Cohomology of Quotients in Real Symplectic Geometry

by

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***"Whoever who fell away from the source,
will seek and toil until returned to course."***

*—Rumi (1207-1273)
Great Persian Poet*

To my beautiful wife

ATOUSA

ABSTRACT

Let (M, ω, G, μ) be a Hamiltonian system where (M, ω) is a compact connected symplectic manifold, G is a compact connected Lie group acting symplectically on M and $\mu : M \to \mathfrak{g}^*$ is a moment map where $\mathfrak{g} = \mathrm{Lie}(G)$. Fix an Ad-invariant inner product on $\mathfrak{g}$ and consider the norm squared of the moment map $f = ||\mu||^2 : M \to \mathbb{R}$. Kirwan has proved that the function f is G-equivariantly perfect over the field of rational numbers and M is G-equivariantly formal. She also gives a recursive formula for the equivariant rational Betti numbers of the subspace $M_0 = \mu^{-1}(0)$.

Suppose that there exists a pair of involutions $(\sigma : M \to M, \phi : G \to G)$ in a Hamiltonian system such that σ is anti-symplectic and they are compatible in a way that the fixed point set of ϕ, denoted by G^ϕ, acts on the fixed point set of σ, denoted by M^σ. If $f^\sigma : M^\sigma \to \mathbb{R}$ is the restriction of f to the real locus M^σ, we prove that under certain conditions, called 2-primitivity and free extension property, the restricted function f^σ is equivariantly perfect over the field $\mathbb{Z}_2$ and the real locus M^σ is G^ϕ-equivariantly formal over the field $\mathbb{Z}_2$. In particular, when $G = \mathrm{U}(n)$ or $\mathrm{SU}(n)$ and the group involution is the complex conjugation, we compute the $\mathbb{Z}_2$-Betti numbers of the quotient space M_0^σ / G^ϕ where $M_0^\sigma = M^\sigma \cap \mu^{-1}(0)$.

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CHAPTER 1

Introduction

> *"I am not really doing research, just trying to cultivate myself."*
>
> *—Alexander Grothendieck (1928-2014)*

1.1 Motivation and Goals

A *Hamiltonian system* $\mathcal{H} = (M, \omega, G, \mu)$ consists of a Lie group G acting symplectically on a symplectic manifold (M, ω) governed by a moment map $\mu : M \to \mathfrak{g}^*$ where $\mathfrak{g} = \mathrm{Lie}(G)$. We will always assume that G and M are compact and connected. If G acts freely on the zero level set of μ, $M_0 = \mu^{-1}(0)$, there exists a symplectic form ω_{red} on the quotient space $M /\!\!/ G = M_0/G$ such that the pair $(M/\!\!/G, \omega_{\mathrm{red}})$ is a symplectic manifold, known as the *symplectic reduction* [19].

A *real Hamiltonian system* is a tuple $\mathcal{RH} = (M, \omega, G, \mu, \sigma, \phi)$ in which (M, ω, G, μ) is a Hamiltonian system, $\phi : G \to G$ is a Lie group automorphism of order two and $\sigma : M \to M$ is an anti-symplectic involution (namely, $\sigma^2 = \mathrm{Id}$ and $\sigma^*\omega = -\omega$) satisfying certain compatible conditions (see Definition 3.5) such that the ϕ-invariant subgroup $G^\phi = \{g \in G \mid \phi(g) = g\}$ (the *real subgroup*) acts on the fixed point set $M^\sigma = \{x \in M \mid \sigma(x) = x\}$ (the *real locus*) which is a Lagrangian submanifold of M. If G acts freely on M_0, the *real reduction space* $M^\sigma/\!\!/G^\phi = M_0^\sigma/G^\phi$ embeds as a Lagrangian submanifold in the symplectic quotient $M/\!\!/G$. The goal of this thesis is to develop the Morse theory techniques to calculate the mode 2 cohomology of the real reduction space $M^\sigma/\!\!/G^\phi$ which are analogues of Kirwan's thorems in [42].

Given an invariant inner product on the Lie algebra $\mathfrak{g}$, one can form the norm squared of the moment map

$$f = ||\mu||^2 : M \to \mathbb{R}. \tag{1.1}$$

Kirwan [42] showed that the critical set C_f of f is a finite collection of disjoint invariant closed subsets $\{C_\beta \mid \beta \in \Lambda\}$ (the *critical subsets*) and the function f is G-equivariantly perfect over the field of the rational numbers $\mathbb{Q}$; i.e.,

$$\mathbf{P}_G(M,t;\mathbb{Q}) = \sum_{C_\beta \subset C_f} t^{d(\beta)} \mathbf{P}_G(C_\beta, t; \mathbb{Q}), \tag{1.2}$$

where $\mathbf{P}_G(M,t;\mathbb{Q})$ is the equivariant Poincaré series of M and $d(\beta)$ is the Morse index of f along C_β. The indexing set Λ is a subset of the Lie algebra $\mathfrak{g}$. As a result of equivariant perfection, she also showed that the inclusion $M_0 \subset M$ determines a map $\kappa : H^*_G(M;\mathbb{Q}) \to H^*_G(M_0;\mathbb{Q})$, known as the *Kirwan map*, which is surjective. When G acts freely on the zero level set M_0, $H^*_G(M_0;\mathbb{Q}) \cong H^*(M/\!\!/G;\mathbb{Q})$.

Kirwan also proved that M is G-equivariantly formal over the field of the rational numbers $\mathbb{Q}$; i.e, the Serre spectral sequence of the fibration $M \hookrightarrow M_G \to BG$, induced by the homotopy quotient space M_G, collapses at page two and thus

$$H^*_G(M;\mathbb{Q}) \cong H^*(BG;\mathbb{Q}) \otimes_{\mathbb{Q}} H^*(M;\mathbb{Q}), \tag{1.3}$$

where BG is the classifying space of the Lie group G. By using these results, Kirwan derived a recursive formula for the equivariant rational Betti numbers of the zero level set $\mu^{-1}(0) = M_0$:

$$\mathbf{P}_G(\mu^{-1}(0),t;\mathbb{Q}) = \mathbf{P}(M,t;\mathbb{Q})\mathbf{P}(BG,t;\mathbb{Q}) - \sum_{C_\beta \neq C_0} t^{d(\beta)} \mathbf{P}_G(C_\beta,t;\mathbb{Q}), \tag{1.4}$$

where $C_0 = \mu^{-1}(0)$. Kirwan showed that for each $\beta \in \Lambda$, there is a Hamiltonian subsystem $\mathcal{H}_\beta = (Z_\beta, \omega, G_\beta, \mu_\beta)$ in which $Z_\beta \subset M$ is a symplectic submanifold, $G_\beta \subset G$ is the stabilizer subgroup of β and $C_\beta = G \times_{G_\beta} \mu_\beta^{-1}(0)$ where $\mu_\beta^{-1}(0) = Z_\beta \cap \mu^{-1}(\beta)$. This reduces the computation of the equivariant Betti numbers of the zero level set $\mu^{-1}(0)$ in a Hamiltonian system $\mathcal{H}$ to the computation of the equivariant Betti numbers of the zero level sets $\mu_\beta^{-1}(0)$ of generated Hamiltonian subsystems $\mathcal{H}_\beta$.

Ananogus of Kirwan's work have been proven by several people (see [10], [28] or [56]). The equivariant perfection and Kirwan surjectivity for real abelian Hamiltonian systems have been investigated by Goldin-Holm in [28]. The equivariant formality for such systems has been proved by Biss-Guillemin-Holm in [10].

1.2 Summary of Results

Consider a real Hamiltonian system $\mathcal{RH} = (M, \omega, G, \mu, \sigma, \phi)$. Our first result is about equivariant perfection for the restricted function $f^\sigma : M^\sigma \to \mathbb{R}$. We first construct a Morse stratification $\{S^\sigma_{\beta_i}\}$ for f^σ with critical subsets $C^\sigma_{\beta_i}$. Along the way, we get the following.

(i) A finite family of real Hamiltonian subsystems

$$\{\mathcal{RH}_{\beta_i} = (Z_{\beta_i}, \omega, G_{\beta_i}, \mu_{\beta_i}, \sigma, \phi) \mid \beta_i \in \mathfrak{g}_-\}$$

where $\mathfrak{g}_- = \{X \in \mathfrak{g} \mid \phi_*(X) = -X\}$, $\phi_* : \mathfrak{g} \to \mathfrak{g}$ is the induced involution on the Lie algebra and $\sigma : Z_{\beta_i} \to Z_{\beta_i}, \phi : G_{\beta_i} \to G_{\beta_i}$ are the restricted maps.

(ii) The critical set of f^σ is a finite collection of G^ϕ-invariant closed subsets $C^\sigma_{\beta_i}$ such that

$$C^\sigma_{\beta_i} \cong G^\phi \times_{G^\phi_{\beta_i}} (Z^\sigma_{\beta_i} \cap \mu^{-1}_{\beta_i}(0)), \tag{1.5}$$

where $Z^\sigma_{\beta_i} = Z_{\beta_i} \cap M^\sigma$ and $G^\phi_{\beta_i} = G^\phi \cap G_{\beta_i}$.

(iii) Each stratum $S^\sigma_{\beta_i} \subset M^\sigma$ is a G^ϕ-invariant locally closed submanifold that deformation retracts onto the critical subset $C^\sigma_{\beta_i}$ of f^σ and has a constant codimension $d(\beta_i, \sigma)$ which is the Morse index of f^σ along the critical subset $C^\sigma_{\beta_i}$.

As expected, the index number $d(\beta_i, \sigma)$ is half of the Morse index $d(\beta_i)$ of f along $C^\sigma_{\beta_i}$ which is an even number; i.e.,

$$d(\beta_i, \sigma) = \frac{1}{2} d(\beta_i) \tag{1.6}$$

Following Kirwan's approach, we use the equivariant Thom-Gysin long exact sequence to prove equivariant perfection for f^σ. Using a partial order on the indexing set, we get the following long exact sequence:

$$\cdots \to H^{*-d(\beta_i,\sigma)}_{G^\phi}(S^\sigma_{\beta_i}; \mathbb{Z}_2) \xrightarrow{i_{\beta_i}} H^*_{G^\phi}(\bigcup_{\alpha \le \beta_i} S^\sigma_\alpha; \mathbb{Z}_2) \xrightarrow{j_{\beta_i}} H^*_{G^\phi}(\bigcup_{\alpha < \beta_i} S^\sigma_\alpha; \mathbb{Z}_2) \to \cdots \tag{1.7}$$

We show that under hypotheses described below the equivariant top Stiefel-Whitney class of the G^ϕ-equivariant normal bundle of each stratum $S^\sigma_{\beta_i}$ is not a zero divisor. This condition forces the map i_{β_i} to be 1-1 and breaks the long exact sequence into short exact sequences:

$$0 \to H^{*-d(\beta_i,\sigma)}_{G^\phi}(S^\sigma_{\beta_i}; \mathbb{Z}_2) \xrightarrow{i_{\beta_i}} H^*_{G^\phi}(\bigcup_{\alpha \le \beta_i} S^\sigma_\alpha; \mathbb{Z}_2) \xrightarrow{j_{\beta_i}} H^*_{G^\phi}(\bigcup_{\alpha < \beta_i} S^\sigma_\alpha; \mathbb{Z}_2) \to 0, \tag{1.8}$$

which gives equivariant perfection by induction.

The first condition we consider is called the *free extension property* and is on the pair (G, ϕ). It states that for any $\beta \in \mathfrak{g}_-$ and any maximal elementary abelian 2-subgroup D_β of the real stabilizer subgroup $G_\beta^\phi = G_\beta \cap G^\phi$, the cohomology ring $H^*(BD_\beta; \mathbb{Z}_2)$ is a free $H^*(BG_\beta^\phi; \mathbb{Z}_2)$-module. Examples of Lie groups with the free extension property are the product of unitary groups with the complex conjugation as the group involution.

The second required condition is called the *2-primitivity property.* This property is about G-vector bundles $V \to X$ which guarantees the existence of an elementary abelian 2-subgroup $E \subset G$ fixing X and no nonzero vectors in V.

By using these two conditions, we formulate a real version of the famous Atiyah-Bott Lemma ([3], Proposition 13.4) which says that if the pair (G, Id) has the free extension property and the vector bundle is 2-primitive, then the equivariant top Stiefel-Whitney class is not a zero divisor.

To prove the equivariant perfection theorem, we need to generalize the 2-primitivity property to real Hamiltonian systems which requires that the $G_{\beta_i}^\phi$-equivariant normal bundle of each submanifold $Z_{\beta_i}^\sigma \cap \mu_{\beta_i}^{-1}(0)$ in the generated real Hamiltonian subsystem $\mathcal{RH}_{\beta_i} = (Z_{\beta_i}, \omega, G_{\beta_i}, \mu_{\beta_i}, \sigma, \phi)$ is 2-primitive as a $G_{\beta_i}^\phi$-vector bundle. These properties guarantee that the injectivity of each map i_{β_i} in (1.8) holds and does not depend on the choice of a maximal elementary abelian 2-subgroup for each real subgroup $G_{\beta_i}^\phi$.

When the injectivity is satisfied, the equivariant Thom-Gysin sequence must break into short exact sequences. By induction, we show that f^σ is equivariantly perfect over the field $\mathbb{Z}_2$; i.e.,

$$\mathbf{P}_{G^\phi}(M^\sigma, t; \mathbb{Z}_2) = \sum_{C_{\beta_i}^\sigma \subset C_{f^\sigma}} t^{d(\beta_i, \sigma)} \mathbf{P}_{G^\phi}(C_{\beta_i}^\sigma, t; \mathbb{Z}_2), \tag{1.9}$$

This yields a recursive formula for the equivariant $\mathbb{Z}_2$-Betti numbers of the real zero level set $M_0^\sigma = M^\sigma \cap \mu^{-1}(0)$ as follows:

$$\mathbf{P}_{G^\phi}(M_0^\sigma, t; \mathbb{Z}_2) = \mathbf{P}_{G^\phi}(M^\sigma, t; \mathbb{Z}_2) - \sum_{C_{\beta_i}^\sigma \neq M_0^\sigma} t^{d(\beta_i, \sigma)} \mathbf{P}_{G_{\beta_i}^\phi}(Z_{\beta_i}^\sigma \cap \mu_{\beta_i}^{-1}(0), t; \mathbb{Z}_2). \tag{1.10}$$

Thus, computing the $\mathbb{Z}_2$-equivariant Betti numbers of M_0^σ reduces to computing the $\mathbb{Z}_2$-equivariant Betti numbers of M^σ and $Z_{\beta_i}^\sigma \cap \mu_{\beta_i}^{-1}(0)$.

An immediate consequence of the equivariant perfection theorem is that the map $\kappa_\mathbb{R} : H^*_{G^\phi}(M^\sigma; \mathbb{Z}_2) \to H^*_{G^\phi}(M_0^\sigma, \mathbb{Z}_2)$ induced by the inclusion $M_0^\sigma \subset M^\sigma$ is surjective. We call this surjection the *real Kirwan map.* When the action of G on M_0 is free, we prove that the real reduction $M^\sigma /\!\!/ G^\phi$ can be considered as a Lagrangian submanifold of the symplectic reduction $M /\!\!/ G$ and since $H^*_{G^\phi}(M_0^\sigma; \mathbb{Z}_2) \cong H^*(M^\sigma /\!\!/ G^\phi, \mathbb{Z}_2)$, the map $H^*_{G^\phi}(M^\sigma; \mathbb{Z}_2) \to H^*(M^\sigma /\!\!/ G^\phi; \mathbb{Z}_2)$ is also a surjection. In other words, we have proved a *Kirwan surjectivity* for real Hamiltonian systems.

Our next result is about the equivariant formality for real Hamiltonian systems. We prove this theorem for those real Hamiltonian systems $\mathcal{RH} = (M, \omega, G, \mu, \sigma, \phi)$ in which the pair (G, ϕ) has the *special free extension property.*

A pair (G,ϕ) has the special free extension property if it has the free extension property and there exists a maximal torus $T \subset G$ such that the restriction of ϕ to T is the inversion map as well as the real torus T^ϕ is a maximal elementary abelian 2-subgroup of G^ϕ. Examples include $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ with the complex conjugation as the involution.

To prove our equivariant formality theorem, we first consider the real abelian Hamiltonian system $(M,\omega,T,\mu_T,\sigma,\phi)$ where $\mu_T : M \to \mathfrak{t}^*$ is the composition of μ with the orthogonal projection onto $\mathfrak{t}^*$ where $\mathfrak{t} = \mathrm{Lie}(T)$. By a theorem of Biss-Guillemin-Holm [10], the real locus M^σ of this abelian system is T^ϕ-equivariantly formal over the field $\mathbb{Z}_2$. That is, the Serre spectral sequence of the fibration $BT^\phi \hookrightarrow M^\sigma_{T^\phi} \to M^\sigma$ collapses at page 2. By combining this result with the special free extension property and considering a commutative diagram, we prove that the Serre spectral sequence of the fibration $BG^\phi \hookrightarrow M^\sigma_{G^\phi} \to M^\sigma$ also collapses at page 2, and thus

$$H^*_{G^\phi}(M^\sigma;\mathbb{Z}_2) \cong H^*(BG^\phi;\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(M^\sigma;\mathbb{Z}_2). \tag{1.11}$$

This is, the real locus M^σ is G^ϕ-equivariantly formal over the field $\mathbb{Z}_2$.

Finally, we apply our results to specific examples of real Hamiltonian systems and give explicit formulas for the $\mathbb{Z}_2$-Betti numbers of their real reductions. The first example we consider is a product of complex projective lines $M = (\mathbb{CP}^1)^n$ on which the unitary group $\mathrm{U}(2)$ acts diagonally and the involution is a product of complex conjugations. In this example, the real locus is a product of real projective lines $M^\sigma = (\mathbb{RP}^1)^n$ and the real group is $G^\phi = \mathrm{O}(2)$, the orthogonal group of rank 2. We show that the 2-primitivity and the special free extension properties are satisfied for this system and by using equivariant perfection and formality theorems, we compute the $\mathbb{Z}_2$-Betti numbers of the real reduction space $(\mathbb{RP}^1)^n /\!\!/ \mathrm{O}(2)$ as follows:

$$\beta_k = \sum_{j=0}^{\min\{k,n-k-3\}} \binom{n-1}{j}, \text{ for } k = 0,...,n-3. \tag{1.12}$$

The other example is a generalization of the first one. We consider the diagonal action of the unitary group $\mathrm{U}(n)$ on a product of complex Grassmannians $M = \mathrm{Gr}_{l_1}(\mathbb{C}^n) \times \cdots \times \mathrm{Gr}_{l_r}(\mathbb{C}^n)$ in which the involution ϕ is again the complex conjugation and the involution σ is induced by the complex conjugation on $\mathbb{C}^n$. Here, the real locus is a product of real Grassmannians $M^\sigma = \mathrm{Gr}_{l_1}(\mathbb{R}^n) \times \cdots \times \mathrm{Gr}_{l_r}(\mathbb{R}^n)$ and the real subgroup is the orthogonal group $\mathrm{O}(n)$. We will see that the equivariant perfection and formality theorems are satisfied provided that n is odd and numbers $\sum_{j=1}^r l_j$ and n are coprime. This provides us with a recursive formula for the equivariant $\mathbb{Z}_2$-Betti numbers of the real reduction.

When we compare our computations with Kirwan's results, we realize that the following relation is satisfied:

$$\mathbf{P}_{G^\phi}(M^\sigma /\!\!/ G^\phi, t;\mathbb{Z}_2) = \mathbf{P}_G(M /\!\!/ G, t^{\frac{1}{2}};\mathbb{Q}). \tag{1.13}$$

One may think (1.13) holds in general real Hamiltonian systems but we present a counterexample in Section 8.2 (see Example 8.6).

1.3 Content Outline

The outline of the rest of this thesis is as follows. In chapter 2, we have gathered the mathematical preliminaries needed for next chapters. It gives theorems and definitions that will be used throughout this thesis. It consists of six sections: Linear Algebra, Algebraic Topology, Differential Topology, Lie Theory, Symplectic Geometry and Differential Homological Algebra.

Chapter 3 gives the main ideas and definitions of real symplectic geometry. In section 3.1, we introduce the idea of real structures on complex vector spaces. In section 3.2, we describe the notion of real structures on symplectic manifolds and then define real Hamiltonian systems. In section 3.3, several examples of real Hamiltonian systems have been provided. Section 3.4 formulates a real version of the symplectic reduction.

In chapter 4, we show that how to obtain a real Morse stratification for the real locus of a real Hamiltonian system. Section 4.1 sums up the main results of Kirwan regarding the Morse stratification induced by the norm squared of the moment map in a Hamiltonian system. In section 4.2, we use the main ideas of section 4.1 to obtain a real version of the Morse stratification for a real Hamiltonian system (new results).

In chapter 5, we discuss the idea of free extension property which is one of the most important tools we need in this thesis. In section 5.1, we first introduce the idea of free extension in module theory and give some examples. In section 5.2, we extend this idea to Lie groups with an involutions and give some properties and examples of them (new results).

Chapter 6 is devoted to the Atiyah-Bott Lemma and describes this notion. In section 6.1, we give the main Atiyah-Bott Lemma. In section 6.2, we state and prove a real version of the Atiyah-Bot Lemma that we need in our work (new results).

Chapter 7 contains two of our main results. Section 7.1 discusses the property of 2-primitivity in a real Hamiltonian system which we need in proving a real version of equivariant perfection. Section 7.2 formulates an equivariant formality theorem for a special class of real Hamiltonian systems (new results).

In chapter 8, we use our results in two examples to show that how they can be applied in real Hamiltonian systems. In section 8.1, we digress to compute some homotopy quotients of the action of the orthogonal groups (new results). In section 8.2, we compute the $\mathbb{Z}_2$-Betti numbers of some real reductions (new results).

The final part consists of three appendices. Appendix A gives some basic facts about the complex projective line and Mobius transformations. Appendix B summarizes some properties of the Grassmannians. Appendix C is about the linear projective groups and some of their important properties.

1.4 Writing Approach

I want to say a few words about the approach I have adopted here to write this thesis. I believe that learning and teaching mathematics is impossible without using pictures, specially in Geometry and Topology. I have always preferred using mathematical books with numerous figures. Pictures help me to imagine the ideas very vividly.

When I quit printing and designing business and came back to mathematical world, I thought I can use my work experience in this area. I have a very good familiarity with two great design software: Corel Draw and Adobe Illustrator (AI). I decided to use them for creating geometric objects in all my mathematical writings since then. I believe this helps people to understand my ideas better and clearer.

To draw my pictures, I usually use AI and MATLAB together. By using AI, one can create two or three dimensional objects and easily get outputs in EPS formats which work nicely with Latex files. MATLAB generates fancy surfaces and we can import those objects in an artboard in AI to customize them based on our needs. Another benefit of working with AI is that we can use LaTeXiT to type in an AI file easily. In this case, we are able to label some parts of the figure or even type some formulas with the same font as in our Latex file. When we insert the figure in our Latex file and produce a pdf file, the output is excellent and beautiful.

CHAPTER 2

Mathematical Preliminaries

"*I know how to control the universe. So tell me, why should I run for a million?*"

—*Grigori Perelman (1966-present)*

In this chapter, we give all the basic machinery that will be used throughout this thesis. The content of the current chapter consists of elementary definitions and theorems from different areas of mathematics. It contains six sections: Linear Algebra, Algebraic Topology, Differential Topology, Lie Theory, Symplectic Geometry and Differential Homological Algebra. In each section, we have listed basic definitions and main theorems that are needed in the future.

2.1 Linear Algebra

"Algebra is the offer made by the devil to the mathematician. All you need to do, is give me your soul: give up geometry!"

—Sir Michael Atiyah (1929-present)

Definition 2.1. Let $T : V \to V$ be a linear operator on an n-dimensional vector space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

1. The **characteristic polynomial** χ_T of T is defined by
$$\chi_T(x) = \det(T - x\mathrm{Id}). \tag{2.1}$$
2. The roots of χ_T are called the **eigenvalues** of T. For any eigenvalue λ, $E_\lambda = \mathrm{Ker}(T - \lambda \mathrm{Id})$ is called the **eigenspace** corresponding to λ. Elements of eigenspaces are called **eigenvectors**.
3. The **minimal polynomial** m_T of T is the unique monic polynomial such that $m_T(T) = 0$ and for any other polynomial q with $q(T) = 0$, m_T divides q.
4. T is **diagonalizable** if there exists some basis $\mathcal{B}$ for V consisting of eigenvectors of T; i.e., the matrix of T with respect to $\mathcal{B}$ is a diagonal matrix.
5. A family of operators $\{T_\alpha : V \to V \mid \alpha \in \Lambda\}$ is called **simultaneously diagonalizable** if there exists some basis $\mathcal{B}$ for V such that the matrix of each T_α with respect to $\mathcal{B}$ is diagonal.

Proposition 2.1. *An operator $T : V \to V$ on an n-vector space V is diagonalizable if and only if the minimal polynomial m_T is a product of linear factors; i.e., $m_T(x) = (x - \lambda_1) \cdots (x - \lambda_k)$, where λ_i are distinct scalars.*

Proof. See [36], Chapter 4, Theorem 5. ■

Corollary 2.1.1. *If T is an **involution**, i.e., $T^2 = \mathrm{Id}$, then T is diagonalizable.*

Proof. Since $T^2 = \mathrm{Id}$, T is a root of the polynomial $x^2 - 1$. Thus, the minimal polynomial divides the polynomial $x^2 - 1 = (x-1)(x+1)$. This implies that m_T is a product of linear factors. Proposition 2.1 proves the claim. ■

Definition 2.2. A **commuting family** of operators is a collection $\{T_\alpha : V \to V \mid \alpha \in \Lambda\}$ of operators such that $T_\alpha T_\beta = T_\beta T_\alpha$, for any $\alpha, \beta \in \Lambda$.

Proposition 2.2. *A commuting family of diagonalizable operators are simultaneously diagonalizable.*

Proof. See [36], Chapter 6, Theorem 8. ■

Corollary 2.2.1. *A commuting family of involutions are simultaneously diagonalizable.*

Proof. This is a combination of Propositions 2.1 and 2.2. ■

Definition 2.3. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $M(n, \mathbb{F})$ be the set of all $n \times n$ matrices with entries in $\mathbb{F}$. Suppose that $A \in M(n, \mathbb{F})$.

1. A is **symmetric** (or **skew-symmetric**) if $A^t = A$ (or $A^t = -A$).

2. A is **Hermitian** (or **skew-Hermitian**) if $A^* = A$ (or $A^* = -A$), where $A^* = \overline{A}^t$.

3. A is **orthogonal** if $AA^t = A^tA = \mathrm{Id}$. The set of all orthogonal matrices is denoted by $\mathrm{O}(n; \mathbb{F})$.

4. A is **unitary** if $A^*A = AA^* = \mathrm{Id}$. The set of all unitary matrices is denoted by U(n).

Remark 2.1. We usually write $\mathrm{O}(n)$ for $\mathrm{O}(n; \mathbb{R})$.

Proposition 2.3 (**Spectral Theorem**). *Let $A \in M(n, \mathbb{F})$. The following are satisfied.*

1. *If $\mathbb{F} = \mathbb{C}$ and A is Hermitian, then there exist a unitary matrix $U \in M(n, \mathbb{C})$ and a real diagonal matrix $D \in M(n, \mathbb{R})$ such that $UAU^* = D$.*

2. *If $\mathbb{F} = \mathbb{R}$ and A is symmetric, then there exist an orthogonal matrix $Q \in M(n, \mathbb{R})$ and a real diagonal matrix $D \in M(n, \mathbb{R})$ such that $QAQ^{-1} = D$.*

3. *If $\mathbb{F} = \mathbb{C}$ and A is symmetric, then there exist a unitary matrix $U \in M(n, \mathbb{C})$ and a real diagonal matrix $D \in M(n, \mathbb{R})$ such that $UAU^{-1} = D$.*

Proof. See [24], Chapter 9, Theorems 9.3 and 9.7 for parts 1,2 and Chapter 10, Theorem 10.6 for part 3. ■

2.2 Algebraic Topology

"It was my lot to plant the harpoon of algebraic topology into the body of the whale of algebraic geometry."

—Solomon Lefschetz (1884-1972)

2.2.1 Topological groups and continuous actions

Definition 2.4. A **topological group** G is a topological space which has a group structure such that the map $G \times G \to G$ defined by $(g, h) \longmapsto gh^{-1}$ is continuous with respect to the topological structure on G and the product topology on $G \times G$. A topological **subgroup** is a subspace which is also a subgroup. A **homomorphism** between two topological groups is a group homomorphism which is also a continuous map. Similarly, an **isomorphism** between topological groups is a group isomorphism which is a homeomorphism. The set of all isomorphisms of a topological group G is a group under composition which is called the **group of automorphisms** of G and denoted by $\mathrm{Aut}(G)$.

Proposition 2.4. *If H is a closed normal subgroup of a topological group G, then the quotient group G/H is a topological group. In particular, the identity component of G, denoted by G_e, is a closed normal subgroup of G and G/G_e is a topological group.*

Proof. See [41], Chapter 1. ■

Definition 2.5. Let X be a topological space and G be a topological group. We say that G **acts on** X on the left or X is a left G**-space** if there exists a continuous map $\psi : G \times X \to X$ denoted by $\psi(g, x) = \psi_g(x)$ such that the following are satisfied.

1. $\psi_e(x) = x$, for all $x \in X$.
2. $\psi_g(\psi_h(x)) = \psi_{gh}(x)$, for all $g, h \in G$ and $x \in X$.

In this case, the map ψ is called the **left action** of G on X. We usually write $\psi_g(x) = gx$. For any point $x \in X$, the **orbit** of x is the set $\mathcal{O}_x = \{gx \mid g \in G\}$ and the **stabilizer subgroup** is $G_x = \{g \in G \mid gx = x\}$. A subset $Y \subset X$ is called G-**invariant** if $gy \in Y$, for any $g \in G$ and $y \in Y$. The **fixed point set** of the action of G on X is the set $X^G = \{x \in X \mid gx = x, \forall g \in G\}$.

Remark 2.2. Similar to Definition 2.5, we can define the **right action** of G on X denoted by $\psi : X \times G \to X$ such that $\psi(x, g) = xg$. In fact, the left and right actions are equivalent by using the rule $gx = xg^{-1}$ (see [41], Section 2.2).

Definition 2.6. Let a topological group G act on a topological space X.

1. The action is **free** if for all $x \in X$, $gx = x$ implies that $g = e$; or equivalently, $G_x = \{e\}$, for all $x \in X$.

2. The action is **locally free** if the stabilizer subgroups are discrete.

3. The action is **transitive** if for any $x, y \in X$, there exists some $g \in G$ such that $y = gx$; or equivalently, there is only one orbit in X.

4. The action is **simply transitive** if for any $x, y \in X$, there exists exactly one $g \in G$ such that $y = gx$.

5. The action is **effective** if for any $g \neq e$ in G, there exists some $x \in X$ such that $gx \neq x$; or equivalently, ψ_g is not the identity map.

Remark 2.3. It is seen from the definitions above that any free action is locally free and effective. Also, an action is simply transitive if and only if it is transitive and free.

Proposition 2.5. *Let X be a G-space and X/G be the set of orbits as well as $\pi : X \to X/G$ be the projection map. If we equip X/G with the quotient topology, then π is an open map. Moreover, when G is compact, the following are satisfied.*

1. *π is a closed map.*

2. *If X is Hausdorff, then so is X/G.*

3. *π is a proper map, that is, the inverse image of compact sets are compact.*

4. *X is compact if and only if X/G is compact.*

5. *If X is Hausdorff, then for any $x \in X$, G/G_x and $\mathcal{O}_x$ are homeomorphic.*

Proof. See [41], Chapter 1. ■

Definition 2.7. Let X and Y be a G-space and H-space respectively and $\theta : G \to H$ e a group homomorphism. A continuous map $f : X \to Y$ is called an **equivariant map** if $f(g.x) = \theta(g).f(x)$, for all $x \in X$ and $g \in G$. When $G = H$ and $\theta = \mathrm{Id}$, we say f is a G**-map**. A G**-homeomorphism** of G-spaces is an equivariant map which is also a homeomorphism.

Proposition 2.6 (**Quotient in Stages**). *Let H be a closed normal subgroup of a topological group G and X be a G-space. Then the G-action on X induces a G/H-action on the orbit space X/H and the natural map $\varphi : X/G \to (X/H)/(G/H)$ is a homeomorphism; i.e.,*

$$X/G \cong \frac{X/H}{G/H}. \tag{2.2}$$

Proof. See [41], Chapter 1. ■

Proposition 2.7. *Let G, K be topological groups such that X is a G-space and Y is a K-space. Then the product space $X \times Y$ is a $(G \times K)$-space and there exists the following homeomorphism between the orbit spaces:*

$$(X \times Y)/(G \times K) \cong (X/G) \times (Y/K). \tag{2.3}$$

Proof. For any $g \in G$, $k \in K$, $x \in X$ and $y \in Y$, define $(g, k).(x, y) = (gx, ky)$. This makes $X \times Y$ into a $G \times K$-space and it is easy to see that the canonical map $\psi : \mathcal{O}_{(x,y)} \longmapsto (\mathcal{O}_x, \mathcal{O}_y)$ is a homeomorphism. ■

Definition 2.8. Let $n > 1$ be a natural number, $G_1, ..., G_n$ be topological groups and $X_0, ..., X_n$ be topological spaces. Suppose that X_0 is a right G_1-space, X_n is a left G_n space and for any $i = 1, ..., n-1$, the space X_i is a left G_i-space and right G_{i+1}-space. For $g_i \in G_i$ and $x_i \in X_i$, set

$$(g_1, ..., g_n).(x_0, x_1, ..., x_{n-1}, x_n) = (x_0 g_1^{-1}, g_1 x_1 g_2^{-1}, ..., g_{n-1} x_{n-1} g_n^{-1}, g_n x_n).$$

This defines an action of $G_1 \times \cdots \times G_n$ on the product space $X_0 \times \cdots \times X_n$. The orbit space of this action is called the **twisted product** of spaces X_i and denoted by $X_0 \times_{G_1} X_1 \times_{G_2} \cdots \times_{G_n} X_n$ In this case, the orbit of each tuple $(x_0, ..., x_n)$ is denoted by $[x_0, ..., x_n]$. In special case, when X is a right G-space and Y is a left G-space, we denote the twisted product by $X \times_G Y$.

Proposition 2.8 (**Associativity of Twisted Product**)**.** *For any $0 \leq i < j \leq n$, consider the following space:*

$$X(i, j) = X_0 \times_{G_1} \cdots \times_{G_{i-1}} X_{i-1} \times_{G_i} \left(X_i \times_{G_{i+1}} \cdots \times_{G_j} X_j \right) \times_{G_{j+1}} \cdots \times_{G_n} X_n.$$

Then $X(i, j)$ is homeomorphic to the twisted product space $X_0 \times_{G_1} X_1 \times_{G_2} \cdots \times_{G_n} X_n$.

Proof. See [41], Chapter 1. ■

Proposition 2.9. *Let G be a topological group. Suppose that X is a right G-space and the action of G on Y is* ***trivial****; that is, $gy = y$, for all $g \in G$ and $y \in Y$. Then we have the following homeomorphism:*

$$X \times_G Y \cong (X/G) \times Y. \tag{2.4}$$

Proof. Consider the natural map $\varphi : X \times_G Y \to (X/G) \times Y$ defined by $\varphi[x, y]_G = ([x]_G, y)$ for each $x \in X$ and $y \in Y$. Define $\psi : (X/G) \times Y \to X \times_G Y$ by $\psi([x]_G, y) = [x, y]_G$. It is easy to see that φ and ψ are continuous maps and $\varphi^{-1} = \psi$. This completes the proof. ■

Proposition 2.10. *Let G be a topological group and X be a right G-space. If G acts on itself by* ***left translations****; i.e., $L_g : G \to G$ where $L_g(h) = gh$, then we have the following G-homeomorphism:*

$$X \times_G G \cong X. \tag{2.5}$$

Proof. See [41], Chapter 1. ■

2.2.2 Fibrations, Fiber Bundles and Principal Bundles

Definition 2.9. A map $p : E \to B$ between topological spaces has the **homotopy lifting property** with respect to a space X if for any map $\widehat{f_0} : X \to E$ and any homotopy $\{f_t : X \to B\}_{t\in[0,1]}$ with $p \circ \widehat{f_0} = f_0$, there exists a homotopy $\{\widehat{f_t} : X \to E\}_{t\in[0,1]}$ such that $p \circ \widehat{f_t} = f_t$, for all $t \in [0, 1]$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\widehat{f_t}"] \arrow[r, "f_t"'] & B \end{tikzcd}$$

Diagram 2.1: Homotopy lifting property

Definition 2.10. A map $p : E \to B$ between topological spaces is called a **fibration** if it has the homotopy lifting property with respect to all topological spaces. In this case, we call B the **base space** and E the **total space**. Also, for any $b \in B$, $p^{-1}(b) = E_b$ is called the **fiber** over b.

Proposition 2.11. *If the base space B in a fibration $p : E \to B$ is path-connected, then the following are satisfied.*

1. *All the fibers E_b have the same homotopy type and there exists an action of the fundamental group of B on the homology and the cohomology of each fiber E_b.*

2. *There exists the following long exact sequence of homotopy groups:*

$$\cdots \to \pi_n(E_b) \to \pi_n(E) \to \pi_n(B) \to \cdots \to \pi_1(B) \to \pi_0(E_b) \to \pi_0(E) \to \pi_0(B)$$

Proof. For part 1, see [48], Proposition 4.18 and for part 2 see [52], Theorem 3.10. ■

Remark 2.4. By Proposition 2.11, we can talk about the fiber F of a fibration $p : E \to B$ over a path-connected space B. In this case, we use the notation $F \overset{i}{\hookrightarrow} E \xrightarrow{p} B$.

Definition 2.11. A **morphism** between fibrations $F \overset{i}{\hookrightarrow} E \xrightarrow{p} B$ and $F \overset{i'}{\hookrightarrow} E' \xrightarrow{p'} B'$ is a pair of maps $(f : B' \to B, \widehat{f} : E' \to E)$ such that $p \circ \widehat{f} = f \circ p'$. Moreover, if $f, \widehat{f}$ are homeomorphisms, then $(f, \widehat{f})$ is called an **isomorphism of fibrations**.

$$\begin{tikzcd} E' \arrow[r, "\widehat{f}"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$$

Diagram 2.2: Morphism between fibrations

Example 2.1. The projection map $p : B \times F \to B$ is a fibration. Any fibration isomorphic to $p : B \times F \to B$ is said to be **trivial**.

Example 2.2. Let $p : E \to B$ be a fibration and $f : X \to B$ be a continuous map. Set $f^*E = \{(x, e) \in X \times E \mid f(x) = p(e)\}$. It is easy to see that the projection map $p_f : f^*E \to X$, defined by $p_f(x, e) = x$, is a fibration and $p_f^{-1}(x) = E_b$. This fibration is called the **pullback** of E by f.

Definition 2.12. Let E, B and F be topological spaces. A **fiber bundle** with **fiber** F is a continuous surjection $p : E \to B$ satisfying the following conditions.

1. For any $b \in B$, the fiber $E_b = p^{-1}(b)$ over b is homeomorphic to F.

2. There exists an open covering $\{U_i \mid i \in I\}$ of B such that for any $i \in I$, we can find a homeomorphism $\varphi_i : p^{-1}(U_i) \to U_i \times F$ for which $p_1 \circ \varphi_i = p$, where p_1 is the projection onto the first component. The pair (U_i, φ_i) is called a **local trivialization**.

$$\begin{tikzcd}
p^{-1}(U_i) \arrow[rr, "\varphi_i"] \arrow[dr, "p"'] & & U_i \times F \arrow[dl, "p_1"] \\
& U_i &
\end{tikzcd}$$

Diagram 2.3: A local trivialization

3. For any $b \in U_i \cap U_j \neq \emptyset$, there exists a homeomorphism $\theta_{ij}(b) : F \to F$ such that $\varphi_i^{-1}(x, b) = \varphi_j^{-1}(\theta_{ij}(b)x, b)$, for all $x \in F$.

In this case, B, E and p are called the **base space**, the **total space** and the **projection**, respectively. The map θ_{ij} is called the **transition map** for the local trivializations (U_i, φ_i) and (U_j, φ_j).

Remark 2.5. A theorem of Hurewicz says that fiber bundles over paracompact base spaces are fibrations (see [20], Chapter 4).

Definition 2.13. Let G be a topological group and $p : E \to B$ be a fiber bundle with fiber F. If there exists an effective G-action on the fiber F such that for any transition map θ_{ij}, we have $\theta_{ij}(b) \in G$, then we call $p : E \to B$ a **G-fiber bundle** with the **structure group** G.

Definition 2.14. Two fiber bundles $p_i : E_i \to B$, $i = 1, 2$, are called **isomorphic** if there exists a homeomorphism $f : E_1 \to E_2$ such that $p_2 \circ f = p_1$. In this case, we write $E_1 \cong E_2$. A fiber bundle $E \to B$ with fiber F is called **trivial** if it is isomorphic to the product space $B \times F$.

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

Diagram 2.4: Isomorphic fiber bundles

Example 2.3. The unit 2-sphere without two poles is an example of a trivial fiber bundle with the unit circle as the base space and the real line as the fiber (see Figure2.1).

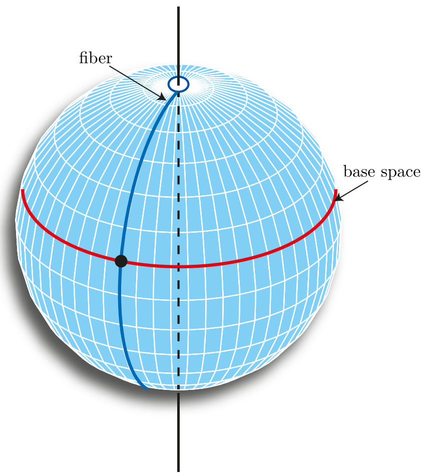

Figure 2.1: A trivial fiber bundle

An example of a nontrivial fiber bundle is the Mobius strip in which the fiber is a line segment and the base space is the circle (see Figure 2.2).

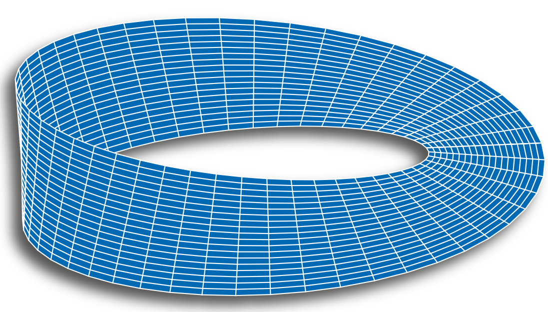

Figure 2.2: Mobius strip as a nontrivial fiber bundle

Example 2.4. Let $p : E \to B$ be a fiber bundle with fiber F and $f : X \to B$ be a continuous map. Set $f^*E = \{(e, x) \in E \times X \mid f(x) = p(e)\}$. Similar to fibrations, the projection on the second component $p_2 : f^*E \to X$ is a fiber bundle with fiber F, called the **pullback** of the fiber bundle $p : E \to B$. A continuous map $s : B \to E$ is called a **section** of the fiber bundle $p : E \to B$ if $p \circ s = \mathrm{Id}_B$.

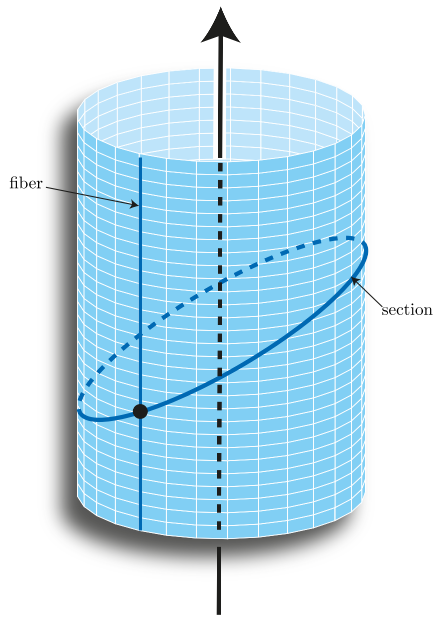

Figure 2.3: Section of a fiber bundle

Definition 2.15. A **principal G-bundle** is a G-fiber bundle $p : E \to B$ in which the fiber is $F = G$ and G acts on F by left translations.

Proposition 2.12. *If $p : E \to B$ is a principal G-bundle, then there exists a canonical free G-action on E such that E/G is homeomorphic to B. If a compact group G acts freely on a completely regular topological space E, then the quotient map $\pi : E \to E/G$ is a principal G-bundle.*

Proof. See [16], Chapter 1. ■

Proposition 2.13. *A principal G-bundle $p : E \to B$ is trivial if and only if there exists a section $s : B \to E$ for p.*

Proof. See [41], Chapter 2. ■

Proposition 2.14 (**Existence of Universal Bundles**)**.** *Let G be a topological group. Then there exists a principal G-bundle $EG \to BG$ with contractible total space EG. Moreover, for any principal G-bundles $E \to B$ over a paracompact space B, there exists a map $f : B \to BG$ such that E is the pullback of EG and if $g : B \to BG$ is another map such that $g^*(EG) = E$, then f and g are homotopic.*

Proof. See [38], Chapter 4. ■

Corollary 2.14.1. *For any topological group G, the base space BG is unique up to homotopy. In particular, if G is connected, then BG is simply connected.*

Proof. First part follows from Proposition 2.14. For the second part, we consider the homotopy long exact sequence of the fibration $G \hookrightarrow EG \to BG$ given in Proposition 2.11. Since EG is contractible the assertion follows. ■

Definition 2.16. Let G be a topological group and R be a commutative ring with unity. The principal G-bundle $p : EG \to BG$ in which the total space EG is contractible is called a **universal G-bundle** for the group G. In this case, $BG = EG/G$ is called the **classifying space** associated to this bundle. Also, for any principal G-bundle $p : X \to B$ and map $f : B \to BG$, the ring $f^*(H^*(BG; R))$ is called the **characteristic ring** of the principal bundle and its elements are called **characteristic classes** of the principal bundle $p : X \to B$.

Remark 2.6. An explicit choice for EG given in [38] is called the **Milnor Join**. It is defined as follows. Let $I_c^\infty = \bigoplus_{n\geq 0} I_n$ where $I_n = [0, 1]$ and set

$$\Delta_\infty = \{t = (t_i)_{i\geq 0} \in I_c^\infty \mid \sum_{i=0}^{\infty} t_i = 1\}.$$

Suppose that G^∞ is the set of all sequences $x = (x_i)_{i\geq 0}$ where $x_i \in G$ and define the equivalence relation $\sim$ on $G^\infty \times \Delta_\infty$ by

$$(x_0, t_0; x_1, t_1; ...) \sim (y_0, s_0; y_1, s_1, ...) \Leftrightarrow (\forall i,\ t_i = s_i) \ \&\ (\text{if } t_i = s_i \neq 0 \Rightarrow x_i = y_i).$$

Denote the quotient space $(G^\infty \times \Delta_\infty)/\sim$ by EG and its elements by $< x, t >$. By putting the direct limit topology on EG, one can see that G acts freely on EG by $g. < x, t >=< g.x, t >$. If we have a group homomorphism $\theta : G \to H$, the structure on EG helps us to define an equivariant map $E\theta : EG \to EH$ by $E\theta < x, t >=< \theta(x), t >$ where $\theta(x) = (\theta(x_i))_{i\geq 0}$. This also induces a map $B\theta : BG \to BH$ on the classifying spaces.

Example 2.5. Let $k \leq n$ be natural numbers. Let $V_k(\mathbb{R}^n)$ be the set of all k-frames in $\mathbb{R}^n$; i.e., k-tuples of orthonormal vectors, and $\mathrm{Gr}_k(\mathbb{R}^n)$ be the set of all k-dimensional subspaces of $\mathbb{R}^n$. It is known that $V_k(\mathbb{R}^n)$ and $\mathrm{Gr}_k(\mathbb{R}^n)$ are manifolds called **Stiefel manifold** and **Grassmannian manifold** respectively (see Appendix B). Consider the natural map $\pi_k : V_k(\mathbb{R}^n) \to \mathrm{Gr}_k(\mathbb{R}^n)$ sending each k-frame to the subspace spanned by it. One can show that this is a principal $\mathrm{O}(n)$-bundle (see [41], Section 3.9). The map $(x_1, ..., x_k) \longmapsto (x_1, ..., x_n, 0)$ embeds $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$. So we can consider k-frames and k-subspaces of $\mathbb{R}^n$ as k-frames and k-subspaces of $\mathbb{R}^{n+1}$ respectively. Therefore, we have the inclusions $V_k(\mathbb{R}^n) \subset V_k(\mathbb{R}^{n+1})$ and $\mathrm{Gr}_k(\mathbb{R}^n) \subset \mathrm{Gr}_k(\mathbb{R}^{n+1})$. Set $V_k(\mathbb{R}^\infty) = \bigcup_{n\geq 1} V_k(\mathbb{R}^n)$ and $\mathrm{Gr}_k(\mathbb{R}^\infty) = \bigcup_{n\geq 1} \mathrm{Gr}_k(\mathbb{R}^n)$ and consider the weak topology on each of them. The natural map $\pi : V_k(\mathbb{R}^\infty) \to \mathrm{Gr}_k(\mathbb{R}^\infty)$ is also a principal $\mathrm{O}(n)$-bundle and $V_k(\mathbb{R}^\infty)$ is contractible (see [32], Chapter 4). So this is a universal bundle for the orthogonal group $\mathrm{O}(n)$ and thus $B\mathrm{O}(n) = \mathrm{Gr}_k(\mathbb{R}^\infty)$. Similarly, we can define complex Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$ and show that $B\mathrm{U}(n) = \mathrm{Gr}_k(\mathbb{C}^\infty) = \cup_{n\geq 1}\mathrm{Gr}_k(\mathbb{C}^n)$.

2.2.3 Equivariant Cohomology

Definition 2.17. Let X be a left G-space and $EG \to BG$ be a fixed universal G-bundle. The twisted product space $X_G = EG \times_G X$ is called the **homotopy quotient** or the **Borel construction** of X with respect to the fixed universal bundle.

Remark 2.7. Although for different universal bundles $EG \to BG$ we get different homotopy quotients, they all have the same homotopy types and thus up to homotopy, our definition is well-defined. In particular, when X is a singleton $\{*\}$, it follows from Proposition 2.9 that $X_G \simeq BG$.

Proposition 2.15. *Let X be a left G-space and $EG \to BG$ be a fixed universal G-bundle. Then we have the following commutative diagram of maps, called **Cartan-Borel diagram**:*

$$\begin{tikzcd}
EG \arrow[d] & EG \times X \arrow[l] \arrow[r] \arrow[d] & X \arrow[d] \\
BG & X_G \arrow[l, "\pi"'] \arrow[r, "\sigma"'] & X/G
\end{tikzcd}$$

Diagram 2.5: Cartan-Borel diagram

Moreover, the map $\pi : X_G \to BG$ is a fiber bundle with fiber X and for the projection $\sigma : X_G \to X/G$, the fiber over $\mathcal{O}_x$ is $\sigma^{-1}(\mathcal{O}_x) = BG_x$, the classifying space of the stabilizer subgroup of $x \in X$. In particular, if X is paracompact and G acts freely on it, then σ is a fibration with contractible fiber EG and therefore a homotopy equivalence.

Proof. See [12], Chapter 4 or [5], Chapter 6. ■

Suppose that topological groups G and H act on the left on spaces X, Y respectively. Let $\theta : G \to H$ be a group homomorphism and $f : X \to Y$ be a continuous map for which $f(gx) = \theta(g)f(x)$. Then, it is easy to see from Remark 2.6 that f induces a map $f_h : X_G \to Y_H$ between homotopy quotients such that $f_h[e, x] = [E\theta(e), f(x)]$ and the following diagram is commutative:

$$\begin{tikzcd}
EG \arrow[dr, "E\theta"] \arrow[dd] & & EG \times X \arrow[ll] \arrow[rr] \arrow[dr, "E\theta\times f"] \arrow[dd] & & X \arrow[dr, "f"] \arrow[dd] & \\
& EH \arrow[dd] & & EH \times Y \arrow[ll] \arrow[rr] \arrow[dd] & & Y \arrow[dd] \\
BG \arrow[dr, "B\theta"] & & X_G \arrow[ll] \arrow[rr] \arrow[dr, "f_h"] & & X/G \arrow[dr, "\bar{f}"] & \\
& BH & & Y_H \arrow[ll] \arrow[rr] & & Y/H
\end{tikzcd}$$

Diagram 2.6: Commutative diagram induced by a group homomorphism

Proposition 2.16. *Let $\theta : G \to H$ and $\eta : H \to K$ be group homomorphisms on topological groups.*

1. *If $B\theta : BG \to BH$, $B\eta : BH \to BK$ and $B(\eta \circ \theta) : BG \to BK$ are the induced maps on corresponding classifying spaces, then two maps $B\eta \circ B\theta$ and $B(\eta \circ \theta)$ are homotopic.*

2. *Suppose that X, Y and Z are left G-space, H-space and K-space respectively as well as for continuous maps $f : X \to Y$ and $g : Y \to Z$, we have $f(gx) = \theta(g)f(x)$ and $g(hy) = \eta(h)g(y)$. If $f_h : X_G \to Y_H$, $g_h : Y_H \to Z_K$ and $(g \circ f)_h : X_h \to Z_K$ are the induced maps on homotopy quotients, then $g_h \circ f_h$ and $(g \circ f)_h$ are homotopic.*

Proof. For part 1, see [21], Chapter 14. Part 2 follows from Part 1, Proposition 2.14 and Diagram 2.7. ■

$$\begin{array}{ccccc}
X_G & \xrightarrow{f_h} & Y_H & \xrightarrow{g_h} & Z_K \\
\downarrow & & \downarrow & & \downarrow \\
BG & \xrightarrow{B\theta} & BH & \xrightarrow{B\eta} & BK
\end{array}$$

Diagram 2.7: Commutative diagram induced by a pair of group homomorphisms

Proposition 2.17. *Let X be a left G-space and K be a closed subgroup of G. Then X is a left K-space and we have the following commutative diagram of fibrations.*

$$\begin{array}{ccccc}
 & & X & \xrightarrow{\mathrm{Id}} & X \\
 & & \downarrow & & \downarrow \\
G/K & \longrightarrow & X_K & \longrightarrow & X_G \\
{\scriptstyle \mathrm{Id}}\downarrow & & \downarrow & & \downarrow \\
G/K & \longrightarrow & BK & \longrightarrow & BG
\end{array}$$

Diagram 2.8: Commutative diagram induced by a closed subgroup

Proof. See [37], Chapter 3. ■

Proposition 2.18 (Homotopy Quotient Extension). *Let K be a closed subgroup of a topological group G and Y be a left K-space. If $X = G \times_K Y$ is the twisted product of G and Y, then the map $g.[h, y] = [gh, y]$ makes X into a left G-space and the homotopy quotients X_G and Y_K are homotopy equivalent.*

Proof. See [41], Chapter 1 and [3], Section 13. ■

Proposition 2.19 (**Homotopy Quotient in Stages**). *Let K be a closed normal subgroup of a topological group G and X be a left G-space. If $Y = X_K$ is the homotopy quotient space of X with respect to K-action, then Y is a left G/K-space and the homotopy quotient spaces X_G and $Y_{G/K}$ are homotopy equivalent; i.e,*

$$X_G \simeq (X_K)_{G/K}. \tag{2.6}$$

Proof. Let $E_1G \to B_1G$ and $E(G/K) \to B(G/K)$ be universal bundles for groups G and G/K respectively and set $EG = E(G/K) \times E_1G$. The group G acts on $E(G/K)$ via the quotient group G/K and EG is contractible, so we can take EG as the total space of a universal bundle $EG \to BG$ of G. The assertion follows from the quotient in stages (Proposition 2.6) and the associativity of the twisted product (Proposition 2.8). ■

Proposition 2.20. *Let G and K be topological groups such that X is a left G-space and Y is a left K-space. The product space $X \times Y$ is a $(G \times K)$-space and there exists the following homotopy equivalence between homotopy quotient spaces:*

$$(X \times Y)_{G\times K} \simeq X_G \times Y_K. \tag{2.7}$$

In particular, two classifying spaces $B(G \times K)$ and $BG \times BK$ are homotopy equivalent.

