SEMIGROUP GRADED RINGS

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by

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A thesis submitted to the

School of Graduate Studies

in partial fulfilment of the

requirements for the degree of

Doctor of Philosophy

Department of Mathematics and Statistics
Memorial University of Newfoundland

June 1993

St. John's Newfoundland

Abstract

Let S be a semigroup. A ring R with a direct sum decomposition $R = \bigoplus_{s \in S} R_s$ such that $R_s R_t \subseteq R_{st}$ for elements s and t in S is called a semigroup graded ring.

In this thesis, we develop techniques for studying such rings based on the structure theory of semigroups. We apply these techniques to investigate various ring theoretic properties of semigroup graded rings.

In many cases, we relate a property of R to the components R_e indexed by idempotent elements e of the grading semigroup S. If S is finite, then R is perfect, semilocal, or semiprimary if and only if the same is true of each such component R_e. We prove that the nilpotency of the Jacobson radical of each R_e is sufficient for the nilpotency of the Jacobson radical of R for rings graded by finite semigroups, and obtain a similar condition for the Jacobson radical to be locally nilpotent for rings graded by locally finite semigroups. We also show that R is a Jacobson ring if each R_e is a Jacobson ring and S is finite.

We show that cancellativity is a necessary condition on a semigroup S in order that the Jacobson radical of each S-graded ring be homogeneous. With certain restrictions on the graded ring, we completely classify commutative semigroups and regular semigroups for which the Jacobson radical of each S-graded ring is homogeneous.

A result of Zelmanov that only finite semigroups admit right Artinian semigroup algebras is generalised to show that, under certain conditions, a right Artinian semigroup graded ring necessarily has finite support.

We find necessary and sufficient conditions for rings graded by elementary Rees matrix semigroups to be semisimple Artinian. These rings are one of the essential pieces in the structure theory of graded rings that we develop herein.

Results on nilpotence and perfectness are generalised to semigroup graded rings with finite support.

Acknowledgements

I am deeply indebted to my supervisor, Dr. Eric Jespers, without whose encouragement, advice, and assistance this work could not have been completed. I am also grateful to him for providing me the opportunity and the means to attend two conferences during the course of my studies.

I acknowledge a further debt to Andrei Kelarev and Angel del Río with whom I had fruitful collaborations which led to some of the results included herein.

I would like to thank the Department of Mathematics and Statistics in general and Drs. Bruce Shawyer and Edgar Goodaire, the past and present Department Heads, in particular for providing me with the facilities that I needed and a friendly atmosphere in which to work.

I am grateful to NSERC for financial support during the three years of my programme.

Finally, I would like to thank my wife Catherine for providing support, encouragement, and welcome distractions from mathematics.

Contents

Introduction

Let S be a semigroup. A ring R with a direct sum decomposition $R = \bigoplus_{s \in S} R_s$ of the additive structure such that $R_s R_t \subseteq R_{st}$ for all $s, t \in S$ is said to be an S graded ring or a semigroup graded ring [55].

The notion of graded ring has long been used as a tool in a variety of areas of mathematics such as algebraic geometry, number theory, topology, and ring theory. For example, a polynomial ring R in commuting or non-commuting variables can be graded by the semigroup of natural numbers $\mathbf{N}$ by letting R_n be the homogeneous polynomials of total degree n. This observation was used to great effect by Golod and Shafarevitch [27, Chapter 8] to find a counterexample to the Burnside problem. Tensor, symmetric, and exterior algebras are all examples of $\mathbf{N}$-graded rings in which the grading plays an important role [47, chapter XVI], while in topology, the singular cohomology of a space can be given a product called the cup product which makes it a $\mathbf{N}$-graded ring [66, Chapter 5].

Considerable interest in more general graded rings arose about 20 years when group graded rings began to be studied. Certain group graded rings, the group crossed products, arise unavoidably in the study of central simple algebras [27] and group algebras [59], and it is often easier to prove results in the more general setting of arbitrary group graded rings. For example, many classical results on polynomial rings, skew polynomial rings, and crossed products are simple consequences of graded results [38]. In the last 10 years, many questions about group graded rings have been answered, particularly with the advent of the duality theorems for group graded rings [16, 61]. A few books dealing largely with group graded rings have appeared [37, 53, 54, 59] and material on group graded rings has been included in several recent books on general ring theory [38, 48, 63].

Although the study of semigroup graded rings has almost as long a history, most authors have concentrated on certain subclasses of semigroups. In particular, much attention has been paid to rings graded by semilattices because such rings play a useful role in tackling problems about semigroup rings of commutative semigroups [10, 36, 67, 68, 70, 71, 73]. Other classes that have been considered are rings graded by commutative semigroups [1, 2, 18, 19, 32, 33, 40], bands [39, 43, 44, 52], inverse semigroups [72], and ordered semigroups [34]. Only recently have a few papers appeared which deal with more general semigroup graded rings [41, 42].

Reasons for studying semigroup graded rings are threefold. Firstly, they are a natural generalisation of group graded rings on the one hand and semigroup rings on the other, classes of rings upon which much attention has been lavished.

Secondly, many disparate ring constructions, viewed in the right light, are seen to be examples of semigroup graded rings. As examples, we cite polynomial rings and skew polynomial rings, monomial algebras [55], group and semigroup crossed products [37, 59], generalised matrix rings [9], Morita contexts, structural matrix rings [64], generalised Rees rings [45], and Munn algebras [15]. By studying semigroup graded rings in general, it may be possible to unify results which have been proved separately for these various ring constructions.

Thirdly, as the roles of semilattice graded rings in the study of semigroup rings of commutative semigroups and group graded rings in the study of group rings have shown, more general semigroup graded rings serve as a useful tool in the investigation of arbitrary semigroup rings. As an example of this, we cite Theorem 8.3.8 in which graded results are used to deduce that certain semigroup algebras are Jacobson rings.

As mentioned earlier, little work has yet been done on rings graded by arbitrary semigroups. This thesis is an attempt to rectify this situation. We develop a general framework for the investigation of semigroup graded rings, and we apply these techniques to study various ring theoretic properties of such rings. These methods are particularly suited to rings graded by finite semigroups or semigroup graded rings with finite support, on which we concentrate most of our attention. We also obtained some results for rings graded by infinite semigroups, but generally with some restrictions on the semigroups considered.

In the study of group graded rings, a common problem is to try to relate the properties of a graded ring R to the component R_1 corresponding to the identity 1 of the group. For many of the properties considered here, we attempt a similar feat, except that instead of relating the properties of a semigroup graded ring R to a single homogeneous component, we must consider all the components R_e corresponding to idempotent elements of the semigroup.

The techniques used are quite different from those encountered in the group graded theory. Using the structure theory of semigroups, we combine results for group graded rings, rings graded by a class of semigroups called the elementary Rees matrix semigroups, and ideal extensions to obtain the general case. This works particularly well for rings graded by finite semigroups.

In Chapter 1 we introduce the necessary elements of the theory of semigroups. In Chapter 2 we explain the basic notions of the theory of graded rings and define some notation that will be used throughout. In Chapter 3, we give several examples of semigroup graded rings. These examples will serve to illustrate some later results.

In Chapter 4 we show how the structure theory of semigroups can be used to tackle semigroup graded rings. The technique is illustrated by studying nilpotency conditions of the Jacobson radical. We show that the Jacobson radical of a ring R graded by a finite semigroup is nilpotent provided that the same is true for each component R_e, where e ranges over the set of idempotents of S. We obtain a similar condition for the Jacobson radical of a ring graded by a locally finite semigroup to be locally nilpotent. The auxiliary results proved in this chapter are used throughout the thesis.

The remaining chapters are relatively independent.

The topic of Chapter 5 is the homogeneity of the Jacobson radical. In particular, we show that cancellativity is a necessary condition on a semigroup S in order that the Jacobson radical of all S-graded rings be homogeneous. We also characterise those commutative and regular semigroups S for which some large classes of S-graded rings have homogeneous Jacobson radical.

We study perfect, semilocal, and semiprimary rings in Chapter 6. For a ring R graded by a finite group, it is known that R is perfect if and only if R_1 is perfect [5]. We extend this result to a ring R graded by a finite semigroup S, obtaining the result that R is perfect if and only if R_e is perfect for each idempotent e of S. Analogous results are obtained for semilocal and semiprimary rings.

In Chapter 7, we study ideals and homogeneous ideals of a class of rings, the contracted graded rings graded by elementary Rees matrix semigroups. These rings are central to the structure theory developed in Chapter 4. We show that there is a one-to-one correspondence between prime and graded prime ideals of such rings, and obtain necessary and sufficient conditions for these rings to be semisimple Artinian.

Jacobson rings are the subject of Chapter 8. We show that a ring R graded by a finite semigroup is a Jacobson ring if R_e is a Jacobson ring for each idempotent e. We also apply graded results to deduce that a semigroup algebra satisfying a polynomial identity is a Jacobson ring provided that the semigroup and all its homomorphic images have finite rank. This generalises a result of Gilmer for commutative semigroup algebras [24].

In Chapter 9, we generalise a theorem of Zelmanov [74] that a semigroup S must be finite if the semigroup algebra $K[S]$ is right Artinian. The analogous statement for graded rings is that a right Artinian semigroup graded ring has finite support. Although not true in general, we obtain two weaker forms of this statement and as a consequence, find a new proof for Zelmanov's theorem.

Finally, in Chapter 10, we try to extend some of our earlier results to semigroup graded rings with finite support. In the absence of certain types of principal factors, we obtain similar results for perfect rings and the nilpotence of the Jacobson radical.

CHAPTER 1

Semigroups

In our investigations of semigroup graded rings we naturally require a great deal of terminology from the theory of semigroups and we make much use of the structure theories of various classes of semigroups.

This first chapter is an attempt to bring together the basic definitions and state the major results that we will need from the theory of semigroups. Other subsidiary results and terminology will be introduced throughout the work as needed.

The reader who is familiar with semigroups can omit this chapter. Our notation and terminology follows closely that of Clifford and Preston [15] from which most of the information conveyed in this chapter is taken.

Those who are less familiar with semigroups should find everything that they need to know in this chapter. While some notions in semigroup theory derive from group theory, semigroup theory in many ways resembles ring theory more closely. Familiar concepts such as idempotent elements, ideals, and regularity are defined for semigroups essentially as for rings.

In order to get an intuitive feeling for semigroups, it is useful to picture a semigroup as a sea in which some groups float as islands. These groups each contain a single idempotent, and each idempotent determines a group. Groups arising from distinct idempotents are disjoint. The structure theory describes how these islands sit in the sea. As will become apparent later, our procedure for investigating properties of semigroup graded rings often involves transferring the problem from the whole ring to subrings graded by the subgroups of the semigroup. Knowing the structure theory of semigroups is crucial to this task.

1.1. Some Basic Definitions

A *semigroup* is a non-empty set S with an associative binary operation. We will usually write the operation by juxtaposition of elements.

We begin by introducing a few names for certain semigroup elements and some classes of semigroups.

1.1.1. Let S be a semigroup. An element $u \in S$ is a *right identity* of S if $su = s$ for all $s \in S$. We can similarly define *left identity* and u is an *identity* of S if it is both a left and right identity. A semigroup S may have multiple right or left identities, but if it has a right identity and a left identity, they must necessarily coincide and in this case S has a unique identity.

Let S be a semigroup with an identity 1. Then u is a *right unit* of S if there is a v such that $uv = 1$. In a similar way, we can define *left unit* and u is a *unit* if it is both a left and a right unit. We write $\mathcal{U}(S)$ for the set of units of S.

A semigroup with an identity element is called a *monoid*. If we further require that every element of the semigroup be a unit, then the semigroup is simply a *group*.

If S is a monoid then $\mathcal{U}(S)$ is a group.

1.1.2. An element $z \in S$ is a *right zero* of S if $sz = z$ for all $s \in S$. Similarly, we can define *left zero* and z is a *zero* of S if it is a left and right zero. As for identity elements, S can have multiple left or right zeros, but if S has a left and a right zero, then they coincide and in fact S has a unique zero element.

If S has a zero element, it will usually be denoted 0.

A *null* semigroup is one in which the product of any two elements is the zero element 0. (So a null semigroup must have a zero.)

A *right zero* semigroup is one in which every element is a right zero. Such a semigroup satisfies $xy = y$ for all x and y, and any set can be made into a right zero semigroup by defining the product that way. Similarly, a *left zero* semigroup satisfies the identity $xy = x$.

1.1.3. If S is a semigroup without an identity, we can adjoin one simply by adding a new element 1 and extending the multiplication by defining $1s = s1 = s$ for all $s \in S \cup \{1\}$. We denote this new semigroup by S^1. If S has an identity already, then we agree that $S^1 = S$.

In a similar way, we can always adjoin a zero element 0, and we write $S^0 = S \cup \{0\}$ for this new semigroup.

Although adjoining zeros and identities in this way is easy, it turns out that adjoining an identity is often not useful in our investigation of semigroup graded rings. Our approach relies on the structure theory of semigroups and involves reductions to subsemigroups and these subsemigroups will not have identities in general. On the other hand, adjoining a zero is harmless and often simplifies arguments. This will become clearer as we proceed.

1.1.4. The classes of left and right zero semigroups defined above are examples of classes of semigroups defined by requiring the semigroup to satisfy certain monomial identities.

The most important example of such a class is the class of semigroups which satisfy the identity $xy = yx$; this is the class of *commutative semigroups*. Note that we will continue to use multiplicative (rather than additive) notation for commutative semigroups.

1.1.5. An element $e \in S$ which satisfies $e = e^2$ is called an *idempotent*. We write $E(S)$ for the set of idempotent elements of a semigroup S. The set $E(S)$ can be partially ordered by $e \leq f$ if and only if $ef = fe = e$.

A semigroup in which every element is idempotent is called a *band*. A commutative band is called a *semilattice*. Note that if we give a semilattice S the partial ordering given above and define the infimum of two elements of S to be their product then this definition agrees with the usual definition of a semilattice.

1.1.6. An element $a \in S$ is *regular* if there is an $x \in S$ such that $a = axa$. Elements a and b are *inverses* if $a = aba$ and $b = bab$.

A *regular* semigroup is one in which every element is regular. An *inverse* semigroup is one in which every element has a unique inverse. Note that uniqueness of the inverse is important here. Indeed a regular element a of a semigroup always has at least one inverse; if $a = axa$ then a and xax are easily seen to be inverses. The following alternative characterisation of inverse semigroups is often useful: S is an inverse semigroup if and only if S is regular and $ef = fe$ for all idempotents $e, f \in E(S)$.

1.1.7. Let A and B be subsets of a semigroup S. We shall write AB for the set of products $\{ab \mid a \in A, b \in B\}$. If one of these sets is a singleton, say $B = \{b\}$ we simply write Ab for the set AB.

1.2. Subsemigroups and Subgroups

1.2.1. A *subsemigroup* T of a semigroup S is a non-empty subset which is closed under multiplication. A *submonoid* is a subsemigroup which has an identity element.

A *subgroup* G of S is a subsemigroup which is a group. Note that the identity of G need not be the identity of S (indeed S need not have an identity); it can be any idempotent element of S.

1.2.2. Let $e \in E(S)$ be an idempotent. Then $eSe = \{ese \mid s \in S\}$ is a submonoid of S. Note that $eSe = \{x \in S \mid ex = xe = x\}$, the set of elements of S for which e is an identity element. Let $H_e = \mathcal{U}(eSe)$, the group of units of the monoid eSe. Then H_e is a subgroup of S and it is the largest subgroup for which e is the identity element. Such subgroups are called the *maximal* subgroups of S. Since e is the unique idempotent element of H_e, there is a one-to-one correspondence between idempotents and maximal subgroups. Distinct maximal subgroups are disjoint.

1.2.3. If A is a non-empty subset of a semigroup S, we write $\langle A\rangle$ for the subsemigroup generated by A. If A is finite, say $A = \{a_1, a_2, \dots, a_n\}$, we often write $\langle a_1, a_2, \dots, a_n\rangle$ instead of $\langle A\rangle$.

A semigroup S is *cyclic* if $S = \langle x\rangle$ for some $x \in S$. An element x of a semigroup S is a *periodic element* if $\langle x\rangle$ is finite. A finite cyclic semigroup always contains an idempotent.

A semigroup is a *periodic semigroup* if every cyclic subsemigroup is finite. A semigroup is *locally finite* if every finitely generated subsemigroup is finite. Clearly, a locally finite semigroup is periodic.

1.3. Homomorphisms, Congruences and Ideals

A *homomorphism* of semigroups is a function $f : S \to T$ from a semigroup S to a semigroup T such that $f(xy) = f(x)f(y)$ for all $x, y \in S$.

In considering homomorphic images of semigroups, congruences fulfill the role played by normal subgroups in the theory of groups. More precisely, the analogue of congruence in the theory of groups is the equivalence relation that is defined by the cosets of a normal subgroup. In another direction, there is also a notion of an ideal of a semigroup, analogous to the situation in ring theory, and then of a quotient semigroup which is a homomorphic image of the original semigroup. As we shall see, an ideal is really a special case of a congruence.

1.3.1. An equivalence relation ρ on a semigroup S is a *right congruence* if $x \rho y$ implies $xz \rho yz$ for all $x, y, z \in S$. We define *left congruence* similarly. A *congruence* is an equivalence relation ρ on S which is both a left and a right congruence.

Since it is a relation, we may regard a congruence ρ as a subset of $S \times S$. This leads to a natural partial ordering of congruences and they form a lattice under this partial ordering. In particular, the meet of any family of congruences is simply their intersection (considered as subsets of $S \times S$); this is easily seen to be a congruence.

If S is a group with identity e and ρ is a congruence on S, it is not difficult to see that the ρ-class of e is a normal subgroup N, and that the other ρ-classes are the cosets of N. In this case, the congruence is completely determined by the class of e. More general semigroups do not exhibit this behaviour. Even if a semigroup S has an identity, the class of the identity may not completely determine a congruence. For example, if S is a semigroup without an identity, then any congruence on S can be extended to a congruence on S^1 by simply creating a new class containing only the identity.

1.3.2. Write S/ρ for the set of equivalence classes of the congruence ρ. We refer to S/ρ as the *quotient* of the semigroup S by the congruence ρ. Write $\bar{s}$ for ρ-class of s.

Then S/ρ is a semigroup with operation $\bar{s}\,\bar{t} = \overline{st}$, and the map $\pi_\rho\colon S \to S/\rho$, $s \mapsto \bar{s}$ is a homomorphism of S onto S/ρ.

Conversely, if $\phi\colon S \to S'$ is a homomorphism of semigroups, define a relation ρ by $x\ \rho\ y$ if and only if $\phi(x) = \phi(y)$. Then ρ is a congruence and ϕ induces $\bar{\phi}\colon S/\rho \to S'$, $\bar{s} \mapsto \phi(s)$ so that $\phi = \bar{\phi}\pi_\rho$. The congruence ρ is called the *kernel* of ϕ.

If we put $T = \phi(S)$, then $\bar{\phi}\colon S/\rho \to T$ is an isomorphism and the lattice of congruences of T is isomorphic in the obvious way to the sublattice of congruences of S which contain ρ.

1.3.3. Just as in group theory, semigroups can be specified by generators and relations. The *free semigroup* on a non-empty set X, denoted $\mathcal{F}_X$, is the set of words in the alphabet X with multiplication by concatenation of words. We write $\mathcal{F}^1_X$ for the semigroup $\mathcal{F}_X$ with an identity adjoined; the identity can be represented as the empty word. We call $\mathcal{F}^1_X$ the *free monoid* on X. Given a set X and a set W of relations of the form $w_1 = w_2$ where w_1 and w_2 are words in the alphabet X, the semigroup generated by X with relations W is the quotient of $\mathcal{F}_X$ by the smallest congruence on $\mathcal{F}_X$ which contains the relations in W. Similarly we can define the monoid generated by X with relations W (here W may include relations of the form $w = 1$ for some word w) as a suitable quotient of $\mathcal{F}^1_X$.

1.3.4. Let I be a non-empty subset of a semigroup S. Then I is a *right ideal* of S if $xs \in I$ for all $s \in S$ and $x \in I$. *Left ideal* is defined similarly, and I is an *ideal* of

S if it is a left and right ideal of S.

If S has a zero element 0, then $\{0\}$ is always an ideal of S.

If $\{I_\alpha \mid \alpha \in A\}$ is a family of ideals of a semigroup S then $\bigcup I_\alpha$ and $\bigcap I_\alpha$ are also ideals of S, the latter provided that it is non-empty. The same is true for families of right or left ideals.

If $a \in S$ then the right ideal generated by a is denoted aS^1; clearly $aS^1 = aS \cup \{a\}$. Similarly, the left ideal generated by a is denoted S^1a. The ideal generated by a is $S^1aS^1 = SaS \cup Sa \cup aS \cup \{a\}$.

1.3.5. For a non-empty subset I of S, define an equivalence relation ρ_I by $x \, \rho_I \, y$ if and only if $x = y$ or both $x, y \in I$. If I is an ideal then ρ_I is a congruence. Similarly, if I is a left or right ideal, then ρ_I is a left or right congruence.

For an ideal I of S, write S/I for the semigroup S/ρ_I. The semigroup S/I is called a *Rees factor* of S. If we identify the elements of $S \setminus I$ with the corresponding ρ_I-classes, then the elements of S/I are just $\{x \mid x \in S \setminus I\} \cup \{I\}$, and I is a zero element of S/I. Intuitively, when we pass from S to S/I, we have identified all the elements of I with zero. Note in particular that the non-zero idempotents of S/I are just the idempotents of S which are elements of $S \setminus I$.

As a special case, it is convenient to agree that $S/\emptyset = S$ even though $\emptyset$ is not an ideal.

There is a lattice isomorphism between the ideals of S containing I and the ideals of S/I. Standard isomorphism theorems apply. If $I \subset J$ are ideals of S then J/I is an ideal of S/I and $(S/I)/(J/I) \cong S/J$. If T is a subsemigroup of S and I is an ideal of S which intersects T, then $T \cap I$ is an ideal of T and $T/(T \cap I) \cong (T \cup I)/I$.

1.3.6. Let S be a semigroup with a zero and let $\{T_\alpha \mid \alpha \in A\}$ be a family of subsets of S. We say that T is a 0-*disjoint union* of the family if $T = \bigcup_{\alpha \in A} T_\alpha$ and $T_\alpha \cap T_\beta = \{\theta\}$ for all $\alpha, \beta \in A$.

Suppose that $\{S_\alpha \mid \alpha \in A\}$ are subsemigroups of S and that $S = \bigcup_{\alpha \in A} S_\alpha$ is a 0-disjoint union. Suppose also that for $\alpha \neq \beta$, $S_\alpha S_\beta = \{\theta\}$. Then each S_α is an ideal of S and we say that S is a *0-direct union* of the semigroups S_α.

1.3.7. We will require the following simple observation about the unit group of a periodic monoid.

LEMMA ([55, Lemma 4.12]). *Let S be a periodic monoid and let $U = \mathcal{U}(S)$ be the group of units of S. If $U \neq S$ then $S \setminus U$ is an ideal of S.*

PROOF. Suppose that $x, y \in S$ and xy is a unit. Then there is a z such that $xyz = 1$. Since S is periodic, there is an n such that $(yz)^n = e$ is an idempotent. Then $1 = x^n(yz)^n = x^n e$ and therefore $e = x^n e^2 = x^n e = 1$. Hence, $1 = x^n e = x^n$ and x is a unit. So if $x \in S \setminus U$, then $xy \in S \setminus U$. Similarly, $yx \in S \setminus U$ and therefore, $S \setminus U$ is an ideal. □

1.4. Simple and 0-Simple Semigroups

The basic building blocks of the structure theory of semigroups are the simple and 0-simple semigroups. If one were to extend the definition of simple group to semigroups, then it would be natural to require that a simple semigroup have no non-trivial congruences. On the other hand, mimicking ring theory, one might use the weaker definition that a simple semigroup have no proper ideals. This second approach is the one taken in semigroup theory. It has the drawback that simple semigroups may have non-trivial homomorphic images, but nevertheless results in a nice structure theory.

1.4.1. Let S be a semigroup. We say that S is a *simple semigroup* if it has no ideals other than S itself.

This definition is not very interesting if S has a zero element θ, for in that case $\{\theta\}$ is always an ideal of S. So we say that a semigroup S with a zero is a *0-simple semigroup* if $S^2 = S$ and S has no ideals other than S and $\{\theta\}$. The condition $S^2 = S$ makes the theory cleaner and excludes only one semigroup, the two element null semigroup.

In an entirely analogous manner, we can define *right simple semigroup*, *right 0-simple semigroup* and the corresponding versions for left ideals. Once again, we exclude the two-element null semigroup. These are stronger conditions.

Note that if S does not have a zero and S is simple, then S^0 is 0-simple. This means that any results about 0-simple semigroups can be easily applied to simple semigroups. For this reason, we will usually restrict our attention to 0-simple semigroups.

Let S be a 0-simple semigroup. If a is a non-zero element of S then the principal ideal S^1aS^1 is non-zero so we must have $S^1aS^1 = S$. In fact it is not difficult to show that $SaS = S$. Since a non-zero ideal always contains principal ideals, the condition that $S = S^1aS^1$ for all $a \neq 0$ characterises 0-simple semigroups.

1.4.2. Let S be a semigroup with a zero. An ideal M of S is a *0-minimal ideal* if there are no non-zero ideals of S properly contained in M. We will need the following fact on occasion: if M is a zero minimal ideal of S, then considered as a semigroup, M is either null or 0-simple.

Similarly, if M is a maximal ideal of S, then S/M contains no non-zero proper ideals and therefore, either S/M is 0-simple, or S/M is a two-element null semigroup.

1.4.3. Let S be a semigroup. Define an equivalence relation $\mathcal{J}$ on S by $x \mathcal{J} y$ if and only if $S^1xS^1 = S^1yS^1$. Note that $\mathcal{J}$ is not a congruence in general. Write $\mathcal{J}(a)$ for the $\mathcal{J}$-class of S containing a.

Let S be a semigroup and adjoin a zero to it if it does not already have one. Consider a principal ideal S^1aS^1 of S, with $a \neq 0$. For convenience, let $J_a = S^1aS^1$. Let $I_a = \{x \in J_a \mid J_x \subsetneq J_a\}$, the set of elements of J_a which do not generate J_a as an ideal. It is easy to see that I_a is an ideal of S. Since each element of $J_a \setminus I_a = \mathcal{J}(a)$ generates J_a as an ideal there can be no ideal of S strictly between I_a and J_a. Hence

J_a/I_a is a 0-minimal ideal of S/I_a and so is either 0-simple or null. (If we do not assume that S has a zero, then I_a may be empty in which case J_a/I_a is simple rather than 0-simple.) The quotient J_a/I_a is called a *principal factor* of S.

1.4.4. Let S be a semigroup with a zero. A strictly decreasing series

$$S = S_1 \supset S_2 \supset \cdots \supset S_n \supset S_{n+1} = \{\theta\}$$

is a *principal series* if each S_i is an ideal of S and there are no ideals of S strictly between S_i and S_{i+1}.

Each factor S_i/S_{i+1} of a principal series is a 0-minimal ideal of S/S_{i+1} and is therefore null or 0-simple.

Note that our definition of principal series differs slightly from that of [15]. Our series ends with the zero ideal rather than the empty set. Of course, this restricts our definition to semigroups with zeros, so we will always adjoin a zero before considering principal series.

If S has a principal series as above then the quotients S_i/S_{i+1} are isomorphic in some order to the non-zero principal factors of S. In the notation of §1.4.3, S_i/S_{i+1} is isomorphic to J_a/I_a where a is any element of $S_i \setminus S_{i+1}$.

Finite semigroups always have principal series.

1.4.5. The *bicyclic semigroup* $C(p,q)$ is the monoid generated by $\{p,q\}$ with the single relation $pq = 1$. This semigroup plays an important role in the theory of

0-simple semigroups; see §1.4.6.

The elements of $C(p,q)$ are all words of the form $q^n p^m$ for $m, n \geq 0$ (with the understanding $p^0 = q^0 = 1$). These words are multiplied as follows:

$$q^n p^m q^k p^l = \begin{cases} q^{n+k-m} p^l & \text{if } m \leq k, \\ q^n p^{l+m-k} & \text{if } m \geq k. \end{cases}$$

It is apparent that $C(p,q)$ is simple, for if $q^n p^m$ is an element of $C(p,q)$, then $1 = p^n(q^n p^m)q^m$ and so $S^1 q^n p^m S^1 = S$.

It is useful to picture some further properties of $C(p,q)$ by arranging the elements in a table:

$$\begin{array}{cccccc} 1 & p & p^2 & p^3 & p^4 & \cdots \\ q & qp & qp^2 & qp^3 & qp^4 & \cdots \\ q^2 & q^2p & q^2p^2 & q^2p^3 & q^2p^4 & \cdots \\ q^3 & q^3p & q^3p^2 & q^3p^3 & q^3p^4 & \cdots \\ q^4 & q^4p & q^4p^2 & q^4p^3 & q^4p^4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Then the elements below any horizontal line drawn through this table form a right ideal and the elements to the right of any vertical line form a left ideal. Further, the elements on the diagonal are all idempotents and the ordering defined in §1.1.5 is $1 > qp > q^2p^2 > q^3p^3 > \cdots$.

If S is a semigroup with elements e, p', q' satisfying $ep' = p'e = p'$, $eq' = q'e = q'$, and $p'q' = e$, then it is clear that the subsemigroup $\langle p', q'\rangle$ is a homomorphic image of $\mathcal{C}(p,q)$ under the map $p \mapsto p'$, $q \mapsto q'$, because these relations are exactly those satisfied by the bicyclic semigroup. If further, $q'p' \neq e$, then this homomorphism is actually an isomorphism.

1.4.6. A semigroup S is a *completely 0-simple semigroup* if it is 0-simple and it contains a *primitive idempotent*, that is a minimal non-zero idempotent with respect to the partial ordering given in §1.1.5.

In fact, if S is completely 0-simple, it turns out that every non-zero idempotent is primitive. Completely 0-simple semigroups are regular.

In Section 1.5, we shall give a structure theorem for completely 0-simple semigroups. Once this structure is known, it is rather easy to prove many things about these semigroups.

Since these semigroups are relatively easily handled, it is of some interest to know when a 0-simple semigroup is completely 0-simple. We state two results of this sort.

1.4.7. The first result tells us something about 0-simple semigroups which are not completely 0-simple.

THEOREM ([15, Theorem 2.54]). *Let S be a 0-simple semigroup which is not completely 0-simple. If e is a non-zero idempotent of S then e is the identity element of a subsemigroup T of S which is isomorphic to the bicyclic semigroup. Conversely, a 0-simple semigroup which contains a subsemigroup isomorhic to the bicyclic semigroup* *is not completely 0-simple.*

This result is less useful than at first it might appear, for there are classes of 0-simple semigroups which contain no non-zero idempotents.

1.4.8. A semigroup S is a *strongly π-regular semigroup* if for every $x \in S$, there is an n such that x^n is contained in some subgroup of S.

THEOREM ([15, Theorem 2.55]). *Let S be a 0-simple semigroup. Then S is completely 0-simple if and only if S is strongly π-regular.*

It is easy to see that periodic semigroups (and hence finite or locally finite semigroups) are strongly π-regular, so in many situations, this theorem will allow us to assume that 0-simple semigroups are completely 0-simple.

1.5. Rees Matrix Semigroups

Rees matrix semigroups are of fundamental importance because all completely 0-simple semigroups are of this type (see Theorem 1.5.2 below). They have a very rigid structure which makes it relatively easy to prove statements about completely 0-simple semigroups.

1.5.1. Let G be a group. Recall that G^0 is the semigroup obtained by adjoining a zero element 0 to G. Let I and Λ be indexing sets, and let P be a $\Lambda \times I$ matrix with entries from G^0.

For $g \in G^0$, $i \in I$, and $\lambda \in \Lambda$, write $(g)_{i\lambda}$ for the $I \times \Lambda$ matrix with (i, λ)-entry g and all other entries θ. For all i and λ, $(\theta)_{i\lambda}$ is the zero matrix; we write this matrix as θ.

Let $\mathfrak{M}^0(G; I, \Lambda; P)$ be the set whose elements are all such $(g)_{i\lambda}$ and define a multiplication on this set by

$$AB = A \circ P \circ B$$

where $A, B \in \mathfrak{M}^0(G; I, \Lambda; P)$ and '$\circ$' denotes ordinary multiplication of matrices. Note that this multiplication makes sense and gives an $I \times \Lambda$ matrix with at most one non-zero entry because A and B are such matrices. This operation is associative since ordinary matrix multiplication is associative, so $\mathfrak{M}^0(G; I, \Lambda; P)$ endowed with this operation is a semigroup. We call $\mathfrak{M}^0(G; I, \Lambda; P)$ a *Rees matrix semigroup*. The matrix P is called the *sandwich matrix*.

Alternatively, we can take the elements $(g)_{i\lambda}$ as formal symbols and define multiplication by

$$(g)_{i\lambda}(h)_{j\mu} = (gp_{\lambda j}h)_{i\mu}$$

where P is the matrix with (λ, i)-entry $p_{\lambda i}$ and the product $gp_{\lambda j}h$ is taken in G^0. It is easy to see that this definition is equivalent to the one given above.

Sometimes when the indexing sets I and Λ are finite, we will write $\mathfrak{M}^0(G; m, n; P)$; in this case the indices are $1 \leq i \leq m$ and $1 \leq \lambda \leq n$.

1.5.2. Our interest in these semigroups stems from the following result. The sandwich matrix P is *regular* if it has non-zero entries in every row and column.

THEOREM ([15, Theorem 3.5]). *A semigroup is completely 0-simple if and only if it is isomorphic to a Rees matrix semigroup $\mathfrak{M}^0(G; I, \Lambda; P)$ with regular sandwich matrix P.*

1.5.3. We introduce some further notation for Rees matrix semigroups. Let $S = \mathfrak{M}^0(G; I, \Lambda; P)$. For $i \in I$ and $\lambda \in \Lambda$, let

$$S_{i\lambda} = \{(g)_{i\lambda} \mid g \in G^0\},$$

$$S_{i\bullet} = \bigcup_{\lambda \in \Lambda} S_{i\lambda},$$

$$\text{and} \qquad S_{\bullet\lambda} = \bigcup_{i \in I} S_{i\lambda}.$$

The following properties follow easily from the explicit form of the multiplication given above.

LEMMA. *Let $S = \mathfrak{M}^0(G; I, \Lambda; P)$ be a Rees matrix semigroup.*

(i) *An element $(g)_{i\lambda} \neq 0$ is idempotent if and only if $p_{\lambda i} \neq 0$ and $g = {p_{\lambda i}}^{-1}$.*

(ii) *Distinct idempotents commute only if their product is zero. Hence every non-zero idempotent is primitive.*

(iii) *Each set $S_{i\lambda}$ is a subsemigroup of S. If $p_{\lambda i} = 0$ then it is a null semigroup; otherwise the map $(g)_{i\lambda} \mapsto gp_{\lambda i}$ is an isomorphism of $S_{i\lambda}$ onto G^0. In the latter* *case,* $S_{i\lambda} \setminus \{0\}$ *is a maximal subgroup of* S *and these subgroups (together with* $\{0\}$*) are all the maximal subgroups of* S.

(iv) *Each set* S_{i*} *is a right ideal of* S, *and* S *is the 0-disjoint union of the family* S_{i*}, $i \in I$. *Similarly, each* $S_{*\lambda}$ *is a left ideal of* S *and* S *is the 0-disjoint union of the family of such left ideals.*

(v) *Each* $S_{i\lambda}$ *is a left ideal of* S_{i*} *and* S_{i*} *is the 0-disjoint union of all* $S_{i\lambda}$, $\lambda \in \Lambda$. *Similarly,* $S_{*\lambda}$ *is the 0-disjoint union of right ideals* $S_{i\lambda}$, $i \in I$.

(vi) *For* $i, j \in I$ *and* $\lambda, \mu \in \Lambda$, *we have* $S_{i\lambda}S_{j\mu} \subseteq S_{i\mu}$. *Further, if* $S_{i\lambda}S_{j\mu} \neq \{0\}$, *then* $(S_{i\lambda} \setminus \{0\})(S_{j\mu} \setminus \{0\}) = (S_{i\mu} \setminus \{0\})$.

(vii) *If* S *is finitely generated, then* I *and* Λ *are finite. Consequently,* S *is locally finite if* G *is locally finite.*

1.5.4. A Rees matrix semigroup of the form $\mathfrak{M}^0(1; I, \Lambda; P)$ where 1 is the trivial group is called an *elementary Rees matrix semigroup*. Clearly these semigroups have only trivial subgroups.

1.5.5. Let $S = \mathfrak{M}^0(G; I, \Lambda; P)$ and let $\phi\colon G \to H$ be a group homomorphism. If we define $\phi(0) = 0$, then ϕ becomes a semigroup homomorphism $\phi\colon G^0 \to H^0$. As before, denote the (λ, i)-entry of P by $p_{\lambda i}$. Let P' be the $\Lambda \times I$ matrix with (λ, i)-entry $p'_{\lambda i} = \phi(p_{\lambda i})$. Then ϕ induces a semigroup homomorphism

$$\Phi\colon \mathfrak{M}^0(G; I, \Lambda; P) \to \mathfrak{M}^0(H; I, \Lambda; P')$$

defined by

$$(g)_{i\lambda} \mapsto (\phi(g))_{i\lambda}.$$

In particular, the trivial homomorphism $G \to \mathbf{1}$ induces a homomorphism of a Rees matrix semigroup $\mathfrak{M}^0(G; I, \Lambda; P)$ onto an elementary Rees matrix semigroup $\mathfrak{M}^0(1; I, \Lambda; P')$. The kernel of this homomorphism is the relation ρ given by

$$(g)_{i\lambda}\ \rho\ (h)_{j\mu} \qquad \text{if and only if} \qquad g = h = 0 \quad \text{or} \quad (i, \lambda) = (j, \mu).$$

1.5.6. Two other classes of Rees matrix semigroups will be of particular interest. The first is the class of completely 0-simple inverse semigroups. It is not difficult to see that the requirement that idempotents commute (see §1.1.6) means that the sandwich matrix of a Rees matrix presentation of such a semigroup must have exactly one non-zero entry in each row and column. In fact, up to isomorphism these are the semigroups of the form $\mathfrak{M}^0(G; I, I; \Delta)$ where Δ is the $I \times I$ identity matrix.

