A Power Comparison of Robust Tests for Monotonic Trends in Recurrent Events

by

© **Jiajia Yue**

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Department of Mathematics and Statistics

Memorial University of Newfoundland

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Abstract

The analysis of the existence and form of time trends in repairable systems is an important issue in reliability studies. Hence, many trend tests have been proposed and studied in the literature. There has been a recent interest in the use of robust trend tests based on estimating functions to test the absence of time trends in the rate functions of recurrent event processes. These tests are appealing because they do not require strong assumptions about the nature of the processes, and are powerful in a wide range of settings. In this study, we consider monotone time trends in recurrent event data from repairable systems, and develop a robust trend test based on rate functions of the power law processes. Our main goal is to discuss the power of robust trend tests as well as to compare their power with other well-known trend tests under various settings. We therefore conducted extensive Monte Carlo simulations to compute and compare the power of these tests under various scenarios. Finally, we analyze two data sets from industry to illustrate the methodology.

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Statement of contribution

Dr. Candemir Cigsar proposed the research question that was investigated throughout this practicum. The overall study was jointly designed by Dr. Candemir Cigsar and Jiajia Yue. The algorithms were implemented, the simulation study was conducted and the manuscript was drafted by Jiajia Yue. The data sets in this study were analyzed by Jiajia Yue. Dr. Candemir Cigsar supervised the study and contributed to the final manuscript.

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Chapter 1

Introduction

The aim of this chapter is to introduce the main goal of this practicum. Some problems involving recurrent events are discussed in Section 1.1 with practical examples. The types of data are briefly introduced in Section 1.2. In Section 1.3, we introduce two motivating examples analyzed later in the practicum. We present a literature review on trend tests in recurrent events in Section 1.4. The outline of the practicum is given in the last section.

1.1 Introduction

In many research fields, a well-defined event may repeatedly occur over randomly time or space. Such an event is called a *recurrent event*, and processes generating recurrent events are called *recurrent event processes*. The data obtained from a recurrent event process are called *recurrent event data*. Recurrent event data often arise in various research areas. For example, in reliability, failures of an air-conditioning equipment or breakdowns of a machine may repeatedly occur over time (Cox and Lewis, 1966); in medicine, patients may have repeated asthma attacks (Duchateau et al., 2013); or in manufacturing, automobile manufacturers collect information on warranty claims

(Lawless and Nadeau, 1995).

In this practicum, we focus on applications from reliability studies of repairable systems. A *repairable system* is a system which can be brought back to the operational condition after a failure occurrence by some repair process rather than by replacing the entire system (Rigdon and Basu, 2000). Some examples of such systems include air-conditioning equipment in aircrafts, printers, automobiles and diesel-operated power generators etc.

Reliability is a crucial concept for manufacturers and system procedures to be competitive. Statistical analysis of failure time data obtained from repairable systems may facilitate reliability enhancement of repairable systems. In repairable systems contexts, any event causing an unwanted system stoppage is usually defined as a failure. Since such failures are recurrent, the statistical analysis of failure time data of repairable systems is conducted under recurrent events framework through models for recurrent event processes (Cox and Lewis, 1966; Cox and Isham, 1980; Rigdon and Basu, 2000; Cook and Lawless, 2007).

One of the most important features of recurrent event processes is *trend*. There are different definitions of a trend in recurrent event settings. We discuss this issue in Section 2.3 in some detail. In most of the situations, a trend usually refers either to a systematic change in the rate of event occurrences over time or to a situation in which the times between event occurrences are stochastically increasing or decreasing (Lawless et al., 2012). A trend is called *monotonic* if the rate of event occurrences is monotonically increasing or decreasing over time. There are also *non-monotonic* trends. Some examples of non-monotonic trends include seasonal fluctuations in the occurrence of failures in power generators or U shaped rate functions. Non-monotonic trends are not as common as monotonic trends in applications involving recurrent events. Therefore, we focus on monotonic trends in this practicum. The reliability of repairable systems can be tracked and quantified by using failure data from a single system or multiple systems. Detection of monotonic trends is a crucial step in the reliability analysis of repairable systems to reveal problems and improve maintenance practices. Because of these reasons, the rate of occurrence of failures in a repairable system is usually monitored in many reliability settings with the goal of revealing the failure patterns.

Many graphical methods and formal tests have been developed in the literature. We outlined the frequently used or historically important ones in Section 1.4. Graphical methods are useful to detect patterns in recurrent event data, especially when a single process is observed over a long period. However, their interpretation is subjective and the patterns revealed by plots may be affected by the choice of the scales of x and y-axes. In addition, graphical methods can be inefficient when too many processes are under observation. Consequently, formal tests for trends have been proposed to detect trends in recurrent event processes. Some of these tests have been adopted as routine checks for trends in reliability settings. The validity of most of these tests depend on the assumed "trend-free" nature of the processes as well as the assumed observation schemes (data accrual period) of the processes (Lawless et al., 2012).

Recently, robust procedures based on the theory of estimating equations have been applied to developed tests for trend in recurrent event settings (Cook and Lawless, 2007; Cigsar, 2010; Lawless, et al, 2012). These tests are easy to implement and do not rely on strong assumptions on the trend-free nature of the processes. The power of these tests has not been investigated and compared in detail. The main goal of this practicum is, therefore, to explore the power of robust trend tests and compare them with other well-known trend tests for recurrent processes.

1.2 Types of Data

Statistical analysis of recurrent event data is usually based on either times of event occurrences or times between event occurrences. The event should be well defined. Times of event occurrences are generally recorded in chronological order. The time variable t denotes the *calendar time* or the *global time*, which means that the event times are recorded as time since the initial startup of the processes (Rigdon and Basu, 2000). Another common way of presenting recurrent event data is to use the times between event occurrences, which is referred to as *waiting times* or *gap times*. The elapsed time since the most recent event time is called the *local time* or *backward recurrence time* (Rigdon and Basu, 2000). Some of these concepts are mathematically defined in Chapter 2.

The use of event times or gap times for the analysis of recurrent events depend on the context of a study. Generally speaking, if the event counts are of interest, recurrent event processes are observed by recording the event occurrence times. In such cases, the global time scale is a natural choice. On the other hand, the analysis is often based on the gap times if the interest is in the prediction of the next event time or in modeling the dependence between gap times. The local time is a canonical choice with the gap time analysis. There are also some models based on both global and local time scales (Lawless and Thiagarajah, 1996). These models can be useful if the failure occurrence are associated with factors related to global time, such as stochastic ageing, or local times, such as residual effects of repairs. We discuss fundamental models for repairable systems in Section 2.2.

In many studies, recurrent event data are recorded along with a set of covariates so that the effects of covariates on the event occurrences can be investigated through regression models. Covariates in recurrent event studies can be time fixed or time varying. We discuss the definitions of different types of covariates in Section 2.2 in more detail. A time varying covariate is called *exogenous* if its values are not affected by the event processes under study (Kalbfleisch and Prentice, 2002). Time fixed covariates are exogenous by their nature. Some examples of fixed covariates in reliability settings include the brand of a power generator, wall thickness of an underlying gas pipeline, design of an electronic hardware, etc. Varying demand of a power generator, seasonality effect for machines working outside and experience of a dragline operator can be considered as exogenous time-varying covariates. In this study, we do not discuss how to adjust the methods with covariates. Our main goal is to compare important tests for monotonic trends in recurrent event processes in terms of their power to detect trends. Some of the tests used frequently in applications as well as graphical methods are not readily available to incorporate covariates. Because of this reason, we focus on the main purpose of trend tests. However, robust tests for monotonic trends can be easily extended to deal with exogenous covariates. A discussion on the trend tests with covariates can be found in Lawless et al. (2012).

Data accrual process in recurrent event studies includes a starting time for the follow-up and an end-of-follow-up time for each process in the study. These times can be the same for all processes under observation or can vary. In many studies involving recurrent events, observation of a process starts at an initial time and ends at a prespecified end-of-followup time. Such an observation scheme is called *Type 1* censoring. In such cases, the observed number of events for a process is random and the last event time is not complete; that is, censored. There is also *Type 2* censoring. In this case, the total number of failures in a process is prespecified. In this case, the end-of-followup time is random. There are many different versions of Type 1 and Type 2 censoring schemes. For details of Type 1 and Type 2 censoring schemes and their ramifications, we refer to Lawless (2003). As discussed in Chapter 3, certain trend tests are based on the assumption that the data are collected under a Type 2 censoring mechanism, which is not a common censoring mechanism in applications comparing with the Type 1 censoring mechanism. It is a common practice to apply those tests when the true censoring mechanism is Type 1. Lawless et al. (2012) discussed the validity of some of these well-known tests under this violation through simulations. Their results showed that these tests may be affected by the violation of this assumption. We would also like to note that robust trend tests requires that the start and ed-of-followup times should be completely independent of event processes. We discuss this issue in Chapter 3 in some detail.

1.3 Motivating Examples

In this section, we discuss two data sets analyzed in Chapter 5 to illustrate the methodology discussed in this practicum. These data sets are from industry, and have been analyzed by many authors as well.

1.3.1 Hydraulic Systems of LHD Machines

Kumar and Klefsjö (1992) presented and analyzed the failure times of a fleet of load-haul-dump (LHD) machines. The main goal of their analysis was to decide the optimal preventive maintenance policies for LHD machines operating in a mine in Sweden. For this reason, they collected the failure time data of the hydraulic systems of LHD machines. They reported the times between successive failures of six hydraulic systems, labelled as LHD 1, LHD 2, LHD 9, LHD 11, LHD 17 and LHD 20. This data set can be found in Table A.1 in Appendix A as the waiting times between successive failures.

Kumar and Klefsjö (1992) categorized the LHD machines into three categories; (*i*) new machines (LHD 17 and LHD 20), (*ii*) medium old machines (LHD 9 and LHD11), and (*ii*) old machines (LHD 1 and LHD 3). The number of failures are 23, 25, 27, 28, 26, and 23 for LHD machines 1, 3, 9, 11, 17, and 20, respectively. We used this data set to apply the trend tests considered in this practicum in Chapter 5. For our analysis, we took the last failure time of each machine (i.e., 2496, 3526, 4743, 2913, 3230, and 3309) as the corresponding end-of-follow-up time.

For the optimization of the replacement policies, it is crucial to detect monotonic trends in the data. Therefore, we applied trend tests to detect the existence or absence of monotonic time trends in the failures of LHD machines. In the analysis presented by Kumar and Klefsjö, graphical methods were used to detect any trends. They applied Cramér-von-Mises test and MIL-HDBK-189 test for single processes to decide the existence of time trends. In our analysis in Chapter 5, we used trends tests with combined data for all six LHD machines. We also applied the power law process model in reliability of repairable systems to analyze this data set.

1.3.2 Failures of Air-Conditioning Equipment

Proschan (1963) analyzed the failure data from air-conditioning equipment in thirteen Boeing 720 aircrafts. This data set is considered as one of the most cited failure time data in the history of reliability (Ascher and Feingold, 1983, p. 146; Lawless, 2000). Cox and Lewis (1966) used this data set to illustrate the trend tests based on rank statistics and exponentially distributed waiting times between successive failures (i.e., Poisson processes). There are only two failures recorded for Aircraft 11 in the original data set. Therefore, Cox and Lewis (1966) discarded this aircraft from their analysis.

Lawless and Thiagarajah (1996) analyzed the failure data from two aircrafts. Their analysis included trend tests as well. Cook and Lawless (2007, Section 5.2.4) considered intensity-based models for a single process to analyze the failure data from

Aircraft 6. They included both trend and residue repair effects together in a single model. This types of clustering and trend features of recurrent event processes were further investigated by Cigsar (2010), who also used the same data set for illustration purposes. In Chapter 5, we considered four aircrafts (Aircrafts 2, 3, 6 and 7), and applied the tests given in Chapter 3 for the combined data set. This data set is presented in Table A.2 in Appendix A as the operating hours between successive failures of air-conditioning equipment in four aircrafts (Aircrafts 2, 3, 6 and 7).

1.4 Literature Review

Recurrent events can be seen in many research areas. Most of the methodological developments in the statistical analysis of recurrent events are based on the theory in counting processes or point processes (e.g., Cox and Isham, 1980; Andersen et al., 1993; Daley and Vere-Jones; 2003). Much of the early work involving recurrent event processes focused on the analysis of trends in recurrent event processes. Therefore, there is a vast literature in the tests for trend in recurrent event processes. As discussed by Cox and Lewis (1966), reasons for the interest in trend testing include that the main goal of many studies is to reveal the existence or absence and the type of a trend and that the most of the statistical methods depend on the presence or absence of trends in recurrent event processes. In this section, we give a summary of the trend detection procedures in the literature. We only consider historically important or most common tests and graphical methods in the literature. Ascher and Feingold (1984) provide an excellent source on the discussion of the trends in repairable systems.

