Rings Whose Cyclics Satisfy a Certain Property

by

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Abstract

In 1964, Osofsky proved that a ring R is semisimple artinian if and only if every cyclic right R-module is injective. Motivated by this result, there have been numerous studies in the rings whose cyclic modules satisfy a certain generalized injectivity condition. An up-to-date account of the literature on this subject can be found in [26].

Following this direction, in Chapter 2, we study the rings whose cyclics are $C3$-modules (or $CC3$-rings). We prove that a ring R is semisimple artinian if every 3-generated right R-module is a $C3$-module. Structure theorems of semiperfect $CC3$-rings and self-injective regular $CC3$-rings are obtained. Applications to rings whose 2-generated modules are $C3$-modules, and whose cyclics are quasi-continuous, are also addressed.

In Chapter 3, we present basic properties of $CD3$-rings, i.e., rings whose cyclics are $D3$-modules. We show that a ring R is semisimple artinian if every 2-generated right R-module is a $D3$-module. Structure of self-injective regular $CD3$-rings is given. We characterize the rings whose cyclic modules are quasi-discrete and, respectively, discrete.

In Chapter 4, we prove that a semiperfect module is lifting if and only if it has a projective cover preserving direct summands. This result is then used to characterize rings whose cyclics are lifting. New characterizations of artinian serial rings with Jacobson radical square-zero are obtained. Furthermore, we show that every cyclic right R-module is $\oplus$-supplemented if and only if every cyclic right R-module is a direct sum of local modules, and that artinian serial rings are exactly these rings for which every left and right module is a direct sum of local modules.

In the last chapter, we present various properties, including a structure theorem and several characterizations, for δ-semiperfect modules. Our method can be adapted to generalize several known results of Mares and Nicholson from projective semiperfect modules to arbitrary semiperfect modules.

In the loving memory of my beloved father,
Nguyen Phu
10/1/1930 - 6/9/1998

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List of Symbols

Symbols	Descriptions
Mod-R	category of right R-modules
$\sigma[M]$	subcategory of Mod-R subgenerated by a module M
$E(M)$	injective envelope of a module M
$\mathrm{Hom}_R(M, N)$	set of R-homomorphisms from M to N
$\mathrm{End}_R(M)$	ring of endomorphisms of an R-module M
$\mathbb{M}_n(R)$	ring of $n \times n$ matrices with entries from a ring R
$m^{\perp}, ann_r(m)$	right annihilator of $m \in M$
$N \subseteq^{\oplus} M$	N is a direct summand of a module M
$N \subseteq_e M$	N is an essential submodule of a module M
$N \ll M$	N is a small submodule of a module M
$N \ll_\delta M$	N is a δ-small submodule of a module M
$rad(M)$	radical of a module M
$soc(M)$	socle of a module M
$J(R)$	Jacobson radical of a ring R
$\delta(R)$	the δ-ideal of a ring R
$\delta(M)$	the δ-submodule of a module M
$R[x]$	ring of polynomials over a ring R
$\mathbb{Z}$	rings of integers
$\mathbb{Z}_n$	ring of integers modulo n

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Introduction

It has been an interesting subject to characterize the rings in terms of their cyclic modules. One of the most important results is the highly non-trivial classical Osofsky's Theorem [50] which states that a ring R is semisimple artinian if every cyclic right R-module is injective. This result promoted a considerable interest in the rings whose cyclics satisfy a certain generalized injectivity condition. For instance, Koehler [34] provided a complete characterization for rings whose cyclic modules are quasi-injective in 1974. In 1978, Jain and Mohamed [25] characterized the rings whose cyclic modules are continuous. The rings for which every cyclic module is quasi-continuous were studied by Goel and Jain in [18]. In 1991, Osofsky and Smith [51], and Huynh, Dung and Wisbauer [23] investigated the rings for which every cyclic module is a $C1$-module. In particular, the following result was obtained: if R is a ring such that every cyclic right R-module is a $C1$-module, then every cyclic right R-module is a finite direct sum of uniform modules.

It is known (and easy to prove) that a ring R is semisimple artinian if and only if every cyclic right R-module is projective. Rings whose cyclics are quasi-projective were studied by Koehler in [33]. Recently, rings whose cyclics are automorphism-invariant and rings whose cyclics are dual automorphism-invariant were discussed in [14] and [35], respectively. One can find in the monograph [26] an up-to-date account of the literature on the subject of determining the structure of rings via their cyclic modules.

Following this direction, in the first three chapters of this dissertation, we carry out a study of the rings whose cyclic modules satisfy the $C3$-, the $D3$- and the $D1$-condition, respectively. The last chapter is a research on δ-semiperfect modules.

As a generalization of quasi-continuous modules, a module is called a $C3$-module if the sum of any two direct summands with zero intersection is again a direct summand. It is easy to show that a ring R is semisimple artinian if and only if every 2-generated right R-module is quasi-continuous. Here we have an example of a non semisimple artinian ring for which every 2-generated right module is a $C3$-module. However, we prove that a ring R is semisimple artinian if every 3-generated right R-module is a $C3$-module. Thus, we are motivated to consider the following two questions: For which rings R, is every cyclic right R-module a $C3$-module? For which rings R, is every 2-generated right R-module a $C3$-module? In Chapter 2, we address these questions. While every 2-generated right module over a ring being a $C3$-module lies strictly between every cyclic right module being a $C3$-module and the ring being semisimple artinian, it is shown that every cyclic right module is a $C3$-module if and only if every cyclic right module satisfies the summand sum property and that every 2-generated right R-module is a $C3$-module if and only if every cyclic right R-module over $\mathbb{M}_2(R)$ is a $C3$-module. Two structure theorems are proved: A semiperfect ring R is such that every cyclic right R-module is a $C3$-module if and only if R is a direct product of a semisimple artinian ring and finitely many local rings. A right self-injective regular ring R is such that every cyclic right R-module is a $C3$-module if and only if R is a direct product of a semisimple artinian ring, a strongly regular ring and the 2×2 matrix ring over a strongly regular ring. These results are applied to the rings whose 2-generated modules are $C3$-modules, and the rings whose cyclic modules are ADS-modules and, respectively, quasi-continuous modules.

As a dual notion of $C3$-modules, a module M is called a $D3$-module if the intersection of any two direct summands whose sum equals M is a direct summand of M. Clearly every quasi-projective module is a $D3$-module. It is also easy to show that a ring R is semisimple artinian if and only if every 2-generated right R-module is quasi-projective. Here we observed that a ring R is semisimple artinian if and only if every 2-generated right R-module is a $D3$-module. In Chapter 3, we carry out a study of the rings whose cyclics are $D3$-modules, which are called $CD3$-rings. We show that a right self-injective regular ring R is a right $CD3$-ring if and only if R is a direct product of a semisimple artinian ring and a strongly regular ring. A sufficient condition for a semiperfect ring to be a right $CD3$-ring is given. We also present characterizations for rings whose cyclic modules are quasi-discrete and, respectively, discrete.

A module M is called a $C1$-module (or extending module) if every submodule of M is essential in a direct summand of M. Dually, a module M is called a $D1$-module (or lifting module) if for every submodule N of M there exists a direct decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$ and $N \cap M_2$ is small in M_2. Rings whose cyclic modules are extending have been extensively studied in [23] and [51]. This naturally leads us to consider rings with the dual condition, i.e., rings whose cyclic modules are lifting. The interest of studying this class of rings is also well motivated by a work of Keskin, Smith and Xue [31, Theorem 3.15], where it is proved that a ring R is artinian serial with Jacobson radical square-zero if and only if every right R-module is lifting, if and only if every 2-generated right R-module is lifting. This question turns out to be well connected to a natural condition on projective covers. A module M is said to have a projective cover preserving direct summands if there is a projective cover

$P \xrightarrow{\eta} M \to 0$ such that $\eta(X)$ is a direct summand of M for every direct summand X of P.

Following Kasch and Mares in [30], an arbitrary module M is called semiperfect if every factor module of M has a projective cover. Chapter 4 begins with a characterization of a semiperfect module that is lifting. We show that a semiperfect module M is lifting if and only if M has a projective cover preserving direct summands. This result is then used to show that every cyclic right module over a ring R is lifting if and only if every cyclic right R-module has a projective cover preserving direct summands, and that R is an artinian serial ring with Jacobson radical square-zero if and only if every (2-generated) right R-module has a projective cover preserving direct summands. We also prove that a ring R is a (semiperfect) right perfect ring if and only if every (cyclic) lifting right R-module has a projective cover preserving direct summands, if and only if every (cyclic) right R-module having a projective cover preserving direct summands is lifting.

As a dual to Osofsky and Smith's result in [51], we show that if every cyclic right R-module is lifting, then every cyclic right R-module is a direct sum of local modules. This is obtained as a consequence of a more general result that every cyclic right R-module is $\oplus$-supplemented if and only if every cyclic right R-module is a direct sum of local modules. Here, as a generalization of lifting modules, a module M is called $\oplus$-supplemented if for every submodule N of M there exists a direct summand K of M such that $M = N + K$ and $N \cap K$ is small in K. The latter result enables us to obtain new characterizations of artinian serial rings and, respectively, rings for which every finitely generated module is a sum of local modules. For example, we can prove that artinian serial rings are exactly these rings for which every left and right module is a direct sum of local modules.

The notions of small submodules and projective covers were invented by Bass in his pioneering paper [6]. Then he introduced semiperfect rings as those rings over which every cyclic module has a projective cover. In [63], as a generalization of small submodules, a submodule N of a module M is called δ-small in M (written N $\ll_\delta M$) if $N + X \neq M$ for any proper submodule X of M with M/X singular. Equivalently, a submodule $N \subseteq M$ is δ-small in M if and only if, whenever $M = X + N$, we have $M = X \oplus Y$ for a projective semisimple submodule Y with $Y \subseteq N$. This concept then leads to the definition of projective δ-covers. A projective δ-cover of a module M is a projective module P with an epimorphism to M whose kernel is δ-small in P. A ring R is called δ-semiperfect if every simple right R-module has a projective δ-cover. Various characterizations of δ-semiperfect rings are presented in [63].

A module M is called δ-semiperfect if every factor module of M has a projective δ-cover. In Chapter 5, we generalize the structure theory of δ-semiperfect rings to modules. We prove various properties, including a structure theorem and several characterizations, for δ-semiperfect modules. Our proofs for δ-semiperfect modules can be adapted to generalize some of the results of Mares [38] and Nicholson [47] from projective semiperfect modules to arbitrary semiperfect modules.

Chapter 1

Preliminaries

This chapter is divided into two sections: (1) Semisimplicity and (2) injectivity and projectivity. Several basic definitions and results in ring and module theory are presented in the first section; these concepts and results are used frequently throughout this dissertation. The next section discusses important properties related to injective modules, projective modules, and their generalizations. Several well-known results are presented for rings characterized by their cyclic modules. The standard references for this chapter are [4], [36] and [41]. All rings considered in this dissertation are associative with unity and all modules are unitary.

1.1 Semisimplicity and their generalizations

Definition 1.1.1. *Let R be a ring, and M be a right R-module.*

1. *M is called a simple module if $M \neq 0$, and M has no proper submodules.*
2. *M is called a semisimple module if every submodule of M is a direct summand of M.*
3. *R is called a semisimple artinian ring if the right R-module R_R (or equivalently, ${}_RR$) is semisimple.*

The sum of all simple submodules of a module M is called the *socle* of M and is denoted by $soc(M)$; we set $soc(M) = 0$ if M does not have simple submodules. It is straightforward to prove that M is semisimple if and only if $M = soc(M)$.

Definition 1.1.2. *Let R be a ring, and M be a right R-module.*

1. *M is called artinian if M has the descending chain condition (DCC) on submodules, i.e., for any descending chain of submodules $M_1 \supseteq M_2 \supseteq \ldots$, there exits an integer $n \geq 1$ such that $M_n = M_{n+1} = \ldots$.*
2. *M is called noetherian if M has the ascending chain condition (ACC) on submodules, i.e., for any ascending chain of submodules $M_1 \subseteq M_2 \subseteq \ldots$, there exits an integer $n \geq 1$ such that $M_n = M_{n+1} = \ldots$.*
3. *R is called right artinian (resp., noetherian) if the module R_R is artinian (resp., noetherian).*

A ring R is called **artinian** (resp., noetherian) if R is both left and right artinian (resp., noetherian). It is well-known that a right (resp., left) artinian ring is always right (resp., left) noetherian. However, an artinian module need not be noetherian. For instance, the Prüfer group $\mathbb{Z}_{p^\infty}$ (p is a prime), as a $\mathbb{Z}$-module, is artinian but not noetherian.

A submodule $N \subseteq M$ is said to be maximal if $N \neq M$ and $N \subseteq X \subseteq M$ with $X \neq N$ implies $X = M$. The **radical** of a module M is defined to be the intersection of all maximal submodules of M, and is denoted by $rad(M)$. We set $rad(M) = M$ if M does not have maximal submodules. For a ring R, $rad(R_R) = rad({}_RR)$, which is called the Jacobson radical of R, and is denoted by $J(R)$. The Jacobson radical $J(R)$ consists of the elements y in R such that $1 + xyz$ is invertible for all $x, z \in R$.

For a module M, M is semisimple and finitely generated if and only if M is artinian and $rad(M) = 0$, if and only if M is semisimple and noetherian (see [4, Proposition 10.15]). A ring R is called semisimple artinian if R satisfies the equivalent conditions in the next theorem.

Theorem 1.1.3. *The following are equivalent for a ring R:*

1. *R_R is a semisimple module.*
2. *${}_RR$ is a semisimple module.*
3. *R is right artinian and $J(R) = 0$.*
4. *R is left artinian and $J(R) = 0$.*
5. *R is artinian and $J(R) = 0$.*

Semisimple artinian rings are characterized by the fundamental Wedderburn-Artin theorem, which is one of the most important structure theorems for rings and algebras. A non-zero ring R is called a **simple ring** if R has no non-trivial two sided ideals. Structure of simple artinian rings (or simple rings with minimal right ideals) is given by

Wedderburn. Simple artinian rings are the simple components of semisimple artinian rings.

Theorem 1.1.4 (Wedderburn Theorem)**.** *A ring R is simple artinian if and only if R is isomorphic to the full matrix ring $\mathbb{M}_n(D)$ for some division ring D and some natural number n.*

Theorem 1.1.5 (Wedderburn-Artin Theorem)**.** *A ring R is semisimple artinian if and only if R is a direct sum of a finite number of simple artinian rings.*

A submodule N of a module M is called an essential (or large) submodule of M if for any non-zero submodule $A \subseteq M$ we have $N \cap A \neq 0$. Dually, we call a submodule N of M a small submodule of M if $M = K + N$ implies $K = M$. We write $N \subseteq_e M$ to mean that N is an essential submodule of M, and write $N \ll M$ to denote that N is a small submodule of M. It is well-known that for a module M, $soc(M)$ is equal to the intersection of all essential submodules of M and $rad(M)$ is equal to the sum of all small submodules of M.

Definition 1.1.6. *Let R be a ring, and M be a non-zero right R-module.*

1. *M is called indecomposable if it has no proper direct summands.*

2. *M is called uniform if every non-zero submodule of M is essential in M.*

3. *M is called hollow if every proper submodule of M is small in M.*

4. *A finitely generated module M is called local if M has a unique maximal submodule.*

5. *R is called a right indecomposable (resp., uniform, local) ring if the module R_R is indecomposable (resp., uniform, local).*

A local module is hollow and its unique maximal submodule has to be the radical. Also hollow modules are always indecomposable. Examples of local $\mathbb{Z}$-modules are simple modules and the modules $\mathbb{Z}_{p^k}$, where p is a prime and $k \in \mathbb{N}$. The Prüfer group $\mathbb{Z}_{p^\infty}$ (p is a prime), as a $\mathbb{Z}$-module, is hollow but not local. The $\mathbb{Z}$-module $\mathbb{Z}$ is indecomposable and uniform but not hollow. A finitely generated module is hollow if and only if it is local. Thus, a ring R is hollow if and only if it is local. Note that a ring R is **local** if and only if $R/J(R)$ is a division ring. A ring R is called a **semilocal** ring if $R/J(R)$ is semisimple artinian.

Definition 1.1.7. *A module M is called uniserial if its lattice of submodules is linearly ordered by inclusion. A ring R is called a right (resp., left) uniserial ring if R_R (resp., ${}_RR$) is a uniserial module. Any direct sum of uniserial modules is called a serial module. A ring R is called a serial ring if R_R and ${}_RR$ are serial modules.*

It is clear that simple modules are uniserial and semisimple modules are serial. Moreover, uniserial modules are hollow but need not be local.

Definition 1.1.8. *A ring R is called a (von Neumann) regular ring if for each $x \in R$, there exists $y \in R$ such that $xyx = x$.*

In [36], it is shown that a ring R is regular if and only if every principal right ideal of R is a direct summand of R_R, if and only if every finitely generated right ideal of R is a direct summand of R_R.

A ring R is called a **unit-regular** if for each $x \in R$, there exists a unit u of R such that $xux = x$. Every semisimple artinian ring is unit-regular, and every unit-regular ring is regular. A ring R is called **abelian** if every idempotent of R is central. An abelian regular ring is called **strongly regular**. Strongly regulars are unit-regular, but semisimple artinian rings need not be strongly regular.

Let I be a right ideal of a ring R. We say that idempotents lift modulo I if, whenever $r^2 - r \in I, r \in R$, there exists $e^2 = e \in R$ such that $e - r \in I$. A ring R is called **semiregular** if $R/J(R)$ is (von Neumann) regular and idempotents lift modulo $J(R)$. A ring R is called an **exchange** ring if idempotents lift modulo every right ideal in R. Various characterizations of exchange rings were given in [48]. For instance, it is shown that a ring R is an exchange ring if and only if, whenever $R = aR + bR$, there is a direct decomposition $R = eR \oplus (1-e)R$ with $e^2 = e \in aR$ and $1 - e \in bR$. It is also proved that a ring R is an exchange ring if and only if $R/J(R)$ is an exchange ring and idempotents can be lifted modulo $J(R)$. Hence, every semiregular ring is an exchange ring.

1.2 Injectivity, projectivity and their generalizations

In this section, we introduce projective modules, injective modules, and their generalizations. These modules are used to classify and characterize many important rings in ring theory.

Definition 1.2.1. *Given two modules M and N, M is called N-projective if for any epimorphism $f : N \to Y$ and any homomorphism $g : M \to Y$, there exists a homomorphism $h : M \to N$ such that $f \circ h = g$. The module M is called projective if it is N-projective for every module N, and M is called self-projective (or quasi-projective) if it is M-projective.*

Semisimple artinian rings can be characterized via their projective modules.

Theorem 1.2.2. *[36, Theorem 2.8] The following are equivalent for a ring R:*

1. *R is semisimple artinian.*

2. *Every right (left) R-module is projective.*

3. *Every cyclic right (left) R-module is projective.*

Projective modules play a central role in modules theory. As shown in [4, Proposition 17.2], a module P is projective if and only if P is isomorphic to a direct summand of a free module. Hence every free module is projective. As each right R-module is a sum of cyclic right R-modules and each cyclic right R-module is an image of R_R, we have

Proposition 1.2.3. *[4, Proposition 17.15] Every module is an epimorphic image of a projective module.*

For some modules M, a stronger assertion is possible: there is a projective module P and an epimorphism $f : P \to M \to 0$ *minimal* in the sense that $f|_L : L \to M$ is not an epimorphism for every proper submodule L of P. This leads to the following important definition.

Definition 1.2.4. *An epimorphism $P \xrightarrow{\eta} M \to 0$ with P projective and $ker(\eta) \ll P$ is called a projective cover of the module M.*

In general, projective covers may not exist for a module M. For instance, the $\mathbb{Z}$-module $\mathbb{Z}_2$ does not have a projective cover. However, if they exist then they are isomorphic (see [4, Lemma 17.17]). In his pioneering work on projective covers, Bass introduced right perfect rings and semiperfect rings. Following [6], a ring R is called right **perfect** if every right R-module has a projective cover, and R is called right **semiperfect** if every cyclic right R-module has a projective cover. Examples of right perfect rings include left or right artinian rings. Moreover, right perfect rings are obviously semiperfect.

In [6, Theorem 2.1], Bass presented various characterizations of right perfect rings and semiperfect rings. For instance, it was shown that a ring R is semiperfect if $R/J(R)$ is semisimple artinian and idempotents lift modulo $J(R)$, and that a ring R is right perfect if and only if $R/J(R)$ is semisimple artinian and $J(R)$ is right T-nilpotent, i.e., for every sequence $a_1, a_2, \dots$ in $J(R)$ there is an n such that $a_n \dots a_2 a_1 = 0$.

The dual of projective modules is the concept known as injective modules.

Definition 1.2.5. *Given two modules M and N, M is called N-injective if for any submodule X of N and any homomorphism $f : X \to M$, there exists a homomorphism $h : N \to M$ that extends f. The module M is called injective if it is N-injective for every module N, and M is called self-injective (or quasi-injective) if it is M-injective. In particular, if R_R is injective, then R is called a right self-injective ring.*

This notion was initially studied by Baer for abelian groups [5]. There is a very useful test for injectivity of a module M, which is known as the Baer's criterion. That is, a right R-module M is injective if and only if M is R_R-injective. The significance of injective modules can be seen in the following proposition.

Proposition 1.2.6. *[4, Proposition 18.6] Every right R-module can be embedded in an injective right R-module.*

The dual of projective covers is the notion of injective envelopes of modules.

Definition 1.2.7. *A monomorphism $0 \to M \xrightarrow{\beta} E$ with E injective and $im(\beta) \subseteq_e E$ is called an injective envelope (or injective hull) of the module M.*

Though not every module has a projective cover, the injective envelope of a module M always exists. The injective envelope E of a module M is denoted by $E(M)$.

Theorem 1.2.8. *[4, Theorem 18.10] Every module has an injective envelope. The injective envelope of a module is unique up to isomorphism.*

Many important properties of projective modules and injective modules are presented in [4]. From the definition of injective modules, every injective submodule E of a module M is always a direct summand of M. Direct sums and direct summands of projective modules are projective, and direct products and direct summands of injective modules are injective (see [4, Propositions 18.2 and 18.7]).

Semisimple artinian rings can be characterized in terms of injectivity.

Theorem 1.2.9. *[36, Theorem 2.9] The following are equivalent for a ring R:*

1. *R is semisimple artinian.*
2. *Every right (left) R-module is injective.*
3. *Every cyclic right (left) R-module is injective.*

The equivalence (1) $\Leftrightarrow$ (3) in Theorem 1.2.9 is known as the Osofsky's Theorem [50], which promoted a considerable interest in rings whose cyclic modules satisfy a certain generalized injectivity condition, such as being quasi-injective, continuous, quasi-continuous, or extending. These concepts will be introduced shortly.

In his work on continuous rings, Utumi [58] identified three conditions on a ring that are satisfied by any self-injective ring. These conditions were extended to modules by Jeremy [27], Mohamed and Bouhy [40] as follows:

- $C1$-condition: Every submodule of M is essential in a direct summand of M.
- $C2$-condition: If a submodule A of M is isomorphic to a direct summand of M, then A is a direct summand of M.

- $C3$-condition: If M_1 and M_2 are direct summands of M with $M_1 \cap M_2 = 0$ then $M_1 \oplus M_2$ is also a direct summand of M.

Here are some generalizations of injective modules and quasi-injective modules.

Definition 1.2.10. *A module M is called a Ci-module if it satisfies the Ci-condition $(i = 1, 2, 3)$. A module M is called continuous if it satisfies both the $C1$- and $C2$-conditions, and quasi-continuous if it satisfies the $C1$- and $C3$-conditions. A ring R is called a right $C1$-ring (resp, $C2$-ring and $C3$-ring) if the module R_R has the corresponding property.*

Every quasi-injective module is continuous, and every $C2$-module is a $C3$-module (see [41, Propositions 2.1 and 2.2]). Let N be a submodule of module M. It is obvious that $N \subseteq_e N$. By applying Zorn's Lemma, there exists a submodule C of M maximal with respect to $N \subseteq_e C$. Such a module C is called a closed submodule in M.

Remark 1.2.11. *In [12], $C1$-modules are called extending module (or CS-modules). The term CS-modules refers to the fact that these modules M are characterized by the property that every closed submodule of M is a direct summand in M.*

For a submodule N of M, there also exist submodules H of M maximal with respect to $N \cap H = 0$. Such a submodule H is called a complement of N in M. A submodule K of M is called a complement in M if there exists a submodule N of M such that K is a complement of N in M. Notice that a submodule K of a module M is closed if and only if K is a complement submodule in M (see [12, p.6]).

A good account of these modules can be found in [41] which covers not only continuous and quasi-continuous modules but their duals discrete and quasi-discrete modules. We include some important properties of these modules here for convenience.

Theorem 1.2.12. *[41, Theorem 2.8] The following are equivalent for a module M:*

1. *M is quasi-continuous.*

2. *$M = X \oplus Y$ for any two submodules X and Y which are complements of each other.*

3. *$e(M) \subseteq M$ for every idempotent $e \in \text{End}(E(M))$.*

4. *$E(M) = \oplus_{i \in I} E_i$ implies $M = \oplus_{i \in I}(M \cap E_i)$.*

In [41, Proposition 2.10], it is shown that if $M = M_1 \oplus M_2$ is quasi-continuous then M_1 is M_2-injective. The endomorphism ring of a continuous module has some important properties. For instance, let M be a module with $S = \text{End}(M)$ be the ring of endomorphisms of M. If we define $\Delta(S) := \{f \in \text{End}(M) \mid ker(f) \subseteq_e M\}$, then the following theorem is proved.

Theorem 1.2.13. *[41, Theorem 3.11] If M is continuous, then $J(S) = \Delta(S)$ and $S/J(S)$ is right continuous regular.*

If we only assume that M is quasi-continuous, some crucial properties of continuous need not be true. For example, the Jacobson radical $J(S)$ might be different from $\Delta(S)$. If we denote $\overline{S} = S/\Delta(S)$ where $S = \text{End}(M)$ is the ring of endomorphisms of M, then we have

Corollary 1.2.14. *[41, Corollary 3.13] Let M be quasi-continuous. Then there is a ring decomposition $\overline{S} = \overline{S_1} \times \overline{S_2}$ such that $\overline{S_1}$ is regular and right self-injective, and $\overline{S_2}$ is reduced.*

Here a ring is called **reduced** if it has no non-zero nilpotent elements. An element x of a ring R is called nilpotent if there exists some positive integer n such that $x^n = 0$. Thus, every reduced ring is abelian, i.e., every idempotent of a reduced ring is central.

Remark 1.2.15. *In [12], quasi-continuous modules are also called π-injective. The term π-injective is due to Goel and Jain [18] and is also used by Wisbauer [60]. It was proved by Johnson and Wong [28] that a module M is quasi-injective if and only if $f(M) \subseteq M$ for every homomorphism $f : E(M) \to E(M)$. Here $E(M)$ is the injective envelope of M. The equivalent condition* (3) *in Theorem 1.2.12 gives a reason for the term π-injective modules.*

There is another reason. Since every $C2$-module is a $C3$-module, quasi-continuity was seen as a weak form of continuity. However, following [41, Proposition 3.15], continuous modules are thought of as being a special class of quasi-continuous modules, in particular they are quasi-continuous modules with rather special endomorphism rings (see Theorem 1.2.13). Thus the term π-injective frees quasi-continuous modules from their earlier subordinate role.

The Ci-conditions were dualized by Oshiro in [49], and by Mohamed, Müller and Singh in [42] as follows:

- $D1$-condition: For every submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq A$ and $A \cap M_2$ is small in M.
- $D2$-condition: If A is a submodule of M such that M/A is isomorphic to a direct summand of M, then A is a direct summand of M.
- $D3$-condition: If M_1 and M_2 are direct summands of M with $M_1 + M_2 = M$ then $M_1 \cap M_2$ is also a direct summand of M.

Definition 1.2.16. *A module M is called a Di-module if it satisfies the Di-condition* $(i = 1, 2, 3)$. *A module M is called discrete if it satisfies both the $D1$- and $D2$-conditions, and quasi-discrete if it satisfies the $D1$- and $D3$-conditions.*

In [41], $D1$-modules are also called lifting modules. Moreover, every quasi-projective module has the $D2$-condition, and every $D2$-module is a $D3$-module (see [41, Lemmas 4.6 and 4.38]).

In summary, we have the following diagram and none of the following implications is reversible.

$$\begin{array}{ccccccccc}
\text{Injective} & \Rightarrow & \text{Quasi-injective} & \Rightarrow & \text{Continuous} & \Rightarrow & \text{Quasi-continuous} & \Rightarrow & C1 \\
 & & & & \Downarrow & & \Downarrow & & \\
 & & & & C2\text{-modules} & \Rightarrow & C3\text{-modules} & &
\end{array}$$

$$\begin{array}{ccccccccc}
\text{Projective} & \Rightarrow & \text{Quasi-projective} & \Rightarrow & D2\text{-modules} & \Rightarrow & D3\text{-modules} & & \\
 & & & & \Uparrow & & \Uparrow & & \\
 & & & & \text{Discrete} & \Rightarrow & \text{Quasi-discrete} & \Rightarrow & D1
\end{array}$$

As a generalization of lifting modules, a module M is called **weakly supplemented** (see [41, Definitions A.1]) if for any submodule N of M there exists a submodule H of M such that H is minimal with respect to $M = N + H$. Such a submodule H is called a supplement (in M) of N, which was introduced by Kasch and Mares in [30]. Moreover, M is called supplemented if for any two submodules A and B with $A + B = M$, B contains a supplement of A.

Extending modules are generalizations of injective modules. However, projective modules need not be lifting modules. Complements of any submodule N of a module M exist by Zorn's Lemma, though the existence of supplements is not guaranteed.

