USING MANIPULATIVES WITH FRACTIONS, DECIMALS, INTEGERS, AND ALGEBRA: A GUIDE FOR THE INTERMEDIATE TEACHER

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USING MANIPULATIVES WITH FRACTIONS, DECIMALS, INTEGERS, AND ALGEBRA:

A GUIDE FOR THE INTERMEDIATE TEACHER

by

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ABSTRACT

Research in recent years has shown that traditional methods of instruction being employed in many intermediate classrooms today are clearly lacking. The National Council of Teachers of Mathematics (NCTM) states that "learning mathematics without understanding has long been a common outcome of school mathematics instruction" (NCTM, 2000, p. 19). According to cognitive learning theorists, true learning involves moving from the concrete to the abstract. These concrete representations, or manipulatives, contribute to the development of well-grounded, interconnected understandings of mathematical ideas (Stein and Bovalino, 2001). As such, the NCTM and the Atlantic Provinces Education Foundation (APEF) have incorporated them into recent mathematics reform at the intermediate level.

The new intermediate mathematics curriculum calls for the regular use of manipulatives by teachers and students (APEF, 1999). This, however, causes problems for the teacher who is unsure of how to use them. Educational methods courses which focus on the use of manipulatives have been restricted to primary and elementary levels. This reinforces the idea that manipulatives are intended only for lower grade instruction.

This paper will address these issues in light of the new APEF curriculum for intermediate mathematics, culminating with a resource for teachers. This guide is intended to facilitate the efficacious integration of manipulatives into the teaching of fractions, decimals, integers, and algebra at the intermediate level.

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Using Manipulatives with Fractions, Decimals, Integers, and Algebra: A Guide for the Intermediate Teacher

Manipulatives have been in the forefront of recent educational reforms of intermediate mathematics teaching and learning. Yeatts (1991) defines them as "objects that students are able to feel, touch, handle, and move" (p. 7). Cognitive learning theorists suggest that they are a necessary component of mathematics, enabling students to move from the concrete into the more abstract areas of mathematics. The importance of manipulatives is supported by the National Council of Teachers of Mathematics (NCTM) and the Atlantic Provinces Education Foundation (APEF), who have incorporated them into recent mathematics reform. This, however, causes problems for the intermediate mathematics teacher who is unsure of how or when to use them.

This paper will address these issues in light of the new APEF curriculum for intermediate mathematics, culminating with a resource for teachers which will facilitate the efficacious integration of manipulatives.

Inside Today's Classroom

Research in recent years suggests that instructional methods being employed in many classrooms fail to focus on understanding. The NCTM states that "learning mathematics without understanding has long been a common outcome of school mathematics instruction" (NCTM, 2000, p. 19). This is further supported by the results of the National Assessment of Educational Progress and the Third International Mathematics and Science Study which illustrated that only sixty percent of students taught using traditional methods of instruction could compute efficiently (as cited in Tsuruda, 1998). This "traditional instruction is based on the outdated assumption that children must internalize in ready-made form the results of centuries of construction by adult mathematicians" (Warrington & Kamii, 1998, p. 343).

The mathematics curriculum being taught in classrooms today tends to emphasize low-level cognitive skills, such as rote memorization, as well as inadequate teaching practices, such as using one method of instruction exclusively or accepting only one answer as being correct (Kolstad, Briggs, and Hughes, 1993). The curriculum has become so packed that this seems to be the most efficient way to get the topics covered. It is unfortunate that true learning is not taking place at the same time.

The driving force behind instruction in the classroom is covering the objectives of the prescribed curriculum. Curriculum goals become more important so teachers rush students to the algorithm. On the surface it appears as though these goals can be realized, however, by looking deeper one can see that doing so can have detrimental effects on the learning that takes place.

> [Students] are unlikely to have the time to contemplate and compare methods, to talk about their work, and to reason through their own processes. This loss of understanding creates a weak foundation for reasoning in current and later mathematics work and destroys students' faith in their own powers of invention. Rather than help them trust their intelligence and thoughtfulness, telling students how to do problems has the long-term effect of disempowering them mathematically. (Corwin, Russell, and Tierney, 1991, p. 13)

The math classroom becomes a place where the objective is to memorize a rule. Cramer and Bezuk (1991) showed that students can easily learn these rules with little conceptual understanding. Later, when these rules are forgotten, students have nothing to fall back on (Hinzman, 1997). When facts or procedures are memorized without understanding, students are often not sure of when or how to use what they know, and such learning is often quite fragile (Bransford, Brown, and Cocking, 1999 as cited in NCTM, 2000). Mack (1990) adds that children's knowledge of algorithms is often faulty and frequently interferes with their thinking. She states that knowing that children's awareness of a rule prevents them from drawing on their own informal knowledge and, hence, they are even more likely to accept the answers they obtained using their faulty procedures rather than those obtained by relying on their own informal knowledge. These deficiencies in the classroom have led the NCTM to suggest that reforms be made to the mathematics curriculum.

A Call for Reform

The NCTM states that "calls for reform in school mathematics suggest that new goals are needed" (NCTM, 1989, p. 3). Ten NCTM Standards describe what and how mathematics should be taught. The five Content Standards explicitly describe the content to be taught, while the five Process Standards suggest ways in which the content can be acquired and used. One of these Process Standards is representation.

> Instructional programs from pre-kindergarten through grade 12 should enable all students to -
>
> - create and use representations to organize, record, and communicate mathematical ideas;
> - select, apply, and translate among mathematical representations to solve problems;
> - use representations to model and interpret physical, social, and mathematical phenomena. (NCTM, 2000, p. 66).

These representations are fundamental to mathematical understanding as they provide students with the opportunity to expand their capacity to think mathematically (NCTM, 2000).

The NCTM (2000) states that multiple representations are necessary to support mathematical understanding and should be emphasized throughout the mathematics curriculum. They propose an intermediate level mathematics curriculum in which representations are used extensively. "Middle-grade students who are taught with this Standard in mind will learn to recognize, compare, and use an array of representational forms for fractions, decimals, percents, and integers." (NCTM, 2000, p. 279).

The NCTM (2000) suggests that repetition of topics from year to year must be replaced with a more in-depth look at the topics when they are first introduced. Students conceptual understanding must be evident before a new topic is introduced. The NCTM (1989) states that in order to achieve this end the strategies presently being used by teachers may need to be reversed and that knowledge should emerge from experience - "knowing mathematics is doing mathematics" (p. 7). "This constructive, active view of the learning process must be reflected in the way much of mathematics is taught." (NCTM, 1989, p. 10).

According to Hinzman (1997), effective teaching of mathematics needs to focus on instruction which promotes students' activity and moves away from lecturing. Students must be "actively engaged each day in the doing of mathematics" (APEF, 1999, p. 6).

> Students, particularly at the middle school level, need to be actively engaged in learning. They need to be able to verbalize their ideas and share them with their classmates; to be given opportunities to build their understanding of mathematical concepts by "doing" mathematics; and to be actively involved in activities designed for collaboration, discussion, thinking, and reflecting. (Tsuruda, 1998, p. 5)

The classroom can no longer be a place filled with empty vessels waiting for knowledge to be transmitted. Students must be actively involved in the construction of their own knowledge.

> Learning should engage students both intellectually and physically. They must become active learners, challenged to apply their prior knowledge and experience in increasingly more difficult situations. Instructional approaches should engage students in the process of learning rather than transmit information for them to receive. (NCTM, 1989, p. 67)

Warrington and Kamii (1998) state that "children will go much further, with depth, pleasure, and confidence, if they are allowed to construct their own mathematics that makes sense to them every step of the way" (p. 343). The learning of mathematics must be an active and constructive process (APEF, 1999). Students should be actively engaged in tasks and experiences designed to deepen and connect their knowledge (NCTM, 2000). These findings point to a constructivist approach.