The graphical methods in the detection of trends in recurrent events have been discussed by many researchers. As noted by Lawless et al. (2012), these methods are especially useful when a trend detection in single process with many failures is of interest. In this practicum, we do not consider single process case, but we briefly discussed some important graphical methods in Section 3.2. A very simple graphical method is based on dot plots. These plots can give some insight about the failure patterns in the history of a process. Dot plots are discussed by Rigdon and Basu (2000). A very useful plot is based on the Nelson-Aalen estimator of the cumulative mean function. Plots based on the Nelson-Aalen estimator have been discussed in many studies and applied in many settings. Among them, we refer to Andersen et al. (1993) for a rigorous treatment of the Nelson-Aalen plots, and Kvaløy and Lindqvist (1998), Lindqvist (2006) and Cook and Lawless (2007) for practical applications. Duane (1964) introduced a plot of cumulative failures against cumulative operating time, and very useful to assess the adequacy of power law processes. This plot is now referred to as Duane plot, which is discussed by Rigdon and Basu (2000). Another important graphical method, called the total time on test (TTT) plot, was developed by Barlow and Campo (1975). The TTT plots was discussed by Klefsjo and Kumar (1992) and Kvaløy and Lindqvist (1998) in detail. They also developed a test for trend in interfailure times of a recurrent event process based on TTT plots.

Cox and Lewis (1966) considered trend testing in the third chapter of their seminal monograph, which is still an excellent source on the analysis of recurrent events and trend testing. They discussed the development of linear rank test and Laplace test for monotone trend in single and multiple processes. Both of these tests are discussed in Chapter 3 of this practicum, and included in our power comparisons in Chapter 4. These tests, especially the Laplace tests, have been later discussed by numerous authors (e.g., Lawless and Thiagarajah, 1996; Kvaløy and Lindqvist, 1998; Lindqvist, 2006; Lawless et al., 2012). A rigorous treatment of rank statistics is given by Hajek and Sidak (1967). It should be noted that, as we discuss in the next two chapters, the Laplace test is based on the null hypothesis that the trend free process is a Poisson process. However, the linear rank test is based on the renewal processes as the model under the null hypothesis. Another such test is called the Lewis-Robinson test, which was developed by Lewis and Robinson (1973). This test is considered as a generalization of the Laplace test in the sense that the trend free can be any renewal process. This test is investigated by many authors as well (e.g., Lawless and Thiagarajah, 1996; Lindqvist 2006; Lawless et al. 2012;). We discuss the Lewis-Robinson test in Section 3.3.2. There are trend tests based on the power law processes. The power law process and its applications in the reliability of a single repairable system are discussed by Rigdon and Basu (1989). Baker (1996) introduced trend tests based on the power law process. She considered both single and multiple systems. We developed a trend test based on power law process in Chapter 3.

There are also trend tests based on robust methods for the analysis of recurrent events. These methods are based on the rate or mean functions of processes and using theory of estimating equations for the development of the tests. The robust procedures based on these functions in recurrent event processes is discussed by Lawless and Nadeau (1995). A very useful introduction about the robust methods in recurrent events, including the trend procedures, can be found in Cook and Lawless (2007, Chapter 3). Also, more rigorous treatment of robust method are given Lin et al. (2000). A more detailed discussion of the robust trend tests is given by Lawless et al. (2012), they extended the Laplace test to the robust version, and called it the generalized Laplace test for monotonic trend. We discuss this test in Chapter 3. Furthermore, we follow the procedures given in Lawless et al. (2012) and developed a robust version of the power law process in Chapter 3. We call this test the generalized power law process test.

We would like to note that many of the papers mentioned in this section include limited power comparisons of the tests discussed. However, a more comprehensive power comparison of the trend tests is still not widely available. Some important papers in this context include Bain et al. (1985), where the power comparison of five trend tests was given for single processes. A more detailed power comparison of three trend tests, including the Lewis-Robinson test, rank test and the generalized Laplace test, is given by Cigsar (2010). In this practicum, we considered seven tests including two robust tests for trend.

1.5 Main Goal and Outline of the Practicum

In this section, we outline the remaining part of the practicum. The main goal of this practicum is to compare important formal statistical tests for the absence of monotonic trends in recurrent event processes in terms of their power in various settings. Therefore, we first introduce important trend tests including Laplace test, rank test, and Lewis-Robinson test which are frequently applied for the detection of monotonic trends in recurrent event studies. Trends due to stochastic ageing in recurrent event processes can be included in some basic models, such as non-homogeneous Poisson process and renewal processes. An important limitation of the many existent trend tests is that they require an assumption for trend-free model, which is usually a homogeneous Poisson process or renewal process. Our primary objective in this practicum is to discuss trend tests based on robust methods, and compare their power with other well-known trend tests. Our hypothesis is that robust trend tests provide good power when the assumed trend-free model is correct or mildly misspecified as long as assumed simple marginal characteristics of the processes such as rate or mean functions are true. Therefore, robust tests can be applied as routine checks for the presence or absence of monotonic trends in recurrent event processes without any detailed model assumptions. A power comparison of tests for trend is to provide an information prospective in order to show the competitive advantage of robust trend tests.

In Chapter 2, we introduce the notation used in this practicum, mathematical background and some foundational models for recurrent event processes. The definition of trends in recurrent event processes are discussed, not only in mathematical and statistical point of views, but also from a practical perspective. We also introduce the simulation procedures, which lead to the results given in Chapter 4.

In Chapter 3, we introduce the trend tests used in power studies. We first review important tests for monotonic trends, which are widely applied in practice (e.g., Laplace test, power law process test, rank test, and Lewis-Robinson test) in identical processes settings. We then discuss the development of two robust trend tests; the generalized Laplace test and the generalized power law process (PLP) test in identical processes settings. To our knowledge, the generalized PLP tests has not been discussed in the literature before. Therefore, we focus mostly on this test. Next, we discuss all of these trend tests in non-identical processes settings.

In Chapter 4, we first introduce our setup for the simulation studies. This include the models under the null hypothesis of the absence of a monotonic trend and the models under the alternative hypothesis of an monotonically increasing trend. In the subsequent sections, we give the steps of simulations in all cases considered in simulation studies. In the simulations, we consider the accuracy of the standard normal approximations under the trend-free models including a homogeneous Poisson process, a renewal process with gamma distribution and delayed renewal processes for identical and non-identical cases. Next, we study the power of the tests, and give detailed explanation of this simulation study. The simulation results are presented in Appendices B and C. We summarize the results of the simulation studies in the final section of this chapter.

We analyze two data sets from industry in Chapter 5. Our main goal with these analyses is to illustrate the methods discussed in the previous chapters. Finally, we give a conclusion and future work on trend testing in Chapter 6.

Chapter 2

Concepts and Terminology

A stochastic process is called a *point process* if it defines random occurrences of point events over time or space. In this practicum, the times of event occurrences are failure times of a repairable system observed as points on the time axis. Recurrent events are usually modeled under the point process framework (Cox and Isham, 1980; Daley and Vere-Jones, 2003). Fundamental models for recurrent events include renewal processes, homogeneous Poisson processes which is also a special case of renewal processes, and non-homogeneous Poisson processes. We introduce these models in Section 2.2, and discuss their roles in testing for trends in recurrent event processes. The goal of this section is to introduce the notation and some concepts that are frequently used in this practicum.

2.1 Terminology and Notation

Suppose that m independent processes are under observation. We let $T_{i1}, T_{i2}, \dots$ denote the event times of the ith process ($i = 1, \dots, m$), where $0 < T_{i1} \leq T_{i2} \leq \dots$. From now on, an event refers to any reason that causes an unexpected system stoppage in a repairable system, and we call it a failure for most of the remaining part of this practicum. Thus, T_{ij} denotes the time of the jth failure in the ith process, for $i = 1, \ldots, m$ and $j = 1, 2, \ldots$. By convention, we also let $T_{i0} = 0$ for $i = 1, \ldots, m$. The jth gap (or waiting) time of the ith process is then defined by $W_{ij} = T_{ij} - T_{i,j-1}$, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots$.

We let the random variable $N_i(t)$ denote the number of failures in the ith process, $i = 1, \ldots, m$, over the time interval $(0, t]$; that is, $N_i(t) = \sum_{j=1}^{\infty} I(T_{ij} \leq t)$, where $I(\cdot)$ is a typical $0 - 1$ valued indicator function and $N_i(0) = 0$. As an extension of our notation, we next define $N_i(s, t)$ to represent the number of failure occurrences over the time interval $(s, t]$; that is, $N(s, t) = N(t) - N(s)$ for any $0 \leq s < t$. We let $\{N_i(t); t \geq 0\}$ denote a counting process. Many properties of counting processes on the positive real line R^+ can be found in point process books (e.g., see Daley and Vere-Jones, 2003, Chapter 3). For example, the counting process $\{N_i(t); t \geq 0\}$ has *the mean function* $\mu_i(t)$, where

$$\mu_i(t) = E\{N_i(t)\}, \qquad t \geq 0, \tag{2.1}$$

which gives the expected number of failures over $[0, t]$. Assuming that $\mu_i(t)$ is differentiable, *the rate function* (also sometimes called *the rate of occurrence of failures* or shortly, *ROCOF*) for process i is then defined as

$$\rho_i(t) = \frac{d\mu_i(t)}{dt} = \mu_i^{'}(t), \qquad t \geq 0. \tag{2.2}$$

It follows that

$$E\{\Delta N_i(t)\} = \rho_i(t)\Delta t + o(\Delta t), \tag{2.3}$$

where $\Delta N_i(t) = N_i((t + \triangle t)^-) - N_i(\triangle t^-)$ represents the number of events in a short interval $[t, t + \Delta t)$, and $o(\Delta t)$ is a quantity that goes to zero more quickly than Δt

when Δt tends to zero.

We say that a point process has *stationary increments* if, for any positive t and h, the number of failures in $(t, t+h]$ has the same distribution as the number of failures in any other interval of length h. We also say that process has *independent increments* if, for any disjoint partition of the interval $[s, t]$ such that $0 \leq l_1 < u_1 \leq \cdots \leq l_K < u_K$, we have $\Pr\{N_i(l_1, u_1) = n_1, \ldots, N_i(l_K, u_K) = n_K\} = \prod_{k=1}^{K} \Pr\{N_i(l_k, u_k) = n_k\}$. Following the independent increments property, we call a point process $\{N_i(t); t \geq 0\}$ *without aftereffect* if, for any $0 < l_1 < l_2 < \cdots < l_K$ and all K, $\Pr\{N_i(l_{K-1}, l_K) = n_K | N_i(l_k) = n_k; k = 1, \ldots, K-1\} = \Pr\{N_i(l_{K-1}, l_K) = n_K\}$, where $n_k = 0, 1, 2, \ldots$, for all $k = 1, \ldots, K$. A point process is without aftereffect if and only if it has independent increments. A proof of this assertion can be found in Thompson (1988, p.21).

We next let $H_i(t) = \{N_i(s); 0 \leq s < t\}$ denote *the history* of the ith process, $i = 1, \ldots, m$. The history $H_i(t)$ includes all observed information about the random variable $N_i(t)$ over the interval $[0, t)$. Another essential concept, which gives the instantaneous probability of an event (failure) that occurs at time t, conditional on the process history $H_i(t)$, is the *intensity function* $\lambda_i(t|H_i(t))$ of a counting process $\{N_i(t) : t \geq 0\}$. It is mathematically defined as follows.

$$\lambda_i(t \mid H_i(t)) = \lim_{\triangle t \downarrow 0} \frac{\Pr\{\triangle N_i(t) = 1 \mid H_i(t)\}}{\triangle t}, \qquad t > 0. \tag{2.4}$$

The intensity function (2.4) completely specifies a counting process in the continuous time scale, in which more than one event cannot occur simultaneously at any time $t > 0$ (Cook and Lawless, 2007, p.10).

The *survival function* of a positive valued continuous random variable W is given by

$$S(w) = \Pr(W > w) = \int_w^\infty f(s)ds = 1 - F(w), \qquad w > 0, \tag{2.5}$$

where $F(w)$ is the cumulative distribution function (c.d.f.) of W and $f(w)$ is the probability density function (p.d.f.) of W. The *hazard function* $h(w)$ gives the conditional probability of a failure at time w in the limit as Δw approaches zero, given that the individual survives up to w, where $w > 0$. That is,

$$h(w) = \lim_{\triangle w \downarrow 0} \frac{\Pr(w < W \le w + \triangle w | W > w)}{\triangle w} = \frac{f(w)}{S(w)}, \qquad w > 0. \tag{2.6}$$

Survivor and hazard functions are especially useful in modeling the gap times.