Let A and B be arbitrary sets and define an operation on the product $A \times B$ by $(a_1, b_1)(a_2, b_2) = (a_1, b_2)$. This forms a semigroup in which every element is idempotent. Such a semigroup is called a *rectangular band*. If we adjoin a zero, the resulting semigroup is isomorphic to the elementary Rees matrix semigroup $\mathfrak{M}^0(1; A, B; P)$ where P is a $B \times A$ matrix in which every entry is 1.

1.6. Commutative Semigroups

There is a nice decomposition theorem for commutative semigroups which can often be applied usefully in the study of rings graded by commutative semigroups. We present this theory in this section.

1.6.1. Let S be a semigroup (not necessarily commutative) and let ϕ be a homomorphism of S onto a semilattice Γ. For $\gamma \in \Gamma$, let $S_\gamma = \phi^{-1}(\gamma)$.

Note that the sets S_γ and S_β are disjoint if $\gamma \neq \beta$ and that $S = \bigcup_{\gamma \in \Gamma} S_\gamma$.

If $x, y \in S_\gamma$ then $\phi(xy) = \phi(x)\phi(y) = \gamma\gamma = \gamma$ so $xy \in S_\gamma$. Hence each set S_γ is a subsemigroup of S. If $x \in S_\gamma$ and $y \in S_\beta$ for some $\gamma, \beta \in \Gamma$ then $\phi(xy) = \phi(x)\phi(y) = \gamma\beta$ so that $xy \in S_{\gamma\beta}$.

In this way, S is a disjoint union of subsemigroups S_γ indexed by a semilattice Γ such that $S_\gamma S_\beta \subseteq S_{\gamma\beta}$ for all $\gamma, \beta \in \Gamma$. Such a decomposition is called a *semilattice decomposition* of S.

Let ρ be the kernel of the homomorphism ϕ. Then $S/\rho \cong \Gamma$ so that ρ satisfies the conditions

$$x^2 \, \rho \, x \quad \text{and} \quad xy \, \rho \, yx \qquad \text{for all } x, y \in S.$$

A congruence ρ which satisfies these conditions is called a *semilattice congruence* and if ρ is a semilattice congruence then S/ρ is a semilattice. There is a one-to-one correspondence between semilattice decompositions of S and semilattice congruences on S.

It is easy to see that the intersection of any family of semilattice congruences on a semigroup S is again a semilattice congruence so taking the intersection of all semilattice congruences on S we have:

LEMMA. *Every semigroup admits a smallest semilattice congruence.*

1.6.2. Let S be a commutative semigroup. We say that x *divides* y and write $x \mid y$ if there is a $z \in S^1$ such that $xz = y$. We say that S is an *Archimedean semigroup* if for all $x, y \in S$, there exists an $n \geq 1$ such that $x \mid y^n$.

1.6.3. Let S be a commutative semigroup. Define a congruence η on S by $x \;\eta\; y$ if and only if there exist $m, n \geq 1$ such that $x \mid y^n$ and $y \mid x^m$. (Note that we may always take $m = n$ if desired.)

THEOREM ([15, Theorem 4.13]). *Let S be a commutative semigroup. Then the congruence η is the smallest semilattice congruence on S. Let $\Gamma = S/\eta$, and let $S = \bigcup_{\gamma \in \Gamma} S_\gamma$ be the semilattice decomposition of S corresponding to the natural map $S \to \Gamma$. Then each subsemigroup S_γ is Archimedean. Furthermore, this is the only decomposition of S as a semilattice of Archimedean subsemigroups.*

This decomposition of a commutative semigroup S is called the *Archimedean decomposition* of S.

1.7. The Group of Fractions

One further construction that we shall need is the group of fractions of a cancellative semigroup. This is analogous to the construction of a ring of quotients for a domain.

1.7.1. A semigroup S is *right cancellative* if $xz = yz$ implies $x = y$ for any elements $x, y, z \in S$. Similarly, in a *left cancellative* semigroup, $zx = zy$ implies $x = y$, and a semigroup is *cancellative* if it is both right and left cancellative.

A finite cancellative semigroup is a group.

1.7.2. Let S be a cancellative semigroup. A *group of right fractions* of S is a group G such that S embeds in G and every element of G can be written in the form st^{-1} with $s, t \in S$. The group of right fractions is unique up to isomorphism if it exists, and we write $G = SS^{-1}$.

A necessary and sufficient condition for a cancellative semigroup S to have a group of right fractions is the right Ore condition:

$$sS \cap tS \neq \emptyset \qquad \text{for all } s, t \in S.$$

If S has a group of right fractions, then the product of elements xy^{-1} and uv^{-1} in SS^{-1} is $(xu')(vy')^{-1}$ where u' and y' are elements of S satisfying $uy' = yu'$.

Of course, there is also a notion of a left group of fractions. If S has a left and right group of fractions then they must coincide and we refer to this group as the *group of fractions* of S.

If S is commutative and cancellative, then it clearly satisfies the left and right Ore conditions. It is easy to see that the group of fractions of such a semigroup is Abelian.

CHAPTER 2

Semigroup Graded Rings

Let S be a semigroup. A ring R with a direct sum decomposition $R = \bigoplus_{s \in S} R_s$ of additive subgroups such that $R_s R_t \subseteq R_{st}$ is said to be an *S-graded ring* (see, for example, [55, Chapter 6]). These rings are the focus of our study.

In this chapter, we introduce the basic terminology and techniques needed to deal with such rings. Because Rees matrix semigroups are fundamental to the theory of semigroups, we introduce special notation for rings graded by such semigroups.

2.1. Basic Terminology

Let S be a semigroup. When we say that R is an S-graded ring, it is understood that we are referring to a fixed decomposition of R, the components of which are written R_s indexed by the elements $s \in S$. Throughout this section, R will be an S-graded ring.

2.1.1. The components R_s are called *homogeneous components* and R_s is called the *s-component* of R.

Every element $r \in R$ has a unique decomposition $r = \sum_{s \in S} r_s$ with $r_s \in R_s$; the element r_s is called the *s-component* of r. If we write "let $r = \sum_{s \in S} r_s$ be an element of R" it is understood that $r_s \in R_s$. An element $x \in R_s$ is called a *homogeneous element* of R.

Let X be a subset of S. We write $R_X = \sum_{s \in X} R_s$; in general R_X is just an additive subgroup of R. If $r = \sum_{s \in S} r_s$ is an element of R, then we write r_X for the sum $r_X = \sum_{s \in X} r_s$. There is a natural epimorphism of additive groups $R \to R_X$ given by $r \mapsto r_X$.

If T is a subsemigroup of S then R_T is a subring of R. We may regard $R' = R_T$ as a T-graded ring by putting $R'_t = R_t$ for $t \in T$ or as an S-graded ring by putting

$$R'_s = \begin{cases} R_s & \text{if } s \in T, \\ 0 & \text{if } s \in S \setminus T. \end{cases}$$

If T is a right, left or two-sided ideal of S then R_T is also a right, left, or two-sided ideal of R.

If S is a semigroup with a zero element θ and R is an S-graded ring with $R_\theta = 0$ then we say that R is a *contracted S-graded ring*.

2.1.2. Let $r = \sum_{s \in S} r_s$ be an element of R. The set $\text{supp}(r) = \{s \in S \mid r_s \neq 0\}$ is called the *support* of r. Of course, $\text{supp}(r)$ is a finite set and $\text{supp}(r) = \emptyset$ if and only if $r = 0$.

If A is a subset of R, then the *support* of A is the set $\text{supp}(A) = \bigcup_{a \in A} \text{supp}(a)$; it need not be finite.

The support of R itself is $\text{supp}(R) = \{s \in S \mid R_s \neq 0\}$. An S-graded ring R is said to have *finite support* if $\text{supp}(R)$ is a finite set.

2.1.3. Let I be an ideal of R. If $I = \sum_{s \in S} I \cap R_s$ then I is called a *homogeneous ideal* of R. This condition is equivalent to the assertion that $r = \sum_{s \in S} r_s \in I$ implies $r_s \in I$ for all $s \in S$.

Note that if I is any ideal of R, then $\sum_{s \in S} I \cap R_s$ is always a homogeneous ideal of R and is in fact the largest homogeneous ideal of R contained in I.

Since this definition makes no use of the multiplicative structure of the ring, we may define *homogeneous left ideal*, *homogeneous right ideal*, *homogeneous subring* and even *homogeneous additive subgroup* in a like manner and similar remarks apply.

If X is a subset of S then R_X is clearly a homogeneous additive subgroup of R.

2.1.4. Let I be a homogeneous ideal of R and let $\bar{R} = R/I$. Write the image of $x \in R$ in R/I as $\bar{x}$. For $s \in S$ put $\bar{R}_s = (R_s + I)/I$. We have $\bar{R}_s \bar{R}_t \subseteq \bar{R}_{st}$. Suppose that $x_s \in R_s$ for each s and that $\sum \overline{x_s} = 0$; then $\sum x_s \in I$ and because I is homogeneous, each $x_s \in I$ and so each $\overline{x_s} = 0$. Hence, $\bar{R} = \sum_{s \in S} \bar{R}_s$ is an S-graded ring. Notice that if we write $\bar{x}_s$ for the s-component of $\bar{x}$, then $\bar{x}_s = \overline{x_s}$; we will normally write the former.

In particular, if T is an ideal of S, then R_T is a homogeneous ideal of R and so the quotient $\bar{R} = R/R_T$ is an S-graded ring. Note that $\bar{R}_s = 0$ if $s \in T$. Remembering that we can identify the non-zero elements of the quotient S/T with the elements of $S \setminus T$, we see that $\bar{R}$ can be regarded as a contracted S/T-graded ring. The homogeneous components of $\bar{R}$ corresponding to non-zero elements of S/T are just the homogeneous components R_s of R with $s \in S \setminus T$.

If S is a semigroup with a zero then R/R_0 is a contracted S-graded ring.

We single out the following observation as a lemma. The proof is straightforward, but this result is the key to many of our reductions.

LEMMA. *Let T be an ideal of a semigroup S and let A be a subsemigroup of S such that $A \cap T = \emptyset$. Let R be an S-graded ring, let $\bar{R} = R/R_T$, and denote the image of an element $x \in R$ by $\bar{x}$. Then the map $x \mapsto \bar{x}$ induces a ring isomorphism $R_A \cong \bar{R}_A$.*

2.1.5. It is often convenient to change the grading semigroup. There are two trivial ways of doing this. Let R be an S-graded ring. If S embeds in a larger semigroup T, then we can give R a T-gradation by setting $R_t = 0$ for $t \in T \setminus S$. If $\phi: S \to S'$ is a homomorphism of semigroups, then we obtain a natural S' gradation on R by putting $R_{s'} = \sum_{\substack{s \in S \\ \phi(s)=s'}} R_s$.

In this situation we sometimes say that an ideal is S-homogeneous or S'-homogeneous to indicate which gradation is being considered.

2.2. Rees Matrix Semigroups

As mentioned in §1.5.2, Rees matrix semigroups play an important role in the structure theory of semigroups. We introduce some special notation for dealing with rings graded by such semigroups; this notation will be used throughout this work.

2.2.1. Let $S = \mathfrak{M}^0(1; I, \Lambda; P)$ be an elementary Rees matrix semigroup and let R be a contracted S-graded ring. Recall from §1.5.1 that non-zero elements of S are written $(1)_{i\lambda}$ for $i \in I$ and $\lambda \in \Lambda$. To avoid cumbersome notation, we shall write the homogeneous component corresponding to $(1)_{i\lambda}$ as $R_{i\lambda}$. In a like manner, we shall write the $(1)_{i\lambda}$-homogeneous component of an element $r \in R$ as $r_{i\lambda}$, so that $r = \sum_{i,\lambda} r_{i\lambda}$. (The 0-component of r is zero because R is a contracted S-graded ring).

Remembering the way multiplication works in such semigroups, it is immediate that

$$R_{i\lambda}R_{j\mu} \subseteq R_{i\mu} \qquad \text{if } p_{\lambda j} \neq 0,$$

$$\text{and} \quad R_{i\lambda}R_{j\mu} = 0 \qquad \text{if } p_{\lambda j} = 0.$$

Note that if $p_{\lambda i} = 0$ then $(R_{i\lambda})^2 = 0$ and otherwise $(R_{i\lambda})^2 \subseteq R_{i\lambda}$. So each component $R_{i\lambda}$ is actually a subring of R.

Let $R_{i\bullet} = \sum_{\lambda} R_{i\lambda}$ and $R_{\bullet\lambda} = \sum_{i} R_{i\lambda}$. From the above it is apparent that $R_{i\bullet}$ is a right ideal of R for each $i \in I$ and $R_{\bullet\lambda}$ is a left ideal of R for each $\lambda \in \Lambda$.

Note also that $R_{i\bullet}R_{\bullet\lambda} \subseteq R_{i\lambda}$; in fact for any subset A of R, $R_{i\bullet}AR_{\bullet\lambda} \subseteq R_{i\lambda}$.

2.2.2. Let $S = \mathfrak{M}^0(G; I, \Lambda; P)$ be an arbitrary Rees matrix semigroup and let R be a contracted S-graded ring. Recall (see §1.5.5) that the homomorphism $G \to 1$ induces a homomorphism $S \to S'$ where S' is an elementary Rees matrix semigroup $\mathfrak{M}^0(1; I, \Lambda; P')$. Hence, R can be regarded as a contracted S'-graded ring in the manner of §2.1.5.

If $(g)_{i\lambda} \in S$, we write $R_{(g)_{i\lambda}}$ for the $(g)_{i\lambda}$-component of R with the S-gradation. Since S' is an elementary Rees matrix semigroup, we follow the notation of §2.2.1 and write $R_{i\lambda}$ for the homogeneous components of R with respect to the S'-gradation.

Note that for each i and λ,

$$R_{i\lambda} = \sum_{g \in G} R_{(g)_{i\lambda}}.$$

If $p_{\lambda i} \neq 0$, the set $S_{i\lambda} \setminus \{0\} = \{(g)_{i\lambda} \mid g \in G\}$ is isomorphic to G by Lemma 1.5.3(iii), so that $R_{i\lambda}$ is a G-graded ring. Note that the g-component of $R_{i\lambda}$ with this gradation is $R_{(gp_{\lambda i}^{-1})_{i\lambda}}$ rather than $R_{(g)_{i\lambda}}$. Otherwise, if $p_{\lambda i} = 0$ then $R_{i\lambda}$ is a ring with zero multiplication and can still be considered to be a G-graded ring since it is a direct sum of components indexed by the elements of G.

In this way, we consider $R = \sum_{i,\lambda} R_{i\lambda}$ as a ring graded by the elementary Rees matrix semigroup S' in which every component $R_{i\lambda}$ is a G-graded ring.

Our approach to such rings will be to consider first rings graded by elementary Rees matrix semigroups and then to combine results so obtained with results for group graded rings to deal with rings graded by arbitrary Rees matrix semigroups.

The notation established in this section will be used throughout. In particular, if $S = \mathfrak{M}^0(G; I, \Lambda; P)$ is an arbitrary Rees matrix semigroup and R is a contracted S-graded ring, then we will write $R_{i\lambda}$ for the components of R with respect to the gradation by the associated elementary Rees matrix semigroup S'.

2.2.3. The following simple result is extremely important for our study of rings graded by elementary Rees matrix semigroups.

LEMMA. *Let $S = \mathfrak{M}^0(1; I, \Lambda; P)$ be an elementary Rees matrix semigroup and let R be a contracted S-graded ring. If X is any subset of R then RXR is a homogeneous ideal of R.*

PROOF. Clearly RXR is an ideal of R. For any i and λ, note that $R_{i*}XR_{*\lambda} \subseteq RXR \cap R_{i\lambda}$ so

$$\begin{aligned} RXR &= \sum_{i,\lambda} R_{i*}XR_{*\lambda} \\ &\subseteq \sum_{i,\lambda} RXR \cap R_{i\lambda} \\ &\subseteq RXR \end{aligned}$$

and therefore $RXR = \sum_{i,\lambda} RXR \cap R_{i\lambda}$ is homogeneous. □

2.3. Group-like Subsemigroups

We present another technique that will be useful at times for transferring problems from a semigroup graded ring to subrings graded by certain special subsemigroups.

2.3.1. Let T be a subsemigroup of a semigroup S. If $st \in T$ and $t \in T$ imply $s \in T$ for all elements $s, t \in S$ then T is called a *right group-like subsemigroup* of S. Similarly, if $ts \in T$ and $t \in T$ imply $s \in T$ then T is a *left group-like subsemigroup*.

This terminology is motivated by the observation that a subgroup H of a group G is both left and right group-like in G.

Let X be a subset of S. Define sets $X^{-1}X$ and XX^{-1} by

$$X^{-1}X = \{s \in S \mid Xs = X\} \qquad \text{and} \qquad XX^{-1} = \{s \in S \mid sX = X\}.$$

If $X = \{x\}$ is a singleton, we write $x^{-1}x$ and xx^{-1} instead of $X^{-1}X$ and XX^{-1}.

These sets provide examples of left and right group-like subsemigroups.

LEMMA. *Let X be a non-empty subset of a semigroup S.*

(i) *If $X^{-1}X \neq \emptyset$, then $X^{-1}X$ is a left group-like subsemigroup of S.*

(ii) *If X is a left simple subsemigroup of S then $X^{-1}X$ is a left group-like subsemigroup of S and X is a right ideal of $X^{-1}X$.*

Analogous statements with right and left interchanged hold for XX^{-1}.

PROOF. Suppose that $X^{-1}X$ is not empty. Let $s, t \in X^{-1}X$; then $Xs = Xt = X$, so $X(st) = (Xs)t = Xt = X$ and $st \in X^{-1}X$. So $X^{-1}X$ is a subsemigroup. If $s \in S$ and $ts, t \in X^{-1}X$, then $Xs = (Xt)s = X(ts) = X$ and therefore $s \in X^{-1}X$. Hence $X^{-1}X$ is a left group-like subsemigroup. This proves (i).

If X is left simple, then $Xx = X$ for all $x \in X$, so $X \subseteq X^{-1}X$. By (i), $X^{-1}X$ is a left group-like subsemigroup of S. That X is a right ideal of $X^{-1}X$ follows immediately from the definition. This proves (ii). □

2.3.2. Left and right group-like subsemigroups are useful because of the following result.

PROPOSITION. *Let T be a subsemigroup of a semigroup S and let R be an S-graded ring. Define a map $\pi_T \colon R \to R_T$ by projection: if $r = \sum_{s \in S} r_s \in R$ then $\pi_T(r) = \sum_{s \in T} r_s$. If T is right group-like then π_T is a right R_T-module homomorphism and R_T is a direct summand of R as a right R_T-module. Similarly, if T is left group-like then π_T is a left R_T-module homomorphism and R_T is a direct summand of R as a left R_T-module.*

PROOF. We prove the assertion for a right group-like subsemigroup T of S. Let $s \in S$ and $t \in T$, and let $a_t \in R_t$ and $r_s \in R_s$. If $s \in T$ then $st \in T$ so $\pi_T(r_s)a_t = r_s a_t = \pi_T(r_s a_t)$. Otherwise, if $s \notin T$ then $st \notin T$ and $\pi_T(r_s)a_t = 0 = \pi_T(r_s a_t)$. Because π_T is an additive group homomorphism, we see that $\pi_T(r)a = \pi_T(ra)$ for all $r \in R$. and $a \in R_T$. Hence, π_T is a right R_T-module homomorphism.

Because π_T splits the inclusion map $R_T \to R$, R_T is a direct summand of R as a right R_T module. □

2.4. Group Graded Rings

We end this chapter with a few remarks about group graded rings. We will need the results of this section in later chapters.

2.4.1. Let G be a group and let R be a G-graded ring. The grading is said to be *non-degenerate* if it satisfies: for all $g \in G$ and all $x \in R_{g^{-1}}$, if $xR_g = 0$ or $R_g x = 0$ then $x = 0$. This notion was introduced in [17]. For our purposes, it will be useful to note the following sufficient condition for non-degeneracy:

LEMMA ([17]). *Let G be a group and let R be a G-graded ring with finite support. If R has no non-zero nilpotent homogeneous ideals then the grading is non-degenerate.*

2.4.2. Non-degenerate group graded rings are much better behaved than arbitrary group graded rings. In particular, they have the following properties:

LEMMA ([17]). *Let G be a group with identity 1 and let R be a non-degenerate G-graded ring.*

(i) *R has an identity if and only if R_1 has an identity, and these identities necessarily coincide.*

(ii) *If* supp(R) *is finite and R_1 is semisimple Artinian then R is right and left Artinian.*

2.4.3. In [26], Grzeszczuk discusses a more general construction than the group graded ring which he calls a G-system. Let G be a group. A ring R is a *G-system* if as an additive group, R is the sum of components R_g, indexed by the elements of G, and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. This definition differs from the definition of G-graded ring only in that the sum $R = \sum_{g \in G} R_g$ need not be direct.

It is apparent that a homomorphic image of a G-graded ring is a G-system. Conversely, any G-system $R = \sum_{g \in G} R_g$ is a homomorphic image of the G-graded ring formed by taking the direct sum of the components R_g.

We require the following result in Chapter 6.

LEMMA ([26]). *Let G be a finite group with identity 1 and let R be a G-system. If R has an identity element 1, then $1 \in R_1$.*

CHAPTER 3

Examples of Semigroup Graded Rings

Before proceeding to discuss ring theoretic properties of semigroup graded rings, we will present a selection of examples of such rings. These serve to illustrate the wide range of semigroup graded rings and to provide applications for our later results.

3.1. Semigroup Rings

Semigroup rings have been extensively studied. See for example, Gilmer's book [23] for commutative semigroup rings, and Okniński's book [55] for the non-commutative case.

After giving the construction, we will present a few basic results and some notation which we will have occasion to use later.

3.1.1. Let A be a ring and let S be a semigroup. The *semigroup ring* $A[S]$ is the ring whose elements are all formal sums

$$\sum_{s\in S} r_s s$$

with each coefficient $r_s \in A$ and all but finitely many of the coefficients equal to zero. Addition is defined component-wise so that

$$\sum_{s\in S} r_s s + \sum_{s\in S} q_s s = \sum_{s\in S} (r_s + q_s)s.$$

Multiplication is given by the rule

$$(r_s s)(q_t t) = (r_s q_t)(st)$$

which is extended distributively so that

$$\Big(\sum_{s\in S} r_s s\Big)\Big(\sum_{s\in S} q_s s\Big) = \sum_{s\in S}\Big(\sum_{uv=s} r_u q_v\Big) s.$$

This is, of course, the natural generalisation of the group ring. If $A = K$ is a field, then $K[S]$ is called a *semigroup algebra*.

Semigroup rings are semigroup graded rings in a natural way: $R = A[S]$ is an S-graded ring with $R_s = As$ for each $s \in S$.

If S has a zero element θ, we write $A_0[S]$ for the quotient $A[S]/A\theta$; $A_0[S]$ is called a *contracted semigroup ring*. Of course, $A_0[S]$ is a contracted S-graded ring in the obvious way.

If T is an ideal of S, then $A[T]$ is an ideal of $A[S]$ and the quotient $A[S]/A[T] \cong A_0[S/T]$ (cf §2.1.4).

3.1.2. Suppose that A is a ring with an identity and S is a semigroup. Let I be an ideal of a semigroup ring $A[S]$. Define a relation ρ_I on S by $s \,\rho_I\, t$ if and only if $s - t \in I$. We list some elementary properties of this congruence which can be found in [55, Chapter 4].

LEMMA. *With the notation above,*

(i) *The relation ρ is a congruence.*

(ii) *The semigroup S/ρ embeds in the multiplicative semigroup of the ring $A[S]/I$. This embedding induces a surjective ring homomorphism $A[S/\rho] \to A[S]/I$*

(iii) *The epimorphism $A[S] \twoheadrightarrow A[S]/I$ factors through $A[S/\rho]$. It is the composition of the homomorphism $A[S/\rho] \twoheadrightarrow A[S]/I$ above and the natural homomorphism $A[S] \twoheadrightarrow A[S/\rho]$.*

3.1.3. Recall from §2.1.5 that an S-graded ring R can be given a new gradation by any semigroup S' which is a homomorphic image of S.

In particular, if $R = A[S]$ is a semigroup ring and $\phi\colon S \twoheadrightarrow S'$ is a surjective semigroup homomorphism, then we may give R an S'-gradation by putting

$$R_r = \sum_{\phi(s)=r} As = \Big\{ \sum_{\phi(s)=r} a_s s \;\Big|\; a_s \in A \Big\}$$

for each $r \in S'$.

This technique has been exploited with particular success in the study of semigroup rings of commutative semigroups. If we write $S = \bigcup_{\gamma\in\Gamma} S_\gamma$ for the Archimedean decomposition of S (cf §1.6.3), and put $R = A[S]$, then R has a Γ-gradation in this way. Furthermore, each R_γ is the semigroup ring $A[S_\gamma]$ of the Archimedean semigroup S_γ. In this way, problems about semigroup rings of commutative semigroups can often be solved by considering first semigroup rings of Archimedean semigroups and then semilattice graded rings.

3.2. Munn Algebras

In his investigations of semigroup algebras of completely 0-simple semigroups [50], Munn introduced what is now termed a Munn algebra.

3.2.1. The construction of a Munn algebra is rather similar to that of a Rees matrix semigroup (cf §1.5.1).

Let K be a field and let A be a K-algebra. Let I and Λ be indexing sets, and let P be a $\Lambda \times I$ matrix with entries in A. We define $R = \mathfrak{M}(A; I, \Lambda; P)$ to be a K-algebra as follows. The elements of R are the $I \times \Lambda$ matrices over A with finitely many non-zero entries. Addition is the usual addition of matrices, and scalar multiplication by elements of K is component-wise. Matrices multiply by insertion of the sandwich matrix P. Specifically, if X and Y are two elements of R, then the product of X and Y is

$$XY = X \circ P \circ Y$$

where '$\circ$' denotes ordinary matrix multiplication.

The K-algebra $R = \mathfrak{M}(A; I, \Lambda; P)$ is called a *Munn algebra*.

If the indexing sets I and Λ are finite, we will often write $\mathfrak{M}(A; m, n; P)$; in this case, the row indices are $1, 2, \ldots, m$ and the column indices are $1, 2, \ldots, n$.

3.2.2. Munn algebras arise naturally from semigroup algebras in the following way. Let $S = \mathfrak{M}^0(G; I, \Lambda; P)$ be a Rees matrix semigroup and let $R = K_0[S]$ be a contracted semigroup algebra for some field K. Then R is isomorphic to the Munn algebra $\mathfrak{M}(K[G]; I, \Lambda; P)$ where we now regard the entries in P as elements of the group algebra $K[G]$. An element $\sum k_{(g)_{i\lambda}}(g)_{i\lambda}$ of $K_0[S]$ corresponds to the matrix whose (i, λ)-entry is the group algebra element $\sum_{g \in G} k_{(g)_{i\lambda}} g$.

If S is a finite semigroup with a zero and S has a principal series

$$S = S_1 \supset S_2 \supset \cdots \supset S_n \supset S_{n+1} = \{\theta\}$$

then the semigroup algebra $K[S]$ has a series of ideals

$$K[S] \supset K[S_2] \supset \cdots \supset K[S_n] \supset K\theta$$

and each factor $K[S_i]/K[S_{i+1}] \cong K_0[S_i/S_{i+1}]$ is either a Munn algebra or a nilpotent algebra. Thus Munn algebras are an important component of the study of semigroup algebras of finite semigroups.

3.2.3. Munn algebras can be graded by elementary Rees matrix semigroups. We use the notation of §2.2.1.

Let $R = \mathfrak{M}(A; I, \Lambda; P)$ be a Munn algebra over a K-algebra A. Write the (λ, i)-entry of P as $p_{\lambda i}$. Let P' be the $\Lambda \times I$ matrix whose (λ, i)-entry is

$$p'_{\lambda i} = \begin{cases} 1 & \text{if } p_{\lambda i} \neq 0 \\ \theta & \text{if } p_{\lambda i} = 0 \end{cases}$$

and let S be the elementary Rees matrix semigroup $S = \mathfrak{M}^0(\mathbf{1}; I, \Lambda; P')$.

Then we may regard R as a contracted S-graded ring by putting $R_{i\lambda}$ to be those matrices with all entries zero except the (i, λ)-entry. For if $X \in R_{i\lambda}$ and $Y \in R_{j\mu}$ then the only possible non-zero entry of the matrix product $X \circ P \circ Y$ is the (i, μ)-entry, and furthermore, $X \circ P \circ Y = 0$ if $p'_{\lambda j} = \theta$. So $R_{i\lambda} R_{j\mu} \subseteq R_{i\mu}$ if $p'_{\lambda j} = 1$ and $R_{i\lambda} R_{j\mu} = 0$ if $p'_{\lambda j} = \theta$ as required.

3.3. Generalised Matrix Rings

The class of generalised matrix rings is a well-known class of rings which can be graded by semigroups in a nice way. This class includes rings which are often studied in other contexts such as the endomorphism rings of finite direct sums of modules and the ring of a Morita context.

These rings and some of the results developed in [72] are the starting point for many of our later results. Generalised matrix rings are also discussed in [9].

3.3.1. Let I be an indexing set. A *ring of $I \times I$ generalised matrices* is a ring R with a decomposition (as an additive group)

$$R = \bigoplus_{i,j \in I} R_{ij}$$

such that $R_{ij}R_{kl} \subseteq R_{il}$ if $j = k$ and $R_{ij}R_{kl} = 0$ if $j \neq k$. We write $R = (R_{ij})$ to indicate that R is a generalised matrix ring.

If I is finite, we usually replace it by its cardinal n and speak of a ring of $n \times n$ generalised matrices with components R_{ij} for $1 \leq i, j \leq n$. If we arrange the components R_{ij} as follows:

$$\begin{pmatrix} R_{11} & R_{12} & \dots & R_{1n} \\ R_{21} & R_{22} & \dots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n1} & R_{n2} & \dots & R_{nn} \end{pmatrix}$$

and we write elements of R in the same fashion, then the multiplication follows the same pattern as ordinary matrix multiplication.

3.3.2. Let $R = (R_{ij})$ be a ring of $I \times I$ generalised matrices.

Define a semigroup S as follows. Let the elements of S be the symbols e_{ij}, for $i, j \in I$ together with a zero element 0. Define multiplication by the rule

$$e_{ij}e_{kl} = \begin{cases} e_{il} & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

(and, of course, $0s = s0 = 0$ for all $s \in S$). This multiplication is associative, so S is a semigroup.

If we put $R_{e_{ij}} = R_{ij}$ for all $i, j \in I$ and $R_0 = 0$, then $R = \bigoplus_{s \in S} R_s$ is an S-gradation of the ring R. In fact, R is a contracted S-graded ring.

Note that the semigroup S is isomorphic to the elementary Rees matrix semigroup $\mathfrak{M}^0(1; I, I; \Delta)$ where Δ is the $I \times I$ identity matrix: we identify the element e_{ij} with $(1)_{ij}$. Thus, S is a completely 0-simple inverse semigroup (cf §1.5.6).

Following the notation of §2.2.1, we will write $R_{i\bullet} = \sum_{j \in I} R_{ij}$ and $R_{\bullet j} = \sum_{i \in I} R_{ij}$.

3.3.3. There is another well-known (see for example [17]) way of grading generalised matrices and that is to grade the diagonals by the integers. Specifically, let $R = (R_{ij})$

be a ring of $n \times n$ generalised matrices. For $k \in \mathbf{Z}$, put

$$R_k = \begin{cases} \sum_{\substack{j=i+k \\ 1 \leq i,j \leq n}} R_{ij} & -n+1 \leq k \leq n-1, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that this is indeed a $\mathbf{Z}$-gradation of R. Of course, R has finite support. With this gradation, R has identity component $R_0 = R_{11} \oplus R_{22} \oplus \cdots \oplus R_{nn}$.

This $\mathbf{Z}$-gradation of R can be made into a $\mathbf{Z}_m$-gradation for any $m > 1$ using the canonical map $\mathbf{Z} \to \mathbf{Z}_m$ in the manner of §2.1.5. In particular, choosing $m = 2n+1$, we obtain a gradation by a finite group in which each diagonal is a separate homogeneous component.

3.4. Sums of Left and Right Ideals

Our final example is a rather simple one; we show that a ring which is a direct sum of one-sided ideals can be given a semigroup gradation in a natural way.

3.4.1. Let R be a ring which is a right R-module direct sum of a family of right ideals; say

$$R = \bigoplus_{\alpha \in A} R_\alpha$$

where each R_α is a right ideal of R.

Suppose that we define a product on A by $\alpha\beta = \alpha$ for all $\alpha, \beta \in A$. Then A becomes a left zero semigroup (cf §1.1.2). Since $R_\alpha R_\beta \subseteq R_\alpha$ for all $\alpha, \beta \in A$, we see that R is an A-graded ring.

Similarly, a ring which is a direct sum of left ideals is graded by a right zero semigroup.

This example is extremely important. If we hope to prove a general result about rings graded by finite semigroups, we must be able to prove it for a ring graded by the two element left zero semigroup, that is a ring which is the direct sum of two right ideals. In some of our investigations, for example of perfect rings (see Chapter 6), this turns out to be the crucial case; once we have proved our result for such rings, the general case for rings graded by arbitrary finite semigroups follows rather easily using the structure theory of semigroups.

3.4.2. Let $S = \{e, f\}$ be a two element left zero semigroup. Note that S^0 is completely 0-simple; in fact it is isomorphic to the elementary Rees matrix semigroup $\mathfrak{M}^0(1; 2, 1; (1\ 1))$ if we identify e with $(1)_{11}$ and f with $(1)_{21}$.

Similarly, if S is a two element right zero semigroup, then S^0 is isomorphic to the completely 0-simple Rees matrix semigroup $\mathfrak{M}^0(1; 1, 2; \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right))$.

If R is a ring graded by either of these semigroups (so that it is a sum of two right or left ideals), then we can regard it as a contracted graded ring over an elementary Rees matrix semigroup, that is, a ring of the type discussed in §2.2.1.

CHAPTER 4

Nilpotency of the Jacobson Radical

In this chapter, we investigate the Jacobson radical of semigroup graded rings. In particular, we study various nilpotency conditions of the Jacobson radical.

The way we proceed in this chapter is illustrative of a general method of attacking problems about semigroup graded rings, at least for finite semigroups. We first consider three separate cases: group graded rings (for which most of the results we need are easily deduced from known results), rings graded by elementary Rees matrix semigroups, and ideal extensions. The results obtained can then be pieced together using the structure theory of semigroups to deal with semigroup graded rings. The techniques developed here will be used in subsequent chapters.

Many of the results obtained in this chapter will appear in [11].

4.1. Radical Properties

We are primarily interested in the Jacobson radical because of its importance in the structure theory of rings. However, it will be convenient to make use of a couple of other radical properties in our investigations.

We begin by reviewing the definitions of the three radicals that are of interest to us and introduce a bit of terminology from radical theory.

4.1.1. Let R be a ring. The *Jacobson radical* of R, written $J(R)$, is an ideal defined by the following equivalent properties:

(i) $J(R)$ is the intersection of the annihilators of all simple right (or left) R-modules.

(ii) $J(R)$ is the intersection of all maximal regular right (or left) ideals of R.

(iii) $J(R)$ is the sum of all quasi-regular right (or left) ideals of R.

(iv) $J(R)$ is the largest quasi-regular ideal of R.

We will make us of (iii) and (iv) particularly.

Recall that an element $x \in R$ is *right quasi-regular* if there is a $y \in R$ such that $x+y+xy = 0$. Similarly, we can define *left quasi-regular* elements. A one- or two-sided ideal is *right quasi-regular* if every element of the ideal is right quasi-regular. While an individual element of R may be right quasi-regular but not left quasi-regular, it turns out that every element of $J(R)$ is both left and right quasi-regular. Since every right (or left) quasi-regular ideal (or one-sided ideal) is contained in $J(R)$, the same is true for such ideals. Hence, we will simply refer to (one- or two-sided) ideals as being *quasi-regular*.

An element x of R is *quasi-regular* if is both right and left quasi-regular. In this case, if $x + y + xy = 0 = x + z + zx$ then $y = z$ is unique; this element is called the *quasi-inverse* of x.

Note also that if a (one- or two-sided) ideal A of R is quasi-regular, then it is easy to see that the quasi-inverse of each element of A is also in A. So A is a quasi-regular ideal of any ring B such that $A \subseteq B \subseteq R$.

4.1.2. The *prime radical* of a ring R, written $B(R)$ is characterised by the following equivalent properties:

(i) $B(R)$ is the intersection of all prime ideals of R.

(ii) $B(R)$ is the smallest semiprime ideal of R.

(iii) $B(R)$ is the set of all $x \in R$ such that every sequence $x_0 = x$, $x_i \in x_{i-1}Rx_{i-1}$ for $i > 0$ is eventually zero.

The sequences constructed in (iii) are called *m-sequences*.

4.1.3. The *Levitzki radical* $L(R)$ of a ring R is the largest locally nilpotent ideal of R. Recall that a subring A of R is *locally nilpotent* if every finite subset of A generates a nilpotent subring. It is also true that $L(R)$ contains every locally nilpotent one-sided ideal of R.

4.1.4. The three radicals given above are examples of a more general notion of a radical property of a ring. For our purposes, we require a few properties of general radicals and a few definitions of properties that are common to the Jacobson, prime, and Levitzki radicals, although not held by all radicals. The reader is referred to [21] for details.

Let $\mathcal{R}$ be a property of rings which defines a class of rings, the *$\mathcal{R}$-rings* or *$\mathcal{R}$-radical rings*. Then $\mathcal{R}$ is a *radical property* if it satisfies:

(i) The class of $\mathcal{R}$-rings is closed under homomorphic images.

(ii) Every ring R has a largest ideal in the class of $\mathcal{R}$-rings; this ideal is written $\mathcal{R}(R)$.

(iii) $\mathcal{R}(R/\mathcal{R}(R)) = 0$.

The ideal $\mathcal{R}(R)$ is called the *$\mathcal{R}$-radical* of R. A ring is *$\mathcal{R}$-radical* if $\mathcal{R}(R) = R$ and is *$\mathcal{R}$-semisimple* if $\mathcal{R}(R) = 0$.

