ENGEL'S THEOREM IN GENERALIZED LIE ALGEBRAS

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by

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Abstract

In this thesis we deal with Engel's Theorem about simultaneous triangulability of the space of nilpotent operators closed under Lie bracket, one of the corner stones of Lie Theory. This theorem was first proven in 1892 by F. Engel in his paper [4]. Since then several various versions of this theorem and its proofs have been suggested ([3], [8], [12]). In some versions the authors deal with weakly closed sets of elements in associative algebras [6]. In the others they look at representations of Lie algebras by nilpotent transformations of vector spaces [3].

Recently people began looking at the version of Engel's Theorem for generalized Lie algebras. Engel's Theorem in the case of ordinary Lie superalgebras was mentioned (without proof) in the fundamental paper of V. Kac [8] devoted to the classification of simple finite-dimensional Lie superalgebras and in the monograph of M. Scheunert [12]. It was quite clear that a similar result should hold also in the case of more general color Lie superalgebras [2]. The most recent development leads to Lie algebras over Hopf algebras. A version of Engel's Theorem for this much more general setting was suggested in the Ph. D. dissertation of V. Linchenko [9].

In this dissertation we choose one of the possible versions of Engel's Theo-

rem, in the spirit of Bourbaki [3], using the approach via representation theory. We demonstrate how this approach can work in the case of the color Lie superalgebras. We also tried the case of so called (H, β)-Lie algebras where β is a bicharacter on a cotriangular Hopf algebra H. The result we give here generalizes the case of ordinary Lie algebras but when restricted to the case of color Lie superalgebras produces a considerably weaker result. The proof of this result and several complementary lemmas was communicated to us by M. Kotchetov.

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Chapter 1

Definitions

The first chapter contains definitions, lemmas, propositions and theorems that contribute to a better understanding of all the ideas and concepts defined in the thesis. A collection of illustrative examples is also exhibited here.

1.1 Algebras

Definition 1.1.1. *Let R be a nonempty set with two binary operations: addition and multiplication, such that R is an abelian group relative to addition and*

$$(a+b)c = ac + bc, \quad a(b+c) = ab + ac,$$

for all $a, b, c \in R$. Then R is said to be a ring.
If there exists an element $e \in R$ such that $ea = ae = a$ for all $a \in R$, then e is called the unity element of R, and R is called a ring with the unity *element.*
If $a(bc) = (ab)c$, for all $a, b, c \in R$, then R is said to be associative.
If $ab = ba$, for all $a, b \in R$, then R is said to be commutative.

Definition 1.1.2. *Let $\Bbbk$ be an associative commutative ring with unity element. An abelian group A is a* $\Bbbk$-module *if there is a mapping $\Bbbk \times A \to A$ such that*

$$ea = a, \tag{1.1.1}$$

$$\alpha(a+b) = \alpha a + \alpha b, \tag{1.1.2}$$

$$(\alpha+\beta)a = \alpha a + \beta a, \tag{1.1.3}$$

$$(\alpha\beta)a = \alpha(\beta a), \tag{1.1.4}$$

for all $\alpha, \beta \in \Bbbk, a, b \in A$, e being the unity element of $\Bbbk$.
A ring R with a structure of a $\Bbbk$-module is said to be a $\Bbbk$ -algebra *if*

$$(\alpha a)b = \alpha(ab) = a(\alpha b),$$

for all $\alpha \in \Bbbk, a, b \in R$.

1.2 Lie algebras

Definition 1.2.1. *Let $\Bbbk$ be a commutative associative ring with the unity element. A $\Bbbk$-algebra L is said to be a* Lie algebra *if for its multiplication $[,]$, called the bracket operation, the following are satisfied:*

L1 *The bracket operation is bilinear,*

L2 $[x, x] = 0$ *for all $x \in L$,*

L3 $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ *(Jacobi identity),*

for all $x, y, z \in L$.

Notice that $L1$ and $L2$ applied to $[x+y, x+y]$ imply

$L2'$ $\quad [x,y] = -[y,x] \qquad$ (anticommutativity).

Example 1.2.1. Let $\Bbbk = \mathbb{R}$. The vector space $\mathbb{R}^3$ is a Lie algebra relative to the multiplication given by the cross product of vectors.

Note 1.2.1. If V is a finite-dimensional vector space over a field $\Bbbk$, denote by $End(V)$ the set of linear transformations from $V \to V$. As a vector space over $\Bbbk$, $End(V)$ has dimension n^2, where $n = dimV$, and $End(V)$ is a ring relative to the usual product operation. Define a new operation $[x,y] = xy - yx$ called the commutator of x and y. With this operation $End(V)$ becomes a Lie algebra over $\Bbbk$. In order to distinguish this new algebra structure from the old associative one, we write $gl(V)$ for $End(V)$ viewed as Lie algebra and call it the general linear algebra.

Any subalgebra of a Lie algebra $gl(V)$ is called a linear Lie algebra. Also, we can identify $gl(V)$ with the set of all $n \times n$ matrices over $\Bbbk$, denoted $gl(n, \Bbbk)$. For reference, we write down the multiplication table for $gl(n, \Bbbk)$ relative to the standard basis consisting of all the matrices e_{ij} (having 1 in the (i,j) position and 0 elsewhere). Since $e_{ij}e_{kl} = \delta_{jk}e_{il}$, it follows that $[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}$.

Definition 1.2.2. *A $\Bbbk$-submodule M of a Lie algebra L is called a* subalgebra *if $[x,y] \in M$, whenever $x, y \in M$; in particular, M is a Lie algebra relative to the inherited operations.*

Example 1.2.2. Let $\Bbbk = \mathbb{R}$. Consider the set L of infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$. On this vector space over $\mathbb{R}$, let the Poisson's bracket be defined in the following way:

$$[f_1, f_2] = f_1 f_2' - f_1' f_2,$$

where f_1', f_2' are the derivatives of f_1, f_2, respectively. Then L is a Lie algebra over $\mathbb{R}$.

Definition 1.2.3. *A* $\mathbb{k}$*-submodule* I *of a Lie algebra* L *is called an* ideal *of* L *if for any* $x \in L$ *and* $y \in I$*, we have* $[x, y] \in I$.

Definition 1.2.4. *A linear transformation* $\phi : L \to L'$*, (*L *and* L' *Lie algebras) is called a* homomorphism *if* $\phi([x, y]) = [\phi(x), \phi(y)]$*, for all* $x, y \in L$.

Definition 1.2.5. *A* representation *of a Lie algebra* L *by linear transformations of a vector space* V *is a homomorphism* $\phi : L \to gl(V)$.

Definition 1.2.6. *By a* derivation *of* V *we mean a* $\mathbb{k}$*-linear map* $\delta : V \to V$ *satisfying the product rule* $\delta(ab) = a\delta(b) + \delta(a)b$.

The collection $DerV$ of all derivations of V is a vector subspace of $EndV$. $DerV$ is a subalgebra of $gl(V)$. Since a Lie algebra L is a $\mathbb{k}$-algebra in the above sense, $DerL$ is defined.

Definition 1.2.7. *If* $x \in L$*, then* $y \to [x, y]$ *is a vector space* endomorphism *and we denote it by* adx.

Definition 1.2.8. *The map* $L \to DerL$ *sending* x *to* adx *is called the* adjoint representation *of* L.

Definition 1.2.9. *Let* $x \in L$*. We say that* adx *is* nilpotent *if* $(adx)^m = 0$*, for some* $m > 0$.

Lemma 1.2.1. ([3]) *Let* $x \in gl(V)$ *be a nilpotent endomorphism. Then* adx *is also nilpotent.*

Theorem 1.2.1. (ENGELS'S THEOREM FOR LIE ALGEBRAS)

Let L be a Lie algebra, V a finite-dimensional space, $\varphi : L \to End(V)$ a representation of L by linear transformations of V. If L consists of nilpotent endomorphisms and $V \neq 0$, then there exists nonzero $v \in V$ for which $\varphi(L)v = 0$.

The proof can be found, for example in [3].

Definition 1.2.10. *A Lie algebra L is called nilpotent if there exists a natural number N such that for any $x_1, x_2, ..., x_N \in L$ we have $(adx_1)(adx_2)...(adx_N) = 0$. In other words, for any $x_0, x_1, x_2, ...x_N$ we have $[...[[x_0, x_1], x_2], ..., x_N] = 0$.*

Corollary 1.2.1. *If L is a finite-dimensional Lie algebra, in which for any x we have adx nilpotent, than L is a nilpotent Lie algebra.*

Proof: We take $\varphi = ad$, $V = L$, in Theorem 1.2.1. Then we find $0 \neq z \in L$ such that $(adx)(z) = [x, z] = 0$, for all $x \in L$. Then z is an element of the center of L and we can form $\bar{L} = L/\mathbb{k}z$. This algebra satisfies the same condition as L and $\dim \bar{L} < \dim L$, so we can apply the induction hypothesis and conclude that $\bar{L}$ is nilpotent, i.e. $[...[\bar{x}_1, \bar{x}_2], ..., \bar{x}_m] = 0$, for any $\bar{x}_i = x_i + \mathbb{k}z$, for some natural number m. This means $[...[x_1, x_2], ..., x_m] + \mathbb{k}z = \mathbb{k}z$, or $[...[x_1, x_2], ..., x_m] \in \mathbb{k}z$. Therefore for any x_{m+1} we have
$[[...[x_1, x_2], ..., x_m], x_{m+1}] = 0$, as required. □

This result is also a corollary of Engel's theorem in the case of generalized Lie algebras, which we are going to study in Chapters 2 and 3. It follows from the "main" Engel's theorem using similar arguments. Therefore we do not formulate this corollary in our future chapters.