Proposition 1.2.17. *[41, Proposition 4.39] A quasi-projective module M is discrete if and only if every submodule of M has a supplement.*

Another concept we should mention here is semiperfect modules. Mares [38] defined a module P to be **semiperfect** if P is projective and every homomorphic image of P has a projective cover. Lately, Kasch and Mares [30] proved that projective semiperfect modules are exactly those projective modules whose every submodule has a supplement. Particularly, semiperfect rings are those rings whose every right ideal has a supplement. The next theorem shows that the investigation of semiperfect modules can be reduced essentially to the projective semiperfect modules.

Theorem 1.2.18. *[30, Theorem 11.1.5] Let $P \xrightarrow{\eta} M \to 0$ be a projective cover of M. The following are equivalent:*

1. *M is semiperfect.*
2. *P is semiperfect.*
3. *Every submodule of P has a supplement.*

Corollary 1.2.19. *[41, Corollary 4.42] A ring R is semiperfect, if and only if R_R is discrete, if and only if every right ideal of R has a supplement.*

As our main focus is on a study of properties of cyclic modules, we recall an important result on the structure of cyclic modules over semiperfect rings.

Lemma 1.2.20. *[4, Lemma 27.3] A cyclic module M_R has a projective cover if and only if $M \cong eR/eI$ for some idempotent $e \in R$ and some right ideal $I \subseteq J(R)$. For e and I satisfying this condition the natural map $eR \to eR/eI \to 0$ is a projective cover.*

Rings R for which every right R-module is extending are precisely the artinian serial rings with Jacobson radical square-zero (see [13, Theorem 11]). Interestingly, this class of rings coincides with the class of rings whose modules are lifting (see [52, Corollary 2.5] and [31, Theorem 3.15]). We present those results in the following theorem.

Theorem 1.2.21. *The following are equivalent for a ring R.*

1. *R is artinian serial and $J(R)^2 = 0$.*
2. *Every right R-module is extending.*
3. *R is semiregular and every 2-generated right R-module is extending.*
4. *Every right R-module is lifting.*
5. *Every 2-generated right R-module is lifting.*

Chapter 2

Rings whose cyclics are $C3$-modules

In this chapter, we carry out a study of right $CC3$-rings, that is, rings whose cyclic modules are $C3$-modules. The motivation is the observation that a ring R is semisimple artinian if and only if every 3-generated right R-module is a $C3$-module. Many basic properties are obtained for $CC3$-rings, and some structure theorems are proved. For instance, it is proved that a semiperfect ring is a right $CC3$-ring if and only if it is a direct product of a semisimple artinian ring and finitely many local rings, and that a right self-injective regular ring is a right $CC3$-ring if and only if it is a direct product of a semisimple artinian ring, a strongly regular ring and a 2×2 matrix ring over a strongly regular ring. Applications to the rings whose 2-generated modules are $C3$-modules, and the rings whose cyclics are ADS or quasi-continuous are addressed.

2.1 Cyclics are C3-modules

We begin with an important property of $C3$-modules which was showed in [3].

Proposition 2.1.1. *[3, Proposition 2.3] Let M be a $C3$-module. If $M = M_1 \oplus M_2$ and if $f : M_1 \to M_2$ is a homomorphism with $ker(f) \subseteq^{\oplus} M_1$, then $im(f) \subseteq^{\oplus} M_2$.*

Proof. We first show that if $f : M_1 \to M_2$ is a monomorphism, then $im(f) \subseteq^{\oplus} M_2$. Let $T = \{a + f(a) \ : \ a \in M_1\}$ be the graph submodule of M. We claim that $M = T \oplus M_2$. For, if $x \in M$, then $x = a + b$ for some $a \in M_1$ and $b \in M_2$. Now $x = (a + f(a)) + (-f(a) + b) \in T + M_2$, and so $M = T + M_2$. If $x \in T \cap M_2$, then $x = a+f(a)$ for some $a \in M_1$, and hence $a = x-f(a) \in M_1 \cap M_2 = 0$. Clearly, $x = 0$, $M = T \oplus M_2$ and $T \subseteq^{\oplus} M$. Next, we show that $M_1 \cap T = 0$. For, if $x \in M_1 \cap T$, then $x = a + f(a)$ for some $a \in M_1$, and consequently, $x - a = f(a) \in M_1 \cap M_2 = 0$. Now since f is a monomorphism, $a = 0$, and hence $x = 0$. Since M is a $C3$-module, $M_1 \oplus T \subseteq^{\oplus} M$. Finally, we show that $M_1 \oplus T = M_1 \oplus im(f)$. For, if $x \in im(f)$, then $x = f(a)$ for some $a \in M_1$, and so $x = -a + a + f(a) \in M_1 + T$, and hence $M_1 \oplus T = M_1 \oplus im(f)$. Since $M_1 \oplus T \subseteq^{\oplus} M$, $im(f) \subseteq^{\oplus} M$, and so $im(f) \subseteq^{\oplus} M_2$. Now let $f : M_1 \to M_2$ be an arbitrary homomorphism with $ker(f) \subseteq^{\oplus} M_1$. If $M_1 = ker(f) \oplus B$ for a submodule $B \subseteq^{\oplus} M_1$, then $M = M_1 \oplus M_2 = ker(f) \oplus B \oplus M_2$, and the restriction map $f|_B : B \to M_2$ is a monomorphism. Since a direct summand of a $C3$-module is again a $C3$-module, we infer from the preceding argument that $im(f) = im(f|_B) \subseteq^{\oplus} M_2$, as required. □

As a consequence of Proposition 2.1.1, the following lemma will be repeatedly used in this chapter.

Lemma 2.1.2. *Suppose that $M_1 \oplus M_2$ is a $C3$-module. If $X \subseteq M_2$ is isomorphic to a direct summand of M_1, then $X \subseteq^{\oplus} M_2$.*

Proof. Suppose that $\alpha : Y \to X$ is an isomorphism where $M_1 = Y \oplus Z$. Then $f = \alpha \circ \pi : M_1 \to M_2$ where $\pi : M_1 \to Y$ is the natural projection. It follows that $X \subseteq^{\oplus} M_2$ by Proposition 2.1.1 as $ker(f) = Z \subseteq^{\oplus} M_1$. □

Semisimple artinian rings can be characterized via their $C3$-modules.

Corollary 2.1.3. *The following are equivalent for a ring R:*

1. *R is semisimple artinian.*

2. *Every 3-generated right R-module is a $C3$-module.*

Proof. (1) $\Rightarrow$ (2). It is clear. (2) $\Rightarrow$ (1). Let N be a cyclic right R-module and let $x \in E(N)$ where $E(N)$ is the injective envelope of N. Then $N \oplus (N + xR)$ is a 3-generated right R-module and hence is a $C3$-module by hypothesis. Thus, by Lemma 2.1.2, $N \subseteq^{\oplus} (N + xR)$. As N is essential in $E(N)$, $N = N + xR$. So $x \in N$, and hence $N = E(N)$. So N is injective. Thus, R is semisimple artinian by Theorem 1.2.9(3). □

The following questions arise naturally.

Questions 2.1.4.

1. *For which rings R, is every cyclic right R-module a $C3$-module?*

2. *For which rings R, is every 2-generated right R-module a $C3$-module?*

In this section, we address these questions, and so we give the following definition.

Definition 2.1.5. *A ring R is called a right $CC3$-ring if every cyclic right R-module is a $C3$-module. A ring R is called a $CC3$-ring if it is both a right and a left $CC3$-ring.*

Before giving some important examples of right $CC3$-rings, we recall a class of modules which can be seen as examples of $C3$-modules.

A right R-module M is said to satisfy the summand sum property if the sum of two direct summands of M is again a summand of M. A module with the summand sum property is called an SSP-module. A ring R is an SSP-ring if the module R_R is an SSP-module. This is a left-right symmetric condition for rings (see [17, Proposition 2.2]).

Lemma 2.1.6. *[21, Theorem 2.3] The following conditions on a right R-module M are equivalent:*

1. *M is an SSP-module.*
2. *If $M = A \oplus B$, for some submodules A and B, then for every R-homomorphism $f : A \to B$, $im(f) \subseteq^{\oplus} B$.*

Proof. Assume that M is an SSP-module. Let $M = A \oplus B$ and $f : A \to B$ be a homomorphism. Let $T = \{a + f(a) \ : \ a \in A\}$. As shown in the proof of Proposition 2.1.1, we have $M = T \oplus B$ and $A + T = A \oplus im(f)$. By hypothesis, $M = (A + T) \oplus L$ for some submodule $L \subseteq M$. So $im(f) \subseteq^{\oplus} M$. Hence $im(f) \subseteq^{\oplus} B$.

For the converse, assume that for every decomposition $M = A \oplus B$ and every homomorphism $f : A \to B$, the image of f is a direct summand of B. Let $M = N \oplus N_1$, $M = K \oplus K_1$ and let $\pi|_{N_1} : M \to N_1$ and $\pi|_K : M \to K$ be the natural epimorphisms. Now, define $h = (\pi|_{N_1} \circ \pi|_K)|_N$. Notice that h is defined from N to N_1. Thus, $im(h) \subseteq^{\oplus} M$. Let $H = im(h)$. We have $H = (N+K_1) \cap (N+K) \cap N_1$ and $H \oplus L = M$ for some $L \subseteq M$. Hence $N_1 = H \oplus (N_1 \cap L)$. Then $(N+K) \cap \big[(N+K_1) \cap (N_1 \cap L)\big] = \big[(N + K) \cap (N + K_1) \cap N_1\big] \cap (N_1 \cap L) = H \cap (N_1 \cap L) = 0$.

To show that $N+K$ is a direct summand of M, it is enough to prove that $M = (N+K) + \big[(N+K_1) \cap (N_1 \cap L)\big]$. Since $H \subseteq N+K$ and $H \subseteq N+K_1$, the modular law and $M = N \oplus N_1 = N \oplus \big[H \oplus (N_1 \cap L)\big] = N \oplus H \oplus (N_1 \cap L)$ imply

$$N + K = (N+K) \cap M = N \oplus H \oplus \big[(N+K) \cap (N_1 \cap L)\big]$$

and

$$N + K_1 = (N+K_1) \cap M = N \oplus H \oplus \big[(N+K_1) \cap (N_1 \cap L)\big].$$

Hence $M = N+K+K_1 = (N \oplus H) + \big[(N+K) \cap (N_1 \cap L)\big] + \big[(N+K_1) \cap (N_1 \cap L)\big] \subseteq (N+K) + \big[(N+K_1) \cap (N_1 \cap L)\big]$. Thus $N+K$ is direct summand and so M is an SSP-module. □

Clearly every SSP-module is a $C3$-module, but a $C3$-module (indeed, an injective module) need not be an SSP-module (see [21, Example 2.6]). However, all factor modules of a module are SSP-modules if and only if all factor modules are $C3$-modules.

Corollary 2.1.7. *A ring R is a right $CC3$-ring if and only if every cyclic right R-module is an SSP-module.*

Proof. The sufficiency is clear. For the necessity, let M be a cyclic right R-module, $M = A \oplus B$ and $f : A \to B$ an R-homomorphism. Set $K := ker(f)$ and consider the monomorphism $g : A/K \to B$ defined by $g(a+K) = f(a)$, for all $a \in A$. By the hypothesis $M/K \cong A/K \oplus B$ is a $C3$-module. So, by Lemma 2.1.2, $im(f) = im(g) \subseteq^{\oplus} B$. Hence M is an SSP-module by Lemma 2.1.6. □

One source of $CC3$-rings is in the class of exchange rings. Recall that a ring R is an exchange ring if idempotents can be lifted modulo every right (equivalently, left) ideal.

A ring R is said to satisfy condition $(*)$ if for any idempotents e and f, $eR+fR=gR$ where $g^2=g\in\langle e,f\rangle$. Here $\langle e,f\rangle$ is the subring of R generated by e,f and 1_R. Clearly, any abelian ring (i.e., every idempotent is central) satisfies condition $(*)$, but a ring with $(*)$ need not be abelian (e.g., $\mathbb{M}_2(\mathbb{Z}_2)$).

Lemma 2.1.8. *[9, Lemma 4.6] Let $e^2=e\in R$ and let I be a right ideal of R. The following conditions are equivalent:*

1. $(R/I)_R=(eR+I)/I\oplus((1-e)R+I)/I$.
2. $eI\subseteq I$.

Proof. (1) $\Rightarrow$ (2). By (1), we have $I=\big(eR+I\big)\cap\big((1-e)R+I\big)=eR\cap\big((1-e)R+I\big)+I$, so $eR\cap\big((1-e)R+I\big)\subseteq I$. For $a\in I, ea=(1-e)(-a)+a\in eR\cap\big((1-e)R+I\big)$, so $ea\in I$.

(2) $\Rightarrow$ (1). It suffices to show that $I=\big(eR+I\big)\cap\big((1-e)R+I\big)$ or $eR\cap\big((1-e)R+I\big)\subseteq I$. Let $x:=ea=(1-e)b+c$ where $a,b\in R$ and $c\in I$. Then $ex=eex=ea=x$, and $ex=e((1-e)b+c)=ec\in eI\subseteq I$. So $x=ex\in I$. $\square$

Theorem 2.1.9. *Any exchange ring R with condition $(*)$ is a right $CC3$-ring.*

Proof. Let I be a right ideal of R. Suppose K/I is a summand of $(R/I)_R$. Then $(R/I)_R=(K/I)\oplus(K'/I)$. Thus $R=K+K'$. Write $1=x+x'$ where $x\in K$ and $x'\in K'$. Then $x-x^2=x'x\in K\cap K'=I$. Since R is an exchange ring, idempotents can be lifted modulo I. Hence there exists $e^2=e\in R$ such that $x-e\in I$, and so $(1-x)-(1-e)\in I$. It follows that $K=eR+I$ and $K'=(1-e)R+I$. Since $(R/I)_R=(K/I)\oplus(K'/I)$, we deduce that $eI\subseteq I$ by Lemma 2.1.8. If L/I is another summand of $(R/I)_R$, then as above $L=fR+I$ where $f^2=f$ and $fI\subseteq I$. By the hypothesis, there exists $g^2=g\in\langle e,f\rangle$ such that $eR+fR=gR$. Thus,

$K/I + L/I = (eR + fR + I)/I = (gR + I)/I$. Inasmuch as $g \in \langle e, f \rangle, eI \subseteq I$ and $fI \subseteq I$, we infer that $gI \subseteq I$. By Lemma 2.1.8, $(gR + I)/I$ is a summand of $(R/I)R$. Hence, $(R/I)_R$ is a $C3$-module. □

Examples of $CC3$-rings can be found in some important classes of rings. For instance, we have

Examples 2.1.10. *1. Every commutative ring is a $CC3$-ring.*

2. Every strongly regular ring is a $CC3$-ring.

3. Every local ring is a $CC3$-ring.

Proof. (1). Let R be a commutative ring. For an ideal I of R, the direct summands of $(R/I)_R$ coincide with the direct summands of $(R/I)_{R/I}$. So $(R/I)_R$ is a $C3$-module if and only if $(R/I)_{R/I}$ is a $C3$-module. As R/I is a commutative ring, we only need to show that every commutative ring is a $C3$-module. Indeed, let S be an arbitrary commutative ring and $x^2 = x, y^2 = y \in S$. Assume $xS \cap yS = 0$, then $xy = 0$. So $xS \oplus yS = (x + y)S$ where $(x + y)^2 = (x + y)$. Hence, $xS \oplus yS$ is a direct summand of S.

(2). A strongly regular ring is an exchange ring and an abelian ring. So the claim follows by Theorem 2.1.9.

(3). Every local ring is an exchange ring and an abelian ring. The claim follows from Theorem 2.1.9. □

The ring of 2×2 matrices over a $CC3$-ring need not be a right $CC3$-ring.

Example 2.1.11. *Let $R = \mathbb{M}_2(\mathbb{Z})$, and consider the following three right ideals:*

$$A = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}, \quad \text{and} \quad B = \left\{ \begin{pmatrix} x & y \\ 2x & 2y \end{pmatrix} : x, y \in \mathbb{Z} \right\}.$$

Clearly, $R = A \oplus C = B \oplus C$, so A, B are direct summands of R_R. However, $A \cap B = 0$ and $A \oplus B = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{pmatrix} \subseteq_e R_R$. Hence $A \oplus B$ is not a direct summand of R, so R is not a right $C3$-ring. Thus, R is not a right $CC3$-ring. But $\mathbb{Z}$ is a $CC3$-ring by Examples 2.1.10(1).

We have a characterization for the matrix ring $\mathbb{M}_n(R)$ to be a right $CC3$-ring.

Lemma 2.1.12. *Let $n \geq 2$. The following are equivalent for a ring R:*

1. *Every n-generated right R-module is a $C3$-module.*

2. *Every cyclic right $\mathbb{M}_n(R)$-module is a $C3$-module.*

Proof. Let $P = (R^n)_R$ and $S = \text{End}(P_R)$. Then $\text{Hom}_R(P,) : N_R \mapsto \text{Hom}_R({}_S P_R, N_R)$ defines a Morita equivalence between Mod-R and Mod-S with inverse equivalence $- \otimes_S P : M_S \mapsto M \otimes P$. For any n-generated right R-module N, $\text{Hom}_R(P, N)$ is a cyclic right S-module, and, for any cyclic right S-module M, $M \otimes_S P$ is an n-generated right R-module. Moreover, a Morita equivalence preserves the $C3$-condition of modules. Thus, every cyclic right S-module is a $C3$-module if and only if every n-generated right R-module is a $C3$-module. □

One sees that $\mathbb{Z}$ is a $CC3$-ring, but not every 2-generated $\mathbb{Z}$-module is a $C3$-module. For instance, the 2-generated $\mathbb{Z}$-module $M = 2\mathbb{Z} \oplus \mathbb{Z}$ is not a $C3$-module (by Lemma

2.1.2 as $2\mathbb{Z}$ is not a direct summand of $\mathbb{Z}$). Some properties of right $CC3$-rings can be given in a few lemmas.

Lemma 2.1.13. *Any factor ring of a right $CC3$-ring is again a right $CC3$-ring.*

Proof. Let $S = R/I$ be a factor ring of a right $CC3$-ring R where $I \subseteq R$ is an ideal. Every cyclic right S-module has a form $M = S/K$ for some right ideal $K \subseteq S$. Since $S = R/I$, we have $K = H/I$ for some right ideal $H \subseteq R$. As $(R/H)_R$ is a $C3$-module, $(R/H)_S$ is a $C3$-module. So $M_S = S/K = (R/I)/(H/I) \cong (R/H)_S$. □

Lemma 2.1.14. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a finite direct product of rings. Then R is a right $CC3$-ring if and only if every R_i is a right $CC3$-ring.*

Proof. ($\Rightarrow$). This is by Lemma 2.1.13.

($\Leftarrow$). We can assume that $n = 2$. Let I be a right ideal of R. Then $M := R_1/IR_1 \times R_2/IR_2$ is a right R-module, where $(r_1 + IR_1, r_2 + IR_2)r = (r_1r + IR_1, r_2r + IR_2)$ for $r_1 \in R_1, r_2 \in R_2$ and $r \in R$. Define the R-isomorphism $\theta : R/I \to M$ by letting $\theta(r + I) = (e_1r + IR_1, e_2r + IR_2)$, where $e_1 = 1_{R_1}$ and $e_2 = 1_{R_2}$. Suppose that $K/I, L/I$ are two direct summands of $(R/I)_R$ with zero intersection. Then $\theta(K/I) = (e_1K/IR_1, e_2K/IR)$, $\theta(L/I) = (e_1L/IR_1, e_2L/IR)$ are two direct summands of M with zero intersection. It follows that, as R_1-modules, $e_1K/IR_1, e_1L/IR_1$ are direct summands of R_1/IR_1 with zero intersection and that, as R_2-modules, $e_2K/IR_2, e_2L/IR_2$ are direct summands of R_2/IR_2 with zero intersection. So, for $i = 1, 2$, $e_iK/IR_i \cap e_iL/IR_i$ is a direct summand of R_i/IR_i as R_i-modules. Hence $\theta(K/I + L/I) = \theta(K/I) + \theta(L/I)$ is a direct summand of M. Thus we have showed that $K/I + L/I$ is a direct summand of $(R/I)_R$. □

Example 2.1.15. *An infinite direct product of right $CC3$-rings need not be a right $CC3$-ring: Let D_i be a division ring for $i \geq 1$. Then $\mathbb{M}_3(D_i)$ is a right $CC3$-ring,* *but* $\prod_{i=1}^{\infty}\mathbb{M}_3(D_i)\cong\mathbb{M}_3\big(\prod_{i=1}^{\infty}(D_i)\big)$ *is not a right* $CC3$*-ring (by Lemma 2.1.12 and Corollary 2.1.3) as* $\prod_{i=1}^{\infty}(D_i)$ *is not semisimple artinian. This example also shows that a unit-regular ring need not be a right* $CC3$*-ring, a contrast to Examples 2.1.10(2).*

Lemma 2.1.16. *Let* e *be an idempotent of* R *with* $ReR=R$*. If* R *is a right* $CC3$*-ring, then so is* eRe*.*

Proof. Let $S=eRe$ and $P=(eR)_R$. Then $\mathrm{Hom}_R(P,-):N_R\mapsto\mathrm{Hom}_R({}_SP_R,N_R)$ defines a Morita equivalence between Mod-R and Mod-S with inverse equivalence $-\otimes_S P:M_S\mapsto M\otimes P$. If R is a right $CC3$-ring, then every quotient module of R_R is a $C3$-module. Since a Morita equivalence preserves the $C3$-condition of modules, every quotient module of the right S-module $\mathrm{Hom}_R({}_SP_R,R)$ is a $C3$-module. But, $\mathrm{Hom}_R({}_SP_R,R)\cong(Re)_S=[(1-e)Re\oplus eRe]_S$. So every quotient module of S_S is a $C3$-module. □

Next we characterize semiperfect rings that are right $CC3$-rings. The next lemma is needed.

Lemma 2.1.17. *Let* R *be a right* $CC3$*-ring and let* e *and* f *be orthogonal idempotents of* R *with* $eaf\neq 0$ *for some* $a\in R$*.*

1. *If* eR *is indecomposable, then* $eafR=eR$*.*
2. *If* eR *and* fR *both are indecomposable, then* eR *and* fR *are isomorphic minimal right ideals of* R*.*

Proof. (1) Let $N=ann_r(ea)\cap fR$. Then $eafR\oplus eR\cong(fR/N)\oplus eR\cong(fR+eR)/N=(f+e)R/N\cong R/[N\oplus(1-f-e)R]$, which is a $C3$-module. By Lemma 2.1.2, $eafR$ is a direct summand of eR, so $eafR=eR$.

(2) By (1), $eafR = eR$. Since $fr \mapsto eafr$ is an epimorphism from fR to the projective module $eafR = eR$, eR is isomorphic to a direct summand of fR and so $fR \cong eR$. Let $0 \neq eb \in eR$. We want to show that $ebR = eR$ and thus eR is simple. If $eb(1-e) \neq 0$, then, by (1), $eb(1-e)R = eR$, showing that $ebR = eR$. If $eb(1-e) = 0$, then $eb = ebe$. We see $ebeR \oplus eR \cong ebeR \oplus fR = ebeR + fR = (ebe + f)R$ is a $C3$-module. By Lemma 2.1.2, $ebR = ebeR = eR$. So, eR is a minimal right ideal of R. □

A primitive idempotent of a ring R is a nonzero idempotent e such that eRe has no nontrivial idempotents, equivalently, $(eR)_R$ is indecomposable. A local idempotent of a ring R is an idempotent e such that eRe is a local ring. Note that, every primitive idempotent of a semiperfect ring is local. Then a ring is local if and only if it is semiperfect with no nontrivial idempotents. It is also well-known that a ring R is semiperfect if and only if $1_R = e_1 + \ldots + e_n$, where the e_i are local, pairwise orthogonal idempotents (i.e., $e_i e_j = 0$ for $i \neq j$).

Theorem 2.1.18. *A semiperfect ring R is a right $CC3$-ring if and only if $R = R_1 \times R_2$, where R_1 is a semisimple artinian ring and R_2 is a finite direct product of local rings.*

Proof. The sufficiency is by Lemma 2.1.14 and Corollary 2.1.3. For the necessity, let R be a right $CC3$-ring. Since R is semiperfect, R_R has an indecomposable direct decomposition. So we can write $R = e_1R \oplus e_2R \oplus \ldots \oplus e_nR$ where e_i are pairwise orthogonal local idempotents of R. Let $[e_tR] = \sum_i \{e_iR : e_iR \cong e_tR\}$. Renumbering if necessary, we may write $R = [e_1R] \oplus [e_2R] \oplus \ldots \oplus [e_kR]$. By Lemma 2.1.17, each $[e_iR]$ is an ideal of R. If $[e_iR]$ contains more than one direct summands, then $[e_iR]$ is a simple artinian ring by Lemma 2.1.17. If $[e_iR]$ consists of exactly one direct summand, then $[e_iR] = e_iR = e_iRe_i$ is a local ring. So, the proof is complete. □

We have been unable to show that $CC3$-ring is a left-right symmetric notion. However, the next corollary is an immediate consequence of Corollary 2.1.7, and Theorem 2.1.18.

Corollary 2.1.19. *The following are equivalent for a semiperfect ring R:*

1. *R is a right $CC3$-ring.*
2. *R is a left $CC3$-ring.*
3. *Every cyclic right R-module is an SSP-module.*
4. *Every cyclic left R-module is an SSP-module.*

Remark 2.1.20. *It is worth to note that the well-known Osofsky's Theorem for rings also has a module version in [51, Corollary 2]). That is, if every quotient of any cyclic submodule of M is injective then M is semisimple.*

Here we also have a module analog of Corollary 2.1.3. We need to recall some notions in [60]. For a module M, we denote by $\sigma[M]$ the full subcategory of Mod-R, whose objects are the submodules of M-generated modules. An R-module N is called M-generated if there exists an epimorphism $M^{(I)} \to N$ for some index set I. We also write $E_M(N)$ for the M-injective envelope of a module N which is the trace of M in the injective envelope $E(N)$ of N, that is, $E_M(N) = \sum \{f(M) : f \in \mathrm{Hom}(M, E(N))\}$. Thus, the same argument in the proof of Corollary 2.1.3 shows

Corollary 2.1.21. *The following are equivalent for a module M:*

1. *M is semisimple.*
2. *Every 3-generated module in $\sigma[M]$ is a $C3$-module.*

In the next part, a structure of right self-injective regular $CC3$-rings is obtained. We start with an observation. Let R be the ring of linear transformations of a vector space V over a division ring D. Then R is a right $CC3$-ring if and only if $\dim(V) < \infty$. Indeed, the sufficiency is obvious. Suppose $\dim(V) = \infty$. Then $V_D \cong V_D^3$, so $R = \mathrm{End}(V_D) \cong \mathrm{End}(V_D^3) \cong \mathbb{M}_3(R)$. Since R is not semisimple artinian, $\mathbb{M}_3(R)$ is not a right $CC3$-ring by Lemma 2.1.12 and Corollary 2.1.3. So, R is not a right $CC3$-ring.

It is known that the ring $R = \mathrm{End}_D(V)$ is a right self-injective regular ring. A motivated question is: Which right self-injective regular rings are right $CC3$-rings? To answer this question, several technical lemmas are needed.

Lemma 2.1.22. *Let I, K be right ideals of a ring R with $I \subseteq K$, and let u be a unit of R. Then $(K/I)_R \subseteq^{\oplus} (R/I)_R$ if and only if $(uK/uI)_R \subseteq^{\oplus} (R/uI)_R$.*

Proof. The mapping $\theta : R/I \to R/uI$ given by $\theta(x+I) = ux+uI$ is an R-isomorphism, and $\theta(K/I) = uK/uI$. So the claim follows. □

Lemma 2.1.23. *Let I, K be right ideals of a ring R with $I \subseteq K$, and let e be a central idempotent of R. Then $(K/I)_R \subseteq^{\oplus} (R/I)_R$ if and only if $(eK/eI)_{eR} \subseteq^{\oplus} (eR/eI)_{eR}$ and $((1-e)K/(1-e)I)_{(1-e)R} \subseteq^{\oplus} ((1-e)R/(1-e)I)_{(1-e)R}$.*

Proof. The mapping $\theta : R/I \to eR/eI \times (1-e)R/(1-e)I$ given by $\theta(x+I) = (ex+eI, (1-e)x+(1-e)I)$ is an R-isomorphism, and $\theta(K/I) = eK/eI \times (1-e)K/(1-e)I$. Hence,

$$\left(\frac{K}{I}\right)_R \subseteq^{\oplus} \left(\frac{R}{I}\right)_R$$

$$\Leftrightarrow \left(\frac{eK}{eI}\right)_R \subseteq^{\oplus} \left(\frac{eR}{eI}\right)_{eR} \text{ and } \left(\frac{(1-e)K}{(1-e)I}\right)_R \subseteq^{\oplus} \left(\frac{(1-e)R}{(1-e)I}\right)_R$$

$\Leftrightarrow \left(\frac{eK}{eI}\right)_{eR} \subseteq^{\oplus} \left(\frac{eR}{eI}\right)_{R}$ and $\left(\frac{(1-e)K}{(1-e)I}\right)_{(1-e)R} \subseteq^{\oplus} \left(\frac{(1-e)R}{(1-e)I}\right)_{(1-e)R}$. □

Lemma 2.1.24. *Let $R = \mathbb{M}_2(S)$ where S is a strongly regular ring. Let $e = \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}$ or $e = \begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix}$ where $h^2 = h \in S$. If I is a right ideal of R with $eI \subseteq I$, then $(eR + yR + I)/I$ is a direct summand of $(R/I)_R$ for any $y^2 = y \in R$ with $yI \subseteq I$.*

Proof. Let $K = eR + yR + I$.