It is not difficult to see that the class of $\mathcal{R}$-radical rings is closed under ideal extensions; that is, if $I \lhd R$ and I, R/I are $\mathcal{R}$-radical then R is also $\mathcal{R}$-radical.

4.1.5. A radical $\mathcal{R}$ is said to be *supernilpotent* if $R/\mathcal{R}(R)$ is semiprime, or equivalently, $\mathcal{R}(R)$ contains all nilpotent ideals of R.

Note that to show that an element $x \in R$ belongs to a supernilpotent radical $\mathcal{R}(R)$, it suffices to show that $RxR \subseteq \mathcal{R}(R)$, even if R does not have an identity. For if $RxR \subseteq \mathcal{R}(R)$ and I is the ideal of R generated by x, then $I^3 \subseteq RxR \subseteq \mathcal{R}(R)$, and since $R/\mathcal{R}(R)$ is semiprime, $x \in I \subseteq \mathcal{R}(R)$. We will frequently make use of this remark.

4.1.6. A radical $\mathcal{R}$ is said to be *hereditary* if every ideal of an $\mathcal{R}$-radical ring is $\mathcal{R}$-radical, or equivalently, if $\mathcal{R}(R) \cap I = \mathcal{R}(I)$ for every ideal I of R.

The Jacobson, prime, and Levitzki radicals are all hereditary and supernilpotent (see [21]). In fact, the prime radical is the smallest supernilpotent radical and for any ring R we have $B(R) \subseteq L(R) \subseteq J(R)$.

4.1.7. One further property of these radicals is worth mentioning.

Suppose that R is a ring without an identity. There is a standard way of embedding R in a slightly larger ring with an identity. Put $R^1 = R \times \mathbb{Z}$, and define operations on R^1 by

$$(x, m) + (y, n) = (x + y, m + n)$$

and

$$(x, m)(y, n) = (xy + my + nx, mn).$$

Then R^1 is a ring and the map $x \mapsto (x, 0)$ embeds R as an ideal of R^1.

For $\mathcal{R} = J$, B, or L, it is true that $\mathcal{R}(R) = \mathcal{R}(R^1)$. This enables us to extend many results that are known for rings with identity to rings without identity.

4.1.8. For a given radical property, it is often possible to construct a graded radical for group graded rings by a suitable modification of the definition of the radical property. This is done by choosing a suitable definition of the radical property and replacing 'ideal' with 'homogeneous ideal' and 'module' with 'graded module'. The drawback of this approach is that different characterisations of the ungraded radical might lead to inequivalent definitions of the graded radical. Nevertheless, for the common radicals, graded versions have been defined in this way and the relationships between the graded and ungraded versions have been extensively investigated, particularly in the case of the Jacobson radical.

We will need graded versions of the Levitzki and Jacobson radicals in this chapter.

The graded Jacobson radical J_{gr} of a group graded ring R is defined to be the intersection of the annihilators of the graded simple right (or left) R-modules. Equivalently, $J_{gr}(R)$ is the largest homogeneous ideal I of R such that $I \cap R_1 \subseteq J(R_1)$ (where $1 \in G$ is the identity) [8]. Note that $J_{gr}(R)$ behaves in a similar way to the ordinary Jacobson radical. For example, if N is a homogeneous nil ideal of R, then $N \cap R_1 \subseteq J(R_1)$ and so $N \subseteq J_{gr}(R)$.

The graded Levitzki radical $L_{gr}(R)$ is the largest homogeneous locally nilpotent ideal of R [6]. It is easy to see that $L_{gr}(R)$ is just the homogeneous part $\sum L(R) \cap R_g$ of $L(R)$.

Beattie and Stewart [6] have developed a theory of reflected radicals which provides a general method for constructing a graded radical $\mathcal{R}_{ref}$ for group graded rings from an ordinary radical $\mathcal{R}$. The details of this theory need not concern us here; however, we will use some of their results in §4.2.4 to characterise the graded Levitzki radical.

4.2. The Jacobson Radical of Group Graded Rings

In this section, we deal with the case of group graded rings. The questions in which we are interested are easily answered for group graded rings using known results. Here we summarise the answers and record a few standard results that will be useful later.

Throughout this section, the identity element of a group will always be denoted by 1.

4.2.1. One slight difficulty that we must overcome is that many authors who have investigated group graded rings assume that their rings have identities. Fortunately, it is usually not difficult to extend results about radicals to rings without identity. We will explain how this works; in cases where this method works we will simply refer back to this explanation.

Let R be a ring graded by a group G, and suppose that R does not have an identity. The usual method of adjoining an identity (see §4.1.7) works especially well here in that the resulting ring R^1 can be given a G-gradation by putting $(R^1)_1 = (R_1)^1$ and $(R^1)_g = R_g$ for $g \neq 1$. In other words, adding an identity to R only changes the 1-component R_1 to $(R_1)^1$.

As mentioned in §4.1.4, the Jacobson, prime, and Levitzki radicals are unchanged by the adjunction of an identity and the same is true for the graded versions of these radicals.

Suppose then that we have a theorem about group graded rings that has been proved for rings with an identity. To extend the theorem to rings without identity, proceed as follows. Let R be a group graded ring without identity which otherwise satisfies the hypotheses of the theorem. Adjoin an identity to R and check that the hypotheses are still satisfied. If so, apply the theorem to R^1. We conclude that R^1 satisfies some property. Finally, check whether we can restrict our attention to R and conclude that it too satisfies the property.

For example, suppose that we know the following for rings with identity: if R is graded by a finite group then $J(R_1)$ nilpotent implies that $J(R)$ is nilpotent. Let R be ring without identity graded by a finite group and suppose $J(R_1)$ is nilpotent. Then $J((R^1)_1) = J((R_1)^1) = J(R_1)$ is nilpotent, so $J(R) = J(R^1)$ is nilpotent.

Of course, this method does not always work. Sometimes theorems cannot be extended to rings without identity and sometimes another approach is needed.

4.2.2. We record a few basic facts about the Jacobson radical of group graded rings.

LEMMA. *Let G be a group and let R be a G-graded ring.*

(i) *If H is a subgroup of G, then $J(R) \cap R_H \subseteq J(R_H)$.*

(ii) $J(R) \cap R_1 \subseteq J(R_1)$.

(iii) $J_{gr}(R) \cap R_1 = J(R_1)$.

Suppose further that $n = |\mathrm{supp}(R)|$ is finite. Then

(iv) $J_{gr}(R) \subseteq J(R)$.

(v) $J(R)^n \subseteq J_{gr}(R)$.

(vi) $J(R) \cap R_1 = J(R_1)$.

PROOF. All these statements can be extended to rings without identity using the method of §4.2.1. We give references to proofs for rings with identity. Statement (i) is well-known; it follows immediately from [58, Lemma 7.1.3]. Statement (ii) is a special case of (i). Statement (iii) is [4, Corollary 3.3]. Statements (iv) and (v) follow from [49, Proposition 4.6] and [49, Corollary 4.4] respectively, and in the case of finite groups from [16, Theorem 4.4]. Finally, (vi) is an immediate consequence of (ii), (iii), and (iv). □

4.2.3. In order to prove the result that we are aiming for on the nilpotence of the Jacobson radical for rings graded by finite semigroups, we first need the corresponding result for group graded rings. The proof of the following proposition uses the argument of [17, Lemma 1.1]. We prove this for graded rings with finite support because we will need this full generality later.

PROPOSITION ([11]). *Let G be a group and let R be a G-graded ring with n $|\mathrm{supp}(R)|$ finite. Suppose that $J(R_1)$ is nilpotent with index of nilpotency d. Then $J(R)^{n^2 d} = 0$. Conversely, if $J(R)$ is nilpotent, so is $J(R_1)$.*

PROOF. The last statement follows immediately from Lemma 4.2.2(vi).

Suppose then that $J(R_1)^d = 0$. We first show that $J_{gr}(R)$ is nilpotent. Let $x_1, x_2, \ldots, x_{nd}$ be homogeneous elements of $J_{gr}(R)$ with $x_i \in R_{g_i}$ for $1 \le i \le nd$. Let $h_0 = 1$ and $h_k = g_1 g_2 \ldots g_k$ for $1 \le k \le nd$. If some $h_k \notin \mathrm{supp}(R)$ then $x_1 x_2 \ldots x_{nd} = 0$. Suppose then that $h_k \in \mathrm{supp}(R)$ for $0 \le k \le nd$. Among the $nd + 1$ elements $h_0, h_1, \ldots, h_{nd}$ of G, there must be $d + 1$ with the same value, say $h_{j_0} = h_{j_1} = \cdots = h_{j_d}$ with $0 \le j_0 < j_1 < \cdots < j_d \le nd$. For $0 \le k \le d - 1$ we have

$$g_{j_k+1} g_{j_k+2} \cdots g_{j_{k+1}} = 1$$

and so by Lemma 4.2.2(iii),

$$x_{jn+1}x_{jn+2}\cdots x_{jn+n} \in J_{gr}(R) \cap R_1 = J(R_1).$$

Hence $x_{jn+1}x_{jn+2}\cdots x_{nd} \in J(R_1)^d = 0$. Because every element in $J_{gr}(R)$ is a sum of homogeneous elements in $J_{gr}(R)$, we have shown that the product of nd elements of $J_{gr}(R)$ is zero; in other words, that $J_{gr}(R)^{nd} = 0$. Finally, by Lemma 4.2.2(v), $J(R)^{n^2d} \subseteq J_{gr}(R)^{nd} = 0$ and the result is proved. □

4.2.4. We now turn our attention to local nilpotence, beginning with a lemma relating the graded Levitzki radical to the identity component of a graded ring. This is analogous to the alternative characterisation of the graded Jacobson radical given in §4.1.8.

LEMMA. *Let R be a ring graded by a locally finite group G. Then $L_{gr}(R)$ is the largest homogeneous ideal I of R such that $I \cap R_1 \subseteq L(R_1)$.*

PROOF. The lemma just requires a combination of results about reflected radicals. Firstly, [6, Theorem 3.3] states that $L_{ref}(R) = L_{gr}(R)$ provided that G is locally finite. Secondly, [7, Corollary 2.6] says that $L_{ref}(R)$ is the largest homogeneous ideal I of R such that $I \cap R_1 \subseteq L(R_1)$. □

4.2.5. We also require a simple result about powers of finitely generated subrings.

LEMMA. *Let T be finitely generated subring of a ring R. Then T^n is also finitely generated for any n.*

PROOF. Let $\{x_1, x_2, \ldots, x_m\}$ be a finite set of generators of A. It is easily verified that A^n is generated by the finite set of words in the x_i of length between n and $2n-1$. □

4.2.6. Our next result is similar to Proposition 4.2.3, except that we consider local nilpotence rather than nilpotence. In this case, the grading group can be locally finite.

PROPOSITION ([11]). *Let G be a locally finite group and let R be a G-graded ring such that $J(R_1)$ is locally nilpotent. Then $J(R)$ is locally nilpotent.*

PROOF. That $J(R_1)$ is locally nilpotent implies that $J(R_1) = L(R_1)$. Let H be a subgroup of G. We know that $J_{gr}(R_H)$ is the largest homogeneous ideal I of R_H such that $I \cap R_1 \subseteq J(R_1)$ so by Lemma 4.2.4, $J_{gr}(R_H) = L_{gr}(R_H)$ and therefore $J_{gr}(R_H)$ is locally nilpotent.

Let A be a finite subset of $J(R)$ and let T be the subring of R generated by A. Let H be the subgroup generated by the support of A; then $n = |H|$ is finite because G is locally finite. By Lemma 4.2.2(i), $J(R) \cap R_H \subseteq J(R_H)$, so $T \subseteq J(R_H)$. By Lemma 4.2.2(v), $T^n \subseteq J_{gr}(R_H)$. But T^n is finitely generated by Lemma 4.2.5, and $J_{gr}(R_H)$ is locally nilpotent. Hence T^n and therefore T are nilpotent subrings of R. So $J(R)$ is locally nilpotent as claimed. □

4.3. Elementary Rees Matrix Semigroups

Our next task is to develop similar results for rings graded by elementary Rees matrix semigroups, or more precisely, contracted graded rings.

4.3.1. We begin with a characterisation of various radicals for these rings. We give a general result that characterises radicals satisfying certain conditions which we then show are satisfied by the Jacobson, prime, and Levitzki radicals. Note that the first characterisation given in the Proposition below was obtained by Kelarev [42] for the Jacobson, Levitzki, and prime radicals in the case of rings graded by rectangular bands; the second characterisation was obtained for these radicals in [11].

Proposition ([11, 42]). *Let $\mathcal{R}$ be a hereditary supernilpotent radical which satisfies the following conditions:*

(A) *If A is a right (or left) ideal of R then $\mathcal{R}(R) \cap A \subseteq \mathcal{R}(A)$.*

(B) *If A and B are right (or left) ideals of R and $A \subseteq \mathcal{R}(B)$ then $A \subseteq \mathcal{R}(R)$.*

Let $S = \mathfrak{M}^0(1; I, \Lambda; P)$ be an elementary Rees matrix semigroup and let R be a contracted S-graded ring. Then:

(i) *$\mathcal{R}(R)$ is the largest ideal I of R such that $I \cap R_{i\lambda} \subseteq \mathcal{R}(R_{i\lambda})$ for all i and λ.*

(ii) $\mathcal{R}(R) = \{x \in R \mid RxR \subseteq \sum_{i,\lambda} \mathcal{R}(R_{i\lambda})\}$.

PROOF. We use the notation and observations of Section 2.2. Applying property (A) twice, we see that for any i and λ,

$$\begin{aligned}\mathcal{R}(R)\cap R_{i\lambda} &= (\mathcal{R}(R)\cap R_{i*})\cap R_{i\lambda}\\ &\subseteq \mathcal{R}(R_{i*})\cap R_{i\lambda}\\ &\subseteq \mathcal{R}(R_{i\lambda}).\end{aligned}$$

Suppose that I is an ideal of R and $I\cap R_{i\lambda}\subseteq \mathcal{R}(R_{i\lambda})$ for all i and λ. Because R is supernilpotent, to prove (i) it suffices to show that $RIR\subseteq \mathcal{R}(R)$.

Note that

$$RIR=\sum_{i,\lambda} R_{i*}IR_{*\lambda}.$$

For any i and λ,

$$R_{i*}IR_{*\lambda}\subseteq I\cap R_{i\lambda}\subseteq \mathcal{R}(R_{i\lambda})\subseteq R_{i*}.$$

But $R_{i*}IR_{*\lambda}$ and $R_{i\lambda}$ are left ideals of R_{i*}, hence $R_{i*}IR_{*\lambda}\subseteq \mathcal{R}(R_{i*})$ by property (B). This holds for any λ, so

$$R_{i*}IR=\sum_{\lambda} R_{i*}IR_{*\lambda}\subseteq \mathcal{R}(R_{i*}).$$

But $R_{i*}IR$ and R_{i*} are right ideals of R, so by property (B) again, $R_{i*}IR\subseteq \mathcal{R}(R)$ and therefore

$$RIR=\sum_{i} R_{i*}IR\subseteq \mathcal{R}(R)$$

as desired. This proves (i).

Let $K = \{x \in R \mid RxR \subseteq \sum_{\iota,\lambda} \mathcal{R}(R_{\iota\lambda})\}$; then K is an ideal of R. Now for any ι and λ,

$$R\mathcal{R}(R)R \cap R_{\iota\lambda} \subseteq \mathcal{R}(R) \cap R_{\iota\lambda} \subseteq \mathcal{R}(R_{\iota\lambda})$$

by (i). Since $R\mathcal{R}(R)R$ is a homogeneous ideal by Lemma 2.2.3,

$$\begin{aligned} R\mathcal{R}(R)R &= \sum_{\iota,\lambda} R\mathcal{R}(R)R \cap R_{\iota\lambda} \\ &\subseteq \sum_{\iota,\lambda} \mathcal{R}(R_{\iota\lambda}). \end{aligned}$$

Hence, by the definition of K, $\mathcal{R}(R) \subseteq K$.

Conversely, $RKR \subseteq \sum_{\iota,\lambda} \mathcal{R}(R_{\iota\lambda})$ and RKR is homogeneous by Lemma 2.2.3, so $RKR \cap R_{\iota\lambda} \subseteq \mathcal{R}(R_{\iota\lambda})$. Therefore, $RKR \subseteq \mathcal{R}(R)$ by (i) and so $K \subseteq \mathcal{R}(R)$ because $\mathcal{R}$ is supernilpotent. This proves (ii). □

4.3.2. We now demonstrate that the Jacobson, prime and Levitzki radicals satisfy conditions (A) and (B) of Proposition 4.3.1.

COROLLARY. *Let S be an elementary Rees matrix semigroup and let R be a contracted S-graded ring. The Jacobson, prime, and Levitzki radicals satisfy properties (A) and (B) of Proposition* 4.3.1. *Hence, these radicals of R are characterised by statements* (i) *and* (ii) *of Proposition* 4.3.1.

PROOF. We need only show that these radicals satisfy properties (A) and (B) of the proposition. We will use right ideals; the proofs for left ideals are analogous. Throughout this proof, let R be a ring and let $A \subseteq B$ be right ideals of R.

Note that $J(R) \cap A$ is a quasi-regular ideal of A, so $J(R) \cap A \subseteq J(A)$. If $A \subseteq J(B)$, then A is a quasi-regular right ideal of B. But A is a right ideal of R, so A is actually a quasi-regular right ideal of R. Hence $A \subseteq J(R)$. This proves that J satisfies (A) and (B).

Similarly, $L(R) \cap A$ is a locally nilpotent ideal of A so $L(R) \cap A \subseteq L(A)$. If $A \subseteq L(B)$, then A is locally nilpotent, and since A is a right ideal of R, $A \subseteq L(R)$. Hence L satisfies (A) and (B).

Let $a \in B(R) \cap A$. Let $\{x_i\}_{i \geq 0}$ be an m-sequence in A with $x_0 = a$. Then $A \subseteq R$ and $a \in B(R)$ implies that $x_k = 0$ for some k. So $a \in B(A)$. This shows that B satisfies (A).

Suppose that $A \subseteq B(B)$. Let $a \in A$, and let $x_0 = a$, $x_i = x_{i-1} r_{i-1} x_{i-1}$ for $i > 0$ be an m-sequence with each $r_i \in R$. Let $y_0 = x_0 r_0$ and $y_i = y_{i-1} x_0 r_i y_{i-1}$ for $i > 0$. Since $x_0 r_i \in B$ for all i, the sequence $\{y_i\}_{i \geq 0}$ is an m-sequence in B starting with $y_0 = x_0 r_0 = a r_0 \in A \subseteq B(B)$. Hence, there is a k such that $y_k = 0$. But it is easy to prove by induction that $x_{i+1} = y_i x_0$ for all i and so $x_{k+1} = 0$. So $a \in B(R)$. Therefore, B satisfies (B). □

We remark that the characterisation (ii) of the Jacobson radical of such rings is a generalisation of similar results for Munn algebras (see, for example, [55, Corollary 5.18]) and for a certain class of rings graded by rectangular bands obtained by Munn [52, Corollary 3.2].

Note that the fact that the Jacobson radical satisfies property (A) of Proposition 4.3.1 is well-known and we shall frequently use this without explicit reference.

4.3.3. We prove a result on the nilpotence of the Jacobson radical for rings graded by elementary Rees matrix semigroups. This is similar to Proposition 4.2.3 except that all idempotents of the semigroup must be considered.

PROPOSITION ([11]). *Let S be a finite elementary Rees matrix semigroup and let R be a contracted S-graded ring. Then $J(R)$ is nilpotent if $J(R_e)$ is nilpotent for all idempotents $e \in S$.*

PROOF. We may suppose that $S = \mathfrak{M}^0(1; I, \Lambda; P)$ where I and Λ are finite sets since S is finite.

Note that with the given hypothesis, we may assume that $J(R_{i\lambda})$ is nilpotent for all i and λ, for if $R_{i\lambda}$ does not correspond to an idempotent element of S then $(R_{i\lambda})^2 = 0$.

Let $n = |S| - 1 = |I||\Lambda|$ and let d be the maximum index of nilpotency of all the rings $J(R_{i\lambda})$.

Let $m = nd + 2$ and let $x_1, x_2, \ldots, x_m$ be a sequence of homogeneous elements of $J(R)$ with $x_j \in R_{i_j\lambda_j}$ for all j. Consider the $nd + 1$ pairs (λ_j, i_{j+1}), $1 \leq j \leq nd + 1$. Since there are only $n = |I||\Lambda|$ possible distinct pairs, one such pair must occur at least $d + 1$ times, say

$$\lambda_{j_1} = \lambda_{j_2} = \cdots = \lambda_{j_{d+1}} = \lambda$$

and

$$i_{j_1+1} = i_{j_2+1} = \cdots = i_{j_{d+1}+1} = i$$

for some indices j_l with $1 \le j_1 < j_2 < \cdots < j_{d+1} \le nd + 1$. Then for $1 \le l \le d$, we have

$$x_{j_l+1} \in R_{i_{j_l+1}\lambda_{j_l+1}} = R_{i\lambda_{j_l+1}}$$

and

$$x_{j_{l+1}} \in R_{i_{j_{l+1}}\lambda_{j_{l+1}}} = R_{i_{j_{l+1}}\lambda}$$

so that

$$x_{j_l+1}x_{j_l+2}\cdots x_{j_{l+1}} \in R_{i\lambda}.$$

But each $x_j \in J(R)$ so

$$x_{j_l+1}x_{j_l+2}\cdots x_{j_{l+1}} \in J(R) \cap R_{i\lambda} \subseteq J(R_{i\lambda}),$$

the last inclusion coming from Proposition 4.3.1 and Corollary 4.3.2. Hence,

$$x_{j_1+1}\cdots x_{j_2}x_{j_2+1}\cdots x_{j_{d+1}} \in J(R_{i\lambda})^d = 0$$

and so $x_1x_2\ldots x_m = 0$.

We have shown that any product of m homogeneous elements of $J(R)$ is zero. By Lemma 2.2.3, $RJ(R)R$ is a homogeneous ideal of R that is contained in $J(R)$, so $(RJ(R)R)^m = 0$. Finally, $J(R)^{3m} \subseteq (RJ(R)R)^m = 0$. □

Combining this result with group-graded results we easily prove a similar result for arbitrary finite Rees matrix semigroups.

COROLLARY ([11]). *Let S be a finite Rees matrix semigroup and let R be a contracted S-graded ring. If $J(R_e)$ is nilpotent for each non-zero idempotent e of S then $J(R)$ is nilpotent.*

PROOF. Write $S = \mathfrak{M}^0(G; I, \Lambda; P)$ and let S' be the homomorphic image of S induced by the trivial homomorphism $G \to 1$ as in §1.5.5. We use the notation of §2.2.2. Let $i \in I$ and $\lambda \in \Lambda$. If $p_{\lambda i} = \theta$ then $(R_{i\lambda})^2 = 0$ and $J(R_{i\lambda})$ is nilpotent. Otherwise $R_{i\lambda}$ is a G-graded subring the identity component of which is R_e for some idempotent $e \in S$ in which case $J(R_{i\lambda})$ is nilpotent by Proposition 4.2.3. Then, regarding R as an S'-graded ring, its homogeneous components $R_{i\lambda}$ all have nilpotent Jacobson radical. Since S' is a finite elementary Rees matrix semigroup, that $J(R)$ is nilpotent follows from Proposition 4.3.3. □

4.3.4. Note that the converse of Corollary 4.3.3 is true. This follows easily from the following lemma.

LEMMA. *Let A be a right (or left) ideal of a ring R. If $J(R)$ is nilpotent then $J(A)$ is nilpotent.*

PROOF. Assume that A is a right ideal; the other case being similar. Notice that $J(A)A$ is a right ideal of R and $J(A)A \subseteq J(A)$. By Corollary 4.3.2, the Jacobson radical satisfies property (B) of Proposition 4.3.1 and so $J(A)A \subseteq J(R)$. Hence, $J(A)^2 \subseteq J(R)$ and so $J(A)$ is nilpotent if $J(R)$ is. □

PROPOSITION. *Let S be a finite Rees matrix semigroup and let R be a contracted S-graded ring. If $J(R)$ is nilpotent then $J(R_e)$ is nilpotent for each non-zero idempotent e of S.*

PROOF. Suppose that $J(R)$ is nilpotent. Let $S = \mathfrak{M}^0(G; I, \Lambda; P)$. Let e be an idempotent element of S, then $e = (g)_{i\lambda}$ for some $g \in G$, $i \in I$, and $\lambda \in \Lambda$. Now, R_{i*} is a right ideal of R and $R_{i\lambda}$ is a left ideal of R_{i*}, so that applying the lemma twice, we see that $J(R_{i\lambda})$ is nilpotent. But $R_{i\lambda}$ is graded by a finite group isomorphic to G and e is the identity of this group. So $J(R_e)$ is nilpotent by Proposition 4.2.3. □

We remark that this proposition is still true if I and Λ are infinite, since the proof makes no use of their finiteness.

4.3.5. We prove a result similar to Proposition 4.3.3 for local nilpotence of the Jacobson radical. In this case there is no need for the semigroup to be finite. Note however that an elementary Rees matrix semigroup is locally finite.

PROPOSITION ([11]). *Let S be an elementary Rees matrix semigroup and let R be a contracted S-graded ring. If $J(R_e)$ is locally nilpotent for all idempotents $e \in S$ then $J(R)$ is locally nilpotent.*

PROOF. Let $S = \mathfrak{M}^0(1; I, \Lambda; P)$.

Once again note that we may assume that $J(R_{i\lambda})$ is locally nilpotent for all i and λ, for if the element $(1)_{i\lambda}$ is not idempotent, then the corresponding component $R_{i\lambda}$

satisfies $(R_{i\lambda})^2 = 0$. Then we have $J(R_{i\lambda}) = L(R_{i\lambda})$ for all i and λ. Since J and L both satisfy the properties (A) and (B) of Proposition 4.3.1 by Corollary 4.3.2, we see immediately using either characterisation of Proposition 4.3.1 that $J(R) = L(R)$. Hence $J(R)$ is locally nilpotent as claimed. □

Once again, we can easily extend this result to a more general class of Rees matrix semigroups.

COROLLARY ([11]). *Let S be a locally finite Rees matrix semigroup and let R be a contracted S-graded ring. If $J(R_e)$ is locally nilpotent for each non-zero idempotent e of S then $J(R)$ is locally nilpotent.*

PROOF. The proof is essentially the same as that of Corollary 4.3.3. We may assume that S contains at least one non-zero idempotent e, for otherwise S is a null semigroup and then $R^2 = 0$. If we write $S = \mathfrak{M}^0(G; I, \Lambda; P)$ then G is locally finite because it is isomorphic to the maximal subgroup H_e of S determined by any non-zero idempotent e. So the result follows by Proposition 4.2.6 and Proposition 4.3.5. □

We do not know if the converse is true, for we do not know if the converse is true in the case of a ring graded by a locally finite group. However, if we restrict our semigroups to be those Rees matrix semigroups which have only finite subgroups, then the converse to the corollary does hold. For in this case, the result is true for the group graded subrings $R_{i\lambda}$ by virtue of Lemma 4.2.2(vi), and we can prove the converse in the same manner as Proposition 4.3.4.

4.4. Ideal Extensions

One final step is needed to complete the set of tools that will enable us to prove results about rings graded by general semigroups. We must show that nilpotence and local nilpotence of the Jacobson radical are preserved by ideal extensions.

4.4.1. We first prove a general result about radicals of ideal extensions. This is a generalisation of a result of Wauters [70, Lemma 1.3] for rings graded by a two-element semilattice and the proof is essentially the same.

LEMMA ([11]). *Let I be an ideal of a ring R and let $\mathcal{R}$ be a hereditary supernilpotent radical. Then $\mathcal{R}(R) = \{r \in R \mid (I + r) \in \mathcal{R}(R/I)$ and $rI \subseteq \mathcal{R}(I)\}$.*

PROOF. Let $K = \{r \in R \mid (I + r) \in \mathcal{R}(R/I)$ and $rI \subseteq \mathcal{R}(I)\}$. Note that if $x \in R$ and $rI \subseteq \mathcal{R}(I)$ then $rxI \subseteq rI \subseteq \mathcal{R}(I)$ and $xrI \subseteq x\mathcal{R}(I) \subseteq \mathcal{R}(I)$ because $\mathcal{R}(I) = I \cap \mathcal{R}(R)$ is an ideal of R. It then follows easily that K is an ideal of R, the other verifications being straightforward.

Because $(I + \mathcal{R}(R))/I \subseteq \mathcal{R}(R/I)$ and $\mathcal{R}(R)I \subseteq I \cap \mathcal{R}(R) = \mathcal{R}(I)$ it follows that $\mathcal{R}(R) \subseteq K$.

For the converse, consider the homomorphism $\varphi \colon K \to R/I$ which is the composition of the projection $R \to R/I$ with the inclusion $K \to R$. The image of φ is the ideal $(I + K)/I$ of R/I and it is contained in $\mathcal{R}(R/I)$ by definition of K. Since $\mathcal{R}$ is hereditary, the ideal $(I + K)/I$ is $\mathcal{R}$-radical.

The kernel of φ is the ideal $K \cap I = \{r \in I \mid rI \subseteq \mathcal{R}(I)\}$. Clearly, $\mathcal{R}(I)$ is contained in this set. We write rI^1 for the right ideal of I generated by an element $r \in I$. If $rI \subseteq \mathcal{R}(I)$ then $(rI^1)^2 \subseteq rI \subseteq \mathcal{R}(I)$; so $r \in rI^1 \subseteq \mathcal{R}(I)$ because $\mathcal{R}$ is supernilpotent. Hence $\ker \varphi = \mathcal{R}(I)$.

So φ induces an isomorphism $K/\mathcal{R}(I) \cong (I + K)/I$ and therefore $K/\mathcal{R}(I)$ is $\mathcal{R}$-radical. Because radical classes are closed under ideal extensions K is $\mathcal{R}$-radical. Hence $K \subseteq \mathcal{R}(R)$. □

Since the Jacobson, Levitzki, and prime radicals are hereditary and supernilpotent, we may apply this result to them.

4.4.2. The following results are well-known.

LEMMA. *Let I be an ideal of a ring R.*

(i) *If $J(I)$ and $J(R/I)$ are nilpotent then $J(R)$ is nilpotent.*

(ii) *If $J(I)$ and $J(R/I)$ are locally nilpotent then $J(R)$ is locally nilpotent.*

PROOF. Suppose that $J(I)^n = 0$ and $J(R/I)^m = 0$. Then $J(R)^m \subseteq I \cap J(R) = J(I)$ so $J(R)^{mn} = 0$. This proves (i).

For (ii), note that the hypotheses say that $J(I) = L(I)$ and $J(R/I) = L(R/I)$. So by Lemma 4.4.1, we have $J(R) = L(R)$ and hence $J(R)$ is locally nilpotent. □

We remark that the converse fails in each part of the lemma. This will be demonstrated by an example in §4.5.6.

4.5. Semigroup Graded Rings

We are now in a position to combine the results of the previous sections to obtain nilpotency results for the Jacobson radical of rings graded by general semigroups.

4.5.1. We first deal with the nilpotence of the Jacobson radical for rings graded by finite semigroups.

THEOREM ([11]). *Let S be a finite semigroup and let R be an S-graded ring. If $J(R_e)$ is nilpotent for all idempotents $e \in S$ then $J(R)$ is nilpotent.*

PROOF. If S does not have a zero then adjoin one and put $R_\theta = 0$. Otherwise, since θ is an idempotent, we are given that $J(R_\theta)$ is nilpotent and it suffices to prove that $J(R/R_\theta)$ is nilpotent by Lemma 4.4.2(i). So we may assume that S has a zero and R is a contracted S-graded ring.

Since S is finite, it has a principal series

$$S = S_1 \supset S_2 \supset \cdots \supset S_n \supset S_{n+1} = \{\theta\}.$$

Furthermore, by Theorem 1.4.8, each 0-simple factor S_i/S_{i+1} is completely 0-simple. We will proceed by induction on the length n of this series.

If $n = 1$ then S is null or completely 0-simple. In the former case, $R^2 \subseteq R_0 = 0$ so $J(R)$ is nilpotent. In the latter case, S can be assumed to be a finite Rees matrix semigroup by Theorem 1.5.2 and so $J(R)$ is nilpotent by Corollary 4.3.3.

Suppose then that we have proved the result for semigroups with principal series of length less than n. Consider the contracted S/S_n-graded ring $\bar{R} = R/R_{S_n}$ (cf §2.1.4). The non-zero idempotents of S/S_n are those idempotents $e \in S \setminus S_n$ (see §1.3.5) and the corresponding homogeneous components of $\bar{R}$ are just the components R_e of R. Since S/S_n has a prinicpal series of length $n-1$, it follows from the induction hypothesis that $J(\bar{R})$ is nilpotent. Since S_n is completely 0-simple or null, $J(R_{S_n})$ is nilpotent by the previous paragraph. Hence $J(R)$ is nilpotent by Lemma 4.4.2(ii).

This completes the proof. □

4.5.2. We prove a similar theorem about the local nilpotence of the Jacobson radical for rings graded by locally finite semigroups.

THEOREM ([11]). *Let S be a locally finite semigroup and let R be an S-graded ring. Suppose that $J(R_e)$ is locally nilpotent for all idempotents $e \in S$. Then $J(R)$ is locally nilpotent.*

PROOF. Just as in the proof of Theorem 4.5.1, we may assume that S has a zero and R is a contracted S-graded ring.

Let $\{a_1, a_2, \ldots, a_n\}$ be a finite subset of $J(R)$ and let A be the subring of $J(R)$ generated by $\{a_1, a_2, \ldots, a_n\}$. Let $B = \bigcup_{i=1}^{n} \text{supp}(a_i) \cup \{\theta\}$ and let T be the subsemigroup of S generated by B. Since S is locally finite, T is finite. Note that T is also generated by the support of A and θ.

We proceed by induction on $|T|$. If $|T| = 1$, then $T = \{\theta\}$ and therefore $A = 0$ because $R_\theta = 0$.

Suppose then that we have shown that all finitely generated subrings of $J(R)$ whose supports generate a subsemigroup strictly smaller than T are nilpotent. We shall find a $k > 1$ such that the support of A^k generates a strictly smaller subsemigroup. By Lemma 4.2.5, A^k is a finitely generated subring of $J(R)$. Hence the induction hypothesis says that A^k, and so also A, is a nilpotent subring of R.

Let I be the ideal of S generated by B. So $T \subseteq I$. Since I is finitely generated, the set of ideals strictly contained in I contains maximal elements; this follows easily from Zorn's Lemma. Let M be a maximal such ideal. Then I/M is a 0-minimal ideal of S/M and as such is null or 0-simple by §1.4.2.

Now, $A \subseteq R_T \subseteq R_I$ and $A \subseteq J(R)$ so $A \subseteq R_I \cap J(R) \subseteq J(R_I)$, the last inclusion holding because J is hereditary and R_I is an ideal of R. Let $\bar{A}$ be the image of A in the contracted I/M-graded ring $\bar{R} = R_I/R_M$. Since $A \subseteq J(R_I)$, it follows that $\bar{A}$ is a finitely generated subring of $J(\bar{R})$.

Note that the homogeneous components of $\bar{R}$ corresponding to non-zero idempotents are precisely those components R_e of R corresponding to idempotents $e \in I \setminus M$. So the contracted I/M-graded ring $\bar{R}$ inherits the hypothesis of the theorem.

If I/M is a null semigroup, then $\bar{R}^2 = 0$ so $J(\bar{R}) = \bar{R}$ is nilpotent.

If I/M is a 0-simple semigroup then it is completely 0-simple by Theorem 1.4.8. Furthermore, I/M is isomorphic to a locally finite Rees matrix semigroup by Theorem 1.5.2, so $J(\bar{R})$ is locally nilpotent by Corollary 4.3.5.

In either case, since $\bar{A}$ is a finitely generated subring of $J(\bar{R})$, there is a $k > 1$ such that $\bar{A}^k = 0$. So $A^k \subseteq R_M$. But $A^k \subseteq R_T$ so $A^k \subseteq R_{T\cap M}$. Since I is generated as an ideal by T and $M \subsetneq I$ we cannot have $T \subseteq M$ and so $T \cap M \subsetneq T$. Then $M \cap T$ contains the support of A^k so the subsemigroup generated by the support of A^k is strictly smaller than T as desired. □

4.5.3. We draw some corollaries from this theorem.

Recall that a band is a semigroup of idempotents. It is known that bands are locally finite; this follows from the fact that finitely generated free bands are finite (see for example [15, Exercise 4.2.9(d)]). So the following corollary is immediate.

COROLLARY ([11]). *Let S be a band and let R be an S-graded ring. If $J(R_s)$ is locally nilpotent for all $s \in S$ then $J(R)$ is locally nilpotent.*

Munn has studied similar properties of the Jacobson radical for semigroup rings over bands [51] and a class of band-graded rings called special band graded rings

[52]. In particular, he shows that for such rings, if the Jacobson radical is nil for each homogeneous component, then the Jacobson radical of the ring itself is nil. Of course, the homogeneous components of a band ring $A[S]$ are all isomorphic to A.

4.5.4. For band rings, Munn also obtains a converse. The key to the converse is the following lemma which is obtained from [51].

LEMMA. *Let S be a band and A be a ring. Then $J(A)$ is isomorphic to a subring of $J(A[S])$.*

PROOF. Let $e \in S$. Suppose first that the ring A has an identity. It follows from [51, Theorem 1] that the map $a \mapsto ae$ embeds $J(A)$ into $J(A[S])$. If A does not have an identity, then adjoin one in the manner of §4.1.7. Then $A[S]$ is an ideal of $A^1[S]$ because A is an ideal of A^1. So, the map $a \mapsto ae$ embeds $J(A) = J(A^1)$ into $J(A^1[S]) \cap A[S] = J(A[S])$. □

If, for example, $J(A[S])$ were locally nilpotent, then it would follow immediately that $J(A)$ were locally nilpotent. We obtain:

COROLLARY ([11]). *Let S be a band and let A be a ring. Then $J(A[S])$ is locally nilpotent if and only if $J(A)$ is locally nilpotent.*

4.5.5. We remark that we cannot obtain a theorem similar to Theorem 4.5.1 for the Jacobson radical to be nil. The impediment is the unknown status of the Köthe conjecture: that the sum of two nil one-sided ideals is nil. Indeed, we can be more precise:

PROPOSITION. *The following are equivalent:*

(i) *The sum of two one-sided nil ideals is nil.*

(ii) *For every finite elementary Rees matrix semigroup S and every contracted S-graded ring R, $J(R)$ is nil provided that $J(R_e)$ is nil for every idempotent $e \in S$.*

PROOF. We will sketch the proof. Suppose first that the Köethe conjecture (i) is true. Let $A = A_1 + A_2$ be a ring which is the sum of two right ideals, each of which has nil Jacobson radical. Consider

$$I = A_1 \cap J(A) + A_2 \cap J(A_2);$$

I is easily seen to be an ideal of A. But $A_i \cap J(A) \subseteq J(A_i)$ for $i = 1, 2$, so by (i), I is a nil ideal. Then

$$J(A)^2 \subseteq AJ(A) = A_1J(A) + A_2J(A) \subseteq A_1 \cap J(A) + A_2 \cap J(A_2) \subseteq I$$

so that $J(A)$ is nil. The same is true if A is a sum of two left ideals.

