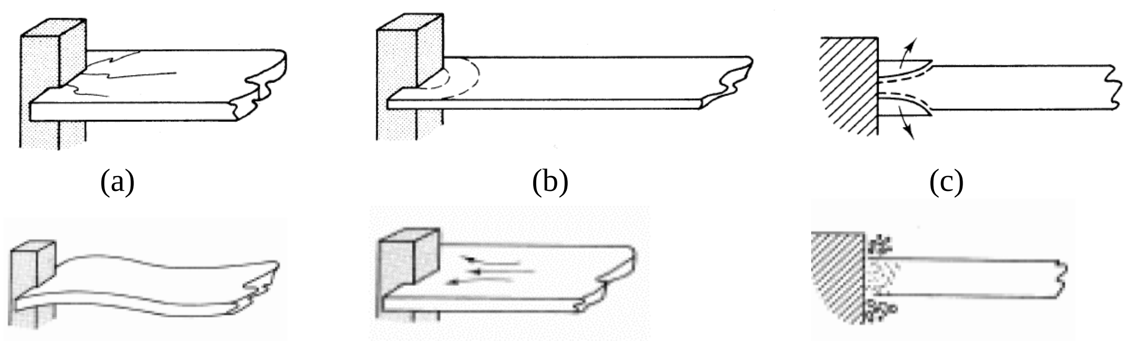
In Chapter 2, a literature review was conducted on ice flexural strength and scale effects associated with it. In Chapter 3, the database of freshwater ice flexural strength measurements is introduced, and the analysis that has been done for scale effects in freshwater ice flexural strength is discussed. Similarly, Chapter 4 describes the database of sea-ice flexural strength measurements and the analysis of scale effects in sea-ice flexural strength. Finally, Chapter 5 includes a discussion, summary of main conclusions and future research ideas.

2 Literature review

In this chapter, the available literature on ice flexural strength and scale effects associated with it, are reviewed. In Section 2.1, ice failure modes are defined, emphasizing flexural loading. Ice flexural strength testing methodologies are reviewed (Section 2.2). The theoretical basis of scale effects in ice mechanics are discussed, and then previous studies that have been carried out to investigate scale effects in flexural strength are reviewed in detail (Section 2.3). Finally, a summary of the chapter is given in Section 2.4.

2.1 Background

Ice can fail in different modes when it interacts with structures: creep, buckling, crushing, spalling, radial cracking, circumferential cracking, or mixed mode (Figure 2.1).



(a) (b) (c)

(d) (e) (f)

Figure 2.1. (a) Radial cracking (b) Circumferential cracking (c) Spalling
(d) Buckling (e) Creep (f) Crushing (Sanderson, 1988).

If the ice is moving slowly, creep loading takes place (Figure 2.1e). This usually happens when land fast ice is subjected to thermal and/or wind stresses (Palmer and Croasdale, 2013). If the ice is thin, it buckles (Figure 2.1d) due to eccentricities in ice loads because of irregularities in ice shape or thickness (Taylor, 2010). When thicker ice meets a vertical-walled structure, it experiences compressive ice failure. Then, it may crush (Figure 2.1f) generating fine-grained particles or spalls due to local edge fractures that run to the top and bottom surfaces (Sanderson, 1988). Also, radial cracking can happen, fracturing the ice floe into pieces (Figure 2.1a).

Circumferential cracks (Figure 2.1b) usually happen when ice is interacting with sloped walled structures, particularly after radial cracks initiate which divides the sheet into segments that fail more easily in flexure. Flexural strength is defined as the ice strength capacity, when the failure mode is bending (Ervik, 2013). It is an important design input for inclined faced structures and ice-class ship design. However, flexural strength tests are indirect because it is not accounting for all factors, and the effects of different conditions such ice and test conditions on flexural strength should be considered.

2.2 Flexural Strength Measurements

Due to the variability in ice associated with variation in distributions of flaw size, temperature, brine pockets and channels and test conditions, flexural strength tests data should be analyzed using a statistical approach. Ideally, the number of repetitions of a certain test should be chosen to get a high confidence level. However, tests are often costly and time consuming to conduct, and there are practical limits to the number of repeats that can be done. Adding data, particularly field-scale helps our understanding of ice flexural failure. For flexural strength testing, some considerations should be taken into account; the beam length should be between 7 to 10 times the ice thicknesses. The beam width should be between 1 to 2 times the ice thickness, and for freshwater ice, should be at least 10 times the ice crystals' diameter in order to avoid the grain size effect (Schwarz et al., 1981). These recommendations are to avoid shear effects in short beams, and plate behavior in long beams, where the beam will have biaxial stresses or rotation around the root for cantilever tests (Frederking and Häusler, 1978; Lavrov, 1971). In addition, the loading rate should be high enough to allow the beam to deform elastically (Tatinclaux and Hirayama, 1982).

Flexural strength is usually calculated using simple beam theory. The main disadvantages of using this theory, are the assumptions associated with it. First, plane sections are assumed to remain plane. Second, deflections are very small compared to the beam thickness. Third, linear elastic behavior is assumed (Schwarz and Weeks, 1977). Fourth, ice is assumed homogeneous and isotropic. In addition, the loading is assumed to remain quasi-static. All are assumed to simplify the calculation. However, ice properties may vary significantly across the thickness of the ice cover (Ervik, 2013). Ice is in fact an inhomogeneous, anisotropic and viscoelastic material (Schwarz and Weeks, 1977). In addition, for anisotropic materials, shear deformations should be accounted for, which to date has not been accounted for by researchers (Lainey and Tinawi, 1981). What is more, existing flaws and air inclusions are not inherent in simple beam theory (ITTC, 2014).

In simple beam theory, the neutral axis is assumed to be located at the center of the specimen, but in reality, it is shifted to the compressed side (ITTC, 2014; Schwarz, 1975).

The neutral axis shifts to the stiffer side in order to make the tension and compression forces equal, due to ice, like many brittle materials, not having the same properties in tension and compression. Moreover, flexural strength tests usually cause non-uniform stress fields over the depth of the ice sample, which is not taken into consideration (Timco and Weeks, 2010; Schwarz and Weeks, 1977). Furthermore, tests usually do not cause a constant bending moment along the whole beam length (Lainey and Tinawi, 1981).

However, despite these complexities and all of the simplifying assumptions, the load versus deflection curves for flexural strength tests are typically linear (see Figure 2.2) which suggests the assumptions are sufficiently valid to permit the use of simple beam theory (Tatinclaux and Wu, 1978; Schwarz, 1975). Therefore, the results of these tests will be good approximate values that can be used as an index of strength (Gow, 1977). Nonetheless, there are many factors that influence the flexural strength of ice and as such, considerable variation is expected in measurements, and analysis of the data using statistical methods to account for such variability is recommended.

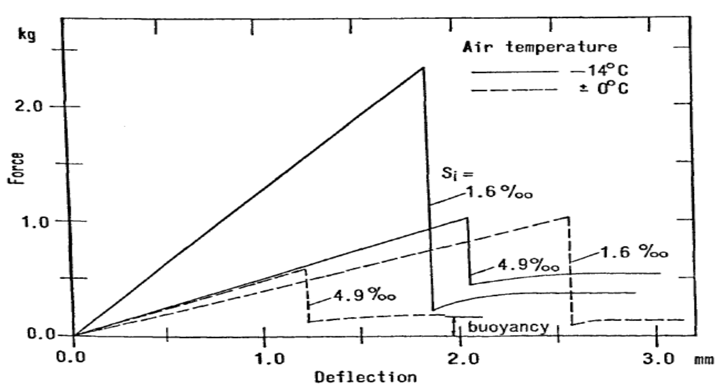


Figure 2.2. Force-deflection curve for saline ice (Schwarz, 1975).

The main approaches that have been used to measure ice flexural strength are cantilever beam, three-point and four-point bending tests; these are discussed in detail below.

2.2.1 Cantilever beam tests

Cantilever tests are usually done *in situ*, and are easy to perform on large beams. Like all *in situ* tests, they have the advantage of maintaining the temperature gradient and variation through the thickness of ice cover by utilizing its full thickness (Ji et al., 2011; Blanchet et al., 1997). The general technique for obtaining ice flexural strength using cantilever tests is as follows: First a U-shaped channel is cut in the ice. This channel isolates an in-place cantilever ice beam with one end attached to the sheet. Both pull-up and push-down tests can be performed on these beams using a vertical load applied to the free end of the ice beam until it fails; see Figure 2.3.

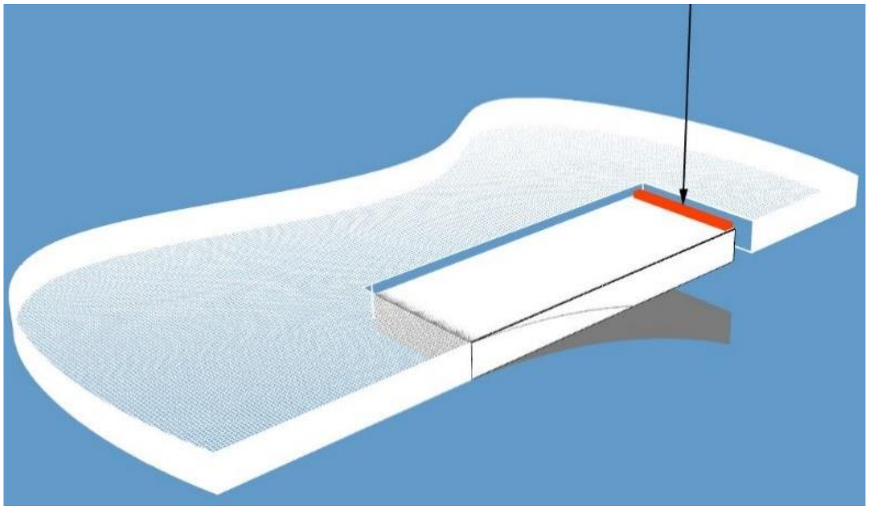


Figure 2.3. Cantilever beam test.

As discussed in more detail later, an important consideration for cantilever tests, is whether or not the beam fails at the root due to stress concentrations. This results in lower strength values than typically obtained in three- and four-point tests (Timco and O'Brien, 1994; Frederking and Häusler, 1978). This effect is more pronounced in freshwater ice, as it is more brittle than sea ice (Timco and O'Brien, 1994). To minimize stress concentrations, circular cuts should be made at the root of the beam. The radius of these circles is suggested to be 1/15th of the beam width (Schwarz et al., 1981). Svec et al.(1985) suggested relieving the stress concentration by drilling holes of a similar radius as a better solution.

For cantilever tests, the flexural strength is calculated using simple elastic beam theory,

$$\sigma_f = \frac{6FL}{bh^2} , \tag{2.1}$$

where F is the maximum force required to break the beam, L is the beam length, b is the beam width, h is the beam thickness. Shear force and bending moment diagrams of cantilever beam tests are shown in Figure 2.4.

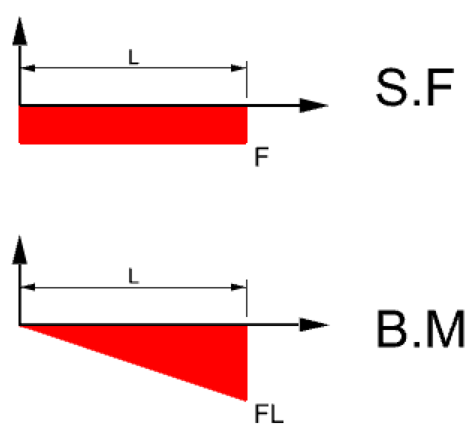


Figure 2.4. Cantilever beam shear force and bending moment diagrams.

2.2.2 Three- and four-point bending tests

For three- and four-point bending, the ice beam is completely cut free from the ice sheet. The ends of this beam are supported, and load is applied at the center in case of three-point bending, and at two equidistant points in case of four-point bending, as is shown in Figure 2.5 and Figure 2.6, respectively.

For three-point bending tests, the flexural strength is calculated as,

$$\sigma_f = \frac{3FL}{2bh^2} \ . \tag{2.2}$$

Shear force and bending moment diagrams of three-point bending beam tests are shown in Figure 2.7.

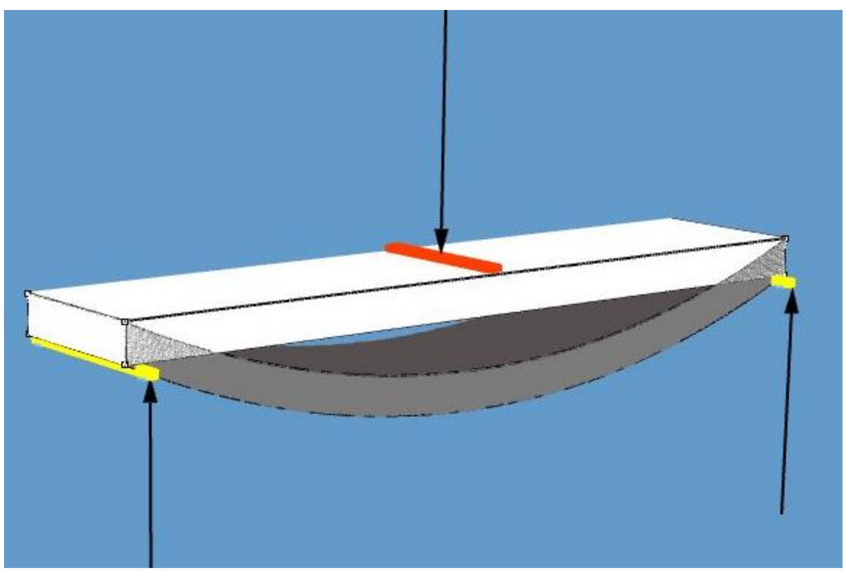


Figure 2.5. Three-point bending test.

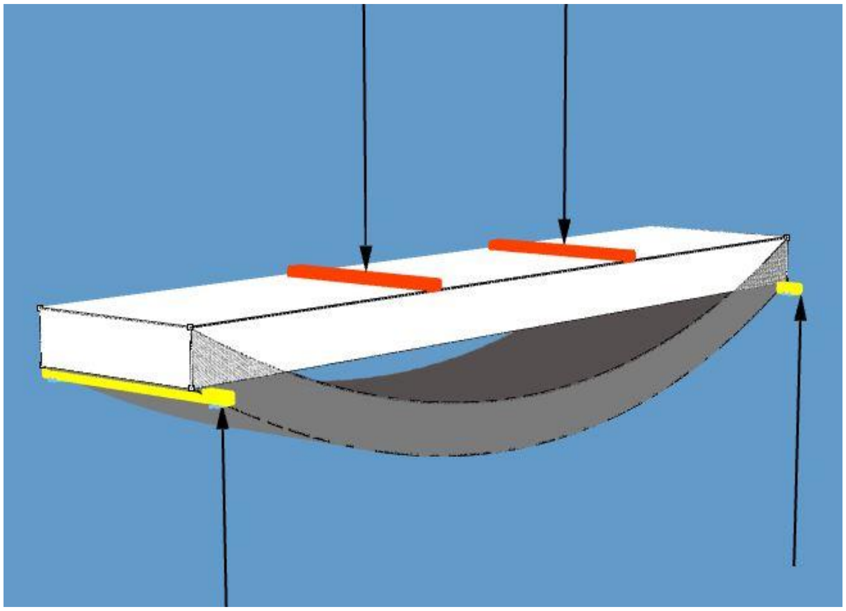


Figure 2.6. Four-point bending beam test.

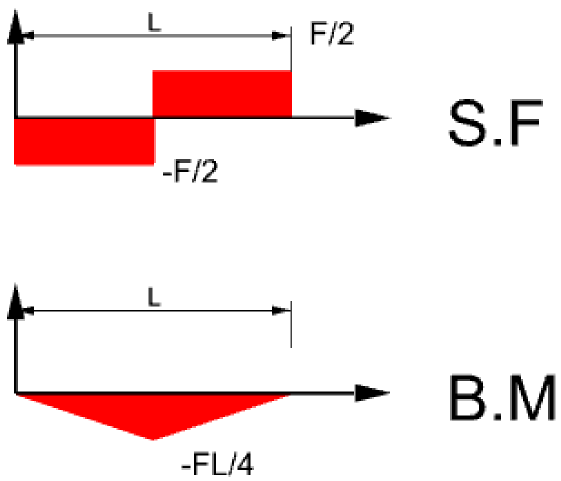


Figure 2.7. Three-point bending tests shear force and bending moment diagrams.

For four-point bending tests, the flexural strength is calculated as,

$$\sigma_f = \frac{3Fc}{bh^2}\ , \tag{2.3}$$

where c is the distance from the loading pin to the end support. Shear force and bending moment diagrams of four-point bending beam tests are shown in Figure 2.8.

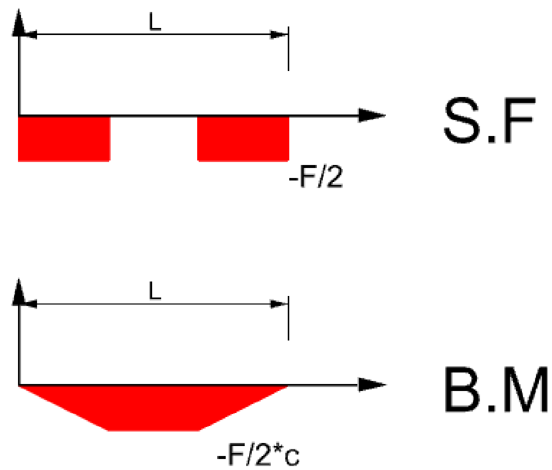


Figure 2.8. Four-point bending tests shear force and bending moment diagrams.

The disadvantage of three-point bending tests is that the beam usually fails at the center, where the maximum moment takes place, preventing the beam from failing at its weakest point. Four-point bending tests result in a large central region of constant moment and zero shear between the loading points allowing the beam to fail at its weakest point, which is generally recommended for brittle materials.

Local indentation effects at the loading and supporting points can be cause for concern in three- and four-point bending. The test apparatus should have round supports to avoid stress concentration or local indentations at these points (ITTC, 2014). The actual point for deflection measurements should be about 10 cm from the center of the beam for 3-point bending tests, to avoid local deformation effects (Gow, 1977).

2.3 Scale Effects in Ice Mechanics

2.3.1 Theoretical Basis for Scale Effects in Ice

Theoretical statistical and probabilistic theories of fracture have been applied to many other brittle materials, such as ceramics (Batdorf and Heinisch, 1978; Evans, 1978), glass (Reid, 1991) and concrete (Bažant, 1998; Mier, 1997) as discussed by (Taylor, 2010). Ice failure can be modeled by Weibull weakest-link theory, where the failure exhibited by a system is governed by the failure of its weakest element. The famously known Weibull three-parameter probability distribution is based on this theory. Parsons and Lal (1991) demonstrated the goodness of fit of Weibull distributions to sea-ice flexural strength data. They concluded this by examining Weibull fit for thirteen experimental datasets (Figure 2.9). Likewise, Tozawa and Taguchi (1986) got the same conclusion for freshwater ice. They conducted three-point bending tests for different specimen sizes, and then evaluated Weibull fit for the tests' results (Figure 2.10, Figure 2.11 and Figure 2.12).

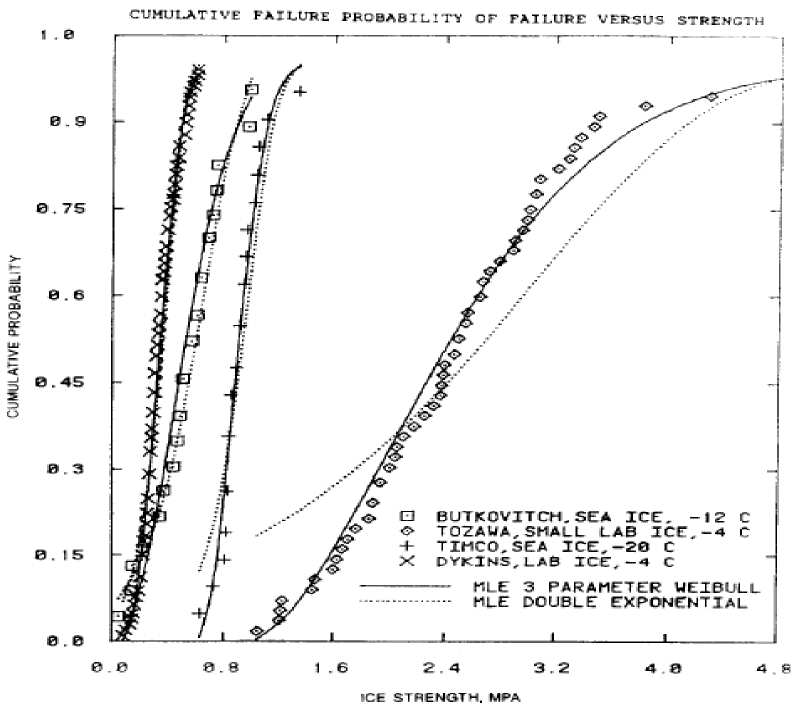


Figure 2.9. Weibull fit of four out of thirteen flexural strength experimental datasets (Parsons and Lal, 1991).

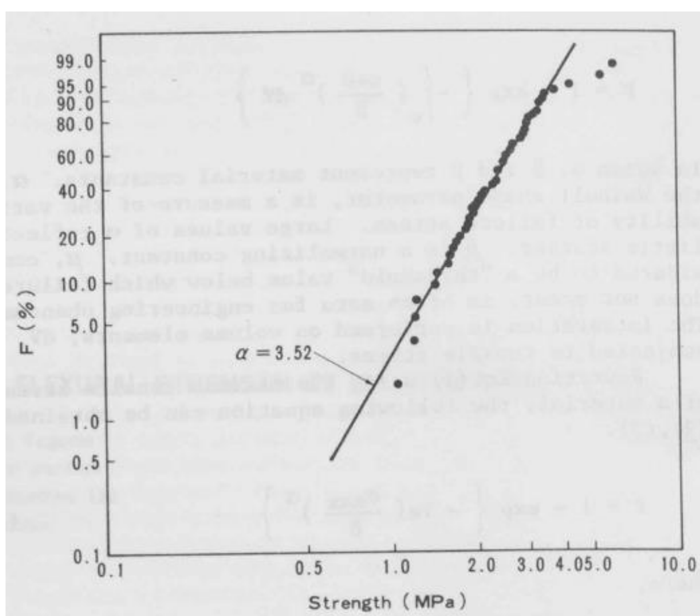


Figure 2.10. Weibull plot for small specimen size data (Tozawa and Taguchi, 1986).

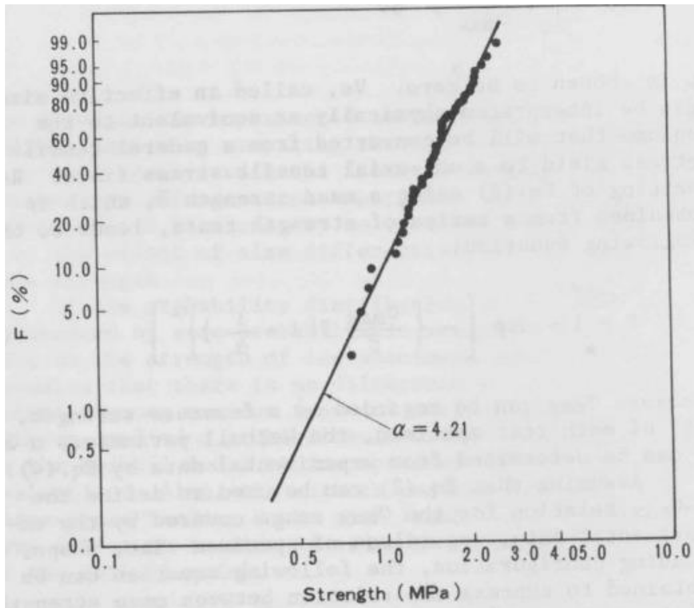


Figure 2.11. Weibull plot for medium specimen size data (Tozawa and Taguchi, 1986).

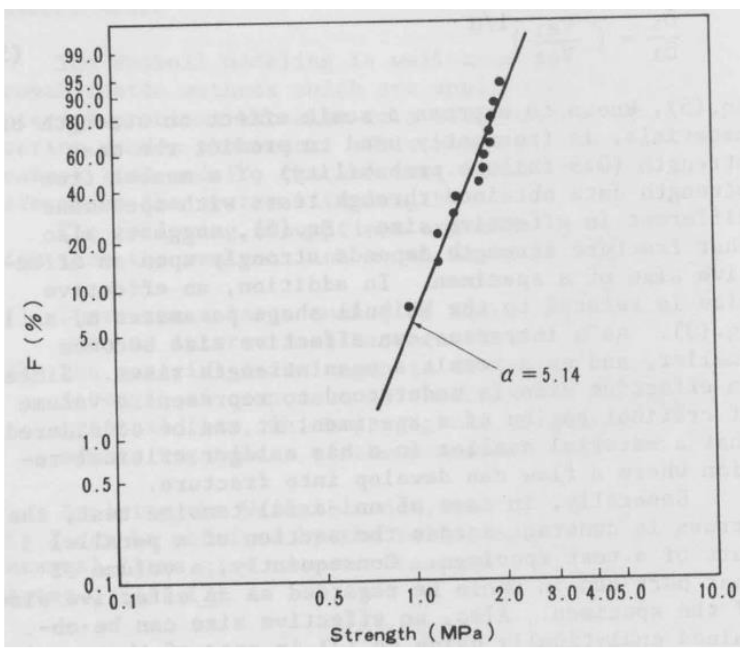


Figure 2.12. Weibull plot for large specimen size data (Tozawa and Taguchi, 1986).

Weibull assumes that the maximum capacity of the system is the minimum of system elements' capacities. Thus, when the demand increases, the system will not fail until the capacity of the limiting weakest element is exceeded. This can be interpreted as the distribution of the minimum of a set of random strengths of system elements (Taylor, 2010). Hence, when an ice beam is subjected to pressure, failure will not occur unless at some location, the stress (demand) exceeds ice strength (capacity of the system). If no failure occurs for a given pressure, the pressure will continue to increase until ice fails at some location (Figure 2.13).

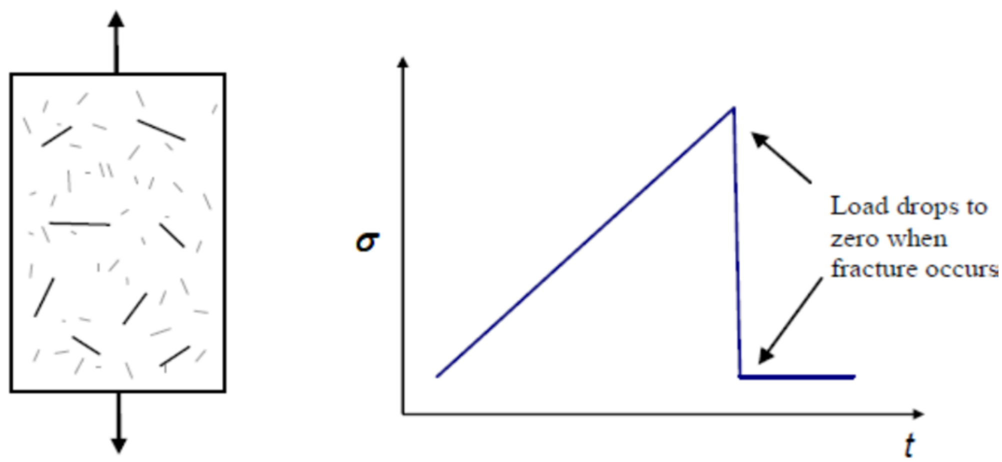


Figure 2.13. Specimen in tension; failure results in total loss of strength (Taylor and Jordaan, 2015).