The Constructivist Classroom

Students in a constructivist classroom are presented with problems and encouraged to invent their own ways to solve them (Warrington and Kamii, 1998). Learning becomes very much a personal affair as students construct their knowledge in different ways through utilizing past experiences, existing knowledge, learning styles and motivation (Post, 1992). As such, they are empowered by the knowledge that the answers lie within them. As their confidence in their mathematical ability increases they are more motivated to learn and become more receptive to new experiences. These learning experiences, which are facilitated by the teacher, enable students to become lifelong learners (Anderson, 1996), thereby achieving one of the societal goals identified by the NCTM in 1989. This role of teacher as facilitator is a central premise of the constructivist philosophy. According to Wentworth and Monroe (1995), the teacher's role is to present student centered tasks and to question students as they work through them in such a way as to guide their learning. Teachers must design engaging and challenging activities which meet the needs of their students, while encouraging free thinking and risk taking (Tsuruda, 1998).

It is important to note that "constructivism is a philosophy of learning, not a methodology of teaching" (Clements, 1997, p. 200). As such, it is important to understand the learning that is taking place, as well as the underlying theories upon which it was founded.

Cognitive Theories of Learning

Cognitive psychology provides the major theoretical rationale for promoting the student as an active participant in the learning process. True understanding is given highest priority in the teaching-learning process and is achieved as students internalize concepts and make them their own (Post, 1992). The works of Piaget, Bruner, and Lesh have contributed greatly to this cognitive psychological perspective. Piaget initially divided learning into four stages of intellectual development - sensorimotor, preoperational, concrete operational, and formal operational (Post, 1992). Students must be provided with developmentally appropriate materials to assist movement through these stages.

Bruner, who was greatly influenced by the work of Piaget, suggested three modes of representational thought (as cited in Post, 1992). The first mode is enactive, which involves hands-on or direct experience. The second, iconic, is based on the use of visual mediums such as pictures and diagrams. The third mode is termed symbolic because the learner uses abstract symbols to represent reality. Textbooks, by their very nature, are exclusively iconic and symbolic. An enactive void is created unless the textbook is supplemented by hands-on activities (Post, 1992).

Lesh's translation model (1979) is an extension of Bruner's work (as cited in Post, 1992). He used "manipulative aids" to refer to enactive, "pictures" to refer to iconic, and "written symbols" to refer to symbolic. Lesh also added two more representations to Bruner's model, namely, "spoken symbols" and "real-world situations". He stressed the interdependence of these five modes and stated that learning must involve various translations within and among them (Post, 1992). Regardless of the terms used in these cognitive psychological perspectives, it is evident that all involve students moving from the concrete through to the abstract.

Manipulatives - The Missing Link

Research shows that manipulatives are the missing link to help students bridge the gap between their own concrete sensory environment and the more abstract levels of mathematics (Yeatts, 1991). They provide students with a concrete way to explore mathematical concepts (Moyer and Jones, 1998). "[Manipulatives] provide a concrete way for students to link new, often abstract information to already solidified and personally meaningful networks of knowledge, thereby allowing students to take in the new information and give it meaning." (Stein and Bovalino, 2001, p. 356).

The new curriculum for intermediate mathematics calls for the regular use of manipulatives by teachers and students (APEF, 1999). Cathcart, Pothier, Vance, and Bezuk (2000) state that "the use of manipulative materials is essential in all mathematics classrooms" (p. 23). It is clearly evident from the research that effective learning should emphasize the construction of knowledge through the use of such manipulatives (Sheffield and Cruikshank, 2001; Stein and Bovalino, 2001; Cathcart et al., 2000; NCTM, 2000; APEF, 1999; Tsuruda, 1998).

Positive Attributes

The successful integration of manipulatives into the mathematics curriculum can have positive consequences for both the teacher and the students. The proper use of manipulative materials can assist teachers and students in attaining the goals put forth by the NCTM and the new APEF curriculum for intermediate mathematics. They can assist teachers in meeting the individual needs of students, whether they be different learning styles or varying mathematical abilities. Manipulatives can also increase students' conceptual understanding and performance, as well as motivating students and decreasing mathematical anxiety.

As stated previously, one of the Process Standards identified by the NCTM is representation. "In order to become deeply knowledgeable about fractions - and many other concepts in school mathematics - students will need a variety of representations that support their understanding." (NCTM, 2000, p. 68). A multitude of manipulatives can be used to provide these multiple representations. For example, Cuisenaire rods, fraction pieces, and pattern blocks, are manipulatives that can be used to effectively represent operations with fractions. The incorporation of these manipulatives, as well as others, into the teaching of various intermediate level topics, such as fractions, decimals, integers, and algebra, will be addressed later in this project.

With the advent of the new APEF curriculum for intermediate mathematics it is crucial that teachers find a way to deal with varying abilities and learning styles within the same class. "Every classroom comprises students at many different cognitive levels. Rather than choosing a certain level at which to teach, a teacher is responsible for tailoring instruction to reach as many of these students as possible." (APEF, 1995, p. 32). Manipulatives can assist the teacher in achieving this goal by providing for individual learning styles (Yeatts, 1991) and allowing teachers to be better able to assess and meet the individual needs of students (Ross and Kurtz, 1993). Toliver (1996) notes that many students are visual or kinesthetic learners and the use of manipulatives makes the traditional, abstract way of thinking become clear for them. They need to manipulate objects in order to internalize the concepts being taught (Sheffield and Cruikshank. 2001). When teachers design open-ended activities which incorporate manipulatives. students can work to their own level without losing faith in their mathematical abilities (Fractions .... 1997). Manipulatives can also assist in meeting the needs of students with learning disabilities. as well as those who are mathematically gifted (Thornton and Wilmot. 1986).

The effective use of manipulatives provides a strong basis for conceptual understanding (NCTM. 2000: Hinzman. 1993: Kolstad. Briggs. and Hughes. 1993: Ross and Kurtz. 1993). "Children seem to learn best when learning begins with a concrete representation." (Cathcart et al.. 2000. p. 23). These concrete representations. or manipulatives. can contribute to the development of well-grounded. interconnected understandings of mathematical ideas (Stein and Bovalino. 2001). When students have a strong conceptual foundation they are better able to understand the formal algorithms because they understand the thinking behind them (Toliver. 1996). It is essential that students make this connection between the conceptual work done with manipulatives and the procedural knowledge it is intended to support (NCTM. 1989).

Students who use manipulatives usually outperform those who do not (Kennedy and Tipps. 2000: Sowell. 1989: Suydam. 1986). A study conducted by Parham in 1983 showed that students using manipulatives achieved in the 85th percentile. while those not using them were in the 50th percentile (as cited in Suydam. 1986). Moser (1986) goes as far as to say that when manipulatives are used properly they may remove the need for later remediation. Manipulative use was proven to improve problem solving skills in general (Canny. 1984 as cited in Suydam. 1986) and to increase scores on retention and problem solving tests (Baroody. 1996 as cited in Clements and McMillen. 1996).

The incorporation of manipulatives also increases student motivation (Hinzman, 1997; Moyer and Jones, 1998; Dutton and Dutton, 1991; Yeatts, 1991) and significantly reduces mathematical anxiety (Vinson, Haynes, Brasher, Sloan, and Gresham, 1997). If students enjoy what they are doing and perceive it to be fun, they will be more willing to participate in classroom activities. Consequently, their attitudes toward mathematics will improve (Moyer and Jones, 1998; Sowell, 1989). In order to foster this positive attitude in students, teachers must exhibit a positive attitude toward mathematics as well. Research shows that teachers' beliefs and attitudes influence their behavior in the classroom which ultimately affects students' beliefs (Benninga, Guskey, and Thornburg, 1982). Self-reflection is needed by the teachers so that they will become more cognizant of the role they play in shaping their students into mathematical learners.

> If we help students develop positive beliefs and attitudes towards mathematics, their performance should improve. If we encourage students to think of mathematical problems as challenges rather than frustrations, they should be better able to control their emotions. If students' affective responses improve, our mathematics classrooms can be much more inviting places for both teaching and learning. (McLeod and Ortega, 1993, p. 33)

Regardless of the amount of research which points to the positive, theory is often overshadowed by the realities of the classroom.