Let $Y_i(t)$ denote the *at risk* indicator function for process i, This is, $Y_i(t) = 1$ if the ith process is under observation and at risk of a failure at time t, and $Y_i(t) = 0$, otherwise. At risk indicator $Y_i(t)$ can be useful when extending the methods to the cases with more complicated observation schemes. For example, in the simplest case, where the process i is continuously observed over $[0, \tau_i]$ and τ_i is a fixed end-of-follow-up time, then $Y_i(t) = I(0 \le t \le \tau_i)$. Following this discussion, the τ_i can be random variables as well. For example, if the ith process is continuously observed over $[0, \tau_i]$ and censored randomly at time τ_i, then $Y_i(t) = I(0 \le t \le \tau_i)$ which makes $Y_i(t)$ a random variable. In a more complicated observation scheme, the τ_i can depend on the history of a process. In this case, they are called stopping times (Daley and Vere-Jones, 2003, p. 49). In this practicum, we consider the simple case in which $Y_i = I(0 \le t \le \tau_i)$ and the τ_i are fixed. However, many of the methods based on the intensity function (2.4) are still valid under more complicated observation schemes as long as the at risk indicator $Y_i(t)$ and the counting process $\{N_i(t); t \ge 0\}$ are conditionally independent, given the history of the process $H_i(t)$ at any time $t > 0$. One exception is the robust procedures based on the mean or rate functions. In this case, it is required that $Y_i(t)$ and $\{N_i(t); t \geq 0\}$ are independent. This condition excludes observation schemes in which the τ_i are stopping times. More discussion on stopping times can be found in Cook and Lawless (2007, Section 2.6).

2.2 Fundamental Models

The nature of repairs undertaken after each failure defines statistical models for the analysis of failure time data in the context of repairable systems. The concept of a *minimal repair* means that the repair conducted on a system after a failure leaves the system in an "*exactly the same*" condition as it was just before the failure (Rigdon and Basu, 2000). A system subject to minimal repair is usually modeled through Poisson processes, which are discussed in Section 2.2.1. Another common assumption about the nature of a repair is called a *perfect repair*, which means that, after each failure, repairs bring the system to a *like new* condition. In other words, the system is renewed after each repair conducted after a failure. Renewal processes are canonical models for modeling perfect repairs. We discuss the renewal processes in Section 2.2.2.

2.2.1 Poisson Processes

Poisson processes are useful when there is an interest in modeling the counts in recurrent event processes. There are different ways to characterize a Poisson process, one way is through the counting process properties for non-overlapping time intervals (Daley and Vere-Jones, 2003, Chapter 2). Another way is through the intensity function given in (2.4). A counting process $\{N_i(t); t \geq 0\}$ is a Poisson process if and only if its intensity function is equal to its rate function (Cook and Lawless, 2007). That is, in a Poisson process

$$\lambda_i(t|H_i(t)) = \rho_i(t), \qquad t \geq 0. \tag{2.7}$$

Therefore, the intensity function of a Poisson process is independent of the history of the process. This fact implies that the probability of a failure in $[t, t+\Delta t)$ does not depend on failures occurred in the past. From the relation in (2.7), we can show that, for Poisson processes,

$$\Pr\{\Delta N_i(t) = 0\} = 1 - \rho_i(t)\Delta t + o(\Delta t), \tag{2.8}$$

$$\Pr\{\Delta N_i(t) = 1\} = \rho_i(t)\Delta t + o(\Delta t), \tag{2.9}$$

and

$$\Pr\{\Delta N_i(t) \geq 2\} = o(\Delta t). \tag{2.10}$$

Poisson processes possess the independent increments property explained in Section 2.1. For any $0 \leq s < t$, in a Poisson process,

$$\Pr\{N_i(s,t) = k\} = \frac{1}{k!}[\mu_i(s,t)]^k e^{-\mu_i(s,t)}, \qquad k = 0, 1, 2, \ldots, \tag{2.11}$$

where $\mu_i(s,t) = \int_s^t \rho_i(u)du$ is the expected number of failures in [s,t]. The result (2.11) implies that, if $\{N_i(t); t \geq 0\}$ is a Poisson process with rate function $\rho_i(t)$, the random variable $N_i(s,t)$ has a Poisson distribution with mean $\mu_i(s,t) = \int_s^t \rho_i(u)du$. In addition, $\{N_i(t); t \geq 0\}$ is a Poisson process if and only if it has independent increments property. The proofs of these well-known results can be found, for example, in Rigdon and Basu (2000, Chapter 2).

If the Poisson process $\{N_i(t); t \geq 0\}$ has a constant intensity or rate function; say, $\rho_i(t) = \rho$, where ρ is a positive constant, the process i is called a *homogeneous Poisson process* (HPP); otherwise, it is called a *non-homogeneous Poisson process* (NHPP). This distinction is important in the context of trend testing. As we discuss in Section 2.3, an HPP model defines a constant rate of occurrence of failures over time, which implies that the model is free of any type of trend in its rate function. On the other hand, a NHPP is a useful model for the analysis of repairable systems, and can incorporate time trends in its rate function.

NHPP models are canonical when repairable systems are subject to minimal repairs and time trends due to stochastic ageing (Lai and Xie, 2006). An important NHPP model is of exponential form, in which the rate function is defined as follows.

$$\rho_i(t;\alpha_i,\beta) = e^{\alpha_i+\beta t}, \qquad t \geq 0, \tag{2.12}$$

where α_i, $\beta \in R$ are unknown parameters. Another common NHPP model is the power law process (PLP) with the rate function

$$\rho_i(t;\theta_i,\beta) = \frac{\beta}{\theta_i}\left(\frac{t}{\theta_i}\right)^{\beta-1}, \qquad t \geq 0, \tag{2.13}$$

where $\beta, \theta_i > 0$, $i = 1, \ldots, m$, are unknown parameters (Rigdon and Basu, 1989).

In many applications, the values of some explanatory variables are recorded along with the failure times. Such covariates can be easily included in Poisson processes as explained by Cook and Lawless (2007, Section 2.2.2). This can be summarized as follows. Let $x_i(t)$ denote a $p \times 1$ vector of external time varying as well as fixed covariates for process i. We consider $X_i(t)$ as a covariate process denoted by $X_i(t) = \{x_i(s); 0 \leq s \leq t\}$. The rate function of a Poisson process is then of the form $\rho_i(t|X_i(\infty))$. Note that the rate function is conditional on the complete path of the covariate process, which is denoted by $X_i(\infty)$. However, since the covariates are assumed to be of exogenous types, at any time t, $\rho_i(t|X_i(\infty)) = \rho_i(t|X_i(t))$. Therefore, the interpretation of the rate function is based on the covariate process $X_i(t)$ for any $t \geq 0$. Following this discussion, the exponential model (2.12) can be extended to include the covariates with the rate function

$$\rho_i(t|X_i(\infty)) = \rho_i(t|x_i(t)) = \rho_{i0}(t;\alpha_i,\beta)\exp(\gamma' x_i(t)), \qquad t \geq 0, \tag{2.14}$$

where $\rho_{i0}(t;\alpha_i,\beta) = \exp(\alpha_i + \beta t)$ is called the baseline rate function with parameters α_i and β, and γ is a $p \times 1$ vector of regression parameters.

We now state some well-known theorems, which will be used later in this practicum. The first one is useful to generate realizations of a HPP. Its proof can be found, for example, in Rigdon and Basu (2000, Section 2.2).

Theorem 2.2.1. *Let $\{N_i(t); t \geq 0\}$ be a counting process with the intensity function $\lambda_i(t|H_i(t))$. Then it is an HPP with $\lambda_i(t|H_i(t)) = \rho$ if and only if the gap times W_{ij}, $j = 1, 2, \ldots$, are independent and identically distributed exponential random variables with mean ρ^{-1}.*

Second theorem is useful to generate realizations of a NHPP. A proof of the theorem can be found in Thompson (1988, p. 59).

Theorem 2.2.2. *Let $\{N_i(t); t \geq 0\}$ be a NHPP with the mean function $\mu_i(t)$. Then W_{ij}, $j = 1, 2, \ldots$, are the gap times of the process $\{N_i(t); t \geq 0\}$ if and only if the random variables $\mu(W_{ij})$, $j = 1, 2, \ldots$, are the gap times of a HPP with the rate function $\rho_i(t) = 1$.*

2.2.2 Renewal Processes

Renewal processes and their extensions provide canonical models when there is an interest in the analysis of gap times in recurrent event processes. A renewal process $\{N_i(t); t \geq 0\}$ is a stochastic process in which the gap times are independent and identically distributed (i.i.d.). Following this definition, for $i = 1, \ldots, m$, if we let

W_{ij}, $j = 1, 2, \ldots$ be i.i.d. non-negative random variables with the c.d.f $F_i(w)$, $w > 0$, then the counting process $\{N_i(t); t \geq 0\}$, where

$$N_i(t) = \max\{k : T_{in_i} = W_{i1} + W_{i2} + \cdots + W_{ik} \leq t\},$$

is called a renewal process. In this case, the intensity function of the process $\{N_i(t); t \geq 0\}$ is given by

$$\lambda_i(t|H_i(t)) = h_i(B_i(t)), \qquad t \geq 0, \tag{2.15}$$

where $B_i(t) = t - T_{iN_i(t^-)}$ is the backward recurrence time, which gives the elapsed time since the most recent failure time. The function $h(\cdot)$ in (2.15) is the hazard function for the gap times W_{ij}, $j = 1, 2, \ldots$, which is defined in Section 2.1.

Many properties of renewal processes are rigorously investigated in point process books (e.g., see Cox and Isham, 1980, Section 3.2; Rigdon and Basu, 2000, Chapter 3; Daley and Vere-Jones, 2003, Chapter 4). We summarize some of the important ones here. First note that the events "$N_i(t) \geq k$" and "$T_{ik} \leq t$", for any $k = 0, 1, 2, \ldots$ and $t > 0$, are equivalent. Therefore, $\Pr\{N_i(t) = k\} = \Pr\{T_{ik} \leq t\} - \Pr\{T_{i,k+1} \leq t\}$. Let $F_{ik}(t)$ be the c.d.f of T_{ik}; that is, $F_{ik}(t) = \Pr\{T_{ik} \leq t\}$. Then, for any renewal process $\{N_i(t); t \geq 0\}$, we have $\mu_i(t) = E\{N_i(t)\} = \sum_{k=1}^{\infty} F_{ik}(t)$. This last result follows from the fact that $\mu_i(t) = E\{N_i(t)\} = \sum_{k=1}^{\infty} k \Pr\{N_i(t) = k\}$.

In renewal processes settings, the mean of the distribution of $N_i(t)$ is usually called the renewal function, which is of interest in many applications. It can be, however, difficult to find a closed form of it. Exceptions include the cases in which the W_{ij} are i.i.d. gamma, exponential, or normal random variables. For example, if the W_{ij} are i.i.d. exponential random variables with rate ρ, then the renewal function of $\{N_i(t); t \geq 0\}$ is $\mu_i(t) = \rho t$, $t \geq 0$. We would like to note that the HPP with the rate function ρ defined in Section 2.2.1 satisfies the definition of a renewal process in which the gap times are i.i.d. exponential random variables with mean ρ^{-1} (Cook and Lawless, 2007). Let μ_i be the expected value of gap time $W_{ij}, j = 1, \ldots$; that is, $E\{W_{ij}\} = \mu_i$ for the ith process, $i = 1, \ldots$. Then, the average number of events on $(0, t]$, $\frac{N_i(t)}{t}$, goes to $\frac{1}{\mu_i}$ with probability 1 as t goes to infinity. Elementary renewal theorem is very useful and important in recurrent events analysis. It can be expressed as the average of the mean of number of events $\mu_i(t)$ on $(0, t]$, $\frac{\mu_i(t)}{t}$, converges to $\frac{1}{\mu_i}$ as t goes to infinity.

If the process i, $i = 1, 2, \ldots$, does not begin at an original point, we can not use a natural renewal process to deal with the case. Cox and Isham (1980) introduced a *modified renewal process* by comparing the distributions of gap times W_{ij}, $j = 1, 2, \ldots$. If the gap times W_{i2}, W_{i3}, $\ldots$ are independent and identically distributed random variables, but the first gap time W_{i1} follows a different distribution. By the definition of renewal process, we can find that the renewal process starts from the second gap time, therefore, we call it as *delayed renewal process* in this practicum. Delayed renewal processes are very common, hence, we consider this case in simulation studies later.

2.3 Definitions of Trends in Recurrent Events Processes

In this section, we discuss the definition of a trend in recurrent event processes. In many settings, a trend is simply defined as a systematic variation in either event occurrence rate of a process or gap times between successive events. Even though it sounds like a simple concept, it is in fact complex. For example, White and Granger (2011) denoted that there is no generally accepted definition of a trend in time series.

Similarly, Ascher and Feingold (1984) and Lawless et al. (2012) discussed the difficulty in stating a general definition of a trend in recurrent event processes. Nonetheless, we give the most common formal definitions of a trend or absence of a trend below. These definitions are based on families of models introduced in Section 2.2.

The first family of models that can incorporate time trends is the Poisson processes. As discussed in the previous section, NHPPs can be used to model time trends in the rate of occurrence of failures. In this case, the trend free model corresponds to a HPP; that is, the rate function (2.2) is constant for all $t > 0$. A monotonic time trend can be included in the rate function by a designated function $g(t)$ as follows. Suppose that the NHPP $\{N_i(t); t \geq 0\}$ has the rate function $\rho(t) = \alpha \exp\{\beta g(t)\}$, where $g(t)$ is a specified function, α is a positive valued parameter and β is a real valued parameter. For example, a choice of $g(t) = t$ for $t > 0$ leads to the well known model behind the Laplace test for trend. In this case, we want to test the trend free null hypothesis $H_0 : \beta = 0$ against the trend alternative hypothesis $H_1 : \beta \neq 0$, where $\alpha > 0$ is a nuisance parameter.