1.2.1 G-graded algebras

Definition 1.2.11. *Let V be a vector space over the field $\Bbbk$ and G an abelian group. The vector space V is said to be* G-graded *if*

$$V = \oplus_{g\in G} V_g,$$

where each V_g is a subspace of V. Any element x of V_g is called homogeneous of degree g *and we write $d(x) = g$. In the case of $G = \mathbb{Z}_2$ the elements of V_0 (resp., V_1) are called even (resp., odd).*

Definition 1.2.12. *A* subspace *U of V is called G*-graded *if it contains the homogeneous components of all its elements , i.e. if*

$$U = \oplus_{g\in G}(U \cap V_g).$$

Definition 1.2.13. *If V is a G-graded space and U is a graded subspace, then $W = V/U$ becomes G-graded if we set $W_g = (V_g + U)/U$.*

Definition 1.2.14. *Let G be a group. We say that a $\Bbbk$-algebra A is G-graded if*

$$A = \oplus_{g\in G} A_g,$$

where A_g are $\Bbbk$-submodules, and

$$A_g A_h \subseteq A_{gh}, \qquad \forall g, h \in G.$$

If A has a unity element 1, we can prove that $1 \in A_0$.

Definition 1.2.15. *An element $a \in A$ is said to be* homogeneous *if $a \in A_g$ for some $g \in G$. (In this case we write $d(a) = g$).*

Definition 1.2.16. *Let A and B be G-graded $\mathbb{k}$-algebras. We say that a homomorphism $\psi : A \to B$ is a* homomorphism of G-graded $\mathbb{k}$-algebras *if $\psi(A_g) \subseteq B_g$, for all $g \in G$.*

Definition 1.2.17. *If I is a G-graded ideal of a G-graded $\mathbb{k}$-algebra A, i.e. I is an ideal of A and $I = \oplus_{g\in G} I_g$, where $I_g = I \bigcap A_g$, then the factor algebra A/I is naturally G-graded and the canonical mapping $\theta : A \to A/I$, $\quad \theta(a) = a + I$ is a homomorphism of G-graded $\mathbb{k}$-algebras.*

Example 1.2.3. Free algebras are graded by the length of words, i.e. the subspace A_i of $A = \mathbb{k}\{X\}$ is defined as the subspace linearly generated by all monomials of degree i. The elements of X are of degree 1. The grading semigroup is the semigroup of non-negative integers.

Example 1.2.4. The polynomial algebra $\mathbb{k}[x_1, ..., x_n]$ is graded as the quotient of the free algebra $A = \mathbb{k}\{x_1, ...x_n\}$ (graded as in the example above) by the ideal I generated by the homogeneous elements $x_i x_j - x_j x_i$ of degree 2, where i and j run over all integers between 1 and n. The quotient grading is the same as the grading by degree.

1.3 Lie superalgebras

Definition 1.3.1. *A $\mathbb{Z}_2$-graded algebra $A = A_0 \oplus A_1$ is called a* superalgebra.

1. If $d(a) = 0$, i.e. $a \in A_0$, then a is said to be even.

2. If $d(a) = 1$, i.e. $a \in A_1$, then a is said to be odd.

Example 1.3.1. Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$-graded finite-dimensional vector space over the field $\Bbbk$, $L(V) = End(V)$ the algebra of all linear transformations of V,

$$L(V)_i = \{f \in L(V) | f(V_j) \subseteq V_{i+j}, j \in \mathbb{Z}_2\}, \quad i \in \mathbb{Z}_2.$$

Then $L(V) = L(V)_0 + L(V)_1$ is an associative superalgebra.

If $\{e_1, ..., e_m\}$ is a basis of V_0 and $\{e_{m+1}, ..., e_n\}$ is a basis of V_1, then each operator f of $L(V)$ has the matrix of a form :

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix},$$

where F_{11} is an $m \times m$ matrix, and F_{22} is an $(n-m) \times (n-m)$ matrix, $d(F) = 0$ if and only if $F_{12} = 0$ and $F_{21} = 0$; $d(F) = 1$ if and only if $F_{11} = 0$ and $F_{22} = 0$.

Definition 1.3.2. *A $\mathbb{Z}_2$-graded algebra $L = L_0 \oplus L_1$, with the multiplication* $[,]$*over* $\Bbbk$ *is called a* Lie superalgebra *if for any homogeneous elements $x, y, z \in L$, the following conditions are satisfied:*

1. $[x, y] = -(-1)^{d(x)d(y)}[y, x]$ *(called super anti-commutativity),*

2. $(-1)^{d(x)d(z)}[x, [y, z]] + (-1)^{d(z)d(y)}[z, [x, y]] + (-1)^{d(y)d(x)}[y, [z, x]] = 0$ *(called super Jacobi identity).*

Example 1.3.2. If *char* $\Bbbk \neq 2$, $A = A_0 \oplus A_1$ is a $\mathbb{Z}_2$-graded associative $\Bbbk$-algebra, then A with the bilinear multiplication defined by

$$[a, b] = ab - (-1)^{d(a)d(b)}ba,$$

where a, b are homogeneous elements, is a Lie superalgebra.

1.4 Color Lie superalgebras

Definition 1.4.1. *Let* $\Bbbk$ *be a commutative associative ring with unity element,* $\Bbbk^*$ *the group of invertible elements of* $\Bbbk$*,* G *an additive abelian group. A mapping* $\epsilon : G \times G \to \Bbbk^*$ *is called a* bicharacter *on* G*, if*

1. $\epsilon(g, h + f) = \epsilon(g, h)\epsilon(g, f)$,

2. $\epsilon(g + h, f) = \epsilon(g, f)\epsilon(h, f)$,

for all $g, h, f \in G$.

If also we have that $\epsilon(g, h)\epsilon(h, g) = 1$, then the bicharacter is a *skew-symmetric* one. Also, we set

1. $G_+ = \{g \in G | \epsilon(g, g) = 1\}$,

2. $G_- = \{g \in G | \epsilon(g, g) = -1\}$.

G_+ is always a subgroup of G of index ≤ 2, if G is a group.

Example 1.4.1. For any abelian group G one can consider the trivial bicharacter ϵ given by $\epsilon(g, h) = 1$ for all $g, h \in G$.

Example 1.4.2. For $G = \mathbb{Z}$, $\epsilon(g, h) = (-1)^{gh}$ for all $g, h \in G$.
For $G = \mathbb{Z}_2$, $\epsilon(0, 0) = \epsilon(1, 0) = \epsilon(0, 1) = 1$ and $\epsilon(1, 1) = -1$.

Example 1.4.3. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2, \Bbbk = \mathbb{C}, f = (f_1, f_2), g = (g_1, g_2) \in G$. The following forms are bicharacters on G:

$$\epsilon_1(f, g) = (-1)^{(f_1+f_2)(g_1+g_2)}, \qquad G_- = \{(0, 1), (1, 0)\};$$

$$\epsilon_2(f, g) = (-1)^{(f_1g_2+f_2g_1)}, \quad G_- = \emptyset, \quad G_+ = G;$$

$$\epsilon_3(f, g) = (-1)^{(f_2g_1-f_1g_2)}, \quad G_- = \emptyset, \quad G_+ = G.$$

Definition 1.4.2. *Let G be an commutative group, $\Bbbk$ a commutative associative ring, with unity, $\Bbbk^*$ the group of invertible elements of $\Bbbk$, $\epsilon : G \times G \to \Bbbk^*$ a bicharacter. We say that a G-graded algebra $L = \oplus_{g \in G} L_g$ over $\Bbbk$ is a* color Lie superalgebra *if*

$$[a, b] = -\epsilon(d(a), d(b))[b, a], \tag{1.4.1}$$

$$\epsilon(d(c), d(a))[a, [b, c]] + \epsilon(d(a), d(b))[b, [c, a]] + \epsilon(d(b), d(c))[c, [a, b]] = 0, \tag{1.4.2}$$

for all homogeneous $a, b, c \in L$. (1.4.1) *and* (1.4.2) *are refered to as* ϵ-anti-commutativity *and* ϵ-Jacobi identity, *which are analogous to anti-commutativity and Jacobi identity.*

Note 1.4.1. Sometimes it is more convenient to use the super Jacobi identity in the following from:

$$[a, [b, c]] = [[a, b], c] + \epsilon(d(a), d(b))[b, [a, c]],$$

or

$$[[a, b], c] = [a, [b, c]] - \epsilon(d(a), d(b))[b, [a, c]].$$

Note 1.4.2. As in the case of ordinary Lie algebras we denote the operator $y \mapsto [x, y]$ by adx.

Example 1.4.4. Given a group G, a skew-symmetric bicharacter ϵ and an associative algebra A graded by G we can define an ϵ-commutator by

$$[a, b]_\epsilon = ab - \epsilon(d(a), d(b))ba, \tag{1.4.3}$$

for all homogeneous $a, b \in A$. In this context ϵ is refered to as the *commutation factor*. For simplicity we write $[,]_\epsilon = [,]$. Then, it is easy to check that $[,]$ satisfies (1.4.1) and (1.4.2), that is A with this bracket becomes a color Lie superalgebra.

Example 1.4.5. Let G be an abelian group and let V be a G-graded vector space over a field $\Bbbk$. By $End_{\Bbbk}^{g}(V)$, where $g \in G$ we understand the set of all linear $\Bbbk$-operators $f \in End_{\Bbbk}(V)$ such that the image of each homogeneous element $x \in V$ is a homogeneous element and $d(f(x)) = g + d(x)$. Let

$$End_{\Bbbk}^{G}(V) = \oplus_{g \in G} End_{\Bbbk}^{g}(V).$$

Then $End_{\Bbbk}^{G}(V)$ is a G-graded associative $\Bbbk$-algebra with the usual composition of operators. The set $End_{\Bbbk}^{G}(V)$ with operation defined in (1.4.3) is called the general linear color Lie superalgebra and it is denoted $gl(V)$. As we have mentioned earlier in a more general setting, the bracket satisfies (1.4.1) and (1.4.2).

Example 1.4.6.

1. If ϵ is the trivial bicharacter, then a color Lie superalgebra is a G-graded Lie algebra.
2. If $G = \mathbb{Z}_2, \epsilon(f, g) = (-1)^{fg}$, then a color Lie superalgebra is a usual Lie superalgebra.

1.5 Hopf algebras

Throughout we let $\Bbbk$ be a field. Tensor products are assumed to be over $\Bbbk$ unless othewise specified. We first express the associative and unit properties of an algebra via maps so that we can dualize them.