Proof. If $EG \to BG$ and $EH \to BH$ are universal bundles for G and H respectively, then $EG \times EH \to BG \times BH$ is a universal bundle for the product group $G \times H$. The first part follows from Propositions 2.7 and 2.8. To prove the second part, set $X = Y = \{*\}$. ■

Definition 2.18. Let X be a left G-space, R a commutative ring with unity and $EG \to BG$ be a universal G-bundle. The **equivariant cohomology** of X is defined as the ordinary cohomology of the homotopy quotient X_G; i.e.,

$$H^*_G(X; R) = H^*(X_G; R). \tag{2.8}$$

Remark 2.8. Corollary 2.14.1 guarantees that Definition 2.18 is well-defined. Moreover, the functorial nature of the construction $X \longmapsto X_G$ enables us to extend all the concepts of ordinary cohomology to the equivariant cohomology in an essentially routine manner.

Example 2.6. Consider the compact Lie groups $G = (\mathbb{Z}_2)^n$ or $(S^1)^n$. It follows from Example 2.5 and Proposition 2.20 that $BG = (\mathbb{RP}^\infty)^n$ or $(\mathbb{CP}^\infty)^n$, respectively.

2.2.4 Vector Bundles

Definition 2.19. A fiber bundle $p : E \to B$ is called a **vector bundle** if the fiber F has a vector space structure and each map $\varphi_i : E_b \to \{b\} \times F$ is a linear isomorphism. In this case, we call the map $p : E \to B$ a **real or complex vector bundle** provided that F is a real or complex vector space. The dimension of the vector space F is called the **rank** of the vector bundle.

Example 2.7. Any product space $B \times \mathbb{R}^n$ with the natural projection $\pi_1 : B \times \mathbb{R}^n \to B$ is a real n-vector bundle. Another example of a real vector bundle of rank n is the tangent bundle of an n-dimensional smooth manifold (see Figure 2.4).

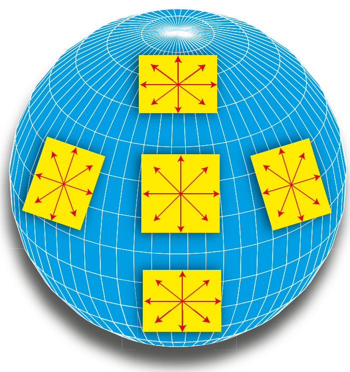

Figure 2.4: Tangent bundle of 2-sphere as a 2-vector bundle

Definition 2.20. Two vector bundles $p : E \to B$ and $p' : E' \to B$ are called **isomorphic** if there exists a homeomorphism $f : E \to E'$ such that $p' \circ f = p$ and the restriction map $f_b : E_b \to E'_b$ is a vector space isomorphism. Any vector bundle isomorphic to the product bundle $B \times \mathbb{R}^n$ or $B \times \mathbb{C}^n$ is called a **trivial vector bundle**.

Example 2.8 (Tautological Bundles). Let $\mathrm{Gr}_k(\mathbb{R}^n)$ be the real Grassmannian. Set

$$\gamma_k^n = \{(L, x) \in \mathrm{Gr}_k(\mathbb{R}^n) \times \mathbb{R}^n \mid x \in L\}. \tag{2.9}$$

One can show that the natural projection $\pi_k : \gamma_k^n \to \mathrm{Gr}_k(\mathbb{R}^n)$ is a nontrivial k-vector bundle over the Grassmannian $\mathrm{Gr}_k(\mathbb{R}^n)$ (see [51], Section 5). Moreover, if we consider the spaces $\mathrm{Gr}_k(\mathbb{R}^\infty) = \bigcup_{n\geq 1} \mathrm{Gr}_k(\mathbb{R}^n)$ and $\gamma_k^\infty = \cup_{n\geq 1}\gamma_k^n$ with the weak topology, then the projection $\pi_\infty : \gamma_k^\infty \to \mathrm{Gr}_k(\mathbb{R}^\infty)$ is a real k-vector bundle (see [33], Lemma 1.15). These vector bundles are called the **tautological vector bundles** over the real Grassmannians. Similarly, we can define the tautological vector bundles over the complex Grassmannians.

Definition 2.21. A **section** of a vector bundle $p : E \to B$ is a continuous map $s : B \to E$ such that $p \circ s = \mathrm{Id}$. The **zero section** $s_0 : B \to E$ is the map sending any point $b \in B$ to the zero vector in the vector space $E_b \cong F$.

Definition 2.22. A **metric** on a vector bundle $p : E \to B$ is a family of maps $\{g_b : E_b \times E_b \to \mathbb{R} \mid b \in B$ such that each g_b is an inner product on the vector space E_b and for any two sections $s, s' : B \to E$, the map $g : E \to \mathbb{R}$ by $g(e) = g_{p(e)}(s(p(e)), s'(p(e)))$ is continuous.

Proposition 2.21 (Pullback). *For any k-vector bundle $p : E \to B$ and any continuous map $f : X \to B$, the set $f^*E = \{(e, x) \in E \times X \mid p(e) = f(x)\}$ with the natural* *projection $p_f : f^*E \to X$ is a k-vector bundle, called the **pullback** of E by f. In the special case, the pullback of a subset $C \subset B$ by the inclusion map $i : C \hookrightarrow B$; i.e. i^*E, is called the **restriction** of $p : E \to B$.*

Proof. See [51], Section 3. ∎

Proposition 2.22 (**Whitney Sum**). *Let $p : E \to B$ and $p' : E' \to B'$ be two vector bundles of ranks k and k' respectively. Then the product map $p \times p' : E \times E' \to B \times B'$ is a $k + k'$-vector bundle. In particular, if $B = B'$ and $\Delta : B \to B \times B$ is the diagonal map, then the pullback vector bundle $\Delta^*(B \times B)$ is a $k + k'$-vector bundle over B which is called the **Whitney sum** of E and E', denoted by $E \oplus E'$.*

Proof. See [33], Chapter 1. ∎

Proposition 2.23. *Let $p : E \to B$ be a k-vector bundle equipped with a metric and F be a **subbundle** of rank k'; i.e., each fiber F_b is a k'-subspace of E_b. Let $F_b^\perp$ be the orthogonal complement of F_b with respect to the inner product on E_p. If $F^\perp$ is the collection of all $F_b^\perp$, for $b \in B$ and $q : F^\perp \to B$ is the natural projection, then $q : F^\perp \to B$ is a $(k - k')$-vector bundle such that $E = F \oplus F^\perp$. This is called the **orthogonal complement bundle** of F in E.*

Proof. See [51], Section 3. ∎

Remark 2.9. When we consider a smooth manifold M as a subset of the Euclidean space $\mathbb{R}^n$, then the orthogonal complement of the tangent bundle TM of M in $\mathbb{R}^n$ is called the **normal bundle** of M, denoted by $\nu(M)$ (see Figure 2.5). Clearly, the rank of $\nu(M)$ is equal to codim (M).

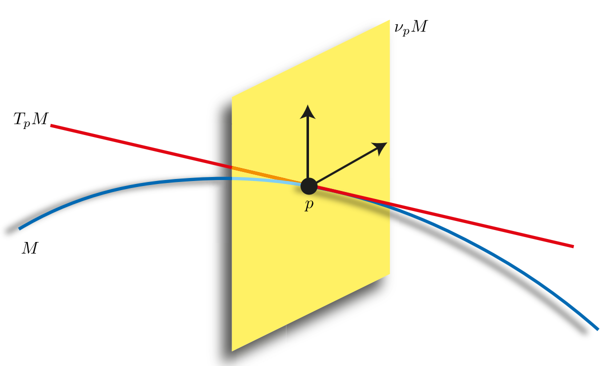

Figure 2.5: Normal bundle as the orthogonal complement of tangent bundle

Definition 2.23. A real n-vector bundle $p : E \to B$ is **orientable** if there exists a function assigning an orientation to each fiber in such a way that for each local trivialization (U, φ), the map $\varphi : p^{-1}(U) \to U \times \mathbb{R}^n$ carries the orientations of fibers in $p^{-1}(U)$ to the standard orientation of $\mathbb{R}^n$ in the fibers of $U \times \mathbb{R}^n$.

2.2.5 Characteristic Classes

Let $p : E \to B$ be a real vector bundle of rank k and E_0 be the complement of the zero section of E. Denote the typical fiber $E_b \cong \mathbb{R}^k$ with F and let $F_0 = E_0 \cap F$. Consider the inclusion of pairs $(F, F_0) \hookrightarrow (E, E_0)$.

Proposition 2.24 (**Thom Isomorphism for $\mathbb{Z}_2$-coefficients**). *The cohomology group $H^n(E, E_0; \mathbb{Z}_2)$ is trivial for $n < k$ and there exists a unique cohomology class $\tau \in H^k(E, E_0; \mathbb{Z}_2)$ such that its restriction to the pair (F, F_0) is the unique nonzero generator of $H^k(F, F_0; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Moreover, the map $T : H^n(B; \mathbb{Z}_2) \to H^{k+n}(E, E_0; \mathbb{Z}_2)$ defined by $T(\alpha) = p^*\alpha \cup \tau$ is an isomorphism.*

Proof. See [51], Section 10. ■

The unique class τ is called the **fundamental class** or the **Thom class** and the isomorphism T is called the **Thom isomorphism**. Note that $\tau = T(1)$.

Proposition 2.25 (**Steenrod Squares**). *For any pair of spaces $Y \subset X$ and non-negative integers n, m, there exist homomorphisms*

$$Sq^m : H^n(X, Y; \mathbb{Z}_2) \to H^{n+m}(X, Y; \mathbb{Z}_2)$$

having the following properties.

1. *For any other pair (X', Y') and map $f : (X, Y) \to (X', Y')$, we have $Sq^m \circ f^* = f^* \circ Sq^m$.*
2. *If $\alpha \in H^n(X, Y; \mathbb{Z}_2)$, then $Sq^0(\alpha) = \alpha$, $Sq^n(\alpha) = \alpha \cup \alpha$ and $Sq^m(\alpha) = 0$, for all $m > n$.*
3. *Whenever $\alpha \cup \beta$ is defined, we have*

$$Sq^p(\alpha \cup \beta) = \sum_{i+j=p} Sq^i(\alpha) \cup Sq^j(\beta). \tag{2.10}$$

Proof. See [51], Section 8. ■

Definition 2.24 (**Stiefel-Whitney Classes**). For a real k-vector bundle $p : E \to B$, its **Stiefel-Whitney classes** are the cohomology classes $w_n(E) \in H^n(B; \mathbb{Z}_2)$ defined by

$$w_n(E) = T^{-1}(Sq^n(\tau)), \ \forall n \geq 0. \tag{2.11}$$

Proposition 2.26 (**Properties of Stiefel-Whitney Classes**). *Let $p : E \to B$ be a real vector bundle of rank k and $w_n(E) \in H^n(B; \mathbb{Z}_2)$ be its Stiefel-Whitney classes. Then the following are satisfied.*

1. *$w_0(E) = 1$ and $w_n(E) = 0$ for $n > k$. In particular, for a trivial vector bundle all the Stiefel-Whitney classes of positive degrees are trivial.*

2. *If* $f : B' \to B$ *is a continuous map, then* $w_n(f^*E) = f^*w_n(E)$*, for all* $n \geq 0$.

3. *If* $p' : E' \to B$ *is another real vector bundle over* B*, then* $w_n(E \oplus E') = \sum_{i=0}^{n} w_i(E)w_{n-i}(E')$*, for all* $n \geq 0$.

4. *For the tautological line bundle* γ_1^2 *over the projective space* $\mathbb{RP}^1$*, the first Stiefel-Whitney class* $w_1(\gamma_1^2) \neq 0$.

5. *If the vector bundle possesses a nowhere zero section, then the top Stiefel-Whitney class is zero.*

Proof. See [51], Sections 4 and 8. ■

Example 2.9. Let $w_1, ..., w_n$ be the Stiefel-Whitney classes of the tautological n-vector bundle $\gamma_n^\infty \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ in Example 2.8. It is known (see [34], Theorem 9.5.8) that the cohomology of classifying spaces of the orthogonal groups $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are the following polynomial rings:

$$\begin{cases} H^*(B\mathrm{O}(n);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, ..., w_n], \\ H^*(B\mathrm{SO}(n);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_2, ..., w_n]. \end{cases} \tag{2.12}$$

If $D(n) = \mathrm{O}(1)^n$ and w is the first Stiefel-Whitney class of the line bundle $\gamma_1^\infty \to \mathrm{Gr}_1(\mathbb{R}^\infty)$, then (2.12) and Proposition 2.20 follow that

$$H^*(BD(n);\mathbb{Z}_2) \cong \bigotimes_{i=1}^{n} H^*(B\mathrm{O}(1);\mathbb{Z}_2) \cong \bigotimes_{i=1}^{n} \mathbb{Z}_2[w] \cong \mathbb{Z}_2[x_1, ..., x_n].$$

Therefore,

$$H^*(BD(n);\mathbb{Z}_2) \cong \mathbb{Z}_2[x_1, ..., x_n], \tag{2.13}$$

where each x_i is a polynomial of degree 1. In fact, one can show that (see [34], Theorem 9.6.1) if $\Delta : D(n) \to \mathrm{O}(n)$ is the map $(\epsilon_1, ..., \epsilon_n) \longmapsto \mathrm{Diag}[\epsilon_1, ..., \epsilon_n]$, then the induced map $\Delta^* : H^*(B\mathrm{O}(n);\mathbb{Z}_2) \to H^*(BD(n);\mathbb{Z}_2)$ is injective and sends each w_i to the i^{th} symmetric polynomial σ_i in $\{x_1, ..., x_n\}$:

$$\begin{cases} \sigma_1 = \sum_{i=1}^{n} x_i, \\ \sigma_2 = \sum_{1 \leq i < j \leq n} x_i x_j, \\ \quad \vdots \\ \sigma_n = x_1 \cdots x_n. \end{cases} \tag{2.14}$$

Similar to Proposition 2.24, there is a Thom isomorphism for the cohomology of oriented real vector bundles with $\mathbb{Z}$-coefficients.

Proposition 2.27. *Let $p : E \to B$ be a real oriented k-vector bundle. Then the cohomology group $H^n(E, E_0; \mathbb{Z}) = (0)$, for $n < k$ and there exists a unique cohomology class $\tau \in H^k(E, E_0; \mathbb{Z})$ (called the Thom class) such that its restriction to the pair (F, F_0) is the generator of $H^k(F, F_0; \mathbb{Z})$ induced by the orientation. Moreover, the map $T : H^n(B; \mathbb{Z}) \to H^{n+k}(E, E_0; \mathbb{Z})$ defined by $T(\alpha) = p^*(\alpha) \cup \tau$ is an isomorphism.*

Proof. See [51], Section 9. ■

Definition 2.25 (Euler Class). Let $p : E \to B$ be a real oriented k-vector bundle. The **Euler class** is the cohomology class $\mathrm{Eul}(E) \in H^k(B; \mathbb{Z})$ such that $p^*(\mathrm{Eul}(E)) = \tau|_E$, where $\tau|_E$ is the image of the Thom class τ under the map $H^k(E, E_0; \mathbb{Z}) \to H^k(E; \mathbb{Z})$ induced by the inclusion $j : (E, \emptyset) \hookrightarrow (E, E_0)$.

Proposition 2.28. *The Euler class of an oriented vector bundle has the following properties.*

1. *If $f : C \to B$ is a continuous map and we equip the pullback with the orientation induced by f, then $\mathrm{Eul}(f^*E) = \mathrm{Eul}(E)$.*

2. *If the orientation is reversed, then the sign of $\mathrm{Eul}(E)$ changes.*

3. *If the rank of vector bundle is odd, then $\mathrm{Eul}(E) = 0$.*

4. *The natural homomorphism $H^k(B; \mathbb{Z}) \to H^k(B; \mathbb{Z}_2)$ sends the Euler class $\mathrm{Eul}(E)$ to the top Stiefel-Whitney class $w_k(E)$.*

5. *If the vector bundle possesses a nowhere zero section, then the Euler class is zero.*

Proof. See [51], Section 9. ■

Remark 2.10. It follows from Definition 2.25 and Proposition 2.28 that

$$p^*(w_k(E)) = j^*(\tau), \tag{2.15}$$

where $w_k(E)$ the top Stiefel-Whitney class and τ is the Thom class of the k-vector bundle $p : E \to B$.

Proposition 2.29 (Gysin Sequence). *To any oriented k-vector bundle $p : E \to B$, there is associated a long exact sequence of the following form:*

$$\cdots \to H^n(B; \mathbb{Z}) \xrightarrow{\phi} H^{n+k}(B; \mathbb{Z}) \xrightarrow{p_0^*} H^{n+k}(E_0; \mathbb{Z}) \to \cdots \tag{2.16}$$

where $\phi(\alpha) = \mathrm{Eul}(E) \cup \alpha$ and $p_0 : E_0 \to B$ is the restriction of p to E_0. If the vector bundle is not oriented and we consider $\mathbb{Z}_2$ instead of $\mathbb{Z}$, then the same sequence is associated with the top Stiefel-Whitney class in the place of the Euler class.

Proof. See [51], Section 12. ■

Remark 2.11. Let Y be a manifold and $X \subset Y$ be a submanifold. For any $x \in X$, we have the tangent spaces $T_xX \subset T_xY$. The quotient space $N_xX = T_xY/T_xX$ gives a new vector bundle over X, known as the **normal bundle** $\nu(X)$ of X in Y. It is easy to show that in the presence of a Riemannian metric, this normal bundle is isomorphic to the orthogonal complement bundle defined in Proposition 2.23 (see Figure 2.6).

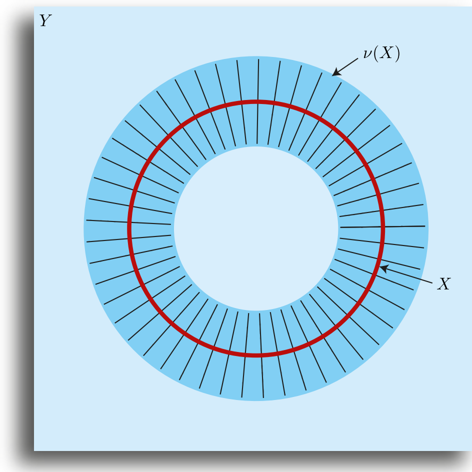

Figure 2.6: Normal bundle of a submanifold

Proposition 2.30 (**Thom-Gysin sequence**)**.** *Let Y be a manifold and X be a submanifold of Y with codimension $d(X)$. Then there exists a long exact sequence in $\mathbb{Z}_2$-cohomology as follows:*

$$\cdots \to H^{*-d(X)}(X;\mathbb{Z}_2) \xrightarrow{\Phi} H^*(Y;\mathbb{Z}_2) \xrightarrow{\Psi} H^*(Y-X;\mathbb{Z}_2) \to \cdots \tag{2.17}$$

Proof. Set $W = Y - X$. Let $\nu(X)$ be the normal bundle of X in Y and $\nu_0(X) = \nu(X) - X$ be the complement of the zero section (see Figure 2.6). Consider the long exact sequence for the triple $(Y, W, Y - \nu(X))$ as follows:

$$\cdots \to H^*(Y,W;\mathbb{Z}_2) \xrightarrow{\alpha^*} H^*(Y;\mathbb{Z}_2) \xrightarrow{\beta^*} H^*(W;\mathbb{Z}_2) \to \cdots \tag{2.18}$$

By using excision for the triple $(Y, W, Y - \nu(X))$, we get the following isomorphism:

$$\varphi : H^*(\nu(X), \nu_0(X);\mathbb{Z}_2) \to H^*(Y,W;\mathbb{Z}_2). \tag{2.19}$$

By Proposition 2.24, we have the Thom isomorphism:

$$T : H^{n-d(X)}(X,\mathbb{Z}_2) \to H^n(\nu(X), \nu_0(X);\mathbb{Z}_2). \tag{2.20}$$

Set $\Phi = \alpha^* \circ \varphi \circ T$ and $\Psi = \beta^*$. By using these maps and the long exact sequence (2.16), we obtain the long exact sequence (2.17). This completes the proof. ∎

Corollary 2.30.1. *If the top Stiefel-Whitney class of the normal bundle of the submanifold X is not a zero divisor, then the long exact sequence (2.17) breaks into short exact sequences.*

Proof. Let $k = d(X)$, $i : \nu(X) \hookrightarrow Y$ be inclusion map and $q : \nu(X) \to X$ be the normal bundle projection. Suppose that $p_w : H^{n-k}(X;\mathbb{Z}_2) \to H^n(X;\mathbb{Z}_2)$ be the cup product with the top Stiefel-Whitney class $w_k(X)$; i.e., $p_w(\gamma) = \gamma \cup w_k(X)$. By (2.15) and the definition of the Thom isomorphism (Proposition 2.24), we have

$$j^*(T(\gamma) = j^*(q^*(\gamma)\cup\tau) = q^*(\gamma)\cup j^*(\tau) = q^*(\gamma)\cup q^*(w_k(X)) = q^*(\gamma\cup w_k(X)) = q^*(p_w(\gamma)).$$

That is, $j^* \circ T = q^* \circ p_w$. This shows that the following diagram induced by the Gysin sequence (2.16) and the long exact sequences for pairs $(Y, Y-X)$ and $(\nu(X), \nu_0(X))$ is commutative:

$$\begin{tikzcd}
\cdots \arrow[r] & H^*(Y, Y-X;\mathbb{Z}_2) \arrow[r, "\alpha^*"] \arrow[d, "\varphi^{-1}"] & H^*(Y;\mathbb{Z}_2) \arrow[r] \arrow[d, "i^*"] & H^*(Y-X;\mathbb{Z}_2) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^*(\nu(X),\nu_0(X);\mathbb{Z}_2) \arrow[r, "j^*"] \arrow[d, "T^{-1}"] & H^*(\nu(X);\mathbb{Z}_2) \arrow[r] \arrow[d, "(q^*)^{-1}"] & H^*(\nu_0(X);\mathbb{Z}_2) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^{*-k}(X;\mathbb{Z}_2) \arrow[r, "p_w"] & H^*(X;\mathbb{Z}_2) \arrow[r] & H^*(\nu_0(X);\mathbb{Z}_2) \arrow[r] & \cdots
\end{tikzcd}$$

Diagram 2.9: Thom-Gysin sequence of normal bundle

On one hand, we saw in proof of Proposition 2.30 that $\Phi = \alpha^* \circ \varphi \circ T$. On the other hand, since $w_k(X)$ is not a zero divisor, p_w is injective and the commutativity implies that α^* is injective. Therefore Φ must be injective and the long exact sequence (2.17) breaks into short exact sequences. This completes the proof. ■

2.2.6 Equivariant Vector Bundles and Characteristic Classes

Definition 2.26. Let $p : E \to X$ be a vector bundle ξ of rank n and G be a topological group. We say ξ is a G-**equivariant vector bundle** of rank n if E and X are G-spaces, p is a G-map and for any $g \in G$ and $x \in X$, the induced map $g : E_x \to E_{gx}$ is a linear isomorphism.

Proposition 2.31. *Let $p : E \to X$ be a G-equivariant vector bundle ξ of rank n. Then the induced map $p_G : E_G \to X_G$ is a vector bundle of rank n, denoted by ξ_G. Moreover, if $p' : E' \to X$ is another G-equivariant vector bundle ξ' of rank m, then their Whitney sum is a G-equivariant vector bundle of rank $n+m$ and we have*

$$(\xi \oplus \xi')_G = \xi_G \oplus \xi'_G. \tag{2.21}$$

Proof. See [34], Chapter 7, Section 5. ■

Definition 2.27. Let $p : E \to X$ be a G-equivariant vector bundle ξ of rank n. The characteristic classes of the induced vector bundle ξ_G are called the **equivariant characteristic classes** of the vector bundle ξ.

Remark 2.12. By using Definition 2.26, we can extend all the ideas of vector bundles to the category of equivariant vector bundles. Therefore, we naturally define the equivariant Thom isomorphism, the equivariant Stiefel-Whitney classes, the equivariant Euler class and the equivariant Thom-Gysin sequence. In particular, the equivariant Euler class $\mathrm{Eul}_G(\xi)$ and the equivariant Stiefel-Whitney classes $w_i^G(\xi)$ of a G-equivariant vector bundle ξ are defined as follows:

$$\begin{cases} \mathrm{Eul}_G(\xi) = \mathrm{Eul}(\xi_G) \in H_G^n(X;\mathbb{Z}), \\ w_i^G(\xi) = w_i(\xi_G) \in H_G^i(X;\mathbb{Z}_2). \end{cases} \tag{2.22}$$

Remark 2.13. Let V be a G-space where V is a k-dimensional vector space and G is a topological group. In this case, we can consider the projection map $p : V \to \{*\}$ as a vector bundle of rank k over a point. This induces a G-vector bundle $p_G : V_G \to BG$ of rank k. The Euler class and the top Stiefel-Whitney class of $p_G : V_G \to BG$ are denoted by $\mathrm{Eul}_G(V)$ and $w_k^G(V)$ such that $\mathrm{Eul}_G(V) \in H^k(BG;\mathbb{Z})$ and $w_k^G(V) \in H^k(BG;\mathbb{Z}_2)$ respectively.

2.3 Differential Topology

"Milnor's arguments were breathtaking. He brought together topology and analysis in a wholly unexpected way, and in doing so initiated the field of differential topology."

—*Donal O'Shea (1952-present)*

2.3.1 Lie Groups

Definition 2.28. A **Lie group** G is a topological group with a smooth structure such that the map $G \times G \to G$ by $(g, h) \longmapsto gh^{-1}$ is a smooth map with respect to smooth structures on G and $G \times G$. A subgroup $K \subset G$ which is also a submanifold is called a **Lie subgroup**.

Example 2.10. The Euclidean space $\mathbb{R}^n$ is a Lie group under addition and $\mathbb{Z}^n$ is a normal discrete subgroup. The quotient group $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is a Lie group, called the n**-torus**. The unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is a Lie group under the multiplication. It is easy to see that $\mathbb{T}^n \cong (S^1)^n$.

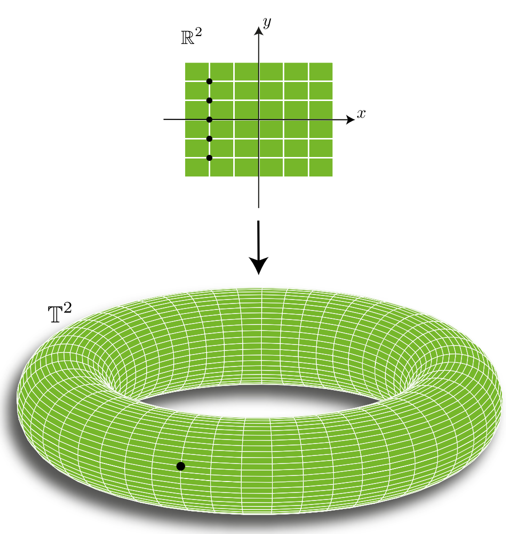

Figure 2.7: 2-torus as a Lie group

Definition 2.29. Let X be a smooth manifold and G be a Lie group. We say that G **acts** on X smoothly if G acts on X continuously and the action map is smooth.

A **Lie group homomorphism** is a group homomorphism which is a smooth map. Similarly, we can define Lie group isomorphism and automorphism as well as the group of automorphisms $\mathrm{Aut}(G)$ (see Definition 2.4).

Proposition 2.32 (**Semi-direct Product**). *Let N and G be two Lie groups and $\theta : G \to \mathrm{Aut}(N)$ by $g \longmapsto \theta_g$ be a Lie group homomorphism. For any $(n, g), (n', g') \in N \times G$, define*

$$(n, g) * (n', g') = (n\theta_g(n'), gg'). \tag{2.23}$$

*Then the pair $(N \times G, *)$ is a Lie group with the identity element (e_N, e_G) and the inverse $(n, g)^{-1} = (\theta_{g^{-1}}(n^{-1}), g^{-1})$. This group is called the **semi-direct product** of N, G with respect to θ and denoted by $N \rtimes_\theta G$.*

Proof. See [41], chapter 3. ■

Let a compact Lie group G act on a manifold M and fix $p \in M$. Suppose that G_p is the stabilizer of p and N_p is the fiber of normal bundle to the submanifold $\mathcal{O}_p$. In this case, for any $g \in G_p$, since $g.p = p$, we get an isomorphism $dg : T_pM \to T_pM$ such that $dg(v) = v$, for all $v \in T_p\mathcal{O}_p$. This induces an automorphism $dg : N_p \to N_p$. Now, consider the map $\pi : G \times_{G_p} N_p \to G/G_p$. It is easy to see that this a vector bundle with fiber N_p and its zero section is G/G_p.

Proposition 2.33 (**Equivariant Tubular Neighborhood**). *Let a compact Lie group G act on a smooth manifold M and $p \in M$. Then there exists an equivariant diffeomorphism $\varphi : U \to V$ in which V is an invariant open neighborhood of the orbit $\mathcal{O}_p$ in M and U is an open neighborhood of the zero section G/G_p in $G \times_{G_p} N_p$ such that $\varphi(gG_p) = g.p$, for all $g \in G$.*

Proof. See [5], Theorem 2.1.1, or [41], Chapter 4.. ■

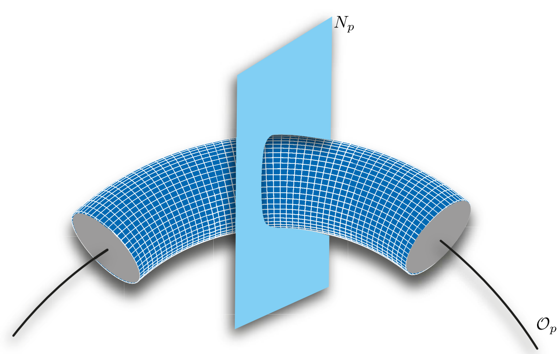

Figure 2.8: Equivariant tubular neighborhood

Corollary 2.33.1. *The fixed point set of the action of a compact Lie group G on a smooth manifold M is a closed submanifold of M.*

Proof. See [41], Chapter 4. ∎

Proposition 2.34 (**Equivariant Thom-Gysin sequence**). *Let Y be a G-manifold and X be an invariant submanifold of Y with codimension $d(X)$. Then there exists a long exact sequence in $\mathbb{Z}_2$-equivariant cohomology as follows:*

$$\cdots \to H_G^{*-d(X)}(X;\mathbb{Z}_2) \xrightarrow{\Phi^*} H_G^*(Y;\mathbb{Z}_2) \xrightarrow{\Psi^*} H_G^*(Y-X;\mathbb{Z}_2) \to \cdots \tag{2.24}$$

Moreover, if the equivariant top Stiefel-Whitney class $w_{d(X)}^G(\nu X)$ of the normal bundle of X in Y is not a zero divisor, then this long exact sequence breaks into short exact sequences.

Proof. See Proposition 2.30 and Corollary 2.30.1. ∎

2.3.2 Vector Fields on Manifolds

Definition 2.30. A **flow** on a smooth manifold M is a smooth action $\psi : \mathbb{R}\times M \to M$ of $\mathbb{R}$ on M.

Remark 2.14. It immediately follows from Definition 2.30 that the map $\psi_t : M \to M$ is a diffeomorphism for each $t \in \mathbb{R}$.

Proposition 2.35. *Let $X \in \mathfrak{X}(M)$ be a smooth vector field on a smooth compact n-manifold M. Then there exists a flow $\psi : \mathbb{R}\times M \to M$ by $\psi(t,p) = \psi_t(p)$ on M such that*

$$\frac{d}{dt}\bigg|_{t=0}\psi_t(p) = X(p),\ \forall p \in M. \tag{2.25}$$

Proof. See [57], Proposition 1.3. ∎

The map ψ is often called the **flow generated by the vector field** X. For any $p \in M$, the induced map $\psi_p : \mathbb{R} \to M$ defined by $\psi_p(t) = \psi(t,p)$ is called the **integral curve** of X through p. The image of ψ_p is called an **orbit** or a **trajectory** of X and denoted by $\mathcal{O}_p = \{\psi_t(p) \mid t \in \mathbb{R}\}$.

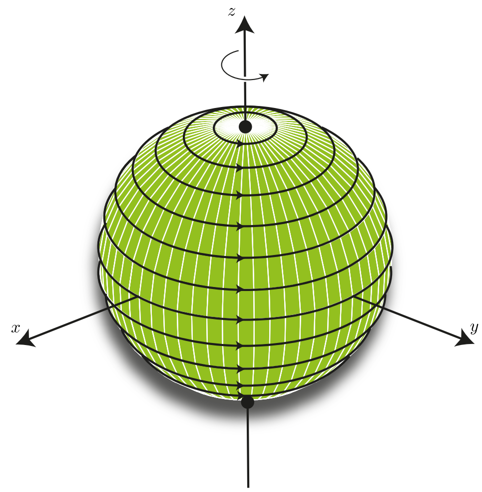

Figure 2.9: Flow of a vector field

Depending on the value of X at each point, we have different types of the orbits of X. If $X(p) = 0$, then the orbit of p is a singleton. In this case, we say that p is a **singularity** or a **zero** of X. If $X(p) \neq 0$, the orbit $\mathcal{O}_p$ of p is an immersed submanifold of M. When the map $\psi_p(t)$ is not injective, the orbit of p is diffeomorphic to the circle S^1, which is called a **closed orbit**. If $\psi_p(t)$ is injective, then the orbit of p is an injective immersion of the real line $\mathbb{R}$, called a **regular orbit** (see Figure 2.10).

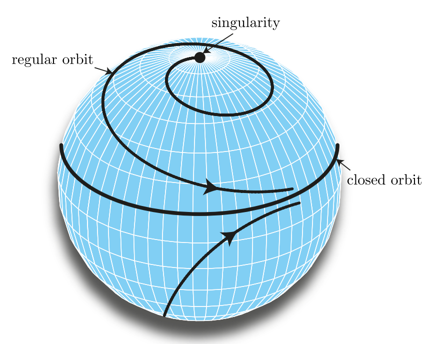

Figure 2.10: Orbits of different types

Definition 2.31. Let $p \in M$ and ψ_t be the flow of a vector field X on M. The **limit set** of p is the set of those points $q \in M$ for which there exists a sequence $t_n \to +\infty$ such that $\psi_{t_n}(p) \to q$, as $n \to +\infty$.

Proposition 2.36 (**Properties of limit sets**)**.** *Let X be a smooth vector field on a compact n-manifold M and $p \in M$. Then the limit set is always nonempty, closed, connected subset which is invariant under the flow generated by X, that is the limit set is a union of orbits of X.*

Proof. See [57], Proposition 1.4. ∎

Remark 2.15. The compactness assumption in Proposition 2.36 is essential. In fact, there exist vector fields on the plane whose limit sets may not be connected. Such an example is shown in Figure 2.11.

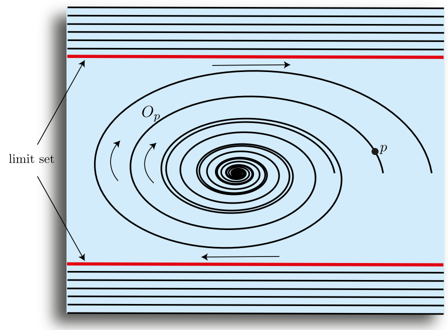

Figure 2.11: A disconnected limit set in the plane

2.3.3 Morse-Bott Theory

Definition 2.32. Let $f : M \to \mathbb{R}$ be a smooth function on a smooth n-manifold and $p \in M$. We say that p is a **critical point** of f if $df(p) = 0$. In this case, $f(p)$ is called a **critical value** of f. The set of all critical points of f is called the **critical set** of f and denoted by C_f.

Let $p \in C_f$ and $(U, x = (x_1, ..., x_n))$ be a coordinate chart around p. Since $df(p) = 0$, we have

$$\frac{\partial f}{\partial x_i}(p) = 0, \ \forall i = 1, ..., n. \tag{2.26}$$

The **Hessian matrix** of f at p with respect to the chart (U, φ) is the symmetric matrix:

$$H_p f(x) = \Big(\frac{\partial^2 f}{\partial x_j \partial x_i}(p)\Big)_{i,j}, \ i, j = 1, ..., n. \tag{2.27}$$

By using this matrix, we can define a linear operator $\widehat{H}_p f : T_pM \to T_pM$ by

$$\widehat{H}_p f(v) = (H_p f)v, \ \forall v \in T_pM.$$

If $(V, y = (y_1, ..., y_n))$ is another chart around p, then from last two equations, we can see that

$$H_p f(y) = J^t H_p f(x) J, \tag{2.28}$$

where $J = d(y \circ x^{-1})$ is the Jacobian of the coordinate change. Since $\det J \neq 0$, from (2.26) and (2.27) it follows that $\det(H_p f(y)) \neq 0$ if and only if $\det H_p f(x) \neq 0$. This enables us to define:

Definition 2.33. A critical point p of a smooth function f is **nondegenerate** if the Hessian matrix of f at p with respect to a coordinate chart is non-singular. The number of negative eigenvalues of $H_p f$ is called the **Morse index** of p and denoted by $\lambda_p(f)$.

Remark 2.16. Since the Hessian is a real symmetric matrix, all of its eigenvalues are real numbers and the index is independent of the choice of a coordinate chart.

Definition 2.34. Let $f : M \to \mathbb{R}$ be a smooth function on a smooth n-manifold M. A connected submanifold $N \subset M$ is called a **nondegenerate critical manifold** for f if it has the following properties.

1. $N \subset C_f$;i.e., elements of N are critical points of f.

2. The linear operator $\widehat{H}_p f$ is nondegenerate on the normal bundle $\nu(N)$ of N in M; or equivalently, the tangent space to N at p is the kernel of the linear operator $\widehat{H}_p f$; i.e., $\ker(\widehat{H}_p f) = T_p N$.

Since $TM = TN \oplus \nu N$, it follows from the above definition that the restriction of the linear operator $\widehat{H}_p f$ to the normal bundle νN, say $\widehat{H}_p^N f$, is non-singular. The number of negative eigenvalues of $\widehat{H}_p^N f$ is called the **Morse index** of f along N and denoted by $\lambda_N(f)$.

Definition 2.35. A smooth function $f : M \to \mathbb{R}$ on a smooth n-manifold M is called a **Morse-Bott function** if the critical set of f is a union of nondegenerate critical manifolds.

Example 2.11. Let $f : S^2 \to \mathbb{R}$ be the square of the height function on 2-sphere; i.e., $f(x, y, z) = z^2$. Then the critical set of f is $C_f = \{C_0, C_1\}$ in which $C_1 = \{N, S\}$, two poles, corresponding to the global maximum 1 and C_0 is the equator corresponding to the global minimum 0. It is easy to see that each critical manifold of f is nondegenerate. So f is a Morse-Bott function. Here, $\lambda_{C_1} = 2$ and $\lambda_{C_0} = 0$ (see Figure 2.12).

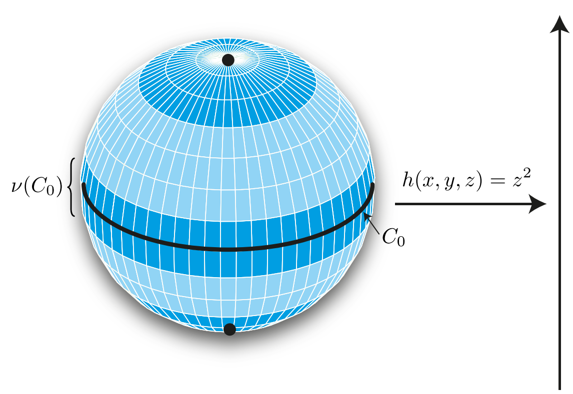

Figure 2.12: A Morse-Bott function on 2-sphere

Remark 2.17. A special kind of Morse-Bott function is the one whose nondegenerate critical manifolds are singletons. Such a function is called a **Morse function**.

Example 2.12. Consider the height function $h : \mathbb{T}^2 \to \mathbb{R}$ on the 2-torus embedded in $\mathbb{R}^3$ as in Figure 2.13. By using a proper local chart, we can see that the critical points of h are p, q, m, n (the tangent space is perpendicular to the z-axis). These critical points are all nondegenerate (there is no 'flat' direction in a neighborhood of these points). So h is a Morse function and $\lambda_p = 0, \lambda_q = \lambda_m = 1, \lambda_n = 2$.

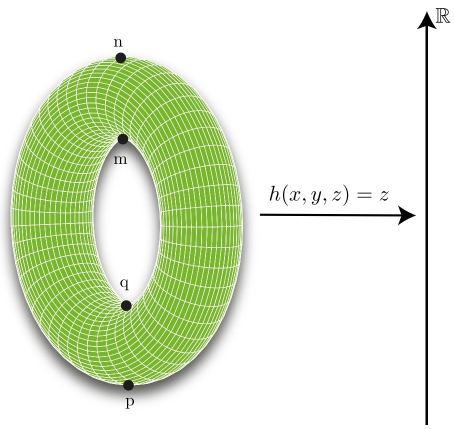

Figure 2.13: A Morse function on 2-torus

Proposition 2.37 (**Morse-Bott Lemma**). *Let $f : M \to \mathbb{R}$ be a Morse-Bott function on an m-manifold M and $N \subset M$ be a nondegenerate critical submanifold of f with dimension n and index $0 \leq \lambda_N(f) = k \leq n$. Then for any $p \in N$, there exists a coordinate chart $(U, (x_1, ..., x_n, y_1, ..., y_k, z_1, ..., z_l))$ around l with $n+k+l = m$ such that in this chart f has the following standard form:*

$$f(x_1, ..., x_n, y_1, ..., y_k, z_1, ..., z_l) = f(p) - y_1^2 - \cdots - y_k^2 + z_1^2 + \cdots + z_l^2. \tag{2.29}$$

Proof. See [8], Chapter 3, Lemma 3.51. ■

Let $f : M \to \mathbb{R}$ be a smooth function on a compact n-manifold M. Choose a Riemannian metric $\{\langle \ , \ \rangle_p \mid p \in M\}$ on M and let $\nabla f : M \to TM$ be the **gradient vector field** of f induced by this metric; i.e., for any $v \in T_pM$

$$\langle \nabla f(p), v \rangle_p = df_p(v). \tag{2.30}$$

It is clear that the critical set of f is the set of singularities (zeros) of the gradient vector field ∇f.

Now, consider the negative of gradient vector field $-\nabla f$. Since M is compact, Proposition 2.35 provides us with a global flow ψ_t generated by the vector field $-\nabla f$ (see Figure 2.14).

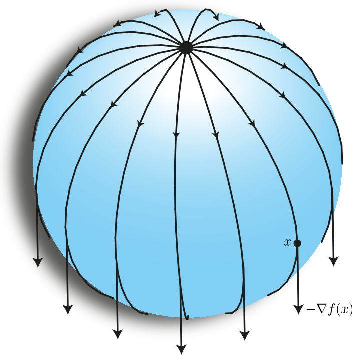

Figure 2.14: Negative gradient flow of the height function

It is known that the limit sets with respect to the gradient vector field of a smooth function are singletons (see [57], Chapter 1, Example 3). So limit sets of $-\nabla f$ are also singletons.

Suppose that f is a Morse-Bott function. Since M is compact, it has a finite number of nondegenerate critical manifolds, say $N_1, ..., N_k$. Define the **stable manifold** of N_i as follows:

$$S_{N_i} = \{x \in M \mid \exists\, t_n \to +\infty \ s.t. \lim_{n \to +\infty} \psi_{t_n}(x) \in N_i\}. \tag{2.31}$$

It is known that any stable manifold S_{N_i} is a locally closed submanifold of M with codimension λ_{N_i} (see [57], Theorem 6.2 or [8], Chapter 4). In fact, the collection $\{S_{N_i} \mid i = 1, ..., k\}$ of stable manifolds has the following properties:

1. $M = \bigcup_{i=1}^k S_{N_i}$.

2. S_{N_i} is a locally closed submanifold of M and its codimension is equal to the Morse index of the corresponding nondegenerate critical manifold N_i.

3. For any i, we have $\overline{S}_{N_i} \subset \bigcup_{N_j \geq N_i} S_{N_j}$, where $N_i < N_j \Leftrightarrow f(N_i) < f(N_j)$.

The collection $\{S_{N_i} \mid i = 1, ..., k\}$ satisfying properties 1-3 above is called the **Morse stratification** of M induced by the Morse-Bott function f. For the Morse-Bott function in Example 2.11, the Morse stratification consists of the finite set $\{N, S\}$, and the 2-dimensional submanifold $S^2 - \{N, S\}$ (see Figure 2.15).

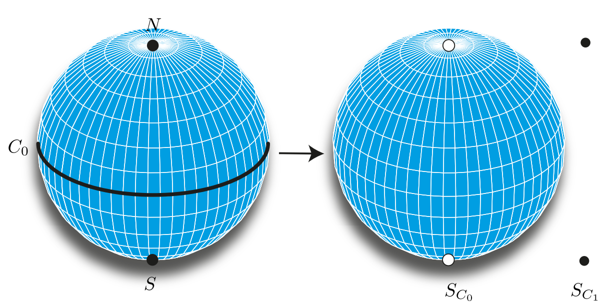

Figure 2.15: Morse stratification of a Morse-Bott function

Recall that for any locally finite manifold M, we define its Poincaré series relative to the field $\mathbb{F}$ to be the generating function

$$\mathbf{P}(M,t;\mathbb{F}) = \sum_{i\geq 0}[\dim H^i(M;\mathbb{F})]t^i.$$

Definition 2.36. Let $f : M \to \mathbb{R}$ be a Morse-Bott function on a compact n-manifold M. If $N_1, ..., N_k$ are the nondegenerate critical manifolds of f with Morse indices $\lambda_1, ..., \lambda_k$, respectively, then the **Morse-Bott series** relative to a field $\mathbb{F}$ is defined as follows:

$$\mathbf{M}(f,t;\mathbb{F}) = \sum_{i=1}^{k} t^{\lambda_i}\mathbf{P}(N_i,t;\mathbb{F}), \tag{2.32}$$

where $\mathbf{P}(N_i,t;\mathbb{F})$ is the Poincaré series of N_i relative to the field $\mathbb{F}$.

Proposition 2.38 (Morse-Bott Inequality). *Let $\mathbb{F}$ be a field and $\mathbf{P}(M,t;\mathbb{F})$ be the Poincaré series of a compact manifold M relative to a field $\mathbb{F}$. If $\mathbf{M}(f,t;\mathbb{F})$ is the Morse-Bott series of a Morse-Bott function on M and the normal bundle of nondegenerate critical manifolds of f are orientable, then*

$$\mathbf{M}(f,t;\mathbb{F}) - \mathbf{P}(M,t;\mathbb{F}) = (1+t)R(t), \tag{2.33}$$

where $R(t)$ is a series with non-negative coefficients.

Proof. See [55], Section 2.3. ■

Definition 2.37. Let $f : M \to \mathbb{R}$ be a Morse-Bott function on a compact manifold M. We say that f is a **perfect Morse-Bott function** on M relative to a field $\mathbb{F}$ if $\mathbf{M}(f,t;\mathbb{F}) = \mathbf{P}(M,t;\mathbb{F})$.

Remark 2.18. Let M be a compact G-Manifold where G is a compact Lie group and fix a G-invariant Riemannian metric on M. If we have an equivariant Morse-Bott function $f : M \to \mathbb{R}$ on M , then nondegenerate critical manifolds of f are G-invariant and the strata of the Morse stratification induced by f are G-invariant locally closed submanifolds. In this case, the Morse stratification induced by f is called the **equivariant Morse stratification**. Moreover, if Λ is the set of nondegenerate critical manifolds of f, then the **equivariant Morse-Bott series** of f is defined as follows:

$$\mathbf{M}_G(f,t;\mathbb{F}) = \sum_{N\in\Lambda} t^{\lambda_N}\mathbf{P}_G(N,t;\mathbb{F}), \tag{2.34}$$

where $\mathbf{P}_G(N,t;\mathbb{F}) = \mathbf{P}(N_G,t;\mathbb{F})$ is the **equivariant Poincaré series** of the nondegenerate critical manifold N relative to a field $\mathbb{F}$ (see [55], Section 2.6 for details).

Definition 2.38. If the equivariant Morse-Bott series $\mathbf{M}_G(f,t;\mathbb{F})$ of an equivariant Morse-Bott function $f : M \to \mathbb{R}$ is equal to the equivariant Poincaré series $\mathbf{P}_G(M,t;\mathbb{F})$ of M, then we say that f is **equivariantly perfect** over the field $\mathbb{F}$.

2.3.4 Morse-Kirwan Functions

Definition 2.39 (**Minimal Degeneracy Property**). A smooth function $f : M \to \mathbb{R}$ on a compact manifold M has the **minimal degeneracy property** if the following hold.

1. The critical set of f is a finite union of disjoint closed subsets $\{C_j\}_{j=1}^k$ on each of which f takes a constant value $f(C_j)$.
2. For any $j = 1, ..., k$, there exists a locally closed submanifold Σ_j containing C_j with orientable normal bundle in M such that
   (i) the restriction of f to Σ_j takes its minimum value on C_j.
   (ii) For any $p \in C_j$, the tangent space $T_p\Sigma_j$ is maximal among all the subspaces of T_pM on which the Hessian H_pf is positive-semidefinite.

In this case, the submanifold Σ_j is called a **minimizing manifold** for f along the critical subset C_j.

Definition 2.40 (**Morse-Kirwan Function**). A smooth function $f : M \to \mathbb{R}$ on a compact manifold M is called a **Morse-Kirwan** function if it has the minimal degeneracy property.

Example 2.13. Let $g : M \to \mathbb{R}$ be a Morse-Bott function on a compact manifold M and define $f : M \to \mathbb{R}$ by

$$f(p) = (g(p) - a)^2,$$

for some $a \in \mathbb{R}$. Then f is a Morse-Kirwan function (see [43] for details). More examples and some properties of Morse-Kirwan functions are given in [43] and [35].

It follows from part (2-ii) of Definition 2.39 that the index of the Hessian of f is equal to the codimension of each minimizing submanifold Σ_{C_j}. We call this constant number the **index of Morse-Kirwan map** f along the critical subset C_j and denote it by λ_{C_j}. In this case, the Morse series of f is defined as (2.32) by replacing N_j by C_j. Now we choose a metric on M and consider the negative gradient vector field $-\nabla f$ on M. If Ψ_t is the flow, then for each critical set C_j, we can define the corresponding stratum by

$$S_{C_j} = \{p \in M \mid \exists\ t_n \to +\infty\ s.t.\ \lim_{n+\to\infty} \Psi_{t_n}(p) \in C_j\}.$$

Kirwan showed the following (see [43], Theorems 10.2 and 10.4):

(MK1) *If the gradient vector field ∇f is tangent to the minimizing submanifold Σ_{C_j}, then the collection $\{S_{C_j}\}$ form a Morse stratification for f such that each stratum S_{C_j} is coincide with Σ_{C_j} in a neighborhood of C_j.*

(MK2) *The Morse inequality is satisfied for the Morse-Kirwan map f.*

(MK3) *If a compact Lie group G acts on M such that f is G-equivariant and the metric is G-invariant, then f induces a G-invariant Morse stratification and satisfies the G-equivariant Morse inequality.*

2.4 Lie Theory

> "*I am certain, absolutely certain that these theories will be recognized as fundamental at some point in the future.*"
>
> —*Sophus Lie (1842-1899)*

2.4.1 Compact Lie Groups

Definition 2.41 (**Lie Algebra of a Lie Group**)**.** A vector field X on a Lie group G is called a **left invariant vector field** if for any $g, h \in G$, $X(L_g h) = dL_g X(h)$, where $L_g : G \to G$ is defined by $L_g h = gh$. The set of all left invariant vector fields on G with the Lie bracket form a Lie algebra, denoted by $\mathrm{Lie}(G) = \mathfrak{g}$ and called the **Lie algebra** of G.

Proposition 2.39. *The Lie algebra $\mathfrak{g}$ of a Lie group G is isomorphic to the tangent space $T_e G$ at the identity element as a vector space.*

Proof. See [19], Section 21.5. ■

Proposition 2.40 (**Exponential Map**)**.** *Let G be a compact Lie group with Lie algebra $\mathfrak{g} = T_e G$. Then there exists a smooth map $\exp : \mathfrak{g} \to G$ having the following properties.*

1. $\exp(0) = e$, *the identity element of G.*
2. $\exp(t+s)X = \exp(tX)\exp(sX)$, *for any $X \in \mathfrak{g}$ and $s, t \in \mathbb{R}$. In particular,* $\exp(-X) = \exp(X)^{-1}$.
3. *The image of* $\exp$ *is contained in G_e, the identity component.*
4. $\exp$ *is a local diffeomorphism at 0.*
5. *If G is an abelian group, then* $\exp$ *is a homomorphism.*
6. *If G is connected, then* $\exp$ *is surjective.*

Proof. See [25], Chapters 3,4. ■

Proposition 2.41 (**Cartan**)**.** *Let G be a compact Lie group and H be a subgroup of G. Then H is a Lie subgroup of G if and only if H is a closed subspace of G.*

Proof. See [25], Chapter 4. ■

Proposition 2.42. *If G is an abelian connected Lie group, then $G \cong \mathbb{T}^n \times \mathbb{R}^m$, where $\mathbb{T}^n \cong (\mathbb{R}/\mathbb{Z})^n$ is the n-torus. In particular, if G is an abelian compact connected Lie group, then G is isomorphic to a torus.*

Proof. See [25], Chapter 4. ∎

Definition 2.42. A Lie subgroup T of a compact Lie group G is called a **maximal torus** if T is a torus and if U is another torus such that $T \subset U$, then $T = U$.

Proposition 2.43. *Let G be a compact Lie group. The following are satisfied.*

1. *Any connected abelian subgroup of G is contained in a maximal torus.*

2. *Any two maximal tori in G are conjugate.*

3. *If G is connected and T is a maximal torus in G, then $Z(G) \subset T$, where $Z(G)$ is the center of G and $Z_G(T) = T$, where $Z_G(T)$ is the centralizer of T.*

Proof. For (1) and (2), see [25], Theorem 6.18 and for (3), see [62], Theorems 5.12 and 5.13. ∎

Definition 2.43. The dimension of a maximal torus of a Lie group G is called the **rank** of G.

Definition 2.44. Let p be a prime number. An abelian group all of whose non-trivial elements have order p is called an **elementary abelian p-group**.

Remark 2.19. One can show that an elementary abelian p-group E can be treated as a $\mathbb{Z}_p$-vector space. In this case, subgroups of E can be regarded as subspaces. Thus any subgroup is also an elementary abelian p-group. In particular, when E is finite, $E \cong (\mathbb{Z}_p)^n$ for some n known as the **rank** of E (see [61], Chapter 4).

Example 2.14. Let E be an elementary abelian p-group of rank n and $E_0 \subset E$ be a subgroup of rank m. We can choose a basis $\mathcal{B}_0$ for E_0 as a $\mathbb{Z}_p$-vector space and extend it to a basis $\mathcal{B}$ for E. If E_1 is the subgroup spanned by the set $\mathcal{B} - \mathcal{B}_0$, it is clear that the rank of E_1 is $n - m$ and $E \cong E_0 \times E_1$.

Definition 2.45. Let G be a compact Lie group and p be a prime number. A Lie subgroup E of G which is an elementary abelian p-group is called an **elementary abelian p-subgroup**. An elementary abelian p-subgroup D of G is **maximal** if there is no elementary abelian p-subgroup containing D other than D.

Example 2.15. Consider the Lie group $\mathbb{T}^n = (S^1)^n$. For any prime number p, let H_p be $\{\theta_k = e^{\frac{2\pi k i}{p}} | k = 0, ..., p-1\}$. It is clear that $\underbrace{H_p \times \cdots \times H_p}_{n}$ is the only maximal elementary abelian p-subgroup of $\mathbb{T}^n$.