Let $S = \mathfrak{M}^0(1; I, \Lambda; P)$ be a finite elementary Rees matrix semigroup and let R be a contracted S-graded ring such that $J(R_e)$ is nil for all idempotents $e \in S$. Then R is a sum of finitely many right ideals R_{i*}, and each R_{i*} is a sum of finitely many left ideals $R_{i\lambda}$. Each $R_{i\lambda}$ is either nilpotent or has nil Jacobson radical by hypothesis, so we conclude that $J(R)$ is nil. Hence (ii) is true.

Conversely, suppose that (ii) holds. Let A and B be nil right ideals of a ring C. Let $S = \{e, f\}$ be a two element left zero semigroup, so that S^0 is an elementary Rees matrix semigroup by §3.4.2. Consider the subring $R = Ae + Bf$ of the semigroup ring $C[S]$. Then R is an S-graded ring and so a contracted S^0-graded ring, with $R_e = Ae$ and $R_f = Bf$. Then $J(R_e) = R_e = Ae$ and $J(R_f) = R_f = Bf$ are both nil. By Corollary 4.3.2, it is easy to see that $J(R) = R$. So by (ii), R is nil. Let $a \in A$ and $b \in B$. Then there is an n such that $(ae + bf)^n = 0$. However, $(ae + bf)^n = a(a+b)^{n-1}e + b(a+b)^{n-1}f$ so that $a(a+b)^{n-1} = b(a+b)^{n-1} = 0$ in C and therefore $(a+b)^n = 0$ in C. This shows that $A + B$ is a nil right ideal of C. A similar argument holds for left ideals, and therefore (i) is true. □

4.5.6. We give an example to show that the converses of both Theorems 4.5.1 and 4.5.2 are false. The example is so constructed that the converse of Lemma 4.4.2 fails: we provide a ring R with $J(R) = 0$ but with a homomorphic image which has non-nil Jacobson radical.

Let K be a field and let $A = K[[X]]$, the ring of power series $\sum_{n=0}^{\infty} k_n X^n$ in one variable X over K. It is well-known that $J(A)$ consists of those power series which have zero constant term, and that such power series are not nilpotent (see [63, Example 2.5.11]). So $J(A)$ is not nil and indeed A contains no nil-ideals. Let $R = A[Y]$, the polynomial ring over A in one variable Y. By a theorem of Amitsur [63, Theorem 2.5.3], $J(R) = 0$ because A has no nil ideals.

Let $S = \{\alpha, \beta\}$ be a two-element semilattice with $\alpha > \beta$. If we put $R_\alpha = A$ and $R_\beta = YA[Y]$, then $R = R_\alpha \oplus R_\beta$ and it is easy to see this is an S-gradation because A is a subring of R and $YA[Y]$ is an ideal of R.

So $J(R) = 0$ but $J(R_\alpha) = J(A)$ is not nil.

Hence, the converses of Theorems 4.5.1 and 4.5.2 fail. Since $R_\alpha \cong R/R_\beta$, this example is also a counter example to the converse of each statement of Lemma 4.4.2.

4.5.7. One might ask whether the nilpotence result, Proposition 4.5.1, holds for infinite semigroups. In general, the answer is no, as the next example, of a ring graded by a locally finite semigroup shows. This example is [21, Example 3].

Let $S = \{x \mid 0 < x < 1,\ x \text{ a real number}\} \cup \{\theta\}$ be a semigroup with operation

$$xy = \begin{cases} x + y & \text{if } x \neq \theta,\ y \neq \theta, \text{ and } x + y < 1, \\ \theta & \text{otherwise.} \end{cases}$$

Then S is a locally finite semigroup, for if x is the smallest real number in a finite subset A of S, then $A^n = \{\theta\}$, where n is chosen so that $\frac{1}{n} < x$. Notice that S contains no idempotents except θ.

Let $R = K_0[S]$ be a contracted semigroup algebra of S over a field K, and regard R as an S-graded ring in the obvious way. Then R has no non-zero homogeneous components corresponding to idempotents. Notice that R is nil. For if x is the smallest element in the support of an element $r = \sum_{0<y} r_y y$ of R, then $r^n = 0$, where again $\frac{1}{n} < x$. Hence, $J(R) = R$. But R is not nilpotent, for the product $\frac{1}{2}\frac{1}{4}\cdots\frac{1}{2^n}$ in S is never zero for any n.

Of course, R is locally nilpotent, as Proposition 4.5.2 says it should be: if x is the smallest element of the support of finite set $\{r_1, r_2, \ldots, r_n\}$ of elements of R, then it is easy to see that $\langle r_1, r_2, \ldots, r_n\rangle^n = 0$, where $\frac{1}{n} < x$.

4.6. The Jacobson Radical and Subrings

Before finishing this chapter we will prove a proposition relating the Jacobson radical of a semigroup graded ring to the radical of subrings graded by certain subsemigroups.

4.6.1. In the first instance, these subsemigroups are left or right group-like (cf §2.3.1).

LEMMA. *Let T be a right (or left) group-like subsemigroup of S and let R be an S-graded ring. Then $J(R) \cap R_T \subseteq J(R_T)$.*

PROOF. Suppose that T is right group-like; the other case is similar. Let $\pi_T : R \to R_T$ be the projection map. By Proposition 2.3.2, π_T is a right R_T-module homomorphism.

Note that $J(R) \cap R_T$ is an ideal of R_T, so we need only show that an element $x \in J(R) \cap R_T$ is left quasi-regular in R_T. Since $x \in J(R)$, there is a $y \in R$ such that

$x + y + yx = 0$. Hence,

$$0 = \pi_T(x + y + yx) = x + \pi_T(y) + \pi_T(y)x$$

and since $\pi_T(y) \in R_T$, x is indeed left quasi-regular in R_T. □

4.6.2. We extend this lemma to any left or right simple subsemigroup T of S.

PROPOSITION. *Let T be a subsemigroup of S which is left or right simple. Then $J(R) \cap R_T \subseteq J(R_T)$. In particular, if e is an idempotent of S, then $J(R) \cap R_e \subseteq J(R_e)$.*

PROOF. Assume that T is right simple. By Lemma 2.3.1(ii), TT^{-1} is a right group-like subsemigroup of S and T is a left ideal of TT^{-1}. So $J(R) \cap R_{TT^{-1}} \subseteq J(R_{TT^{-1}})$ by Lemma 4.6.1 and therefore $J(R) \cap R_T \subseteq J(R_{TT^{-1}}) \cap R_T \subseteq J(R_T)$. Since $\{e\}$ is a right simple subsemigroup for any idempotent e, the last statement is a special case of the general statement. □

Note that we have already proved this for a contracted elementary Rees semigroup graded ring (Proposition 4.3.1 and Corollary 4.3.2), and for group graded rings, this is Lemma 4.2.2(i). This result was also proved in [41] in the case T is a subgroup using a different method.

CHAPTER 5

Homogeneity of the Jacobson Radical

Let S be a semigroup. If $J(R)$ is homogeneous for all S-graded rings, then we say that the Jacobson radical is *S-homogeneous*. In this chapter, we investigate which semigroups have this property, obtaining a necessary condition that S be cancellative. We also introduce some related but weaker homogeneity conditions. We are able to completely determine which semigroups satisfy these weaker conditions, at least for the classes of commutative and regular semigroups.

The S-homogeneity of the Jacobson radical has received much attention. Even in the case that S is a group, the problem has not been completely solved. It is not difficult to see that a group G must be torsion free if the Jacobson radical is G-homogeneous. For if H is a cyclic subgroup of G of prime order p and K is a field of characteristic p, then $R = K[H]$ is G-graded by putting $R_g = Kg$ if $g \in H$ and $R_g = 0$ otherwise. But $J(K[H])$ is the augmentation ideal of $K[H]$ (cf [58, Lemma 3.1.6]) which is not homogeneous. On the other hand, it is not known if the Jacobson radical is G-homogeneous for all torsion free groups G or even for ordered or u.p. groups. The first positive result was obtained by Bergman [8] who showed that the Jacobson radical is homogeneous for each $\mathbf{Z}$-graded ring. This was extended by Jespers and Puczyłowski [35] to free groups and torsion free nilpotent groups.

In [34], Jespers, Krempa, and Puczyłowski studied the homogeneity of various radicals for u.p. and t.u.p. semigroups; affirmative results were obtained for the Levitzki and prime radicals, but the problem for the Jacobson radical was not solved. More recently, Kelarev [40] solved the problem completely for commutative semigroups: the Jacobson radical is S-homogeneous for a commutative semigroup S if and only if S is torsion free and cancellative.

It was conjectured in [40] that in general, if the Jacobson radical is S-homogeneous, then S is embeddable in a group. Our result that such semigroups are cancellative adds credence to this conjecture. There are, however, examples of cancellative semigroups which are not embeddable in a group (see [15, §12.6]).

The results of this chapter were obtained in collaboration with Andrei Kelarev and can be found in [14].

5.1. Homogeneity Conditions

The condition that the Jacobson radical is S-homogeneous defines a class of semigroups. If we restrict the clas f S-graded rings for which we require homogeneity of the radical, we include more semigroups. In this section we will define a few related classes of semigroups in this way, and give some elementary closure properties of these classes.

5.1.1. Let S be a semigroup. We say that S is a *JH-semigroup* if for each S-graded ring R, $J(R)$ is homogeneous. If S is a group which is a JH-semigroup, we will call it a JH-group.

Let S be a semigroup with a zero. Then S is a *JH_0-semigroup* if for each contracted S-graded ring R, $J(R)$ is homogeneous. We introduce this definition for two reasons. Firstly, as we have already seen, we often use chains of ideals of a semigroup to reduce a problem about a semigroup graded ring to contracted semigroup graded rings. Secondly, some well-known ring constructions, for example generalised matrix rings, can be regarded as contracted semigroup graded rings. Indeed, the Jacobson radical of a generalised matrix ring is homogeneous [9, 72].

As mentioned above, which groups are JH-groups has not been completely determined. For this reason, it is convenient, when considering the homogeneity of $J(R)$ for an S-graded ring R, to assume that $J(R_G)$ is homogeneous for each subgroup G of S. This makes the results easier to state, and results without this assumption can easily be recovered (cf §5.1.4). In addition, for some ring constructions, for example semigroup algebras over the complex numbers, we may know *a priori* that the subrings $J(R_G)$ are homogeneous. Accordingly, we say that S is a *JGH-semigroup* if $J(R)$ is homogeneous for each S-graded ring R such that $J(R_G)$ is homogeneous for all subgroups G of S. Similarly, if S has a zero, then S is a *JGH_0-semigroup* if $J(R)$ is homogeneous for each contracted S-graded ring R such that $J(R_G)$ is homogeneous for all subgroups G of S.

5.1.2. It is easy to see that a subsemigroup of a JH-semigroup is a JH-semigroup. For the other classes of semigroups, we must be a bit careful about which subsemi groups we take.

Following [41], we say that a subsemigroup T of S is *group closed* if for every subgroup G of S, either $T \cap G = \emptyset$ or $T \cap G$ is a group.

LEMMA. *Let S be a semigroup and let T be a subsemigroup of S. If S has a zero, denote it by 0.*

(i) *If S is a JH-semigroup then T is a JH-semigroup.*

(ii) *If S is a JH_0-semigroup and $0 \in T$ then T is a JH_0-semigroup.*

(iii) *If S is a JH_0-semigroup and $0 \notin T$ then T is a JH-semigroup.*

Suppose further that T is group-closed.

(iv) *If S is a JGH-semigroup then T is a JGH-semigroup.*

(v) *If S is a JGH_0-semigroup and $0 \in T$ then T is a JGH_0-semigroup.*

(vi) *If S is a JGH_0-semigroup and $0 \notin T$ then T is a JGH-semigroup.*

PROOF. Let R be a T-graded ring; then R can be graded by S by putting $R_s = 0$ for $s \in S \setminus T$. So (i), (ii), and (iii) follow immediately from the definitions. For (iv), (v), and (vi), we need only note additionally that the condition that T be group-closed ensures that $J(R_G) = J(R_{G \cap T})$ is homogeneous for each subgroup G of S, provided that the same is true for every subgroup of T. □

5.1.3. In a similar way, we can pass to Rees factors.

LEMMA. *Let S be a semigroup with a zero and let I be an ideal of S. If S is a JH_0-semigroup (resp. a JGH_0-semigroup), then S/I is also a JH_0-semigroup (resp. a JGH_0-semigroup).*

PROOF. If R is a contracted S/I-graded ring, then R becomes a contracted S-graded ring by putting $R_s = 0$ for $s \in I$. Furthermore, if G is a subgroup of S such that $R_G \neq 0$, then $G \subset S \setminus I$ and we may identify G with a subgroup of S/I. □

5.1.4. The next lemma relates JGH-semigroups to JH-semigroups and JGH_0-semigroups to JH_0-semigroups.

LEMMA. *Let S be a semigroup.*

(i) *S is a JH-semigroup if and only if S is a JGH-semigroup and every subgroup G of S is a JH-group*

(ii) *S is a JH_0-semigroup if and only if S is a JGH_0-semigroup and every subgroup G of S is a JH-group..*

PROOF. We prove only (i), the proof of (ii) being similar.

Suppose that S is a JH-semigroup. It is clear from the definitions that being a JH-semigroup is stronger that being a JGH-semigroup. If G is a subgroup of S and R is a G-graded ring, then $J(R)$ must be homogeneous because R can be considered to be an S-graded ring. Hence G is a JH-group.

Conversely, suppose that S is a JGH-semigroup and every subgroup G of S is a JH-group. If R is an S-graded ring, then for each subgroup G of S, $J(R_G)$ is homogeneous because G is a JH-group. Then $J(R)$ is homogeneous because S is a JGH-semigroup. □

5.2. A Necessary Condition

In this section, we prove that JH-semigroups are cancellative. In fact, we prove the stronger result that JGH-semigroups are cancellative. We also deduce a more technical result for JGH_0-semigroups; this will be needed in Sections 5.4 and 5.5.

5.2.1. The first results on the homogeneity of the Jacobson radical for group graded rings were obtained by Bergman [8].

THEOREM ([8]). *Torsion free Abelian groups are JH-groups.*

5.2.2. For a semigroup ring $A[S]$, the augmentation homomorphism is defined to be $\omega: A[S] \to A$, $\sum_{s\in S} a_s s \mapsto \sum_{s\in S} a_s$.

We single out the following trivial observation because we shall use it twice in this section.

LEMMA. *Suppose that R is a subring of a semigroup ring $A[S]$ such that $\omega(R) = A$. If $J(A) = 0$ and $x \in J(R)$, then $\omega(x) = 0$.*

5.2.3. We first establish a necessary condition somewhat weaker than cancellativity.

LEMMA. *Let S be a JGH-semigroup. Then S does not have two distinct elements e and f such that $ef = f^2$ and $fe = e^2$ (or $ef = e^2$ and $fe = f^2$).*

PROOF. We will prove the first case; the other following by symmetry. Suppose that S does contain two such elements. We must construct an S-graded ring R which does not have homogeneous Jacobson radical.

Let V be the subsemigroup generated by e and f. Clearly,

$$V = \{e^n, f^n \mid n \geq 1\}.$$

Let K be a field of characteristic zero, and consider the semigroup ring $R = K[V]$. Then R is an S-graded ring if we put $R_s = Ks$ for $s \in V$ and $R_s = 0$ for $s \in S \setminus V$.

Now take any subgroup G of S. We will show that $J(R_G)$ is homogeneous. If $G \cap V = \emptyset$, then $R_G = 0$ and $J(R_G)$ is certainly homogeneous.

Otherwise, let $P = G \cap V$. Since $V = \text{supp}(R)$, $R_G = R_P$. Choose p to be the smallest power of e or f in P; without loss of generality we may assume that $p = e^m$ for some $m \geq 1$. Denote by E the subsemigroup generated by e. We claim that $P \subseteq E$. For suppose that $f^k \in P$. Then $k \geq m$ by choice of p. Let 1_G be the identity of the group G. Since $e^m \in P \subseteq G$, there is a $u \in G$ such that $e^m u = 1_G$. Also, $e^m 1_G = e^m$ and $f^k 1_G = f^k$ because $e^m, f^k \in G$; the former implies that $e^k 1_G = e^k$

since $k \geq m$. But $fe = e^2$ implies $e^k e^m = f^k e^m$. Hence,

$$f^k = f^k 1_G = f^k e^m u = e^k e^m u = e^k 1_G = e^k,$$

so that $f^k \in E$.

If e is not periodic, then E is an infinite cyclic semigroup, and so $P \subseteq E$ is isomorphic to a subsemigroup of **Z**. By Theorem 5.2.1 and Lemma 5.1.2(i), $J(R_G) = J(R_P)$ is homogeneous.

If e is periodic, then $P = E \cap G$ is a finite group since E is finite and G is cancellative. But $R_G = R_P = K[P]$, and by Maschke's Theorem [58, Theorem 2.4.2], $J(R_G) = 0$.

So $J(R_G)$ is homogeneous for all subgroups G of S.

Put $d = e - f \in R$. Then for any $v \in V$, the equality $ev = fv$ holds, implying $dR = 0$ and hence $d \in J(R)$. But by Lemma 5.2.2, $e \notin J(R)$, since $R = K[V]$ and e has augmentation 1. Hence, $J(R)$ is not homogeneous and therefore S is not a JGH-semigroup. □

5.2.4. As a trivial consequence of Lemma 5.2.3 we have:

LEMMA. *Let S be a JGH-semigroup. If $u, v, w \in S$ and $u \neq v$, the equalities $uw = vw$ and $wu = wv$ are equivalent.*

PROOF. Suppose that $uw = vw$. Put $e = wu$ and $f = wv$. Then $ef = wuwv = wvwv = f^2$ and $fe = wvwu = wuwu = e^2$. By Lemma 5.2.3, we must have $e = f$.

Similarly, $wu = wv$ implies $uw = vw$. □

5.2.5. With this last lemma, we are ready to prove that JGH-semigroups are cancellative. Notice that this lemma says that left and right cancellativity are equivalent for JGH-semigroups.

THEOREM ([14]). *JGH-semigroups are cancellative.*

PROOF. Suppose that S is a JGH-semigroup and S is not cancellative. Then there exist elements $u, v, w \in S$ such that $u \neq v$ but $uw = vw$. We will construct an S-graded ring such that $J(R_G)$ is homogeneous for all subgroups G of S, but $J(R)$ is not homogeneous, contradicting our supposition.

Denote by M the ring of 2×2 matrices over the complex numbers $\mathbf{C}$. Let $e_{12} = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ be the standard matrix unit and put $N = \mathbf{C}e_{12}$; then $N^2 = 0$.

Let $W = S^1wS^1$ be the ideal of S generated by w. Define a subring R of the semigroup ring $M[S]$ by

$$R = NS + MW = \{\sum_{s \in S} a_s s \in M[S] \mid a_s \in N \text{ if } s \notin W\};$$

this is a subring because W is an ideal of S. Since R is a homogeneous subring of $M[S]$, it inherits the usual S-gradation from $M[S]$.

Let G be a subgroup of S. Since W is an ideal of S, either $G \cap W = \emptyset$ or $G \subseteq W$. In the former case, $R_G = N[G]$, so $R_G{}^2 = 0$ and therefore $J(R_G) = R_G$. In the latter case, R_G is the group ring $M[G]$ which is isomorphic to the ring of 2×2 matrices over the group algebra $\mathbb{C}[G]$. But $J(\mathbb{C}[G]) = 0$ (by [58, Theorem 7.1.1]) and therefore $J(R_G) \cong J(M_2(\mathbb{C}[G])) = 0$ (by [58, Theorem 7.2.6]). So $J(R_G)$ is homogeneous for all subgroups G of S.

Consider the element $d = e_{12}(u - v)$ of the ring R. Take any $s \in W$, say $s = awb$, where $a, b \in S^1$. Since $uwb = vwb$, Lemma 5.2.4 yields $wbu = wbv$, so $su = awbu = awbv = sv$, and then $us = vs$ by Lemma 5.2.4 again. Therefore,

$$dR_s = e_{12}(u - v)Ms = e_{12}M(u - v)s = 0$$

because $(u - v)s = 0$. If $s \in S \setminus W$, then

$$dR_s = e_{12}(u - v)Ns = e_{12}N(u - v)s = 0$$

because $e_{12}N \subseteq N^2 = 0$. So $dR = 0$ and therefore $d \in J(R)$.

If $J(R)$ were homogeneous, we would have $e_{12}u \in J(R)$. Let $\omega\colon M[S] \to M$ be the augmentation homomorphism. Then $\omega(R) = M$, and $\omega(e_{12}u) = e_{12} \notin J(M) = 0$. So by Lemma 5.2.2, $e_{12}u \notin J(R)$ and we conclude that $J(R)$ is not homogeneous. Hence S cannot be a JGH-semigroup. □

5.2.6. We can immediately deduce necessary conditions for S to be a JH-semigroup, a JH_0-semigroup, or a JGH_0-semigroup.

COROLLARY ([14]). *Let S be a semigroup.*

(i) *If S is a JH-semigroup then S is cancellative.*

(ii) *If S is a JH_0-semigroup and T is a subsemigroup of S not containing 0, then T is cancellative.*

(iii) *If S is a JGH_0-semigroup and T is a group-closed subsemigroup of S not containing 0, then T is cancellative.*

PROOF. These statements follow easily from Theorem 5.2.5 using Lemma 5.1.4(i), Lemma 5.1.2(vi), and Lemma 5.1.2(iii) respectively. □

5.3. The Jacobson Radical of Semilattice Graded Rings

Before proceeding to discuss the homogeneity of the Jacobson radical for rings graded by commutative semigroups, we pause to give a characterisation of the radical for semilattice graded rings. We shall need this in Section 5.4.

5.3.1. We begin with an elementary lemma about semilattices. Recall from §1.1.5 that a semilattice is a commutative semigroup in which every element is idempotent and that the elements of a semilattice are partially ordered by $\alpha \geq \beta$ if and only if $\alpha\beta = \beta$.

LEMMA. *Let Γ be a semilattice and let $\gamma \in \Gamma$. Define*

$$N(\gamma) = \{\alpha \in \Gamma \mid \alpha \not\geq \gamma\}.$$

Then $N(\gamma)$ is an ideal of Γ and $\big(N(\gamma) \cup \{\gamma\}\big)/N(\gamma)$ is an ideal of $\Gamma/N(\gamma)$.

PROOF. If $\alpha\beta \geq \gamma$, then $\alpha \geq \alpha\beta \geq \gamma$. So $\alpha \not\geq \gamma$ implies $\alpha\beta \not\geq \gamma$ and therefore $N(\gamma)$ is an ideal.

If $\beta \in \Gamma$, then $\beta\gamma \leq \gamma$ so that $\beta\gamma \in N(\gamma) \cup \{\gamma\}$. Hence, $N(\gamma) \cup \{\gamma\}$ is an ideal of Γ and $\big(N(\gamma) \cup \{\gamma\}\big)/N(\gamma)$ is an ideal of $\Gamma/N(\gamma)$. $\square$

5.3.2. The following theorem is actually a special case of [41, Theorem 1].

THEOREM. *Let Γ be a semilattice and let R be a Γ-graded ring. The Jacobson radical of R is the largest ideal I of R satisfying the following property: for all $x \in I$ and each $\gamma \in \Gamma$ which is a maximal element of* supp(x), *we have $x_\gamma \in J(R_\gamma)$.*

PROOF. First we show that $J(R)$ satisfies the given property. Let $x \in J(R)$ and let γ be a maximal element of supp(x). Let $N(\gamma)$ be the ideal of Lemma 5.3.1, let $\bar{R} = R/R_{N(\gamma)}$ and write $\bar{r}$ for the image in $\bar{R}$ of an element $r \in R$. By choice of γ, the ideal $N(\gamma)$ includes every element of supp(x) except γ. Also, by Lemma 5.3.1, $\bar{R}_\gamma$ is an ideal of $\bar{R}$. So

$$\bar{x}_\gamma = \bar{x} \in J(\bar{R}) \cap \bar{R}_\gamma \subseteq J(\bar{R}_\gamma).$$

But by Lemma 2.1.4, the map $r \mapsto \bar{r}$ induces an isomorphism $R_\gamma \cong \bar{R}_\gamma$. We conclude that $x_\gamma \in J(R_\gamma)$.

Suppose now that I is an ideal of R satisfying the given property and that $x \in I$. We must show that x is quasi-regular in R.

Let $A = \langle \text{supp}(x) \rangle$ be the subsemigroup generated by supp(x). Then A is finite because semilattices are easily seen to be locally finite. Since $R_A x R_A \subseteq I$, the former satisfies the property of the theorem in the A-graded ring R_A. If we show that $R_A x R_A \subseteq J(R_A)$, it will follow that x is quasi-regular in R. Replacing A by Γ, R_A by R and $R_A x R_A$ by I, we may assume that Γ is a finite semilattice.

We proceed by induction on $|\Gamma|$. If $|\Gamma| = 1$, say $\Gamma = \{\gamma\}$, then $x = x_\gamma \in J(R_\gamma)$ is quasi-regular.

Suppose that the result is true for all rings graded by smaller semilattices. Let γ be a maximal element of Γ. Then $N(\gamma) = \Gamma \setminus \{\gamma\}$ is an ideal of Γ.

Let

$$\bar{R} = \frac{R}{R_{N(\gamma)}} = \frac{\left(R_\gamma + R_{N(\gamma)}\right)}{R_{N(\gamma)}} \cong R_\gamma$$

and let $r \mapsto \bar{r}$ be the projection $R \to \bar{R}$. Since $x_\gamma \in J(R_\gamma)$, we see that $\bar{x} = \bar{x}_\gamma \in J(\bar{R})$.

Since $I \cap R_{N(\gamma)} \subseteq I$, it satisfies the property of the theorem in the $N(\gamma)$-graded ring $R_{N(\gamma)}$. So by induction, $x R_{N(\gamma)} \subseteq I \cap R_{N(\gamma)} \subseteq J(R_{N(\gamma)})$.

By Lemma 4.4.1, we conclude that $x \in J(R)$ as required. □

5.3.3. We deduce a much simpler criterion for a homogeneous ideal to be contained in the Jacobson radical of a semilattice graded ring.

COROLLARY. *Let Γ be a semilattice and let R be a Γ-graded ring. Let I be a homogeneous ideal of R. Then $I \subseteq J(R)$ if and only if $I \cap R_\gamma \subseteq J(R_\gamma)$ for each $\gamma \in \Gamma$.*

PROOF. Suppose that $I \subseteq J(R)$. Let $\gamma \in \Gamma$. Then $I \cap R_\gamma \subseteq J(R) \cap R_\gamma \subseteq J(R_\gamma)$ by Proposition 4.6.2.

Conversely, suppose that $I \cap R_\gamma \subseteq J(R_\gamma)$ for each $\gamma \in \Gamma$. Let $x \in I$ and let γ be a maximal element of supp(x). Since I is homogeneous, $x_\gamma \in I$. Hence, $x_\gamma \in I \cap R_\gamma \subseteq J(R_\gamma)$. We conclude by Theorem 5.3.2 that $I \subseteq J(R)$. □

5.4. Commutative Semigroups

Our goal in this section is to completely describe commutative JH_0-semigroups and JGH_0-semigroups.

5.4.1. The question of which commutative semigroups are JH-semigroups has already been answered by Kelarev in [40]. We will deduce this result from Theorem 5.2.5.

A commutative semigroup S is *torsion free* if for all $x, y \in S$, if $x^n = y^n$ for some $n > 0$, then $x = y$. If S is an Abelian group, this coincides with the usual definition of torsion free for groups.

THEOREM ([40]). *Let S be a commutative semigroup. Then S is a JH-semigroup if and only if S is torsion free and cancellative.*

PROOF. Suppose that $x \neq y$ in S but $x^n = y^n$ for some least $n > 0$. By replacing x and y by a suitable power, we may assume that n is prime. Let K be a field of characteristic n, and let $R = K[S]$; then R is a commutative ring. Since K

has characteristic n, $(x-y)^n = x^n - y^n = 0$. Hence, $x - y \in J(R)$. But clearly $x \notin J(R)$ because it has augmentation 1. Therefore, R does not have homogeneous Jacobson radical. Combining this with Corollary 5.2.6(i) shows that commutative JH-semigroups are torsion free and cancellative.

Conversely, let S be a commutative cancellative torsion free semigroup. Since S is commutative and cancellative, it has an Abelian group of fractions G (cf §1.7.2). Furthermore, if $g = st^{-1}$ is an element of G with $s, t \in S$ and $g^n = 0$ for some $n > 0$, then $s^n = t^n$ in S, so $s = t$ and $g = 1$. Hence, G is a torsion free Abelian group. It follows that S is a JH-semigroup by Theorem 5.2.1 and Lemma 5.1.2(i). □

5.4.2. Our first task is to extend this last result to JGH-semigroups. As one might expect, the only change is to include all Abelian groups in addition to the cancellative torsion free semigroups.

PROPOSITION ([14]). *Let S be a commutative semigroup. Then S is a JGH-semigroup if and only if S is cancellative and torsion free or S is a group.*

PROOF. The sufficiency of these conditions follows at once from Theorem 5.4.1 for cancellative torsion free semigroups and from the definition of JGH-semigroup for the group case.

Suppose that S is a JGH-semigroup. By Theorem 5.2.5, S is cancellative.

If all subgroups of S are torsion free, then it follows from Theorem 5.2.1 and Lemma 5.1.4(i) that S is a JH-semigroup and from Theorem 5.4.1 that S is cancellative and torsion free.

Otherwise, there is an element $x \in S$ such that $\langle x \rangle$ is a non-trivial finite subgroup, say of order n. Let e be the identity of $\langle x \rangle$. If $s \in S$, then $es = e^2 s$ and so $s = es$ because S is cancellative. So e is an identity element for S. Let $G = H(S)$ be the group of units of S. Then as we showed above for e, an idempotent element of S must be the identity of S. So G contains all subgroups of S. We will show that $S = G$.

If $S \neq G$, there is an element $y \notin G$. Let $z = xy$, then $z \notin G$, because otherwise and y would be units. Note that $y \neq z$ since $x \neq e$. But $z^n = x^n y^n = e y^n = y^n$.

Let $T = \langle y, z \rangle$. If $T \cap G \neq \emptyset$, then $y^k z^l$ is a unit for some k and l, whence at least one of y and z is a unit, a contradiction. So $T \cap G = \emptyset$, and therefore T is a group-closed subsemigroup of S. By Lemma 5.1.2(iv), T is a JGH-semigroup. But T contains no subgroups, so T is actually a JH-semigroup, and T must be torsion free by Theorem 5.4.1. This contradicts the assertion that $y^n = z^n$ while $y \neq z$. □

5.4.3. We begin our investigations of JGH_0-semigroups with torsion free commutative semigroups. This is perhaps the most interesting case, and, as we shall see, it is not difficult to proceed from this case to the case of arbitrary commutative semigroups.

Let S be a semigroup with a zero. We say that S is *0-cancellative* if the set of non-zero elements forms a cancellative subsemigroup. We say that S is *almost cancellative* if for all elements $x, y, z \in S$, if $xz = yz \neq 0$ then $x = y$. Clearly, 0-cancellative semigroups are almost cancellative.

THEOREM ([14]). *Let S be a commutative torsion free semigroup with a zero. Let $S = \bigcup_{\gamma \in \Gamma} S_\gamma$ be the Archimedean decomposition of S; then Γ has a zero element γ_0 and $S_{\gamma_0} = \{0\}$. The following are equivalent:*

(i) *S is a JGH_0-semigroup.*

(ii) *S is a JH_0-semigroup.*

(iii) *For $\gamma_0 \neq \gamma \in \Gamma$, the subsemigroup $\bigcup_{\alpha \geq \gamma} S_\alpha$ is cancellative.*

(iv) *S is almost cancellative.*

PROOF. If we let γ_0 be the element of Γ such that $0 \in S_{\gamma_0}$, then it is clear that γ_0 is a zero element of Γ. Let $s \in S_{\gamma_0}$. Since S_{γ_0} is an Archimedean semigroup, $0 \mid s^n$ for some $n > 0$ and therefore, $s^n = 0 = 0^n$. Since S is torsion free, we conclude that $s = 0$. So $S_{\gamma_0} = \{0\}$ as claimed.

The equivalence of (i) and (ii) follows at once from Lemma 5.1.4(ii) since every subgroup of S is torsion free and is therefore a JH-group by Theorem 5.2.1.

Suppose that S satisfies (iii). Let $x, y, z \in S$ such that $xz = yz \neq 0$ with $x \in S_\alpha$, $y \in S_\beta$, $z \in S_\gamma$, and $xz \in S_\delta$. Then $\alpha, \beta, \gamma \geq \delta \neq \gamma_0$, and so $x, y, z \in \bigcup_{\epsilon \geq \delta} S_\epsilon$ which is cancellative. Hence $xz = yz$ implies $x = y$. So S is almost cancellative. Conversely, suppose that S is almost cancellative. If $\gamma \neq \gamma_0$ and $x, y, z \in \bigcup_{\alpha \geq \gamma} S_\alpha$ are elements such that $xz = yz$, then we cannot have $xz = yz = 0$ since $0 \notin \bigcup_{\alpha \geq \gamma} S_\alpha$. Hence, $x = y$

and $\bigcup_{\alpha \geq \gamma} S_\alpha$ is cancellative. We have shown that (iii) and (iv) are equivalent.

Let $\gamma_0 \neq \gamma \in \Gamma$. Then $\bigcup_{\alpha \geq \gamma} S_\alpha$ is a subsemigroup of S not containing θ. By Corollary 5.2.6(ii), we see that (ii) implies (iii).

It remains to show that (iii) implies (ii). Suppose that S satisfies (iii). Let R be a contracted S-graded ring and let $r \in J(R)$. We must show that for $t \in \operatorname{supp}(r)$, $r_t \in J(R)$. Let $I = Rr_tR$; it suffices to show that $I \subseteq J(R)$. Let α be the element of Γ such that $t \in S_\alpha$.

There is a homomorphism $S \to \Gamma$ coming from the Archimedean decomposition of S (cf §1.6.1) and this gives R a Γ-gradation (cf §2.1.5) $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$, where $R_\gamma = \bigoplus_{s \in S_\gamma} R_s$. To avoid confusion, we shall use the terms Γ-homogeneous and Γ-supp when referring to the Γ-gradation of R.

Now, I is a Γ-homogeneous ideal of R because $r_t \in R_t \subseteq R_\alpha$ is certainly a Γ-homogeneous element. So by Corollary 5.3.3, it suffices to show that $I \cap R_\gamma \subseteq J(R_\gamma)$ for all $\gamma \in \Gamma$.

Note that if $\gamma \in \Gamma\text{-}\operatorname{supp}(I)$, then $\gamma \leq \alpha$ because $r_t \in R_\alpha$. So if $\gamma \not\leq \alpha$, then $I \cap R_\gamma = 0 \subseteq J(R_\gamma)$.

Suppose that $\gamma \leq \alpha$. We may assume that $\gamma \neq \gamma_0$ since $R_{\gamma_0} = R_\theta = 0$. Let $A = \{\beta \mid \beta \geq \gamma\}$; if $N(\gamma)$ is the ideal defined in Lemma 5.3.1, then $\Gamma = A \cup N(\gamma)$ is a disjoint union. So

$$\frac{R}{R_{N(\gamma)}} = \frac{(R_A + R_{N(\gamma)})}{R_{N(\gamma)}} \cong R_A$$

and the map $x \mapsto x_A$ is a ring homomorphism of R onto R_A. In particular, $r_A \in J(R_A)$. Let $T = \bigcup_{\beta \geq \gamma} S_\beta$. Then T is a commutative cancellative torsion free semigroup by (iii). But

$$R_A = \bigoplus_{\beta \geq \gamma} R_\beta = \bigoplus_{s \in T} R_s$$

is T-graded, so by Theorem 5.4.1, $J(R_A)$ is homogeneous for the T-gradation. Since $r_A \in J(R_A)$, we conclude that $r_t \in J(R_A)$.

Note that $I \cap R_\gamma = Rr_tR \cap R_\gamma = R_Ar_tR_A \cap R_\gamma$ since $\delta \alpha \epsilon = \gamma$ implies $\delta, \epsilon \geq \gamma$. Hence,

$$I \cap R_\gamma = R_Ar_tR_A \cap R_\gamma \subseteq J(R_A) \cap R_\gamma = J(R_\gamma)$$

because R_γ is an ideal of R_A.

We conclude by Corollary 5.3.3, that $I \subseteq J(R)$ and therefore $r_t \in J(R)$, so $J(R)$ is homogeneous. Hence S is a JH_0-semigroup and (ii) holds. □

5.4.4. Note that a 0-direct union (cf §1.3.6) of almost cancellative semigroups is almost cancellative. In particular, a semigroup S which is a 0-direct union of a family $\{S_i \mid i \in I\}$ of commutative 0-cancellative torsion free subsemigroups is almost cancellative. In this case, a contracted S-graded ring R can be written as a direct sum $R = \bigoplus_{i \in I} R_{S_i}$ of subrings graded by the subsemigroups S_i, and the homogeneity of $J(R)$ follows immediately from the fact that $J(R_{S_i})$ is homogeneous for each i by Theorem 5.4.1. (Since each R_{S_i} is contracted S_i-graded, it is graded by the cancellative semigroup $T_i = S_i \setminus \{\theta\}$.)