An important disadvantage of the trend tests based on Poisson processes is that the trend free process needs to be a HPP. However, there are trend free processes which are not Poisson. This includes renewal processes introduced in Section 2.2.2. Let $\{N_i(t) : t \geq 0\}$ be a renewal process with the rate function $\rho(t), t \geq 0$. In this case, by definition, the gap times, $W_{ij}, j = 1, 2, \ldots,$ are i.i.d. random variables with c.d.f. $F(w), w > 0$. A trend test then can be based on the null hypothesis H_0 : the W_{ij} $(j = 1, 2, \ldots)$ are i.i.d. In other words, we want to test whether the process $\{N_i(t) : t > 0\}$ is a renewal process or not. It should be noted that, since a HPP is a special case of renewal processes in which the W_{ij} are i.i.d. exponential random variables with a constant rate function, the tests based on renewal processes can be also used to test HPPs under the null hypothesis. We would like to note that the rate function $\rho(t)$ of a renewal process $W_{i1}, W_{i2}, \ldots$ converges to $E\{W_{ij}\}^{-1}$, which is a constant, as t approaches infinity. Therefore, the trend tests based on constant rate functions in the null hypothesis can be used for testing the trend in renewal processes observed long enough. However, the rate function of a renewal process may fluctuate significantly for small values of t (Lawless et al, 2012), and the tests based on constant rate functions may lead to the wrong conclusion about trends for such renewal processes. In the renewal process framework, the monotonic trend alternative usually includes a model in which the gap times W_{ij} either stochastically increases or decreases as j ($j = 1, 2, \ldots$) increases.

Many trend tests proposed in the literature have been focusing on the above two trend free null hypotheses. Other definitions of the absence of a trend in a process can also be given. For example, a process which produces identically distributed gap times W_{ij} ($j = 1, 2, \ldots$) can be considered as a trend free process. An exponential autoregressive process of order 1 (EAR 1) where the W_{ij} ($j = 1, 2, \ldots$) are identically distributed but not independent can be given as an example (Cox and Isham, 1980, p. 62). The trend tests based on renewal processes under the null hypothesis may not be valid in such cases.

A more general definition of a trend free process can be obtained by checking for stationarity of the processes. As defined by Cox and Isham (1980, Section 2.2), a point process is called stationary if the translation of the time axis does not alter its structure. In this sense, any point process that is stationary with respect to certain characteristics can be considered as a trend free process. Many properties of stationary processes have been discussed by Cox and Lewis (1966) and Cox and Isham (1980). In this practicum, our goal is to compare the trend tests frequently used in practice with trend tests based on robust methods. We focus on monotonic time trends. We therefore consider the definition of trend free process either a renewal process or the process has a constant rate function. HPP is a special case of renewal processes, but because of its importance, we investigate it as a separate case. We therefore consider three trend free null hypothesis

(1) the process is a HPP with rate function $\alpha > 0$.

(2) the process is a renewal process but not a HPP.

(3) the process has a constant rate function. (We do not specify any model in this case).

In Chapter 3, we discuss important trend tests developed under above null hypotheses for multiple processes under observation simultaneously. Their asymptotic properties and power were discussed in Chapter 4.

2.4 Likelihood Function for Recurrent Event Processes and Related Procedures

Likelihood function for recurrent event processes can be written in terms of intensity functions. Suppose that m independent processes are under observation. Let $\{N_i(t), t \geq 0\}$ be the ith recurrent event process with the associated intensity function $\lambda_i(t|H_i(t))$, $i = 1, \ldots, m$. As discussed in Section 2.1, the intensity function $\lambda_i(t|H_i(t))$ completely specifies the process $\{N_i(t); t \geq 0\}$ in the continuous time scale. Suppose that the ith process is observed over the time interval $[\tau_{0i}, \tau_i]$, where τ_{0i} and τ_i are prespecified positive values such that $0 \leq \tau_{0i} < \tau_i$. We now state the likelihood function for the outcome "$n_i \geq 0$" events observed at times $t_{i1} \leq \cdots \leq t_{in_i}$ over the interval $[\tau_{0i}, \tau_i]$ for m independent processes.

Theorem 2.4.1. *Following the setting stated above, the likelihood function for the outcome "$n_i \geq 0$ events observed at times $t_{i1} < \cdots < t_{in_i}$ over $[\tau_{0i}, \tau_i]$, $i = 1, \ldots, m$"*

conditional on $H_i(\tau_{0i})$ *is given by*

$$L = \prod_{i=1}^{m} L_i, \tag{2.16}$$

where

$$L_i = \{\prod_{j=1}^{n_i} \lambda_i(t_{ij}|H_i(t_{ij}))\} \exp\{-\int_{\tau_{0i}}^{\tau} \lambda_i(u|H_i(\mu))du\} \tag{2.17}$$

A sketch proof of the above theorem is given by Cook and Lawless (2007, Section 2.1). A more rigorous treatment of the likelihood function for recurrent event processes can be found in Daley and Vere-Jones (2003, Chapter 7). From now on, we assume that the observation of a process starts at time 0 and so we take $\tau_{0i} = 0, i = 1, \ldots, m$. Unless stated otherwise, the methods also work when the τ_{0i} are not equal to zero.

Let the intensity function be parameterized by a $p \times 1$ vector of parameters $\theta = (\theta_1, \ldots, \theta_p)^t$, where t stands for a vector transpose. Then, the likelihood function is given by

$$L(\theta) = \prod_{i=1}^{m} L_i(\theta), \tag{2.18}$$

where

$$L_i(\theta) = \{\prod_{j=1}^{n_i} \lambda_i(t_{ij} H(t_{ij}); \theta)\} \exp\{-\int_{0}^{\tau_i} \lambda_i(u|H_i(u); \theta)du\} \tag{2.19}$$

The log likelihood function $l(\theta) = logL(\theta)$ is

$$l(\theta) = \sum_{i=1}^{m} l_i(\theta), \tag{2.20}$$

where

$$l_i(\theta) = \sum_{j=1}^{n_i} log\lambda_i(t_{ij}|H(t_{ij}); \theta) - \int_{0}^{\tau_i} \lambda_i(u|H_i(u); \theta)du. \tag{2.21}$$

The $p \times 1$ score vector $U(\theta) = (U_1(\theta), \ldots, U_p(\theta))^t$ includes the components of the form

$$U_k(\theta) = \partial l(\theta)/\partial \theta_k, \qquad k = 1, \ldots, p, \tag{2.22}$$

which are called score functions. Under some regularity conditions, an estimate of θ, denoted by, $\hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_p)^t$ can be found by solving $U(\theta) = \mathbf{0}$ for $\theta = \hat{\theta}$, where $\mathbf{0}$ is a $p \times 1$ vector of zeros. The $U_k(\theta) = \mathbf{0}$, $k = 1, \ldots, p$, are called the maximum likelihood equations and the $\hat{\theta}_k$ are called the maximum likelihood estimators of the θ_k, $k = 1, \ldots, p$.

We next let $I(\theta)$ denote the $p \times p$ information matrix with components $I_{kl}(\theta) = -\partial^2 l(\theta)/\partial\theta_k \partial\theta_l$ for $k, l = 1, \ldots, p$. Similarly, we let $J(\theta)$ denote the $p \times p$ expected or Fisher information matrix. That is, $J(\theta) = E\{I(\theta)\}$. Suppose that we want to test the null hypothesis $H_0 : \theta = \theta_0$ where $\theta_0 = (\theta_{01}, \ldots, \theta_{0p})$ are specified values of parameters. The score test statistic for testing H_0 is then defined by

$$\delta = U^t(\theta_0) J(\theta_0)^{-1} U(\theta_0). \tag{2.23}$$

Under regularity conditions specified by Andersen et al (1993, Section VII 2.2), the asymptotic distribution of the test statistic (2.23) is a chi-squared distribution with p degrees of freedom under the null hypothesis as m approaches infinity (Cook and Lawless, 2007).

Now, let the parameter vector θ be partitioned as $\theta = (\alpha^t, \beta^t)^t$, where $\alpha = (\alpha_1, \ldots, \alpha_r)^t$ is an $r \times 1$ vector of parameters and $\beta = (\beta_1, \ldots, \beta_q)$ is a $q \times 1$ vector of parameters and $r + q = p$. We next define the score vectors $U_\alpha(\theta)$ and $U_\beta(\theta)$, where

$$U_\alpha(\theta) = \left[\frac{\partial l(\theta)}{\partial \alpha_1}, \ldots, \frac{\partial l(\theta)}{\partial \alpha_r}\right]^t, \tag{2.24}$$

and

$$U_\beta(\theta) = \left[\frac{\partial l(\theta)}{\partial \beta_1}, \ldots, \frac{\partial l(\theta)}{\partial \beta_q}\right]^t. \tag{2.25}$$

We next partition the information and Fisher information matrices accordingly. For example, the partitioned information matrix is given by

$$I(\theta) = \begin{bmatrix} I_{\alpha\alpha}(\theta) & I_{\alpha\beta}(\theta) \\ I_{\beta\alpha}(\theta) & I_{\beta\beta}(\theta) \end{bmatrix}, \tag{2.26}$$

where $I_{\alpha\alpha}(\theta)$ is an $r \times r$ matrix with components $I_{kl}(\theta) = -\partial^2 l(\theta)/\partial\alpha_k \partial\alpha_l$ for $k, l = 1, \ldots, r$, $I_{\alpha\beta}(\theta)$ is an $r \times q$ matrix with components $I_{kl}(\theta) = -\partial^2 l(\theta)/\partial\alpha_k \partial\beta_l$ for $k = 1, \ldots, r$ and $l = 1, \ldots, q$, etc. We also define the inverse of (2.26) as

$$I^{-1}(\theta) = \begin{bmatrix} I^{\alpha\alpha}(\theta) & I^{\alpha\beta}(\theta) \\ I^{\beta\alpha}(\theta) & I^{\beta\beta}(\theta) \end{bmatrix}. \tag{2.27}$$

Similarly, the expected or Fisher information matrix is then $J(\theta) = E[I(\theta)]$ and given by

$$J(\theta) = \begin{bmatrix} J_{\alpha\alpha}(\theta) & J_{\alpha\beta}(\theta) \\ J_{\beta\alpha}(\theta) & J_{\beta\beta}(\theta) \end{bmatrix}, \tag{2.28}$$

and its inverse matrix is denoted by

$$J^{-1}(\theta) = \begin{bmatrix} J^{\alpha\alpha}(\theta) & J^{\alpha\beta}(\theta) \\ J^{\beta\alpha}(\theta) & J^{\beta\beta}(\theta) \end{bmatrix}. \tag{2.29}$$

Now, suppose that β_0 is a specified value of β. Let $\widetilde{\alpha}(\beta_0)$ be the value of α that maximizes $L(\alpha, \beta)$ or equivalently $l(\alpha, \beta)$. The functions $L(\widetilde{\alpha}(\beta), \beta_0)$ and $l(\widetilde{\alpha}(\beta), \beta_0)$ are called profile likelihood and profile log likelihood functions for β, respectively. Let's consider the null hypothesis $H_0 : \beta = \beta_0$ and let α be vector of nuisance parameters.

We also let $\widetilde{\theta}_0 = (\widetilde{\alpha}(\beta_0)^t, \beta_0^t)^t$. A partial score test statistic for testing H_0 is then defined by

$$S_2 = U_\beta^t(\widetilde{\theta}_0) J^{\beta\beta}(\widetilde{\theta}_0) U_\beta(\widetilde{\theta}_0). \tag{2.30}$$

A test for H_0 based on (2.30) is called a partial score test. Under the null hypothesis H_0 and some regularity conditions, the asymptotic distribution of S_2 is a chi squared with q degrees of freedom as m goes to infinity (Boos, 1992). It should be noted that the same asymptotic result holds for any consistent estimator of $J^{\beta\beta}(\theta_0)$ replaced with $J^{\beta\beta}(\widetilde{\theta}_0)$ in (2.30). We are going to use these results later in Chapter 3 in the development of a robust trend test.

2.5 Simulation Procedures

In this section, we first discuss how to generate realizations of a recurrent event process with a given intensity function. We then provide steps of a general algorithm used in this practicum, and discuss the methods based on the simulations.

2.5.1 Simulation of a Recurrent Event Process

Simulation of stochastic processes has been discussed by many authors. A procedure which is sufficient for our purposes in this practicum is given by Daley and Vere-Jones (2003) and Cook and Lawless (2007). We used this procedure to generate realizations of recurrent event processes. It can be explained as follows.