Case 1: $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Let $I_1 = \left\{ \begin{pmatrix} r & s \\ 0 & 0 \end{pmatrix} \in R : \begin{pmatrix} r & s \\ w & t \end{pmatrix} \in I \text{ for some } w, t \in S \right\}$ and $I_2 = \left\{ \begin{pmatrix} 0 & 0 \\ r & s \end{pmatrix} \in R : \begin{pmatrix} w & t \\ r & s \end{pmatrix} \in I \text{ for some } w, t \in S \right\}$. As $eI \subseteq I, I = I_1 \oplus I_2$. Write $y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$. Then $y_3S + y_4S = tS$ where $t^2 = t \in S$. We have that $K = eR + yR + I = \begin{pmatrix} S & S \\ tS & tS \end{pmatrix} + I$. Let $L = \begin{pmatrix} 0 & (1-t)S \\ 0 & (1-t)S \end{pmatrix} + I$. Then $K + L = R$. If $x \in K \cap L$, write $x = \begin{pmatrix} r_1 & r_2 \\ tr_3 & tr_4 \end{pmatrix} + a_0 = \begin{pmatrix} 0 & 0 \\ (1-t)r_5 & (1-t)r_6 \end{pmatrix} + b_0$ where $a_0, b_0 \in I$. Since $f := \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$ is central in R, $(1-f)I \subseteq I$. So $(1-f)(a_0 - b_0) = \begin{pmatrix} -(1-t)r_1 & -(1-t)r_2 \\ (1-t)r_5 & (1-t)r_6 \end{pmatrix} \in I$, showing that $\begin{pmatrix} 0 & 0 \\ (1-t)r_5 & (1-t)r_6 \end{pmatrix} \in I_2 \subseteq I$. Hence $x \in I$. This shows that $K \cap L = I$, so $(K/I)_R \oplus (L/I)_R = (R/I)_R$.

Case 2: $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Let $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $ueu^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Since $uK =$

$(ueu^{-1})R+(uyu^{-1})R+uI, (ueu^{-1})(uI) \subseteq uI$ and $(uyu^{-1})(uI) \subseteq uI$, we have $(uK/uI)_R \subseteq^{\oplus} (R/uI)_R$ by Case 1. So $(K/I)_R \subseteq^{\oplus} (R/I)_R$ by Lemma 2.1.22.

Case 3: $e = \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}$ with $h^2 = h$. Let $f = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}$. Then f is a central idempotent of R, $fR = \mathbb{M}_2(hSh)$, and $(1-f)R = \mathbb{M}_2((1-h)S(1-h))$. We have $fK = eR + (fy)R + fI, e(fI) \subseteq fI$ and $(fy)(fI) \subseteq fI$. So $(fK/fI)_{fR} \subseteq^{\oplus} (fR/fI)_{fR}$ by Case 1. On the other hand, we have $(1-f)K = (1-f)yR + (1-f)I$. Since $(1-f)y \cdot (1-f)I \subseteq (1-f)I, (1-f)K/(1-f)I$ is a direct summand of $((1-f)R/(1-f)I)_{(1-f)R}$ by Lemma 2.1.8. Hence K/I is a direct summand of $(R/I)_R$ by Lemma 2.1.23.

Case 4: $e = \begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix}$ with $h^2 = h$. The proof is similar to Case 3. □

Lemma 2.1.25. *Let $R = \mathbb{M}_2(S)$ where S is a strongly regular ring. Then R is a right $CC3$-ring.*

Proof. Let I be a right ideal of R. Suppose that $A/I, B/I$ are direct summands of $(R/I)_R$. We show that $(A+B)/I$ is a direct summand of $(R/I)_R$. As in the proof of Theorem 2.1.9, there exists $x^2 = x \in R$ such that $A = xR + I$ and $A' := (1-x)R + I$ and $A/I \oplus A'/I = R/I$. So $xI \subseteq I$ by Lemma 2.1.8. Similarly, there exists $y^2 = y \in R$ such that $B = yR + I$ with $yI \subseteq I$. Let $K = A + B = xR + yR + I$. Next we show that $(K/I)_R \subseteq^{\oplus} (R/I)_R$.

Henriksen in [22] showed that every square matrix over a unit-regular ring is equivalent to a diagonal matrix. Since S is strongly regular, it follows that x is equivalent to a diagonal matrix. In [56], Song and Guo showed that an idempotent matrix over a ring is similar to a diagonal matrix if and only if it is equivalent to a diagonal matrix. So there exists a unit u in R such that $x_1 := uxu^{-1} = \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}$ where g, h are idempotents of S. Let $I_1 = uI$ and $y_1 = uyu^{-1}$, and $K_1 := uK = x_1R + y_1R + I_1$. By Lemma 2.1.22, $(K/I)_R \subseteq^{\oplus} (R/I)_R \Leftrightarrow (K_1/I_1)_R \subseteq^{\oplus} (R/I_1)_R$. So it suffices to show that $(K_1/I_1)_R \subseteq^{\oplus} (R/I_1)_R$. Note that $x_1I_1 \subseteq I_1$ and $y_1I_1 \subseteq I_1$.

Let $e = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$ and $f = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}$. Then e, f are central idempotents of R. By Lemma 2.1.23, $(K_1/I_1)_R \subseteq^{\oplus} (R/I_1)_R$ if and only if

$$\Big(\frac{(1-e)K_1}{(1-e)I_1}\Big)_{(1-e)R} \subseteq^{\oplus} \Big(\frac{(1-e)R}{(1-e)I_1}\Big)_{(1-e)R} \tag{2.1}$$

and

$$\Big(\frac{eK_1}{eI_1}\Big)_{eR} \subseteq^{\oplus} \Big(\frac{eR}{eI_1}\Big)_{eR}. \tag{2.2}$$

Again by Lemma 2.1.23, (2.2) holds if and only if

$$\Big(\frac{(e-ef)eK_1}{(e-ef)eI_1}\Big)_{(e-ef)eR} \subseteq^{\oplus} \Big(\frac{(e-ef)eR}{(e-ef)eI_1}\Big)_{(e-ef)eR} \tag{2.3}$$

and

$$\Big(\frac{efK_1}{efI_1}\Big)_{efR} \subseteq^{\oplus} \Big(\frac{efR}{efI_1}\Big)_{efR}. \tag{2.4}$$

So we only need to verify (2.1), (2.3) and (2.4). Since $efK_1 = efR$, we see that (2.4) holds. Note that $(1-e)K_1 = (1-e)x_1R + (1-e)y_1R + (1-e)I_1$, and that $(1-e)x_1 \cdot (1-e)I_1 \subseteq (1-e)I_1$ and $(1-e)y_1 \cdot (1-e)I_1 \subseteq (1-e)I_1$. Since $(1-e)x_1 = \begin{pmatrix} 0 & 0 \\ 0 & (1-g)h \end{pmatrix}$ and $(1-e)R = \mathbb{M}_2((1-g)S(1-g))$, (2.1) holds by Lemma 2.1.24. As for (2.3), we have that $(e-ef)eK_1 = (e-ef)x_1R + (e-ef)y_1R + (e-ef)I_1$, and that $(e-ef)x_1 \cdot (e-ef)I_1 \subseteq (e-ef)I_1$ and $(e-ef)y_1 \cdot (e-ef)I_1 \subseteq (e-ef)I_1$.

Because $(e - ef)x_1 = \begin{pmatrix} g(1-h) & 0 \\ 0 & 0 \end{pmatrix}$ and $(e-ef)R = \mathbb{M}_2(g(1-h)Sg(1-h))$, (2.3) holds by Lemma 2.1.8. □

We need to recall some notions from [19, pp. 111-115] for the proof of the next result. A ring R is called directly finite if $ab = 1$ in R implies $ba = 1$ for all $a, b \in R$. An idempotent e in a regular ring R is called an abelian idempotent if the ring eRe is abelian, and is called a directly finite idempotent if the ring eRe is directly finite. An idempotent e in a regular right self-injective ring is called a faithful idempotent if 0 is the only central idempotent orthogonal to e. A regular right self-injective ring is: of Type I_f if it contains a faithful abelian idempotent and is directly finite; of Type II_f if it contains a faithful directly finite idempotent but contains no non-zero abelian idempotents and is directly finite; and purely infinite if it contains no nonzero directly finite central idempotents.

As shown in Goodearl [19], a right self-injective regular ring decomposes into direct products of rings of certain basic types. We include those results here for convenience.

Theorem 2.1.26. *[19, Theorems 10.16, 10.22 and 10.24, Proposition 10.28] Let R be a right self-injective regular ring.*

1. *R is purely infinite if and only if $R^n \cong R$ for all positive integer n.*

2. *R is uniquely a direct product of purely infinite rings and rings of Types I_f and II_f.*

3. *R is Type I_f if and only if there exist right self-injective regular rings $R_1, R_2, \ldots$ such that $R \cong \prod R_n$ and each R_n is an $n \times n$ matrix ring over a strongly regular ring.*

4. if R contains no non-zero abelian idempotents, then there exist idempotents $e_1, e_2, \ldots \in R$ *such that* $\oplus_{i=1}^{n} e_i R \cong R_R$ *for all* n.

Now we come to the structure of right self-injective regular rings that are $CC3$-rings.

Theorem 2.1.27. *Let* R *be a right self-injective regular ring. Then* R *is a right* $CC3$*-ring if and only if* R *is a direct product of a semisimple artinian ring, a strongly regular ring and a* 2×2 *matrix ring over a strongly regular ring.*

Proof. ($\Leftarrow$) This is by Corollary 2.1.3, Example 2.1.10(2), Lemmas 2.1.25 and 2.1.14.

($\Rightarrow$) By Theorem 2.1.26(2), there exists a decomposition of rings $R = A \times B \times C$ where A is of type I_f , B is of type II_f, and C is purely infinite. Thus, $C_C \cong (C \oplus C \oplus C)_C$ by Theorem 2.1.26(1). So $C \cong \mathbb{M}_3(C)$, which is also a right $CC3$-ring by Lemma 2.1.14. Thus, C is semisimple artinian by Corollaries 2.1.21 and 2.1.3. So it must be that $C = 0$.

Since B is of type II_f , $B_B \cong (eB \oplus eB \oplus eB)_B$ where $e^2 = e \in B$ by Theorem 2.1.26(4). Thus $B \cong \mathbb{M}_3(eBe)$, which is a right $CC3$-ring by Lemma 2.1.14. So eBe is semisimple artinian, and hence B is semisimple artinian. So $B = 0$.

By Theorem 2.1.26(3), $A \cong \prod_{i=1}^{\infty} \mathbb{M}_i(S_i)$, where each S_i is strongly regular. Let $S = \prod_{i=3}^{\infty} \mathbb{M}_i(S_i)$. For each $i \geq 3$, let e_i be the matrix in $\mathbb{M}_i(S_i)$ whose $(1,1)-$, $(2,2)-$, and $(3,3)$-entries are 1 and all other entries are zero, and let $e = (e_i) \in S$. Then $e^2 = e$ and $SeS = S$. By Lemma 2.1.16, eSe is a right $CC3$-ring. However, $eSe \cong \prod_{i \geq 3} \mathbb{M}_3(S_i) \cong (\prod_{i \geq 3} S_i)$. So, by Lemma 2.1.12 and Corollary 2.1.3, we deduce that $\prod_{i \geq 3} S_i$ is semisimple artinian. Hence $S_i = 0$ for almost all $i \geq 3$. So S is semisimple artinian, and $A \cong S_1 \oplus \mathbb{M}_2(S_2) \oplus S$. $\square$

Recall that a module is continuous if it is both a $C1$-module and a $C2$-module. A ring R is called right continuous if R_R is a continuous module.

Corollary 2.1.28. *Let R be a right continuous regular ring. Then R is a right $CC3$-ring if and only if R is a direct product of a semisimple artinian ring, a strongly regular ring and a 2×2 matrix ring over a strongly regular ring.*

Proof. ($\Leftarrow$) This is by Corollary 2.1.3, Example 2.1.10(2), Lemmas 2.1.25 and 2.1.14.

($\Rightarrow$) Since R is right continuous, by Corollary 1.2.14 we have $R = R_1 \times R_2$, where R_1 is right self-injective regular and R_2 is reduced. So R_2 is strongly regular. By Lemma 2.1.14, R_1 is a right $CC3$-ring. So, by Theorem 2.1.27, R_1 is a direct product of a semisimple artinian ring, a strongly regular ring and a 2×2 matrix ring over a strongly regular ring. Hence the claim follows. □

2.2 2-Generated modules are $C3$-modules

We begin this section by giving an example to show that the rings whose 2-generated modules are $C3$-modules lie strictly between regular rings and semisimple artinian rings. By Lemma 2.1.12, R is a ring whose 2-generated R-modules are $C3$-modules if and only if the matrix ring $\mathbb{M}_2(R)$ is a $CC3$-ring. By Lemma 2.1.25, the 2×2 matrix ring over a strongly regular ring is a right $CC3$-ring. Thus, if R is a strongly regular ring, then every 2-generated R-module is a $C3$-module.

Moreover, $\mathbb{Z}$ is a $CC3$-ring, but, by Example 2.1.11, not every 2-generated $\mathbb{Z}$-module is a $C3$-module. Here is another example. Let R be any ring and M be any (R, R)-bimodule. We form $S := R \oplus M$, and define a multiplication on S by the rule $(r, a)(s, b) = (rs, rb + as)$. Then S is a ring with identity $(1, 0)$, having $R = R \oplus 0$ as a subring. The ring S constructed in this way is called the trivial extension of R by M, denoted as $S = R \propto M$.

Example 2.2.1. *Let $R = \mathbb{Z}_2 \propto \mathbb{Z}_2$ be the trivial extension of the ring $\mathbb{Z}_2$ by the $\mathbb{Z}_2$-module $\mathbb{Z}_2$. Then R is local and hence is a right $CC3$-ring. But, since $I := (0) \propto \mathbb{Z}_2$ is not a direct summand of R_R, the 2-generated right R-module $I \oplus R$ is not a $C3$-module by Lemma 2.1.2.*

Next, we present properties of rings whose 2-generated modules are $C3$-modules.

Proposition 2.2.2. *The following are equivalent for a ring R:*

1. *R is a regular ring.*
2. *Every finitely generated submodule of a projective right R-module is a direct summand.*
3. *Every finitely generated submodule of a projective right R-module is a $C3$-module.*
4. *Every 2-generated submodule of a projective right R-module is a $C3$-module.*

Proof. (1) $\Rightarrow$ (2). This is a well-known result of Kaplansky (see [29, Lemma 4]).

(2) $\Rightarrow$ (3) $\Rightarrow$ (4). These are clear.

(4) $\Rightarrow$ (1). If $a \in R$, then $aR \oplus R$ is a $C3$-module by (4). By Lemma 2.1.2, $aR \subseteq^{\oplus} R$. So R is a regular ring. $\square$

A module M is said to have *Goldie dimension* (or uniform dimension) n, denoted as $\mathrm{Gdim}(M) = n$, if the injective envelope $E(M)$ is a direct sum of n uniform submodules. Equivalently, if M has an essential submodule which is a direct sum of n uniform submodules.

Lemma 2.2.3. *Suppose that every 2-generated right R-module is a $C3$-module. Then:*

1. *Every finite uniform dimensional right R-module is injective and semisimple.*

2. *Every finitely generated submodule of a cyclic right R-module is a direct summand.*

Proof. (1). It suffices to show that every uniform module is injective. Let N be a uniform module. For $0 \neq x \in E(N)$ and $0 \neq B \subseteq xR$, take $0 \neq y \in B$. Then $xR \oplus yR$ is a $C3$-module. By Lemma 2.1.2, yR is a direct summand of xR. Thus, $yR = xR$ since xR is indecomposable, and so $B = xR$. This shows that xR is semisimple. It follows that $E(N)$ is semisimple. So $N = E(N)$ is injective.

(2). Let $N = aR$ be a cyclic module and $X = x_1R + \ldots + x_nR \subseteq N$. Since $x_1R \oplus aR$ is a $C3$-module, x_1R is a direct summand of aR by Lemma 2.1.2. Write $aR = x_1R \oplus N_1$ where N_1 is a cyclic submodule of N and write $x_2 = x_1r + n_1$ where $r \in R$ and $n_1 \in N_1$. As above, n_1R is a direct summand of N_1. Write $N_1 = n_1R \oplus N_2$. Then $aR = (x_1R \oplus n_1R) \oplus N_2 = (x_1R + x_2R) \oplus N_2$. Continue the process and see that X is a direct summand of N. □

Recall that a ring R is a right V-ring if every simple right R-module is injective. Also a ring R is called I-finite if R contains no infinite set of orthogonal idempotents, or equivalently, R_R has the ACC (or DCC) on direct summands.

Corollary 2.2.4. *Suppose that every 2-generated right R-module is a C3-module. The following hold:*

1. *R is a regular right V-ring.*

2. *R is I-finite if and only if R is a semisimple artinian ring.*

Proof. (1) follows from Lemma 2.2.3, and (2) follows from (1). □

We close this section by giving a characterization of continuous rings whose 2-generated modules are $C3$-modules.

Theorem 2.2.5. *Let R be a right continuous ring. Then every 2-generated right R-module is a $C3$-module if and only if R is a direct product of a semisimple artinian ring and a strongly regular ring.*

Proof. ($\Rightarrow$) By Corollary 2.2.4, R is regular. So, by Corollary 2.1.28, $R \cong R_1 \times R_2 \times \mathbb{M}_2(R_3)$, where R_1 is semisimple artinian and R_2, R_3 are strongly regular. By Lemma 2.1.12, $\mathbb{M}_2(R)$ is a right $CC3$-ring, so $\mathbb{M}_4(R_3) \cong \mathbb{M}_2\big(\mathbb{M}_2(R_3)\big)$ (as a factor ring of the ring $\mathbb{M}_2(R)$) is a right $CC3$-ring by Lemma 2.1.13. So R_3 is semisimple artinian by Corollary 2.1.3, and hence $\mathbb{M}_2(R_3)$ is semisimple artinian.

($\Leftarrow$) This is by Lemmas 2.1.12, 2.1.14 and 2.1.25. □

2.3 Applications

A ring R is called a right cc-ring if every cyclic right R-module is continuous [25], and a ring R is called a right πc-rings if every cyclic right R-module is quasi-continuous [18]. In this section, some main results in [18] and [25] are reproved as consequences of what we obtained in previous sections. In [18, Theorems 2.4 and 2.9], Goel and Jain proved that, for a right self-injective or a semiperfect ring R, R is a right πc-ring if and only if $R = R_1 \times R_2$, where R_1 is a semisimple artinian ring and R_2 is a finite direct product of right valuation rings.

Here a ring R is a right valuation ring if for every pair of elements x and y in the ring, either $x \in yR$ or $y \in xR$.

Lemma 2.3.1. *The following are equivalent for a local ring R:*

1. *R is a right* πc*-ring.*

2. *R is a right valuation ring.*

3. *Every cyclic right R-module is a C1-module.*

Proof. (1) $\Rightarrow$ (2). Let R be a local ring. Then R is semiperfect and indecomposable. So R must be either simple artinian ring or a right valuation ring by [18, Theorem 2.4]. Assume that R is a simple artinian ring. Then R is a division ring, so a right valuation ring.

(2) $\Rightarrow$ (3). Assume R is a right valuation local ring. Notice that R is a uniform ring. We show that every cyclic right R-module is uniform. For, let $M = R/I$ where I is a right ideal of R. Assume we have $I \subset A, B \subset R$. Then $A/I, B/I$ are non-trivial submodules of M. It follows that there exist elements $x \in A \setminus I, y \in B \setminus I$ such that either $x \in yR$ or $y \in xR$. Then we have either $0 \neq (xR + I)/I \subseteq A/I \cap B/I$ or $0 \neq (yR + I)/I \subseteq A/I \cap B/I$.

(3) $\Rightarrow$ (1). This is clear as we already showed in Examples 2.1.10(3) that every local ring is a $CC3$-ring. Hence, R is a πc-ring. □

The result of Goel and Jain [18, Theorem 2.4] for semiperfect rings can be extended to a semipotent ring. Here a ring R is semipotent if any right (or left) ideal not contained in $J(R)$ contains a nonzero idempotent.

Corollary 2.3.2. *Let R be a semipotent ring. Then R is a right πc-ring if and only if $R = R_1 \times R_2$, where R_1 is a semisimple artinian ring and R_2 is a finite direct product of right valuation rings.*

Proof. The sufficiency follows from Lemma 2.3.1 and the fact that a finite direct product of rings is a right πc-ring if and only if each direct summand is a right πc-ring (see [18, Lemma 2.2]). For the necessity, since every cyclic right R-module is a $C1$-module, R_R has an indecomposable decomposition (see [51]), and thus is finite uniform dimensional, so that R is I-finite. Hence R is semiperfect (being semipotent). Thus, the claim follows from Theorem 2.1.18 and Lemma 2.3.1. □

A right duo ring is a ring in which every right ideal is an ideal. A one-sided (or two-sided) ideal I of a ring R is said to be nil if I consists of nilpotent elements; I is said to be nilpotent if $I^n = 0$ for some natural number n.

Corollary 2.3.3. *[25] A ring R is a right cc-ring if and only if $R = R_1 \times R_2$, where R_1 is a semisimple artinian ring and R_2 is a finite direct product of right valuation, right duo ring with nil Jacobson radical.*

Proof. Note that a local ring is a right *cc*-ring if and only if it is a right valuation, right duo ring with nil Jacobson radical by [25, Proposition 2.2], and that a finite direct product of rings is a right *cc*-ring if and only if each direct summand is a right *cc*-ring. Moreover, every right *cc*-ring is I-finite and right continuous; so it is semiperfect. Thus, the claim follows from these and Theorem 2.1.18. □

A module M is called ADS (the absolute direct summand property) if for any decomposition $M = A \oplus B$ and for any complement B' of A in M we have $M = A \oplus B'$ [16]. Note that M is ADS if and only if, whenever $M = A \oplus B$, A and B are relatively injective (see [7, Proposition 1.1]). Clearly, every quasi-continuous module is ADS, and every ADS module is a $C3$-module. We call a ring R right fully ADS if every cyclic right R-module is ADS. Note that, in [1], a module is called completely ADS if each of its subfactors is ADS. Thus, if R_R is completely ADS then R is right fully ADS, but the converse is false.

Example 2.3.4. *Let $R = \mathbb{Z} \propto (\mathbb{Z}_2 \oplus \mathbb{Z}_4)$ be the trivial extension of $\mathbb{Z}$ by the $\mathbb{Z}$-module $\mathbb{Z}_2 \oplus \mathbb{Z}_4$. Since R is commutative, it is right fully ADS. Consider the subfactor $M := 0 \propto (\mathbb{Z}_2 \oplus \mathbb{Z}_4)$ of R_R. Then $M_R = X_R \oplus Y_R$, where $X = 0 \propto (\mathbb{Z}_2 \oplus 0)$ and $Y = 0 \propto (0 \oplus \mathbb{Z}_4)$. Clearly, X_R is not Y_R-injective, so M_R is not ADS.*

Corollary 2.3.5. *A semiperfect ring is right fully ADS if and only if it is a direct product of a semisimple artinian ring and finitely many local rings.*

Proof. ($\Rightarrow$). This is by Theorem 2.1.18.

($\Leftarrow$). Note that local rings are right fully ADS and that a finite direct product of right fully ADS rings is right fully ADS. □

Lemma 2.3.6. *Let R be a ring. The following are equivalent:*

1. *Every 2-generated right R-module is ADS.*
2. $\mathbb{M}_2(R)$ *is right fully ADS.*
3. *R is semisimple artinian.*

Proof. (1) $\Leftrightarrow$ (2). Let $P = (R^2)_R$ and $S = \text{End}(P_R)$. Then $\text{Hom}_R(P,) : N_R \to \text{Hom}_R({}_SP_R, N_R)$ defines a Morita equivalence between Mod-R and Mod-S with inverse equivalence $\otimes_S P : M_S \to M \otimes P$. It is clear that, for any 2-generated right R-module N, $\text{Hom}_R(P, N)$ is a cyclic right S-module, and, for any cyclic right S-module M, $M \otimes_S P$ is a 2-generated right R-module. Moreover, a Morita equivalence preserves the ADS-condition of modules. Thus, every cyclic right S-module is ADS if and only if every 2-generated right R-module is ADS.

(1) $\Rightarrow$ (3). Let $M = mR$ be an arbitrary cyclic right R-module. Then $N = mR \oplus R$ is a 2-generated right R-module and also an right ADS-module by hypothesis. So M is

R-injective, and hence M is injective by the Baer's criterion. Hence, R is semisimple artinian by Corollary 1.2.9(3).

(3) $\Rightarrow$ (1). It is obvious. □

Corollary 2.3.7. *A right continuous regular ring R is right fully ADS if and only if it is a direct product of a semisimple artinian ring and a strongly regular ring.*

Proof. ($\Rightarrow$). By Corollary 2.1.28, $R \cong R_1 \times R_2 \times \mathbb{M}_2(S)$, where R_1 is semisimple artinian, R_2 and S are strongly regular. As $\mathbb{M}_2(S)$ is right fully ADS, S is semisimple artinian by Lemma 2.3.6, so $\mathbb{M}_2(S)$ is semisimple artinian.

($\Leftarrow$). Note that strongly regular rings are right fully ADS and that a finite direct product of right fully ADS rings is right fully ADS. □

Examples 2.3.8. *1. Every right cc-ring is a right πc-ring; but the converse is false. The ring $\mathbb{Z}$ is clearly not a right cc-ring. To show that $\mathbb{Z}$ is a right πc-ring, let I be a right ideal of $\mathbb{Z}$. Then $I = n\mathbb{Z}$ for some $n \in \mathbb{Z}$. It is easy to see that $(\mathbb{Z}/I)_{\mathbb{Z}}$ is quasi-continuous if $n = 0$ or $n = 1$. Let $n \geq 2$, and write $n = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}$, where the s_i's are positive integers and the p_i's are distinct prime numbers. Then $R/I \cong \mathbb{Z}_{p_1^{s_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{s_k}}$. By [41, Corollary 2.14], a direct sum $\oplus_1^n M_i$ of modules is quasi-continuous if and only if each M_i is quasi-continuous and M_j-injective for all $j \neq i$. It follows that R/I is quasi-continuous $\mathbb{Z}$-module. So $\mathbb{Z}$ is a right πc-ring.*

2. *Every right πc-ring is right fully ADS; but the converse is false. Let $R = \mathbb{Z}_2 \propto (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ be the trivial extension of the ring $\mathbb{Z}_2$ by the $\mathbb{Z}_2$-module $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. It is easily seen that every commutative ring is fully ADS. So R is right fully ADS. But, since R is not a right valuation ring, R is not a right πc-ring by Lemma 2.3.1.*

3. *Every right fully ADS ring is a right CC3-ring; but the converse is false. In fact, if R is a strongly regular ring that is not semisimple artinian, then $\mathbb{M}_2(R)$ is a right CC3-ring that is not fully ADS (by Lemma 2.3.6).*

Remark 2.3.9. *The material in this chapter is taken from [24].*

Chapter 3

Rings whose cyclics are $D3$-modules

As a dual notion of $C3$-modules, a module M is called a $D3$-module if for any two direct summands M_1 and M_2 of M with $M = M_1 + M_2$, $M_1 \cap M_2$ is a direct summand of M. This chapter is concerned with right $CD3$-rings, i.e., rings whose cyclic right modules are $D3$-modules. The motivation is the observation that a ring R is semisimple artinian if and only if every 2-generated right R-module is a $D3$-module. In the first section, various basic properties of right $CD3$-rings are presented, and it is proved that a right self-injective regular ring is a right $CD3$-ring if and only if it is a direct product of a semisimple artinian ring and a strongly regular ring. Semiperfect rings need not be $CD3$-rings in general. We gives a sufficient condition for a semiperfect ring to be a right $CD3$-ring. The last section discusses two related classes of rings. We completely characterize the rings whose cyclic modules are quasi-discrete and, respectively, discrete. A number of illustrated examples are also given.

3.1 Cyclics are D3-modules

We start with an important property of $D3$-modules which was proved in [2].

Proposition 3.1.1. *[2, Proposition 4] Let M be a D3-module. If $M = M_1 \oplus M_2$ and if $f : M_1 \to M_2$ is a homomorphism with $im(f) \subseteq^{\oplus} M_2$, then $ker(f) \subseteq^{\oplus} M_1$.*

Proof. We first show that if $f : M_1 \to M_2$ is an epimorphism, then $ker(f) \subseteq^{\oplus} M_1$. Let $T = \{a + f(a) \ : \ a \in M_1\}$ be the graph submodule of M. As shown in Proposition 2.1.1, we have $M = T \oplus M_2$.

Next, we prove that $M = M_1 + T$. If $x \in M = M_1 \oplus M_2$, then $x = a + b$ for some $a \in M_1$ and $b \in M_2$. Since f is an epimorphism, there exists $a_1 \in M_1$ such that $b = f(a_1)$. Therefore, $x = a + f(a_1) = (a - a_1) + (a_1 + f(a_1)) \in M_1 + T$. Since M is a $D3$-module, $T \cap M_1 \subseteq^{\oplus} M$. Finally, we only need to show that $T \cap M_1 = ker(f)$. If $x \in T \cap M_1$, then $x = a = a_1 + f(a_1)$ where $a, a_1 \in M_1$. Therefore, $a - a_1 = f(a_1) \in M_1 \cap M_2 = 0$, and so $f(x) = f(a_1) = 0$ and $T \cap M_1 \subseteq ker(f)$. If $x \in ker(f)$, then $x = x + f(x) \in T \cap M_1$, and so $ker(f) = T \cap M_1 \subseteq^{\oplus} M$, as required. Now, let $f : M_1 \to M_2$ be an arbitrary homomorphism with $im(f) \subseteq^{\oplus} M_2$. If $M_2 = im(f) \oplus B$ for a submodule $B \subseteq M_2$, then $M_1 \oplus im(f) \subseteq^{\oplus} M$. Since a direct summand of a $D3$-module is again a $D3$-module, then by applying the preceding argument to the module $M_1 \oplus im(f)$, we infer that $ker(f) \subseteq^{\oplus} M_1$, as required. □

As a consequence of Proposition 3.1.1, the following lemma will be repeatedly used.