Putting Theory into Practice

There is often great difficulty in putting theory into practice. Changes suggested for mathematics education reform must go well beyond the documents and materials.

No document, no exhortation, no program or set of materials can, by itself, change what goes on in classrooms. Change depends on teachers' working alone and together to teach in ways that help all students develop mathematical literacy and power, and to improve teaching as envisioned by the Professional Teaching Standards. (Ball and Schroeder, 1985, p. 67)

"[Manipulatives] have little intrinsic educational value....the real value of manipulatives in the classroom lies in the ways in which they are incorporated into lessons." (Ross and Kurtz, 1993, p. 256). To this end, Ross and Kurtz identify four ways to promote the success of manipulative use.

1. Manipulatives have to be chosen to support the lesson's objectives; it is not enough to make them available.
2. Students need to be made aware of the rules and procedures regarding the use of manipulatives in the classroom.
3. During a lesson each student must be actively involved.
4. Evaluation should reflect an emphasis on conceptualization and understanding.

Concern for individual needs must govern the use of manipulative materials (Suydam, 1984). It is important to recognize that students may differ in their need for manipulatives, hence, they should be able to make the choice of whether or not they use them (Sheffield and Cruikshank, 2001; Clements and McMillen, 1996). Manipulatives should not be locked away in a cupboard and used only for specific lessons. They should be made available to students to use at their own discretion. This gives students a sense of control, as well as enabling them to see the manipulatives as tools, rather than toys (Moyer and Jones, 1998). Some students will find it easier to work with one type of manipulative over another (Sheffield and Cruikshank, 2001). If students are restricted to working with the manipulative chosen by the teacher it may make the problem seem more difficult (Touger, 1986). Consequently, it is important for the teacher to have a variety of manipulatives available to the students (Sheffield and Cruikshank, 2001; Moyer and Jones, 1998). Each time a new manipulative is introduced it is important to allow time for free exploration (Joyner, 1990). Sheffield and Cruikshank (2001) suggest interviewing the students frequently to determine the best way to utilize each manipulative for the students. Once the teacher is confident that the students have achieved an adequate level of proficiency using each type, it should be left to the student which type they will choose, if any.

It is important for teachers to become proficient with the range of manipulatives (Clements and McMillen, 1996). They must learn to use the manipulatives as tools for conceptual understanding, rather than just for games or problem solving (Moyer and Jones, 1998). Students need to see teachers model the use of these manipulatives (Joyner, 1990). Teachers who do so are opening many doors for students who struggle with the abstract nature of mathematics (Moyer and Jones, 1998). Students' conceptualization will improve only if teachers are knowledgeable about the use of the manipulatives (Sowell, 1989).

In order for manipulatives to be effectively incorporated into the curriculum, students must discuss their thoughts while working through problems (Robitaille and Travers, 1992). Students should be encouraged to reflect on and justify their solutions to these problems (Clements and McMillen, 1996). This self-reflection is an important part of the lesson because it promotes learning and decreases anxiety (Heuser, 2000).

Students must be committed to expressing their learning in meaningful ways before using manipulatives can be productive (Thompson, 1992). The teacher can help facilitate this by providing students with different ways of articulating their learning. Journals can be an effective means of recording their thoughts about using the manipulatives to solve a particular problem. Teachers can design worksheets which correlate with the lesson so that students can keep track of their progress throughout the class. Questioning the student in an interview type situation can prove to be an effective way for teachers to assess students' understanding on an ongoing basis.

Obstacles to Effective Implementation

Regardless of the research pointing to the positive aspects of manipulative use in intermediate mathematics, there still tends to be a widespread belief that manipulatives are intended only for lower grade instruction (Moyer and Jones, 1998; Tooke et al., 1992). A study conducted by Scott in 1983 showed that the percentage of teachers using manipulatives declined each year after the first grade, with that being fewer than 60% (as cited in Suydam, 1986). This is perhaps due, in great part, to the fact that few teachers beyond the elementary level have been trained in the use of manipulatives. Instructional methods courses that were offered to today's intermediate and secondary mathematics teachers did not focus on the utilization of manipulatives. It is no wonder, then, that intermediate mathematics teachers in today's classrooms hold this opinion of manipulative materials.

Teachers are also uncertain about how to use manipulatives (Tooke et al., 1992). They buy into "the fallacy of assuming that students will automatically draw the conclusions their teachers want simply by interacting with particular manipulatives" (Ball, 1992, p. 17). It is important to remember that "manipulatives do not magically carry mathematical understanding" (Stein and Bovalino, 2001, p. 356). It is not enough to make them available (Edwards, 2000; NCTM, 2000). Bohan and Shawaker (1994) stated that "sometimes in our fervor to use manipulatives, we lose sight of the fact that they are a means to an end, not an end in themselves"

(p. 246). This results in obliterating the pedagogical value of using them in the first place. Students see using manipulatives as little more than play time.

Even though discussion is an essential component of successful implementation, a study conducted by Stigler and Barnes (1988) determined that manipulatives are often used by teachers as a substitute for discussion (as cited in Robitaille and Travers, 1992). Students are often given the manipulatives and left to their own devices to complete an assigned task. Without discussion, it becomes unclear as to whether learning is taking place.

Students are also introduced to the algorithm too soon (Moyer and Jones, 1998). Teachers often assume that the manipulatives can be used the first day, for the introduction to the new topic, and then be taken away the next to proceed to the algorithm (Sheffield and Cruikshank, 2001). Students must understand the concept being taught before symbols are introduced or teachers will only be "adding abstractness to the abstraction" (Kolstad, Briggs, and Hughes, 1993, p. 183).

Another obstacle identified by Tooke et al (1992) is the textbook itself. They noted that the nature of textbooks does not correlate with manipulative use. Teachers are being inundated with new mathematics curricula, but are provided with little, if any, support (Ball, 1992). Professional development is at a minimum and textbooks do not match the new objectives. So, once again, teachers are left to their own devices to establish and maintain an effective way of implementing the new curriculum. Unfortunately, only those truly dedicated to their profession are willing to take the time needed to learn how to use manipulatives effectively (Tooke et al, 1992).

Conclusion

All is not lost however. The advantages of implementing manipulatives into the intermediate mathematics curriculum far outweigh any obstacles that might be encountered. It is crucial that this faith in the use of manipulatives be held by all levels within the educational system. Only then will teachers get the necessary support and professional development to successfully integrate manipulatives. "If manipulatives are to find their appropriate and fruitful place among the many possible improvements to mathematics education, there will have to be more opportunities for individual reflection and professional discourse." (Ball, 1992, p. 47).

With faith in the educational system and a willingness on the part of teachers to adapt to the new curriculum, these obstacles can be overcome and the efficacious integration of manipulatives into intermediate mathematics can be fully realized.

CONTEXT AND OVERVIEW OF GUIDE

Based on the Principles and Standards identified by the NCTM, the APEF has designed a new intermediate mathematics curriculum which is to begin implementation in September, 2001. This curriculum takes a new approach to the teaching of intermediate mathematics, placing a much stronger emphasis on the regular use of manipulatives. Most mathematics teachers at this level are high school trained, and therefore have little, if any, experience with manipulatives. Consequently, there is a need for teachers to become familiar with using them. This guide is designed to support the teacher's transition into the new curriculum by illustrating how manipulatives can be utilized.

It is intended to be used by teachers to supplement other instructional materials. It describes the various types of manipulatives suggested by the APEF and illustrates how they can be incorporated to meet the objectives of the intermediate curriculum. The guide consists of a terminology section, as well as five content sections devoted to multiple representations, fractions, decimals, integers, and algebra. The content sections cover the majority of the objectives of the intermediate curriculum, however, they are not exhaustive. Although this guide is geared toward the intermediate level, it can be used at any level where these topics are taught.