Suppose that m independent processes are under observation and observation over the time interval $[0, \tau_i]$, $i = 1, \ldots, m$. Let $\lambda_i(t|H_i(t))$ be the associated intensity function of the ith process. It can be shown that

$$\Pr\{N_i(s,t) = 0 | H_i(s^+)\} = \exp\{-\int_s^t \lambda_i(u|H_i(u))du\}, \tag{2.31}$$

where $H_i(u) = \{H_i(s^+), N_i(s,u) = 0; 0 \leq s < u\}$ in the integral. A proof of this assertion is given by Cook and Lawless (2007, p.30). Note that the events "$N_i(t_{i,j-1}, t_{i,j-1+w}) = 0|H_i(t_{i,j-1})$" and "$W_{ij} > w|T_{i,j-1}, H_i(t_{i,j-1})$" are equivalent. Therefore, from the result in (2.31), we have

$$\Pr\{W_{ij} > w|T_{i,j-1} = t_{i,j-1}, H_i(t_{i,j-1})\} = \exp\{-\int_{t_{i,j-1}}^{t_{i,j-1+w}} \lambda_i(u|H_i(u))du\}, \quad (2.32)$$

which follows a standard uniform distribution denoted by $U(0,1)$. The conditional distribution of W_{ij} in (2.32) given the previous event history can be used to generate realizations of a recurrent event process by considering the gap times as follows. Let E_{ij} $(i = 1, \ldots, m,\ j = 1, 2, \ldots)$ be a random variable defined by

$$E_{ij} = \int_{t_{i,j-1}}^{t_{i,j-1}+W_{ij}} \lambda_i(u|H_i(u))du. \quad (2.33)$$

Then, from the result in (2.32), we have that $U_{ij} = \exp\{-E_{ij}\}$ for $i = 1, \ldots, m$ and $j = 1, 2, \ldots$, which follows a standard uniform distribution. Therefore, given $H(t_{i,j-1})$ and $t_{i,j-1}$, the random variable $E_{ij} = -logU_{ij}$ has a standard exponential distribution denoted by $E_{ij} \sim Exp(1)$. We next provide a general algorithm to generate realizations of a recurrent event process for a given intensity function.

2.5.2 Algorithms

In this practicum, we generate realizations of multiple recurrent event processes with and without monotonic time trends. The major processes considered include HPPs, RPs and delayed RPs for trend free processes and their trend alternatives.

A general algorithm for generating realizations of a recurrent event process with the intensity function $\lambda_i(t|H_i(t))$ is given below. In Chapter 4, we give the algorithms used for each simulation scenarios separately. All of those algorithms are based on this general one. The algorithm used in this study is then given as follows:

1. Set Pseudo-random number j=1 and initialize $t_{i0} = 0$ for process i.

2. Generate $U_{ij} \sim U(0, 1)$.

3. By $T_{ij} \leftarrow t_{i0} - log(U_{ij})$, set the transformation as $E_{ij} = -log(U_{ij})$.

4. Calculate W_{ij} by solving $E_{ij} = \int_{t_{i,j-1}}^{t_{i,j-1}+W_{ij}} \lambda_i(t|H_i(t))dt$. And the jth event time $T_{ij} = t_{i,j-1} + W_{ij}$.

5. Deliver $t_{i,j-1} = T_{i,j-1}$ and $j = j + 1$ if $T_{ij} < \tau$. Otherwise, stop and $W_{i,n_i+1} = \tau - T_{i,n_i}$

6. Go to Step (1)

The gap times W_{ij} are equal to $\frac{-log(U_{ij})}{\rho}$ if we generate the data from a HPP with the rate function as ρ. It should be noted that in the above algorithm, we may need to solve the integral $E_{ij} = \int_{t_{i,j-1}}^{t_{i,j-1}+W_{ij}} \lambda_i(u|H_i(u))du$ numerically. A numeric method for this purpose is explained by Lawless and Thiagarajah (1996). In this practicum, we use R software in all simulations.

2.5.3 The Use of Simulations

We used simulations to study accuracy of the limiting distributions of trend tests in finite sample sizes and obtain power of the trend tests considered in this practicum. Using the algorithm explained previously, we generated D realizations of m (m > 1) independent recurrent event processes under a trend free null hypothesis. Let Z be one of the trend tests considered in this practicum. For $d = 1, \ldots, D$, we calculated the value of the test statistic, denoted by Z_d, and values of other test statistics by using the same generated data. We then kept their values in D dimensional vectors. We used these vectors to study the distribution of the test statistics under the null hypotheses by using the normal quantile-quantile (Q-Q) plots.

As discussed in Section 2.4, the partial score statistic given in (2.30) is asymptotically $\chi^2_{(p)}$ under the null hypothesis $H_0 : \beta = \beta_0$, where β is a $p \times 1$ vector of parameters. A p-value of the test can be calculated by using this approximation for large values of m. If the $\chi^2_{(p)}$ approximation is not accurate, a p-value can be obtained with simulations as explained next.

Let $Z^\star$ be the value of the test statistic calculated using the observed data. We generate D data sets under the null hypothesis with the algorithm given previously. For each generated data set, we calculate the test statistic, denoted by Z_d, $d = 1, \ldots, D$. The estimated p-value is then given by

$$\frac{\sum_{d=1}^{D} I(Z_d > Z^\star)}{D}. \tag{2.34}$$

In this practicum, we considered the cases in which the degree of freedom of the chi-square distribution is 1. We, therefore, investigated the accuracy of the $Z \sim N(0,1)$ approximations in Chapter 4 under various finite sample scenarios.

We also calculated the power of the tests with simulations. The power function of the hypothesis $H_0 : \beta = \beta_0$ against $H_1 : \beta \neq \beta_0$ is given by $\Pr(\beta_1) = \Pr\{\text{reject } H_0 \mid \beta = \beta_1\}$, where $\beta_1 \neq \beta_0$. We, therefore, generated D realizations of independent processes under the alternative hypothesis for various scenarios in Chapter 4. For $d = 1, \ldots, D$, we calculated the values of test statistic Z, say Z_d. We kept the values of test statistics in D dimensional vectors. The empirical power of the test was then calculated for a nominal test size. We used cut off points based on asymptotic distributions of test statistics as well as empirical quantiles of test statistics based on 10,000 simulation runs under the null hypothesis. For example, let the nominal size of the test be 0.05 and cutoff1 and cutoff2 be the 0.025 quantile and 0.975 quantile of the values of 10,000 Z statistic calculated under the null hypothesis. The empirical power of the test is then given by

$$\widehat{\Pr}(\beta_1) = \sum_{d=1}^{D} \{I(Z_d < \text{cutoff1}) + I(Z_d > \text{cutoff2})\}/D$$

We presented the empirical power of the tests in Chapter 4.

Chapter 3

Testing for Trend in Recurrent Event Processes

In Chapter 2, we have introduced some fundamental concepts in recurrent event processes. In this chapter, we first briefly discuss some graphical methods for determining trends in Section 3.2. In Section 3.3, we introduce common model based tests for monotonic trends, as well as robust trend tests, when m identical processes are under observation. In particular, we focus on a robust trend test based on the rate function of a power law process (PLP). We call this test the generalized PLP (GPLP) test. In Section 3.4, we extend the tests introduced in Section 3.3 to deal with non-identical recurrent event processes.

3.1 Introduction

Many formal tests for trend have been developed in the literature. Our main goal in this practicum is to compare the power of some important tests for monotonic trends in recurrent event processes. To do this, we included tests that can be easily applied in routine checks for the presence or absence of trend before developing any elaborate model for the analysis. In the next section, we first introduce some simple graphical methods for the detection of trends. As discussed in the previous concepts, graphical methods can be useful and recommended as a starting point for any trend analysis. However, they have important limitations since graphical methods are not of interest in this practicum, our introduction is very brief and not comprehensive. In the following sections of this chapter, we introduced the tests that more compared in this practicum. We use first the model based procedures. That is, those procedures require a fully specification of the trend free model under the null hypothesis. We next consider the robust methods based on some marginal characteristics of recurrent event processes. Trend tests based on these robust procedures do not require a fully specification of a model, and can be applied in a wide range of applications. In the robust trend tests sections, we introduce a robust test statistic based on power law processes (PLPs), which has not been discussed in detail in the literature. The development of this test statistic is based on a procedure recommended by Lawless et al. (2012). We discussed the trend tests first when m identical processes are under observation in Section 3.3 and then when m non-identical processes are under observation in Section 3.4.

3.2 Graphical Methods

The graphical methods may provide some insight about the presence and form of trends in recurrent event processes. These methods can be very useful especially then a large number of events are observed in a single process. However, the detection of trends in recurrent event processes can be challenging with graphical methods when the observed number of events is small per processes or multiple processes are simultaneously of interest.

The first graphical method that can be used for the detection of trends is the dot plot. Cook and Lawless (2007) described a graphical method which is called *event plots*, in this practicum, we call it as *dot plots*. Dot plots is placing a dot on the time point when an event occur, and it is possible occur more than one event on the same time. Cox and Lewis (1966) gave failure data for air-conditioning equipment which displays in Chapter 5. The number of failures for aircraft 6, with the operating hours is shown in Fig 3.1.

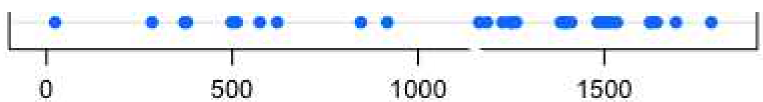

Figure 3.1: Time dot plot for air-conditioning equipment for aircraft 6 (Cox Lewis, 1966).

The next plot is based on the cumulative sample mean function as defined below. Suppose that m independent processes are under observation. Let $\{N_i(t); t \geq 0\}$ be a counting process observed over $[0, \tau_i]$. The cumulative sample mean function is then given by

$$\hat{\mu}(t) = \frac{1}{m} \sum_{i=1}^{m} N_i(t), \qquad t \in [0, \tau_i], \tag{3.1}$$

which is a nonparametric estimator of the mean function $\mu(t)$, that is assumed to be common for all processes. A plot of $\hat{\mu}(t)$ against cumulative time scale t can be useful to reveal trends. For example, a convex or a concave shape indicates an increasing or a decreasing trend in recurrent event processes, respectively (Kvaløy and Lindqvist, 1998). A roughly straight line indicates the absence of a trend. A similar plot can be used for single processes ($m = 1$) by plotting $N_i(t)$ against t. A more generalized version of the plot based on cumulative sample mean function is called the Nelson-Aalen estimate (Cook and Lawless, 2007). This plot can be used with more complicated observation schemes as well. In this practicum, we used the dot plots and the cumulative mean function plots for graphical checks for trend in Chapter 5.

We next introduce two graphical procedures that are sometimes applied for the detection of trends in applications. Duane (1964) introduced a scatter plot of $\frac{N(t_{ij})}{t_{ij}}$ versus t_{ij} on double logarithmic paper, $i = 1, \ldots, m$; $j = 1, \ldots, n_i$. That is, to plot *cumulative failures* versus *cumulative time* graphically, where $\frac{N(t_{ij})}{t_{ij}}$ is the *cumulative failure rate*. If the cumulative number of failures on the graph looks like concave surface, then the inter-arrival times of an improving (deteriorating) system tend to be larger (smaller). Note that Duane plots can be used to assess the adequacy of power law process, a Duane plot should be roughly linear under power law process. Barlow and Campo (1975) acknowledged Total time on test (TTT) plots for recurrent events. TTT plots by plotting the ordered pairs $(\frac{i}{N}, \frac{r(T_i)}{r(\tau)})$ for non-repairable system, where i is the individual can be checked, N is the total units, and $r(T_i)$ represents the total time that all units have been on test at the times of the ith failure, and $r(\tau)$ is the total time on test at time t. It expects that the plots should converge to the diagonal of the unit square. Therefore, TTT plots can be used as a good-fit-test for the exponential distribution. Klefsjö and Kumar (1992) have developed a method that suggested how to apply TTT plots to a data set from repairable system by the properties of power law processes.

3.3 Testing for Trend in Identical Processes

In this section, we assume m identical processes are under observation. We introduce the test statistics below under three categories: (i) Tests based on PPs, (ii) tests based on RPs and (iii) robust tests based on marginal characteristics of recurrent event processes. In many reliability studies, it is reasonable to assume that the processes are identical. However, in the next section, we also consider the same test statistic for the nonidentical processes. The tests considered in each category include:

(i) Tests on PPs: (1) Laplace test, (2) Power law process (PLP) test.

(ii) Tests based on RPs: (3) Lewis-Robinson test, (4) Rank test.

(iii) Robust tests: (5) the generalized Laplace test, (6) the generalized PLP test.

3.3.1 Tests Based on Poisson Processes

We introduced HPPs and NHPPs in Section 2.2.1. Following our notation, we let $\{N_i(t) : t \geq 0\}$ be a PP with the intensity function $\lambda_i(t|H_i(t)) = \rho_i(t)$, $t > 0$, for $i = 1, \ldots, m$. A trend free process with the family of Poisson processes indicates a constant intensity function. On the other hand, a monotonic time trend can be incorporated through NHPPs. A very useful NHPP model incorporating a monotonic trend is given by

$$\lambda_i(t|H_i(t)) = \rho_i(t; \alpha, \beta) = \alpha \exp(\beta g(t)), \qquad t \geq 0, \tag{3.2}$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are parameters and $g(t)$ is a function that specifies the time trend. The model (3.2) can be easily extended to include external fixed or time varying covariates if external covariates are of interest. Let $Z(t)$ be a $p \times 1$ vector of external covariates and γ be a $p \times 1$ vector of regression parameters. Then, the extended model is $\rho_i(t; \alpha, \beta, \gamma) = \alpha \exp(\beta g(t) + \gamma^t Z(t))$, $t \geq 0$. In the following discussion, we do not consider external covariates but methods can be easily extended to deal with covariates as well.