Lemma 3.1.2. *Suppose that $M_1 \oplus M_2$ is a D3-module. If M_1/X is isomorphic to a direct summand of M_2 where $X \subseteq M_1$, then $X \subseteq^{\oplus} M_1$.*

Proof. Suppose that $\alpha : M_1/X \to Y$ is an isomorphism where $M_2 = Y \oplus Z$. Then $f = \alpha \circ \pi : M_1 \to M_2$ where $\pi : M_1 \to M_1/X$ is the natural projection. It follows that $X \subseteq^{\oplus} M_1$ by Proposition 3.1.1 as $im(f) = Y \subseteq^{\oplus} M_2$. □

Semisimple artinian rings can be characterized in terms of $D3$-modules.

Corollary 3.1.3. *The following are equivalent for a ring R:*

1. *R is semisimple artinian.*
2. *Every 2-generated right R-module is a $D3$-module.*

Proof. (1) $\Rightarrow$ (2). It is clear. (2) $\Rightarrow$ (1). We show that the right R-module R_R is semisimple. Indeed, let I be a right ideal of R and consider the 2-generated right R-module $M = R \oplus R/I$. Then M is a $D3$-module by hypothesis. Thus, by Lemma 3.1.2, $I \subseteq^{\oplus} R$. So, R is semisimple artinian. □

The motivated question is which rings R have the property that every cyclic right R-module is a $D3$-module. In this chapter, we study rings whose cyclics are $D3$-modules, and so we give the following definition.

Definition 3.1.4. *A ring R is called a right $CD3$-ring if every cyclic right R-module is a $D3$-module. A ring R is called a $CD3$-ring if it is both a right and a left $CD3$-ring.*

The next lemma is an equivalent condition for the matrix ring to be a right $CD3$-ring.

Lemma 3.1.5. *Let $n \geq 2$. The following are equivalent for a ring R:*

1. *Every n-generated right R-module is a $D3$-module.*
2. *Every cyclic right $\mathbb{M}_n(R)$-module is a $D3$-module.*

Proof. Let $P = (R^n)_R$ and $S = \text{End}(P_R)$. Then $\text{Hom}_R(P, -) : N_R \mapsto \text{Hom}_R({}_SP_R, N_R)$ defines a Morita equivalence between Mod-R and Mod-S with inverse equivalence

$- \otimes_S P : M_S \mapsto M \otimes P$. For any n-generated right R-module N, $\mathrm{Hom}_R(P, N)$ is a cyclic right S-module, and, for any cyclic right S-module M, $M \otimes_S P$ is an n-generated right R-module. Moreover, a Morita equivalence preserves the $D3$-condition of modules. Thus, every cyclic right S-module is a $D3$-module if and only if every n-generated right R-module is a $D3$-module. □

The following corollary is a quick consequence of Corollary 3.1.3 and Lemma 3.1.5

Corollary 3.1.6. *Let $n \geq 2$. The following are equivalent for a ring R:*

1. *R is a semisimple artinian ring.*
2. *Every cyclic right module over $\mathbb{M}_n(R)$ is a $D3$-module.*

Let $n > 1$. Koehler [33] proved that every cyclic right module over $\mathbb{M}_n(R)$ is quasi-projective if and only if R is semisimple artinian, and Xue [61] proved that every cyclic right module over $\mathbb{M}_n(R)$ is a $D2$-module if and only if R is semisimple artinian. These are immediate consequences of Corollary 3.1.6.

Before giving further examples of right $CD3$-rings, we recall a class of modules which can be considered as examples of $D3$-modules. A module M is said to satisfy the summand intersection property if the intersection of any two direct summands of M is again a direct summand (see [17]). Again, recall that an abelian ring is a ring for which every idempotent is central. The next proposition gives some important examples of right $CD3$-rings.

Proposition 3.1.7. *Let R be an abelian exchange ring. Then every cyclic right R-module satisfies the summand intersection property. So R is a right $CD3$-ring.*

Proof. Let I be a right ideal of R. Suppose K/I is a direct summand of $(R/I)_R$. Then $(R/I)_R = (K/I) \oplus (K'/I)$. Thus $R = K + K'$. Write $1 = x + x'$ where

$x \in K$ and $x' \in K'$. Then $x - x^2 = x'x \in K \cap K' = I$. Since R is an exchange ring, idempotents can be lifted modulo I. Hence there exists $e^2 = e \in R$ such that $x - e \in I$, and so $(1-x) - (1-e) \in I$. It follows that $K = eR + I$. If L/I is another direct summand of $(R/I)_R$, then as above $L = fR + I$ where $f^2 = f$ and $fI \subseteq I$. Assume $ea + r = fb + s \in K \cap L$ where $a, b \in R$ and $r, s \in I$. Then $ea+r = e(ea)+r = e(fb+s-r)+r = efb+(es-er+r) \in efR+I$. This shows that $K/I \cap L/I = (K \cap L)/I = (efR + I)/I$. As ef is a central idempotent, $(efR+I)/I$ is a direct summand of $(R/I)_R$ with $((1-ef)R+I)/I$ a complement. □

Examples for $CD3$-rings can be found in some important rings. For instance, we have

Examples 3.1.8. *1. Every commutative ring R is a $CD3$-ring.*

2. Every strongly regular ring is a $CD3$-ring.

3. Every local ring is a $CD3$-ring.

Proof. Following the arguments as shown in Examples 2.1.10 and Proposition 3.1.7, we have (2) and (3). For commutative rings, we have a stronger result, that is, every cyclic module over a commutative ring R is quasi-projective. In particular, every commutative ring is a $CD3$-ring. Indeed, let $M =: xR$ be a cyclic right R-module, L a right R-module and $f : xR \to L$ an R-homomorphism. Consider the following diagram

$$\begin{array}{ccc} & xR & \\ & \downarrow f & \\ xR & \xrightarrow{g} \quad L & \to 0 \end{array}$$

with g an R-epimorphism. Set $f(x) = y \in L$. Since g is an epimorphism, $y = g(xa)$ for some $a \in R$. Now define $\lambda : xR \longrightarrow xR$ by $\lambda(xr) = x(ar)$. Since R is commutative, λ is a well-defined homomorphism and $g\lambda(xr) = g(xar) = g(xa)r = yr = f(x)r = f(xr)$ for all $xr \in xR$. Therefore M is quasi-projective, as required. □

By applying a similar argument as we showed in the proofs of Lemmas 2.1.13, 2.1.14 and 2.1.16 for $CC3$-rings, the following three lemmas can be proved.

Lemma 3.1.9. *Any factor ring of a right $CD3$-ring is again a right $CD3$-ring.*

Lemma 3.1.10. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a finite direct product of rings. Then R is a right $CD3$-ring if and only if every R_i is a right $CD3$-ring.*

Example 3.1.11. *An infinite direct product of right $CD3$-rings need not be a right $CD3$-ring: Let D_i be a division ring for $i \geq 1$. Then $\mathbb{M}_2(D_i)$ is a right $CD3$-ring, but $\prod_{i=1}^{\infty} \mathbb{M}_2(D_i) \cong \mathbb{M}_2\big(\prod_{i=1}^{\infty}(D_i)\big)$ is not a right $CD3$-ring as $\prod_{i=1}^{\infty}(D_i)$ is not semisimple artinian. This example also shows that a unit regular ring need not be a right $CD3$-ring, a contrast to Examples 3.1.8(2).*

Lemma 3.1.12. *Let e be an idempotent of R with $ReR = R$. If R is a right $CD3$-ring, then so is eRe.*

It is known that the ring $R = \mathrm{End}_D(V)$ of linear transformations of a vector space V over a division ring D is a right self-injective regular ring. Moreover, it is not difficult to show that R is a right $CD3$-ring if and only if $dim(V) < \infty$. Thus, a motivated question is: Which right self-injective regular rings are right $CD3$-rings?

Next we determine the structure of right self-injective regular $CD3$-rings. Note that we already have a characterization of right self-injective regular $CC3$-rings as a direct product of a semisimple artinian ring, a strongly regular ring and a 2×2 matrix ring over a strongly regular ring (see Theorem 2.1.27). Following Corollary 3.1.6, if R is a strongly regular which is not semisimple artinian, then $\mathbb{M}_n(R)$ is not a $CD3$-ring for $n \geq 2$.

Theorem 3.1.13. *Let R be a right self-injective, regular ring. Then R is a right $CD3$-ring if and only if R is a direct product of a semisimple artinian ring and a strongly regular ring.*

Proof. ($\Leftarrow$). This is by Lemma 3.1.10, Corollary 3.1.3 and Examples 3.1.8(3).

($\Rightarrow$). By Theorem 2.1.26(2), there exists a decomposition of rings $R = A \times B \times C$ where A is of type I_f, B is of type II_f, and C is purely infinite. Thus, $C_C \cong (C \oplus C)_C$ (see Theorem 2.1.26(1)). So $C \cong \mathbb{M}_2(C)$, which is also a right $CD3$-ring by Lemma 3.1.9. Thus, C is semisimple artinian by Corollary 3.1.3. So it must be that $C = 0$.

Since B is of type II_f, $B_B \cong (eB \oplus eB)_B$ where $e^2 = e \in B$ (see Theorem 2.1.26(4)). Thus $B \cong \mathbb{M}_2(eBe)$, which is a right $CD3$-ring by Lemma 3.1.9. So eBe is semisimple artinian, and hence B is semisimple artinian. So $B = 0$.

By Theorem 2.1.26(3), $A \cong \prod_{i=1}^{\infty} \mathbb{M}_i(S_i)$, where each S_i is strongly regular. Let $S = \prod_{i=2}^{\infty} \mathbb{M}_i(S_i)$. For each $i \geq 2$, let e_i be the matrix in $\mathbb{M}_i(S_i)$ whose $(1,1)$- and $(2,2)$-entries are 1 and all other entries are zero, and let $e = (e_i) \in S$. Then $e^2 = e$ and $SeS = S$. By Lemma 3.1.9, S is a right $CD3$-ring. So, by Lemma 3.1.12, $eSe \cong \prod_{i\geq 2} \mathbb{M}_2(S_i) \cong \mathbb{M}_2\big(\prod_{i\geq 2} S_i\big)$ is a right $CD3$-ring. By Corollary 3.1.3, we deduce that $\prod_{i\geq 2} S_i$ is semisimple artinian. So $S_i = 0$ for almost all $i \geq 2$. Hence S is semisimple artinian, and $R \cong S_1 \times S$. □

Corollary 3.1.14. *Let R be a right continuous regular ring. Then R is a right $CD3$-ring if and only if R is a direct product of a semisimple artinian ring and a strongly regular ring.*

Proof. ($\Leftarrow$). This is by Corollary 3.1.3, Examples 3.1.8(3) and Lemma 3.1.10.

($\Rightarrow$). Since R is right continuous, by Corollary 1.2.14 we have $R = R_1 \times R_2$, where R_1 is right self-injective regular and R_2 is reduced. So R_2 is strongly regular. By Lemma

3.1.10, R_1 is a right $CD3$-ring. So, by Theorem 3.1.13, R_1 is a direct product of a semisimple artinian ring and a strongly regular ring. Hence the claim follows. □

Semiperfect rings need not to be $CD3$-rings in general. We next give a sufficient condition for a semiperfect ring to be a right $CD3$-ring.

Definition 3.1.15. *Let K be a right ideal of a ring R. We say that idempotents are relatively invariant with respect to K if, whenever $R = eR + fR$ where e, f are idempotents with $eK \subseteq K$ and $fK \subseteq K$, we have $(ef)^n \in fR$ or $(fe)^n \in eR$ for some $n \geq 1$.*

Lemma 3.1.16. *Let I be a right ideal of a ring R and $e^2 = e \in R$ such that idempotents lift modulo eI.*

1. *Let A be a right ideal of R. Then A/eI is a direct summand of $(eR/eI)_R$ if and only if $A = xeR + eI$ where $x^2 = x \in eR$ with $xeI \subseteq eI$.*
2. *If eI is small in eR and idempotents are relatively invariant with respect to eI, then $(eR/eI)_R$ is a $D3$-module.*

Proof. (1) If $A = xeR + eI$ where $x^2 = x \in eR$ with $xeI \subseteq eI$, then one can easily see that $(eR/eI)_R = (A/eI) \oplus \big((e - xe)R/eI\big)$.

Assume that A/eI is a direct summand of eR/eI, and write $(eR/eI)_R = (A/eI) \oplus (B/eI)$. Then $eR = A + B$ and $A \cap B = eI$. Write $e = a + b$ with $a \in A$ and $b \in B$. As $a = ea = a^2 + ba$, $a - a^2 = ba \in A \cap B = eI$. As idempotents lift modulo eI, there exists $x^2 = x \in R$ such that $a - x \in eI$. Then $xe \in A$ and $e - xe = (e - a)e + (a - x)e \in B$. As $eR/eI = (xeR + eI)/eI + ((e - xe)R + eI)/eI$, it follows from $(eR/eI)_R = (A/eI) \oplus (B/eI)$ that $A/eI = (xeR + eI)/eI$ and $B/eI =$

$((e-xe)R+eI)/eI$. Hence $A = xeR+eI$. For $c \in I$, $xec = -(e-xe)c+ec \in A \cap B$, so $xec \in eI$. That is, $xeI \subseteq eI$.

(2) To show that $(eR/eI)_R$ is a $D3$-module, assume that $K/eI, L/eI$ are direct summands of $(eR/eI)_R$ with $(eR/eI) = (K/eI) + (L/eI)$. We need to show that $(K/eI) \cap (L/eI)$ is a direct summand of $(eR/eI)_R$. By (1), there exist idempotents x and y of R such that $K = xeR + eI$, $L = yeR + eI$ with $xeI \subseteq eI$ and $yeI \subseteq eI$. Then $eR = K + L = xeR + yeR + eI = xeR + yeR$ as eI is small in eR. So $R = xeR + yeR + (1-e)R = xeR + (ye+1-e)R$, where xe and $ye+1-e$ are idempotents with $(xe)eI \subseteq eI$ and $(ye+1-e)eI \subseteq eI$. By the hypothesis, there exists $n \geq 1$ such that $\big(xe(ye+1-e)\big)^n \subseteq (ye+1-e)R$ or $\big((ye+1-e)xe\big)^n \subseteq xeR$. That is, $(xy)^n e \subseteq yeR$ or $(yx)^n e \subseteq xeR$. Without loss of generality, we may assume that $(xy)^n e \subseteq yeR$.

We have that $K \cap L = (xeR+eI) \cap (yeR+eI) = (xyeR+eI) \cap (yeR+eI) = (yxyeR+eI) \cap (xeR+eI) = (xyxyeR+eI) \cap (yeR+eI) = \cdots = ((xy)^n eR+eI) \cap (yeR+eI) = (xy)^n eR+eI$. Write $(xy)^n e = yet$ for some $t \in R$. Then $(xy)^n e \cdot (xy)^n e = (xy)^n \cdot yet = (xy)^n et = (xy)^{n-1} \cdot xyet = (xy)^{n-1} \cdot yet = \cdots = xy \cdot yet = xyet = yet = (xy)^n e$, and $(xy)^n e \cdot eI = (xy)^n eI \subseteq eI$. So, by (1), $((xy)^n e \cdot eR + eI)/eI$ is a direct summand of $(eR/eI)_R$. □

Theorem 3.1.17. *Let R be a semiperfect ring. If idempotents are relatively invariant with respect to every small right ideal of R, then R is a right $CD3$-ring.*

Proof. As R is semiperfect, every cyclic module over R is isomorphic to eR/eI for some idempotent $e \in R$ and some right ideal I in $J(R)$ (see Lemma 1.2.20). It follows from Lemma 3.1.16 that $(eR/eI)_R$ is a $D3$-module. □

We next give an application of Theorem 3.1.17. A Morita context is a 4-tuple

$\begin{pmatrix} A & M \\ N & B \end{pmatrix}$, where A, B are rings, ${}_AM_B$ and ${}_BN_A$ are bimodules, and there exist context products $M \times N \to A$ and $N \times M \to B$ written multiplicatively as $(w, z) \mapsto wz$ and $(z, w) \mapsto zw$, such that $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is an associative ring with the obvious matrix operations. A Morita context $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is called trivial if the context products are trivial, i.e., $MN = 0$ and $NM = 0$ (see [39, p.1993]). For bimodules ${}_AM_B$ and ${}_BN_A$, the 4-tuple $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is a ring with addition defined componentwise and with multiplication given by

$$\begin{pmatrix} a_1 & x_1 \\ y_1 & b_1 \end{pmatrix} \begin{pmatrix} a_2 & x_2 \\ y_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & a_1x_2 + x_1b_2 \\ y_1a_2 + b_1y_2 & b_1b_2 \end{pmatrix}.$$

This ring is certainly a trivial Morita context. Moreover, every trivial Morita context is obtained this way. Formal triangular matrix rings are obvious examples of trivial Morita contexts.

Proposition 3.1.18. *Let* $R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ *be a trivial Morita context where* A, B *are local rings and* ${}_AM_B$, ${}_BN_A$ *are bimodules. Then* R *is a* $CD3$*-ring.*

Proof. The ring R is semiperfect (see [57, Theorem 2.7]). The non-trivial idempotents of R are $\begin{pmatrix} 1 & x \\ y & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & x \\ y & 1 \end{pmatrix}$, where $x \in M$ and $y \in N$, and $\begin{pmatrix} 1 & x \\ y & 0 \end{pmatrix} R = \left\{ \begin{pmatrix} a & ya \\ z & 0 \end{pmatrix} : a \in A, z \in M \right\}$ and $\begin{pmatrix} 0 & x \\ y & 1 \end{pmatrix} R = \left\{ \begin{pmatrix} 0 & xb \\ w & b \end{pmatrix} : b \in B, w \in N \right\}$.

If $R = eR + fR$ where e, f are non-trivial idempotents of R, then without loss of generality we may assume that $e = \begin{pmatrix} 1 & x \\ y & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & z \\ w & 1 \end{pmatrix}$ for some $x, z \in M$ and $y, w \in N$, so $(ef)^2 = 0 \in fR$. Therefore, idempotents are relatively invariant with respect to every (small) right ideal of R. So R is a right $CD3$-ring by Theorem 3.1.17. Similarly, R is a left $CD3$-ring. □

Corollary 3.1.19. *Let* $R = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ *be a formal triangular matrix ring where* A, B *are local rings and* ${}_AM_B$ *is bimodule. Then* R *is a* $CD3$*-ring.*

We have a structure of semiperfect rings in terms of $CD3$-rings as follows:

Proposition 3.1.20. *Let* R *be a semiperfect ring with* $R/J(R)$ *simple. Then* R *is a right* $CD3$*-ring if and only if* R *is simple artinian or local.*

Proof. The sufficiency is clear. For the necessity, as R is a semiperfect ring with $R/J(R)$ simple, $R \cong \mathbb{M}_n(S)$ for some local ring S (see [36, 23.10]). If $n = 1$, then R is local. If $n \geq 2$, then $J(S) = 0$ by Corollary 3.1.6 as $\mathbb{M}_n(S)$ is a right $CD3$-ring. Hence R is simple artinian. □

A family of semiperfect rings that are not right $CD3$-rings can be given as follows.

Example 3.1.21. *Let* $R = \mathbb{M}_n(D)$ *where* D *is a division ring and* $n \geq 2$*. Then* $S = R[x]/(x^k)$ *is not a right* $CD3$*-ring for all* $k \geq 2$*.*

Proof. It is clear that S is a semiperfect ring. Moreover, $J(S) \neq 0$ and $S/J(S)$ is not a division ring. So S is neither simple artinian nor local. Thus, S is not a $CD3$-ring by Proposition 3.1.20. □

3.2 Cyclics are (quasi-)discrete modules

In this section, we provide a complete characterization of rings whose cyclics are quasi-discrete. It is well-known (and easy to prove) that a ring R is semisimple artinian if and only if every cyclic right R-module is projective. Koehler [33] defined a ring R to be a right q^*-ring if every cyclic right R-module is quasi-projective.

Theorem 3.2.1. *[33, Theorem 2.1] A semiperfect ring R is a right q^*-ring if and only if every right ideal contained in $J(R)$ is an ideal.*

Corollary 3.2.2. *Let R be a right continuous regular ring. The following are equivalent:*

1. *R is a right $CD3$-ring.*
2. *R is a right q^*-ring.*
3. *R is a direct product of a semisimple artinian ring and a strongly regular ring.*

Proof. (2) $\Rightarrow$ (1). It is clear.

(1) $\Leftrightarrow$ (3). This is by Corollary 3.1.14.

(3) $\Rightarrow$ (2). A semisimple artinian ring is certainly a right q^*-ring. We only need to show that every strongly regular ring R is a right q^*-ring. By assumption, each principal right ideal of R is generated be a central idempotent and so is a two-sided ideal. Hence all right ideals, which are a sum of principal ideals, are two-sided ideals. Let $M = R/I$ be a cyclic right R-module and $\pi : R/I \to R/K$ be an epimorphism, where $I \subseteq K \subseteq R$ are two-sided ideals of R. Suppose $f : R/I \to R/K$ is a homomorphism with $f(1+I) = a+K$ for some $a \in R$. Since π is an epimorphism, there exists $b \in R$ such that $\pi(b+I) = a+K$. Define a map $g : R/I \to R/I$ such that $g(1+I) = b+I$. We have g is a homomorphism and $\pi \circ g(1+I) = a+K$. So R is a right q^*-ring. $\square$

Recall that a module is discrete if it is both a $D1$-module and a $D2$-module, and quasi-discrete if it is both a $D1$-module and a $D3$-module. The next characterization of quasi-discrete modules is needed.

Lemma 3.2.3. *[41, Proposition 4.45] Let N be quasi-discrete, and let $f : N \to M$ be an epimorphism with a small kernel. Then M is quasi-discrete if and only if $ker f$ is invariant under every idempotent of* End(N).

Proof. Assume that M is quasi-discrete and let e be an idempotent of End(N). Write $N = A_1 \oplus A_2$ where $A_1 = e(N)$ and $A_2 = (1-e)(N)$. Then $M = f(A_1)+f(A_2)$, and by quasi-discreteness, $M = B_1 \oplus B_2$ with $B_i \subseteq f(A_i)$ for $i = 1, 2$. Hence, $N = f^{-1}(B_1)+ f^{-1}(B_2)$. Now $f^{-1}(B_i) = A_i \cap f^{-1}(B_i) + ker(f)$ for $i = 1, 2$. Since $ker(f) \ll N$, $N = A_1 \oplus A_2 = A_1 \cap f^{-1}(B_1) \oplus A_2 \cap f^{-1}(B_2)$. Thus, $A_i = A_i \cap f^{-1}(B_i)$, and so $A_i \subseteq f^{-1}(B_i)$, and consequently $f(A_i) = B_i$ for $i = 1, 2$. Hence, $M = f(A_1) \oplus f(A_2)$, which implies that $e(ker(f)) \subseteq ker(f)$. Indeed, assume $x = ea_1 + (1 - e)a_2 \in ker(f)$ for some $a_1 \in A_1, a_2 \in A_2$. It follows that $f(x) = f(ea_1) = 0$. So $ex = ea_1 \in ker(f)$, as required.

Conversely, assume that $ker(f)$ is invariant under every idempotent of End(N). It is straightforward to check that if $N = A_1 \oplus A_2$ then $M = f(A_1) \oplus f(A_2)$ for any direct decomposition of $N = A_1 \oplus A_2$. Let A be an arbitrary submodule of M. Since N satisfies the $D1$-condition, $N = N_1 \oplus N_2$ with $N_1 \subseteq f^{-1}(A)$ and $N_2 \cap f^{-1}(A) \ll N$. Then $M = f(N_1) \oplus f(N_2)$ with $f(N_1) \subseteq A$ and $f(N_2) \cap A = f(N_2 \cap f^{-1}(A)) \ll M$ as f is an epimorphism. So M has the $D1$-condition.

Now let M_1, M_2 be direct summands of M such that $M = M_1 + M_2$. Then $N = f^{-1}(M_1) + f^{-1}(M_2)$, and by quasi-discreteness, $N = N_1 \oplus N_2$ with $N_i \subseteq f^{-1}(M_i)$ for $i = 1, 2$. Hence, $M = f(N_1) \oplus f(N_2)$ with $f(N_i) \subseteq M_i$ for $i = 1, 2$. By modular law, $M_1 = f(N_1) \oplus (f(N_2) \cap M_1)$ and $M_2 = f(N_2) \oplus (f(N_1) \cap M_2)$. So $M_1 \cap M_2 =$

$(f(N_2) \cap M_1) \oplus (f(N_1) \cap M_2)$ is a direct summand of M. Hence, M has the $D3$-condition. □

We now characterize the rings whose cyclics are quasi-discrete.

Theorem 3.2.4. *The following are equivalent for a ring R:*

1. *Every cyclic right R-module is quasi-discrete.*

2. *R is a semiperfect ring such that $ea \in aR$ for all $e^2 = e \in R$ and $a \in J(R)$.*

Proof. (1) $\Rightarrow$ (2). By (1), R_R is quasi-discrete. As R_R is a $D2$-module, R_R is discrete, so R is semiperfect by Corollary 1.2.19. Let $I = aR$ with $a \in J(R)$ and let $e^2 = e \in R$. Then R_R with the natural homomorphism to $(R/I)_R$ is a projective cover of $(R/I)_R$. As both R_R and $(R/I)_R$ are quasi-discrete, I_R is invariant under every idempotent of $\mathrm{End}(R_R)$ by Lemma 3.2.3. It follows that $eI \subseteq I$.

(2) $\Rightarrow$ (1). Let M_R be a cyclic module. As R is semiperfect, $M \cong eR/eI$ where $e^2 = e \in R$ and $I \subseteq J(R)$ is a right ideal of R by Lemma 1.2.20. Thus, eR with the natural homomorphism to eR/eI is a projective cover of $(eR/eI)_R$. If $g^2 = g \in \mathrm{End}(eR_R)$, then $g(e)$ is an idempotent of R, so $g(eI) = g(e)(I) \subseteq I$ by the hypothesis. Thus eI is invariant under the idempotents of $\mathrm{End}(eR_R)$. As eR is quasi-discrete, by Lemma 3.2.3, we deduce that $(eR/eI)_R$ is quasi-discrete, so M_R is quasi-discrete. □

Corollary 3.2.5. *Let R be an abelian ring. Then every cyclic right R-module is quasi-discrete if and only if R is semiperfect ring.*

To characterize rings whose cyclics are discrete, let us call a ring R a right $CD2$-ring if every cyclic right R-module is a $D2$-module.

Lemma 3.2.6. *The following statements hold for a ring R:*

1. *If R is a right $CD2$-ring, then, for any $u \in U(R)$ and any right ideal I in $J(R)$, uI is not properly contained in I.*

2. *If R is a semiperfect, right $CD2$-ring, then, for any $u \in U(R)$ and any right ideal I of R, uI is not properly contained in I.*

Proof. (1) Assume that $uI \subsetneqq I$ where $u \in U(R)$ and I is a right ideal in $J(R)$. Then $(R/uI)/(I/uI) \cong R/I \cong uR/uI = R/uI$. As $(R/I)_R$ is a $D2$-module, $(R/uI)_R$ is a $D2$-module, so it follows that $(I/uI)_R$ is a direct summand of $(R/uI)_R$. But, as I_R is small in R_R, $(I/uI)_R$ is small in $(R/uI)_R$, so we deduce that $I/uI = 0$, a contradiction.

(2) Assume that $uI \subsetneqq I$ where $u \in U(R)$ and I is a right ideal of R. Then $(R/uI)/(I/uI) \cong R/I \cong uR/uI = R/uI$. As $(R/I)_R$ is a $D2$-module, $(R/uI)_R$ is a $D2$-module, so it follows that $(I/uI)_R$ is a direct summand of $(R/uI)_R$. As claimed in the proof of Theorem 3.2.4, $I = (fR + uI)/uI$ where $f^2 = f \in I$ with $fuI \subseteq uI$. As $u^{n+1}I \subsetneqq u^nI$ for any $n \in \mathbb{Z}$, $R/J(R)$ being semisimple artinian implies that $u^{n+1}I + J(R) = u^nI + J(R)$ for some $n \in \mathbb{Z}$. It follows that $uI + J(R) = I + J(R)$. So, $fR \subseteq uI + J(R)$, and hence $fR = fuI + fJ(R)$. As $fJ(R)$ is small in fR, we deduce that $fR = fuI$. As $fuI \subseteq uI$, we have $I = fR + uI = uI$, a contradiction. □

Theorem 3.2.7. *The following are equivalent for a ring R:*

1. *Every cyclic right R-module is discrete.*

2. *R is a semiperfect ring such that, for any $e^2 = e \in R$ and $a \in J(R)$, $ea \in aR$ and that, for any $u \in U(R)$ and any right ideal I of R, uI is not properly contained in I.*

Proof. (1) ⇒ (2). By Theorem 3.2.4 and Lemma 3.2.6.

(1) $\Leftarrow$ (2). Let M_R be a cyclic module. We show that M is discrete. Note that M is quasi-discrete by Theorem 3.2.4. It is proved in [41, Theorem 4.15] that every quasi-discrete is a direct sum of hollow modules. So $M = \oplus_{i\in I} H_i$ where each H_i is a hollow module. It is proved in [41, Theorem 5.2] that if $\oplus_{i\in I} M_i$ is a quasi-discrete module where each M_i is a hollow module, then $\oplus_{i\in I} M_i$ is discrete if and only if each M_i is discrete. Hence, to show that M is discrete, it suffices to show that each H_i is discrete. Thus we can assume that M is hollow. Now we only need to show that M is a $D2$-module.