Weibull weakest-link theory will be discussed below in the context of ice statistical fracture and failure modeling for both homogeneous, and inhomogeneous stress states where stress varies across the beam.

2.3.1.1 Weibull Theory and Associated Scale Effects

Jordaan (2005) described the Weibull (1951) weakest-link model as a chain of elements. If this chain is composed of a series of *n* elements, the chain will fail if one of its elements fails. If T_i is the strength of the *i*th element, and the strengths of the elements are independent and identically distributed (*iid*), which have a distribution function $F_T(t)$ for each element i = 1,2,3,.., n. Then, an ice beam fails when its weakest element fails. We denote the strength of the whole beam as R. Thus, R = min ($T_1, T_2, T_3, .., T_i, .., T_n$) or $R = min_n T_i$, and the failure probability of the beam $F_R(r)$ can be expressed by,

$$F_R(r) = 1 - [1 - F_T(r)]^n. \tag{2.4}$$

This also can be written using exponential and natural logarithm functions as,

$$F_R(r) = 1 - \exp\{n \ln[1 - F_T(r)]\}. \tag{2.5}$$

For an ice beam of volume V composed of elements each of volume ν_0, then n = ν/ν_0 elements, and Equation 2.5 can be written as,

$$F_R(r) = 1 - exp\left\{\frac{V}{\nu_0} \ln[1 - F_T(r)]\right\}. \tag{2.6}$$

Weibull suggested using a power-law material function $m(r)$, which is an empirical function to replace the term $\{-\ln[1 - F_T(r)]\}$ where,

$$m(r) = \left(\frac{r-r_0}{r_1}\right)^{\alpha}. \tag{2.7}$$

In this expression, α and r_1 are constants representing the distribution shape and scale parameters respectively. The constant r_0 represents the lower limit for ice strength. By substituting Equation 2.7 into Equation 2.6 we get,

$$F_R(r) = 1 - exp\left\{-\frac{V}{\nu_0}\left(\frac{r - r_0}{r_1}\right)^{\alpha}\right\}. \tag{2.8}$$

If this is compared with the standard general three-parameter Weibull distribution,

$$F_X(x) = 1 - exp\left\{-\left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right\}, \tag{2.9}$$

we get r $\equiv x$, $(\frac{v_0}{V})^{\frac{1}{\alpha}} r_1 \equiv \beta$ and $r_0 \equiv \gamma$. In most cases, r_0 is assumed to be zero since this is a natural limit for strength, which simplifies the expression for Equation 2.9 to,

$$F_R(r) = 1 - exp\left\{-\frac{V}{v_0}\left(\frac{r}{r_1}\right)^{\alpha}\right\}. \tag{2.10}$$

2.3.1.2 Inhomogeneous Stress State

Jordaan (2005) suggested a way to modify Weibull theory to account for inhomogeneous stress case. Taylor (2010) simplified this method, so an ice beam having an inhomogeneous state of stress is approximated to contain *n* homogeneously stressed elemental volumes ΔV_i, where *i*= 1,2,3,⋯, *n.* For the small volume ΔV_i, the center coordinates of the element are given by x_i, and the elements have stresses $\sigma(x_i)$ at positions x_i.

In order to represent the stress at each element $\sigma(x_i)$, we use the formula,

$$\sigma(x_i) = r\phi(x_i), \tag{2.11}$$

where *r* represents a reference value, usually the maximum value in the body. The parameter $\phi(x_i)$ is a function of position, which represents the variation of stress across the body due to inhomogeneity.

Using Equation 2.5 again, but replacing *r* by $\sigma(x_i)$, and the volume υ is divided into n small elements each with volume ΔV_i , where *i* =1,2,3,⋯, *n.*

Then the failure probability of the specimen is,

$$F_R(\sigma) = 1 - \exp\left[\frac{1}{\nu_0}\sum_{i=1}^{n}\left(\Delta V_i \ln\{1 - F_T\left(\sigma(x_i)\right)\right)\right], \tag{2.12}$$

and by substituting $\sigma(x_i) = r\phi(x_i)$,

$$F_R(\sigma) = 1 - \exp\left[\frac{1}{\nu_0}\sum_{i=1}^{n}\left(\Delta V_i \ln\{1 - F_T\left(r\phi(x_i)\right)\right)\right]. \tag{2.13}$$

If the sum is replaced by an integral, the expression will be,

$$F_R(r) = 1 - \exp\left[\frac{1}{\nu_0}\int_V \ln\left\{1 - F_T\left(r\phi(x_i)\right)\right\} d\nu\right]. \tag{2.14}$$

By using the power-law material function,

$$m(r) = \left(\frac{r\phi(x_i) - r_0}{r_1}\right)^{\alpha}, \tag{2.15}$$

suggested by Weibull instead of the term $\{-\ln[1 - F_T(\sigma(x_i))]\}$ as before, we get,

$$F_R(r) = 1 - exp\left[\frac{1}{\nu_0}\int_V \left(\frac{r\phi(x_i) - r_0}{r_1}\right)^{\alpha} d\nu\right]. \tag{2.16}$$

By simplification and setting $r_0 = 0$ as suggested before,

$$F_R(r) = 1 - exp\left[\frac{1}{\nu_0}\left(\frac{r}{r_1}\right)^{\alpha}\int_V \phi^{\alpha}(x_i)\, d\nu\right]. \tag{2.17}$$

The integral in this equation is called "**reduced volume**" and can be found by,

$$v^* = \int_V \phi^{\alpha}(x_i)\, d\nu. \tag{2.18}$$

Then,

$$F_R(r) = 1 - exp\left[\frac{\nu^*}{\nu_0}\left(\frac{r}{r_1}\right)^{\alpha}\right]. \tag{2.19}$$

The value ν_0 is a reference volume, such as that of a standard test specimen (Bolotin, 1969).

Equations 2.10 and 2.19 are the same except Equation 2.19 uses the 'reduced volume' concept ν^* due to the inhomogeneous stress state. Note that the 'reduced volume' is usually less than the total volume of the body, which means that only a portion of the body is subjected to tensile stress, hence the terminology.

Weibull theory accounts only for tensile strength, as it does not account for negative values of compression stress. This is because it is assumed that cracks only grow when subjected to tension. This is a problem for the study of compressive strength failure; however, Parsons and Lal (1991) suggest that this will not matter in case of flexural failure. This due to the fact that when a beam fails in flexure, the crack begins on the tensile surface of the beam, and then propagates causing failure. Therefore, the Weibull model is expected to provide good approximation of flexural strength.

Ice has many flaws and cracks, as the specimen size increases, the probability of encountering such flaws increases leading to a decrease in the strength (Sanderson. 1988; Tozawa and Taguchi, 1986). The statistical distribution of flaws and the probability of critical ones becoming unstable is the dominant factor in ice failure. Larger ice samples have a higher probability of containing critical flaws, so it is probable that these larger specimens would fail at lower stress levels as shown in Figure 2.14 (Taylor, 2010).

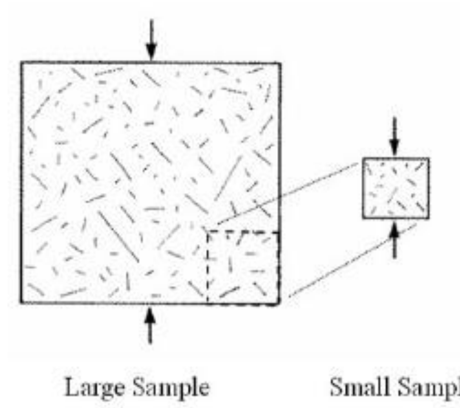


Figure 2.14. Critical flaws distributed through samples (Taylor, 2010).

From Weibull theory (Jordaan, 2005), the mean value of Equation 2.10 is,

$$\langle R\rangle = r_0 + r_1\left(\frac{V}{v_0}\right)^{\frac{-1}{\alpha}} \Gamma\left(1+\frac{1}{\alpha}\right), \tag{2.20}$$

where the lower limit value of strength r_0 equals zero, and $\Gamma()$ is the gamma function.

By dividing the means of strengths of the two volumes V_1 and V_2 as,

$$\frac{\langle R\rangle_1}{\langle R\rangle_2} = \frac{r_0+r_1\left(\frac{V_1}{v_0}\right)^{\frac{-1}{\alpha}} \Gamma(1+\frac{1}{\alpha})}{r_0+r_1\left(\frac{V_2}{v_0}\right)^{\frac{-1}{\alpha}} \Gamma(1+\frac{1}{\alpha})}\ , \tag{2.21}$$

then,

$$\frac{\langle R\rangle_1}{\langle R\rangle_2} = \left(\frac{V_2}{V_1}\right)^{\frac{1}{\alpha}}. \tag{2.22}$$

From the relation in (2.22), we can conclude that statistically, ice strength is inversely proportional to the power of beam volume . The same relation applies for reduced volumes.

Weibull scale effects model can be used to theoretically compare average strengths $\langle R \rangle$ of two volumes ν_1 and ν_2.

Tozawa and Taguchi (1986) confirmed the validity of Weibull model for describing scale effects in ice flexural strength. They found that the mean flexural strength values from three-point bending tests on freshwater ice specimens were in agreement with the ones predicted from Weibull for different beam volumes (see Figure 2.15). Jordaan et al. (2007) also fitted flexural strength datasets of freshwater ice in Figure 2.16, and found that Weibull scaling relationship gives a good estimation of flexural strength with varying beam volume.

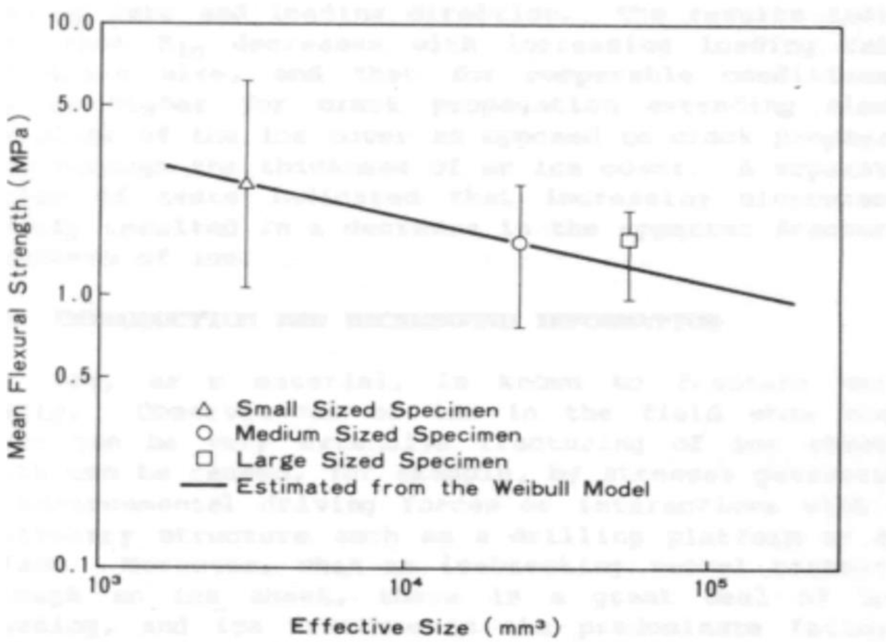


Figure 2.15. Comparing test results with Weibull model estimations (Tozawa and Taguchi, 1986).

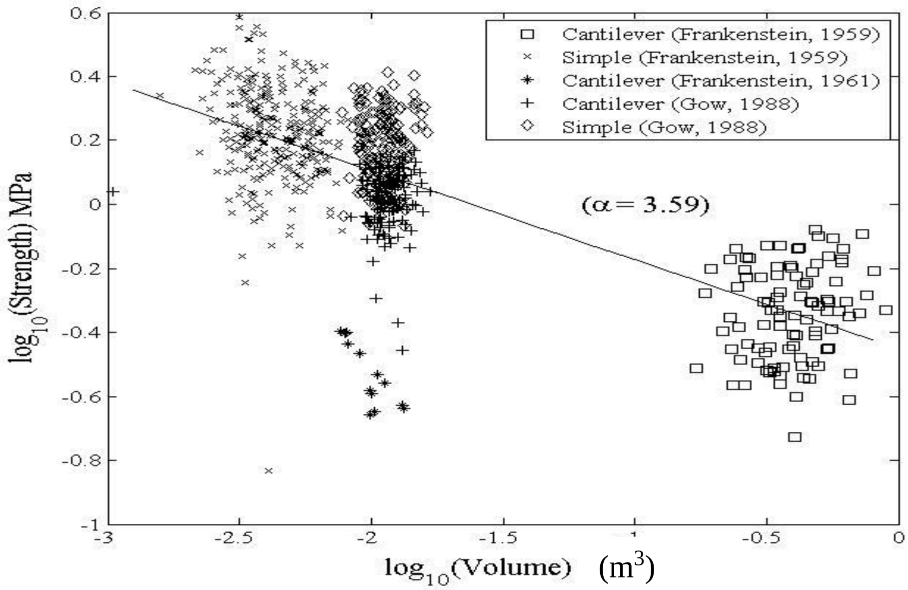


Figure 2.16. Scale effect in freshwater ice (Jordaan et al., 2007).

2.3.2 Scale Effect in Compressive Strength

For compressive ice strength there is general agreement on the existence of scale effects (*e.g.* Taylor and Jordaan, 2015; Jordaan et al., 2012; Sanderson, 1988). Where ice compressive ice is the limit stress, the estimation of ice loads depends on the presence of a scale effect. The average pressure on the structure decreases with increasing contact area (Sanderson, 1988); see Figure 2.17 and Figure 2.18. Measurements from the STRICE project in the Baltic Sea done by Kärnä and Qu (2006) in Figure 2.19, showed a decrease in average pressure with increasing ice thickness. For ice that is loaded in compression, the occurrence of fracture under shear (wing cracks) and under tension (lateral tension cracks) both contribute to local edge failure (Taylor, 2010). Consequently, local ice fracture processes are proven to be scale dependent, since larger beam volumes would be expected to contain a larger flaw and would fail at a lower nominal stress.

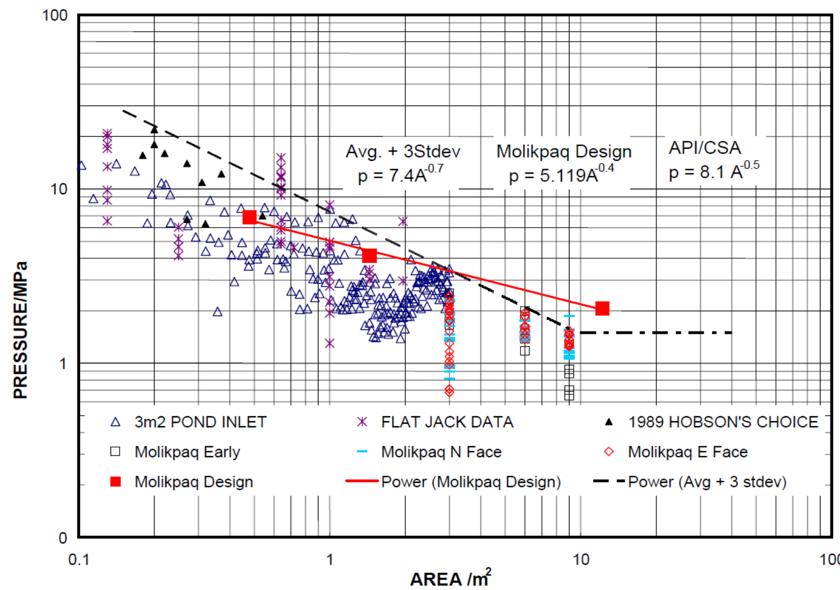


Figure 2.17. Ice pressure vs. area from combined data (Masterson et al., 2007).

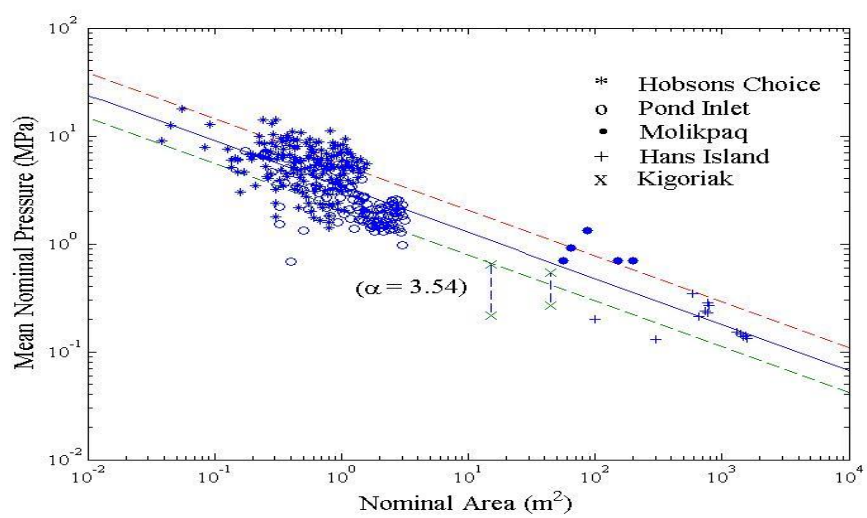


Figure 2.18. Weibull fit of compressive ice failure data showing scale effects (Taylor, 2010).

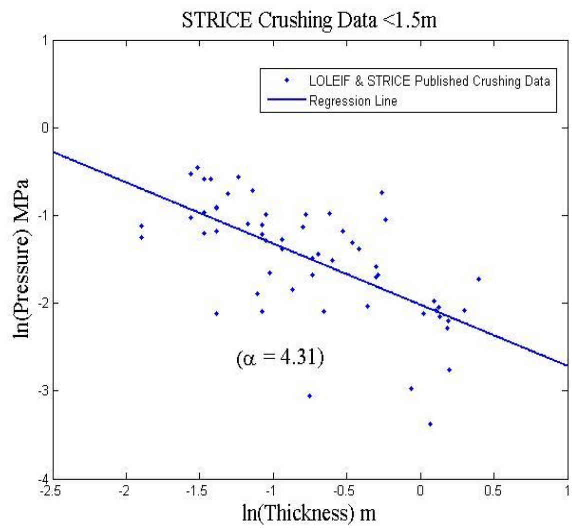


Figure 2.19. Data from the STRICE project (Jordaan et al., 2007).

Scale effects in ice compressive failure are attributed to two main reasons: ice fracture and probabilistic averaging. Ice fracture depends on the probability of encountering flaws and cracks which increases with increasing specimen size. On the other hand, probabilistic averaging happens where the local pressures on local areas are averaged out to lower global pressures (Taylor, 2010). For flexural strength, fracture mechanisms are expected to dominate scale effects in beam failure. This is because the localized failure processes which are responsible for probabilistic averaging are not typically present. In short, flexural strength is detrmined by first crack, where compressive strength is determined from local failure process. In the present analysis, the effects of localized compressive failure at the point of local application are assumed to be negligible for beam tests. No further consideration of compressive failure is given in this thesis. For full-scale scenarios, consideration of the non-simultaneous nature of point loads that occur at the ice-structure interface for ice failing in flexure against a sloped structure should also be considered.

2.3.3 Past Work on Scale Effect in Flexural Strength

Williams and Parsons (1994) found a clear trend of decreasing ice flexural strength with increasing specimen size for both sea ice and freshwater ice. They concluded for sea ice that, after brine volume, specimen size will have the second greatest influence on flexural strength. For freshwater ice, specimen size has the strongest influence. They based their conclusion on statistical correlation analyses for five ice flexural strength parameters: brine volume, beam volume, grain diameter, temperature, and strain rate. They implemented their analyses on a database compiled of 1771 sea ice and 650 freshwater flexural strength measurements.

In their analysis, the authors excluded all freshwater ice cantilever tests from their database due to the stress concentrations phenomena, which has been suggested to lower the flexural strength of ice through the presence of stress raisers at the root of the beam. Using regression analysis on their database, they developed an empirical two-parameter model of sea-ice flexural strength as a function of brine and beam volumes as follows,

$$\sigma_f = 1760(e^{-5.395}\sqrt{v_b})(\frac{V}{V_1})^{-0.057}\ , \tag{2.23}$$

(see Figure 2.20 and Figure 2.22). For freshwater ice, only beam volume was considered as the main factor controlling flexural strength,

$$\sigma_f = 1629(\frac{V}{V_1})^{-0.084}, \tag{2.24}$$

where σ_f is in kPa, V_1 is a reference volume (it was suggested to be 0.01 m^3), and V is the beam volume in m^3 (Figure 2.21 and Figure 2.22).

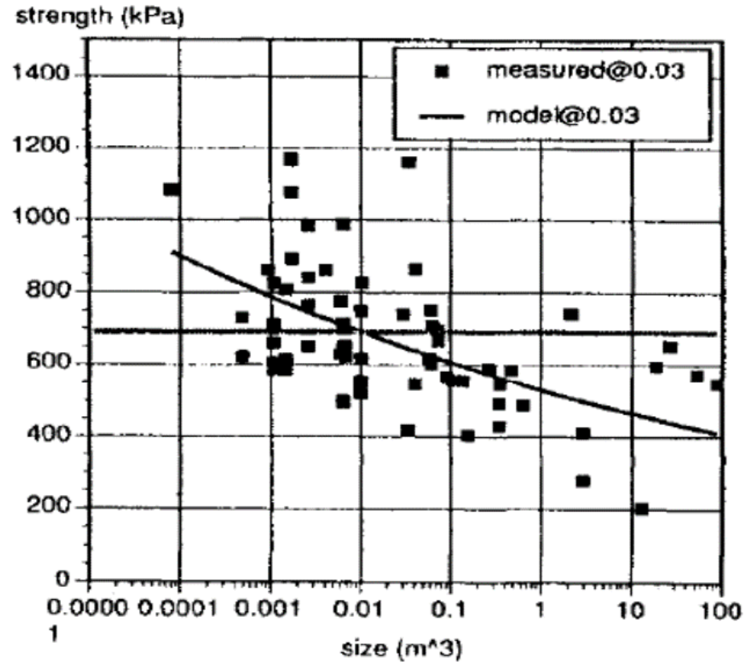


Figure 2.20. Measured and model flexural strength vs beam size for sea-ice beams near $v_b = 0.03$ (Williams and Parsons, 1994).

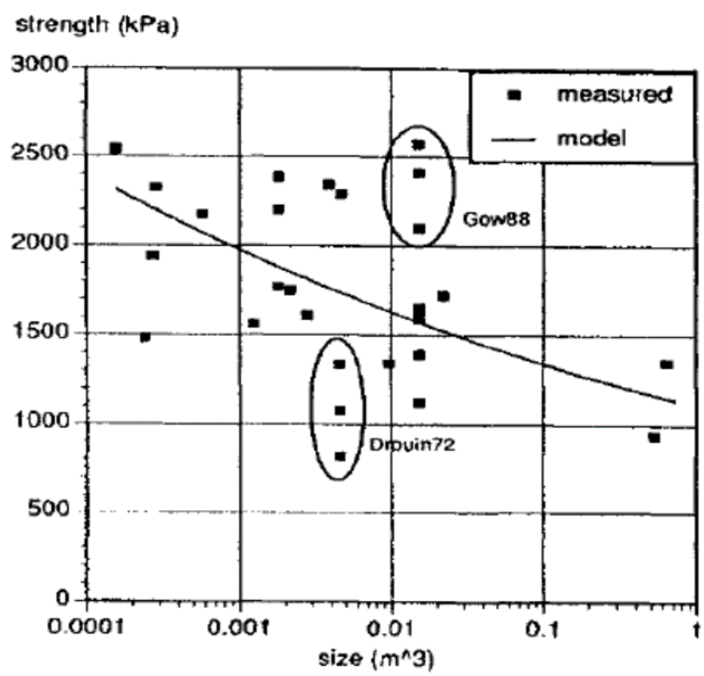


Figure 2.21. Measured and model flexural strength vs beam size for simple freshwater beams (Williams and Parsons, 1994).

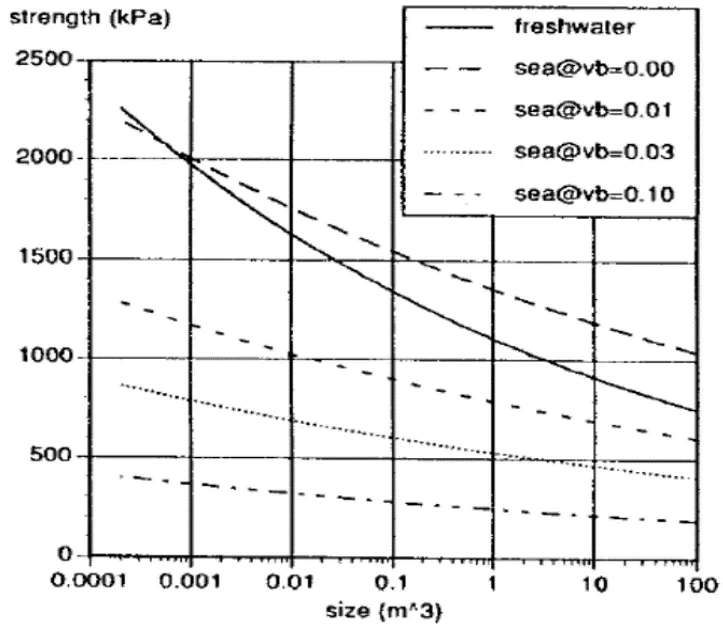


Figure 2.22. Flexural strength dependence on beam size for freshwater ice and sea ice of different brine volumes (Williams and Parsons, 1994).

Lau et al. (2001) added their data to that of Williams and Parsons (1994) for both sea and freshwater ice during a study on how to take scale effects in ice strength into consideration during centrifuge model testing; they came to the same conclusion. Lavrov (1971) also found from experiments that sea ice and freshwater ice flexural strength decreases as beam thickness increases. Maattanen (1975) attributed observed scale effect behavior to the stress field across the beam. He suggested that stress field converts to two dimensions as beam size increases. As a result, it will be easier for fracture to take place between ice crystals, causing the beam to fail under lower loads.

Frederking and Sudom (2013) also found that the flexural strength of multi-year sea ice decreases as the specimen size increases. They found this result by analyzing simple beam (three and four-point) test data for large and small beams cut from a multi-year sea-ice ridge. They also analyzed ship ram data that was taken when traversing through multi-year sea-ice floes, and found that flexural strength decreased as the thicknesses of the ice cover increased (see Figure 2.23 and Figure 2.24).

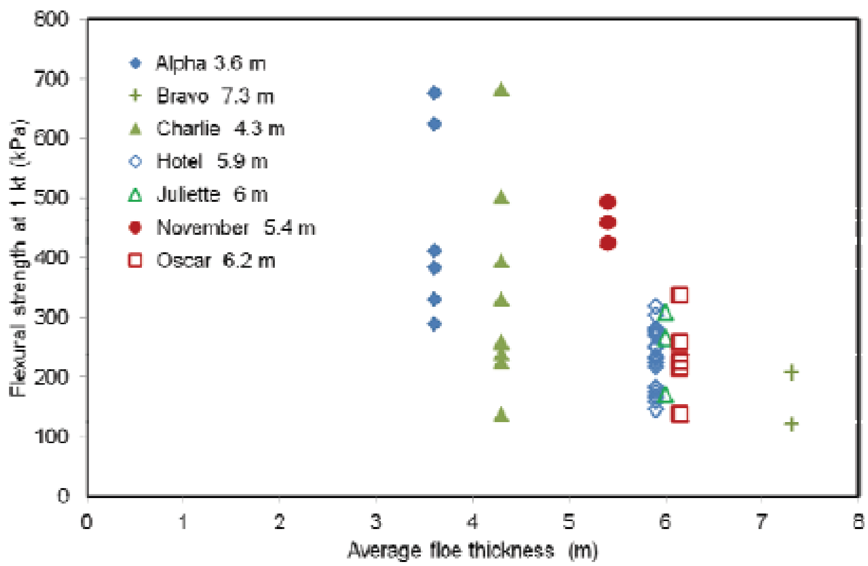


Figure 2.23. Flexural strength normalized to a 1 knot (kt) ramming speed as a function of floe thickness (Edge breaking mode) (Frederking and Sudom, 2013).

