Equivariant Cohomology and GKM-sheaves

by

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Dedication

To

the Memory of

My father.

I also would like to dedicate this thesis to:

Judi and Rashid

My mother

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Abstract

If a topological group T acts on a topological space X, we may define the equivariant cohomology ring $H_T^*(X)$. Due to its importance, several techniques have been developed to study equivariant cohomology. Goresky, Kottwitz, and MacPherson proved that of T torus action with a certain condition (GKM-manifold) the equivariant cohomology ring $H_T^*(X)$ has a combinatorial description. More recently, T. Baird applied GKM-methods to general equivariantly formal compact T-manifold X. He developed a new class of sheaves (GKM-sheaves), and proved that the equivariant cohomology of X is isomorphic to the global sections of a GKM-sheaf $\mathcal{F}_X$. The purpose of this thesis is studying the GKM-theory and GKM-sheaves. In particular, we study the higher cohomology of GKM-sheaves and generalize the theory to compact T-manifolds for which $H_T^*(X)$ is reflexive.

Chapter 1

Introduction

Let $T = (\mathbb{S}^1)^r$ be a compact torus Lie group and let X be a compact T-manifold. The equivariant cohomology $H_T^*(X)$ is a graded ring. The constant map $r : X \to pt$, where pt is the one element set and induces a map in equivariant cohomology $H_T^*(pt) \to H_T^*(X)$, which gives $H_T^*(X)$, the structure of an $H_T^*(pt)$-module[1]. If $H_T^*(X)$ is a free $H^*(BT)$-module then the T-space X is called *equivariantly formal*. In this thesis we aim to study the equivariant cohomology of a T-space X in different aspects.

Many researchers have studied equivariant cohomology in different areas. M. Goresky, R. Kottwitz and R. MacPherson proposed a technique to simplify the computation of $H_T^*(X)$, commonly known as GKM- theory. They proved that the equivariant cohomology ring $H_T^*(X)$ of a torus action with a certain condition (GKM-manifold) has a combinatorial description. GKM-theory has been studied in detail by V. Guillemin and C. Zara [13, 14, 15]. In **chapter two** of this thesis, we will provide an introduction to both equivariant cohomology and GKM-theory.

On the other side, sheaf theory is a powerful area in algebraic topology that is concerned with consistent local information to recover some global information. T. Braden and R. MacPherson [5] have found applications in sheaf theory for computing and studying equivariant cohomology by using what they call a moment graph. Re-

[1] We denote $R = H_T^*(pt)$ in the rest of this thesis.

cently, T. Baird [3] developed a new class of sheaves called GKM sheaves and used them to generalize the GKM theory to all smooth, compact T-manifolds X. He proved that if a smooth compact T-manifold X is an equivariantly formal space, then the equivariant cohomology of the T-space X is isomorphic to the global sections of a GKM-sheaf[2] $\mathcal{F}_X$ which states as follows:

Theorem 1.0.1. *Let X be a smooth compact T-manifold. If X is equivariantly formal, then*

$$H_T^*(X) \cong H^0(\mathcal{F}_X).$$

In **chapter three** we recall all the necessary background of sheaf theory, cohomology of sheaves and GKM-sheaves.

In **chapter four**, we study the higher cohomology of GKM-sheaves. We prove that $H^n(\mathcal{F}) = 0$ for $n \geq 2$ (see Proposition 4.1.1) and produce chain complexes to calculate $H^1(\mathcal{F})$ (see Proposition 4.2.1).

Let $T = (\mathbb{S}^1)^r = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ be a compact torus of rank r, and X be a compact, smooth T-manifold. Let X_i be the union of all orbits of dimension less than or equal to i, i.e, $X_i = \{x \in X, dim(T.x) \leq i\}$. We call X_i the i-**skeleton** of T-space X. In particular, $X_{-1} = \phi$, $X_0 = X^T$, and $X_r = X$, where r is the rank of T. In **chapter five** we show that the global section of GKM-sheaf $H^0(\mathcal{F}_X)$ is reflexive (a 2-syzygy) see (Proposition 5.1.1) and use this result to generalize the Theorem 1.0.1 (see Theorem 5.1.1) since reflexive is more general than the equivarantly formal, and prove the following theorem:

Theorem 1.0.2. *If X is a compact T-manifold, and $H_T^*(X)$ is reflexive, then there is a natural exact sequence*

$$0 \to H^0(\mathcal{F}_X) \to H_T^*(X_0) \xrightarrow{\delta} H_T^{*+1}(X_1, X_0) \to H^1(\mathcal{F}_X) \to 0. \tag{1.1}$$

[2]A GKM-sheaf that is associated to any smooth T-manifold X is denoted by $\mathcal{F}_X$.

Finally, this thesis contains : **Appendix A** is a survey of fibre bundles and classifying spaces.

Chapter 2

Equivariant Cohomology

2.1 Equivariant Cohomology and Equivariant Formality

Let G be a compact, connected Lie group, and X be a smooth manifold with a C^∞-action of the group G. If G acts freely on X, then the quotient space X/G is a manifold, and we define the equivariant cohomology of X to be the singular cohomology (with complex coefficients) of X/G

$$H_G^*(X) = H^*(X/G). \tag{2.1}$$

If the action of G on X is not free, then the quotient space X/G is not guaranteed to be a manifold. To avoid this case, *Borel* proposed an infinite dimensional manifold which is homotopy equivalent to X, and G acts freely on this manifold, its called Borel construction and is outlined below.

The universal bundle $EG \to BG$ is a principal G-bundle for G, for which EG is contractible and G acts freely on EG. BG is called the *classifying space* of G.

Let X be a G-space and let G act diagonally on the product space $EG \times X$, that is, for $g \in G$ and $(e, x) \in EG \times X$, the diagonal action is $g.(e, x) = (g.e, g.x)$. The action on EG is free, so G acts freely on $EG \times X$. We define the equivariant cohomology to be the singular cohomology of $(EG \times X)/G$,

$$H_G^*(X) = H^*((EG \times X)/G) = H^*(EG \times_G X). \tag{2.2}$$

More generally, if X and Y are G-spaces with $Y \subseteq X$, the relative equivariant cohomology of a pair (X, Y) defined as follows:

$$H_G^*(X, Y) = H^*((X \times EG)/G, (Y \times EG)/G). \tag{2.3}$$

For more details of fibre bundles and classifying space, see **Appendix A**.

Example 2.1.1. *Let $\mathbb{S}^1 = \{z \in \mathbb{C} \ ||z| = 1\}$ be the unit circle in $\mathbb{C}$, and $\mathbb{S}^{2n+1} = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \ ||z_0|^2 + \dots + |z_n|^2 = 1\}$ be the unit sphere in $\mathbb{C}^{n+1}$. Consider the action of $\mathbb{S}^1$ on $\mathbb{S}^{2n+1}$ by left multiplication. Observe that this action is free and the quotient space $\mathbb{S}^{2n+1}/\mathbb{S}^1$ is the n-dimensional complex projective space $\mathbb{C}P^n$. The inclusions $\mathbb{C}^n \subseteq \mathbb{C}^{n+1}$, $(z_0, \dots, z_n) \longmapsto (z_0, \dots, z_n, 0)$ determine inclusions $\mathbb{S}^{2n-1} \subseteq \mathbb{S}^{2n+1}$. Define $\mathbb{S}^\infty = \cup_{n=1}^{\infty} \mathbb{S}^{2n+1}$, the infinite sphere $\mathbb{S}^\infty$ is an infinite dimensional CW-complex and is contractible[17]. Also, we have $\mathbb{S}^\infty/\mathbb{S}^1 = \mathbb{C}P^\infty$. Hence, $\mathbb{S}^\infty \to \mathbb{C}P^\infty$ is a universal principal $\mathbb{S}^1$-bundle.*

In this thesis, we are interested in a compact torus $T = (\mathbb{S}^1)^r = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$. Let X be a smooth T-manifold.

According to the above example, we may take $ET = \mathbb{S}^\infty \times ... \times \mathbb{S}^\infty$. The classifying space of T is the orbit space:

$$BT = ET/T = \mathbb{C}P^\infty \times ... \times \mathbb{C}P^\infty. \tag{2.4}$$

By the Künneth Theorem,

$$H^*(BT) = H^*(\mathbb{C}P^\infty) \otimes \cdots \otimes H^*(\mathbb{C}P^\infty) = \mathbb{C}[x_1] \otimes \cdots \otimes \mathbb{C}[x_r] = \mathbb{C}[x_1, ..., x_r]$$

where each class x_i has degree two. The equivariant cohomology of a point is

$$H^*_T(pt) = H^*(ET/T) = H^*(BT). \tag{2.5}$$

Given a T-space X, the equivariant map $r : X \to pt$ induces a map in equivariant cohomology $r^* : H^*(BT) \to H^*_T(X)$ which gives $H^*_T(X)$ the structure of an $H^*(BT)$-module. Consider the associated bundle $X \overset{i}{\hookrightarrow} X \times_T ET \overset{\pi}{\to} BT$ with the fibre X. Therefore, we have the following sequence

$$H^*(BT) \xrightarrow{\pi^*} H^*(X \times_T ET) \xrightarrow{i^*} H^*(X). \tag{2.6}$$

Theorem 2.1.1. [1] *Let T be a compact torus. The T-space X is equivariantly formal if and only if any of the following equivalent conditions hold:*

1. *$H^*_T(X)$ is a free module over the ring $H^*(BT)$.*

2. *There is an isomorphism of graded $H^*(BT)$-modules from $H^*(X) \otimes_{\mathbb{C}} H^*(BT)$ to $H^*_T(X)$.*

3. *The morphism i^* in (2.6) is surjective.*

Proof. See [10] and [25]. □

The following theorem states a sufficient condition for X is to be an equivariantly formal.

Theorem 2.1.2. *Let X be a compact T-manifold. If the ordinary cohomology of X vanishes in odd degrees, then X is equivariantly formal.*

[1] Theorem 2.1.1 holds if we replace a compact torus T by any compact, connected Lie group.

Proof. See [21] and [22]. □

Example 2.1.2. *Let T be a compact torus acting on T-space X trivially, that is, all the points of X are fixed by this action. Thus, $H_T^*(X) = H^*(X \times_T ET) = H^*(X \times BT) = H^*(X) \otimes_{\mathbb{C}} H^*(BT)$, by the Künneth Theorem.*

Example 2.1.3. *Let $\mathbb{S}^1$ be a unit circle acting on a unit sphere $\mathbb{S}^2$ by rotation. Explicitly,*

$$\mathbb{S}^1 \times \mathbb{S}^2 \to \mathbb{S}^2$$
$$(x, (y, z)) \longrightarrow (y, xz)$$

identifying $\mathbb{S}^2 := \{(y, z) \in \mathbb{R} \times \mathbb{C} \mid \ y^2 + |z|^2 = 1\}$. The ordinary cohomology of $\mathbb{S}^2$ vanishes in odd degrees. By Theorem 2.1.2, this implies that $\mathbb{S}^2$ is an equivariantly formal space.

Furthermore, we can determine the module structure of $H^*_{\mathbb{S}^1}(\mathbb{S}^2)$ over $H^*(B\mathbb{S}^1)$:

$$\begin{aligned} H^*_{\mathbb{S}^1}(\mathbb{S}^2) &\cong H^*(B\mathbb{S}^1) \otimes_{\mathbb{C}} H^*(\mathbb{S}^2) \\ &\cong \mathbb{C}[x] \otimes_{\mathbb{C}} \langle 1, \alpha \rangle \\ &= \mathbb{C}[x] \oplus \mathbb{C}[x] \cdot \alpha \end{aligned}$$

where α is an element of degree 2.