As R is semiperfect, $M \cong eR/eI$ where $e^2 = e \in R$ and $I \subseteq J(R)$ is a right ideal of R by Lemma 1.2.20. To show that eR/eI is a $D2$-module, suppose that $(eR/eI)/(eK/eI) \cong B \subseteq^{\oplus} (eR/eI)_R$, where $K \supseteq I$ is a right ideal of R. We need to show that $(eK/eI)_R$ is a direct summand of $(eR/eI)_R$. We can assume that $B \neq 0$, so $B = eR/eI$ is hollow as a right R-module. We have $(eR/eK)_R \cong (eR/eI)_R$, and let us assume that the isomorphism is θ.

Let $x = e + eK \in eR/eK$ and $y = e + eI \in eR/eI$. Then $eR/eK = xR$ and $eR/eI = yR = \theta(x)R$. Write $\theta(x) = yu$ and $y = \theta(x)v$ with $u, v \in R$. Then $\theta(x) = yu = yeue$ and $y = \theta(x)v = \theta(x)eve$, so we can assume that $u = eue$ and $v = eve$. We have $y = yuv$, i.e. $y(e - uv) = 0$ or $e - uv \in eI$, showing that $e - uv \in eJ(R)e$. So $uv \in U(eRe)$, and hence $u, v \in U(eRe)$ (as eRe is semiperfect).

From $eK \subseteq ann_r(x) = \theta(x)^{\perp} = (yu)^{\perp}$, we have $yueK = 0$ or $ueK \subseteq eI$. From $eI \subseteq y^{\perp} = (\theta(x)v)^{\perp}$, we have $\theta(x)veI = 0$. So $xveI = 0$ or $veI \subseteq eK$. From $ueK \subseteq eI$ and $veI \subseteq eK$, we obtain that $uveI \subseteq ueK \subseteq eI$ and $vueK \subseteq veI \subseteq eK$. Let $w = (1 - e) + uv$. Then $w \in U(R)$ and $weI = uveI \subseteq eI$. As eI is a small right ideal of R, we deduce that $uveI = weI = eI$ by the hypothesis. It follows that $ueK = eI \subseteq eK$. Note that, as $u \in U(eRe)$, the map $eR \to eR$ given by $a \mapsto ua$ is an isomorphism of right R-modules. Thus, as $ueK = eI$ and eI is small in eR, we see that eK is small in eR and hence in R_R. Let $t = (1-e)+u$. Then $t \in U(R)$ and $teK = ueK \subseteq eK$, so $teK = eK$ by the hypothesis. It follows that $eK = eI$, and hence eK/eI is a direct summand of $(eR/eI)_R$. $\square$

In summary, we have the following implications and none of them is reversible, as shown by the next examples.

Semisimple artinian	$\Rightarrow$	Cyclics are discrete	$\Rightarrow$	Cyclics are quasi-discrete
$\Downarrow$		$\Downarrow$		$\Downarrow$
Cyclics are quasi-projective	$\Rightarrow$	$CD2$	$\Rightarrow$	$CD3$

Example 3.2.8. *The local ring $\mathbb{Z}_4$ is a right q^*-ring and every cyclic right module over $\mathbb{Z}_4$ is discrete. But $\mathbb{Z}_4$ is not semisimple artinian.*

The next example gives a local ring R for which every cyclic right R-module is discrete, but R is not a right q^*-ring.

Example 3.2.9. *Let $K = k(t)$, where k is a field and t is an indeterminate. Then $\theta : K \to K$, $f(t) \mapsto f(t^2)$ is a monomorphism such that $\dim(K_{\theta(K)}) = 2$. Let ${}_KM = {}_KK$ as a left K-module, and M be a right K-module as defined by $x \circ k = \theta(k)x$ where $x \in M$ and $k \in K$, with which M is a (K,K)-bimodule. Let $R = K \propto M$ be the trivial extension of K by M. Then R is a local ring. Moreover, $L := 0 \propto \theta(K)$ is a right ideal in $J(R)$, which is not an ideal. So R is not a right q^*-ring by Theorem 3.2.1. But, since $\dim(K_{\theta(K)}) = 2$, R does not have an infinite chain of right ideals. So, uI is not properly contained in I for any $u \in U(R)$ and any right ideal I of R. Hence, by Theorem 3.2.7, every cyclic right R-module is discrete.*

The next example shows that there exists a right $CD2$-ring (respectively, a right $CD3$-ring) over which some cyclic right module is not discrete (respectively, quasi-discrete).

Example 3.2.10. *Let $R=\mathbb{Z}$. Then R_R is not a $D1$-module. Next we show that R is a right $CD2$-ring. It is clear that R_R is a $D2$-module. Let $I=n\mathbb{Z}$ with $n\geq 2$. It suffices to show that $(R/I)_R$ is a $D2$-module. Write $n=p_1^{s_1}p_2^{s_2}\cdots p_k^{s_k}$, where the s_i's are positive integers and the p_i's are distinct prime numbers. Then $R/I\cong Z_{p_1^{s_1}}\oplus\cdots\oplus\mathbb{Z}_{p_k^{s_k}}$, which is a quasi-discrete R-module by [41, Corollary 4.50]. As each $\mathbb{Z}_{p_k^{s_k}}$ is a discrete R-module, $(R/I)_R$ is discrete by [41, Theorem 5.2]. So $(R/I)_R$ is a $D2$-module.*

Next we construct a local ring R such that uI is properly contained in I for some $u\in U(R)$ and for some right ideal I of R.

Example 3.2.11. *Consider the field $K=F(y_1,y_2,\ldots)$ with F a field and define an endomorphism $\rho:K\to K$ by $\rho(y_i)=y_{i+1}$ and $\rho(c)=c$ for all $c\in F$ (see [54]). Then $L:=\rho(K)$ is a proper subfield of K and $\rho:K\to L$ is an isomorphism. Let $K[x;\rho]$ be the ring of skew left polynomials over K where $xk=\rho(k)x$ for all $k\in K$. Set $R=K[x;\rho]/(x^2)$. Then $R=\{a+bx: a,b\in K\}$ where $x^2=0$ and $xk=\rho(k)x$ for $k\in K$. Clearly, R is a local ring. Let $u=y_1$, and $I=(\sum_{i\geq 0}u^iL)x$. Then $u\in U(R)$ and I is a right ideal of R. Moreover, $uI\subseteq I$. Assume that $uI=I$. Then $(uL+u^2L+\cdots)x=(L+uL+u^2L+\cdots)x$, implying that $uL+u^2L+\cdots=L+uL+u^2L+\cdots$. Take $0\neq a\in L$. Then $a=ua_1+u^2a_2+\cdots+u^na_n$ where $n\geq 1$ and $a_i\in L$ (for $i=1,\ldots,n$) with $a_n\neq 0$. It follows that $y_1^n+y_1^{n-1}b_{n-1}+\cdots+y_1b_1+b_0=0$, where $b_0=-a/a_n$ and $b_i=a_i/a_n$ $(i=1,\ldots,n)$ are all in L. This is a contradiction. So, we deduce that $uI\subsetneqq I$.*

The ring R given in Example 3.2.11 is a right $CD3$-ring that is not a right $CD2$-ring. Also, by Example 3.2.11, there exists a ring R such that every cyclic right R-module is quasi-discrete but some cyclic right R-module is not discrete.

The next example gives a local ring that is a left q^*-ring but it is not a right $CD2$-ring.

Example 3.2.12. *Let R be the ring given in Example 3.2.11. Then R is a local ring that is not a right $CD2$-ring. Next we show that R is a left q^*-ring. Note that R has only three left ideals: 0, $J(R)$ and R. If $I = 0$, then ${}_R(R/I)$ is projective. If $I = J(R)$, then ${}_R(R/I)$ is simple, so is quasi-projective. If $I = R$, ${}_R(R/I)$ is certainly quasi-projective. Therefore, R is a left q^*-ring.*

Thus, Example 3.2.12 shows the following: a left $CD2$-ring need not be a right $CD2$-ring; every cyclic left R-module being discrete does not imply that every cyclic right R-module is discrete; a right q^*-ring need not be a left q^*-ring (this was observed by Koehler in [33]).

The next example gives a ring R for which every cyclic right R-module is quasi-discrete, but not every cyclic left R-module is quasi-discrete. But we have been unable to find a right $CD3$-ring that is not a left $CD3$-ring.

Example 3.2.13. *Let $R = \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$. This is the ring given by Koehler [33, Example 3.8]. Then R is semiperfect, and is a right q^*-ring by Theorem 3.2.1. So every cyclic right R-module is quasi-discrete by Theorem 3.2.4. But for $a = \begin{pmatrix} \bar{2} & \bar{1} \\ 0 & 0 \end{pmatrix} \in J(R)$ and $e = \begin{pmatrix} \bar{1} & 0 \\ 0 & 0 \end{pmatrix} \in R$, $ae = \begin{pmatrix} \bar{2} & 0 \\ 0 & 0 \end{pmatrix} \notin Ra$. So, by Theorem 3.2.4, some cyclic left R-module is neither quasi-discrete nor quasi-projective.*

There is a semiperfect, $CD3$-ring R for which not every cyclic right R-module is quasi-discrete.

Example 3.2.14. *Let* $R = \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$. *Then* R *is semiperfect, and is a* $CD3$-*ring by Corollary 3.1.19. But for* $a = \begin{pmatrix} 0 & \bar{2} \\ 0 & \bar{2} \end{pmatrix} \in J(R)$ *and* $e = \begin{pmatrix} \bar{1} & 0 \\ 0 & 0 \end{pmatrix} \in R$, $ea = \begin{pmatrix} 0 & \bar{2} \\ 0 & 0 \end{pmatrix} \notin aR$. *So, by Theorem 3.2.4, some cyclic right* R-*module is not quasi-discrete.*

Remark 3.2.15. *The material in this chapter is taken from [45].*

Chapter 4

Rings whose cyclic modules are lifting and $\oplus$-supplemented

Section 1 begins with a characterization of a semiperfect module that is lifting. We show that a semiperfect module M is lifting if and only if M has a projective cover preserving direct summands. This property is then used to characterize rings whose cyclic modules are lifting, and also artinian serial rings with Jacobson radical square-zero.

In section 2, we show that if every cyclic right R-module is lifting, then every cyclic right R-module is a direct sum of local modules. This is obtained as a consequence of a more general result that every cyclic right R-module is $\oplus$-supplemented if and only if every cyclic right R-module is a direct sum of local modules. The latter result enables us to obtain new characterizations of artinian serial rings and, respectively, rings for which every finitely generated module is a sum of local modules. For instance, we can prove that artinian serial rings are exactly these rings for which every left and right module is a direct sum of local modules.

4.1 Cyclics are lifting modules

Let A, P be submodules of a module M. Recall that P is called a supplement of A in M if it is minimal with respect to the property $M = A + P$; equivalently, $M = A + P$ and $A \cap P \ll P$. A module M is called supplemented if for any two submodules A and B with $A + B = M$, B contains a supplement of A. More generally, a module M is called weakly supplemented if every submodule of M has a supplement. Recall that a module is lifting (or satisfies $D1$) if it satisfies the following equivalent conditions.

Lemma 4.1.1. *[41, Proposition 4.8] The following are equivalent for a module M:*

1. *For every submodule N of M, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$ and $N \cap M_2 \ll M_2$.*
2. *Every submodule N of M can be written as $N = M_1 \oplus S$ with $M_1 \subseteq^{\oplus} M$ and $S \ll M$.*
3. *M is supplemented and every supplement submodule of M is a direct summand.*

Proof. (1) $\Rightarrow$ (2). M has a decomposition $M = M_1 \oplus M_2$ with $M_1 \subseteq N$ and $N \cap M_2 \ll M_2$. Then $N = M_1 \oplus (N \cap M_2)$, and the result follows with $S = N \cap M_2$.

(2) $\Rightarrow$ (3). Let $M = X + Y$. We show that Y contains a supplement of X. By assumption, we may assume $Y \subseteq^{\oplus} M$. Now $X \cap Y = Y_1 \oplus S$ such that $Y_1 \subseteq^{\oplus} M$ and $S \ll M$. Since $Y \subseteq^{\oplus} M, S \ll M$. Write $Y = Y_1 \oplus Y_2$, and let π denote the projection $Y_1 \oplus Y_2 \to Y_2$. Then $X \cap Y = Y_1 \oplus (X \cap Y \cap Y_2)$, and $X \cap Y_2 = X \cap Y \cap Y_2 = \pi(X \cap Y) = \pi(Y_1 + S) = \pi(S)$. Hence $X \cap Y_2 \ll Y_2$. Now $M = X + Y = X + Y_1 + Y_2 = X + Y_2$. So Y_2 is a supplement of X.

Now let P be a supplement submodule of M. Then there exists $K \subseteq M$ such that P is minimal with the property $K + P = M$. Since $P = L \oplus T$ with $L \subseteq^{\oplus} M$ and

$T \ll M$, $M = K + L$. Then the minimality of P implies $P = L$.

(3) $\Rightarrow$ (1). Let $A \subseteq M$. Then A has a supplement B and B has a supplement M_1 such that $M_1 \subseteq A$ and $M_1 \subseteq^{\oplus} M$. Write $M = M_1 \oplus M_2$. Then $A = M_1 \oplus (A \cap M_2)$. Also $M = M_1 + B$ and so $A = M_1 + (A \cap B)$. Let π denote the projection $M_1 \oplus M_2 \to M_2$. Then $A \cap M_2 = \pi(A) = \pi(A \cap B)$. Since B is a supplement of A, $A \cap B \ll M$ and hence $A \cap M_2 \ll M$. □

By Bass [6], a ring R is semiperfect if and only if every cyclic right R-module has a projective cover. Lately, Kasch and Mares [30] called an arbitrary module M semiperfect if every homomorphic image of M has projective cover. Thus, a ring R is semiperfect if and only if every cyclic right R-module is semiperfect. Next we study rings such that every cyclic right module is lifting.

Lemma 4.1.2. *Let M be a module with a projective cover $P \stackrel{\eta}{\to} M \to 0$. If M is lifting, then $\eta(X) \subseteq^{\oplus} M$ for all $X \subseteq^{\oplus} P$.*

Proof. Suppose $P = P_1 \oplus P_2$. We will show $\eta(P_i)$ $(i = 1, 2)$ are direct summands of M. Indeed, $P = P_1 \oplus P_2$ implies $M = \eta(P_1) + \eta(P_2)$. Since M is lifting, by Lemma 4.1.1, there exist direct summands M_1, M_2 of M and small submodules S_1, S_2 in M such that $\eta(P_1) = M_1 \oplus S_1$ and $\eta(P_2) = M_2 \oplus S_2$. Thus $M = \eta(P_1) + \eta(P_2) = M_1 + S_1 + M_2 + S_2 = M_1 + M_2$, where $M_i \subseteq \eta(P_i)$ for $i = 1, 2$. We next show that $\eta(P_i) \subseteq M_i$ for $i = 1, 2$.

Firstly, as η is onto, $M_i = \eta(\eta^{-1}(M_i))$ $(i = 1, 2)$. It follows that $M_i = \eta(\eta^{-1}(M_i)) \cap \eta(P_i) \supseteq \eta(\eta^{-1}(M_i) \cap P_i)$. Let $x \in \eta(\eta^{-1}(M_i)) \cap \eta(P_i)$ and write $x = \eta(a) = \eta(b)$ with $a \in \eta^{-1}(M_i)$ and $b \in P_i$. Then $b \in \eta^{-1}(M_i) \cap P_i$, so $x = \eta(b) \in \eta(\eta^{-1}(M_i) \cap P_i)$. Hence $M_i = \eta(\eta^{-1}(M_i)) \cap \eta(P_i) = \eta(\eta^{-1}(M_i) \cap P_i)$.

Now $\eta(\eta^{-1}(M_i)) = M_i = \eta(\eta^{-1}(M_i) \cap P_i)$ implies $\eta^{-1}(M_i) = \eta^{-1}(M_i) \cap P_i + ker\eta$. With

$M = M_1 + M_2$ we have $P = \eta^{-1}(M_1) + \eta^{-1}(M_2) = \eta^{-1}(M_1) \cap P_1 + \eta^{-1}(M_2) \cap P_2 + ker\eta$. As $ker\eta \ll P$, $P = \eta^{-1}(M_1) \cap P_1 + \eta^{-1}(M_2) \cap P_2$. It follows that $\eta^{-1}(M_i) \cap P_i = P_i$ for $i = 1, 2$. So, $\eta(P_i) \subseteq M_i$ $(i = 1, 2)$. As $M_i \subseteq \eta(P_i)$, we obtain that $\eta(P_i) = M_i$ for $i = 1, 2$. □

Note that in general a semiperfect module need not be lifting. However, we have

Lemma 4.1.3. *Every semiperfect module is supplemented.*

Proof. Let M be a semiperfect module. Assume that $M = A + B$. We show that B contains a supplement of A. By hypothesis, M/A has a projective cover $P \xrightarrow{\eta} M/A \rightarrow 0$. Let $f : B \rightarrow M/A$ be the natural homomorphism. Since P is projective, there exists $g : P \rightarrow B$ such that $fg = \eta$. Thus, $(fg)(P) = \eta(P)$ and $A \cap g(P) = g(ker(\eta))$. Hence, $M = A + B = A + g(P)$ and $A \cap g(P) = g(ker(\eta))$. Since $ker(\eta)$ is small in P, $g(ker(\eta))$ is small in $g(P)$, and so $A \cap g(P)$ is small in $g(P)$. This shows that $g(P)$, contained in B, is a supplement of A. □

Lemma 4.1.4. *Let M be a semiperfect module with a projective cover $P \xrightarrow{\eta} M \rightarrow 0$. If N is a supplement submodule of M, then $N = \eta(X)$ for some $X \subseteq^{\oplus} P$.*

Proof. As N is a supplement submodule of M, there exists a submodule A such that N is minimal with respect to $M = A + N$. By Theorem 1.2.18, P is semiperfect. So P is lifting by Proposition 1.2.17. From $M = A + N$, it follows that $P = \eta^{-1}(A) + \eta^{-1}(N)$. Since P is lifting, we have $P = \eta^{-1}(A) + X$ with $X \subseteq \eta^{-1}(N), X \subseteq^{\oplus} P$ and $\eta^{-1}(A) \cap X \ll X$. Hence $M = A + \eta(X)$. As $\eta(X) \subseteq N$, the minimality of N shows that $N = \eta(X)$. □

Definition 4.1.5. *A module M is said to have a projective cover preserving direct summands if there is a projective cover $P \xrightarrow{\eta} M \rightarrow 0$ such that $\eta(X) \subseteq^{\oplus} M$ for all $X \subseteq^{\oplus} P$.*

Now we have a characterization of a semiperfect module that is lifting.

Theorem 4.1.6. *A semiperfect module M is lifting if and only if M has a projective cover preserving direct summands.*

Proof. ($\Rightarrow$). The implication follows from Lemma 4.1.2.

($\Leftarrow$). Suppose that M has a projective cover $P \xrightarrow{\eta} M \to 0$ such that $\eta(X) \subseteq^{\oplus} M$ for all $X \subseteq^{\oplus} P$. By Lemma 4.1.3, M is supplemented. If N is a supplement submodule of M, then, by Lemma 4.1.4, $N = \eta(X)$ for some $X \subseteq^{\oplus} P$. Thus, N is a direct summand of M by assumption. So M is lifting by Lemma 4.1.1(3). □

As an immediate consequence of Theorem 4.1.6, we have the following characterizations of rings whose cyclics are lifting.

Corollary 4.1.7. *The following are equivalent for a ring R:*

1. *Every cyclic right R-module is lifting.*
2. *Every cyclic right R-module has a projective cover preserving direct summands.*
3. *R is semiperfect and, whenever $I \subseteq J(R)$, $e^2 = e \in R$ and $f^2 = f \in eR$, $(fR + eI)/eI \subseteq^{\oplus} eR/eI$.*
4. *R is semiperfect and, whenever $I \subseteq J(R)$, $e^2 = e \in R$ and $f^2 = f \in eR$, we have $fR + eI = xeR + eI$ and $xeI \subseteq eI$ for some $x^2 = x \in eR$.*

Proof. (1) $\Leftrightarrow$ (2). Note that either (1) or (2) implies that R is semiperfect and so every cyclic right R-module is semiperfect. Thus, the equivalence follows from Theorem 4.1.6.

(2) $\Rightarrow$ (3). We see that R is semiperfect. Let $M = eR/eI$ where $e^2 = e \in R$ and $I \subseteq J(R)$ (see Lemma 1.2.20). By (2), M has a projective cover $P \xrightarrow{\eta} M \to 0$ such

that $\eta(X) \subseteq^{\oplus} M$ for all $X \subseteq^{\oplus} P$. As $eI \ll eR$, $eR \xrightarrow{\pi} M \to 0$ is also a projective cover where π is the natural homomorphism. Then, by the uniqueness of projective covers, there exists an isomorphism $\alpha : eR \to P$ such that $\eta\alpha = \pi$. Let $f^2 = f \in eR$. As $fR \subseteq^{\oplus} eR$, $\alpha(fR) \subseteq^{\oplus} P$, so $\eta(\alpha(fR)) \subseteq^{\oplus} M$. That is, $\pi(fR) \subseteq^{\oplus} M$, or $(fR + eI)/eI \subseteq^{\oplus} eR/eI$.

(3) $\Leftrightarrow$ (4). It is by Lemma 3.1.16.

(3) $\Rightarrow$ (1). Let M be a cyclic right R-module. As R is semiperfect, $M \cong eR/eI$ where $e^2 = e \in R$ and $I \subseteq J(R)$. Then M is supplemented by Lemma 4.1.3. So, by Lemma 4.1.1, to show that M is lifting, it suffices to show that every supplement submodule of eR/eI is a direct summand. Let N be a supplement submodule of eR/eI. By Lemma 4.1.4, $N = (fR + eI)/eI$ where $f^2 = f \in eR$. By hypothesis, N is a direct summand of eR/eI, as required. □

Corollary 4.1.8. *A commutative ring R is semiperfect if and only if every cyclic R-module is lifting.*

The next example shows that, over a semiperfect ring, a cyclic module need not be lifting.

Example 4.1.9. *Let* $R = \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$ *and* $y = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$. *Then* $K := yR = \left\{ \begin{pmatrix} 0 & 2a \\ 0 & 2a \end{pmatrix} : a \in \mathbb{Z}_4 \right\} \subseteq J(R)$. *Let* $f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. *Then* $f^2 = f \in R$, *and* $fR + K = \begin{pmatrix} 0 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$. *We verify that* $(fR+K)/K$ *is not a direct summand of* $(R/K)_R$. *Assume that* $(fR+K)/K$ *is a direct summand of* $(R/K)_R$. *Then, by Lemma 3.1.16, there exists* $g^2 = g \in R$ *such that* $fR + K = gR + K$ *and* $gK \subseteq K$. *As* $g \in fR + K$, *it must be that* $g = \begin{pmatrix} 0 & 2b \\ 0 & 1 \end{pmatrix}$

where $b \in \mathbb{Z}_4$. *However,* $g \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \notin K$. *So* $(fR + K)/K$ *is not a direct summand of* $(R/K)_R$. *By Lemma 4.1.2,* $(R/K)_R$ *is not lifting.*

The next example shows that there exists a ring such that every cyclic right R-module is lifting, but not every cyclic left R-module is lifting.

Example 4.1.10. *Let* $R = \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$. *Then, as shown in Example 3.2.13, every cyclic right R-module is self-projective. As R is semiperfect, every right ideal contained in* $J(R)$ *is an ideal of* R *by Theorem 3.2.1. Hence, by Theorem 3.2.4, every cyclic right* R*-module is quasi-discrete and hence lifting. Let* $y = \begin{pmatrix} [2] & \bar{1} \\ 0 & 0 \end{pmatrix}$. *Then* $I := Ry = \left\{0, \begin{pmatrix} [2] & \bar{1} \\ 0 & 0 \end{pmatrix} \right\} \subseteq J(R)$. *Let* $f = \begin{pmatrix} 0 & 0 \\ 0 & \bar{1} \end{pmatrix}$. *Then* $f^2 = f \in R$, *and* $Rf + I = \begin{pmatrix} 2\mathbb{Z}_4 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$. *We verify that* $(Rf + I)/I$ *is not a direct summand of* ${}_R(R/I)$. *Assume that* $(Rf + I)/I$ *is a direct summand of* ${}_R(R/I)$. *Then, by Lemma 3.1.16, there exists* $g^2 = g \in R$ *such that* $Rf + I = Rg + I$ *and* $Ig \subseteq I$. *As* $g \in Rf + I$, *it must be that* $g = \begin{pmatrix} 0 & \bar{b} \\ 0 & \bar{1} \end{pmatrix}$. *However,* $\begin{pmatrix} [2] & \bar{1} \\ 0 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & \bar{1} \\ 0 & \bar{1} \end{pmatrix} \notin I$. *So* $(Rf + I)/I$ *is not a direct summand of* ${}_R(R/I)$. *By Lemma 4.1.2,* ${}_R(R/I)$ *is not lifting.*

From Corollary 4.1.7(2), the following question arises: for which rings R, does every (finitely generated) right R-module have a projective cover preserving direct summands? As another corollary of Theorem 4.1.6, we show next that these rings are exactly artinian serial rings with Jacobson radical square-zero. By Theorem 1.2.21, a ring R is artinian serial with $J(R)^2 = 0$ if and only if every right R-module is lifting and, if and only if every 2-generated right R-module is lifting.

Corollary 4.1.11. *The following are equivalent for a ring R:*

1. *R is artinian serial with $J(R)^2 = 0$.*

2. *Every right R-module has a projective cover preserving direct summands.*

3. *Every 2-generated right R-module has a projective cover preserving direct summands.*

Proof. (1) $\Rightarrow$ (2). By Theorem 1.2.21, every right R-module is lifting. As R is perfect, every right R-module is semiperfect. So (2) follows from Theorem 4.1.6.

(2) $\Rightarrow$ (3). The implication is clear.

(3) $\Rightarrow$ (1). Again, by Theorem 1.2.21, it suffices to show that every 2-generated right R-module M is lifting. By (3), every 2-generated right R-module is semiperfect. So, by Theorem 4.1.6, every 2-generated right R-module is lifting. □

Corollary 4.1.12. *The following are equivalent for a ring R:*

1. *R is a (semiperfect) right perfect ring.*

2. *Every (cyclic) lifting right R-module has a projective cover preserving direct summands.*

3. *Every (cyclic) right R-module having a projective cover preserving direct summands is lifting.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3). If R is (semiperfect) right perfect, then every (cyclic) right R-module has a projective cover, so every (cyclic) right R-module is a semiperfect module. Thus, by Theorem 4.1.6, a (cyclic) right R-module is lifting if and only if it has a projective cover preserving direct summands. So (2) and (3) hold.

(2) $\Rightarrow$ (1). It is known that every semisimple right R-module is lifting. So by (2), every (simple) semisimple right R-module has a projective cover. Hence, by [55, Theorem 5], R is (semiperfect) right perfect.

(3) $\Rightarrow$ (1). As a projective module has a projective cover preserving direct summands, we infer that every (cyclic) projective right R-module P is lifting by (3). So, for any submodule N of P, there is a direct decomposition $P = P_1 \oplus P_2$ such that $P_1 \subseteq N$ and $P_2 \cap N \ll P_2$. Then $P_2 \to P_2/P_2 \cap N \cong P/N$ gives a projective cover of P/N. Hence, every (cyclic) projective right R-module is semiperfect. Because an image of a semiperfect module is again semiperfect and every (cyclic) right R-module is an image of a (cyclic) projective right R-module, it follows that every (cyclic) right R-module is semiperfect. So R is (semiperfect) right perfect. □

4.2 Cyclics are $\oplus$-supplemented modules

As a generalization of lifting modules, Mohamed and Müller [41, Definition A.1] define a module M to be $\oplus$-supplemented if for every submodule N of M there exists a direct summand K of M such that $M = N + K$ and $N \cap K \ll K$. Every $\oplus$-supplemented module is weakly supplemented. Rings whose (finitely generated) modules are $\oplus$-supplemented were studied in [31].

In this section we first show that every cyclic right module is $\oplus$-supplemented if and only if every cyclic right module is a direct sum of local modules. We begin with some important observations. Note that a module M is local if and only if M is hollow with $rad(M) \neq M$. The Prüfer group $\mathbb{Z}_{p^\infty}$ (p is a prime), as a $\mathbb{Z}$-module, is hollow but not local.