With the NHPP model (3.2), a test for trend can be developed by considering the hypothesis $H_0 : \beta = 0,\ \alpha > 0$ against $H_1 : \beta \neq 0,\ \alpha > 0$. Note that under the null hypothesis, the model is a HPP with the constant rate function α. A partial score test for the absence of trend can be developed as explained in Section 2.4. However, as discussed by Cox and Lewis (1966), α is a nuisance parameter and a simple test for the absence of trend can be developed by conditioning on the value of the sufficient statistic for α, which is the observed number of failures in the ith process; that is, $N_i(\tau_i) = n_i$, $i = 1, \ldots, m$. The following discussion is given by Lawless et al (2012).

Suppose that m independent processes are under observation. The process $\{N_i(t);\ t \geq 0\}$ with the intensity function (3.2) is observed over $[0, \tau_i]$, $i = 1, \ldots, m$. Let $t_{i1} < t_{i2} < \ldots t_{in_i}$ be the failure times of the ith process over $[0, \tau_i]$. From Theorem 2.4.1 in Section 2.4, the likelihood function of the outcome "n_i failures are observed at times $t_{i1}, \ldots, t_{in_i}$ over $[0, \tau_i]$, for $i = 1, \ldots, m$" is given by

$$L(\alpha, \beta) = \prod_{i=1}^{m} \{\alpha^{n_i} e^{\beta \sum_{i=1}^{n_i} g(t_{ij})} e^{\int_0^{\tau_i} \alpha e^{g(u)} du}\}. \tag{3.3}$$

Since $N_i(t)$ is a Poisson random variable with the mean $\mu(t) = \int_0^t \alpha e^{g(u)} du$, we have

$$\Pr\{N_i(\tau_i) = n_i\} = \frac{\alpha^{n_i} (\int_0^{\tau_i} e^{g(u)} du) e^{-\int_0^{\tau_i} \alpha e^{g(u)} du}}{n_i!} \tag{3.4}$$

Therefore, the conditional likelihood function $L_c(\beta)$ of the failure times $t_{i1}, \ldots, t_{in_i}$, $i = 1, \ldots, m$, given $N_i(\tau_i) = n_i$ is proportional to

$$L_c(\beta) = \prod_{i=1}^{m} \prod_{j=1}^{n_i} \left\{ \frac{e^{\beta g(t_{ij})}}{\int_0^{\tau_i} e^{\beta g(u)} du} \right\}. \tag{3.5}$$

Note that the nuisance parameter α does not appear in the conditional likelihood function (3.5). There are different choices available for the function g(t) in (3.2). If we let $g(t) = t$, $t \geq 0$ and apply a score procedure based on $L_c(\beta)$ given in (3.5), we obtain the Laplace test for trend. This test is discussed next. Another choice of $g(t) = \log(t)$ leads to the Military Handbook test (Rigdon and Basu, 2000).

3.3.1.1 The Laplace Test

We follow the setup given above and let $g(t) = t$ in (3.2). From (3.5), the conditional score statistic $U_c(\beta) = \frac{\partial l_c(\beta)}{\partial \beta}$, where $l_c(\beta) = logL_c(\beta)$, is given by

$$U_c(\beta) = \sum_{i=1}^{m} \left\{ \sum_{j=1}^{n_i} t_{ij} - \sum_{j=1}^{n_i} \frac{\int_0^{\tau_i} ue^{\beta u} du}{\int_0^{\tau_i} e^{\beta u} du} \right\}. \tag{3.6}$$

The variance of $U_c(\beta)$ can be found as follows:

$$Var(U_c(\beta)) = E \left\{ \frac{\sum_{i=1}^{m} n_i^2 \tau_i^2}{12} \right\} + Var \left\{ \sum_{i=1}^{m} \frac{n_i \tau_i}{2} - \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{\int_0^{\tau_i} ue^{\beta u} du}{\int_0^{\tau_i} e^{\beta u} du} \right\} \tag{3.7}$$

Under the null hypothesis, $H_0 : \beta = 0$, (3.6) and (3.7) give

$$U_c(0) = \sum_{i=1}^{m} \left\{ \sum_{j=1}^{n_i} t_{ij} - \frac{n_i \tau_i}{2} \right\}, \tag{3.8}$$

and

$$Var_c(0) = \sum_{i=1}^{m} \frac{n_i \tau_i^2}{12} \tag{3.9}$$

respectively. Therefore, the Laplace test statistic for testing the null hypothesis $H_0 : \beta = 0, \alpha > 0$ is given by

$$Z_{LT} = \frac{U_c(0)}{\sqrt{Var_c(0)}} = \frac{\sum_{i=1}^{m} \{\sum_{j=1}^{n_i} t_{ij} - n_i \tau_i / 2\}}{\{\sum_{i=1}^{m} n_i \tau_i^2 / 12\}^{1/2}}. \tag{3.10}$$

The trend test on (3.10) is called the Laplace test. The development and properties of the Laplace test can be found in Cox and Lewis (1966, Section 3.3).

3.3.1.2 The Power Law Process Trend Test

We introduced the power law process (PLP) in Section 2.2.1. It is a NHPP with the intensity function

$$\lambda_i(t \mid H_i(t)) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}, \qquad t \geq 0, \tag{3.11}$$

where $\beta > 0$ and $\theta > 0$ are unknown parameters. This process is sometimes called the Weibull Poisson process, but as discussed by Rigdon and Basu (1989), we prefer to call it the PLP.

It is easy to see from (3.11) that the PLP becomes a HPP with the rate function θ^{-1} when $\beta = 1$. Therefore, a test for trend can be developed for the hypothesis $H_0 : \beta = 1$ against $H_1 : \beta \neq 1$. When $\beta > 1$, the intensity function (3.11) is an increasing function of t.This case corresponds to a monotonically increasing trend. When $0 < \beta < 1$, the intensity function (3.11) is a decreasing function of t, which can model a monotonically decreasing trend in the rate of event occurrences. The properties of the PLP and its use as a model for single repairable systems have been discussed by Rigdon and Basu (1989). In this section, we focus its role in the development of a test for the absence of monotonic trends in $m > 1$ independent and identical systems.

Following the previous notation, we let $\{N_i(t); t \geq 0\}$ be a counting process with the associated intensity function (3.11). The process $\{N_i(t), t \geq 0\}$ is observed over the time interval $(0, \tau_i]$ for $i = 1, \ldots, m$. From Section 2.4, the likelihood function of β and θ is given by

$$L(\beta, \theta) = \prod_{i=1}^{m} \left\{ \left[\prod_{j=1}^{n_i} \frac{\beta}{\theta} \left(\frac{t_{ij}}{\theta}\right)^{\beta-1} \right] \exp\left(-\frac{\tau_i^{\beta}}{\theta^{\beta}}\right) \right\}, \tag{3.12}$$

and the log likelihood function $l(\beta, \theta) = \log L(\beta, \theta)$ is

$$l(\beta, \theta) = \sum_{i=1}^{m} \{n_i \log \beta - n_i \beta \log \theta + (\beta - 1) \sum_{j=1}^{n_i} \log t_{ij} - (\frac{\tau_i}{\theta})^{\beta}\} \tag{3.13}$$

The score functions are defined by $U_\theta(\beta, \theta) = \frac{\partial l(\beta,\theta)}{\partial \theta} = \sum_{i=1}^{m} U_{\theta i}(\beta, \theta)$ and $U_\beta(\beta, \theta) = \frac{\partial l(\beta,\theta)}{\partial \beta} = \sum_{i=1}^{m} U_{\beta i}(\beta, \theta)$, where

$$U_{\theta i}(\beta, \theta) = -\frac{\beta n_i}{\theta} + \frac{\beta \tau_i^{\beta}}{\theta^{\beta+1}}, \tag{3.14}$$

and

$$U_{\beta i}(\beta, \theta) = \frac{n_i}{\beta} - n_i \log \theta + \sum_{j=1}^{n_i} \log t_{ij} - (\frac{\tau_i}{\theta})^{\beta} \log(\frac{\tau_i}{\theta}) \tag{3.15}$$

Under the null hypothesis of the absence of a trend in a PLP with the intensity function (3.11); that is, under $H_0 : \beta = 1$, we have

$$U_\theta(1, \theta) = \frac{\tau.}{\theta^2} - \frac{n.}{\theta}, \tag{3.16}$$

where $\tau. = \sum_{i=1}^{m} \tau_i$ and $n. = \sum_{i=1}^{m} n_i$. By solving $U_\theta(1, \theta) = 0$ for $\theta = \tilde{\theta}$, we obtain an constant estimator of θ that maximizes $l(1, \theta)$ as follows

$$\tilde{\theta} = \frac{\tau.}{n.}. \tag{3.17}$$

From (3.15), the function $U_\beta(1, \tilde{\theta})$ can be rewritten as follows.

$$U_\beta(1, \tilde{\theta}) = \sum_{i=1}^{m} U_{\beta i}(1, \tilde{\theta}) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \log t_{ij} - \frac{n.}{\tau.} \sum_{i=1}^{m} \tau_i \log \tau_i + n.. \tag{3.18}$$

From Section 2.4, a variance estimate of $U_\beta(1,\tilde\theta)$ is given by

$$\widehat{Var}\{U_\beta(1,\tilde\theta)\} = I_{\beta\beta}(1,\tilde\theta) - I_{\beta\theta}(1,\tilde\theta)I_{\theta\theta}^{-1}(1,\tilde\theta)I_{\theta\beta}(1,\tilde\theta), \tag{3.19}$$

where $I_{\beta\beta}(\beta,\theta) = -\frac{\partial^2 l(\beta,\theta)}{\partial\beta^2}$, $I_{\beta\theta}(\beta,\theta) = I_{\theta,\beta}(\beta,\theta) = -\frac{\partial^2 l(\beta,\theta)}{\partial\beta\theta}$ and $I_{\theta\theta}(\beta,\theta) = -\frac{\partial^2 l(\beta,\theta)}{\partial\theta^2}$. Therefore,

$$\widehat{Var}\{U_\beta(1,\tilde\theta)\} = \sum_{i=1}^{m}\int_0^{\tau_i}\left[q(u) - \frac{G.}{\tau.}\right]^2 \frac{n.}{\tau.}du, \tag{3.20}$$

where $G. = \sum_{i=1}^m \int_0^{\tau_i}(1 - \log\frac{\tau.}{n.} + \log u)\,du$. That is,

$$I_{\beta\beta}(\beta,\theta) = \frac{n.}{\beta^2} + \sum_{i=1}^{m}(\frac{\tau_i}{\theta})^\beta[\log(\frac{\tau_i}{\theta})]^2, \tag{3.21}$$

$$I_{\beta\theta}(\beta,\theta) = \frac{-\sum_{i=1}^m \tau_i^\beta - \beta\sum_{i=1}^m \tau_i^\beta \log(\frac{\tau_i}{\theta}) + n.\theta^\beta}{\theta^{\beta+1}}, \tag{3.22}$$

and

$$I_{\theta\theta}(\beta,\theta) = -\frac{\beta n.}{\theta^2} + \frac{\beta(\beta+1)\sum_{i=1}^m \tau_i^\beta}{\theta^{\beta+2}}. \tag{3.23}$$

We therefore obtain

$$I_{\beta\beta}(1,\tilde\theta) = n. + \sum_{i=1}^{m}\frac{\tau_i}{\tilde\theta}[\log(\frac{\tau_i}{\tilde\theta})]^2, \tag{3.24}$$

$$I_{\beta\theta}(1,\tilde\theta) = \frac{-\sum_{i=1}^m \tau_i log(\frac{\tau_i}{\tilde\theta})}{\tilde\theta^2}, \tag{3.25}$$

and

$$I_{\theta\theta}(1,\tilde\theta) = -\frac{n.}{\tilde\theta^2} + \frac{2\tau.}{\tilde\theta^3}. \tag{3.26}$$

Once again, the variance estimate of $U_\beta(1,\tilde\theta)$ can be simplified as

$$\widehat{Var}\{U_\beta(1,\tilde\theta)\} = n. + \frac{\tau.(\sum_{i=1}^m[\log\tau_i]^2) - (\sum_{i=1}^m \tau_i\log\tau_i)^2}{\tau.\tilde\theta}, \tag{3.27}$$

where $\tilde{\theta} = \frac{\tau_{\cdot}}{n_{\cdot}}$. In simulation part, we define $\tau_i = \tau$, then $\widehat{Var}\{U_\beta(1, \tilde{\theta})\} = n_{\cdot}$. A partial score test statistic for testing the null hypothesis $H_0 : \beta = 1$, $\theta > 0$ against the alternative hypothesis $H_1 : \beta \neq 1$, $\theta > 0$ is then given by

$$Z_{PLP} = \frac{U_\beta(1, \tilde{\theta})}{\sqrt{\widehat{Var}\{U_\beta(1, \tilde{\theta})\}}}, \tag{3.28}$$

where $U_\beta(1, \tilde{\theta})$ is given in (3.18) and $\widehat{Var}\{U_2(1, \tilde{\theta})\}$ is given in (3.27). We call a test based on the test statistic Z_{PLP} given in (3.28) a PLP trend test. By the discussion in Section 2.4, the asymptotic distribution of the test statistic Z_{PLP} is a standard normal distribution as $m \to \infty$ for fixed τ_i values.