In this thesis we fix a compact torus $T = (\mathbb{S}^1)^r$ with complexified Lie algebra $\mathbf{t}$=$Lie(T) \otimes \mathbb{C}$ and dual $\mathbf{t}^* = Hom(\mathbf{t}, \mathbb{C})$. The *weight lattice* is $\Lambda := Hom(T, \mathbb{S}^1) \cong \mathbb{Z}^r$, the isomorphism $\psi : \mathbb{Z}^r \xrightarrow{\cong} Hom(T, \mathbb{S}^1)$, is defined by

$$\psi((n_1, n_2, ..., n_r))(z_1, z_2, ..., z_r) = z_1^{n_1} z_2^{n_2} ... z_r^{n_r} \tag{2.7}$$

where $(z_1, z_2, ..., z_r) \in T$. Since $H^*(BT) \cong \mathbb{C}[x_1, \cdots, x_r]$ where each class x_i has degree two. The embedding $\mathbb{Z}^r \overset{i}{\hookrightarrow} \mathbb{C}[x_1, \cdots, x_r]$ defined by

$$i((n_1, ..., n_r)) = n_1x_1 + n_2x_2 + ... + n_rx_r \tag{2.8}$$

determines a natural embedding $i \circ \psi^{-1} : \Lambda \to H^2(BT)$.

2.2 Borel Localization Theorem

We have seen that $H^*_T(X)$ has a module structure over $H^*(BT)$.

Suppose the compact torus T acts on a compact manifold X, and let X^T be the fixed points of the T-action. The inclusion map $i : X^T \hookrightarrow X$ induces a map $i^* : H^*_T(X) \to H^*_T(X^T)$. The Borel localization theorem asserts that i^* is an isomorphism modulo torsion. Let us recall some notions of algebra.

If M is a finitely generated $H^*(BT)$-module, we define the *annihilator ideal* of M as follows:

$$I_M = \{f \in H^*(BT) |\ fM = 0\} \tag{2.9}$$

since $H^*(BT) \cong \mathbb{C}[x_1, ..., x_r]$, this will allows us to consider elements of $H^*(BT)$ as polynomials in $\mathbb{C}[x_1, ..., x_r]$. The support of M is:

$$suppM = \{x \in \mathbb{C}^n \ | \ f(x) = 0 \quad \text{for all} \quad f \in I_M\} \tag{2.10}$$

Clearly, for a nontrivial free module M, the support of M is $\mathfrak{t} \otimes \mathbb{C} \cong \mathbb{C}^n$. An element $m \in M$ is defined to be a torsion element if $fm = 0$ for some $f \neq o$. The set of torsion elements is a submodule of M, and M is called a torsion module if this submodule is M itself, i.e., if every element is a torsion element. It is clear from the definition that M is a torsion module if and only if the support of M is a proper subset of $\mathfrak{t} \otimes \mathbb{C}$.

The next theorem allows us to study the kernel and cokernel of the restriction map $i^* : H^*_T(X) \to H^*_T(X^T)$.

Theorem 2.2.1. *Let T act on a compact manifold X and let Y be a T-invariant submanifold. Then the support of modules $H_T^*(X,Y)$ and $H_T^*(X-Y)$ are contained in the set*

$$\bigcup_{K\subset T} \mathfrak{k}\otimes\mathbb{C} \tag{2.11}$$

where $Lie(K)=\mathfrak{k}$ and K runs over the finite set of all isotropy subgroups of T of points $X-Y$.

Proof. See Theorem 11.4.1 in [12]. □

In particular, let $Y = X^T$. We already know that T acts on X^T trivially, hence $H_T^*(X^T)$ is a free $H^*(BT)$-module. Therefore, we have an important consequence.

Corollary 2.2.1. *Let X be a compact T-manifold. The kernel and cokernel of the map $i^*: H_T^*(X)\to H_T^*(X^T)$ are supported in*

$$\bigcup_{K\subset T} \mathfrak{k}\otimes\mathbb{C} \tag{2.12}$$

where $Lie(K)=\mathfrak{k}$ and K runs over the finite set of all subgroups of T which occur as isotropy groups of points of $X-X^T$. In particular, $\ker(i^*)$ and coker(i^*) are torsion $H^*(BT)$-modules.

Proof. From the exact sequence of the pair (X,X^T),

$$H_T^*(X,X^T)\to H_T^*(X)\to H_T^*(X^T)\to H_T^{*+1}(X,X^T) \tag{2.13}$$

the kernel and cokernel are quotient and submodule of $H_T^*(X,X^T)$, thus their supports are contained in the support of $H_T^*(X,X^T)$ which by Theorem 2.2.1 is contained in (2.12). The space X is compact, so the set of all subgroups of T which occur as isotropy groups of points in X is finite which implies the union (2.12) is finite and thus a proper subset of $\mathfrak{t}\otimes\mathbb{C}$. Since the kernel and cokernel are supported in (2.12), they are both torsion modules. □

Corollary 2.2.2. *Let $i : X^T \hookrightarrow X$ be the inclusion map, and suppose that $H_T^*(X)$ is torsion free. Then the induced map*

$$i^* : H_T^*(X) \to H_T^*(X^T) \tag{2.14}$$

is injective.

2.2.1 Atiyah-Bredon Sequence

Let $R = H^*(BT) = \mathbb{C}[x_1, \cdots, x_r]$. A finitely generated R-module M is said to be a j-th syzygy if there is an exact sequence

$$0 \to M \to F^1 \to F^2 \to \cdots \to F^j$$

where the $\{F^i\}_{i\in\{1,..,j\}}$ are finitely generated free R-modules.

Remark 2.2.1. *Observe that a 1-syzygy is torsion free. If the R-module M is free, then it is j-syzygy for all $j > 0$.*

Remark 2.2.2. *We say that the finitely generated R-module M is* ***reflexive*** *if it is a 2-syzygy. This is equivalent to the double dual map*

$$M \to Hom_R(Hom_R(M, R), R) \tag{2.15}$$

being an isomorphism [1].

Let $T = (\mathbb{S}^1)^r = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ be a compact torus of rank r, and X be a compact, smooth T-manifold. Let X_i be the union of all orbits of dimension less than or equal to i, i.e, $X_i = \{x \in X, dim(T.x) \le i\}$. We call X_i the i-**skeleton** of T-space X. In particular, $X_{-1} = \phi$, $X_0 = X^T$, and $X_r = X$, where r is the rank of T.

Under the above hypothesis, C. Allay, M. Franz, and V. Puppe [1] proved the following theorem.

Theorem 2.2.2. *Let $j \geqslant 0$ and let T be a torus of rank r, and X be a compact T-manifold. Consider the sequence*

$$0 \to H_T^*(X) \to H_T^*(X_0) \xrightarrow{\delta} H_T^{*+1}(X_1, X_0) \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_r} H_T^{*+r}(X_r, X_{r-1}) \to 0, \qquad (2.16)$$

where δ_i is the boundary map of the triple (X_{i-1}, X_i, X_{i+1}). Then (2.16) is exact for all the positions $i \leqslant j-2$ if and only if $H_T^*(X)$ is j-th syzygy. In particular, the Chang-Skjelbred sequence

$$0 \to H_T^*(X) \to H_T^*(X_0) \xrightarrow{\delta} H_T^{*+1}(X_1, X_0) \qquad (2.17)$$

is exact if and only if $H_T^*(X)$ is a 2-syzygy.

Proof. See Theorem 5.7 in [1]. □

Remark 2.2.3. *Observe that the sequence (2.17) implies*

$$H_T^*(X) \cong ker(\delta), \qquad (2.18)$$

whenever $H_T^*(X)$ is a 2-syzygy. GKM theory is concerned with calculating $ker(\delta)$.

Remark 2.2.4. *If $H_T^*(X)$ is a free module over the ring $H^*(BT)$, then it is j-syzygy for all $j > 0$. Thus, the Atiyah Bredon sequence (2.16) and the Chang-Skjelbred sequence (2.17) are exact.*

2.2.2 The Chang-Skjelbred Theorem

Let X be a compact smooth T-manifold and let $H_T^*(X)$ be a 1-syzygy, that is, torsion free. Corollary 2.2.2 says that the map $i^* : H_T^*(X) \to H_T^*(X^T)$ is injective. So, the image of i^* is isomorphic to $H_T^*(X)$. Our goal of recalling the Chang-Skjelbred theorem is to describe the image of i^*. Taking a codimension one subtorus H of T. The fixed point set X^H contains X^T [12], this implies

$$\begin{array}{ccc} X^T & \xrightarrow{i_H} & X^H \\ & {\scriptstyle i}\searrow \quad \swarrow{\scriptstyle r_H} & \\ & X & \end{array}$$

and induces a commutative diagram in equivariant cohomology as follows,

$$\begin{array}{ccc} H_T^*(X) & \xrightarrow{i_H^*} & H_T^*(X^H) \\ & {\scriptstyle i^*}\searrow \quad \swarrow{\scriptstyle r_H^*} & \\ & H_T^*(X^T) & \end{array}$$

Theorem 2.2.3. *The image of i^* is the set*

$$\bigcap_{H \subset T} r_H^*(H_T^*(X^H)) \tag{2.19}$$

where the intersection is taken over all codimensional one subtori H of T which occur as identity components of isotropy groups.

Proof. See Theorem 11.5.1 [12]. □

Remark 2.2.5. *Section 2.3 and 2.4 are not necessary for understanding the remainder of the thesis.*

2.3 Hamiltonian Actions and Moment map

In symplectic geometry, F. Kirwan [19] studied Hamiltonian action $\mathbb{S}^1$ on symplectic manifolds X. She proved that if a compact, connected Lie group G acts on a symplectic manifold in a Hamiltonian fashion, then $H_G^*(X)$ is a free $H^*(BG)$-module (Proposition 5.8 in [19]).

To introduce these results, we need some of the basic concepts in equivariant symplectic geometry.

A *symplectic manifold* is a pair (X, ω) where X is a smooth $2n$-manifold and ω is a 2-form, which is *closed* and *nondegenrate*. Let G be a compact, connected Lie group with Lie algebra $\mathfrak{g}$. For any $\xi \in \mathfrak{g} \cong T_eG$, the corresponding right-invariant vector field $\xi^{\#}$ on X defined by

$$\xi^{\#}(p) = \frac{d}{dt}(e^{t\xi}.p)|_{t=0}$$

If there exist a moment G-map $\psi : X \to \mathfrak{g}^*$, and

$$d\psi^{\xi} = i_{\xi^{\#}}\omega$$

where $\psi^{\xi} : X \to \mathbb{R}$ defined by $p \mapsto \psi_p(\xi)$. This action is called *Hamiltonian* action and ψ is *moment* map.

The next result due to F. Kirwan [19].

Theorem 2.3.1. *If X is a compact symplectic manifold and the action is Hamiltonian, then X is equivariantly formal.*

As a consequence of Theorem 2.3.1, we have:

Corollary 2.3.1. *Suppose a torus T acts on a compact symplectic manifold X and the action is Hamiltonian, and let X^T be the set of fixed points with inclusion map $i : X^T \to X$. Then the induced map in equivariant cohomology*

$$i^* : H_T^*(X) \longrightarrow H_T^*(X^T)$$

is injective.

Suppose that $\psi : X \to \mathfrak{t}^*$ is a moment map and *zero* is a regular value of ψ. The preimage of zero, $\psi^{-1}(0)$ is a submanifold of X and T acts freely on $\psi^{-1}(0)$. Then the orbit space $\psi^{-1}(0)/T$ is a manifold, this space is called the reduced space and is denoted by $X//T$. The inclusion map $\kappa : \psi^{-1}(0) \hookrightarrow X$ induces a map in equivariant cohomology

$$\kappa^* : H_T^*(M) \longrightarrow H_T^*(\psi^{-1}(0)) \tag{2.20}$$

F.Kirwan [19] studied the surjectivity of this map, and she obtained the following result:

Theorem 2.3.2. *Let (X, ω, ψ) be a T-Hamiltonian manifold, zero a regular value of the moment map ψ, and a torus T act freely on $\psi^{-1}(0)$. Then the induced map*

$$\kappa^* : H_T^*(M) \longrightarrow H_T^*(\psi^{-1}(0))$$

is surjective.

Also, Tolman and Weitsman [28] computed the kernel of the map κ^* and proved Chang-Skjelbred theorem by using the concept of i-skeleton, their construction relates the equivariant cohomology of X_1 and equivariant cohomology of the T-space X. Since we have the inclusion maps $j : X_0 \hookrightarrow X$ and $i : X_0 \hookrightarrow X_1$, we obtain the commutative diagram:

$$\begin{array}{ccc} H_T^*(X) & \longrightarrow & H_T^*(X_1) \\ & {}_{i^*}\searrow \quad \swarrow_{j^*} & \\ & H_T^*(X_0) & \end{array}$$

such that $im(j^*) = im(i^*)$.