It is also set up in such a way that it can easily be used by any teacher with little or no mathematics background. It is assumed that teachers have a facility with pencil and paper computations. The guide, therefore, is not intended to teach the various concepts, but rather to explain them in a different way.

USING MANIPULATIVES

WITH FRACTIONS, DECIMALS, INTEGERS, AND ALGEBRA

A Guide for the Intermediate Teacher

by Tina M. Smith

INTRODUCTION

This guide is set up to be used as a stand alone resource. However, it can be supplemented by other resources, such as elementary methods texts. Based on the assumption that teachers at this level have very little experience with manipulatives, there is a section devoted to simply describing the manipulative materials and defining associated terminology. This section is designed to be used as a reference and could easily be omitted by the teacher who is familiar with the manipulatives. In such instances, the section may be used as a glossary. The section on multiple representations should be read before those of fractions and decimals, but the sections on integers and algebra can be covered independently. Before doing the algebra section, however, a familiarity with the *zero principle* is necessary, so it may be wise to read the section on integers first.

When working through the examples in the guide, it is suggested that one do so with a partner or in a group, if possible. The notion of multiple representations becomes increasingly evident when the guide is used in this manner. In contrast to a traditional approach, there are a multitude of ways to approach mathematics using manipulatives. Consequently, keep in mind that this guide is meant to introduce teachers to their use, not to be an exhaustive account of the various ways to do so.

An effort has been made to draw the diagrams accurately, however, the representations may not be exact. The illustrations are intended as instructional aids only. The original guide was prepared in color. Such a copy may be obtained from the author. The black and white copies contain the identical text, however, in the case of the algebra tiles, those tiles that represent negative quantities have been denoted with a negative sign. It is expected that users of the guide will work with actual manipulatives. Teachers are encouraged to model the various representations with different manipulatives and to work through the questions that have been included periodically throughout the guide.

TERMINOLOGY

Algebra Tiles

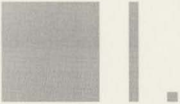
Algebra tiles are available commercially or can easily be made using paper. They consist of larger squares (x by x, or x^2), rectangles (x by 1, or x), and smaller squares (1 by 1, or 1). To represent positive and negative values, two different colors are used. Algebra tiles can be used to represent algebraic expressions. For example, $3x^2 + 2x + 4$ can be represented as follows:

If negative values are used, a different color can be chosen for the representation. For the purposes of this guide, black and red will be used to indicate positive and negative, respectively.

For example, $2x^2 - x - 3$ can be represented as follows:

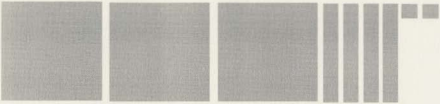
Base Ten Blocks

Similar to algebra tiles, base ten blocks consist of larger squares (10 by 10), rectangles (1 by 10), and smaller squares (1 by 1).



The larger square is often referred to as the "flat" and the rectangle is the "rod". Base ten blocks also come with a large cube (10 by 10 by 10). Base ten blocks are often used in primary and elementary grades to represent whole numbers, the large cube being 1000, the flat representing 100, the rod 10 and the small cube 1. For example, the number 1243 could be represented using 1 large cube, 2 flats, 4 rods, and 3 small cubes.

When representing decimals, the base ten blocks used will depend on what shape represents the unit. It is the fixed scaling factor of 10 that makes base ten blocks useful for representing decimals. For example, if the flat represents 1, then 3.42 can be modeled using 3 flats, 4 rods, and 2 small cubes.

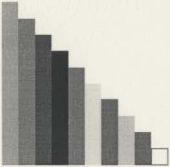


If , however, the large cube represents 1, then 3.421 can be modeled using 3 large cubes, 4 flats, 2 rods, and 1 small cube.

Cuisenaire rods

Cuisenaire rods are three-dimensional plastic or wooden rods which range in length from 1 cm to 10 cm, each length being represented by a different color. Commercially packaged sets contain the following colors:

white (1)
red (2)
light green (3)
purple (4)
yellow (5)
dark green (6)
black (7)
brown (8)
blue (9)
orange (10)

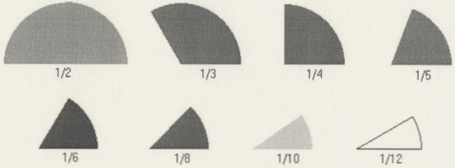


When using Cuisenaire rods it is helpful to line them up as pictured above, so that students have a quick reference to the color coding.

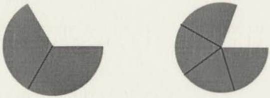
Cuisenaire rods can be used to represent fractions. For example, 3/5 and 5/8 can be represented as follows:

Fraction Pieces

Fraction pieces, or fraction circles, are circles which are subdivided into halves, thirds, quarters, fifths, sixths, eighths, tenths, and twelfths.



They are commercially available or can be duplicated using the blackline masters that accompany most teacher texts. Fraction pieces can be used to represent different fractions, such as 2/3 and 4/5.



Integer Counters

Commercial integer counters are bicolored circles with red on one side and yellow on the other. If they are unavailable, however, any object available in two colors can be used as integer counters. One color represents positive and the other represents negative quantities. Typically, each positive counter represents +1, and each negative counter represents -1. Integers can be modeled as follows:

Pattern Blocks

Pattern blocks are similar to fraction pieces, but are based on the hexagonal unit, rather than the circular unit. They are comprised of four basic shapes, namely hexagons, trapezoids, parallelograms, and triangles.

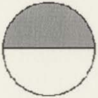
The relative areas of the pieces facilitate various fractional representations, such as 1/2, 1/3, 2/3, and 1/6, respectively.

MULTIPLE REPRESENTATIONS

Representing Fractions and Decimals

One of the Process Standards identified by the National Council for Teachers of Mathematics is representation. "Students can develop and deepen their understanding of mathematical concepts and relationships as they create, compare, and use various representations." (NCTM, 2000, p.279). These multiple representations are a crucial component in the teaching of fractions and decimals. A multitude of concrete manipulatives exist which can be successfully utilized to increase students' conceptual understanding. Cuisenaire rods, fraction pieces, pattern blocks, and base ten blocks can be quite effective in teaching fractions and decimals. The diagrams below show how to represent different fractions and decimals using these manipulatives.

1/2 can be easily represented using either the fraction pieces, the pattern blocks, or the Cuisenaire rods.

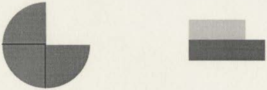


In cases where 1/2 is to be considered as 50%, the base ten blocks may be useful. As such, half of a flat could be covered as follows:

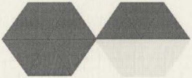
Students may also use other representations of 1/2, such as one of two identical objects.

It is important to note, however, that some manipulatives readily lend themselves to certain fractions, while others do not. The following example illustrates this.

The fraction 3/4 can be represented easily using fraction pieces or Cuisenaire rods.

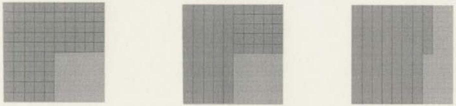


It becomes more difficult, however, to represent 3/4 using the pattern blocks because the hexagon can not be easily divided into 4 equal parts. Here it becomes helpful to think of the whole in terms of multiples of six. Consequently, 3/4 can be represented as 9/12. Therefore, two hexagons are used to represent the whole, and nine triangles are used to represent the fraction. Note that three trapezoids or other suitable combinations can also cover the area.



Students should be given the opportunity to work through the different representations and learn to make their own decisions regarding the most effective manipulative materials to use in each case. This will depend, in large part, on the purpose for which the fraction is being modeled. When working with two fractions it is important to look at the representation for each. The manipulative that can be used to model both fractions should be chosen. Although Cuisenaire rods are the most flexible, it is important that students learn to manipulate the other materials as well.