As discussed previously, a simple test for the null hypothesis $H_0 : \beta = 1$, $\theta > 0$ can be based on the conditional likelihood function for β given the value of the sufficient statistic of θ, and the following the score procedures. This test is called the Military Handbook test (Kvaløy and Lindqvist, 1998; Lawless et al., 2012). We do not consider the Military Handbook test in this practicum because power of this tests has been investigated and compared with some other tests considered in this practicum by other authors (e.g. Bain et al. 1985; Cohen and Sackrowitz, 1993).

3.3.2 Tests Based on Renewal Processes

An important drawback about the trend tests based on Poisson processes is that the trend free model needs to be a HPP. Therefore, these tests may falsely reject the absence of a trend when a process is trend free but not a HPP. For example, as introduced in Section 2.2.2, the gap times in a renewal process are i.i.d. Thus, if a process is a renewal process but not a HPP, which is a special case of renewal processes, then trend tests based on Poisson processes may lead to false rejection of the absence of trend. We therefore introduce two tests for monotonic trends in recurrent event processes for testing the null hypothesis which states the process is a renewal process against monotonic trend alternatives.

3.3.2.1 The Rank Test for Trend

The rank test for monotonic trends was introduced by Cox and Lewis (1996). Write the rank test, our aim is to test the null hypothesis H_0 : the W_{ij} $(j = 1, 2, \dots)$ are i.i.d. for each i. The rank test avoids parametric assumptions on the distribution of W_{ij} for each i $(i = 1, \dots, m)$. Once again, we consider m independent processes. The process $\{N_i(t); t \geq 0\}$ is followed over $[0, \tau_i]$, $i = 1, \dots, m$. The T_{ij} and W_{ij} $(i = 1, \dots, m$ and $j = 1, 2, \dots)$ denote the failure times and gap times, respectively. It should be noted that the rank test is developed under a Type 2 censoring mechanism; that is, the values of n_i are prespecified rather than the τ_i. This type of observation is not as common as Type 1 censoring mechanism, where the values of τ_i are prespecified rather than the n_i values. A discussion of this issue can be found in Lawless et al. (2012). We therefore do not discuss this here anymore. In the development of the test statistic, we take the last event time as the end-of-followup time τ_i as it is common in applications.

Following the approach of Cox and Lewis (1996, Section 3.4), we use exponential ordered scores denoted by S_{ij}, where

$$S_{ij} = \frac{1}{n_i} + \cdots + \frac{1}{n_i - r_{ij} + 1} \qquad j = 1, \dots, n_i, \tag{3.29}$$

and r_{ij} is the rank of the gap time W_{ij} in $W_{i1}, \dots, W_{in_i}$. Therefore, the exponential ordered scores S_{ij} in (3.29) are functions of the ranks r_{ij} of the gap times W_{ij} for any prespecified n_i. The rank test is a test of the absence of association between the gap times W_{ij} $(j = 1, \dots, n_i)$ for the i th process and a specifically designed variable, denoted by Z_{ij}, $j = 1, \ldots, n_i$, for the ith process, $i = 1, \ldots, m$. Cox and Lewis (1996) used $Z_{ij} = j$ $(j = 1, \ldots, n_i)$ in the development of the rank test. We also specify the same values for the Z_{ij}.

Let $\bar{Z}_i = \sum_{j=1}^{n_i} Z_{ij}/n_i$, $\bar{S}_i = \sum_{j=1}^{n_i} S_{ij}/n_i$ and

$$U_i = \sum_{j=1}^{n_i} S_{ij}(Z_{ij} - \bar{Z}_i) \tag{3.30}$$

Then, the rank test for monotonic trend is given by

$$Z_R = \frac{\sum_{i=1}^{m} U_i}{\sqrt{\sum_{i=1}^{m} Var\{U_i\}}}, \tag{3.31}$$

where

$$Var(U_i) = \left\{ \sum_{j=1}^{n_i} (Z_{ij} - \bar{Z}_i)^2 \right\} \left\{ \sum_{j=1}^{n_i} \frac{(S_{ij} - \bar{S}_i)^2}{n_i - 1} \right\}. \tag{3.32}$$

The asymptotic distribution of the rank test statistic (3.31) is a standard normal as m approaches infinity as well as as n_i approaches infinity for a fixed m (Lawless et al., 2012). The use of the exponential ordered scores (3.29) is especially useful in the comparison of the rank test with trend tests based on Poisson processes where the gap times are i.i.d. exponential random variables under the null hypothesis. However, other choices are also possible. More details on rank tests can be found in Hajek and Sidak (1967).

3.3.2.2 The Lewis-Robinson Test

The Lewis-Robinson test is another well know test for trend. It was proposed by Lewis and Robinson (1974) as a modification of the Laplace test to deal with overdispersed processes with respect to the Poisson process. They used an ad-hoc method based on the coefficient of variation C(X), where $C(X) = \sqrt{Var(X)}/E(X)$, assuming that

$Var(X)$ exists for a random variable X. Note that this quantity is equal to 1 when the gap times W_{ij} are exponentially distributed, which corresponds to HPPs. However, Lewis and Robinson (1974) noted that the coefficient of variation of the distribution of the gap times W_{ij} is always greater than 1 for other renewal processes. Therefore, they proposed the following test statistic for testing the null hypothesis of a renewal process where W_{ij} $(j = 1, 2, \dots)$ are i.i.d. for a single process $\{N_i(t); t \geq 0\}$.

$$Z_{LRi} = \frac{\bar{W}_i}{\hat{\sigma}_i} \left\{ \frac{\sum_{j=1}^{n_i-1} T_{ij} - \frac{(n_i-1)}{2} T_{in_i}}{T_{in_i} (\frac{(n_i-1)}{12})^{1/2}} \right\}, \tag{3.33}$$

where $\bar{W}_i$ is the average of $W_{i1}, \dots, W_{in_i}$ and $\hat{\sigma}_i$ is any consistent estimate of the standard deviation of $W_{i1}, \dots, W_{in_i}$. A pooled version of the test statistic (3.33) for m independent processes is then given by

$$Z_{LR} = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} Z_{LRi}. \tag{3.34}$$

The test statistic (3.34) has been investigated by Kvaloy and Lindqvist (2003), where they used

$$\hat{\sigma_i^2} = \frac{1}{2(n_i - 1)} \sum_{j=1}^{n_i-1} (W_{i,j+1} - W_{ij})^2, \tag{3.35}$$

for some reasons. However, in this practicum, we used the usual estimator $S_i^2 = \sum_{j=1}^{n_i} (W_{ij} - \bar{W}_i)^2/(n_i - 1)$, $i = 1, \dots, m$, in (3.34). For any consistent estimator of the Lewis-Robinson test statistic (3.34) is asymptotically standard normal under the null hypothesis when m approaches infinity as well as when n_i approaches infinity for a fixed m (Lawless et al., 2012).

Lawless et al. (2012) gave an alternative derivation of the Lewis-Robinson test statistic. Their approach resulted in a slightly modified version of the Lewis-Robinson test statistic (3.34). We call this test statistic the adjusted Lewis-Robinson test statistic, which is given by

$$Z_{ALR} = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} Z_{ALRi} = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} \sqrt{\frac{n_i}{n_i + 1}} Z_{LRi}, \tag{3.36}$$

where Z_{LRi} is given in (3.33). They showed through simulations that Z_{ALR} performs better than Z_{LR} when the sample sizes n_i are small. They also showed that Z_{ALR} is asymptotically standard normal as m approaches infinity. In the next chapter, we only considered Lewis-Robinson test and adjusted Lewis-Robinson test in our simulations.

3.3.3 Robust Trend Tests Based on Rate Functions

Robust procedures based on marginal characteristics of recurrent events have been summarized by Cook and Lawless (2007, Section 3.6). Following their discussion, Lawless et al. (2012) proposed a robust test for monotonic trends. We first introduce this procedure and then two robust tests for monotonic trends in this section.

Suppose that m independent processes are under observation. We let $\{N_i(t), t \geq 0\}$, $i = 1, \ldots, m$, be the ith process with the intensity function $\lambda_i(t|H_i(t))$ and the rate function $\rho_i(t)$. We also define $dN_i(t)$ as the number of observed failures in the ith process over the time interval $(t - dt, t]$, $t \geq 0$. In the remaining part of this section, we assume that the rate function of a process is correctly specified so that $E\{dN_i(t)\} = \rho_i(t)\, dt$. However, we do not assume any model such as a Poisson process or renewal process for processes.

It should be noted that the development of robust tests requires that the counting process $\{N_i(t); t \geq 0\}$ and the observation scheme of a process should be completely independent (Cook and Lawless, 2007, Section 3.6). Since in this study we apply an observation scheme in which the process $\{N_i(t); t \geq 0\}$ is continuously observed over the interval $[0, \tau_i]$, $i = 1, \ldots, m$, and the τ_i are prespecified, this requirement is satisfied. However, robust tests for trends based on marginal characteristics of processes may not be applied in some situations. For example, an important exclusion is the case where a Type 2 censoring mechanism is applied. In this type of censoring mechanism, the τ_i are stopping times and the observation of a process and the counting processes are not completely independent. More on this discussion can be found in Cook and Lawless (2007, Section 3.6).

We discussed the likelihood function and score procedures for recurrent event processes in Section (2.4). For $i = 1, \ldots, m$ and $t \geq 0$, let $\rho_i(t) = \rho_i(t; \alpha, \beta)$ $= \alpha \exp\{\beta\, g(t)\}$, where $g(t)$ is monotonically increasing or decreasing function. Following our setup in this chapter and the likelihood and log likelihood functions given in Section (2.4), we obtain the following score functions.

$$U_\alpha(\alpha, \beta) = \frac{\partial \ell(\alpha, \beta)}{\partial \alpha} = \frac{\sum_i^m n_i}{\alpha} - \sum_{i=1}^{m} \int_0^{\tau_i} \alpha\, e^{\beta\, g(u)}\, du, \tag{3.37}$$

and

$$U_\beta(\alpha, \beta) = \frac{\partial \ell(\alpha, \beta)}{\partial \beta} = \sum_i^m \sum_{j=1}^{n_i} g(t_{ij}) - \alpha \sum_{i=1}^{m} \int_0^{\tau_i} g(u)\, e^{\beta\, g(u)}\, du, \tag{3.38}$$

where $\ell(\alpha, \beta) = \log L(\alpha, \beta)$ is the log likelihood function and the likelihood function $L(\alpha, \beta)$ is introduced in Section 2.4. We let $\tilde{\alpha}$ be the value of α which maximizes $\ell(\alpha, 0)$. Solving $U_\alpha(\alpha, 0) = 0$, we find $\tilde{\alpha} = (\sum_{i=1}^m n_i)/(\sum_{i=1}^m \tau_i)$. If we replace (α, β) in 3.38, we obtain

$$U_\beta(\tilde{\alpha}, 0) = \sum_i^m \sum_{j=1}^{n_i} g(t_{ij}) - \left(\frac{\sum_{i=1}^m n_i}{\sum_{i=1}^m \tau_i} \right) \sum_{i=1}^{m} \int_0^{\tau_i} g(u)\, du. \tag{3.39}$$

The function $U_\beta(\tilde{\alpha}, 0)$ in (3.39) can be written as follows (Cook and Lawless, 2007,

p. 117).

$$U_{\beta}(\tilde{\alpha},0)=\sum_{i=1}^{m}U_{\beta i}(\tilde{\alpha},0)=\sum_{i=1}^{m}\int_{0}^{\tau_i}\left(g(u)-\frac{g_{\cdot}}{\sum_{i=1}^{m}\tau_i}\right)[dN_i(u)-\alpha\,du]\,, \tag{3.40}$$

where $g_{\cdot}=\sum_{i=1}^{m}\int_0^{\tau_i}g(u)\,du$. It is clear that the expectation of (3.40) is zero under the null hypothesis $H_0:\beta=0$ as long as the rate function $\rho_i(t)$ is correctly specified; that is, $E\{dN_i(t)\}=\rho_i(t)\,dt$. This result does not require the process $\{N_i(t),t\geq 0\}$ to be a Poisson process. Furthermore, the expectation of (3.40) is greater or less than zero if $g(t)$ is an increasing or decreasing function of t, respectively. Since the $U_{\beta i}(\tilde{\alpha},0)$ terms in (3.40) are independent, a variance estimate of $U_{\beta}(\tilde{\alpha},0)$ in (3.40) is given by (see, Cook and Lawless, 2007, p. 117)

$$\widehat{Var}(U_{\beta}(\tilde{\alpha},0))=\sum_{i=1}^{m}U_{\beta i}(\tilde{\alpha},0)^2. \tag{3.41}$$

Therefore, a robust test for monotonic trends in recurrent event processes can be based on the standardized test statistic

$$Z_{Robust}=\frac{\sum_{i=1}^{m}U_{\beta i}(\tilde{\alpha},0)}{\left(\sum_{i=1}^{m}U_{\beta i}(\tilde{\alpha},0)^2\right)^{1/2}}. \tag{3.42}$$

It should be noted that the robust variance estimator (3.41) is obtained from the theory of estimating functions. To see this results, we note that, under some regularity conditions, as m increases, $(1/\sqrt{m})\,[U_{\alpha}(\alpha,\beta),U_{\beta}(\alpha,\beta)]'$ converges in distribution to a bivariate normal distribution with a 2×1 zero mean vector and a 2×2 asymptotic variance matrix $B(\alpha,\beta)$ (Cook and Lawless, 2007, pp. 342–343). Let $\hat{\alpha}$ and $\hat{\beta}$ be the maximum likelihood estimators of α and β, respectively. An estimate of $B(\alpha,\beta)$ is given by $\frac{1}{m}\sum_{i=1}^{m}[U_{\alpha i}(\hat{\alpha},\hat{\beta}),U_{\beta i}(\hat{\alpha},\hat{\beta})]\,[U_{\alpha i}(\hat{\alpha},\hat{\beta}),U_{\beta i}(\hat{\alpha},\hat{\beta})]'$. This estimate of the asymptotic variance (3.41) is valid as long as the rate functions of the processes are correctly specified, and thus, does not require the Poisson assumption. Similarly, this result also holds under the null hypothesis $H_0 : \beta = 0$, where $(\hat{\alpha}, \hat{\beta})$ is replaced with $(\tilde{\alpha}, 0)$, and the robust variance estimator (3.41) is obtained.