Lemma 4.2.1. *A cyclic module M is hollow if and only if M is local, if and only if* *there exists a maximal submodule of M which is also a small submodule of M.*

Proof. Let M be a cyclic hollow module and A be a maximal submodule of M. Assume A is also a small submodule. We show that M is local. Equivalently, A is the only maximal submodule of M. Indeed, let B be another submodule of M. If B is not contained in A then $M = A + B$. As $A \ll M$, we have $B = M$. So A is the only maximal submodule. □

Proposition 4.2.2. *Let R be a semiperfect ring and M be a cyclic right R-module. Then every supplement of a maximal submodule of M is a local module.*

Proof. As R is semiperfect, we can write $M = eR/eI$ with $e^2 = e \in R$ and $I \subseteq J(R)$ (see Lemma 1.2.20). Let $N = A/eI$ be a maximal submodule of M. If N is small in M, then M is hollow by Lemma 4.2.1 and hence M is the only supplement of N. As M is cyclic, M is a local module. Assume that N is not small in M. By Lemma 4.1.3, M is supplemented. So there exists a proper submodule P of M which is a supplement of N. Moreover, P is a hollow module. To see this, let X be a proper submodule of P and $P = X + Y$. As P is minimal with respect to $M = N + P$, $N + X \neq M$, so $N + X = N$ because N is a maximal submodule of N. Thus, $M = N + P = (N + X) + Y = N + Y$. By the minimality of P, we see that $Y = P$. So X is small in P for every proper submodule X of P. Hence P is hollow. As R is semiperfect, eR is lifting by Lemma 4.1.3 and Proposition 1.2.17. So, by Lemma 4.1.4, $P = (K + eI)/eI$ where $K \subseteq^{\oplus} eR$. So P is cyclic and hollow, and hence P is a local module. □

Lemma 4.2.3. *Let R be a semiperfect ring. Let $M = eR/eI$ where $e^2 = e \in R$ and $I \subseteq J(R)$. If $M = M_1 \oplus M_2$, then there exists $x^2 = x \in eR$ with $xeI \subseteq eI$ such that $M_1 = (xeR+eI)/eI$ and $M_2 = ((1-x)eR+eI)/eI$. Moreover, $eR = xeR \oplus (1-x)eR$.*

Proof. The first part is contained in the proof of Lemma 3.1.16. For the second part, we see that $eR = xeR + (1-x)eR + eI$. As $I \subseteq J(R)$, $eI \ll eR$, so $eR = xeR + (1-x)eR$. As $xeR \cap (1-x)eR = 0$, it follows that $eR = xeR \oplus (1-x)eR$. □

Lemma 4.2.4. *Let R be a semiperfect ring. Let $M = eR/eI$ where $e^2 = e \in R$ and $I \subseteq J(R)$. Then M satisfies the descending chain condition on direct summands.*

Proof. Assume that M does not satisfy the descending chain condition on direct summands. Then there exists a strictly descending chain of direct summands: $M = M_0 \supset M_1 \supset M_2 \supset \cdots$. Write $M = M_1 \oplus M_1^{'}$. By Lemma 4.2.3, there exists $x^2 = x \in eR$ such that $M_1 = (xeR + eI)/eI$, $M_1^{'} = ((1-x)eR + eI)/eI$ and $eR = xeR \oplus (1-x)eR$. Let $e_1 = xe, e_1^{'} = (1-x)e$. Then $e_1 e_1^{'} = e_1^{'} e_1 = 0$, $eR = e_1 R \oplus e_1^{'} R$, and $M_1 = (e_1 R + eI)/eI$. Let $L_1 = e_1 R/(e_1 R \cap eI) = e_1 R/e_1 I_1$ where $I_1 = e_1 R \cap eI \subseteq J(R)$. Then $M_1 \cong L_1$ and there is a strictly descending chain of direct summands (in L_1) $L_1 \supset L_2 \supset \cdots$. Write $L_1 = L_2 \oplus L_2^{'}$. As argued above, there exist orthogonal idempotents $e_2, e_2^{'}$ in $e_1 R$ such that $e_1 R = e_2 R \oplus e_2^{'} R$, $L_2 = (e_2 R + e_1 I_1)/e_1 I_1$. Let $N_2 = e_2 R/(e_2 R \cap e_1 I_1) = e_2 R/e_2 I_2$ where $I_2 = e_2 R \cap e_1 I_1 \subseteq J(R)$. Then $L_2 \cong N_2$ and there is a strictly descending chain of direct summands (in N_2) $N_2 \supset N_3 \supset \cdots$. This process continues, and we obtain a strictly descending chain of direct summands in R_R: $eR \supset e_1 R \supset e_2 R \supset \cdots$. This means that R contains a family of infinite orthogonal nonzero idempotents, a contradiction to the hypothesis that R is semiperfect. □

Lemma 4.2.5. *[20, Theorem 1.4] For any ring R, every finite direct sum of $\oplus$-supplemented R-modules is $\oplus$-supplemented.*

Proof. Let n be any positive integer and let M_i be a $\oplus$-supplemented R-module for each $1 \leq i \leq n$. Let $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$. To prove that M is $\oplus$-supplemented it is sufficient by induction on n to prove this is the case when $n = 2$. Thus, suppose $n = 2$.

Let L be any submodule of M. Then $M = M_1 + M_2 + L$. As M_2 is $\oplus$-supplemented, let H be a supplement of $M_2 \cap (M_1 + L)$ in M_2 such that H is a direct summand of M_2. It follows that $M_2 = M_2 \cap (M_1 + L) + H$ with $H \cap (M_1 + L) \ll H$. Hence, $M = (M_1 + L) + H$ with $H \cap (M_1 + L) \ll H$. So H is a supplement of $M_1 + L$ in M. Let K be a supplement of $M_1 \cap (L+H)$ in M_1 such that K is a direct summand of M_1. Again, we have K is a supplement of $H + L$ in M. Since H is a direct summand of M_2 and K is a direct summand of M_1 it implies that $H \oplus K$ is a direct summand of M. Moreover, $M = L+H+K$ and $L \cap (H+K) \subseteq [H \cap (L+K)] + [K \cap (H+L)] \ll H \oplus K$. Thus, $H \oplus K$ is a supplement of L in M, as required. □

Theorem 4.2.6. *Every cyclic right R-module is $\oplus$-supplemented if and only if every cyclic right R-module is a direct sum of local modules.*

Proof. ($\Rightarrow$). Let M be a cyclic right R-module. If M is hollow then we are done. Assume that M is not hollow. Let N be a maximal submodule of M. Then, by Lemma 4.2.1, N is not a small submodule of M. Since M is $\oplus$-supplemented, N has a supplement M_1 in M which is also a proper direct summand of M. By Proposition 4.2.2, M_1 is a local module. Write $M = M_1 \oplus M_1'$. If M_1' is hollow, we are done. If M_1' is not hollow, M_1' is also $\oplus$-supplemented (as it is cyclic). Hence, we can apply the same argument to M_1' to obtain a decomposition $M_1' = M_2 \oplus M_2'$ with M_2 local. This process continues, but will stop after a finite steps because M satisfies the descending chain condition on direct summands (by Lemma 4.2.4). So we have $M = \oplus_{i=1}^{n} M_i$ where M_i is a local module for all i.

($\Leftarrow$). Note that every local module is hollow and hence $\oplus$-supplemented. So the implication follows from Lemma 4.2.5. □

We doubt whether every cyclic $\oplus$-supplemented module is a direct sum of local modules. From the proof of Theorem 4.2.6, one sees that, for a cyclic right module M over a semiperfect ring, if every direct summand of M is $\oplus$-supplemented, then M is a direct sum of local modules.

There is a ring R such that every cyclic right R-module is $\oplus$-supplemented, but not every cyclic left R-module is $\oplus$-supplemented.

Example 4.2.7. *Let* $R = \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$. *By Example 4.1.10, every cyclic right R-module is lifting and hence* $\oplus$*-supplemented. Let* $y = \begin{pmatrix} [2] & \bar{1} \\ 0 & 0 \end{pmatrix}$. *Then* $K := Ry = \left\{0, \begin{pmatrix} [2] & \bar{1} \\ 0 & 0 \end{pmatrix}\right\} \subseteq J(R)$. *We next show that the cyclic left R-module* $M := R/K$ *is not* $\oplus$*-supplemented. Let* $f = \begin{pmatrix} 0 & 0 \\ 0 & \bar{1} \end{pmatrix}$. *Then* $Rf + K = \begin{pmatrix} 2\mathbb{Z}_4 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$. *Consider the submodule* $N := (Rf + K)/K$ *of* M. *Assume that* M *is* $\oplus$*-supplemented. Then* N *has a supplement* N' *in* M *that is a direct summand. Thus, by Lemma 4.2.3, there exists* $g^2 = g \in R$ *such that* $N' = (Rg + K)/K$ *and* $Kg \subseteq K$. *But* $Kg \subseteq K$ *implies that* g *is trivial. As* $g \neq 0$, *it must be that* $g = 1_R$. *Thus,* $N' = M$, *and it follows that* $N = N \cap N' \ll N' = M$. *However, this cannot be true because* M *is the sum of* N *and* $(R(1-f) + K)/K$ *and* $M \neq (R(1-f) + K)/K$. *This contradiction shows that* M *is not* $\oplus$*-supplemented.*

The next result is needed for the rest of this section.

Theorem 4.2.8. *[31, Theorem 3.11] A ring R is artinian serial if and only if every right R-module and left R-module is* $\oplus$*-supplemented.*

The next example shows that there is a ring R such that every cyclic right R-module is $\oplus$-supplemented, but not every cyclic right R-module is lifting.

Example 4.2.9. *Let* $R = \begin{pmatrix} F & F & F \\ 0 & F & F \\ 0 & 0 & F \end{pmatrix}$, *where F is a field. Then R is artinian serial, so by Theorem 4.2.8 every cyclic right R-module is $\oplus$-supplemented. Consider the right ideal* $I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix} \subseteq J(R)$, *and the idempotent* $e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in R$. *We show that the cyclic module $M = R/I$ is not lifting. By Lemma 4.1.2, it suffices to show that $(eR + I)/I$ is not a direct summand of M. Note that* $eR + I = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & a & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in F \right\}$. *Assume that $(eR + I)/I$ is a direct summand of M. Then, by Lemma 4.2.3, there exists $x^2 = x \in eR + I$ such that $eR + I = xR + I$ and $xI \subseteq I$. Write* $x = \begin{pmatrix} 0 & a & b \\ 0 & a & c \\ 0 & 0 & 0 \end{pmatrix}$. *As $x \neq 0$, we see* $x = \begin{pmatrix} 0 & 1 & c \\ 0 & 1 & c \\ 0 & 0 & 0 \end{pmatrix}$ *with $c^2 = c$. But* $xI = \left\{ \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} : b \in F \right\} \nsubseteq I$. *The contradiction shows that $(eR + I)/I$ is not a direct summand of M.*

Recall that a module M is called quasi-discrete if M is a lifting module satisfying the $D3$-condition. The next example gives a ring R such that every cyclic right R-module is lifting, but not every cyclic right R-module is quasi-discrete.

Example 4.2.10. *Let $R = \mathbb{M}_n(D)[x]/(x^k)$, where D is a division ring and $n, k \geq 2$. Then, by Example 3.1.21, not every cyclic right R-module is quasi-discrete. Note that*

$R \cong S$ where $S = \mathbb{M}_n(D[x]/(x^k))$. We have $S_S = M_1 \oplus M_2 \oplus \cdots \oplus M_k$, where

$$M_1 = \begin{pmatrix} \sum_{i=0}^{k-1} x^i D & \cdots & \sum_{i=0}^{k-1} x^i D \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 0 & \cdots & 0 \\ \sum_{i=0}^{k-1} x^i D & \cdots & \sum_{i=0}^{k-1} x^i D \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix},$$

$$\vdots$$

$$M_k = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ \sum_{i=0}^{k-1} x^i D & \cdots & \sum_{i=0}^{k-1} x^i D \end{pmatrix}.$$

Each M_i is a uniserial S-module. For instance, M_1 is uniserial because the submodules of M_1 form a chain

$$M_1 \supset \begin{pmatrix} \sum_{i=1}^{k-1} x^i D & \cdots & \sum_{i=1}^{k-1} x^i D \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \supset \begin{pmatrix} \sum_{i=2}^{k-1} x^i D & \cdots & \sum_{i=2}^{k-1} x^i D \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \supset \cdots$$

$$\cdots \supset \begin{pmatrix} \sum_{i=k-2}^{k-1} x^i D & \cdots & \sum_{i=k-2}^{k-1} x^i D \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \supset \begin{pmatrix} x^{k-1} D & \cdots & x^{k-1} D \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

Similarly, ${}_S S$ is a direct sum of uniserial S-modules. So S and hence R is artinian serial. By Theorem 4.2.8, every cyclic R-module is $\oplus$-supplemented.

If $k = 2$, then $J(R)^2 = 0$. So, by Theorem 1.2.21, every cyclic right R-module is lifting.

The next example gives a commutative local domain R such that some 2-generated R-module is not $\oplus$-supplemented. Note that a ring R is local if and only if every cyclic right (left) R-module is local.

Example 4.2.11. *Let F be a field and $R = F[[X, Y]]$, the ring of formal power series over F in the indeterminates X, Y. Then R is a commutative noetherian local domain. So every cyclic R-module is local and hence lifting. However, the 2-generated ideal $J = RX + RY$ is the unique maximal ideal of R and is uniform, so J is not a $\oplus$-supplemented R-module. In particular, J is not a direct sum of cyclic modules.*

Corollary 4.2.12. *Let R be a ring and let n be a positive integer. Then every n-generated right R-module is $\oplus$-supplemented if and only if every n-generated right R-module is a direct sum of local modules.*

Proof. It is proved in [31, Corollary 2.5] that every n-generated right R-module is $\oplus$-supplemented if and only if every cyclic right R-module is $\oplus$-supplemented and every n-generated right R-module is a direct sum of cyclic modules. The claim follows from this, Lemma 4.2.5 and Theorem 4.2.6. □

Corollary 4.2.13. *Let R be a ring. Then every finitely generated right R-module is $\oplus$-supplemented if and only if every finitely generated right R-module is a direct sum of local modules.*

Proof. It is proved in [31, Corollary 2.6] that every finitely generated right R-module is $\oplus$-supplemented if and only if every cyclic right R-module is $\oplus$-supplemented and every finitely generated right R-module is a direct sum of cyclic modules. The claim follows from this, Lemma 4.2.5 and Theorem 4.2.6. □

Various characterizations of an artinian serial ring are found in [31]. The next result gives a noteworthy characterization of an artinian serial ring. We should remark that it remains unknown what is the structure of noncommutative rings each of whose modules is a direct sum of cyclic modules. However, our characterization of an artinian serial ring is somehow related to this question.

Corollary 4.2.14. *A ring R is artinian serial if and only if every left R-module and right R-module is a direct sum of local modules.*

Proof. ($\Rightarrow$). Suppose that R is artinian serial. Then, by Theorem 4.2.8, every left and right R-module is $\oplus$-supplemented. Moreover, it is a well-known result of Nakayama [43, Theorem 17] that every left and right module over an artinian serial ring is a direct sum of cyclic modules. Therefore, it follows from Theorem 4.2.6 that every left and right R-module is a direct sum of local modules.

($\Leftarrow$). Since every local module is cyclic, from the hypothesis it follows that every left and right R-module is a direct sum of cyclic modules. It is a well-known result of Chase [10, Theorem 4.4] that if every left R-module is a direct sum of finitely generated modules, then R is left artinian. So, R is artinian and hence perfect. Since every local module is lifting, we also see from the hypothesis that every left and right

R-module is a direct sum of lifting modules. It is proved in [31, Corollary 2.13] that, over a right perfect ring, any module that is a direct sum of lifting modules is a $\oplus$-supplemented module. Hence, we deduce that every left and right R-module is $\oplus$-supplemented. So R is artinian serial by Theorem 4.2.8. $\square$

In [51], Osofsky and Smith proved that if every cyclic right R-module is extending, then every cyclic right R-module is a direct sum of uniform modules. The following result is an immediate consequence of Theorem 4.2.6.

Corollary 4.2.15. *Suppose that every cyclic right R-module is lifting. Then every cyclic right R-module is a direct sum of local modules.*

A direct summand X of a module M is called a local summand if X is a local module.

Lemma 4.2.16. *Let R be a semiperfect ring and $M = eR/eI$ where $e^2 = e \in R$ and $I \subseteq J(R)$. Let $P = eR$ and $\pi : P \to M$ be the natural homomorphism. If $\pi(X) \subseteq^{\oplus} M$ for every local summand X of P, then M is $\oplus$-supplemented.*

Proof. By Lemma 4.1.3, M is supplemented. If M is hollow then we are done. Otherwise, let $N = A/eI$ be a maximal submodule of M. Then by Proposition 4.2.2, every supplement submodule K of N in M is a local module. Moreover, by Lemma 4.1.4, there exists a direct summand X of P such that $K = \pi(X) = (X + eI)/eI$. Write $X = fR$ with $f^2 = f \in eR$, so $K = (fR + eI)/eI \cong fR/(fR \cap eI) = fR/fL$ where $L = fR \cap eI \subseteq J(R)$. Note that $fL \subseteq fJ(R) = rad(fR) \subset fR$. Since K is local, $fJ(R)$ is the unique maximal submodule of fR, so $f \in R$ is a local idempotent. That is, fR is a local summand of eR. By hypothesis, $K = (fR + eI)/eI$ is a direct summand of M. Write $M = M_1 \oplus M_2$ with $M_1 = K$ local. Now we can apply the same argument to M_2. As M satisfies the descending chain conditions on direct summands, this process will yield a decomposition $M = \oplus_{i=1}^{n} M_i$ where each M_i is a local module.

Since every local module is hollow and hence $\oplus$-supplemented, M is $\oplus$-supplemented by Lemma 4.2.5. $\square$

Proposition 4.2.17. *If every cyclic right R-module M has a projective cover $P \xrightarrow{\eta} M \to 0$ such that, for any local summand X of P, $\eta(X)$ is a direct summand of M, then every cyclic right R-module is $\oplus$-supplemented.*

Proof. Note that R is semiperfect. Let M be a cyclic right R-module. Then $M = eR/eI$ where $e^2 = e \in R$ and $I \subseteq J(R)$. By hypothesis, M has a projective cover $P \xrightarrow{\eta} M \to 0$ such that $\eta(X) \subseteq^{\oplus} M$ for all local summands X of P. As $eI \ll eR$, $eR \xrightarrow{\pi} M \to 0$ is also a projective cover where π is the natural homomorphism. Then, by the uniqueness of projective covers, there exists an isomorphism $\alpha : eR \to P$ such that $\eta\alpha = \pi$. Let f be a local idempotent in eR. As fR is a local summand of eR, $\alpha(fR) \subseteq^{\oplus} P$, so $\eta(\alpha(fR)) \subseteq^{\oplus} M$. That is, $\pi(fR) \subseteq^{\oplus} M$, or $(fR+eI)/eI \subseteq^{\oplus} eR/eI$. So M is $\oplus$-supplemented by Lemma 4.2.16. $\square$

In view of Corollary 4.1.7, one would think that every cyclic right R-module is $\oplus$-supplemented if and only if every cyclic right R-module M has a projective cover $P \xrightarrow{\eta} M \to 0$ such that $\eta(X) \subseteq^{\oplus} M$ for all local summands X of P. The sufficiency is verified by Proposition 4.2.17, but the necessity is not true by the next example.

Example 4.2.18. *Let R be the ring given in Example 4.2.9. Then every cyclic right R-module is $\oplus$-supplemented. Consider the cyclic right R-module $M := R/I$ where $I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}$ is a right ideal of R contained in $J(R)$. Let $\pi : R \to M$ be the natural homomorphism. Then $R \xrightarrow{\pi} M \to 0$ is a projective cover of M. For the* *idempotent element* $e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $eRe = \left\{ \begin{pmatrix} 0 & a & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix} : a \in F \right\} \cong F$. *So* eR *is a local summand of* R_R. *But* $\pi(eR) = (eR + I)/I$ *is not a direct summand of* M, *as verified in Example 4.2.9.*

Remark 4.2.19. *The material in this chapter is taken from [46].*

Chapter 5

On δ-semiperfect modules

Semiperfect rings and semiperfect modules have been extensively studied in [6], [30], [38], [47] and [55]. In [63], the concepts of δ-small submodules, projective δ-covers, and δ-semiperfect rings are introduced. A ring R is called δ-semiperfect if every simple right R-module has a projective δ-cover. Various characterizations of δ-semiperfect rings are presented in [63]. This chapter is a research on δ-semiperfect modules. The goal here is to generalize the structure theory of δ-semiperfect rings to modules. Our proofs can be adapted to generalize some of the results of Mares [38] and Nicholson [47] from projective semiperfect modules to arbitrary semiperfect modules.

5.1 Characterizations of δ-semiperfect modules

For any module M_R, the singular submodule $Z(M)$ consists of elements $m \in M$ for which the right annihilator $ann_r(m)$ is an essential submodule in R_R. The module is called nonsingular if $Z(M) = 0$, and singular if $Z(M) = M$. It is known that a right R-module M_R is singular if and only if there exist two R-modules $A \subseteq_e B$ such that $M \cong B/A$ (see [37, p.247]). Recall that a submodule N of M is δ-small in M (written N $\ll_\delta M$) if $N + X \neq M$ for any proper submodule X of M with M/X singular. A projective δ-cover of M is a projective module P with an epimorphism $\alpha : P \to M$ such that $ker(\alpha)$ is δ-small in P (see [63]). A module M is δ-semiperfect[1] if every factor module of M has a projective δ-cover. This notion was introduced by [59], but not much information on this class of modules is readily available.

A module M is called δ-lifting if for any submodule N of M, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$ and $N \cap M_2$ is δ-small in M (see [32, p.54]). Equivalently, a module M is δ-lifting if and only if every submodule N of M can be written as $N = M_1 \oplus S$ with $M_1 \subseteq^\oplus M$ and $S \ll_\delta M$ (this is [63, Lemma 3.4]). We present some basic properties of δ-small submodules, which will be used repeatedly. The following lemma is [63, Lemmas 1.2 and 1.3].

Lemma 5.1.1. *Let M be a module.*

1. *A submodule $N \subseteq M$ is δ-small in M if and only if, whenever $M = X + N$, we have $M = X \oplus Y$ for a projective semisimple submodule Y with $Y \subseteq N$.*

2. *For submodules N, K, L of M with $K \subseteq N$, we have*

   (a) *N is δ-small in M if and only if K is δ-small in M and N/K is δ-small in M/K.*

[1] This terminology has been used differently in [53]

(b) $(N+L)$ *is* δ*-small in* M *if and only if* N *is* δ*-small in* M *and* L *is* δ*-small in* M*.*

3. *If* K *is* δ*-small in* M *and* $f : M \to N$ *is a homomorphism, then* $f(K)$ *is* δ*-small in* N*.*

4. *Let* $K_1 \subseteq M_1 \subseteq M, K_2 \subseteq M_2 \subseteq M$ *and* $M = M_1 \oplus M_2$*. Then* $K_1 \oplus K_2$ *is* δ*-small in* $M_1 \oplus M_2$ *if and only if* K_1 *is* δ*-small in* M_1 *and* K_2 *is* δ*-small in* M_2*.*

In [63], $\delta(M)$ is defined to be the reject in M of the class of all singular simple modules, i.e. $\delta(M) = \cap\{N \subseteq M \mid M/N \text{ is singular simple}\}$. The following lemma is [63, Lemmas 1.5 and 1.9]

Lemma 5.1.2. *Let* M *and* N *be modules.*

1. $\delta(M)$ *is the sum of all* δ*-small submodules of M.*

2. *If* $f : M \to N$ *is a homomorphism, then* $f(\delta(M)) \subseteq \delta(N)$*. Therefore,* $\delta(M)$ *is a fully invariant submodule of* M *and* $M\delta(R) \subseteq \delta(M)$*.*

3. *If* $M = \oplus_{i\in I} M_i$*, then* $\delta(M) = \oplus_{i\in I}\delta(M_i)$*.*

4. *If every proper submodule of M is contained in a maximal submodule of M, then* $\delta(M)$ *is the unique largest* δ*-small submodule of M.*

5. *If* P *is a projective module, then* $\delta(P) = P\delta(R)$ *and* $\delta(P)$ *is the intersection of all essential maximal submodules of P.*

We should mention another result about the projective δ-covers of a module. That is, unlike projective covers, the projective δ-covers of a module are not unique up to isomorphism. However they differ by only a projective semisimple direct summand.

Lemma 5.1.3. *[63, Lemma 2.3] Let $p : P \to M$ be a projective δ-cover. If Q is projective and $q : Q \to M$ is an epimorphism, then there exist decompositions $P = A \oplus B$ and $Q = X \oplus Y$ such that*

1. $A \cong X$,
2. $p|_A : A \to M$ *is a projective δ-cover,*
3. $q|_X : X \to M$ *is a projective δ-cover,*
4. B *is a projective semisimple module with* $B \subseteq ker(p)$ *and* $Y \subseteq ker(q)$.

Let A, B be submodules of a module M. We call B a δ-supplement of A in M if $M = A + B$ and $A \cap B \ll_\delta B$. A module M is called δ-supplemented if for any two submodules A and B with $A + B = M$, B contains a δ-supplement of A. More generally, a module M is called weakly δ-supplemented[2] if every submodule of M has a δ-supplement. A module M is called $\oplus$-δ-supplemented if every submodule of M has a δ-supplement that is a direct summand of M. It is clear that every δ-lifting module is $\oplus$-δ-supplemented and every $\oplus$-δ-supplemented module is weakly δ-supplemented.

We first give some characterizations of a projective δ-semiperfect module in terms of δ-lifting and δ-supplemented modules. Note that every δ-semiperfect module is δ-weakly supplemented by [59, Theorem 4.6].

Lemma 5.1.4. *If a module M is δ-semiperfect or δ-lifting, then M is δ-supplemented.*

Proof. Assume M is δ-semiperfect and let $M = A + B$. So M/A has a projective δ-cover $P \xrightarrow{\alpha} M/A \to 0$. Let $\pi : B \to M/A$ be the natural homomorphism. As P is projective, there exists $g : P \to B$ such that $\alpha = \pi \circ g$. Thus, $\pi(g(P)) = \pi(B)$, so $B = g(P) + A \cap B$. Hence, $M = A + g(P)$ and $A \cap g(P) = g(ker(\alpha))$. As $ker(\alpha) \ll_\delta P$,

[2] This notion has been termed as δ-supplemented module in [8], [32] and [59]

$g(ker(\alpha)) \ll_\delta g(P)$, so $A \cap g(P) \ll_\delta g(P)$. Hence $g(P)$ is a δ-supplement of A, contained in B. So M is δ-supplemented.

Suppose M is δ-lifting. Let $M = X + Y$. We show that Y contains a δ-supplement of X in M. Since M is δ-lifting and $Y \subseteq M$, by the definition of δ-lifting module, we can write $Y = Y_1 \oplus Y_2$, where $Y_1 \subseteq^{\oplus} M$ and $Y_2 \ll_\delta M$. Then $M = X + Y_1 + Y_2$. As $Y_2 \ll_\delta M$, we have $M = (X+Y_1) \oplus Y_3$, where $Y_3 \subseteq Y_2$ and Y_3 is projective semisimple. It follows that $M = X+Z$, where $Z = Y_1 \oplus Y_3$. Write $M = Y_1 \oplus Y_1'$, and $X \cap Z = Z_1 \oplus S$ where $Z_1 \subseteq^{\oplus} M$ and $S \ll_\delta M$, and let $\pi_1 : M \to Y_1$ be the natural projection. Then $\pi_1(S) \ll_\delta Y_1$, and $S \subseteq Z = Y_1 \oplus Y_3$. Since Y_3 is a projective semisimple module, $Y_3 \ll_\delta Y_3$, so $S \subseteq \pi_1(S) \oplus Y_3 \ll_\delta Z = Y_1 \oplus Y_3$. Furthermore, write $Z = Z_1 \oplus Z_1'$ with the projection $\pi : Z \to Z_1'$. By modular law, we have $X \cap Z = Z_1 \oplus (X \cap Z_1')$ and, moreover, $X \cap Z_1' = X \cap Z \cap Z_1' \subseteq \pi(X \cap Z) = \pi(Z_1 \oplus S) = \pi(S) \ll_\delta Z_1'$. Therefore, $M = X+Y = X+Z = X+Z_1'$ with $X \cap Z_1' \ll_\delta Z_1' \subseteq Z \subseteq Y$. So Z_1' is a δ-supplement of X contained in Y, and hence M is δ-supplemented. □

Lemma 5.1.5. *The following are equivalent for a projective module P:*

1. *P is a δ-semiperfect module.*
2. *P is δ-lifting.*
3. *P is $\oplus$-δ-supplemented.*
4. *P is δ-supplemented.*
5. *P is weakly δ-supplemented.*

Proof. (1) $\Rightarrow$ (2). Let P be a projective module and $N \subseteq P$. Assume P/N has a projective δ-cover $q : Q \to P/N$. Let $p : P \to P/N$ be the canonical epimorphism. By Lemma 5.1.3, there exists a decomposition $P = X \oplus Y$ such that $p|_X : X \to P/N$

is a projective δ-cover and $Y \subseteq ker(p) = N$. Thus, $X \cap N = ker(p|_X) \ll_\delta X$. Since X is a direct summand of P, $X \cap N$ is δ-small in P by Lemma 5.1.1(4). So P is δ-lifting by the definition.

(2) $\Rightarrow$ (1). Suppose P is δ-lifting and $N \subseteq P$. Then there exists a decomposition $P = P_1 \oplus P_2$ such that $P_1 \subseteq N$ and $N \cap P_2$ is δ-small in P. Let $p : P_2 \to P/N$ be the canonical epimorphism. Then $ker(p) = N \cap P_2$ is δ-small in P and hence δ-small in P_2 by Lemma 5.1.1(4). So $(p; P_2)$ is a projective δ-cover of P/N.

The implication (2) $\Rightarrow$ (3) is trivial, and (1) $\Rightarrow$ (4) follows by Lemma 5.1.4. The implications (3) $\Rightarrow$ (5), and (4) $\Rightarrow$ (5) are obvious. Since P is projective, the implication (5) $\Rightarrow$ (2) holds by [32, Proposition 3.2], which says that a projective module M is δ-lifting if and only if every submodule of M has a δ-supplement. □

The next theorem shows that the investigation of δ-semiperfect modules can essentially be reduced to projective δ-semiperfect modules. Its analog for semiperfect modules was proved by Mares (see Theorem 1.2.18).

Theorem 5.1.6. *Let $P \overset{\alpha}{\to} M \to 0$ be a projective δ-cover of M. The following are equivalent:*

1. *M is δ-semiperfect.*

2. *P is δ-semiperfect.*

Proof. (2) $\Rightarrow$ (1). The implication is clear by the definition of δ-semiperfect modules.