Definition 1.5.1. *A* $\mathbb{k}$-algebra *(with unit) is a* $\mathbb{k}$*-vector space* A *together with two* $\mathbb{k}$*-linear maps, multiplication* $m : A \otimes A \to A$ *and unit* $u : \mathbb{k} \to A$*, such that the following diagrams are commutative:*

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "m \otimes id"] \arrow[d, "id \otimes m"'] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"'] & A
\end{tikzcd}$$

Figure 1.1: Associativity

$$\begin{tikzcd}
& A \otimes A \arrow[dd, "m"] & \\
\mathbb{k} \otimes A \arrow[ur, "u \otimes id"] \arrow[dr] & & A \otimes \mathbb{k} \arrow[ul, "id \otimes u"'] \arrow[dl] \\
& A &
\end{tikzcd}$$

Figure 1.2: Unit

The two lower maps in Figure 1.2 are given by scalar multiplication. We say that A is commutative if $m \circ \tau = m$. We use $\circ$ for the composition of maps.

Definition 1.5.2. *For any* $\mathbb{k}$*-spaces* V *and* W*, the map* $\tau : V \otimes W \to W \otimes V$ *given by* $\tau(v \otimes w) = w \otimes v$ *is called the* twist map.

Definition 1.5.3. *A* $\mathbb{k}$-coalgebra *(with counit) is a* $\mathbb{k}$*-vector space* C *together with two* $\mathbb{k}$*-linear maps, comultiplication* $\Delta : C \to C \otimes C$ *and counit* $\epsilon : C \to \mathbb{k}$*, such that the following diagrams are commutative:*

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta \otimes id"'] \\
C \otimes C \arrow[r, "id \otimes \Delta"'] & C \otimes C \otimes C
\end{tikzcd}$$

Figure 1.3: Coassociativity

$$\begin{tikzcd}
& C \arrow[dl, "1\otimes"'] \arrow[dr, "\otimes 1"] \arrow[dd, "\Delta"] & \\
\mathbb{k} \otimes C & & C \otimes \mathbb{k} \\
& C \otimes C \arrow[ul, "\epsilon \otimes id"] \arrow[ur, "id \otimes \epsilon"'] &
\end{tikzcd}$$

Figure 1.4: Counit

The two upper maps in Figure 1.4 are given by $c \mapsto 1 \otimes c$ and $c \mapsto c \otimes 1$, for any $c \in C$. We say that C is cocommutative if $\tau \circ \Delta = \Delta$.

Definition 1.5.4. *Let C and D be coalgebras, with comultiplications Δ_C and Δ_D, and counits ϵ_C and ϵ_D, respectively. A linear map $f : C \to D$ is a* coalgebra morphism *if $\Delta_D \circ f = (f \otimes f)\Delta_C$ and $\epsilon_D \circ f = \epsilon_C$.*

We recall Lemma 1.2.2 from [11]. If C is a coalgebra, then C^* is an algebra, with multiplication $m = \Delta^*$ and unit $u = \epsilon^*$. By C^* we denote the dual space of C. We mention that $\Delta^* : C^* \otimes C^* \to C^*$ is defined as $\Delta^*(f \otimes g)(c) = (f \otimes g)\Delta(c)$, where $f, g \in C^*, c \in C$ and that $u : \Bbbk \to C^*$ and it is defined by $u(\alpha)(c) = \alpha\epsilon(c)$, for $\alpha \in \Bbbk, c \in C$.
Now, we combine the definitions of algebra and coalgebra.

Definition 1.5.5. *A $\Bbbk$-space B is a* bialgebra *if (B, m, u) is an algebra, (B, Δ, ϵ) is a coalgebra, and either of the following (equivalent) conditions holds:*

1. *Δ and ϵ are algebra morphisms,*

2. *m and u are coalgebra morphisms.*

Example 1.5.1. If B is any bialgebra, we can form a new bialgebra by taking the opposite of either the algebra or coalgebra structure. Thus B^{cop} has the same multiplication but the opposite comultiplication, B^{op} has the opposite multiplication but the same comultiplication.

Note 1.5.1. Let C be any coalgebra with comultiplication $\Delta : C \to C \otimes C$. The *sigma notation* for Δ is given as follows: for any $c \in C$, we write

$$\Delta c = \sum c_{(1)} \otimes c_{(2)}.$$

Example 1.5.2. The field $\Bbbk$, with its algebra structure, and with the canonical coalgebra structure ($\Delta(\alpha) = \alpha(1 \otimes 1)$, $\epsilon(\alpha) = \alpha 1$), is a bialgebra.

Example 1.5.3. If G is a monoid, then the semigroup algebra $\Bbbk G$, endowed with a coalgebra structure $\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$ for any $g \in G$, is a bialgebra.

Definition 1.5.6. *Let C be a coalgebra and A be an algebra. Then $Hom_{\Bbbk}(C, A)$ becomes an algebra under the* convolution product*:*

$$(f * g)(c) = \sum f(c_{(1)})g(c_{(2)}),$$

for all $f, g \in Hom_{\Bbbk}(C, A)$, $\quad c \in C$.

Definition 1.5.7. *Let $(H, m, u, \Delta, \epsilon)$ be a bialgebra. Then H is a* Hopf algebra *if there exists an element $S \in Hom_{\Bbbk}(H, H)$ which is an inverse to id_H under the convolution product. S is called an* antipode *for H.*

Definition 1.5.8. *Let C be a coalgebra. C is* pointed *if every simple subcoalgebra is one-dimensional.*

Remark 1.5.1. In a Hopf algebra, the antipode is unique. The fact that $S : H \to H$ is the antipode is written as $S * I = I * S = u\epsilon$, and using the sigma notation: $\sum S(h_1)h_2 = \sum h_1 S(h_2) = \epsilon(h)1$ for any $h \in H$.

Note 1.5.2. For a bialgebra B, recall that B^{cop} is a bialgebra. It may happen that B^{cop} is a Hopf algebra, with antipode $\bar{S}$. Then
$\sum(\bar{S}(h_2))h_1 = \sum h_2(\bar{S}(h_1)) = \epsilon(h)1$.

Lemma 1.5.1. ([11]) *Let B be a bialgebra. Then B is a Hopf algebra with an invertible antipode S under $\circ \Leftrightarrow B^{cop}$ is a Hopf algebra with invertible antipode $\bar{S}$. In this situation, $S \circ \bar{S} = \bar{S} \circ S = id$.*

Corollary 1.5.1. ([11]) *If H is commutative or cocommutative, then $S^2 = id$.*

Examples of Hopf algebras:

Example 1.5.4. The group algebra.
Let G be a group, and $H = \Bbbk G$ the associated group algebra. Then by defining $S(g) = g^{-1}$ for each $g \in G$, we obtain that H is a Hopf algebra, recall (Example 1.5.3).

Example 1.5.5. Sweedler's 4-dimensional Hopf algebra.
Assume that *char* $\Bbbk \neq 2$. Let H be the algebra given by generators and relations as follows: H is generated as a $\Bbbk$-algebra by c and x satisfying the relations

$$c^2 = 1, \quad x^2 = 0, \quad xc = -cx.$$

Then H has dimension 4 as a $\Bbbk$-vector space, with basis $\{1, c, x, cx\}$.
The coalgebra structure is induced by

$$\Delta(c) = c \otimes c, \quad \Delta(x) = c \otimes x + x \otimes 1, \quad \epsilon(c) = 1, \quad \epsilon(x) = 0.$$

In this way, H becomes a bialgebra, which also has an antipode S given by $S(c) = c^{-1}$, $S(x) = -cx$. This was the first example of a non-commutative and non-cocommutative Hopf algebra.

Example 1.5.6. The Taft algebras.
Let $n \geq 2$ be an integer, and λ a primitive nth root of unity. Consider the algebra $H_{n^2}(\lambda)$ defined by the generators c and x with relations

$$c^n = 1, \quad x^n = 0, \quad xc = \lambda cx.$$

On this algebra we can introduce a coalgebra structure induced by

$$\Delta(c) = c \otimes c, \quad \Delta(x) = c \otimes x + x \otimes 1, \quad \epsilon(c) = 1, \quad \epsilon(x) = 0.$$

In this way $H_{n^2}(\lambda)$ becomes a bialgebra of dimension n^2, having the basis $\{c^i x^j | 0 \leq i, j \leq n-1\}$. The antipode is defined by $S(c) = c^{-1}$,
$S(x) = -c^{-1}x$. We note that for $n = 2$ and $\lambda = -1$ we obtain Sweedler's 4-dimensional Hopf algebra.

Recall that we defined the notion of commutation factors for groups. Now we generalize it for Hopf algebras.

Definition 1.5.9. *Let H be a Hopf algebra.*

1. *A function $\beta : H \otimes H \to \Bbbk$ is called a* bicharacter *on H if β is bilinear and for all $h, k, l \in H$:*

   (a) $\beta(hk, l) = \sum \beta(h, l_1)\beta(k, l_2)$,

   (b) $\beta(h, kl) = \sum \beta(h_2, k)\beta(h_1, l)$,

   (c) β is normal, i.e. $\beta(h, 1) = \beta(1, h) = \epsilon(h)$,

   (d) β is convolution invertible.