We give an example of an almost cancellative torsion free commutative semigroup which does not decompose trivially in this way.

Let S be the commutative semigroup generated by elements x, y, and z satisfying relations $yz = xyz = y^2z = yz^2$. Then $0 = yz$ is a zero element of S. Furthermore, an element $x^a y^b z^c$ of S satisfies $x^a y^b z^c = 0$ if and only if $b > 0$ and $c > 0$. So elements $s, t, u \in S$ can only satisfy $st = su \neq 0$ if in fact s, t, u are contained in one of the subsemigroups $\langle x, y \rangle$ or $\langle x, z \rangle$ generated by x and y or x and z respectively. These subsemigroups are free commutative semigroups and are therefore cancellative.

Suppose S has a 0-direct union decomposition $A \cup B$. We may assume without loss of generality that $x \in A$. Since xy and xz are both non-zero, we must have $y, z \in A$ and so $S = A$. Hence, $B = \{0\}$. So S does not admit such a decomposition in a non-trivial way.

Thus S is an almost cancellative semigroup which is not a non-trivial 0-direct union of 0-cancellative semigroups.

5.4.5. Now we relax the restriction that S be torsion free. Our next result shows that in this case, the bottom piece of the Archimedean decomposition of S may be nil.

LEMMA. *Let S be a commutative semigroup with a zero, and let N be the ideal of nilpotent elements of S. Then S is a JGH_0-semigroup if and only if S/N is a JGH_0-semigroup.*

PROOF. Note that the set of nilpotent elements does form an ideal because S is commutative.

Suppose that S/N is a JGH_0-semigroup. Let R be a contracted S-graded ring such that $J(R_G)$ is homogeneous for all subgroups G of S. Let $\bar{R}$ be the contracted S/N-graded ring R/R_N. If we identify a (non-zero) subgroup G of S/N with the corresponding subgroup of S, then $\bar{R}_G \cong R_G$ and therefore $J(\bar{R}_G)$ is homogeneous. So $J(\bar{R})$ is homogeneous.

We claim that R_N is a nil ideal of R. Let $X = \{s_1, s_2, \ldots, s_n\}$ be a finite subset of N with $s_i^{k_i} = \theta$ for integers $k_1, k_2, \ldots, k_n$. Then because the s_i commute, it is easy to see that $X^k = \{\theta\}$ where $k = \sum_{i=1}^{n} k_i$. In particular, if we take $X = \mathrm{supp}(r)$ for an element $r \in R_N$, then $\mathrm{supp}(r^k) \subseteq X^k = \{\theta\}$ for some k and therefore $r^k = 0$ because $R_\theta = 0$.

Hence, $R_N \subseteq J(R)$, so that $J(\bar{R}) = J(R)/R_N$. Since $J(\bar{R})$ is homogeneous,

$$J(\bar{R}) = \sum_{s \in S \setminus N} J(\bar{R}) \cap \bar{R}_s = \sum_{s \in S \setminus N} \frac{\big(J(R) \cap R_s\big) + R_N}{R_N}$$

and so $J(R) = \sum_{s \in S \setminus N} \big(J(R) \cap R_s\big) + R_N$ is homogeneous.

The converse follows from Lemma 5.1.3. □

5.4.6. We are now in a position to describe all commutative JGH_0-semigroups.

THEOREM ([14]). *Let S be a commutative semigroup with a zero. Let N be the ideal of nilpotent elements of S. Then S is a JGH_0-semigroup if and only if S/N*

is a 0-direct union of Abelian groups with zeros and almost cancellative torsion free commutative semigroups.

PROOF. As mentioned in §5.4.4, a 0-direct union of almost cancellative semigroups is almost cancellative. This is also true for torsion free semigroups. So we may assume that there is only one almost cancellative torsion free piece in the decomposition of S/N.

By Lemma 5.4.5, we may assume that $N = \{\theta\}$.

If $S = \bigcup_{j \in J} T_j$ is a 0-direct union, then a contracted S-graded ring is simply a direct sum of a family of contracted T_j-graded rings. So the sufficiency of the given condition follows from Theorem 5.4.3 for the almost cancellative pieces and trivially for the group pieces.

To prove necessity, consider the Archimedean decomposition $S = \bigcup_{\alpha \in \Gamma} S_\alpha$ of S. As in the proof of Theorem 5.4.3, Γ has a zero element γ_0. It is easy to see that S_{γ_0} consists of all the nilpotent elements of S and so $S_{\gamma_0} = \{\theta\}$.

Note that groups are Archimedean semigroups. So a subgroup G of S is entirely contained within one of the components S_γ. Hence any subsemigroup of S which is a union of some of the Archimedean components of S must be group-closed.

Suppose that some S_α is not torsion free, $\alpha \neq \gamma_0$. Since $\bigcup_{\gamma \geq \alpha} S_\gamma$ is a group closed subsemigroup of S not containing θ, it follows from Lemma 5.1.2(vi) and Proposition 5.4.2 that $\bigcup_{\gamma \geq \alpha} S_\gamma$ is in fact a group. Therefore $\bigcup_{\gamma \geq \alpha} S_\gamma = S_\alpha$ and α is a maximal element of

Γ. Similarly, if $\gamma_0 \neq \beta \leq \alpha$, then $\bigcup\limits_{\gamma \geq \beta} S_\gamma$ is a group and therefore $\bigcup\limits_{\gamma \geq \beta} S_\gamma = S_\alpha$ and $\alpha = \beta$. So α is both a maximal element of Γ and a minimal non-zero element of Γ.

Let $A = \{\alpha \mid S_\alpha \text{ is not torsion free}\}$ and let $T = \bigcup\limits_{\gamma \notin A} S_\gamma$. If $\beta, \gamma \notin A$ but $\beta\gamma = \alpha \in A$, then by the maximality of α, we get $\beta = \gamma = \alpha$, a contradiction. So T is a subsemigroup of S. If $\gamma \notin A$ and $\alpha \in A$, then $\gamma\alpha = \gamma_0$ by the maximality and minimality of α. Hence $\left(\bigcup\limits_{\alpha \in A} S_\alpha^0\right) \cup T$ is a 0-direct union.

For each $\alpha \in A$, the above remarks show that S_α^0 is a group with a zero adjoined. We claim that T is torsion free. If $x^n = y^n$ in T, then $x \mid y^n$ and $y \mid x^n$ so that x and y belong to the same Archimedean component $S_\gamma \subseteq T$. But S_γ must be torsion free since $\gamma \notin A$. Hence, $x = y$. Since T is also group-closed, we conclude from Lemma 5.1.2(v) and Theorem 5.4.3 that T is almost cancellative.

So S has a 0-direct union of the required form. □

5.4.7. As a corollary, we describe the commutative JII_0-semigroups.

COROLLARY ([14]). *Let S be a commutative semigroup with a zero and let N be the ideal of nilpotent elements of S. Then S is a JII_0-semigroup if and only if S/N is an almost cancellative torsion free commutative semigroup.*

PROOF. Suppose that S is a JII_0-semigroup. By Lemma 5.1.4(ii) and Theorem 5.2.1 the groups appearing in the decomposition of Theorem 5.4.6 must be torsion free. Since the 0-direct union of almost cancellative torsion free semigroups remains almost cancellative and torsion free, S/N has the desired form. Conversely, S is at least a JGH_0-semigroup by Theorem 5.4.6. Since N has no non-trivial subgroups and S/N can only have torsion free Abelian subgroups, S is a JH_0-semigroup by Lemma 5.1.4(ii) and Theorem 5.2.1. □

5.5. Regular Semigroups

We now turn our attention to regular semigroups. Note that a cancellative regular semigroup is a group, so Theorem 5.2.5 yields immediately that regular JGH-semigroups are just groups. The class of regular JGH_0-semigroups is more interesting. It turns out that these are just 0-direct unions of completely 0-simple inverse semigroups (cf §1.5.6). Because all groups are regular, we cannot obtain any further information on regular JH_0-semigroups except to say that they have the form above and that their subgroups must be JH-groups.

5.5.1. The basis for the work of this section is the observation in [9] and [72] that the Jacobson radical of a generalised matrix ring is homogeneous. We refer to §3.3.1 for the definition and notation for generalised matrix rings. As pointed out in §3.3.2, a ring of $I \times I$ generalised matrices can be graded by the completely 0-simple inverse semigroup $\mathfrak{M}^0(1; I, I; \Delta)$ where Δ is the $I \times I$ identity matrix. Homogeneity is with respect to this gradation.

LEMMA ([9, 72]). *Let $R = (R_{ij})$ be a ring of $I \times I$ generalised matrices. Then $J(R)$ is homogeneous.*

PROOF. Let $x = \sum_{i,j\in I} x_{ij} \in J(R)$. Recall that $R_{ij}R_{kl} = 0$ unless $j = k$. Fix $i, j \in I$. Then

$$Rx_{ij}R = R_{*i}x_{ij}R_{j*} = R_{*i}xR_{j*} \subseteq J(R),$$

and therefore $x_{ij} \in J(R)$. □

5.5.2. There is a nice relationship between $J(R)$ and the Jacobson radicals of the diagonal components R_{ii}.

LEMMA ([72]). *Let $R = (R_{ij})$ be a ring of $I \times I$ generalised matrices. For each $i \in I$, $J(R) \cap R_{ii} = J(R_{ii})$.*

PROOF. For each $i \in I$,

$$J(R) \cap R_{ii} = \big(J(R) \cap R_{i*}\big) \cap R_{ii} \subseteq J(R_{i*}) \cap R_{ii} \subseteq J(R_{ii})$$

since R_{i*} is a right ideal of R and R_{ii} is a left ideal of R_{i*}.

Conversely, we must show that $J(R_{ii}) \subseteq J(R)$. It suffices to show that $J(R_{ii})R$ is a right quasi-regular right ideal of R. Let $x = \sum_{j\in I} x_{ij} \in J(R_{ii})R$. Then $x_{ii} \in J(R_{ii})$ and so has a quasi inverse $y_{ii} \in R_{ii}$. Let

$$z = x + y_{ii} + xy_{ii} = \sum_{j\in I} x_{ij} + y_{ii} + x_{ii}y_{ii} = \sum_{j\neq i} x_{ij}.$$

Then $z^2 = 0$ and a calculation shows that $y_{ii} - z - y_{ii}z$ is a right quasi-inverse for x. □

5.5.3. We use this last result to get some information about the off-diagonal elements of $J(R)$.

LEMMA. *Let $R = (R_{ij})$ be a ring of $I \times I$ generalised matrices. Let $x \in R_{ij}$. Then $x \in J(R)$ if and only if $xR_{ji} \subseteq J(R_{ii})$.*

PROOF. If $x \in J(R)$ then

$$xR_{ji} \subseteq J(R) \cap R_{ii} = J(R_{ii})$$

by Lemma 5.5.2. If $xR_{ji} \subseteq J(R_{ii})$, then

$$(xR)^2 = xR_{ji}xR \subseteq J(R_{ii})R \subseteq J(R)$$

by Lemma 5.5.2 and so $x \in J(R)$. □

5.5.4. Using these lemmas, we quickly conclude that completely 0-simple inverse semigroups are JGH_0-semigroups.

LEMMA. *Let $S = \mathfrak{M}^0(G; I, I; \Delta)$ be a completely 0-simple inverse semigroup. Then S is a JGH_0-semigroup.*

PROOF. Let R be a contracted S-graded ring such that $J(R_H)$ is homogeneous for each subgroup H of S.

We will follow the notation of §2.2.2, so that the $(g)_{ij}$-component of R is written $R_{(g)_{ij}}$ and $R_{ij} = \bigoplus_{g \in G} R_{(g)_{ij}}$. Because Δ is the $I \times I$ identity matrix, $R = (R_{ij})$ is a ring of $I \times I$ generalised matrices. For an element $x \in R$, we will also write $x_{(g)_{ij}}$ for the

S-homogeneous components of x and x_{ij} for the generalised matrix ring components of x.

Recall from Lemma 1.5.3(iii) that the maximal subgroups of S are the sets $H = \{(g)_{ii} \mid g \in G\}$, for $i \in I$, and these are isomorphic to G. So regarding $R_{ii} = \bigoplus_{g \in G} R_{(g)_{ii}}$ as a G-graded ring, $J(R_{ii})$ is homogeneous.

Let $x \in J(R)$. We must show that each S-homogeneous component of x is also in $J(R)$. By Lemma 5.5.1, we have $x_{ij} \in J(R)$ for all $i, j \in I$. Fix elements $i, j \in I$ and $g \in G$.

If $i = j$, then $x_{ii} \subseteq J(R) \cap R_{ii} \subseteq J(R_{ii})$ by Lemma 5.5.2. Then $x_{(g)_{ii}} \in J(R_{ii})$ since $J(R_{ii})$ is homogeneous and so $x_{(g)_{ii}} \in J(R)$ by Lemma 5.5.2.

Otherwise, choose $h \in G$ and an element $y \in R_{(h)_{ji}}$. By Lemma 5.5.3, $x_{ij}y \in J(R_{ii})$. Since $J(R_{ii})$ is homogeneous for the G-gradation and y is a homogeneous element, we see that $x_{(g)_{ij}}y \in J(R_{ii})$. But h and y are arbitrary, so $x_{(g)_{ij}}R_{ji} \subseteq J(R_{ii})$. We conclude by Lemma 5.5.3 once more that $x_{(g)_{ij}} \in J(R)$.

Hence $J(R)$ is homogeneous and S is a JGH_0-semigroup. □

5.5.5. Of course, 0-direct unions of completely 0-simple inverse semigroups are also JGH_0-semigroups. The next theorem shows that these are the only regular JGH_0-semigroups.

THEOREM ([14]). *Let S be a regular semigroup. Then S is a JGH_0-semigroup if and only if S is a 0-direct union of completely 0-simple inverse semigroups.*

PROOF. That the given condition is sufficient follows from Lemma 5.5.4 and the argument used in the proof of Theorem 5.4.6.

Suppose then that S is a JGH_0-semigroup. We first show that S must be an inverse semigroup. Let $a \in S$. Then a has at least one inverse; we must show that this element is unique (cf §1.1.6). Suppose that b and c are inverses of a. Then

$$a = aba = aca, \quad b = bab, \quad \text{and } c = cac.$$

If $a = 0$, then $b = c = 0$. Assume that $a \neq 0$; then $b, c \neq 0$ also. Let $e = ab$ and $f = ac$. Then $e^2 = fe = e$ and $f^2 = ef = f$. Let $T = \{e, f\}$; then T is a subsemigroup of S which is group-closed because e and f are idempotents. By Corollary 5.2.6(iii), T is cancellative, so that $e = f$, that is $ab = ac$. Similarly, $ba = ca$ and therefore $b = bab = bac = cac = c$.

For $x \in S$, we denote by $\mathcal{J}(x)$ the $\mathcal{J}$-class of S containing x (cf §1.4.3). Write x' for the inverse of x. Let $C_x = \mathcal{J}(x) \cup \{0\}$. We claim that each C_x is a subsemigroup and that S is the 0-direct union of the distinct subsemigroups C_x. Suppose that $x, y \in S$ and $xy \neq 0$. Let $d = x'x$ and $e = yy'$. Since $xd = xx'x = x$ and $ey = yy'y = y$, we see that $\mathcal{J}(x) = \mathcal{J}(d)$ and $\mathcal{J}(y) = \mathcal{J}(e)$. Note also that $de = x'xyy'$ and $xy = xdey$ so that $\mathcal{J}(de) = \mathcal{J}(xy)$; in particular, this means that $de \neq 0$. But $d^2 = x'xx'x = x'x = d$ and similarly, $e^2 = e$. Since idempotents in an inverse semigroup commute (cf §1.1.6), we see that $T = \{d, e, de\}$ is a subsemigroup of S not containing 0, and it is group-closed because each element of T is an idempotent. By

Corollary 5.2.6(iii), T is cancellative, so $d = e = de$. Hence $\mathcal{J}(x) = \mathcal{J}(y) = \mathcal{J}(xy)$. This means that if $C_x \neq C_y$, then $C_xC_y = \{0\}$ and $C_xC_x \subseteq C_x$, so that S is a 0-disjoint union of the subsemigroups C_x. Since $\mathcal{J}(x) = \mathcal{J}(x')$ we see that $x' \in C_x$ and so each semigroup C_x is an inverse semigroup.

It remains to show that each C_x is completely 0-simple. Let $0 \neq y \in C_x$. Then $\mathcal{J}(x) = \mathcal{J}(y)$ which is to say that $S^1xS^1 = S^1yS^1$. But because S is a 0-direct union of the sets C_x, we see that $y \in C_x^1yC_x^1 = S^1yS^1 = S^1xS^1 = C_x^1xC_x^1$. Since y is arbitrary, this means that $C_x^1yC_x^1 = C_x$ for all $0 \neq y \in C_x$. Hence, C_x is 0-simple. Furthermore, C_x contains the idempotent xx'. If C_x is not completely 0-simple, then it must contain an idempotent which is not primitive (cf §1.4.6), that is it must contain distinct non-zero idempotents e and f such that $ef = fe = f$. But then $\{e, f, ef\}$ is a group-closed subsemigroup of S not containing 0 which is not cancellative. This contradicts Corollary 5.2.6(iii). So C_x must be a completely 0-simple inverse semigroup. □

CHAPTER 6

Perfect, Semiprimary, and Semilocal Semigroup Graded Rings — Finite Semigroups

In this chapter, we prove necessary and sufficient conditions for a ring graded by a finite semigroup to be perfect, semiprimary, or semilocal.

The conditions that we obtain are a generalisation of those obtained for rings graded by finite groups by Beattie and Jespers [5] (see also [31, 62, 65]): a ring graded by a finite group is right perfect if and only if the identity component is right perfect. At the same time, our result also generalises one obtained by Wauters [70], that a ring graded by a finite semilattice is right perfect if and only if each homogeneous component is right perfect.

These two results are in some sense the extreme cases of our result. On the one hand, a group has but a single idempotent, the identity element, while on the other, every element of a semilattice is idempotent. Our result is Theorem 6.7.1 below: a ring graded by a finite semigroup is right perfect if and only if each homogeneous component corresponding to an idempotent of the semigroup is right perfect. We also obtain analogous results for left perfect, semiprimary, and semilocal semigroup graded rings.

To prove our desired result, we follow the method developed in Chapter 4, combining results for group graded rings, rings graded by elementary Rees matrix semigroups, and ideal extensions.

Most of the results in this chapter will appear in [12].

6.1. Perfect, Semilocal, and Semiprimary Rings

We define the conditions semiprimary, semilocal, and left and right perfect.

6.1.1. Recall that a ring R is *right T-nilpotent* if for every sequence $x_1, x_2, x_3, \ldots$ of elements of R, there exists an n such that $x_n x_{n-1} \ldots x_1 = 0$. Similarly, R is *left T-nilpotent* if for every such sequence there exists an n such that $x_1 x_2 \ldots x_n = 0$.

A ring R is *semilocal* if the quotient $R/J(R)$ is Artinian. R is *semiprimary* if R is semilocal and $J(R)$ is nilpotent. R is *right perfect* (or *left perfect*) if R is semilocal and $J(R)$ is right T-nilpotent (or left T-nilpotent).

Note that for a ring R with an identity, R is right perfect if and only if R has the descending chain condition on principal left ideals [3, Theorem P]. For rings without identity, this is not true. For example, if R is a ring with additive group $\mathbf{Z}$ but with zero-multiplication (the product of every two elements is zero), then R is nilpotent and therefore perfect. But R does not satisfy the descending chain condition on principal left ideals.

We have the following relations between these properties:

right or left Artinian $\Rightarrow$ semiprimary $\Rightarrow$ right or left perfect $\Rightarrow$ semilocal

6.2. Ideal Extensions

We show that the properties semilocal, semiprimary, and left and right perfect are inherited by ideals, homomorphic images and are preserved under ideal extensions. These results are well-known although we cannot find references for all of them in the literature.

Note that ideals and homomorphic images of semisimple Artinian rings are semisimple Artinian and that an extension of a semisimple Artinian ring by a semisimple Artinian ideal is semisimple Artinian.

6.2.1. We first prove a simple result about semilocal rings.

LEMMA. *Let R be a ring and let A be an ideal of R such that $A \subseteq J(R)$. Then R is semilocal if and only if R/A is semilocal.*

PROOF. Since $A \subseteq J(R)$, we have $J(R/A) = J(R)/A$ and therefore

$$\frac{R}{J(R)} \cong \frac{R/A}{J(R)/A} = \frac{R/A}{J(R/A)}.$$

The result is now immediate from the definition of semilocal. □

6.2.2. Let e be an idempotent of a ring R not necessarily with an identity. If A is a subset of R, we shall write

$$(1-e)A = \{x - ex \mid x \in A\} \qquad \text{and} \qquad A(1-e) = \{x - xe \mid x \in A\}.$$

If R does not have an identity, then these are formal notations. If R does have an identity, then these definitions agree with the usual products of the element $(1-e)$ with the set A in the ring R.

These notations behave much as one would want them to. For example, if A is a right ideal then so is $(1-e)A$. We also have a decomposition $R = eR \oplus (1-e)R$ of R as the sum of two right ideals, or as the sum of two ideals if e is central.

We shall make much use of this notation in this chapter.

6.2.3. We will also need the following remark. If A is an ideal of a ring R and A has an identity element e (regarding A as a ring) then e is a central idempotent of R. For if $x \in R$, then $ex, xe \in A$ and since e is an identity element of A, we see that $ex = (ex)e = e(xe) = xe$.

6.2.4. We prove the desired result for semilocal rings.

LEMMA. *Let I be an ideal of a ring R. Then R is semilocal if and only if I and R/I are semilocal.*

PROOF. Suppose first that R is semilocal.

Note that

$$\frac{I}{J(I)} = \frac{I}{J(R) \cap I} \cong \frac{I + J(R)}{J(R)}$$

and the latter is an ideal of $R/J(R)$ so is semisimple Artinian. Hence I is semilocal.

For R/I, note that

$$\frac{R}{J(R)+I} \cong \frac{R/I}{(J(R)+I)/I}.$$

The former is a homomorphic image of $R/J(R)$ and $(R/I)/J(R/I)$ is a homomorphic image of the latter because $(J(R)+I)/I \subseteq J(R/I)$. So $(R/I)/J(R/I)$ is a homomorphic image of $R/J(R)$ and is therefore semisimple Artinian. Hence R/I is semilocal.

Now suppose that I and R/I are semilocal. Because $J(I) = J(R) \cap I$ is an ideal of R it suffices by Lemma 6.2.1 to prove that $R/J(I)$ is semilocal. So we may assume that $J(I) = 0$. Then I is a semisimple Artinian ring and therefore has an identity element e. As noted in §6.2.3, e is a central idempotent of R, so by §6.2.2, we may decompose $R = Re \oplus R(1-e)$ as a sum of ideals. Note that $Re = I$ and therefore $R(1-e) \cong R/I$. Factoring out $J(R)$ it is evident that

$$\frac{R}{J(R)} \cong I \oplus \frac{R/I}{J(R/I)}$$

and therefore $R/J(R)$ is semisimple Artinian and R is semilocal. $\square$

6.2.5. It is easy now to extend the last lemma to perfect and semiprimary rings.

LEMMA. *Let I be an ideal of a ring R. Then R is right perfect (resp. left perfect, semiprimary) if and only if I and R/I are right perfect (resp. left perfect, semiprimary).*

PROOF. Because of Lemma 6.2.4, we need only prove that the Jacobson radicals have the desired nilpotency properties.

Suppose first that R is right perfect, so that $J(R)$ is right T-nilpotent. Then $J(I) = J(R) \cap I$ is also right T-nilpotent, so I is right perfect. Now, $R/I \cong (R/J(I))/(I/J(I))$ so we may assume that $J(I) = 0$ and therefore that I is semisimple Artinian. As in the proof of Lemma 6.2.4, we obtain $R \cong I \oplus R/I$ and therefore $J(R/I)$ is isomorphic to a subring of $J(R)$ and is right T-nilpotent. Hence R/I is right perfect. The same argument holds if R is left perfect or semiprimary.

If R/I and I are semiprimary, then $J(R)$ is nilpotent by Lemma 4.4.2(i) and in view of Lemma 6.2.4, R is semiprimary.

To deal with perfect rings, we need a result analogous to Lemma 4.4.2(i) for T-nilpotence of the Jacobson radical. This, together with Lemma 6.2.4 will complete the proof. We present the argument for right T-nilpotence, the left handed version being entirely similar.

Accordingly, suppose that $J(I)$ and $J(R/I)$ are right T-nilpotent and consider a sequence $x_1, x_2, x_3, \ldots$ of elements of $J(R)$. Write y_i for the image of the element x_i in R/I; and note that $y_i \in J(R/I)$ for each i. Given a non-negative integer k, the right T-nilpotence of $J(R/I)$ implies that there must be an $l > k$ such that $y_l y_{l-1} \ldots y_{k+1} = 0$. Hence, starting with $k_0 = 0$, we construct a sequence of integers

$k_0 < k_1 < k_2 < \cdots$ such that

$$y_{k_l} y_{k_l - 1} \cdots y_{k_{l-1}+1} = 0$$

for $l > 0$. Then $z_l = x_{k_l} x_{k_l - 1} \ldots x_{k_{l-1}+1} \in J(R) \cap I = J(I)$ for $l > 0$ and the right T-nilpotence of $J(I)$ gives an n such that $x_{k_n} x_{k_n - 1} \ldots x_1 = z_n z_{n-1} \ldots z_1 = 0$. Hence $J(R)$ is right T-nilpotent as desired. $\square$

We remark that we may deduce results similar to Lemma 6.2.1 for perfect and semiprimary rings from this lemma. For example, if I is a right T-nilpotent ideal of R then R is right perfect if and only if R/I is right perfect; this is an immediate consequence of Lemma 6.2.5 because the ideal I is trivially a right perfect ring. We will use this lemma in this way without further comment.

6.3. Group Graded Rings

As mentioned above, several authors have obtained the group-graded version of our result, at least for right and left perfect and semiprimary rings. The most general result is perhaps that of Beattie and Jespers [5] who treated not just finite groups but rings with finite support graded by arbitrary groups. Our task in this section is to extend this result to rings without identity and to deduce a similar result for semilocal rings.

Throughout this section, the identity element of a group will be denoted by 1.

6.3.1. We state the result of Beattie and Jespers [5, Theorem 1.4].

THEOREM ([5]). *Let G be a group and let R be a G-graded ring with an identity such that* supp(R) *is finite. Then R is right perfect (resp. left perfect, semiprimary) if and only if R_1 is right perfect (resp. left perfect, semiprimary).*

6.3.2. Our first task is to prove a similar result for semilocal rings. This can be deduced from Theorem 6.3.1 and in one direction we are able to prove it for rings not necessarily having identities.

PROPOSITION. *Let G be a group and let R be a G-graded ring such that* supp(R) *is finite. Then R is semilocal if R_1 is semilocal. Conversely, if G is locally finite or totally ordered or if R has an identity, then R_1 is semilocal if R is semilocal.*

PROOF. Let $\bar{R} = R/J_{gr}(R)$. Since $J_{gr}(R) \cap R_1 = J(R_1)$ by Lemma 4.2.2(iii), we see that $\bar{R}_1 \cong R_1/J(R_1)$. Note that $\bar{R}$ has no non-zero homogeneous nilpotent ideals and therefore the grading is non-degenerate by Lemma 2.4.1. In particular, this implies that $\bar{R}$ has an identity if and only if $\bar{R}_1$ has an identity (and these identities necessarily coincide) by Lemma 2.4.2(i).

Suppose that R_1 is semilocal. Then $\bar{R}_1$ is semisimple Artinian and therefore semiprimary. Furthermore, $\bar{R}_1$ has an identity, being semisimple Artinian, and therefore $\bar{R}$ has an identity. We conclude by Theorem 6.3.1 that $\bar{R}$ is semiprimary. Since $J_{gr}(R) \subseteq J(R)$ by Lemma 4.2.2(iv), it follows that $R/J(R) \cong \bar{R}/J(\bar{R})$ is semisimple

Artinian and therefore R is semilocal as desired.

Conversely, suppose that R is semilocal. Then $\bar{R}$, being a homomorphic image of R is also semilocal. But $J(\bar{R}) = J(R)/J_{gr}(R)$ by Lemma 4.2.2(iv) and this is nilpotent by Lemma 4.2.2(v) so $\bar{R}$ is in fact semiprimary.

We claim that $\bar{R}$ has an identity if G is locally finite or totally ordered. If G is ordered, it follows from [49, Proposition 4.8(3)] that $J_{gr}(R) = J(R)$ (finite support is needed here) and therefore $\bar{R} = R/J(R)$ is semisimple Artinian whence it has an identity.

If G is locally finite, we may replace it by the subgroup generated by supp(R) and assume that G is finite. Note that $R/J(R)$ has an identity because it is semisimple Artinian. But $R/J(R)$ is a G-system because it is a homomorphic image of the G-graded ring $\bar{R}$ (cf §2.4.3), and we conclude from Lemma 2.4.3 that the ring $(R_1 + J(R))/J(R)$ has an identity. Since $J(R) \cap R_1 = J(R_1)$ by Lemma 4.2.2(vi), we see that

$$\bar{R}_1 \cong \frac{R_1}{J(R_1)} = \frac{R_1}{J(R) \cap R_1} \cong \frac{R_1 + J(R)}{J(R)},$$

so that $\bar{R}_1$ has an identity. By our earlier remarks $\bar{R}$ has an identity.

So in all cases covered by the statement of the converse, the ring R has an identity. It now follows from Theorem 6.3.1 that $\bar{R}_1$ is semiprimary and since $\bar{R}_1 \cong R_1/J(R_1)$ that R_1 is semilocal. □

6.3.3. We extend the last result to perfect and semiprimary rings so that (with some additional hypotheses in one direction) Theorem 6.3.1 also holds for rings without identities. This requires a result analogous to Proposition 4.2.3 for T-nilpotence which was essentially obtained in [5].

LEMMA. *Let G be a group and let R be a G-graded ring with finite support. Then $J(R)$ is right (or left) T-nilpotent if and only if $J(R_1)$ is right (or left) T-nilpotent.*

PROOF. Let $n = |\mathrm{supp}(R)|$. We prove the result for right T-nilpotence. It follows from [5, Lemma 1.1] that $J_{gr}(R)$ is right T-nilpotent if and only if $J(R_1)$ is right T-nilpotent (for rings with or without identity by the methods of §4.2.1). Since $J_{gr}(R) \subseteq J(R)$ and $J(R)^n \subseteq J_{gr}(R)$ by Lemma 4.2.2, we see that $J(R)$ is right T-nilpotent if and only if $J_{gr}(R)$ is right T-nilpotent. □

COROLLARY. *Let G be a group and let R be a G-graded ring such that* $\mathrm{supp}(R)$ *is finite. Then R is right perfect (resp. left perfect, semiprimary) if R_1 is right perfect (resp. left perfect, semiprimary). Conversely, if G is locally finite or totally ordered or if R has an identity then R right perfect (resp. left perfect, semiprimary) implies that R_1 is right perfect (resp. left perfect, semiprimary).*

PROOF. The assertions of the corollary for the case of semiprimary rings follow at once from Proposition 6.3.2 and Proposition 4.2.3 and for perfect rings from Proposition 6.3.2 and Lemma 6.3.3. □

6.4. Rees Matrix Semigroups

Our approach to rings graded by Rees matrix semigroups is somewhat different than that of Section 4.3. Rather than dealing at once with the general case, we prove results for rings which are sums of right (or left) ideals and then combine these results to handle Rees matrix semigroups.

6.4.1. We state the result that we are aiming for on rings which are sums of one-sided ideals. This result is reminiscent of those for two-sided ideals given by Lemma 6.2.4 and Lemma 6.2.5.

PROPOSITION. *Let $A_1, A_2, \ldots, A_n$ be right (or left) ideals of a ring R such that*

$$R = A_1 + A_2 + \cdots + A_n.$$

Then R is semilocal (resp. right perfect, left perfect, semiprimary) if and only if each A_i (regarded as a ring) is semilocal (resp. right perfect, left perfect, semiprimary).

Note that if R were actually a direct sum of n right ideals, then we could regard R as a ring graded by an n-element left zero semigroup (see §3.4.1). So in this case, the proposition is a special case of the general result Theorem 6.7.1 below.

We shall prove this result in Section 6.6.

6.4.2. We first show how Proposition 6.4.1 can be used to prove Theorem 6.7.1 in the case of contracted graded rings graded by finite Rees matrix semigroups.

PROPOSITION ([12]). *Let S be a finite Rees matrix semigroup and let R be a contracted S-graded ring. Then R is semilocal (resp. right perfect, left perfect, semiprimary) if and only if each component R_e corresponding to a non-zero idempotent $e \in S$ is semilocal (resp. right perfect, left perfect, semiprimary).*

PROOF. We may take $S = \mathfrak{M}^0(G; I, \Lambda; P)$ where G is a finite group and I and Λ are finite indexing sets. We use the notation of §1.5.3 and §2.2.2. We will give the proof for right perfect rings; the other cases are dealt with in an identical manner.

Since $R = \sum_{i \in I} R_{i*}$ and each R_{i*} is a right ideal of R, Proposition 6.4.1 tells us that R is right perfect if and only if each R_{i*} is right perfect. But for each i, we have $R_{i*} = \sum_{\lambda \in \Lambda} R_{i\lambda}$ and each $R_{i\lambda}$ is a left ideal of R_{i*}. So by another application of Proposition 6.4.1 we deduce that R is right perfect if and only if each $R_{i\lambda}$ is right perfect.

Fix $i \in I$ and $\lambda \in \Lambda$.

If $p_{\lambda i} = \theta$ then $(R_{i\lambda})^2 = 0$ and $R_{i\lambda}$ is certainly right perfect. In this case, the support of $R_{i\lambda}$ is the set $S_{i\lambda} \setminus \{\theta\}$ which contains no idempotents because $S_{i\lambda}$ is a null semigroup.

Otherwise, $p_{\lambda i} \neq 0$ and the corresponding subsemigroup $S_{i\lambda}$ contains exactly one non-zero idempotent by Lemma 1.5.3(i); we shall denote this idempotent by f. As explained in §2.2.2, $R_{i\lambda}$ is a G-graded ring with identity component R_f. By Corollary 6.3.3, $R_{i\lambda}$ is right perfect if and only if R_f is perfect.

Since all the non-zero idempotents of S arise in this way, we may combine these remarks to conclude that R is right perfect if and only if for each non-zero idempotent e of S, R_e is right perfect. □

6.5. Generalised Matrix Rings

Before proceeding with the proof of Proposition 6.4.1, we present a result about generalised matrix rings which will be needed in the proof. We also show how a ring containing an idempotent can be written as generalised matrix ring in a nice way. We refer to §3.3.1 for definitions and notations concerning generalised matrix rings.

6.5.1. The first result gives a sufficient condition for a generalised matrix ring to be semisimple Artinian. This is part of [72, Proposition 2.10] which gives necessary and sufficient conditions for a generalised matrix ring to be semisimple Artinian.

For our immediate purposes, however, we require only a sufficient condition, and we provide a simple proof, which is somewhat different from that of [72].

We will return to [72, Proposition 2.10] in Chapter 7 at which point a complete proof will be furnished.

LEMMA. *Let $R = (R_{ij})$ be a ring of $n \times n$ generalised matrices. If R is semiprime and each R_{ii} is semisimple Artinian, then R is semisimple Artinian.*

PROOF. It suffices to show that R is right Artinian, since a right Artinian semiprime ring is semisimple Artinian.

Regard R as a $\mathbb{Z}$-graded ring in the manner of §3.3.3. Then R has finite support and the grading is non-degenerate by Lemma 2.4.1 because a semiprime ring has no non-zero nilpotent ideals. The identity component for the $\mathbb{Z}$-gradation is $R_0 = R_{11} \oplus R_{22} \oplus \cdots \oplus R_{nn}$ which is semisimple Artinian. Hence, by Lemma 2.4.2(ii), R is right Artinian. □

6.5.2. Let R be a ring and let e be an idempotent of R. Now, R has a Peirce decomposition

$$R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e)$$

with respect to the idempotent e. Here, we use the notation of §6.2.2. This decomposition allows us to write R as a ring of 2×2 generalised matrices as follows:

$$R \cong \begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix}.$$

It is easy to check that these components do indeed multiply in the manner required for a ring of generalised matrices.

6.6. Sums of One-sided Ideals

In this section, we prove Proposition 6.4.1.

6.6.1. We begin by establishing the necessity of the condition of Proposition 6.4.1. In fact, we prove a stronger result.

LEMMA. *Let A be a right (or left) ideal of a ring R. If R is semilocal (resp. right perfect, left perfect, semiprimary) then so too is A.*

PROOF. We will suppose that A is a right ideal, the left-handed case being similar. We may assume that R is at least semilocal, that being a weaker condition than each of the others.

Consider the ideal $J(R) \cap A$ of A; it is contained in $J(A)$ and so A is semilocal if $A/(J(R) \cap A)$ is semilocal by Lemma 6.2.1. If R is, for example, right perfect, then $J(R) \cap A$ is right T-nilpotent, and so A is right perfect if $A/(J(R) \cap A)$ is right perfect by Lemma 6.2.5 and the remarks following it. Similar statements apply in the other cases.

But $A/(J(R) \cap A) \cong (A+J(R))/J(R)$ is a right ideal of $R/J(R)$ which is semisimple Artinian. So it suffices to prove that a right ideal of a semisimple Artinian ring is semiprimary, since semiprimary is stronger than each of the other conditions.

Accordingly, let R be a semisimple Artinian ring and let Q be a right ideal of R. Then there is an idempotent e of Q such that $Q = eR$. Then $Q = eRe \oplus eR(1-e)$ as an additive group. But eRe is semisimple Artinian and $eR(1-e)$ is a nilpotent ideal of Q. Hence $J(Q) = eR(1-e)$ is nilpotent and $Q/J(Q) \cong eRe$ is semisimple Artinian. Therefore Q is semiprimary as desired. □

6.6.2. In order to prove the sufficiency of the condition of Proposition 6.4.1 we first handle the case of a ring which is the sum of only two semilocal one-sided ideals

LEMMA. *Let A_1 and A_2 be right (or left) ideals of R such that $R = A_1 + A_2$. Suppose that A_1 and A_2 are semilocal. Then R is semilocal.*

PROOF. We must show that $R/J(R)$ is semisimple Artinian. First note that $R/J(R) = A_1' + A_2'$ where $A_i' = (A_i + J(R))/J(R)$ for $i = 1, 2$. But each A_i' is semilocal by Lemma 6.2.4 being a homomorphic image of A_i. Hence it suffices to prove that $R = A_1 + A_2$ is semisimple Artinian when $J(R) = 0$ and A_1 and A_2 are semilocal. We assume these hypotheses until the end of the proof.