2.4 GKM-Manifolds

GKM theory has been introduced to compute the torus equivariant cohomology on some *nice* T-spaces called **GKM-manifolds**. A great achievement due to M. Goresky, R. Kottwitz, and R. MacPherson [10] since they proposed a powerful technique for computing $H_T^*(X)$ is called GKM-theory. It works under certain assumptions, namely, the fixed point set X_0 is finite and X_1 is 2-dimensional, and X is an orientable space.

In this section we recall the main definitions and results about GKM-manifolds and the GKM-graph Γ which comes from that manifold. The i-skeleton X_i of X is the set $X_i = \{x \in X : dim(T_x) \geq r - i\}$, where T_x is the stabilizer of x.

2.4.1 GKM-Spaces and GKM-Graphs

Let X be an oriented, compact, connected manifold with an effective smooth action of a torus T^n. Let T acts on a manifold X, and T_p is the corresponding isotropy subgroups at the point $p \in X$, then the isotropy representation $Is_p : T_p \to GL(T_pX)$ associates with each $h \in T_p$ the differential $Is_p(h) = dh_p$ of the transformation h at p

Definition 2.4.1. *We say that X is a* ***GKM-manifold*** *if the following conditions are satisfies:*
i) $H_T^*(X)$ is a free R-module.
ii) X^T is finite.
iii) For every $p \in X^T$ the weights

$$\{\alpha_{i,p}\}, i = 1, ..., m$$

of the isotropy representation of T on T_p are pairwise linearly independent.

The condition iii) is equivalent to the following: For every codimension one subtorus H of T, the dimension of connected components X_H of X^H are at most two dimensional. If the dimension of X_H equals two, then it is diffeomorphic to a two dimensional sphere $\mathbb{S}^2$, and the diffeomorphism conjugates the action of T/H with the standard $\mathbb{S}^1$ action on $\mathbb{S}^2$ given by the rotation about the z axis. The Lie algebra of the codimension one subtorus is given by

$$\mathfrak{h} = \ker(\alpha) = \{\xi \in \mathfrak{t} | \; \alpha(\xi) = 0\} \tag{2.21}$$

the inclusion map $\mathfrak{h} \hookrightarrow \mathfrak{t}$ induces the map

$$r_H : S(\mathfrak{t}^*) \to S(\mathfrak{h}^*). \tag{2.22}$$

The next theorem gives a description of the equivariant cohomology in terms of elements in $H_T^*(X^T)$.

Theorem 2.4.1. *The equivariant cohomology ring $H_T^*(\mathbb{S}^2)$ is the subring of $S(\mathfrak{t}^*) \bigoplus S(\mathfrak{t}^*)$ consisting of all pairs:*

$$(f_1, f_2) \in S(\mathfrak{t}^*) \bigoplus S(\mathfrak{t}^*) \tag{2.23}$$

satisfying

$$f_1 - f_2 = \alpha \cdot \lambda \quad \textit{for some } \lambda \in S(\mathfrak{t}^*). \tag{2.24}$$

Proof. See Theorem 11.7.2 [12]. □

Let $\{H_i\}$ be the collection of codimension one subtori of T, the connected components of X^{H_i} are two-spheres or points, each two dimension sphere intersects with X^T at two points. This can be encoded in a graph, called the **GKM graph** $\Gamma = (\mathcal{V}^\Gamma, \mathcal{E}^\Gamma)$ whose edges $\mathcal{E}^\Gamma$ are the connected components labelled by $\alpha : T \longrightarrow \mathbb{S}^1$, such that T acts by rotation through α, and the set of vertices $\mathcal{V}^\Gamma$ is given by the fixed point set X^T. Let $\Gamma(\mathcal{V}, \mathcal{E})$ be a GKM-graph, for any $e \in \mathcal{E}$ we will denote the initial and terminal vertex $i(e)$ and $t(e)$, respectively[2]. The edge e^{-1} represents the edge e with opposite direction, that is, $i(e) = t(e^{-1})$ and $t(e) = i(e^{-1})$.

Recall that the weight lattice $\Lambda := Hom(T, \mathbb{S}^1) \cong \mathbb{Z}^r$, embeds naturally into $H^2(BT)$.

[2]We assume that the graph has no loop, i.e $i(e) \neq t(e)$

An axial function is a map:

$$\mathcal{E}^{\Gamma} \to Hom(T, \mathbb{S}^1)$$
$$e \longrightarrow \alpha_e$$

that satisfies the following conditions:

- $\alpha_e = -\alpha_{e^{-1}}$
- Let $\mathcal{E}_p^{\Gamma} = \{e \in \mathcal{E}^{\Gamma} |\, i(e) = p\}$, for any vertex, the set $\{\alpha_e |\, e \in \mathcal{E}_p^{\Gamma}\} \subseteq H^2(BT)$ is pairwise linearly independent. where $\mathcal{E}_p^{\Gamma} = \{e \in \mathcal{E}^{\Gamma}\; i(e) = p\}$.

Now, let $\{H_i\}$ be a family of codimesion one subtori of T, and X_{H_i} is the connected component of X^{H_i}. We have already known the connected components $X_{H_i} \cong \mathbb{S}^2$ or a point and $\ker(\alpha_i)$ is a Lie algebra, for some $\alpha_i \in \mathfrak{t}^*$. Clearly, that we can find the equivariant cohomology for each component X_{H_i} as follows:

$$r^*_{H_i} : H^*_{\mathbb{S}^1}(\mathbb{S}^2) \longrightarrow H^*_{\mathbb{S}^1}(\{N, S\})$$

where $\{N, S\}$ are fixed points. Any element $f \in H^*_{\mathbb{S}^1}(\{N, S\})$ in the image of $r^*_{H_i}$ satisfies the following property:

$$f_N - f_S \in x \cdot \mathbb{C}[x].$$

M. Goresky, R. Kottwitz, R. MacPherson presented a combinatorial description of equivariant cohomology of X using the description of $H^*_T(X_H)$

Theorem 2.4.2. *Let X ba a compact, symplectic manifold with a Hamiltonian action of a compact torus T. Each connected component X_H has dimension 0 or 2 and $X^T = \{p_1, ..., p_n\}$ are fixed points. Let $\pi_H : \mathfrak{t}^* \to \mathfrak{h}^*$ induced by the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{t}$. Then the map*

$$i^* : H^*_T(X) \to H^*_T(X^T)$$

has image $(f_1, ...f_n)$ *such that*

$$\pi_H(f_i) = \pi_H(f_j) \tag{2.25}$$

whenever $\{p_i, p_j\} = X_H \cap X^T$, *for some path component* X_H *of* X^H.

Proof. See [10]. □

In fact, Guillemin and Zara [13] stated Theorem (2.4.2) in terms of GKM-graph as:

Theorem 2.4.3. *Given a GKM-manifold* X, *let* $H_T^*(X)$ *be the equivariant cohomology ring of* X. *Let* Γ *be the GKM-graph associated to* X, *and* $H_T^*(\Gamma, \alpha)$ *the cohomology ring of* Γ. *Then as rings, we have*

$$H_T^*(X) \cong H_T^*(\Gamma, \alpha). \tag{2.26}$$

such that $H_T^*(\Gamma, \alpha)$ *is given by*

$$H_T^*(\Gamma, \alpha) = \{f : \mathcal{V}^\Gamma \to H^*(BT) \mid f(p) - f(q) \equiv 0 \, mod \, \alpha(pq)\} \tag{2.27}$$

where $pq \in \mathcal{E}^\Gamma$ *and* $\alpha(pq) \in H^2(BT)$.

Observe that the graph cohomology $H_T^*(\Gamma)$ is unchanged if we replace $\alpha(pq)$ with $\lambda \cdot \alpha(pq)$ for a non-zero scalar $\lambda \in H^*(BT)$. In what follows, we will consider weights equivalent if they are scalar multiples each other.

Remark 2.4.1. *For all what we discussed before we considered the case that the* $\dim X_H \leq 2$, *for more general situation. R. Goldin and T. Holm [9] restate Theorem (2.4.2) in the case in which* $\dim X_H \leq 4$.

Theorem 2.4.4. *Let* X *be a compact, connected symplectic manifold with an effective Hamiltonian* T *action and has fixed points* $X^T = \{p_1, ..., p_n\}$. *In case* $\dim X_H \leq 4$

for all $H \subset T$ of codimesion one subtorus. Let $f_i \in H_T^*(pt)$ denote the restriction of $f \in H_T^*(X)$ to the fixed point P_i. Then

$$i^* : H_T^*(X) \to H_T^*(X^T)$$

has image $(f_1, ...f_n) \in S(\mathbf{t}^*)$ which satisfiy

$$\begin{aligned} &\pi_H(f_{i_j}) = \pi_H(f_{i_k}) && \text{if } \{p_{i_1}, ..., p_{i_l}\} = X_H^T \\ &\text{and } \Sigma_{j=1}^{l} \frac{f_{ij}}{\alpha_1^{i_j} \alpha_2^{i_j}} && \text{if } \{p_{i_1}, ..., p_{i_l}\} = X^T \text{ and } \dim X_H \leq 4 \end{aligned}$$

where $\alpha_1^{i_j}$ and $\alpha_2^{i_j}$ are the linearly independent weights of the T action on $T_{p_{ij}} X_H$

Proof. See [9]. □

2.4.2 Examples

Example 2.4.1. *Let $\mathbb{S}^2$ be a 2-dimensional sphere. And $\mathbb{S}^1$ acts on $\mathbb{S}^2$ by rotation. The standard action of $\mathbb{S}^1$ on $\mathbb{S}^2$ has two fixed points "north and south". And $\mathbb{S}^2$ is an equivariantly formal space.*

We have two fixed points and the corresponding weights as follows:

Fixed points	weights
$p_1 = (1, 0, 0)$	x
$p_2 = (0, 0, 1)$	$-x$

The GKM-graph associated to this action is:

Recall Theorem 2.4.3, the image of the equivariant cohomology $H_{\mathbb{S}^1}^*(\mathbb{S}^2)$ in $H_{\mathbb{S}^1}^*(\{p_1, p_2\}) \cong \mathbb{C}[x] \oplus \mathbb{C}[x]$, generated by the pair functions (f_1, f_2) such that

$$f_1 - f_2 \in x \cdot \mathbb{C}[x]$$

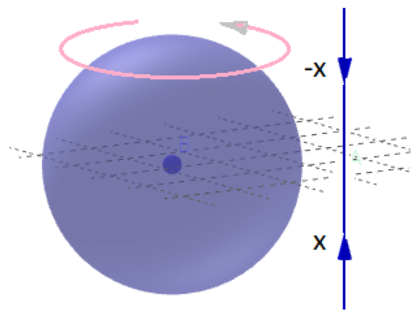

Figure 2.1: $\mathbb{S}^1 \curvearrowright \mathbb{S}^2$

Example 2.4.2. ***Complex projective plane***. *Let T^n act on $\mathbb{C}P^n$, and the action defined by*

$$\psi : T^n \times \mathbb{C}P^n \longrightarrow \mathbb{C}P^n$$

$$((t_1, t_2, \ldots .t_n), [z_0 : z_1 : \ldots .z_n]) \to [z_0 : t_1 z_1 : \ldots .t_n z_n].$$

The GKM-graph in this example is the complete graph of $n+1$ vertices. Consider the case $n = 2$.

The 0-dimensional orbits, which are the three points $X_0 = (\mathbb{C}P^2)^T = \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$, and the set of 1-dimensional orbits are $X_1 = \{[0 : z_1 : z_2], [z_0 : 0 : z_2], [z_0 : z_1 : 0]\} \cong \mathbb{C}P^1 \cup \mathbb{C}P^1 \cup \mathbb{C}P^1$. The orbit space X_1/T yields three edges.