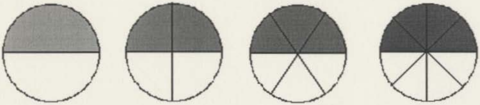
Base ten blocks can be used to represent different decimals. This will lend itself nicely when parts of 100, or percent, are introduced. As noted previously, multiple representations also exist using the base ten blocks. For example, 0.75 can be represented using: 75 small cubes; 5 rods and 25 small cubes; or 7 rods and 5 small cubes. In fact, any random covering of 75 out of the 100 unit squares would suffice.



Renaming Fractions and Decimals

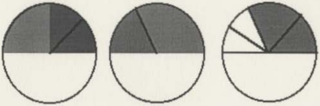
The ability to rename fractions and decimals is fundamental if students are to progress to using various operations with them. Students must understand the concept of equivalency in order to compare fractions and decimals. They need to know how to get equivalent fractions in order to understand the addition and subtraction of fractions. The following diagrams illustrate how the renaming of fractions and decimals can be carried out concretely.

Using fraction pieces, students can cover the shaded area using different pieces. The concept of equivalency is more accessible if identical pieces are used. For example, the following diagrams represent the same area.



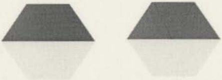
The same area is shaded, showing that 1/2 is equal to 2/4, 3/6, and 4/8.

Some students may choose to cover the area as follows.



These representations do not lend themselves nicely to equivalent fractions, however, they could prove beneficial when introducing addition, as students could be asked to determine different ways of making 1/2.

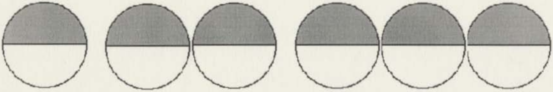
The same holds true when pattern blocks are used. However, it is important to note that the fractions obtained may be different, depending on how students choose to cover the area.

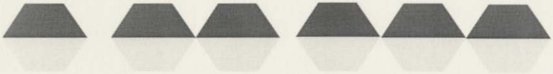


The same area is shaded, showing that 1/2 is equal to 3/6.

☞ How many ways can you represent 3/4?

Another way to illustrate equivalent fractions is achieved through repeating the same ratio. The following diagrams illustrate equivalency using fraction pieces, pattern blocks, and Cuisenaire rods.





These diagrams illustrate that 1/2 is equal to 2/4, 3/6, and so on.

Students must also understand how to rename fractions as decimals, and vice versa. This can be illustrated using the base-ten blocks.

0.3 can be represented as follows:

Considering the flat as the whole, 0.3 covers 30 blocks, which students should see as 0.30, 30 out of 100, or 3/10 of the area. Hence, 0.3 = 0.30 = 30/100 = 3/10.

A familiarity with the blocks can be established by having the students start with the flat and cover the correct area using the smaller pieces. For example, 1/4 can be represented as follows:

Students should note that 25 blocks are covered, or 0.25. Hence, 1/4 = 25/100 = 0.25.

Using the above method would also be very effective for teaching percent.

FRACTIONS

Limitations

Although operations involving fractions can be modeled using fraction pieces, pattern blocks, and Cuisenaire rods, it is important to keep in mind that some fractions lend themselves more toward one type of manipulative. A familiarity with all three is essential, however, if students are to learn to make appropriate choices regarding which is the most efficient.

The concept of lowest common denominator can be restrictive when representing fractions concretely. In some instances, it may be more effective to think in terms of a more efficient common multiple, which is not necessarily the lowest. Regardless of how the denominator is chosen, it may be impractical to model the fractions if it is a large number.

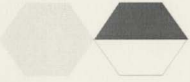
It is assumed that teachers and students will have an understanding of the previous section on multiple representations before proceeding.

Addition and Subtraction

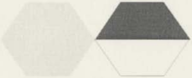
Common Misconceptions

It is important to stress to students that in order to add or subtract fractions the "whole" must be the same. The following example illustrates the common misconception of merely adding the parts.

Students should understand that both of the diagrams illustrate 1/2. Ask them what would happen if you added them together. If they add the pieces together they get the following, which represents 3/4 or 3/2, depending on which shape represents the whole.



3/4 if the whole is

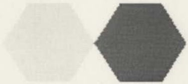


3/2, or 1 1/2, if the whole is

Most students will know that 1/2 + 1/2 is 1. This can be correctly modeled as follows:

First, both must be represented as part of the same whole.

Because the whole is the same, the pieces can be combined.



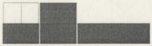
Some students may only represent the pieces themselves, while others may find it beneficial to represent the whole separately and then cover with the pieces. This reinforces the concept of the "whole".

A similar example using Cuisenaire rods is effective in illustrating the common misconception of adding numerators and denominators together. Before you begin, make sure that students understand that 2/4 is the same as 1/2.

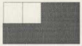
Again, ask students how these should be added together. If the pieces are combined you get the following, which represents 3/6, or 1/2.

This will surely puzzle the students. Question students about the whole being compared. One rod is only 2, while the other is 4. Both have to be the same in order for the fractions to be added together.

Now the rods can be combined. Students may suggest combining the top and the bottom.



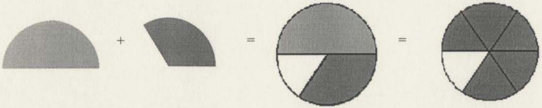
This results in 4 over 8, which still represents one half. Remind students of the concept of "parts of a whole". The whole that we started with was 4, so now we have 4 over 4, which represents 1.

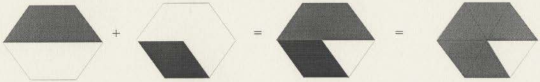


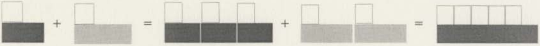
Again, the concept of whole can be reinforced if students model it separately with a purple rod, and then place both white rods and the red rod on top.

Addition

The following diagrams illustrate how 1/2 + 1/3 can be represented concretely, using the various manipulatives. Again, it is important to note that the whole must be the same, that is, fractions must have the same denominators.





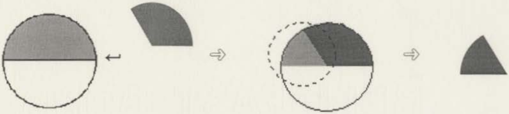


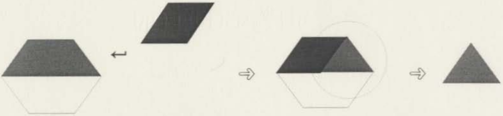
The three previous representations illustrate how to model 1/2 + 1/3 to obtain 5/6, utilizing the fraction pieces, pattern blocks, and Cuisenaire rods, respectively..

☞ Use the various manipulatives to show that 2/3 + 1/4 = 11/12.

Subtraction

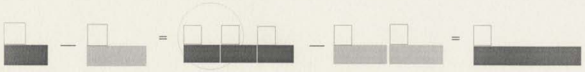
Subtraction can be done in much the same manner as addition. When using fraction pieces and pattern blocks it is more efficient to represent the first fraction, and then cover it with the second fraction. The area remaining uncovered would be the answer. The following diagrams illustrate 1/2 - 1/3.





Both examples illustrate that the area remaining uncovered represents 1/6.

When using Cuisenaire rods to model subtraction it is easier to model both fractions first, keeping in mind the concept of a common whole (or denominator). Once the fractions are represented, the appropriate pieces are removed. The previous example would be represented as follows:

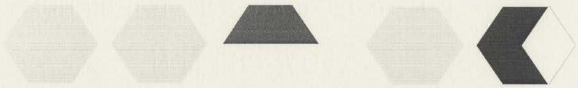


☞ Use the various manipulatives to show that 3/4 - 2/3 = 1/12.

Mixed Numbers

The following diagrams illustrate addition and subtraction when mixed numbers are involved. It is important that students consider both fractions to determine the most efficient manipulative to model them.

For example, 2 1/2 and 1 2/3 can easily be represented using pattern blocks.



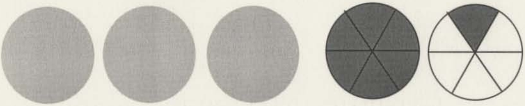
To add, simply combine the pieces.