(1) $\Rightarrow$ (2). By Lemma 5.1.5, it suffices to show that P is weakly δ-supplemented. Let $A \subseteq P$, and consider the epimorphism $g = \pi \circ \alpha : P \overset{\alpha}{\to} M \overset{\pi}{\to} M/\alpha(A)$ where π is the natural homomorphism. By hypothesis, $M/\alpha(A)$ has a projective δ-cover. So, by Lemma 5.1.3, we have $P = P_1 \oplus P_2$ such that $g_1 = g|_{P_1} : P_1 \to M/\alpha(A)$ is a projective

δ-cover and $P_2 \subseteq ker(g)$. From $g(P) = g(P_1)$, it follows that $P = P_1 + ker(g)$. As $ker(g) = ker(\pi \circ \alpha) = \alpha^{-1}(ker\pi) = \alpha^{-1}(\alpha(A)) = A + ker(\alpha)$, it follows that $P = P_1 + A + ker(\alpha)$. Since $ker(\alpha)$ is δ-small in P we have $P = (P_1 + A) \oplus Y$ for a projective semisimple submodule Y with $Y \subseteq ker(\alpha)$. Next we verify that $P_1 \oplus Y$ is a δ-supplement of A, i.e., $A \cap (P_1 \oplus Y) \ll_\delta P_1 \oplus Y$. We first note that $A \cap (P_1 \oplus Y) = A \cap P_1$. Indeed, if $a = p_1 + y$ where $a \in A, p_1 \in P_1$ and $y \in Y$, then $y = a - p_1 \in A + P_1$, so $y \in Y \cap (A + P_1) = 0$.

We check that $P_1 \cap (A + ker(\alpha)) \subseteq ker(g_1)$. Let $p_1 = a + k$ where $p_1 \in P_1, a \in A$ and $k \in ker(\alpha)$. Then $g_1(p_1) = g(p_1) = g(a) + g(k) = g(a) = \pi \circ \alpha(a) = 0$. Hence $P_1 \cap (A + ker(\alpha)) \subseteq ker(g_1)$. As $ker(g_1) \ll_\delta P_1$, we have $P_1 \cap (A + ker(\alpha)) \ll_\delta P_1$. Since $A \cap (P_1 \oplus Y) = A \cap P_1$ and $A \cap P_1 \subseteq P_1 \cap (A + ker(\alpha))$, it follows that $A \cap (P_1 \oplus Y) \ll_\delta P_1 \oplus Y$, as required. □

Note that in general a δ-lifting module need not be δ-semiperfect. For instance, the simple module $\mathbb{Z}_p$ (p is a prime) over $\mathbb{Z}$ is δ-lifting, but it does not have a projective δ-cover.

Every nonzero projective module contains a maximal submodule (see [4, Proposition 17.14]). As argued in the proof of [4, Proposition 17.14], one can show that every non-semisimple projective module contains an essential maximal submodule, that is,

Lemma 5.1.7. *Let P be a projective right R-module. Then $P = \delta(P)$ if and only if P is semisimple. In particular, if $P = X \oplus Y$ with $X \subseteq \delta(P)$, then X is semisimple.*

Proof. Let P be a projective semisimple right R-module. Following Lemma 5.1.1(1), we have xR is δ-small in P for all $x \in P$. So $P = \delta(P)$ by Lemma 5.1.2(1). For the other direction, assume $P = \delta(P)$, we will show that P is semisimple. Indeed, take a suitable projective module Q such that $P \oplus Q$ is a free R-module $F = \oplus_{i \in \Omega} e_i R$. By

Lemma 5.1.2(5), $\delta(P) = P\delta(R)$. Let $x = \sum_{i=1}^{n} e_i r_i \in P$. To simplify notion, we shall use integers for elements of the indexing set Ω. Let π be the projection of $F = P \oplus Q$ onto P. Since $P = P\delta(R)$, we can write $\pi(e_i) = \sum_{j=1}^{m} e_j a_{ij}$ where $a_{ij} \in \delta(R)$ and $m \geq n$ is an integer. Then

$$x = \pi(x) = \sum_{i=1}^{n} \pi(e_i) r_i = \sum_{j=1}^{m} e_j \Big(\sum_{i=1}^{n} a_{ij} r_i \Big)$$

Comparing this with $x = \sum_{j=1}^{n} e_j r_j$, we get a system of n equations

$$\begin{pmatrix} 1 - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & 1 - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & 1 - a_{nn} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{5.1}$$

For $a \in R$, we write $\overline{a} = a + \mathrm{soc}(R_R) \in R/\mathrm{soc}(R_R)$. Then we have

$$\begin{pmatrix} \overline{1} - \overline{a_{11}} & -\overline{a_{12}} & \cdots & -\overline{a_{1n}} \\ -\overline{a_{21}} & \overline{1} - \overline{a_{22}} & \cdots & -\overline{a_{2n}} \\ \vdots & \vdots & \vdots & \vdots \\ -\overline{a_{n1}} & -\overline{a_{n2}} & \cdots & \overline{1} - \overline{a_{nn}} \end{pmatrix} \begin{pmatrix} \overline{r_1} \\ \overline{r_2} \\ \vdots \\ \overline{r_n} \end{pmatrix} = \begin{pmatrix} \overline{0} \\ \overline{0} \\ \vdots \\ \overline{0} \end{pmatrix} \tag{5.2}$$

Note that $J\big(R/\mathrm{soc}(R_R)\big) = \delta\big(R\big)/\mathrm{soc}\big(R_R\big)$ by [53, Proposition 2.13]. The coefficient matrix of the linear system (5.2) is $\overline{I_n} - \begin{pmatrix} \overline{a_{11}} & \overline{a_{12}} & \cdots & \overline{a_{1n}} \\ \overline{a_{21}} & \overline{a_{22}} & \cdots & \overline{a_{2n}} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{a_{n1}} & \overline{a_{n2}} & \cdots & \overline{a_{nn}} \end{pmatrix}$, which is in

$1 + J\Big(\mathbb{M}_n\big(R/\mathrm{soc}(R_R)\big)\Big)$. So it is invertiable, and hence $\begin{pmatrix} \overline{r_1} \\ \overline{r_2} \\ \vdots \\ \overline{r_n} \end{pmatrix} = \overline{0}$. It follows that $r_i \in \mathrm{soc}(R_R)$ for $i = 1, \ldots, n$. So $xR \subseteq \sum_{i=1}^{n} e_i \cdot \mathrm{soc}(R_R)$. Note that each $e_i \cdot \mathrm{soc}(R_R)$ is the image of the semisimple module $\mathrm{soc}(R_R)$. Then xR is a semisimple module. Thus P is semisimple. In particular, if $P = X \oplus Y$ with $X \subseteq \delta(P)$, then $X = X \cap \delta(P) = \delta(X)$, so X is semisimple. □

Corollary 5.1.8. *Let P be a projective δ-semiperfect module. Then $\delta(P)$ is δ-small in P.*

Proof. If $P = \delta(P)$, then P is (projective) semisimple by Lemma 5.1.7, so $\delta(P) = P \ll_\delta P$. Otherwise, we have $\delta(P) \neq P$. Since P is δ-lifting (Lemma 5.1.5), we have a decomposition $P = P_1 \oplus P_2$ with $P_1 \subseteq \delta(P)$ and $\delta(P) \cap P_2 \ll_\delta P_2$. The natural projection $\pi : P \to P_1$ sends $\delta(P)$ to $\delta(P_1)$, so $P_1 = \delta(P_1)$. By Lemma 5.1.7, P_1 is semisimple, so $P_1 \ll_\delta P_1$. Thus, $\delta(P) = P_1 \oplus (\delta(P) \cap P_2) \ll_\delta P_1 \oplus P_2 = P$. □

Lemma 5.1.9. *Let $f : M \to N \to 0$ be an epimorphism of modules.*

1. *If $ker(f) \ll M$, then $f^{-1}(X) \ll M$ for any $X \ll N$.*
2. *If $ker(f) \ll_\delta M$, then $f^{-1}(X) \ll_\delta M$ for any $X \ll_\delta N$.*

Proof. (1) Let X be a small submodule of N. Assume $M = f^{-1}(X) + A$ where $A \subseteq M$. Then $N = X + f(A) = f(A)$, so $M = A + ker(f) = A$ (as $ker(f) \ll M$). Thus, $f^{-1}(X)$ is small in M, as required.

(2) Let X be a δ-small submodule of N. Assume $M = f^{-1}(X) + A$ where M/A singular. Then $N = X + f(A)$. As f is an epimorphism, it induces an epimorphism

$M/A \to N/f(A) \to 0$, and it follows that $N/f(A)$ is singular. As X is δ-small in N, we have $N = f(A)$. So $M = A + ker(f)$. As $ker(f) \ll_\delta M$, $M = A \oplus S$ where $S \subseteq ker(f)$ is projective semisimple. Thus, $S \cong M/A$ is singular, and it must be that $S = 0$. So $A = M$, and hence $f^{-1}(X)$ is δ-small in M. □

Lemma 5.1.10. *If $f : M \to N$ is an epimorphism and $ker(f) \ll_\delta \delta(M)$, then $f(\delta(M)) = \delta(N)$.*

Proof. This is by Lemmas 5.1.2(2) and 5.1.9(2). □

Proposition 5.1.11. *If M is a δ-semiperfect module, then $\delta(M) \ll_\delta M$ and $M/\delta(M)$ is semisimple.*

Proof. Let $P \overset{\alpha}{\to} M \to 0$ be a projective δ-cover of M. By Theorem 5.1.6, P is δ-semiperfect, so $\delta(P) \ll_\delta P$ by Corollary 5.1.8. Since $ker(\alpha) \ll_\delta P$, we have $\alpha(\delta(P)) = \delta(M)$ by Lemma 5.1.10. Hence, by Lemma 5.1.1, δ(M) is δ-small in M.

By Lemma 5.1.4, M is δ-supplemented. Let $\delta(M) \subseteq N \subseteq M$. There exists $X \subseteq M$ such that $M = N + X$ and $N \cap X \ll_\delta X$. So $N \cap X \ll_\delta M$. Then $M/\delta(M) = N/\delta(M) + (X + \delta(M))/\delta(M) = N/\delta(M) \oplus (X + \delta(M))/\delta(M)$ because $N \cap (X + \delta(M)) = (N \cap X) + \delta(M) = \delta(M)$. So $M/\delta(M)$ is semisimple. □

A module X is said to has the exchange property if, for any module M and any two direct sum decompositions $M = X \oplus Y = \oplus_{i \in I} A_i$, there exist submodules $A_i' \subseteq A_i$ such that $M = X \oplus (\oplus_{i \in I} A_i')$. If this condition holds for finite sets I, the module X is said to have the finite exchange property (see [11]).

Lemma 5.1.12. *Let $P \overset{\alpha}{\to} M \to 0$ be a projective δ-cover and $M = \oplus_{i \in I} M_i$ be a direct sum decomposition. If, for each $i \in I$, there exist a projective module Q_i and a homomorphism $\theta_i : Q_i \to M_i \to 0$ where $ker(\theta_i) \subseteq \delta(Q_i)$, then the decomposition $M = \oplus_{i \in I} M_i$ can be lifted to a direct sum decomposition of P.*

Proof. Let $Q = \oplus_{i\in I} Q_i$ and $\theta = \oplus_{i\in I}\theta_i$. As Q is projective, there exists $h : Q \to P$ such that $\alpha \circ h = \theta$. Since θ is onto, $P = h(Q) + ker(\alpha)$. As $ker(\alpha)$ is δ-small in P, $P = h(Q) \oplus S$ where $S \subseteq ker(\alpha)$ is semisimple. Thus, $h(Q)$ is projective, so $h : Q \to h(Q)$ splits, i.e., there exists $h' : h(Q) \to Q$ such that $h \circ h' = 1_{h(Q)}$. Hence $Q = ker(h) \oplus Q'$ with $Q' = h'(h(Q))$. Note that $ker(h) \subseteq ker(\theta) = \oplus_{i\in I} ker(\theta_i) \subseteq \oplus_{i\in I}\delta(Q_i) = \delta(Q)$. By Lemma 5.1.7, $ker(h)$ is semisimple, and hence is quasi-injective. By a result of Fuchs [15, Theorem 3] that every quasi-injective module satisfies the exchange property, $ker(h)$ is a module satisfying the exchange property. Therefore, from $Q = ker(h) \oplus Q' = \oplus_{i\in I} Q_i$, we have $Q = ker(h) \oplus (\oplus_{i\in I} A_i)$ where $A_i \subseteq Q_i$ for all $i \in I$. Let $A = \oplus_{i\in I} A_i$. One easily sees that $h|_A : A \to h(A)$ is an isomorphism, so $h(Q) = h(A) = \oplus_{i\in I} h(A_i)$. We check that $M = \oplus_{i\in I} M_i = \theta(Q) = \theta(A) = \sum \theta(A_i)$ with $\theta(A_i) \subseteq M_i$ for all $i \in I$. It follows that $M_i = \theta(A_i) = \pi(h(A_i))$ for all $i \in I$. Take $i_0 \in I$, and let $K = I \setminus \{i_0\}$. Then $P = h(Q) \oplus S = (S \oplus h(A_{i_0})) \oplus (\oplus_{i\in K} h(A_i))$, and $M_{i_0} = \alpha(S \oplus h(A_{i_0}))$ and $M_i = \alpha(h(A_i))$ for all $i \in K$. The claim is verified. □

Corollary 5.1.13. *Let $P \overset{\alpha}{\to} M \to 0$ be a projective δ-cover of M. If M is δ-semiperfect, then every direct sum decomposition of M can be lifted to a direct sum decomposition of P.*

Proof. Let $M = \oplus_{i\in I} M_i$ be a direct sum decomposition. As M is δ-semiperfect, each M_i has a projective δ-cover $Q_i \overset{\theta_i}{\to} M_i \to 0$, so $ker(\theta_i) \subseteq \delta(Q_i)$. Hence, by Lemma 5.1.12, the decomposition $M = \oplus_{i\in I} M_i$ can be lifted to a direct sum decomposition of P. □

The following characterization of a δ-semiperfect module is of interest both for the structure of such a module and for determining whether a given module is δ-semiperfect.

Theorem 5.1.14. *Let M be a module and $\pi : M \to M/\delta(M)$ be the natural homomorphism. The following are equivalent:*

1. *M is δ-semiperfect.*

2. *$\delta(M)$ is δ-small in M, $M/\delta(M)$ is semisimple, and there is a projective δ-cover $P \stackrel{\alpha}{\to} M \to 0$ such that every direct summand of $M/\delta(M)$ is the image of a direct summand of P under $\pi \circ \alpha$.*

3. *$\delta(M)$ is δ-small in M, $M/\delta(M)$ is semisimple, and there is a projective δ-cover $P \stackrel{\alpha}{\to} M \to 0$ such that every direct sum decomposition of $M/\delta(M)$ can be lifted to a direct sum decomposition of P via $\pi \circ \alpha$.*

Proof. (1) $\Rightarrow$ (3). By Proposition 5.1.11, $\delta(M)$ is δ-small in M and $M/\delta(M)$ is semisimple. Let $P \stackrel{\alpha}{\to} M \to 0$ be a projective δ-cover. It remains to show that every direct sum decomposition of $M/\delta(M)$ can be lifted to a direct sum decomposition of P. As $M/\delta(M)$ is δ-semiperfect (being an image of M), by Corollary 5.1.13, it suffices to show that $P \stackrel{\pi\circ\alpha}{\to} M/\delta(M) \to 0$ is a projective δ-cover. That is, $ker(\pi \circ \alpha)$ is δ-small in P. Note that $ker(\pi \circ \alpha) = \alpha^{-1}(\delta(M))$. As $\delta(M) \ll_\delta M$ (see Proposition 5.1.11) and $ker(\alpha) \ll_\delta P$, we have $\alpha^{-1}(\delta(M)) \ll_\delta P$ by Lemma 5.1.9. This is, $ker(\pi \circ \alpha)$ is δ-small in P.

(3) $\Rightarrow$ (2). The implication is clear.

(2) $\Rightarrow$ (1). By Lemma 5.1.5 and Theorem 5.1.6, it suffices to show that P is weakly δ-supplemented. By Lemma 5.1.10, $\alpha(\delta(P)) = \delta(M)$. Moreover, as seen in the proof of "(1) $\Rightarrow$ (3)", $ker(\pi \circ \alpha) = \alpha^{-1}(\delta(M))$ is δ-small in P. It follows that $\delta(P) = \alpha^{-1}(\delta(M))$. So

$$\bar{\alpha} : P/\delta(P) \to M/\delta(M);\ x + \delta(P) \ \mapsto\ \alpha(x) + \delta(M)$$

is an isomorphism. In particular, $P/\delta(P)$ is semisimple.

Let $X \subseteq P$. We have $P/\delta(P) = (X+\delta(P))/\delta(P) \oplus Y/\delta(P)$. So, $\alpha(Y)/\delta(M) = \bar{\alpha}(Y/\delta(P))$ is a direct summand of $M/\delta(M)$. By hypothesis, there exists a direct summand $B \subseteq^{\oplus} P$ such that $\alpha(Y)/\delta(M) = (\pi \circ \alpha)(B) = (\alpha(B)+\delta(M))/\delta(M) = \alpha(B+\delta(P))/\delta(M)$. This gives $\alpha(Y) = \alpha(B+\delta(P))$, implying $Y = B+\delta(P)$. Thus, $P = X+Y+\delta(P) = X+B+\delta(P)$. Since $\delta(P) \ll_\delta P$ (as seen above), there exists a projective semisimple submodule H of $\delta(P)$ such that $P = (X+B)\oplus H = X+(B\oplus H)$. We show that $B \oplus H$ is a δ-supplement of X in P, i.e., $X \cap (B \oplus H) \ll_\delta B \oplus H$. Indeed, from the decomposition $P/\delta(P) = (X+\delta(P))/\delta(P) \oplus Y/\delta(P)$, we have $(X+\delta(P)) \cap Y \subseteq \delta(P)$, implying that $X \cap B \subseteq X \cap (B+\delta(P)) = X \cap Y \subseteq \delta(P)$. Note that $X \cap (B \oplus H) = X \cap B$, because $x = b+h$ with $x \in X, b \in B$ and $h \in H$ gives that $h = x-b \in H \cap (X+B) = 0$. So we have $X \cap (B \oplus H) \subseteq \delta(P) \ll_\delta P$. Since B is a direct summand of P, it follows that $X \cap (B \oplus H) = X \cap B \ll_\delta B$. Thus $X \cap (B \oplus H) = X \cap B \ll_\delta B \oplus H$, as required. □

Corollary 5.1.15. *The following are equivalent for a projective module P:*

1. *P is δ-semiperfect.*

2. *$\delta(P)$ is δ-small in P, $P/\delta(P)$ is semisimple, and every direct summand of $P/\delta(P)$ is the image of a direct summand of P under the natural homomorphism $P \to P/\delta(P)$.*

3. *$\delta(P)$ is δ-small in P, $P/\delta(P)$ is semisimple, and every direct sum decomposition of $P/\delta(P)$ can be lifted to a direct sum decomposition of P.*

For a ring R, it is always true that $\delta(R)$ is δ-small in R_R. So Corollary 5.1.15 naturally extends the structure of a δ-semiperfect ring which, contained in [63, Theorem 3.6], says that a ring R is δ-semiperfect if and only if $R/\delta(R)$ is semisimple and idempotents lift modulo $\delta(R)$.

Another characterization of a projective δ-semiperfect module is given below.

Theorem 5.1.16. *Let P be a projective module. Then P is δ-semiperfect if and only if*

1. *every proper submodule of P is contained in a maximal submodule of P, and*
2. *every simple factor module of P has a projective δ-cover.*

Proof. ($\Rightarrow$). Let A be a proper submodule of P. Since P is δ-lifting, there exists a decomposition $P = P_1 \oplus P_2$ such that $P_1 \subseteq A$ and $A \cap P_2 \ll_\delta P$. So, $A = P_1 \oplus (A \cap P_2)$.

If $\delta(P_2) = P_2$ then P_2 is semisimple. As a proper submodule of P_2, $A \cap P_2$ is contained in a maximal submodule of P_2. So A is contained in a maximal submodule of P.

If $\delta(P_2) \neq P_2$, then P_2 has an essential maximal submodule N. Let $f : P \to P/A$ be the natural homomorphism. Then $f|_{P_2} : P_2 \to P/A$ is a projective δ-cover and $ker(f|_{P_2}) \subseteq \delta(P_2) \subseteq N$. Thus, $f(N)$ is a maximal submodule of P/A, so A is contained in a maximal submodule of P. So (1) holds. We have (2) by the definition of a δ-semiperfect module.

($\Leftarrow$). By Lemma 5.1.2, $\delta(P)$ is δ-small in P. Next we show that $P/\delta(P)$ is semisimple. By Lemma 5.1.7, we can assume that $\delta(P) \neq P$. Let $\pi : P \to P/\delta(P)$ be the natural epimorphism. Assume that $P/\delta(P)$ is not semisimple. Then $P/\delta(P)$ has a proper essential submodule K, so $\pi^{-1}(K)$ is a proper essential submodule of P. By hypothesis, $\pi^{-1}(K)$ is contained in a maximal submodule A of P and P/A has a projective δ-cover. By Lemma 5.1.3, we have $P = X \oplus Y$ such that $\pi|_X : X \to P/A$ is a projective δ-cover and $Y \subseteq ker(\pi|_X)$. So $X \cap A = ker(\pi|_X) \subseteq \delta(X) \subseteq \delta(P)$ and $Y \subseteq \delta(P)$. Therefore, $P/\delta(P) = (A + X)/\delta(P) = A/\delta(P) \oplus (X + \delta(P))/\delta(P)$. As $A/\delta(P)$ is essential in $P/\delta(P)$, $X \subseteq \delta(P)$. It follows that $P = \delta(P)$. This is a contradiction.

To finish the proof, by Corollary 5.1.15 it suffices to show that every direct sum decomposition of $P/\delta(P)$ can be lifted to a direct sum decomposition of P. Let $P/\delta(P) = \oplus_{i\in I}C_i$ be a direct sum decomposition. As $P/\delta(P)$ is semisimple, each C_i is semisimple, so write $C_i = \oplus_{j\in J}S_j$ as a direct sum of simples. As a simple factor of P, S_j has a projective δ-cover $P_j \xrightarrow{f_j} S_j \to 0$, so $ker(f_j) \subseteq \delta(P_j)$. Hence, with $Q_i = \oplus_{j\in J}P_j$ and $\theta = \oplus_{j\in J}f_j$, we have $Q_i \xrightarrow{\theta_i} C_i \to 0$ with Q_i projective and $ker(\theta_i) \subseteq \delta(Q_i)$. By Lemma 5.1.12, the decomposition $P/\delta(P) = \oplus_{i\in I}C_i$ can be lifted to a direct sum decomposition of P. □

Theorem 5.1.16 has the following self-strengthening.

Theorem 5.1.17. *A module M is δ-semiperfect if and only if*

1. *M has a projective δ-cover, and*

2. *every proper submodule of M is contained in a maximal submodule of M, and*

3. *every simple factor module of M has a projective δ-cover.*

Proof. ($\Rightarrow$). Suppose that M is δ-semiperfect. Then M has a projective δ-cover, say $P \xrightarrow{\alpha} M \to 0$. By Theorem 5.1.6, P is δ-semiperfect. So, by Theorem 5.1.16, every simple factor module of P has a projective δ-cover and every proper submodule of P is contained in a maximal submodule of P. It follows that every simple factor module of M has a projective δ-cover. If X is a proper submodule of M, then $\alpha^{-1}(X)$ is a proper submodule of P, and so $\alpha^{-1}(X)$ is contained in a maximal submodule Q of P. Note that $ker(\alpha) \subseteq \alpha^{-1}(X)$. We deduce that $\alpha(Q)$ is a maximal submodule of M and $X \subseteq \alpha(Q)$.

($\Leftarrow$). Let $P \xrightarrow{\alpha} M \to 0$ be a projective δ-cover of M. By Theorem 5.1.6, it suffices to show that P is δ-semiperfect. We will verify this by showing that P satisfies (1) and

(2) of Theorem 5.1.16. Let A be a proper submodule of P. If $\alpha(A) \neq M$, then $\alpha(A)$ is contained in a maximal submodule N of M. So $\alpha^{-1}(N)$ is a maximal submodule of P and $A \subseteq \alpha^{-1}(N)$. If $\alpha(A) = M$, then $P = A + ker(\alpha)$. As $ker(\alpha) \ll_\delta P$, $P = A \oplus B$ where $B \subseteq ker(\alpha)$ is projective semisimple. It follows that A is contained in a maximal submodule of P.

Let X be a maximal submodule of P. If $ker(\alpha) \subseteq X$, then P/X is a simple factor module of M, so it has a projective δ-cover. If $ker(\alpha) \nsubseteq X$, then $P = X + ker(\alpha)$, so $P = X \oplus S$ where $S \subseteq ker(\alpha)$ is projective simple. It follows that P/X is projective, which has a projective δ-cover. Hence, by Theorem 5.1.16, P is δ-semiperfect. □

In [38, Theorem 5.2], Mares proved that a direct sum $P = \oplus_{i \in I} P_i$ of projective semiperfect modules P_i is semiperfect if and only if $rad(P)$ is small in P. Here we have

Theorem 5.1.18. *A direct sum $M = \oplus_{i \in I} M_i$ of modules is δ-semiperfect if and only if every M_i is δ-semiperfect, M has a projective δ-cover and $\delta(M) \ll_\delta M$.*

Proof. The necessity is clear. To prove the sufficiency, let $P \overset{\alpha}{\to} M \to 0$ be a projective δ-cover. By Theorem 5.1.6, it suffices to show that P is δ-semiperfect.

Let $\pi : M \to M/\delta(M)$ be the natural homomorphism. As $\delta(M) \ll_\delta M$, $\alpha^{-1}(\delta(M)) \ll_\delta P$ by Lemma 5.1.9. As $ker(\pi \circ \alpha) = \alpha^{-1}(\delta(M))$, $P \overset{\pi\circ\alpha}{\to} M/\delta(M) \to 0$ is a projective δ-cover of $M/\delta(M)$. Moreover, $\alpha(\delta(P)) = \delta(M)$ implies that $\delta(P) \subseteq ker(\pi \circ \alpha)$, so $\delta(P) = ker(\pi \circ \alpha)$ is δ-small in P and $P/\delta(P) \cong M/\delta(M) \cong \oplus_{i \in I} M_i/\delta(M_i)$ is semisimple, as each M_i is δ-semiperfect.

To finish the proof, by Corollary 5.1.15 it suffices to show that every direct sum decomposition of $P/\delta(P)$ can be lifted to a direct sum decomposition of P. The argument is almost the same as the one in the last part of the proof of Theorem 5.1.16.

Let $P/\delta(P) = \oplus_{i \in I} C_i$ be a direct sum decomposition. As $P/\delta(P)$ is semisimple, each C_i is semisimple, so write $C_i = \oplus_{j \in J} S_j$ as a direct sum of simples. Each S_j must be a factor module of M_k for some $k \in I$, so it has a projective δ-cover $P_j \xrightarrow{f_j} S_j \to 0$, with $ker(f_j) \subseteq \delta(P_j)$. Hence, with $Q_i = \oplus_{j \in J} P_j$ and $\theta_i = \oplus_{j \in J} f_j$, we have $Q_i \xrightarrow{\theta_i} C_i \to 0$ with Q_i projective and $ker(\theta_i) \subseteq \delta(Q_i)$. By Lemma 5.1.12, the decomposition $P/\delta(P) = \oplus_{i \in I} C_i$ can be lifted to a direct sum decomposition of P. □

Corollary 5.1.19. *A direct sum $M = \oplus_{i \in I} M_i$ of projective δ-semiperfect modules M_i is δ-semiperfect if and only if $\delta(M) \ll_\delta M$.*

The proof of Theorem 5.1.14 can be adapted to generalize Mares Theorem [38, Theorem 5.1] as follows.

Theorem 5.1.20. *Let M be a module and $\pi : M \to M/rad(M)$ be the natural homomorphism. The following are equivalent:*

1. *M is semiperfect.*
2. *$rad(M)$ is small in M, $M/rad(M)$ is semisimple, and there is a projective cover $P \xrightarrow{\alpha} M \to 0$ such that every direct summand of $M/rad(M)$ is the image of a direct summand of P under $\pi \circ \alpha$.*
3. *$rad(M)$ is small in M, $M/rad(M)$ is semisimple, and there is a projective cover $P \xrightarrow{\alpha} M \to 0$ such that every direct sum decomposition of $M/rad(M)$ can be lifted to a direct sum decomposition of P via $\pi \circ \alpha$.*

The proof of Theorem 5.1.17 can be adapted to generalize Nicholson's Theorem [47, Theorem] as follows.

Theorem 5.1.21. *A module M is semiperfect if and only if*

1. *M has a projective cover, and*

2. *every proper submodule of M is contained in a maximal submodule of M, and*

3. *every simple factor module of M has a projective cover.*

The proof of Theorem 5.1.18 can be adapted to generalize Mares' result [38, Theorem 5.2] as follows.

Theorem 5.1.22. *A direct sum $M = \oplus_{i \in I} M_i$ is semiperfect if and only if every M_i is semiperfect, M has a projective cover and $rad(M) \ll M$.*

Note that a δ-semiperfect module need not be projective. Indeed, if R is a right δ-semiperfect ring that is not semisimple Artinian, then R/I is not projective for some right ideal I of R. But R/I is δ-semiperfect.