Several researchers disagree with, or have neglected the presence of a scale effect in the flexural failure of ice. Timco and O'Brien (1994) developed a correlation between flexural strength and brine volume using a database compiled of 2495 tests (1556 freshwater and 933 sea ice). They found that strength at times fluctuated by an order of magnitude for the same brine volume. For freshwater ice, their analysis showed strong scatter in the data at exact or close ice temperatures for the same test type. This suggests that there are other factors that strongly affect the flexural strength. However, they attributed that larger beams generally had lower strengths than smaller ice beams due to that large beam volumes contain larger brine volume.

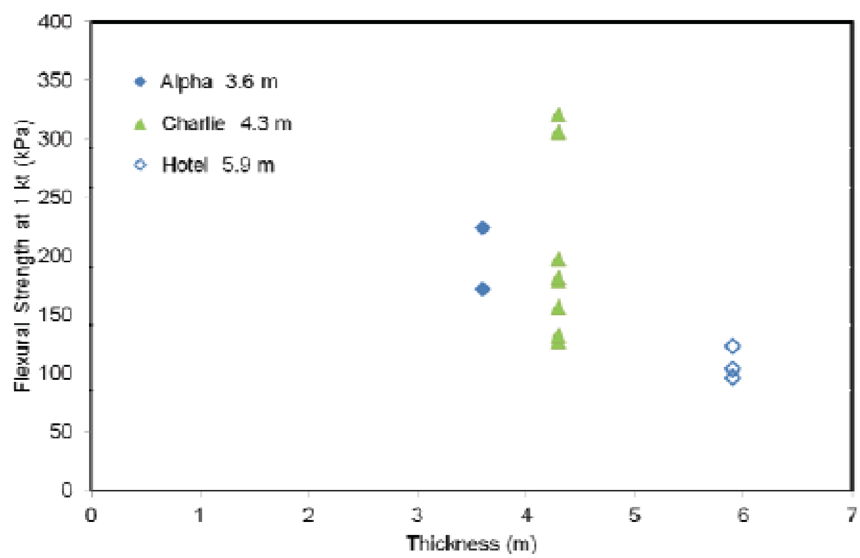


Figure 2.24. Flexural strength normalized to a 1 kt ramming speed as a function of floe thickness (Continuous icebreaking mode) (Frederking,and Sudom, 2013).

Parsons and Lal (1991) did not observe significant a scale effect in their analysis of 13 datasets to check the goodness-of-fit of the Weibull and double exponential distributions for sea and freshwater ice flexural strength data. Parsons et al. (1992) found that for the relatively small dataset they considered for first-year sea ice, the influence of specimen size was not strongly evident. Using regressing analysis on experimental results from three-point bending tests, they observed that sea-ice flexural strength depended only weakly on beam volume for their data, according to the relation $\sigma_f \alpha\, V^{-1/12}$. As for freshwater ice, the dataset they considered showed less decrease in flexural strength with increasing volume, and the authors suggested that scale effect can be completely neglected. Their dataset included 127 sea-ice tests between 0.008 to 8 m^3 volumes, and 80 freshwater tests between 0.027 to 2.197 m^3 volumes.

It is important to note that given the high scatter inherent in ice data, this is not surprising since sufficiently large ranges and number of data are needed to see trends more distinctly.

2.4 Summary

As reviewed in previous sections, there is both strong theoretical basis for why scale effects are expected, and strong empirical evidence that they exist, but yet current ice flexural strength models do not account for their presence. An ice beam will not fail until the strength at some location is exceeded. When a beam fails in flexure, the crack begins on the tensile surface. On this basis, from Weibull theory, we would expect scale effects in ice flexural strength to exist.

The main methods for measuring ice flexural strength are cantilever beam, three-point and four-point bending tests. Ideally one should use four-point bending tests since they produce a state of pure bending with constant moment and zero shear in the region between the two applied forces. This allows the beam to fail at its weakest point, rather than the loading point, which is desirable to provide more representative flexural strength measurements. Moreover, given the small amount of such data available, all test types should be considered, given appropriate assessment of the effects of other factors such as temperature and stress concentrations.

Frederking and Sudom (2013), Lau et al.(2001), Williams and Parsons (1994), Maattanen (1975) and Lavrov (1971) found a clear trend of decreasing ice flexural strength with increasing specimen size for both sea ice and freshwater ice. Prior analysis that did report significant scale effects (Parsons et al., 1992, Parsons and Lal, 1991) focused on small datasets over a limited range. To provide an updated treatment of scale effects, and help clarify issues associated with use of data from multiple sources, a detailed review and analysis of ice flexural strength data for freshwater ice are presented in Chapter 3, and for sea ice in Chapter 4.

3 Freshwater Ice Flexural Strength

The absence of brine in freshwater ice leads to distinct differences from sea ice. Consequently, this chapter only considers freshwater ice behavior; sea ice is considered in Chapter 4. Freshwater ice flexural strength depends on physical parameters such as grain size, crystal orientation and type (granular, columnar, discontinuous columnar or frazil), temperature and specimen size. In addition to external parameters, such as test conditions (cantilever, three-point or four-point bending), loading direction and loading rate will affect the strength properties of ice in flexure (Timco and O'Brien, 1994). In Section 3.1, the database of freshwater ice flexural strength measurements is introduced. Flexural strength dependencies are then reviewed (Section 3.2). The analysis that has been done for scale effects in freshwater ice flexural strength is discussed in Section 3.3. In Section 3.4, a residual analysis was conducted to model variability in these data. Finally, a summary of the chapter is given in Section 3.5. The work included in this chapter was also presented in a paper entitled 'Scale Effect in Freshwater Ice Flexural Strength' at the ASME 37th International Conference on Ocean, Offshore & Arctic Engineering in Madrid, Spain (see Appendix A for details).

3.1 Freshwater Flexural Strength Database

To thoroughly examine flexural failure of freshwater ice, an updated database has been compiled, which includes data from 2073 freshwater ice beam tests between 1.6 x 10^{-5} to 2.197 m^3, making this database the most comprehensive, to the best of the author's knowledge. The data were obtained from 16 papers from the literature as summarized in

Table 3.1. The table lists information about each source, including the authors, test type, number of tests, location (field or laboratory), beam volume, ice temperature and flexural strength. Flexural strength measurements are plotted in Figure 3.1 against beam volume; both are on logarithmic scale. Data points are given symbols according to their sources listed in Table 3.1.

Table 3.1. Summary of Freshwater Ice Data

Symbol	**Author**	**No. of tests**	**Location**	**Test type**	**Beam volume (m^3)**	**Temperature (℃)**	**Flexural strength (kPa)**
▲	Parsons et al. (1992)	80	Laboratory	3-pt bending	0.027 - 2.197	-4.8,-5.5,-0.5	1805.9 ± 97
▲	Frederking and Timco (1983)	67	Field	Cantilever	0.00092-0.01593	-3	790.1 ± 205.8
▼	Lavrov (1971)	180	Field	Cantilever and 3-pt bending	0.00029 - 0.102	-5.5 to -0.5	1645 ± 555.4
▼	Dempsey et al. (1988)	15	Laboratory	4-pt bending	0.00092 - 1.012	-10	2169.3 ± 999.6
▶	Barrette (2011)	56	Laboratory	4-pt bending	0.001	-9,-5.5,-0.5	1254.7 ± 561.8
◀	Tatinclaux and Wu (1978)	15	Laboratory	3 and 4-pt bending	0.00053	-5	2025.9 ± 444.2
▷	Tozawa and Taguchi (1986)	112	Laboratory	3-pt bending	0.00016-0.00281	-2	2047 ± 486.6
◀	Tabata (1967)	40	Laboratory	4-pt bending	0.00024	-15,-55	2810.5 ± 1347.5
◆	Frankenstein (1959)	228	Field	Cantilever	0.0296-0.899	-9.7 to 0	496.9 ± 175.9
◆	Gow and Langston (1975)	123	Field	Cantilever	1.219-0.0133	-	531.3 ± 369.7
●	Gow et al. (1978)	62	Field	Cantilever and 3-pt bending	0.532-1.38	-1,-3.5	756.8 ± 267.9
○	Gow et al. (1988)	706	Laboratory	Cantilever and 3-pt bending	0.00259-0.00117	-19 to -1	1226.6 ± 486.9
✶	Timco and Frederking (1983)	28	Laboratory	4-pt bending	0.0026	-3	867.9 ± 129.3
✶	Frederking and Sudom (2013)	6	Field	3-pt bending	0.0018	-21	2327
■	Drouin and Michel (1972)	331	Laboratory	4-pt bending	0.00457-0.724	-15, -1	1411.5 ± 479.5
□	Williams (1990)	22	Laboratory	3-pt bending	0.0355	-20	1715.1 ± 340.4

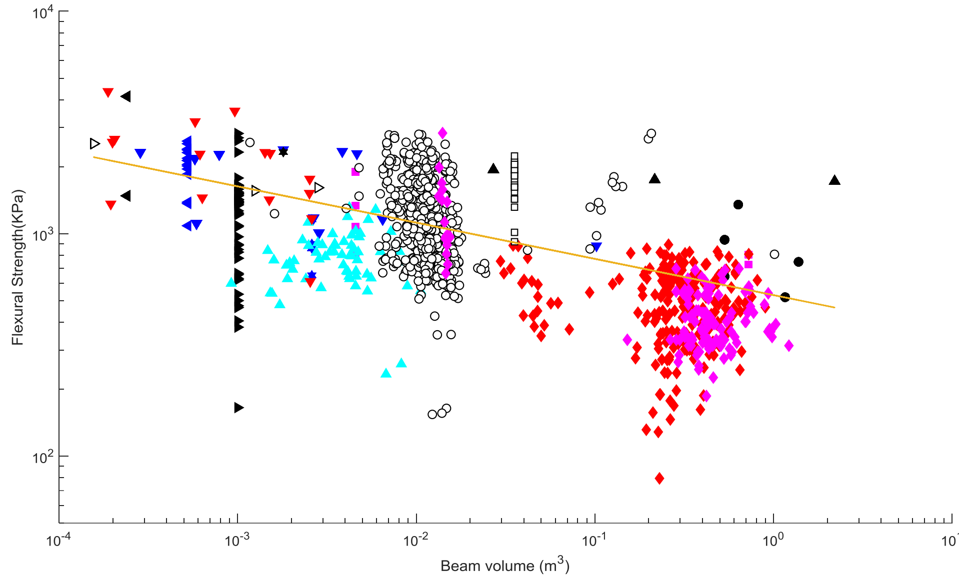


Figure 3.1. Freshwater ice flexural strength vs. beam size.

Specimen volume in this database was chosen to be represented by beam volume as has been used by many researchers as an appropriate way to study scale effects. Williams and Parsons (1994) also suggested using beam volume to represent specimen size, mainly because there are not enough details over the range of data in the literature to study the effect of each beam dimension separately. The same approach has been used in the present analysis.

3.2 Flexural Strength Dependencies

For temperature, Timco and O'Brien (1994) and Tatinclaux and Wu (1978) found no significant effect on flexural strength as shown in Figure 3.2 and Figure 3.3.

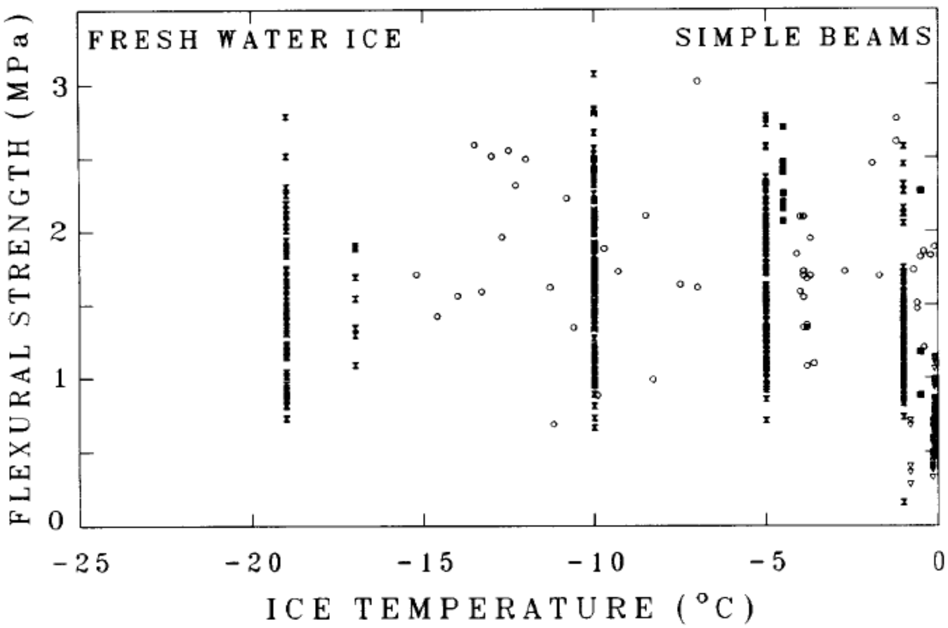


Figure 3.2. Freshwater ice flexural strength vs. ice temperature measured using simple beam tests (Timco and O'Brien, 1994).

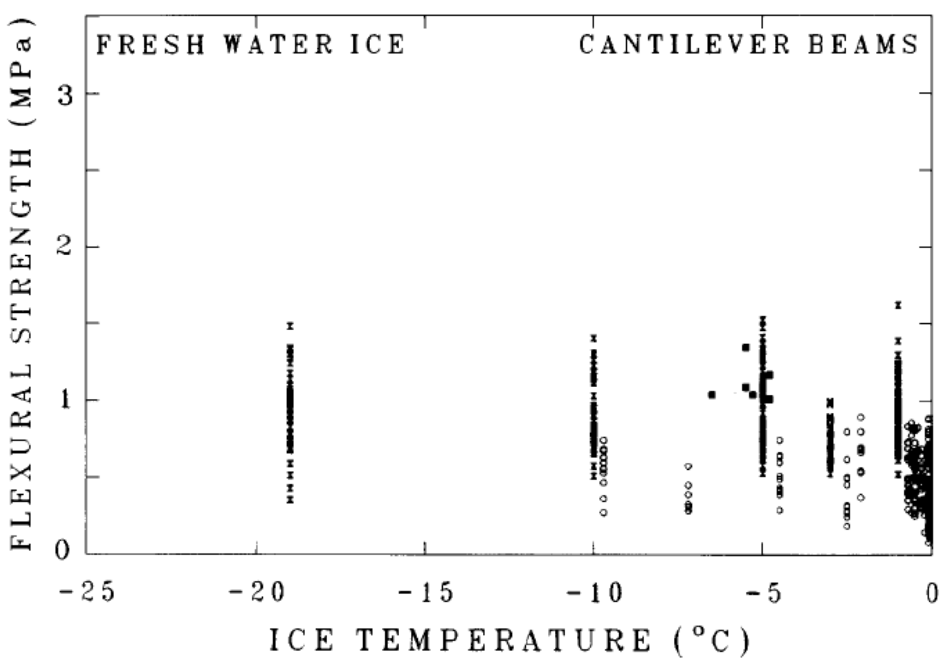


Figure 3.3. Freshwater ice flexural strength vs. ice temperature measured using cantilever beam tests (Timco and O'Brien, 1994).

From the collected database in this study, the flexural strength of each test was plotted against ice temperature in Figure 3.4. Based on these observations, it is evident that for freshwater ice, there is a high degree of variability over the entire temperature range, and flexural strength does not depend significantly on ice temperature over the range typically of interest for engineering applications. This result is consistent with the work of Timco and O'Brien (1994) and Tatinclaux and Wu (1976). On this basis temperature was neglected as a factor in the regression analysis.

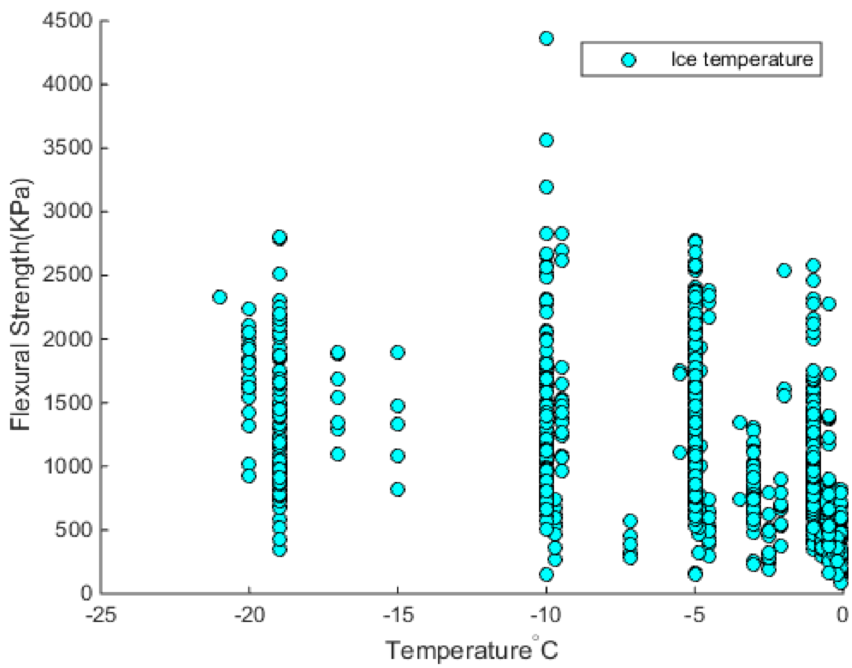


Figure 3.4. Freshwater ice flexural strength vs. ice temperature.

Loading rate is reported inconsistently or not at all in many cases. Some researchers reported stress rate where others reported strain rate, and the relationship between them is not well determined for ice (Timco and Weeks, 2010; Timco and O'Brien, 1994). Timco and Frederking (1983) found that the effect of stress rate is limited for flexural strength. When strain rate was measured, Maattanen (1975), Tabata et al. (1975) and Tabata (1967) showed that for a broad range of strain rates, sea-ice flexural strength generally increases with increasing strain rate. At very low strain rates, viscous behavior usually takes place, while at high strain rates, beam inertia, shear and water or wave effects (water displacement) appear (Ervik, 2013). To get accurate values of flexural strength, independent of the loading rate, it has been recommended to use a fixed test time of about 1 second (Timco and Weeks, 2010; Schwarz, 1981).

The most pronounced effects of strain rate occur where they result in a change in failure behavior (for example, for compression ductile failure is prominent for $\dot{\varepsilon} < 10^{-3}$, and brittle behaviour at higher rates). Here it is assumed that if the beams are failing in a brittle manner, then this is sufficient for inclusion in the present analysis since ice drifts over a wide range of speeds in nature and an overall approach which capture the scatter over the brittle domain is deemed a reasonable approach.

Ice can be loaded in either upward, downward or sideways orientations. In general, flexural strength represents the tensile strength of the extreme fiber, which is in the cold upper part in case of push-down, and of the warm bottom in case of pull-up configuration. The loading direction is usually vertical to stimulate the reaction of ice to loading from an icebreaker or a structure with inclined faces. For freshwater ice, Gow et al. (1978) concluded that loading direction has no effect on flexural strength. However, it is worth mentioning that

Tatinclaux and Wu (1978) found that the loading direction has a significant effect. This was attributed to the difference of crystal size between the upper and bottom surfaces of the beam.

Other parameters are considered to have less influence on ice flexural strength. Based on the above considerations, the effects of ice microstructure, loading direction, temperature and loading rate are not considered further here. Emphasis here is placed on the effect of specimen size.

3.3 Scale Effects

In Figure 3.1, where all measurements are plotted, there is a general trend of decrease in freshwater ice flexural strength with increasing beam volume. Williams and Parsons (1994) suggested that strengths should be averaged for tests that have the same beam volume (and similar tests conditions) to avoid biasing the data towards small-scale strength measurements, which are represented in significantly more reported tests than larger-scale measurements.

To be consistent with this approach, the average strength values are given in Figure 3.5, along with the best fit line suggested by Williams and Parsons (1994). The figure shows that while there is a general scale effect, the Williams and Parsons (1994) model line does not fully capture the scale effects trend, suggesting that further analysis is needed.

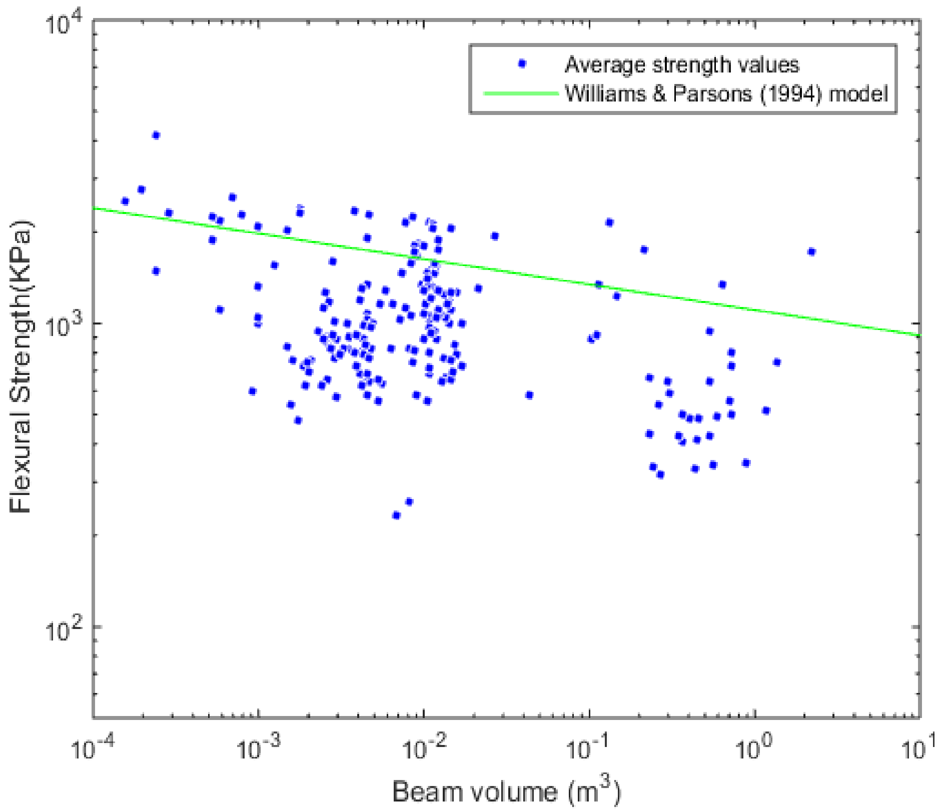


Figure 3.5. Freshwater ice flexural strength vs. beam volume using average values of strength for all tests with same beam volume.

Test location (field or laboratory) and test type (cantilever, three-point or four-point bending) have an influence on flexural strength and the scale effects associated with it. To examine this, freshwater ice flexural strength measurements were plotted against beam volume whilst differentiating between test location in Figure 3.6 and test type in Figure 3.7.

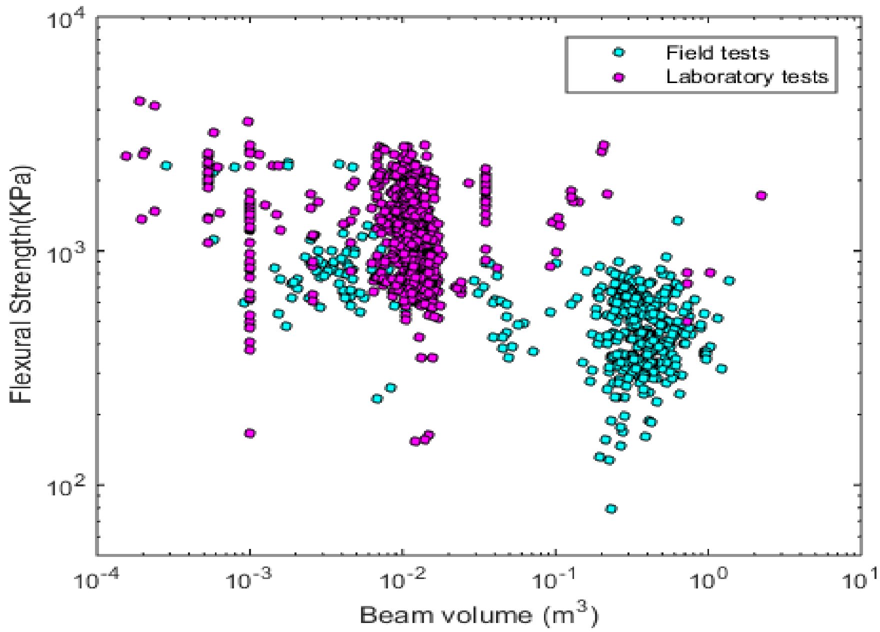


Figure 3.6. Plot of all freshwater ice flexural strength vs. beam volume data grouped to indicate test location as either field or laboratory.

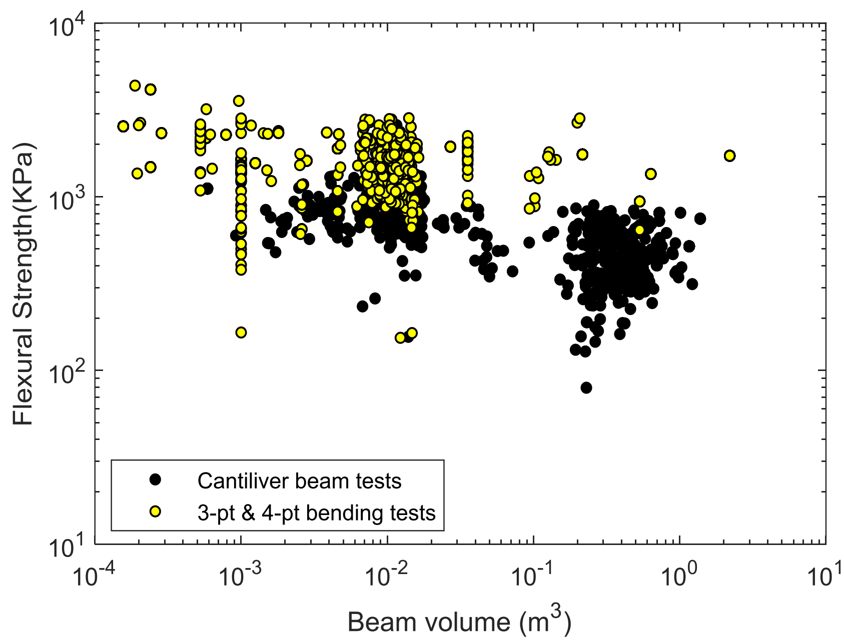


Figure 3.7. Plot of all freshwater ice flexural strength vs. beam volume data grouped according to test type.

Differentiating between test locations (Figure 3.6) is of interest here because field data correspond to ice that is more representative of ice in full-scale applications. Ice in the field has many naturally occurring flaws that are not present in laboratory ice. Laboratory test specimens are usually selected to ensure they have minimal flaws. Furthermore, they have smaller volumes which may not be large enough to account for large grain sizes found in some ice environments.