Fixed points	weights
$p_1 = [1 : 0 : 0]$	$x,\ y$
$p_2 = [0 : 1 : 0]$	$-y,\ x - y$
$p_2 = [0 : 0 : 1]$	$-x,\ -x + y$

The image of the equivariant cohomology $H^*_{T^2}(\mathbb{C}P^2)$ in $H^*_{T^2}(\{p_1, p_2, p_3\}) \cong \bigoplus_{i=1}^{3} \mathbb{C}[x, y]$ generated by the triple functions (f_1, f_2, f_3) such that

$$f_1 - f_2 \in (y) \cdot \mathbb{C}[x, y]$$

$$f_1 - f_3 \in (x) \cdot \mathbb{C}[x, y]$$

$$f_2 - f_3 \in (y - x) \cdot \mathbb{C}[x, y]$$

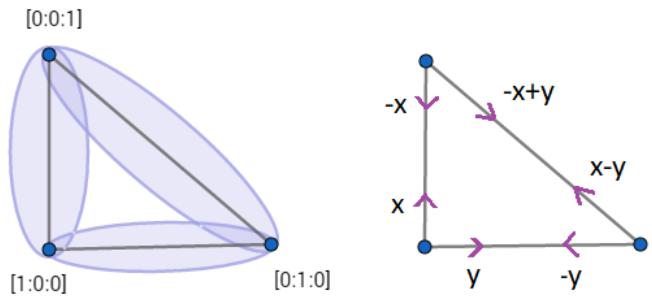

Figure 2.2: The GKM-graph associated to $T^2 \curvearrowright \mathbb{C}P^2$.

Example 2.4.3. *Let $\mathbb{S}^1$ acts on $\mathbb{C}P^2$ and the action defined by:*

$$\psi : \mathbb{S}^1 \times \mathbb{C}P^2 \longrightarrow \mathbb{C}P^2$$

$$(e^{i\theta}, [z_0 : z_1 : z_2]) \to [e^{i\theta} z_0 : z_1 : e^{-i\theta} z_2].$$

We have two fixed points and the corresponding weights as follows:

Fixed points	Weights
$p_1 = [1 : 0 : 0]$	$x,\ 2x$
$p_2 = [0 : 1 : 0]$	$-x,\ x$
$p_3 = [0 : 0 : 1]$	$-2x,\ -x$

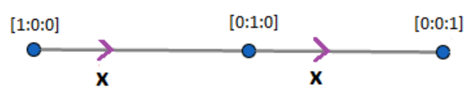

Figure 2.3: The GKM-graph associated to $\mathbb{S}^1 \curvearrowright \mathbb{C}P^2$.

Recall Theorem 2.4.4 the image of the equivariant cohomology $H^*_{\mathbb{S}^1}(\mathbb{C}P^2)$ in $H^*_{\mathbb{S}^1}(\{p_1, p_2, p_3\}) \cong \bigoplus_{i=1}^{3} \mathbb{C}[x]$ generated by the triple functions (f_1, f_2, f_3) such that
$f_i - f_j \in x \cdot \mathbb{C}[x] \ \ \forall\, i, j \in \{1, 2, 3\}$ and $\dfrac{f_1}{(2x)(x)} + \dfrac{f_2}{(-x)(x)} + \dfrac{f_3}{(-2x)(-x)} \in \mathbb{C}[x]$.

Chapter 3

Sheaf Theory

As an extension of the GKM theory, T. Baird [3] developed a new class of sheaves, called GKM-sheaves. Under the condition of equivariantly formality he proved that the global sections of the GKM-sheaf on a hypergraph is isomorphic to the equivariant cohomology of smooth, compact T-manifold X.

3.1 Basics on Sheaves

Definition 3.1.1. *Let X be a topological space and let $\mathbf{C}$ be an abelian category, in our use $\mathbf{C}$ will be the category of graded R-modules. A presheaf on X with values in $\mathbf{C}$ assigns to every open set $U \subset X$ an object $\mathcal{F}(U)$ in $\mathbf{C}$ and to every pair of open sets (U, V) with $V \subset U$ a morphism*

$$\rho^U_V : \mathcal{F}(U) \longrightarrow \mathcal{F}(V) \tag{3.1}$$

which is called the restriction map, such that

- *For every open set $U \subset X$ we have $\rho^U_U = id$;*

- *For all open sets $U, V, W \subset X$ such that $U \subset V \subset W$, we have*

$$\rho_U^V \circ \rho_V^W = \rho_U^W. \tag{3.2}$$

For any open set $U \subset X$ an element $s \in \mathcal{F}(U)$ is called a local section of $\mathcal{F}$ over U and if an open set $W \subset U$, then we denote $\rho_W^U(s)$ by $s|_W$. We call it the restriction of s to W.

We can rewrite Definition 3.1.1 of presheaf as follows:

Definition 3.1.2. *Given a topological space X and an abelian category $\mathbf{C}$. A presheaf on X with values in $\mathbf{C}$ consists of an assignment of some objects $\mathcal{F}(U)$ in $\mathbf{C}$ to every open subset U of X and of a map*

$$\mathcal{F}(i) : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$

to every inclusion $i : V \to U$ of open subsets $V \subset U \subset X$ such that $\mathcal{F}(id_U) = id_{\mathcal{F}(U)}$ and $\mathcal{F}(i \circ j) = \mathcal{F}(j) \circ \mathcal{F}(i)$ for any two inclusions $i : V \to U$ and $j : W \to V$, with $W \subset V \subset U$.

The above definition says that a presheaf is contravariant functor from the category of open sets of X to $\mathbf{C}$. Some examples of presheaves:

Example 3.1.1. *The constant presheaf $\mathcal{A}_X$ with value in $\mathcal{A} \in \mathbf{C}$, defined as: $\mathcal{A}_X(U) = \mathcal{A}$ for all open subsets U of X, and ρ_V^U is the identity function of $\mathcal{A}$ for all open subsets U, V such that $V \subset U$.*

Example 3.1.2. *Let $X = \mathbb{R}^n$ with usual metric topology. The functor $C_X^0(U) := \{f : U \to \mathbb{R}|\ f \text{ is continuous }\}$ is a presheaf of abelian groups such that $(f+g)(x) = f(x) + g(x)$, where $x \in U$ and $f, g \in C_X^0(U)$. For any pair of open subsets (U, V), $V \subset U$ and the restriction map: $\rho_V^U : C_X^0(U) \longrightarrow C_X^0(V)$ defined as $\rho_V^U(f) = f|_V$.*

Example 3.1.3. *Similarly, let M be a smooth manifold. The functor*

$C^\infty_M(U) := \{f : U \to \mathbb{R} | \ f \text{ is } C^\infty\}$ *is a presheaf of abelian groups.*

We define a morphism $\phi : \mathcal{F} \to \mathcal{G}$ between two presheaves $\mathcal{F}$ and $\mathcal{G}$ on X to consist of a family of maps $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for any subset U of X such that the following diagram commutes for every open subsets U, V such that $V \subset U \subset X$

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \\ {\scriptstyle (\rho_{\mathcal{F}})^U_V}\downarrow & & \downarrow{\scriptstyle (\rho_{\mathcal{G}})^U_V} \\ \mathcal{F}(V) & \xrightarrow[\phi_V]{} & \mathcal{G}(V) \end{array} .$$

Definition 3.1.3. *A sheaf is a presheaf $\mathcal{F}$ whose sections are determined by local data in the following sense:*

- *(Uniqueness property) If U is an open set in X, $\{U_i\}_{i\in I}$ a family of open sets covering U,i.e., $U = \cup_{i\in I} U_i$ and if $s, t \in \mathcal{F}(U)$ such that $\rho^U_{U_i}(s) = \rho^U_{U_i}(t)$ for all $i \in I$, then $s = t$.*

- *(Gluing property) For any family $\{U_i\}_{i\in I}$ of open subsets of X, and for any family of sections $\{s_i\}_{i\in I}$ where $s_i \in \mathcal{F}(U_i)$ with the property that for each $i, j \in I$,*

$$\rho^{U_i}_{U_i\cap U_j}(s_i) = \rho^{U_j}_{U_j\cap U_j}(s_j) \tag{3.3}$$

then there is a unique $s \in \mathcal{F}(U)$, such that $\rho^U_{U_i}(s) = s_i$ for all $i \in I$.

The presheaves in Examples 3.1.2 and 3.1.3 are sheaves. But Example 3.1.1 illustrates that a presheaf may not be a sheaf (does not possess the gluing property). This leads to construct the sheaf $\widetilde{\mathcal{F}}$ associated to a presheaf $\mathcal{F}$, called the ***sheafification*** of $\mathcal{F}$. For the purpose of constructing $\widetilde{\mathcal{F}}$ we need to define the concept of ***stalks***. We can think of the stalk $\mathcal{F}_x$ as capturing the local properties of a presheaf near (around) a point $x \in X$

Definition 3.1.4. *If $\mathcal{F}$ is a presheaf on X with values in an abelian category* $\mathbf{C}$*, we define the stalk $\mathcal{F}_x$ at $x \in X$ as follows:*

$$\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U) \tag{3.4}$$

where the direct limit is taken over open subsets U of X such that $x \in U$.

Given $s \in \mathcal{F}(U)$ determines an element $s_x \in \mathcal{F}_x$. To be precise, consider $\langle s, U \rangle$ as a pair, where U is a neighbourhood of x and s is section in $\mathcal{F}(U)$. Now, let us define the relation $\sim$ between two pairs as the following: $\langle s, U \rangle \sim \langle s', U' \rangle$ if there exists a neighbourhood V at x, $V \subset U \cap U'$ and

$$\rho_V^U(s) = \rho_V^{U'}(s').$$

Then, $\mathcal{F}_x$ is the set of equivalence classes of $\langle s, U \rangle$ with respect to the relation $\sim$.

This equivalence class is called a ***germ*** at x. If $\mathcal{F}$ takes values in abelian groups, then we can define addition on $\mathcal{F}_x$, let $[s]_x, [t]_x \in \mathcal{F}_x$ such that $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$, let $W \subset U \cap V$ be an open set in X. We define the addition as follows:

$$[s]_x + [t]_x = [(\rho_W^U)(s) + (\rho_W^V)(t)]. \tag{3.5}$$

This operation is well-defined and $\mathcal{F}_x$ is an abelian group which follows from the definition (3.4). A morphism $\phi : \mathcal{F} \to \mathcal{G}$ between two presheaves $\mathcal{F}, \mathcal{G}$ on X induces a map of stalks

$$\phi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x \tag{3.6}$$

defined as

$$\phi_x([s]_x) = (\phi_U(s))_x \tag{3.7}$$

for all $x \in X$, where $[s]_x \in \mathcal{F}_x$ is the equivalence class of some $s \in \mathcal{F}(U)$ with some open subset U of X containing x.

Now, we are ready to explain the concept of *sheafification*. For any presheaf $\mathcal{F}$ on a topological space X, we construct the sheaf $\widetilde{\mathcal{F}}$ associated to $\mathcal{F}$ by using the following steps:

Step 1 Define the stalk space $\coprod_{x\in X} \mathcal{F}_x$ and, called **etale space**, denoted as:

$$Et(\mathcal{F}) = \coprod_{x\in X} \mathcal{F}_x.$$

Step 2 Define a topology on $Et(\mathcal{F})$ by using the basic open sets of the form

$$B_s(U) = \{[s]_x \mid x \in U\}$$

for all open $U \subseteq X$ and $s \in \mathcal{F}(U)$.

Step 3 We have the projection map:

$$p : \coprod_{x\in X} \mathcal{F}_x \longrightarrow X$$

defined by $p([s]_x) = x$.

Step 4 For every open set $U \subseteq X$ the sheaf $\widetilde{\mathcal{F}}$ associated to $\mathcal{F}$ is:

$$\widetilde{\mathcal{F}}(U) := \{\widetilde{s} : U \longrightarrow \textstyle\coprod_{x\in X} \mathcal{F}_x | \ p \circ \widetilde{s} = id_U \text{ and } \widetilde{s} \text{ is continuous on } U\}.$$

That is, $\widetilde{\mathcal{F}}(U)$ is the set of continuous sections of p over U. Given any presheaf $\mathcal{F}$ on a topological space X, there is a natural morphism $\eta : \mathcal{F} \longrightarrow \widetilde{\mathcal{F}}$ having the following universal property.