2. *β is called skew-symmetric under $*$ if $(\beta)^{-1} = \beta \circ \tau$.*

Definition 1.5.10. *For a $\Bbbk$-algebra A, a* left A-module *is a $\Bbbk$-space M with a $\Bbbk$-linear map $\gamma : A \otimes M \to M$ such that the following diagrams commute:*

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "m \otimes id"] \arrow[d, "id \otimes \gamma"'] & A \otimes M \arrow[d, "\gamma"'] \\ A \otimes M \arrow[r, "\gamma"'] & M \end{tikzcd} \qquad \begin{tikzcd} \mathbb{k} \otimes M \arrow[r, "u \otimes id"] \arrow[dr, "\text{scalar mult.}"'] & A \otimes M \arrow[d, "\gamma"] \\ & M \end{tikzcd}$$

Figure 1.5 Figure 1.6

Definition 1.5.11. *For a $\mathbb{k}$-coalgebra C, a* right C-comodule *is a pair (M, ρ), where M is a $\mathbb{k}$-vector space, $\rho : M \to M \otimes C$ is a morphism of $\mathbb{k}$-vector spaces such that the following diagrams are commutative:*

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\rho"'] & M \otimes C \arrow[d, "id \otimes \Delta"] \\ M \otimes C \arrow[r, "\rho \otimes id"'] & M \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\rho"] \arrow[dr, "\otimes 1"'] & M \otimes C \arrow[d, "id \otimes \epsilon"] \\ & M \otimes \mathbb{k} \end{tikzcd}$$

Figure 1.7 Figure 1.8

Note 1.5.3. In other words, we write this in the sigma notation for right comodules: $\rho(m) = \sum m_{(0)} \otimes m_{(1)} = \sum m_0 \otimes m_1 \in M \otimes C$. If M_1, M_2 are H-comodules, with the structure maps $\rho_{M_1} : m \to \sum m_0 \otimes m_1$ and

$\rho_{M_2} : n \to \sum n_0 \otimes n_1$, then $M_1 \otimes M_2$ becomes a H-comodule with the structure map $\rho_{M_1 \otimes M_2} : m \otimes n \to \sum m_0 \otimes n_0 \otimes m_1 n_1$.

Definition 1.5.12. *A vector subspace $N \subset M$ is called a* subcomodule *if $\rho(N) \subset N \otimes C$.*

The least subcomodule containing a subset X is denoted by $\langle X \rangle$. If X is a one-element set, say $X = \{b\}$, then we simply write $\langle X \rangle = \langle b \rangle$. An important property of comodules is the following one.

Proposition 1.5.1. ([11]) *Any subcomodule $\langle X \rangle$ generated by a finite set X is finite-dimensional.*

Example 1.5.7. ([11]) Let $C = \Bbbk G$. Then M is a right $\Bbbk G$-comodule if and only if M is a G-graded $\Bbbk$-space; that is $M = \oplus_{g \in G} M_g$. Here, $M_g = \{m \in M | \rho(m) = m \otimes g\}$.

Definition 1.5.13. *Let H be a Hopf algebra and A a $\Bbbk$-algebra. We say that H coacts on A (or that A* is a right *H*-comodule algebra*) if the following conditions are fulfilled:*

1. *A is a right H-comodule, with structure map*
   $\rho : A \to A \otimes H, \quad \rho(a) = \sum a_0 \otimes a_1,$
2. $\sum (ab)_0 \otimes (ab)_1 = \sum a_0 b_0 \otimes a_1 b_1$, *for all $a, b \in A$,*
3. $\rho(1) = 1_A \otimes 1_H$.

Left H-comodule algebras are defined similarly.

Definition 1.5.14. *A Hopf algebra is called* coquasitriangular(CQT) *if there exists a bicharacter* $\langle | \rangle : H \otimes H \to \Bbbk$*, such that for all* $h, m, l \in H$,

$$\sum \langle h_1 | m_1 \rangle m_2 h_2 = \sum h_1 m_1 \langle h_2 | m_2 \rangle.$$

A CQT *Hopf algebra with a skew-symmetric bicharacter is called a* cotriangular Hopf algebra.

Example 1.5.8. Let H be any commutative Hopf algebra. Then H is CQT by taking $\langle h|m \rangle = \epsilon(h)\epsilon(m)$, the trivial braiding.

Example 1.5.9. ([11]) Let $H = \Bbbk G$ the group algebra. Because H is cocommutative and $\langle | \rangle$ is invertible, the conditions in Definition 1.5.14 become:

1. $\langle h|gl \rangle = \langle h|g \rangle \langle h|l \rangle$,

2. $\langle hg|l \rangle = \langle h|l \rangle \langle g|l \rangle$,

for all $h, g, l \in G$. That is, G is abelian and the form $\langle | \rangle$ is a bicharacter on G. See Definition 1.4.1 with addition replaced by multiplication.

Note 1.5.4. It follows from above that if $H = \Bbbk G$, for G a non-abelian group, then H can not be CQT.

1.6 (H, β)-Lie algebras

Definition 1.6.1. *Assume that* (H, β) *is cotriangular. A* (right) (H,β)-Lie algebra *is a (right)* H*-comodule* L *together with a* β*-Lie bracket* $[,] : L \otimes L \to L$ *which is an* H*-comodule morphism satisfying, for all* $a, b, c \in L$:

1. *β-anticommutativity:* $[a, b] = -\beta(a_1, b_1)[b_0, a_0]$,

2. *β-Jacobi identity:*

$$\beta(c_1,a_1)[[a_0,b],c_0]+\beta(b_1,c_1)[[c_0,a],b_0]+\beta(a_1,b_1)[[b_0,c],a_0]=0.$$

Note 1.6.1. As usual, for any fixed $x \in L$ and any $y \in L$, the map $y \mapsto [x,y]$ is denoted by adx.

Definition 1.6.2. *Let H be a pointed cocommutative Hopf algebra with a skew-symmetric bicharacter β. We can define the decompositon $H = H_+ \oplus H_-$, where H_+ is as normal subHopf algebra and H_- a subcoalgebra, and the* sign bicharacter β_0 *on H as follows: for any homogeneous $h,k \in H$,*

$$\beta_0(h,k)=\begin{cases} -\epsilon(h)\epsilon(k) & \text{if } h,k \in H_-, \\ \epsilon(h)\epsilon(k) & \text{otherwise.}\end{cases}$$

Example 1.6.1.

1. Let $\beta=\epsilon\otimes\epsilon$, be the trivial bicharacter. Then an $(H,\epsilon\otimes\epsilon)$-Lie algebra is an ordinary Lie algebra L with an H-comodule structure such that $[,]$ is an H-comodule morphism.

2. Let $\beta=\beta_0$, the sign bicharacter. Then an (H,β_0)-Lie algebra is an ordinary Lie superalgebra $L=L_0\oplus L_1$, such that L is an H-comodule and $[,]$ is an H-comodule morphism.

3. Let A be an H-comodule algebra, β any bicharacter on H. Define $[a,b]_\beta = ab - \sum \beta(a_1,b_1)a_0b_0$.

Throughout this section $\Bbbk$ will be a commutative ring, and the Hopf algebra H will be an arbitrary Hopf algebra, unless othewise specified. In order to differentiate between coactions and comultiplication, when A is a right H-comodule with coaction $\rho_A : A \to A\otimes H$, we write $\rho_A(a)=\sum a_{(0)}\otimes a_{(1)}$.

In particular, note that definition of right comodule means, in this notation, that:

$$\sum a_{(0)} \otimes (a_{(1)})_{(1)} \otimes (a_{(1)})_{(2)} = \sum (a_{(0)})_{(0)} \otimes (a_{(0)})_{(1)} \otimes a_{(1)}.$$

An important concept that we deal with here is that of an algebra or a Hopf algebra in a category; this means that the structure maps are morphisms in the category. Denote the category of right H-modules by $\mathcal{M}_{\mathcal{H}}$, and that of right H-comodules by $\mathcal{M}^{\mathcal{H}}$. In these terms, the right H-module algebra becomes an algebra in $\mathcal{M}_{\mathcal{H}}$, and a right H-module coalgebra becomes a coalgebra in $\mathcal{M}_{\mathcal{H}}$. A right H-comodule algebra is actually an algebra in $\mathcal{M}^{\mathcal{H}}$, and so satisfies

$$\rho_A(ab) = \sum a_{(0)}b_{(0)} \otimes a_{(1)}b_{(1)},$$

and

$$\rho_A(1_A) = 1_A \otimes 1_H.$$

Similarly, a right H-comodule coalgebra is a coalgebra in $\mathcal{M}^{\mathcal{H}}$, and the fact that Δ_C and ϵ_C are comodule maps is expressed by:

$$\sum (c_1)_{(0)} \otimes (c_2)_{(0)} \otimes (c_1)_{(1)}(c_2)_{(1)} = \sum (c_{(0)})_1 \otimes (c_{(0)})_2 \otimes c_{(1)},$$

and

$$\sum \epsilon_C(c_{(0)})\epsilon_H(c_{(1)}) = \epsilon_C(c).$$

The following lemma, was first stated in [5], as Lemma 2.10. In that paper, the proof is given for left H-comodules and the proof makes use of $\bar{S}$, the antipode for H^{cop}. Here we give the proof in the case of right comodules. We will assume that the comodules are right, unless otherwise stated.

Lemma 1.6.1. *If V is a finite-dimensional object in $\mathcal{M}^{\mathcal{H}}$, then $End(V)$ is an algebra in $\mathcal{M}^{\mathcal{H}}$, and the evaluation map $\varphi : End(V) \otimes V \to V$ is an $\mathcal{M}^{\mathcal{H}}$-morphism.*

Proof: The finite dimensionality of V is the key point here, for when V is finite-dimensional, not only is $End(V) \cong V \otimes V^*$, but also the coaction $\rho : V \to V \otimes H$ has its image in $V \otimes C$, where C is a finite-dimensional subcoalgebra of H. Thus we may define a left H-coaction on V^* as follows. Given the coaction $\rho : V \to V \otimes C$, there is a standard dual action $\alpha : C^* \otimes V \to V$ defined by $\alpha(f \otimes v) = \sum f(v_{(1)})v_{(0)}$, for all $v \in V, f \in C^*$; its dual α^* defines a left coaction $\alpha^* : V^* \to (C^* \otimes V)^* \cong C \otimes V^*$, via

$$\alpha^*(v^*)(f \otimes v) = v^*(\alpha(f \otimes v)) = v^*(\sum f(v_{(1)})v_{(0)}) = \sum f(v_{(1)})v^*(v_{(0)}).$$

Writing this in our usual comodule notation gives:

$$\alpha^*(v^*)(f \otimes v) = \sum((v_*)_{(1)} \otimes (v_*)_{(0)})(f \otimes v) = \sum(v^*)_{(-1)}(f)(v^*)_{(0)}(v),$$

for all $v^* \in V^*, v \in V, f \in C^*$. Hence we have the equality:

$$\sum f(v_{(1)})v^*(v_{(0)}) = \sum(v^*)_{(-1)}(f)(v^*)_{(0)}(v).$$

Thus, for all $f \in H^*$, $f(\sum v^*(v_{(1)})(v_{(0)})) = f(\sum(v^*)_{(-1)}(v^*)_{(0)}(v))$. The non-degeneracy of evaluation $H^* \otimes H \to \Bbbk$ now implies

$$\sum v^*(v_{(1)})(v_{(0)}) = \sum(v^*)_{(-1)}(v^*)_{(0)}(v), \tag{1.6.1}$$

However, we are interested in right H-comodules; V^* now becomes a right H-comodule via the coaction

$$\beta : V^* \to V^* \otimes H, \qquad v^* \mapsto \sum v^*_{(0)} \otimes S(v^*_{(-1)})$$

and so $V \otimes V^*$ is a right H-comodule via

$$v \otimes v^* \mapsto \sum v_{(0)} \otimes v^*_{(0)} \otimes v_{(1)} S(v^*_{(-1)}), \tag{1.6.2}$$

This is seen to be an algebra coaction in the usual way; and thus $End(V) \cong V \otimes V^*$ is an algebra in $\mathcal{M}^{\mathcal{H}}$.