Note that when the pieces are combined, 1/6 overlaps. Hence, 2 1/2 + 1 2/3 = 4 1/6.

Another approach to modeling addition involves exchanging pieces for more practical, smaller ones. Rather than determining the overlapping section, the smaller pieces are used to form the whole, and the remaining pieces determine the fractional part.

This method is illustrated below as 2 1/2 + 1 2/3 is modeled with fraction pieces.

The fractional parts being added, namely 1/2 and 2/3, have a common multiple of six. Consequently, they will be exchanged for 1/6 pieces, seven in total. Note, however, that different combinations of pieces could be used to cover the area.

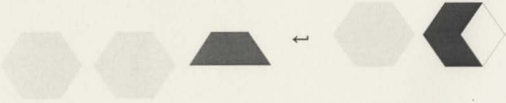


The above representation illustrates that 2 1/2 + 1 2/3 = 4 1/6.

☞ Use the various manipulatives to show that 1 3/4 + 2 5/6 = 4 7/12.

One strategy for subtraction is to represent the larger quantity before covering a portion of it with the smaller amount. The remaining uncovered area is the result.

For example, to model 2 1/2 - 1 2/3, represent 2 1/2 first. Then cover it with 1 2/3.



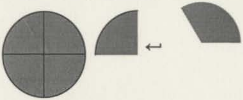
The illustration above shows that the are remaining uncovered is 1/3 + 1/2, or 5/6.

Another approach to subtracting is to begin with the smaller quantity and then add pieces to reach the larger amount. The result would be the total of the pieces that were added. Therefore, if representing the previous example, 1 2/3 would be modeled first. Pieces would then be added until the total was 2 1/2.

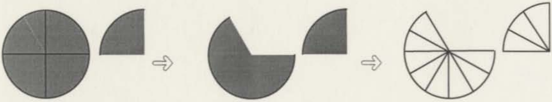
☞ Using this approach, can you show that 2 1/2 - 1 2/3 = 5/6?

Both previous strategies involved modeling the whole numbers, as well as the fractions. This, however, is not always efficient, nor necessary. Once students have a better conceptual understanding of adding and subtracting fractions, only the fraction portion of a mixed number should be represented concretely. When subtracting, however, it may become necessary to trade in a whole and represent it differently.

For example, 2 1/4 - 1 1/3 can not be done concretely if only 1/4 - 1/3 is modeled. Consequently, 2 1/4 must be thought of as 1 5/4 (one of the wholes is traded for four quarters). Now it is only necessary to model 5/4 and 1/3, keeping in mind that the wholes are not being represented.



As explained previously, the first fraction is covered by the second, and the uncovered area is determined.

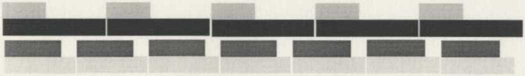


The above illustrations show that 2 1/4 - 1 1/3 = 11/12.

☞ How would you model the previous example using the pattern blocks?

Fractions are often easier to model with the Cuisenaire rods because their linear nature makes them less restrictive with respect to denominators.

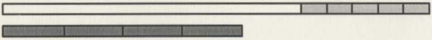
For example, 3/7 and 4/5 would be difficult to model with the fraction pieces or pattern blocks. However, using the Cuisenaire rods they can be modeled as follows.



Note that the fractions are represented side by side in order to facilitate obtaining a common multiple, in this case 35.

As stated previously, one need only consider the parts of the whole to add mixed numbers. Consequently, if modeling 2 3/7 + 1 4/5, the diagram above would still suffice. The 5 green rods and 7 purple rods can be combined, resulting in 43 parts out of 35. Students should see this as 1 whole and 8 parts left over, in other words, 1 8/35. Remind students that only the fractional parts of the mixed numbers were modeled, so 3 more wholes must be added. Hence, the final result can be written as 4 8/35.

When subtracting 2 3/7 - 1 4/5 it becomes necessary to trade. Instead of representing only 3/7, as in the addition example, 1 3/7 will be represented concretely. Since the common multiple used to represent the whole is 35, 35 blocks must be added to the representation above. Now 1 3/7 is represented by a total of 50 blocks (35 for the whole plus 5 green rods). As illustrated above, 4/5 can be represented using a total of 28 blocks (7 purple rods). To subtract, line up these rods and determine the remaining portion.

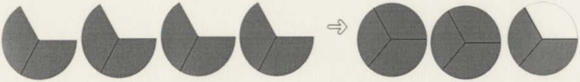


There are 22 blocks left over. Once again, the whole number portion of the mixed numbers were not modeled. Because trading took place, 2 3/7 was represented as 1 50/35. Hence,
2 3/7 - 1 4/5 = 1 50/35 - 1 28/35 = 22/35.

Multiplication

Multiplying by a Scalar

Multiplying fractions by a scalar quantity can be shown by using repeated addition. The following diagrams illustrate how the various manipulatives can be used to show 4 x 2/3. As with addition, students may wish to represent the whole separately and then cover with the pieces.



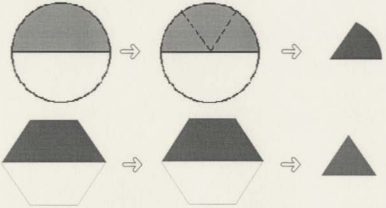
Each of the above illustrations show that 4 x 2/3 = 2 2/3. When using Cuisenaire rods, however, the mixed number is not as readily seen. Consequently, students may prefer to represent it as 8/3.

☞ Use the various manipulatives to model 3 x 1 3/4.

Multiplying by a Fraction

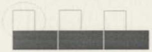
Before proceeding to multiplying two fractions together, it is important that students understand that 1/3 means 1 out of 3. Therefore, when taking 1/3 of 9, 9 must first be divided into three equal groups of 3. The answer, then, would be one of those groups, or 3. Similarly, if taking 2/3 of 9, the answer would be 2 groups, or 6.

The following diagrams illustrate 1/3 of 1/2 using fraction pieces and pattern blocks. The result in both cases is 1/6.



represent 1/2 divide into 3 groups take 1 out of 3

If the fraction can not be divided into smaller pieces, equivalent fractions can be used. For example, when using the Cuisenaire rods, students can not divide 1/2 into 3 equal groupings. However, they can use the concept of equivalent fractions and begin with 3/6.



Now it becomes easier to see that when you take one of the three groups you get one block out of six, or 1/6.

☞ Use the various manipulatives to show that 3/4 x 2/3 = 1/2.

Multiplying by a Mixed Number

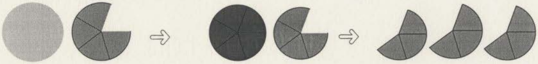
The concept of equivalency is further illustrated in the following example: 2/3 x 1 2/5.

1 2/5 can not be divided into three equal groups so it is modeled three times. As stated previously, it may be more helpful to represent mixed numbers as improper fractions when using Cusienaire rods. For this example, 1 2/5 is represented as 7/5.

To obtain 2/3, take two of those groupings, keeping in mind that the whole is represented by 15.

Hence, the answer is 14/15.

Modeling 2/3 x 1 4/5 would have been much easier. 1 4/5 is equal to 9/5, which can be divided into three groups.



Taking 2 of those groups, one gets 6/5, or 1 1/5.

☞ Model the above example with Cuisenaire rods.

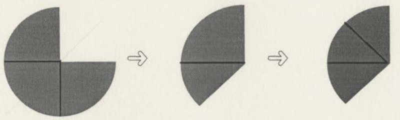
☞ Which type of manipulative would most readily model 3/4 x 1 1/2? Try each of them and see if you agree with your choice.

Division

Dividing by a Scalar

When modeling division by a scalar, the fraction is represented first, and is then divided accordingly.

For example, 3/4 ÷ 2 would be modeled as follows.