Proposition 3.1.1. *For any sheaf $\mathcal{G}$, and any morphism $\phi : \mathcal{F} \longrightarrow \mathcal{G}$, there is a unique* *morphism* $\widetilde{\phi} : \widetilde{\mathcal{F}} \longrightarrow \mathcal{G}$ *such that* $\widetilde{\phi} \circ \eta = \phi$. *Equivalently, the diagram below commute:*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\eta} & \widetilde{\mathcal{F}} \\ & \overset{\phi}{\searrow} & \downarrow{\scriptstyle \widetilde{\phi}} \\ & & \mathcal{G} \end{array}$$

If we have a morphism $\phi : \mathcal{F} \to \mathcal{G}$ of presheaves on a topological space X,

Definition 3.1.5. *The kernel of* ϕ *is defined as follows: for every open subset* U *of* X*,*

$$(\ker \phi)(U) = \{s \in \mathcal{F}(U) \mid \phi_U(s) = 0\}. \tag{3.8}$$

Definition 3.1.6. *If* $\phi : \mathcal{F} \to \mathcal{G}$ *is a morphism of presheaves on a space* X*, for every open subset* U *of* X *we define* $(\mathrm{im}\phi)(U)$ *by*

$$(\mathrm{im}\phi)(U) = \{t \in \mathcal{G}(U) \mid (\exists s \in \mathcal{F}(U))(\phi_U(s) = t)\}. \tag{3.9}$$

Moreover, the cokernel is defined by

$$coker\, \phi(U) = \mathcal{G}(U) \Big/ (\mathrm{im}\phi)(U). \tag{3.10}$$

Remark 3.1.1. *If* $\phi : \mathcal{F} \to \mathcal{G}$ *is a morphism of sheaves on a topological space* X*, then* $\ker \phi$ *is a sheaf. In general, the image and cokernel are presheaves, but may not be sheaves.*

Definition 3.1.1. *If* $\phi : \mathcal{F} \to \mathcal{G}$ *is a morphism of sheaves on a topological space* X*, then the cokernel sheaf* $\widetilde{coker}(\phi)$ *is the sheafification of the cokernel presheaf* $coker(\phi)$.

Recall that a morphism of presheaves on X induces a map of stalks.

Proposition 3.1.2. *The morphism* $\phi : \mathcal{F} \to \mathcal{G}$ *of sheaves is an isomorphism if and only if the map of stalks* $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ *is an isomorphism for every* $x \in X$.

Proof. See Proposition 1.1 in [16].

□

Definition 3.1.7. *The sequence of sheaves*

$$\cdots \to \mathcal{F}^{i-1} \xrightarrow{\phi^{i-1}} \mathcal{F}^{i} \xrightarrow{\phi^{i}} \mathcal{F}^{i+1} \to \cdots \to \dots \tag{3.11}$$

is exact if and only if for each $x \in X$ the corresponding sequence of stalks

$$\cdots \to \mathcal{F}_x^{i-1} \xrightarrow{\phi_x^{i-1}} \mathcal{F}_x^{i} \xrightarrow{\phi_x^{i}} \mathcal{F}_x^{i+1} \to \cdots \to \dots \tag{3.12}$$

is exact as a sequence of abelian groups.

3.2 The Godement Resolution

In this section, we will explain a particular resolution of sheaves called the Godement resolution and use it to compute the sheaf cohomology.

Given a sheaf $\mathcal{F}$ on X, we construct sheaves $\mathcal{F}^n$ and $C^n\mathcal{F}$ for all $n \geqslant 0$ as follows. First $\mathcal{F}^0 = \mathcal{F}$, and the sheaf $C^0\mathcal{F}$ on X is given by

$$C^0\mathcal{F}(U) = \prod_{x \in U} \mathcal{F}_x \tag{3.13}$$

for all open sets $U \subseteq X$ with the obvious restriction morphisms.

There is a natural injective homomorphism of sheaves

$$\mathcal{F} \longrightarrow C^0\mathcal{F}. \tag{3.14}$$

Let $\mathcal{F}^1$ denote the cokernel sheaf of $\mathcal{F} \to C^0\mathcal{F}$. Inductively, define $\mathcal{F}^n$ to be the cokernel sheaf of $\mathcal{F}^{n-1} \to C^0\mathcal{F}^{n-1}$, and define $C^n\mathcal{F} = C^0\mathcal{F}^n$. We get short exact sequences of sheaves

$$0 \to \mathcal{F}^n \xrightarrow{\alpha} C^n\mathcal{F} \xrightarrow{\beta} \mathcal{F}^{n+1} \to 0 \tag{3.15}$$

for all $n \geq 0$.

Let $d = \alpha \circ \beta$ be the composition,

$$C^n\mathcal{F} \xrightarrow{\beta} \mathcal{F}^{n+1} \xrightarrow{\alpha} C^{n+1}\mathcal{F} \tag{3.16}$$

Theorem 3.2.1. *The sequence of sheaves*

$$0 \to \mathcal{F} \to C^0\mathcal{F} \xrightarrow{d} C^1\mathcal{F} \xrightarrow{d} \dots \xrightarrow{d} C^n\mathcal{F} \xrightarrow{d} \cdots \tag{3.17}$$

is exact.

Proof. See [18]. □

The sequence (3.17) is called **Godement resolution**, denoted by $C^\bullet\mathcal{F}$.

3.2.1 Cohomology of Sheaves

Given a topological space X, denote by $\mathbf{Sh}(X)$ the category of sheaves taking values in an abelian category $\mathbf{C}$. The global section functor is

$$\Gamma(X, -) : \mathbf{Sh}(X) \longrightarrow \mathbf{C}$$

defined by $\Gamma(X, \mathcal{F}) = \mathcal{F}(X)$.

We define the sheaf cohomology by using Godement resolution. Applying the global sections determines a cochain complex in $\mathbf{C}$:

$$0 \to \Gamma(C^0\mathcal{F}) \xrightarrow{d_0} \Gamma(C^1\mathcal{F}) \xrightarrow{d_1} \dots . \xrightarrow{d_n} \Gamma(C^n\mathcal{F}) \to \cdots \tag{3.18}$$

that is, $d_n \circ d_{n-1} = 0$, for all $n \geq 0$.

Define

$$H^i(X, \mathcal{F}) = \frac{\ker(d_i)}{\operatorname{im}(d_{i-1})}.$$

Because the global section functor is left exact,

$$0 \to \Gamma(\mathcal{F}) \to \Gamma(C^0\mathcal{F}) \xrightarrow{d_0} \Gamma(C^1\mathcal{F})$$

is exact, which implies $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

Given an open set $U \subseteq X$, we get a cochain complex

$$0 \to \Gamma(U, C^0\mathcal{F}) \xrightarrow{\widetilde{d_0}} \Gamma(U, C^1\mathcal{F}) \xrightarrow{\widetilde{d_1}} \ldots . \xrightarrow{\widetilde{d_n}} \Gamma(U, C^n\mathcal{F}) \to \cdots . \tag{3.19}$$

We define the cohomology of a sheaf $\mathcal{F}$ restricted to U as

$$H^i(U, \mathcal{F}) = \frac{\ker(\widetilde{d_i})}{\operatorname{im}(\widetilde{d_{i-1}})}.$$

Given a closed subset $A \subseteq X$, we can define the local cohomology groups $H^i_A(\mathcal{F})$ as follows:

Definition 3.2.1. *For a sheaf $\mathcal{F}$ on X and closed $A \subseteq X$ we define*

$$\Gamma_A(X, \mathcal{F}) = \{s \in \Gamma(X, \mathcal{F}) \mid supp(s) \subseteq A\} \tag{3.20}$$

where $supp(s) = \{x \in X \mid s_x \in \mathcal{F}_x,\ \ s_x \neq 0\}$.

By analogy with the above construction, we define the sheaf local cohomology. By applying this functor (3.20) to the Godement resolution determines a chain complex of abelian groups

$$0 \to \Gamma_A(C^0\mathcal{F}) \xrightarrow{d_0} \Gamma_A(C^1\mathcal{F}) \xrightarrow{d_1} \ldots . \xrightarrow{d_n} \Gamma_A(C^n\mathcal{F}) \to \cdots . \tag{3.21}$$

Define

$$H^i_A(X, \mathcal{F}) = \frac{\ker(d_i)}{\operatorname{im}(d_{i-1})}.$$

The next proposition is a crucial fact in this section, and we will use it in chapter four.

Proposition 3.2.1. *Let A be closed in X. A sheaf $\mathcal{F}$ on X gives rise to a long exact sequence*

$$0 \to H^0_A(X, \mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(X - A, \mathcal{F}) \xrightarrow{\delta} H^1_A(X, \mathcal{F}) \to H^1(X, \mathcal{F}) \to H^1(X - A, \mathcal{F}) \to \cdots$$

Proof. See Proposition 9.2 in [18]. □

3.3 GKM-Sheaves

T. Baird [3] generalized the GKM-theory to all smooth, compact T-manifolds X. He introduced the notion of an abstract GKM-sheaf $\mathcal{F}$ over a GKM-hypergraph Γ, and a GKM-sheaf associated to a smooth T-manifold X, denoted by $\mathcal{F}_X$, and proved that if X is an equivariantly formal space, then

$$H^*_T(X) \cong H^0(\mathcal{F}_X).$$

In the following subsection, we review this theory.

3.3.1 GKM-Hypergraphs and GKM-Sheaves

We begin with the definition of the GKM-hypergraph and some of its properties. Let

$$\Lambda := Hom(T, \mathbb{S}^1)$$

defined as in (2.7) and define the *projective weights* $\mathbb{P}(\Lambda)$ to be the set of non-zero weights in Λ modulo scalar multiplication. The elements of $\mathbb{P}(\Lambda)$ are in one to one correspondence with the codimension one subtori of T by the rule

$$\alpha \in \mathbb{P}(\Lambda) \leftrightarrow \ker_\circ(\tilde{\alpha}) \subset T$$

where $\tilde{\alpha} \in \Lambda$ is a representative of α, and $\ker_\circ(\tilde{\alpha})$ is the identity component of the kernel of $\tilde{\alpha} : T \longrightarrow S^1$. We denote $\ker(\alpha) = \ker_\circ(\tilde{\alpha})$.

Definition 3.3.1. *A* ***GKM-hypergraph*** Γ *consists of:*

1. *A finite set of vertices* $\mathcal{V} = \mathcal{V}_\Gamma$.

2. *An equivalence relation* $\sim_\alpha$ *on* $\mathcal{V}$ *for each* $\alpha \in \mathbb{P}(\Lambda)$.

Example 3.3.1. *To a given compact T-manifold X, we associate a GKM-hypergraph Γ_X as follows: $\mathcal{V}_X = \mathcal{V}_{\Gamma_X}$ is the set of path components of the fixed point set X^T, and the equivalence relations are defined as follows: if $v_1, v_2 \in \mathcal{V}_X$ then $v_1 \sim_\alpha v_2$ if and only if they are in the same path component of $X^{\ker(\alpha)}$, the fixed point set of* $\ker(\alpha)$.

Given a GKM-hypergraph Γ, the set of hyperedges is defined to be

$$\mathcal{E} = \mathcal{E}_\Gamma := \{(S, \alpha) \in \wp(\mathcal{V}) \times \mathbb{P}(\Lambda) \mid S \textit{ is an equivalence class for } \sim_\alpha\}$$

where $\wp(\mathcal{V})$ is the power set of $\mathcal{V}$.

Projection defines maps

- $\alpha : \mathcal{E} \to \mathbb{P}(\Lambda)$ the *axial function*, and

- $I : \mathcal{E} \to \wp(\mathcal{V})$ the *incidence function*.

Thus, each hyperedge $e \in \mathcal{E}_\Gamma$ has associated to it a projective weight $\alpha(e)$ and a non-empty subset $I(e) \subseteq \mathcal{V}$. We say a vertex $v \in \mathcal{V}$ is *incident* to $e \in \mathcal{E}$ if $v \in I(e)$. Define $\mathrm{Top}(\Gamma)$ to be the topological space on the set $\mathcal{V}_\Gamma \cup \mathcal{E}_\Gamma$ generated by basic open sets $U_v = \{v\}$ for $v \in \mathcal{V}_\Gamma$, and $U_e = \{e\} \cup I(e)$ for $e \in \mathcal{E}_\Gamma$.