Example 2.16. Let $\mathrm{O}(n)$ be the orthogonal group of order n and $D(n)$ be the subgroup of diagonal matrices in $\mathrm{O}(n)$. Then

$$A \in D(n) \Leftrightarrow A = \mathrm{Diag}[a_1, ..., a_n] \ \&\ AA^t = \mathrm{Id}.$$

This implies that $a_i = \pm 1$ for each $i = 1, ..., n$. Therefore,

$$D(n) = \left\{ \begin{pmatrix} \varepsilon_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varepsilon_n \end{pmatrix} \mid \varepsilon_i \in \{\pm 1\} \right\}, \tag{2.35}$$

which implies that $D(n) \cong (\mathbb{Z}_2)^n$. That is, $D(n)$ is an elementary abelian 2-subgroup. In addition, since the rank of $D(n)$ is equal to the rank of $\mathrm{O}(n)$, then $D(n)$ is maximal. Now, let D' be another maximal abelian 2-subgroup of $\mathrm{O}(n)$. Since D' is abelian and all of its elements have order two, then it follows from Proposition 2.2 that they are simultaneously diagonalizable. That is, there exists a matrix A such that $D' = AD(n)A^{-1}$. This shows that $\mathrm{O}(n)$ has a unique conjugacy class of maximal elementary abelian 2-subgroups.

Example 2.17. Let $\mathrm{SO}(n)$ be the special orthogonal group and $D_s(n)$ be the subgroup of diagonal matrices in $\mathrm{SO}(n)$. Define $f : D(n) \to \mathbb{Z}_2$ by $f(A) = \det A$, determinant of A. Clearly, f is a surjective group homomorphism and $\mathrm{Ker}\ (f) = D_s(n)$. Therefore, $D_s(n) \cong (\mathbb{Z}_2)^{n-1}$; i.e., it is an elementary abelian 2-subgroup. Since $D(n)$ is maximal in $\mathrm{O}(n)$, so $D_s(n)$ is a maximal elementary abelian 2-subgroup. A similar argument to the one in Example 2.16 shows that $\mathrm{SO}(n)$ also has a unique conjugacy class of maximal elementary abelian 2-subgroups.

Example 2.18. Let $G = G_1 \times \cdots \times G_k$ where $G_i = \mathrm{O}(n_i)$ or $\mathrm{SO}(n_i)$ and D_i be the maximal elementary abelian 2-subgroup in G_i consisting of diagonal matrices. Clearly, $D = D_1 \times \cdots \times D_k$ is a maximal elementary abelian 2-subgroup of G. If we consider any other maximal elementary abelian 2-subgroup D' of G, then they are simultaneously diagonalizable and thus D' is conjugate to D. Hence, D determines a unique conjugacy class of maximal elementary abelian 2-subgroups.

2.4.2 Representation Theory

Definition 2.46. Let G be a compact Lie group. A **real (complex) representation** of G is a continuous homomorphism $\pi : G \to \mathrm{GL}(V)$, where V is a finite dimensional real (complex) vector space. In this case, we denote each isomorphism $\pi(g) : V \to V$ by $\pi_g : V \to V$ or simply $g : V \to V$.

Definition 2.47. Let G be a compact Lie group and $\pi : G \to \mathrm{GL}(V)$ be a representation. We say a subspace W of V is **invariant** if $gW \subset W$, for all $g \in G$.

Definition 2.48. Let G be a compact Lie group and $\pi : G \to \mathrm{GL}(V)$ be a representation. We say that the representation is **irreducible** if V has no non-trivial invariant subspace.

Proposition 2.44. *Any representation of a compact Lie group is a direct sum of irreducible representations.*

Proof. See [25], Chapter 7. ■

Proposition 2.45. *Let G be an abelian compact Lie group and $\pi : G \to \mathrm{GL}(V)$ be an irreducible complex representation. Then V is one-dimensional.*

Proof. See [25], Chapter 7. ■

Example 2.19 (**Real Representations of 2-group** $(\mathbb{Z}_2)^n$)**.** Let $\pi : G \to \mathrm{GL}(V)$ be an irreducible real representation of $G = (\mathbb{Z}_2)^n$ where V is a real vector space. Since any element of G has order two, any linear isomorphism $\pi_g : V \to V$ is an involution; i.e., $\pi_g^2 = \mathrm{Id}$. On the other hand, they all commute, so Corollary 2.1.1 implies that the operators π_g are simultaneously diagonalizable. Since V is irreducible, we must have $\dim V = 1$. That is, any irreducible real representation of G is one-dimensional. On the other hand, since G is a compact Lie group, it follows from Proposition 2.44 that any real representation of G is a direct sum of irreducible representations. In conclusion, any real representation of $(\mathbb{Z}_2)^n$ is a direct sum of one-dimensional real representations.

Definition 2.49. Let G be a compact Lie group, $\pi : G \to \mathrm{GL}(V)$ be a representation of G and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

1. A Hermitian inner product $\langle\ ,\ \rangle : V \times V \to \mathbb{F}$ is called **invariant** if for any $g \in G$ and $u, v \in V$, we have $\langle gu, gv\rangle = \langle u, v\rangle$.

2. A representation $\pi : G \to \mathrm{GL}(V)$ of G is called **unitary** if there exists an invariant Hermitian inner product on V.

Proposition 2.46. *Let G be a compact Lie group. Then the following are satisfied.*

1. *Any representation of G is unitary.*

2. *G is isomorphic to a closed subgroup of the unitary group $\mathrm{U}(n)$, for some n.*

Proof. See [62], Chapters 2 and 3. ■

Let G be a compact Lie group and $\mathfrak{g} = T_eG$ be its Lie algebra. For any $g \in G$, the conjugate map is the diffeomorphism $\iota_g : G \to G$ defined by $\iota_g(h) = ghg^{-1}$, for all $h \in G$. Since $\iota_g(e) = e$, we get an isomorphism $d\iota_g(e) : \mathfrak{g} \to \mathfrak{g}$. We denote this map by Ad_g. These diffeomorphisms define a new action of G on the Lie algebra $\mathfrak{g}$ as follows:

$$\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g},\ \ \mathrm{Ad}(g, X) = \mathrm{Ad}_g(X),\ \ \forall g \in G, X \in \mathfrak{g}. \tag{2.36}$$

This action is called the **adjoint action** of Lie group G on its Lie algebra $\mathfrak{g}$. Similarly, we can define the **coadjoint action** of G on the dual of Lie algebra $\mathfrak{g}^*$

$$\begin{cases} \mathrm{Ad}^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*,\ \mathrm{Ad}^*(g,\xi) = \mathrm{Ad}^*_g(\xi),\ \forall g \in G, \xi \in \mathfrak{g}^*; \\ \mathrm{Ad}^*_g(\xi)(X) = \xi\Big(\mathrm{Ad}_{g^{-1}}(X)\Big),\ \forall X \in \mathfrak{g}. \end{cases} \tag{2.37}$$

Definition 2.50. Let G be a compact Lie group. The **adjoint representation** of G over the Lie algebra $\mathfrak{g}$ is the map $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ with $\mathrm{Ad}(g) = \mathrm{Ad}_g$, for all $g \in G$.

Definition 2.51. Let G be a compact Lie group and $\mathfrak{g}$ be its Lie algebra. The **adjoint representation** of $\mathfrak{g}$ on itself is the map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ defined by $X \longmapsto \mathrm{ad}_X$, for all $X \in \mathfrak{g}$, where $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ is given by $\mathrm{ad}_X Y = [X,Y]$, the Lie bracket of X and Y, for all $Y \in \mathfrak{g}$. Similarly, we can define the **coadjoint representation** of $\mathfrak{g}$ on dual Lie algebra $\mathfrak{g}^*$:

$$\begin{cases} \mathrm{ad}^* : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*,\ \mathrm{ad}^*(X,\xi) = \mathrm{ad}^*_X(\xi),\ \forall X \in \mathfrak{g}, \xi \in \mathfrak{g}^*; \\ \mathrm{ad}^*_X(\xi)(Y) = \xi[Y,X],\ \forall X, Y \in \mathfrak{g}. \end{cases} \tag{2.38}$$

Proposition 2.47. *Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Then For any $X \in \mathfrak{g}$ and $t \in \mathbb{R}$,*

$$\begin{cases} \mathrm{Ad}_{\exp(tX)} = \exp \circ\ \mathrm{ad}_{(tX)}, \\ d_e(\mathrm{Ad}) = \mathrm{ad}. \end{cases} \tag{2.39}$$

Moreover, for any $\beta \in \mathfrak{g}^*$, if $\mathcal{O}_\beta$ is the coadjoint orbit of β and $\alpha \in \mathcal{O}_\beta$, then

$$T_\alpha \mathcal{O}_\beta = \{\mathrm{ad}^*_X \alpha \mid X \in \mathfrak{g}\}. \tag{2.40}$$

Proof. See [25], Chapter 5. ∎

Definition 2.52. Let G be a compact Lie group, T be a maximal torus and $\mathbb{C}^\times = \mathbb{C} - \{0\}$. Any continuous homomorphism $\alpha : T \to \mathbb{C}^\times$ is called a **character**. The set of all characters is called the **character group** of T and denoted by $\widehat{T}$.

Now, let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Fix a maximal torus T with Lie algebra $\mathfrak{t}$. Let $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$ and $\mathfrak{t}_\mathbb{C} = \mathfrak{t} \otimes_\mathbb{R} \mathbb{C}$ be the complexifications of $\mathfrak{g}$ and $\mathfrak{t}$, respectively. The adjoint representation induces a representation of G on $\mathfrak{g}_\mathbb{C}$, denoted by $\mathrm{Ad} \otimes I : G \to \mathrm{Aut}(\mathfrak{g}_\mathbb{C})$. Consider the restricted representation $\Psi = \mathrm{Ad} \otimes I|_T : T \to \mathrm{Aut}(\mathfrak{g}_\mathbb{C})$. Since T is abelian, all the operators $\Psi(t)$ are simultaneously diagonalizable and therefore there are nonzero homomorphism $\alpha_j : T \to \mathbb{C}^\times$, for $j = 1, ..., k$ such that

$$\Psi(t) = \mathrm{Diag}[1, ..., 1, \alpha_1(t), ..., \alpha_k(t)],$$

where the number of 1 is equal to $\dim T$. The above homomorphisms α_j are called the **roots** of the compact Lie group G and the set of roots is denoted by $\Delta(\mathfrak{g}_\mathbb{C})$. They form a finite subset of the character group $\widehat{T}$. In this case, for each root α_j, we have $id\alpha_j \in \mathfrak{t}^*$, the dual Lie algebra of $\mathfrak{t}$, and

$$\alpha(\exp X) = e^{id\alpha(X)},\ \forall X \in \mathfrak{t}. \tag{2.41}$$

Let Λ_T be the lattice generated by the roots; i.e., $\Lambda_T = \bigoplus_{\alpha\in\Delta(\mathfrak{g}_\mathbb{C})} \mathbb{Z}\alpha$. This lattice is called the **root lattice** of G.

Proposition 2.48. *Let G be a compact Lie group with a maximal torus T. The root lattice Λ_T is isomorphic to a sublattice of the character group $\widehat{T}$.*

Proof. See [62], Chapter 6. ■

Definition 2.53. Let G be a compact Lie group with maximal torus T. The **Weyl group** of G is $W = N(T)/T$, where $N(T)$ is the normalizer of T in G. For any root $\alpha \in \mathfrak{t}^*$, let L_α be the kernel of α. The connected components of $\mathfrak{t} - \cup_\alpha L_\alpha$ are called the **Weyl chambers**.

Proposition 2.49. *The Weyl group W of a compact Lie group G acts transitively and simply on the set of the Weyl chambers.*

Proof. See [25], Chapter 8. ■

Definition 2.54. Fix a Weyl chamber $C \subset \mathfrak{t}$ of a compact Lie group G. We say that an element $\alpha \in \mathfrak{t}^*$ is **positive** with respect to C if $\alpha(X) > 0$, for all $X \in C$. In this case, C is called a **positive Weyl chamber** and denoted by $\mathfrak{t}_+$.

2.5 Symplectic Geometry

"Proofs are to mathematics what spelling (or even calligraphy) is to poetry. Mathematical works do consist of proofs, just as poems do consist of words."
—*Vladimir Arnold (1937-2010)*

2.5.1 Symplectic Vector Spaces

Definition 2.55. Let V be a real n-vector space. A bilinear map $\Omega : V \times V \to \mathbb{R}$ is called a **symplectic form** if it has the following properties.

1. Ω is **alternating**: $\Omega(u, v) = -\Omega(v, u)$, for all $u, v \in V$.
2. Ω is **nondegenerate**: if $\Omega(u, v) = 0$ for all $v \in V$, then $u = 0$.

In this case, the pair (V, Ω) is called a **symplectic vector space**.

Example 2.20. Let V be a complex vector space of dimension n and consider a **Hermitian structure** $\langle\ ,\ \rangle : V \times V \to \mathbb{C}$ on it; i.e., for any $\lambda, \mu \in \mathbb{C}$ and $u, v, w \in V$, we have

1. $\langle \lambda u + \mu v, w\rangle = \lambda\langle u, w\rangle + \mu\langle v, w\rangle$;
2. $\langle w, \lambda u + \mu v\rangle = \overline{\lambda}\langle w, u\rangle + \overline{\mu}\langle w, v\rangle$;
3. $\langle u, v\rangle = \overline{\langle v, u\rangle}$;
4. $\langle u, u\rangle \geq 0$.

It is known that (see [1], Appendix 3) the real part of this form is an inner product on V and its imaginary part is a symplectic form on V such that

$$\mathrm{Re}\langle u, v\rangle = \mathrm{Im}\langle u, iv\rangle,\ \forall u, v \in V, \tag{2.42}$$

where $i = \sqrt{-1}$. Therefore, any Hermitian structure on a complex vector space gives rise to a symplectic structure.

Example 2.21. Let $\mathbb{C}^n$ be the standard complex n-vector space. The **canonical Hermitian structure** on $\mathbb{C}^n$ is defined by

$$\langle z, w\rangle = w^* z = \sum_{j=1}^{n} z_j \overline{w}_j,\ \forall z, w \in \mathbb{C}^n. \tag{2.43}$$

By Example 2.20, the imaginary part $\omega(z, w) = \mathrm{Im}\langle z, w\rangle$ defines a symplectic form on $\mathbb{C}^n$, called the **standard symplectic form**.

Proposition 2.50. *Let (V, Ω) be a symplectic n-vector space. Then $n = 2m$ for some m and there exists a basis $B = \{e_1, ..., e_m, f_1, ..., f_m\}$ such that $\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0$ and $\Omega(e_i, f_j) = \delta_{ij}$, for all $i, j = 1, ..., m$. In particular, for any $u, v \in V$, we have*

$$\Omega(u, v) = u^t J v, \tag{2.44}$$

where J is the following square matrix of order $2m$:

$$J = \begin{pmatrix} 0_m & I_m \\ -I_m & 0_m \end{pmatrix}. \tag{2.45}$$

Proof. See [19], Chapter 1. ∎

Proposition 2.51. *Let (V, Ω) be a symplectic $2n$-vector space and W be a subspace of V. Suppose that $W^\Omega = \{v \in V \mid \Omega(v, w) = 0,\ \forall w \in W\}$. Then W^Ω is a subspace of V, known as the* ***symplectic complement*** *of W and has the following properties.*

1. $\dim V = \dim W + \dim W^\Omega$.
2. $(W^\Omega)^\Omega = W$.
3. *For two subspaces U, W, we have $U \subset W \Leftrightarrow W^\Omega \subset U^\Omega$.*

Proof. See [19], Chapter 1. ∎

Definition 2.56. Let (V, Ω) be a symplectic $2m$-vector space and W be a subspace of V.

1. W is a **symplectic subspace** if $\Omega|_W : W \times W \to \mathbb{R}$ is nondegenerate; or equivalently, $V = W \oplus W^\Omega$.
2. W is an **isotropic subspace** if $\Omega(u, v) = 0$, for all $u, v \in W$; or equivalently, $W \subset W^\Omega$.
3. W is a **Lagrangian subspace** if W is isotropic and $\dim W = m$; or equivalently, $W = W^\Omega$.

Example 2.22. Consider $2n$-dimensional Euclidean space $\mathbb{R}^{2n} = \{(x, y) \mid x, y \in \mathbb{R}^n\}$. Define the map $\Omega_0 : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ as follows:

$$\Omega_0\Big((x, y), (x', y')\Big) = \sum_{i=1}^{n} \det \begin{pmatrix} x_i & y_i \\ x'_i & y'_i \end{pmatrix} = \sum_{i=1}^{n} (x_i y'_i - x'_i y_i), \tag{2.46}$$

In differential forms language, we usually write $\Omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$. It is easy to see that $(\mathbb{R}^2, \Omega_0)$ is a symplectic vector space, known as the **standard symplectic $2n$-vector space**. It is also clear that n-dimensional subspace $L = \{(x, 0) \mid x \in \mathbb{R}^n\}$ is Lagrangian, because $\Omega_0(x, 0) = 0$.

Definition 2.57. A linear isomorphism $\varphi : (V, \Omega) \to (V', \Omega')$ between symplectic vector spaces is called a **symplectomorphism** if $\varphi^*\Omega' = \Omega$, where $\varphi^*(\Omega')(u, v) = (\varphi(u), \varphi(v)))$, for all $u, v \in V$.

Definition 2.58. Let V be a real vector space.

1. A **complex structure** on V is a linear map $J : V \to V$ such that $J^2 = -\mathrm{Id}$.

2. Let Ω be a symplectic form on V. A complex structure J on V is said to be **compatible** with Ω if the map $G_J : V \times V \to \mathbb{R}$ defined by $G_J(u, v) = \Omega(u, Jv)$ is an inner product on V.

Proposition 2.52. *Any symplectic vector space has a compatible complex structure.*

Proof. See [19], Chapter 12. ∎

2.5.2 Symplectic Manifolds

Definition 2.59. Let M be a $2n$-dimensional smooth manifold. A differential 2-form $\omega \in \Omega^2(M)$ is called a **symplectic form** if the following are satisfied.

1. For each $p \in M$, ω_p is a symplectic form on the tangent space T_pM.

2. ω is a closed de Rham form: $d\omega = 0$.

In this case, the pair (M, ω) is called a **symplectic manifold**.

Definition 2.60. Let (M, ω) be a symplectic $2n$-manifold. A submanifold L of M is called a **Lagrangian submanifold** if for each $p \in L$, T_pL is a Lagrangian subspace of T_pM.

Remark 2.20. It is easy to see that a submanifold L of M is Lagrangian if and only if $\dim L = \frac{1}{2} \dim M$ and $i^*\omega = 0$, where $i : L \hookrightarrow M$ is the inclusion map.

Example 2.23. Let S^2 be the unit sphere in 3-space. For each $p \in S^2$, we can use the standard inner product in $\mathbb{R}^3$ to define the tangent space of S^2 at p as $T_pS^2 = \{u \in \mathbb{R}^3 \mid \langle p, u \rangle = 0\}$. Define a 2-form $\omega_p : T_pS^2 \times T_pS^2 \to \mathbb{R}$ as follows:

$$\omega_p(u, v) = \langle p, u \times v \rangle, \ \forall u, v \in T_pS^2. \tag{2.47}$$

Clearly, ω is a bilinear form and closed because it is of top degree on S^2. In addition, if for any $u \neq 0$, we choose $v = u \times p$, then $\omega_p(u, v) \neq 0$. That is, ω is nondegenerate. In this case, the equator $E = \{(x, y, 0) \mid x^2 + y^2 = 1\}$, is a Lagrangian submanifold of S^2. The form ω is usually called the **area form** on the 2-sphere and denoted by ω_S (see Figure 2.16).

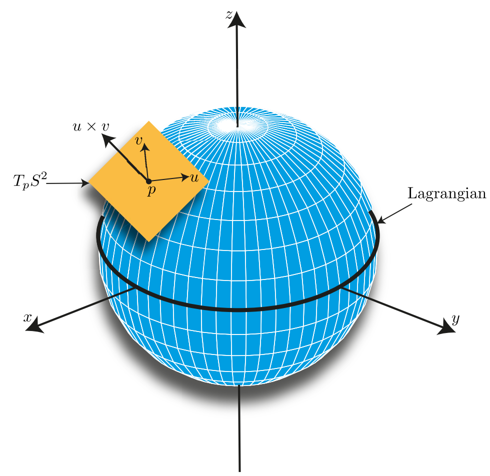

Figure 2.16: Area form on 2-sphere

Example 2.24. Let $\mathbb{C}^{n+1}$ be the complex vector space. Define an equivalence relation $\sim$ in $\mathbb{C}^{n+1} - \{0\}$ by

$$(z_0, ..., z_n) \sim (w_0, ..., w_n) \Leftrightarrow \exists \lambda \in \mathbb{C}^{\times} \text{ s.t. } z_i = \lambda w_i, \ \forall i = 0, ..., n. \tag{2.48}$$

The quotient space $\mathbb{C}^{n+1} - \{0\}/\sim$ is called the **complex projective space** and denoted by $\mathbb{CP}^n$. It is known that $\mathbb{CP}^n$ is a complex manifold of complex dimension n (see [19], Section 15). Now, let $q : \mathbb{C}^{n+1} - \{0\} \to \mathbb{CP}^n$ be the quotient map and $z \in \mathbb{C}^{n+1} - \{0\}$. In this case, any tangent vector $\xi' \in T_{q(z)}\mathbb{CP}^n$ has the form $dq(z)\xi$, for some $\xi \in T_z\mathbb{C}^{n+1}$. For $\xi_1' = dq(z)\xi_1$ and $\xi_2' = dq(z)\xi_2$, define $\Omega : T_{p(z)}\mathbb{CP}^n \times T_{p(z)}\mathbb{CP}^n \to \mathbb{C}$ by

$$\Omega(\xi_1', \xi_2') = \frac{\langle \xi_1, \xi_2 \rangle \langle z, z \rangle - \langle \xi_1, z \rangle \langle z, \xi_2 \rangle}{|z|^4}. \tag{2.49}$$

One can show (see [1], Appendix 3) that this is a Hermitian structure on $T_{q(z)}\mathbb{CP}^n$ whose imaginary part $\mathrm{Im}(\Omega)$ defines a symplectic form on $\mathbb{CP}^n$. Define

$$\omega_{FS}(q(z)) = -\frac{1}{\pi}\mathrm{Im}(\Omega)(q(z)). \tag{2.50}$$

This is also a symplectic form, known as the **Fubini-Study form** and denoted by ω_{FS}. We usually use this symplectic structure on the complex projective space, $(\mathbb{CP}^n, \omega_{FS})$.

Proposition 2.53 (Darboux Theorem). *Let (M, ω) be a symplectic $2n$-manifold and $p \in M$. Then there exists a coordinate chart $(U, (x, y) = (x_1, ..., x_n, y_1, ..., y_n))$ centered at p such that in U, we have*

$$\omega(x, y) = \sum_{i=1}^{n} dx_i \wedge dy_i. \tag{2.51}$$

Proof. See [19], Chapter 8. ∎

Definition 2.61. Let M be a smooth manifold.

1. An **almost complex structure** on M is a smooth field J on M such that for any $x \in M$, J_x is a complex structure on the tangent space T_xM.

2. Let (M, ω) be a symplectic manifold. An almost complex structure J on M is said to be **compatible** with ω if the assignment $x \xrightarrow{g} G_{J_x}$ is a Riemannian metric on M where
$$\begin{cases} G_{J_x} : T_xM \times T_xM \to \mathbb{R} \\ (u, v) \to \omega_x(u, J_xv). \end{cases}$$
In this case, the triple (ω, g, J) is called a **compatible triple** .

Proposition 2.54. *Let (M, ω) be a symplectic manifold and g be a Riemannian metric on M. Then there exists a canonical almost complex structure J on M which is compatible: $g(u, v) = \omega(u, Jv)$. In particular, any symplectic manifold has a compatible almost complex structure.*

Proof. See [19], Chapter 12. ∎

Definition 2.62. A **complex manifold** of complex dimension n is a smooth manifold equipped with an atlas whose coordinate charts take values in $\mathbb{C}^n$ and have holomorphic transition maps. An almost complex structure J on a manifold M is called **integrable** if it is induced by a structure of a complex manifold on M.

Proposition 2.55. *Any complex manifold M has a canonical almost complex structure J_0.*

Proof. See [19], Proposition 15.2. ∎

Definition 2.63. An almost complex structure J on a manifold M is called **integrable** if it is induced by a structure of a complex manifold on M. A **Kähler manifold** is a symplectic manifold (M, ω) equipped with an integrable compatible almost complex structure. In this case, the symplectic form ω is called a **Kähler form**.

Example 2.25. The complex projective space $\mathbb{CP}^n$ with the Fubini-Study form ω_{FS} is a Kähler manifold.

2.5.3 Moment Maps and Hamiltonians

Definition 2.64. Let a Lie group G act on a symplectic manifold (M, ω). We say that the action is a **symplectic action** if for each $g \in G$, $g^*\omega = \omega$, where g^* is the pullback induced by the diffeomorphism $g : M \to M$.

Example 2.26. Let the Lie group S^1 act on the 2-sphere by rotations around the z-axis. If we consider the area form ω_S on 2-sphere, it is clear that the action preserves ω_S. In fact, the rotation doesn't change the area of any region on 2-sphere (see Figure 2.17). Hence, this action is symplectic.

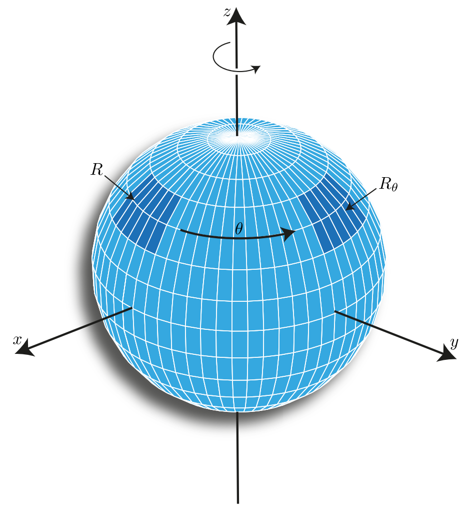

Figure 2.17: Rotations preserve the area form on 2-sphere

Definition 2.65. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Suppose that G acts on a symplectic manifold (M, ω) symplectically. A map $\mu : M \to \mathfrak{g}^*$ is called a **moment map** if the following are satisfied.

1. For any $X \in \mathfrak{g}$, we have
$$d\mu^X = \iota_{X^\#}\omega, \tag{2.52}$$
where the **component function** $\mu^X : M \to \mathbb{R}$ is defined by the natural pairing $\mu^X(p) = \langle \mu(p), X \rangle$, for all $p \in M$ and $X^\#$ is the vector field on M generated by the action of G; i.e.,
$$X^\#(p) = \frac{d}{dt}\Big|_{t=0}\Big(\exp(tX).p\Big). \tag{2.53}$$

2. μ is an equivariant map; i.e.,
$$\mu(g.p) = \mathrm{Ad}_g^*\Big(\mu(p)\Big), \ \forall g \in G, \forall p \in M. \tag{2.54}$$

In this case, the tuple (M, ω, G, μ) is called a **Hamiltonian G-system** or a **Hamiltonian G-space**.

Example 2.27. Consider the tuple (S^2, ω_S, S^1, h), where S^1 acts on S^2 by rotations and $h : S^2 \to \mathbb{R}$ is the height function on the 2-sphere. If we identify the Lie algebra of S^1 with the real line $\mathbb{R}$, then it is easy to see that h is a moment map (see [19], Chapter 21) and thus the tuple (S^2, ω_S, S^1, h) is a Hamiltonian system.

Example 2.28. Consider the symplectic manifold $(\mathbb{C}^n, \omega)$, where ω is the standard symplectic form on $\mathbb{C}^n$ as in Example 2.24. First, note that the standard action of

$\mathrm{U}(n)$ on $\mathbb{C}^n$ is symplectic. In fact, since $\omega(z,w) = \mathrm{Im}(w^*z)$, then for any $A \in \mathrm{U}(n)$ and $z, w \in \mathbb{C}^n$, we have

$$\omega(Az, Aw) = \mathrm{Im}\Big((Aw)^*Az\Big) = \mathrm{Im}(w^*A^*Az) = \mathrm{Im}(w^*z) = \omega(z,w).$$

Let $\mathfrak{u}(n)$ be the Lie algebra of $\mathrm{U}(n)$, the space of all skew-Hermitian matrices (that is $X^* = -X$) and consider the standard inner product $\langle A, B\rangle = -\mathrm{Tr}(AB)$ on $\mathfrak{u}(n)$. By using the identification $\mathfrak{u}(n)^* \cong \mathfrak{u}(n)$, we define $\mu : \mathbb{C}^n \to \mathfrak{u}(n)$ by

$$\mu(z) = \frac{i}{2}zz^*, \ \forall z \in \mathbb{C}^n. \tag{2.55}$$

We show that μ is a moment map. First, it is an equivariant map because

$$\mu(Az) = \frac{i}{2}Az(Az)^* = \frac{i}{2}Azz^*A^* = A(\frac{i}{2}zz^*)A^* = A(\frac{i}{2}zz^*)A^{-1} = \mathrm{Ad}_A\mu(z).$$

On the other hand, for any $X \in \mathfrak{u}(n)$ and $z \in \mathbb{C}^n$, we have

$$\begin{aligned}
\mu^X(z) &= \langle \mu(z), X\rangle \\
&= \langle \frac{i}{2}zz^*, X\rangle \\
&= -\frac{i}{2}\mathrm{Tr}(X^*zz^*) \\
&= \frac{i}{2}\sum_{j=1}^{n}(X^*zz^*)_{jj} \\
&= \frac{i}{2}\sum_{j=1}^{n}\Big(\sum_{k=1}^{n}X^*_{jk}(zz^*)_{kj}\Big) \\
&= \frac{i}{2}\sum_{j,k=1}^{n}\overline{X}_{kj}z_k\overline{z}_j \\
&= \frac{i}{2}z^*X^*z.
\end{aligned}$$

Thus,

$$\mu^X(z) = \frac{i}{2}z^*X^*z, \ \forall z \in \mathbb{C}^n. \tag{2.56}$$

Suppose that $w \in \mathbb{C}^n$. It is clear from (2.56) that

$$\begin{aligned}
d\mu^X(z)(w) &= \frac{d}{dt}\Big|_{t=0}\Big(\mu^X(z+tw)\Big) \\
&= \frac{i}{2}\frac{d}{dt}\Big|_{t=0}\Big((z+tw)^*X^*(z+tw)\Big) \\
&= \frac{i}{2}\Big([\frac{d}{dt}(z^*+tw^*)X^*](z+tw)\Big)\Big|_{t=0} + \frac{i}{2}\Big((z^*+tw^*)X^*[\frac{d}{dt}(z+tw)]\Big)\Big|_{t=0} \\
&= \frac{i}{2}w^*X^*z + \frac{i}{2}z^*X^*w.
\end{aligned}$$

Hence, we get

$$d\mu^X(z)(w) = \frac{i}{2}w^*X^*z + \frac{i}{2}z^*X^*w, \ \forall z, w \in \mathbb{C}^n. \tag{2.57}$$

For any fixed $X \in \mathfrak{u}(n)$, the generated vector field $X^\sharp$ on $\mathbb{C}^n$ is as follows:

$$X^\sharp(z) = Xz, \ \forall z \in \mathbb{C}^n. \tag{2.58}$$

So for any $z, w \in \mathbb{C}^n$, from (2.58) and $X^* = -X$, we have

$$\begin{aligned}
\iota_{X^\sharp}\omega|_z(w) &= \omega(X^\sharp(z), w) \\
&= \omega(Xz, w) \\
&= \mathrm{Im}(w^*Xz) \\
&= \frac{1}{2i}(w^*Xz - (w^*Xz)^*) \\
&= \frac{1}{2i}(-w^*X^*z - z^*X^*w) \\
&= \frac{-1}{2i}(w^*X^*z + z^*X^*w) \\
&= \frac{i}{2}(w^*X^*z + z^*X^*w).
\end{aligned}$$

That is,

$$\iota_{X^\sharp}\omega|_z(w) = \frac{i}{2}(w^*X^*z + z^*X^*w), \ \forall z \in \mathbb{C}^n. \tag{2.59}$$

It follows from (2.57) and (2.59) that $d\mu^X = \iota_{X^\sharp}\omega, \ \forall X \in \mathfrak{u}(n)$, which implies that μ is a moment map for the action of $\mathrm{U}(n)$ on the symplectic manifold $(\mathbb{C}^n, \omega)$. Therefore, $(\mathbb{C}^n, \omega, \mathrm{U}(n), \mu)$ is a Hamiltonian system.

Remark 2.21. Throughout this thesis, we use the standard inner product $\langle A, B\rangle = -\mathrm{Tr}(AB)$ to identify $\mathfrak{u}(n)$ with its dual $\mathfrak{u}(n)^*$.

Proposition 2.56. *Let $\mu : M \to \mathfrak{g}^*$ be a moment map for the symplectic action of a Lie group G on a symplectic manifold (M, ω). If G_p is the stabilizer subgroup of $p \in M$ and $\mathfrak{g}_p$ is its Lie algebra, then the following are satisfied.*

1. Ker $(d\mu_p) = (T_p\mathcal{O}_p)^{\omega_p}$, *the symplectic complement of $T_p\mathcal{O}_p$. In particular, if G acts freely on M_0, then* Ker $(d\mu_p) = T_pM_0$.

2. Im $(d\mu_p) = \mathfrak{g}_p^0$, *the annihilator of $\mathfrak{g}_p$ in the dual of Lie algebra.*

3. *The action is locally free at p if and only if p is a regular point of μ.*

4. *If $[\mathfrak{g}, \mathfrak{g}]$ is the commutator ideal of the Lie algebra $\mathfrak{g}$, then for any $c \in [\mathfrak{g}, \mathfrak{g}]^0$, the map $\mu_c = \mu + c$ is also a moment map for the system. In particular, if $\mathfrak{g}$ is commutative, then for any $c \in \mathfrak{g}^*$, the map $\mu_c = \mu + c$ is a moment map.*

Proof. See [19], Chapters 23 and 26. ■

Proposition 2.57. *Let $(M, \omega, \mathbb{T}^n, \mu)$ be a Hamiltonian system in which $\mathbb{T}^n$ is the real n-torus and M is a compact connected $2m$-manifold. Then for any $X \in \mathfrak{t}$, the Lie algebra of $\mathbb{T}^n$, the component function $\mu^X : M \to \mathbb{R}$ is a Morse-Bott function. Moreover, each non-degenerate critical manifold of μ^X is an even-dimensional submanifold with even index.*

Proof. See [19], Chapter 27. ■

Proposition 2.58. *Let $(M_i, \omega_i, G, \mu_i)$, $i = 1, ..., n$, be Hamiltonian G-systems. Suppose that $M = M_1 \times \cdots \times M_n$ and $\pi_i : M \to M_i$ is the projection on i^{th} component. Set $\omega = \pi_1^*\omega_1 + \cdots + \pi_n^*\omega_n$(called the **product form**) and $\mu = \mu_1 \circ \pi_1 + \cdots + \mu_n \circ \pi_n$. Then the tuple (M, ω, G, μ) is a Hamiltonian G-system.*

Proof. See [19], Chapter 3. ■

Example 2.29. Let $n, k > 1$ be natural numbers and $M = M_{k\times n}(\mathbb{C}) \cong (\mathbb{C})^{k\times n}$, the space of complex matrices of order $k \times n$. Consider the diagonal action of the unitary group $\mathrm{U}(k)$ on M. Let $\pi_j : M \to \mathbb{C}^k$ be the projection onto the j^{th} component and $\omega = \sum_{j=1}^{n} \pi_j^*\omega_j$ be the product form, where ω_j is the standard symplectic form on $\mathbb{C}^k$. Then by Example 2.28 and Proposition 2.58, it is clear that $(M, \omega, \mathrm{U}(k), \mu_k)$ is a Hamiltonian system such that for any $(z_1, ..., z_n) \in M$, we have

$$\mu_k(z_1, ..., z_n) = \frac{i}{2}\sum_{j=1}^{n} z_j z_j^*. \tag{2.60}$$

Any n-tuple $z = (z_1, ..., z_n)$ forms a $k \times n$ matrix whose columns are k-vectors z_j. In this case, the moment map $\mu_k : M \to \mathfrak{u}(k)$ has the following matrix form:

$$\mu_k(z) = \frac{i}{2} zz^*, \tag{2.61}$$

where z^* is the complex conjugate of the matrix z. Therefore, $(M_{k\times n}(\mathbb{C}), \omega, \mathrm{U}(k), \mu_k)$ is a Hamiltonian system with moment map μ. According to Proposition 2.56, by adding the central element $\frac{1}{2i}\mathrm{Id}_k$ to μ_k, we obtain a new moment map

$$\overline{\mu}_k(z) = \frac{i}{2} zz^* + \frac{1}{2i}\mathrm{Id}_k. \tag{2.62}$$

Example 2.30. Let $M = M_{k\times n}(\mathbb{C})$ be the space of complex matrices of order $k \times n$ and ω be the standard symplectic form on M. The product group $\mathrm{U}(k) \times \mathrm{U}(n)$ acts symplectically on M by

$$(A, B).z = AzB^{-1}.$$

We can define a map $\mu : M \to \mathfrak{u}(k) \times \mathfrak{u}(n)$ by

$$\mu(z) = (\overline{\mu}_k(z), \mu_n(z)) = \Big(\frac{i}{2} zz^* + \frac{1}{2i}\mathrm{Id}_k, \frac{i}{2} z^* z\Big). \tag{2.63}$$

Let $B \in \mathrm{U}(n)$ and $z \in M$. Since $BB^* = \mathrm{Id}$, from (2.62) we have

$$\overline{\mu}_k(B.z) = \frac{i}{2}(zB^{-1})(zB^{-1})^* + \frac{1}{2i}\mathrm{Id}_k = \frac{i}{2}z(BB^*)^{-1}z^* + \frac{1}{2i}\mathrm{Id}_k = \frac{i}{2}zz^* + \frac{1}{2i}\mathrm{Id}_k = \overline{\mu}_k(z).$$

That is, $\overline{\mu}_k$ is $\mathrm{U}(n)$-invariant. Similarly, μ_n is $\mathrm{U}(k)$-invariant. These two facts and Example 2.29 show that μ is a moment map for the action of $\mathrm{U}(k) \times \mathrm{U}(n)$ on M and therefore $(M_{n,k}(\mathbb{C}), \omega, \mathrm{U}(k) \times \mathrm{U}(n), \mu)$ is a Hamiltonian system.

2.5.4 Symplectic Reduction

Let (M, ω, G, μ) be a Hamiltonian G-system where G is a compact Lie group. Suppose that $M_0 = \mu^{-1}(0)$ is the zero level set of μ and $i : M_0 \hookrightarrow M$ is the inclusion map. Since M_0 is G-invariant, G acts on M_0. Denote the orbit space M_0/G by $M/\!\!/G$.

Proposition 2.59. *If M_0 is nonempty and G acts freely on M_0, then the orbit space $M/\!\!/G$ is a smooth manifold with $\dim M/\!\!/G = \dim M - 2\dim G$. Moreover, there exists a symplectic form ω_{red} on $M/\!\!/G$ such that $i^*\omega = q^*\omega_{\mathrm{red}}$, where $q : M_0 \to M_0/G$ is the quotient map.*

Proof. See [19], Chapter 23 or [51], Section 5.4. ∎

Definition 2.66. The symplectic form ω_{red} is called the **reduction form** and the pair $(M_{\mathrm{red}} = M/\!\!/G, \omega_{\mathrm{red}})$ is called the **symplectic reduction** or **symplectic quotient** of the Hamiltonian system (M, ω, G, μ).

Example 2.31. Let S^1 act on the symplectic manifold $(\mathbb{C}^{n+1}, \omega)$ by

$$e^{i\theta}.(z_0, ..., z_n) = (e^{i\theta}z_0, ..., e^{i\theta}z_n). \tag{2.64}$$

This action is symplectic because

$$(e^{i\theta})^*\omega(z, w) = \omega(iz, iw) = \mathrm{Im}[(iz)(iw)^*] = \mathrm{Im}(zw^*) = \omega(z, w).$$

Define $\mu : \mathbb{C}^{n+1} \to \mathbb{R}$ by

$$\mu(z) = -\frac{1}{2}|z|^2 + \frac{1}{2}. \tag{2.65}$$

It is easy to see that this is a moment map for this action (see [19], Chapter 26). The zero level set $\mu^{-1}(0)$ is the unit sphere $S^{2n+1} = \{z \in \mathbb{C}^{n+1} \mid |z| = 1\}$. Therefore, the symplectic quotient $\mu^{-1}(0)/S^1$ is the usual orbit space S^{2n+1}/S^1 which is the complex projective space $\mathbb{CP}^n$. In this case, the quotient form ω_{red} is equal to the standard Fubini-Study form ω_{FS}; i.e., $q^*\omega = \omega_{FS}$, where $q : M_0 \to M_0/G$ is the quotient map.

Example 2.32. Let $n > k$ be natural numbers and M be the space of $k \times n$ complex matrices $M_{k\times n}(\mathbb{C})$. As we saw in Example 2.29, $(M, \omega, \mathrm{U}(k), \mu)$ is a Hamiltonian system with the moment map $\mu : M \to \mathfrak{u}(k)$ defined by

$$\mu(z) = \frac{i}{2}zz^* + \frac{1}{2i}\mathrm{Id}_k, \ \forall z \in M. \tag{2.66}$$

In this case, $\mu^{-1}(0) = \{z \in M \mid zz^* = \mathrm{Id}_k\}$ which is diffeomorphic to the Stiefel manifold $V_k(\mathbb{C}^n)$. Since the action of $\mathrm{U}(k)$ on $V_k(\mathbb{C}^n)$ is free, the symplectic quotient $\mu^{-1}(0)/\mathrm{U}(k)$ exists and the orbit space is the complex Grassmanian $\mathrm{Gr}_k(\mathbb{C}^n)$, k-planes in $\mathbb{C}^n$. Therefore, we can equip the complex Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$ with the quotient form ω_{red}. We denote this symplectic form by $\omega_{k,n}$. So $(\mathrm{Gr}_k(\mathbb{C}^n), \omega_{k,n})$ is a symplectic manifold.

Let G_1 and G_2 be two compact connected Lie groups with Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Set $G = G_1 \times G_2$ and $\mathrm{Lie}(G) = \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Suppose that (M, ω, G, μ) is a Hamiltonian G-system with the moment map $\mu : M \to \mathfrak{g}^*$. In this case, $\mu = (\mu_1, \mu_2)$ where $\mu_i : M \to \mathfrak{g}_i^*$ for $i = 1, 2$.

Proposition 2.60 (**Reduction in Stages**). *Let (M, ω, G, μ) be a Hamiltonian G-system where $G = G_1 \times G_2$ and G_1, G_2 are compact connected Lie groups. If G_1 acts freely on $\mu_1^{-1}(0)$, $M_{\mathrm{red}} = \mu_1^{-1}(0)/G_1$ and ω_{red} is the reduction form for this action, then G_2 acts symplectically on the reduction space $(M_{\mathrm{red}}, \omega_{\mathrm{red}})$ and there exists a moment map $\mu_{\mathrm{red}} : M_{\mathrm{red}} \to \mathfrak{g}_2^*$ for this action such that $\mu_{\mathrm{red}} \circ q_1 = \mu_2 \circ i$ where $q_1 : \mu_1^{-1}(0) \to M_{\mathrm{red}}$ is the projection map and $i : \mu_1^{-1}(0) \hookrightarrow M$ is the inclusion map. In particular, $(M_{\mathrm{red}}, \omega_{\mathrm{red}}, G_2, \mu_{\mathrm{red}})$ is a Hamiltonian G_2-system.*

Proof. See [19], Section 24. ■

Example 2.33. Consider the Hamiltonian system $(M_{k\times n}(\mathbb{C}), \omega, \mathrm{U}(k) \times \mathrm{U}(n), \mu)$ in Example 2.30. Since $\mathrm{U}(k)$ acts freely on $\mu_k^{-1}(0) \cong V_k(\mathbb{C}^n)$, the reduction space in this case is $(\mathrm{Gr}_k(\mathbb{C}^n), \omega_{k,n})$ where $\omega_{k,n}$ is the reduction form (see Example 2.32). It follows from Proposition 2.60 that there exists a moment map $\mu_{k,n} : \mathrm{Gr}_k(\mathbb{C}^n) \to \mathfrak{u}(n)^*$ for the action of $\mathrm{U}(n)$ on the symplectic manifold $(\mathrm{Gr}_k(\mathbb{C}^n), \omega_{k,n})$ such that $\mu_{k,n}(q(z)) = \mu_n(z)$ for any $z \in V_k(\mathbb{C}^n)$, where $q : V_k(\mathbb{C}^n) \to \mathrm{Gr}_k(\mathbb{C}^n)$ is the quotient map. Therefore, $(\mathrm{Gr}_k(\mathbb{C}^n), \omega_{k,n}, \mathrm{U}(n), \mu_{k,n})$ is a Hamiltonian system. Let $L \in \mathrm{Gr}_k(\mathbb{C}^n)$ and $\mathrm{Pr}_L : \mathbb{C}_n \to \mathbb{C}^n$ be the orthogonal projection onto L. Choose an orthonormal basis $\{b_1, ..., b_k\}$ for L and suppose that B is the matrix whose columns are vectors b_j. It is easy to see that

$$\mu_{k,n}(L) = \frac{i}{2} BB^* = \frac{i}{2} \mathrm{Pr}_L.$$

We may also consider the following moment map for this Hamiltonian system:

$$\mu_{k,n}(L) = \frac{i}{2} \mathrm{Pr}_L + \frac{k}{2ni} \mathrm{Id}_n. \tag{2.67}$$

2.5.5 Equivariant Perfection and Kirwan Surjectivity

Let (M, ω, G, μ) be a Hamiltonian system in which M is a compact connected manifold and G is a compact Lie group. Fix an Ad-invariant inner product on $\mathrm{Lie}(G) = \mathfrak{g}$. Consider the function $f : M \to \mathbb{R}$ defined by $f(x) = ||\mu(x)||^2$, where the norm is induced by the inner product. Kirwan has investigated the Morse theory of this function in [42]. She has proved the following:

EP1. *The norm squared of the moment map* $f = ||\mu(x)||^2$ *has the minimal degeneracy property; i.e., it is a Morse-Kirwan map (see Proposition 4.15 in [42]).*

EP2. *The map* $f = ||\mu(x)||^2$ *is* G*-equivariantly perfect over the field* $\mathbb{Q}$ *(see Theorem 4.16 in [42]).*

Now, suppose that $M_0 = \mu^{-1}(0)$ and $i : M_0 \hookrightarrow M$ is the inclusion. An immediate consequence of EP2 shows that the induced map $i^* : H_G^*(M;\mathbb{Q}) \to H_G^*(M_0;\mathbb{Q})$ is a surjection. We call this surjective map the **Kirwan map** and denote it by

$$\kappa : H_G^*(M;\mathbb{Q}) \to H_G^*(M_0;\mathbb{Q}). \tag{2.68}$$

When the action of G on M_0 is free, Proposition 2.15 provides us with an isomorphism $\psi : H_G^*(M_0;\mathbb{Q}) \to H^*(M_0/G;\mathbb{Q})$. Therefore, the composite map $\psi \circ i^*$ is also a surjection. Many people call this composite map as the Kirwan map.

In her thesis, Kirwan proves that the limit set of any point under the gradient flow of a Morse-Kirwan map is a subset of the corresponding critical subset. She mentions that although the limit set is not a singleton for a general Morse-Kirwan set, it is unlikely that this happens for the norm squared of a moment map. In fact, this has been proved by Duistermaat [22]. Let (M, ω, G, μ) be a Hamiltonian G-system in which G is a compact Lie group and M is a compact connected manifold. Choose an invariant metric on M and let Ψ_t be the flow of $-\nabla f$.

Proposition 2.61 (**Duistermaat**). *If* C *is a critical set of the norm squared of moment map and* S_C *is the corresponding stratum induced by the flow* Ψ_t*, then*

1. *for any* $p \in M$*, the limit set of the trajectory of* p *is a singleton* $\Psi_\infty(p) \in M$*.*

2. *The map* $\Psi : [0, \infty] \times S_C \to S_C$ *defined by*

$$\Psi(t,p) = \begin{cases} \Psi_t(p) & \text{if } 0 \le t < \infty \\ \Psi_\infty(p) & \text{if } t = \infty \end{cases} \tag{2.69}$$

induces a deformation retraction of S_C *onto* C*.*

A detailed proof for this theorem has been provided by Lerman in [46].

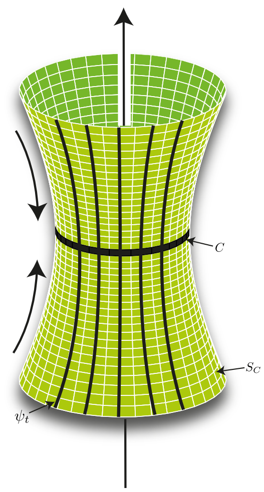

Figure 2.18: Stratum S_C corresponding to a critical subset C

Remark 2.22. Note that part 1 of Proposition 2.61 is not true for general gradient flows. For example, the unit circle in the plane is a limit set for the gradient vector filed of the following function $f : \mathbb{R}^2 \to \mathbb{R}$ (see [57], Chapter 1, Example 3, for details):

$$f(r,\theta) = \begin{cases} e^{\frac{1}{r^2-1}}, & if\ 0 < r < 1, \\ 0, & if\ r = 1, \\ \sin(\dfrac{1}{r-1} - \theta)e^{-\frac{1}{r^2-1}}, & if\ r > 1. \end{cases} \tag{2.70}$$

In Figure 2.19, we have pictured a trajectory of the vector field ∇f at some point p whose limit set is the unit circle.

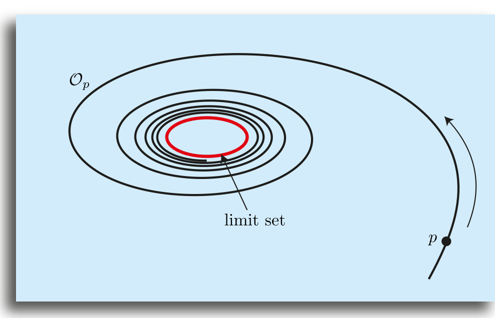

Figure 2.19: A nontrivial limit set of a trajectory

Remark 2.23. It is known from the dynamical system theory that for a *real analytic function* $f : M \to \mathbb{R}$ on a compact manifold, the limit sets of the gradient flow are single points. Here, f being real analytic means that it is a smooth function and for each $p \in M$, there exists an open set U around it on which f converges to its Taylor series. The key point in Lerman's proof for Proposition 2.61 is that he shows that the norm squared of a moment map in a Hamiltonian system with a compact Lie group is a *locally real analytic function* and for such functions the limit sets are still singletons.

2.6 Differential Homological Algebra

> "*The application to topology of homological algebra leads to somewhat different developments which may be included under the heading of differential homological algebra.*"
>
> —*John C. Moore (1923-2016)*

2.6.1 Tor Functor

Definition 2.67. Let R be a commutative ring with unity.

1. A **graded (or bigraded) R-module** M is a collection of R-modules $\{M_n \mid n \in \mathbb{Z}\}$ (or $\{M_{n,m} \mid n, m \in \mathbb{Z}\}$). If $x \in M_n$ (or $x \in M_{n,m}$), we say the **degree** (or **bidegree**) of x is n (or (n, m)) and write $\deg(x) = k$ (or $\text{bideg}(x) = (n, m)$).

2. A **homomorphism of graded R-modules** M and N of degree n is a family of R-module homomorphisms $\{f_k : M_k \to N_{k+n} \mid k \in \mathbb{Z}\}$. Similarly, a **homomorphism of bigraded R-modules** M and N of bidegree (s, t) is a collection of homomorphisms $\{f_{p,q} : M_{p,q} \to N_{p+s,q+t}\}$.

3. A **graded R-submodule** of a graded R-module M is a graded R-module N such that N_k is a submodule of M_k for each k. A **bigraded submodule** of a bigraded module is defined similarly.

4. If M is a graded R-module and N is a graded R-submodule of M, then the **quotient graded R-module** M/N is a graded R-module for which $(M/N)_k = M_k/N_k$ for each k. The **quotient bigraded module** is defined similarly.

5. The **tensor product of graded R-modules** M and N is the graded R-module $M \otimes_R N$ such that
$$(M \otimes_R N)_k = \bigoplus_{p+q=k} (M_p \otimes_R N_q).$$
The **tensor product of bigraded R-modules** M and N is a bigraded R-module $M \otimes_R N$ such that
$$(M \otimes_R N)_{p,q} = \bigoplus_{r+s=p} \bigoplus_{t+u=q} (M_{r,s} \otimes_R N_{t,u}).$$

Definition 2.68. Let R be a commutative ring with unity and M be a graded R-module. A **filtration** for M is a family $F^*(M) = \{F^n(M) \mid n \in \mathbb{Z}\}$ of submodules such that $F^p(M) \subset F^q(M)$ if $p > q$. In this case, we say $(M, F^*(M))$ is a **filtered graded module**. A filtration $F^*(M)$ for M is **bounded** if for each n, there exist numbers $s(n)$ and $t(n)$ such that

$$(0) = F^{s(n)}(M_n) \subset \cdots \subset F^{t(n)}(M_n) = M_n.$$

A filtration is called **exhaustive** if $\cup_{n\in\mathbb{Z}} F^n(M) = M$.

Definition 2.69. Let R be a commutative ring with unity and $(M, F^*(M))$ be a filtered graded R-module. The **associated graded module** of M is defined by $AG(M) = \oplus_{n\in\mathbb{Z}} AG(M)^n$ where

$$AG(M)^n = \frac{F^n(M)}{F^{n+1}(M)}.$$

Remark 2.24. The associated graded module of a filtered graded module $(M, F^*(M))$ is actually bigraded. For each p, q, we set

$$AG(M)^{p,q} = \frac{F^p(M) \cap M_{p+q}}{F^{p+1}(M) \cap M_{p+q}}.$$

In this case, $AG(M)^n = \oplus_{p+q=n} AG(M)^{p,q}$.

Definition 2.70. Let R be a commutative ring with unity.

1. A **graded R-algebra** A is a graded R-module with an associative multiplication such that $A_pA_q \subset A_{p+q}$ and there exists an element $1_A \in A_0$ with $1_A.a = a.1_A = a$ for any $a \in A_p$. The graded algebra A is **graded commutative** if for any $x, y \in A$

$$xy = (-1)^{\deg x \deg y} yx.$$

2. The **tensor product of graded R-algebras** A_1 and A_2 is the tensor product of graded R-modules $A_1 \otimes_R A_2$ with the multiplication

$$(a_1 \otimes a_2).(b_1 \otimes b_2) = (a_1b_1) \otimes (a_2b_2),$$

and the unit is defined by $1_{A_1\otimes A_2} = 1_{A_1} \otimes 1_{A_2}$.

3. A **homomorphism between graded R-algebras** A_1 and A_2 is a homomorphism $f : A_1 \to A_2$ of graded R-modules such that $\deg f = 0$, $f(1_{A_1}) = 1_{A_2}$ and $f(a_1a_2) = f(a_1)f(a_2)$.

Definition 2.71. Let R be a commutative ring with unity.

1. A **differential graded module or DG-module** (M, d_M) is a graded R-module M with a homomorphism $d_M : M \to M$ of graded R-modules such that $\deg(d_M) = 1$ and $d_M^2 = 0$. Similarly, a **differential bigraded R-module** (M, d_M) is a bigraded R-module M with a homomorphism $d_M : M \to M$ of bidegree (s, t) such that $s + t = 1$ and $d_M^2 = 0$.

2. A **homomorphism of DG-modules** of degree p between (M, d_M) and (N, d_N) is a homomorphism $f : M \to N$ of graded R-modules such that $\deg(f) = p$ and $f \circ d_M = d_N \circ f$.

3. The **tensor product of DG-modules** (M, d_M) and (N, d_N) is the usual tensor product $M \otimes_R N$ with the differential map $d_M \otimes d_N$ defined by

$$d_M \otimes d_N(m \otimes n) = d_M(m) \otimes n + (-1)^{\deg m} m \otimes d_N(n).$$

4. A **filtration** for a differential graded module (M, d_M) is a filtration $F^*(M)$ for the graded module M such that $d_M(F^p(M)) \subset F^p(M)$.