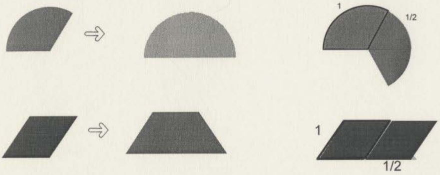


represent 3/4 divide by 2 rename area

☞ Model the above example using the pattern blocks.

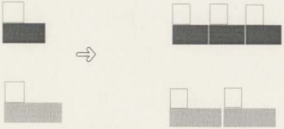
Dividing by a Fraction

When teaching the division of fractions, students should understand the underlying concepts of division. For example, 10 ÷ 2 means "How many times can 2 fit into 10?". Likewise, 1/2 ÷ 1/3 means "How many times can 1/3 fit into 1/2?". This example is illustrated below.



In both instances, the third piece fit in one time, and then half of another. Hence, the answer is 1 1/2.

When using Cuisenaire rods, the notion of common multiple is reinforced. For example, 1/2 ÷ 1/3 can only be modeled when the whole is 6 (or a multiple of 6).



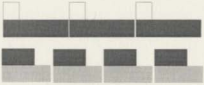
Now, we must determine how many times the two blocks (representing 1/3) can fit into three (representing 1/2). It may help students if these individual blocks are replaced by the corresponding rods, red and light green, respectively. Students should see that the red rod can fit into the green rod once, and half of another. Hence, the answer is 1 1/2.

☞ Can you use each manipulative to model 5/6 ÷ 2/3?

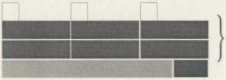
Dividing Mixed Numbers

Cuisenaire rods can also be utilized in modeling the division of mixed numbers. The following diagrams illustrate 2 1/4 ÷ 1 2/3.

First, both 1/4 and 2/3 are modeled to determine a common multiple.

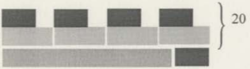


The common multiple is 12, so 2 1/4 can be represented as

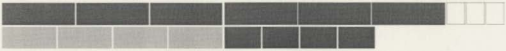


27

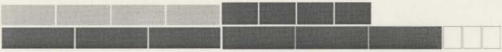
and 1 2/3 can be represented as



Now students must determine how many times 20 fits into 27. Likewise, they can determine how many times 27 covers 20. It covers once, with 7 left over 20, or 1 7/20.



Similarly, if the expression was 1 2/3 ÷ 2 1/4, one would determine how many times 20 covered 27, modeled below as 20/27.



☞ Model 2 3/5 ÷ 1 1/6.

DECIMALS

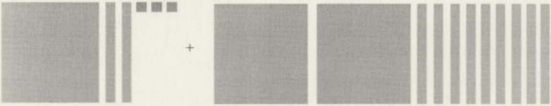
Limitations

Base ten blocks can be utilized to model all four operations with decimals, however, addition and subtraction are more user friendly. The area model will be used to briefly introduce multiplication. The same model can represent division, but often the remainder becomes problematic for students. For this reason, division will not be addressed.

A basic familiarity with representing decimals is assumed for this section.

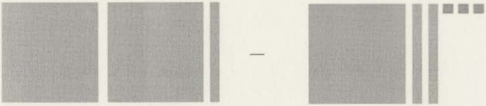
Addition and Subtraction

Decimals can be added by grouping like terms together, again keeping in mind that the whole must be the same. For example, 1.23 + 2.9 can be added concretely as follows:

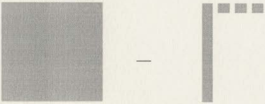


The resulting accumulation consists of 3 flats, 11 rods, and 3 cubes. 10 rods can be trading for a flat, so 4 flats, 1 rod, and 3 cubes remain. This can be represented symbolically as 4.13.

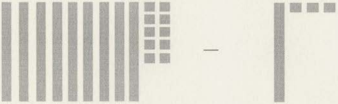
Similarly, subtraction of decimals can be modeled concretely by removing like terms. The diagram below illustrates 2.1 - 1.23.



Removing like terms on each side leaves



The flat can be traded for 9 rods and 10 squares in order to continue removing pieces.

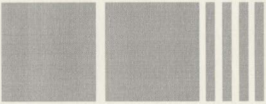


Removing the remaining like terms leaves 8 rods and 7 cubes, or 0.87.

Alternatively, students may prefer to model both decimals and cover the first decimal with the one being subtracted. The area remaining uncovered would be the answer.

Once students have a better conceptual understanding of subtracting decimals, they may prefer to model only the first number, and then remove the appropriate pieces. Trading may be necessary in some instances. The following diagrams illustrate this approach, representing 2.4 - 1.56.

Begin by modeling 2.4 with 2 flats and 4 rods.



1 flat, 5 rods, and 6 cubes must be removed; therefore, one of the flats must be traded. Students may use a variety of representations to show this.

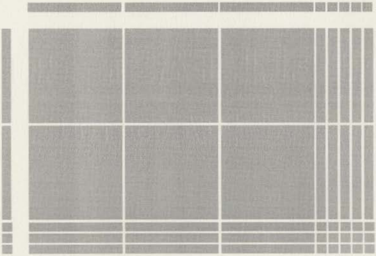
One such representation for 2.4 is

After removing the appropriate pieces, 8 rods and 4 cubes remain. This represents 0.84.

Multiplication

The area model is effective when using manipulatives to model the multiplication of decimals. This involves constructing a rectangle with dimensions equal to the numbers being multiplied. The resulting area is the product.

For example, 2.3 x 3.5 can be represented as follows:



The resulting rectangle is comprised of 6 flats, 19 rods, and 15 small cubes, that is, 6 + 1.9 + 0.15. Once the appropriate trading is done, 8 flats, 0 rods, and 5 cubes remain. This decimal can be symbolically represented as 8.05.

☞ Use the base ten blocks to show that:

1.48 + 0.63 = 2.11   2.41 - 1.67 = 1.74   1.3 x 2.4 = 4.12

INTEGERS

Limitations

Bicolored counters will be used in this section to model integers. However, as noted previously, any object in two different colors is just as effective. It is essential that students know which color represents positive quantities and which represents negative quantities. For the purposes of this unit, yellow will represent positive and red will represent negative. This distinction should be made when the counters are first introduced and should remain consistent. Although multiplication and division can be represented concretely, only addition and subtraction will be addressed.

An understanding of the *zero principle* is required before proceeding to addition and subtraction.

The Zero Principle

The *zero principle* states that "the sum of opposites is zero". This implies that a positive and negative cancel each other out and can be added or removed without changing the value.

For example, each of the following represent -3.

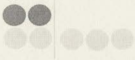
Similarly, each of the following represent +2.

☞ Use 5 counters to represent -1. Represent +5 with 7 counters.

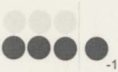
Addition

When adding integers merely combine the counters and then use the *zero principle* to simplify, if necessary.

For example, (-2) + (+5) can be modeled using 2 red counters and 5 yellow counters. Once the zeros are removed, 3 yellow counters remain. Hence, the answer is +3.



Similarly, (+3) + (-4) can be concretely represented as follows:



☞ Model (-3) + (+4) and (+5) + (-1).

Subtraction

When subtracting integers, the first integer is represented concretely. Then, the second integer is removed.

For example, to concretely represent (-3) - (-2), first model -3 using 3 red counters, then, remove 2 red counters.

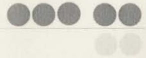
There is 1 red counter remaining; hence, the answer is -1.

If there are not enough counters to remove the second integer, the *zero principle* can be used to add pairs of red and yellow counters without changing the value.

For example, (-3) - (-5) can be modeled as follows:

First, represent -3 using 3 red counters.

Because 5 red counters can not be removed, more counters must be added in the form of zero, resulting in 5 red counters and 2 yellow counters.



Now the 5 red counters can be removed, leaving 2 yellow counters, representing +2. Hence, (-3) - (-5) = +2.

To model (-5) - (+2), first represent -5 using 5 red counters.

Because 2 yellow counters can not be removed, more must be added.