Definition 3.3.2. *[3] A GKM-sheaf $\mathcal{F}$ is a sheaf of finitely generated, $\mathbb{Z}$-graded R-modules*[1] *over Top*(Γ)*, satisfying the following conditions.*

1. *$\mathcal{F}$ is locally free (i.e, for every basic open set U_x, the stalk $\mathcal{F}(U_x) = \mathcal{F}_x$ is a free R-module).*

2. *For every hyperedge $e \in \mathcal{E}_\Gamma$, the restriction map $res_e : \mathcal{F}(U_e) \to \mathcal{F}(I(e))$ is an isomorphism upon inverting $\alpha(e)$:*

$$\mathcal{F}(U_e) \otimes_R R[\alpha(e)^{-1}] \cong \mathcal{F}(I(e)) \otimes_R R[\alpha(e)^{-1}].$$

3. *$res_e : \mathcal{F}(U_e) \to \mathcal{F}(I(e))$ is an isomorphism for all but a finite number of $e \in \mathcal{E}_\Gamma$.*

Condition 3) is a finiteness condition. It means that although there are an infinite number of hyperedges, we only really care about finitely many for any particular GKM-sheaf.

Because of the simple topology of $Top(\Gamma)$, a sheaf is completely determined by its stalks and the restriction maps at edges.

Let X be a compact T-manifold, recall that we defined a GKM-hypergraph Γ_X in Example 3.3.1: the vertices $\mathcal{V}$ correspond to a path components of X^T and the hyperedges $\mathcal{E}$ correspond to a path components of $X^{\ker(\alpha)}$. Define a GKM-sheaf $\mathcal{F}_X$ over Γ_X, as follows[2]: $\mathcal{F}_X(U_v) = H_T^*(v)$, and $\mathcal{F}_X(U_e) = \mathcal{F}_X(e \cup I(e)) = H_T^*(e)/Tor(H_T^*(e))$, where $Tor(H_T^*(e))$ denote the torsion submodule of $H_T^*(e)$. Since $e^T \subset e$, therefore, the restriction maps $res_e : \mathcal{F}_X(U_e) \to \mathcal{F}_X(I(e))$ are identified with the natural map $H_T^*(e)/Tor(H_T^*(e)) \to H_T^*(e^T)$. Now, we need to show that $\mathcal{F}_X(U_v)$ and $\mathcal{F}_X(U_e)$ are free R-modules: T acts on $v \subseteq X^T$ trivially, which implies that

$$\mathcal{F}_X(U_v) = H_T^*(v) = H_T^*(ET \times_T v) = H^*(BT) \otimes H^*(v)$$

[1]We denote $R = H_T^*(pt) = H^*(BT)$.
[2]The hyperedge $e \in \mathcal{E}$ and the vertex $v \in \mathcal{V}$ are subspaces.

by Example 2.1.2. We conclude that $H_T^*(v)$ is a free R-module.

T acts on $e \in \mathcal{E}$, then there is a codimension one subtori $H \leqslant T$ such that H act on e trivially. Find $\mathbb{S}^1 \leqslant T$ such that $H \cap \mathbb{S}^1 = id_T$, then $T \cong H \times \mathbb{S}^1$.

$$\begin{array}{ccc} T \times e & \xrightarrow{\phi \times id} & \mathbb{S} \times e \\ & \searrow \quad \swarrow & \\ & e & \end{array}$$

where $\phi : T \to \mathbb{S}^1$ is a homomorphism with kernel H.

$$\begin{aligned} H_T^*(e) &= H^*(e \times_T ET) \\ &= H^*(e \times_{(H \times \mathbb{S}^1)} E(H \times \mathbb{S}^1)) \\ &= H^*((e \times_{\mathbb{S}^1} E\mathbb{S}^1) \times BH) \\ &= H^*((e \times_{\mathbb{S}^1} E\mathbb{S}^1)) \otimes_{\mathbb{C}[x]} H^*(BH). \end{aligned}$$

Since $\mathbb{C}[x]$ is principal ideal domain (PID), by the fundamental theorem of finitely generated modules over a PID, $H_T^*(e)$ is a direct sum of a free module and a torsion module. Therefore, $\mathcal{F}_X(U_e) = H_T^*(e)/Tor(H_T^*(e))$ is a free R-module.

The following results proven by T. Baird [3], relates the degree zero sheaf cohomology of $\mathcal{F}_X$ with the equivariant cohomology of X.

Proposition 3.3.1. *Let X be a smooth compact T-manifold. The space of global sections $H^0(\mathcal{F}_X)$ fits into an exact sequence of graded R-modules*

$$0 \to H^0(\mathcal{F}_X) \xrightarrow{r} H_T^*(X_0) \xrightarrow{\delta} H_T^{*+1}(X_1, X_0) \tag{3.22}$$

for which r is a homomomorphism of R-algebras.

Proof. See Proposition 2.7 in [3]. □

Theorem 3.3.1. *Let X be a smooth compact T-manifold. If X is equivariantly formal,* *then*

$$H_T^*(X) \cong H^0(\mathcal{F}_X).$$

Proof. Combine Proposition 3.3.1 with Chang-Skjelbred sequence (2.17). □

For later use, we state the following results from [3].

Lemma 3.3.1. *Let X be a smooth compact T-manifold and $H \subset T$ is a codimension one subtorus, then $H_T^*(X^H)$ is the direct sum of a free and a torsion R-module. If $H_T^*(X)$ is torsion free, then $H_T^*(X^H)$ is free.*

Proof. See Lemma 2.6 in [3]. □

Lemma 3.3.2. *Let X_1' be all the components of X_1 that do not intersect with X_0. Then, we decompose $H_T^{*+1}(X_1, X_0)$ into*

$$H_T^{*+1}(X_1, X_0) \cong \bigoplus_{e \in \mathcal{E}} H_T^{*+1}(e, e^T) \oplus H_T^{*+1}(X_1'). \tag{3.23}$$

Proof. See Proposition 2.7 in [3]. □

3.3.2 Examples

Example 3.3.2. *Let X be path connected and let T act on X trivially. Hence, $X_0 = X_1 = X$. This implies that the set of vertices is a point, i.e., $\mathcal{V} = \{X\}$ and the set of hyperedges $\mathcal{E}$ is $\{e_\alpha | \ \alpha \in \mathbb{P}(\Lambda)\}$. So we have the hypergraph $\Gamma_X = \{v\} \cup \{e_\alpha\}_{\alpha \in \mathbb{P}(\Lambda)}$. The GKM-sheaf $\mathcal{F}_X$ over Γ_X is defined as follows:*

$$\mathcal{F}_X(\{v\}) = H_T^*(X),$$
$$\mathcal{F}_X(\{v, e_\alpha\}) = H_T^*(X).$$

The restriction map $res_e : \mathcal{F}_X(\{v, e_\alpha\}) \to \mathcal{F}_X(\{v\})$ is the identity map. Consequently $H^0(\mathcal{F}_X) = H_T^*(X)$.

Example 3.3.3. *Let $\mathbb{S}^2$ be a 2-dimensional sphere, and let $\mathbb{S}^1$ act on $\mathbb{S}^2$ by rotation. The standard action of $\mathbb{S}^1$ on $\mathbb{S}^2$ has two fixed points "north pole and south pole". This implies that $\mathcal{V} = \{v_1, v_2\}$ and $\mathcal{E} = \{e\}$. The hypergraph is $\Gamma_X = \{v_1, v_2\} \cup \{e\}$. The GKM-sheaf $\mathcal{F}_X$ over Γ_X is defined as follows:*

$$\mathcal{F}_X(\{v\}) = H^*_{\mathbb{S}^1}(v_1) = \mathbb{C}[x],$$

$$\mathcal{F}_X(\{v\}) = H^*_{\mathbb{S}^1}(v_2) = \mathbb{C}[x],$$

$$\mathcal{F}_X(U_e) = \mathcal{F}_X(\{v_1, v_2, e\}) = H^*_{\mathbb{S}^1}(\mathbb{S}^2).$$

The restriction map $res_e : \mathcal{F}_X(U_e) \to \mathcal{F}_X(\{v_1, v_2\})$ is the localization map.

Chapter 4

The Higher Cohomology of GKM-Sheaves

4.1 The Godement Chain Complex for GKM-Sheaves

In this chapter, we will study the higher cohomology of GKM-sheaves.

Let $\mathcal{F}^n$ be as defined in Section 3.2.1.

Proposition 4.1.1. *If Γ is a GKM-hypergraph and $\mathcal{F}$ is a sheaf on $Top(\Gamma)$, then $C^n\mathcal{F} = 0$, for all $n \geq 2$.*

Proof. The basic open sets of the topology of a GKM-hypergraph Γ are

(i) $U_v := \{v\}$ for $v \in \mathcal{V}_\Gamma$;

(ii) $U_e := \{e\} \cup I(e)$ for $e \in \mathcal{E}_\Gamma$.

Let $U_v := \{v\}$ be a vertex of a GKM-hypergraph Γ. For a sheaf $\mathcal{F}$ on $Top(\Gamma)$, we have $\mathcal{F}(U_v) = \mathcal{F}_v$, and since U_v contains only one element and

$$(C^0\mathcal{F})_v := \prod_{x \in U_v} \mathcal{F}_x = \mathcal{F}_v. \tag{4.1}$$

The restriction map $\mathcal{F}(U_v) \to \prod_{x \in U} \mathcal{F}_v = \mathcal{F}_v$, is an isomorphism. Therefore, the cokernel of this map is identically zero. So, we conclude that $\mathcal{F}_v^1 = 0$. Similarly, $\mathcal{F}_v^n = 0$ for all $n \geq 0$.

Also, $\mathcal{F}^2_e$ is the cokernel of the restriction map

$$\mathcal{F}^1_e \to \mathcal{F}^1_e \times \mathcal{F}^1_{v_1} \times ... \times \mathcal{F}^1_{v_k}. \tag{4.2}$$

Since $\mathcal{F}^1_{v_i} = 0$ for all $i = 1, ..., k$, then (4.2) is an isomorphism, hence $\mathcal{F}^2_e = 0$. We conclude $\mathcal{F}^2 = 0$ since all its stalks vanish. Consequently, $C^n\mathcal{F} = 0$ for all $n \geq 2$. □

Comment: In the rest of this thesis, we will use δ instead of d_0 in (3.18).

Corollary 4.1.1. *Let $\mathcal{F}$ be any GKM-sheaf, and consider the coboundary map*

$$\Gamma(C^0\mathcal{F}) \xrightarrow{\delta} \Gamma(C^1\mathcal{F}).$$

Then,

$$H^n(Top(\Gamma), \mathcal{F}) = \begin{cases} ker\ \delta & ; \quad n = 0 \\ coker\ \delta & ; \quad n = 1 \\ 0 & ; \quad n \geq 2 \end{cases}.$$

Proof. By Proposition 4.1.1, $C^n(\mathcal{F}) = \mathcal{F}^n = 0$ for all $n \geq 2$, and the global section of the chain complex of sheaves associated with $\mathcal{F}$ is

$$0 \to \Gamma(C^0\mathcal{F}) \xrightarrow{\delta} \Gamma(C^1\mathcal{F}) \to 0 \to \cdots \to 0. \tag{4.3}$$

So, the chain complex contains only two non-zero terms. □

In the next proposition we study the stalks of $C^1\mathcal{F}$.

Proposition 4.1.2. *The sheaf $C^1\mathcal{F}$ has stalks as follows: $C^1\mathcal{F}_v = 0$ for all vertices, and*

$$(C^1\mathcal{F})_e \cong \prod_{i=1}^{k} \mathcal{F}_{v_i} = \mathcal{F}(I(e))$$

for all hyperedeges.

Proof. From Proposition 4.1.1, $\mathcal{F}^1_{v_i} = 0$ for all incident vertices. Thus, $C^1\mathcal{F}_v = 0$.

Given a hyperedge e, recall $U_e := \{e\} \cup I(e)$ where $I(e) = \{v_1, v_2, ...v_k\}$, is the set of finite number of incident vertices. Now the stalk of $\mathcal{F}^1$ at e is the cokernel of a map

$$\mathcal{F}_e \xrightarrow{\varepsilon} \prod_{x \in U_e} \mathcal{F}_x = \mathcal{F}_e \times \mathcal{F}_{v_1} \times \cdots \times \mathcal{F}_{v_k} \tag{4.4}$$

the map ε is defined as $\varepsilon(s_e) = (s_e, res_{(e,v_1)}(s_e), ..., res_{(e,v_k)}(s_e))$.