By comparing Figure 3.6 and Figure 3.7, it may be observed that the field tests were mainly done using the cantilever technique, while most of the three- and four-point bending tests were conducted in the lab. For freshwater ice, cantilever beam tests are generally believed to yield lower strength values than other measurement methodologies. This was confirmed by plotting the flexural strength field data against beam size while differentiating between cantilever beam tests and those for the three-point and four-point bending tests, as shown in Figure 3.8. This is mainly attributed to stress concentrations formed at the root of the beam. This behavior was studied and confirmed by several researchers, including Svec et al. (1985), Svec and Frederking (1981) and Schwarz and Weeks (1977).

To avoid excluding the cantilever tests from this analysis, as was done by Williams and Parsons (1994), a correction factor was instead used to account for reduction in strength in cantilever field tests. This was done by first fitting lines of best fit to the log-log plots (Figure 3.8), using least-squares regression method for the cantilever and grouped three- and four-point test field data, respectively.

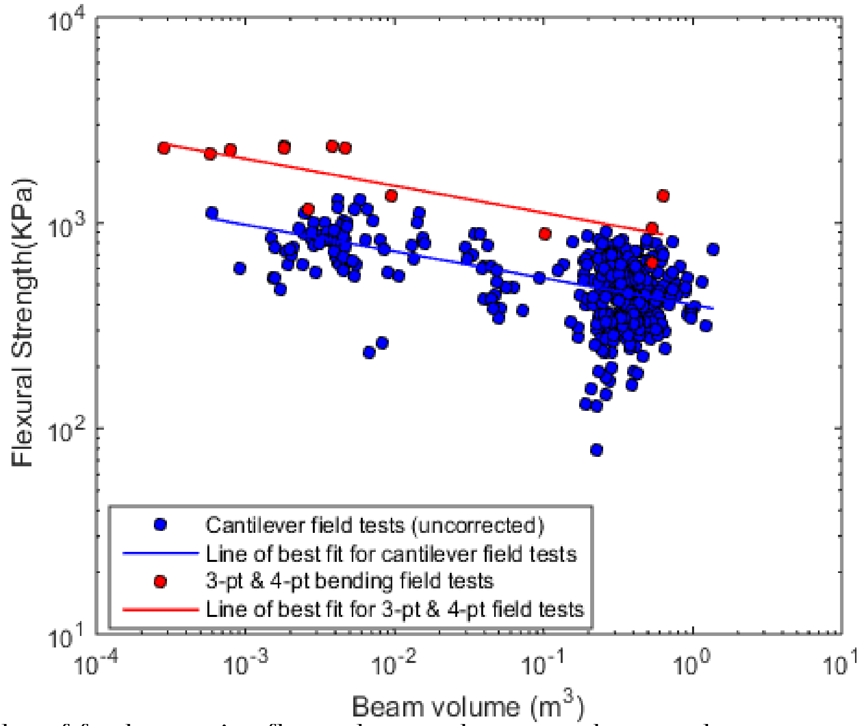


Figure 3.8. Plot of freshwater ice flexural strength tests vs. beam volume grouped by test type (field data only).

This produced the flexural strength equation for field cantilever tests:

$$\sigma_{f\ Cantilever} = 400(V/V_1)^{-0.13}. \tag{3.1}$$

Similarly, a flexural strength equation for the grouped field three- and four-point bending tests was obtained,

$$\sigma_{f\ 3-4\ Point} = 828(V/V_1)^{-0.13}, \tag{3.2}$$

where σ_f is the flexural strength in kpa, V is the beam volume in m^3 and V_1 is a reference volume (1 m^3). The exponents in the previous equations were rounded from -0.1296 and

-0.1311 respectively to -0.13 to simplify calculations. Taking a ratio of Equation 3.2 to Equation 3.1 yields a correction factor of about 2. This is consistent with Gow (1977), who conducted a number of cantilever and three-point bending tests to explore this difference, and found correction factors in the range of 1.2 to 2. The corrected cantilever beam field datasets have been combined with the three-point and four-point field data sets and plotted in Figure 3.9.

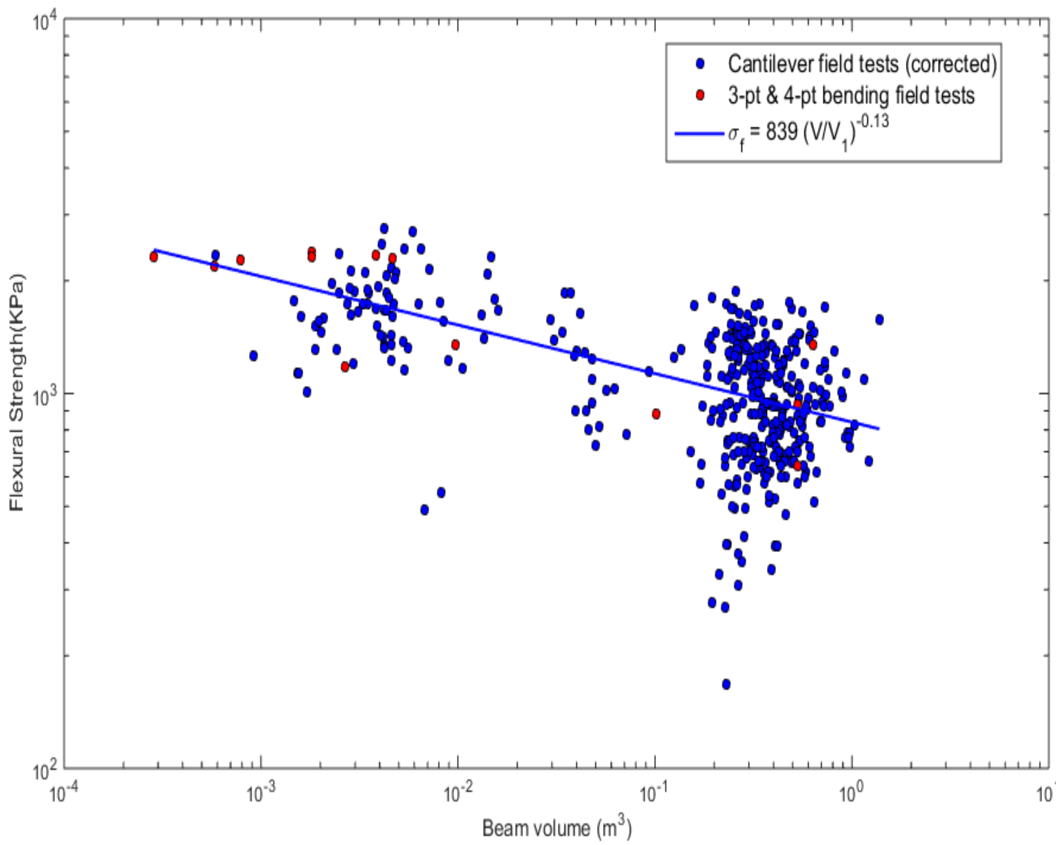


Figure 3.9. Freshwater ice beam flexural strength vs. beam volume for all field tests including corrected cantilever test data.

Using the same fitting method, the combined and corrected field data were fit by the relationship,

$$\sigma_f = 839(V/V_1)^{-0.13}. \tag{3.3}$$

By implementing a statistical t-test for the linear regression of freshwater ice data in Figure 3.9, it can be noted that the p-values for the coefficients are less than 0.005. This indicates that the fitted line slope is significantly different from zero. Thus the observed scale effects have a significant statistical basis, and are not just based on visual conclusion. This expression can be used to assess how such scale effects may influence ice loads on ships and structures under different conditions, leading to potential opportunities for refinement of current design methodology.

3.4 Residuals Analysis

The model developed in Section 3.3 can be used for estimating the mean freshwater ice strength values that would be expected. However, there is also a need to capture the scatter and variability that usually exist in ice strength data. Therefore, a residual analysis was implemented, and the residuals were calculated, and based on the values of the same volumes predicted by Equation 3.3. These Log values were fitted by a three-parameter Weibull distribution as best fit (scale parameter η = 3.622, shape parameter β = 13.02, location parameter γ = -3.484); see Figure 3.10 and Figure 3.11. It noted here that, as discussed by Neter et al. (1996) and Minitab (2017), as long as n > 40, the regression analysis is valid and not sensitive to the normality of the residuals.

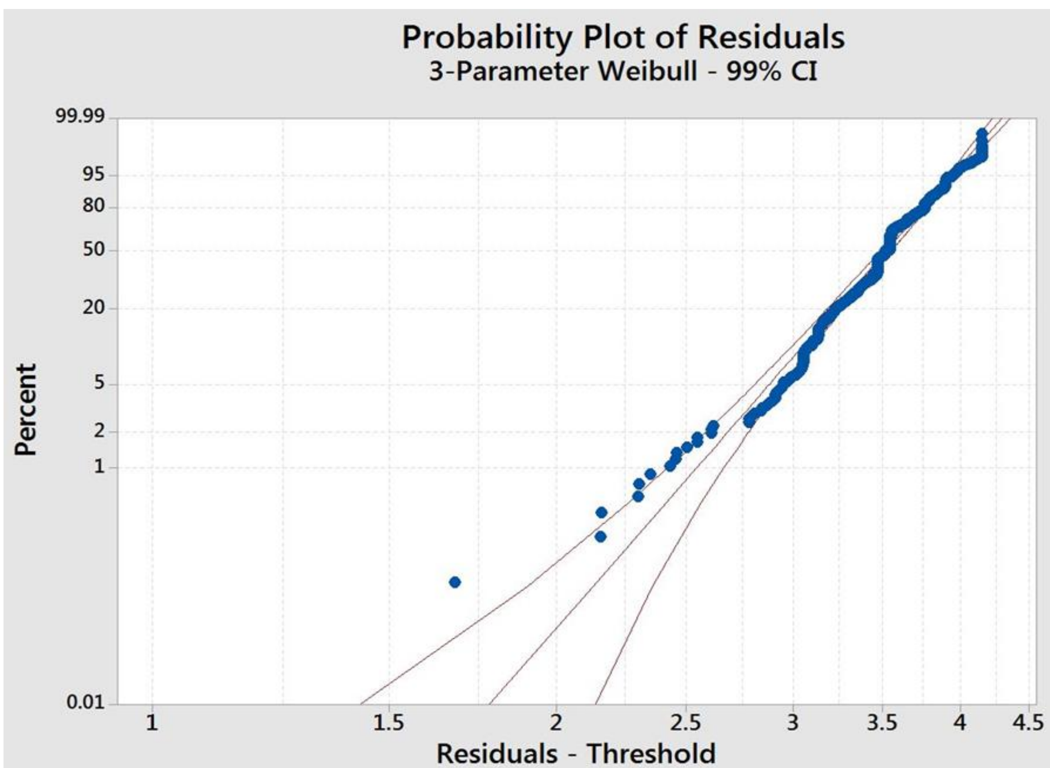


Figure 3.10. Probability plot of residuals for three-parameter Weibull distribution.

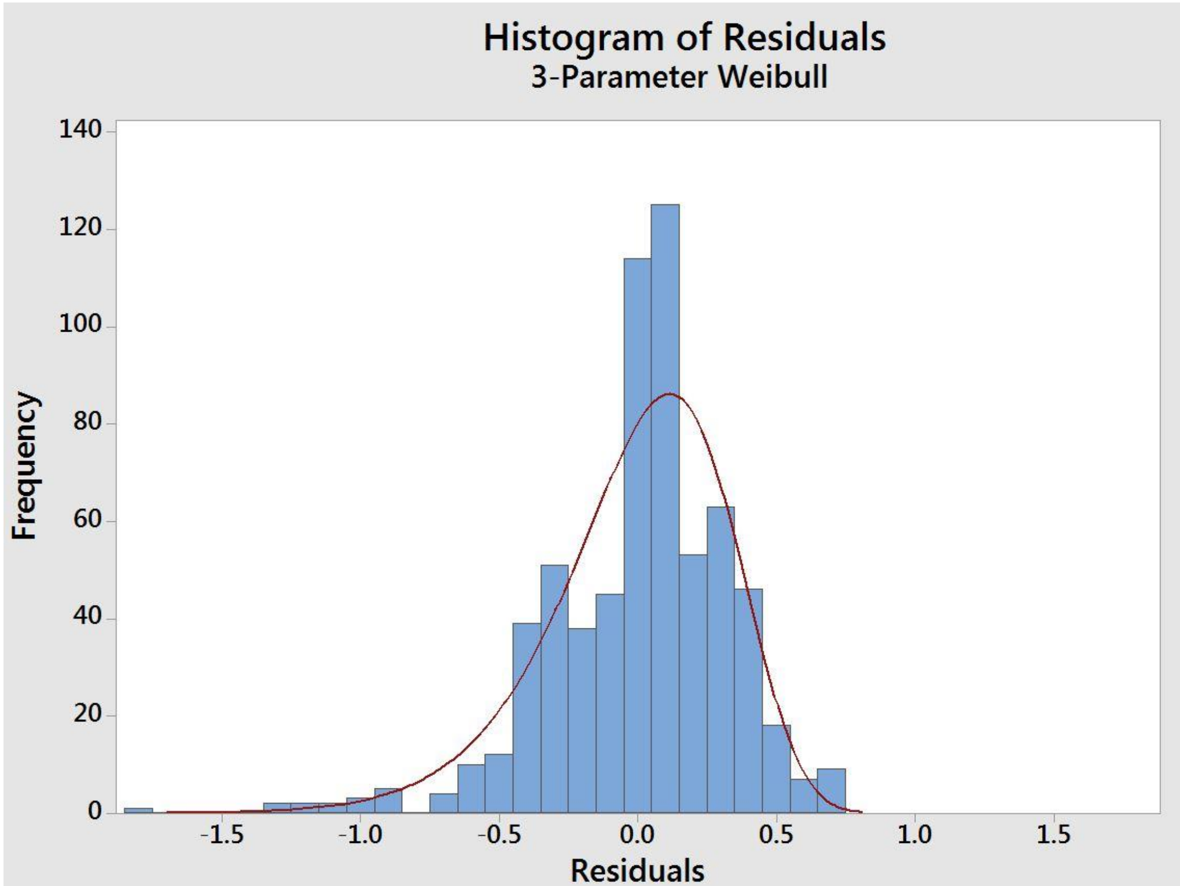


Figure 3.11.Weibull distribution histogram of residuals for freshwater ice.

As shown in Figure 3.10, the probability plot of residuals for the three-parameter Weibull distribution shows a good fit to the extreme strength values, which of an interest to design, and also to the mean values for operational use. Consequently, a probabilistic model based on the empirical relationship in Equation 3.3, and accounting for contribution of residuals is givens as,

$$\sigma_f = 840(V/V_1)^{-0.13} \cdot e^r, \tag{3.4}$$

where r is the residual, which can be sampled from the stated Weibull distribution. This model can be used to enhance probabilistic calculations and Monte Carlo simulations for ice loads where freshwater ice flexural strength is an input.

3.5 Summary

For freshwater ice, ice temperature was observed to have a limited effect on flexural strength. Other factors, such as ice microstructure, loading direction, temperature and loading rate, were not found to have a significant effect, and were not considered. The data show a considerable decrease in flexural strength as specimen size increases. When examined over a large size range, scale effects were observed to be a dominant factor affecting flexural strength. When considered separately, laboratory test data for ice flexural strength are observed to contain higher values and exhibit less pronounced scale effects than are expected in natural ice at full-scale. This is due to the smaller beam size and exclusion of specimens containing flaws from laboratory test programs, since the presence of natural flaws ice is an important consideration to scale effects.

In the field, for freshwater ice cantilever beam tests may give lower flexural strength values compared to other testing techniques due to stress concentrations at the root of the beam. However, these results should be corrected to avoid excluding them since they are highly important in representing large-scale beams of ice formed under natural conditions. A correction factor of about 2 was observed when field data for cantilever and three- and four-point tests were compared.

Based on the analysis presented in this chapter, a new empirical relationship given by Equation 3.3 above was developed to account for the effect of specimen volume on freshwater ice flexural strength. In addition, a probabilistic model based on the empirical equation was developed based on an analysis of the residuals, given by Equation 3.4.

4 Sea Ice Flexural Strength

For sea ice, brine volume is also an important parameter influencing flexural strength in addition to specimen size, grain size, crystal orientation and type (granular, columnar, discontinuous columnar or frazil), temperature, salinity, test conditions (cantilever, three-point or four-point bending), loading direction and loading rate (Timco and O'Brien, 1994). The compiled database of sea-ice flexural strength measurements is presented in Section 4.1. In Section 4.2, the parameters affecting sea-ice flexural strength are discussed. Brine volume influence is covered in Section 4.3. Scale effects in sea-ice flexural strength are examined in Section 4.4. In Section 4.5, a residual analysis is implemented to capture the scatter in sea-ice strength data. Finally, a summary of the chapter is given in Section 4.6.

4.1 Sea Ice Flexural Strength Database

To thoroughly examine flexural failure in case of sea ice, an updated database has been compiled, which includes data from 2843 sea-ice beam tests between 0.00048 to 59.87 m^3, making this database the most comprehensive to date for sea-ice flexural strength measurements, to the best of the author’s knowledge. The data were obtained from 36 papers from the literature as summarized in Table 4.1. The table lists information about each source, including the authors, test type, number of tests, location (field or laboratory), beam volume, ice temperature and flexural strength. Flexural strength measurements are plotted in Figure 4.1 against beam volume; both are on logarithmic scale. Data points are given symbols according to their sources listed in Table 4.1.

Table 4.1. Summary of Sea Ice Data

Symbol	Author	No. of tests	Location	Test type	Beam volume (m^3)	Avg. brine Volume (ppt)	Flexural strength (kPa)
▲	Murat and Tinwai (1977)	148	Laboratory	3-pt bending	0.00068-0.0013	25.8	622 ± 196.9
▲	Ervik (2013)	12	Field	Cantilever	0.323-1.86	70.2	225.33± 80.8
▼	Lau and Browne (1989)	5	Field	3-pt bending	0.00079	28.5	1040.28 ± 272.1
▼	Vaudrey (1975)	434	Lab and Field	Cantilever & 3-pt bending	0.001-1.92	33.6	572.6± 236.6
◀	Saeki et al. (1978)	41	Field	4-pt bending	0.004	60.8	649.9 ± 176.5
◀	Borek et al. (1988)	27	Field	3-pt bending	0.029-0.0729	18	1773.4 ± 507.7
▶	Butkovich (1956)	88	Field	Cantilever & 3-pt bending	0.0034-0.159	53	699 ± 299.8
▶	Butkovich (1959)	70	Field	3-bending	0.0014	24.9	1467.8 ± 920.5
△	Frankenstein, Guenther and Garner (1970)	82	Field	3-pt bending	0.0017	30.8	1091.9 ± 347.3
▲	Frederking and Häusler (1978)	11	Field	Cantilever	0.0008-0.003	105	464.5 ± 229.3
▽	Kayo et al. (1983)	90	Field	Cantilever & 3-pt bending	0.035-0.64	108	386 ± 81
▽	Lainey and Tinawi (1981)	22	Laboratory	4-pt bending	0.0064	21	1068.3 ± 714.9
◁	Michailidis (1981)	9	Field	Cantilever	5.314-8.28	14	266.8 ± 65.8
◁	Parsons et al. (1992)	127	Field	Cantilever	0.008-8	10	1215.5 ± 260.64
▷	Tabata (1967)	39	Laboratory	4-pt bending	0.00048	8	1065 ± 86
▷	Tabata et al. (1967)	24	Field	Cantilever	0.35	197.4	109.1 ± 19.85
□	Frederking et al. (1982)	21	Field	4-pt bending	0.024	15.5	909.9 ± 158.9
□	Weeks and Anderson (1958)	208	Field	Cantilever	0.034-0.138	124	210.2 ± 47.6
■	Williams et al. (1991)	43	Field	Cantilever	0.8618-2.42	45	232.6 ± 66.2
■	Williams et al. (1992)	71	Field & Laboratory	3-pt bending	0.0067-0.6867	56.1	566.5 ± 157.3
○	Williams et al. (1993)	38	Field	3-pt bending	0.0088-0.4234	20.2	898 ± 214.8
○	Williams et al. (1993)	8	Field	3-pt bending	0.00792-0.0136	45	418.5 ± 102.8
●	Airaksinen (1974)	27	Field	Cantilever	0.0282-0.0903	46	438 ± 70.8

●	Saeki et al. (1981)	31	Field	3-pt bending	0.006-0.1851	49.4	539 ± 208
□	Dykins (1971)	285	Field and Laboratory	3-pt & 4-pt bending	0.001-59.87	71.5	406.5 ± 149
◇	Marchenko (2017)	2	Laboratory	Cantilever	0.0079	104	112.6 ± 21.2
◇	Dykins (1968)	37	Laboratory	3-pt bending	0.0011	8.7	1150.3 ± 41.9
◇	Kujala et al. (1990)	34	Field	4-pt bending	0.495-0.7189	45.1	571.4 ± 116.2
◆	Blanchet et al. (1997)	41	Field	Cantilever	0.0037-6	22.6	937.4 ± 235.7
✶	Butkovich (1959)	100	Field	3-pt bending	0.0014	-	2172.2 ± 108.6
✡	Ji et al. (2011)	153	Field	3-pt bending	0.0039	84.5	861.9 ± 478.4
✡	Christensen (1986)	6	Field	3-pt bending	0.079-0.0956	14.3	598.3 ± 199
✶	Tatinclaux and Wu (1978)	13	Laboratory	4-pt bending	0.0005	16.9	347.9 ± 382.6
X	Shapiro and Weeks (1995)	137	Field	4-pt bending	-	0.0011	509.4 ± 117.2
X	C-CORE	7	Field	4-pt bending	-	0.7-3.85	346 ± 145.6
X	Frederking and Sudom (2013)	10	Field	3-pt bending	-	0.0018-0.0281	686.8 ± 14.3

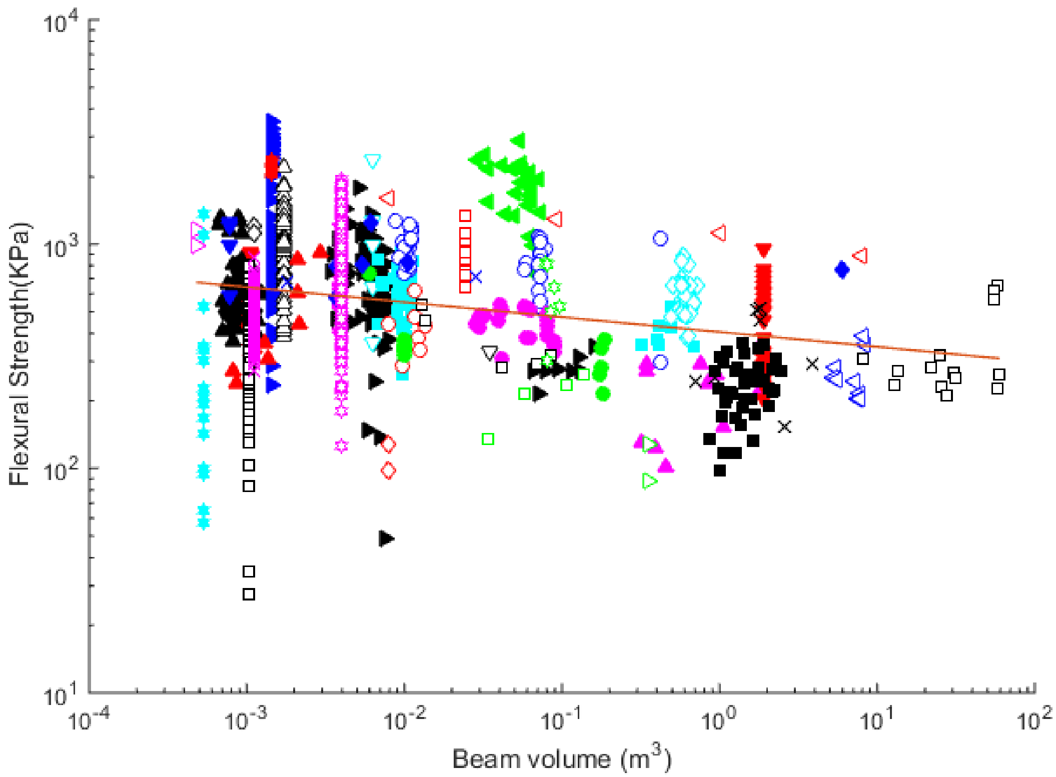


Figure 4.1. Sea ice flexural strength vs. beam size.

Similar to the case of freshwater ice, specimen volume in the sea-ice database was chosen to correspond to beam volume as has been used by many researchers as a representative way to study scale effects. Williams and Parsons (1994) suggested representing specimen size by beam volume, mainly because there are not enough details over the range of data in the literature to study the effect of each beam dimension separately. The same approach has been used in the present analysis.

4.2 Flexural Strength Dependencies

Loading rate, as mentioned before, is reported inconsistently or not at all in many cases. Timco and Weeks (2010) and Timco and O'Brien (1994) stated that there are not enough studies to give accurate facts. However, for sea ice Blanchet et al. (1997) found that the stress rate has very little effect on flexural strength. Ji et al. (2011), Tabata et al. (1975) and Gagnon and Gammon (1995) suggested that the average flexural strength slightly increases with increasing stress rate (see Figure 4.2). However, the maximum values of flexural strength show a decreasing trend with stress rate, and a clear trend is not evident in the data.

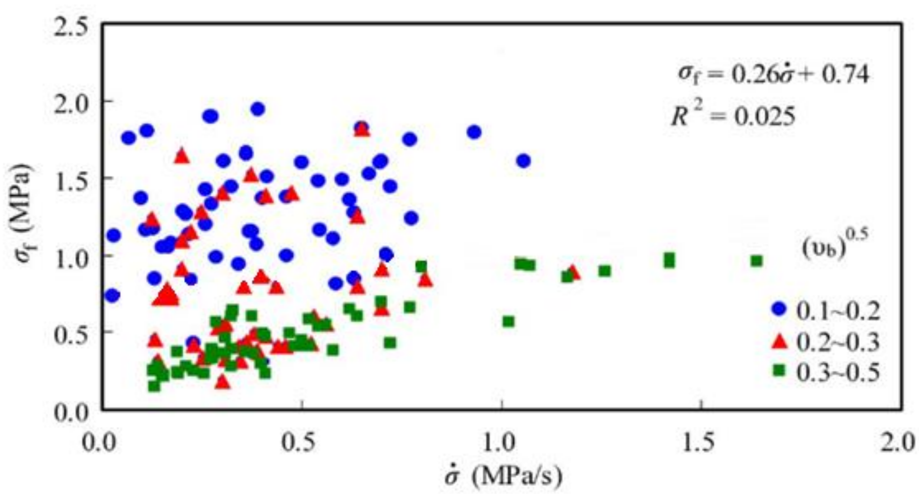


Figure 4.2. Flexural strength-stress rate relationship (modified after Ji et al., 2011).

At very low strain rates, viscous behavior usually takes place, while at high strain rates, beam inertia, shear and water or wave effects (water displacement) appear (Ervik, 2013). To get accurate values of flexural strength index, independent of the loading rate, tests should be done at test time of around 1 second (Timco and Weeks, 2010; Schwarz et al., 1981) or a correction factor should be added to account for the beam mass and the hydrodynamic effect of water (Maattanen, 1975). Insufficient information is currently available for hydrodynamic effects, so this has not been considered further here.