5. The **homology** $H(M, d_M)$ of a DG-module (M, d_M) is defined by

$$H^p(M, d_M) = \frac{\text{Ker } (d_M^p : M_p \to M_{p+1})}{\text{Im } (d_M^{p-1} : M_{p-1} \to M_p)}.$$

Remark 2.25. Let R be a commutative ring with unity and $(M, d_M, F^*(M))$ be a filtered differential graded R-module. Each inclusion $i_p : F^p(M) \hookrightarrow M$ induces a map in homology $H^p(i_p) : H(F^p(M), d_M) \to H(M, d_M)$. If we set

$$F^p H(M, d_M) = \text{Im } H^p(i_p),$$

then it is easy to see that $\{F^p H(M, d_M)\}$ is a filtration for the homology $H(M, d_M)$.

To formulate a proper version of Tor functor, we need to define the category of modules over a graded algebra.

Definition 2.72. Let R be a commutative ring with unity and fix a graded R-algebra A.

1. A **left module over a graded R-algebra** A is a DG-module (M, d_M) with a bilinear map $A \times M \to M$ such that for any $a, b \in A$ and $m \in M$, we have

   (i) $A_p M_q \subset M_{p+q}$;

   (ii) $a.(b.m) = (a.b).m$;

   (iii) $1_A.m = m$.

   A right module over A is defined in a similar way.

2. A **homomorphism of left modules over a graded algebra** is a homomorphism $f : M \to N$ of DG-modules which is compatible with the multiplication.

3. Let M be a right module over A and N be a left module over A. The **tensor product** $M \otimes_A N$ of M and N is the quotient space $(M \otimes_R N)/I$ where $M \otimes_R N$ is the the usual tensor product of DG-modules M, N and I is generated by the elements $(m.a) \otimes n - m \otimes (a.n)$ for $m \in M$, $n \in N$ and $a \in A$.

Definition 2.73. Consider a sequence of left modules over a graded algebra A:

$$\cdots \to (M_{n-1}, d_{n-1}) \xrightarrow{\delta_{n-1}} (M_n, d_n) \xrightarrow{\delta_n} (M_{n+1}, d_{n+1}) \to \cdots$$

We say this sequence is **exact** if Ker $\delta_n = \text{Im } \delta_{n-1}$, for any n.

Definition 2.74. Let M be a left module over a graded algebra A. A **resolution** for M is a exact sequence like the following:

$$\cdots \to P_{-n} \xrightarrow{\delta_{-n}} P_{-n+1} \xrightarrow{\delta_{-n+1}} \cdots \to P_{-1} \xrightarrow{\delta_{-1}} P_0 \xrightarrow{\varepsilon} M \to (0). \tag{2.71}$$

Definition 2.75. Let A be a graded R-algebra.

1. A left module P over A is **projective** if for any homomorphism $f : P \to N$ and any surjective homomorphism $g : M \to N$, there exists a homomorphism $\widehat{f} : P \to M$ such that $g \circ \widehat{f} = f$.

$$\begin{tikzcd} & P \arrow[dl, "\widehat{f}"'] \arrow[d, "f"] \\ M \arrow[r, "g"'] & N \end{tikzcd}$$

Diagram 2.10: Projective module

2. A **projective resolution** of a left module M over a graded algebra is an exact sequence like (2.71) in which all the modules P_{-n} are projective for all $n \geq 0$.

Proposition 2.62. *Every left module over a graded algebra has a projective resolution.*

Proof. See [48], Chapter 7. ■

Now, consider a left module (N, d_N) and a right module (M, d_M) over a graded algebra A. Fix a projective resolution $P(N)$ for N like (2.71). In this case, $P_{-n} = \bigoplus_{m\in\mathbb{Z}} P^m_{-n}$ and $\delta_{-n} = \bigoplus_{m\in\mathbb{Z}} \delta^m_{-n}$ where $\delta^m_{-n} : P^m_{-n} \to P^{-n+1}_m$. If we apply the functor $M \otimes_A -$ to this projective resolution, we get a sequence $S = M \otimes_A P(N)$ of differential graded modules. If we delete the last term of the sequence S, we get the following sequence:

$$\cdots \to M \otimes_A P_{-n} \xrightarrow{\text{Id}_M \otimes \delta_{-n}} M \otimes_A P_{-n+1} \xrightarrow{\text{Id}_M \otimes \delta_{-n+1}} \cdots \xrightarrow{\text{Id}_M \otimes \delta_{-1}} M \otimes_A P_0, \tag{2.72}$$

where $\text{Id}_M \otimes \delta_{-n} = \bigoplus_{q\in\mathbb{Z}}(\bigoplus_{i+j=q} \text{Id}_{M_i} \otimes \delta^j_{-n})$. Form the bigraded differential module (L, D) such that

$$\begin{cases} L_{p,q} = (M \otimes_A P_p)_q = \displaystyle\bigoplus_{i+j=q} M_i \otimes P^j_p, \\ D_{p,q} = \displaystyle\bigoplus_{i+j=q} (\text{Id}_{M_i} \otimes \delta^j_p). \end{cases} \tag{2.73}$$

Remark 2.26. The differential graded module (L, D) can be filtered as follows. Let $F^1 = (0)$ and for any $n \geq 0$, set

$$F_p^{-n} = \bigoplus_{i+j=p, i \geq -n} P_i^j$$

Then $\mathcal{F}^* = \{M \otimes_A F^{-n} \mid n \geq -1\}$ is a filtration for the differential bigraded module (L, D).

It is know that the homology of the differential bigraded module (L, D) is independent of the choice of the projective resolution $P(N)$ (see [48], Chapter 7 or [9], Section 2). This enables us to define the following.

Definition 2.76 (Tor Functor). The bigraded module $\mathrm{Tor}_A(M, N)$ is defined as the homology of the differential bigraded module (L, D) defined by (2.73); i.e.,

$$\mathrm{Tor}_A^{-p,q}(M, N) = \frac{\text{Ker } (D_{-p,q} : L_{-p,q} \to L_{-p+1,q})}{\text{Im } (D_{-p-1,q} : L_{-p-1,q} \to L_{-p,q})}. \tag{2.74}$$

In this case, we call $\mathrm{Tor}_A(M, N)$ the **torsion product** of A-modules M and N.

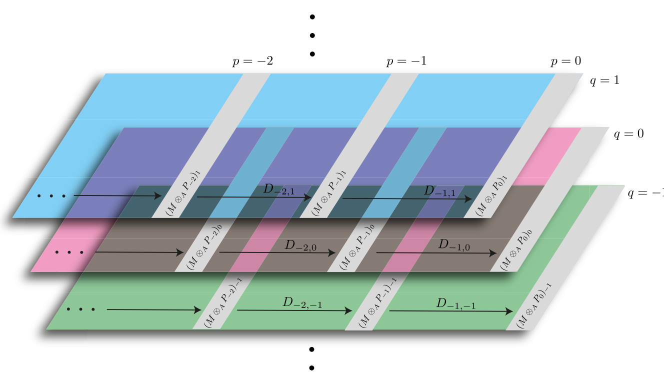

Figure 2.20: Differential bigraded module (L, D)

Remark 2.27. The filtration $\mathcal{F}^* = \{M \otimes_A F^{-n} \mid n \geq -1\}$ for the graded differential module (L, D) induces a filtration $\mathbf{F}$ for the torsion product $\mathrm{Tor}_A(M, N)$. In particular, when N is a projective A-module, we can take $P_0 = N$ and $P_{-n} = (0)$ for $n > 0$ in (2.71) which implies that $F^{-n} = N$ for $n \geq 0$ and $F^1 = (0)$. This follows that the induced filtration of $\mathrm{Tor}_A(M, N)$ is $\mathbf{F}^1 = (0)$ and $\mathbf{F}^{-n} = M \otimes_A N$ for any $n \geq 0$.

In the following proposition, we list the most important properties of the Tor functor which will be used throughout this exposition.

Proposition 2.63. *Let M and N be right and left modules over a graded algebra A respectively. Then the following are satisfied.*

1. *If $p > 0$, then for any $q \in \mathbb{Z}$*
$$\operatorname{Tor}_A^{p,q}(M,N) = (0). \tag{2.75}$$
2. *If $p = 0$, then*
$$\bigoplus_{q\in\mathbb{Z}} \operatorname{Tor}_A^{0,q}(M,N) = M \otimes_A N. \tag{2.76}$$
3. *If M or N is a free A-module, then for any $p < 0$ and $q \in \mathbb{Z}$*
$$\operatorname{Tor}_A^{p,q}(M,N) = (0). \tag{2.77}$$
In particular, $\operatorname{Tor}_A(M,N) = M \otimes_A N$.

Proof. See [48], Chapter 7 and [9], Section 2. ∎

2.6.2 Spectral Sequences

Definition 2.77. Let R be a commutative ring with unity. A **cohomology spectral sequence** is a sequence of pairs $\{(E_r, d_r)\}_{r\geq 1}$ such that the following are satisfied.

1. Each pair (E_r, d_r) consists of a differential bigraded R-module E_r with differential map d_r of bidegree $(r, 1-r)$; i.e.,
$$\begin{cases} E_r = \displaystyle\bigoplus_{p,q\in\mathbb{Z}} E_r^{p,q}, \\ d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}. \end{cases} \tag{2.78}$$
2. For each $r \geq 1$, $E_{r+1} = H(E_r, d_r)$, the homology of the chain complex (E_r, d_r).

The bigraded R-module E_r is called the **E_r-term** or **E_r-page** of the spectral sequence.

Remark 2.28. One can show that (see [48], Section 2.1) a spectral sequence $\{(E_r, d_r)\}_{r\geq 1}$ provides us with an infinite tower of bigraded submodules $\{B_n\}$ and $\{Z_n\}$ of E_2 satisfying the following:
$$B_2 \subset B_3 \subset \cdots \subset B_n \subset \cdots \subset Z_n \subset \cdots \subset Z_3 \subset Z_2 \subset E_2,$$
and for each n, $E_{n+1} \cong Z_n/B_n$ as well as the differential map d_{n+1} can be taken as a map $Z_n/B_n \to Z_n/B_n$ whose kernel is Z_{n+1}/B_n and whose image is B_{n+1}/B_n. If we set $Z_\infty = \bigcap_n Z_n$ and $B_n = \bigcup_n B_n$, then $B_\infty \subset Z_\infty$ and the quotient $E_\infty = Z_\infty/B_\infty$ is a bigraded module which is called the **limit term of the spectral sequence**.

Remark 2.29. There are two special classes of spectral sequences. A spectral sequence $\{(E_r, d_r)\}_{r\geq 1}$ is called **first quadrant** if for all $r \geq 1$, $E_r^{p,q} = (0)$ when $p < 0$ or $q < 0$. It is called **second quadrant** if for all $r \geq 1$, $E_r^{p,q} = (0)$ when $p > 0$ or $q < 0$. One can show that in a first quadrant spectral sequence, for each fixed p, q, there is always $r(p,q) \geq 1$ such that $E_r^{p,q} = E_{r(p,q)}^{p,q}$ for all $r \geq r(p,q)$. This implies that the sequence is stationary after a finite number of steps and the limit page E_∞ exists.

Definition 2.78. A spectral sequence $\{(E_r, d_r)\}_{r\geq 1}$ is said to **collapse at the N^{th}-term** if $d_r = 0$, for all $r \geq N$. In this case, $E_\infty^{p,q} = E_N^{p,q}$, for all $p, q \in \mathbb{Z}$; i.e., the limit page is the same as the E_N-page.

Definition 2.79. A spectral sequence $\{(E_r, d_r)\}_{r\geq 1}$ of R-modules is said to **converge** to a graded R-module M if there exists a filtration $F(M)$ for M such that its associated graded module $AG(M)$ is isomorphic to the limit term E_∞ of the spectral sequence; i.e.,

$$E_\infty^{p,q} \cong AG^{p,q}(M) = F^p M_{p+q}/F^{p+1}M_{p+q}, \tag{2.79}$$

In this case, we use the notation $E_r \Rightarrow M$ for the convergence.

Proposition 2.64. *Each filtered differential graded module $(M, d_M, F^*(M))$ determines a spectral sequence $\{(E_r, d_r)\}_{r\geq 1}$ such that each differential d_r has bidegree $(r, 1-r)$ and*

$$E_1^{p,q} \cong H^{p+q}(F^p/F^{p+1}).$$

Moreover, if the filtration is bounded, then the spectral sequence converges to $H(M, d_M)$; that is,

$$E_\infty^{p,q} \cong F^p[H^{p+q}(M, d_M)]/F^{p+1}[H^{p+q}(M, d_M)], \tag{2.80}$$

where $F^*(H(M, d))$ is the induced filtration for $H(M, d_M)$.

Definition 2.80. Let $V = \bigoplus_{n\geq 0} V_n$ be a graded vector space over a field $\mathbb{F}$. We say that V is **locally finite** if $\dim_{\mathbb{F}} V_n < \infty$ for any $n \geq 0$. The **Poincaré series** of V is defined to be

$$\mathbf{P}(V, t; \mathbb{F}) = \sum_{n\geq 0} (\dim_{\mathbb{F}} V_n) t^n. \tag{2.81}$$

It is clear from the above definition that for two locally finite graded vector spaces V and W, the tensor product is also a locally finite bigraded vector space and

$$\mathbf{P}(V \otimes W, t; \mathbb{F}) = \mathbf{P}(V, t; \mathbb{F})\mathbf{P}(W, t; \mathbb{F}), \tag{2.82}$$

where multiplication on the right-hand side is the Cauchy product of power series.

Proposition 2.65. *Let $\{(E_r, d_r)\}_{r\geq 1}$ be a first quadrant spectral sequence of vector spaces converging to a graded vector space V such that E_2 is locally finite over a field $\mathbb{F}$. Then V is locally finite over $\mathbb{F}$ and $\mathbf{P}(V, t; \mathbb{F}) = \mathbf{P}(E_2, t; \mathbb{F})$ if and only if the spectral sequence collapses at E_2-page.*

Proof. See [48], Chapter 1. ■

Remark 2.30. If V is a locally finite graded vector space and F is a filtration for V, then the associated graded vector space $AG(V)$ induced by F is isomorphic to V because they have the same dimensions. Thus in this case we can recover V from its associated graved vector space $AG(V)$ up to isomorphism.

Now, we state two of the most important spectral sequences which are needed in what follows: *Leray-Serre spectral sequence* and *Eilenberg-Moore spectral sequence*. We formulate the versions that meet our needs here.

I. Leray-Serre Spectral Sequence

Consider a fibration $F \overset{i}{\hookrightarrow} E \overset{\pi}{\to} B$ where F is connected and B is path-connected. One might ask how to compute the cohomology of the total space E with respect to the cohomologies of F and B. An answer to this question is provided by the Leray-Serre spectral sequence in the following proposition.

Proposition 2.66 (**Leray-Serre Cohomology Spectral Sequence**). *Let $F \overset{i}{\hookrightarrow} E \overset{\pi}{\to} B$ be a fibration where the base space B is path-connected and the fiber F is connected. If $\pi_1(B)$ acts trivially on $H^*(F;\mathbb{F})$ for a field $\mathbb{F}$, then there exists a first quadrant cohomology spectral sequence $\{(E_r, d_r)\}_{r\geq 1}$ with the following properties.*

1. $E_2^{p,q} \cong H^p(B;\mathbb{F}) \otimes_{\mathbb{F}} H^q(F;\mathbb{F})$*, for all* $p, q \geq 0$*;*

2. $E_r \Rightarrow H^*(E;\mathbb{F})$.

Proof. See [48], Chapter 5. ∎

Definition 2.81. A topological space X is of **finite type** over a field $\mathbb{F}$ if all the cohomology groups of X are finite dimensional vector spaces: $\dim H^q(X,\mathbb{F}) < \infty$, for all $q \geq 0$.

Proposition 2.67 (**Leray-Hirsch Theorem**). *Let $F \overset{i}{\hookrightarrow} E \overset{\pi}{\to} B$ be a fibration with F connected, B of finite type over a field $\mathbb{F}$ and path-connected. Then the induced map $i^* : H^*(E;\mathbb{F}) \to H^*(F;\mathbb{F})$ is surjective if and only if the Serre spectral sequence corresponding to this fibration collapses at the E_2-term. In such a case, we have*

$$H^*(E;\mathbb{F}) \cong H^*(B;\mathbb{F}) \otimes_{\mathbb{F}} H^*(F;\mathbb{F}), \tag{2.83}$$

as vector spaces.

Proof. See [48], Chapter 5. ∎

II. Eilenberg-Moore Spectral Sequence

Let R be a commutative ring with unity and $F \overset{i}{\hookrightarrow} E \overset{\pi}{\to} B$ be a fibration where F is connected and B is path connected. For any continuous map $f : X \to B$, let $E_f = f^*E$ be the pullback and consider the pullback fibration $F \overset{j}{\hookrightarrow} E_f \overset{\pi_f}{\longrightarrow} X$ as in the following commutative diagram:

$$\begin{tikzcd}
 & F \arrow[dl, "j"'] \arrow[dr, "i"] & \\
E_f \arrow[rr, "p_2"] \arrow[d, "\pi_f"'] & & E \arrow[d, "\pi"] \\
X \arrow[rr, "f"'] & & B
\end{tikzcd}$$

Diagram 2.11: A pullback diagram

Let $C^*(X;R)$ denote the cochain complex with coefficients in R. The cup product structures on $C^*(X;R)$ and $C^*(B;R)$ together with the induced maps f^* and π^* make each of them into a differential graded module over the graded algebra $C^*(B;R)$. Similarly, we can define $H^*(B;R)$-module structures on $H^*(X;R)$ and $H^*(B;R)$. The question of how to compute the cohomology of E_f with respect to the cohomologies of B, X and F is answered by the following Eilenberg-Moore spectral sequence.

Proposition 2.68 (**Eilenberg-Moore Spectral Sequence**). *Let $\mathbb{F}$ be a field and $F \overset{i}{\hookrightarrow} E \xrightarrow{\pi} B$ be a fibration with F connected and B path-connected. If the fundamental group $\pi_1(B)$ acts trivially on $H^*(F;\mathbb{F})$ and $f : X \to B$ is a continuous map with pullback E_f, then there exists a second quadrant cohomology spectral sequence $\{(E_r, d_r)\}_{r\geq 1}$ with the following properties.*

1. $E_2^{p,q} \cong \mathrm{Tor}^{p,q}_{H^*(B;\mathbb{F})}\Big(H^*(X;\mathbb{F}), H^*(E;\mathbb{F})\Big)$, *for all $p \leq 0$ and $q \geq 0$.*
2. $E_r \Rightarrow H^*(E_f;\mathbb{F})$.

Proof. See [48], Chapter 7. ∎

Remark 2.31. In fact, there exists an exhaustive filtration $F^*(H^*(E_f;\mathbb{F}))$ such that its associated graded module is isomorphic to the limit term E_∞. That is,

$$E_\infty^{p,q} \cong F^p(H^{p+q}(E_f;\mathbb{F}))/F^{p+1}(H^{p+q}(E_f;\mathbb{F})).$$

Corollary 2.68.1. *If $H^*(E;\mathbb{F})$ is a free $H^*(B;\mathbb{F})$-module, then*

$$H^*(E;\mathbb{F}) \cong H^*(B;\mathbb{F}) \otimes_{\mathbb{F}} H^*(F;\mathbb{F}), \tag{2.84}$$

as $H^*(B;\mathbb{F})$-modules.

Proof. See [48], Chapter 7. ∎

CHAPTER 3

Real Symplectic Geometry

"*One geometry cannot be more true than another; it can only be more convenient.*"

—Henri Poincaré (1854-1912)

In this chapter, we start with the idea of real structures on complex vector spaces and use them to define a real structure on a symplectic manifold. Then we introduce the notion of a real pair in a Hamiltonian system to define a special class of Hamiltonians known as real Hamiltonian systems. The idea of real pair is due to O'Shea and Sjamaar [56], [63]. After introducing the main properties, we give several examples of real Hamiltonian systems which mostly have been constructed based on the examples in Chapter 2. Finally, we define the real reduction space for real Hamiltonian systems and prove a real version of the reduction in stages for the real Hamiltonian systems whose group is a product of two Lie groups.

3.1 Real Structures on Complex Vector Spaces

In this section, we define the notion of a real structure on a complex vector space and give some of its basic ideas.

Definition 3.1. Let V be a complex n-vector space. A **real structure** on V is a map $\sigma : V \to V$ having the following properties.

1. σ is antilinear: $\sigma(\lambda v + \mu w) = \overline{\lambda}\sigma(v) + \overline{\mu}\sigma(w)$, for all $\lambda, \mu \in \mathbb{C}$ and $v, w \in V$.

2. σ is an involution: $\sigma^2 = \mathrm{Id}$.

Definition 3.2. Let V be a complex n-vector space and $\sigma : V \to V$ be a real structure on it. The **real locus** $V_{\mathbb{R}}$ of V is the fixed point set of σ; i.e.,

$$V_{\mathbb{R}} = \{v \in V \mid \sigma(v) = v\}. \tag{3.1}$$

Also, for any subspace W of V, we set $W_{\mathbb{R}} = W \cap V_{\mathbb{R}}$.

Proposition 3.1. *Let (V, σ) be a complex n-vector space with a real structure σ. Then the following are satisfied.*

1. *The real locus $V_{\mathbb{R}}$ is a real n-vector space and*

$$V = V_{\mathbb{R}} \oplus iV_{\mathbb{R}}. \tag{3.2}$$

2. *The complexification of $V_{\mathbb{R}}$ is isomorphic to V; i.e.,*

$$V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong V. \tag{3.3}$$

Proof. Straightforward. ∎

Remark 3.1. Because of Proposition 3.1, the real locus $V_{\mathbb{R}}$ of V is also called the **real subspace** of V.

Example 3.1. Consider the standard complex n-vector space $\mathbb{C}^n$. Define $\sigma : \mathbb{C}^n \to \mathbb{C}^n$ by $\sigma(z_1, ..., z_n) = (\overline{z}_1, ..., \overline{z}_n)$, where $\overline{z}_j$ means the complex conjugate of z_j. It is easily seen that σ is a real structure on $\mathbb{C}^n$ and its real subspace is $\mathbb{R}^n$. In this case, we have $\mathbb{C}^n \cong \mathbb{R}^n \oplus i\mathbb{R}^n$. Also, for any complex subspace V of $\mathbb{C}^n$, we have $V_{\mathbb{R}} = V \cap \mathbb{R}^n$.

Proposition 3.2. *Let $n > k$ be natural numbers and consider the complex conjugation as a real structure on $\mathbb{C}^n$. If $V \subset \mathbb{C}^n$ is a complex k-subspace, then $\overline{V} = \{\overline{v} \mid v \in V\}$ is a complex k-subspace and the following are equivalent.*

1. $\overline{V} = V$.

2. $\dim_{\mathbb{C}} V = \dim_{\mathbb{R}} V_{\mathbb{R}}$.

Proof. See [18], Theorem 4.12. ∎

Example 3.2. By Example 3.1 and Proposition 3.2, the complex conjugation map $\sigma : \mathbb{C}^n \to \mathbb{C}^n$ induces an involution on the complex Grassmannian $\sigma_{k,n} : \mathrm{Gr}_k(\mathbb{C}^n) \to \mathrm{Gr}_k(\mathbb{C}^n)$ such that $\sigma_{k,n}(V) = \overline{V}$. Let F be the fixed point set of $\sigma_{k,n}$ and define a map $\Psi : F \to \mathrm{Gr}_k(\mathbb{R}^n)$ by $\Psi(V) = V_{\mathbb{R}}$. It follows from part 1 of Proposition 3.1 that Ψ is 1-1. Let $V \in \mathrm{Gr}_k(\mathbb{R}^n)$ and set $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. Since $V_{\mathbb{C}} = \overline{V_{\mathbb{C}}}$, we have $V_{\mathbb{C}} \in F$. On the other hand, by (3.3) we have $\Psi(V_{\mathbb{C}}) = V$ which proves the surjectivity. Therefore, Ψ is a bijection. By using this bijection, we can identify the real locus F with the real Grassmannian $\mathrm{Gr}_k(\mathbb{R}^n)$.

3.2 Real Structures on Symplectic Manifolds

In this section, we define the notion of a real structure on a symplectic manifold and generalize it to the class of Hamiltonian systems.

Definition 3.3. Let (M, ω) be a symplectic manifold. An **anti-symplectic involution** on M is a diffeomorphism $\sigma : M \to M$ having the following properties.

1. $\sigma^2 = \mathrm{Id}$.
2. $\sigma^* \omega = -\omega$.

In this case, the triple (M, ω, σ) is called a **real symplectic manifold**.

Example 3.3. Define $\sigma : \mathbb{C}^n \to \mathbb{C}^n$ by $\sigma(z_1, ..., z_n) = (\overline{z}_1, ..., \overline{z}_n)$, where bar denotes the complex conjugate. Clearly, $\sigma^2 = \mathrm{Id}$. Let $\omega : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$ be the standard symplectic form on $\mathbb{C}^n$; i.e., $\omega(z, w) = \mathrm{Im}(w^* z)$, $\forall z, w \in \mathbb{C}^n$, where w^* is the transpose of complex conjugate of w. A simple calculation shows that $d_z\sigma(w) = \overline{w}$, for any $z, w \in \mathbb{C}^n$. Hence, we have

$$\begin{aligned}
\sigma^* \omega_z(w, w') &= \omega_z(d_z\sigma(w), d_z\sigma(w')) \\
&= \omega_z(\overline{w}, \overline{w'}) \\
&= \mathrm{Im}(\overline{w'}^* \overline{w}) \\
&= -\mathrm{Im}(w'^* w) \\
&= -\omega_z(w, w').
\end{aligned}$$

Therefore, σ is an anti-symplectic involution and $(\mathbb{C}^n, \omega, \sigma)$ is a real symplectic manifold.

Example 3.4. Consider the symplectic manifold (S^2, ω_S), where S^2 is the unit sphere in $\mathbb{R}^3$ and ω_S is the area form (see Example 2.23). Let $\sigma : S^2 \to S^2$ be the reflection with respect to the xz-plane. If we use the cylindrical coordinates (θ, z) on S^2, then the area form ω_S and the involution σ have the following forms:

$$\begin{cases} \sigma(\theta, z) = (-\theta, z), \\ \omega_S(\theta, z) = d\theta \wedge dz. \end{cases} \tag{3.4}$$

It is clear from (3.4) that

$$\sigma^* \omega_S(\theta, z) = \omega_S(-\theta, z) = -d\theta \wedge dz = -\omega_S(\theta, z).$$

That is, σ is an anti-symplectic involution on the 2-sphere and thus (S^2, ω_S, σ) is a real symplectic manifold.

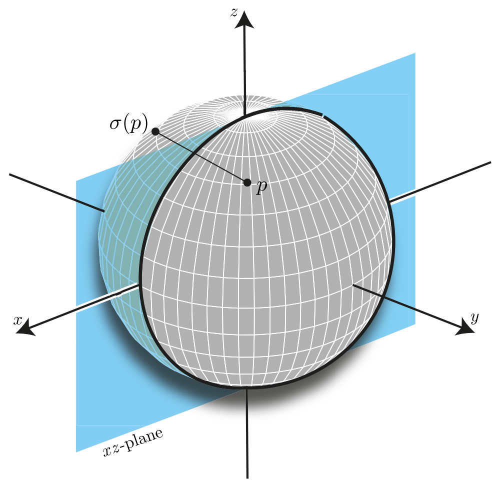

Figure 3.1: Reflection relative to xz-plane

Example 3.5. Consider the complex projective space $\mathbb{CP}^n$ with Fubini-Study form ω_{FS}. Let $\widehat{\sigma}$ be the map induced by the anti-symplectic involution $\sigma : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ in Example 3.3. If $q : \mathbb{C}^{n+1} - \{0\} \to \mathbb{CP}^n$ is the quotient map, then $q \circ \sigma = \widehat{\sigma} \circ q$ and $q^*\omega_{FS} = \omega$, where ω is the standard form on $\mathbb{C}^{n+1}$. In this case, we have

$$q^*(\widehat{\sigma}^*\omega_{FS}) = \sigma^*(q^*\omega_{FS}) = \sigma^*(\omega) = -\omega = -q^*\omega_{FS}.$$

Since q^* is injective, we get $\widehat{\sigma}^*\omega_{FS} = -\omega_{FS}$. Thus, $\widehat{\sigma}$ is an anti-symplectic involution and therefore $(\mathbb{CP}^n, \omega_{FS}, \widehat{\sigma})$ is a real symplectic manifold.

Definition 3.4. Let (M, ω, σ) be a real symplectic manifold. The fixed point set of σ, $M^\sigma = \{x \in M \mid \sigma(x) = x\}$, is called the **real locus**.

Example 3.6. The real loci of the real symplectic manifolds in Examples 3.3, 3.4 and 3.5 are $\mathbb{R}^n$, S^1 and $\mathbb{RP}^n$ respectively.

One of the most interesting properties of real loci of real symplectic manifolds is given in the following proposition due to Duistermaat [22].

Proposition 3.3. *If the real locus M^σ of a real symplectic manifold (M, ω, σ) is non-empty, then M^σ is a Lagrangian submanifold of M and $T_pM^\sigma = \mathrm{Ker}(\mathrm{Id} - d\sigma_p)$, for all $p \in M^\sigma$.*

Proof. Consider $G = \{\mathrm{Id}, \sigma\}$. Since $\sigma^2 = \mathrm{Id}$, $G \cong \mathbb{Z}_2$ is a compact Lie group acting on M by σ. In this case, the fixed point set of the action is exactly the real locus M^σ. So by Corollary 2.33.1, M^σ is a closed submanifold of M.

Let $p \in M^\sigma$ and $D = d\sigma(p) : T_pM \to T_pM$. Since $D^2 = \mathrm{Id}$, it has two eigenvalues 1 and -1 with corresponding eigenspaces $E_1 = \mathrm{Ker}(\mathrm{Id} - D)$ and $E_{-1} = \mathrm{Ker}(\mathrm{Id} + D)$ such that

$$T_pM = E_1 \oplus E_{-1}. \tag{3.5}$$

If $u, v \in E_1$, then $D(u) = u$, $D(v) = v$ and

$$\omega_p(u, v) = \omega_p(D(u), D(v)) = \sigma^*\omega_p(u, v) = -\omega_p(u, v).$$

That is, $\omega_p(u, v) = 0$ and thus, E_1 is isotropic. Similarly, E_{-1} is isotropic and hence $\dim E_1 \leq n$ and $\dim E_{-1} \leq n$. Since $\dim E_1 + \dim E_{-1} = n$, we must have

$$\dim E_1 = \dim E_{-1} = n, \tag{3.6}$$

which shows that E_1 and E_{-1} are both Lagrangian subspaces of T_pM. On the other hand, an easy computation shows that

$$T_pM^\sigma = E_1. \tag{3.7}$$

Therefore, M^σ is a Lagrangian submanifold of M and the proof is complete. ■

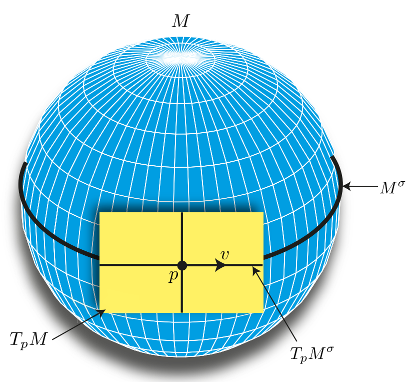

Figure 3.2: Tangent space of the real locus

The next definition is due to Sjamaar [63].

Definition 3.5. A **real structure** on a Hamiltonian G-system (M, ω, G, μ) is a pair of smooth maps $\sigma : M \to M$ and $\phi : G \to G$ such that the following are satisfied.

1. ϕ is a Lie group involution; i.e., a Lie group automorphism of order two.

2. σ is an anti-symplectic involution.

3. σ and ϕ satisfy the following **compatibility conditions**:

$$\begin{cases} \sigma \circ g = \phi(g) \circ \sigma, \ \forall g \in G, \\ \mu \circ \sigma = -\phi^* \circ \mu. \end{cases} \tag{3.8}$$

Here, $\phi^* : \mathfrak{g}^* \to \mathfrak{g}^*$ is the dual of the involution $\phi_* = d\phi : \mathfrak{g} \to \mathfrak{g}$ induced by ϕ.

In this case, the pair (σ, ϕ) is called a **real pair** and the tuple $\mathcal{RH} = (M, \omega, G, \mu, \sigma, \phi)$ is called a **real Hamiltonian system**.

Remark 3.2. Since $\phi : G \to G$ is a group involution, it induces two involutions $\phi_* : \mathfrak{g} \to \mathfrak{g}$ and $\phi_* : \mathfrak{g}^* \to \mathfrak{g}^*$ where $\mathfrak{g} = \mathrm{Lie}(G)$. In this case, we get the following direct sum decompositions:

$$\begin{cases} \mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-, \\ \mathfrak{g}^* = \mathfrak{g}^*_+ \oplus \mathfrak{g}^*_-, \end{cases} \tag{3.9}$$

where " $+$ " and " $-$ " denote the eigenspaces corresponding to eigenvalues $+1$ and -1. It is also clear that $\mathfrak{g}_+ = \mathrm{Lie}(G^\phi)$.

Remark 3.3. It follows from the compatibility conditions that if $p \in M^\sigma$, then

$$\mu(p) = \mu(\sigma(p)) = -\phi^*(\mu(p)).$$

This means $\mu(p) \in \mathfrak{g}^*_-$. Therefore the image of the real locus M^σ under the moment map is contained $\mathfrak{g}^*_-$ and we have $\mu : M^\sigma \to \mathfrak{g}^*_-$.

The next lemma shows that the Lie subalgebra $\mathfrak{g}^*_-$ is invariant under the action of real subgroup. We will use this in Chapter 4.

Lemma 3.4. *The subspace $\mathfrak{g}^*_- = \{\xi \in \mathfrak{g}^* \mid \phi^*(\xi) = -\xi\}$ is invariant under the restricted G^ϕ-coadjoint action.*

Proof. Let $g \in G$ and $X \in \mathfrak{g}$. If $\iota_g : G \to G$ is the inner automorphism, then a simple computation shows that $\phi \circ \iota_g = \iota_{\phi(g)} \circ \phi$. This follows that

$$\begin{cases} \mathrm{Ad}^*_{\phi(g)} \circ \phi^* = \phi^* \circ \mathrm{Ad}^*_g \\ \mathrm{ad}^*_{\phi_*(X)} \circ \phi^* = \phi^* \circ \mathrm{ad}^*_X. \end{cases} \tag{3.10}$$

If $g \in G^\phi$ and $\xi \in \mathfrak{g}^*_-$, then (3.10) implies that

$$\phi^*(\mathrm{Ad}^*_g(\xi)) = \mathrm{Ad}^*_{\phi(g)}(\phi^*(\xi)) = -\mathrm{Ad}^*_g(\xi).$$

Thus, $\mathrm{Ad}^*_g(\xi) \in \mathfrak{g}^*_-$ which completes the proof. ■

3.3 Examples of Real Hamiltonians

This section gives several examples regarding real Hamiltonian systems some of which will be used in next sections.

Example 3.7. Let $S^1 \cong \mathrm{U}(1)$ act on the sphere $S^2 \cong \mathbb{CP}^1$ by rotations around z-axis and consider the Hamiltonian system $(S^2, \omega_S, S^1, \mu)$, where ω_S is the area form and μ is the height function. In the cylindrical coordinates (θ, z) on S^2, the area form is $\omega_S = d\theta \wedge dz$. Define $\sigma : S^2 \to S^2$ by $\sigma(\theta, z) = (-\theta, z)$ and $\phi : S^1 \to S^1$ by $\phi(\alpha) = -\alpha$. It is evident that σ and ϕ are involutions and we saw in Example 3.4 that σ is anti-symplectic. Now, let $\alpha \in S^1$ and $(\theta, z) \in S^2$. Then

$$\sigma(\alpha.(\theta, z)) = \sigma(\alpha + \theta, z) = (-\alpha - \theta, z)) = \phi(\alpha).(-\theta, z) = \phi(\alpha).(\sigma(\theta, z)).$$

That is, the first condition in (3.8) holds. Since $\phi_* = -\mathrm{Id}$, we have $\phi^* = -\mathrm{Id}$. On the other hand, since $\mu(\theta, z) = z$, we can write

$$\mu \circ \sigma(\theta, z) = \mu(-\theta, z) = z = \mu(\theta, z) = -\phi^*(\mu(\theta, z)).$$

Thus, $\mu \circ \sigma = -\phi^* \circ \mu$ and the second condition in (3.8) is satisfied. This proves that the pair (σ, ϕ) is a real structure and the tuple $(S^2, \omega_S, S^1, \mu, \sigma, \phi)$ is a real Hamiltonian system. Moreover, the real locus of this system is the great circle $M^\sigma = \{(x, 0, z)^t \mid x^2 + z^2 = 1\} \cong \mathbb{RP}^1$ and the real subgroup is $\mathrm{O}(1) \cong \mathbb{Z}_2$, which acts on the real locus by reflection with respect to the z-axis in the xz-plane (see Figure 3.3).

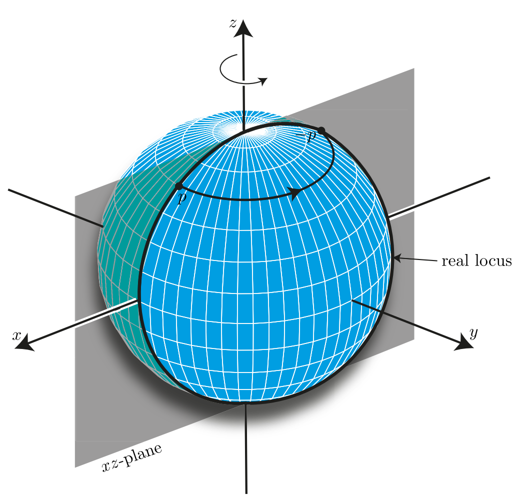

Figure 3.3: Action of $\mathbb{Z}_2$ on the real locus

Example 3.8. Consider the Hamiltonian system $(\mathbb{C}^n, \omega, \mathrm{U}(n), \mu)$ in Example 2.28. Define two maps $\sigma : \mathbb{C}^n \to \mathbb{C}^n$ and $\phi : \mathrm{U}(n) \to \mathrm{U}(n)$ as follows:

$$\begin{cases} \sigma(z) = \overline{z}, \ \forall z \in \mathbb{C}^n, \\ \phi(A) = \overline{A}, \ \forall A \in \mathrm{U}(n). \end{cases} \tag{3.11}$$

Clearly, σ and ϕ are involutions and we saw in Example 3.3 that σ is anti-symplectic. Also,

$$\sigma(Az) = \overline{Az} = \overline{A}\,\overline{z} = \phi(A)\sigma(z).$$

So the first condition in (3.8) holds. On the other hand, if $\phi_* = d\phi(I)$, then $\phi_*(A) = \overline{A}$, for all $A \in \mathfrak{u}(n)$, because

$$\phi_*(A) = \frac{d}{dt}\Big|_{t=0}\Big(\phi(e^{tA})\Big) = \frac{d}{dt}\Big|_{t=0}(e^{t\overline{A}}) = \overline{A}.$$

This follows that for any $z \in \mathbb{C}^n$, we have

$$\mu(\sigma(z)) = \mu(\overline{z}) = \frac{i\overline{z}\ \overline{z}^*}{2} = -\overline{(\frac{izz^*}{2})} = -\overline{\mu(z)} = -\phi_*(\mu(z)). \tag{3.12}$$

That is, $\mu \circ \sigma = -\phi^* \circ \mu$ and the second condition in (3.8) holds. Thus the pair (σ, ϕ) is a real structure on the Hamiltonian system $(\mathbb{C}^n, \omega, \mathrm{U}(n), \mu)$. In this case, the real locus is the n-real subspace $\mathbb{R}^n$ and the real subgroup is the real orthogonal group $\mathrm{O}(n)$, which acts on $\mathbb{R}^n$ in the standard way.

Example 3.9. Consider the Hamiltonian system $(M_{k\times n}(\mathbb{C}), \omega, \mathrm{U}(k) \times \mathrm{U}(n), \mu)$ in Example 2.30. Let $\sigma : M_{k\times n}(\mathbb{C}) \to M_{k\times n}(\mathbb{C})$, $\phi_k : \mathrm{U}(k) \to \mathrm{U}(k)$ and $\phi_n : \mathrm{U}(n) \to \mathrm{U}(n)$ be the corresponding complex conjugations. If $\phi = \phi_k \times \phi_n$, then a similar calculation as in Example 3.8 shows that (σ, ϕ) is a real pair and $(M_{k\times n}(\mathbb{C}), \omega, \mathrm{U}(k) \times \mathrm{U}(n), \mu, \sigma, \phi)$ is a real Hamiltonian system. In this case, the real locus is $M_{k\times n}(\mathbb{R})$ and the real subgroup is the product of orthogonal groups $G^\phi = \mathrm{O}(k) \times \mathrm{O}(n)$.

Example 3.10. Consider the symplectic manifold $(\mathbb{CP}^n, \omega_{FS})$, where ω_{FS} is the Fubini-Study form. By Example 3.8, the induced action of unitary group $\mathrm{U}(n+1)$ on $\mathbb{CP}^n$ is symplectic. Now, for any $z \in \mathbb{C}^{n+1} - \{0\}$, denote its class in $\mathbb{CP}^n$ by $q(z) = [z]$, where q is the quotient map and define the map $\mu : \mathbb{CP}^n \to \mathfrak{u}(n+1)$ by

$$\mu[z] = \frac{zz^*}{2\pi i|z|^2}. \tag{3.13}$$

We show that this is a moment map. First, note that for any $A \in \mathrm{U}(n+1)$ and $[z] \in \mathbb{CP}^n$, since $A^* = A^{-1}$, we have

$$\mu(A.[z]) = \mu[Az] = \frac{Azz^*A^*}{2\pi i|Az|^2} = A(\frac{zz^*}{2\pi i|z|^2})A^* = \mathrm{Ad}_A(\mu[z]).$$

Thus, μ is equivariant. Moreover, we can write

$$\mu^A[z] = \langle \mu[z], A\rangle = \mathrm{Tr}((\mu[z])^* A) = \frac{-1}{2\pi i|z|^2}\mathrm{Tr}(zz^*A) = \frac{-1}{2\pi i|z|^2} z^*Az. \tag{3.14}$$

Let $w \in T_z\mathbb{C}^{n+1} = \mathbb{C}^{n+1}$. Since $dq(z)w \in T_{[z]}\mathbb{CP}^n$, (3.14) follows that

$$\begin{aligned} d\mu^A[z](dq(z)w) &= \frac{d}{dt}\Big|_{t=0}\Big(\mu^A[z+tw]\Big) \\ &= \frac{-1}{2\pi i}\left(\frac{d}{dt}\Big|_{t=0}\Big(\frac{(z^*+tw^*)A(z+tw)}{|z+tw|^2}\Big)\right) \\ &= \frac{-1}{2\pi i|z|^4}\Big((w^*Az + z^*Aw)|z|^2 - (z^*w + w^*z)z^*Az\Big). \end{aligned}$$

So

$$d\mu^A[z](dq(z)w) = \frac{-1}{2\pi i|z|^4}\Big((w^*Az + z^*Aw)|z|^2 - (z^*w + w^*z)z^*Az\Big). \tag{3.15}$$

On the other hand, for any $A \in \mathfrak{u}(n+1)$ and $[z] \in \mathbb{CP}^n$, we have $A^\#[z] = dq(z)(Az)$. Hence, by the definition of the Fubini-Study form ω_{FS}, for any $w \in T_z\mathbb{C}^{n+1} = \mathbb{C}^{n+1}$, we can write

$$\begin{aligned} \iota_{A^\#[z]}\omega_{FS}\Big(dq(z)(w)\Big) &= \omega_{FS}\Big(dq(z)(Az), dq(z)(w)\Big) \\ &= \frac{-1}{\pi|z|^4}\left(\mathrm{Im}\Big(\langle Az, w\rangle|z|^2 - \langle Az, z\rangle\langle z, w\rangle\Big)\right) \\ &= \frac{-1}{2\pi i|z|^4}\Big(|z|^2(w^*Az - z^*A^*w) - z^*Azw^*z + z^*wz^*A^*z\Big) \\ &= \frac{-1}{2\pi i|z|^4}\Big((w^*Az + z^*Aw)|z|^2 - (w^*z + wz^*)z^*Az\Big). \end{aligned}$$

Thus

$$\iota_{A^\#[z]}\omega_{FS}\Big(dq(z)(w)\Big) = \frac{-1}{2\pi i|z|^4}\Big((w^*Az + z^*Aw)|z|^2 - (w^*z + wz^*)z^*Az\Big). \tag{3.16}$$

It follows from (3.15) and (3.16) that $d\mu^A = \iota_{A^\#}\omega_{FS}$. That is, μ is a moment map.

Now, define $\phi : \mathrm{U}(n+1) \to \mathrm{U}(n+1)$ by $\phi(A) = \overline{A}$, for all $A \in \mathrm{U}(n+1)$ and $\sigma : \mathbb{CP}^n \to \mathbb{CP}^n$ by $\sigma[z] = [\overline{z}]$, for all $z \in \mathbb{C}^{n+1}$. Clearly, ϕ and σ are involutions and by Example 3.5, the map σ is anti-symplectic.

For any $A \in \mathrm{U}(n+1)$ and $[z] \in \mathbb{CP}^n$, since $A[z] = [Az]$, we have

$$\sigma(A[z]) = \sigma([Az]) = [\overline{Az}] = [\overline{A}\ \overline{z}] = \overline{A}[\overline{z}] = \phi(A)\sigma[z].$$

So the first condition in (3.8) holds. Also, since $\phi_*(A) = \overline{A}$, a similar argument as in Example 3.8 shows that $\mu \circ \sigma = -\phi^* \circ \mu$. Thus $(\mathbb{CP}^n, \omega_{FS}, \mathrm{U}(n+1), \mu, \sigma, \phi)$ is a real Hamiltonian system. In this case, the real locus is the real projective space $\mathbb{RP}^n$ and the real subgroup is the orthogonal group $\mathrm{O}(n+1)$ which acts on the real locus in the standard way.

Example 3.11. Consider the real Hamiltonian system in Example 3.10. If we add the central element $\frac{i}{4\pi n}\mathrm{Id}$ to the moment map, by Proposition 2.56 we get a new moment map as follows:

$$\mu[z] = \frac{1}{2\pi i}\Big(\frac{zz^*}{|z|^2} - \frac{1}{2n}\mathrm{Id}\Big). \tag{3.17}$$

Similar to computation in Example 3.10, it is easy to see that $(\mathbb{CP}^n, \omega_{FS}, \mathrm{U}(n+1), \mu, \sigma, \phi)$ is a real Hamiltonian system where μ is defined by (3.17).

Example 3.12. Consider the special case $n = 1$ in Example 3.11. We get the real Hamiltonian system $(\mathbb{CP}^1, \omega_{FS}, \mathrm{U}(2), \mu, \sigma, \phi)$. In this case, for $z = (z_1, z_2)^t \in \mathbb{C}^2 - \{(0,0)\}$, the moment map has the following form:

$$\mu\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \frac{1}{2\pi i(|z_1|^2 + |z_2|^2)}\begin{pmatrix} |z_1|^2 - |z_2|^2 & z_1\overline{z}_2 \\ z_2\overline{z}_1 & |z_2|^2 - |z_1|^2 \end{pmatrix}. \tag{3.18}$$

By Proposition A.1 in Appendix A, we know that the space $\mathbb{CP}^1$ is diffeomorphic to the unit sphere S^2 by a diffeomorphism Φ such that

$$\begin{cases} \Phi(x,y,z)^t = \begin{bmatrix} 1+z \\ x-yi \end{bmatrix}, \ if \ z \neq -1, \\ \Phi(x,y,z)^t = \begin{bmatrix} 1-z \\ x+yi \end{bmatrix}, \ if \ z \neq 1. \end{cases} \tag{3.19}$$

It is easy to see that

$$\mathfrak{u}(2) = \Big\{\begin{pmatrix} ci & -b+ia \\ b+ia & di \end{pmatrix} \mid a,b,c,d \in \mathbb{R}\Big\}. \tag{3.20}$$

Define a map $\Psi : \mathfrak{u}(2) \to \mathbb{R}^4$ by

$$\Psi\begin{pmatrix} ci & -b+ia \\ b+ia & di \end{pmatrix} = (a,b,c,d)^t. \tag{3.21}$$

By using this map, we can identify the unitary group $\mathfrak{u}(2)$ with the Euclidean space $\mathbb{R}^4$. Let $(x, y, z)^t \in S^2$. Then by (3.18), (3.19) and (3.21), we get

$$\Psi \circ \mu \circ \Phi(x, y, z)^t = \frac{-1}{4\pi}(x, y, 2z, -2z)^t. \tag{3.22}$$

By using these identifications, we can consider the moment map $\mu : S^2 \to \mathbb{R}^4$ defined by (3.22). On the other hand, for any $A \in \mathfrak{u}(2)$ and $(x, y, z)^t \in S^2$, by using (3.18) and a routine calculation, we see that

$$\mu^A(x, y, z) = \frac{-1}{4\pi}\Big(2ax + 2by + (c - d)z + c + d\Big). \tag{3.23}$$

Also, the corresponding anti-symplectic involution $\sigma : S^2 \to S^2$ is the reflection with respect to the xz-plane in the 3-space; i.e.,

$$\sigma(x, y, z)^t = (x, -y, z)^t. \tag{3.24}$$

The real subgroup is $\mathrm{O}(2)$ and the real locus is $M^\sigma = \{(x, 0, z)^t \mid x^2 + z^2 = 1\}$ which is diffeomorphic to $\mathbb{RP}^1$. In this case, the real subgroup $\mathrm{O}(2)$ acts on the real locus $\mathbb{RP}^1$ in the standard way.

Example 3.13. Let $n > 1$ be a natural number and set $M = (\mathbb{CP}^1)^n$. Consider the diagonal action of $\mathrm{U}(2)$ on M. By using the construction in Example 3.10 and projections $\pi_i : M \to \mathbb{CP}^1$, we can define the product form $\omega = \pi_1^*\omega_{FS} + \cdots + \pi_n^*\omega_{FS}$ on M (see Proposition 2.58). It follows from Example 3.10 that this action is symplectic. If we identify $\mathbb{CP}^1$ with S^2 and the Lie algebra $\mathfrak{u}(2)$ with $\mathbb{R}^4$ as in Example 3.12, then the moment map $\mu : M \to \mathbb{R}^4$ is as follows:

$$\mu(X_1, ..., X_n) = \frac{-1}{4\pi}\Big(\sum_{j=1}^n x_j, \sum_{j=1}^n y_j, 2\sum_{j=1}^n z_j, -2\sum_{j=1}^n z_j\Big)^t, \quad \forall X_j \in S^2. \tag{3.25}$$

Also, for any $A = \begin{pmatrix} ci & -b + ia \\ b + ia & di \end{pmatrix} \in \mathfrak{u}(2)$, the component of moment map is

$$\mu^A(X_1, ..., X_n) = \frac{-1}{4\pi}\Big(2a\sum_{j=1}^n x_j + 2b\sum_{j=1}^n y_j + (c - d)\sum_{j=1}^n z_j + nc + nd\Big). \tag{3.26}$$

Let ϕ and σ be the same involutions as in Example 3.10. Set $\sigma_n = \sigma \times \cdots \times \sigma$, n times. By doing an argument similar to the one in Example 3.10, we can see that the pair (σ_n, ϕ) is a real pair. In this case, the real subgroup is $\mathrm{O}(2)$ and the real locus is the n-torus $S^1 \times \cdots \times S^1 \cong (\mathbb{RP}^1)^n$ on which $\mathrm{O}(2)$ acts diagonally.

Remark 3.4. It is worth noting that the zero level set $M_0 = \mu^{-1}(0)$ in Example 3.13 can be identified with the set of n points on the 2-sphere whose center of gravity is the origin (see Figure 3.4).

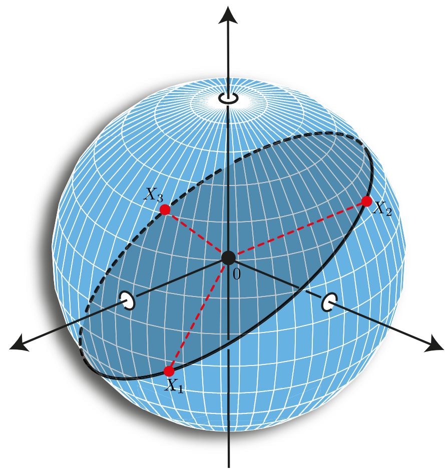

Figure 3.4: Center of gravity of three points on the 2-sphere

In this case, if n is an odd number, one can show that the action of O(2) on the zero level set M_0 is locally free. To see that, let $X = (X_1, ..., X_n) \in M_0$ and $AX = X$ for $A \in \mathrm{O}(2)$ and $f_A : S^2 \to S^2$be the corresponding Mobius transformation (see Appendix A). Clearly, $f_A(X_j) = X_j$ for $j = 1, ..., n$. Since $X_1 + \cdots + X_n = 0$ and n is odd, at least three of X_j must be different. Thus f_A has more than two fixed points and it follows from Proposition A.1 that f_A is the identity. That is, A is a diagonal matrix. Since A is orthogonal, we have $A = \{\pm \mathrm{Id}\}$ and hence the stabilizer of any point in M_0 is the subgroup $\{\pm \mathrm{Id}\}$. Since the central subgroup $\{\pm \mathrm{Id}\}$ acts trivially on M_0, it follows that the projective group $\mathrm{PO}(2) = \mathrm{O}(2)/\{\pm \mathrm{Id}\}$ acts freely. We will use this observation in chapter 8.

3.4 Real Symplectic Reduction

In this section, we describe the idea of symplectic reduction in real symplectic geometry. In particular, we prove a real version of symplectic reduction in stages.

Let $\mathcal{RH} = (M, \omega, G, \mu, \sigma, \phi)$ be a real Hamiltonian system where G is a compact Lie group and M is a compact connected manifold. Let M^σ be the real locus, $M_0^\sigma = M^\sigma \cap \mu^{-1}(0)$ and G^ϕ be the real subgroup which acts on the real locus M^σ. The inclusion map $i_0 : M_0^\sigma \hookrightarrow M^\sigma$ induces a map $i_0^* : H_{G^\phi}(M^\sigma; \mathbb{Z}_2) \to H^*_{G^\phi}(M_0^\sigma; \mathbb{Z}_2)$ in the G^ϕ-equivariant cohomology with coefficients in $\mathbb{Z}_2$. Clearly, the real subgroup G^ϕ acts on the *real zero level set* M_0^σ.

Definition 3.6. Let $\mathcal{RH} = (M, \omega, G, \mu, \sigma, \phi)$ be a real Hamiltonian system where G is a compact Lie group and M is a compact connected manifold. The orbit space M_0^σ / G^ϕ is called the **real reduction** and denoted it by $M^\sigma /\!\!/ G^\phi$.

We know that when G acts freely on M_0, the reduction space M_0/G is a symplectic manifold (see Propositions 2.59 and 3.3). In this case, we will see that the real reduction M_0^σ / G^ϕ embeds as a Lagrangian submanifold of the reduction M_0/G (Proposition 3.6). Foth [26] has investigated real reductions and characterized them based on certain conjugacy classes of involutions on G.