Note that $J(A_i)A_i \subseteq J(A_i)$, so $J(A_i)A_i$ is a quasi-regular right ideal of R. Then $J(R) = 0$ implies that $J(A_i)A_i = 0$ for $i = 1, 2$. In particular, $J(A_i)^2 = 0$ so A_1 and A_2 are in fact semiprimary.

Now, $A_1/J(A_1)$ is semisimple Artinian, so it has an identity. But $J(A_1)$ is a nilpotent ideal and therefore we may lift this identity to an idempotent e of A_1 (see, for example, [63, Corollary 1.1.28], the proof of which does not require rings with identities). Then $(1-e)A_1 \subseteq J(A_1)$, so $(1-e)A_1 = 0$ since it is a quasi-regular right ideal of R. This means that e is a left identity element for A_1.

Using this idempotent e we express R as a ring of 2×2 generalised matrices as in §6.5.2. Let

$$R_{11} = eRe, \qquad R_{12} = eR(1-e),$$

$$R_{21} = (1-e)Re, \qquad \text{and} \quad R_{22} = (1-e)R(1-e)$$

so that

$$R \cong \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}.$$

Since e is a left identity for A_1 we see that $eR \subseteq A_1 = eA_1 \subseteq eR$ and so $eR = A_1$. Also $(1-e)A_1 = 0$ so $(1-e)R = (1-e)A_2$. We have:

$$R_{11} = A_1 e, \qquad R_{12} = A_1(1-e),$$

$$R_{21} = (1-e)A_2 e, \qquad \text{and} \qquad R_{22} = (1-e)A_2(1-e).$$

We claim that R_{11} and R_{22} are semisimple Artinian. $R_{11} = A_1 e$ is a left ideal of A_1, so is semilocal by Lemma 6.6.1. Similarly, $A_2(1-e)$ is semilocal being a left ideal of A_2 and therefore $R_{22} = (1-e)A_2(1-e)$ is semilocal as it is the homomorphic image of $A_2(1-e)$ under the map $x \mapsto x - ex$, which is easily seen to be a ring homomorphism. But by Lemma 5.5.2, $J(R_{ii}) \subseteq J(R) = 0$ for $i = 1, 2$ and therefore R_{11} and R_{22} are semisimple Artinian as claimed.

Since $J(R) = 0$, R is certainly semiprime. From the above and Lemma 6.5.1, it follows that R is semisimple Artinian as desired. □

6.6.3. In order to extend the last lemma to perfect and semiprimary rings we require that the sum of nilpotent or T-nilpotent one-sided ideals is again nilpotent or T-nilpotent. This is the content of the next lemma. The first statement is well-known; however we will give a brief proof.

LEMMA. *Let A_1 and A_2 be right (or left) ideals of a ring R.*

(i) *If A_1 and A_2 are nilpotent then $A_1 + A_2$ is nilpotent.*

(ii) *If A_1 and A_2 are right (resp. left) T-nilpotent then $A_1 + A_2$ is right (resp. left) T-nilpotent.*

PROOF. Suppose that $A_1^n = A_2^m = 0$. We will show that the product of any $m + n$ elements of $A_1 \cup A_2$ is zero; since elements of $A_1 + A_2$ are sums of such elements this suffices to show that $(A_1 + A_2)^{m+n} = 0$. Let $x_1, x_2, \ldots, x_{m+n}$ be elements of $A_1 \cup A_2$. Either at least n of these elements are in A_1 or at least m of these elements are in A_2. Without loss of generality, assume the former, so that $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$ are elements of A_1 where $1 \leq i_1 < i_2 < \cdots < i_n \leq m + n$. For $1 \leq j < n$, let

$$y_j = x_{i_j} x_{i_j+1} \ldots x_{i_{j+1}-1};$$

then $y_j \in A_1$ because $x_{i_j} \in A_1$ and A_1 is a right ideal of R. But $A_1^n = 0$ so that $y_1 y_2 \ldots y_{n-1} x_{i_n} = 0$ and therefore $x_1 x_2 \ldots x_{m+n} = 0$. This proves (i).

Suppose now that A_1 and A_2 are right T-nilpotent but $A_1 + A_2$ is not right T-nilpotent. Then there is a sequence $z_1, z_2, z_3, \ldots$ of elements of $A_1 + A_2$ such that for all n, we have $z_n z_{n-1} \ldots z_1 \neq 0$. Write $z_i = x_i + y_i$ where $x_i \in A_1$ and $y_i \in A_2$. Then there must be a sequence $w_i \in \{x_i, y_i\}$ such that for all n, we have $w_n w_{n-1} \ldots w_1 \neq 0$. Now, one of the two right ideals A_1 and A_2 must contain infinitely many of the elements w_i; without loss of generality assume that A_1 contains the subsequence $w_{i_1}, w_{i_2}, w_{i_3}, \ldots$ where $i_1 < i_2 < i_3 < \cdots$. Let $u_j = w_{i_{j+1}} w_{i_{j+1}-1} \ldots w_{i_j+1}$ for each j. Then $u_j \in A_1$ since A_1 is a right ideal of R and $w_{i_{j+1}} \in A_1$. But the sequence

$u_1, u_2, u_3, \ldots$ violates the right T-nilpotency of A_1 because $u_n u_{n-1} \ldots u_1 = 0$ for some n would imply that $w_{i_{n+1}} w_{i_{n+1}-1} \ldots w_1 = 0$. This contradiction shows that $A_1 + A_2$ must be right T-nilpotent. A similar argument deals with the left T-nilpotent case. This proves (ii). □

6.6.4. It is now a simple matter to prove a version of Lemma 6.6.2 for perfect and semiprimary rings.

LEMMA. *Let A_1 and A_2 be right (or left) ideals of R such that $R = A_1 + A_2$. Suppose that A_1 and A_2 are right perfect (resp. left perfect, semiprimary). Then R is right perfect (resp. left perfect, semiprimary).*

PROOF. In view of Lemma 6.6.2 we need only show that $J(R)$ inherits the necessary nilpotency conditions.

Suppose that A_1 and A_2 are right perfect right ideals so that $J(A_1)$ and $J(A_2)$ are right T-nilpotent. For $i = 1$ or 2, we have $A_i J(R) \subseteq A_i \cap J(R)$ because A_i is a right ideal of R. But $A_i \cap J(R)$ is a quasi-regular ideal of A_i so that $A_i J(R) \subseteq J(A_i)$, and therefore $A_i J(R)$ is a right T-nilpotent right ideal of R. Note that,

$$J(R)^2 \subseteq RJ(R) = (A_1 + A_2)J(R) = A_1 J(R) + A_2 J(R).$$

By Lemma 6.6.3(ii), $A_1 J(R) + A_2 J(R)$ is right T-nilpotent and so therefore is $J(R)$.

The proof in the other cases is similar. □

6.6.5. Finally, we prove Proposition 6.4.1 which we restate for the reader's convenience.

PROPOSITION ([12]). *Let $A_1, A_2, \ldots, A_n$ be right (or left) ideals of a ring R such that*

$$R = A_1 + A_2 + \cdots + A_n.$$

Then R is semilocal (resp. right perfect, left perfect, semiprimary) if and only if each A_i (regarded as a ring) is semilocal (resp. right perfect, left perfect, semiprimary).

PROOF. Suppose that $R = A_1+A_2+\cdots+A_n$ is a sum of one-sided ideals. That each A_i inherits the semilocal, semiprimary and perfect properties from R is the content of Lemma 6.6.1. The converse statement, that R is semilocal, right perfect, left perfect or semiprimary if each A_i is semilocal, right perfect, left perfect or semiprimary follows by induction from Lemma 6.6.2 and Lemma 6.6.4. □

6.7. Finite Semigroups

6.7.1. We now have all the necessary pieces to deal with rings graded by finite semigroups.

THEOREM ([12]). *Let S be a finite semigroup and let R be an S-graded ring. Then R is semilocal (resp. right perfect, left perfect, semiprimary) if and only if for each idempotent e of S, the component R_e is semilocal (resp. right perfect, left perfect, semiprimary).*

PROOF. We will give the proof for the right perfect case only. The proofs in the other cases follow in the same way.

If S does not have a zero, adjoin one and put $R_\theta = 0$. Otherwise, θ is an idempotent of S and R_θ is an ideal of R, and by Lemma 6.2.5, R is right perfect if and only if R_θ and R/R_θ are right perfect. So we may assume that S has a zero and that R is a contracted S-graded ring.

If S is a null semigroup, then it has only one idempotent, the zero element θ. Since R is a contracted S-graded ring, $R_\theta = 0$ which is trivially right perfect. Furthermore, $R^2 \subseteq R_\theta = 0$ so that R is a nilpotent ring and is therefore right perfect. So the theorem holds in this case.

If S is a 0-simple semigroup, then it is completely 0-simple by Theorem 1.4.8. In this case, we may take S to be a Rees matrix semigroup by Theorem 1.5.2 and the assertions of the theorem follow at once from Proposition 6.4.2.

For general finite semigroups, we proceed by induction on $|S|$. If $|S| = 1$, then $S = \{\theta\}$ and S is a null semigroup so the theorem holds. Suppose then that the result is known for semigroups which are strictly smaller than S. If S is 0-simple or null, we have already demonstrated the result. So assume that S is not of these types. Then S has a proper non-zero ideal T, and $|T|, |S/T| < |S|$. By induction, R_T is right perfect if and only if R_e is right perfect for each idempotent $e \in T$. In addition, $\bar{R} = R_S/R_T$ is a contracted S/T-graded ring and so $\bar{R}$ is right perfect if and only if R_e is right perfect for all non-zero idempotents $e \in S/T$ (we need not consider R_θ since $\bar{R}$ is a contracted S/T-graded ring). But the non-zero idempotents of S/T are just the idempotents of S which are not in T (see §1.3.5) and the corresponding components of $\bar{R}$ are just the components R_e of R where e is an idempotent in $S \setminus T$ (see §2.1.4). So, $\bar{R}$ is right perfect if and only if R_e is right perfect for each idempotent e in the set $S \setminus T$. Finally, by Lemma 6.2.5, R is right perfect if and only if R_T and $\bar{R}$ are right perfect, which yields the result. □

6.8. Corollaries

6.8.1. As mentioned previously, Wauters obtained our result for semilattices in [70]. Since every element of a semilattice is idempotent, the result becomes:

COROLLARY ([70]). *Let Γ be a finite semilattice and let R be a Γ-graded ring. Then R is semilocal (resp. right perfect, left perfect, semiprimary) if and only if each component R_γ, $\gamma \in \Gamma$ is semilocal (resp. right perfect, left perfect, semiprimary).*

In fact Wauters proved more: with the additional assumption that $J(R_\gamma) \neq R_\gamma$ for each component, he showed that if a semilattice graded ring is semilocal then the semilattice must be finite.

6.8.2. We apply the theorem to finite generalised matrix rings.

COROLLARY ([12]). *Let $R = (R_{ij})$ be a ring of $n \times n$ generalised matrices. Then R is semilocal (resp. right perfect, left perfect, semiprimary) if and only if for $1 \leq i \leq n$,* *the subring R_{ii} is semilocal (resp. right perfect, left perfect, semiprimary).*

PROOF. As explained in §3.3.2, we may regard R as a contracted S-graded ring where $S = \mathfrak{M}^0(1; n, n; \Delta)$ is a completely 0-simple inverse semigroup. The components of R corresponding to non-zero idempotents of S are the diagonal components R_{ii}. So this result is a special case of Theorem 6.7.1. □

6.8.3. For certain generalised matrix rings we can obtain a rather stronger result, at least for right and left perfect rings.

Recall (see, for example, [63, §4.1]) that a ring of 2×2 generalised matrices

$$\begin{pmatrix} A & P \\ Q & B \end{pmatrix}$$

is called a Morita context. If A and B are rings with identities and furthermore, $AP = P = PB$, $QA = Q = BQ$, $PQ = A$, and $QP = B$, then by Morita's Theorem (for example, [63, Theorem 4.1.17]), the categories Mod-A and Mod-B of right A-modules and right B-modules respectively are equivalent. This means that any property of the ring A which can be defined in terms of right A-modules in a categorical way can be transferred to the ring B, and vice versa. Such rings are said to be Morita equivalent.

Now, for rings with identity, there is a categorical way of defining perfectness, namely a ring R is right perfect if and only if each right R-module has a projective cover (see, for example, [3]). So in the example above, A is right perfect if and only if R is right perfect.

We can extend this to larger generalised matrix rings as follows:

COROLLARY. *Let $R = (R_{ij})$ be a ring of $n \times n$ generalised matrices. Suppose that $R_{ij}R_{jk} = R_{ik}$ for all i, j, and k, and that the subrings R_{ii} each have an identity. Then R is right (or left) perfect if and only if any one of the subrings R_{ii} is right (or left) perfect.*

PROOF. Of course, if R is right perfect each of the components R_{ii} is right perfect by Corollary 6.8.2.

Conversely, suppose that R_{ii} is right perfect. For each $j \neq i$, the components R_{ii}, R_{ij}, R_{ji}, and R_{jj} form a subring of 2×2 generalised matrices:

$$\begin{pmatrix} R_{ii} & R_{ij} \\ R_{ji} & R_{jj} \end{pmatrix}$$

which defines a Morita equivalence between the rings R_{ii} and R_{jj}. By the remarks above, right perfectness is preserved by Morita equivalences, so that R_{jj} is right perfect. In this way, we see that all the diagonal components R_{jj} are right perfect. Hence R is right perfect by Corollary 6.8.2. □

CHAPTER 7

Semisimple Artinian Rings Graded by Elementary Rees Matrix Semigroups

In this chapter, we give necessary and sufficient conditions for a contracted graded ring graded by an elementary Rees matrix semigroup to be semisimple Artinian. This result generalises both Munn's result on Munn algebras [15, 50] and a result of Wauters and Jespers on generalised matrix rings [72].

We first investigate the relationship between ideals and homogeneous ideals of such rings. This enables us to relate semiprime rings to graded semiprime rings.

Throughout this chapter, unless stated otherwise, $S = \mathfrak{M}^0(1; I, \Lambda; P)$ is an elementary Rees matrix semigroup and R is a contracted S-graded ring. We will use the notation of §2.2.2.

The results of this chapter will appear in [11].

7.1. Ideals and Homogeneous Ideals

We will investigate the relationship between ideals of R and homogeneous ideals of R. In particular, there is a one-to-one correspondence between prime (resp. semiprime) ideals of R and graded prime (resp. graded semiprime) ideals of R (see §7.2.1 for definitions of these terms). The work of this section and the next is a generalisation of a similar analysis of ideals of Munn algebras in [55, Chapter 5].

7.1.1. Recall that the set of ideals of R forms a lattice which we will write as $\mathcal{L}(R)$. Furthermore, the set of homogeneous ideals of R forms a sublattice (since the intersection and sum of homogeneous ideals is a homogeneous ideal) denoted $\mathcal{L}_{gr}(R)$.

We will relate homogeneous ideals and ideals by means of a pair of maps between these lattices. In one direction, we use the map which takes an arbitrary ideal to its homogeneous part. Define

$$(-)_h : \mathcal{L}(R) \longrightarrow \mathcal{L}_{gr}(R)$$
$$N \longmapsto N_h = \sum_{i,\lambda} N \cap R_{i\lambda}.$$

An obvious choice for the other map is the inclusion $\mathcal{L}_{gr}(R) \hookrightarrow \mathcal{L}(R)$ which maps a homogeneous ideal to itself. However, this map does not have the properties that we want; for example, a graded prime ideal is not necessarily prime. Instead, define

$$(-)^* : \mathcal{L}_{gr}(R) \longrightarrow \mathcal{L}(R)$$
$$M \longmapsto M^* = \{x \in R \mid RxR \subseteq M\}.$$

7.1.2. We list some basic properties of the maps defined above.

LEMMA. *For all $N \in \mathcal{L}(R)$ and all $M \in \mathcal{L}_{gr}(R)$*

(i) *The maps $(\ )_h$ and $(\ )^*$ are order preserving.*

(ii) $N_h \subseteq N$.

(iii) $RM^*R \subseteq M \subseteq M^*$.

(iv) $M_h = M$. *Hence $(\ )_h$ is surjective.*

(v) $N \subseteq (N_h)^*$.

(vi) $M \subseteq (M^*)_h$.

PROOF. Statements (i), (ii), (iii), and (iv) follow immediately from the definitions. By Lemma 2.2.3, RNR is a homogeneous ideal of R, so $RNR \subseteq N_h$, and therefore $N \subseteq (N_h)^*$. Hence (v). By (iii), $M \subseteq M^*$. Applying (i) and (iv), we see that $M = M_h \subseteq (M^*)_h$. Hence (vi). □

In fact, it is true that the maps $(-)_h$ and $(-)^*$ are $\wedge$-complete semilattice homomorphisms (that is, they preserve arbitrary intersections), but the assertions of the lemma suffice for our purposes.

7.2. Prime and Graded Prime Ideals

We will be mainly interested in applying the maps introduced above to prime and semiprime ideals of R. To do this, we must introduce the corresponding notions of graded prime ideals and graded semiprime ideals.

7.2.1. Although we will only make use of these concepts in the present situation of a ring R graded by an elementary Rees matrix semigroup S, the definitions introduced here (and the statements made about them) are all valid for a ring R graded by any semigroup S.

The ring R is said to be a *graded prime ring* if for all homogeneous ideals A and B of R, if $AB = 0$ then $A = 0$ or $B = 0$. The ring R is a *graded semiprime ring* if it has no non-zero nilpotent homogeneous ideals. Note that we may replace ideal with right ideal or left ideal and obtain equivalent definitions. Of course, a graded prime ring is graded semiprime.

An ideal P of R is called a *graded prime ideal* (resp. *graded semiprime ideal*) if it is a homogeneous ideal and R/P is a graded prime ring (resp. graded semiprime ring).

There are several other equivalent ways of defining graded prime and graded semiprime ideals just as there are for the ungraded versions. For example, P is a graded prime ideal if and only if it is homogeneous and for all homogeneous elements x and y of R, if $xRy \subseteq P$ then $x \in P$ or $y \in P$. Similarly, an ideal is graded semiprime if and only if it is an intersection of graded prime ideals. The proofs of these equivalences are essentially the same as for their corresponding ungraded versions, using homogeneous elements and homogeneous ideals throughout.

The *graded prime radical* of R, written $B_{gr}(R)$, is defined to be the intersection of all the graded prime ideals of R, or equivalently, the smallest graded semiprime ideal of R.

7.2.2. We first show that the composition of the two maps of §7.1.1 in either order is the identity when restricted to the set of semiprime or graded semiprime ideals as appropriate.

LEMMA. *Let $M \in \mathcal{L}_{gr}(R)$ and $N \in \mathcal{L}(R)$.*

(i) *If R is a graded semiprime ring and x is a non-zero homogeneous element of R then $RxR \neq 0$.*

(ii) *If M is a graded semiprime ideal then $(M^*)_h = M$.*

(iii) *If N is a semiprime ideal then $(N_h)^* = N$.*

Proof. Suppose that R is graded semiprime, x is a homogeneous element of R, and $RxR = 0$. Then $(R^1xR^1)^3 \subseteq RxR = 0$. Since R^1xR^1 is a homogeneous ideal, $R^1xR^1 = 0$ and therefore $x = 0$. This proves (i).

For (ii), let $x \in (M^*)_h$ be a homogeneous element. Since $(M^*)_h \subseteq M^*$ by Lemma 7.1.2(ii), it follows from the definition of M^*, that $RxR \subseteq M$. Applying (i) to the graded semiprime ring R/M, we conclude that $x \in M$. Hence, $(M^*)_h \subseteq M$. The reverse inclusion is Lemma 7.1.2(vi).

For (iii), note that $((N_h)^*)^3 \subseteq R(N_h)^*R \subseteq N_h \subseteq N$ by Lemma 7.1.2(ii),(iii). Since N is semiprime, $(N_h)^* \subseteq N$. The reverse inclusion is Lemma 7.1.2(v). □

7.2.3. Now we prove the desired one-to-one correspondence between prime (resp. semiprime) ideals of R and graded prime (resp. graded semiprime) ideals of R.

Proposition ([11]). *The maps $(\ )_h$ and $(\ )^*$ establish a one-to-one correspondence between the prime (resp. semiprime) ideals of R and the graded prime (resp. graded semiprime) ideals of R.*

Proof. Let N be a prime ideal of R and let M be a graded prime ideal of R. We first show that N_h is graded prime and that M^* is prime.

Suppose that A and B are ideals of R such that $AB \subseteq M^*$. Let $x \in A_h$ and $y \in B_h$ be homogeneous elements. Then xy is a homogeneous element of AB, that is $xy \in (AB)_h$. Hence

$$A_h B_h \subseteq (AB)_h \subseteq (M^*)_h = M$$

(by Lemma 7.2.2(ii)). Since M is graded prime, it follows that one of A_h and B_h is contained in M, say $A_h \subseteq M$. Then

$$A \subseteq (A_h)^* \subseteq M^*$$

by Lemma 7.1.2(i),(v). This proves that M^* is a prime ideal. If instead, M were merely graded semiprime, then putting $A = B$ in the above argument, M^* would be a semiprime ideal.

Suppose that C and D are graded ideals of R such that $CD \subseteq N_h$. Let $c_1, c_2 \in C^*$ and $d_1, d_2 \in D^*$. Then

$$Rc_1d_1c_2d_2R \subseteq (Rc_1R)(Rd_2R) \subseteq CD \subseteq N_h.$$

Hence $(C^*D^*)^2 \subseteq (N_h)^* = N$ (by Lemma 7.2.2(iii)). Since N is prime, it follows that either C^* or D^* is contained in N, say $C^* \subseteq N$. By Lemma 7.1.2(i),(vi), $C \subseteq (C^*)_h \subseteq N_h$. So N_h is graded prime. Similarly, taking $C = D$ in the above argument would prove that N is semiprime implies N_h is graded semiprime.

Because $(N_h)^* = N$ and $(M^*)_h = M$ by Lemma 7.2.2, it now follows that the maps $(-)_h$ and $(-)^*$ are surjective and injective when restricted to the sets of semiprime and graded semiprime ideals (or prime and graded prime ideals) respectively. □

7.2.4. As an immediate corollary, we have

COROLLARY ([11]). *Let R be a contracted S-graded ring.*

(i) *R is a prime (resp. semiprime) ring if and only if R is a graded prime (resp. graded semiprime) ring and $RxR = 0$ implies $x = 0$ for $x \in R$.*

(ii) *$B(R)_h = B_{gr}(R)$ and $B_{gr}(R)^* = B(R)$.*

PROOF. Write 0 for the zero ideal of R. The condition $RxR = 0$ implies $x = 0$ is equivalent to the assertion that $0^* = 0$.

We will prove (i) in the prime case; the semiprime case is proved in the same way. If R is a prime ring, then 0 is a prime ideal, and by Proposition 7.2.3, $0_h = 0$ is a graded prime ideal, so R is a graded prime ring. By Lemma 7.2.2(iii), $0^* = (0_h)^* = 0$. Conversely, if R is a graded prime ring and $0^* = 0$, then by Proposition 7.2.3, 0 is a prime ideal of R and R is a prime ring. This proves (i).

Statement (ii) follows immediately from Proposition 7.2.3, Lemma 7.1.2(i), and the fact that $B(R)$ and $B_{gr}(R)$ are respectively the smallest semiprime and graded semiprime ideals of R. □

We remark that the fact that R is a contracted S-graded ring where S is an elementary Rees matrix semigroup enters into the results of this section only through Lemma 2.2.3, the assertion that RXR is a homogeneous ideal for any subset X, which is used to prove Lemma 7.1.2(v). However, Lemma 2.2.3 is a strong result which does not hold for general semigroups. Indeed, it can be shown that the only 0-simple semigroups for which a result like Lemma 7.1.2 holds are elementary Rees matrix semigroups.

7.3. Semisimple Artinian Rings

Let $S = \mathfrak{M}^0(1; I, \Lambda; P)$ be an elementary Rees matrix semigroup and let R be a contracted S-graded ring. In this section we obtain necessary and sufficient conditions for R to be semisimple Artinian. In some sense, the conditions we obtain are disappointing, but the class of rings graded by elementary Rees matrix semigroups is large and we give examples to show that none of the conditions can be dispensed with.

7.3.1. The result we obtain follows easily from the work of the previous section and Chapter 6.

Proposition ([11]). *Let S be a finite elementary Rees matrix semigroup and let R be a contracted S-graded ring. Then R is semisimple Artinian if and only if it satisfies the following conditions:*

(i) *R is graded semiprime.*

(ii) *For any $x \in R$, if $RxR = 0$ then $x = 0$.*

(iii) *Each component R_e corresponding to a non-zero idempotent e of S is semiprimary.*

PROOF. A ring is semisimple Artinian if and only if it is semiprime and semiprimary. By Corollary 7.2.4(i), conditions (i) and (ii) are equivalent to R being semiprime. By Theorem 6.7.1, condition (iii) is equivalent to R being semiprimary. □

7.3.2. We remark that the restriction to finite semigroups in the proposition is not really a restriction.

For suppose $S = \mathfrak{M}^0(1; I, \Lambda; P)$ is an elementary Rees matrix semigroup and R is a contracted S-graded ring which is semisimple Artinian. Then R must have an identity element 1. Write $1 = \sum_{i,\lambda} 1_{i\lambda}$ as a sum of homogeneous components. Since the support of 1 is finite, there are finite sets $I_0 \subseteq I$ and $\Lambda_0 \subseteq \Lambda$ such that $1_{i\lambda} = 0$ if $i \notin I_0$ or $\lambda \notin \Lambda_0$.

Suppose $j \notin I_0$ and choose an element $r_{j\mu} \in R_{j\mu}$ for some $\mu \in \Lambda$. We require that

$$r_{j\mu} = 1r_{j\mu} = \sum_{i,\lambda} 1_{i\lambda} r_{j\mu}.$$

Taking homogeneous components in $R_{j\mu}$ on each side, we get

$$r_{j\mu} = \sum_{\lambda} 1_{j\lambda} r_{j\mu}.$$

But $1_{j\lambda} = 0$ for all λ by choice of j, so we conclude that $r_{j\mu} = 0$. Since μ was arbitrary, we see that $R_{j\cdot} = 0$. A similar argument shows that $R_{\cdot\mu} = 0$ for all $\mu \notin \Lambda_0$.

Let P_0 be the $\Lambda_0 \times I_0$ matrix obtained from P by deleting all rows and columns except those indexed by elements of Λ_0 and I_0. Then $T = \mathfrak{M}^0(1; I_0, \Lambda_0; P_0)$ is a subsemigroup of S if we identify elements in the obvious way. Since I_0 and Λ_0 are finite, T is finite. But we have shown that $\operatorname{supp}(R) \subseteq T$, so we may consider R to be a T-graded ring.

7.3.3. We give an example to show that we cannot improve condition (iii) of Proposition 7.3.1 to say that each idempotent component be semisimple Artinian.

Let K be a field and let $R = M_3(K)$ be the ring of 3×3 matrices over K. Let $P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and let $S = \mathfrak{M}^0(1; 2, 2; P)$. We will express R as an S-graded ring by partitioning R as follows:

$$\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = \left(\begin{array}{c|cc} K & K & K \\ K & K & K \\ \hline K & K & K \end{array}\right)$$

so that, for instance, R_{21} is the set of matrices with the only non-zero entry in the $(3, 1)$-position. It is easy to verify that these components $R_{i\lambda}$ multiply in the manner required for $R = \bigoplus_{i,\lambda=1}^{2} R_{i\lambda}$ to be a contracted S-graded ring.

Then R is a semisimple Artinian ring, but the components $R_{i\lambda}$ each have non-zero Jacobson radical.

7.3.4. Let $S = \{e, f\}$ be a semigroup with $e^2 = fe = e$ and $f^2 = ef = f$. Then S^0 is isomorphic to an elementary Rees matrix semigroup (cf §3.4.2). Let $R = \mathbf{Q}[S]$, the semigroup algebra over S with rational coefficients. Putting $R_e = \mathbf{Q}e$, $R_f = \mathbf{Q}f$, and $R_\theta = 0$, we see that R is a contracted S-graded ring.

We claim that R is graded semiprime. Suppose that I is a nilpotent homogeneous ideal of R. Then $I \cap R_e$ is a nilpotent ideal of R_e so $I \cap R_e = 0$ since R_e is isomorphic to $\mathbf{Q}$. Similarly, $I \cap R_f = 0$ and therefore $I = 0$.

Since R_e and R_f are isomorphic to $\mathbf{Q}$, they are certainly semiprimary.

But R is not semisimple Artinian. Indeed, it is neither semisimple nor Artinian. For $J(R) = \mathbf{Q}(e - f)$ and

$$\mathbf{Z}(e-f) \supset 2\mathbf{Z}(e-f) \supset \cdots \supset 2^n\mathbf{Z}(e-f)\cdots$$

is a strictly descending chain of right ideals of R; both of these claims follow easily once it is noted that $(e - f)R = 0$.

So condition (ii) of Proposition 7.3.1 is required.

7.4. Munn Algebras and Generalised Matrix Rings

We show how Proposition 7.3.1 generalises Munn's characterisation of semisimple Artinian Munn algebras and the work of Wauters and Jespers on generalised matrix rings.

7.4.1. We refer to §3.2.1 for the definition of a Munn algebra and recall from §3.2.3 that a Munn algebra can be graded by an elementary Rees matrix semigroup. In [50], Munn proved the following theorem giving necessary and sufficient conditions for a finite dimensional Munn algebra to be semisimple Artinian. We will briefly indicate how this theorem can be derived from Proposition 7.3.1.

THEOREM ([50]). *Let K be a field and let A be a finite dimensional K-algebra. Let $R = \mathfrak{M}(A; m, n; P)$ be a Munn algebra over A. Then R is semisimple Artinian if and only if the following are satisfied:*

(i) *A is semisimple Artinian.*

(ii) *$m = n$ and P is an invertible matrix in $M_n(A)$.*

PROOF. We will use the notation of §3.2.1. In particular, for matrices X and Y, the product XY is their product in a Munn algebra and $X \circ Y$ is their ordinary matrix product.

To demonstrate the necessity of conditions (i) and (ii), we will show that they follow from conditions (i), (ii), and (iii) of Proposition 7.3.1. Accordingly, suppose that R satisfies conditions (i), (ii), and (iii) of Proposition 7.3.1.

Let N be an ideal of A and suppose that $N^2 = 0$. Then the set

$$\hat{N} = \{r \in R \mid r_{i\lambda} \in N \text{ for all } i, \lambda\}$$

is a homogeneous ideal of R. If $X, Y \in \hat{N}$ then $XY = X \circ P \circ Y = 0$ since $x_{i\lambda} p_{\lambda j} y_{j\mu} \in N^2 = 0$ for all i, j, λ, and μ. So $\hat{N}^2 = 0$ and therefore by Proposition 7.3.1(i), $\hat{N} = 0$. Hence $N = 0$. So A is semiprime. But A is semiprimary by Proposition 7.3.1(iii), so A is in fact semisimple Artinian, and (i) holds.

If $m \neq n$ or P is not invertible, then there exists an $m \times n$ matrix X over A such that either $P \circ X = 0$ or $X \circ P = 0$ (see [15, Corollary 5.10 and Theorem 5.11]). Then regarding X as an element of R, we see that for any $Y, Z \in R$,

$$YXZ = Y \circ P \circ X \circ P \circ Z = 0.$$

Hence X is a non-zero element of R satisfying $RXR = 0$. So Proposition 7.3.1(ii) implies (ii).

Conversely, suppose that R satisfies (i) and (ii). Then A has an identity because it is semisimple Artinian. Let Q be the inverse of P in $M_n(A)$ so that $P \circ Q = Q \circ P = I_n$, the $n \times n$ identity matrix over A. Of course, Q may be regarded as an element of R.

Rather than demonstrating that R satisfies conditions (i), (ii), and (iii) of Proposition 7.3.1, it is easier to show that R is semisimple Artinian directly. For the map $\mathfrak{M}(A; n, n; P) \to M_n(A)$, $X \mapsto X \circ P$ is easily seen to be an algebra homomorphism and it is an isomorphism because it has inverse $Y \mapsto Y \circ Q$. But $M_n(A)$ is semisimple Artinian because A is semisimple Artinian. Hence R is semisimple Artinian. □

7.4.2. Another related result of which Proposition 7.3.1 is a generalisation is the characterisation due to Wauters and Jespers of semisimple Artinian generalised matrix rings. We will show how this characterisation can be derived from Proposition 7.3.1.

Note that the implication (ii) implies (i) of the theorem was proved earlier as Lemma 6.5.1 and this lemma was used to prove Theorem 6.7.1 which was in turn used to prove Proposition 7.3.1.

Let $R = (R_{ij})$ be a ring of $n \times n$ generalised matrices. Following [72], we will say that R satisfies the *annihilator condition* if it satisfies:

(ANN) If $x \in R_{ji}$ and $R_{ij}x = 0$ then $x = 0$.

THEOREM ([72]). *Let $R = (R_{ij})$ be a ring of $n \times n$ generalised matrices. The following are equivalent:*

(i) *R is semisimple Artinian.*

(ii) *R is semiprime and all R_{ii} are semisimple Artinian.*

(iii) *R satisfies the annihilator condition (ANN) and all R_{ii} are semisimple Artinian.*

PROOF. We introduce two more conditions and show that all five conditions are equivalent.

(iv) R is graded semiprime, each R_{ii} is semiprimary, and if $RxR = 0$ for some $x \in R$ then $x = 0$.

(v) R is semiprime and each R_{ii} is semiprimary.

First note that (i) and (iv) are equivalent by Proposition 7.3.1 and (iv) and (v) are equivalent by Corollary 7.2.4(i).

Suppose that R satisfies (v). Fix $1 \leq i \leq n$ and suppose that $J(R_{ii})^m = 0$. Note that

$$J(R_{ii})RJ(R_{ii}) = J(R_{ii})R_{ii}J(R_{ii}) \subseteq J(R_{ii})^2$$

because for elements x and y of R_{ii}, we have $xR_{jk}y = 0$ if $j \neq i$ or $k \neq i$. Applying this result several times we get $(RJ(R_{ii}))^m \subseteq RJ(R_{ii})^m = 0$. Since R is semiprime, this implies $RJ(R_{ii}) = 0$ and hence $J(R_{ii}) = 0$. So R_{ii} is semisimple Artinian. Thus (v) and (ii) are equivalent, the other implication being trivial.

We will show now that (iv) and (ii) imply (iii). Clearly, we need only verify that R satisfies (ANN). Suppose that $x \in R_{ji}$ and $R_{ij}x = 0$. Then $xRx = xR_{ij}x = 0$ so that $x = 0$ because R is graded semiprime. Hence (ANN) holds.

Finally, suppose that R satisfies (iii). We will show that (iv) holds. Clearly, each R_{ii} is semiprimary since it is semisimple Artinian.

Let I be a nilpotent homogeneous ideal of R. Then for each i, $I \cap R_{ii}$ is a nilpotent ideal of R_{ii} so that $I \cap R_{ii} = 0$. Fix $1 \leq i, j \leq n$. Then $(I \cap R_{ij})R_{ji} \subseteq I \cap R_{ii} = 0$ and therefore $I \cap R_{ij} = 0$ by (ANN). Because I is homogeneous, $I = 0$. Hence R is graded semiprime.

Suppose that $RxR = 0$ for some $x = \sum_{i,j=1}^{n} x_{ij} \in R$. Fix $1 \leq i, j \leq n$. Then $0 = R_{ij}xR_{ii} = R_{ij}x_{ji}R_{ii}$. But $R_{ij}x_{ji} \subseteq R_{ii}$ and R_{ii} is semisimple Artinian, so $R_{ij}x_{ji}R_{ii} = 0$

implies that $R_{ij}x_{ji} = 0$. Then $x_{ji} = 0$ by (ANN). Since i and j were arbitrary, it follows that $x = 0$. So R satisfies (iv) as claimed. □

CHAPTER 8

Jacobson Rings

Our goal in this chapter is to prove that a ring graded by a finite semigroup is a Jacobson ring provided that all the homogeneous components corresponding to idempotents are Jacobson rings. Once again, we follow the method of Chapter 4, combining results for group graded rings, ideal extensions, and rings graded by elementary Rees matrix semigroups.

We apply this result to find a sufficient condition for a semigroup algebra satisfying a polynomial identity to be a Jacobson ring.

These results will appear in [11].

8.1. Ideal Extensions

After briefly reviewing the definition of Jacobson ring, we begin as usual with ideal extensions.

8.1.1. A ring R is called a *Jacobson ring* if it satisfies one of the following equivalent conditions:

(i) The prime and Jacobson radicals of every homomorphic image of R coincide.

(ii) Every prime homomorphic image of R is Jacobson semisimple.

Historically, interest in Jacobson rings arose in commutative algebra. If $R = K[x_1, x_2, \ldots, x_n]$ is a polynomial ring over an algebraically closed field K, then the assertion that R is a Jacobson ring is equivalent to the strong form of Hilbert's Nullstellensatz (see [25]). For this reason, commutative Jacobson rings are often called *Hilbert rings*.

8.1.2. Note that nilpotent rings are Jacobson rings since such rings have no non-trivial prime homomorphic images.

8.1.3. The following lemma is due to Watters [69]; we will provide a short proof since we have obtained the essential ingredients in Chapter 4.

LEMMA ([69]). *Let I be an ideal of a ring R. Then R is a Jacobson ring if and only if both I and R/I are Jacobson rings.*

PROOF. Suppose first that R is a Jacobson ring. Since any homomorphic image of R/I is a homomorphic image of R, it follows from the definition that R/I is a Jacobson ring.

Let P be a prime ideal of I. We claim that P is also an ideal of R. For if $x \in R$, then $xPx \subseteq I$ so that $xPxP \subseteq P$. But xP is a right ideal of I, and so $xP \subseteq P$. Similarly, $Px \subseteq P$. So P is indeed an ideal of R.