Regarding ice microstructure, if ice samples are taken from different positions through thickness of ice cover, Timco and Frederking (1982) found that sea-ice flexural strength is higher in the upper region where ice is granular, and lower in the lower region where the ice is columnar (see Figure 4.3). When there is evidence of c-axis alignment, sea-ice flexural strength is higher along the hard-fail direction (perpendicular to c-axis) than easy-fail direction (parallel to c-axis) by about 50%. In addition, beams cut vertically from the ice sheet have 2-5 times higher strength than horizontal ones (Kayo et al., 1983). Due to the lack of reporting in the data, further treatment of the effects of ice microstructure are beyond the scope of this work.

The loading direction (upward/downward) for sea-ice cantilever beam tests has been found to have little effect on sea-ice flexural strength (Timco and O'Brien, 1994; Weeks and Assur, 1967; Brown and Kingery, 1963; Weeks and Anderson, 1958). On this basis, loading direction is not considered further here.

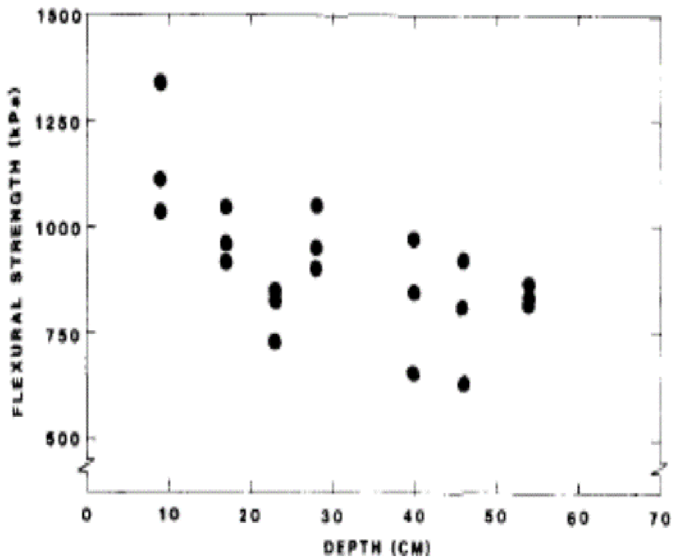


Figure 4.3. Flexural strength vs. depth (Timco and Frederking, 1982).

Unlike in freshwater ice, temperature has a major influence on sea-ice flexural strength. As the temperature decreases the flexural strength increases; see Figure 4.4 (Blanchet et al., 1997; Timco and O'Brien, 1994; Weeks and Anderson, 1958). This is attributed to two reasons. First, the decrease in temperature causes the stress required to activate dislocation motion to increase exponentially, which causes the ice to become more brittle with decreasing temperature (Goodman et al., 1981). Second, as temperature decreases, the brine volume (and porosity) decreases. This increases the ice-to-ice contacts leading to stronger ice (Timco and O'Brien, 1994).

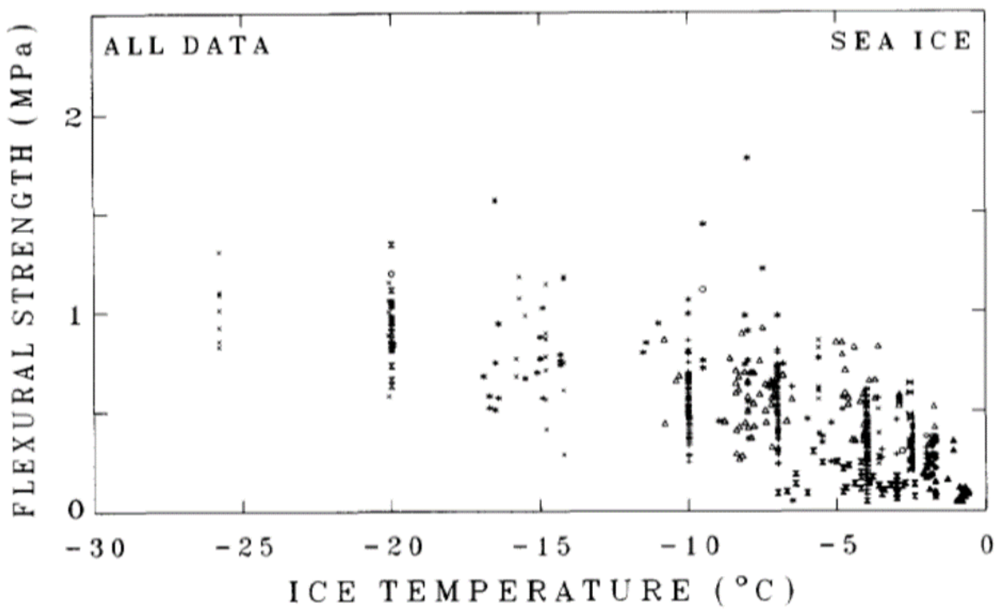


Figure 4.4. Flexural strength vs temperature for sea ice (Timco and O'Brien, 1994).

Frankenstein and Garner (1967) suggested the following formula to calculate brine volume between the temperatures -0.5°C and -22.9°C as function of salinity S (in ppt) and temperature T (in °C),

$$v_b = S\left(\frac{49.185}{T} + 0.532\right), \tag{4.1}$$

which assumes a constant density of 0.926 Mgm^{-3} for sea ice. Timco and Frederking (1996) pointed out that density of sea ice is not constant and ranges from 0.84-0.94 Mgm^{-3}. Cox and Weeks (1983) developed the following equations to calculate the total ice porosity (brine volume plus the volume occupied by air),

$$v_b = \frac{\rho_{si}\, S}{F_1(T)}, \tag{4.2}$$

$$v_a = 1 - \frac{\rho_{si}}{\rho_i} + \rho_{si} S \frac{F_2(T)}{F_1(T)}, \quad (4.3)$$

where ρ_{si} is the sea-ice density (in Mgm^{-3}), S is ice salinity (in ppt), ρ_i is the density of pure ice (in ppt), and $F_1(T)$ and $F_2(T)$ are functions for the temperature dependence of density and salinity on brine and solid salt content (Cox and Weeks, 1983). However, ice density is rarely reported, making it harder to calculate total porosity (Timco and Weeks, 2010). The relation between ice flexural strength and air porosity should be investigated more in the future, but is beyond the scope of the current analysis.

A number of researchers including Barrette et al. (1999), Blanchet et al. (1997), Timco and O'Brien (1994), Borek et al. (1988), Schwarz and Weeks (1977), Frankenstein, Guenther and Garner (1970) and Weeks and Assur (1969) confirmed from experimental results for sea ice that flexural strength decreases as brine volume increases. Timco and O’Brien (1994) summarized 2495 flexural strength test points for both freshwater and sea ice, and concluded that flexural strength has a negative exponential relationship to the square root of brine volume (see Figure 4.5).

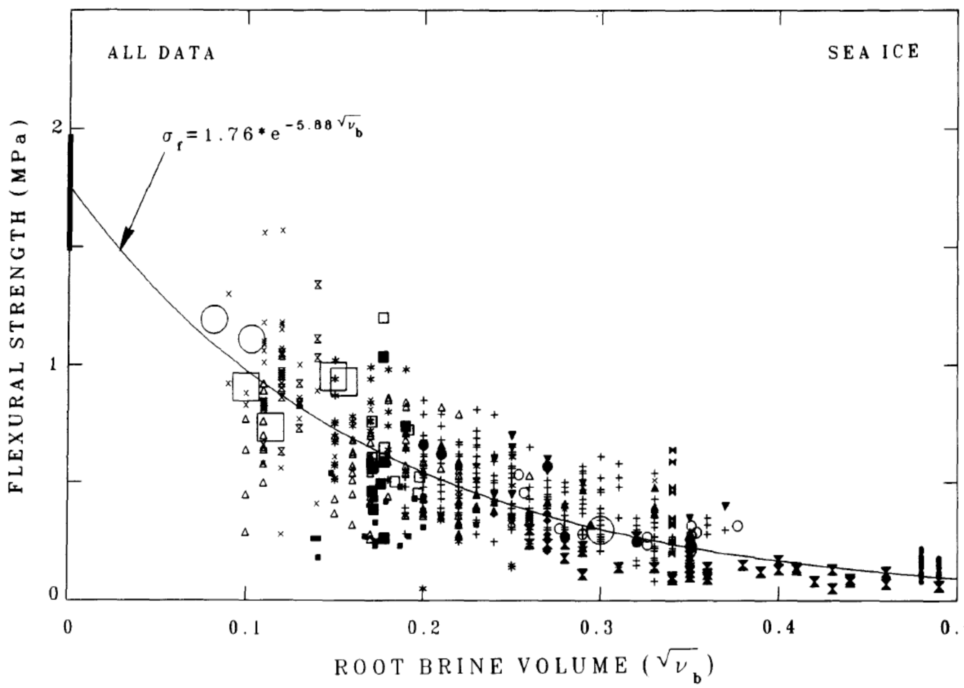


Figure 4.5. Flexural strength vs square root of brine volume (Timco and O'Brien, 1994).

This relation has a correlation coefficient $r^2 = 0.77$ between the flexural strength and brine volume as follows,

$$\sigma_f = 1.76e^{-5.88\sqrt{v_b}}, \quad (4.4)$$

where σ_f is flexural strength in MPa and v_b is brine volume. The 1.76 MPa for zero brine volume is close to the average strength of freshwater ice, which supports the usage of this relation.

4.3 Brine Volume

From the database collected in this chapter, the flexural strength of each test was plotted against brine volume in Figure 4.6. It is observed that flexural strength decreases as brine volume increases, and has a negative exponential relationship to the square root of brine volume as follows,

$$\sigma_f = 1.73e^{-4.89\sqrt{v_b}}. \tag{4.5}$$

From the freshwater ice measurements database, all data for temperatures below -4.5°C have been averaged to give a value of 1.7 $\pm$ 0.6 MPa. In Figure 4.6, this range of the average value plus and minus one standard deviation is indicated by a solid bar on the zero brine volume axis. The strength value for zero brine volume from the model agrees with the average value determined from the freshwater ice database. This indicates that the model gives good approximation to sea-ice flexural strength as a function of brine volume. The same approach was used by Timco and O'Brien (1994) to check the validity of the brine volume model. Figure 4.6 and Equation 4.5 show good agreement between the new model and the Timco and O'Brien (1994) model. However, the new model is recommended for use to estimate flexural strength as function of brine volume only, because it is based on more updated and comprehensive database. Flexural strength was plotted against brine volume whilst differentiating between test location in Figure 4.7 and test type in Figure 4.8. However, it is difficult to form any conclusion about the effect of test type or location due to the influence of beam size from these plots. These effects are examined in more detail in the next section.

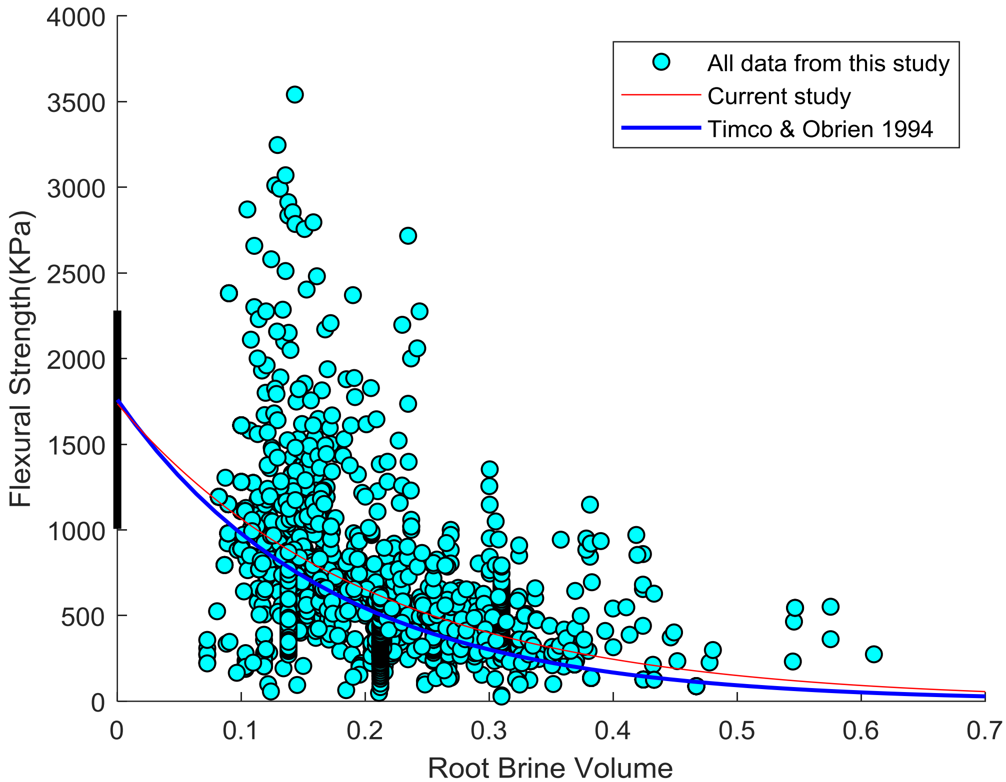


Figure 4.6. Flexural strength vs square root of brine volume.

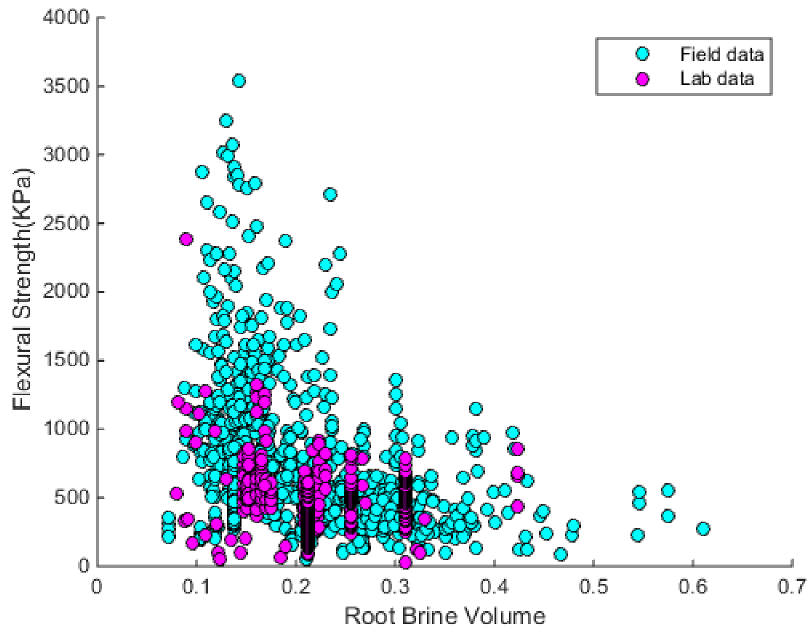


Figure 4.7. Flexural strength vs square root of brine volume grouped to indicate test location as either field or laboratory.

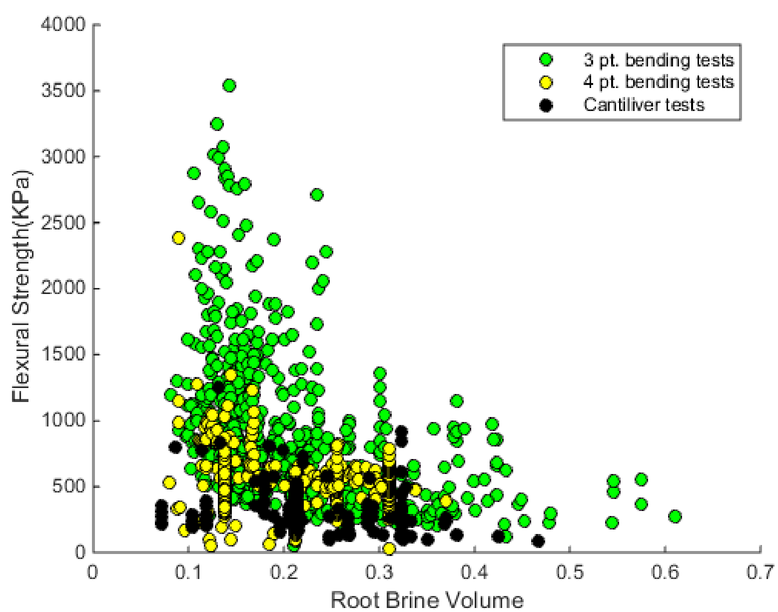


Figure 4.8. Flexural strength vs square root of brine volume grouped according to test type.

4.4 Scale Effects

Figure 4.1, where all measurements are plotted, shows a decrease in sea-ice flexural strength with increasing beam volume. However, brine volume should be considered before developing a regression equation. This suggests that further detailed analysis is needed. From the available literature, few tests were done on multi-year sea ice (Frederking and Sudom, 2013; Gladwell, 1977). Flexural strength of multi-year ice is much higher than first-year sea ice, because of brine drainage. Accordingly, multi-year ice tests were added to freshwater ice tests (Figure 4.9), since they are more comparable. From this figure, it is observed that multi-year data fit well within the range of the freshwater ice data. Use of the freshwater data may be more appropriate for modelling multi-year ice; however, more data for very large volumes of interest in multi-year interactions are needed to validate the applicability of this freshwater ice curve for multi year ice load models.

For first-year ice, few tests were done in locations where ice is brackish (low salinity ice). These tests were plotted with the rest of sea-ice flexural strength tests in Figure 4.10. As expected, there is generally good agreement between brackish ice and first-year ice, but for consistency brackish ice tests have been excluded from the rest of the analysis.

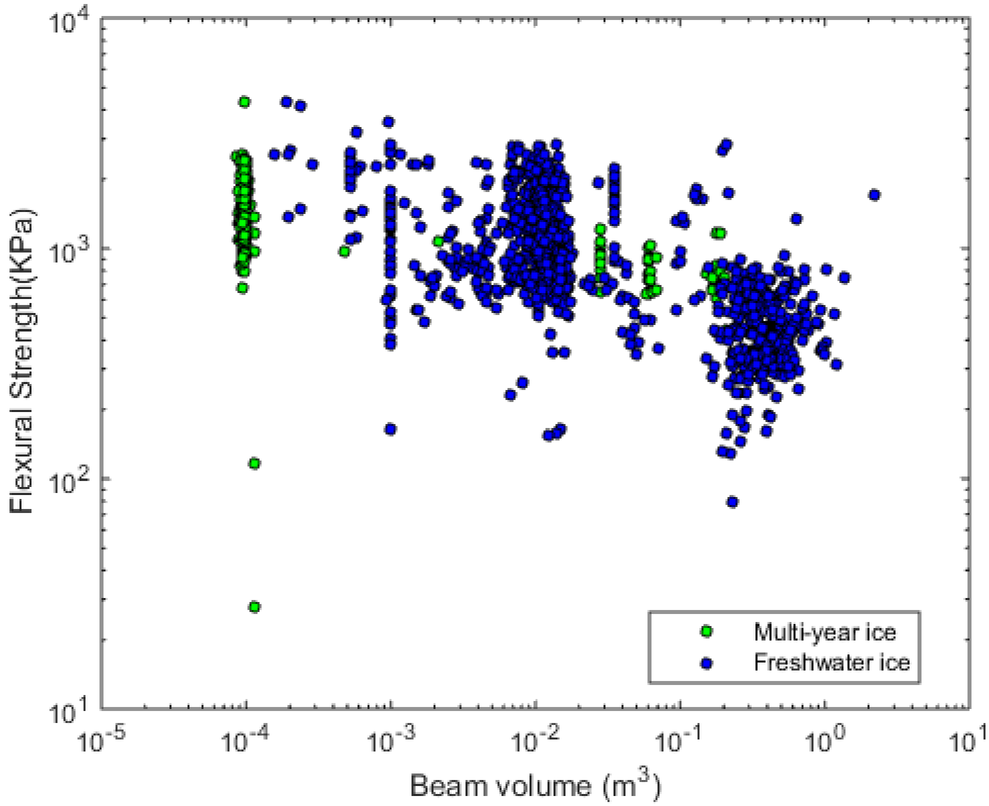


Figure 4.9. Multi-year sea ice and freshwater ice flexural strength vs. beam size.

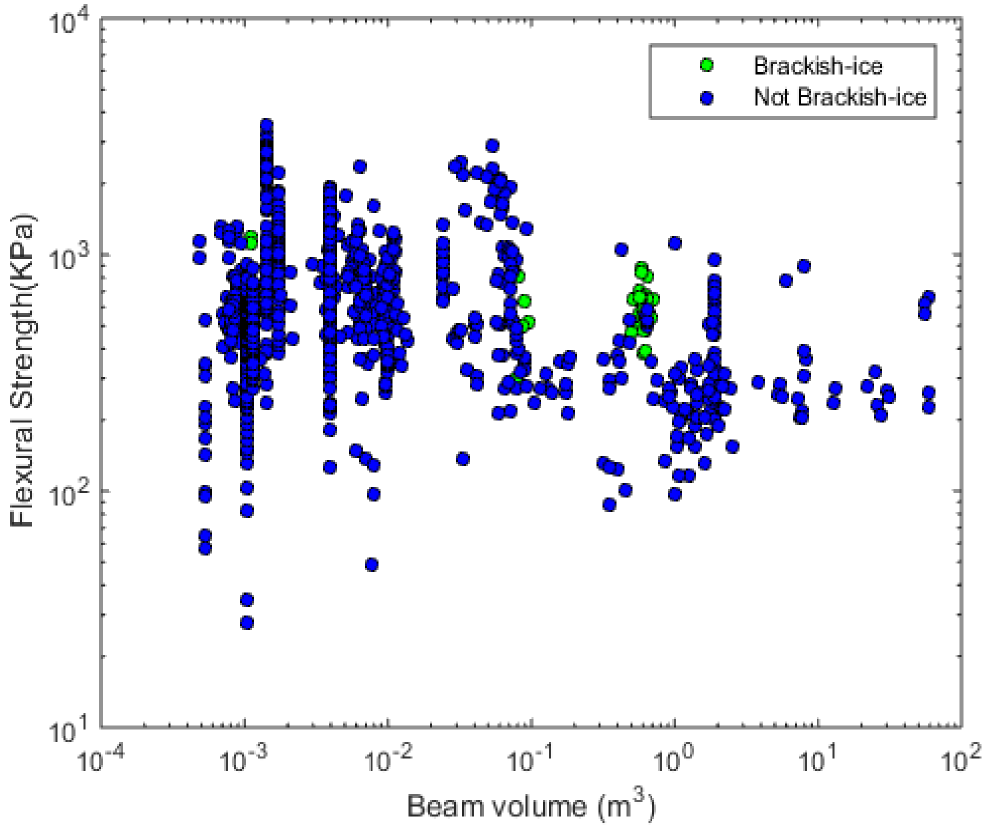


Figure 4.10. Brackish and sea-ice flexural strength vs. beam size.

Test location (field or laboratory) and test type (cantilever, three-point or four-point bending) have an influence on flexural strength and the scale effects associated with it. To examine this, sea-ice flexural strength measurements were plotted against beam volume whilst differentiating between test location in Figure 4.11, and test type in Figure 4.12.

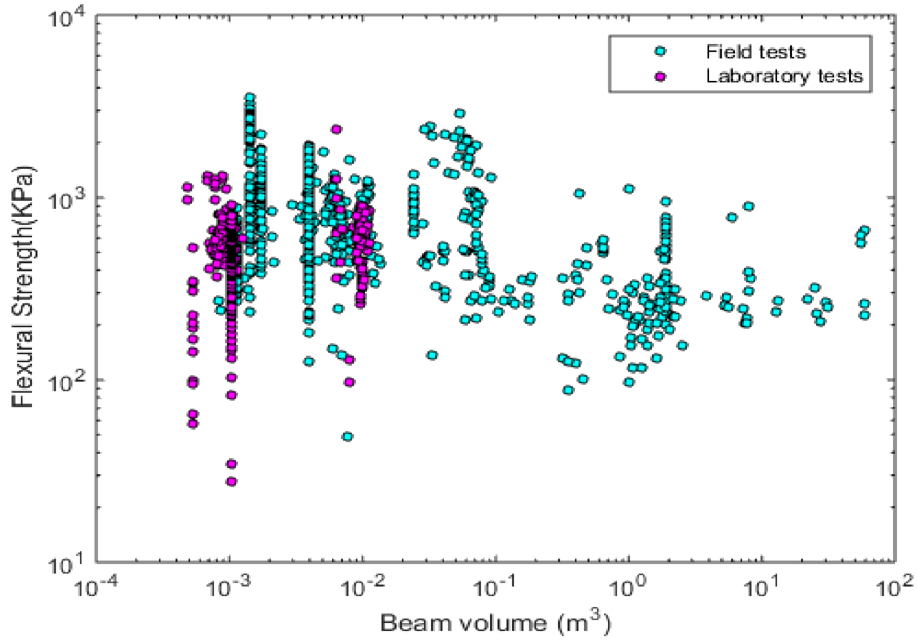


Figure 4.11. Plot of all sea-ice flexural strength vs. beam volume data grouped to indicate test location as either field or laboratory.

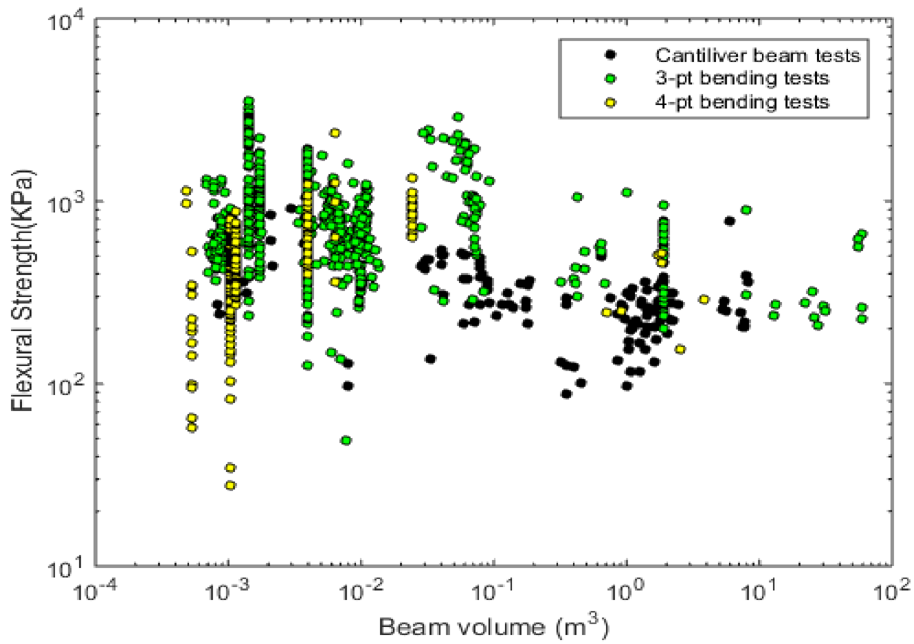


Figure 4.12. Plot of all sea-ice flexural strength vs. beam volume data grouped according to test type.

As shown in Figure 4.11, laboratory data mostly have small volumes, which makes them not representative of full-scale ice applications. Furthermore, their volumes may not be large enough to account for grain size effects. In addition, laboratory test specimens are selected to ensure they have minimal flaws, where ice in the field has many naturally occurring flaws. Consequently, the focus of this analysis will be on field tests. By differentiating between test types for all data (Figure 4.12) and considering only field tests (Figure 4.13), it is observed that cantilever tests sometimes have lower strength values than other measurement methodologies. However, this mainly attributed to scale, because cantilever tests are usually done in field, and have larger volumes.