Lemma 3.5. *The anti-symplectic involution $\sigma : M \to M$ descents to an anti-symplectic involution $\sigma_{\text{red}} : M_{\text{red}} \to M_{\text{red}}$ on the symplectic reduction manifold $(M_{\text{red}}, \omega_{\text{red}})$.*

Proof. It follows from $\mu(\sigma(x)) = -\phi^*(\mu(x))$ that if $\mu(x) = 0$, then $\mu(\sigma(x)) = 0$. So we can define a map $\sigma_{\text{red}} : M_{\text{red}} \to M_{\text{red}}$ by using the following commutative diagram:

$$\begin{array}{ccc} M_0 & \xrightarrow{\sigma} & M_0 \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ M_{\text{red}} & \xrightarrow[\sigma_{\text{red}}]{} & M_{\text{red}} \end{array}$$

Diagram 3.1: Commutative diagram induced by an involution

where $q : M_0 \to M_{\text{red}}$ is the quotient map. That σ_{red} is an involution is clear. Since $q^*\omega_{\text{red}} = i^*\omega$ and $\sigma^*\omega = -\omega$, by using the commutativity of Diagram 3.1, we can write

$$\begin{aligned} q^*(\sigma_{\text{red}}^*\omega_{\text{red}}) &= (\sigma_{\text{red}} \circ q)^*\omega_{\text{red}} \\ &= (q \circ \sigma)^*\omega_{\text{red}} \\ &= \sigma^*(q^*\omega_{\text{red}}) \\ &= \sigma^*(i^*\omega) \\ &= -i^*\omega \\ &= -q^*\omega_{\text{red}} \\ &= q^*(-\omega_{\text{red}}). \end{aligned}$$

Therefore, $q^*(\sigma_{\mathrm{red}}^*\omega_{\mathrm{red}}) = q^*(-\omega_{\mathrm{red}})$. The injectivity of q^* implies that $\sigma_{\mathrm{red}}^*\omega_{\mathrm{red}} = -\omega_{\mathrm{red}}$. That is, σ_{red} is anti-symplectic and the proof is complete. ■

Proposition 3.6. *Let $(M, \omega, G, \mu, \sigma, \phi)$ be a real Hamiltonian system where G is a compact Lie group and M is a compact connected manifold. If G acts freely on $M_0 = \mu^{-1}(0)$, then the real reduction $M^\sigma /\!\!/ G^\phi$ embeds as a Lagrangian submanifold of the real symplectic manifold $(M_{\mathrm{red}}, \omega_{\mathrm{red}}, \sigma_{\mathrm{red}})$ contained in the real locus $(M_{\mathrm{red}})^{\sigma_{\mathrm{red}}}$.*

Proof. First, we see from Lemma 3.5 that the induced map $\sigma_{\mathrm{red}} : M_{\mathrm{red}} \to M_{\mathrm{red}}$ is an anti-symplectic involution. Hence, by Proposition 3.3, the real locus $(M_{\mathrm{red}})^{\sigma_{\mathrm{red}}}$ is a Lagrangian submanifold of M_{red}.

The inclusion map $i_0 : M_0^\sigma \hookrightarrow M_0$ descents to a map $\psi : M_0^\sigma/G^\phi \to M_0/G$ which commutes the following diagram:

$$\begin{tikzcd} M_0^\sigma \arrow[r, "i_0"] \arrow[d, "q_\sigma"'] & M_0 \arrow[d, "q"] \\ M_0^\sigma/G^\phi \arrow[r, "\psi"'] & M_0/G \end{tikzcd}$$

Diagram 3.2: Commutative diagram induced by the inclusion

In this diagram, two maps $q : M_0 \to M_0/G$ and $q_\sigma : M_0^\sigma \to M_0^\sigma/G$ are quotient maps. It follows from commutative Diagrams 3.1 and 3.2 that $\mathrm{Im}(\psi) \subset (M_{\mathrm{red}})^{\sigma_{\mathrm{red}}}$ and

$$dq \circ di_0 = d\psi \circ dq_\sigma. \tag{3.27}$$

Because the action of G on M_0 is free, it easy to see that for any $x \in M_0^\sigma$, we have

$$\mathrm{Ker}\ (dq_\sigma) = \mathrm{Ker}\ (dq) \cap T_x M_0^\sigma. \tag{3.28}$$

If $v \in T_x M_0^\sigma$ and $d\psi(dq_\sigma(v)) = 0$, then it follows from (3.27) that

$$0 = d\psi(dq_\sigma(v)) = d(\psi \circ q_\sigma)(v) = d(q \circ i_0)(v) = dq(v).$$

Thus $dq(v) = 0$ and (3.28) implies that $dq_\sigma(v) = 0$. This means that $d\psi$ is 1-1 and ψ is an immersion.

Suppose that $x, y \in M_0^\sigma$ and $\psi(q_\sigma(x)) = \psi(q_\sigma(y))$. The commutativity of Diagram (3.2) shows that $q(x) = q(y)$ and thus there exists some $g \in G$ for which $y = gx$. This implies that

$$gx = y = \sigma(y) = \sigma(gx) = \phi(g)\sigma(x) = \phi(g)x.$$

But the action of G on M_0 is free, so $g = \phi(g)$; or equivalently, $g \in G^\phi$. It follows that $q_\sigma(x) = q_\sigma(y)$ which shows that ψ is injective. Since domain and codomain are compact manifolds, then ψ is a closed map and therefore ψ must be an embedding. This implies that ψ embeds the real reduction $M^\sigma /\!\!/ G^\phi$ into the real locus $(M_{\mathrm{red}})^{\sigma_{\mathrm{red}}}$ of the symplectic reduction M_{red}. Therefore the image of $M^\sigma /\!\!/ G^\phi$ under ψ is isotropic.

It remains to show that

$$\dim(M^\sigma /\!\!/ G^\phi) = \frac{1}{2}\dim M_{\text{red}}. \tag{3.29}$$

We saw in Remark 3.3 that the restriction of the moment map to the real locus is $\mu : M^\sigma \to \mathfrak{g}_-^*$. Denote the differential map of this restricted map by $\Phi : T_xM^\sigma \to \mathfrak{g}_-^*$; i.e., $\Phi(v) = d\mu_x(v)$. Since G acts freely on M_0, 0 is a regular value of μ and thus the map $d_x\mu : T_xM \to \mathfrak{g}^*$ is onto for any $x \in M_0$. This implies that Φ is surjective. On the other hand, be Proposition 2.56 $v \in \text{Ker }(\Phi)$ if and only if $v \in T_xM^\sigma$ and $v \in \text{Ker }(d\mu_x) = T_xM_0$. Therefore, $\text{Ker }(\Phi) = T_xM_0^\sigma$. By the rank-nullity theorem for the linear transformation Φ and Propositions 3.3 and 2.56, we can write

$$\begin{aligned}
\dim(M^\sigma /\!\!/ G^\phi) &= \dim M_0^\sigma - \dim G^\phi \\
&= \dim T_xM_0^\sigma - \dim \mathfrak{g}_+^* \\
&= \dim \text{Ker }(\Phi) - \dim \mathfrak{g}_+^* \\
&= \dim T_xM^\sigma - \dim \mathfrak{g}_-^* - \dim \mathfrak{g}_+^* \\
&= \frac{1}{2}(\dim T_xM - 2\dim \mathfrak{g}^*) \\
&= \frac{1}{2}(\dim M - 2\dim G) \\
&= \frac{1}{2}\dim M_{\text{red}}.
\end{aligned}$$

This proves (3.29). Therefore, we can consider the real reduction $M^\sigma /\!\!/ G^\phi$ as a Lagrangian submanifold of the symplectic reduction $(M_{\text{red}}, \omega_{\text{red}})$ and the proof is complete. ■

Definition 3.7. Let $i_0 : M_0^\sigma \hookrightarrow M^\sigma$ be the inclusion map. The induced map $i_0^* : H_{G^\phi}^*(M^\sigma; \mathbb{Z}_2) \to H_{G^\phi}^*(M_0^\sigma; \mathbb{Z}_2)$ is called the **real Kirwan map**.

When the action of G on M_0 is free, the projection map $\pi_{G^\phi} : EG^\phi \times_{G^\phi} M_0^\sigma \to M_0^\sigma/G^\phi$ induces the following natural isomorphism (see Proposition 2.15):

$$\pi_{G^\phi}^* : H_{G^\phi}^*(M_0^\sigma; \mathbb{Z}_2) \xrightarrow{\cong} H^*(M_0^\sigma/G^\phi; \mathbb{Z}_2). \tag{3.30}$$

By composing two maps $\pi_{G^\phi}^*$ and $i_0^* : H_{G^\phi}^*(M^\sigma; \mathbb{Z}_2) \to H_{G^\phi}^*(M_0^\sigma; \mathbb{Z}_2)$, we get a map

$$\pi_{G^\phi}^* \circ i_0^* : H_{G^\phi}^*(M^\sigma; \mathbb{Z}_2) \to H^*(M^\sigma /\!\!/ G^\phi; \mathbb{Z}_2). \tag{3.31}$$

In this case, we usually consider this composite map as the real Kirwan map.

Now, we formulate a real version of the reduction in stages for the product of Lie groups. Suppose that $G = G_1 \times G_2$ where G_i is a compact connected Lie group with Lie algebra $\mathfrak{g}_i^*$. Let $(M, \omega, G, \mu, \sigma, \phi)$ be a real Hamiltonian system. In this case, we can assume $\phi = (\phi_1, \phi_2)$ and $\mu = (\mu_1, \mu_2)$ in which $\phi_i : G_i \to G_i$ and $\mu_i : M \to \mathfrak{g}_i^*$. Set $Z_1 = \mu_1^{-1}(0)$, $M_{\text{red}} = Z_1/G_1$, $q_1 : Z_1 \to M_{\text{red}}$ and $i : Z_1 \hookrightarrow M$.

Proposition 3.7 (**Real Reduction in Stages**). *If G_1 acts freely on Z_1, then the action of G_2 on the reduction space $(M_{\text{red}}, \omega_{\text{red}})$ is symplectic and there exist two maps $\mu_{\text{red}} : M_{\text{red}} \to \mathfrak{g}_2^*$ and $\sigma_{\text{red}} : M_{\text{red}} \to M_{\text{red}}$ such that $\mu_{\text{red}} \circ q_1 = \mu_2 \circ i$ and $\sigma_{\text{red}} \circ q_1 = q_1 \circ \sigma$. In particular, $(M_{\text{red}}, \omega_{\text{red}}, G_2, \mu_{\text{red}}, \sigma_{\text{red}}, \phi_2)$ is a real Hamiltonian system.*

Proof. Since μ is equivariant, the map $\mu_{\text{red}} : M_{\text{red}} \to \mathfrak{g}_2^*$ defined by $\mu_{\text{red}}(q_1(x)) = \mu_2(x)$ is well-defined. By Proposition 2.60, $(M_{\text{red}}, \omega_{\text{red}}, G_2, \mu_{\text{red}})$ is a Hamiltonian system. Since $\mu \circ \sigma = -\phi^* \circ \mu$, it follows that $\mu_i \circ \sigma = -\phi_i^* \circ \sigma$. Thus, the map $\sigma_{\text{red}} : M_{\text{red}} \to M_{\text{red}}$ defined by $\sigma_{\text{red}}(q_1(x)) = q_1(\sigma(x))$ is also well-defined. We show that $(\sigma_{\text{red}}, \phi_2)$ is a real pair. Since G_1-action and G_2-action on M commute, we have $g_2 \circ q_1 = q_1 \circ g_2$ for any $g_2 \in G_2$ and thus

$$\sigma_{\text{red}}(g_2 q_1(x)) = \sigma_{\text{red}}(q_1(g_2 x)) = q_1(\sigma(g_2 x)) = q_1(\phi_2(g_2)\sigma(x)) = \phi_2(g_2)\sigma_{\text{red}}(q_1(x)).$$

On the other hand,

$$\mu_{\text{red}}(\sigma_{\text{red}}(q_1(x))) = \mu_{\text{red}}(q_1(\sigma(x))) = \mu_2(\sigma(x)) = -\phi_2^*(\mu_2(x)) = -\phi_2^* \circ \mu_{\text{red}}(q_1(x)).$$

Therefore, the compatibility conditions in (3.8) are satisfied and $(\sigma_{\text{red}}, \phi_2)$ is a real pair. Thus the tuple $(M_{\text{red}}, \omega_{\text{red}}, G_2, \mu_{\text{red}}, \sigma_{\text{red}}, \phi_2)$ is a real Hamiltonian system. This completes the proof. ∎

Example 3.14. Consider the real Hamiltonian system in Example 3.9:

$$(M_{k\times n}(\mathbb{C}), \omega, \mathrm{U}(k) \times \mathrm{U}(n), \mu_k \times \mu_n, \sigma, \phi_k \times \phi_n).$$

Since $\mathrm{U}(k)$ acts freely on $\mu_k^{-1}(0) = V_k(\mathbb{C}^n)$, the symplectic reduction of the $\mathrm{U}(k)$-action on $\mu_k^{-1}(0)$ is $(\mathrm{Gr}_k(\mathbb{C}^n), \omega_{k,n})$ (see Example 2.32). Thus Proposition 3.7 follows that there exist a moment map $\mu_{k,n} : \mathrm{Gr}_k(\mathbb{C}^n) \to \mathfrak{u}(n)^*$ and an anti-symplectic involution $\sigma_{k,n} : \mathrm{Gr}_k(\mathbb{C}^n) \to \mathrm{Gr}_k(\mathbb{C}^n)$ such that $(\mathrm{Gr}_k(\mathbb{C}^n), \omega_{k,n}, \mathrm{U}(n), \mu_{k,n}, \sigma_{k,n}, \phi_n)$ is a real Hamiltonian system. In this case, the moment map is defined by (2.67) and the induced involution $\sigma_{k,n}$ is defined by $\sigma_{k,n}(V) = \overline{V}$ for any $V \in G_k(\mathbb{C}^n)$, where $\overline{V}$ is the complex conjugate of subspace V. Moreover, the real subgroup is the orthogonal group $\mathrm{O}(n)$ and the real locus is the real Grassmanian $\mathrm{Gr}_k(\mathbb{R}^n)$ (see Example 3.2).

Example 3.15. Let $r, n > 1$ be natural numbers and $M = \mathrm{Gr}_{l_1}(\mathbb{C}^n) \times \cdots \times \mathrm{Gr}_{l_r}(\mathbb{C}^n)$ be a product of complex Grassmannians where $1 \leq l_1 \leq \cdots \leq l_r < n$ are positive integers. Consider the diagonal action of the unitary group $\mathrm{U}(n)$ on M. As we saw in Example 3.14, for any $j = 1, ..., r$, we have a real Hamiltonian system

$$\Big(\mathrm{Gr}_{l_j}(\mathbb{C}^n), \omega_{l_j,n}, \mathrm{U}(n), \mu_{l_j,n}, \sigma_{l_j,n}, \phi\Big).$$

So by Proposition 2.58, the diagonal action of $\mathrm{U}(n)$ on M is symplectic and the map $\mu : M \to \mathfrak{u}(n)$ defined by

$$\mu(V_1, ..., V_r) = \sqrt{-1}\sum_{j=1}^{r} \mathrm{Pr}_{V_j} + (\frac{\sum_{j=1}^{r} l_j}{n\sqrt{-1}})\mathrm{Id}_n, \tag{3.32}$$

is a moment map. In this case, the zero level set $M_0 = \mu^{-1}(0)$ is

$$M_0 = \Big\{(V_1, ..., V_r) \in M \mid \sum_{j=1}^{r} \mathrm{Pr}_{V_j} = (\frac{\sum_{j=1}^{r} l_j}{n})\mathrm{Id}_n\Big\}. \tag{3.33}$$

Let $\phi : \mathrm{U}(n) \to \mathrm{U}(n)$ be the complex conjugation and $\sigma : M \to M$ be defined as follows:

$$\sigma(V_1, ..., V_r) = (\overline{V}_1, ..., \overline{V}_r), \ \forall (V_1, ..., V_r) \in M, \tag{3.34}$$

where $\overline{V}_j$ is the complex conjugate of V_j. An easy computation shows that (σ, ϕ) is a real pair and the tuple

$$\Big(M = \mathrm{Gr}_{l_1}(\mathbb{C}^n) \times \cdots \times \mathrm{Gr}_{l_r}(\mathbb{C}^n), \omega, \mathrm{U}(n), \mu, \sigma, \phi\Big)$$

is a real Hamiltonian system. In this case, the real subgroup is the orthogonal group $\mathrm{O}(n)$ and the real locus M^σ is a product of real Grassmannians:

$$M^\sigma = \mathrm{Gr}_{l_1}(\mathbb{R}^n) \times \cdots \times \mathrm{Gr}_{l_r}(\mathbb{R}^n). \tag{3.35}$$

Remark 3.5. Suppose that the numbers $\sum_{j=1}^{r} l_j$ and n are coprime in Example 3.15. Let $A \in \mathrm{U}(n)$ be an element of the stabilizer subgroup of an element $V = (V_1, ..., V_r)$ in the zero level set M_0 and E_λ be the eigenspace of an eigenvalue λ of A. Since $AV_j = V_j$ and A is unitary, we have $AV_j^\perp = V_j^\perp$. This follows that $A\mathrm{Pr}_{V_j} = \mathrm{Pr}_{V_j}A$ which implies that $\mathrm{Pr}_{V_j}(E_\lambda) \subset E_\lambda$. As a result of this, we see that

$$\mathrm{Pr}_{E_j}\mathrm{Pr}_{V_j} = \mathrm{Pr}_{E_\lambda \cap V_j}. \tag{3.36}$$

By (3.33) and (3.36), we get

$$\sum_{j=1}^{r} \mathrm{Pr}_{E_\lambda \cap V_j} = (\frac{\sum_{j=1}^{r} l_j}{n})\mathrm{Pr}_{E_\lambda}. \tag{3.37}$$

Let $\dim E_\lambda = m_\lambda$ and $\dim E_\lambda \cap V_j = m_{\lambda,j}$. By taking trace of (3.37), we get

$$\sum_{j=1}^{r} m_{\lambda,j} = (\frac{\sum_{j=1}^{r} l_j}{n})m_\lambda. \tag{3.38}$$

Since $\sum_{j=1}^{r} l_j$ and n are coprime and $m_\lambda \leq n$, it follows from (3.38) that the only possible value for m_λ is n; i.e., all vectors in $\mathbb{C}^n$ are eigenvectors of A. This means A is a scalar matrix. Thus we have shown that the stabilizer subgroup of any element $V \in M_0$ is the central subgroup $D\mathrm{U}(n) = \{\lambda \mathrm{Id} \mid \lambda \in S^1\}$, and therefore the action of the projective unitary group $P\mathrm{U}(n) = \mathrm{U}(n)/D\mathrm{U}(n)$ on the zero level set M_0 is free. We will use this observation in Chapter 8.

CHAPTER 4

Morse stratification for Real Hamiltonians

> "*Every mathematician has a secret weapon. Mine is Morse theory.*"
>
> —*Raoul Bott (1923-2005)*

In this chapter, we construct a real Morse stratification for a real Hamiltonian system. To do so, we adapt Kirwan's approach which concerns the norm squared of the moment map as a Morse function. In the real case, besides the invariance, we need to impose extra conditions on involutions and metrics. Here, we prove several technical lemmas and propositions which are needed for our main theorem in this chapter (Theorem 4.6). This theorem is our first goal in this thesis and is the first step to prove a real equivariant perfection theorem.

4.1 Morse Stratification for Hamiltonians

This section summarizes the main results of Kirwan's work on the Morse stratification induced by the norm squared of the moment map in a Hamiltonian system. We will use these ideas to get a real Morse stratification for a real Hamiltonian system.

Let $\mathcal{H} = (M, \omega, G, \mu)$ be a Hamiltonian G-system in which G is a compact connected Lie group and M is a compact connected $2n$-manifold. Choose an Ad-invariant inner product on the Lie algebra $\mathfrak{g}$ and a G-invariant Riemannian metric on M compatible with ω. Let $||.||$ be the norm induced by the inner product and define the norm squared of the moment map $f : M \to \mathbb{R}$ by

$$f(p) = ||\mu(p)||^2, \ \forall p \in M. \tag{4.1}$$

Also for any $X \in \mathfrak{g}$, define the component of the moment map along X, $\mu^X : M \to \mathbb{R}$, by

$$\mu^X(p) = \langle \mu(p), X \rangle, \ \forall p \in M. \tag{4.2}$$

Here, $\langle \ , \ \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the natural pairing between the Lie algebra and its dual.

Kirwan [42] showed that the function $f = ||\mu||^2$ is a Morse-Kirwan function and generates a smooth Morse stratification for M (see [42], Theorem 10.4). Here, we list Kirwan's main results about the norm squared of a moment map $f = ||\mu||^2$ that will be used throughout this chapter.

Let $T \subset G$ be a maximal torus and $\beta \in \mathfrak{t} = \mathrm{Lie}(T)$. By using our invariant inner product on G, we can identify the Lie algebra $\mathfrak{g}$ with the dual Lie algebra $\mathfrak{g}^*$. By Proposition 2.57, the component map $\mu^\beta : M \to \mathbb{R}$ is a Morse-Bott function whose nondegenerate critical manifolds are even dimensional with even indices. Let C_{μ^β} be the critical set of μ^β and Z_β (possibly disconnected) be the union of those nondegenerate critical manifolds of μ^β on each of which μ^β takes the value $||\beta||^2$; i.e.,

$$Z_\beta = C_{\mu^\beta} \cap \Big[(\mu^\beta)^{-1}(||\beta||^2)\Big]. \tag{4.3}$$

For any $m = 0, ..., 2n = \dim M$, set $Z_{\beta,m} = \{x \in Z_\beta \mid \mathrm{Ind}_x(\mu^\beta) = m\}$ and let $G_\beta = \{g \in G \mid \mathrm{Ad}_g \beta = \beta\}$ be the stabilizer subgroup of G induced by β under the Ad-action. It is easy to see that $Z_{\beta,m}$ is G_β-invariant. Denote the Lie algebra of G_β by $\mathfrak{g}_\beta$ and consider the orthogonal projection $\mathrm{Pr}_\beta : \mathfrak{g}^* \to \mathfrak{g}^*_\beta$. Kirwan has proved the following results in her doctoral thesis [42].

K1. *The critical set C_f of f is a finite collection of disjoint invariant closed (possibly disconnected) subsets $\{C_{\beta,m} \mid \beta \in \Lambda, \ 0 \le m \le \dim M\}$ such that the indexing set Λ is a finite subset of a positive Weyl chamber in the Lie algebra of the maximal torus T:*

$$C_f = \coprod_{\beta,m} C_{\beta,m}, \tag{4.4}$$

In this case, f takes a constant value on each critical subset $C_{\beta,m}$ and the Morse index of f along $C_{\beta,m}$ is a constant number $d(\beta,m)$. Also the image of $C_{\beta,m}$ under the moment map μ is the coadjoint orbit of β in $\mathfrak{g}^*$, denoted by $\mathcal{O}_\beta$.

K2. *For any β and m, there exists a Hamiltonian subsystem $\mathcal{H}_{\beta,m} = (Z_{\beta,m}, \omega, G_\beta, \mu_{\beta,m})$ where $\mu_{\beta,m} : Z_{\beta,m} \to \mathfrak{g}^*_\beta$ is defined by $\mu_{\beta,m} = \mathrm{Pr}_\beta \circ \mu - \beta$. In this case, $\mu^{-1}_{\beta,m}(0) = Z_{\beta,m} \cap \mu^{-1}(\beta)$ and*

$$C_{\beta,m} = G.\mu^{-1}_{\beta,m}(0) = G.[Z_{\beta,m} \cap \mu^{-1}(\beta)]. \tag{4.5}$$

K3. *Suppose that the collection of diffeomorphisms $\{\psi_t : M \to M | t \in \mathbb{R}\}$ is the negative gradient flow of f with respect to the G-invariant metric. For each β and m, set*

$$S_{\beta,m} = \{x \in M \mid \exists\ t_n \to +\infty\ s.t.\ \lim_{t\to+\infty} \psi_{t_n}(x) \in C_{\beta,m}\}. \tag{4.6}$$

Then $\{S_{\beta,m}\}$ is a G-invariant Morse stratification of M such that each $S_{\beta,m}$ is a locally closed invariant submanifold of M whose codimension is equal to the Morse index of f along $C_{\beta,m}$. In addition, there exists a partial order on the indexing set Λ such that

$$\beta_1 < \beta_2 \Leftrightarrow f(C_{\beta_1}) < f(C_{\beta_2}), \tag{4.7}$$

where $C_\beta = \coprod_{m=0}^{2n} C_{\beta,m}$.

K4. *For any critical subset $C_{\beta,m}$, there exists a symplectic invariant submanifold $\Sigma_{\beta,m} \subseteq M$ with orientable normal bundle which contains $C_{\beta,m}$ and coincides with the stratum $S_{\beta,m}$ in an open neighborhood of $C_{\beta,m}$. In particular, the restriction of f to $\Sigma_{\beta,m}$ takes its minimum along $C_{\beta,m}$.*

K5. *For any $\beta \in \Lambda$ and $m \in \{0, ..., \dim M\}$, the index of f along the critical subset $C_{\beta,m}$ is*

$$d(\beta,m) = m - \dim G + \dim G_\beta. \tag{4.8}$$

Moreover, there exists an isomorphism in G-equivariant cohomology with rational coefficients:

$$H^*_G(S_{_{\beta,m}}; \mathbb{Q}) \cong H^*_G(C_{\beta,m}; \mathbb{Q}). \tag{4.9}$$

In fact, by the Duistermaat theorem (see Proposition 2.61), each $S_{\beta,m}$ deformation retracts onto corresponding critical subset $C_{\beta,m}$.

K6. *There exists a recursive formula for the rational equivariant Betti numbers of the submanifold $M_0 = \mu^{-1}(0)$ as follows:*

$$\begin{aligned}\mathbf{P}_G(\mu^{-1}(0), t; \mathbb{Q}) = {} & \mathbf{P}(M, t; \mathbb{Q})\mathbf{P}(BG, t; \mathbb{Q}) \\ & - \sum_{\substack{\beta\neq 0 \\ 0\leq m\leq \dim M}} t^{d(\beta,m)} \mathbf{P}_{G_\beta}(\mu^{-1}_{\beta,m}(0), t; \mathbb{Q}),\end{aligned} \tag{4.10}$$

where $d(\beta, m)$ is the Morse index of f along $C_{\beta,m}$ and $\mu_{\beta,m}$ is the moment map of the Hamiltonian subsystem $\mathcal{H}_{\beta,m} = (Z_{\beta,m}, \omega, G_\beta, \mu_{\beta,m})$ defined in K2.

Remark 4.1. The great advantage of the recursive formula (4.10) is that to compute the equivariant Betti numbers of the zero level set of the moment map in a Hamiltonian system, we just need to compute the equivariant Betti numbers of the zero level sets of the moment maps of a finite number of Hamiltonian subsystems.

4.2 Morse Stratification for Real Hamiltonians

In this section, we use the results in the previous section and give a Morse stratification for a real Hamiltonian system.

Consider a real Hamiltonian G-system $\mathcal{H} = (M, \omega, G, \mu, \sigma, \phi)$ where M is a compact connected $2n$-manifold and G is a compact connected Lie group. In this case, we have compatible conditions in (3.8).

Lemma 4.1. *There exist a G-invariant Riemannian metric on M compatible with ω and an* Ad*-invariant inner product on $\mathfrak{g}$ which make σ and ϕ into isometries respectively.*

Proof. Define the group homomorphism $\Phi : \mathbb{Z}_2 \to \mathrm{Aut}(G)$ by $\Phi(1) = \mathrm{Id}_G$ and $\Phi(-1) = \phi$. Set $K = \mathbb{Z}_2 \rtimes_\Phi G$, the semi-direct product with respect to Φ. By Proposition 2.32, K is a compact Lie group. This group acts on G and M via ϕ and σ respectively:

$$\begin{cases} (1,g).h = gh \text{ and } (-1,g).h = \phi(gh), & \forall g, h \in G, \\ (1,g).p = gp \text{ and } (-1,g).p = \sigma(gp), & \forall g \in G, \ \forall p \in M. \end{cases} \tag{4.11}$$

Clearly, $\phi(g) = (-1,e).g$ and $\sigma(p) = (-1,e).p$. Since K is compact, by Proposition 2.46 there exists an Ad-invariant inner product on $\mathfrak{k}$ and a K-invariant Riemannian metric $\{(\ , \ )_p : T_pM \times T_pM \to \mathbb{R} \mid p \in M\}$ on M which is compatible with ω. Clearly, the inner product and the metric are invariant under ϕ and σ respectively. This proves the lemma. ∎

From now on, we fix a G-invariant metric $\{(\ , \ )_p : T_pM \times T_pM \to \mathbb{R} \mid p \in M\}$ compatible with ω and an Ad-invariant inner product $\langle \ , \ \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ as in Lemma 4.1. Denote the induced norm on $\mathfrak{g}$ by $|| \ ||$. We identify the Lie algebra with its dual by using this inner product, $\mathfrak{g} \cong \mathfrak{g}^*$.

Let M^σ be the real locus and G^ϕ be the real subgroup. It easily follows from (3.8) that M^σ is invariant under the action of the real subgroup G^ϕ. Let $p \in M$. Since ϕ^* is an isometry, it preserves the norm. Thus it follows from (3.8) that

$$f \circ \sigma(p) = ||\mu \circ \sigma(p)||^2 = || - \phi^*(\mu(p))||^2 = ||\mu(p)||^2 = f(p).$$

Hence, we get

$$f \circ \sigma = f. \tag{4.12}$$

Denote the restriction of f to the real locus M^σ by $f^\sigma : M^\sigma \to \mathbb{R}$ and let ψ_t be the negative gradient flow of f. We denote the critical sets of f and f^σ by C_f and C_{f^σ} respectively.

Proposition 4.2. *Let f, M^σ, f^σ, C_f and C_{f^σ} be as above. Then the following are satisfied.*

1. *For any critical subset $C_{\beta,m}$ of f, defined in K1, if $C_{\beta,m} \cap M^\sigma \neq \emptyset$, then $\sigma(C_{\beta,m}) = C_{\beta,m}$. In particular, the critical set C_f is preserved by σ.*

2. *For the critical set C_{f^σ}, we have $C_{f^\sigma} = C_f \cap M^\sigma$.*

3. *C_{f^σ} is a finite union of disjoint G^ϕ-invariant closed subsets of M^σ on each of which f^σ takes a constant value.*

4. *M^σ is invariant under the flow ψ_t of the negative gradient $-\nabla f$. In particular, if $p \in M^\sigma$, then the limit point of p is in M^σ.*

5. *Let $S^\sigma_{\beta,m} = S_{\beta,m} \cap M^\sigma$ and $S_{C^\sigma_{\beta,m}}$ be the stratum of f^σ induced by $C^\sigma_{\beta,m} = C_{\beta,m} \cap M^\sigma$. If $C^\sigma_{\beta,m} \neq \emptyset$, then $S^\sigma_{\beta,m} = S_{C^\sigma_{\beta,m}}$; i.e., the strata of f^σ are the intersection of the strata of f with the real locus M^σ.*

Proof. By (4.12), $df = df \circ d\sigma$. Since $d\sigma(p)$ is an isomorphism, then $df_p = 0$ if and only if $df_{\sigma(p)} \circ d\sigma_p = 0$. Since σ sends path-components to path-components, it follows that $C_{\beta,m} = \sigma(C_{\beta,m})$. This clearly follows that C_f is preserved by σ which proves part 1.

Since f^σ is the restriction of f to M^σ, $C_f \cap M^\sigma \subset C_{f^\sigma}$. Conversely, let $p \in C_{f^\sigma}$ and $v \in T_pM$. Then Proposition 3.3, (4.12) and the fact that $v + d\sigma_p(v) \in T_pM^\sigma$ imply that

$$\begin{aligned} df_p(v) &= df_p\Big(v + d\sigma_p(v) - d\sigma_p(v)\Big) \\ &= df_p\Big(v + d\sigma_p(v)\Big) - df_p\Big(d\sigma_p(v)\Big) \\ &= 0 - d(f \circ \sigma)_p(v) \\ &= -df_p(v). \end{aligned}$$

Therefore $df_p = 0$ which follows that $C_{f^\sigma} \subset C_f \cap M^\sigma$. This proves part 2.

Since $C^\sigma_{\beta,m} = C_{\beta,m} \cap M^\sigma \neq \emptyset$, each $C^\sigma_{\beta,m}$ is a closed G^ϕ-invariant subset of M^σ and from part 2 and (4.4), it follows that

$$C_{f^\sigma} = \coprod_{\beta,m} C^\sigma_{\beta,m}, \tag{4.13}$$

also f^σ takes the same constant value $f(C_{\beta,m})$ on $C^\sigma_{\beta,m}$. This proves part 3.

Let $p \in M$. Since $df_p = df_{\sigma(p)} \circ d\sigma_p$, $d\sigma_p \circ d\sigma_{\sigma(p)} = \mathrm{Id}$ and σ is an isometry, a simple calculation shows that

$$d\sigma_p(\nabla f(p)) = \nabla f(\sigma(p)), \tag{4.14}$$

which implies that

$$\sigma(\psi_t(p)) = \psi_t(\sigma(p)), \ \forall t \in \mathbb{R}. \tag{4.15}$$

If $p \in M^\sigma$, then (4.14) follows that $\nabla f(p) = d\sigma_p(\nabla f(p))$ which implies that $-\nabla f(p) \in T_pM^\sigma$. Therefore, M^σ is invariant under the flow ψ_t of $-\nabla f$. But M^σ is closed, so the limit point of p is also in M^σ. This proves part 4.

By the definition, it is clear that $S_{C^\sigma_{\beta,m}} \subset S^\sigma_{\beta,m}$. Conversely, let $p \in S^\sigma_{\beta,m}$. So $p \in M^\sigma$ and there exists a sequence $t_n \to +\infty$ such that $\lim_{n\to+\infty} \psi_{t_n}(p) \in C_{\beta,m}$. Since $\sigma(p) = p$, it follows from (4.15) that $\lim_{n\to+\infty} \psi_{t_n}(p) \in M^\sigma$, and thus $\lim_{n\to+\infty} \psi_{t_n}(p) \in C_{\beta,m} \cap M^\sigma = C^\sigma_{\beta,m}$. That is, $p \in S_{C^\sigma_{\beta,m}}$ which implies that $S^\sigma_{\beta,m} \subset S_{C^\sigma_{\beta,m}}$. Therefore, we have $S_{C^\sigma_{\beta,m}} = S^\sigma_{\beta,m}$. This proves part 5 and the proof is complete. ∎

Proposition 4.3. *The collection $\{S^\sigma_{\beta,m} \mid \beta, m\}$ is an invariant Morse stratification for the restricted function f^σ and each stratum $S^\sigma_{\beta,m}$ deformation retracts onto $C^\sigma_{\beta,m}$. Moreover, the codimension of each stratum $S^\sigma_{\beta,m}$ is a constant number and equals to*

$$\operatorname{codim}\ S^\sigma_{C_{\beta,m}} = \frac{1}{2} \operatorname{codim}\ S_{\beta,m}. \tag{4.16}$$

In particular, if $d(C_{\beta,m})$ and $d(C^\sigma_{\beta,m})$ are the indices of functions f and f^σ along $C_{\beta,m}$ and $C^\sigma_{\beta,m}$ respectively, then

$$d(C^\sigma_{\beta,m}) = \frac{1}{2} d(C_{\beta,m}). \tag{4.17}$$

Proof. By part 3 of Proposition 4.2, $p \in S^\sigma_{\beta,m}$ if and only if for some sequence $t_n \to +\infty$, $\lim_{n\to+\infty} \psi_{t_n}(p) \in C^\sigma_{\beta,m}$. On the other hand, since M^σ is closed and $\{S_{\beta,m} \mid \beta, m\}$ is a G-invariant Morse stratification for M, we have

$$\begin{aligned}
\overline{S^\sigma}_{\beta,m} &\subset M^\sigma \cap \overline{S}_{\beta,m} \\
&\subset M^\sigma \cap \Big(\bigcup_{(\beta',m')\geq(\beta,m)} S_{\beta',m'} \Big) \\
&\subset \bigcup_{(\beta',m')\geq(\beta,m)} (M^\sigma \cap S_{\beta',m'}) \\
&\subset \bigcup_{(\beta',m')\geq(\beta,m)} S^\sigma_{\beta',m'}).
\end{aligned}$$

It follows from the definition that $\{S^\sigma_{\beta,m} \mid \beta, m\}$ is a G^ϕ-invariant Morse stratification for M^σ. By Proposition 2.61 and part 1 of Proposition 4.2, the restriction of the flow ψ_t to the real locus M^σ induces a deformation retraction of $S^\sigma_{\beta,m}$ onto $C^\sigma_{\beta,m}$.

According to K4, $S_{\beta,m}$ coincides with a symplectic manifold $\Sigma_{\beta,m}$ in an open neighborhood of $C_{\beta,m}$. So for any $p \in C^\sigma_{\beta,m}$, the triple $(T_p S_{\beta,m}, \omega_p, d\sigma_p)$ is an anti-symplectic vector space. Let $(T_p S_{\beta,m})^\sigma$ be the real locus of $T_p S_{\beta,m}$. It is easy to see that

$$(T_p S_{\beta,m})^\sigma = T_p(S^\sigma_{\beta,m}). \tag{4.18}$$

By Proposition 3.3, $(T_p S_{\beta,m})^\sigma$ is a Lagrangian subspace of $T_p S_{\beta,m}$ and (4.18) implies that

$$\dim\ S_{\beta,m} = \dim\ T_p S_{\beta,m} = 2 \dim\ (T_p S_{\beta,m})^\sigma = 2 \dim\ T_p(S^\sigma_{\beta,m}) = 2 \dim\ S^\sigma_{\beta,m}.$$

This proves (4.16).

Since the indices of f and f^σ along $C_{\beta,m}$ and $C^\sigma_{\beta,m}$ are the codimensions of $S_{\beta,m}$ and $S^\sigma_{\beta,m}$, respectively, (4.16) implies (4.17) and the proof is complete. ∎

Let $\mathfrak{g}^*_- = \{\xi \in \mathfrak{g}^* \mid \phi^*(\xi) = -\xi\}$ be the -1-eigenspace. For any $\beta \in \mathfrak{g}^*$, let $\mathcal{O}^-_\beta = \{\mathrm{Ad}^*_g\beta \mid g \in G^\phi\}$ and $\mathcal{O}_\beta = \{\mathrm{Ad}^*_g\beta \mid g \in G\}$ be the orbits of β with respect to the coadjoint actions of G^ϕ and G, respectively.

Proposition 4.4. *If $\beta \in \mathfrak{g}^*$ and $\mathcal{O}_\beta \cap \mathfrak{g}^*_- \neq (0)$, then the following statements are satisfied.*

1. *There exist $\beta_1, ..., \beta_{k_\beta} \in \mathfrak{g}^*_- \cap \mathcal{O}_\beta$ such that*

$$\mathcal{O}_\beta \cap \mathfrak{g}^*_- = \coprod_{i=1}^{k_\beta} \mathcal{O}^-_{\beta_i}, \tag{4.19}$$

where $\beta_i = \mathrm{Ad}^*_{g_i}\beta$, for some $g_i \in G$.

2. *If for each β_i and m we set*

$$\begin{cases} Z_{\beta_i,m} = g_i Z_{\beta,m} = \{g_i x \mid x \in Z_{\beta,m}\}, \\ Z^\sigma_{\beta_i,m} = Z_{\beta_i,m} \cap M^\sigma, \end{cases} \tag{4.20}$$

then for the critical subset $C_{\beta,m}$ we have

$$\mu^{-1}(\mathcal{O}^-_{\beta_i}) \cap C^\sigma_{\beta,m} = G^\phi.(Z^\sigma_{\beta_i,m} \cap \mu^{-1}(\beta_i)), \tag{4.21}$$

and in particular

$$C^\sigma_{\beta,m} = \coprod_{i=1}^{k_C} G^\phi.(Z^\sigma_{\beta_i,m} \cap \mu^{-1}(\beta_i)). \tag{4.22}$$

Proof. Choose $\alpha \in \mathcal{O}_\beta \cap \mathfrak{g}^*_-$. We claim that

$$T_\alpha\mathcal{O}^-_\alpha = (T_\alpha\mathcal{O}_\beta) \cap \mathfrak{g}^*_-. \tag{4.23}$$

Suppose that $\xi \in (T_\alpha\mathcal{O}_\beta) \cap \mathfrak{g}^*_-$. Then $\xi \in \mathfrak{g}^*_-$ and it follows from Proposition 2.47 that $\xi = \mathrm{ad}^*_X\alpha$, for some $X \in \mathfrak{g}$. Since $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, we have $X = X_+ + X_-$ where $X_+ \in \mathfrak{g}_+$ and $X_- \in \mathfrak{g}_-$. Thus $\xi = \mathrm{ad}^*_{X_+}\alpha + \mathrm{ad}^*_{X_-}\alpha$. Since $\mathrm{ad}^*_{X_+}\alpha \in \mathfrak{g}^*_-$ and $\mathrm{ad}^*_{X_-}\alpha \in \mathfrak{g}^*_+$, it follows that $\mathrm{ad}^*_{X_-}\alpha = 0$ and thus $\xi \in T_\alpha\mathcal{O}^-_\alpha$. Therefore, we have $(T_\alpha\mathcal{O}_\beta) \cap \mathfrak{g}^*_- \subset T_\alpha\mathcal{O}^-_\alpha$. Conversely, let $\xi \in T_\alpha\mathcal{O}^-_\alpha$. So for some $X \in \mathfrak{g}_+$, we have $\xi = \mathrm{ad}^*_X(\alpha)$. It follows from (3.10) that

$$\phi^*(\xi) = \phi^* \circ \mathrm{ad}^*_X(\alpha) = \mathrm{ad}^*_{\phi_*(X)}(\phi^*(\alpha)) = -\mathrm{ad}^*_X(\alpha) = -\xi.$$

This implies that $T_\alpha\mathcal{O}^-_\beta \subset (T_\alpha\mathcal{O}_\beta) \cap \mathfrak{g}^*_-$ which proves (4.23) (see Figure 4.1).

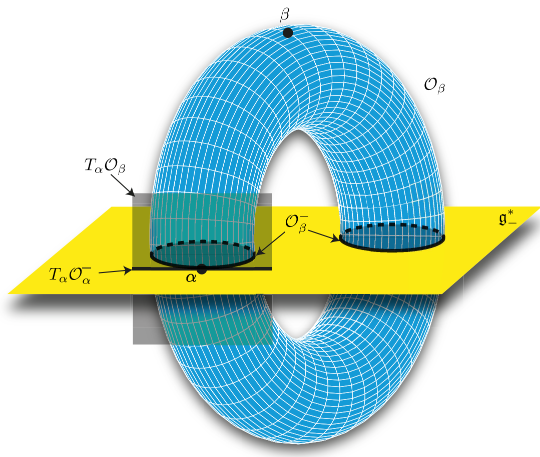

Figure 4.1: Tangent space of coadjoint orbit

On one hand, (4.23) says that $\mathcal{O}_\alpha^-$ is open in $\mathcal{O}_\beta \cap \mathfrak{g}_-^*$. On the other hand, $\mathcal{O}_\alpha^-$ is a closed subset because it is an orbit of a compact Lie group. Thus $\mathcal{O}_\alpha^-$ is a union of connected components of $\mathcal{O}_\beta \cap \mathfrak{g}_-^*$. Since $\mathcal{O}_\beta$ is compact so $\mathcal{O}_\beta \cap \mathfrak{g}_-^*$ is also compact. Therefore, the number of connected components of $\mathcal{O}_\beta \cap \mathfrak{g}_-^*$ must be finite. So we can choose a finite number of elements $\beta_1, ..., \beta_{k_\beta}$ in $\mathcal{O}_\beta \cap \mathfrak{g}_-^*$ such that $\mathcal{O}_\beta \cap \mathfrak{g}_-^* = \coprod_{i=1}^{k_\beta} \mathcal{O}_{\beta_i}^-$. Since $\beta_i \in \mathcal{O}_\beta$, there exists some $g_i \in G$ for which $\mathrm{Ad}_{g_i}^* \beta = \beta_i$. This proves part 1.

Since $\mu : M^\sigma :\to \mathfrak{g}_-^*$ is G^ϕ-equivariant, we can easily see that $\mu^{-1}(\beta_i) = g_i \mu^{-1}(\beta)$ and $\mu^{-1}(\mathcal{O}_{\beta_i}^-) = G^\phi.\mu^{-1}(\beta_i)$. Thus

$$G^\phi.(Z_{\beta_i,m}^\sigma \cap \mu^{-1}(\beta_i)) \subseteq G^\phi.\mu^{-1}(\beta_i) = \mu^{-1}(\mathcal{O}_{\beta_i}^-). \tag{4.24}$$

Also

$$\begin{aligned} G^\phi.(Z_{\beta_i,m}^\sigma \cap \mu^{-1}(\beta_i)) &\subseteq [G^\phi.(Z_{\beta_i,m} \cap \mu^{-1}(\beta_i))] \cap M^\sigma \\ &\subseteq [G.(Z_{\beta,m} \cap \mu^{-1}(\beta))] \cap M^\sigma \\ &= C_{\beta,m} \cap M^\sigma \\ &= C_{\beta,m}^\sigma. \end{aligned}$$

So

$$G^\phi.(Z_{\beta_i,m}^\sigma \cap \mu^{-1}(\beta_i)) \subseteq C_{\beta,m}^\sigma. \tag{4.25}$$

It follows from (4.24) and (4.25) that $G^\phi.(Z_{\beta_i,m}^\sigma \cap \mu^{-1}(\beta_i)) \subseteq \mu^{-1}(\mathcal{O}_{\beta_i}^-) \cap C_{\beta,m}^\sigma$.

For the other inclusion, we can write

$$\begin{aligned} \mu^{-1}(\mathcal{O}_{\beta_i}^-) \cap C_{\beta,m}^\sigma &= [G^\phi.\mu^{-1}(\beta_i)] \cap C_{\beta,m}^\sigma \\ &= [G^\phi.\mu^{-1}(\beta_i)] \cap [G.(Z_{\beta,m} \cap \mu^{-1}(\beta))] \cap M^\sigma \\ &= [G^\phi.\mu^{-1}(\beta_i)] \cap [G.(Z_{\beta_i,m} \cap \mu^{-1}(\beta_i))] \cap M^\sigma \\ &\subseteq G^\phi.(Z_{\beta_i,m} \cap \mu^{-1}(\beta_i)) \cap M^\sigma \\ &= G^\phi.(Z_{\beta_i,m}^\sigma \cap \mu^{-1}(\beta_i)). \end{aligned}$$

This proves (4.21). Since $\mu(C_{\beta,m}) \subseteq \mathcal{O}_\beta$ and $\mu(M^\sigma) \subset \mathfrak{g}_-^*$, it follows that

$$C_{\beta,m}^\sigma = C_{\beta,m} \cap M^\sigma \subseteq \mu^{-1}(\mathcal{O}_\beta) \cap \mu^{-1}(\mathfrak{g}_-^*) = \mu^{-1}(\mathcal{O}_\beta \cap \mathfrak{g}_-^*) = \mu^{-1}\Big(\coprod_{i=1}^{k_C} \mathcal{O}_{\beta_i}^-\Big) = \coprod_{i=1}^{k_C} \mu^{-1}(\mathcal{O}_{\beta_i}^-).$$

Thus, $C_{\beta,m}^\sigma \subset \coprod_{i=1}^{k_C} \mu^{-1}(\mathcal{O}_{\beta_i}^-)$ which implies that

$$C_{\beta,m}^\sigma = \coprod_{i=1}^{k_C} \Big(\mu^{-1}(\mathcal{O}_{\beta_i}^-) \cap C_{\beta,m}^\sigma\Big). \tag{4.26}$$

From (4.21) and (4.26), it follows that (4.22) is satisfied. This proves part 2 and the proof is complete. ∎

As we saw in Kirwan's result K2 (see Section 4.1), for each β and m, there exists a Hamiltonian subsystem $\mathcal{H}_{\beta,m} = (Z_{\beta,m}, \omega, G_\beta, \mu_{\beta,m})$. It is easy to see that these Hamiltonian subsystems induce real Hamiltonian subsystems.

Proposition 4.5. *Let $C_{\beta,m}$ be a critical subset of f and $\beta_i \in \mathfrak{g}_-^*$ be as in Proposition 4.4. Suppose that $G_{\beta_i} = g_i G_\beta g_i^{-1}$, $\mathfrak{g}_i = \mathrm{Lie}(G_{\beta_i})$ and $\mu_{\beta_i,m} : Z_{\beta_i,m} \to \mathfrak{g}_i^*$ is defined by $\mu_{\beta_i,m} = \mathrm{Pr}_{\beta_i} \circ \mu - \beta_i$ where Pr_{β_i} is the orthogonal projection onto $\mathfrak{g}_i^*$. If $\sigma : Z_{\beta_i,m} \to Z_{\beta_i,m}$ and $\phi : G_{\beta_i} \to G_{\beta_i}$ denote the restrictions of σ and ϕ respectively, then the tuple $\mathcal{RH}_{\beta_i,m} = (Z_{\beta_i,m}, \omega, G_{\beta_i}, \mu_{\beta_i,m}, \sigma, \phi)$ is a real Hamiltonian subsystem such that its real subgroup is $G_{\beta_i}^\phi = G^\phi \cap G_{\beta_i}$ and its real locus is $Z_{\beta_i,m}^\sigma$. In particular,*

$$\mu_{\beta_i,m}^{-1}(0) = Z_{\beta_i,m} \cap \mu^{-1}(\beta_i). \tag{4.27}$$

Proof. We first show that the restricted maps $\sigma : Z_{\beta_i,m} \to Z_{\beta_i,m}$ and $\phi : G_{\beta_i} \to G_{\beta_i}$ are well-defined involutions and also σ is anti-symplectic. Since $\phi^*(\beta_i) = -\beta_i$ and μ is an equivariant map, a simple computation shows that

$$\mu^{\beta_i} = \mu^{\beta_i} \circ \sigma. \tag{4.28}$$

Now, let $X \in \mathfrak{g}$ and $t \in \mathbb{R}$. Since $\phi \circ \exp(tX) = \exp(t\phi_* X)$, again a simple computation shows that $\phi_* \circ \mathrm{Ad}_g = \mathrm{Ad}_{\phi(g)} \circ \phi_*$ which follows that

$$\phi^* \circ \mathrm{Ad}_g^* = \mathrm{Ad}_{\phi(g)}^* \circ \phi^*. \tag{4.29}$$

Suppose that $x \in Z_{\beta_i,m}$, so $d\mu^{\beta_i}(x) = 0$ and $\mu^{\beta_i}(x) = ||\beta_i||^2$. Since $d\sigma(x)$ is an isomorphism, it follows from (4.28) that $d\mu^{\beta_i}(\sigma(x)) = 0$, $\mathrm{Ind}_{\sigma(x)}(\mu^{\beta_i}) = m$ and $\mu^{\beta_i}(\sigma(x)) = ||\beta_i||^2$. That is, $\sigma(x) \in Z_{\beta_i,m}$ which implies that $\sigma(Z_{\beta_i,m}) \subset Z_{\beta_i,m}$. Now, let $g \in G_{\beta_i}$; i.e., $\mathrm{Ad}_g^* \beta_i = \beta_i$. Since, $\phi^*(\beta_i) = -\beta_i$, it follows from (4.29) that

$$\mathrm{Ad}_{\phi(g)}^* \beta_i = -\mathrm{Ad}_{\phi(g)}^*(\phi^*(\beta_i)) = -\phi^*(\mathrm{Ad}_g^* \beta_i) = -\phi^*(\beta_i) = \beta_i.$$

That is, $\phi(g) \in G_{\beta_i}$ which proves the claim.

Clearly, for any $g \in G_{\beta_i}$ and $x \in Z_{\beta_i,m}$, we have $\sigma(gx) = \phi(g)\sigma(x)$. Let $\mathrm{Pr}_{\beta_i} : \mathfrak{g}^* \to \mathfrak{g}_i^*$ be the orthogonal projection onto $\mathfrak{g}_i^*$. Since $\mathrm{Pr}_{\beta_i} \circ \phi^* = \phi^* \circ \mathrm{Pr}_{\beta_i}$ and $\phi^*(\beta_i) = -\beta_i$, we can write

$$\begin{aligned} \mu_{\beta_i,m}(\sigma(x)) &= \mathrm{Pr}_{\beta_i}(\mu(\sigma(x))) - \beta_i \\ &= \mathrm{Pr}_{\beta_i}(-\phi^*(\mu(x))) - \beta_i \\ &= -\phi^*(\mathrm{Pr}_{\beta_i}(\mu(x))) - \beta_i \\ &= -\phi^*(\mu_{\beta_i,m}(x) + \beta_i) - \beta_i \\ &= -\phi^*(\mu_{\beta_i,m}(x)) + \beta_i - \beta_i \\ &= -\phi^* \circ \mu_{\beta_i,m}(x). \end{aligned}$$

This shows that the conditions in (3.8) are satisfied and thus (σ, ϕ) is a real pair. Obviously, the real locus is $Z^\sigma_{\beta_i,m} = Z_{\beta_i,m} \cap M^\sigma$ and the real subgroup is $G^\phi_{\beta_i} = G_{\beta_i} \cap G^\phi$. Finally, (4.27) follows from the definition of $\mu_{\beta_i,m}$. This completes the proof. ■

Now, we have all the necessary ingredients to state our first main theorem concerning the existence of a real Morse stratification for the restriction of the norm squared of the moment map to the real locus.

Theorem 4.6 (**Real Morse Stratification**). *Let $(M, \omega, G, \mu, \sigma, \phi)$ be a real Hamiltonian G-system where G is a compact connected Lie group and M is a compact connected manifold. Choose a G-invariant metric on M and an* Ad*-invariant inner product on Lie algebra $\mathfrak{g}$ such that involutions σ and ϕ are isometries. Let M^σ be the real locus, G^ϕ be the real subgroup, $f = ||\mu||^2 : M \to \mathbb{R}$ and $f^\sigma = f|_{M^\sigma}$. Then the following are satisfied.*

1. *The critical set C_{f^σ} of f^σ is a finite collection of disjoint G^ϕ-invariant closed subsets $\{C^\sigma_{\beta_i,m} \mid \beta_i \in \Lambda_\sigma, m = 0, ..., \dim M^\sigma\}$ on each of which f^σ takes a constant value. Moreover, the indexing set Λ_σ is a subset of $\mathfrak{g}^*_-$ and the Morse index of f^σ along each $C^\sigma_{\beta_i,m}$ is a constant number $d(C^\sigma_{\beta_i,m})$.*

2. *If $S^\sigma_{\beta_i,m}$ is the stratum of f^σ corresponding to $C^\sigma_{\beta_i,m}$ under the flow of $-\nabla f^\sigma$, then the collection $\{S^\sigma_{\beta_i,m} \mid \beta_i \in \Lambda_\sigma, m = 0, ..., \dim M^\sigma\}$ is a smooth G^ϕ-invariant Morse stratification of M^σ such that each stratum $S^\sigma_{\beta_i,m}$ has the constant codimension $d(C^\sigma_{\beta_i,m})$ and deformation retracts onto the corresponding critical subset $C^\sigma_{\beta_i,m}$. In addition, there exists a partial order $<$ on Λ_σ such that $C^\sigma_{\beta_i,m} < C^\sigma_{\gamma_{i'},m'}$ if and only if $f^\sigma(C^\sigma_{\beta_i,m}) < f^\sigma(C^\sigma_{\gamma_{i'},m'})$.*

3. *For each $\beta_i \in \Lambda_\sigma$ and $m \in \{0, ..., \dim M^\sigma\}$, there exist a symplectic submanifold $Z_{\beta_i,m}$ of M such that $\mathcal{RH}_{\beta_i,m} = (Z_{\beta_i,m}, \omega, G_{\beta_i}, \mu_{\beta_i,m}, \sigma, \phi)$ is a real Hamiltonian subsystem where σ, ϕ are the restricted maps onto $Z_{\beta_i,m}$ and G_{β_i} respectively and $\mu_{\beta_i,m} : Z_{\beta_i,m} \to \mathfrak{g}_i^*$ is defined by $\mu_{\beta_i,m} = \mathrm{Pr}_{\beta_i} \circ \mu - \beta_i$ in which $\mathrm{Pr}_{\beta_i} : \mathfrak{g}^* \to \mathfrak{g}_i^*$ is the orthogonal projection on $\mathfrak{g}_i^*$. In particular, the real subgroup of $\mathcal{RH}_{\beta_i,m}$ is $G^\phi_{\beta_i} = G^\phi \cap G_{\beta_i}$ and the real locus is $Z^\sigma_{\beta_i,m} = M^\sigma \cap Z_{\beta_i,m}$ such that*

$$C^\sigma_{\beta_i,m} \cong G^\phi \times_{G^\phi_{\beta_i}} (Z^\sigma_{\beta_i,m} \cap \mu^{-1}_{\beta_i,m}(0)). \tag{4.30}$$

Proof. Part 1 follows from Proposition 4.2 and Part 2 follows from Propositions 4.3 and 4.4. On the other hand, Proposition 4.5 implies the existence of the real Hamiltonian subsystems

$$\mathcal{RH}_{\beta_i,m} = (Z_{\beta_i,m}, \omega, G_{\beta_i}, \mu_{\beta_i,m}, \sigma, \phi).$$

To prove, (4.30), consider the map $H_i : G^\phi \times (Z^\sigma_{\beta_i,m} \cap \mu^{-1}_{\beta_i,m}(0)) \to C^\sigma_{\beta_i,m}$ defined by $H(g,p) = gp$, for $g \in G^\phi$ and $p \in Z^\sigma_{\beta_i,m} \cap \mu^{-1}_{\beta_i,m}(0)$. Let $G^\phi \times_{G^\phi_{\beta_i,m}} (Z^\sigma_{\beta_i,m} \cap \mu^{-1}_{\beta_i,m}(0))$ be the twisted product of G^ϕ and $Z^\sigma_{\beta_i,m} \cap \mu^{-1}_{\beta_i,m}(0)$ where $G^\phi_{\beta_i}$ is the stabilizer of β_i in the real subgroup G^ϕ. Clearly, H_i descends to a map $\widehat{H}_i : G^\phi \times_{G^\phi_{\beta_i}} (Z^\sigma_{\beta_i,m} \cap \mu^{-1}_{\beta_i,m}(0)) \to C^\sigma_{\beta_i,m}$ such that

$$\widehat{H}_i[g,p] = gp, \ \forall g \in G^\phi, p \in Z^\sigma_{\beta_i,m} \cap \mu^{-1}_{\beta_i,m}(0), \tag{4.31}$$

where $[g,p]$ is the $G^\phi_{\beta_i}$-orbit of (g,p). Let $p, q \in Z^\sigma_{\beta_i,m} \cap \mu^{-1}_{\beta_i,m}(0)$ and $g, h \in G^\phi$. If $gp = hq$, then $(h^{-1}g)p = q$. Since $\mu(p) = \mu(q) = \beta_i$, for $g_1 = h^{-1}g \in G^\phi$, we have

$$\mathrm{Ad}^*_{g_1}\beta_i = \mathrm{Ad}^*_{g_1}\mu(p) = \mu(g_1 p) = \mu(q) = \beta_i.$$

Thus, $g_1 \in G^\phi_{\beta_i}$ which follows that

$$g_1.(g,p) = (gg_1^{-1}, g_1 p) = (h,q).$$

Namely, $[g,p] = [h,q]$ and therefore $\widehat{H}_i$ is 1-1. The surjectivity is obvious. Since, the spaces are compact and Hausdorff, $\widehat{H}_i$ is a homeomorphism. This prove (4.30) and completes the proof of the theorem. ■

4.3 Examples

In this section, we find the critical subsets of the restricted map f^σ for two real Hamiltonian systems which we will use in the next chapters.