Finally, we show that the evaluation map φ is in $\mathcal{M}^{\mathcal{H}}$, by verifying commutativity of the diagram:

$$\begin{tikzcd}
V \otimes V^* \otimes V \arrow[r, "\varphi"] \arrow[d, "\rho_{V \otimes V^* \otimes V}"'] & V \arrow[d, "\rho_V"] \\
V \otimes V^* \otimes V \otimes H \arrow[r, "\varphi \otimes id"'] & V \otimes H
\end{tikzcd}$$

Figure 1.9

Let $v, w \in V, v^* \in V^*$. Then:

$$\rho_V \circ \varphi(v \otimes v^* \otimes w) = \rho_V(v^*(w)v) = \sum v_{(0)} \otimes v^*(w) v_{(1)}.$$

On the other hand,

$$\begin{aligned}
& (\varphi \otimes id)(\rho_{V \otimes V^* \otimes V})(v \otimes v^* \otimes w) \\
= \; & (\varphi \otimes id)(\textstyle\sum v_{(0)} \otimes (v^*)_{(0)} \otimes w_{(0)} \otimes v_{(1)} S(v^*_{(-1)}) w_{(1)} \\
= \; & \textstyle\sum v^*_{(0)}(w_{(0)}) v_{(0)} \otimes v_{(1)} S((v^*)_{(-1)}) w_{(1)} \\
= \; & \textstyle\sum v_{(0)} \otimes v_{(1)} S(v^*_{(-1)}) v^*_{(0)}(w_{(0)}) w_{(1)} \\
= \; & \textstyle\sum v_{(0)} \otimes v_{(1)} S((v^*_{(-1)})(v^*_{(0)})_{(0)}(w)(v^*_0)_{(-1)} \\
= \; & \textstyle\sum v_{(0)} \otimes v_{(1)} S(v^*_{(-2)}) v^*_{(0)}(w) v^*_{(-1)} \\
= \; & \textstyle\sum v_{(0)} \otimes v_{(1)} v^*(w),
\end{aligned}$$

as required. On the fifth step we use (1.6.1). □

Chapter 2

Color Lie superalgebras

In this chapter, we state and prove Engel's theorem for the color Lie superalgebra case.

2.1 Preliminaries

Lemma 2.1.1. *Let L be a color Lie superalgebra, $x, y, z \in L$, homogeneous in L, $d(x) \in G_+, d(y) \in G_-$, $2,3 \in \mathbb{k}^*$. Then*

1. $[x, x] = 0$,
2. $[[y, y], y] = 0$,
3. $[[y, y], z] = 2[y, [y, z]]$.

Proof: We note $\beta(d(x), d(x)) = 1$. We make use of (1.4.1) and Note 1.4.1.

1. Since $[x, x] = -\beta(d(x), d(x))[x, x] = -[x, x] \Rightarrow 2[x, x] = 0 \Rightarrow [x, x] = 0$.

2. $[[y, y], y] = [y, [y, y]] - \beta(h, h)[y, [y, y]] = 2[y, [y, y]] = -2\beta(h, 2h)[[y, y], y] = -2\beta(h, h)\beta(h, h)[[y, y], y] = -2[[y, y], y] \Rightarrow$
$3[[y, y], y] = 0 \Rightarrow [[y, y], y] = 0$ for any $y \in L_h$, where h is the degree of y.

3. For $y \in L_h, z \in L$ we derive
$[[y, y], z] = [y, [y, z]] - \beta(h, h)[y, [y, z]] = 2[y, [y, z]]$. □

2.2 About graded vector spaces

We will present in this subsection two lemmas related to vector spaces.

Lemma 2.2.1. *Let V be a nonzero G-graded vector space and $b : V \to V$ a graded transformation that is nilpotent, i.e. $b^n = 0$, for some n. Then there exists $0 \neq w \in V$, such that w is homogeneous and $bw = 0$.*

Proof: Consider $b : V \to V$ a graded nilpotent transformation and its kernel $W = Kerb = \{w \in V | bw = 0\}$. W is a nonzero subspace, since b is nilpotent. Also, W is a graded subspace, i.e. $W = \oplus_{g \in G} W_g$, because b is a graded transformation. Hence, $W_g \neq 0$ for some $g \in G$ and we can pick $0 \neq w \in W_g$.

□

Lemma 2.2.2. *Let $V \neq 0$ be a G-graded vector space over $\mathbb{k}$, G a group. If x is a nilpotent endomorphism of V, then the map $ad(x)$is nilpotent.*

Proof: Using the definition of adx, see (1.4.3), and replacing ϵ by β, we compute:
$(ad(x))^m(y) = (ad(x))^{m-1}(xy - \beta(d(x), d(y))yx)$

$= (ad(x))^{m-2}(x^2y - \beta(d(x)+d(y), d(x))xyx - \beta(d(x), d(y))xyx + \beta(d(x), d(x)+d(y))yx^2) = ... = \sum_{i+j=m} \lambda_{i,j} x^i y x^j$, where $\lambda_{i,j} \in \Bbbk$.

Since x is nilpotent, we obtain $x^l = 0$ for some l. Setting $m = 2l-1$ in the formula above, we see that all the terms $\lambda_{i,j} x^i y x^j$ vanish since either i or j must be $\geq l$. Therefore, $(ad(x))^{2l-1}(y) = 0$ for any y. This proves that $ad(x)$ is nilpotent. □

2.3 Engel's theorem

Theorem 2.3.1. (Engel's theorem for color Lie superalgebras)

Let G be an abelian group, V a G-graded vector space, $\beta : G \times G \to \Bbbk^*$ a bicharacter, char $\Bbbk \neq 2, 3$ and let L be a finite-dimensional graded β-Lie subalgebra of $gl(V)$, whose homogeneous elements are nilpotent. If $V \neq \{0\}$, then there exists a nonzero homogeneous $u \in V$, such that $xu = 0$, for all $x \in L$.

Proof: The proof goes by induction on $\dim L$. If $\dim L = 0$, then the claim is obvious. Assume that $\dim L > 0$. We have that $L = \oplus_{g \in G} L_g$, where $L_g = (gl(V))_g \cap L$. Consider a maximal proper graded subalgebra M of L. Hence $M = \oplus_{g \in G} M_g$, where $M_g = L_g \cap M$. For $x \in M$ we have $ad_L x : L \to L, \quad y \mapsto [x, y]$. M is subalgebra, so M is an invariant subspace of L under the action $ad_M : M \to M, \quad y \mapsto [x, y]$. We consider M as an M-submodule of L, so we can induce L/M to be an M factor module with action $ad_{L/M} : L/M \to L/M, \quad y + M \mapsto [x, y] + M$, where $x \in M, y \in L$.

Note 2.3.1. M is adx-invariant, so $ad_{L/M} x$ is well-defined.

Proof (of the note):

We show that if $y, y' \in L$ and $y - y' \in M$, then $[x, y] - [x, y'] \in M$. But, $[x, y] - [x, y'] = [x, y - y'] \subseteq M$ (because M is a subalgebra).

Now, we also have that $M = \oplus_{g \in G} M_g$ and $L = \oplus_{g \in G} L_g$, so L/M is a G-graded space, i.e. $L/M = \oplus_{g \in G}(L_g + M)/M$. Moreover,

Note 2.3.2. $ad_{L/M}M$ is a graded subalgebra of $gl(L/M)$.

Proof (of the note):

Consider $ad_{L/M}x \in ad_{L/M}M$. Decompose $x = \sum_{g \in G} x_g, x_g \in M_g$. Then $ad_{L/M}x = ad_{L/M}\sum_{g \in G} x_g = \sum_{g \in G} ad_{L/M}x_g$; $ad_{L/M}x_g \in ad_{L/M}M$ and $ad_{L/M}x_g$ is a homogeneous element of degree g , since: $ad_L x_g : L_h \to L_{h+g}$ and $ad_M x_g : M_h \to M_{h+g}$ implies $ad_{L/M}x_g : (L/M)_h \to (L/M)_{h+g}$, and so, $ad_{L/M}x_g \in gl(L/M)_g$.

Also, we have

$$\dim ad_{L/M}M \leq \dim M < \dim L. \tag{2.3.1}$$

From Lemma 2.2.2, we have that $ad_L x$ is nilpotent, hence

$$ad_{L/M}x \qquad \text{is nilpotent.} \tag{2.3.2}$$

By (2.3.1) and (2.3.2), we apply the induction hypothesis to $ad_{L/M}M \subset gl(L/M)$ and obtain the existence of a nonzero homogeneous $\bar{a} \in L/M$, such that $(ad_{L/M}x)(\bar{a}) = 0$ for all $x \in M$. We express $\bar{a} = b + M, b \in L$. Because $\bar{a}$ is homogeneous of degree g, $\bar{a} \in L_g + M/M \Rightarrow \bar{a} = b + M$, $b \in L_g \Rightarrow b$ is homogeneous.