Claim 4.1.1.

$$\mathcal{F}^1_e \cong \prod_{i=1}^{k} \mathcal{F}_{v_i}$$

Proof. Consider, the map $h : \mathcal{F}_e \times \prod_{v \in I(e)} \mathcal{F}_v \longrightarrow \prod_{v \in I(e)} \mathcal{F}_v$ is given by

$$h(s_e, s_{v_1}, s_{v_2}, ...s_{v_k}) = (res_{(e,v_1)}(s_e) - s_{v_1}, res_{(e,v_2)}(s_e) - s_{v_2}, ..., res_{(e,v_k)}(s_e) - s_{v_k}). \tag{4.5}$$

The map h is a homomorphism and surjective, so

$$\mathcal{F}_e \times \prod_{v \in I(e)} \mathcal{F}_v \Big/ \ker(h) \cong \prod_{v \in I(e)} \mathcal{F}_v.$$

The kernel of the map h is the same of the image of the map ε. Thus,

$$\mathcal{F}^1_e := \mathcal{F}_e \times \prod_{v \in I(e)} \mathcal{F}_v \Big/ \mathrm{im}(\varepsilon) \cong \prod_{v \in I(e)} \mathcal{F}_v.$$

□

Since $\mathcal{F}^1_{v_i} = 0$ for all vertices, we have

$$(C^1\mathcal{F})_e := \prod_{x \in U_e} \mathcal{F}^1_x = \mathcal{F}^1_e \times \mathcal{F}^1_{v_1} \times \mathcal{F}^1_{v_2} \times .... \times \mathcal{F}^1_{v_k} = \mathcal{F}^1_e. \tag{4.6}$$

□

The next result gives a concrete description of this chain complex (4.3).

Proposition 4.1.3. *Let $\mathcal{F}$ be a GKM-sheaf and $\delta : \Gamma(C^0\mathcal{F}) \longrightarrow \Gamma(C^1\mathcal{F})$. Then there exists a commutative diagram,*

$$\begin{array}{ccc}
\Gamma(C^0\mathcal{F}) & \xrightarrow{\quad\delta\quad} & \Gamma(C^1\mathcal{F}) \\
\Big\downarrow{\scriptstyle\phi} & & \Big\downarrow{\scriptstyle\psi} \\
\prod\limits_{x\in\mathcal{V}\cup\mathcal{E}} \mathcal{F}_x & \xrightarrow[\quad\widetilde{\delta}\quad]{} & \prod\limits_{e\in\mathcal{E}}\prod\limits_{v\in I(e)} \mathcal{F}_v
\end{array}$$

where the maps ϕ, ψ are isomorphisms, and $\tilde{\delta} := \psi\circ\delta\circ\phi^{-1}$. Furthermore, The concrete formula of the map $\tilde{\delta}$ in terms of elements $s = (s_x)_{x\in\mathcal{V}\cup\mathcal{E}}$ is given by

$$\tilde{\delta}(s)_{(e,v)} = res_{(e,v)}(s_e) - s_v. \tag{4.7}$$

In particular, $H^0(\mathcal{F}) \cong \ker(\widetilde{\delta})$ and $H^1(\mathcal{F}) \cong coker(\widetilde{\delta})$.

Proof. The map ϕ is the defining identity, see (3.13). Also, we have seen

$$(C^1\mathcal{F})_e \cong \prod_{i=1}^{k} \mathcal{F}_{v_i}$$

and

$$(C^1\mathcal{F})_v = 0.$$

By taking the global section for all $e \in \mathcal{E}$ and by Proposition 4.1.2, then we have

$$\Gamma(C^1\mathcal{F}) := \prod_{x\in\mathcal{V}\cup\mathcal{E}} (C^1\mathcal{F})_x = \prod_{e\in\mathcal{E}} (C^1\mathcal{F})_e \cong \prod_{e\in\mathcal{E}}\prod_{i=1}^{k} \mathcal{F}_{v_i}.$$

Therefore, there exists a commutative diagram and we have such a map $\tilde{\delta}$. The next step is to clarify the concrete formula of $\tilde{\delta}$.

Let $s \in \prod\limits_{x\in\mathcal{V}\cup\mathcal{E}} \mathcal{F}_x$ such that $s = (s_x)_{x\in\mathcal{V}\cup\mathcal{E}}$, where $s_x \in \mathcal{F}_x$ for all $x \in \mathcal{V}\cup\mathcal{E}$, and by using the definition of the map δ in Subsection 3.2 which describes as follows:

$(C^0\mathcal{F})_e = \mathcal{F}_e \times \mathcal{F}_{v_1} \times ... \times \mathcal{F}_{v_k}$ and $(C^1\mathcal{F})_e = \mathcal{F}^1_e \times \mathcal{F}^1_{v_1} \times ... \times \mathcal{F}^1_{v_k} = \mathcal{F}^1_e \cong \prod_{i=1}^{k} \mathcal{F}_{v_i}$.
Let $d = \alpha \circ \beta$ be the composition,

$$(C^0\mathcal{F}) \xrightarrow{\beta} \mathcal{F}^1 \xrightarrow{\alpha} (C^0\mathcal{F}^1). \tag{4.8}$$

Observe that α is an isomorphism map, because it restricts to isomorphisms on stalks

$$\Gamma(C^0\mathcal{F}) \xrightarrow{\delta} \Gamma(C^1\mathcal{F}) = \Gamma(\mathcal{F}^1). \tag{4.9}$$

Thus, the description of the map

$$\prod_{x\in\mathcal{V}\cup\mathcal{E}} \mathcal{F}_x \xrightarrow{\tilde{\delta}} \prod_{e\in\mathcal{E}} \prod_{v\in I(e)} \mathcal{F}_v \tag{4.10}$$

is given by

$$\tilde{\delta}(s)_{(e,v)} = res_{(e,v)}(s_e) - s_v. \tag{4.11}$$

□

The set of hyperedges $\mathcal{E}$ is partitioned $\mathcal{E} = \mathcal{E}^d \cup \mathcal{E}^{nd}$, where $\mathcal{E}^{nd}$ is the finite set of hyperedges for which the restriction map $res_e : \mathcal{F}(U_e) \to \mathcal{F}(I(e))$ is not an isomorphism.

The following proposition computes $H^0(\mathcal{F})$ and $H^1(\mathcal{F})$ as kernel and cokernel of a morphism of finitely generated free R-modules.

Proposition 4.1.4. *Let $\mathcal{F}$ be a GKM-sheaf and $\beta : \bigoplus_{x\in\mathcal{E}^{nd}} \mathcal{F}_x \to \bigoplus_{e\in\mathcal{E}^{nd}} \bigoplus_{v\in I(e)} \mathcal{F}_v$ be the morphism of finitely generated free R-modules defined as $\beta(s)_{(e,v)} = res_{(e,v)}(s_e) - s_v$. Then $H^0(\mathcal{F}) \cong \ker(\beta)$ and $H^1(\mathcal{F}) \cong coker(\beta)$.*

Proof. We have the following commutative diagram of R-modules with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod_{x\in\mathcal{E}^d}\mathcal{F}_x & \xrightarrow{\psi} & \prod_{x\in\mathcal{V}\cup\mathcal{E}}\mathcal{F}_x & \xrightarrow{\phi} & \bigoplus_{x\in\mathcal{E}^{nd}}\mathcal{F}_x & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\widetilde{\delta}} & & \downarrow{\scriptstyle\beta} & & \\
0 & \longrightarrow & \prod_{e\in\mathcal{E}^d}\prod_{v\in I(e)}\mathcal{F}_v & \xrightarrow{\psi'} & \prod_{e\in\mathcal{E}}\prod_{v\in I(e)}\mathcal{F}_v & \xrightarrow{\phi'} & \bigoplus_{e\in\mathcal{E}^{nd}}\bigoplus_{v\in I(e)}\mathcal{F}_v & \longrightarrow & 0
\end{array}$$

where ϕ, ϕ' are projections and ψ, ψ' are inclusions; and γ is defined by commutativity. By the Snake Lemma (see [16]) there is an exact sequence

$$0\to\ker\gamma\xrightarrow{\overline{\psi}}\ker(\widetilde{\delta})\xrightarrow{\overline{\phi}}\ker(\beta)\xrightarrow{\delta}coker(\gamma)\xrightarrow{\overline{\psi'}}coker(\widetilde{\delta})\xrightarrow{\overline{\phi'}}coker(\beta)\to 0. \qquad (4.12)$$

It is clear by definition of $\mathcal{E}^d$ that γ is an isomorphism. Thus,

$$0\to 0\to\ker(\widetilde{\delta})\xrightarrow{\overline{\phi}}\ker(\beta)\xrightarrow{\delta}0\to coker(\widetilde{\delta})\xrightarrow{\overline{\phi'}}coker(\beta)\to 0. \qquad (4.13)$$

So, $\overline{\phi}$ and $\overline{\phi'}$ are isomorphisms, as desired. $\square$

4.2 Local Cohomology of a GKM-Sheaf

We can apply Proposition 3.2.1 for an abstract GKM-sheaf $\mathcal{F}$ over $Top(\Gamma)=\mathcal{V}\cup\mathcal{E}$ because the set of vertices $\mathcal{V}$ is an open set and the set of edges $\mathcal{E}$ is a closed set. This gives rise to a long exact sequence

$$0\to H^0(Top(\Gamma),\mathcal{F})\to H^0(\mathcal{V},\mathcal{F}|_{\mathcal{V}})\to H^1_{\mathcal{E}}(Top(\Gamma),\mathcal{F})\to H^1(Top(\Gamma),\mathcal{F})\to\cdots$$

The subspace $\mathcal{V}$ is discrete, so sheaf cohomology vanishes in positive degree. That is, $H^i(\mathcal{V},\mathcal{F}|_{\mathcal{V}})=0$ for all $i\geq 1$. Therefore, we obtain an exact sequence

$$0\to H^0(Top(\Gamma),\mathcal{F})\to H^0(\mathcal{V},\mathcal{F}|_{\mathcal{V}})\to H^1_{\mathcal{E}}(Top(\Gamma),\mathcal{F})\to H^1(Top(\Gamma),\mathcal{F})\to 0. \qquad (4.14)$$

The next proposition determines $H^0(\mathcal{V}, \mathcal{F}|_{\mathcal{V}})$ and $H^1_{\mathcal{E}}(Top(\Gamma), \mathcal{F})$ in terms of stalks.

Proposition 4.2.1. *If Γ is a GKM-hypergraph and $\mathcal{F}$ is a GKM-sheaf on $Top(\Gamma)$, then $H^0(\mathcal{V}, \mathcal{F}|_{\mathcal{V}}) \cong \bigoplus_{v\in\mathcal{V}} \mathcal{F}_v$ and $H^1_{\mathcal{E}}(Top(\Gamma), \mathcal{F}) \cong \bigoplus_{e\in\mathcal{E}^{nd}} coker(res_e)$.*

Proof. The subspace $\mathcal{V}$ is discrete, which implies $H^0(\mathcal{V}, \mathcal{F}|_{\mathcal{V}}) = \prod_{v\in\mathcal{V}} \mathcal{F}(\{v\}) = \bigoplus_{v\in\mathcal{V}} \mathcal{F}_v$.