Example 4.1. Let $n > 1$ be a natural number and $M = (\mathbb{CP}^1)^n$. Consider the real Hamiltonian system $(M, \omega, \mathrm{U}(2), \mu, \sigma, \phi)$ in Example 3.13. We saw in Examples 3.12 and 3.13 that by identifying M with $(S^2)^n$, the moment map $\mu : M \to \mathbb{R}^4$ is defined by (3.25) and thus the norm squared of the moment map $f : M \to \mathbb{R}$ is

$$f(X_1, ..., X_n) = \frac{1}{16\pi^2}\left[\Big(\sum_{j=1}^n x_j\Big)^2 + \Big(\sum_{j=1}^n y_j\Big)^2 + 8\Big(\sum_{j=1}^n z_j\Big)^2\right]. \tag{4.32}$$

Let T be the maximal torus of $\mathrm{U}(2)$ consisting of diagonal matrices. The Lie algebra of T is

$$\mathfrak{t} = \left\{ \begin{pmatrix} i\theta_1 & 0 \\ 0 & i\theta_2 \end{pmatrix} \mid \theta_1, \theta_2 \in \mathbb{R} \right\}. \tag{4.33}$$

We can identify $\mathfrak{t}$ with $\mathbb{R}^2$ by using the identification in Example 3.12:

$$\begin{pmatrix} i\theta_1 & 0 \\ 0 & i\theta_2 \end{pmatrix} \longleftrightarrow (\theta_1, \theta_2). \tag{4.34}$$

The action of any diagonal matrix $\mathrm{Diag}[e^{i\theta_1}, e^{i\theta_2}]$ on M is the rotation around the z-axis with angle $\theta_2 - \theta_1$. Thus, the fixed point set of the T-action is

$$M^T = \{(X_1, ..., X_n) \mid X_j \in \{S, N\}\}, \tag{4.35}$$

where $S = (0, 0, -1)$ and $N = (0, 0, 1)$.

Following Kirwan's approach [42], we can see that the indexing set Λ of the critical subsets of f is as follows:

$$\Lambda = \Big\{ \Big(\frac{-(2r-n)}{2\pi}, \frac{2r-n}{2\pi} \Big) \mid \frac{n}{2} \leq r \leq n \Big\}. \tag{4.36}$$

By using the identification in (4.34), to any $\beta = 2r - n \neq 0$, we can correspond the element $A_\beta \in \mathfrak{t}$ such that

$$A_\beta = \begin{pmatrix} -\dfrac{\beta}{2\pi} i & 0 \\ 0 & \dfrac{\beta}{2\pi} i \end{pmatrix}. \tag{4.37}$$

The norm squared of A_β with respect to the inner product $\langle A, B \rangle = -\mathrm{Tr}(AB)$ is

$$||A_\beta||^2 = \frac{\beta^2}{2\pi^2}. \tag{4.38}$$

Let T_β be the subtorus of T generated by A_β. It is easily seen that the fixed point set of T_β-action on M is equal to M^T; i.e.,

$$M^{T_\beta} = \{(X_1, ..., X_n) \mid X_j \in \{S, N\}\}. \tag{4.39}$$

By using the identifications in Example 3.12, if we denote the component map μ^{A_β} by μ^β, then we have

$$\mu^\beta(X_1, ..., X_n) = \frac{\beta}{2\pi^2} \Big(\sum_{j=1}^{n} z_j \Big), \tag{4.40}$$

where $X_j = (x_j, y_j, z_j)^t$. Hence, (4.38) and (4.40) imply that

$$(\mu^\beta)^{-1}(||A_\beta||^2) = \{(X_1, ..., X_n) \in M \mid z_1 + \cdots + z_n = \beta\}. \tag{4.41}$$

Since the critical set of μ^β is equal to M^{T_β}, it follows from (4.40) and (4.41) that the corresponding symplectic submanifold $Z_\beta = (\mu^\beta)^{-1}(||A_\beta||^2) \cap M^{T_\beta}$ has the form

$$Z_\beta = \{(X_1, ..., X_n) \in M \mid X_j \in \{N, S\}, \ \ \#(N) = r \text{ and } \#(S) = n - r\}, \tag{4.42}$$

where $\#(N) = r$ means the number of north pole N in the tuple $(X_1, ..., X_n)$ is equal to r. Clearly, Z_β is a finite set with cardinality $m_\beta = \binom{n}{r}$. So $Z_\beta = \coprod_{k=1}^{m_\beta} Z_{\beta,k}$, a union of singletons $Z_{\beta,k}$. As the index of the height function on the 2-sphere S^2 at N and S is 2 and 0 respectively, then by the definition of μ^β, we can see that the index of μ^β is $2r$ at each point of Z_β. Thus in this example $Z_{\beta,m} = \emptyset$ for $m \neq 2r$ and $Z_\beta = Z_{\beta,2r}$.

The critical subsets of the function $f = ||\mu||^2$ are $C_\beta = \mathrm{U}(2).(Z_\beta \cap \mu^{-1}(A_\beta))$, so by (4.41) and (4.42), we have

$$(X_1, ..., X_n) \in Z_\beta \Rightarrow z_1 + \cdots + z_n = 2r - n = \beta.$$

That is, $Z_\beta \subset \mu^{-1}(A_\beta)$ and thus

$$C_\beta = \mathrm{U}(2).Z_\beta. \tag{4.43}$$

Since the action of U(2) on S^2 is transitive, (4.42) and (4.43) follow that

$$C_\beta = \{(X_1, ..., X_n) \in M \mid X_j \in \{\pm X_0\}, X_0 \in S^2, \#(X_0) = r, \#(-X_0) = n-r\}. \tag{4.44}$$

Also since $Z_{\beta,2r} = Z_\beta$, (4.8) follows that the index of f along C_β is $2r - 2$; i.e.,

$$d(C_\beta) = 2r - 2. \tag{4.45}$$

In this example, the real subgroup is $G^\phi = \mathrm{O}(2)$ and the real locus is

$$M^\sigma = \{(X_1, ..., X_n) \in M \mid X_j = (x_j, 0, z_j)^t\} \cong (\mathbb{RP}^1)^n. \tag{4.46}$$

Therefore the restricted function $f^\sigma = f : M^\sigma \to \mathbb{R}$ is

$$f^\sigma(X_1, ..., X_n) = \frac{1}{16\pi^2}\left[\Big(\sum_{j=1}^n x_j\Big)^2 + 8\Big(\sum_{j=1}^n z_j\Big)^2\right]. \tag{4.47}$$

We know from Proposition 4.2 that the critical subsets of f^σ are $C_\beta^\sigma = C_\beta \cap M^\sigma$. Thus, 4.44 and 4.46 imply that

$$C_\beta^\sigma = \{(X_1, ..., X_n) \in M^\sigma \mid X_j \in \{\pm X_0\}, X_0 \in E, \#(X_0) = r, \#(-X_0) = n-r\}, \tag{4.48}$$

where $E = \{(x, y, z)^t \in S^2 \mid y = 0\}$.

It is easy to see that C_β^σ is a disjoint union of finite number of subsets $C_{\beta,k}^\sigma$ each of which is isomorphic to S^1. In Figure 4.2, we have shown the case in which $n = 3$, $r = 2$, $\beta = 1$ and $k = 1$. Here $Z_{\beta,k} = \{(N, N, S)\}$ and $C_{\beta,k}^\sigma = \{(X_0, X_0, -X_0) \mid X_0 \in E\}$.

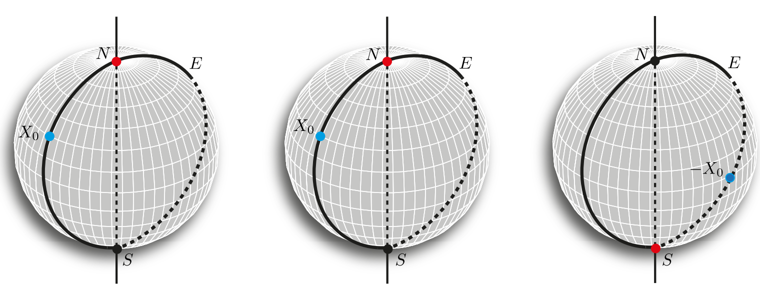

Figure 4.2: Elements of the critical subset $C_{\beta,k}^\sigma$

Thus, we have $C^\sigma_\beta = \coprod_{k=1}^{m_\beta} C^\sigma_{\beta,k}$. Since the action of $\mathrm{O}(2)$ on S^1 is transitive and $Z_{\beta,k} \subset M^\sigma$, we have $C^\sigma_{\beta,k} = \mathrm{O}(2).Z_{\beta,k}$ which implies that

$$C^\sigma_\beta = \mathrm{O}(2).Z_\beta. \tag{4.49}$$

This follows that

$$C^\sigma_\beta \cong \mathrm{O}(2) \times_{\mathrm{O}(2)_\beta} Z_\beta, \tag{4.50}$$

where $\mathrm{O}(2)_\beta$ is the stabilizer of β (actually A_β) in $\mathrm{O}(2)$. Finally, by (4.45) and Proposition 4.3, the index of $f^\sigma : M^\sigma \to \mathbb{R}$ along C^σ_β is

$$d(C^\sigma_\beta) = r - 1. \tag{4.51}$$

Example 4.2. Consider the real Hamiltonian system

$$\Big(M = \mathrm{Gr}_{l_1}(\mathbb{C}^n) \times \cdots \times \mathrm{Gr}_{l_r}(\mathbb{C}^n), \omega, \mathrm{U}(n), \mu, \sigma, \phi\Big)$$

in Example 3.15 where the real subgroup G^ϕ is the orthogonal group $\mathrm{O}(n)$ and the real locus M^σ is a product of real Grassmannians:

$$M^\sigma = \mathrm{Gr}_{l_1}(\mathbb{R}^n) \times \cdots \times \mathrm{Gr}_{l_r}(\mathbb{R}^n). \tag{4.52}$$

Let $k = \sum_{j=1}^r l_j$. For any $1 \le s \le n$, set

$$P^0(n,s) = \Big\{(m_i)_{i=1}^s \in \mathbb{Z}^s \;\Big|\; 0 \le m_i \le n \text{ and } \sum_{i=1}^s m_i = n\Big\}, \tag{4.53}$$

and

$$P(n,s) = \Big\{(m_i)_{i=1}^s \in P^0(n,s) \;\Big|\; m_i > 0\Big\}. \tag{4.54}$$

Define

$$\Lambda^s = \Bigg\{\Big(\underbrace{\frac{k_1}{m_1}, \dots, \frac{k_1}{m_1}}_{m_1}, \dots, \underbrace{\frac{k_s}{m_s}, \dots, \frac{k_s}{m_s}}_{m_s}\Big) \in \mathbb{R}^n \;\Big|\; (m_i)_{i=1}^s \in P(n,s), \\ (k_i)_{i=1}^s \in P^0(k,s) \text{ and } \frac{k_1}{m_1} > \cdots > \frac{k_s}{m_s}\Bigg\}. \tag{4.55}$$

For any $\beta = (\frac{k_1}{m_1}, \dots, \frac{k_1}{m_1}, \dots, \frac{k_s}{m_s}, \dots, \frac{k_s}{m_s}) \in \Lambda^s$, set

$$\Gamma^s_\beta = \Big\{l = (l_{ij})_{\substack{1\le i\le s\\ 1\le j\le r}} \in M_{s,r}(\mathbb{Z}) \;\Big|\; \sum_{i=1}^s l_{ij} = l_j \text{ and } \sum_{j=1}^r l_{ij} = k_i\Big\}. \tag{4.56}$$

We can think of $l = (l_{ij})$ as a matrix of order $s \times r$ such that the entries of i^{th}-row are partitions of k_i and the entries of j^{th}-column are the partitions of l_j. Let $\{e_1, ..., e_n\}$ be the standard basis of the complex n-vector space $\mathbb{C}^n$ and consider the standard inner product $\langle z, w\rangle = w^* z$ on it. For each $\beta \in \Lambda^s$, define

$$\mathcal{F}^s_\beta = \Big\{(L_1, ..., L_s) \in \mathrm{Gr}_{m_1}(\mathbb{C}^n) \times \cdots \times \mathrm{Gr}_{m_s}(\mathbb{C}^n) \;\Big|\; L_i \perp L_{i'} \text{ for } i \neq i'\Big\}, \tag{4.57}$$

where $L_i \perp L_{i'}$ means that for any $z \in L_i$ and $z' \in L_{i'}$, we have $\langle z, z'\rangle = 0$.

It follows from (3.32) that the norm squared of moment map $f = ||\mu||^2 : M \to \mathbb{R}$ has the following form:

$$f(V_1, ..., V_r) = 2 \sum_{1 \leq j < j' \leq r} \mathrm{Tr}(\mathrm{Pr}_{V_j} \mathrm{Pr}_{V_{j'}}) + \frac{nk - k^2}{n}, \tag{4.58}$$

where $\mathrm{Pr}_{V_j} : \mathbb{C}^n \to \mathbb{C}^n$ is the orthogonal projection onto the complex subspace V_j. Let T be the maximal torus of $\mathrm{U}(n)$ consisting of the diagonal matrices. The Lie algebra of the maximal torus T is

$$\mathfrak{t} = \Big\{\mathrm{Diag}\Big[\theta_1\sqrt{-1}, \cdots, \theta_n\sqrt{-1}\Big] \;\Big|\; \theta_j \in \mathbb{R}\Big\}. \tag{4.59}$$

According to Kirwan (see [42]), the indexing set Λ for the critical subsets of f is

$$\Lambda = \bigcup_{s=1}^{n} \Big\{(\beta, l) \mid \beta \in \Lambda^s \& \; l \in \Gamma^s_\beta\Big\}. \tag{4.60}$$

For any element $\beta = (\frac{k_1}{m_1}, ..., \frac{k_1}{m_1}, ..., \frac{k_s}{m_s}, ..., \frac{k_s}{m_s})$ in Λ^s, set

$$D_\beta = \mathrm{Diag}\Big[\frac{k_1}{m_1}\mathrm{Id}_{m_1}, ..., \frac{k_s}{m_s}\mathrm{Id}_{m_s}\Big], \tag{4.61}$$

so $\sqrt{-1}D_\beta$ is an element in the Lie algebra $\mathfrak{t}$.

Suppose that $E_i = \mathrm{Ker}(D_\beta - \frac{k_i}{m_i}\mathrm{Id})$ is the eigenspace of D_β corresponding to the eigenvalue $\frac{k_i}{m_i}$. It is clear that $\mathbb{C}^n = E_1 \oplus \cdots \oplus E_s$ and also

$$E_1 = \mathrm{Span}_{\mathbb{C}}\langle e_1, ..., e_{m_1}\rangle, ..., E_s = \mathrm{Span}_{\mathbb{C}}\langle e_{m_{s-1}+1}, ..., e_{m_s}\rangle. \tag{4.62}$$

Let $E_{i,\mathbb{R}}$ be the real subspace of E_i; i.e.,

$$E_{1,\mathbb{R}} = \mathrm{Span}_{\mathbb{R}}\langle e_1, ..., e_{m_1}\rangle, ..., E_{s,\mathbb{R}} = \mathrm{Span}_{\mathbb{R}}\langle e_{m_{s-1}+1}, ..., e_{m_s}\rangle. \tag{4.63}$$

If T_β is the subtorus generated by D_β, Proposition B.3 in Appendix B follows that the critical set of the Morse-Bott function $\mu^\beta = \mu^{\sqrt{-1}D_\beta} : M \to \mathbb{R}$ is equal to

$$C_{\mu^\beta} = \{(V_1, ..., V_r) \in M \mid V_j = \bigoplus_{i=1}^{s}(E_i \cap V_j)\}. \tag{4.64}$$

For any $(V_1, ..., V_r) \in M$, we have

$$\mu^\beta(V_1, ..., V_r) = \sum_{j=1}^{r} \mathrm{Tr}(\mathrm{Pr}_{V_J} D_\beta) - \frac{k^2}{n}. \tag{4.65}$$

Also by (4.61)

$$||D_\beta||^2 = \frac{k_1^2}{m_1} + \cdots + \frac{k_s^2}{m_s}. \tag{4.66}$$

In this case, the symplectic manifold $Z_\beta = C_{\mu^\beta} \cap (\mu^\beta)^{-1}(||D_\beta||^2)$ has the following form:

$$Z_\beta = \Big\{(V_1, ..., V_r) \in M \mid V_j = \bigoplus_{i=1}^{s}(E_i \cap V_j),\ \sum_{j=1}^{r} \dim_{\mathbb{C}}(E_i \cap V_j) = k_i\Big\}, \tag{4.67}$$

and the index of μ^β along Z_β is a constant number.

If for any $l = (l_{ij}) \in \Lambda_\beta^s$, we set

$$Z_{\beta,l} = \Big\{(V_1, ..., V_r) \in M \mid V_j = \bigoplus_{i=1}^{s}(E_i \cap V_j),\ \dim_{\mathbb{C}}(E_i \cap V_j) = l_{ij}\Big\}, \tag{4.68}$$

then it follows from (4.67) and (4.68) that

$$Z_\beta = \coprod_{l \in \Lambda_\beta^s} Z_{\beta,l}. \tag{4.69}$$

Let $Z_{\beta,l}^\sigma = Z_{\beta,l} \cap M^\sigma$ and $Z_\beta^\sigma = Z_\beta \cap M^\sigma$. It is implied from (4.68) and (4.69) that

$$Z_{\beta,l}^\sigma = \Big\{(V_1, ..., V_r) \in M^\sigma \mid V_{j,\mathbb{R}} = \bigoplus_{i=1}^{s}(E_{i,\mathbb{R}} \cap V_{j,\mathbb{R}}),\ \dim_{\mathbb{R}}(E_{i,\mathbb{R}} \cap V_{j,\mathbb{R}}) = l_{ij}\Big\}, \tag{4.70}$$

and

$$Z_\beta^\sigma = \coprod_{l \in \Lambda_\beta^s} Z_{\beta,l}^\sigma. \tag{4.71}$$

For each $l \in \Lambda_\beta^s$, let

$$C_{\beta,l} = \mathrm{U}(n).(Z_{\beta,l} \cap \mu^{-1}(\beta)). \tag{4.72}$$

We know that the critical subset corresponding to Z_β is $C_\beta = \mathrm{U}(n).(Z_\beta \cap \mu^{-1}(\beta))$. So (4.72) follows that

$$C_\beta = \coprod_{l \in \Lambda_\beta^s} C_{\beta,l}. \tag{4.73}$$

We can see from (4.68) and (4.72) that

$$C_{\beta,l} = \bigcup_{(L_1,\ldots,L_s)\in\mathcal{F}^s_\beta} \Bigg\{ (V_1,\ldots,V_r) \in M \;\Big|\; V_j = \bigoplus_{i=1}^{s} (L_i \cap V_j), \\ \dim_{\mathbb{C}}(L_i \cap V_j) = l_{ij} \text{ and } \sum_{j=1}^{r} \mathrm{Pr}_{(L_i\cap V_j)} = (\frac{k_i}{m_i})\mathrm{Id}_{m_i} \Bigg\}, \tag{4.74}$$

where $\mathcal{F}^s_\beta$ is defined by (4.57). If $C^\sigma_{\beta,l} = C_{\beta,l} \cap M^\sigma$, (4.74) implies that

$$C^\sigma_{\beta,l} = \bigcup_{(L_1,\ldots,L_s)\in\mathcal{F}^s_\beta} \Bigg\{ (W_1,\ldots,W_r) \in M^\sigma \;\Big|\; W_{j,\mathbb{R}} = \bigoplus_{i=1}^{s} (L_{i,\mathbb{R}} \cap W_{j,\mathbb{R}}), \\ \dim_{\mathbb{R}}(L_{i,\mathbb{R}} \cap W_{j,\mathbb{R}}) = l_{ij} \text{ and } \sum_{j=1}^{r} \mathrm{Pr}_{(L_{i,\mathbb{R}}\cap W_{j,\mathbb{R}})} = (\frac{k_i}{m_i})\mathrm{Id}_{m_i} \Bigg\}, \tag{4.75}$$

where $L_{i,\mathbb{R}} = L_i \cap \mathbb{R}^n$ is the real subspace of L_i. We claim that

$$C^\sigma_{\beta,l} = \mathrm{O}(n).(Z^\sigma_{\beta,l} \cap \mu^{-1}(\beta)). \tag{4.76}$$

To prove this, we first note that $\mathrm{O}(n).(Z^\sigma_{\beta,l} \cap \mu^{-1}(\beta)) \subset C^\sigma_{\beta,l}$. Conversely, let $W = (W_1,\ldots,W_r) \in C^\sigma_{\beta,l}$. So there exists $(L_1,\ldots,L_s) \in \mathcal{F}^s_\beta$ such that conditions in (4.75) hold. Since $E_{i,\mathbb{R}}$ and $L_{i,\mathbb{R}}$ are mutually orthogonal subspaces with the same real dimension m_i, we can choose orthogonal matrices $A_i \in \mathrm{O}(m_i)$ such that $L_{i,\mathbb{R}} = A_i E_{i,\mathbb{R}}$. Set $A = \mathrm{Diag}[A_1,\ldots,A_s] \in \mathrm{O}(n)$ and $V_j = A^{-1}W_j$. Since A^{-1} is an orthogonal matrix, we can see from above relations that $V_j = \bigoplus_{i=1}^{s} E_{i,\mathbb{R}} \cap V_j$ and $\dim_{\mathbb{R}}(E_{i,\mathbb{R}} \cap V_j) = l_{ij}$. This means $V = (V_1,\ldots,V_s) \in Z^\sigma_{\beta,l}$. On the other hand, since $\mathrm{Pr}_{(E_{i,\mathbb{R}}\cap V_{j,\mathbb{R}})} = \mathrm{Pr}_{(L_{i,\mathbb{R}}\cap W_{j,\mathbb{R}})}$, we have $\mu(V) = D_\beta$ which implies that $V \in Z^\sigma_{\beta,l} \cap \mu^{-1}(\beta)$. Since $W = AV$, we see that $W \in \mathrm{O}(n).(Z^\sigma_{\beta,l} \cap \mu^{-1}(\beta))$ which completes the proof of (4.76).

It follows from (4.76) that

$$C^\sigma_{\beta,l} = \mathrm{O}(n) \times_{\mathrm{O}(n)_\beta} (Z^\sigma_{\beta,l} \cap \mu^{-1}(\beta)), \tag{4.77}$$

where $\mathrm{O}(n)_\beta$ is the stabilizer of D_β in $\mathrm{O}(n)$. Finally, from Kirwan [42], we know that the index of f along C_β is

$$d(\beta) = \sum_{1\le i<i'\le s} 2(k_i - m_i)m_{i'}. \tag{4.78}$$

So (4.78) and Proposition 4.3 imply that the index of $f^\sigma : M^\sigma \to \mathbb{R}$ along C^σ_β is

$$d(\beta,\sigma) = \sum_{1\le i<i'\le s} (k_i - m_i)m_{i'}. \tag{4.79}$$

Remark 4.2. We can see from our computation that the generated real Hamiltonian subsystems in this example are tuples $(Z_{\beta,l}, \omega, \mathrm{U}(n)_\beta, \mu_{\beta,l}, \sigma, \phi)$ such that $\mu_{\beta,l} : Z_{\beta,l} \to \mathrm{Lie}(\mathrm{U}(n)_\beta)$ is defined by $\mu_{\beta,l} = \mathrm{Pr}_\beta \circ \mu - D_\beta$ as well as the involutions σ and ϕ are the usual complex conjugation. Clearly the real locus is $Z^\sigma_{\beta,l}$ and the real subgroup is $\mathrm{O}(n)_\beta$. If for any $i = 1, ..., s$, we define

$$Z_{\beta,l,i} = \{(V_1 \cap E_i, ..., V_r \cap E_i) \mid (V_1, ..., V_r) \in Z_{\beta,l}\}, \tag{4.80}$$

then it follows from (4.68) that each $Z_{\beta,l,i}$ is a product of Grassmannians and

$$Z_{\beta,l} \cong \prod_{i=1}^{s} Z_{\beta,l,i}. \tag{4.81}$$

In this case, if we consider the real Hamiltonian subsystems $(Z_{\beta,l,i}, \omega, \mathrm{U}(m_i), \mu_{\beta,l,i}, \sigma, \phi)$ in which the moment maps $\mu_{\beta,l,i} : Z_{\beta,l,i} \to \mathrm{U}(m_i)$ are given by

$$\mu_{\beta,l,i}(W_1, ..., W_r) = \sqrt{-1} \sum_{j=1}^{r} \mathrm{Pr}_{W_j} + \frac{1}{\sqrt{-1}} \left(\frac{k_i}{m_i} + \frac{k}{n}\right) \mathrm{Id}_{m_i}, \tag{4.82}$$

then we have

$$\mu^{-1}_{\beta,l}(0) \cong \prod_{i=1}^{s} \mu^{-1}_{\beta,l,i}(0). \tag{4.83}$$

As we will see, it is better to work with these new real Hamiltonian subsystems $(Z_{\beta,l,i}, \omega, \mathrm{U}(m_i), \mu_{\beta,l,i}, \sigma, \phi)$ in computations, because the generated real Hamiltonian subsystems $(Z_{\beta,l}, \omega, \mathrm{U}(n)_\beta, \mu_{\beta,l}, \sigma, \phi)$ decomposes as the product of these new ones. We will use this in Chapter 8.

Remark 4.3. In particular, one can see that when $s = 1$, we have $\Lambda^1 = \{\beta_0 = (\frac{k}{n}, ..., \frac{k}{n})\}$, $\Gamma^1_{\beta_0} = \{l_0 = (l_1, ..., l_r)\}$ and thus by (4.68)

$$Z_{\beta_0,l_0} = \mathrm{Gr}_{l_1}(\mathbb{C}^n) \times \cdots \times \mathrm{Gr}_{l_r}(\mathbb{C}^n) = M. \tag{4.84}$$

That is, $M_0 = \mu^{-1}(0) = \mu^{-1}_{\beta_0,l_0}(0)$. Thus the singleton set $\{(\beta_0, l_0)\}$ is corresponding to the zero level set M_0 which is the critical subset consisting of the points on which f takes its global minimum.

CHAPTER 5

Free Extension Property

"Obvious is the most dangerous word in mathematics."
—Eric Temple Bell (1883-1960)

In this chapter, we introduce the notion of free extension for the category of modules and then apply it to the class of real Hamiltonian systems. This property is one of the main conditions we need to consider in order to give a real version of Atiyah-Bott Lemma in the next chapter.

5.1 Definition and Examples

This section gives the definition and some examples of the free extension property.

Proposition 5.1 (**Restriction of Scalars**). *Let $\varphi : R \to S$ be a homomorphism between commutative rings with unity and M be a left S-module. Then the map $R \times M \to M$ defined by $(r, m) = \varphi(r)m$ defines a left R-module structure on M. In particular, S itself has a left R-module structure.*

Proof. Straightforward. ■

Example 5.1. Let X be a G-space and R be a commutative ring with unity. The projection map $p : X \to \{*\}$ induces a ring homomorphism in equivariant cohomology $p^* : H^*(BG; R) \to H^*_G(X; R)$. By using p^*, we can make $H^*_G(X; R)$ into a $H^*(BG; R)$-module.

Example 5.2. Let G be a compact Lie group and K be a closed subgroup. The inclusion map $i : K \hookrightarrow G$ induces a map $Bi : BK \to BG$ between classifying spaces. So we get a ring homomorphism $Bi^* : H^*(BG; R) \to H^*(BK; R)$. This gives $H^*(BK; R)$ a $H^*(BG; R)$-module structure.

Proposition 5.2 (**Extension of Scalars**). *Let $\varphi : R \to S$ be a homomorphism between commutative rings with unity and M be a left R-module. Then the map $S \times (S \otimes_R M) \to S \otimes_R M$ induced by $(s, s' \otimes m) \longmapsto ss' \otimes m$ gives the abelian group $S \otimes_R M$ a unique left S-module structure which is compatible with the left R-module structure induced by φ; i.e., for any $r \in R$ and $x \in S \otimes_R M$, we have $rx = \varphi(r)x$. Moreover, if M is a free left R-module, then $S \otimes_R M$ is a free left S-module.*

Proof. See [17], Theorems 6.4 and 6.6. ■

Definition 5.1. A ring homomorphism $\varphi : R \to S$ is called a **free extension** if S is a free left R-module with respect to the left module structure induced by φ.

Remark 5.1. One can easily see that if $\varphi : R \to S$ and $\psi : S \to T$ are two free extensions, then the composite map $\psi \circ \varphi : R \to T$ is also a free extension.

Example 5.3. Consider the Lie group $\mathrm{U}(1) \cong S^1$ and its subgroup $\mathrm{O}(1) \cong \mathbb{Z}_2$. Let $Bi^* : H^*(B\mathrm{U}(1); \mathbb{Z}_2) \to H^*(B\mathrm{O}(1); \mathbb{Z}_2)$ be the map induced by the inclusion map. We know that $H^*(B\mathrm{U}(1)) \cong \mathbb{Z}_2[u]$ with $\deg u = 2$ and $H^*(B\mathrm{O}(1)) \cong \mathbb{Z}_2[v]$ with $\deg v = 1$ such that $Bi^*(u) = v^2$ (see [52], Chapter 3, Theorem 5.11). It is easy to see that for any polynomial $f \in \mathbb{Z}_2[v]$, there are two polynomials $f_1(u), f_2(u) \in \mathbb{Z}_2[u]$ such that

$$f(v) = Bi^*(f_1(u)).1 + Bi^*(f_2(u)).v.$$

This implies that the set $\{1, v\}$ is a basis for the $H^*(B\mathrm{O}(1); \mathbb{Z}_2)$ as a $H^*(B\mathrm{U}(1); \mathbb{Z}_2)$-module. Therefore, Bi^* is a free extension.

Example 5.4. Let G be a compact connected Lie group with the maximal torus T such that the integral cohomology of G has no 2-torsion. The inclusion map $i : T \hookrightarrow G$ induces a ring homomorphism in cohomology $Bi^* : H^*(BG;\mathbb{Z}_2) \to H^*(BT;\mathbb{Z}_2)$. It is known that $H^*(BT;\mathbb{Z}_2) \cong H^*(BG;\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(G/T;\mathbb{Z}_2)$ (see [48], Theorem 8.3). Since $H^*(G/T;\mathbb{Z}_2)$ is a $\mathbb{Z}_2$-free module (vector space), it follows from Proposition 5.2 that $H^*(BT;\mathbb{Z}_2)$ is a free $H^*(BG;\mathbb{Z}_2)$-module. Therefore, Bi^* is a free extension.

Example 5.5. Let G be the orthogonal group $\mathrm{O}(n)$ or the special orthogonal group SO(n). Suppose that $D(n)$ is the subgroup of diagonal matrices in G. The inclusion map $i : D(n) \hookrightarrow G$ induces a homomorphism $Bi^* : H^*(BG;\mathbb{Z}_2) \to H^*(BD(n);\mathbb{Z}_2)$. A theorem of Borel (see [13], Theorem 22.7) says that $H^*(BD(n);\mathbb{Z}_2) \cong H^*(BG;\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(F_n;\mathbb{Z}_2)$ where $F_n = G/D(n)$. Since $H^*(F_n;\mathbb{Z}_2)$ is a vector space, Proposition 5.2 implies that Bi^* is a free extension.

Suppose that a compact Lie group G acts on a compact connected manifold X. Let K be a closed subgroup of G with the inclusion map $i : K \hookrightarrow G$. Denote the corresponding universal bundles of G and K by $EG \to BG$ and $EK \to BK$, respectively (since K is a subgroup of G, we can take $EK = EG$). Let X_G and X_K be the homotopy quotients with respect to G-action and K-action, respectively. In this case, we have the following commutative diagram:

$$\begin{tikzcd} X_K \arrow[r, "i_X"] \arrow[d, "q_K"'] & X_G \arrow[d, "q_G"] \\ BK \arrow[r, "Bi"'] & BG \end{tikzcd}$$

Diagram 5.1: Pullback diagram of homotopy quotients

Here, maps i_X and Bi are induced by the inclusion map i and maps q_K and q_G are the projections in the relative fiber bundles. The following proposition is a generalization of Proposition 1 in [37] in which K is a maximal torus and the coefficient ring is the set of rational numbers.

Proposition 5.3. *Let $\mathbb{F}$ be a field and X be a connected G-space where G is a compact Lie group. Suppose that K is a closed subgroup with the inclusion map $i : K \hookrightarrow G$. If the induced map $Bi^* : H^*(BG;\mathbb{F}) \to H^*(BK;\mathbb{F})$ is a free extension and the action of $\pi_1(BG)$ on $H^*(X;\mathbb{F})$ is trivial, then the map $i_X^* : H_G^*(X;\mathbb{F}) \to H_K^*(X;\mathbb{F})$ is a free extension. In particular, i_X^* is injective.*

Proof. Since the action of $\pi_1(BG)$ on $H^*(X;\mathbb{F})$ is trivial, the Eilenberg-Moore spectral sequence can be applied to Diagram 5.1 and Proposition 2.68 implies that there exists a second quadrant spectral sequence $\{E_r, d_r\}_{r\geq 1}$ converging to $H_K^*(X;\mathbb{F})$ such that

$$\begin{cases} E_2^{p,q} \cong \mathrm{Tor}^{p,q}_{H^*(BG;\mathbb{F})}\Big(H^*(BK;\mathbb{F}), H_G^*(X;\mathbb{F})\Big), \ \forall p \leq 0, \ \forall q \geq 0 \\ E_\infty^{p,q} \cong AG^{p,q}(H_K^*(X;\mathbb{F})), \end{cases} \tag{5.1}$$

where $AG(H^*_K(X;\mathbb{F}))$ is the associated graded module induced by an exhaustive filtration of $H^*_K(X;\mathbb{F})$. Since $Bi^* : H^*(BG;\mathbb{F}) \to H^*(BK;\mathbb{F})$ is a free extension, Definition 5.1 follows that $H^*(BK;\mathbb{F})$ is a free $H^*(BG;\mathbb{F})$-module and therefore, by Proposition 2.63, we get

$$\begin{cases} E_2^{p,*} = (0), \ \forall p < 0, \\ E_2^{0,*} = H^*(BK;\mathbb{F}) \otimes_{H^*(BG;\mathbb{F})} H^*_G(X;\mathbb{F}). \end{cases} \tag{5.2}$$

This implies that all the negative columns in the E_2-term of the second quadrant spectral sequence are trivial and thus the page E_2 reduces to the zeroth column. This follows that all the differentials d_2 on the page E_2 are trivial. So $d_r = d_2 = 0$, for all $r \geq 2$ and therefore the spectral sequence collapses at page E_2; that is,

$$E_\infty^{*,*} = E_2^{*,*}. \tag{5.3}$$

Let $\mathbf{F} = \{\mathbf{F}^{-n} \mid n \geq -1\}$ be the exhaustive filtration of $H^*_K(X;\mathbb{F})$ for which (5.1) holds. It follows from (5.2) and (5.3) that

$$AG^n(H^*_K(X;\mathbb{F})) \cong \begin{cases} (0), & if\ n \neq 0, \\ H^*(BK;\mathbb{F}) \otimes_{H^*(BG;\mathbb{F})} H^*_G(X;\mathbb{F}), & if\ n = 0. \end{cases} \tag{5.4}$$

This implies that

$$\mathbf{F}^{-n} \cong \begin{cases} (0), & if\ n = -1, \\ H^*(BK;\mathbb{F}) \otimes_{H^*(BG;\mathbb{F})} H^*_G(X;\mathbb{F}), & if\ n \geq 0. \end{cases} \tag{5.5}$$

Since the filtration is exhaustive, (5.5) follows that

$$H^*_K(X;\mathbb{F}) = \bigcup_{n\geq -1} \mathbf{F}^{-n} = \mathbf{F}^0 \cong H^*(BK;\mathbb{F}) \otimes_{H^*(BG;\mathbb{F})} H^*_G(X;\mathbb{F}).$$

Therefore, $H^*_K(X;\mathbb{F}) = H^*(BK;\mathbb{F}) \otimes_{H^*(BG;\mathbb{F})} H^*_G(X;\mathbb{F})$. Since $H^*(BK;\mathbb{F})$ is a free $H^*(BG;\mathbb{F})$-module, it follows from Proposition 5.2 that $i^*_X : H^*_G(X;\mathbb{F}) \to H^*_K(X;\mathbb{F})$ is a free extension. This completes the proof. ∎

Example 5.6. Let the special orthogonal group $\mathrm{SO}(n)$ act on a space X and consider the subgroup $D(n)$ of the diagonal matrices in $\mathrm{SO}(n)$. On one hand, by Example 5.5, the induced map $H^*(B\mathrm{SO}(n);\mathbb{Z}_2) \to H^*(BD(n);\mathbb{Z}_2)$ is a free extension. On the other hand, $B\mathrm{SO}(n)$ is simply connected, so the action of $\pi_1(B\mathrm{SO}(n)) = (0)$ on $H_*(X;\mathbb{Z}_2)$ is clearly trivial. Therefore, Proposition 5.3 implies that the induced map $H^*_{\mathrm{SO}(n)}(X;\mathbb{Z}_2) \to H^*_{D(n)}(X;\mathbb{Z}_2)$ is a free extension.

Example 5.7. Let M be a compact connected G-space where $G = \mathrm{U}(n)$ or $\mathrm{SU}(n)$. Consider the subgroup $K = \mathrm{O}(n)$ or $\mathrm{SO}(n)$ and let $N \subset M$ be a K-invariant subspace. Suppose that D is the subgroup of diagonal matrices in K. We saw in Example 5.5 that the induced map $H^*(BK;\mathbb{Z}_2) \to H^*(BD;\mathbb{Z}_2)$ is a free extension. On the other hand, BG is simply connected and thus the action of $\pi_1(BG)$ on $H^*(M;\mathbb{Z}_2)$ is trivial. Hence the action of $\pi_1(BK)$ on $H^*(N;\mathbb{Z}_2)$ is also trivial. Now, Proposition 5.3 follows that the induced map $H^*_K(N;\mathbb{Z}_2) \to H^*_D(N;\mathbb{Z}_2)$ is a free extension.

5.2 Free Extension Property for Involutive Lie Groups

This section deals with a specific property for a Lie group with an involution. This property is one of the main conditions which plays a key role in proving our main theorems in the subsequent chapters.

Definition 5.2. An **involutive Lie group** is a pair (G, ϕ) in which G is a Lie group and $\phi : G \to G$ is a Lie group involution (an automorphism of order 2). The fixed point set of the involution is denoted by $G^\phi = \{g \in G \mid \phi(g) = g\}$.

Let (G, ϕ) be an involution Lie group. In this case, the involution ϕ induces an involution on the Lie algebra $\phi_* : \mathfrak{g} \to \mathfrak{g}$ with two eigenvalues 1 and -1. Denote the corresponding eigenspaces with $\mathfrak{g}_+$ and $\mathfrak{g}_-$. Clearly, $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ where $\mathfrak{g}_- = \{X \in \mathfrak{g} \mid \phi_*(X) = -X\}$. For any $\beta \in \mathfrak{g}_-$, the real stabilizer subgroup G^ϕ_β is

$$G^\phi_\beta = \{g \in G^\phi \mid \mathrm{Ad}_g\beta = \beta\}. \tag{5.6}$$

Definition 5.3. Let (G, ϕ) be an involutive Lie group. We say the pair (G, ϕ) has the **free extension property** if for any $\beta \in \mathfrak{g}_-$ and any maximal elementary abelian 2-subgroup (see Definition 2.45) $D_\beta \subset G^\phi_\beta$, the induced morphism $H^*(BG^\phi_\beta; \mathbb{Z}_2) \to H^*(BD_\beta; \mathbb{Z}_2)$ is a free extension in the sense of Definition 5.1. We also say that a compact Lie group G has the free extension property if the pair (G, Id) has the free extension property.

Remark 5.2. Let G be a compact Lie group and consider the identity map $\mathrm{Id} : G \to G$ as the **trivial involution** on G. It is clear from the above definition that $\mathfrak{g}_- = (0)$ and the pair (G, Id) has the free extension property if and only if for any maximal elementary abelian 2-subgroup $D \subset G$, the induced morphism $H^*(BG; \mathbb{Z}_2) \to H^*(BD; \mathbb{Z}_2)$ is a free extension. Note that by Example 5.5, the compact Lie groups $\mathrm{U}(1)$, $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ have the free extension property.

Remark 5.3. Let an involutive Lie group (G, ϕ) have the free extension property. It follows from the definition that (G, ϕ) has the free extension property if and only if for any $\beta \in \mathfrak{g}_-$, the subgroup G^ϕ_β has the free extension property as a Lie group. In particular, $G^\phi = G^\phi_0$ has the free extension property as a Lie group.

Proposition 5.4. *If two involutive Lie groups (G_1, ϕ_1) and (G_2, ϕ_2) have the free extension property, then the involutive Lie group $(G_1 \times G_2, \phi_1 \times \phi_2)$ has also the free extension property.*

Proof. Let $G = G_1 \times G_2$ and $\phi = \phi_1 \times \phi_2$. Clearly, the pair (G, ϕ) is an involutive Lie group, $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $G^\phi = G_1^{\phi_1} \times G_2^{\phi_2}$. Let $\beta = \beta_1 + \beta_2 \in \mathfrak{g}_-$. A simple calculation shows that

$$G^\phi_\beta \cong G^{\phi_1}_{1,\beta_1} \times G^{\phi_2}_{2,\beta_2}, \tag{5.7}$$

where $G^{\phi_i}_{i,\beta_i}$ is the stabilizer of $\beta_i \in \mathfrak{g}_{i,-}$ in the real subgroup $G^{\phi_i}_i$. Let $D \subset G^\phi$ be a maximal elementary abelian 2-subgroup. So there are maximal elementary abelian 2-subgroups D_i in $G^{\phi_i}_{i,\beta_i}$ such that the induced morphism $H^*(BG^{\phi_i}_{i,\beta_i};\mathbb{Z}_2) \to H^*(BD_i;\mathbb{Z}_2)$ is a free extension. Therefore, it follows from Corollary 2.68.1 that

$$H^*(BD_i;\mathbb{Z}_2) \cong H^*(BG^{\phi_i}_{i,\beta_i};\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(F_i;\mathbb{Z}_2), \tag{5.8}$$

where $F_i = G^{\phi_i}_{i,\beta_i}/D_i$. Let $F = F_1 \times F_2$. By Proposition 2.20 and (5.8), we have

$$\begin{aligned}
H^*(BD;\mathbb{Z}_2) &\cong H^*(BD_1;\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(BD_2;\mathbb{Z}_2)\\
&\cong H^*(BG_1^{\phi_1};\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(F_1;\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(BG_2^{\phi_2};\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(F_2;\mathbb{Z}_2)\\
&\cong [H^*(BG_1^{\phi_1};\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(BG_2^{\phi_2};\mathbb{Z}_2)] \otimes_{\mathbb{Z}_2} [H^*(F_1;\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(F_2;\mathbb{Z}_2)]\\
&\cong H^*(B(G_1^{\phi_1} \times G_2^{\phi_2});\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(F_1 \times F_2;\mathbb{Z}_2)\\
&\cong H^*(BG^{\phi};\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(F;\mathbb{Z}_2).
\end{aligned}$$

That is, $H^*(BD;\mathbb{Z}_2)$ is a free $H^*(BG^\phi;\mathbb{Z}_2)$-module which implies that the induced map $H^*(BG^\phi;\mathbb{Z}_2) \to H^*(BD;\mathbb{Z}_2)$ is a free extension. Thus, the pair $(G_1 \times G_2, \phi_1 \times \phi_2)$ has the free extension property and the proof is complete. ■

Example 5.8. Let $\mathbb{T}^n = \mathrm{U}(1) \times \cdots \times \mathrm{U}(1)$, n-times, be the n-torus and $D(n) = \mathrm{O}(1) \times \cdots \times \mathrm{O}(1)$, n-times. Clearly, $D(n)$ is a maximal elementary abelian 2-subgroup of $\mathbb{T}^n$ that determines a unique conjugacy class. By Example 5.3 and Proposition 5.4, we see that $H^*(B\mathbb{T}^n;\mathbb{Z}_2) \to H^*(BD(n);\mathbb{Z}_2)$ is a free extension. Thus the n-torus $\mathbb{T}^n$ has the free extension property. Also, it follows from Remark 5.1 and Example 5.4 that the map $H^*(B\mathrm{U}(n);\mathbb{Z}_2) \to H^*(BD(n);\mathbb{Z}_2)$ is a free extension. This implies that the Lie group $\mathrm{U}(n)$ has the free extension property.

Let $G = \mathrm{U}(n)$ and $\phi(A) = \overline{A}$ be the complex conjugation. So the real subgroup is the real orthogonal group $\mathrm{O}(n)$ and an easy calculation shows that $\phi_*(B) = \overline{B}$, for all $B \in \mathfrak{u}(n)$. It is clear that

$$\mathfrak{u}(n)_- = \{B = \sqrt{-1}B_0 \mid B_0 \in M(n;\mathbb{R}) \ \&\ B_0^t = B_0\}. \tag{5.9}$$

For any element $B \in \mathfrak{u}(n)_-$, let $\mathrm{U}(n)_B$ be the usual stabilizer subgroup and $\mathrm{O}(n)_B$ be the real stabilizer subgroup; i.e.,

$$\mathrm{O}(n)_B = \{A \in \mathrm{O}(n) \mid AB = BA\}. \tag{5.10}$$

Lemma 5.5. *If $B \in \mathfrak{u}(n)_-$, then there exist natural numbers $k_1, ..., k_p$ with $k_1 + \cdots + k_p = n$ such that*

$$\mathrm{O}(n)_B \cong \mathrm{O}(k_1) \times \cdots \times \mathrm{O}(k_p). \tag{5.11}$$

Proof. Since $B \in \mathfrak{u}(n)_-$, we have $B = \sqrt{-1}B_0$, where B_0 is a real symmetric matrix. By part 2 of the spectral theorem (Proposition 2.3), there exist a real orthogonal matrix

$D \in \mathrm{O}(n)$, distinct real numbers $\lambda_1, ..., \lambda_p$ and natural numbers $k_1, ..., k_p$ with $k_1 + \cdots + k_p = n$ such that

$$DB_0D^{-1} = \mathrm{Diag}[\lambda_1 \mathrm{Id}_{k_1}, \lambda_2 \mathrm{Id}_{k_2}, ..., \lambda_p \mathrm{Id}_{k_p}], \tag{5.12}$$

where Id_{k_j} is the identity matrix of order k_j. It is easy to see that the map $\Psi : \mathrm{O}(n)_B \to \mathrm{O}(n)_{DB_0D^{-1}}$ defined by $\Psi(A) = DAD^{-1}$ is an isomorphism. So

$$\mathrm{O}(n)_B \cong \mathrm{O}(n)_{DB_0D^{-1}}. \tag{5.13}$$

It follows easily from (5.13) that

$$\mathrm{O}(n)_{DB_0D^{-1}} \cong \mathrm{O}(k_1) \times \cdots \times \mathrm{O}(k_p), \tag{5.14}$$

where $k_1 + \cdots + k_p = n$. This completes the proof. ■

Let $\mathrm{SU}(n) = \{A \in \mathrm{U}(n) \mid \det A = 1\}$ be the special unitary group with the skew-Hermitian traceless matrices $\mathfrak{su}(n)$ as its Lie algebra. The restriction of $\phi : \mathrm{SU}(n) \to \mathrm{SU}(n)$ is a group involution. Here, again $\phi_*(A) = \overline{A}$, the real subgroup is $\mathrm{SO}(n)$, the real special orthogonal group, and

$$\mathfrak{su}(n)_- = \{B \in \mathfrak{su}(n) \mid \overline{B} = -B\}. \tag{5.15}$$

Lemma 5.6. *For any $B \in \mathfrak{su}(n)_-$, there exist natural numbers $k_1, ..., k_p$ with $k_1 + \cdots + k_p = n$ such that*

$$\mathrm{SO}(n)_B \cong \{A \in \mathrm{O}(k_1) \times \cdots \times \mathrm{O}(k_p) \mid \det A = 1\}. \tag{5.16}$$

In particular, if one of k_i (say k_1) is an odd number, then

$$\mathrm{SO}(n)_B \cong \mathrm{SO}(k_1) \times \mathrm{O}(k_2) \times \cdots \times \mathrm{O}(k_p). \tag{5.17}$$

Proof. Suppose that $B \in \mathfrak{su}(n)_-$. Equation (5.16) is a direct consequence of (5.11). Let k_1 be odd and define $\Psi : \mathrm{SO}(n)_B \to \mathrm{SO}(k_1) \times \mathrm{O}(k_2) \times \cdots \times \mathrm{O}(k_p)$ by

$$\Psi(A) = ((\det A_1)A_1, A_2, ..., A_p). \tag{5.18}$$

A simple argument shows that Ψ is an isomorphism. This proves (5.17) and completes the proof. ■

Proposition 5.7. *Let $\phi : M(n;\mathbb{C}) \to M(n;\mathbb{C})$ be the complex conjugation. Then the involutive Lie groups $(\mathrm{U}(n), \phi)$ and $(\mathrm{SU}(n), \phi)$ have the free extension property.*

Proof. Consider $\phi : \mathrm{U}(n) \to \mathrm{U}(n)$. So the real subgroup is $\mathrm{U}(n)^\phi = \mathrm{O}(n)$. Let $D(n)$ be the subgroup of diagonal matrices in $\mathrm{O}(n)$. We know from Example 2.16 that $D(n)$ is a maximal elementary abelian 2-subgroup of $\mathrm{O}(n)$ that determines a unique conjugacy class. On the other hand, for any $C \in \mathfrak{u}(n)_-$, Lemma 5.5 implies that

$$\mathrm{U}(n)^\phi_C = \mathrm{O}(n_1) \times \cdots \times \mathrm{O}(n_k),$$

where $n_1 + \cdots + n_k = n$. Set

$$D = D(n_1) \times \cdots \times D(n_k),$$

in which each $D(n_i)$ is a maximal elementary abelian 2-subgroup of each $\mathrm{O}(n_i)$ that determines a unique conjugacy class. So D is also a maximal elementary abelian 2-subgroup that determines a unique conjugacy class. By Proposition 5.4 and Remark 5.2, the map $H^*(B\mathrm{U}(n)^\phi_C; \mathbb{Z}_2) \to H^*(BD; \mathbb{Z}_2)$ is a free extension. This shows that the involutive Lie group $(\mathrm{U}(n), \phi)$ has the free extension property.

Now consider $\phi : \mathrm{SU}(n) \to \mathrm{SU}(n)$ and let $C \in \mathfrak{su}(n)_-$, then Lemma 5.6 implies that

$$\mathrm{SU}(n)^\phi_C = \{A \in \mathrm{O}(n)_C \mid \det A = 1\}.$$

If D_s is the subgroup of diagonal matrices in $\mathrm{SU}(n)^\phi_C$, then $D_s = D \cap \mathrm{SU}(n)^\phi_C$. In this case, we can easily see that the quotient groups $\mathrm{U}(n)^\phi_C/D$ and $\mathrm{SU}(n)^\phi_C/D_s$ are the same, say F. So we have the following commutative diagram of classifying spaces:

$$\begin{tikzcd}
 & F \arrow[dl, "j"'] \arrow[dr, "i"] & \\
BD_s \arrow[rr] \arrow[d] & & BD \arrow[d] \\
B\mathrm{SU}(n)^\phi_C \arrow[rr] & & B\mathrm{U}(n)^\phi_C
\end{tikzcd}$$

Diagram 5.2: Commutative diagram of classifying spaces

Here, F is a product of quotient groups. By first part, the right hand side fibration is multiplicative in cohomology, so the induced map i^* is surjective. The commutativity implies that j^* is also surjective. Thus the Leray-Hirsch theorem shows that

$$H^*(BD_s; \mathbb{Z}_2) \cong H^*(B\mathrm{SU}(n)^\phi_C; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(F; \mathbb{Z}_2),$$

which means the induced map $H^*(B\mathrm{SU}(n)^\phi_C; \mathbb{Z}_2) \to H^*(BD_s; \mathbb{Z}_2)$ is a free extension. Thus the involutive Lie group $(\mathrm{SU}(n), \phi)$ has the free extension property and the proof is complete. ∎

Proposition 5.8. *Let $G = G_1 \times \cdots \times G_k$ where $G_i = \mathrm{U}(n_i)$ or $\mathrm{SU}(n_i)$ and $\phi = \phi_1 \times \cdots \times \phi_k$ where $\phi_i : G_i \to G_i$ is the complex conjugation. Then (G, ϕ) has the free extension property.*

Proof. By Proposition 5.7, each pair (G_i, ϕ_i) has the free extension property. Proposition 5.4 follows that (G, ϕ) has the free extension property. ∎

Example 5.9. Let $G = \mathrm{U}(n) \times \mathrm{U}(n)$ and $\phi : G \to G$ be defined by $\phi(A, B) = (\overline{B}, \overline{A})$. Clearly (G, ϕ) is an involutive Lie group and

$$G^\phi = \{(A, \overline{A}) \mid A \in \mathrm{U}(n)\} \cong \mathrm{U}(n). \tag{5.19}$$

Since $\mathfrak{g} = \mathfrak{u}(n) \times \mathfrak{u}(n)$, an easy computation shows that the induced involution $\phi_* : \mathfrak{g} \to \mathfrak{g}$ is defined by $\phi_*(X, Y) = (\overline{Y}, \overline{X})$. Therefore,

$$\mathfrak{g}_- = \{(\sqrt{-1}X, \sqrt{-1}X) \mid X \in M(n; \mathbb{R}) \ \& \ X^t = X\}. \tag{5.20}$$

Let $\beta = (\sqrt{-1}X, \sqrt{-1}X) \in \mathfrak{g}_-$. Since $X^t = X$, the spectral theorem for real symmetric matrices shows that X is a block diagonal matrix. A similar argument as in Lemma 5.5 implies that there are natural numbers $n_1, ..., n_k$ with $n_1 + \cdots + n_k = n$ such that

$$G_\beta^\phi \cong \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_k). \tag{5.21}$$

That is, the real subgroup is a product of unitary groups. If $D(n_j) \subset \mathrm{U}(n_j)$ is the subgroup of elements of order two and $D = D(n_1) \times \cdots \times D(n_k)$, then it is easy to see that each $D(n_j)$ is a maximal elementary abelian 2-subgroup that determines a unique conjugacy class (see example 2.16). Thus $D \subset G_\beta^\phi$ is also a maximal elementary abelian 2-subgroup that determines a conjugacy class. On the other hand, by Example 5.8, each map $H^*(B\mathrm{U}(n_j); \mathbb{Z}_2) \to H^*(BD(n_j); \mathbb{Z}_2)$ is a free extension. So Proposition 5.4 shows that the map $H^*(G_\beta^\phi; \mathbb{Z}_2) \to H^*(BD; \mathbb{Z}_2)$ is also a free extension. Therefore the involutive Lie group $(\mathrm{U}(n) \times \mathrm{U}(n), \phi)$ has the free extension property.