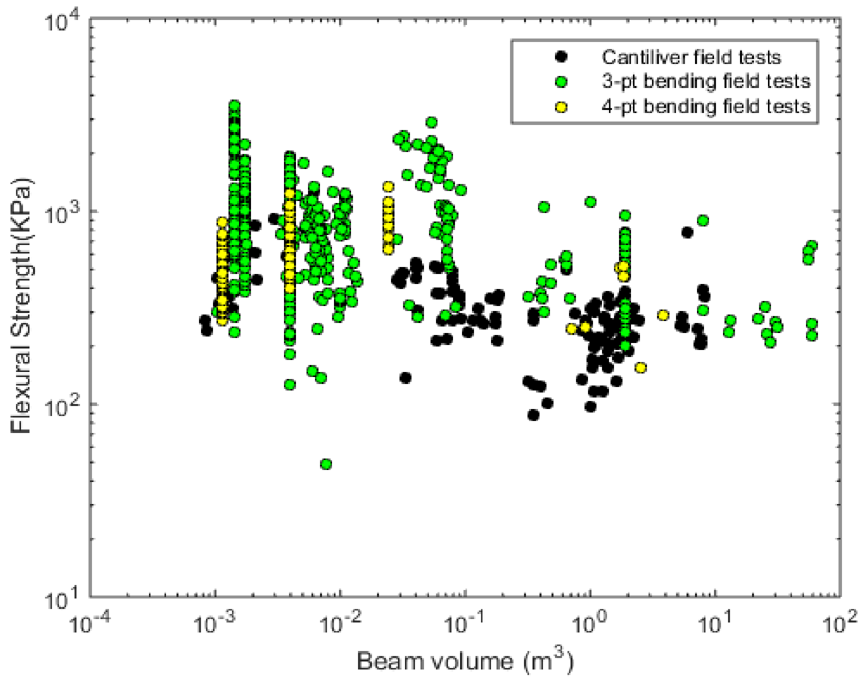


Figure 4.13. Plot of field sea-ice flexural strength vs. beam volume data grouped according to test type.

It is believed that stress concentrations have very limited effect on sea-ice beams (Williams and Parsons, 1994; Timco and O'Brien, 1994; Schwarz, et al., 1981; Schwarz and Weeks, 1977; Maattanen, 1975; Frankenstein, 1966). This mainly due to sea ice being more ductile than freshwater ice, which relieves stress concentrations (Schwarz and Weeks, 1977; Frankenstein, 1966). Schwarz and Weeks (1977) and Maattanen (1975) stated that sea-ice microstructure also contribute to less stress concentration effects. As a result, sea-ice cantilever tests are not corrected in this analysis as they were for the case of freshwater ice analysis.

To investigate influence of scale effects in sea-ice flexural strength without the influence of brine volume, all flexural strength measurements were normalized against a reference value. A reference value of 744.7 kPa was chosen based on a typical ice temperature of -10 °C and salinity of 5 ppt using Equations 4.1 and 4.5. The formula used for the normalization is as follows,

$$\sigma_{Normalized} = \sigma_{v_b} \times \frac{\sigma_{Ref_{v_b,Ref}}}{\sigma_{calculated,v_b}} , \tag{4.6}$$

where $\sigma_{Normalized}$ is strength corrected to a reference brine volume, σ_{v_b} is the strength at certain measured brine volume, $\sigma_{calculated,v_b} = 1.73e^{-4.89\sqrt{v_b}}$ is the strength calculated at the measured brine and $\sigma_{Ref_{v_b,Ref}} = 1.73e^{-4.89\sqrt{v_{b,Ref}}}$ is the strength at reference brine volume, where $v_{b,Ref} = \mathrm{S}\left(\frac{49.185}{T} + 0.532\right)$ is the reference brine volume calculated using an ice temperature $T = -10$ °C and salinity $S = 5\ ppt$.

The normalized flexural strength values were plotted against beam volume in Figure 4.14. The figure shows that even when the data has been normalised to same brine volume, a decrease in flexural strength with increasing volume is still evident. In addition, by implementing a statistical t-test for the linear regression of sea-ice data in Figure 4.14, it can be noted that the p-values for the coefficients are less than 0.005, which indicates that the fitted line slope is significantly different from zero. This supports the observation that scale effects are real and based on statistical rationale, not just visual conclusion. Since brine volume and beam volume are the dominant parameters controlling flexural strength of sea ice, a two-parameter model was developed using least-squares regression method,

$$\sigma_f = 1324\left(\frac{V}{V_1}\right)^{-0.054} e^{-4.969\sqrt{v_b}}. \tag{4.7}$$

This relationship can be used to enhance ships and structures design methodologies by evaluating scale effects in flexural strength for ice load calculations in case of sea ice.

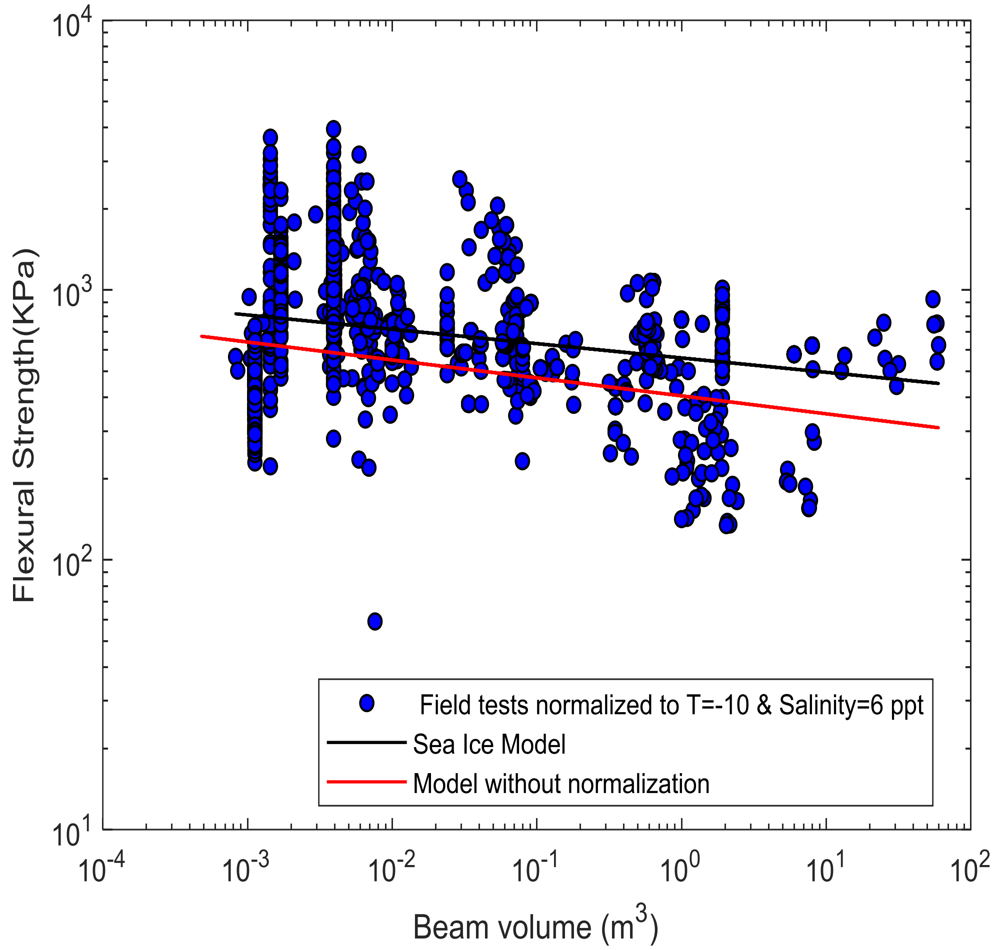


Figure 4.14. Plot of normalized field sea-ice flexural strength vs. beam volume.

4.5 Residuals Analysis

The model developed in Section 4.4 can be used to evaluate the mean trend of flexural strength in case of sea ice. Nonetheless, to model the variability that usually exist in ice strength data, an analysis of residuals was conducted by fitting the residuals of the fitted line in Log values by normal distribution as best fit (mean $\mu = 0$, standard deviation $\sigma = 0.4934$) in Figure 4.15 and Figure 4.16. The residuals appear randomly scattered around the zero-mean indicating that the linear model is a good fit. Since the normal distribution is unbounded, an upper bound residual limit of +1.68 is recommended based on the data.

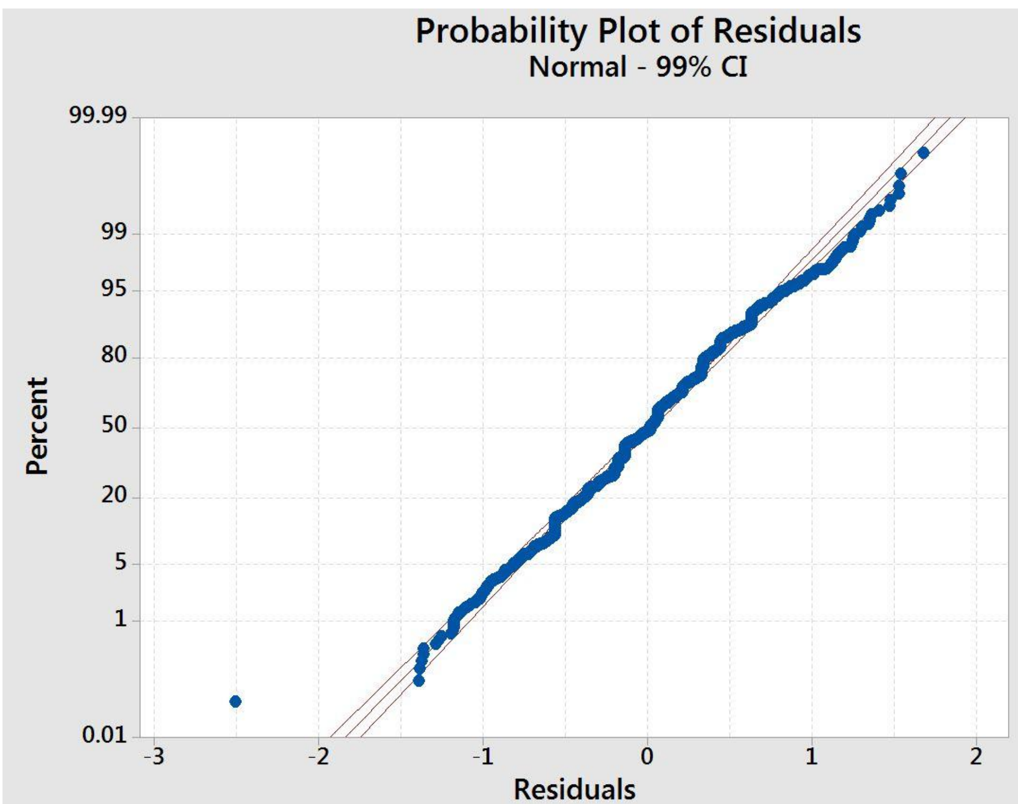


Figure 4.15. Probability plot of residuals for normal distribution.

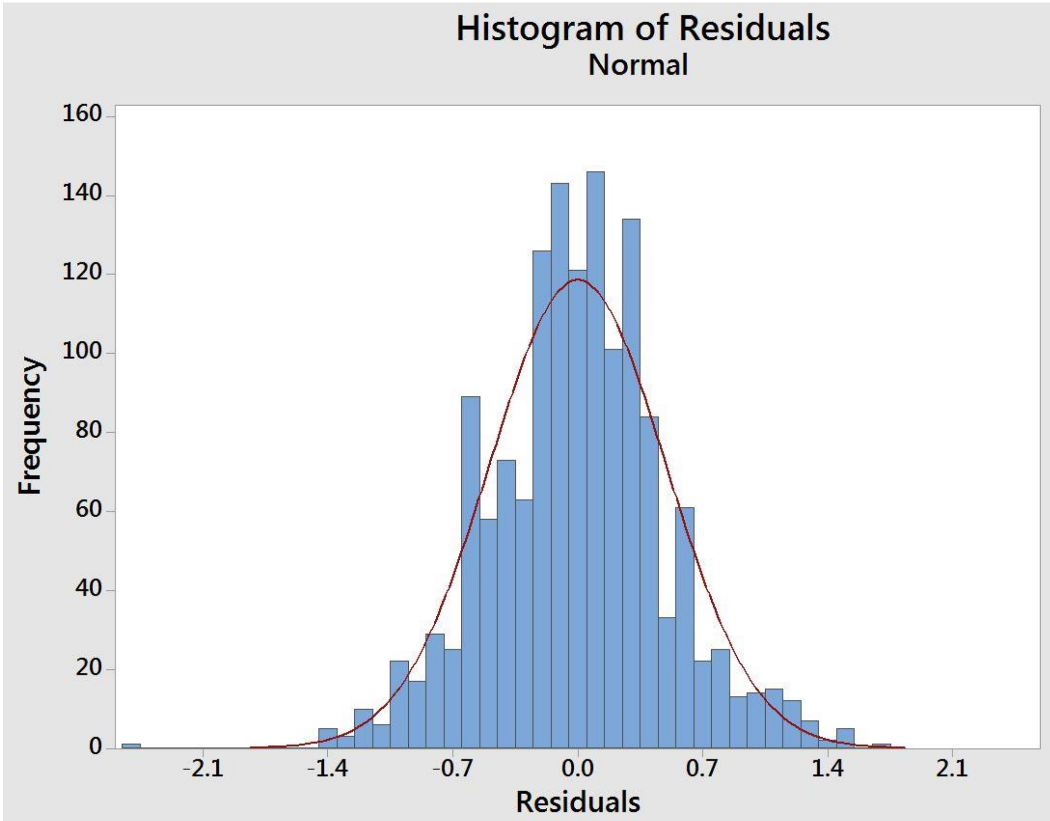


Figure 4.16. Normal distribution histogram of residuals for sea ice.

As shown in Figure 4.15, the normal distribution shows a good fit to the extremes, which of interest to design, and also to the mean values for operational use. Consequently, a probabilistic model based on the empirical equation was developed as,

$$\sigma_f = 1324(\frac{V}{V_1})^{-0.054} e^{-4.969\sqrt{v_b}}.e^r, \tag{4.8}$$

where r is the residual, which can be sampled from the normal distribution having the parameter values reported above. This model can be used in probabilistic calculations and Monte Carlo simulations for ice loads where sea-ice flexural strength is an input.

4.6 Summary

For sea ice, brine volume and beam volume were observed to have the greatest influence on flexural strength. The data confirmed that flexural strength decreases as brine volume increases. A negative exponential relationship of flexural strength to the square root of brine volume was developed. This relationship is similar to the Timco and O'Brien (1994) equation, but the relationship developed here is recommended for use since it is based on a more updated and comprehensive database, and indicates a somewhat higher strength values than Timco and O'Brien (1994) over part of the range of brine volumes.

Flexural strength of multi-year ice was found to be much higher than first-year sea ice, and more comparable to freshwater ice. On this basis, the freshwater curve may be more appropriate for multi-year ice although data are needed for larger beam volumes of interest in multi-year ice interactions. Brackish ice data were excluded from this flexural strength analysis, but may be estimated using Equation 4.7 using an adjusted salinity value.

Field data are more representative of full-scale ice applications, due to their larger volumes and naturally occurring flaws. While the sea-ice cantilever tests have somewhat lower strength values than other measurement methodologies. This has been mainly attributed to the fact that they have larger volumes, not to stress concentrations. It is believed that stress concentrations have more limited effect on sea-ice beams, because sea ice is more ductile, which relieves stress concentrations. A two-parameter empirical relationship was developed to account for the effect of specimen volume on sea-ice flexural strength as a function of brine volume and specimen volume (Equation 4.7). In addition, a probabilistic model based on the empirical relationship that was modified to account for the distribution of residuals was established (Equation 4.8).

5 Conclusions and Recommendations

5.1 Conclusions and Discussion

There is both strong theoretical basis for why scale effects are expected, and strong empirical evidence that they exist, but yet current ice flexural strength models do not account for their presence. The main methods to measure ice flexural strength are cantilever beam, three-point and four-point bending tests. Ideally one should use four-point bending tests since they produce a state of pure bending with constant moment and zero shear in the region between the two applied forces. This allows the beam to fail at its weakest point, rather than the loading point, which is desirable to provide more representative flexural strength measurements. Frederking and Sudom (2013), Lau et al. (2001), Williams and Parsons (1994), Maattanen (1975) and Lavrov (1971) found a clear trend of decreasing ice flexural strength with increasing specimen size for both sea ice and freshwater ice.

An updated, comprehensive study of scale effects in flexural strength was completed as major part of this thesis using a new database compiled for this study which contains 2073 freshwater ice and 2843 sea-ice flexural strength measurements. For freshwater ice, the data show a considerable decrease in flexural strength as specimen size increases. When examined over a large size range, scale effects were observed to be a dominant factor affecting flexural strength. When considered separately, laboratory test data for ice flexural strength are observed to contain higher values and exhibit less pronounced scale effects than are expected in natural ice at full-scale. This is due to the smaller beam size and exclusion of specimens containing flaws from laboratory test programs. In the field, freshwater ice cantilever beam tests usually give lower flexural strength values compared to other testing techniques due to stress concentrations at the root of the beam. However, these results should be corrected to avoid excluding them since they are highly important in representing large-scale beams of ice formed under natural conditions. A correction factor of about 2 was observed when field data for cantilever and three- and four-point tests were compared. A new empirical relationship given by,

$$\sigma_f = 839(V/V_1)^{-0.13}, \tag{5.1}$$

to account for the effect of specimen volume on freshwater ice flexural strength, where V_1 is a reference volume of 1 m^3.

For sea ice, brine volume and beam volume were observed to have the greatest influence on flexural strength.

The data confirmed that flexural strength decreases strongly as brine volume increases. A negative exponential relationship of flexural strength to the square root of brine volume was developed,

$$\sigma_f = 1.73e^{-4.89\sqrt{v_b}}. \tag{5.2}$$

This relationship is close to the Timco and O'Brien (1994); however, the new relationship is based on more updated and comprehensive database. Emphasis here has been placed on field data since they are more representative of full-scale ice applications, due to its larger volumes and naturally occurring flaws. It may be noted that sea-ice cantilever tests may have lower strength values than other measurement methodologies, but this is mainly attributed to their larger volumes rather than stress concentrations. Scale effects appeared to be more significant when sea-ice flexural strength measurements were normalized against a reference strength value calculated at reference ice conditions. Using these data, a two-parameter empirical relationship was developed,

$$\sigma = 1324(\frac{V}{V_1})^{-0.054} e^{-4.969\sqrt{v_b}} \ , \tag{5.3}$$

to account for the effect of specimen volume on sea-ice flexural strength as a function of brine volume v_b and specimen volume V, where V_1 is a reference volume of 1 m^3.

These models can be used to assess how such scale effects may influence ice loads on ships and structures under different conditions, leading to potential opportunities for refinement of current design methods. It should be noted that these models have an upper limit for the validity of their usage as strength will not decrease to reach zero value. Extrapolation of these models outside the range of the database should be done with care, as such models can be invalid outside this range. Probabilistic ice strength models based on the empirical equations and incorporating an analysis of residuals were developed. For freshwater ice,

$$\sigma_f = 840(V/V_1)^{-0.13} \cdot e^r \ , \tag{5.4}$$

and for sea ice,

$$\sigma_f = 1324(\frac{V}{V_1})^{-0.054} e^{-4.969\sqrt{v_b}} . e^r \ , \tag{5.5}$$

where r is the residual, which can be sampled from the Weibull distribution (scale parameter η = 3.622, shape parameter β = 13.02, location parameter γ = -3.484) in case of freshwater ice, and normal distribution (mean $\mu = 0$, standard deviation $\sigma = 0.4934$ with an upper limit of 1.68) in case of sea ice. These models can be used to enhance probabilistic calculations and Monte Carlo simulations for ice loads where ice flexural strength is an input.

Comparing the fitted lines for mean freshwater ice and normalized sea-ice flexural strength (Figure 5.1), it may be observed that the scale effects in freshwater ice are more pronounced. This mainly attributed to the dominance of the brine volume effect in sea ice. Moreover, brine inclusions in sea ice causes it to always exhibits lower strength values than freshwater ice. It is interesting to note that at large beam volumes freshwater flexural strengths approach those of sea ice, possibly suggesting that, for sea ice, flaws and cracks may dominate failure mechanisms at large scales and that the brine volume effect becomes less dominant in terms of governing the strength of sea ice for very large scales. Further work to explore these effects and the practical implications for design are recommended.

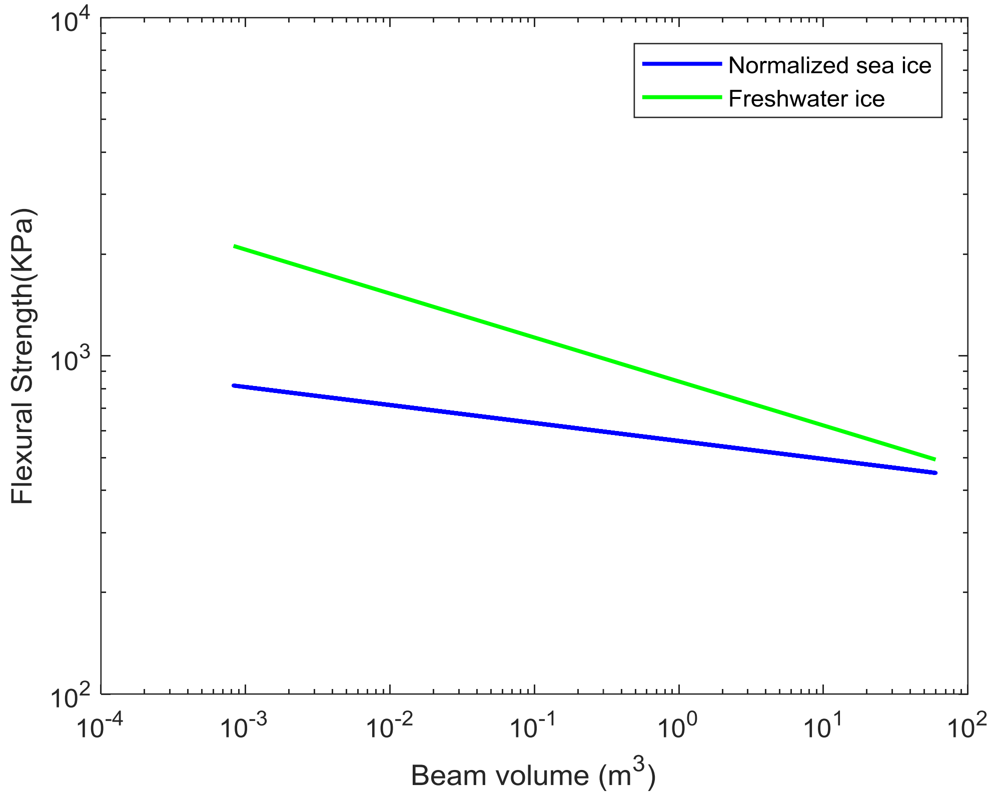


Figure 5.1. Comparison between normalized sea ice and freshwater ice models.

5.2 Recommendations for future research

In order to develop a deeper understanding of scale effects and ice flexural strength in general, the following recommendations are made:

- For future testing programs, it is recommended to use four-point bending tests since they produce a state of pure bending with constant moment and zero shear in the region between the two applied forces. This allows the beam to fail at its weakest point, rather than the loading point, which is desirable to provide more representative flexural strength measurements;

- To have a consistently reported database, and get accurate values of flexural strength index, further tests should be done at a test time of around 1 second (Timco and Weeks, 2010; Schwarz, 1981). This will standardize testing, and eliminate the effects of differing beam mass and hydrodynamic effect of water.
- The relation between ice flexural strength and air porosity should be investigated more in the future. Ice total porosity is a more significant and inclusive parameter than brine volume.
- Expand models to include other parameters as needed, should these parameters become more consistently reported (e.g. loading rate and direction, and grain size).
- Additional data are needed to help improve the database, particularly for large field-scale tests. For these data, loading rate and loading direction should be reported.
- Other more robust methods for normalizing flexural strength against brine content can be examined if new methods became available.
- Weibull modelling and associated scaling relationships should be more thoroughly examined for modelling ice flexural strength data. Further analysis is needed to explore how parameters relate to microstructure, temperature, salinity, volume, and loading rate.
- Further analysis to evaluate scale effects as a function of cross-sectional area and also considering effective volumes are recommended.

- More detailed consideration for design are recommended in future work to extend and apply this work to account for the effects of water acting as an elastic foundation, as well as the interplay between radial and circumferential cracks that result in flexural failure. The effects of surrounding ice cover on this process for first-year and multi-year ridges, as well as the relationship between breaking loads associated with flexural strength and other forces present during rubble formation and accumulation should be studied in further detail.
- Further investigation for how the empirical and probabilistic models could incorporated into methodology used in ice-class ship and structures design codes is needed. The interpretation of beam volume to be a function of the structure dimensions and the thickness of ice sheet needs to be studied, to allow incorporation into the design methodology used in existing codes and standards.

In summary, ice flexural strength was observed to exhibit clear scale effects for the case of both freshwater and sea ice. Through the application of more representative ice flexural strength models, such as these proposed in this research, improvements to engineering design methodologies may be possible.

REFERENCES

Airaksinen, K. (1974). Free beam tests and friction tests at pond inlet, NWT. *Polarforschung, 44*(1), 71-75.

Aly, M., Taylor, R., Dudley, E., & Turnbull, I. (2018). Scale effect in freshwater ice flexural strength. Paper presented at the *31th International Conference on Ocean, Offshore & Arctic Engineering,* 17-22 June 2018, Madrid, Spain.

Barrette, P. D. (2011). A laboratory study on the flexural strength of white ice and clear ice from the rideau canal skateway. *Canadian Journal of Civil Engineering, 38*(12), 1435-1439.

Barrette, P., Phillips, R., Clark, J., Crocker, G., & Jones, S. (1999). Flexural behavior of model sea ice in a centrifuge. *Journal of Cold Regions Engineering, 13*(3), 122-138.

Batdorf, S. B., & Heinisch, H. L. (1978). Weakest link theory reformulated for arbitrary fracture criterion. *Journal of the American Ceramic Society, 61*(7), 355-358.

Bažant, Z.,P. (1998). *Fracture and size effect in concrete and other quasibrittle materials*. Boca Raton, FL: CRC Press.

Blanchet, D., Abdelnour, R., & Comfort, G. (1997). Mechanical properties of first-year sea ice at tarsiut island. *Journal of Cold Regions Engineering, 11*(1), 59-83.

Borek, M., Parsons, B.,L., & Hill, B. (1988). Simple beam tests on type S3 sea ice. Paper presented at the *7th International Conference on Offshore Mechanics and Arctic Engineering,* Houston, TX. 63-68.

Brown, J. H., & Kingery, W. D. (1963). *Elasticity and strength of sea ice*. Cambridge, Massachusetts, USA: M.I.T. Press.

Butkovich, T. R. (1959). *On the mechanical properties of sea ice, thule, greenland, 1957.* (No. 54). Hanover, NH: U.S. Army CRREL.

Butkovich, T. R. (1959). *Some physical properties of ice from the TUTO tunnel and ramp, thule, greenland.* (No. 47). Hanover, NH: U.S. Army CRREL.

Butkovich, T. R. (1956). *Strength studies of sea ice.*Wilmette, IL: Snow Ice and Permafrost Research Establishment, Corps of Engineers, U.S. Army.

Christensen, F. (1986). Sea ice strength measurements from the inner danish waters in early 1985. Paper presented at the *Ice technology; 1st International Conference,* 247-253.