Observation 2.3.1. $alg\{b\} = Span\{b, [b, b]\}$.

Proof:

1. $[b,b] \in alg\{b\}$,
2. $[b,[b,b]] = 0$ (using Lemma 2.1.1),
3. $[[b,b],b] = -[b,[b,b]] = 0$,
4. $[[b,b],[b,b]] = [[[b,b],b],b] + \beta(2b,b)[b,[[b,b],b]] = 0$.

Now, $M + alg\{b\}$ is a graded space, which is a subalgebra of L, because $alg\{b\}$ is a subalgebra and:

1. If $x, y \in M \Rightarrow [x,y] \in M$,
2. If $x \in M, b \in alg\{b\} \Rightarrow [x,b] \in M$,
3. If $x \in M$, then $[x,[b,b]] \in M$, because $[x,[b,b]] = [[x,b],b] + \beta(x,b)[b,[x,b]] \in M$.

Hence we have proved that $M + alg\{b\}$ is a graded subalgebra of L. By maximality of M, we get $M + alg\{b\} = L$. Using the fact that $\dim M < \dim L$, we apply the induction hypothesis to $M \subset gl(V)$. Consider $W = \{v \in V | xv = 0, \quad \forall x \in M\}$. We have that W is a G-graded space of V because:

1. $\oplus_{g\in G}(W \cap V_g) = \oplus_{g\in G} W_g \subset W$ (which is obvious);
2. Let $v \in W \Rightarrow xv = 0, x \in M$ homogeneous of degree h. Hence, $x(\sum_g v_g) = \sum_g xv_g = 0 \Rightarrow xv_g = 0$, for all $v_g \in W$. So, W is a direct sum of W_g and $xv_g \in W_{g+h}$.

W is b-invariant, because: $xb(v) = [x,b]v + \beta(x,b)bx(v) = 0$, ($[x,b] \in M$) for all homogeneous $x \in M$. Thus $b' = b_{|W}$ is nilpotent and homogeneous. By Lemma 2.2.1, applied to b', we get the existence of a $w \in W$ homogeneous such that $b'w = 0$. Hence $bw = 0$, $[b,b]w = 0$, but $Mw = 0$, so we finally get $Lw = 0$, since $L = M + alg\{b\}$. □

Note 2.3.3. Now, we state Engel's theorem for the particular case of Lie superalgebras: Let V be a $\mathbb{Z}_2$-graded vector space over $\Bbbk$ and L be a finite-dimensional $\mathbb{Z}_2$-graded subalgebra of $gl(V)$ whose homogeneous elements are nilpotent endomorphisms of V. If $V \neq 0$, then there exists a nonzero homogeneous $v \in V$, such that $xv = 0$, for all $x \in L$.

Note 2.3.4. If L is an ordinary graded Lie algebra, then we obtain the ordinary Engel's Theorem 1.2.1. , with an additional property that v is homogeneous.

Corollary 2.3.1. *Let V be a finite-dimensional G-graded vector space, G an abelian group, β skew-symmetric bicharacter $\beta : G \times G \to \Bbbk^*$, L be a graded finite-dimensional subspace of $gl(V)$ closed under the β-commutator, consisting of nilpotent elements. Then there exists a basis of V, such that all elements of L have strictly lower triangular matrices.*

Proof: Consider $\dim V = n$. By Theorem 2.3.1. we can find a homogeneous $v = v_n \in V$, $v_n \neq 0$ such that $Lv_n = 0$. Because v_n is homogeneous, $\Bbbk v_n$ is graded.

Consider $\tilde{V} = V/\Bbbk v_n$. We denote $\tilde{v} = v + \Bbbk v_n$, where $\tilde{v} \in \tilde{V}$ for $v \in V$.

Define $\tilde{x}(\tilde{v}) = xv + \Bbbk v_n$. Consider the homomorphism map $\varphi : L \to gl(\tilde{V})$ sending x to $\tilde{x}$. Consider the new algebra $\tilde{L} = \langle \tilde{x} | x \in L \rangle \subset End(\tilde{V})$. For any

$x \in L$, $\tilde{x}$ is nilpotent because

$$(\tilde{x})^n(\tilde{v}) = x^n v + \Bbbk v_n = \Bbbk v_n = 0_{\tilde{V}}.$$

So, $\tilde{L}$ consists of nilpotent transformations, $\tilde{L}$ is G-graded , so we can apply induction, and we obtain a basis $\{\tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_{n-1}\}$ for $\tilde{V}$ such that matrices of elements of $\tilde{L}$ are lower triangular i.e.

$$\tilde{x}(\tilde{v}_i) \in Span\{\tilde{v}_{i+1}, ..., \tilde{v}_{n-1}\}. \tag{2.3.3}$$

For any $x \in L$ we get

$$\tilde{x}(\tilde{v}_i) = xv_i + \Bbbk v_n. \tag{2.3.4}$$

So, by (2.3.3) and (2.3.4) we get

$$x(v_i) + \Bbbk v_n \in Span\{v_{i+1}, ..., v_{n-1}\} + \Bbbk v_n,$$

hence, $x(v_i) \in Span\{v_{i+1}, ..., v_{n-1}, v_n\}$. For $i \in \{1, 2, ..., n-1\}$ we have $xv_n = 0$, hence the matrix of x is strictly lower triangular in the basis $\{v_1, ...v_n\}$. □

Corollary 2.3.2. *Let V be a finite-dimensional G-graded vector space over a field $\Bbbk$ and a skew-symmetric bicharacter $\beta : G \times G \to \Bbbk^*$ on an abelian group. Let $L \subset gl(V)$ to be a subalgebra of $gl(V)$ under β commutator and R an associative subalgebra generated by L. If L consists of nilpotent operators, then R is nilpotent.*

Proof: By Corollary 2.3.1, we can find a basis in V such that all matrices of elements of L are strictly lower triangular. Then all the matrices of R are also strictly lower triangular. So, R is nilpotent. □

Chapter 3

(H, β)-Lie algebras

In this chapter, we state and prove Engel's theorem for the (H, β)-Lie algebra case.

3.1 Some auxiliary results

Observation 3.1.1. Fix a basis $\{e_j\}$ in a right H-comodule W. Then we can write $\rho(e_j) = \sum_i e_i \otimes c_{ij}$, where c_{ij} are uniquely defined elements of H. We claim that:

1. $\Delta c_{ij} = \sum_k c_{ik} \otimes c_{kj}$,
2. $\epsilon(c_{ij}) = \delta_{ij}$.

Proof: $(1 \otimes \Delta)\rho(e_j) = (\rho \otimes 1)\rho(e_j)$. But, $(1 \otimes \Delta)\rho(e_j) = \sum_i e_i \otimes \Delta c_{ij}$, and $(\rho \otimes 1)\rho(e_j) = \sum_k \rho(e_k) \otimes c_{kj} = \sum_i \sum_k e_i \otimes c_{ik}c_{kj}$. Hence,

$$\Delta c_{ij} = \sum_k c_{ik} \otimes c_{kj}.$$

$(1\otimes\epsilon)\rho(w)=w$ for any $w\in W$, and $\rho(e_j)=\sum_i e_i\otimes c_{ij}$.

Hence, by letting $w=e_j$, we obtain:

$$(1\otimes\epsilon)\rho(e_j)=\sum_i\epsilon(c_{ij})e_i=e_i\Rightarrow\epsilon(c_{ij})=\delta_{ij}.$$

Also, $\sum c_1Sc_2=\sum(Sc_1)c_2=\epsilon(c)$. Take $c=c_{ij}$. Hence,

$$\sum(c_{ij})_1S(c_{ij})_2=\sum(S(c_{ij})_1)(c_{ij})_2=\epsilon(c_{ij})=\delta_{ij}\Leftrightarrow$$

$$\sum_k c_{ik}Sc_{kj}=\sum_k(Sc_{ik})c_{kj}=\delta_{ij}.$$ □

Lemma 3.1.1. *Let H be a Hopf algebra, W any finite-dimensional H-comodule. Consider $End(W)$ with the comodule structure defined in Lemma 1.6.1. If $D\in End(W)$ and $T\in End(W)\otimes H$, then $\rho''(D)=T$ if and only if for any $w\in W$, $\rho(Dw)=T\rho(w)$.*

Proof: Due to the finite dimensionality of W, consider $W=Span\{e_1,...,e_n\}$. Choose the dual basis $\{e^1,...,e^n\}$ in W^* and consider the coaction $\rho':W^*\to H\otimes W^*$, corresponding to α^* as in Lemma 1.6.1. Suppose $\rho'(e^i)=\sum_j b_{ij}\otimes e^j$. Then (1.6.1) gives for $v^*=e^i$ and $v=e_j$ and $\sum(e_j)_0\otimes(e_j)_1=\sum_k e_k\otimes c_{kj}$ and $\sum(e^i)_{-1}\otimes(e^i)_0=\sum_k b_{ik}\otimes e^k$ that $\sum_k e^k(e_j)b_{ik}=\sum_k e^i(e_k)c_{kj}$. Therefore, $b_{ij}=c_{ij}$. Then by (1.6.2), and taking into account that $E_{ij}=e_i\otimes e^j$ upon identification of $End(W)$ with $W\otimes W^*$, we obtain: $\rho(e_i)=\sum_k e_k\otimes c_{ki}$ and $\rho'(e^j)=\sum_l c_{jl}\otimes e^l$. That is $\rho''(e_i\otimes e^j)=\sum_{k,l}e_k\otimes e^l\otimes c_{ki}S(c_{lj})$. Hence $\rho''(E_{ij})=\sum_{k,l}E_{kl}\otimes c_{ki}S(c_{jl})$.

Also we know that

$$E_{ij}e_j=e_i. \tag{3.1.1}$$

Let $D \in End(W)$, expressed as

$$D = \sum d_{kl} E_{kl}. \tag{3.1.2}$$

By (3.1.1) and (3.1.2), we get that $De_k = \sum d_{sk} e_s$.