CHAPTER 6

Atiyah-Bott Argument

> "*Perfect spheres do not exist in the real world, but they do have reality. They exist in the human imagination —and that's the most important world there is.*"
>
> *—Sir Michael Atiyah (1929-present)*

In this chapter, we discuss the Atiyah-Bott Lemma. This lemma is of crucial importance in proving equivariant perfection. It gives some sufficient conditions guaranteeing that the equivariant Euler class of an equivariant vector bundle is not a zero divisor. To prove the real equivariant perfection theorem, we need to formulate a real version of the Atiyah-Bott Lemma which gives the conditions guaranteeing that the equivariant top Stiefel-Whitney class of an equivariant vector bundle is not a zero divisor.

6.1 Atiyah-Bott Lemma for Hamiltonians

This section summarizes the main ideas of the Atiyah-Bott argument from [42].

Consider a Hamiltonian system $\mathcal{H} = (M, \omega, G, \mu)$ and fix an invariant inner product on the Lie algebra. Let $\{S_\beta\}$ be the G-invariant stratification for the norm squared of the moment map μ described in Section 4.1. We can use the equivariant Thom-Gysin long exact sequence for the strata. Apply Proposition 2.34 to the pair $Y = \bigcup_{\alpha \leq \beta} S_\alpha$ and $X = \bigcup_{\alpha < \beta} S_\alpha$, we get the following long exact sequence:

$$\cdots \to H_G^{*-d_\beta}(S_\beta; \mathbb{Q}) \xrightarrow{i_\beta} H_G^*(\bigcup_{\alpha \leq \beta} S_\alpha; \mathbb{Q}) \xrightarrow{j_\beta} H_G^*(\bigcup_{\alpha < \beta} S_\alpha; \mathbb{Q}) \to \cdots \tag{6.1}$$

where $d_\beta = \text{codim } S_\beta$, the codimension of the stratum S_β. An interesting property of sequence (6.1) is that if the equivariant Euler class of the normal bundle of the stratum S_β is not a zero divisor in the G-equivariant cohomology with rational coefficients, then a similar argument as in Proposition 2.34 shows that i_β is an injection and the long exact sequence (6.1) breaks into short exact sequences as follows:

$$0 \to H_G^{*-d_\beta}(S_\beta; \mathbb{Q}) \xrightarrow{i_\beta} H_G^*(\bigcup_{\alpha \leq \beta} S_\alpha; \mathbb{Q}) \xrightarrow{j_\beta} H_G^*(\bigcup_{\alpha < \beta} S_\alpha; \mathbb{Q}) \to 0 \tag{6.2}$$

The exactness of sequence (6.2) implies that each map j_β is a surjection. This is the key point in the proof of equivariant perfection and Kirwan surjectivity which is based on the condition that the equivariant Euler class of the normal bundle is not a zero divisor. In general, the conditions for having a nonzero divisor equivariant Euler class for equivariant vector bundles are provided by a Lemma due to Atiyah and Bott ([3], Proposition 13.4). We state it here:

Proposition 6.1 (**Atiyah-Bott Lemma for $\mathbb{Q}$-cohomology**). *Let $\pi : E \to X$ be a G-equivariant vector bundle where G is a compact Lie group and X is a connected space. If there exists a subtorus T_0 of G that acts trivially on X and fixes no nonzero vectors, then the equivariant Euler class $\mathrm{Eul}_G(E)$ of E is not a zero divisor in the equivariant cohomology ring $H_G^*(X; \mathbb{Q})$.*

An important part of the proof of Proposition 6.1 is that the natural map $H_T^*(X; \mathbb{Q}) \to H_G^*(X; \mathbb{Q})$ is always injective, for any maximal torus $T \subset G$ which contains the subtorus T_0. Since all maximal tori are conjugate, this injectivity doesn't depend on the choice of a maximal torus. In the real version, we use the coefficient field $\mathbb{Z}_2$ instead of the rational numbers and this makes things more complicated. The role of the equivariant Euler class is played by the equivariant top Stiefel-Whitney class of the equivariant vector bundle. In order to formulate a real version of this lemma we need two things. Firstly, instead of a subtorus, we demand an elementary abelian 2-subgroup $D_0 \subset G$ that acts trivially on X and fixes no non-zero vectors in the normal bundle. Secondly, we have to find a maximal elementary abelian 2-subgroup D containing D_0 the natural map $H^*_G(X;\mathbb{Z}_2) \to H^*_D(X;\mathbb{Z}_2)$ is injective and the injectivity doesn't depend on the choice of D. The injectivity property does not hold for compact Lie groups in general. Because of these conditions, we have to consider two properties. The first one is called the 2-primitivity which guarantees the existence of an elementary abelian 2-subgroup $D_0 \subset G$ and the second one is the very free extension property which provides us with maximal elementary abelian 2-subgroups implying injectivity. We will discuss these properties in the next section.

6.2 Atiyah-Bott Lemma for Real Hamiltonians

Our goal in this section is to prove a version of Atiyah-Bott Lemma for $\mathbb{Z}_2$-cohomology. Here, the role of a subtorus is played by an elementary abelian 2-subgroup.

Definition 6.1. Let G be a compact Lie group. An equivariant G-vector bundle $E \to X$ over a connected space X is called **2-primitive** if the action of $\pi_1(BG)$ on $H^*(X;\mathbb{Z}_2)$ is trivial and there exists an elementary abelian 2-subgroup D_0 of G that acts trivially on X and fixes no nonzero vectors in E.

Example 6.1. Consider the subgroup

$$G = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \mid A \in \mathrm{SO}(2) \right\} \subset \mathrm{SO}(3).$$

The standard action of SO(3) on S^2 induces an action of G on S^2 (see Figure 6.1). Let $X = \{N, S\}$, two poles and consider the normal bundle $\nu(X)$ of X which is an equivariant G-vector bundle. Let $J = \mathrm{Diag}[-1,-1,1] \in G$ and consider the 2-subgroup $D_0 = \{\mathrm{Id}, J\} \subset G$. It is clear that $Jv = -v$ for any $v \in \nu(X)$. Thus the elementary abelian 2-subgroup D_0 fixes X and no nonzero vectors in the normal bundle $\nu(X)$. Since G is connected, the action of $\pi_(BG) = (0)$ on $H^*(X;\mathbb{Z}_2)$ is trivial and thus the equivariant vector bundle $\nu(X) \to X$ is 2-primitive.

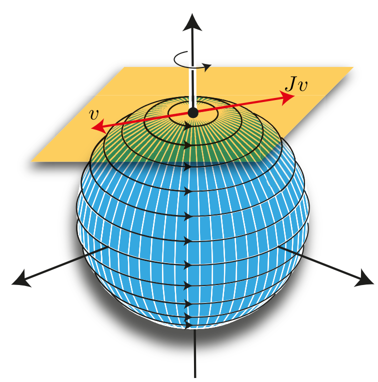

Figure 6.1: 2-primitive vector bundle

Proposition 6.2 (**Atiyah-Bott Lemma for $\mathbb{Z}_2$-cohomology**). *Let G be a compact Lie group and $\pi : E \to X$ be a G-vector bundle of rank m over a connected manifold X. If G has the free extension property and the G-vector bundle is 2-primitive, then the equivariant top Stiefel-Whitney class $w_m^G(E)$ is not a zero-divisor in $H_G^*(X;\mathbb{Z}_2)$.*

Proof. By the assumptions, the vector bundle $E \to X$ is 2-primitive, so $\pi_1(BG)$ acts trivially on $H^*(X;\mathbb{Z}_2)$ and there exists an elementary abelian 2-subgroup $D_0 \cong (\mathbb{Z}_2)^p$

of G that acts trivially on X and fixes no nonzero vectors in E. Choose a maximal elementary abelian 2-subgroup $D \cong (\mathbb{Z}_2)^n$ containing D_0. The inclusion map $i : D \hookrightarrow G$ induces the following commutative diagram of homotopy quotients:

$$\begin{array}{ccc} X_D & \xrightarrow{i_D} & X_G \\ {\scriptstyle q_D}\downarrow & & \downarrow{\scriptstyle q_G} \\ BD & \xrightarrow[Bi]{} & BG \end{array}$$

Diagram 6.1: Commutative diagram of homotopy quotients

By functorial properties of Stiefel-Whitney classes, it follows from Diagram 6.1 that

$$w_m^D(E) = i_D^*(w_m^G(E)). \tag{6.3}$$

Since G has the free extension property and D is a maximal elementary abelian 2-subgroup of G, it follows from Remark 5.2 that the map $H^*(BG;\mathbb{Z}_2) \to H^*(BD;\mathbb{Z}_2)$ is a free extension. Since $\pi_1(BG)$ acts trivially on $H^*(X;\mathbb{Z}_2)$, Proposition 5.3 implies that $i_G^* : H_D^*(X;\mathbb{Z}_2) \to H_D^*(X;\mathbb{Z}_2)$ is injective. Thus, if we show that the equivariant top Stiefel-Whitney class $w_m^D(E)$ is not a zero divisor, then the equivariant top Stiefel-Whitney class $w_m^G(E)$ is not a zero divisor too.

Since $D_0 \subset D$, there exists a subgroup $D_1 \subset D$ such that $D = D_0 \times D_1$ where $D_1 = (\mathbb{Z}_2)^q$ and $p + q = n$ (see Example 2.14). The action of D_0 on X is trivial, so by Proposition 2.9, we have

$$X_D \cong BD_0 \times X_{D_1}. \tag{6.4}$$

By the Kunneth formula, it follows from (6.4) that

$$H_D^*(X;\mathbb{Z}_2) \cong H^*(BD_0;\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H_{D_1}^*(X;\mathbb{Z}_2), \tag{6.5}$$

where $H^*(BD_0;\mathbb{Z}_2) = \mathbb{Z}_2[\sigma_1, ..., \sigma_p]$ (see Example 2.9). This formula makes $H_D^*(X;\mathbb{Z}_2)$ into a bigraded ring. Since $w_m^D(E) \in H_D^m(X;\mathbb{Z}_2)$, (6.5) implies that there exist some $\alpha_0 \in H^m(BD_0;\mathbb{Z}_2)$ and $\alpha' \in \bigoplus_{i=1}^m H^{m-i}(BD_0;\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H_{D_1}^i(X;\mathbb{Z}_2)$ such that

$$w_m^D(E) = \alpha_0 \otimes 1 + \alpha'. \tag{6.6}$$

We claim that if α_0 is nonzero, then $w_m^D(E)$ is not a zero divisor in $H_D^*(X;\mathbb{Z}_2)$. Assume that this is the case and without loss of generality choose a nonzero homogeneous element β in $[H^*(BD_0;\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H_{D_1}^*(X;\mathbb{Z}_2)]^k$. So by (6.5) there exist elements $\beta_{i,j} \in H^i(BD_0;\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H_{D_1}^j(X;\mathbb{Z}_2)$ with $i + j = k$ such that

$$\beta = \sum_{i+j=k} \beta_{i,j}. \tag{6.7}$$

Let (i_0, j_0) be such that $\beta_{i_0,j_0} \neq 0$ and $i_0 \geq i$ for all i. If $\beta' = \beta - \beta_{i_0,j_0}$, then (6.6) and (6.7) imply that

$$w_m^D(E) \cup \beta = (\alpha_0 \otimes 1) \cup \beta_{i_0,j_0} + (\alpha_0 \otimes 1) \cup \beta' + \alpha' \cup \beta \tag{6.8}$$

Since $H^*(BD_0;\mathbb{Z}_2)=\mathbb{Z}_2[\sigma_1,...,\sigma_p]$ is an integral domain and α_0 is a nonzero element, then the product map $\alpha_0\cup(-):H^*(BD_0;\mathbb{Z}_2)\to H^*(BD_0;\mathbb{Z}_2)$ is 1-1. Therefore, the tensor product map $(\alpha_0\otimes 1)\cup(-):H^*_D(X;\mathbb{Z}_2)\to H^*_D(X;\mathbb{Z}_2)$ is also 1-1. This implies that the term $(\alpha_0\otimes 1)\cup\beta_{i_0,j_0}$ in (6.8) is nonzero. On the other hand, since

$$\begin{cases}(\alpha_0\otimes 1)\cup\beta_{i_0,j_0}\in H^{m+i_0}(BD_0;\mathbb{Z}_2)\otimes_{\mathbb{Z}_2}H^{j_0}_{D_1}(X;\mathbb{Z}_2)\\ (\alpha_0\otimes 1)\cup\beta'+\alpha'\cup\beta\in\displaystyle\bigoplus_{i+j=k+m,i\neq i_0+m}H^i(BD_0;\mathbb{Z}_2)\otimes_{\mathbb{Z}_2}H^j_{D_1}(X;\mathbb{Z}_2),\end{cases}\tag{6.9}$$

it is clear that the component $(\alpha_0\otimes 1)\cup\beta_{i_0,j_0}$ of $w^D_m(E)\cup\beta$ is always nonzero. Thus (6.8) follows that $w^D_m(E)\cup\beta\neq 0$ and therefore $w^D_m(E)$ is not a zero-divisor in $H^*_D(X;\mathbb{Z}_2)$. This proves the first part.

It remains to show that $\alpha_0\neq 0$. Choose a point $x\in X$. The inclusion maps $i_x:\{x\}\hookrightarrow X$ and $i_0:D_0\hookrightarrow D$ induce the following pullback diagram:

$$\begin{tikzcd}(E_x)_{D_0} \arrow[r] \arrow[d] & E_{D_0} \arrow[r] \arrow[d] & E_D \arrow[d]\\ BD_0 \arrow[r, "Bi_x"'] & X_{D_0} \arrow[r, "i_{D_0}"'] & X_D\end{tikzcd}$$

Diagram 6.2: Commutative diagram of vector bundles

where E_x is the fiber over x. Since α_0 is the component of $w^D_m(E)$ in $H^*(BD_0;\mathbb{Z}_2)$, it follows from Diagram 6.2 that

$$\alpha_0=(i_{D_0}\circ Bi_x)^*\Big(w^D_m(E)\Big).\tag{6.10}$$

Thus, α_0 is the top Stiefel-Whitney class of the vector bundle $(E_x)_{D_0}\to BD_0$. Since D_0 fixes X, we see that E_x is a representation of D_0. But the subgroup D_0 is a finite product of $\mathbb{Z}_2$, so Example 2.19 implies that E_x is a direct sum of 1-dimensional representations; i.e.,

$$(E_x)_{D_0}=(E^1_x)_{D_0}\oplus\cdots\oplus(E^m_x)_{D_0},\tag{6.11}$$

where we can consider each of these 1-dimensional representations as an equivariant line bundle. Proposition 2.26 and (6.11) imply that

$$\alpha_0=w^{D_0}_m(E_x)=\prod_{i=1}^m w^{D_0}_1(E^i_x),\tag{6.12}$$

where each $w^{D_0}_1(E^i_x)$ is the top Stiefel-Whitney class of the corresponding line bundle $(E^i_x)_{D_0}\to BD_0\approx(\mathbb{RP}^\infty)^p$. Let $\pi_i:(\mathbb{RP}^\infty)^p\to\mathbb{RP}^\infty$ be the projection on the i^{th} component and $\gamma^\infty_1\to\mathbb{RP}^\infty$ be the tautological line bundle (see Example 2.8). Consider the pullback diagram

$$\begin{tikzcd}
(E^i_x)_{D_0} \arrow[r] \arrow[d] & \gamma_1^\infty \arrow[d] \\
(\mathbb{RP}^\infty)^p \arrow[r, "\pi_i"'] & \mathbb{RP}^\infty
\end{tikzcd}$$

Diagram 6.3: Commutative diagram of line bundles

It follows from the last diagram that

$$w_1^{D_0}(E^i_x) = \pi_i^*(w_1(\gamma_1^\infty)).$$

Since $w_1(\gamma_1^\infty) \neq 0$ (see Proposition2.26) and π_i^* is injective, we see that each Stiefel-Whitney class $w_1^{D_0}(E^i_x)$ is nonzero; i.e,

$$w_1^{D_0}(E^i_x) \neq 0,\ \forall i = 1, ..., m. \tag{6.13}$$

Since $H^*(BD_0; \mathbb{Z}_2) = \mathbb{Z}_2[\sigma_1, ..., \sigma_p]$ is an integral domain, it follows from (6.12) and (6.13) that $\alpha_0 \neq 0$. This completes the proof. ■

CHAPTER 7

Real Equivariant Perfection and Formality

"If one must choose between rigor and meaning, I shall unhesitatingly choose the latter."

—Rene Thom (1923-2002)

In this chapter, we formulate and prove our real equivariant perfection and formality theorems. After defining some specific properties, we use our previous results about the real Morse stratification to prove the equivariant perfection theorem (Theorem 7.1). At the end, we prove the equivariant formality for real Hamiltonians which we use mostly in our computations (Theorem 7.3).

7.1 Real Equivariant Perfection

In this section, we prove a real version of equivariant perfection for the restriction of the norm squared of the moment map to the real locus in a real Hamiltonian system.

Let $\mathcal{RH} = (M, \omega, G, \mu, \sigma, \phi)$ be a real Hamiltonian system where G is a compact connected Lie group and M is a compact connected manifold. Fix an Ad-invariant inner product on the Lie algebra $\mathfrak{g}$ and suppose that $f^\sigma : M^\sigma \to \mathbb{R}$ is the restriction of the norm squared of the moment map to the real locus M^σ. By Theorem 4.6, there exist a finite number of real Hamiltonian subsystems $\mathcal{RH}_{\beta_i,m} = (Z_{\beta_i,m}, \omega, G_{\beta_i}, \mu_{\beta_i,m}, \sigma, \phi)$ such that

- $\beta_i \in \mathfrak{g}^*_-$ and $m \in \{0, ..., \dim M\}$;
- the real locus is $Z^\sigma_{\beta_i,m}$ and the real subgroup is $G^\phi_{\beta_i}$;
- the critical set of f^σ is

$$C_{f^\sigma} = \coprod_{\beta_i,m} G^\phi \times_{G^\phi_{\beta_i}} (Z^\sigma_{\beta_i,m} \cap \mu^{-1}_{\beta_i,m}(0)). \tag{7.1}$$

Definition 7.1. Consider a real Hamiltonian system $\mathcal{RH} = (M, \omega, G, \mu, \sigma, \phi)$. We call the finite family $\{\mathcal{RH}_{\beta_i,m}\}$ induced by Theorem 4.6 the **generated real Hamiltonian subsystems** of $\mathcal{RH}$. A real Hamiltonian system $\mathcal{RH}$ is **2-primitive** if for any generated real Hamiltonian subsystem $\mathcal{RH}_{\beta_i,m}$ of $\mathcal{RH}$, the $G^\phi_{\beta_i}$-equivariant normal bundle of the real zero level set $Z^\sigma_{\beta_i,m} \cap \mu^{-1}_{\beta_i,m}(0)$ is a 2-primitive equivariant vector bundle in the sense of Definition 6.1.

Example 7.1. Consider the real Hamiltonian system $(M = (\mathbb{CP}^1)^n, \omega, \mathrm{U}(2), \mu, \sigma, \phi)$ in Example 3.13 with the Ad-invariant inner product $\langle A, B\rangle = -\mathrm{Tr}(AB)$ on the Lie algebra $\mathfrak{u}(2)$. Let $f^\sigma : M^\sigma \to \mathbb{R}$ be defined by $f^\sigma(p) = ||\mu(p)||^2$. We saw in Example 4.1 that the real locus of each generated real Hamiltonian subsystem $Z^\sigma_\beta = Z_\beta$ is a finite set which is defined by (4.42). Also the real stabilizer subgroup $\mathrm{O}(2)_\beta \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is the real stabilizer of matrix A_β defined by (4.37). Let

$$J_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.2}$$

It is easy to see that $J_\beta \in \mathrm{O}(2)_\beta$ and it fixes Z_β. In addition, for any $X = (x, y, z)^t \in S^2 - \{N, S\}$, we have

$$J_\beta(x, y, z)^t = (-x, -y, z)^t \tag{7.3}$$

Since $z \neq \pm 1$, neither x nor y is zero and thus (7.3) implies that $J_\beta X \neq X$. This means that the elementary abelian 2-subgroup $E_\beta = \{\mathrm{Id}, J_\beta\}$ fixes no nonzero vectors in the normal bundle of Z^σ_β. On the other hand, the action of $\pi_0(\mathrm{O}(2)_\beta)$ on the cohomology of finite space Z^σ_β is clearly trivial. Therefore each equivariant vector bundle $\nu(Z^\sigma_\beta) \to Z^\sigma_\beta$ is 2-primitive. That is, the real Hamiltonian system in this example is 2-primitive.

Example 7.2. Consider the real Hamiltonian system

$$(M = \mathrm{Gr}_{l_1}(\mathbb{C}^n) \times \cdots \times \mathrm{Gr}_{l_r}(\mathbb{C}^n), \omega, \mathrm{U}(n), \mu, \sigma, \phi)$$

in Example 3.15. Fix the Ad-invariant inner product $\langle A, B\rangle = -\mathrm{Tr}(AB)$ on the Lie algebra $\mathfrak{u}(n)$ and consider the restricted function $f^\sigma : M^\sigma \to \mathbb{R}$ where M^σ is the real locus $\mathrm{Gr}_{l_1}(\mathbb{R}^n) \times \cdots \times \mathrm{Gr}_{l_r}(\mathbb{R}^n)$. As we saw in Example 4.2, the real locus of each generated real Hamiltonian subsystem $Z^\sigma_{\beta,l}$ defined by (4.70) is compact. The real stabilizer subgroup in this example is $\mathrm{O}(n)_\beta \cong \mathrm{O}(m_1) \times \cdots \times \mathrm{O}(m_s)$ which is the real stabilizer of the matrix D_β defined by (4.61). It is easy to see that $\pi_1(\mathrm{O}(n)_\beta)$ acts trivially on $H^*(Z^\sigma_{\beta,l} \cap \mu^{-1}_{\beta,l}(0); \mathbb{Z}_2)$ (see [7], Proposition 6).

For $\beta = (\frac{k_1}{m_1}, ..., \frac{k_1}{m_1}, ..., \frac{k_s}{m_s}, ..., \frac{k_s}{m_s}) \in \Lambda^s$ and $l = (l_{ij}) \in \Gamma^s_\beta$, consider the subgroup H_β of the real stabilizer group $\mathrm{O}(n)_\beta$ as follows:

$$H_\beta = \left\{ \mathrm{Diag}\Big[\varepsilon_1 \mathrm{Id}_{m_1}, ..., \varepsilon_s \mathrm{Id}_{m_s}\Big] \;\Big|\; \varepsilon_i \in \{\pm 1\} \right\}. \tag{7.4}$$

Fix $V = (V_1, ..., V_r) \in Z^\sigma_{\beta,l}$. Since the normal bundle $N_V(Z^\sigma_{\beta,l})$ is the quotient space $T_V(M^\sigma)/T_V(Z^\sigma_{\beta,l})$, it follows from Proposition B.3 in Appendix B that

$$N_V(Z^\sigma_{\beta,l}) \cong \bigoplus_{\substack{1\le j\le r\\ 1\le i\ne i'\le s}} \mathrm{Hom}\Big(E_i \cap V_j, E_{i'}/(E_{i'} \cap V_j)\Big). \tag{7.5}$$

It is clear from (7.4), (7.5) and part 3 of Proposition B.3 in Appendix B that each element $J_\beta \in H_\beta$ acts on each element $T = \bigoplus T_{j,i,i'} \in N_V(Z^\sigma_{\beta,l})$ in such a way that

$$T_{j,i,i'} \longmapsto (\epsilon_i \epsilon_{i'}) T_{j,i,i'}.$$

Therefore, for any $T \in N_V(Z^\sigma_{\beta,l})$, we can always find an element $J_\beta \in H_\beta$ for which $J_\beta T \ne T$. This says that the action of the elementary abelian 2-subgroup H_β of $\mathrm{O}(n)_\beta$ on the normal bundle of $Z^\sigma_{\beta,l}$ fixes no nonzero vectors. Therefore the equivariant normal bundle of each real locus $Z^\sigma_{\beta,l}$ is 2-primitive. This means that the real Hamiltonian system in this example is 2-primitive.

Now, we state and prove our second main theorem which is a real version of the equivariant perfection theorem. We consider two main conditions regarding the group action (free extension property) and the manifold (2-primitivity).

Theorem 7.1 (**Real Equivariant Perfection**). *Let $(M, \omega, G, \mu, \sigma, \phi)$ be a real Hamiltonian system where G is a compact Lie group and M is a compact connected manifold. Fix an* Ad*-invariant inner product on $\mathfrak{g} = \mathrm{Lie}(G)$ and let $f^\sigma : M^\sigma \to \mathbb{R}$ be the restriction of the norm squared of the moment map to the real locus M^σ. If the pair (G, ϕ) has the free extension property and the real Hamiltonian system is 2-primitive, then the function f^σ is equivariantly perfect over the field $\mathbb{Z}_2$.*

Proof. By parts 1 and 2 of Theorem 4.6, the critical set of f^σ is a finite collection $\{C^\sigma_{\beta_i,m}\}$ of disjoint closed G^ϕ-invariant subsets and there exists a smooth invariant Morse stratification $\{S^\sigma_{\beta_i,m}\}$ such that each stratum $S^\sigma_{\beta_i,m}$ has a constant codimension $d(\beta_i,m)$ and deformation retracts onto the corresponding critical set $C^\sigma_{\beta_i,m}$. Consider the generated real Hamiltonian subsystems $(Z_{\beta_i,m},\omega,G_{\beta_i},\mu_{\beta_i,m},\sigma,\phi)$. By part 3 of Theorem 4.6, we have

$$C^\sigma_{\beta_i,m} \cong G^\phi \times_{G^\phi_{\beta_i}} (Z^\sigma_{\beta_i,m} \cap \mu^{-1}_{\beta_i,m}(0)), \tag{7.6}$$

where $Z^\sigma_{\beta_i,m} = M^\sigma \cap Z_{\beta_i,m}$ and $G^\phi_{\beta_i} = G^\phi \cap G_{\beta_i}$.

Apply the equivariant Thom-Gysin sequence to the invariant Morse stratification $\{S^\sigma_{\beta_i,m}\}$ regarding the partial order: $C^\sigma_{\beta_i,m} < C^\sigma_{\gamma_j,m'}$ if $f^\sigma(C^\sigma_{\beta_i,m}) < f^\sigma(C^\sigma_{\gamma_j,m'})$. For any $C^\sigma_{\beta_i,m}$, set

$$Y_{\beta_i,m} = \bigcup_{C^\sigma_{\beta_i,m} \le C^\sigma_{\gamma_j,m'}} S_{\gamma_j,m'},\ X_{\beta_i,m} = S^\sigma_{\beta_i,m},\ W_{\beta_i,m} = \bigcup_{C^\sigma_{\beta_i,m} < C^\sigma_{\gamma_j,m'}} S_{\gamma_j,m'}. \tag{7.7}$$

By applying Proposition 2.30 to the pair $(Y_{\beta_i,m}, X_{\beta_i,m})$, we get the following long exact sequence in the equivariant cohomology with $\mathbb{Z}_2$-coefficients:

$$\cdots \to H^{*-d(\beta_i,m)}_{G^\phi}(S^\sigma_{\beta_i,m};\mathbb{Z}_2) \xrightarrow{\Phi_{\beta_i,m}} H^*_{G^\phi}(Y_{\beta_i,m};\mathbb{Z}_2) \xrightarrow{\Psi_{\beta_i,m}} H^*_{G^\phi}(W_{\beta_i,m};\mathbb{Z}_2) \to \cdots \tag{7.8}$$

On one hand, by Proposition 2.30, the map $\Phi_{\beta_i,m}$ is injective if the G^ϕ-equivariant top Stiefel-Whitney class $w^{G^\phi}(S^\sigma_{\beta_i,m})$ of the normal bundle of the stratum $S^\sigma_{\beta_i,m}$ in M^σ is not a zero divisor. On the other hand, since $S^\sigma_{\beta_i,m}$ deformation retracts onto $C^\sigma_{\beta_i,m}$, from (7.6) and Proposition 2.18, it follows that

$$H^*_{G^\phi}(S^\sigma_{\beta_i,m};\mathbb{Z}_2) \cong H^*_{G^\phi}(C^\sigma_{\beta_i,m};\mathbb{Z}_2) \cong H^*_{G^\phi_{\beta_i}}(Z^\sigma_{\beta_i,m} \cap \mu^{-1}_{\beta_i,m}(0);\mathbb{Z}_2). \tag{7.9}$$

Thus, the equivariant top Stiefel-Whitney class $w^{G^\phi}(S^\sigma_{\beta_i,m})$ can be identified with the $G^\phi_{\beta_i}$-equivariant top Stiefel-Whitney class $w^{G^\phi_{\beta_i}}(Z^\sigma_{\beta_i,m} \cap \mu^{-1}_{\beta_i,m}(0))$. If we show that this class is not a zero divisor, then (7.9) implies that $w^{G^\phi}(S^\sigma_{\beta_i,m})$ is not a zero divisor.

By the assumptions, the real Hamiltonian is 2-primitive, so the equivariant normal bundle $\nu(Z^\sigma_{\beta_i,m} \cap \mu^{-1}_{\beta_i,m}(0)) \to Z^\sigma_{\beta_i,m} \cap \mu^{-1}_{\beta_i,m}(0)$ is 2-primitive. On the other hand, (G,ϕ) has the free extension property, hence each real subgroup $G^\phi_{\beta_i}$ has the free extension property. 2-primitivity implies that the action of $\pi_1(BG^\phi_{\beta_i})$ on $H^*(Z^\sigma_{\beta_i,m} \cap \mu^{-1}_{\beta_i,m}(0);\mathbb{Z}_2)$ is trivial. Therefore, the conditions of the real Atiyah-Bott Lemma (Proposition 6.2) are satisfied for each $G^\phi_{\beta_i}$-equivariant vector bundle $\nu(Z^\sigma_{\beta_i,m} \cap \mu^{-1}_{\beta_i,m}(0)) \to Z^\sigma_{\beta_i,m} \cap \mu^{-1}_{\beta_i,m}(0)$. This means the equivariant top Stiefel-Whitney class of this normal bundle is not a zero divisor in $H^*_{G^\phi_{\beta_i}}(Z^\sigma_{\beta_i,m} \cap \mu^{-1}_{\beta_i,m}(0);\mathbb{Z}_2)$. Proposition 2.30 now follows that each map $\Phi_{\beta_i,m}$ is an injection. Since the sequence in (7.8) is exact and $\Phi_{\beta_i,m}$ is injective, the sequence breaks into short exact sequences and thus we get a finite number of direct sums as follows:

$$H^*_{G^\phi}(Y_{\beta_i,m};\mathbb{Z}_2) \cong H^*_{G^\phi}(W_{\beta_i,m};\mathbb{Z}_2) \oplus H^{*-d(\beta_i,m)}_{G^\phi}(S^\sigma_{\beta_i,m};\mathbb{Z}_2), \tag{7.10}$$

Since $M^\sigma = \cup S^\sigma_{\beta_i,m}$, it follows by induction from (7.10) that

$$\dim H^*_{G^\phi}(M^\sigma;\mathbb{Z}_2) = \sum_{\beta_i,m} \dim H^{*-d(\beta_i,m)}_{G^\phi}(S^\sigma_{\beta_i,m};\mathbb{Z}_2). \tag{7.11}$$

From (7.9) and (7.11), it is implied that

$$\mathbf{P}_{G^\phi}(M^\sigma,t;\mathbb{Z}_2) = \sum_{\beta_i,m} t^{d(\beta_i,m)}\mathbf{P}_{G^\phi}(C^\sigma_{\beta_i,m},t;\mathbb{Z}_2) = \mathbf{M}_{G^\phi}(f^\sigma,t;\mathbb{Z}_2).$$

This means that f^σ is G^ϕ-equivariantly perfect over the field $\mathbb{Z}_2$ and the proof is complete. ■

Corollary 7.1.1 (**Real Surjectivity**). *The real Kirwan map $\kappa_\mathbb{R} : H^*_{G^\phi}(M^\sigma;\mathbb{Z}_2) \to H^*_{G^\phi}(M^\sigma_0;\mathbb{Z}_2)$ induced by the inclusion $i_0 : M^\sigma_0 \hookrightarrow M^\sigma$ is a surjection.*

Proof. Let $C^\sigma_0 < C^\sigma_1 < \cdots < C^\sigma_k$ be the critical subsets of f^σ. On one hand, we saw in the proof of Theorem 7.1 that for any critical subset C^σ_i of f^σ, the map $\Phi_{C^\sigma_i}$ in sequence (7.8) is injective. It follows from exactness that $\Psi_{C^\sigma_i}$ is surjective. On the other hand, we have the following sequence of inclusions:

$$M^\sigma_0 = S_{C^\sigma_0} \overset{j_1}{\hookrightarrow} S_{C^\sigma_0} \cup S_{C^\sigma_1} \overset{j_2}{\hookrightarrow} \cdots \overset{j_k}{\hookrightarrow} S_{C^\sigma_0} \cup \cdots \cup S_{C^\sigma_k} = M^\sigma, \tag{7.12}$$

such that $i_0 = j_k \circ \cdots \circ j_1$. Since $\Psi_{C^\sigma_i} = j^*_i$, the induced map $i^*_0 : H^*_{G^\phi}(M^\sigma;\mathbb{Z}_2) \to H^*_{G^\phi}(M^\sigma_0;\mathbb{Z}_2)$ is the composition of maps $\Psi_{C^\sigma_i}$; i.e.,

$$i^*_0 = \Psi_{C^\sigma_m} \circ \cdots \circ \Psi_{C^\sigma_1}. \tag{7.13}$$

Since each $\Psi_{C^\sigma_i}$ is surjective, (7.13) follows that $\kappa_\mathbb{R} = i^*_0 : H^*_{G^\phi}(M^\sigma;\mathbb{Z}_2) \to H^*_{G^\phi}(M^\sigma_0;\mathbb{Z}_2)$ is a surjection which completes the proof. ■

Remark 7.1. In the proof of Theorem 7.1, we saw that

$$\mathbf{P}_{G^\phi}(M^\sigma,t;\mathbb{Z}_2) = \sum_{\beta_i,m} t^{d(\beta_i,m)}\mathbf{P}_{G^\phi}(C^\sigma_{\beta_i,m},t;\mathbb{Z}_2), \tag{7.14}$$

and by (7.11) we have

$$\mathbf{P}_{G^\phi}(C^\sigma_{\beta_i,m},t;\mathbb{Z}_2) = \mathbf{P}_{G^\phi_{\beta_i}}(Z^\sigma_{\beta_i,m} \cap \mu^{-1}_{\beta_i,m}(0),t;\mathbb{Z}_2). \tag{7.15}$$

If $C^{\sigma}_{0,0}$ is the critical set corresponding to the global minimum 0 of f^{σ}, then from (7.14) and (7.15), it follows that

$$\begin{aligned}\mathbf{P}_{G^{\phi}}(M^{\sigma}\cap\mu^{-1}(0),t;\mathbb{Z}_2) = \mathbf{P}_{G^{\phi}}(M^{\sigma},t;\mathbb{Z}_2)\\ -\sum_{C^{\sigma}_{\beta_i,m}\neq C^{\sigma}_{0,0}} t^{d(\beta_i,m)}\mathbf{P}_{G^{\phi}_{\beta_i}}(Z^{\sigma}_{\beta_i,m}\cap\mu^{-1}_{\beta_i,m}(0),t;\mathbb{Z}_2).\end{aligned} \tag{7.16}$$

Therefore, (7.16) gives us a recursive formula by which we can compute the $\mathbb{Z}_2$- equivariant Betti numbers of the real zero level set $M^{\sigma}_0 = M^{\sigma}\cap\mu^{-1}(0)$ provided that we know the equivariant Poincaré series of the real locus M^{σ} and the equivariant Poincaré series of all real zero level sets $Z^{\sigma}_{\beta_i,m}\cap\mu^{-1}_{\beta_i,m}(0)$ of the generated real Hamiltonian subsystems.

7.2 Real Equivariant Formality

In this section, we state and prove equivariant formality for the real locus in a real Hamiltonian system. To do so, we need a special kind of the free extension property and a result due to Biss-Guillemin-Holm [10] regarding the equivariant formality for abelian real Hamiltonians whose group involution is the inversion map.

We know that a compact G-space X is equivariantly formal over a field $\mathbb{F}$ if the Serre spectral sequence of the fibration $X \hookrightarrow X_G \to BG$ collapses at the E_2-term; i.e.,

$$H^*_G(X;\mathbb{F}) = H^*(BG;\mathbb{F}) \otimes_{\mathbb{F}} H^*(X;\mathbb{F}). \tag{7.17}$$

By Proposition 2.65, this is equivalent to saying that the corresponding Poincaré series over the field $\mathbb{F}$ satisfies the following equality:

$$\mathbf{P}_G(X,t;\mathbb{F}) = \mathbf{P}(BG,t;\mathbb{F})\mathbf{P}(X,t;\mathbb{F}). \tag{7.18}$$

A theorem of Kirwan (see [42], Theorem 5.8) shows that a compact Hamiltonian G-system is always equivariantly formal over the field $\mathbb{Q}$ if G is a compact connected Lie group. For a real Hamiltonian system in which the group is a torus and the group involution is the inversion map, equivariant perfection has been proved by Biss-Guillemin-Holm (see [10], Theorem B). They have proved the following:

Proposition 7.2 (**Biss-Guillemin-Holm**). *Let $(M,\omega,T,\mu,\sigma,\phi)$ be a real Hamiltonian system in which M is a compact connected manifold, T is a torus and $\phi(g) = g^{-1}$ for any $g \in T$. If M^σ is the real locus and T^ϕ is the real subgroup, then*

$$H^*_{T^\phi}(M^\sigma;\mathbb{Z}_2) \cong H^*(BT^\phi;\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(M^\sigma;\mathbb{Z}_2) \tag{7.19}$$

as graded modules over $H^*(BT^\phi;\mathbb{Z}_2)$.

We use this result to prove a real version of equivariant perfection for nonabelian real Hamiltonian systems. First, we have the following definition.

Definition 7.2. We say that a compact connected involutive Lie group (G,ϕ) has the **special free extension property** if the following hold.

1. The pair (G,ϕ) has the free extension property.
2. There exists a maximal torus $T \subset G$ such that $\phi(g) = g^{-1}$ for all $g \in T$ and T^ϕ is a maximal elementary abelian 2-subgroup of G^ϕ.

Example 7.3. Consider the involutive compact connected Lie group (G,ϕ) where $G = \mathrm{U}(n)$ or $\mathrm{SU}(n)$ and $\phi : G \to G$ is the complex conjugation. First note that by Proposition 5.7, (G,ϕ) has the free extension property. Let $T \subset G$ and $D \subset G^\phi$ be the subgroups of diagonal matrices respectively. We can easily see that $T \subset G$ is a maximal torus and $D = T^\phi$ is a maximal elementary abelian 2-subgroup. Also, we have $\phi(A) = A^{-1}$ for any $A \in T$. Thus the pair (G,ϕ) has the special free extension property.

Theorem 7.3 (**Real Equivariant Formality**). *Let $(M, \omega, G, \mu, \sigma, \phi)$ be a real Hamiltonian system where G is a compact connected Lie group and M is a compact connected manifold. If (G, ϕ) has the special free extension property, then the real locus M^σ is G^ϕ-equivariantly formal over the field $\mathbb{Z}_2$.*

Proof. Since (G, ϕ) has the special free extension property, there exists a maximal torus $T \subset G$ such that $\phi(g) = g^{-1}$ for any $g \in T$. Let $\mu_T : M \to \mathfrak{t}^*$ be the composition of μ with the orthogonal projection $\mathrm{Pr}_{\mathfrak{t}^*} : \mathfrak{g}^* \to \mathfrak{t}^*$ onto $\mathfrak{t}^*$. It is easy to see that $(M, \omega, T, \mu_T, \sigma, \phi_T)$ is a real Hamiltonian system such that $\phi_T : T \to T$ is the inversion map. Thus by the Biss-Guillemin-Holm theorem, (7.19) holds for the real locus M^σ of this real Hamiltonian system. Since (G, ϕ) has the free extension property, by Remark 5.3 the real subgroup G^ϕ has the free extension property. But T^ϕ is a maximal elementary abelian 2-subgroup of G^ϕ, thus the induced map $H^*(BG^\phi; \mathbb{Z}_2) \to H^*(BT^\phi; \mathbb{Z}_2)$ is a free extension and we have

$$H^*(BT^\phi; \mathbb{Z}_2) \cong H^*(BG^\phi; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(G^\phi/T^\phi; \mathbb{Z}_2), \tag{7.20}$$

as graded $H^*(BG^\phi; \mathbb{Z}_2)$-modules. Consider the following commutative diagram:

$$\begin{tikzcd}
 & & M^\sigma \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\
 & M^\sigma_{T^\phi} \arrow[rr] \arrow[dd] & & M^\sigma_{G^\phi} \arrow[dd] \\
G^\phi/T^\phi \arrow[ur, "j_1"] \arrow[dr, "j_2"'] & & & \\
 & BT^\phi \arrow[rr] & & BG^\phi
\end{tikzcd}$$

Diagram 7.1: Commutative diagram of fibrations

It follows from (7.20) that the Serre spectral sequence of the fibration $G^\phi/T^\phi \hookrightarrow BT^\phi \to BG^\phi$ collapses at page 2. So by Proposition 2.68, the induced map j_2^* is surjective. The commutativity of Diagram 7.1 implies that the induced map j_1^* is also surjective and thus by Proposition 2.68, the Serre spectral sequence of the fibration $G^\phi/T^\phi \hookrightarrow M^\sigma_{T^\phi} \to M^\sigma_{G^\phi}$ must collapse at page two; i.e.,

$$H^*_{T^\phi}(M^\sigma; \mathbb{Z}_2) \cong H^*_{G^\phi}(M^\sigma; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(G^\phi/T^\phi; \mathbb{Z}_2), \tag{7.21}$$

as graded $H^*_{G^\phi}(M^\sigma; \mathbb{Z}_2)$-modules. By (7.19), (7.20) and (7.21), we have

$$\mathbf{P}(M^\sigma_{G^\phi}, t; \mathbb{Z}_2)\mathbf{P}(G^\phi/T^\phi, t; \mathbb{Z}_2) = \mathbf{P}(M^\sigma, t; \mathbb{Z}_2)\mathbf{P}(BG^\phi, t; \mathbb{Z}_2)\mathbf{P}(G^\phi/T^\phi, t; \mathbb{Z}_2), \tag{7.22}$$

which implies that $\mathbf{P}(M^{\sigma}_{G^{\phi}}, t; \mathbb{Z}_2) = \mathbf{P}(M^{\sigma}, t; \mathbb{Z}_2)\mathbf{P}(BG^{\phi}, t; \mathbb{Z}_2)$. Now, Proposition 2.65 follows that the Serre spectral sequence of the fibration $M^{\sigma} \hookrightarrow M^{\sigma}_{G^{\phi}} \to BG^{\phi}$ collapses at page 2 and thus

$$H^*_{G^{\phi}}(M^{\sigma}; \mathbb{Z}_2) \cong H^*(BG^{\phi}; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(M^{\sigma}; \mathbb{Z}_2), \tag{7.23}$$

as graded $H^*(BG^{\phi}; \mathbb{Z}_2)$-modules. This means that M^{σ} is G^{ϕ}-equivariantly formal over the field $\mathbb{Z}_2$ and the proof is complete. ■

CHAPTER 8

Computation of Mod Two Betti Numbers

> “*Throughout my whole life as a mathematician, the possibility of making explicit, elegant computations has always come out by itself, as a byproduct of a thorough conceptual understanding of what was going on.*"
>
> —*Alexander Grothendieck (1928-2014)*

In this chapter, by using our results, we compute the $\mathbb{Z}_2$-Betti numbers of the real reduction of two real Hamiltonian systems. The first one is a real Hamiltonian system in which the unitary group $\mathrm{U}(2)$ acts on a product of complex projective lines with complex conjugation as the group involution and the involution on the space is induced by the complex conjugation on 2-complex vector space. The real locus in this system is a product of real projective lines and the real subgroup is the orthogonal group $\mathrm{O}(2)$. The other example is a generalization of the first one. We consider the action of the unitary group $\mathrm{U}(n)$ on a product of complex Grasmannians. By using complex conjugation as involutions, the real subgroup is $\mathrm{O}(n)$ acting on a product of real Grassmannians. In both examples, we give an explicit formula for the $\mathbb{Z}_2$-Betti numbers of the real reductions.

8.1 Technical Lemmas

To consider our examples in next section, we need to compute the cohomology of some homotopy quotients whose action groups are orthogonal projective groups.

Let $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ be the orthogonal and the special orthogonal groups respectively and denote their central subgroups consisting of diagonal matrices by $D\mathrm{O}(n)$ and $D\mathrm{SO}(n)$ respectively. By the definition (see Appendix C), the corresponding projective groups are $\mathrm{PO}(n) = \mathrm{O}(n)/D\mathrm{O}(n)$ and $\mathrm{PSO}(n) = \mathrm{SO}(n)/D\mathrm{SO}(n)$ respectively.

Lemma 8.1. *Let n be an odd number and the orthogonal group $\mathrm{O}(n)$ act on a smooth manifold X such that the subgroup $D_2 = \{\pm\mathrm{Id}\}$ acts trivially and the projective group $\mathrm{PO}(n)$ acts freely. Then the homotopy quotient space $X_{\mathrm{O}(n)}$ is homotopy equivalent to the product space $BD_2 \times (X/\mathrm{O}(n))$; i.e.,*

$$X_{\mathrm{O}(n)} \simeq \mathbb{RP}^\infty \times (X/\mathrm{O}(n)). \tag{8.1}$$

Proof. Since n is odd, $\mathrm{PSO}(n) = \mathrm{SO}(n)$ and by Propositions C.1 and C.2 in Appendix C, we have $\mathrm{O}(n) \cong D_2 \times \mathrm{SO}(n)$ and thus $\mathrm{PO}(n) \cong \mathrm{SO}(n)$ as Lie groups. Fix universal bundles $ED_2 \to BD_2$ and $E\mathrm{SO}(n) \to B\mathrm{SO}(n)$ for Lie groups D_2 and $\mathrm{SO}(n)$ respectively. Consider the universal bundle $ED_2 \times E\mathrm{SO}(n) \to BD_2 \times B\mathrm{SO}(n)$ for the product group $D_2 \times \mathrm{SO}(n)$. In this case, since D_2 acts trivially on X, by Proposition 2.7, we have

$$\begin{aligned} X_{\mathrm{O}(n)} &\simeq (X \times E\mathrm{SO}(n) \times ED_2)/(\mathrm{SO}(n) \times D_2) \\ &\simeq [(X \times E\mathrm{SO}(n))/\mathrm{SO}(n)] \times (ED_2/D_2) \\ &\simeq X_{\mathrm{SO(n)}} \times BD_2. \end{aligned}$$

That is,

$$X_{\mathrm{O}(n)} \simeq BD_2 \times X_{\mathrm{SO}(n)}. \tag{8.2}$$

By the assumptions, $\mathrm{PO}(n) \cong \mathrm{SO}(n)$ acts freely on X, so we have the homotopy equivalence $X_{\mathrm{SO}(n)} \simeq X/\mathrm{SO}(n)$. On the other hand, since D_2 acts trivially on X, we have

$$X/\mathrm{O}(n) \simeq X/\mathrm{PO}(n) = X/\mathrm{SO}(n) \simeq X_{\mathrm{SO}(n)}. \tag{8.3}$$

Equations (8.2) and (8.3) follow (8.1) which completes the proof. ■

Example 8.1. Consider the real Hamiltonian system

$$(M = \mathrm{Gr}_{l_1}(\mathbb{C}^n) \times \cdots \times \mathrm{Gr}_{l_r}(\mathbb{C}^n), \omega, \mathrm{U}(n), \mu, \sigma, \phi)$$

in Example 3.15 and suppose that n is an odd number and numbers $\sum_{j=1}^r l_j$ and n are coprime. The real subgroup $\mathrm{O}(n)$ acts on the real locus

$$M^\sigma = \mathrm{Gr}_{l_1}(\mathbb{R}^n) \times \cdots \times \mathrm{Gr}_{l_r}(\mathbb{R}^n)$$

such that the action of subgroup $\{\pm\mathrm{Id}\}$ on M^σ is trivial. Let $M_0^\sigma = M^\sigma \cap \mu^{-1}(0)$ which is invariant under the action of $\mathrm{O}(n)$. By Remark 3.5, the projective group $\mathrm{PO}(n)$ acts freely on M_0^σ, so Lemma 8.1 follows that $(M_0^\sigma)_{\mathrm{O}(n)} \cong \mathbb{RP}^\infty \times \Big(M_0^\sigma/\mathrm{O}(n)\Big)$ and therefore

$$\mathbf{P}(M_0^\sigma/\mathrm{O}(n), t; \mathbb{Z}_2) = (1-t)\mathbf{P}_{\mathrm{O}(n)}(M_0^\sigma, t; \mathbb{Z}_2). \tag{8.4}$$

Lemma 8.2. *Let n be an even number and X be a compact $\mathrm{O}(n)$-manifold such that the subgroup $D_2 = \{\pm\mathrm{Id}\}$ acts trivially and the projective group $\mathrm{PO}(n)$ acts freely. Suppose that there exist a section $s : X/\mathrm{PSO}(n) \to X$ for the principal bundle $\mathrm{PSO}(n) \hookrightarrow X \xrightarrow{\pi} X/\mathrm{PSO}(n)$ and an element $A \in \mathrm{O}(n) - \mathrm{SO}(n)$ such that $A^2 = \mathrm{Id}$ and $s[Ax] = As[x]$ where $\pi(x) = [x]$. Then we have the following homotopy equivalence:*

$$X_{\mathrm{O}(n)} \simeq \mathbb{RP}^\infty \times (X/\mathrm{O}(n)). \tag{8.5}$$

Proof. Since n is even, the central subgroups $D\mathrm{O}(n)$ and $D\mathrm{SO}(n)$ are D_2 and thus $\mathrm{PO}(n) = \mathrm{O}(n)/D_2$ and $\mathrm{PSO}(n) = \mathrm{SO}(n)/D_2$. Set $H_2 = \mathrm{PO}(n)/\mathrm{PSO}(n)$. It is clear that $H_2 \cong \langle A\rangle$, the cyclic subgroup generated by A. Suppose that $Y = X/\mathrm{PSO}(n)$ is the orbit space. Since $As[x] = s[Ax]$, it is easy to see that for any $B \in \mathrm{O}(n) - \mathrm{SO}(n)$

$$\pi(Bs[x]) = [Ax]. \tag{8.6}$$

The product group $D_2 \times \langle A\rangle$ acts on the product space $\mathrm{O}(n) \times Y$ as follows:

$$\begin{cases} (\pm\mathrm{Id}, \mathrm{Id}).(B, [x]) = (\pm B, [x]), \\ (\pm\mathrm{Id}, A).(B, [x]) = (\pm BA, [Ax]). \end{cases} \tag{8.7}$$

Denote the orbit of any $(B, [x]) \in \mathrm{O}(n) \times Y$ under this action by $[\![B, [x]]\!]$ and define the following map:

$$\begin{cases} \Psi : \mathrm{O}(n) \times_{(D_2\times H_2)} Y \to X \\ \Psi[\![B, [x]]\!] = Bs[x]. \end{cases} \tag{8.8}$$

Note that for any orbit $[\![B, [x]]\!] \in \mathrm{O}(n) \times_{(D_2\times H_2)} Y$, we have

$$[\![B, [x]]\!] = \Big\{(B, [x]), (-B, [x]), (BA, [Ax]), (-BA, [Ax])\Big\}, \tag{8.9}$$

which implies that

$$[\![B, [x]]\!] = [\![-B, [x]]\!]. \tag{8.10}$$

We claim that Ψ is an $\mathrm{O}(n)$-equivariant homeomorphism. By the assumptions, since $s[Ax] = As[x]$ and $A^2 = \mathrm{Id}$, we can easily see that Ψ is well-defined and continuous. Since $\mathrm{O}(n)$ acts on the product space $\mathrm{O}(n) \times_{(E_2\times H_2)} Y$ by

$$C.[\![B, [x]]\!] = [\![CB, [x]]\!], \quad \forall C \in \mathrm{O}(n),$$

it follows from (8.8) that Ψ is an $\mathrm{O}(n)$-equivariant map.

Let $\Psi[\![B_1, [x_1]]\!] = \Psi[\![B_2, [x_2]]\!]$. So $B_1 s[x_1] = B_2 s[x_2]$. It follows from (8.6) that $[Ax_1] = [Ax_2]$. Since s is 1-1, the last equality implies that $[x_1] = [x_2]$. So, we get $B_2^{-1} B_1 s[x_1] = s[x_1]$. Since the action of $\mathrm{PS(O)}(n)$ on X is free, we must have $B_2 = \pm B_1$. Therefore, (8.10) implies that $[\![B_1, [x_1]]\!] = [\![B_2, [x_2]]\!]$ which means Ψ is 1-1.