Recall that a module M is local if M has a unique maximal submodule which is also a small submodule of M. It is known that a ring R is semiperfect if and only if R_R is a direct sum of local modules. It is also known that every projective semiperfect module P has a direct sum decomposition of local modules (which is what is lifted from the direct sum decomposition of the semisimple module $P/rad(P)$ as a direct sum of simple modules). Combining this with Mares' result [38, Theorem 5.2] for direct sum of projective semiperfect modules, we have that a module P is projective semiperfect if and only if P is a direct sum of projective local modules and $rad(P)$ is small in P. An analogous result for projective δ-semiperfect modules can be reported. In [62], an epimorphism $f : P \to M$ is called a cover of module M if $ker(f) \ll P$. Noting that a module M is local if and only if M is a cover of a simple module, we call a module M a δ-local[3] module if M is a δ-cover of a simple module, i.e., there is an

[3]This terminology has been used differently in [8]

epimorphism from M to a simple module with δ-small kernel. Then every projective δ-local module is δ-semiperfect, and we have:

Corollary 5.1.23. *A module P is projective δ-semiperfect if and only if P is a direct sum of projective δ-local modules with $\delta(P) \ll_\delta P$.*

Proof. The sufficiency follows from Corollary 5.1.19. For the necessity, P is a projective δ-cover of $P/\delta(P)$ via the natural homomorphism π. Being semisimple, $P/\delta(P)$ has a direct sum decomposition as a direct sum of simple modules. By Corollary 5.1.15, this direct sum decomposition can be lifted to a direct sum decomposition of P, which is a direct sum of projective δ-local modules. □

5.2 δ-semiperfect modules that are δ-lifting

Recall that a module M is lifting if for any submodule N of M, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$ and $N \cap M_2$ is small in M. It is proved in Theorem 4.1.6 that a semiperfect module is lifting if and only if the module has a projective cover preserving direct summands. In this section, the δ-version of this result is proved. We need the following characterization of a δ-lifting module. Note that it is known that a module is lifting if and only if it is supplemented and every supplement submodule is a direct summand (see Lemma 4.1.1(3)).

Proposition 5.2.1. *The following are equivalent for a module M:*

1. *M is δ-lifting.*

2. *M is δ-supplemented and every δ-supplement submodule of M is a direct sum of a direct summand of M and a projective semisimple module.*

Proof. (1) $\Rightarrow$ (2). By Lemma 5.1.4, M is δ-supplemented. Let P be a δ-supplement submodule of M. There exists $K \subseteq M$ such that $M = K+P$ with $P \cap K \ll_\delta P$. Write $P = L \oplus T$ with $L \subseteq^\oplus M$ and $T \ll_\delta M$. It follows that $M = (K+L)+T = (K+L)\oplus T'$, where $T' \subseteq T$ and T' is projective semisimple. Then $M = K + (L \oplus T')$. By modular law, we have $P = P \cap M = P \cap K + (L \oplus T')$. As $P \cap K \ll_\delta P$, we have $P = P' \oplus L \oplus T'$, where $P' \subseteq (P \cap K)$ and P' is projective semisimple. Hence $P = L \oplus (P' \oplus T')$, where $L \subseteq^\oplus M$ and $P' \oplus T'$ is projective semisimple.

(2) $\Rightarrow$ (1). Let $A \subseteq M$. By hypothesis, there exists $B \subseteq M$ such that $M = A+B$ and $A \cap B \ll_\delta B$, and further A contains a δ-supplement M_1 of B. That is, $M = M_1 + B$ where $M_1 \subseteq A$ and $B \cap M_1 \ll_\delta M_1$. Again by hypothesis, we have $M_1 = D \oplus D'$ where $D \subseteq^\oplus M$ and D' is projective semisimple. Write $M = D \oplus M_2$ with the projection $\pi : M \to M_2$. By modular law, we have $A = A \cap (D \oplus M_2) = D \oplus (A \cap M_2)$, and $A = A \cap (M_1 + B) = M_1 + A \cap B$. Hence, $A \cap M_2 = \pi(A) = \pi(M_1) + \pi(A \cap B) = \pi(D') + \pi(A \cap B)$. Being projective semisimple, D' is δ-small in M. Moreover, $A \cap B \ll_\delta M$ (as $A \cap B \ll_\delta B$). Thus we see that $\pi(D')$ and $\pi(A \cap B)$ are δ-small submodules in M_2. So $A = D \oplus (A \cap M_2)$ where $D \subseteq^\oplus M$ and $A \cap M_2 = \pi(D') + \pi(A \cap B)$ is δ-small in M_2. □

Corollary 5.2.2. *Let A be a submodule of M. If $A = N \oplus S$, where $N \subseteq^\oplus M$ and S is projective semisimple submodule, then A is a δ-supplement submodule of M.*

Proof. Write $M = N \oplus N'$ with the projection $\pi : M \to N'$. We show that A is a δ-supplement of N'. Indeed, we have $M = A + N'$ and $A \cap N' \subseteq \pi(A) = \pi(N) + \pi(S) = \pi(S)$, which is a projective semisimple module. So $A \cap N'$ is projective semisimple, and hence $A \cap N' \ll_\delta A$. □

Thus, for a δ-lifting module M, a submodule of M is a δ-supplement if and only if it is a direct sum of a direct summand of M and a projective semisimple submodule.

Lemma 5.2.3. *Let M be a δ-semiperfect module with a projective δ-cover $P \overset{\alpha}{\to} M \to 0$. If N is a δ-supplement submodule of M, then $N = \alpha(X) \oplus N'$ where $X \subseteq^{\oplus} P$ and $N' \subseteq M$ is projective semisimple.*

Proof. Let N be a δ-supplement submodule of M. Then there exists $A \subseteq M$ such that $M = A + N$ and $A \cap N \ll_\delta N$. Thus, $P = \alpha^{-1}(A) + \alpha^{-1}(N)$, and $\alpha^{-1}(A \cap N) = \alpha^{-1}(A) \cap \alpha^{-1}(N)$ is δ-small in P by Lemma 5.1.9. By Lemma 5.1.5 and Theorem 5.1.6, P is δ-lifting. By Proposition 5.2.1, $\alpha^{-1}(N) = X \oplus Y$ where $X \subseteq^{\oplus} P$ and $Y \ll_\delta P$. Then $P = \alpha^{-1}(A) + X + Y = (\alpha^{-1}(A) + X) \oplus Y'$ where $Y' \subseteq Y$ is projective semisimple. So $\alpha^{-1}(N) = (X + Y') + \alpha^{-1}(A) \cap \alpha^{-1}(N) = (X + Y') + \alpha^{-1}(A \cap N)$. Hence, $N = \alpha(\alpha^{-1}(N)) = \alpha(X) + \alpha(Y') + A \cap N$. As $A \cap N \ll_\delta N$, by Lemma 5.1.1 we have $N = (\alpha(X) + \alpha(Y')) \oplus B$ where $B \subseteq A \cap N$ is projective semisimple. So $\alpha(Y') + B$ is projective semisimple, and hence $N = \alpha(X) \oplus Z$ for a projective semisimple submodule Z of N. □

Lemma 5.2.4. *Let M be a module with a projective δ-cover $P \overset{\alpha}{\to} M \to 0$. If M is δ-lifting, then $\alpha(X)$ is a direct sum of a direct summand of M and a projective semisimple module, for all $X \subseteq^{\oplus} P$.*

Proof. Suppose $P = P_1 \oplus P_2$. Then $M = \alpha(P_1) + \alpha(P_2)$. Since M is δ-lifting, by Proposition 5.2.1, there exist direct summands M_1, M_2 of M and δ-small submodules S_1, S_2 in M such that $\alpha(P_1) = M_1 \oplus S_1$ and $\alpha(P_2) = M_2 \oplus S_2$. Thus $M = (M_1 + M_2 + S_2) + S_1 = (M_1 + M_2 + S_2) \oplus Z_1$, where $Z_1 \subseteq S_1$ is projective semisimple. Moreover, $M = ((M_1 + M_2) \oplus Z_1) + S_2 = ((M_1 + M_2) \oplus Z_1) \oplus Z_2$, where $Z_2 \subseteq S_2$ is projective semisimple. So $M = (M_1 \oplus Z1) + (M_2 \oplus Z_2)$. Next we show that $\alpha(P_1) = M_1 \oplus Z_1'$ where Z_1' is projective semisimple.

Note that $M_i \oplus Z_i = \alpha(\alpha^{-1}(M_i \oplus Z_i))$, so $M_i \oplus Z_i = \alpha(\alpha^{-1}(M_i \oplus Z_i)) \cap \alpha(P_i) \supseteq \alpha(\alpha^{-1}(M_i \oplus Z_i) \cap P_i)$. If $x \in M_i \oplus Z_i$, write $x = \alpha(y)$ with $y \in P_i$. Then $y \in \alpha^{-1}(M_i \oplus Z_i) \cap P_i$, so $x = \alpha(y) \in \alpha(\alpha^{-1}(M_i \oplus Z_i) \cap P_i)$. Hence, $M_i \oplus Z_i = \alpha(\alpha^{-1}(M_i \oplus Z_i) \cap P_i)$, i.e., $\alpha(\alpha^{-1}(M_i \oplus Z_i)) = \alpha(Q_i)$ with $Q_i = \alpha^{-1}(M_i \oplus Z_i) \cap P_i$. It follows that $\alpha^{-1}(M_i \oplus Z_i) = Q_i + ker(\alpha)$. So, from $M = (M_1 \oplus Z_1) + (M_2 \oplus Z_2)$, we obtain $P = \alpha^{-1}(M_1 \oplus Z_1) + \alpha^{-1}(M_2 \oplus Z_2) = (Q_1 \oplus Q_2) + ker(\alpha)$. As $ker(\alpha) \ll_\delta P$, $P = Q_1 \oplus Q_2 \oplus Z$, where $Z \subseteq ker(\alpha)$ is projective semisimple. Then $P_1 = Q_1 \oplus (P_1 \cap (Q_2 \oplus Z))$.

We verify that $P_1 \cap (Q_2 \oplus Z)$ is projective semisimple. To see this, let $p_i : P \to P_i$ be the natural projection. Then $p_i(Z)$ is semisimple, as Z is semisimple. If $x \in P_1 \cap (Q_2 \oplus Z)$, write $x = y + z$ where $y \in Q_2$ and $z \in Z$. Then $z = x - y$ with $x \in P_1$ and $y \in P_2$. Thus $x \in p_1(Z)$, and so xR is semisimple. Hence $P_1 \cap (Q_2 \oplus Z)$ is projective semisimple, so $\alpha(P_1 \cap (Q_2 \oplus Z))$ is projective semisimple.

As $P_1 = Q_1 \oplus (P_1 \cap (Q_2 \oplus Z))$, we have $M_1 \oplus S_1 = \alpha(P_1) = \alpha(Q_1) + \alpha(P_1 \cap (Q_2 \oplus Z)) \subseteq M_1 + Z_1 + \alpha(P_1 \cap (Q_2 \oplus Z)) = M_1 + N$, where $N = Z_1 + \alpha(P_1 \cap (Q_2 \oplus Z))$ is projective semisimple. So $\alpha(P_1) = M_1 + \alpha(P_1) \cap N$. As $\alpha(P_1) \cap N$ is projective semisimple, $\alpha(P_1) = M_1 \oplus Z_1'$ where Z_1' is projective semisimple. □

Theorem 5.2.5. *Let M be a δ-semiperfect module. The following are equivalent:*

1. *M is δ-lifting.*

2. *M has a projective δ-cover $P \xrightarrow{\alpha} M \to 0$ such that $\alpha(X)$ is a direct sum of a direct summand of M and a projective semisimple submodule of M for any $X \subseteq^{\oplus} P$.*

3. *M has a projective δ-cover $P \xrightarrow{\alpha} M \to 0$ such that $\alpha(X)$ is a direct sum of a direct summand of M and a projective semisimple submodule of M for any δ-supplement submodule X of P.*

Proof. (1) $\Rightarrow$ (3). Let $P \overset{\alpha}{\to} M \to 0$ be a projective δ-cover for M, and let Z be a δ-supplement submodule of P. By Lemma 5.1.5 and Theorem 5.1.6, P is δ-lifting. So, by Proposition 5.2.1, $Z = X \oplus S$ where $X \subseteq^{\oplus} P$ and S is projective semisimple. Then $\alpha(Z) = \alpha(X) + \alpha(S)$. By Lemma 5.2.4, $\alpha(X) = A \oplus B$ where $A \subseteq^{\oplus} M$ and B is projective semisimple. So $\alpha(Z) = A + (B + \alpha(S))$. As $B + \alpha(S)$ is projective semisimple, $\alpha(Z) = A \oplus C$ where C is projective semisimple.

(3) $\Rightarrow$ (2). The implication is clear.

(2) $\Rightarrow$ (1). Since M is δ-semiperfect, M is δ-supplemented by Lemma 5.1.4. Let N be a δ-supplement submodule of M. By Lemma 5.2.3, $N = \alpha(X) \oplus N_1$ where $X \subseteq^{\oplus} P$ and $N_1 \subseteq M$ is projective semisimple. By hypothesis, $\alpha(X) = A \oplus N_2$ where $A \subseteq^{\oplus} M$ and N_2 is projective semisimple. So, $N = A \oplus (N_1 \oplus N_2)$ is a direct sum of a direct summand of M and a projective semisimple submodule of M. Hence, by Proposition 5.2.1, M is δ-lifting. □

Corollary 5.2.6. *The following are equivalent for a ring R:*

1. *Every cyclic right R-module is δ-lifting.*
2. *R is a right δ-semiperfect ring and every cyclic right R-module M has a projective δ-cover $P \overset{\alpha}{\to} M \to 0$ such that $\alpha(X)$ is a direct sum of a direct summand of M and a projective semisimple submodule of M for any $X \subseteq^{\oplus} P$.*

Proof. Note that either (1) or (2) implies that R is right δ-semiperfect and so every cyclic right R-module is δ-semiperfect. Thus, the equivalence follows from Theorem 5.2.5. □

Recall that a ring R is semilocal if $R/J(R)$ is semisimple artinian. A module M is called semilocal if $M/rad(M)$ is semisimple. In [8], Büyükasik and Lomp proved that a ring R is semiperfect if and only if it is semilocal, δ-semiperfect.

Proposition 5.2.7. *The following are equivalent for a projective module P:*

1. *P is semiperfect.*
2. *P is δ-semiperfect and $P/rad(P)$ is semisimple.*
3. *P is δ-semiperfect and $\delta(P) = P_1 \oplus rad(P)$, where $P_1 \subseteq^{\oplus} P$.*

Proof. (1) $\Rightarrow$ (3). It is well-known that every projective semiperfect module is lifting. Thus, there is a direct sum decomposition $P = P_1 \oplus P_2$ such that $P_1 \subseteq \delta(P)$ and $\delta(P) \cap P_2 \ll P$. As $rad(P) \subseteq \delta(P)$, it follows that $\delta(P) = P_1 + rad(P)$. But P_1 is (projective) semisimple by Lemma 5.1.7, so $P_1 \cap rad(P) = 0$. Hence $\delta(P) = P_1 \oplus rad(P)$.

(3) $\Rightarrow$ (2). It suffices to show that $P/rad(P)$ is semisimple. Write $P = P_1 \oplus P_2$. Then $rad(P) = rad(P_1) \oplus rad(P_2)$ and $\delta(P) = \delta(P_1) \oplus \delta(P_2)$. So $\delta(P) = P_1 \oplus rad(P) = P_1 + rad(P_1) + rad(P_2) = P_1 \oplus rad(P_2)$. It follows that $\delta(P_2) = rad(P_2)$ and $P_1 = \delta(P_1)$. So P_1 is (projective) semisimple by Lemma 5.1.7, and hence $rad(P) = rad(P_1) \oplus rad(P_2) = rad(P_2)$. As P_2 is δ-semiperfect, $P_2/\delta(P_2)$ is semisimple by Proposition 5.1.11. Hence, $P/rad(P) \cong P_1 \oplus P_2/rad(P_2) = P_1 \oplus P_2/\delta(P_2)$ is semisimple.

(2) $\Rightarrow$ (1). By Theorem 1.2.18, a projective module is semiperfect if and only if it is weakly supplemented, i.e., every submodule has a supplement. By Lemma 5.1.5, a projective module is δ-semiperfect if and only if it is weakly δ-supplemented. It is shown in [8, Proposition 4.2] that a projective semilocal, weakly δ-supplemented module with small radical is weakly supplemented. Hence, to show the implication, it suffices to show that $rad(P)$ is small in P. Assume that $P = X + rad(P)$ for a submodule X of P. As $rad(P) \subseteq \delta(P)$ and $\delta(P) \ll_{\delta} P$ (by Corollary 5.1.8),

$rad(P) \ll_\delta P$. So, by Lemma 5.1.1, $P = X \oplus Y$ where $Y \subseteq rad(P)$ is projective semisimple. It follows that $Y = 0$. Hence, $P = X$, and so $rad(P) \ll P$. □

Remark 5.2.8. *The assumption that P is projective in Proposition 5.2.7 can not be removed. Indeed, if F is a field and $Q = \prod_{i=1}^{\infty} F_i$ is the direct product of rings with each $F_i = F$, let $R = \langle \oplus_i F_i, 1_Q \rangle$ be the subring of Q generated by $\oplus_i F_i$ and 1_Q, and let $M_R = R/I$ where $I = \oplus_i F_i$. Then M_R is a singular simple module, so $\delta(M) = rad(M) = 0$. As $\delta(R_R) = I$, R_R is a projective δ-cover of M, so M_R is δ-semiperfect. Note that M_R and $(F_i)_R$ $(i = 1, 2, \ldots)$ are all the simple R-modules, and each $(F_i)_R$ is a projective cover of itself. As R is not semiperfect, some simple R-module does not have a projective cover. So we deduce that M_R does not have a projective cover. Hence M_R is not semiperfect.*

Let P be a projective semiperfect module and $S = \text{End}(P)$. Mares in [38] proved that $J(S) = \{f \in \text{End}(P) : f(P) \ll P\}$, idempotents of $S/J(S)$ can be lifted to S , and $S/J(S) \cong \text{End}(P/rad(P))$ with $P/rad(P)$ semisimple, so $\text{End}(P/rad(P))$ is a (von Neumann) regular ring.

For a projective δ-semiperfect module P with $S = \text{End}(P)$, one can show that $\nabla_\delta := \{f \in \text{End}(P) : f(P) \ll_\delta P\}$ is an ideal of S, idempotents of S/∇_δ can be lifted to S, and $S/\nabla_\delta \cong \text{End}(P/\delta(P))$ with $P/\delta(P)$ semisimple, so $\text{End}(P/\delta(P))$ is a (von Neumann) regular ring. But we do not know whether $\delta(S)$ coincides with ∇_δ.

Remark 5.2.9. *The material in this chapter is taken from [44].*

Questions for further consideration

Recall that a right R-module M is said to satisfy the summand sum property if the sum of two direct summands of M is again a direct summand of M. A module with the summand sum property is called an SSP-module. It is clear that every SSP-module is a $C3$-module. A ring R is an SSP-ring if the module R_R is a SSP-module. This is a left-right symmetric condition for rings. However, there exists a ring R such that R_R is a $C3$-module but ${}_RR$ is not. We have been unable to show that $CC3$-ring (respectively $CD3$-ring) is a left-right symmetric concept.

Question 1. *Is a right $CC3$-ring (resp. $CD3$-ring) a left $CC3$-ring (resp. $CD3$-ring)?*

Recall that a module M is called a $C2$-module if any submodule M isomorphic to a direct summand of M is itself a direct summand of M. Because of Theorem 2.1.27, the structure of a regular right self-injective ring whose cyclic modules are $C2$-modules relies on the answer to the next question.

Question 2. *Let R be a strongly regular ring. Is every cyclic right module over $\mathbb{M}_2(R)$ a $C2$-module?*

The next question is motivated by Corollary 2.1.21, which is shown that a module M is semisimple if and only if every 3-generated module in $\sigma[M]$ is a $C3$-module.

Question 3. *Characterize the module M whose every factor module M/N is a $C3$-module.*

Following Theorem 2.1.18, a semiperfect ring has all cyclics $C3$-modules if and only if it is a direct product of a semisimple artinian ring and finitely many local rings. However, we only have a sufficient condition for a semiperfect ring to be a $CD3$-ring in Theorem 3.1.17. Thus, we wonder

Question 4. *What is the structure of semiperfect $CD3$-rings?*

Following [63], a submodule N of M is called δ-small in M (written N $\ll_\delta M$) if $N + X \neq M$ for any proper submodule X of M with M/X singular. A module M is called δ-lifting if for any submodule N of M, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$ and $N \cap M_2$ is δ-small in M (see [32, p.54]). Equivalently, module M is δ-lifting if and only if every submodule N of M can be written as $N = M_1 \oplus S$ with $M_1 \subseteq^\oplus M$ and $S \ll_\delta M$ (see [63, Lemma 3.4]).

Clearly every lifting module is δ-lifting. As shown in Theorem 1.2.21, rings R for which every R-module is lifting are precisely the artinian serial rings with Jacobson radical square-zero. We have a characterization for a δ-semiperfect module to be δ-lifting (see Theorem 5.2.5). By using this result, we are able to show that over an artinian serial ring with Jacobson radical cube-zero, not every module is δ-lifting (see Example 4.2.18).

Question 5. *Characterize rings for which every module is δ-lifting.*

Bibliography

[1] A. Alahmadi, S.K. Jain and A. Leroy, ADS modules, *J. Algebra*, **352**(2012), 215-222.

[2] I. Amin, Y. Ibrahim and M.F. Yousif, $D3$-Modules, *Comm. Algebra*, **42** (2014), 578-592.

[3] I. Amin, Y. Ibrahim and M. Yousif, $C3$-modules, *Algebra Colloq.* **22** (2015), 655670.

[4] F. W. Anderson, K. R. Fuller, Rings and Categories of Modules. Berlin and New York: Springer-Verlag. 1974.

[5] R. Baer, Abelian groups which are direct summands of every containing group, *Bull. Am. Math. Soc.*, **46**(1940), 800-806.

[6] H. Bass, Finitistic homological dimension and a homological generalization of semi-primary rings, *Trans. Amer. Math. Soc.***95** (1960), 466-488.

[7] W. D. Burgess and R. Raphael, On modules with the absolute direct summand property, in *Ring Theory* (World Scientific, River Edge, NJ, 1993). pp. 137148.

[8] E. Büyükasik and C. Lomp, When δ-semiperfect rings are semiperfect, *Turkish J. Math.* **34** (3) (2010), 317-324.

[9] V. Camillo, Y. Ibrahim, M. Yousif and Y. Zhou, Simple-direct-injective modules, *J. Algebra*, **420** (2014), 3953.

[10] S.U. Chase, Direct product of modules, *Trans. Amer. Math. Soc.* **97** (1960), 457-473.

[11] P. Crawley and B. Jónsson, Refinements for infinite direct decompositions of algebraic systems, *Pacific J. Math.* **14** (1964), 797-855.

[12] N.V. Dung, D.V. Huynh, P.F. Smith and R. Wisbauer, Extending modules, Pitman Research Note in Math., Series 313 Longman Harlow, 1994.

[13] N.V. Dung and P.F Smith, Rings for which certain modules are CS, *J. Pure Appl. Algebra* **102** (1995), 273-287.

[14] N. Er, S. Singh, A.K. Srivastava, Rings and modules which are stable under automorphisms of their injective hulls, *J. Algebra*, **379**(2013), 223-229.

[15] L. Fuchs, On quasi-injective modules, *Ann. Scuola Norm. Sup. Pisa* **23** (3) (1969), 541-546.

[16] L. Fuchs, *Infinite Abelian Groups*, Vol. 1, Pure and Applied Mathematics, Vol. 36 (Academic Press, New York, 1970).

[17] J. L. Garcia, Properties of direct summands of modules, *Comm. Algebra* **17**(1) (1989), 7392.

[18] V.K. Goel and S.K. Jain, $\pi-$ injective modules and rings whose cyclic are $\pi-$injective, *Comm. Algebra*, **6**(1)(1978), 59-73.

[19] K.R. Goodearl, Von Neumann Regular Rings, Krieger Publishing Company, Malabar, Florida, 1991.

[20] A. Harmanci, D. Keskin and P.F. Smith, On $\oplus$-supplemented modules, *Acta Math. Hungar.* **83**(1-2) (1999), 161-169.

[21] A. Hamdouni, A. C. Özcan and A. Harmanci, Characterization of modules and rings by the summand intersection property and the summand sum property, *JP J. Algebra Number Theory Appl.* **5** (2005), 469490.

[22] M. Henriksen, On a class of regular rings that are elementary divisor rings, *Arch. Math. (Basel)* **24** (1973) 133141.

[23] D.V. Huynh, N.V. Dung and R. Wisbauer, On modules with finite uniform and Krull dimension, *Arch. Math.*, **57**(1991), 122-132.

[24] Y. Ibrahim, X.H. Nguyen, M.F. Yousif and Y. Zhou, Rings whose cyclics are C3-modules, *J. Algebra Appl.*, **15**(8)(2016), 1650152 (18 pages).

[25] S.K. Jain and S. Mohamed, Rings whose cyclic modules are continuous, *J. Indian Math. Soc.*, **42**(1978), 197-202.

[26] S.K. Jain, A.K. Srivastava and A.A. Tuganbaev, Cyclic Modules and the Structure of Rings, Oxford Math. Monogr., Oxford Univ. Press, 2012.

[27] L. Jeremy, Modules et anneaux quasi-continus, *Canad. Math. Bull.*, **17**(1974), 217-228.

[28] R.E. Johnson and E.T. Wong, Quasi-injective modules and irreducible rings, *J. London Math. Soc.*, **36**(1961), 260-268.

[29] I. Kaplansky, Projective modules, *Ann. of Math.* (2) **68** (1958), 372377.

[30] F. Kasch and E.A. Mares, Eine Kennzeichnung semi-perfekter Moduln, *Nagoya Math. J.* **27** (1966), 525-529.

[31] D. Keskin, P.F. Smith and W. Xue, Rings whose modules are $\oplus$-supplemented, *J. Algebra* **218**(2) (1999), 470-487.

[32] M.T. Kosan, δ-lifting and δ-supplemented modules, *Algebra Colloq.* **14** (1) (2007), 53-60.

[33] A. Koehler, Rings for which every cyclic module is quasi-projective, *Math. Ann.*, **189**(1970), 311-316.

[34] A. Koehler, Rings with quasi-injective cyclic modules, *Quart. J. Math. Oxford*, **25**(1974), 51-55.

[35] M. Tamer Kosan, Nguyen Thi Thu Ha, and Truong Cong Quynh, Rings for which every cyclic module is dual automorphism-invariant, *J. Algebra Appl.* **15**(4) (2016), 1650078 (11 pages).

[36] T.Y. Lam, A First Course in Noncommutative Rings, Graduate Texts in Mathematics, **131**. Springer-Verlag, New York, 1991.

[37] T.Y. Lam, Lectures on Modules and Rings, Graduate Texts in Math., Vol. **189**, Springer-Verlag, 1999.

[38] E.A. Mares, Semi-perfect modules, *Math. Z.*, **82** (1963), 347-360.

[39] M. Marianne, Rings of quotients of generalized matrix rings, *Comm. Alg.* **15**(1987), 1991-2015.

[40] S. Mohamed and T. Bouhy, Continuous Modules, *Arabian J. Sci. Eng.*, **2**(1977), 107-122.

[41] S. Mohamed and B.J. Müller, Continuous and Discrete Modules, Cambridge University Press, 1990.

[42] S. Mohamed, B.J. Müller and S. Singh, Quasi-dual continuous modules, *J. Austral. Math. Soc.*, **24**(1985), 287-299.

[43] T. Nakayama, On Frobenius algebras, *Ann. Math.* **42** (1941), 1-21.

[44] X.H. Nguyen, T. Kosan and Y. Zhou, On δ-semiperfect modules, accepted by *Comm. Algebra*.

[45] X.H. Nguyen, M.F. Yousif and Y. Zhou, Rings whose cyclics are $D3$-modules, *J. Algebra Appl.* **16** (8) (2017), 1750184 (15 pages).

[46] X.H. Nguyen and Y. Zhou, Rings whose cyclic modules are lifting and ⊕-supplemented, accepted by *Comm. Algebra*.

[47] W.K. Nicholson, On semiperfect modules, *Canad. Math. Bull.* **18** (1) (1975), 77-80.

[48] W.K. Nicholson, Lifting idempotents and exchange rings, *Trans. Amer. Math. Soc.* **229** (1977), 269278.

[49] K. Oshiro, Semiperfect modules and quasi-semiperfect modules, *Osaka J. Math.*, **20**(1983), 337-372.

[50] B.L. Osofsky, Rings all of whose finitely generated modules are injective, *Pacific J. Math.*, **14**(1964), 645-650.

[51] B.L. Osofsky and P.F. Smith, Cyclics modules whose quotients have all complement submodules direct summands, *J. Algebra*, **139**(1991), 342-354.

[52] K. Oshiro and R. Wisbauer, Modules with every subgenerated module lifting, *Osaka J. Math.* **32** (1995), 513-519.

[53] A. Özcan and M. Alkan, Semiperfect modules with respect to a preradical, *Comm. Algebra* **34** (2006), 841-856.

[54] A. Rosenberg and D. Zelinsky, Finiteness of the injective hulls, *Math. Z.* **70** (1959), 372-380.

[55] F.L. Sandomierski, On semiperfect and perfect rings, *Proc. Amer. Math. Soc.*, **21** (1969), 205-207.

[56] G. Song and X. Guo, Diagonability of idempotent matrices over noncommutative rings, *Linear Algebra Appl.* **297** (1999) 17.

[57] G. Tang, C. Li and Y. Zhou, Study of Morita contexts, *Comm. Alg.* **42**(2014), 1668-1681.

[58] Y. Utumi, On continuous regular rings, *Canad. Math. Bull.*, **4**(1961), 63-69.

[59] Y.D. Wang, δ-small submodules and δ-supplemented modules, *Int. J. Math. Math. Sci.*, ID58132 (2007).

[60] R. Wisbauer, Foundations of Module and Ring Theory, Gordon and Breach Science Publishers, Philadelphia, PA, 1991.

[61] W. Xue, Characterization of rings using direct-projective modules and direct-injective modules, *J. Pure Appl. Algebra* **87**(1993), 99-104.

[62] W. Xue, Characterization of Semiperfect and Perfect Rings, *Publ. Mat.* **40** (1996), 115-125.

[63] Y. Zhou, Generalizations of perfect, semiperfect and semiregular rings, *Algebra Colloq.* **7** (3) (2000), 305-318.