Also, because $T \in EndW \otimes H$ we represent T uniquely as $T = \sum E_{kl} \otimes h_{kl}$. Then $\rho''(D) = T$ is equivalent to

$$h_{kl} = \sum_{ij} c_{ki} S(c_{jl}) d_{ij}. \tag{3.1.3}$$

Now, $\rho(Dw) = T\rho(w)$ for any $w \in W$ is equivalent to

$$\begin{aligned}
&\sum_{kl} E_{kl} w_0 \otimes h_{kl} w_1 = \sum (Dw)_0 \otimes (Dw)_1 \Leftrightarrow \\
\forall i \quad &\sum E_{kl}(e_i)_0 \otimes h_{kl}(e_i)_1 = \sum (De_i)_0 \otimes (De_i)_1 \Leftrightarrow \\
\forall i \quad &\sum_{k,l,j} E_{kl} e_j \otimes h_{kl} c_{ji} = \sum_s (\sum d_{si} e_s)_0 \otimes (\sum d_{si} e_s)_1 \Leftrightarrow \\
\forall i \quad &\sum_{kj} e_k \otimes h_{kj} c_{ji} = \sum_{ks} d_{si} e_k \otimes c_{ks} \Leftrightarrow \\
\forall k, i \quad &\sum_j h_{kj} c_{ji} = \sum_s d_{si} c_{ks}.
\end{aligned}$$

We want to show:

$\forall k, l \;\; h_{kl} = \sum_{ij} c_{ki} S(c_{jl}) d_{ij} \Leftrightarrow \forall k, i \quad \sum_j h_{kj} c_{ji} = \sum_s d_{si} c_{ks}.$

Now,

$$\forall k, i \quad \sum_j h_{kj} c_{ji} = \sum_s d_{si} c_{ks} \Rightarrow$$

$$\forall k,l \quad \sum_i \sum_j h_{kj} c_{ji} S(c_{il}) = \sum_i \sum_s d_{si} c_{ks} S(c_{il}) \Leftrightarrow$$

$$\forall k,l \quad \sum_j h_{kj} \delta_{jl} = \sum_i \sum_s d_{si} c_{ks} S(c_{il}) \Leftrightarrow$$

$$h_{kl} = \sum_i \sum_s d_{si} c_{ks} S(c_{il}).$$

On the other hand,

$$\forall k,l \ \ h_{kl} = \sum_{sj} c_{ks} S(c_{jl}) d_{sj} \Rightarrow$$

$$\forall k,i \ \ \sum_l h_{kl} c_{li} = \sum_{s,j,l} c_{ks} S(c_{jl}) d_{sj} c_{li} = \sum_{sj} c_{ks} d_{sj} \delta_{ij} = \sum_s c_{ks} d_{si}. \qquad \square$$

Lemma 3.1.2. *Let V be a finite-dimensional H-comodule and H a Hopf algebra. Consider $l : End(V) \to End(End(V))$, l given by $l(A) = l_A$. Then l is a comodule map.*

Proof: We want to show that for all $A \in End(V)$, we have

$$\sum l_{A_0} \otimes A_1 = \rho(l_A). \tag{3.1.4}$$

Now, apply Lemma 3.1.1 to $W = End(V), D = l_A$,
$T = \sum l_{A_0} \otimes A_1 \in End(End(V)) \otimes H$, i.e. for all $B \in End(V)$, we want to show that (3.1.4) is equivalent to:

$$\rho(l_A(B)) = (\sum l_{A_0} \otimes A_1)\rho(B) \Leftrightarrow$$

$$\rho(AB) = (\sum l_{A_0} \otimes A_1)(\sum B_0 \otimes B_1) = \sum l_{A_0}(B_0) \otimes A_1 B_1 = \sum A_0 B_0 \otimes A_1 B_1.$$

$\square$

Lemma 3.1.3. *Let $r : End(V) \to End(End(V))$, where r acts as $r(A) = r_A$ and assume that H is a commutative Hopf algebra and V is a finite-dimensional H-comodule. Then r is a comodule map.*

Proof: We want to show that for all $A \in End(V)$, we have

$$\sum r_{A_0} \otimes A_1 = \rho(r_A). \tag{3.1.5}$$

Now, apply Lemma 3.1.1 to $W = End(V), D = r_A$,
$T = \sum r_{A_0} \otimes A_1 \in End(End(V)) \otimes H$, i.e. for all $B \in End(V)$, we want to show that (3.1.5) is equivalent to:

$$\rho(r_A(B)) = (\sum r_{A_0} \otimes A_1)\rho(B) \Leftrightarrow$$

$$\rho(BA) = (\sum r_{A_0} \otimes A_1)(\sum B_0 \otimes B_1) = \sum r_{A_0}(B_0) \otimes A_1 B_1 = \sum B_0 A_0 \otimes A_1 B_1.$$

But, we know that $\rho(BA) = \sum B_0 A_0 \otimes B_1 A_1$, from Lemma 1.6.1, and using commutativity of H, we obtain the desired equality. □

Lemma 3.1.4. *Assume that V is a finite-dimensional H-comodule. Let (H,β) be a cotriangular Hopf algebra and consider $ad : End(V) \to End(End(V))$, where ad acts as $ad(A)(B) = AB - \sum \beta(A_1 B_1) B_0 A_0$. Then ad is a comodule map.*

Proof: We want to show that for all $A \in End(V)$ we have:

$$\sum ad_{A_0} \otimes A_1 = \rho(ad_A).$$

Apply Lemma 3.1.1 to: $W = End(V), D = ad_{A_0}$,
$T = \sum ad_{A_0} \otimes A_1 \in End(End(V)) \otimes H$, i.e. for all $B \in End(V)$, we want to show that

$$\rho(ad_A(B)) = (\sum ad_{A_0} \otimes A_1)\rho(B).$$

This is equivalent to:

$$\rho(ad(A)(B)) = (\sum ad_{A_0} \otimes A_1)(\sum B_0 \otimes B_1) \Leftrightarrow$$

$$\rho(AB - \sum \beta(A_1, B_1)B_0A_0) = \sum ad(A_0)(B_0) \otimes A_1B_1 \Leftrightarrow$$

$$\rho(AB - \sum \beta(A_1, B_1)B_0A_0) = \sum(A_0B_0 - \sum \beta(A_{0_1}, B_{0_1})B_{0_0}A_{0_0}) \otimes A_1B_1.$$

Now,

$$\rho(AB - \sum \beta(A_1, B_1)B_0A_0) = \rho(AB) - \sum \beta(A_1, B_1)\rho(B_0A_0) =$$

$$\sum A_0B_0 \otimes A_1B_1 - \sum \beta(A_2, B_2)B_0A_0 \otimes B_1A_1. \qquad (3.1.6)$$

And,

$$\sum(A_0B_0 - \sum \beta(A_{0_1}, B_{0_1})B_{0_0}A_{0_0}) \otimes A_0B_0 =$$

$$\sum A_0B_0 \otimes A_1B_1 - \sum \beta(A_1, B_1)B_0A_0 \otimes A_2B_2. \qquad (3.1.7)$$

But both (3.1.6) and (3.1.7) are equal, following from the assumption that H is cotriangular. Hence, we did prove our result. $\square$

Definition 3.1.1. *Suppose A is an algebra and $U, V \subset A$ are subspaces. We denote by $UV = \{\sum_i u_iw_i | u_i \in U, w_i \in W\}$. A subspace $U \subset A$ is said to be nilpotent if $U^n = 0$ for some n.*

Lemma 3.1.5. *Let V be a finite-dimensional comodule over a cocommutative Hopf algebra $H, \beta : H \otimes H \to \Bbbk$ a skew-symmetric bicharacter. If $x \in End(V)$ and the subcomodule generated by x, i.e. $\langle x \rangle$, is nilpotent of degree n, then $\langle ad(x) \rangle$ is nilpotent of degree at most $2n - 1$.*

Proof:

$$\langle ad(x)\rangle = ad\langle x\rangle \tag{3.1.8}$$

is true because ad is an H-comodule map. Let

$$x^1, ..., x^{2n-1} \in \langle x\rangle. \tag{3.1.9}$$

$$\begin{aligned}
(adx^1)(adx^2)(y) &= (adx^1)(x^2y - \sum\beta(x_1^2, y_1)y_0x_0^2) \\
&= x^1(x^2y - \sum\beta(x_1^2, y_1)y_0x_0^2)
\end{aligned}$$

$$-\sum\beta\left(x_1^1, x_1^2y_1 - \sum\beta(x_1^2, y_1)y_{0_1}x_{0_1}^2\right)(x_0^2y_0 - \sum\beta(x_1^2, y_1)y_{0_0}x_{0_0}^2)x_0^1$$

$$= x^1x^2y - \sum\beta(x_1^2, y_1)x^1y_0x_0^2 - \sum\beta\left(x_1^1, x_1^2y_1 - \sum\beta(x_2^2, y_2)y_1x_1^2\right)x_0^2y_0x_0^1 +$$

$$\sum\beta\left(x_1^1, x_1^2y_1 - \sum\beta(x_2^2, y_2)y_1x_1^2\right)\sum\beta(x_1^2, y_1)y_0x_0^2x_0^1.$$

The expression of $(adx^1)...(adx^{2n-1})(y)$ is obtained by similar computations as above. Using the fact that $\langle x\rangle$ is nilpotent of degree n, we obtain $(adx^1)...(adx^{2n-1})(y) = 0$, because every term contains a product of at least n elements from $\langle x\rangle$, hence adx is nilpotent. By using (3.1.8), we get that $ad\langle x\rangle$ is nilpotent, hence by (3.1.9) we obtain that $\langle ad(x)\rangle$ is nilpotent. □

3.2 Engel's theorem

Theorem 3.2.1. (ENGEL'S THEOREM FOR (H,β)-LIE ALGEBRAS)
Let H be a commutative and cocommutative Hopf algebra, $V \neq 0$ a finite-dimensional H-comodule, β a skew-symmetric bicharacter. Suppose that we have a representation φ of a β- Lie algebra L. If for all $x \in L$, the subcomodule $\langle\varphi(x)\rangle$ is nilpotent, then there exists a nonzero $v \in V$, such that $\varphi(x)\langle v\rangle = 0$, for all $x \in L$.