Dempsey, J. P., Nigam, D., Cole, D. M., Dempsey, J. P., Nigam, D., Cole, D. M., . . . Sinha, N. K. (1988). The flexure and fracture of macrocrystalline S1 type freshwater ice. Paper presented at the *7th International Conference on Offshore Mechanics and Arctic Engineering,* Houston, TX. 39-46.

Drouin, M., & Michel, B. (1972). *La resistance en flexion de la glace du st-laurent détermiée en nature.* Laboratoire de Mécanique des Glaces, Département de Génie Civil, Faculté des Sciences, Université Laval.

Dykins, J. (1968). Tensile and flexural properties of saline ice. Paper presented at the *International Symposium on Physics of Ice,* 251-270.

Dykins, J. (1971). *Ice engineering--material properties of saline ice for a limited range of conditions*. Port Hueneme, CA: U.S. Naval Civil Engineering Laboratory.

Ervik, Å. (2013). *Experimental and numerical investigations of cantilever beam tests in floating ice covers.* (Master's thesis).

Evans, A. G. (1978). A general approach for the statistical analysis of multiaxial fracture. *Journal of the American Ceramic Society, 61*(7), 302-308.

Frankenstein, G., & Garner, R. (1970). Dynamic young's modulus and flexural strength of sea ice. *U.S. Army Cold Regions Research and Engineering Laboratory.*

Frankenstein, G. E. (1959). *Strength data on lake ice*. Wilmette, IL.: U.S. Army Snow Ice and Permafrost Research Establishment, Corps of Engineers.

Frankenstein, G. E. (1966). *Strength of ice sheets.* Hanover, N. H: U.S. CRREL.

Frankenstein, G., & Garner, R. (1967). Equations for determining the brine volume of sea ice from −0.5° to −22.9°C. *Journal of Glaciology, 6*(48), 943-944.

Frederking, R., & Sudom, D. (2013). Review of flexural strength of multi-year ice. Paper presented at the *23th International Offshore and Polar Engineering Conference.*

Frederking, R. M. W., & Häusler, F. U. (1978). Flexural behaviour of ice from in situ cantilever beams tests. Paper presented at the *IAHR Symposium on Ice Problems,* Luleå, Sweden. 197-215.

Frederking, R. M. W., & Timco, G. W. (1983). On measuring flexural properties of ice using cantilever beams. *Annals of Glaciology, 4*, 58-65.

Frederking, R. M. W., Timco, G. W., Gulf, R. C., Dome, P. L., & Canada Oil and Gas,Lands Administration. (1982). *NRC ice property measurements at tarsiut island, beaufort sea, winter 1981-82.* National Research Council of Canada.

Gagnon, R. E., & Gammon, P. H. (1995). Characterization and flexural strength of iceberg and glacier ice. *Journal of Glaciology, 41*(137), 103-111.

Gladwell, R. (1977). *Field studies of the strength and physical properties of a multi-year ice pressure ridge in the southern beaufort sea*. Calgary, Canada: Arctic Petroleum Operators' Association.

Gow, A. J. (1977). Flexural strength of ice on temperate lakes. *Journal of Glaciology, 19*(81), 247-256.

Gow, A. J., & Langston, D. (1975). *Flexural strength of lake ice in relation to its growth structure and thermal history.* (No. CRREL-RR-349). Hanover, NH: U.S. Army Cold Regions Research and Engineering Laboratory.

Gow, A. J., Ueda, H. T., Govoni, J. W., & Kalafut, J. (1988). *Temperature and structure dependence of the flexural strength and modulus of freshwater model ice.* (No. CRREL-88-6). Hanover, NH: U.S. Army Cold Regions Research and Engineering Laboratory.

Gow, A. J., Ueda, H. T., & Ricard, J. A. (1978). *Flexural strength of ice on temperate lakes: Comparative tests of large cantilever and simply supported beams.* (No. CRREL-78-9). Hanover, NH: U.S. Army Cold Regions Research and Engineering Laboratory.

Gudmestad, O., Alhimenko, A., Løset, S., Shkhinek, K., Tørum, A., & Jensen, A. (2007). *Engineering aspects related to arctic offshore developments*. Student's Book for Institutes of Higher Education.

International Association of Classifications Societies (IACS). (2006). *Polar class ships rules.*

ISO 19906. (2010). *International standard for arctic offshore structure.*

ITTC. (2014). Test methods for model ice properties. Paper presented at the *International Towing Tank Conference.*

Ji, S., Wang, A., Su, J., & Yue, Q. (2011). Experimental studies on elastic modulus and flexural strength of sea ice in the bohai sea. *Journal of Cold Regions Engineering, 25*(4), 182-195.

Jordaan, I., Taylor, R., & Derradji-Aouat, A. (2012). Scaling of flexural and compressive ice failure. Paper presented at the *31st International Conference on Ocean, Offshore and Arctic Engineering,* 589-595.

Jordaan, I., Taylor, R., & Reid, S. (2007). Fracture, probabilistic averaging and the scale effect in ice-structure interaction. Paper presented at the *International Conference on Port and Ocean Engineering under Arctic Conditions.*

Jordaan, I. J. (2005). *Decisions under uncertainty: Probabilistic analysis for engineering decisions*. Cambridge; New York: Cambridge University Press.

Kärnä, T., & Qu, Y. (2006). *Analysis of the size effect in ice crushing.* (No. RTEU50-IR-6/2005).VTT Technical Research Centre of Finland.

Kayo, Y., Kawasaki, T., Minami, T., Tozawa, S., Tanaka, A., & Abdelnour, R. (1983). Field study on mechanical strength of sea ice at east coast of hokkaido. Paper presented at the *The 7th International Conference on Port and Ocean Engineering under Arctic Conditions,* Espoo, Finland. *, 27* 109-118.

Kujala, P., Riska, K., Varsta, P., Koskivaara, R., & Nyman, T. (1990). Results from in situ four point bending tests with baltic sea ice. Paper presented at the *IAHR Symposium on Ice Problems,* Helsinki, Finland. 261-278.

Lainey, L., & Tinawi, R. (1981). Parametric studies of sea ice beams under short and long term loadings. Paper presented at the *IAHR International Symposium on Ice,* Québec, Canada. 607-619.

Lau, M., Phillips, R., McKenna, R., & Jones, S. (2001). Size effect on the flexural strength of ice in model testing. Paper presented at the *16th International Conference on Port and Ocean Engineering under Arctic Conditions, 12-17 August 2001,* Ottawa, ON. *, 3* 1413-1421.

Lau, P., & Browne, C. (1989). Bending and compression properties of young sea ice. Paper presented at the *Oceans'89,* Seattle, WA. *, 4* 1292-1296.

Lavrov, V. V. (1971). *DEFORMATION AND STRENGTH OF ICE [deformatsiya i prochnost' l' da] (english).*

Maattanen, M. (1975). On the flexural strength of brackish water ice by in-situ tests. Paper presented at the *Port and Ocean Engineering under Arctic Condition,* Alaska, USA. , *75* 349-359.

Marchenko A., Karulina M., Karulin E., Chistyakov P., & Sakharov A. (2017). Flexural strength of ice reconstructed from field tests with cantilever beams and laboratory tests with beams and disks. Paper presented at the *International Conference on Port and Ocean Engineering Under Arctic Conditions,* Busan, Korea., *1* 1-10.

Masterson, D., Frederking, R., Wright, B., Kärnä, T., & Maddock, W. (2007). A revised ice pressure-area curve. Paper presented at the *19th International Conferencce on Port and Ocean Engineering Under Arctic Conditions, 1* 305-314.

Michailidis, M. (1981). *M.V. arctic probe into lake melville labrador.* (No. LTR -SH-301). Arctic Vessel and Marine Research Institute (AVMRI).

Mier, J. G. M. (1997). *Fracture processes of concrete: Assesment of material parameters for fracture models*. Boca Raton, FL: CRC Press.

Murat, J. R., & Tinwai, R. (1977). Sea ice testing in flexure. Paper presented at the *POAC 77. International Conference on Port and Ocean Engineering Under Arctic Conditions,* St. John's, NL. , *2* 638-653.

Neter, J., Kutner, M.H., Nachtsheim, C.J., & Wasserman, W. (1996). A*pplied linear statistical models.* Chicago: Irwin.

Palmer, A., & Croasdale, K. (2013). In Croasdale K., ProQuest (Firm) (Eds.), *Arctic offshore engineering*. World Scientific Publications.

Parsons, B. L., Lal, M., Williams, F. M., Dempsey, J. P., Snellen, J. B., Everard, J., . . . Wllliams, J. (1992). The influence of beam size on the flexural strength of sea ice, freshwater ice and iceberg ice. *Philosophical Magazine A, 66*(6), 1017-1036.

Parsons, B. L., & Lal, M. (1991). Distribution parameters for flexural strength of ice. *Cold Regions Science and Technology, 19*(3), 285-293.

Reid, S. (1991). Flaws in the failure prediction model of glass strength. Paper presented at the *6th International Conference on Applications of Statistics and Probability in Civil Engineering (ICASP6),* Mexico City, Mexico. 111-117.

Saeki, H., Nomura, T., & Ozaki, A. (1978). Experimental study on the testing methods of strength and mechanical properties for sea ice. Paper presented at the *4th IAHR Symposium on Ice,* Lulea, Sweden. , *1* 135-149.

Saeki, H., Ozaki, A., & Kubo, Y. (1981). Experimental study on flexural strength and elastic modulus of sea ice. Paper presented at the *6th International Conference on Port and Ocean Engineering Under Arctic Conditions,* Quebec, Canada. *, 2* 536-547.

Sanderson, T. J. (1988). *Ice mechanics and risks to offshore structures*. Norwell, MA: Kluwer Academic Publishers.

Schwarz, J. (1975). On the flexural strength and elasticity of saline ice. Paper presented at the *3rd International Symposium on Ice Problems, Committee on Ice Problems,* Hanover, NH. 373-385.

Schwarz, J. N. (1981). Sea ice investigation in the weddell sea. *Polarforschung, 51*(1), 21-37.

Schwarz, J., Frederking, R., Gavrillo, V., Petrov, I. G., Hirayama, K. I., Mellor, M., . . . Vaudrey, K. D. (1981). Standardized testing methods for measuring mechanical properties of ice. *Cold Regions Science and Technology, 4*(3), 245-253.

Schwarz, J., & Weeks, W. F. (1977). Engineering properties of sea ice. *Journal of Glaciology, 19*(81), 499-531.

Shapiro, L., & Weeks, W. (1995). Controls on the flexural strength of small plates and beams of first-year sea ice. *ASME Applied Mechanics Division (AMD), 207*, 179-179.

Svec, O. J., & Frederking, R. (1981). Cantilever beam tests in an ice cover: Influence of plate effects at the root. *Cold Regions Science and Technology, 4*(2), 93-101.

Svec, O. J., Thompson, J., & Frederking, R. (1985). Stress concentrations in the root of an ice cover cantilever: Model tests and theory. *Cold Regions Science and Technology, 11*(1), 63-73.

Tabata, T., Suzuki, Y., & Aota, M. (1975). Ice study in the gulf of bothnia: II. measurement of flexural strength. *Low Temperature Science, 33*, 199-206.

Tabata, T., Fujino, K., & Aota, M. (1967). Studies of the mechanical properties of sea ice ＸＩ: The flexural strength of sea ice in situ. Paper presented at the *International Conference on Low Temp Science; Physics of Snow and ice,* Sapporo, Japan. *, 1*(1) 539-550.

Tabata, T. (1967). Studies of the mechanical properties of sea ice X. the flexural strength of small sea ice beams. Paper presented at the *International Conference on Low Temp Science; Physics of Snow and ice,* Sapporo, Japan. 481-497.

Tatinclaux, J., & Hirayama, K. (1982). Determination of the flexural strength and elastic modulus of ice from in situ cantilever-beam tests. *Cold Regions Science and Technology, 6*(1), 37-47.

Tatinclaux, J., & Wu, C. (1976). *Laboratory study of the flexural strength and elastic modulus of freshwater and saline ice*. Iowa City, IA: Iowa Institute of Hydraulic Research, University of Iowa.

Tatinclaux, J. C., & Wu, C. Y. (1978). On flexural strength of freshwater and saline ice. Paper presented at the *Conference on Applied Techniques for Cold Environments,* Anchorage, Alaska. *, 1* 295-306.

Taylor, R. S. (2010). *Analysis of scale effect in compressive ice failure and implications for design.* (PhD thesis).

Taylor, R., & Jordaan, I. (2015). Probabilistic fracture mechanics analysis of spalling during edge indentation in ice. *Engineering Fracture Mechanics, 134*, 242-266.

Timco, G. W., & Frederking, R. (1982). Comparative strengths of fresh water ice. *Cold Regions Science and Technology, 6*(1), 21-27.

Timco, G. W., & Frederking, R. M. W. (1983). Flexural strength and fracture toughness of sea ice. *Cold Regions Science and Technology, 8*(1), 35-41.

Timco, G. W., & O.'Brien, S. (1994). Flexural strength equation for sea ice. *Cold Regions Science and Technology, 22*(3), 285-298.

Timco, G. W., & Weeks, W. F. (2010). A review of the engineering properties of sea ice. *Cold Regions Science and Technology, 60*(2), 107-129.

Tozawa, S., & Taguchi, Y. (1986). A preliminary study of scale effect on flexural strength of ice specimen. Paper presented at the *5th International Offshore Mechanics and Arctic Engineering, 4* 336-340.

Vaudrey, K. (1975). *Ice engineering: Elastic property studies on compressive and flexural sea ice specimens.* Port Hueneme, CA: U.S. Naval Construction Battalion Center.

Vaudrey, K. D. (1983). *Characterization of offshore and coastal ice conditions.* New York, United States: American Society of Civil Engineers.

Weeks, W. F., & Assur, A. (1969). *Fracture of lake and sea ice.* Hanover, NH: U.S. Army Cold Regions Research and Engineering.

Weeks, W. F., & Assur, A. (1967). Mechanical properties of sea ice. *Cold Regions Research and Engineering Laborator.*

Weeks, W. F., & Anderson, D. L. (1958). An experimental study of strength of young sea ice. *American Geophysical Union, 39*(4), 641-647.

Weibull, W. (1951). Wide applicability. *Journal of Applied Mechanics, 103*(730), 293-297.

Williams, F., Crocker, G., & Butt, S. (1993). *Northumberland strait ice properties measurements.* (No. TR-1993-06). St. John's, Canada: National Research Council, Institute for Marine Dynamic.

Williams, F., Everard, J., & Butt, S. (1992). *Ice and snow measurements in support of the operational evaluation of the nathanial B. palmer in the antarctic winter environment.* (No. TR-1992-14). St. John's, NL: National Research Council Canada. Institute for Marine Dynamics.

Williams, F., Spencer, D., Parsons, B., Hackett, P., Gagnon, R., Everard, J., & Butt, S. (1991). *Full-scale ice breaker trials CCGS sir john franklin indian arm/little burnt bay 1991.* (No. TR-1991-03). St. John's, NL: National Research Council Canada. Institute for Marine Dynamics.

Williams, J. L. (1990). *Three point loading of freshwater ice beams.* (No. LM-1990-13). St. John's, NL: National Research Council Canada. Institute for Marine Dynamics.

Williams, F. M., Kirby, C., & Slade, T. (1993). *Strength and fracture toughness of first year arctic sea ice.* (No. TR-1993-12). St. John's, NL: NRC Institute for Ocean Technology; National Research Council Canada.

Williams, F. M., & Parsons, B. L. (1994). Size effect in the flexural strength of ice. Paper presented at the *International Conference on Offshore Mechanics and Arctic Engineering, 13th,* Houston, Texas. 15-22.

Appendix A: Conference Paper

Proceedings of the ASME 37th International Conference on Ocean, Offshore & Arctic Engineering
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SCALE EFFECT IN FRESHWATER ICE FLEXURAL STRENGTH

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ABSTRACT

Ice flexural strength is an important parameter in the assessment of ice loads on the hulls of ice-class ships, sloped offshore structures or sloped bridge piers. While scale effects are well known for compressive ice strength, there has been debate as to whether or not scale effects in ice flexural strength exist. To investigate scale effects during flexural failure of freshwater ice, a comprehensive up-to-date database of beam flexural strength measurements has been compiled. The data show a considerable decrease in flexural strength as the specimen size increases, when examined over a large range of scales. An empirical model of freshwater ice flexural strength as a function of beam volume has been developed using regression analysis.

INTRODUCTION

Ice loads are a key design consideration for ship and offshore structure design for the ice-prone waters. Ice loads depend on the failure mode of the ice, which can include crushing, bending, buckling, or mixed mode. Ice flexural strength is an important parameter in the bending failure mode, and the assessment of ice loads on the hulls of ice-class ships, sloped offshore structures, or sloped bridge piers and lighthouses. Moreover, flexural strength is essential in the study of ice ridging and rafting phenomena, and for calculating the bearing capacity of ice cover, which is critical in the design of winter roads, as well as other on-ice operations. Ships or structures that break the ice in bending typically exhibit much lower loads than others where ice fails in crushing. For example Vaudrey (1983) concluded that the flexural strength of ice is around 10 % to 50% of its compressive strength. This reinforces the need for more investigation of ice flexural strength in general.

The flexural strength of ice depends on physical parameters such as total porosity, specimen size, grain size, crystal orientation and type (granular, columnar, discontinuous columnar or frazil), and in the case of sea ice, also on temperature and salinity. In addition, test conditions (cantilever, three-point or four-point bending), loading direction and loading rate will

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affect the strength properties of ice in flexure (Timco & O'Brien, 1994).

Timco & Weeks (2010) suggested that the total porosity is the most significant parameter influencing flexural strength, particularly for sea ice. A number of researchers including Barrette et al. (1999); Blanchet et al. (1997); Borek et al. (1988); Frankenstein and Garner (1970); Schwarz and Weeks (1977); Tatinclaux and Wu (1978); Weeks and Assur (1969); Weeks and Assur (1967); Timco and O'Brien (1994) confirmed from experimental results for sea ice that flexural strength decreases as brine volume increases, which is a function of temperature and salinity of the ice. However, when Timco and O'Brien (1994) developed a correlation between flexural strength and brine volume in a compiled database of 2495 tests of sea and freshwater ice, strength at times fluctuated by an order of magnitude at the same brine volume value. For freshwater ice, their analysis showed strong scatter in the data at exact or close ice temperatures for the same test type. This suggests that there are other parameters that strongly affect the flexural strength.

Ice, as a geophysical material, contains many flaws and cracks, so it is expected that the probability of encountering such flaws increases with increasing specimen size, which would cause a decrease in strength (Tozawa and Taguchi, 1986). For compressive ice strength there is general agreement on such scale effects (e.g., Taylor and Jordaan, 2015; Jordaan et al., 2012).

Williams and Parsons (1994) found a clear trend of decreasing ice flexural strength with increasing specimen size for both sea ice and freshwater ice. They concluded that, after brine volume, specimen size will have the second greatest influence on flexural strength. They based their conclusion on statistical correlation analyses for five ice flexural strength parameters: brine volume, beam volume, grain diameter, temperature, and strain rate. They implemented their analyses on a database compiled of 1771 sea ice and 650 freshwater flexural strength measurements. They excluded all cantilever tests from their database due to the stress concentrations phenomena, which has been suggested to lower the flexural strength of ice through the presence of stress risers at the root of the beam. Using regression analysis on their database, they developed an empirical two-parameter model of sea ice flexural strength as a function of brine and beam volume. For freshwater ice, only beam volume was considered as the main factor controlling flexural strength as shown below:

$$\sigma_f = 1629\left(\frac{V}{V_1}\right)^{-0.084} \quad (1)$$

where σ_f is in kPa, V_1 is a reference volume (it was suggested to be 0.01 m^3) and V is the beam volume in m^3. Lau et al. (2001) added their data to Williams and Parsons (1994) for both sea and freshwater ice during a study on how to take scale effects in ice strength into consideration during centrifuge model testing; they came to the same conclusion. Lavrov (1971) also found from experiments that sea ice and freshwater ice flexural strength decreases as beam thickness increases.

Frederking and Sudom (2013) also found that the flexural strength of multi-year sea ice decreases as the specimen size increases. They found this result by analyzing simple beam (three and four-point) test data for large and small beams quarried from a multi-year sea ice ridge. They also analyzed ship ram data that was taken when traversing through multi-year sea ice floes and found that flexural strength decreased as the thicknesses of the ice cover increased.

Maattanen (1975) attributed the scale effect behavior to the stress field across the beam, which he suggested converts to two dimensions as beam size increases. As a result, it will be easier for fracture to take place between ice crystals, causing the beam to fail under lower loads.

Several researchers disagree with, or have neglected the presence of a scale effect in the flexural failure of ice. For instance, scale effects for flexural strength are not currently considered in the International Association of Classification Societies (IACS) Polar Class ships rules and the International Standard for Arctic Offshore Structures (ISO 19906). Parsons and Lal (1991) did not find a scale effect when they analyzed 13 datasets to check the goodness-of-fit of the Weibull and double exponential distributions for sea and freshwater ice flexural strength data. Parsons et al. (1992) found that for first-year sea ice, the influence of specimen size is very limited. Using regressing analysis on experimental results from three-point bending tests, they determined that sea ice flexural strength depends very weakly on beam volume, according to the relation $\sigma_f \alpha\, V^{-1/12}$. As for freshwater ice, the specimens showed less decrease in flexural strength with increasing volume, and the authors suggested that scale effect can be completely neglected. Timco and O'Brien (1994) found that larger beams generally had lower strengths than smaller ice beams, yet they attributed this result to the larger brine volume in large beam volumes and not to the beam size.

It is clear from the above that there is still much debate as to whether a scale effect should be considered for ice failing in flexure, which is surprising considering the importance of this parameter for ice-class ships and offshore structure design. Williams and Parsons (1994) suggested that the flexural strength encountered by a specific icebreaker or offshore platform when failing in bending is probably 50% of the measured flexural strength from small-scale beam tests. The last extensive study on the subject was carried out by Williams and Parsons (1994) and since then a wealth of new data has been collected or made public. All of this necessitates a more updated investigation of scale effects, which is the goal of this study.

To thoroughly examine scale effects in ice flexural failure, an updated database has been compiled, which includes data from 2073 freshwater ice beam tests, making this database the most comprehensive to the authors knowledge. Similar work on scale effects in sea ice flexural strength measurements is ongoing. This paper considers only scale effects in flexural strength of freshwater ice; sea ice will be considered in future publications.

DATA SOURCES

The data were obtained from 16 papers from the literature as summarized in Table 1. The table lists information about each source, including the authors, test type, number of tests, location (field or laboratory), beam volume, ice temperature and flexural strength. Flexural strength measurements are plotted in Figure 1 against beam volume; both are on logarithmic scale. Data points are given symbols according to their sources listed in Table 1.

Specimen volume in this database was chosen to be represented by beam volume as has been used by many researchers as a comprehensive way to study scale effects. Williams and Parsons (1994) also suggested representing specimen size by beam volume, mainly because there are not enough details over the range of data in the literature to study the effect of each beam dimension separately. The same approach has been used in the present analysis.

The main approaches that have been used to measure ice flexural strength are cantilever beam, three-point and four-point bending tests. Cantilever tests are usually done *in situ*, and are easy to perform on large beams. They have the advantage of maintaining the temperature gradient and variation through the thickness of ice cover by utilizing its full thickness (Blanchet et al., 1997; Ji et al., 2011).

Table 1: Summary of Data

Symbol	Author	No. of tests	Location	Test type	Beam volume (m^3)	Temperature (°C)	Flexural strength (KPa)
▲	Parsons et al. (1992)	80	Laboratory	3-pt bending	0.027 - 2.197	-4.8,-5.5,-0.5	1805.9 ± 97
▲	Frederking and Timco (1983)	67	Field	Cantilever	0.00092-0.01593	-3	790.1 ± 205.8
▼	Lavrov (1971)	180	Field	Cantilever & 3-pt bending	0.00029 -0.102	-5.5 to -0.5	1645 ± 555.4
▼	Dempsey et al. (1988)	15	Laboratory	4-pt bending	0.00092 -1.012	-10	2169.3 ± 999.6
►	Barrette (2011)	56	Laboratory	4-pt bending	0.001	-9,-5.5,-0.5	1254.7 ± 561.8
◄	Tatinclaux and Wu (1978)	15	Laboratory	3 & 4-pt bending	0.00053	-5	2025.9 ± 444.2
▷	Tozawa and Taguchi (1986)	112	Laboratory	3-pt bending	0.00016-0.00281	-2	2047 ± 486.6
◄	Tabata (1967)	40	Laboratory	4-pt bending	0.00024	-15,-55	2810.5 ± 1347.5
◆	Frankenstein (1959)	228	Field	Cantilever	0.0296-0.899	-9.7 to 0	496.9 ± 175.9
◆	Gow and Langston (1975)	123	Field	Cantilever	1.219-0.0133	-	531.3 ± 369.7
●	Gow et al. (1978)	62	Field	Cantilever & 3-pt bending	0.532-1.38	-1,-3.5	756.8 ± 267.9
○	Gow et al. (1988)	706	Laboratory	Cantilever & 3-pt bending	0.00259-0.00117	-19 to -1	1226.6 ± 486.9
✱	Timco and Frederking (1983)	28	Laboratory	4-pt bending	0.0026	-3	867.9 ± 129.3
✱	Frederking and Sudom (2013)	6	Field	3-pt bending	0.0018	-21	2327
■	Drouin and Michel (1972)	331	Laboratory	4-pt bending	0.00457-0.724	-15, -1	1411.5 ± 479.5
□	Williams (1990)	22	Laboratory	3-pt bending	0.0355	-20	1715.1 ± 340.4

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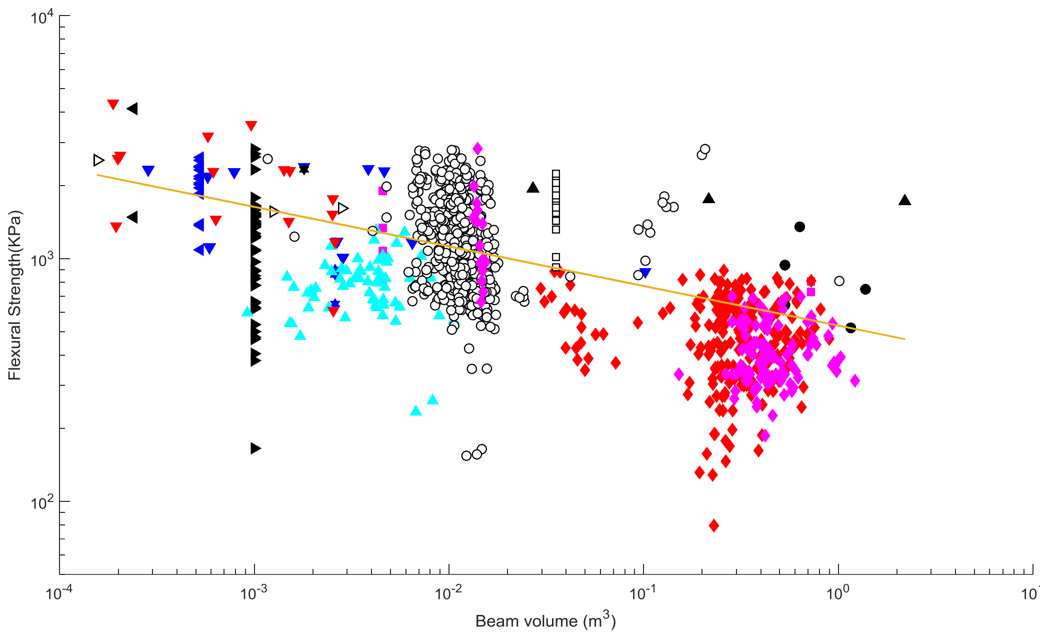

Figure 1. Freshwater ice flexural strength vs. beam size

The general technique for obtaining ice flexural strength using cantilever tests is as follows: First a U-shaped channel is cut in the ice. This channel isolates an in-place cantilever ice beam with one end attached to the sheet. Both pull-up and push-down tests can be performed on these beams using a vertical load applied to the free end of the ice beam until it fails; see Figure 2. As is discussed below, an important consideration for cantilever tests is that failure of the beam occurs at the root of the beam due to stress concentrations, which results in lower strength values than are typically obtained for three- and four-point tests (Frederking and Hausler, 1978; Timco and O'Brien, 1994).