Now, let $x \in X$ and $[x] \in Y$. Since $\pi(s[x]) = \pi(x)$, there exists some $C \in \mathrm{SO}(n)$ such that $s[x] = Cx$. Thus

$$\Psi[\![C^{-1}, [x]]\!] = C^{-1} s[x] = C^{-1} C x = x,$$

which says that Φ is onto. Since spaces are compact and Hausdorff, Φ must be a homeomorphism and thus we get the following $\mathrm{O}(n)$-homeomorphism:

$$X \cong \mathrm{O}(n) \times_{(D_2 \times H_2)} Y. \tag{8.11}$$

Since $\mathrm{PO}(n)$ acts freely on X, then H_2 acts freely on Y and thus by quotient in stages we have $Y_{H_2} \simeq Y/H_2 \simeq X/\mathrm{O}(n)$. On the other hand, by the homotopy quotient in stages (Proposition 2.19) and homotopy quotient extension (Proposition 2.18), it follows from (8.11) that

$$\begin{aligned} X_{\mathrm{O}(n)} &\simeq \Big(\mathrm{O}(n) \times_{(D_2 \times H_2)} Y\Big)_{\mathrm{O}(n)} \\ &\simeq Y_{(D_2 \times H_2)} \\ &\simeq \Big(Y_{H_2}\Big)_{D_2} \\ &\simeq \Big(Y/H_2\Big)_{D_2} \\ &\simeq \Big(X/\mathrm{O}(n)\Big)_{D_2}. \end{aligned}$$

Since D_2 acts trivially on X and $BD_2 \simeq \mathbb{RP}^\infty$, the last equality implies that

$$X_{\mathrm{O}(n)} \simeq BD_2 \times (X/\mathrm{O}(n)) \simeq \mathbb{RP}^\infty \times (X/\mathrm{O}(n)).$$

This proves (8.5) and the proof is complete. ■

Example 8.2. Let $(M = (\mathbb{CP}^1)^n, \omega, \mathrm{U}(2), \mu, \sigma, \phi)$ be the real Hamiltonian system in Example 3.13. The real subgroup $\mathrm{O}(2)$ acts on the real locus $M^\sigma = (\mathbb{RP}^1)^n$ such that the action of the subgroup $\{\pm \mathrm{Id}\}$ on M^σ is trivial. We saw in Remark 3.4 that when n is an odd number, the projective group $\mathrm{PO}(2)$ act freely on the zero level set $M_0^\sigma = M^\sigma \cap \mu^{-1}(0)$. Since the action of $\mathrm{SO}(2) \cong \mathrm{PSO}(2)$ on the real locus M^σ is free, we have the following pullback diagram of principal bundles:

$$\begin{tikzcd}
 & \mathrm{PSO}(2) \arrow[dl] \arrow[dr] & \\
M_0^\sigma \arrow[rr] \arrow[d, "\pi_0"'] & & M^\sigma \arrow[d, "\pi"] \\
M_0^\sigma/\mathrm{PSO}(2) \arrow[rr] & & M^\sigma/\mathrm{PSO}(2)
\end{tikzcd}$$

Diagram 8.1: Pullback diagram of principal bundles

Suppose that $N = \{[1:0]\} \times \mathbb{RP}^1 \times \cdots \times \mathbb{RP}^1 \subset M^\sigma$ and define a map $s : M^\sigma/S^1 \to M^\sigma$ as follows. Let $x \in M^\sigma$ with the orbit $\mathcal{O}_x$. The intersection $\mathcal{O}_x \cap N$ contains a unique point $\widehat{x}$ (see Figure 8.1). Set $s(\mathcal{O}_x) = \widehat{x}$. Clearly, s is a continuous map and $\pi(s(\mathcal{O}_x)) = \pi(\widehat{x}) = \mathcal{O}_{\widehat{x}} = \mathcal{O}_x$. That is, s is a section for the principal bundle $\mathrm{SO}(2) \hookrightarrow M^\sigma \to M^\sigma/\mathrm{SO}(2)$. On the other hand, let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8.12}$$

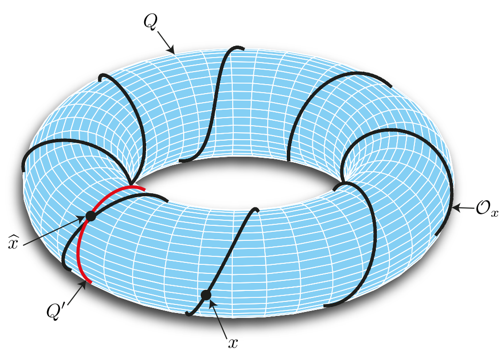

Figure 8.1: A trivial principal bundle

Clearly, $A \in \mathrm{O}(2) - \mathrm{SO}(2)$ and $A\binom{1}{0} = \binom{1}{0}$. This implies that $s(A\pi(x)) = As(\pi(x))$. The same is also true for the induced section $s_0 : M_0^\sigma/\mathrm{PSO}(2) \to M_0^\sigma$ of the principal subbundle $\mathrm{PSO}(2) \hookrightarrow M_0^\sigma \to M_0^\sigma/\mathrm{PSO}(2)$ and thus the assumptions of Lemma 8.2 are satisfied for the action of $\mathrm{O}(2)$ on the space M_0^σ. Therefore, we get the following homotopy equivalence:

$$(M_0^\sigma)_{\mathrm{O}(2)} \simeq \mathbb{RP}^\infty \times (M_0^\sigma/\mathrm{O}(2)), \tag{8.13}$$

which implies that

$$\mathbf{P}(M_0^\sigma/\mathrm{O}(2), t; \mathbb{Z}_2) = (1-t)\mathbf{P}_{\mathrm{O}(2)}(M_0^\sigma, t; \mathbb{Z}_2). \tag{8.14}$$

8.2 Betti Numbers of Real Reductions

In this section, we compute the $\mathbb{Z}_2$-Betti numbers of the real reductions in two real Hamiltonian systems: a product of projective lines and a product of Grassmannians.

Example 8.3. Let $n > 1$ be a natural number and consider the real Hamiltonian system $(M = (\mathbb{CP}^1)^n, \omega, \mathrm{U}(2), \mu, \sigma, \phi)$ in Example 3.13. In this case, the real locus is the real Grassmannian $M^\sigma = (\mathbb{RP}^1)^n$ and the real subgroup is $\mathrm{O}(2)$. Let $f^\sigma : M^\sigma \to \mathbb{R}$ be the restriction of the norm squared of the moment map induced by the standard invariant inner product on the Lie algebra $\mathfrak{u}(2)$. By Proposition 5.7, the pair $(\mathrm{U}(2), \phi)$ has the special free extension property and by Example 7.1 the real locus M^σ is 2-primitive. So we can apply Theorems 7.1 and 7.3 to this system. We also saw in Example 4.1 that each Z_β is a finite set with $\binom{n}{r}$ elements, $Z_\beta^\sigma = Z_\beta$, $\mu_\beta^{-1}(0) \subset Z_\beta$ and $d(\beta) = r - 1$. It follows from (7.16) that

$$\begin{aligned}\mathbf{P}_{\mathrm{O}(2)}(M_0^\sigma, t; \mathbb{Z}_2) = \mathbf{P}(M^\sigma, t; \mathbb{Z}_2)&\mathbf{P}(\mathrm{BO}(2), t; \mathbb{Z}_2) \\ &- \sum_{\beta \neq 0} t^{r-1} \mathbf{P}(Z_\beta, t; \mathbb{Z}_2)\mathbf{P}(\mathrm{BO}(2)_\beta, t; \mathbb{Z}_2).\end{aligned} \tag{8.15}$$

where $M_0^\sigma = M^\sigma \cap \mu^{-1}(0)$ and each β is in the indexing set Λ defined by (4.36). As we saw in Example 4.1, to each β it is associated a diagonal matrix A_β defined by (4.37).

Since A_β has two eigenvalues, $\mathrm{O}(2)_\beta \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and thus $B\mathrm{O}(2)_\beta \simeq B\mathbb{Z}_2 \times B\mathbb{Z}_2$. Since $\mathbf{P}(B\mathbb{Z}_2, t; \mathbb{Z}_2) = \frac{1}{1-t}$, we obtain

$$\mathbf{P}(Z_\beta, t; \mathbb{Z}_2)\mathbf{P}(\mathrm{BO}(2)_\beta, t; \mathbb{Z}_2) = \binom{n}{r} \frac{1}{(1-t)^2}. \tag{8.16}$$

On the other hand, $\mathrm{BO}(2) = G_2(\mathbb{R}^\infty)$, the infinite real Grassmanian and by Proposition B.1 in Appendix B, $\mathbf{P}(\mathrm{BO}(2), t; \mathbb{Z}_2) = (1-t)^{-1}(1-t^2)^{-1}$. Since $\mathbf{P}((\mathbb{RP}^1)^n, t; \mathbb{Z}_2) = (1+t)^n$, we get

$$\mathbf{P}(M^\sigma, t; \mathbb{Z}_2)\mathbf{P}(\mathrm{BO}(2), t; \mathbb{Z}_2) = \frac{(1+t)^n}{(1-t)(1-t^2)}. \tag{8.17}$$

By (8.16) and (8.17), we can rewrite (8.15) as follows:

$$\mathbf{P}_{\mathrm{O}(2)}(M_0^\sigma, t; \mathbb{Z}_2) = \frac{(1+t)^n}{(1-t)(1-t^2)} - \sum_{r > \frac{n}{2}}^{n} \binom{n}{r} \frac{t^{r-1}}{(1-t)^2}, \tag{8.18}$$

or

$$\mathbf{P}_{\mathrm{O}(2)}(M_0^\sigma, t; \mathbb{Z}_2) = \frac{1}{(1-t)^2} \sum_{r=0}^{n-1} \binom{n-1}{r} t^r - \sum_{r > \frac{n}{2} - 1}^{n-1} \binom{n}{r+1} \frac{t^r}{(1-t)^2}. \tag{8.19}$$

Suppose that m is the biggest natural number less than or equal to $\frac{n}{2}$. Then by simplifying the last equation, we get

$$\mathbf{P}_{\mathrm{O}(2)}(M_0^\sigma, t; \mathbb{Z}_2) = \frac{1}{(1-t)^2}\left[\sum_{r=0}^{m-1}\binom{n-1}{r}t^r - \sum_{r=m}^{n-1}\binom{n-1}{r+1}t^r\right]. \tag{8.20}$$

By Example 8.2 when n is odd, $\mathbf{P}(M_0^\sigma/\mathrm{O}(2), t; \mathbb{Z}_2) = (1-t)\mathbf{P}_{\mathrm{O}(2)}(M_0^\sigma, t; \mathbb{Z}_2)$. So (8.20) becomes

$$\mathbf{P}(M_0^\sigma/\mathrm{O}(2), t; \mathbb{Z}_2) = \frac{1}{(1-t)}\left[\sum_{r=0}^{m-1}\binom{n-1}{r}t^r - \sum_{r=m}^{n-1}\binom{n-1}{r+1}t^r\right]. \tag{8.21}$$

We know that

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots \tag{8.22}$$

and since $n = 2m+1$, we have

$$\binom{n-1}{m-p} = \binom{n-1}{m+p}, \text{ for } p = 0, ..., m. \tag{8.23}$$

By using (8.22) and (8.23), we can simplify (8.21) to see that $\mathbf{P}(M^\sigma /\!\!/ \mathrm{O}(2), t; \mathbb{Z}_2)$ is a polynomial in t of degree $n-3$ and the $\mathbb{Z}_2$-Betti numbers β_k of the real reduction $(\mathbb{RP}^1)^n /\!\!/ \mathrm{O}(2)$ are as follows:

$$\beta_k = \sum_{j=0}^{\min\{k, n-k-3\}} \binom{n-1}{j}, \qquad \text{for } k = 0, ..., n-3. \tag{8.24}$$

Example 8.4. Consider the real Hamiltonian system in Example 3.15:

$$\Big(M = \mathrm{Gr}_{l_1}(\mathbb{C}^n) \times \cdots \times \mathrm{Gr}_{l_r}(\mathbb{C}^n), \omega, \mathrm{U}(n), \mu, \sigma, \phi\Big).$$

Fix the Ad-invariant inner product $\langle A, B\rangle = -\mathrm{Tr}(AB)$ on the Lie algebra $\mathfrak{u}(n)$ and consider the restriction of the norm squared of the moment map $f^\sigma : M^\sigma \to \mathbb{R}$, where $M^\sigma = \mathrm{Gr}_{l_1}(\mathbb{R}^n) \times \cdots \times \mathrm{Gr}_{l_r}(\mathbb{R}^n)$ is the real locus. We saw in Example 4.2 and Remark 4.3 that the generated real Hamiltonian subsystems are tuples $(Z_{\beta,l}, \mathrm{U}(n)_\beta, \mu_{\beta,l}, \sigma_{\beta,l}, \phi_\beta)$ and the zero level set $M_0 = \mu^{-1}_{\beta_0,l_0}(0)$ where $\beta_0 = (\frac{k}{n}, ..., \frac{k}{n})$, $k = l_1 + \cdots + l_r$ and $l_0 = (l_1, ..., l_r)$.

By Example 7.2 , the real Hamiltonian is 2-primitive and by proposition 5.7, we know that $(\mathrm{U}(n), \phi)$ has the special free extension property. Therefore, it follows from (7.16) that

$$\begin{aligned}\mathbf{P}_{\mathrm{O}(n)}(M_0^\sigma, t; \mathbb{Z}_2) = {} & \mathbf{P}_{\mathrm{O}(n)}(M^\sigma, t; \mathbb{Z}_2) \\ & - \sum_{\substack{2\le s\le n \\ \beta\in\overline{\Lambda}^s, \overline{l}\in\Gamma^s_\beta}} t^{d(\beta,\sigma)}\mathbf{P}_{\mathrm{O}(n)}(Z^\sigma_{\beta,l} \cap \mu^{-1}_{\beta,l}(0), t; \mathbb{Z}_2),\end{aligned} \tag{8.25}$$

where the sum is taking over $s = 2, ..., n$, $\beta \in \Lambda^s$ and $l \in \Gamma^s_\beta$ as well as $d(\beta, \sigma)$ is the index of the restricted map f^σ along C^σ_β:

$$d(\beta, \sigma) = \sum_{1 \le i < i' \le s} (k_i - m_i) m_{i'}. \tag{8.26}$$

By Theorem 7.3, we know that M^σ is $\mathrm{O}(n)$-equivariantly formal over the field $\mathbb{Z}_2$ and therefore

$$H^*_{\mathrm{O}(n)}(M^\sigma; \mathbb{Z}_2) = H^*(\mathrm{BO}(n); \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(M^\sigma; \mathbb{Z}_2). \tag{8.27}$$

It follows from (8.25) and (8.27) that

$$\begin{aligned} \mathbf{P}_{\mathrm{O}(n)}(M^\sigma_0, t; \mathbb{Z}_2) = {} & \mathbf{P}(B\mathrm{O}(n), t; \mathbb{Z}_2) \mathbf{P}(M^\sigma, t; \mathbb{Z}_2) \\ & - \sum_{\substack{2 \le s \le n \\ \beta \in \Lambda^s, l \in \Gamma^s_\beta}} \mathbf{P}_{\mathrm{O}(n)_\beta}(Z^\sigma_{\beta,l} \cap \mu^{-1}_{\beta,l}(0), t; \mathbb{Z}_2). \end{aligned} \tag{8.28}$$

By Remark 4.2, there are a finite family of real Hamiltonian subsystems $\mathcal{RH}_{\beta,l,i} = (Z_{\beta,l,i}, \omega, \mathrm{U}(m_i), \mu_{\beta,l,i}, \sigma, \phi)$ such that that

$$\begin{cases} Z_{\beta,l,i} \cong \prod_{j=1}^r \mathrm{Gr}_{l_{ij}}(E_i) \\ Z_{\beta,l} = \prod_{i=1}^s Z_{\beta,l,i} \\ \mu^{-1}_{\beta,l}(0) = \prod_{i=1}^s \mu^{-1}_{\beta,l,i}(0). \end{cases} \tag{8.29}$$

By Lemma 5.5, we know that $\mathrm{O}(n)_{D_\beta} \cong \mathrm{O}(m_1) \times \cdots \times \mathrm{O}(m_s)$. Thus, by Proposition 2.20, we have

$$\mathbf{P}_{\mathrm{O}(n)_\beta}\Big(Z^\sigma_{\beta,l} \cap \mu^{-1}_{\beta,l}(0), t; \mathbb{Z}_2\Big) = \prod_{i=1}^s \mathbf{P}_{\mathrm{O}(m_i)}\Big(Z^\sigma_{\beta,l,i} \cap \mu^{-1}_{\beta,l,i}(0), t; \mathbb{Z}_2\Big). \tag{8.30}$$

Since $M^\sigma = \mathrm{Gr}_{l_1}(\mathbb{R}^n) \times \cdots \times \mathrm{Gr}_{l_r}(\mathbb{R}^n)$, by Proposition B.1 we have

$$\mathbf{P}(M^\sigma, t; \mathbb{Z}_2) = \prod_{j=1}^r \left(\frac{\prod_{p=n-l_j+1}^n (1 - t^p)}{\prod_{p=1}^{l_j} (1 - t^p)} \right). \tag{8.31}$$

By using (8.29), (8.30) and (8.31), we can simplify (8.28) to get

$$\begin{aligned} \mathbf{P}_{\mathrm{O}(n)}(M^\sigma_0, t; \mathbb{Z}_2) = {} & \Big(\frac{1}{\prod_{p=1}^n (1 - t^p)} \Big) \left[\prod_{j=1}^r \left(\frac{\prod_{p=n-l_j+1}^n (1 - t^p)}{\prod_{p=1}^{l_j} (1 - t^p)} \right) \right] \\ & - \sum_{\substack{2 \le s \le n \\ \beta \in \Lambda^s, l \in \Gamma^s_\beta}} t^{d(\beta,\sigma)} \left[\prod_{i=1}^s \mathbf{P}_{\mathrm{O}(m_i)}\Big(Z^\sigma_{\beta,l,i} \cap \mu^{-1}_{\beta,l,i}(0), t; \mathbb{Z}_2\Big) \right]. \end{aligned} \tag{8.32}$$

This recursive formula provides us with the $\mathrm{O}(n)$-equivariant $\mathbb{Z}_2$-Betti numbers of the zero level set $M_0^\sigma = M^\sigma \cap \mu^{-1}(0)$.

Now, suppose that n is an odd number and numbers $\sum_{j=1}^r l_j$ and n are coprime. We saw in Example 8.1 that

$$\mathbf{P}(M_0^\sigma/\mathrm{O}(n), t; \mathbb{Z}_2) = (1-t)\mathbf{P}_{\mathrm{O}(n)}(M_0^\sigma, t; \mathbb{Z}_2). \tag{8.33}$$

It follows from (8.32) and (8.33) that

$$\begin{aligned}\mathbf{P}(M^\sigma /\!\!/ \mathrm{O}(n), t; \mathbb{Z}_2) = &\left(\frac{1}{\prod_{p=2}^n (1-t^p)}\right) \prod_{j=1}^r \left(\frac{\prod_{p=n-l_j+1}^n (1-t^p)}{\prod_{p=1}^{l_j}(1-t^p)}\right) \\ &- \sum_{\substack{2\leq s\leq n \\ \beta\in\tilde{\Lambda}^s, \bar{l}\in\Gamma_\beta^s}} (1-t)t^{d(\beta,\sigma)} \left[\prod_{i=1}^s \mathbf{P}_{\mathrm{O}(m_i)}\Big(Z_{\beta,l,i}^\sigma \cap \mu_{\beta,l,i}^{-1}(0), t; \mathbb{Z}_2\Big)\right].\end{aligned} \tag{8.34}$$

Formula (8.34) gives us a recursive formula for the $\mathbb{Z}_2$-Betti numbers of the real reduction $M^\sigma /\!\!/ \mathrm{O}(n)$ when n is odd.

Example 8.5. As a special case of Example 8.4, consider the action of $\mathrm{U}(3)$ on the space $M = (\mathbb{CP}^2)^5$, so $n = 3$, $r = 5$, $l_1 = \cdots = l_5 = 1$. In this case, the real subgroup is $\mathrm{O}(3)$ and the real locus is $M^\sigma = (\mathbb{RP}^2)^5$. By using the formulas in Example 4.2, we can see that the indices β corresponding to the critical subsets C_β^σ of the restricted function f^σ, their Morse indices $d(\beta,\sigma)$ and the real stabilizer subgroups $\mathrm{O}(3)_\beta$ are given in the following table:

s	$(m_1,\dots,m_s)$	$(k_1,\dots,k_s)$	β	$d(\beta,\sigma)$	$\mathrm{O}(3)_\beta$
1	3	5	$(\frac{5}{3},\frac{5}{3},\frac{5}{3})$	0	$\mathrm{O}(3)$
2	$(2,1)$	$(5,0)$	$(\frac{5}{2},\frac{5}{2},0)$	3	$\mathrm{O}(2)\times\mathrm{O}(1)$
		$(4,1)$	$(2,2,1)$	2	$\mathrm{O}(2)\times\mathrm{O}(1)$
	$(1,2)$	$(5,0)$	$(5,0,0)$	8	$\mathrm{O}(1)\times\mathrm{O}(2)$
		$(4,1)$	$(4,\frac{1}{2},\frac{1}{2})$	6	$\mathrm{O}(1)\times\mathrm{O}(2)$
		$(3,2)$	$(3,1,1)$	4	$\mathrm{O}(1)\times\mathrm{O}(2)$
		$(2,3)$	$(2,\frac{3}{2},\frac{3}{2})$	2	$\mathrm{O}(1)\times\mathrm{O}(2)$
3	$(1,1,1)$	$(4,1,0)$	$(4,1,0)$	6	$\mathrm{O}(1)\times\mathrm{O}(1)\times\mathrm{O}(1)$
		$(3,2,0)$	$(3,2,0)$	5	$\mathrm{O}(1)\times\mathrm{O}(1)\times\mathrm{O}(1)$

Table 8.1: Critical subsets and their indexes

Here the vector $(\frac{5}{3}, \frac{5}{3}, \frac{5}{3})$ is correspondence to the global minimum of the function f^σ and the stabilizer subgroups are product of orthogonal groups with smaller ranks.

By using the information in Table 8.1 and formulas in Example 8.3, we can find the generated real Hamiltonian subsystems in this example. To do so, we first compute the corresponding sequences $l = (l_{ij})$ satisfying (4.56) and then find the real loci of the corresponding real Hamiltonian subsystems $Z_{\beta,l,i}$. By using (4.71) and (4.81), we list the real locus of each real Hamiltonian subsystem Z_β as in the following table:

β	$l = (l_{ij})$	$Z^\sigma_{\beta,l,i}$	$Z^\sigma_{\beta,l}$	Z^σ_β
$(\frac{5}{3}, \frac{5}{3}, \frac{5}{3})$	$l_{11} = 5$	$(\mathbb{RP}^2)^5$	$(\mathbb{RP}^2)^5$	$(\mathbb{RP}^2)^5$
$(\frac{5}{2}, \frac{5}{2}, 0)$	$\sum_{j=1}^5 l_{1j} = 5$	$(\mathbb{RP}^1)^5$	$(\mathbb{RP}^1)^5 \times \{*\}$	$(\mathbb{RP}^1)^5 \times \{*\}$
	$\sum_{j=1}^5 l_{2j} = 0$	$\{*\}$		
$(2, 2, 1)$	$\sum_{j=1}^5 l_{1j} = 4$	$(\mathbb{RP}^1)^4$	$(\mathbb{RP}^1)^4 \times \{*\}$	$\coprod_{p=1}^5 \Big((\mathbb{RP}^1)^4 \times \{*\}\Big)$
	$\sum_{j=1}^5 l_{2j} = 1$	$\{*\}$		
$(5, 0, 0)$	$\sum_{j=1}^5 l_{1j} = 5$	$\{*\}$	$\{*\} \times \{*\}$	$\{*\} \times \{*\}$
	$\sum_{j=1}^5 l_{2j} = 0$	$\{*\}$		
$(4, \frac{1}{2}, \frac{1}{2})$	$\sum_{j=1}^5 l_{1j} = 4$	$\{*\}$	$\{*\} \times \mathbb{RP}^1$	$\coprod_{p=1}^5 \Big(\{*\} \times \mathbb{RP}^1\Big)$
	$\sum_{j=1}^5 l_{2j} = 1$	$\mathbb{RP}^1$		
$(3, 1, 1)$	$\sum_{j=1}^5 l_{1j} = 3$	$\{*\}$	$\{*\} \times (\mathbb{RP}^1)^2$	$\coprod_{p=1}^{10} \Big(\{*\} \times (\mathbb{RP}^1)^2\Big)$
	$\sum_{j=1}^5 l_{2j} = 2$	$(\mathbb{RP}^1)^2$		
$(2, \frac{3}{2}, \frac{3}{2})$	$\sum_{j=1}^5 l_{1j} = 2$	$\{*\}$	$\{*\} \times (\mathbb{RP}^1)^3$	$\coprod_{p=1}^{10} \Big(\{*\} \times (\mathbb{RP}^1)^3\Big)$
	$\sum_{j=1}^5 l_{2j} = 3$	$(\mathbb{RP}^1)^3$		
$(4, 1, 0)$	$\sum_{j=1}^5 l_{1j} = 4$	$\{*\}$	$\{*\} \times \{*\} \times \{*\}$	$\coprod_{p=1}^5 \Big(\{*\} \times \{*\} \times \{*\}\Big)$
	$\sum_{j=1}^5 l_{2j} = 1$	$\{*\}$		
	$\sum_{j=1}^5 l_{3j} = 0$	$\{*\}$		
$(3, 2, 0)$	$\sum_{j=1}^5 l_{1j} = 3$	$\{*\}$	$\{*\} \times \{*\} \times \{*\}$	$\coprod_{p=1}^{10} \Big(\{*\} \times \{*\} \times \{*\}\Big)$
	$\sum_{j=1}^5 l_{2j} = 2$	$\{*\}$		
	$\sum_{j=1}^5 l_{3j} = 0$	$\{*\}$		

Table 8.2: Real loci of the generated real Hamiltonian subsystems

Now, we can use data in two previous tables and calculate the equivariant Poincaré series of the real zero level set of each real Hamiltonian subsystems. Next, by applying our formulas, we also can compute the contribution of each real zero level set of Hamiltonian subsystem to the Poincaré series of the main real zero level set M_0^σ. We have given all the related information in the following table:

β	$\mathbf{P}_{\mathrm{O}(3)_\beta}(Z_\beta^\sigma \cap \mu_{\beta,l}^{-1}(0), t; \mathbb{Z}_2)$	Contribution to $\mathbf{P}_{\mathrm{O}(3)}(M^\sigma, t; \mathbb{Z}_2)$
$(\frac{5}{3}, \frac{5}{3}, \frac{5}{3})$	$\mathbf{P}_{\mathrm{O}(3)}(M_0^\sigma, t; \mathbb{Z}_2)$	$\mathbf{P}_{\mathrm{O}(3)}(M_0^\sigma, t; \mathbb{Z}_2)$
$(\frac{5}{2}, \frac{5}{2}, 0)$	$(t^2+5t+1)(1-t)^{-2}$	$(t^5+5t^4+t^3)(1-t)^{-2}$
$(2,2,1)$	$(-t^2+3t+1)(1-t)^{-3}$	$(-5t^4+15t^3+5t^2)(1-t)^{-3}$
$(5,0,0)$	$(1-t)^{-1}(1-t^2)^{-1}$	$t^8(1-t)^{-2}(1-t^2)^{-1}$
$(4, \frac{1}{2}, \frac{1}{2})$	0	0
$(3,1,1)$	$(1-t)^{-3}$	$10t^4(1-t)^{-3}$
$(2, \frac{3}{2}, \frac{3}{2})$	$(1-t)^{-2}$	$10t^2(1-t)^{-2}$
$(4,1,0)$	$(1-t)^{-3}$	$5t^6(1-t)^{-3}$
$(3,2,0)$	$(1-t)^{-3}$	$10t^5(1-t)^{-3}$

Table 8.3: Contribution of real Hamiltonian subsystems to the Poincaré series of M_0^σ

By using data in Table 8.3 and formula (8.32), we can find the O(3)-equivariant Poincaré series of M_0^σ as follows:

$$\begin{aligned}\mathbf{P}_{\mathrm{O}(3)}(M_0^\sigma, t; \mathbb{Z}_2) = \frac{(1+t+t^2)^5}{(1-t)(1-t^2)(1-t^3)} - \Bigg(& \frac{t^5+5t^4+t^3}{(1-t)^2} \\ & + \frac{-5t^4+15t^3+5t^2}{(1-t)^3} + \frac{t^8}{(1-t)^2(1-t^2)} \\ & + \frac{10t^4}{(1-t)^3} + \frac{10t^2}{(1-t)^2} + \frac{5t^6}{(1-t)^3} + \frac{10t^5}{(1-t)^3}\Bigg).\end{aligned}$$

By simplifying these fractions, we get the following formula:

$$\mathbf{P}_{\mathrm{O}(3)}(M_0^\sigma, t; \mathbb{Z}_2) = \frac{t^2+5t+1}{1-t}. \tag{8.35}$$

Since $n = 3$ is odd and numbers $\sum_{j=1}^5 l_j = 5$ and 3 are coprime, it follows from (8.33), (8.34) and (8.35) that

$$\mathbf{P}(M^\sigma /\!\!/ \mathrm{O}(3), t; \mathbb{Z}_2) = t^2 + 5t + 1, \tag{8.36}$$

which gives the $\mathbb{Z}_2$-Betti numbers of the real reduction space $M^\sigma /\!\!/ \mathrm{O}(3) = (\mathbb{RP}^2)^5 /\!\!/ \mathrm{O}(3)$.

Remark 8.1. It is interesting that the $\mathbb{Z}_2$-Poincaré series of the real reduction in both Examples 8.3 and 8.4 as well as the $\mathbb{Q}$-Poincaré series of symplectic reduction computed by Kirwan in [42] satisfy the following relation:

$$\mathbf{P}(M^\sigma /\!\!/ G^\phi, t; \mathbb{Z}_2) = \mathbf{P}(M /\!\!/ G, t^{\frac{1}{2}}; \mathbb{Q}). \tag{8.37}$$

One may guess that this holds for real Hamiltonian systems in general. In fact, this is not true in general by the following counterexample.

Example 8.6. Consider the real Hamiltonian system $(\mathbb{CP}^1, \omega_{FS}, \mathrm{U}(1), h, \sigma, \phi)$ in Example 3.7 and the real Hamiltonian system $(\mathbb{T}^2, \omega_S, \mathrm{U}(1), \mu_0, \sigma', \phi)$ where ω_S is the area form and $\mathrm{U}(1)$ acts trivially on the 2-torus $\mathbb{T}^2$ with the trivial moment map $\mu_0 \equiv 0$ as well as the involution σ' is the reflection with respect to the xz-plane (see Figure 8.2). The product of these two real Hamiltonian systems generates a new real Hamiltonian system $(\mathbb{T}^2 \times \mathbb{CP}^1, \omega, \mathrm{U}(1), \mu, \sigma \times \sigma', \phi)$ where ω is the product symplectic form and the moment map $\mu : \mathbb{T}^2 \times \mathbb{CP}^1 \to \mathbb{R}$ is $\mu(x, y) = h(y)$.

An easy computation shows that the symplectic reduction space is $M /\!\!/ \mathrm{U}(1) \cong \mathbb{T}^2$ and the real reduction space is $M^\sigma /\!\!/ \mathrm{O}(1) \cong S^1 \coprod S^1$. Therefore

$$\mathbf{P}(M /\!\!/ \mathrm{U}(1), t; \mathbb{Q}) = (1+t)^2 \quad \text{and} \quad \mathbf{P}(M^\sigma /\!\!/ \mathrm{O}(1), t; \mathbb{Z}_2) = 2(1+t),$$

which clearly shows that the equality (8.37) is not satisfied.

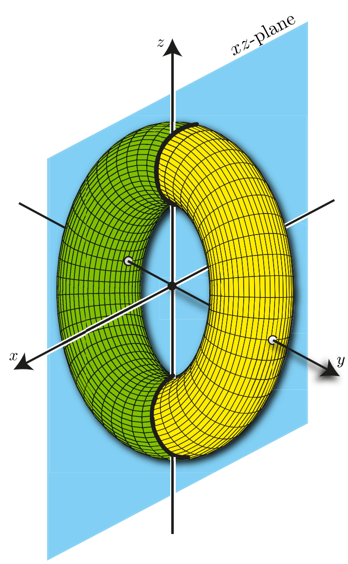

Figure 8.2: A real Hamiltonian system on 2-torus with trivial S^1-action

APPENDIX A

Complex Projective Line

Let $\mathbb{C}$ be the set of complex numbers. By using the natural map $x + iy \longmapsto (x, y)$, we can give $\mathbb{C}$ a smooth structure and make it into a 2-dimensional manifold. The one-point compactification of this space is denoted by $\mathbb{C}_\infty$ and called the **extended plane**. By using the stereographic projection, it is easy to see that $\mathbb{C}_\infty$ is a compact connected 2-dimensional manifold diffeomorphic to the unit sphere S^2 (see Figure A.1). Because of this, the space $\mathbb{C}_\infty$ is also called the **Riemann sphere**.

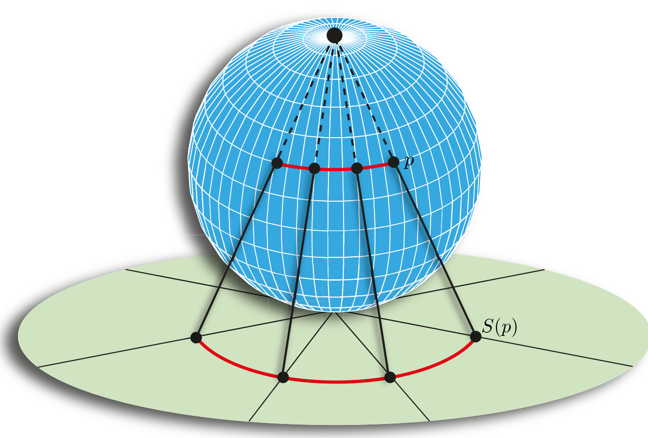

Figure A.1: Stereographic projection

Let $\mathrm{GL}(2; \mathbb{C})$ be the general linear group of order 2; i.e., the group of non-singular complex 2×2-matrices. Suppose that $A \in \mathrm{GL}(2; \mathbb{C})$. So for complex numbers a, b, c, d, we have

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; ad - bc \neq 0. \tag{A.1}$$

We define a map $f_A : \mathbb{C}_\infty \to \mathbb{C}_\infty$ as follows:

$$f_A(z) = \begin{cases} \dfrac{az+b}{cz+d} & \text{if } c \neq 0, \; z \neq \infty \\ \dfrac{a}{c} & \text{if } c \neq 0, \; z = \infty \\ \dfrac{az+b}{d} & \text{if } c = 0, \; z \neq \infty \\ \infty & \text{if } c = 0, \; z = \infty \end{cases} \tag{A.2}$$

The map f_A is called the **Mobius transformation** induced by the matrix A and we usually write $f_A(z) = \dfrac{az+b}{cz+d}$.

If $b = c = 0$ and $a = d$, then we get $f_A = \mathrm{Id}$, the identity map. Thus, the only matrices that induces the identity Mobius transformation are the diagonal matrices $\lambda\mathrm{Id}$ for $\lambda \in \mathbb{C} - \{0\}$. We have the following property for the Mobius transformations:

Proposition A.1. *Any Mobius transformation with more than two distinct fixed points must be the identity map.*

Proof. See [27], Section 3.4. ∎

Let $\mathbb{CP}^1$ be the one dimensional complex projective space; i.e., the set of all complex lines passing through the origin in the complex plane $\mathbb{C}^2$. Any vector $(z, w) \in \mathbb{C}^2 - \{(0,0)\}$ defines a unique line in the complex plane, denoted by $[z : w]$.

Proposition A.2. *The complex projective line $\mathbb{CP}^1$ as a 2-real manifold is diffeomorphic to the unit 2-sphere S^2.*

Proof. Set $U_1 = \mathbb{CP}^1 - \{[0:1]\}$ and $U_2 = \mathbb{CP}^1 - \{[1:0]\}$. Define

$$\begin{cases} \varphi_1 : U_1 \to \mathbb{R}^2, \; \varphi_1[1 : x+iy] = (x, y), \\ \varphi_2 : U_2 \to \mathbb{R}^2, \; \varphi_1[x+iy : 1] = (x, y). \end{cases} \tag{A.3}$$

It is clear that $\varphi_1(U_1 \cap U_2) = \varphi_2(U_1 \cap U_2) = \mathbb{R}^2 - \{(0,0)\}$ and

$$\varphi_2 \circ \varphi_1^{-1}(x, y) = (\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}). \tag{A.4}$$

This defines a 2-dimensional smooth structure on $\mathbb{CP}^1$. Now, let $N = (0,0,1), S = (0,0,-1)$ and set

$$V_1 = S^2 - \{N\}, \; V_2 = S^2 - \{S\}. \tag{A.5}$$

Consider the corresponding stereographic projections

$$\begin{cases} \psi_1 : V_1 \to \mathbb{R}^2, \ \varphi_1(x,y,z) = (\frac{x}{1-z}, \frac{y}{1-z}), \\ \psi_2 : V_2 \to \mathbb{R}^2, \ \psi_2(x,y,x) = (\frac{x}{1+z}, \frac{-y}{1+z}). \end{cases} \tag{A.6}$$

In this case, for any $z = x + iy \neq 0$, we have

$$\begin{aligned} \varphi_2^{-1} \circ \psi_2 \circ \psi_1^{-1} \circ \varphi_1[1:z] &= \varphi_2^{-1} \circ \psi_2 \circ \psi_1^{-1}(\varphi_1[1:z]) \\ &= \varphi_2^{-1} \circ \psi_2 \circ \psi_1^{-1}(x,y) \\ &= \varphi_2^{-1} \circ \psi_2(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2}) \\ &= \varphi_2^{-1}(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}) \\ &= [\frac{x-iy}{x^2+y^2} : 1] \\ &= [1 : x+iy] \\ &= [1:z]. \end{aligned}$$

Therefore,

$$\psi_1^{-1} \circ \varphi_1(p) = \psi_2^{-1} \circ \varphi_2(p), \ \forall p \in \mathbb{R}^2 - \{(0,0)\}. \tag{A.7}$$

Now, define $f : \mathbb{CP}^1 \to S^2$ and $g : S^2 \to \mathbb{CP}^1$ by

$$f(p) = \begin{cases} \psi_1^{-1} \circ \varphi_1(p), & if \ p \in U_1, \\ \psi_2^{-1} \circ \varphi_2(p), & if \ p \in U_2, \end{cases} \tag{A.8}$$

and

$$g(p) = \begin{cases} \varphi_1^{-1} \circ \psi_1(p), & if \ p \in V_1, \\ \varphi_2^{-1} \circ \psi_2(p), & if \ p \in V_2, \end{cases} \tag{A.9}$$

By (A.7), it is clear that f, g are well-defined smooth functions and $g = f^{-1}$. Therefore, f is a diffeomorphism. ■

Consider a Mobius transformation f_A where $A \in \mathrm{GL}(2, \mathbb{C})$. The natural action of A on $\mathbb{C}^2$ induces an action of A on $\mathbb{CP}^1$ as follows:

$$A.[z:w] = [az + bw : cz + dw]. \tag{A.10}$$

By using the above identifications, we can see that the map $A : \mathbb{CP}^1 \to \mathbb{CP}^1$ can be identified with the Mobius transformation $f_A : \mathbb{C}_\infty \to \mathbb{C}_\infty$. Therefore, they have the same number of fixed points.

APPENDIX B

Grassmannians

Let V be an n-dimensional vector space over the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The Grassmannian $\mathrm{Gr}_k(V)$ is the set of all k-dimensional subspaces of V. It is known that this space is a compact manifold of dimension $k(n-k)$ (see [45] or [51]).

Let $\mathbb{F}^\infty$ be the space of all sequences $\{\lambda_m\}_{m\geq 1}$ such that for some m_0, $\lambda_m = 0$, $\forall m \geq m_0$. This is clearly a vector space over $\mathbb{F}$. Suppose that $\mathrm{Gr}_k(\mathbb{F}^\infty)$ is the set of all k-dimensional subspaces. We can equip this set with a CW structure such that each Grassmannian $\mathrm{Gr}_k(\mathbb{F}^n)$ can be regarded as a subcomplex (see [51]).

Proposition B.1. *For the cohomology with $\mathbb{Z}_2$-coefficients, the Poincaré series of complex Grassmannians are as follows:*

$$\mathbf{P}(\mathrm{Gr}_k(\mathbb{C}^n), t; \mathbb{Z}_2) = \frac{\prod_{p=n-k+1}^{n}(1-t^{2p})}{\prod_{p=1}^{k}(1-t^{2p})}, \tag{B.1}$$

and

$$\mathbf{P}(\mathrm{Gr}_k(\mathbb{C}^\infty), t; \mathbb{Z}_2) = \frac{1}{\prod_{p=1}^{k}(1-t^{2p})}. \tag{B.2}$$

For the real Grasmannians, we have

$$\mathbf{P}(\mathrm{Gr}_k(\mathbb{R}^n), t; \mathbb{Z}_2) = \frac{\prod_{p=n-k+1}^{n}(1-t^{p})}{\prod_{p=1}^{k}(1-t^{p})}, \tag{B.3}$$

and

$$\mathbf{P}(\mathrm{Gr}_k(\mathbb{R}^\infty), t; \mathbb{Z}_2) = \frac{1}{\prod_{p=1}^{k}(1-t^{p})}. \tag{B.4}$$

Proof. See [15], [51] and [34]. ■

Now, consider the natural action of $\mathrm{U}(n)$ on the complex Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$; i.e., for any unitary matrix A and k-dimensional subspace V of $\mathbb{C}^n$, the image of V under the operator $A : \mathbb{C}^n \to \mathbb{C}^n$ is defined as the action of A on V. It is easy to see that this action is transitive and the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$ is isomorphic to the adjoint orbit of the subspace V spanned by the first k basis vectors $\{e_1, ..., e_k\}$ of the standard basis of $\mathbb{C}^n$ (see [66] for details). In this case, the stabilizer subgroup of V is isomorphic to $\mathrm{U}(n-k) \times \mathrm{U}(k)$ and therefore

$$\mathrm{Gr}_k(\mathbb{C}^n) \cong \frac{\mathrm{U}(n)}{\mathrm{U}(k) \times \mathrm{U}(n-k)}. \tag{B.5}$$

Similarly, the real Grassmannian $\mathrm{Gr}_k(\mathbb{R}^n)$ is an adjoint orbit of the natural action of $\mathrm{SO}(n)$ on the real Grassmannian $\mathrm{Gr}_k(\mathbb{R}^n)$. Thus,

$$\mathrm{Gr}_k(\mathbb{R}^n) \cong \frac{\mathrm{SO}(n)}{\mathrm{S}\Big(\mathrm{O}(k) \times \mathrm{O}(n-k)\Big)}, \tag{B.6}$$

where $\mathrm{S}\Big(\mathrm{O}(k) \times \mathrm{O}(n-k)\Big) = \mathrm{SO}(n) \cap \Big(\mathrm{O}(k) \times \mathrm{O}(n-k)\Big)$. Since $\mathrm{U}(n)$ and $\mathrm{SO}(n)$ are compact and connected, we have the following

Proposition B.2. *The real and complex Grassmannians* $\mathrm{Gr}_k(\mathbb{R}^n)$ *and* $\mathrm{Gr}_k(\mathbb{C}^n)$ *are compact and connected manifolds.*

Let V and W be vector spaces over a field $\mathbb{F}$. The set of all linear maps between V and W is a vector space and denoted by $\mathrm{Hom}_{\mathbb{F}}(V, W)$.

Proposition B.3. *Let* $\mathrm{Gr}_k(E)$ *be the Grassmannian of* k*-dimensional subspaces of* E *where* $E = \mathbb{R}^n$ *or* $\mathbb{C}^n$ *and consider the natural action of the general linear group* $\mathrm{GL}(n; \mathbb{F})$ *on* $\mathrm{Gr}_k(E)$ *where* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$.

1. *If* $T_V(\mathrm{Gr}_k(E))$ *is the tangent space of* $\mathrm{Gr}_k(E)$ *at* V, *then*

$$T_V(\mathrm{Gr}_k(E)) \cong \mathrm{Hom}_{\mathbb{F}}(V, E/V), \tag{B.7}$$

   where $E/V = \{u + V \mid u \in E\}$ *is the quotient space.*

2. *Let* $B \in \mathrm{GL}(n; \mathbb{F})$ *and* $\Psi_B : \mathrm{Gr}_k(E) \to \mathrm{Gr}_k(E)$ *be the diffeomorphism induced by the action. Then the derivative of* Ψ_B *at* V *is* $d\Psi_B : \mathrm{Hom}_{\mathbb{F}}(V, E/V) \to \mathrm{Hom}_{\mathbb{F}}(BV, E/BV)$ *such that for any* $T \in \mathrm{Hom}_{\mathbb{F}}(V, E/V)$, *we have*

$$d\Psi_B(T) = \overline{B} T B_V^{-1}, \tag{B.8}$$

   where $B_V : V \to BV$ *is the restricted isomorphism and* $\overline{B} : E/V \to E/BV$ *is defined by* $\overline{B}(u + V) = Bu + BV$.

Proof. See [59], Chapter 2. ∎

Let D be a real matrix. For any $V \in \mathrm{Gr}_k(\mathbb{C}^n)$, denote by $P_V : \mathbb{C}^n \to \mathbb{C}^n$ the orthogonal projection onto V. Define the function $f_D : \mathrm{Gr}_k(\mathbb{C}^n) \to \mathbb{R}$ by

$$f_D(V) = -\mathrm{Tr}(\mathrm{P}_V D). \tag{B.9}$$

Let $D = \mathrm{Diag}[b_1, ..., b_1, ..., b_l, ..., b_l]$ such that $b_1 > \cdots > b_l \geq 0$ and multiplicity of each b_j is n_j with $n_1 + \cdots + n_l = n$. Let $\mathbb{C}^n = E_1 \oplus \cdots \oplus E_l$, where E_j is the eigenspace of D corresponding to the eigenvalue b_j. Clearly, $\dim E_j = n_j$, for $j = 1, ..., l$.

Proposition B.4. *The function $f_D : \mathrm{Gr}_k(\mathbb{C}^n) \to \mathbb{R}$ is a Morse-Bott function whose critical set is*

$$C_{f_D} = \{V \in \mathrm{Gr}_k(\mathbb{C}^n) \mid V = (V \cap E_1) \oplus \cdots \oplus (V \cap E_l)\}, \tag{B.10}$$

that is, the set of all k-dimensional complex subspaces V of $\mathbb{C}^n$ spanned by the eigenvectors of D. Moreover, if M_V is the nondegenerate critical manifold containing V, then

$$M_V \cong \mathrm{Gr}_{k_1}(\mathbb{C}^{n_1}) \times \cdots \times \mathrm{Gr}_{k_l}(\mathbb{C}^{n_l}), \tag{B.11}$$

where $k_j = \dim(V \cap E_j)$, for $j = 1, ..., l$.

Proof. See [29]. ■

APPENDIX C

Projective Linear Groups

Let $\mathrm{GL}(n;\mathbb{C})$ be the complex general linear group of order n. The center of $\mathrm{GL}(n;\mathbb{C})$ is the subgroup of all nonzero scalar matrices $\lambda\mathrm{Id}$ denoted by $D(n;\mathbb{C})$.

Definition C.1. Let K be a linear group of order n; i.e., a subgroup of $\mathrm{GL}(n;\mathbb{C})$ and $DK = K \cap D(n;\mathbb{C})$. The quotient group $PK = K/DK$ is called the **projective linear group** of K.

Example C.1. For the unitary group $\mathrm{U}(n)$, we have $D\mathrm{U}(n) = \{\lambda\mathrm{Id} \mid |\lambda| = 1\} \cong S^1$ and for special unitary group we have $D\mathrm{SU}(n) = \{\lambda\mathrm{Id} \mid |\lambda| = 1, \lambda^n = 1\} \cong \mathbb{Z}_n$. Similarly, for orthogonal group $\mathrm{O}(n)$, we have $D\mathrm{O}(n) = \{\pm\mathrm{Id}\} \cong \mathbb{Z}_2$. For the special orthogonal group $\mathrm{SO}(n)$, $D\mathrm{SO}(n) = \{\mathrm{Id}\}$, if n is odd and $D\mathrm{SO}(n) = \{\pm\mathrm{Id}\} \cong \mathbb{Z}_2$, if n is even.

Proposition C.1. *For $\mathrm{O}(n)$ and $\mathrm{SO}(n)$, the following are satisfied.*

1. *If n is odd, then $\mathrm{O}(n)$ and $\mathbb{Z}_2 \times \mathrm{SO}(n)$ are isomorphic as Lie groups.*
2. *If n is even, then $\mathrm{O}(n)$ and $\mathbb{Z}_2 \times \mathrm{SO}(n)$ are homeomorphic as topological spaces.*

Proof. Let $D : \mathbb{Z}_2 \times \mathrm{SO}(n) \to \mathrm{O}(n)$ be defined by

$$D(\epsilon, A) = \epsilon A$$

where $\epsilon = \pm 1$. It is easy to see that D is a Lie group isomorphism when n is odd. This proves case 1. Define $\sigma : \mathbb{Z}_2 \to \mathrm{O}(n)$ by $\sigma(1) = \mathrm{Id}$ and $\sigma(-1) = \mathrm{Diag}[-1, 1, ..., 1]$. Now consider a map $\psi : \mathrm{O}(n) \to \mathbb{Z}_2 \times \mathrm{SO}(n)$ defined by $\psi(A) = (\det A, \sigma(\det A)A)$. It is easily seen that ψ is a homeomorphism. This proves case 2. ■

Proposition C.2. *For $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ the following are satisfied.*

1. *If n is odd, then $\mathrm{SO}(n) = \mathrm{PSO}(n) = \mathrm{PO}(n)$.*

2. *If $n = 2m$ is even, then we have the following commutative diagram:*

$$\begin{tikzcd}
 & \mathbb{Z}_2 \arrow[dl, "i"'] \arrow[dr, "i"] & & \\
\mathrm{SO}(2m) \arrow[rr, "j"] \arrow[dd, "q_2"] & & \mathrm{O}(2m) \arrow[dd, "q_1"] \arrow[dr, "D"] & \\
 & & & \mathbb{Z}_2 \\
\mathrm{PSO}(2m) \arrow[rr, "Pj"] & & \mathrm{PO}(2m) \arrow[ur, "PD"] &
\end{tikzcd}$$

Diagram C.1: Commutative diagram induced by projective groups

where i, j are natural inclusions, q_1, q_2 are quotient maps, D is the determinant map and Pj, PD are induced maps.

3. *For special case $n = 2$, the projective group* $\mathrm{PSO}(2)$ *is isomorphic to the circle group S^1.*

Proof. Parts 1,2 follow easily from Proposition C.1. For part 3, we note that $\mathrm{SO}(2) \cong S^1$ and the map $\varphi : S^1 \to S^1$ by $\varphi(z) = z^2$ is a surjective homomorphism with $\mathrm{Ker}\ \varphi = \{\pm 1\}$. This induces an isomorphism $S^1/\mathbb{Z}_2 \cong S^1$. Therefore, $\mathrm{PSO}(2) \cong \mathrm{SO}(2)/\mathbb{Z}_2 \cong S^1$. This completes the proof. ∎

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List of Notations

$\pi_1(X)$	fundamental group of a space X
$\nu(N)$	normal bundle of a submanifold N
γ_k^∞	tautological k-vector bundle
$\chi(T)$	characteristic polynomial of a linear operator T
$\nabla f(p)$	gradient of a function f at a point p
ω_{FS}	Fubini-Study symplectic form
κ	Kirwan map
$\kappa_{\mathbb{R}}$	real Kirwan map
μ^X	component of a moment map along X
A^t	transpose of a matrix
$\overline{A}$	complex conjugate of a complex matrix A
A^*	conjugate transpose of a complex matrix A
Ad_g	Adjoint action
$AG(M)$	associated graded module of a filtered module M
$\mathrm{Aut}(V)$	automorphisms of a linear space V
BG	classifying space of a group G
$\mathbb{C}$	set of complex numbers
$\mathbb{C}^\times$	set of nonzero complex numbers
$\mathbb{C}^n$	n-dimensional Hermitian vector space
$\mathrm{codim}(N)$	codimension of a submanifold N
$\det(A)$	determinant of a matrix A
$\mathrm{Diag}[a_1, ..., a_n]$	diagonal matrix with entries $a_1, ..., a_n$
$\dim_{\mathbb{F}} V$	dimension of a linear space V over a field $\mathbb{F}$
$\dim M$	dimension of a manifold M
EG	total space of the universal bundle for a group G
$\mathrm{Eul}(E)$	Euler class of a vector bundle
$\mathrm{Eul}_G(E)$	equivariant Euler class of a G-vector bundle

$\exp(X)$	exponential of X
$\mathbb{F}$	a field of scalars
$F(M)$	filtration of a module M
$\mathbb{F}[x_1, ..., x_n]$	polynomial ring in variables $x_1, ..., x_n$ with coefficients in a field $\mathbb{F}$
f^*E	pullback of a bundle by a map f
G_p	stabilizer subgroup of a point p in a G-space
$\mathrm{GL}(n; \mathbb{F})$	general linear group of non-singular matrices with entries in a field $\mathbb{F}$
$\mathrm{Gr}_k(V)$	Grassmannian of k-subspaces of a linear space V
$\mathcal{H}$	a Hamiltonian system
$\mathcal{H}_\beta$	Hamiltonian subsystem of a Hamiltonian $\mathcal{H}$
$H^*(M; R)$	ordinary cohomology of a space M relative to a ring R
$H^*_G(M; R)$	equivariant cohomology of a space M relative to a ring R
$H_p(f)$	Hessian of a function f at a point p
$\mathrm{Hom}_{\mathbb{F}}(V, W)$	homomorphisms between $\mathbb{F}$-linear spaces V and W
Id	identity map
Id_n	identity matrix of order n
$\mathrm{Im}(f)$	image of a map f
$\mathrm{Im}(z)$	imaginary part of a complex number z
$\mathrm{Ind}_p(f)$	Morse index of a function f at a point p
$\mathrm{Ker}(T)$	kernel of a linear map T
$\mathbf{M}(f, t; \mathbb{F})$	Morse series of a function f relative to a field $\mathbb{F}$
$\mathbf{M}_G(f, t; \mathbb{F})$	equivariant Morse series of a G-function f relative to a field $\mathbb{F}$
$\mathrm{Lie}(G)$	Lie algebra of a Lie group G
$M_{n,k}(\mathbb{F})$	matrices of order $n \times k$ with entries in a field $\mathbb{F}$
M_{red}	reduction space of a symplectic manifold M
M/G	orbit space of a G-action on a space M
$M/\!\!/G$	symplectic quotient in a Hamiltonian G-system
m_T	minimal polynomial of a linear operator T
$\mathbb{N}$	set of natural numbers

$N_p(Q)$	normal bundle of a submanifold Q at a point p
$\mathcal{O}_p$	orbit of a point p
$\mathrm{O}(n)$	real orthogonal group of order n
$\mathrm{O}(n;\mathbb{F})$	orthogonal group of order n with entries in a field $\mathbb{F}$
$\mathbf{P}(M,t;\mathbb{F})$	Poincaré series of a space M relative to a field $\mathbb{F}$
$\mathbf{P}_G(M,t;\mathbb{F})$	equivariant Poincaré series of a G-space M relative to a field $\mathbb{F}$
$\mathbf{P}(V,t;\mathbb{F})$	Poincaré series of a graded $\mathbb{F}$-vector space V
$\mathrm{PO}(n)$	projective orthogonal group of order n
Pr_V	orthogonal projection onto a linear subspace V
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathrm{Re}(z)$	real part of a complex number z
$\mathcal{RH}$	a real Hamiltonian system
$\mathcal{RH}_\beta$	real Hamiltonian subsystem of a real Hamiltonian $\mathcal{RH}$
S^n	unit n-sphere in $\mathbb{R}^{n+1}$
$\mathrm{SO}(n)$	real special orthogonal group of order n
$\mathrm{Span}_{\mathbb{F}}\langle B\rangle$	$\mathbb{F}$-linear span of a subset B in a linear $\mathbb{F}$-space
$\mathrm{SU}(n)$	special unitary group of order n
$\mathbb{T}^n$	n-dimensional torus
T_pM	tangent space of a manifold M at a point p
$\mathrm{Tor}_R^{p,q}(M,N)$	torsion product of R-modules M and N
$\mathrm{Tr}(A)$	trace of a matrix A
$\mathrm{U}(n)$	unitary group of order n
$\mathrm{U}(n;\mathbb{F})$	unitary group of order n with entries in a field $\mathbb{F}$
$w_i(E)$	the i^{th}-Stiefel-Whitney class of a vector bundle E
$w_i^G(E)$	the equivariant i^{th}-Stiefel-Whitney class of a G-vector bundle E
X^G	fixed point set of a G-action on a space X
X_G	homotopy quotient of a G-space X

$X \times_G Y$	twisted product of G-spaces X and Y
$\mathfrak{X}(M)$	vector fields on a manifold M
$\mathbb{Z}$	set of integer numbers
$\mathbb{Z}_k$	cyclic group of order k
$Z(G)$	center of a group G

Index