Since R is a Jacobson ring, we have $J(R/P) = B(R/P)$. But both J and B are hereditary radicals (cf §4.1.6) so

$$J(I/P) = J(R/P) \cap I/P = B(R/P) \cap I/P = B(I/P) = 0.$$

Hence I is a Jacobson ring.

Conversely, suppose that I and R/I are Jacobson rings. Let A be an ideal of R, let $\bar{R} = R/A$ and $\bar{I} = (A + I)/A$. We must show that $J(\bar{R}) = B(\bar{R})$.

Since $\bar{I}$ is a homomorphic image of the Jacobson ring I, we have $J(\bar{I}) = B(\bar{I})$. Note that $\bar{R}/\bar{I} \cong R/(I + A)$ which is a homomorphic image of the Jacobson ring R/I, so that $J(\bar{R}/\bar{I}) = B(\bar{R}/\bar{I})$. Since Lemma 4.4.1 applies to J and B, we conclude that $J(\bar{R}) = B(\bar{R})$. □

8.1.4. As an immediate corollary, we deduce that the adjunction of an identity in the manner of §4.1.7 preserves the property of being a Jacobson ring.

COROLLARY. *Let R be a ring without an identity. Then R is a Jacobson ring if and only if R^1 is a Jacobson ring.*

PROOF. This follows from the lemma, noting that R is an ideal of R^1 and $R^1/R \cong \mathbb{Z}$ which is easily seen to be a Jacobson ring. □

8.2. Graded Rings

Once again, we combine results for group graded rings, rings graded by elementary Rees matrix semigroups, and ideal extensions to obtain a theorem about rings graded by arbitrary finite semigroups.

8.2.1. For rings graded by finite groups, the result we require is known, at least for rings with identities. There is a complete proof in [59]; see also [32]. By Corollary 8.1.4, we may extend this to rings without identity in the manner outlined in §4.2.1.

THEOREM ([59]). *Let G be a finite group with identity 1 and let R be a G-graded ring. Then R is a Jacobson ring if and only if R_1 is a Jacobson ring.*

8.2.2. The next step is to prove the result we are aiming for for elementary Rees matrix semigroups.

PROPOSITION ([11]). *Let $S = \mathfrak{M}^0(1; I, \Lambda; P)$ be an elementary Rees matrix semigroup and let R be a contracted S-graded ring. If R_e is a Jacobson ring for each non-zero idempotent $e \in S$, then R is a Jacobson ring.*

PROOF. We adopt the notation of §2.2.1. Write $P = (p_{\lambda i})$. If $p_{\lambda i} = 0$ for some i and λ, then $R_{i\lambda}{}^2 = 0$ and therefore $R_{i\lambda}$ is a Jacobson ring by §8.1.2. So we may assume that $R_{i\lambda}$ is a Jacobson ring for all i and λ.

Let Q be a prime ideal of R and let Q_h be the homogeneous part of Q (cf §7.1.1). By Lemma 2.2.3, $Q^3 \subseteq Q_h$ so that $Q/Q_h \subseteq B(R/Q_h) \subseteq J(R/Q_h)$. Hence,

$$J(R/Q) \cong J\left(\frac{R/Q_h}{Q/Q_h}\right) = \frac{J(R/Q_h)}{Q/Q_h}$$

and a similar statement holds for $B(R/Q)$. So to prove that $J(R/Q) = B(R/Q) = 0$, it suffices to show that $J(R/Q_h) = B(R/Q_h)$.

Let $\bar{R} = R/Q_h$; then $\bar{R}$ is also a contracted S-graded ring because Q_h is a homogeneous ideal. Note that $\bar{R}_{i\lambda} = R_{i\lambda}/(Q \cap R_{i\lambda})$. Since $R_{i\lambda}$ is a Jacobson ring, we see that

$J(\bar{R}_{i\lambda}) = B(\bar{R}_{i\lambda})$. This is true for all i and λ, so by Corollary 4.3.2, $J(R) = B(R)$ as desired. □

8.2.3. Combining Theorem 8.2.1 and Proposition 8.2.2 using the techniques of §2.2.2, the following is immediate.

COROLLARY ([11]). *Let S be a Rees matrix semigroup with no infinite subgroups. Let R be a contracted S-graded ring. If R_e is a Jacobson ring for every non-zero idempotent $e \in S$, then R is a Jacobson ring.*

8.2.4. It is now a simple matter to obtain:

THEOREM ([11]). *Let S be a finite semigroup and let R be an S-graded ring. If R_e is a Jacobson ring for each idempotent $e \in S$, then R is a Jacobson ring.*

PROOF. The proof proceeds in the same way as the proofs of Theorem 4.5.1 and Theorem 6.7.1, so we will omit many of the details.

By adjoining a zero if necessary or appealing to Lemma 8.1.3, we may assume that S has a zero and R is a contracted S-graded ring.

If S is null, then R is a nilpotent ring and is therefore a Jacobson ring. If S is 0-simple, it is completely 0-simple by Theorem 1.4.8, and therefore R is a Jacobson ring by Corollary 8.2.3.

Otherwise, S has a non-zero proper ideal T. By induction on the size of the grading semigroup, R_T and R/R_T are Jacobson rings, and therefore R is a Jacobson ring by

Lemma 8.1.3. []

8.2.5. We do not know if the converse of Theorem 8.2.4 is true. The obstacle to proving the converse is the case of a ring graded by an elementary Rees matrix semigroup; we do not know if the converse of Proposition 8.2.2 is true.

Note that it would suffice, to prove the converse, to show that a right (or left) ideal of a Jacobson ring is a Jacobson ring. In fact, the truth of the converse depends entirely on this statement as the next result shows.

PROPOSITION. *The following are equivalent:*

(i) *The converse of Proposition* 8.2.2 *is true.*

(ii) *If A is a right (or left) ideal of a Jacobson ring then A is a Jacobson ring.*

PROOF. It is easy to see that (ii) implies (i). For if $S = \mathfrak{M}^0(1; I, \Lambda; P)$ is a Rees matrix semigroup and R is a contracted S-graded ring, then for each i and λ, $R_{i\lambda}$ is a left ideal of $R_{i\cdot}$ and $R_{i\cdot}$ is a right ideal of R.

For the other implication, we will show how to construct a counter example to (i) given a counter example to (ii). Suppose then that R is a Jacobson ring and A is a right ideal of R which is not a Jacobson ring.

Let $S = \{e, f\}$ be a two element semigroup with $e^2 = ef = e$ and $f^2 = fe = f$. Then S^0 is isomorphic to an elementary Rees matrix semigroup (cf §3.4.2).

Consider the subring $B = Ae + Rf$ of the semigroup ring $R[S]$. Then B is a contracted S^0-graded ring, putting $B_e = Ae$ and $B_f = Rf$. Working in $R^1[S]$ for the moment, note that $e(e-f) = f(e-f) = 0$ and $(e-f)e = e - f = (e-f)f$. Hence, $A(e-f)$ is a nilpotent ideal of B.

Further, the map $\psi\colon R[S] \to R$ given by $xe + yf \mapsto x + y$ is a ring homomorphism, for if $x, y, z, w \in R$, then

$$\begin{aligned}\psi\big((xe+yf)(ze+wf)\big) &= \psi\big((xz+xw)e + (yz+yw)f\big)\\ &= xz + xw + yz + yw\\ &= (x+y)(z+w).\end{aligned}$$

If we restrict ψ to B, then the kernel is clearly the ideal $A(e-f)$ and the image is R.

We have shown that $B/A(e-f) \cong R$, and therefore conclude that B is a Jacobson ring by Lemma 8.1.3.

But $B_e = Ae \cong A$ is not a Jacobson ring. □

8.3. PI Semigroup Algebras

In this section we use results for graded rings to derive a theorem showing that certain semigroup algebras satisfying polynomial identities are Jacobson rings. This theorem extends a result of Gilmer for commutative semigroup algebras.

This theorem requires rather more technical results from the theory of semigroup algebras than elsewhere in this thesis. We will endeavour to explain the results used as we go. We refer the reader to [55] in which all the required results can be found.

8.3.1. Let K be a field and let R be a K-algebra. Write $K\{x_1, x_2, \ldots, x_n\}$ for the algebra of polynomials in non-commuting variables $x_1, x_2, \ldots, x_n$ with coefficients in K. R is a *PI-algebra* if it satisfies a polynomial identity, that is, there is a non-zero polynomial $f(x_1, x_2, \ldots, x_n) \in K\{x_1, x_2, \ldots, x_n\}$ such that for all n-tuples $(r_1, r_2, \ldots, r_n)$ of elements of R, we have $f(r_1, r_2, \ldots, r_n) = 0$.

The theory of PI-algebras is well developed (see, for example, [60]). Many results have been generalised from commutative rings to PI-algebras. This is what we will do in this section.

Clearly subalgebras and quotients of PI-algebras are also PI-algebras. One other fact that we shall need is that the class of PI-algebras is closed under ideal extensions. Indeed, if R/I satisfies a polynomial identity $f(x_1, x_2, \ldots, x_n)$ and I satisfies a polynomial identity $g(x_1, x_2, \ldots, x_m)$ then R satisfies the identity

$$g\big(f(x_1, x_2, \ldots, x_n), f(x_{n+1}, x_{n+2}, \ldots, x_{2n}), \ldots, f(x_{mn-n+1}, x_{mn-n+2}, \ldots, x_{mn})\big).$$

8.3.2. The following theorem about prime PI-algebras is crucial; it is a simple consequence of Posner's Theorem [58, Theorem 5.4.10].

THEOREM ([55, Theorem 18.1]). *Let R be a prime PI-algebra over a field K. Then there is a field extension L of K and an integer n such that R embeds in the matrix ring $M_n(L)$.*

8.3.3. Following Okniński [55], we define the *rank* of a semigroup S, written $\mathrm{rk}(S)$ to be

$$\sup\{n \mid S \text{ contains a free commutative semigroup on } n \text{ generators}\}.$$

Here, a free commutative semigroup on generators $X = \{x_1, x_2, \ldots, x_n\}$ is the free semigroup on X with relations $W = \{x_i x_j = x_j x_i \mid i \neq j\}$ (cf §1.3.3); alternatively, it's the set of monomials $x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}$ with at least one k_i non-zero.

If S has free commutative subsemigroups of all finite ranks, then, of course, $\mathrm{rk}(S) = \infty$.

Note that if G is an Abelian group, then this definition of the rank of G agrees with the usual torsion free rank of an Abelian group.

Rank has the following elementary properties:

LEMMA. *Let S be a semigroup with subsemigroup T and ideal I.*

(i) $\mathrm{rk}(T) \leq \mathrm{rk}(S)$.

(ii) $\mathrm{rk}(S/I) \leq \mathrm{rk}(S)$.

PROOF. Property (i) is immediate from the definition of rank. For (ii), note that a free commutative subsemigroup of S/I does not contain the zero element and can therefore be regarded as a subsemigroup of S. □

However, the rank of a homomorphic image S/ρ where ρ is a congruence may be larger than the rank of S. For example, let S be a free (non-commutative) semigroup on a set of generators $\{x_i \mid i \in I\}$. Then it is true that $\mathrm{rk}(S) = 1$. But if ρ is the congruence generated by the relations $x_i x_j = x_j x_i$ for all $i, j \in I$, then S/ρ is a free commutative semigroup with generators $\{x_i \mid i \in I\}$ and $\mathrm{rk}(S)$ may even be infinite if I is infinite.

8.3.4. Let S be the multiplicative semigroup of the ring $M_n(K)$ of $n \times n$ matrices over a field K. Regarding an element $x \in S$ as a matrix, we may speak of the "rank of x", meaning the dimension of the image of the linear transformation corresponding to x. (This has nothing to do with the rank of a semigroup introduced in §8.3.3.) Such semigroups have a nice principal series.

THEOREM ([55, Theorem 1.6]). *Let K be a field and let S be the multiplicative semigroup of the ring $M_n(K)$. Then S has a principal series*

$$S = I_n \supset I_{n-1} \supset \cdots \supset I_1 \supset I_0 = \{0\}$$

where I_k is the set of matrices of rank k or less. Each quotient I_k/I_{k-1} is completely 0-simple.

8.3.5. The problem of which semigroups have PI semigroup algebras has been extensively investigated, but not completely solved. However, this question has been answered by Passman and Isaacs [30, 57] for group algebras, and by Zelmanov [74] for cancellative semigroups.

We state these results and a related result below.

The commutator subgroup of a group A is denoted A'.

PROPOSITION. *Let K be a field and let S be a cancellative semigroup such that $K[S]$ is a PI-algebra. Then*

(i) *S has a group of fractions G and $K[G]$ is a PI-algebra.*

(ii) *G has a normal subgroup A of finite index such that A' is a finite p-group if $\text{char}(K) = p > 0$ and $A' = 1$ if $\text{char}(K) = 0$.*

(iii) $\text{rk}(S) = \text{rk}(G) = \text{rk}(A/A')$.

PROOF. Statement (i) is [55, Theorem 20.1]. The condition in (ii) is actually necessary and sufficient for $K[G]$ to be a PI-algebra, see [55, Theorem 18.3]. Finally, (iii) follows from [55, Propositions 19.1 and 23.2]. □

8.3.6. The result that we want to improve is due to Gilmer [24].

THEOREM ([24]). *Let K be a field and let S be a commutative monoid of rank α.*

(i) *If α is finite, then $K[S]$ is a Jacobson ring.*

(ii) *If α is infinite, then $K[S]$ is a Jacobson ring if and only if $\alpha < |K|$.*

Note that Gilmer allows the rank to be any cardinal number, but we are only interested in whether or not the rank is finite.

8.3.7. We begin by extending Theorem 8.3.6(i) to PI semigroup algebras of cancellative semigroups.

THEOREM ([11]). *Let K be a field and let S be a cancellative semigroup of finite rank such that $K[S]$ is a PI-algebra. Then $K[S]$ is a Jacobson ring.*

PROOF. Let $p = \text{char}(K)$. By Proposition 8.3.5, we know that S has a group of fractions G and that G has a normal subgroup A of finite index such that A' is a finite p-group if $p > 0$ or is trivial if $p = 0$.

Write $\omega(K[H])$ for the augmentation ideal of a group algebra $K[H]$ (cf [58, §1.1]). Let $I = \omega(K[A'])K[G] = K[G]\omega(K[A'])$, the kernel of the canonical map $K[G] \mapsto K[G/A']$. If $p = 0$, then $I = 0$. Otherwise, A' is a finite p-group and so $\omega(K[A'])$ is nilpotent by [58, Lemma 3.1.6] and hence I is nilpotent. Then $I \cap K[S]$ is a nilpotent ideal of $K[S]$.

Let $\rho_I = \rho_{(I \cap K[S])}$ be the congruence of §3.1.2. By Lemma 3.1.2(ii), $K[S]/(I \cap K[S])$ is a homomorphic image of $K[S/\rho_I]$. Hence to prove that $K[S]$ is a Jacobson ring, it suffices (by Lemma 8.1.3 and the fact that nilpotent rings are Jacobson rings) to prove that $K[S/\rho_I]$ is a Jacobson ring.

Write $\bar{S} = S/\rho_I$, $\bar{G} = G/A'$, and $\bar{A} = A/A'$. Because I is the kernel of the map $K[G] \to K[G/A']$, we see that for $s, t \in S$, $sA' = tA'$ implies $s - t \in I$ and therefore $s \rho_I t$. Hence $\bar{S}$ embeds in $\bar{G}$. Furthermore, $\bar{A}$ is an Abelian normal subgroup of $\bar{G}$ of finite index, and by Proposition 8.3.5(iii), $\text{rk}(\bar{A})$ is finite.

Now $\bar{S} \cap \bar{A}$ is a commutative cancellative semigroup of finite rank. By Theorem 8.3.6(i), $K[\bar{S} \cap \bar{A}]$ is a Jacobson ring. (If $T = \bar{S} \cap \bar{A}$ does not have an identity, adjoin one. Then $K[T^1]$ still satisfies the hypothesis of Theorem 8.3.6, and $K[I]$ is an ideal of $K[T^1]$ so is a Jacobson ring.)

The canonical map $\bar{G} \to \bar{G}/\bar{A}$ gives $K[\bar{G}]$ a $\bar{G}/\bar{A}$-gradation (cf §2.1.5). Since $K[S]$ is a homogeneous subring with respect to this gradation, we may regard $K[S]$ as a $\bar{G}/\bar{A}$-graded ring. The identity component for this gradation is $K[S \cap A]$. By Theorem 8.2.1, we conclude that $K[\bar{S}]$ is a Jacobson ring. □

8.3.8. Now we extend this result further to general semigroups.

Let S be a semigroup of finite rank and let S' be a homomorphic image of S. As pointed out in §8.3.3, S' might have larger rank than S, and if $\mathrm{rk}(S') = \infty$, then Theorem 8.3.6(ii) shows that $K[S']$ might not be a Jacobson ring. So in order that $K[S]$ be a Jacobson ring, we will require that S and all homomorphic images of S have finite rank.

THEOREM ([11]). *Let K be a field and let S be a semigroup such that $K[S]$ is a PI-algebra. Suppose that every homomorphic image of S has finite rank. Then $K[S]$ is a Jacobson ring.*

PROOF. Let P be a prime ideal of $K[S]$ and let ρ_P be the corresponding congruence on S defined in §3.1.2. By Lemma 3.1.2(ii), $K[S]/P$ is a homomorphic image of $K[S/\rho_P]$, so it suffices to prove that $K[S/\rho_P]$ is a Jacobson ring. By assumption, S/ρ_P has finite rank, and $K[S/\rho_P]$ is a PI-algebra since it is a homomorphic image of $K[S]$. Further, S/ρ_P embeds in $K[S]/P$ by Lemma 3.1.2(ii) and $K[S]/P$ embeds in a matrix ring $M_n(L)$ for some field L by Theorem 8.3.2.

So replacing S by S/ρ_P, it suffices to show that $K[S]$ is a Jacobson ring when $K[S]$ is a PI-algebra, S is a subsemigroup of the multiplicative semigroup $M_n(L)$, and S has finite rank. If S does not contain the zero matrix θ, then adjoin it. Then S^0 still has finite rank and $K[S^0]/K\theta \cong K[S]$ so that $K[S^0]$ is an extension of $K\theta \cong K$ by $K[S]$ and is still a PI-algebra. So we may assume that $\theta \in S$.

Define ideals I_k of $M_n(L)$ for $0 \leq k \leq n$ as in Theorem 8.3.4. Then S has a chain of ideals

$$S = S_n \supseteq S_{n-1} \supseteq \cdots \supseteq S_1 \supseteq S_0 = \{\theta\}$$

where $S_k = S \cap I_k$ and each quotient S_k/S_{k-1} embeds in I_k/I_{k-1}. Now, $K[S]$ has a corresponding chain of ideals

$$K[S] = K[S_n] \supseteq K[S_{n-1}] \supseteq \cdots \supseteq K[S_1] \supseteq K[S_0] \cong K.$$

We need only show that each quotient $K[S_k]/K[S_{k-1}] \cong K_0[S_k/S_{k-1}]$ is a Jacobson ring by Lemma 8.1.3. Note that each ring $K_0[S_k/S_{k-1}]$ is still a PI-algebra and that $\mathrm{rk}(S_k/S_{k-1})$ is finite by Lemma 8.3.3.

So we may assume that S is a subsemigroup of a completely 0-simple semigroup T, that the rank of S is finite, and that $K_0[S]$ is a PI-algebra. Also we may assume by Theorem 1.4.6 that T is a Rees matrix semigroup. Let T' be the elementary Rees matrix semigroup which is a homomorphic image of T in the manner described in §1.5.5. Because $S \subseteq T$, $K_0[S]$ is a contracted T-graded ring, and because T' is a homomorphic image of T, we may regard $K_0[S]$ as a contracted T'-graded ring. With respect to the T'-gradation, the homogeneous components corresponding to non-zero idempotents are the rings $K[S \cap G]$ where G is a maximal subgroup of T'. By Proposition 8.2.2, it suffices to prove that these rings $K[S \cap G]$ are Jacobson rings. But $S \cap G$ is a cancellative semigroup of finite rank and $K[S \cap G] \subseteq K_0[S]$ is a PI-algebra so the rings $K[S \cap G]$ are indeed Jacobson rings by Theorem 8.3.7. □

CHAPTER 9

Artinian Semigroup Graded Rings

Having considered several finiteness conditions in Chapter 6, we turn our attention to the most basic finiteness condition, the Artinian condition. However, we cannot expect to obtain a result similar to those of Chapter 6 for Artinian rings; for instance, the example of §7.3.4 is not Artinian, yet each homogeneous component is a field.

Instead, we consider the following question: *If R is a right Artinian ring graded by a semigroup S, is* supp(R) *finite?*

This question has been answered positively for semigroup algebras by Zelmanov [74], and for band graded rings by Kelarev [44].

Unfortunately, the general answer is no. There is an example due to Passman [56] of a twisted group ring over an infinite group which is actually a field. This raises the interesting question of what additional assumptions are required for the Artinian condition on a group graded ring to imply that the group is finite; in the cited paper, Passman gives some sufficient conditions for twisted group rings, and more recently, Saorín [65] has addressed this question for strongly graded rings. However, we shall not pursue this further here.

Instead we shall show that the existence of infinite subgroups is the major obstacle to an affirmative answer to our question. We offer two results.

THEOREM. *Let S be a semigroup that has no infinite subgroups and in which every*

0-simple principal factor is completely 0-simple. Let R be an S-graded ring. If R is right Artinian, then the support of R is finite.

Our proof of this theorem relies on the structure theory of semigroups, and is based on Kelarev's proof for the band case [44]. Unfortunately, these methods do not seem sufficient to deal with 0-simple factors which are not completely 0-simple, since the structure of such semigroups is much less well understood.

However, if we require that all but finitely many components R_s are non-zero, then we can show that all 0-simple factors must be completely 0-simple. Hence we obtain our second result.

THEOREM. *Let S be a semigroup with no infinite subgroups. Let R be an S graded ring such that $R_s \neq 0$ for all but finitely many elements $s \in S$. If R is right Artinian, then S is finite.*

Using this second theorem, we obtain an independent proof of Zelmanov's result for semigroup algebras [74].

This work was completed jointly with Eric Jespers and Andrei Kelarev [15].

9.1. Preliminaries

We begin with a couple of preliminary lemmas.

9.1.1. The first lemma was originally stated for band graded rings in [44]. The proof is essentially unchanged. The lemma states the intuitively obvious result that an Artinian ring cannot have an infinite family of small right ideals whose sum is very large. Recall that the support of a not necessarily homogeneous ideal of a graded ring is the union of the supports of its elements (cf §2.1.2).

LEMMA ([44]). *Let S be a semigroup and let R be an S-graded ring. Suppose $\mathcal{F}$ is a family of right ideals of R satisfying the following:*

(i) *There is a natural number k such that $|\mathrm{supp}(I)| \leq k$ for all $I \in \mathcal{F}$.*

(ii) $\bigcup_{I \in \mathcal{F}} \mathrm{supp}(I)$ *is infinite.*

Then R is not right Artinian.

PROOF. It suffices to find a sequence $I_1, I_2, I_3, \dots$ of right ideals in $\mathcal{F}$ such that $\mathrm{supp}(I_n) \not\subseteq \bigcup_{m>n} \mathrm{supp}(I_m)$ for all n; then

$$\sum_{i=1}^{\infty} I_n \supsetneq \sum_{i=2}^{\infty} I_n \supsetneq \sum_{i=3}^{\infty} I_n \supsetneq \cdots$$

is an infinite strictly descending chain of right ideals.

So the argument is purely set-theoretic. We give an inductive construction of such a sequence, at the heart of which is the following: given a family $\mathcal{F}_n$ satisfying (i) and (ii), we find an $I_{n+1} \in \mathcal{F}_n$ and a subfamily $\mathcal{F}_{n+1} \subseteq \mathcal{F}_n$ satisfying (i) and (ii) and such that $\mathrm{supp}(I_{n+1}) \not\subseteq \bigcup_{I \in \mathcal{F}_{n+1}} \mathrm{supp}(I)$.

Since $\bigcup_{I \in \mathcal{F}_n} \mathrm{supp}(I)$ is infinite, there is a finite set $\{H_1, H_2, \dots, H_l\} \subseteq \mathcal{F}_n$ such that $|\bigcup_{i=1}^{l} \mathrm{supp}(H_i)| > k$. Let $X = \bigcup_{i=1}^{l} \mathrm{supp}(H_i)$; then X is finite. For $x \in X$, let $\mathcal{E}_x$ be the subfamily of $\mathcal{F}_n$ consisting of all I such that $x \notin \mathrm{supp}(I)$. Note that for any

$I \in \mathcal{F}_n$, we have $X \not\subseteq \mathrm{supp}(I)$ because $|\mathrm{supp}(I)| \geq k$, so $I \in \mathcal{E}_i$ for some x. Hence, $\mathcal{F}_n = \bigcup_{x \in X} \mathcal{E}_x$, and there must be an x such that $\bigcup_{I \in \mathcal{E}_x} \mathrm{supp}(I)$ is infinite. Choose i such that $x \in \mathrm{supp}(H_i)$. Then $I_{n+1} = H_i$ and $\mathcal{F}_{n+1} = \mathcal{E}_x$ have the required properties.

To begin the induction, put $\mathcal{F}_0 = \mathcal{F}$. It is easy to see that the sequence $I_1, I_2, I_3, \ldots$ so constructed has the desired properties. □

9.1.2. The second lemma is also from [44]. In this case, we give a different proof because the proof in [44] made use of the fact that bands are locally finite which is, of course, not true of a general semigroup.

LEMMA ([44]). *Let S be a semigroup and let R be a right Artinian S-graded ring. Let I be a nilpotent homogeneous ideal of R such that there are only finitely many $s \in S$ with $R_s \not\subseteq I$. Then* $\mathrm{supp}(I)$ *and hence* $\mathrm{supp}(R)$ *are finite.*

PROOF. Let n be the index of nilpotency of I. We proceed by induction on n.

Suppose $n = 2$. For $x \in \mathrm{supp}(I)$, let A_x be the right ideal of R generated by $I \cap R_x$, so that $A_x = (I \cap R_x) + (I \cap R_x)R$. Let m be the number of $s \in S$ with $R_s \not\subseteq I$. Because $I^2 = 0$, it follows that $|\mathrm{supp}(A_x)| \leq m + 1$. Now each A_x is a right ideal of R and $x \in \mathrm{supp}(A_x)$. If $\mathrm{supp}(I)$ were infinite, then applying Lemma 9.1.1 to the family $\{A_x \mid x \in \mathrm{supp}(I)\}$ yields that R is not right Artinian, a contradiction. So $\mathrm{supp}(I)$ and hence $\mathrm{supp}(R)$ are finite.

Suppose now that the nilpotency index of I is $n > 2$ and the result holds for smaller nilpotency index. Consider the ring $\bar{R} = R/I^2$ with ideal $\bar{I} = I/I^2$. $\bar{R}$ and $\bar{I}$ satisfy the hypotheses of the lemma and $\bar{I}^2 = 0$. Hence, by the $n = 2$ case, supp$(\bar{R})$ is finite. But this implies that there are only finitely many $s \in S$ such that $R_s \not\subseteq I^2$. Since I^2 has nilpotency index less than n, we conclude by the inductioı hypothesis that supp(R) is finite. □

9.2. 0-minimal Ideals

In this section, we prove a rather technical result which is the key step in the reduction of the general case. This result concerns 0-minimal completely 0-simple ideals of a semigroup.

9.2.1. Let S be a semigroup and let $M = \mathfrak{M}^0(G; I, \Lambda; P)$ be a 0-minimal ideal of S which is completely 0-simple. As in §1.5.3, we put

$$M_{i\lambda} = \{(g)_{i\lambda} \mid g \in G^0\},$$

$$M_{i*} = \bigcup_{\lambda \in \Lambda} M_{i\lambda},$$

and $$M_{*\lambda} = \bigcup_{i \in I} M_{i\lambda}.$$

We begin with a lemma which shows that we can reduce the subgroups of M to the trivial group.

LEMMA. *Let S and M be as above. Define a relation ρ on S by $s \rho t$ if and only if $s = t$, or $s, t \in M_{i\lambda} \setminus \{\theta\}$ for some $i \in I$ and $\lambda \in \Lambda$.*

(i) *Each M_{i*} is a right ideal of S and each $M_{*\lambda}$ is a left ideal of S.*

(ii) *The relation ρ is a congruence on S. Furthermore, the image $\bar{M}$ of M in S/ρ is still a completely 0-simple 0-minimal ideal of S/ρ, and for $i \in I$ and $\lambda \in \Lambda$, the image $\bar{M}_{i\lambda}$ of $M_{i\lambda}$ has exactly two elements: 0 and a non-zero element* $(1)_{i\lambda}$.

PROOF. Let $t = (g)_{i\lambda} \in M_{i\lambda}$ for some i and λ and let s in S. Because M is completely 0-simple, the matrix P has a non-zero entry in each row and column, by Theorem 1.5.2. In particular, there is a $j \in I$ such that the (λ, j)-entry of P is $h \neq 0$. Let $e = (h^{-1})_{j\lambda}$. Then $te = t$. Now, $es \in M$ because M is an ideal of S. Hence $ts = tes \in M_{i*}$ since $t \in M_{i*}$ which is a right ideal of M by Lemma 1.5.3(iv). This proves that M_{i*} is a right ideal of S. Similarly, $M_{*\lambda}$ is a left ideal, demonstrating (i).

It is clear that ρ is an equivalence relation. Suppose that $s, t \in M_{i\lambda} \setminus \{0\}$ for some i and λ. Let e be the element constructed above; since e depends only on the choice of λ, we have $se = s$ and $te = t$. Let $x \in S$. Since M is an ideal, $ex \in M$. We may suppose that $ex \in M_{j\mu}$ for some j and μ. By Lemma 1.5.3(vi), either $sex = tex = 0$ or $sex, tex \in M_{i\mu} \setminus \{0\}$. Since $sx = sex$ and $tx = tex$, we conclude that $sx \,\rho\, tx$. Similarly, $xs \,\rho\, xt$ and ρ is a congruence.

Since the restriction of ρ to $M \times M$ is exactly the kernel of the homomorphism of M onto the associated elementary Rees semigroup $\mathfrak{M}^0(1; I, \Lambda; P'')$ as explained in §1.5.5, the other assertions of (ii) follow. $\square$

9.2.2. Since we will be much concerned with the complement of the support of an ideal of a graded ring, it is useful to introduce some notation. Specifically, if I is an ideal of an S-graded ring R, we will write $\overline{\mathrm{supp}}(I) = \mathrm{supp}(R) \setminus \mathrm{supp}(I)$. Note that if T is an ideal of S, then $\overline{\mathrm{supp}}(R_T) = \mathrm{supp}(R) \cap (S \setminus T)$. Of course, this notation applies equally well if I is a one-sided ideal of R.

The assertion we require for 0-minimal ideals of the type introduced in §9.2.1 is the following:

PROPOSITION. *Let S be a semigroup and let M be a 0-minimal ideal of S which is completely 0-simple and has no infinite subgroups. Let R be a right Artinian S-graded ring such that $\overline{\mathrm{supp}}(R_M)$ is finite. Then $\mathrm{supp}(R)$ is finite.*

The rest of this section consists of the proof of this proposition. The proof is by contradiction, so we start by assuming that $\mathrm{supp}(R)$ is infinite. As we proceed, we will introduce further assumptions in each subsection until a final contradiction is reached. It is implicit that at each stage, all the extra assumptions introduced thus far still hold. We will use the notation introduced in §9.2.1.

9.2.3. *We may assume that R is a contracted S-graded ring.*

To do so, we replace R by R/R_θ, which is right Artinian being a homomorphic image of R. Of course, $\mathrm{supp}(R/R_\theta)$ is infinite.

9.2.4. *We may assume that all subgroups of M are trivial.*

Applying Lemma 9.2.1(ii), we may regrade R by the homomorphic image S/ρ of S. Since M has no infinite subgroups, each ρ-class is finite, and the support of R with the S/ρ-gradation is still infinite. Replace S by S/ρ.

For $i \in I$ and $\lambda \in \Lambda$, put

$$R_{i\lambda} = R_{M_{i\lambda}} = R_{(1)_{i\lambda}}$$

$$R_{i\bullet} = \sum_{\lambda\in\Lambda} R_{i\lambda} = R_{M_{i\bullet}}$$

and $$R_{\bullet\lambda} = \sum_{i\in I} R_{i\lambda} = R_{M_{\bullet\lambda}}.$$

9.2.5. *There is an integer l such that $|\text{supp}(R_{\bullet\lambda})| \leq l$ for all $\lambda \in \Lambda$.*

Since M is the 0-disjoint union of the $M_{i\bullet}$ and $R_\theta = 0$ by 9.2.3, we see that R contains the direct sum $R_M = \bigoplus_{i\in I} R_{i\bullet}$. But each $R_{i\bullet}$ is a right ideal of R since $M_{i\bullet}$ is a right ideal of S by Lemma 9.2.1(i). Because R is right Artinian, $R_{i\bullet} = 0$ for all but finitely many, say l, of the elements $i \in I$. So for $\lambda \in \Lambda$, we see that $\text{supp}(R_{\bullet\lambda})$ includes at most those elements $(1)_{i\lambda}$ for which $R_{i\bullet} \neq 0$.

9.2.6. $R_{\bullet\lambda} \neq 0$ *for infinitely many* $\lambda \in \Lambda$.

Since $S = (S \setminus M) \cup \left(\bigcup_{\lambda\in\Lambda} M_{\bullet\lambda}\right)$ and each of these sets intersects $\text{supp}(R)$ finitely, there must be an infinite number of $\lambda \in \Lambda$ such that $M_{\bullet\lambda}$ intersects $\text{supp}(R)$.

9.2.7. $J(R) \cap R_{i\lambda} = 0$ *for all* $i \in I$ *and* $\lambda \in \Lambda$.

Let $K = \sum_{i,\lambda} J(R) \cap R_{i\lambda}$. Then K is the homogeneous part of the ideal $J(R) \cap R_M$ and is therefore a homogeneous ideal. Since $K \subseteq J(R)$ and R is right Artinian, K is nilpotent. By Lemma 9.1.2, there must be infinitely many $s \in S$ such that $R_s \not\subseteq K$. So R/K is a right Artinian S-graded ring with infinite support. Replacing R by R/K, we have $J(R) \cap R_{i\lambda} = 0$ as claimed. Note that 9.2.5 still holds and therefore 9.2.6 is also still true.

9.2.8. $R_{MI}J(R_{MI})R_{MI} = 0$.

This follows at once from Proposition 4.3.1(ii) and Corollary 4.3.2.

9.2.9. *There is an infinite subset* $\{\lambda_1, \lambda_2, \lambda_3, \dots\}$ *of* Λ *such that* $R_{*\lambda_j} \not\subseteq J(R)$.

Suppose that $R_{*\lambda} \subseteq J(R)$. Then for each $i \in I$, $R_{i\lambda} \subseteq J(R) \cap R_{i\lambda} = 0$ by 9.2.7. Hence, $R_{*\lambda} = \sum_{i \in I} R_{i\lambda} = 0$. By 9.2.6, it follows that there is an inifinite number of $\lambda \in \Lambda$ such that $R_{*\lambda} \not\subseteq J(R)$.

9.2.10. *There exists a* p *such that for all* $q \geq p$.

$$R_{*\lambda_1} + \cdots + R_{*\lambda_p} + J(R) = R_{*\lambda_1} + \cdots + R_{*\lambda_q} + J(R).$$

By Lemma 9.2.1(i), each $R_{*\lambda}$ is a left ideal of R. But $R/J(R)$ is semisimple Artinian and hence left Noetherian, so the chain of left ideals

$$R_{*\lambda_1} + J(R) \subseteq R_{*\lambda_1} + R_{*\lambda_2} + J(R) \subseteq R_{*\lambda_1} + R_{*\lambda_2} + R_{*\lambda_3} + J(R) \subseteq \cdots$$

must stabilise.

9.2.11. With these assumptions in hand, we can obtain a contradiction. Let $L = R_{*\lambda_1} + R_{*\lambda_2} + \cdots + R_{*\lambda_p}$, a left ideal of R and note that $|\text{supp}(L)| \leq pl$.

Let $q > p$. Since $R_{\bullet\lambda_q} \not\subseteq J(R)$ and $R_{\bullet\lambda_q}$ is a left ideal of R it is not nil. In particular, there exists $r_q \in R_{\bullet\lambda_q}$ such that $r_q{}^2 \neq 0$. By 9.2.10, we may write $r_q = x_q + y_q$ for some $x_q \in L$ and $y_q \in J(R) \cap (L + R_{\bullet\lambda_q})$. Note that $|\mathrm{supp}(r_q)| \le l$ and $|\mathrm{supp}(y_q)| \le (p+1)l$.

Consider $0 \neq r_q{}^2 = r_q x_q + r_q y_q$. We have $r_q{}^2 \in R_{\bullet\lambda_q}$ and $r_q x_q \in L$. Since $R_{\bullet\lambda_q} \cap L = 0$, we must have $r_q y_q \neq 0$ and therefore $M_{\bullet\lambda_q} \cap \mathrm{supp}(r_q y_q) \neq \emptyset$.

Let I_q be the right ideal of R generated by $r_q y_q$. Then

$$I_q = \mathbf{Z} r_q y_q + r_q y_q R_M + r_q y_q R_{S \setminus M}.$$

By 9.2.8, $r_q y_q R_M \subseteq R_M J(R_M) R_M = 0$. So

$$|\mathrm{supp}(I_q)| \le |\mathrm{supp}(r_q y_q)| + |\mathrm{supp}(r_q y_q R_{S\setminus M})|$$

$$\le l^2(p+1) + l^2(p+1)n.$$

Since $M_{\bullet\lambda_q} \cap \mathrm{supp}(r_q y_q) \neq \emptyset$, the union $\bigcup_{q>p} \mathrm{supp}(I_q)$ is infinite. But this contradicts Lemma 9.1.1.

Hence our initial assumption that $\mathrm{supp}(R)$ is infinite must be incorrect, completing the proof of Proposition 9.2.2.

9.3. Main Theorem

In this section we will prove the first main theorem of this chapter.