Claim 4.2.1. *The Godement chain complex of $\mathcal{F}|_{\mathcal{E}}$ is given by*

$$0 \to \prod_{e\in\mathcal{E}} \mathcal{F}_e \xrightarrow{\prod_{e\in\mathcal{E}} res_e} \prod_{e\in\mathcal{E}} \prod_{v\in I(e)} \mathcal{F}_v \to 0. \tag{4.15}$$

Proof. Using Definition 3.2.1, we observe

$$\Gamma_{\mathcal{E}}(C^0\mathcal{F}) = \{s \in \Gamma(C^0\mathcal{F}) \mid \ s_v = 0, \forall v \in \mathcal{V}\} = \prod_{e\in\mathcal{E}} \mathcal{F}_e \tag{4.16}$$

and

$$\Gamma_{\mathcal{E}}(C^1\mathcal{F}) = \{s \in \Gamma(C^1\mathcal{F}) \mid \ s_v = 0, \forall v \in \mathcal{V}\} = \prod_{e\in\mathcal{E}} \prod_{v\in I(e)} \mathcal{F}_v. \tag{4.17}$$

So,

$$0 \to \Gamma_{\mathcal{E}}(C^0\mathcal{F}) \to \Gamma_{\mathcal{E}}(C^1\mathcal{F}) \to 0 \tag{4.18}$$

becomes

$$0 \to \prod_{e\in\mathcal{E}} \mathcal{F}_e \xrightarrow{\prod_{e\in\mathcal{E}} res_e} \prod_{e\in\mathcal{E}} \prod_{v\in I(e)} \mathcal{F}_v \to 0. \tag{4.19}$$

□

From the above claim, we obtain

$$\begin{aligned} H^1_{\mathcal{E}}(Top(\Gamma), \mathcal{F}) = & \ \text{coker}(\prod_{e\in\mathcal{E}} res_e) \\ &= \prod_{e\in\mathcal{E}} \ \text{coker}(res_e) \\ &= \bigoplus_{e\in\mathcal{E}^{nd}} \ \text{coker}(res_e). \end{aligned}$$

□

Chapter 5

Reflexive Modules

5.1 Syzygy and Reflexive Modules

We have already defined the j-th syzygy and reflexive modules in Section 2.2.1. In this section we will show that the global section of GKM-sheaf $H^0(\mathcal{F}_X)$ is reflexive (i.e. a 2-syzygy) and generalize Theorem 3.3.1. Eventually, we find a geometric interpretation of $H^1_{\mathcal{E}}(\mathcal{F}_X)$ in terms of the Atiyah-Bredon sequence (2.16).

Proposition 5.1.1. *If $\mathcal{F}$ is a GKM-sheaf, then $H^0(\mathcal{F})$ is reflexive.*

Proof. This follows immediately from Proposition 4.1.4. □

Theorem 5.1.1. *Let X be a compact, smooth T-manifold. $H^*_T(X)$ is reflexive if and only if*

$$H^*_T(X) \cong H^0(\mathcal{F}_X).$$

Proof. Suppose that $H^*_T(X)$ is reflexive, i.e., a 2-syzygy. By using Theorem 2.2.2 and Remark 2.2.3 the Chang-Skjelbred sequence

$$0 \to H^*_T(X) \to H^*_T(X_0) \xrightarrow{\delta} H^{*+1}_T(X_1, X_0)$$

is exact, and this yields

$$H_T^*(X) \cong ker(\delta).$$

By Proposition 3.3.1, $H^0(\mathcal{F}_X) \cong ker(\delta)$, so

$$H_T^*(X) \cong H^0(\mathcal{F}_X).$$

Conversely, suppose $H_T^*(X) \cong H^0(\mathcal{F}_X)$. By Proposition 5.1.1, we conclude $H_T^*(X)$ is reflexive. □

The rest of this chapter is dedicated to finding a geometric interpretation of $H^1(\mathcal{F}_X)$.

Lemma 5.1.1. *Let $X_1' \subseteq X_1$ be the union of path components that do not intersect X_0. Suppose $H_T^*(X)$ is torsion free. Then $X_1' = \emptyset$.*

Proof. Observe that X_1 can be written as follows:

$$X_1 = \bigcup_{H \leqslant T} X^H \tag{5.1}$$

where the union is indexed by H is a codimension one subtori. By hypothesis, $H_T^*(X)$ is torsion free, so by Lemma 3.3.1, $H_T^*(X^H)$ is a free R-module. By the Localization Theorem(see Theorem 11.6.1 in [12]) every path component of X^H intersects X_0. Thus, $X_1' = \emptyset$. □

Lemma 5.1.2. *Suppose $H_T^*(X)$ is torsion free. Then $H^0(\mathcal{V}, \mathcal{F}|_{\mathcal{V}}) \cong H_T^*(X_0)$ and $H^1_{\mathcal{E}}(Top(\Gamma), \mathcal{F}_X) \cong H_T^{*+1}(X_1, X_0)$.*

Proof. The finite set of vertices $\mathcal{V}$ corresponds to a set of path components of X_0, and the set hyperedges $\mathcal{E}$ corresponds to a set of path components of $X^{\ker(\alpha)}$ that intersect with X_0. $\mathcal{F}_X$ is a GKM-sheaf over Γ and defined for the basic open set $U = \{v\}$, as, $\mathcal{F}_v = \mathcal{F}(U_v) = H_T^*(v)$. Therefore,

$$H^0(\mathcal{V}, \mathcal{F}|_{\mathcal{V}}) = \bigoplus_{v \in \mathcal{V}} \mathcal{F}_v = \bigoplus_{v \in \mathcal{V}} H_T^*(v) = H_T^*(X_0). \tag{5.2}$$

By Lemma 3.3.2, decompose $H_T^{*+1}(X_1, X_0)$ into

$$H_T^{*+1}(X_1, X_0) \cong \bigoplus_{e \in \mathcal{E}} H_T^{*+1}(e, e^T) \oplus H_T^{*+1}(X_1'). \tag{5.3}$$

By Lemma 5.1.1, decomposition of the cohomology in (5.3) becomes

$$H_T^{*+1}(X_1, X_0) \cong \bigoplus_{e \in \mathcal{E}} H_T^{*+1}(e, e^T). \tag{5.4}$$

Claim 5.1.1. *If $H_T^*(X)$ is torsion free, then $\mathcal{F}_X(U_e) \cong H_T^*(e)$.*

Proof. $H_T^*(X)$ is a submodule of finitely generated free R-module, so it is torsion free. Recall the Lemma 3.3.1, we conclude that $H_T^*(e)$ is free and $Tor(H_T^*(e)) = 0$. By the definition of $\mathcal{F}_X(U_e) = H_T^*(e)/\mathrm{Tor}(H_T^*(e))$, so $\mathcal{F}_X(U_e) = H_T^*(e)$. □

Claim 5.1.2. $H^1_{\mathcal{E}}(Top(\Gamma), \mathcal{F}_X) \cong \bigoplus_{e \in \mathcal{E}} H_T^{*+1}(e, e^T)$.

Proof. The map $H_T^*(e) \to H_T^*(e^T)$ is identical with $res_e : \mathcal{F}(U_e) \to \mathcal{F}(I(e))$ and it is injective. The long exact sequence for the pair $(e,\ e^T)$ implies

$$H_T^{*+1}(e, e^T) = \mathrm{coker}(res_e). \tag{5.5}$$

By Proposition 4.2.1, we have seen that the

$$H^1_{\mathcal{E}}(Top(\Gamma), \mathcal{F}) = \bigoplus_{e \in \mathcal{E}^{nd}} \mathrm{coker}(res_e). \tag{5.6}$$

Combining with (5.4), (5.5) and (5.6), we conclude

$$H^1_{\mathcal{E}}(Top(\Gamma), \mathcal{F}) \cong H_T^{*+1}(X_1, X_0). \tag{5.7}$$

□

□

Theorem 5.1.2. *Let X be a compact, smooth T-manifold. If $H^*_T(X)$ is reflexive, then $H^0(\mathcal{F}_X)$ and $H^1(\mathcal{F}_X)$ fit into an exact sequence*

$$0 \to H^0(\mathcal{F}_X) \to H^*_T(X_0) \xrightarrow{\delta} H^{*+1}_T(X_1, X_0) \to H^1(\mathcal{F}_X) \to 0. \tag{5.8}$$

Proof. Since $H^*_T(X)$ is reflexive, Theorem 5.1.1 implies that

$$0 \to H^0(\mathcal{F}_X) \to H^*_T(X_0) \xrightarrow{\delta} H^{*+1}_T(X_1, X_0) \tag{5.9}$$

is exact.

From Lemma 5.1.2 and (4.14) we have an isomorphism of exact sequences.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^*_T(X) & \longrightarrow & H^*_T(X_0) & \xrightarrow{\delta} & H^{*+1}_T(X_1, X_0) & \longrightarrow & H^1(\mathcal{F}_X) & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & H^0(\mathcal{F}_X) & \longrightarrow & H^0(\mathcal{V}, \mathcal{F}_X|_V) & \longrightarrow & H^1_{\mathcal{E}}(Top(\Gamma), \mathcal{F}|_X) & \longrightarrow & H^1(\mathcal{F}_X) & \longrightarrow & 0
\end{array}$$

□

Appendix A

Fibre Bundles

In this appendix, we recall some basic definitions and properties of fibre bundle theory. The main references are [27],[20].

Definition A.0.1. *A topological space G is said to be a topological group if it has a group structure such that the multiplication $\phi : G \times G \to G$ by $\phi(g, h) = g \cdot h$, and the inversion map $i : G \to G$, $i(g) = g^{-1}$, are continuous maps.*

Remark A.0.1. *Let G be a smooth manifold. We say G is* **a Lie group** *if the multiplication and inversion maps are smooth maps.*

An action of the Lie group G on X is a continuous map

$$\psi : G \times X \to X$$

often written as $\psi(g, x) = \psi_g(x) = g \cdot x$, that satisfies

1. $\psi_e = id_X$ where e is an identity of G;
2. $\psi_g \circ \psi_h = \psi_{gh}$ for all $g, h \in G$.

In this case, we say that the space X is a **G-*space***. Also, for any $x \in X$, the set $O_x = \{g \cdot x \mid g \in G\}$ is the **orbit** of x, and the **stabilizer** of x or the isotropy subgroup of x is the set $G_x = \{g \in G \mid \ g \cdot x = x\}$.

We say G acts *effectively* on X if $\psi_g = id_X$ only if g is the identity in G. Equivalently, for each $g \in G$ corresponds to a different homeomorphism $\psi_g : X \to X$. The action is transitive if there exists a point $x \in X$ such that $O_x = X$.
Given two G-spaces, X and Y a continuous map $f : X \to Y$ is called an **equivariant map** if it preserves the action, that is,

$$f(g \cdot x) = g \cdot f(x).$$

Definition A.0.2. *Let F, E, and B be topological spaces. A fibre bundle over the base B with fibre F is a collection $\{E,\ B,\ F,\ \pi\}$ with the following properties:*

1. *$\pi : E \to B$ is continuous and surjective;*
2. *For any $x \in B$, there is a neighborhood $U_i \subseteq B$ of x and a homeomorphism*

$$\phi_i : \pi^{-1}(U_i) \to U_i \times F,$$

called a local trivilization, such that for each $(p, x) \in U_i \times F$, $(\pi \circ \phi_i^{-1})(p, x) = p$.

Definition A.0.3. *A principal G-bundle is a fibre bundle E that satisfies the following property:*
Let G be a topological group acting on E. For each $x \in E$ there is a neighborhood $V_x \subseteq E$ and a topological space U_x such that a homeomorphism

$$\phi_x : V_x \to U_x \times G$$

is an equivariant map, we define an action G on $U_x \times G$ by $g_1 \cdot (x, g_2) = (x, g_1 \cdot g_2)$.

Equivalently, a fibre bundle $\pi : E \to B$ is called a principal G-bundle if there is an action of G on E that preserves the fibers of E, that is, if $x \in B$ and $p \in \pi^{-1}(x)$, then $p \cdot g \in \pi^{-1}(x)$ for all $g \in G$, and G acts freely and transitively on each fibre.

For any principal G-bundle $\{E,\ B,\ G,\ \pi\}$ and continuous map $f : B' \to B$, where B' is a topological space, we can define a principal G-bundle over B' as follows:

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{f} & B \end{array}$$

where $f^*E = \{(x, y) \in B' \times E \mid\ f(x) = \pi(y)\}$ and $\pi'(x, y) = x$. The fibre bundle $\{f^*E,\ B',\ G,\ \pi'\}$ is called the **pullback bundle**.

Let X be a topological space, and P a principal G-bundle over X. If $f, g : B \to X$ be homotopic maps, then the pullback bundles f^*P and g^*P are isomorphic as principal G-bundles over B.

If $E \to B$ is a principal G-bundle for which E is contractible, then $E \to B$ is called a universal principal G-bundle and the base space B is often called the **classifying space** for G. In general, we denote the classifying space as BG, and the universal bundle over BG as EG.

Theorem A.0.1. *Let G be any topological group. Then there exists a classifying space for G.*

Proof. See [24]. □

The following example is the essential example in this thesis.

Example A.0.1. *The classifying space of $\mathbb{S}^1$ is $\mathbb{C}P^\infty$ and the universal bundle $E\mathbb{S}^1 = \mathbb{S}^\infty$, see Example 2.1.1.*

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