When the 2 yellow counters are removed, 7 red counters remain. Hence, (-5) - (+2) = -7.

☞ Use the counters to model each of the following:

(-3) - (-4)	(+2) - (-1)
(-4) - (+2)	(+1) - (+3)

ALGEBRA

Limitations

Algebra tiles can be very effective in teaching various algebraic concepts. However, the tiles themselves often prove to be a stumbling block for some students. Their similarity with the base ten blocks makes it difficult for students to think of the algebra tiles as having an unknown, or variable, dimension. The concrete representation is still abstract.

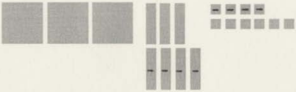
Algebra tiles can be used to model each of the basic operations. However, much of the methodology needed to perform these operations has been addressed elsewhere in this guide. For this reason, addition and multiplication will be addressed briefly. This section will focus on solving linear equations and factoring trinomials.

Addition and Subtraction

Addition with algebra tiles uses the concepts associated with adding decimals and integers. When adding polynomials, like terms are grouped and simplified according to the *zero principle*. Note that the black tiles are used to represent positive values and the red tiles are used to represent negative values. However, any two colors will suffice.

For example, $(2x^2 + 3x - 4) + (x^2 - 4x + 6)$ can be modeled as follows:

Group like terms together as illustrated below.



The *zero principle* is used to simplify, resulting in $3x^2 - x + 2$.

Subtraction of polynomials uses the methods addressed in the integers section. In fact, subtraction principles are employed in the above example. That is, $3x + (-4x)$ is actually $3x - 4x$.

☞ Use the algebra tiles to model $(3x^2 - 5x + 2) - (4x^2 - 6x + 3)$.

Solving Linear Equations

To solve an equation, both sides are first modeled with the algebra tiles. Note that each side is separated by a vertical line which represents the equals sign.

For example, $x + 3 = 2$ is represented as

To solve algebraic equations, the variable, x, must be isolated in order to determine its value. Therefore, when modeling the process, only the rectangular tiles must remain on one side. To do so, the same tiles can be added to each side.

The *zero principle* can now be used to simplify.

Only one red square tile remains on the right; hence, $x = -1$.

Often times, it may be more efficient to remove the same tiles from each side, rather than add them.

For example, $x + 3 = 5$ can be represented as

Removing 3 square tiles from each side leaves

Hence, $x = 2$.

More often than not, however, a combination of both methods will be used by students. Such is the case in solving $x + 3 = 2x + 5$, as illustrated below.

Hence, $-2 = x$, or $x = -2$. Note that the x can be isolated on either the right or the left.

When solving some algebraic equations, more than one x remains. Consequently, both sides must be divided into groups.

For example, $2x + 1 = 5$, is modeled as follows

Removing one square tile from each side leaves

To determine the value for x, both sides are divided by 2.

Hence, $x = 2$.

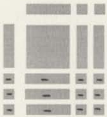
Note, however, that $2x + 1 = 4$ would be problematic because it would simplify to $2x = 3$. The concrete representation merely allows for a simpler form of the equation to be produced first. The equation could then be solved by dividing both sides by 2, resulting in $x = 3/2 = 1\ 1/2$.

☞ Use the algebra tiles to solve $3x - 1 = 8$.

Multiplication

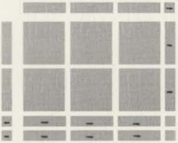
The area model described in multiplying decimals is also used to multiply binomials. It is important to remind students of the rules associated with multiplying positive and negative values so that the appropriate pieces are selected to represent the product.

The following diagram illustrates the modeling of $(x + 2)(x - 3)$. Note that red pieces are selected when a positive quantity is multiplied by a negative quantity.



Using the *zero principle*, the above rectangular area simplifies to $x^2 - x - 6$.

The following diagram is a concrete representation of $(3x - 2)(2x - 1)$. Note that black tiles are selected when two negative quantities are multiplied.



This representation simplifies to $6x^2 - 8x + 2$.

☞ Model $(2x + 3)(x - 1)$.

Factoring

Students should have a conceptual understanding of the area model for multiplication, as well as the *zero principle*, before proceeding with factoring.

The Area Model

To model factoring, the algebraic expression is represented concretely. For example, to factor $x^2 + 5x + 6$, the expression must be modeled.

These tiles are then arranged to form a rectangle.

The resulting dimensions are the factors. Hence, $x^2 + 5x + 6 = (x + 2)(x + 3)$.

The factoring of $x^2 - 3x + 2$ is modeled below.

This illustrates that $x^2 - 3x + 2 = (x - 2)(x - 1)$.

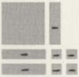
☞ Using algebra tiles, factor $x^2 - 4x + 3$ and $3x^2 + 4x + 1$.

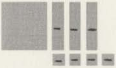
☞ Try using the base ten blocks to model the factoring of expressions in which $x = 10$. For example, consider $(10)^2 - 4(10) + 3$.

The Zero Principle

When some expressions are represented concretely they can not form a rectangle. For example, $x^2 - 3x - 4$ can be represented as

Regardless of how they are arranged, a rectangle can not be formed.





Note that all of the tiles must be used to represent the expression as a product. However, the *zero principle* can be used to add two rectangular tiles at a time - one red and one black. The representation on the left above is missing only one rectangular tile; therefore, it can not be used. The one on the right, however, is short two rectangular tiles so a black and red tile can be added.

It is important to note that positive and negative tiles of the same shape must not be combined along the same side when forming a rectangle. In other words, red rectangles can not be combined with black rectangles along the same side. However, they can be combined with black squares. This is related to the concept of grouping like terms, addressed previously.

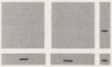
Applying this strategy to the above representation results in the following diagram.

Hence, $x^2 - 3x - 4 = (x - 4)(x + 1)$.

☞ Using the algebra tiles, factor $x^2 + x - 2$ and $2x^2 - x - 3$.

Determining Validity

The rules for multiplying integers become important in determining the validity of the rectangles being formed. For example, $2x^2 - 3x - 1$ can be represented as follows:



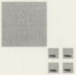
Although these pieces can form a rectangle, they do not follow the rules for multiplying positive and negative quantities. Note that the unit, represented by the small square tile is negative. This can only occur if a positive and negative quantity are multiplied. In this case, both rectangular tiles represent negative quantities; hence, the unit, or small square, would have to be positive.

The *zero principle* can be used to add more rectangular tiles, however, regardless of how they are arranged, a rectangle can not be formed. Hence, this expression can not be factored.

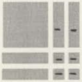
☞ Show why $x^2 - 2x - 1$ can not be factored.

Tips for Factoring

To determine how many rectangular tiles to add, it may be helpful to begin with the large square tiles and the small square tiles. For example, when factoring $x^2 - 4$, begin by arranging the small square tiles. Two representations are possible.



The representation on the left needs five rectangular tiles, so it is invalid. The representation on the right, however, needs four rectangular tiles (two red and two black). The following representation results.



Hence, $x^2 - 4 = (x - 2)(x + 2)$.

When beginning with the square tiles, it is quite evident that $x^2 - 4x + 1$ can not be factored.

Clearly, there is no way to place the four x's, or rectangular tiles, to form a rectangle.

The representation of $x^2 + 6x + 4$ illustrates the advantage of beginning with the square tiles.

The representation on the left has 2 extra rectangular tiles, whereas, the one on the right has an extra rectangular tiles. Therefore, this expression can not be factored by this method.

☞ Factor each of the following, if possible.

$x^2 + 4x + 5$ $\qquad$ $x^2 - 9$ $\qquad$ $2x^2 + 5x - 6$

Common Factors

Often times more than one rectangle can be formed. For example, $2x^2 + 6x + 4$ can form two different rectangles.

The rules for multiplying positive and negative quantities have been followed; therefore, both are valid. This is due to the fact that there is a common factor of 2. Examples such as this are helpful in illustrating the concept of multiple representations.

☞ Model the multiple representations for $8x^2 - 4x - 12$.

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