Proof: We prove the theorem by induction on $\dim L$. We denote by $(\dim L, \varphi, V)$ the situation where $\varphi : L \to EndV$ is the representation of L in an H-comodule V. For $\dim L = 0$, we get $L = 0$, hence for any $0 \neq v \in V$ we have $\varphi(L)\langle v\rangle = 0$. Now assume that $\dim L > 0$. Replacing L with $\varphi(L)$, we can assume without loss of generality that L is a subalgebra of $gl(V)$. Let M be a maximal proper (H,β)-Lie subalgebra of L (this is also an H-subcomodule). Due to the fact that L is an H-comodule and M is an H-subcomodule, we get that L/M is an H-comodule. Now, $End(L/M)$ is an algebra in $\mathcal{M}^{\mathcal{H}}$ (the result is due to Lemma 1.6.1). Consider the map $\sigma : M \to End(L/M)$, which is defined as: $\sigma(x)(y+M) = [x,y] + M$, for all $x \in M, y \in L$. We want to prove that σ is a representation of the β-Lie algebra M. By the definition of the representation we have to show that:

1. ad is a homomorphism of H-comodules and

2. ad is a homomorphism of algebras.

The first statement follows from Lemma 3.1.4 and the second statement is reduced to:

$$\sigma([a,b]) = [\sigma(a), \sigma(b)]_\beta.$$

But, $\sigma([a,b])(y+M) = [[a,b],y] + M$ and

$$[\sigma(a),\sigma(b)]_\beta(y+M) = \sigma(a)\sigma(b)(y+M) - \sum \beta(\sigma(a)_1, \sigma(b)_1)\sigma(b)_0\sigma(a)_0(y+M) =$$

$$\sigma(a)([b,y]+M) - \sum \beta(\sigma(a)_1, \sigma(b)_1)\sigma(b)_0([a_0,y]+M) =$$

$$[a,[b,y]] - \sum \beta(\sigma(a)_1, \sigma(b)_1)[b_0,[a_0,y]] + M = [[a,b],y] + M.$$

For the last equality we use Jacobi indentity.

We can now apply the induction hypothesis to $(M, \sigma, L/M)$. Then there exists a nonzero $\bar{a} \in L/M$ such that $\sigma(x)\langle\bar{a}\rangle = 0$, for all $x \in M$. Let $\bar{a} = b + M$, $b \in L$. So, $[x, \langle b\rangle] \in M$ for all $x \in M$.

Observation 3.2.1. The H-Lie subalgebra, $alg\langle b\rangle$, generated by $\langle b\rangle$ is spanned by monomials w, where w are words composed of elements of $\langle b\rangle$ with a certain bracket structure. Let Γ be the set of monomials w defined by induction on the degree as follows. If $\deg w = 1$, then $w \in \langle b\rangle$, and if $\deg w = n > 1$ then $w = [w_1, w_2]$, where $w_1, w_2 \in \Gamma, \deg w_1, \deg w_2 < n$. If we denote by $alg\langle b\rangle$ the H-Lie subalgebra generated by $\langle b\rangle$, then $alg\langle b\rangle = Span\Gamma$. We can show that $Span\Gamma$ is an H-subcomodule by induction on the degree of the w that span $Span\Gamma$. If $\deg w = 1$, then $w \in \langle b\rangle$, hence $\rho(w)\langle b\rangle \in \Gamma$. If $\deg w > 1$, let $w = [w_1, w_2]$ and express $\rho(w_1) = \sum w_i \otimes h_i$, $\rho(w_2) = \sum w_j \otimes h_j$, where $w_i, w_j \in \Gamma$ and $h_i, h_j \in H$. Therefore $\rho(w) = [\rho(w_1), \rho(w_2)] = [\sum w_i \otimes h_i, \sum w_j \otimes h_j] = \sum [w_i, w_j] \otimes h_i h_j \in Span\Gamma \otimes H$.

Now, because M and $alg\langle b\rangle$ are H-subcomodules of L, then $M + alg\langle b\rangle$ is an H-subcomodule of L. The fact that $M + alg\langle b\rangle$ is a Lie subalgebra is shown by induction as follows. The basis of induction is: for any $x \in M$, $l \in \langle b\rangle$, $[x, l] \in M$.

Now, consider $l = [n, m] \in alg\langle b\rangle$. Then, express $[x, [n, m]] = [[x, n], m] - \sum \beta(n_1, m_1)[[x, m_0], n_0]$. Let $n = [b^{(1)}, ..., b^{(k)}]$ and $m = [\bar{b}^{(1)}, ..., \bar{b}^{(k)}]$, where $b^{(i)}, \bar{b}^{(j)} \in \langle b\rangle$. Also, $\rho(n) = \sum [b_0^{(1)}, ..., b_0^{(k)}] \otimes b_1^{(1)} ... b_1^{(k)}$ and $\rho(m) = \sum [\bar{b_0}^{(1)}, ..., \bar{b_0}^{(k)}] \otimes \bar{b_1}^{(1)} ... \bar{b_1}^{(k)}$. Hence

$$\sum \beta(n_1, m_1)[[x, m_0], n_0] = \sum \beta(n_1, m_1)[[x, [b_0^{(1)}, ..., b_0^{(k)}]], [\bar{b_0}^{(1)}, ..., \bar{b_0}^{(k)}]],$$

where $[b_0^{(1)}, ..., b_0^{(k)}]$ is a commutator of degree k in $\langle b\rangle$ and $[\bar{b_0}^{(1)}, ..., \bar{b_0}^{(k)}]$ is a commutator of degree l in $\langle b\rangle$. Therefore, the induction hypothesis applies and we have that $[x,[n,m]] \in M$.

Hence, we have that $M + alg\langle b\rangle = L$ by maximality of M. Consider $V_0 = \{v \in V | x\langle v\rangle = 0, \forall x \in M\}$, the subspace of V. Now, $V_0 \neq 0$, by the induction hypothesis applied to M. Also, V_0 is stable under b', for any $b' \in \langle b\rangle$, since $\forall x \in M$, $x\langle v\rangle = 0$ implies that

$$xb'\langle v\rangle = [x,b']\langle v\rangle + \sum \beta(x_1,b'_1)b'_0x_0\langle v\rangle = 0, \quad \forall x \in M.$$

Hence $b'v \in V_0$. Now, due to the fact that the restriction of $\langle b\rangle$ to V_0 is a nilpotent subspace of $End(V_0)$, the following lemma gives a nonzero vector $v \in V_0$, such that $\langle b\rangle\langle v\rangle = 0$. Then, obviously, $alg\langle b\rangle\langle v\rangle = 0$. Since $L = M + alg\langle b\rangle$, we obtain that $L\langle v\rangle = 0$. □

Lemma 3.2.1. *Suppose that $V \neq 0$ is a finite-dimensional H-comodule and $b \in EndV$ is such that $\langle b\rangle$ is nilpotent. Then there exists $0 \neq v \in V$ such that $\langle b\rangle\langle v\rangle = 0$.*

Proof: Consider $W = \{v \in V | \langle b\rangle v = 0\}$. $W \neq 0$ since $\langle b\rangle$ is nilpotent. Pick $v \neq 0$ in V. If $\langle b\rangle v = 0$, we are done. Otherwise, take $b' \in \langle b\rangle$ such that $b'v \neq 0$ and replace v by $b'v$ and so on. This process ends after at most $m-1$ steps, where m is a number such that $\langle b\rangle^m = 0$.

We show that W is a subcomodule. Consider $W = Span\{e_1, ..., e_k\}$, and extend it to $V = Span\{e_1, ..., e_n\}$, $k \leq n$. From Lemma 3.1.1 we have $\rho(e_j) = \sum_i e_i \otimes c_{ij}$, $\Delta(c_{ij}) = \sum_l c_{il} \otimes c_{lj}$, $\rho(E_{ij}) = \sum_{s,t} E_{st} \otimes c_{si}S(c_{jt})$. We know that $\rho(W) \subset W \otimes H$. Fix $b' \in \langle b\rangle$. We know $\rho(b') \in \langle b\rangle \otimes H$. Hence

$$\rho(b')(W \otimes 1) = 0. \qquad (3.2.1)$$

We want to prove that

$$(b' \otimes id)\rho(e_j) = 0, \quad \forall j \leq k. \tag{3.2.2}$$

Express $b' = \sum \lambda_{pq} E_{pq}$. Then (3.2.2) becomes

$$\sum_{i,p,q} \lambda_{pq} E_{pq} e_i \otimes c_{ij} = 0 \Leftrightarrow \sum_{p,q} \lambda_{pq} e_p \otimes c_{qj} = \sum_p e_p \otimes \sum_q \lambda_{pq} c_{qj} = 0.$$

From (3.2.1) we have:

$$\sum_{p,q} \lambda_{pq} \sum_{s,t} E_{st} \otimes c_{sp} S(c_{qt})(e_j \otimes 1) = 0, \quad j \leq k \Rightarrow$$

$$\sum_{p,q,s} \lambda_{pq} e_s \otimes c_{sp} S(c_{qj}) = 0 \Rightarrow \forall s, \ \sum_{p,q} \lambda_{pq} c_{sp} S(c_{qj}) = 0 \Rightarrow$$

$$\sum_{p,q,s} \lambda_{pq} (S(c_{ms})) c_{sp} S(c_{qj}) = 0, \quad \forall m \Rightarrow \sum_q \lambda_{mq} S(c_{qj}) = 0, \quad \forall m.$$

Since S is bijective, $\sum_q \lambda_{mq} c_{qj} = 0, \quad \forall m.$ □

Note 3.2.1. If we apply Theorem 3.2.1. to ordinary Lie algebras with $H = \mathbb{k}$, we get the usual Engel's Theorem 1.2.1. For H a group algebra of an abelian group, we do not get Engel's Theorem for color Lie superalgebras 2.3.1., because our comodule hypothesis is stronger than what we assume in Theorem 2.3.1., where the nilpotency condition is imposed only on homogeneous elements.

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