For three- and four-point bending, the ice beam is completely cut free from the ice sheet. The ends of this beam are supported and load is applied at the center in case of three-point bending, and at two equidistant points in case of four-point bending, as is shown in Figure 3 and Figure 4, respectively. Three-point bending tests have the disadvantage that the beam usually fails at the center where the maximum moment takes place, preventing the beam from failing at its weakest point. Four-point bending tests result in a large central region of constant moment and zero shear between the loading points, which is generally recommended for brittle materials.

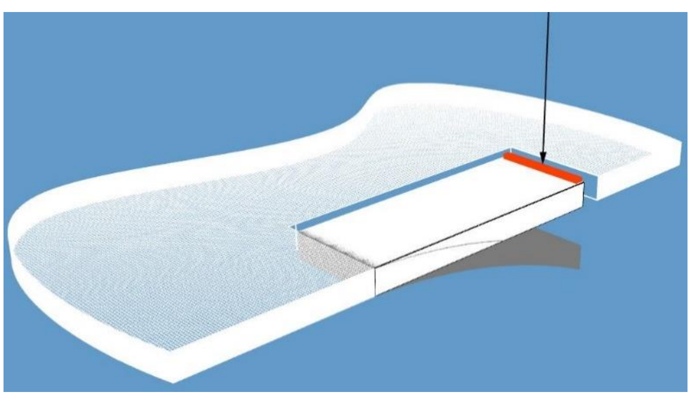

Figure 2: Cantilever beam test

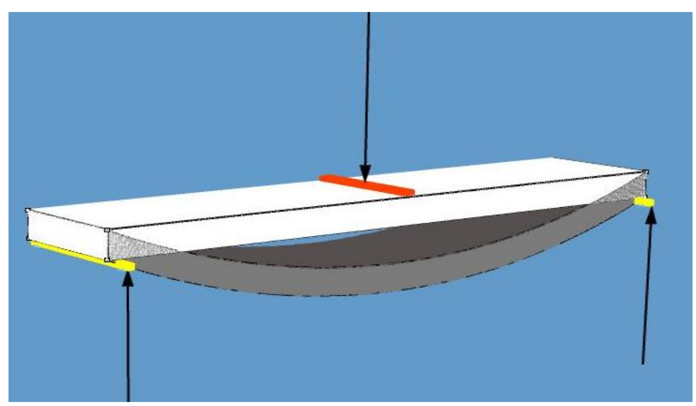

Figure 3: Three-point bending test

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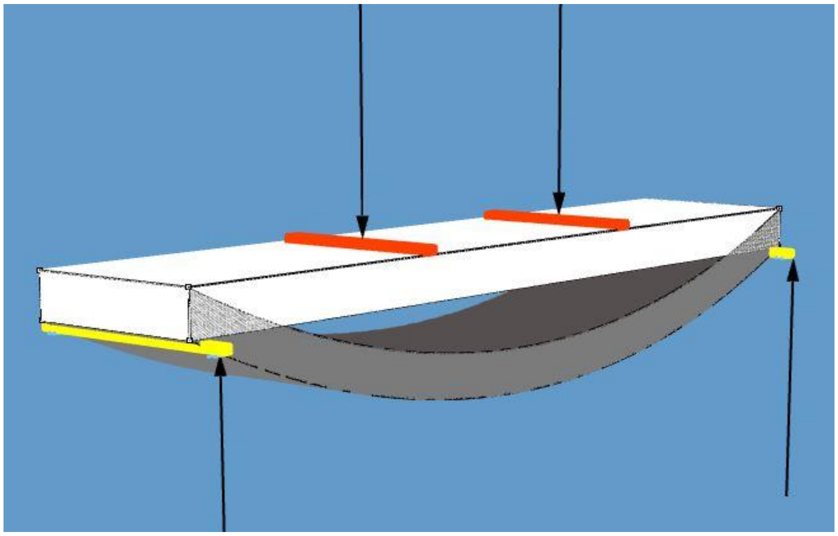

Figure 4: Four-point beam test

RESULTS AND DISCUSSION

The flexural strength of each test was plotted against ice temperature in Figure 5, as well as the average strengths at each temperature. It is clear that for freshwater ice, flexural strength does not depend significantly on ice temperature over the range of values typically of interest for engineering applications. This is consistent with the work of Timco and O'Brien (1994). Some other parameters that can affect flexural strength are harder to study; for example, loading rate is reported inconsistently or not at all in many cases. Thus, we shall only focus on the effect of specimen size herein.

Figure 1, where all measurements are plotted, shows that there is an obvious trend of decrease in freshwater ice flexural strength with increasing beam volume. Williams and Parsons (1994) suggested that strengths should be averaged for tests that have the same beam volume (and similar tests conditions) to avoid biasing the data towards small-scale strength measurements, which are represented in significantly more reported tests than larger-scale measurements.

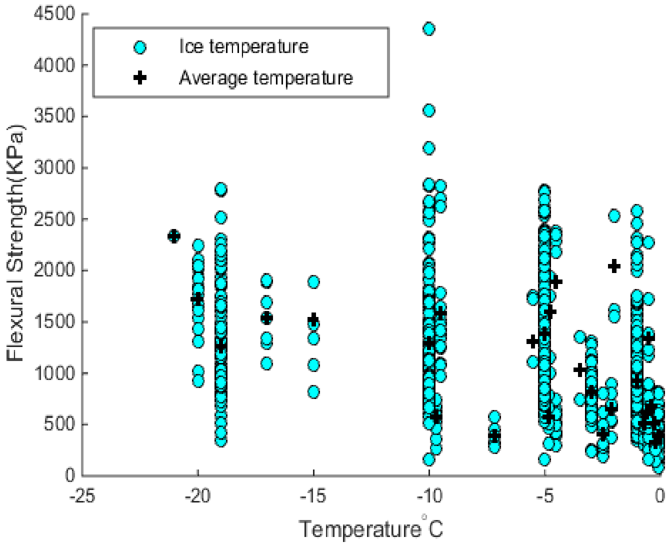

Figure 5: Freshwater ice flexural strength vs. ice temperature

To be consistent with this approach, the average strength values are given in Figure 6, along with the best fit line suggested by Williams and Parsons (1994). The figure shows that while there is a clear scale effect, the Williams and Parsons (1994) model line does not fully capture the scale effects trend, suggesting that further analysis is needed.

Test location (field or laboratory) and test type (cantilever, three-point or four-point bending) have an influence on flexural strength and the scale effects associated with it. To examine this, freshwater ice flexural strength measurements were plotted against beam volume whilst differentiating between test location in Figure 7 and test type in Figure 8.

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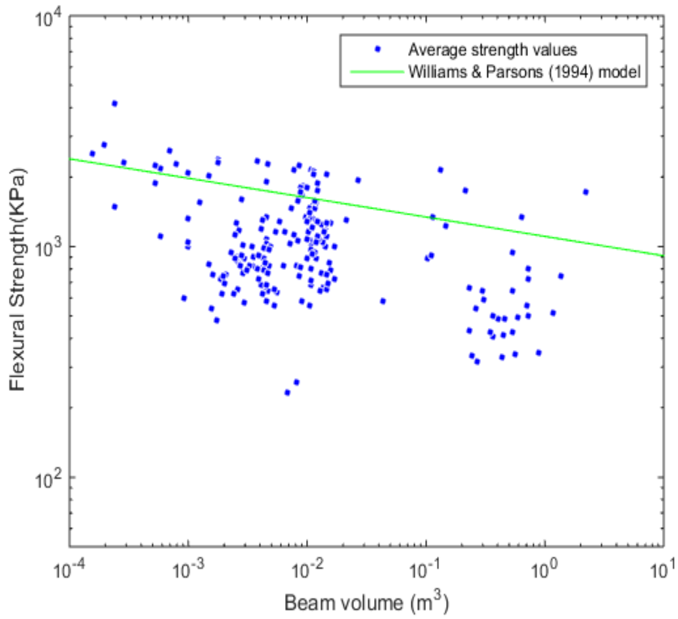

Figure 6: Freshwater ice flexural strength vs. beam volume using average values of strength for all tests with same beam volume

Differentiating between test locations (Figure 7) is of interest here because field data correspond to ice that is more representative of ice in full-scale applications. Ice in the field has many naturally occurring flaws that are not present in laboratory ice. Laboratory test specimens are usually selected to ensure they have minimal flaws. Furthermore, they have smaller volumes which may not be large enough to account for large grain sizes found in some ice environments.

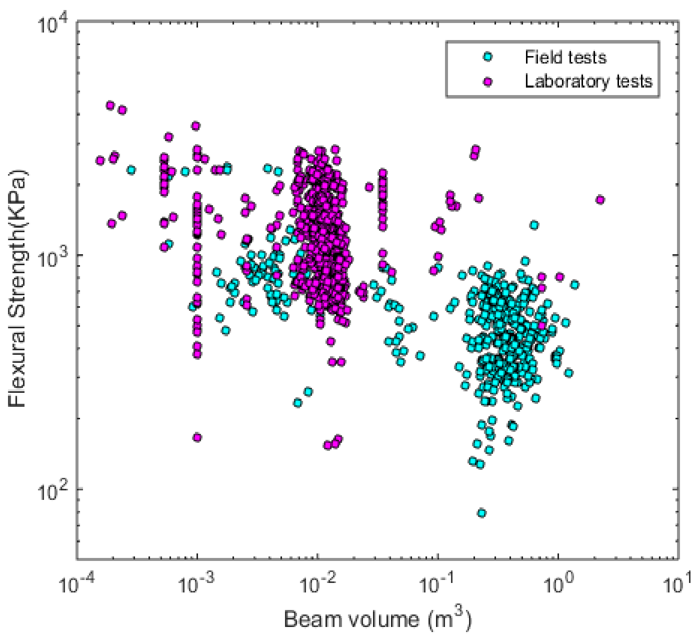

Figure 7: Plot of all freshwater ice flexural strength vs. beam volume data grouped to indicate test location as either field or laboratory

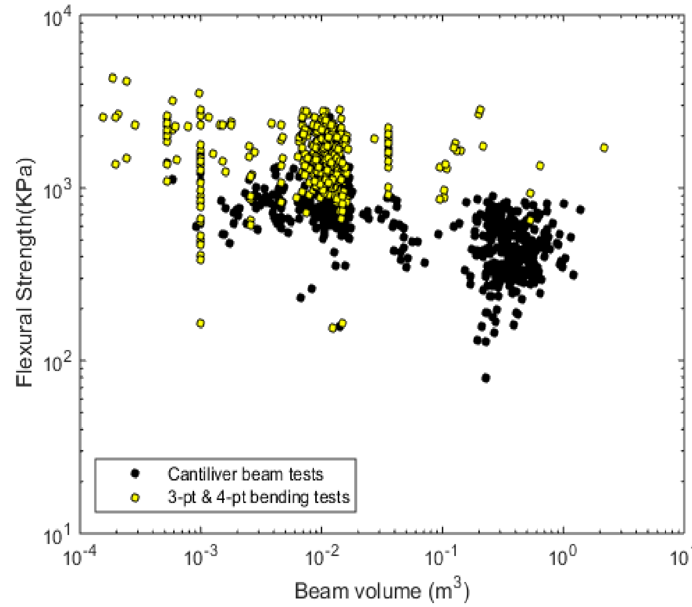

Figure 8: Plot of all freshwater ice flexural strength vs. beam volume data grouped according to test type

By comparing Figure 7 and Figure 8, it may be observed that that the field tests were mainly done using the cantilever technique, while most of the three- and four-point bending tests were conducted in the lab. As previously discussed, cantilever beam tests are generally believed to yield lower strength values than other measurement methodologies. This was confirmed by plotting the flexural strength field data against beam size while differentiating between cantilever beam tests and those for the three-point and four-point bending tests, as shown in Figure 9. This is mainly attributed to stress concentrations formed at the root of the beam. This behavior was studied and confirmed by several researchers, including Schwarz and Weeks (1977), Svec and Frederking (1981) and Svec et al. (1985).

To avoid excluding the cantilever tests from this analysis, as was done by Williams and Parsons (1994), a correction factor was used to account for reduction in strength in cantilever field tests. This was done by first fitting lines of best fit to the log-log plots (Figure 9), using non-linear least-squares regression method for the cantilever and grouped three- and four-point test field data, respectively. This produced the flexural strength equation for field cantilever tests:

$$\sigma_{f\ Cantilever} = 400(V/V_1)^{-0.13} \qquad (2)$$

Similarly, a flexural strength equation for the grouped field three- and four-point bending tests was obtained:

$$\sigma_{f\ 3-4\ Point} = 828(V/V_1)^{-0.13} \qquad (3)$$

where σ_f is the flexural strength in kPa, V is the beam volume in m^3 and V_1 is a reference volume (1 m^3). The exponents in the previous equations were rounded from -0.1296 and -0.1311 respectively to -0.13 to simplify calculations. Taking a ratio of Eq. (3) to Eq. (2) yields a correction factor of about 2. This is consistent with Gow (1977), who conducted a number of

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cantilever and three-point bending tests to explore this difference, and found correction factors in the range of 1.2 to 2.

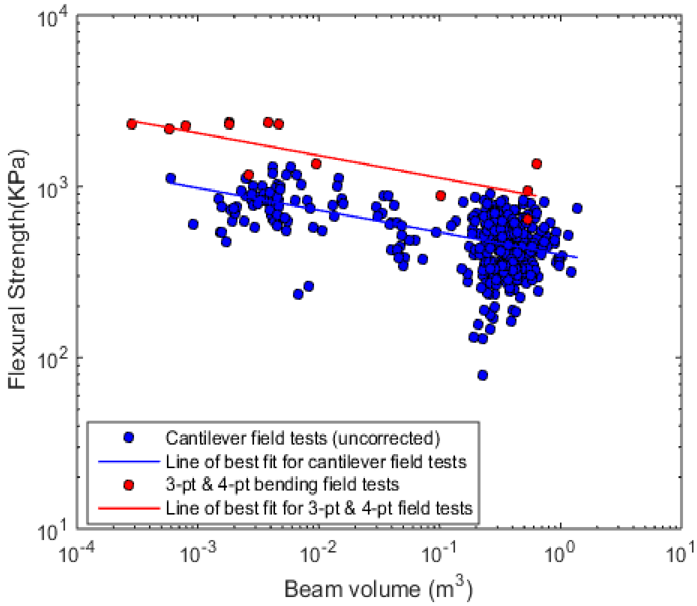

Figure 9: Plot of freshwater ice flexural strength tests vs. beam volume grouped by test type (field data only)

The corrected cantilever beam field data sets have been combined with the three-point and four-point field data sets and plotted in Figure 10. Using the same fitting method, the combined and corrected field data was fit by the relationship:

$$\sigma_f = 839(V/V_1)^{-0.13} \quad (4)$$

CONCLUSIONS AND RECOMMENDATIONS

For freshwater ice, the data show a considerable decrease in flexural strength as specimen size increases. Ice temperature has a limited effect on flexural strength. When examined over a large size range, scale effects will be a dominant factor affecting flexural strength. When considered separately, laboratory test data for ice flexural strength are observed to contain higher values and exhibit less pronounced scale effects than are expected in natural ice at full-scale. This is due to the smaller beam size and exclusion of specimens containing flaws from laboratory test programs. In the field, cantilever beam tests usually give lower flexural strength values compared to other testing techniques due to stress concentrations at the root of the beam. However, these results should be corrected to avoid excluding them since they are highly important in representing large-scale beams of ice formed under natural conditions. A correction factor of about 2 was observed when field data for cantilever and three- and four-point tests were compared. For future testing programs, it is recommended to use four-point bending tests since they produce a state of pure bending with constant moment and zero shear in the region between the two applied forces. This allows the beam to fail at its weakest point, rather than the loading point, which is desirable to provide more representative flexural strength measurements.

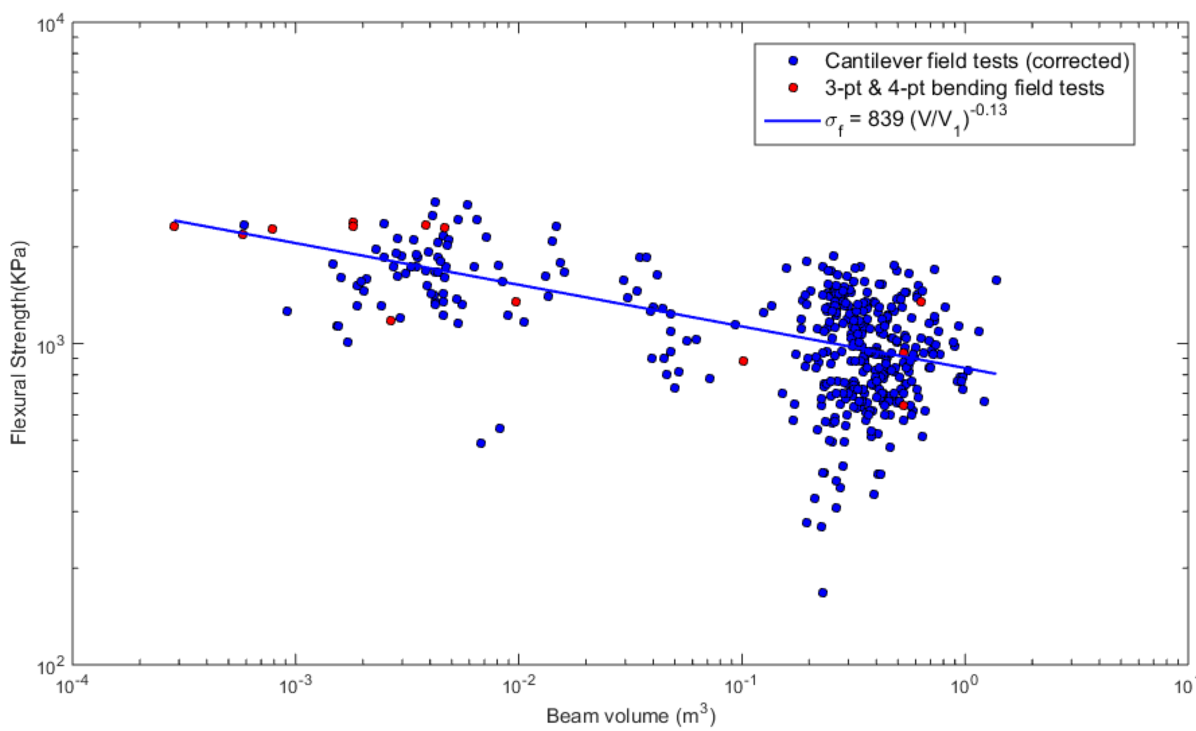

Figure 10: Freshwater ice beam flexural strength vs. beam volume for all field tests including corrected cantilever test data

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Based on the analysis presented in this paper, a new empirical relationship given by Eq. (4) above was developed to account for the effect of specimen volume on freshwater ice flexural strength. This expression can be used to assess how such scale effects may influence ice loads on ships and structures under different conditions, leading to potential opportunities for refinement of current design methodology. Similar analysis on a sea ice measurements database is taking place to get more understanding and insights into scale effects in ice flexural strength.

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REFERENCES

Barrette, P. D. (2011). "A Laboratory Study on the Flexural Strength of White Ice and Clear Ice from the Rideau Canal Skateway." *Canadian Journal of Civil Engineering,* 38(12), 1435-1439.

Barrette, P., Phillips, R., Clark, J., Crocker, G., Jones, S. (1999). "Flexural Behavior of Model Sea Ice in a Centrifuge." *Journal of Cold Regions Engineering*, 13(3), 122-138.

Blanchet, D., Abdelnour, R., Comfort, G. (1997). "Mechanical Properties of First-Year Sea Ice at Tarsiut Island." *Journal of Cold Regions Engineering,* 11(1), 59-83.

Borek, M., Parsons, B.,L., Hill, B. (1988). "Simple beam tests on type S3 sea ice." *Proceedings of the 7th International Conference on Offshore Mechanics and Arctic Engineering,* Houston, TX.

Dempsey, J. P., Nigam, D., Cole, D. M., Dempsey, J. P., Nigam, D., Cole, D. M., Sodhi, D. S., Luk, C. H., Sinha, N. K. (1988). "The flexure and fracture of macrocrystalline S1 type freshwater ice." *Proceedings of the 7th International Conference on Offshore Mechanics and Arctic Engineering,* Houston, Texas, 39-46.

Drouin, M., and Michel, B. (1972). *La Resistance En Flexion De La Glace Du St-Laurent Détermiée En Nature,* Laboratoire de Mécanique des Glaces, Département de Génie Civil, Faculté des Sciences, Université Laval,

Frankenstein, G., and Garner, R. (1970). "Dynamic Young's Modulus and Flexural Strength of Sea Ice." *Cold Regions Research and Engineering Lab.*

Frankenstein, G. E. (1959). *Strength Data on Lake Ice,* U.S. Army Snow Ice and Permafrost Research Establishment, Corps of Engineers, Wilmette, IL.

Frederking, R., and Sudom, D. (2013). "Review of flexural strength of multi-year ice." *Proceedings of the Twenty-Third International Offshore and Polar Engineering Conference,* International Society of Offshore and Polar Engineers.

Frederking, R. M. W., and Timco, G. W. (1983). "On Measuring Flexural Properties of Ice using Cantilever Beams." *Annals of Glaciology,* 4, 58-65.

Frederking, R.M.W., and Häusler, F.U. (1978). "Flexural behaviour of ice from in situ cantilever beams tests." *Proceedings of the IAHR Symposium on Ice Problems,* Luleå, Sweden, 197-215.

Gow, A.J., Ueda, H.T., Govoni, J.W., Kalafut, J. (1988). "Temperature and Structure Dependence of the Flexural Strength and Modulus of Freshwater Model Ice." *Cold Regions Research and Engineering Lab.*

Gow, A.J. (1977). "Flexural Strength of Ice on Temperate Lakes." *Journal of Glaciology,* 19(81), 247-256.

Gow, A.J., and Langston, D. (1975). "Flexural Strength of Lake Ice in Relation to its Growth Structure and Thermal History." *Cold Regions Research and Engineering Lab.*

Gow, A.J., Ueda, H.T., Ricard, J.A. (1978). "Flexural Strength of Ice on Temperate Lakes: Comparative Tests of Large Cantilever and Simply Supported Beams." *Cold Regions Research and Engineering Lab.*

Ji, S., Wang, A., Su, J., Yue, Q. (2011). "Experimental Studies on Elastic Modulus and Flexural Strength of Sea Ice in the Bohai Sea." *Journal of Cold Regions Engineering,* 25(4), 182-195.

Jordaan, I.J., Taylor, R.S., and Derradji-Aouat, A., (2012). “Scaling of Flexural and Compressive Ice Failure.” *Proceedings of OMAE 2012*, Rio de Janeiro, Brazil.

Lau, M., Phillips, R., McKenna, R., Jones, S. (2001). "Size effect on the flexural strength of ice in model testing." *Proceedings of the 16th International Conference on Port and Ocean Engineering Under Arctic Conditions, 12-17 August 2001, Ottawa, ON.*

`

Lavrov, V. V. (1971). "DEFORMATION AND STRENGTH OF ICE [Deformatsiya i Prochnost' L' Da] (English)." *DEFORMATION AND STRENGTH OF ICE [Deformatsiya i Prochnost' L' Da]*.

Maattanen, M. (1975). "On the flexural strength of brackish water ice by in-situ tests." *Proceedings of the 3rd International Conference on Port and Ocean Engineering Under Arctic Conditions (POAC),* 349-359.

Parsons, B.L., Lal, M., Williams, F.M., Dempsey, J.P., Snellen, J.B., Everard, J., Slade, T., Wllliams, J. (1992). "The Influence of Beam Size on the Flexural Strength of Sea Ice, Freshwater Ice and Iceberg Ice." *Philosophical Magazine. A,* 66(6), 1017-1036.

Parsons, B. L., and Lal, M. (1991). "Distribution Parameters for Flexural Strength of Ice." *Cold Regions Science and Technology,* 19(3), 285-293.

Schwarz, J., and Weeks, W. F. (1977). "Engineering Properties of Sea Ice." *Journal of Glaciology,* 19(81), 499-531.

Svec, O. J., Thompson, J., Frederking, R. (1985). "Stress Concentrations in the Root of an Ice Cover Cantilever: Model Tests and Theory." *Cold Regions Science and Technology,* 11(1), 63-73.

Svec, O.J., and Frederking, R. (1981). "Cantilever Beam Tests in an Ice Cover: Influence of Plate Effects at the Root." *Cold Cold Regions Science and Technology,* 4(2), 93-101.

Tabata, T. (1967). "Studies of the Mechanical Properties of Sea Ice X. the Flexural Strength of Small Sea Ice Beams." *Studies of the Mechanical Properties of Sea Ice X. the Flexural Strength of Small Sea Ice Beams,* 481-497.

Tatinclaux, J. C., and Wu, C. Y. (1978). "On flexural strength of freshwater and saline ice." *Proceedings of the Conference on Applied Techniques for Cold Environments,* Anchorage, Alask, 295-306.

Taylor, R.S., Jordaan, I.J., (2015). “Probabilistic fracture mechanics analysis of spalling during edge indentation in ice.” *Engineering Fracture Mechanics,* 134 (2015) 242-266.

Timco, G.W., and Weeks, W.F. (2010). "A Review of the Engineering Properties of Sea Ice." *Cold Regions Science and Technology*, 60(2), 107-129.

Timco, G.W., and O’Brien, S. (1994). "Flexural Strength Equation for Sea Ice." *Cold Regions Science and Technology,* 22(3), 285-298.

Timco, G.W., and Frederking, R.M.W. (1983). "Flexural Strength and Fracture Toughness of Sea Ice." *Cold Regions Science and Technology*, 8(1), 35-41.

Tozawa, S., and Taguchi, Y. (1986). "A preliminary study of scale effect on flexural strength of ice specimen." *Proceedings of the Fifth International Offshore Mechanics and Arctic Engineering Symposium,* American Society of Mechanical Engineers, 336-340.

Vaudrey, K.D. (1983). "Characterization of offshore and coastal ice conditions." *Proceedings of Design for Ice Forces,* ASCE, 3-16.

Weeks, W.F., and Assur, A. (1969). "Fracture of Lake and Sea Ice." *Cold Regions Research and Engineering Lab.*

Weeks, W. F., and Assur, A. (1967). "Mechanical Properties of Sea Ice." *Cold Regions Research and Engineering Laboratory.*

Williams, J. L. (1990). *Three Point Loading of Freshwater Ice Beams,* National Research Council Canada. Institute for Marine Dynamics, St. John's, NL.

Williams, F. M., and Parsons, B. L. (1994). "Size effect in the flexural strength of ice." *Proceedings of the 13th International Conference on Offshore Mechanics and Arctic Engineering,* Houston, Texas, 15-22.