9.3.1. Let S be a semigroup. Following the notation of §1.4.3 we will write $J_a = S^1 a S^1$ for the principal ideal generated by a, and $I_a = \{x \in J_a \mid J_x \subsetneq J_a\}$ for the ideal of non-generators of J_a. Recall that the relation $\mathcal{J}$ is defined on S by $a \mathcal{J} b$ if and only if $J_a = J_b$, and the equivalence class of a under this relation is denoted $\mathcal{J}(a)$. Recall that $\mathcal{J}(a) = J_a \setminus I_a$.

Note that we can partially order the $\mathcal{J}$-classes of S by inclusion of the corresponding principal ideals. Specifically, for elements $a, b \in S$, we will write $\mathcal{J}(a) \leq \mathcal{J}(b)$ if $J_a \subseteq J_b$.

9.3.2. Our aim is to prove that, with certain restrictions on the grading semigroup S, a right Artinian S-graded ring R must have finite support. With the technical results of Sections 9.1 and 9.2 at our disposal, we will only need to consider ideals of the form R_A for ideals A of S. The next lemma allows us in certain circumstances to make A smaller while retaining the property that $\overline{\text{supp}}(R_A)$ is finite.

LEMMA. *Let S be a semigroup with no infinite subgroups and such that every 0-simple principal factor is completely 0-simple. Let R be a right Artinian S-graded ring. Let T be an ideal of S and let a be an element of $\overline{\text{supp}}(R_T)$ with $\mathcal{J}(a)$ minimal among the $\mathcal{J}$-classes meeting $\overline{\text{supp}}(R_T)$. If $\overline{\text{supp}}(R_{T \cup J_a})$ is finite, then $\overline{\text{supp}}(R_T)$ is finite.*

PROOF. By choice of a, we see that I_a cannot contain elements of $\text{supp}(R) \setminus T$. Hence $\text{supp}(R) \cap I_a \subseteq \text{supp}(R) \cap T$. Let $T' = T \cup I_a$; then by the previous inclusion we have $R_{T'} = R_T$.

Note that $I_a \subseteq T' \cap J_a \subsetneq J_a$, for $a \notin T'$ by choice of a. Since there are no ideals strictly between I_a and J_a, we have $I_a = T' \cap J_a$. Then $M = (T' \cup J_a)/T' \sim J_a/I_a$. By the hypotheses on S, M is either completely 0-simple or null.

Let $\bar{S} = S/T'$. Then $\bar{R} = R/R_{T'}$ is a contracted $\bar{S}$-graded ring with ideal $\bar{R}_M$ $R_{T' \cup J_a}/R_{T'}$. From the hypothesis $\overline{\text{supp}}(R_{T' \cup J_a})$ is finite, we conclude that $\text{supp}(\bar{R}_M)$ is finite.

If M is completely 0-simple, then $\text{supp}(\bar{R})$ is finite by Proposition 9.2.2. If M is null, then $(\bar{R}_M)^2 = 0$, and by Lemma 9.1.2, $\text{supp}(\bar{R})$ is finite.

By definition of $\bar{R}$, we conclude that $\overline{\text{supp}}(R_T) = \overline{\text{supp}}(R_{T'})$ is finite. □

9.3.3. We restate the first theorem for convenience.

THEOREM ([13]). *Let S be a semigroup that has no infinite subgroups and in which every 0-simple principal factor is completely 0-simple. Let R be an S-graded ring. If R is right Artinian, then the support of R is finite.*

PROOF. By factoring out R_θ or adjoining a zero to S, we may assume that S has a zero and R is a contracted S-graded ring.

Suppose to the contrary that R has infinite support. Consider the family of ideals

$$F = \{R_T \mid T \text{ is an ideal of } S \text{ and } \overline{\text{supp}}(R_T) \text{ is infinite}\}.$$

Then $R_\theta \in F$ so that F is non-empty.

Now, $R/J(R)$ is semisimple Artinian and therefore is right Noetherian. So F contains maximal elements modulo $J(R)$; let R_T be such a maximal element.

Suppose that U is another ideal of S with $T \subseteq U$ and that $R_U + J(R) = R_T + J(R)$. We claim that $\overline{\text{supp}}(R_U)$ is infinite also. For intersecting each side of the above equality with R_U we obtain

$$R_U = (R_T + J(R)) \cap R_U = R_T + (J(R) \cap R_U),$$

the second equality holding since $R_T \subseteq R_U$. Hence R_U/R_T is a homomorphic image of $J(R) \cap R_U$ which is nilpotent because R is right Artinian. So in the S/T-graded ring R/R_T, R_U/R_T is a nilpotent ideal. But $\text{supp}(R/R_T)$ is infinite since $\overline{\text{supp}}(R_T)$ is infinite. By Lemma 9.1.2, $\overline{\text{supp}}(R_U/R_T)$ must be infinite and therefore $\overline{\text{supp}}(R_U)$ is also infinite as claimed.

In particular, if we take U to be the union of all ideals T' of S such that $T \subseteq T'$ and $R_{T'} + J(R) = R_T + J(R)$, then $R_U + J(R) = R_T + J(R)$ and $\overline{\text{supp}}(R_U)$ is infinite.

Now, R is right Artinian so we can choose $a \in \overline{\text{supp}}(R_U)$ such that $R_{U \cup J_a}$ is minimal. Then $\mathcal{J}(a)$ is minimal among the $\mathcal{J}$-classes intersecting $\overline{\text{supp}}(R_U)$. For if $b \in \overline{\text{supp}}(R_U)$ and $\mathcal{J}(b) \leq \mathcal{J}(a)$, then $J_b \subseteq J_a$ and by choice of a, $R_{U \cup J_b} = R_{U \cup J_a}$. Since $a \in \text{supp}(R_{U \cup J_a}) = \text{supp}(R_{U \cup J_b})$ and $a \notin U$, we must have $a \in J_b$ and hence $\mathcal{J}(a) = \mathcal{J}(b)$.

If $\overline{\text{supp}}(R_{U \cup J_a})$ were infinite, then $R_{U \cup J_a} \in F$ and by choice of T, $R_{U \cup J_a} + J(R) = R_T + J(R)$. But then $U \cup J_a \subseteq U$ by definition of U, a contradiction, since $a \notin U$. So

$\overline{\text{supp}}(R_{U' \cup J_n})$ is finite. By Lemma 9.3.2, we conclude that $\overline{\text{supp}}(R_{U'})$ is finite. But it was earlier stated that $\overline{\text{supp}}(R_{U'})$ is infinite. This contradiction means that our initial assumption that supp(R) is infinite is false, so supp(R) must be finite. □

9.4. Second Theorem

In this section, we deduce a second theorem from Theorem 9.3.3 which allows us to dispense with the assumption that all 0-simple principal factors are completely 0-simple at the expense of requiring that the support of the ring is almost the whole semigroup. We will see that in this circumstance, the semigroup is in fact periodic.

9.4.1. We begin with a "nil implies nilpotent" result concerning certain subrings of Artinian rings.

LEMMA. *Let U be a nil subsemigroup of the multiplicative semigroup of a right Artinian ring R. Then the subring generated by U is nilpotent.*

PROOF. Let $\bar{U}$ be the image of U in $R/J(R)$; the latter is a semisimple Artinian ring, so $R/J(R) = \bigoplus_{i=1}^{r} M_{n_i}(D_i)$ for some n_i and division rings D_i. If U_i is the projection of U onto the component $M_{n_i}(D_i)$, then $\bar{U} \subseteq \prod_{i=1}^{r} \bar{U}_i$. Each nil subsemigroup U_i of $M_{n_i}(D_i)$ is nilpotent of index at most n_i (see [22, 17.20]) so U is nilpotent of index at most the maximum of the n_i. Since R is right Artinian, $J(R)$ is nilpotent and U is nilpotent. If A is the subring of R generated by U then $A = \{\sum_{i=1}^{t} x_i | x_i \in U\}$ is nilpotent of the same index of nilpotency as U. □

9.4.2. We investigate the homogeneous components coming from non-periodic semigroup elements.

LEMMA. *Let S be a semigroup and let R be a right Artinian S-graded ring. Let s be a non-periodic element of S. Then R_s consists of nilpotent elements.*

PROOF. Let S^1 be the semigroup with an identity 1 adjoined. Denote by R^1 the ring obtained by adjoining an identity 1 to R in the usual way. If we let R_1 be the subring generated by 1 (so that $R_1 \cong \mathbf{Z}$), then $R^1 = \bigoplus_{t \in S^1} R_t$ is an S^1-graded ring.

Suppose to the contrary that there exists an $r \in R_s$ which is not a nilpotent element. For any non-negative integer m, denote by I_m the right ideal of R^1 generated by $1 - r^{2^m}$. Then $I_m \supseteq I_{m+1}$ because $1 - r^{2^{m+1}} = (1 - r^{2^m})(1 + r^{2^m})$. Given that R is right Artinian, there is some $k \geq 0$ such that $I_k \cap R = I_{k+1} \cap R = \cdots$. But $R^1 = I_k + R = I_{k+1} + R = \cdots$. So for $m > n \geq k$, we see that

$$I_n = (I_m + R) \cap I_n = I_m + (R \cap I_n)$$

$$= I_m + (R \cap I_m) = I_m.$$

Hence $I_k = I_{k+1} = \cdots$.

Since r is not nilpotent, neither is r^{2^k}. Replacing s by s^{2^k} and r by r^{2^k} we may assume that $k = 0$ and that $I_0 = I_1 = \cdots$.

Since $1 - r^2 \in I_1 = I_0$, there exists an element $w \in R^1$ such that $1 - r = (1 - r^2)w$. Denote by B the subsemigroup generated in S^1 by 1 and s, and let $C = S^1 \setminus B$. Note that $R^1 = R_B \oplus R_C$, so for any $x \in R^1$, we can write $x = x_B + x_C$ uniquely with $x_B \in R_B$ and $x_C \in R_C$. Consider the equation $1 - r = (1 - r^2)(w_B + w_C)$. Since 1, r, and w_B are elements of the subring R_B, if we take components in R_C, we see that $w_C = (r^2 w_C)_C$. But r^2 is a homogeneous element of R, so $|\operatorname{supp}(r^2 w_C)_C| \leq |\operatorname{supp} r^2 w_C| \leq |\operatorname{supp} w_C|$. This and the previous equality imply that $w_C = r^2 w_C$, and so $1 - r = (1 - r^2) w_B$.

Now write $w_B = w_0 + w_1 + w_2 + \cdots$ where $w_0 \in R_1$ and $w_n \in R_{s^n}$ for $n > 0$. Comparing the homogeneous summands in the equation

$$1 - r = (1 - r^2)(w_0 + w_1 + w_2 + \cdots)$$

we see that $w_0 = 1$ and $w_n = \pm r^n$ for $n > 0$. Since $w_n = 0$ for n sufficiently large, this contradicts the assumption that r was not nilpotent. □

9.4.3. We expand upon the last lemma to show that each cyclic subsemigroup intersects the support of a right Artinian graded ring finitely.

LEMMA. *Let S be a semigroup and let R be a right Artinian S-graded ring. Let $s \in S$. Then only a finite number of the components $R_s, R_{s^2}, R_{s^3}, \ldots$ are non-zero.*

PROOF. Suppose to the contrary that there are infinitely many distinct non-zero components $R_s, R_{s^2}, R_{s^3}, \ldots$. Then s is a non-periodic element of S.

Denote by T the set of all $t \in S$ such that $s^{m+n} t = s^m$ for some positive integers m and n. It is easy to see that T is a subsemigroup of S. If $s^{m+n} t = s^m$ then for any $p, q > 0$ we get $s^{m+n(p+q)pq} = s^m$ and $s^{m+n(p+q)p} = s^{m+nq}$; hence $t^{p+q} \neq t^p$ because $s^m \neq s^{m+nq}$. So T is a semigroup of non-periodic elements. By Lemma 9.4.2 the union $M = \bigcup_{t\in T} R_t$ is a nil subsemigroup of the multiplicative semigroup of R. By Lemma 9.4.1, M generates a nilpotent subring N of R. Note that $N = \sum_{t\in T} R_t$; letting $P = \sum_{s\in S\setminus T} R_s$ we have $R = N \oplus P$.

For a positive integer i, put $S_i = \{s^i, s^{i+1}, s^{i+2}, \ldots\}$, put $M_i = \bigoplus_{t\in S_i} R_t$, and denote by I_i the right ideal of R generated by M_i. Since infinitely many of the components R_{s^i} are non-zero, it follows that each M_i is non-zero.

We claim that there are infinitely many integers i such that R_{s^i} is not contained in I_{i+1}. If not, let n be an integer greater than all such i. Let $m \geq n$. Then $R_{s^m} \subseteq I_{m+1}$ by choice of n. Now, $I_{m+1} = M_{m+1} + M_{m+1}R$. Since $R_{s^m} \cap M_{m+1} = 0$ by definition of M_{m+1} and since M_{m+1} and $M_{m+1}R$ are homogeneous (as additive subgroups of R), we must have $R_{s^m} \subseteq M_{m+1}R = M_{m+1}N + M_{m+1}P$. By definition of T, we see that $R_{s^m} \cap M_{m+1}P = 0$. Hence by homogeneity of $M_{m+1}N$ and $M_{m+1}P$, we get $R_{s^m} \subseteq M_{m+1}N \subseteq M_n N$. (We remark that these inclusions hold trivially if $R_{s^m} = 0$.) This holds for all $m \geq n$, so that $M_n \subseteq M_n N$. Then $M_n \subseteq M_n N^k$ for all $k > 0$. Since N is a nilpotent subring, it follows that $M_n = 0$, a contradiction. So the claim holds.

Now, the right ideals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ form an infinite descending chain. We have shown that there are infinitely many i such that $R_{s^i} \not\subseteq I_{i+1}$ while $R_{s^i} \subseteq I_i$ always. So this chain cannot be eventually constant, contradicting the assumption that R is right Artinian. □

9.4.4. It is now easy to prove the second result.

THEOREM ([13]). *Let S be a semigroup with no infinite subgroups. Let R be an S-graded ring such that $R_s \neq 0$ for all but finitely many elements $s \in S$. If R is right Artinian, then S is finite.*

PROOF. Since only finitely many components R_s can be zero, Lemma 9.4.3 tells us that S must be periodic. Hence every 0-simple principal factor of S is completely 0-simple by Theorem 1.4.8, and the hypotheses of Theorem 9.3.3 are satisfied. So supp(R) and also S are finite. □

9.5. Corollaries

We conclude this chapter with a couple of known results which follow easily from our work.

9.5.1. Since bands are periodic and have only trivial subgroups, right Artinian band graded rings satisfy the hypotheses of Theorem 9.3.3.

COROLLARY ([44]). *Let B be a band and let R be a right Artinian B-graded ring. Then the support of R is finite.*

9.5.2. Finally we deduce a well-known result for semigroup algebras ([55, 74]). Our proof is independent of that of Zelmanov and also of the alternative proof, essentially at the semigroup level, which is furnished by a deep result of Hotzel [29].

COROLLARY ([74]). *Let $K[S]$ be a semigroup algebra over a field K. If $K[S]$ is right Artinian then S is finite.*

PROOF. By Theorem 9.4.4, it suffices to show that S cannot have infinite subgroups in this situation. Note that S is periodic by Lemma 9.4.3.

Let G be a maximal subgroup of S with identity e. Then the semigroup algebra $K[eSe]$ is right Artinian because $K[eSe] \cong eK[S]e$, and e is an idempotent. If $G = eSe$, then $K[G]$ is right Artinian. Otherwise, $I = eSe \setminus G$ is an ideal of eSe by Lemma 1.3.7 and therefore $eSe/I \cong G^0$. Hence, $K[eSe]/K[I] \cong K_0[G^0] \cong K[G]$ and therefore $K[G]$ is right Artinian. But this implies that G is finite (see [58, Theorem 10.1.1]). □

CHAPTER 10

Semigroup Graded Rings With Finite Support

In this chapter we attempt to extend some of our earlier results from rings graded by finite semigroups to rings graded by arbitrary semigroups with finite support.

The chief difficulty here is dealing with semigroups which have 0-simple principal factors which are not completely 0-simple. We demonstrate that in the absence of such factors, our earlier proofs in many cases go through with little modification. In particular, this is the case for nilpotency of the Jacobson radical and perfectness.

We also obtain one result without this restriction: that a semigroup graded ring with finite support is nilpotent if each component corresponding to an idempotent semigroup element is nilpotent.

Of course, this begs the question of whether we could somehow regrade an S-graded ring with finite support by another semigroup S' which is finite or at least has only completely 0-simple principal factors. We provide examples in Section 10.1 to show that this is not possible in general.

The results of this chapter were obtained in collaboration with Eric Jespers and Angel del Río.

10.1. Rings Which Cannot be Regraded

10.1.1. Let S be a semigroup and let R be an S-graded ring. As we have seen earlier, there are many ways of looking at R as a ring graded by some other semigroup

T. For example, if T is a homomorphic image of S, then R can be regarded as a T-graded ring using the method of §2.1.5. However, this process may involve amalgamating several homogeneous components for the S-gradation into one component for the T-gradation. Since we want to see whether the homogeneous components for the S-gradation determine certain properties of the ring, we are interested in a more restricted form of changing the grading semigroup — one which leaves the homogeneous components unchanged.

Let T be another semigroup. We say that R can be *regraded faithfully* by T if there is an injective map ψ: $\mathrm{supp}(R) \to T$ such that $R' = R$ is a T-graded ring if we put $R'_t = R_{\psi^{-1}(t)}$ for $t \in \psi\big(\mathrm{supp}(R)\big)$ and $R'_t = 0$ otherwise. Essentially, we relabel the non-zero homogeneous components of R by some elements of T in such a way that the multiplication of homogeneous elements of R is compatible with the semigroup multiplication in T.

We must examine more closely this process to see when it is possible. Suppose then that we are in the situation outlined above, and $s, t \in S$ are elements such that $R_s R_t \neq 0$. Then $R_s R_t \subseteq R_{st}$. Hence in the ring R' we require that $R'_{\psi(s)} R'_{\psi(t)} \subseteq R'_{\psi(st)}$ and because $R'_{\psi(s)} R'_{\psi(t)} \neq 0$, we must have $R'_{\psi(st)} = R'_{\psi(s)\psi(t)}$ so that $\psi(s)\psi(t) = \psi(st)$.

10.1.2. The first example is a ring graded by an infinite semigroup, with finite support, which cannot be regraded by a finite semigroup. In fact, the grading semigroup is a group, so this example solves the corresponding question for group graded rings at the same time.

This example is essentially due to del Río, Dăscălescu, Năstăsescu, and Van Oystaeyen [20], who found a finite subset of an infinite group which could not be embedded in a finite group in such a way that all products were preserved. They did not however, provide an example of a graded ring for which this set was the support.

10.1.3. Let G be an infinite simple group which is finitely presented. Such groups exist, see for example [28]. Specifically, let X be a finite set of symbols, let F be the free group on X, let R be a finite set of words in F, and let N be the normal subgroup of F generated by R such that $F/N \cong G$. For an element $w \in F$ we will write $\bar{w}$ for the image of w in G.

Define a set A of F by

$$A = \{w \mid w \text{ is a subword of an element of } R\} \cup X \cup X^{-1} \cup \{1\}.$$

where, as usual, we denote the identity element of a group by 1, and $X^{-1} = \{x^{-1} \mid x \in X\}$. Note that we include the elements of R itself in A. Because R and X are finite, A is finite. Write $\bar{A}$ for the image of A in G.

10.1.4. We claim that there is no injective map $\psi: \bar{A} \to S$ into a finite semigroup S such that $\psi(\bar{u})\psi(\bar{v}) = \psi(\bar{u}\bar{v})$ whenever $u, v, uv \in A$.

Suppose there is such a map. We first claim that we can replace S by a group H. For if $g \in \bar{A}$, then $g = g1 = 1g$ so that $\psi(g) = \psi(g)\psi(1) = \psi(1)\psi(g)$, and also $\psi(1)\psi(1) = \psi(1)$. So $\psi(\bar{A})$ is contained in the monoid $\psi(1)S\psi(1)$. Let w be an element of A. If w is a subword of an element of R, then there are elements $u, v \in A$ such that uwv is an element of R and so $\bar{u}\bar{w}\bar{v} = 1$. If $w \in X \cup X^{-1} \cup \{1\}$, then take $u = 1$ and $v = w^{-1}$ (which are elements of A) and we also have $uwv = 1$. Furthermore, we have $uw \in A$ in each case. Hence,

$$\psi(1) = \psi(\bar{u}\bar{w}\bar{v}) = \psi(\bar{u}\bar{w})\psi(\bar{v}) = \psi(u)\psi(w)\psi(v).$$

But $\psi(1)S\psi(1)$ is a finite monoid, so left units are also right units and it follows that $\psi(\bar{w})$ is a unit. So $\psi(\bar{A})$ is contained in the group $H = H\big(\psi(1)S\psi(1)\big)$. Henceforth we shall write $\psi(1) = 1$.

Now, the restriction $\psi|_X$ induces a group homomorphism $\phi \colon F \to H$, since F is a free group generated by X. We claim that for $w \in A$, $\phi(w) = \psi(w)$. This is certainly true if $w \in X$ or $w = 1$. If $w = x^{-1} \in X^{-1}$, then

$$1 = \psi(1) = \psi(\bar{x}\bar{x}^{-1}) = \psi(\bar{x})\psi(\bar{x}^{-1}) = \phi(x)\psi(x^{-1})$$

so that $\psi(\bar{x}^{-1}) = \phi(x)^{-1} = \phi(x^{-1})$. For the other elements of A, we proceed by induction on the length of the word. We have already proved the statement for words of length one. Suppose then that w is a subword of an element of R and that $w = ux$ where $x \in X \cup X^{-1}$ is the last symbol in the word w. If the result holds for words shorter than w, then

$$\phi(w) = \phi(u)\phi(x) = \psi(\bar{u})\psi(\bar{x}) = \psi(\bar{u}\bar{x}) = \psi(w)$$

since $u, x, w \in A$.

In particular, since $R \subseteq A$, we see that $\phi(w) = \psi(\bar{w}) = \psi(1) = 1$ for all $w \in R$. Hence, $\phi(N) = 1$ and ϕ induces a homomorphism $\bar{\phi}$: $G \to H$. But $\bar{A}$ certainly has more than two elements and ψ is injective, so the image of ψ and hence the image of ϕ are non-trivial. This is not possible, since G is an infinite simple group and H is finite.

This final contradiction shows that there is no such map ψ.

10.1.5. We now construct a G-graded ring with finite support which cannot be faithfully regraded by a finite semigroup.

Let n be the maximum length of a word in R and let P be a ring with an ideal Q such that $Q^{2n+1} = 0$ but $Q^{2n} \neq 0$. (For example, take P to be the ring of upper triangular $2n+1 \times 2n+1$ matrices over a field and let Q be the ideal of strictly upper triangular matrices.)

Let R be the subring of the group ring $P[G]$ generated by the set

$$P1 \cup \left(\bigcup_{x \in X} Q\bar{x} \right) \cup \left(\bigcup_{x \in X^{-1}} Q\bar{x} \right).$$

Since R is generated by homogeneous elements, it is a homogeneous subring of $P[G]$. For $g \in G$, the g-component of R is $R_g = R \cap Pg$.

Let $g \in G$. If k is the length of the shortest product of elements $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k$ with each $x_i \in X \cup X^{-1}$ such that $g = \bar{x}_1\bar{x}_2 \ldots \bar{x}_k$, then $R_g = Q^k g$. This means that $\mathrm{supp}(R)$ is finite, since $Q^{2n+1} = 0$ and only finitely many elements of G can be written as such products of length less than $2n + 1$.

Let $w \in A$. If $w \in X \cup X^{-1}$, then $R_w = Q\bar{w}$. If $w = 1$, then $R_w = P1$. Otherwise, w is a subword of length k of some element of B, and by choice of n, $0 \neq Q^k \bar{w} \subseteq R_w$. Hence $\bar{A} \subseteq \mathrm{supp}(R)$. Furthermore, if $u, v, uv \in A$, then $Q^k \bar{u} \subseteq R_u$ and $Q^l \bar{v} \subseteq R_v$ for some $k, l \leq n$, so that $0 \neq Q^{k+l}\bar{u}\bar{v} \subseteq R_{uv}$.

If R can be faithfully regraded by a finite semigroup S, then there is an injective map ϕ: $\mathrm{supp}(R) \to S$ such that $\phi(g)\phi(h) = \phi(gh)$ whenever $R_g R_h \neq 0$ (cf §10.1.1). But by the above, the restriction of ϕ to $\bar{A}$ satisfies the properties of the map ψ of §10.1.4. Since such a map ψ does not exist, we conclude that R cannot be faithfully regraded by a finite semigroup.

10.1.6. Our second example, of a ring graded by a 0-simple semigroup which cannot be regraded by a completely 0-simple semigroup is somewhat simpler.

Let $S = C(p, q)$ be a bicyclic semigroup (cf §1.4.5). Let I be an ideal of a ring A with $I^2 \neq 0$ but $I^3 = 0$. Let R be the subring of $A[S]$ generated by the homogeneous elements $A1 \cup Ip \cup Iq$, and let R inherit the S-gradation of $A[S]$. Suppose that ψ: $\mathrm{supp}(R) \to T$ is a map that faithfully regrades R. Then it is easy to see, by considering products of homogeneous components of R, that the following identities hold in T:

$$\psi(1)^2 = \psi(1)$$

$$\psi(1)\psi(p) = \psi(p)\psi(1) = \psi(p)$$

$$\psi(1)\psi(q) = \psi(q)\psi(1) = \psi(q)$$

$$\psi(p)\psi(q) = \psi(1)$$

$$\psi(q)\psi(p) = \psi(qp).$$

Write $1 = \psi(1)$, $p' = \psi(p)$, and $q' = \psi(q)$ and let $T'' = \langle p', q' \rangle$. Because we require ψ to be injective, we have $q'p' \neq 1$. But this means that T'' is actually a bicyclic semigroup (cf §1.4.5). Therefore, T cannot be completely 0-simple by Theorem 1.4.7. We may even say more: T must have a principal factor which is 0-simple but not completely 0-simple. For if J is an ideal of T, and $T'' \cap J \neq \emptyset$, then $T'' \subseteq J$ because T'' is simple. Hence, using the notation of §1.4.3, $T'' \subseteq J_t/I_t$ for an element $t \in T$, and therefore, the principal factor J_t/I_t contains a subsemigroup isomorphic to T''.

So in considering general semigroup graded rings with finite support, we will encounter principal factors which are 0-simple but not completely 0-simple.

10.2. Nilpotent Rings

Let S be a semigroup and let R be an S-graded ring with finite support. In this section, we show that if R_e is a nilpotent ring for each idempotent $e \in S$, then R is also nilpotent.

10.2.1. The proof of the first result is essentially combinatorial in nature. If we multiply a large number of homogeneous elements together, we must somehow bracket the product so that several successive subproducts fall into the same homogeneous component corresponding to an idempotent. In the picture below, given homogeneous elements x_t, we hope to somehow bracket them such that the products y_t are all elements of some R_e for an idempotent e, and of course, r exceeds the index of nilpotency of R_e.

$$\cdots x_{i-1} \underbrace{x_i x_{i+1} \cdots x_{j-1}}_{y_1} \underbrace{x_j x_{j+1} \cdots x_{k-1}}_{y_2} \cdots \underbrace{x_l x_{l+1} \cdots x_{m-1}}_{y_r} x_m \cdots$$

This process does not really depend upon the specific ring elements; only on their supporting elements in the semigroup.

10.2.2. The combinatorial theory that we need for the proof is the theory of repetitive mappings [46, Chapter 4].

Let X be a non-empty set and let $\mathcal{F}_X$ be the free semigroup with generators X (cf §1.3.3). Let $\phi\colon \mathcal{F}_X \to E$ be a map of $\mathcal{F}_X$ to a set E. A word $w \in \mathcal{F}_X$ is said to *contain a k-th power modulo* ϕ (for $k \geq 1$) if there are words $w_1, w_2, \ldots, w_k \in \mathcal{F}_X$ and $u, v \in \mathcal{F}_X^1$ such that $uw_1w_2 \ldots w_kv = w$ and $\phi(w_1) = \phi(w_2) = \cdots = \phi(w_k)$. We say that ϕ is *repetitive* if for each $k \geq 1$, there is an $l \geq 1$ such that every word $w \in \mathcal{F}_X$ of length at least l contains a k-th power modulo ϕ.

The result we require is the following:

THEOREM ([46, Theorem 4.1.1]). *Let X be a non-empty set and let $\phi\colon \mathcal{F}_X \to E$ be a map to a finite set E. Then ϕ is repetitive.*

10.2.3. We apply this theorem to semigroup graded rings.

THEOREM. *Let S be a semigroup and let R be an S-graded ring with $\operatorname{supp}(R)$ finite. Suppose that R_e is nilpotent for each idempotent $e \in \operatorname{supp}(R)$. Then R is nilpotent.*

PROOF. Let $s \in \operatorname{supp}(R)$. If $s^p \notin \operatorname{supp}(R)$ for some p, then $(R_s)^p \subseteq R_{s^p} = 0$. Otherwise, the cyclic semigroup $\langle s \rangle$ is finite since it is entirely contained in $\operatorname{supp}(R)$. Hence, there is an idempotent e such that $s^p = e$ for some p. But $(R_e)^l = 0$ for some l so that $(R_s)^{pl} = 0$. Since $\operatorname{supp}(R)$ is finite, there is an n such that $(R_s)^n = 0$ for all $s \in \operatorname{supp}(R)$.

Let $X = \operatorname{supp}(R)$ and extend the inclusion $X \hookrightarrow S$ to a homomorphism $\psi\colon \mathcal{F}_X \to S$. Let $\bigstar$ be some symbol not in S, and define a map $\phi\colon \mathcal{F}_X \to \operatorname{supp}(R) \cup \{\bigstar\}$ by

$$\phi(w) = \begin{cases} \psi(w) & \text{if } \psi(w) \in \operatorname{supp}(R). \\ \bigstar & \text{otherwise.} \end{cases}$$

By Theorem 10.2.2, ϕ is repetitive, so there exists an m such that every word $w \in \mathcal{F}_X$ of length at least m contains an n-th power modulo ϕ.

Let $r_1, r_2, \ldots, r_m$ be homogeneous elements of R with $r_i \in R_{x_i}$ for elements $x_i \in \operatorname{supp}(R)$. Then the word $x_1 x_2 \ldots x_m$ in $\mathcal{F}_X$ contains an n-th power modulo ϕ; that is there are indices $1 \leq i_1 < i_2 < \cdots < i_{n+1} \leq m+1$ and an element $z \in \operatorname{supp}(R) \cup \{\bigstar\}$ such that $\phi(x_{i_k} x_{i_k+1} \ldots x_{i_{k+1}-1}) = z$ for $1 \leq k \leq n$.

If $z = \bigstar$, then the product $x_{i_1} x_{i_1+1} \ldots x_{i_2-1}$ in S is not in $\operatorname{supp}(R)$ and therefore $r_{i_1} r_{i_1+1} \ldots r_{i_2-1} = 0$ and $r_1 r_2 \ldots r_m = 0$.

If $z \in \operatorname{supp}(R)$, then we have $x_{i_k} x_{i_k+1} \ldots x_{i_{k+1}-1} = z$ in S. So $r_{i_k} r_{i_k+1} \ldots r_{i_{k+1}-1} \in R_z$ for $1 \leq k \leq n$. Hence $r_{i_1} r_{i_1+1} \ldots r_{i_{n+1}-1} \in (R_z)^n = 0$ and so $r_1 r_2 \ldots r_m = 0$.

Since the product of m homogeneous elements of R is zero, it follows that $R^m = 0$. □

10.2.4. As an immediate consequence of Theorem 10.2.3 we have the following interesting result.

COROLLARY. *Let R be a ring graded by a semigroup S. If the support of R is finite and contains no idempotents, then R is a nilpotent ring.*

We remark that for group graded rings, the analogous statement is that a group graded ring with finite support not containing the identity of the group is nilpotent; this was proved by Cohen and Rowen [17, Proposition 1.2(1)].

10.3. Perfect Rings and Nilpotence of the Radical Revisited

In this final section, we show how some of our earlier results can be extended to graded rings with finite support, with certain restrictions on the grading semigroup

Our aim is to extend results such as Theorem 4.5.1 on the nilpotence of the Jacobson radical, or Theorem 6.7.1 for perfect rings to semigroup graded rings with finite support. However, we are unable to prove these results in full generality, the impediment being principal factors which are 0-simple but not completely 0-simple. In fact, we are able to weaken this restriction slightly, for Corollary 10.2.4 enables us to handle 0-simple principal factors which contain no non-zero idempotents.

10.3.1. For nilpotence of the radical, we obtain the following result.

THEOREM. *Let S be a semigroup which has no subsemigroups isomorphic to the bicyclic semigroup. Let R be an S-graded ring with* supp(R) *finite and with $J(R_e)$ nilpotent for each idempotent $e \in$* supp(R). *Then $J(R)$ is nilpotent.*

PROOF. As in the proof of Theorem 4.5.1 we may assume that S has a zero and R is a contracted S-graded ring. Write $\text{supp}(R)^0$ for $\text{supp}(R) \cup \{\theta\}$. We may replace S by $\langle \text{supp}(R)^0 \rangle$, the subsemigroup generated by $\text{supp}(R)^0$.

We proceed by induction on $|\text{supp}(R)|$. If $|\text{supp}(R)| = 1$, then $R = R_x$ for some $x \in S$. If x is idempotent, then $J(R) = J(R_x)$ is nilpotent by hypothesis. Otherwise, there are no idempotents in the support of R and R itself is nilpotent by Corollary 10.2.4.

Suppose that the result is true for rings with support smaller than supp(R). Since S is finitely generated, it contains a maximal ideal M by a standard argument using Zorn's lemma. If $\text{supp}(R) \cap M \neq \emptyset$ and $\text{supp}(R) \cap S \setminus M \neq \emptyset$, then both the M-graded ring R_M and the contracted S/M-graded ring R/R_M have smaller support than R and by induction, $J(R_M)$ and $J(R/R_M)$ are nilpotent. By Lemma 4.4.2(i), we conclude that $J(R)$ is nilpotent. Otherwise, we must have $\text{supp}(R) \subseteq M$ or $\text{supp}(R) \subseteq S \setminus M$. But the former is not possible since M is a proper ideal of S, $\theta \in M$, and S is generated by $\text{supp}(R)^0$. In the latter case, $R \cong R/R_M$ is a contracted S/M-graded ring. Note that S/M is either 0-simple or null (cf §1.4.2).

Replacing S by S/M we may assume that S is 0-simple or null. In the latter case, $R^2 = 0$ because $R_\theta = 0$. So assume that S is 0-simple. If S contains no non-zero idempotents, R is nilpotent by Corollary 10.2.4. Otherwise, since we are assuming that S does not contain a bicyclic subsemigroup, S must be completely 0-simple by Theorem 1.4.7. We may take $S = \mathfrak{M}^0(G; I, \Lambda; P)$ by Theorem 1.5.2 and I and Λ must be finite by Lemma 1.5.3(vii) since S is finitely generated.

With the notation of §2.2.2, each $R_{i\lambda}$ is either nilpotent or a G-graded ring with finite support. Hence $J(R_{i\lambda})$ is nilpotent for all $i \in I$ and $\lambda \in \Lambda$ by Proposition 4.2.3. Using Proposition 4.3.3, we conclude that $J(R)$ is nilpotent. □

10.3.2. We next turn our attention to right perfect rings, obtaining a similar theorem. We will not give the proof, since the method is the same as that of the proof of Theorem 10.3.1, requiring Lemma 6.2.5 to handle ideal extensions and Proposition 6.4.2 and Corollary 6.3.3 to deal with the completely 0-simple principal factors that arise. Of course, similar results hold for left perfect, semilocal, and semiprimary rings.

THEOREM. *Let S be a semigroup which has no subsemigroups isomorphic to the bicyclic semigroup. Let R be an S-graded ring with* $\text{supp}(R)$ *finite and with R_e right perfect for each idempotent $e \in$* $\text{supp}(R)$. *Then R is right perfect.*

Note that we do not obtain the converse statement in this instance since we were unable to prove it in the group graded case (cf Corollary 6.3.3).

10.3.3. Although subsemigroups isomorphic to the bicyclic semigroup are the problem case in Theorem 10.3.1 and Theorem 10.3.2, the case that S is itself the bicyclic semigroup is surprisingly easy to handle.

PROPOSITION. *Let R be a ring graded by the bicyclic semigroup $C(p,q)$. Suppose that* $\text{supp}(R)$ *is finite.*

(i) *If $J(R_e)$ is nilpotent for each idempotent $e \in$* $\text{supp}(R)$, *then $J(R)$ is nilpotent.*

(ii) *If R_e is right perfect for each idempotent $e \in$* $\text{supp}(R)$, *then R is right perfect.*

PROOF. We use the notation of §1.4.5. Note that the map $\phi\colon q^m p^n \mapsto m - n$ is a semigroup homomorphism of $C(p,q)$ onto the additive group **Z** (if we regard $1 = q^0p^0$). So we may give R a **Z**-gradation using this map (cf §2.1.5).

The elements of $C(p,q)$ which ϕ maps to 0 are the elements $1, qp, q^2p^2, \ldots$ which are precisely the idempotents of $C(p,q)$. Since R has finite support, there is a largest n such that $q^np^n \in \text{supp}(R)$. Let $\Gamma = \{1, qp, q^2p^2, \ldots, q^np^n\}$; then Γ is a finite subsemigroup of $C(p,q)$.

If we write R'_n for the n-component of R with the $\mathbb{Z}$-gradation, then $R'_0 = R_1 \oplus R_{qr} \oplus \cdots \oplus R_{q^n p^n}$ is a Γ-graded ring.

Suppose then that each $J(R_{q^k p^k})$ is nilpotent. By Theorem 4.5.1, $J(R'_0)$ is nilpotent. Since R has finite support for the $\mathbb{Z}$-gradation, it follows from Proposition 4.2.3 that $J(R)$ is nilpotent. This proves (i)

Similarly, (ii) follows from Theorem 6.7.1 and Corollary 6.3.3. []

In spite of this result, we are unable to handle the case of a general 0-simple semigroup with non-zero idempotents which is not completely 0-simple. Although Theorem 1.4.7 tells us that such semigroups contain bicyclic semigroups, we do not know enough about the structure of such semigroups to exploit this fact.

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