In the remainder of this section, we first introduce a robust trend test based on the conditional approach explained in Section 3.3.1. We call this test the *generalized Laplace test* (GL). Then, we will discuss another robust trend test which is based on a similar method explained above in this section, but we will use the rate function $\rho_i(t) = (\beta/\theta)(t/\theta)^{\beta-1}$, where $\beta > 0$ and $\theta > 0$. We call this test the *generalized power law process test* (GPLP). To our knowledge, this test has not been studied in the literature before.

3.3.3.1 The Generalized Laplace Test

The development of the generalized Laplace test is given by Lawless et al. (2012). In this section, we will summarize this procedure and introduce the generalized Laplace test statistic used in the next chapters.

We consider m independent processes with the rate functions $\rho_i(t)$, $i = 1, \ldots, m$. We consider testing the hypothesis $H_0 : \rho_i(t) = \alpha$, where $\alpha > 0$ and $t > 0$. In the development, we use the model $\rho_i(t) = \alpha \exp\{\beta g(t)\}$, $t > 0$. From the conditional likelihood function $L_c(\beta)$ given in (3.5) in Section 3.3.1, we obtain

$$U_c(\beta) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} g(t_{ij}) - \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{\int_0^{\tau_i} g(u)\, e^{\beta g(u)}\, du}{\int_0^{\tau_i} e^{\beta g(u)}\, du}. \tag{3.43}$$

When $\beta = 0$, this score function can be written as follows.

$$U_c(0) = \sum_{i=1}^{m} U_i(0) = \sum_{i=1}^{m} \int_0^{\tau_i} \left(g(u) - \frac{1}{\tau_i} \int_0^{\tau_i} g(u)\, du \right) dN_i(u). \tag{3.44}$$

The expectation of $U_c(0)$ in (3.44) under the null hypothesis is zero. This result holds as long as $E\{dN_i(t)\} = \alpha\, dt$ even when the processes $\{N_i(t), t \geq 0\}$ are not Poisson. Since $U_i(0)$, $i = 1, \ldots, m$, in (3.44) are independent, an estimate of the variance of $U_c(0)$ is

$$\widehat{Var}\{U_c(0)\} = \sum_{i=1}^{m} U_i(0)^2. \tag{3.45}$$

From (3.44) and (3.45), we obtain the standardized test statistic

$$Z = \sum_{i=1}^{m} U_i(0) \Big/ \left\{ \sum_{i=1}^{m} U_i(0)^2 \right\}^{1/2} \tag{3.46}$$

for testing the null hypothesis $H_0 : \rho(t) = \alpha$. As $m \to \infty$, the distribution of Z in (3.46) is asymptotically standard normal under the null hypothesis when the τ_i are finite and the function $g(t)$ is integral over the observation periods (Lawless et al., 2012).

The generalized Laplace test is obtained by replacing $g(t) = t$ in the above development. In this case, from (3.44), the score function is given by

$$U_c(0) = \sum_{i=1}^{m} U_i(0) = \sum_{i=1}^{m} \int_0^{\tau_i} \left(u - \frac{\tau_i}{2} \right) dN_i(u), \tag{3.47}$$

where $\int_{i=1}^{\tau_i} u dN_i(u) = \sum_{j=1}^{n_i} t_{ij}$ and $\int_{i=1}^{\tau_i} dN_i(u) = n_i$ for a process $\{N_i(t), t \geq 0\}$ with n_i events observed over $(0, \tau_i]$ at times $0 < t_{i1} < \cdots < t_{in_i}$. If we use (3.47) in (3.46), we obtain the generalized Laplace test Z_{GL}, which is given by

$$Z_{GL} = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n_i} t_{ij} - \frac{n_i \tau_i}{2}}{\sqrt{\sum_{i=1}^{m} (\sum_{j=1}^{n_i} t_{ij} - \frac{n_i \tau_i}{2})^2}}. \tag{3.48}$$

The asymptotic distribution of the generalized Laplace test Z_{GL} is the same with the test statistic given in (3.46). The properties of Z_{GL} has been discussed by Lawless et al. (2012). One drawback about the generalized Laplace test is that it cannot be used when a single process is of interest. Lawless et al. (2012) recommended the generalized Laplace test when m is large but the observed number of events (failures) per process is small.

3.3.3.2 The Generalized Power Law Process Test

In this section, we develop a robust test for monotonic trends based on the generalized power law processes. Our approach is based on a discussion given in Cook and Lawless (2007, Chapter 3) and Lawless et al. (2012).

Derivations of the score functions $U_\theta(\beta, \theta)$ and $U_\theta(\theta, \beta))$ for the power law process with the rate function $(\beta/\theta)(t/\theta)^{\beta-1}$, where $t > 0$, $\beta > 0$ and $\theta > 0$, are given in Section 3.3.1.2. Note that, from (3.14), we can rewrite the score function $U_\theta(\beta, \theta)$ as

$$U_\theta(\beta, \theta) = \sum_{i=1}^{m} \int_0^{\tau_i} \left(-\frac{\beta}{\theta}\right) \left[dN_i(t) - \frac{\beta}{\theta}\left(\frac{u}{\theta}\right)^{\beta-1} du\right]. \tag{3.49}$$

Also, from (3.15), we can obtain the score function $U_\beta(\beta, \theta)$ as follows.

$$U_\beta(\beta, \theta) = \sum_{i=1}^{m} \int_0^{\tau_i} \left(\frac{1}{\beta} - \log\theta + \log u\right) \left[dN_i(t) - \frac{\beta}{\theta}\left(\frac{u}{\theta}\right)^{\beta-1} du\right]. \tag{3.50}$$

Assuming the rate functions are correctly specified and the τ_i are not random, It is easy to see that the expectations of $U_\theta(\beta, \theta)$ and $U_\beta(\beta, \theta)$ are zero. This means that $U_\theta(\beta, \theta)$ and $U_\beta(\beta, \theta)$ are unbiased estimating functions. We can therefore apply the robust procedures explained in in Section 3.3.3.

Under the null hypothesis $H_0 : \beta = 1$, $\theta > 0$, we have

$$U_\theta(1, \theta) = \frac{\tau.}{\theta^2} - \frac{n.}{\theta}, \tag{3.51}$$

where $\tau. = \sum_{i=1}^{m} \tau_i$ and $n. = \sum_{i=1}^{m} n_i$. By solving $U_\beta(1, \theta) = 0$ for θ, we obtain a consistent (under the assumption $\beta = 1$) estimator $\tilde{\theta}$ that maximizes $l(1, \theta)$ as follows

$$\tilde{\theta} = \frac{\tau.}{n.}. \tag{3.52}$$

Let $g(t) = 1 - \log(\tilde{\theta}) + \log(t)$. Plugging (3.52) and $\beta = 1$ in (3.50), we obtain

$$U_\beta(1, \tilde{\theta}) = \sum_{i=1}^{m} U_{\beta i}(1, \tilde{\theta}) = \sum_{i=1}^{m} \int_0^{\tau_i} g(t) \left[dN_i(t) - \tilde{\theta}^{-1}\, dt \right]. \tag{3.53}$$

Now, note that

$$U_\beta(1, \tilde{\theta}) = \sum_{i=1}^{m} \int_0^{\tau_i} g(t) \left[dN_i(t) - \frac{1}{\theta} dt \right] + \sum_{i=1}^{m} \int_0^{\tau_i} \frac{g(t)}{\theta} dt - \sum_{i=1}^{m} \int_0^{\tau_i} \frac{g(t)}{\tilde{\theta}} dt. \tag{3.54}$$

Let $A = \sum_{i=1}^{m} \int_0^{\tau_i} g(t) \left[dN_i(t) - \frac{1}{\theta} dt \right]$. Then,

$$U_\beta(1, \tilde{\theta}) = A + \sum_{i=1}^{m} \int_0^{\tau_i} \frac{\left(\sum_{j=1}^{m} \int_0^{\tau_j} du \right)}{\theta\, \tau.} g(t) dt - \sum_{i=1}^{m} \int_0^{\tau_i} \frac{\left(\sum_{j=1}^{m} \int_0^{\tau_j dN_j(u)} \right)}{\tau.} g(t) dt, \tag{3.55}$$

$$= A + \frac{1}{\theta \tau.} \sum_{i=1}^{m} \int_0^{\tau_i} \left(\sum_{j=1}^{m} \int_0^{\tau_j} g(u) du \right) dt - \frac{1}{\tau.} \sum_{i=1}^{m} \int_0^{\tau_i} \left(\sum_{j=1}^{m} \int_0^{\tau_j} g(u) du \right) dN_i(t) \tag{3.56}$$

If we let $G. = \sum_{j=1}^{m} \int_0^{\tau_j} g(u)\, du$ in (3.56), we obtain

$$U_\beta(1, \tilde{\theta}) = A - \frac{1}{\tau.} \sum_{i=1}^{m} \int_0^{\tau_i} G.\, dN_i(t) + \frac{1}{\tau.} \sum_{i=1}^{m} \int_0^{\tau_i} G. \frac{1}{\theta}\, dt, \tag{3.57}$$

$$= \sum_{i=1}^{m} \int_0^{\tau_i} g(t) \left[dN_i(t) - \frac{1}{\theta} dt \right] - \sum_{i=1}^{m} \int_0^{\tau_i} \frac{G.}{\tau.} \left[dN_i(t) - \frac{1}{\theta} dt \right] \tag{3.58}$$

Therefore, we obtain

$$U_\beta(1,\tilde{\theta}) = \sum_{i=1}^{m} \int_0^{\tau_i} \left(g(t) - \frac{G.}{\tau.}\right) \left[dN_i(t) - \frac{1}{\theta}dt\right]. \tag{3.59}$$

Note that, under $H_0 : \beta = 1, \theta > 0$, $E\{U_\beta(1,\tilde{\theta})\} = 0$ because $E\{dN_i(t)\} = \frac{1}{\theta}\,dt$. This result is true as long as the true rate function under the null hypothesis is $1/\theta$, $\theta > 0$, even if the model is not a Poisson process.

A robust estimate of the the score function can be based on the theory of estimating functions as discussed in Section 3.3.3. Following this discussion, since $U_\beta(1,\tilde{\theta})$ is an unbiased estimating function, under the null hypothesis and some regularity conditions (Cook and Lawless, 2007), $\frac{1}{\sqrt{m}}U_\beta(1,\tilde{\theta})$ converges in distribution to a normal distribution with mean 0 and variance $B(1,\theta) = \lim_{m\to\infty} E\{B_m(1,\theta)\}$, where $B_m(1,\theta) = \frac{1}{m}\sum_{i=1}^{m} U_{\beta i}(1,\theta)^2$, as m approaches infinity. A robust variance estimate of $U_\beta(1,\tilde{\theta})$ is then given by $\frac{1}{m}\sum_{i=1}^{m} U_{\beta i}(1,\tilde{\theta})^2$.

From the above results, we obtain the standardized test statistic

$$Z_{GPLP} = \frac{\sum_{i=1}^{m} U_{\beta i}(1,\tilde{\theta})}{\sqrt{\sum_{i=1}^{m} U_{\beta i}(1,\tilde{\theta})^2}}. \tag{3.60}$$

for testing the null hypothesis $H_0 : \beta = 1$. We call the test statistic Z_{GPLP} in (3.60) the generalized power law process test statistic, and any test based on it a generalized power law process test.

3.4 Testing for Trend in Non-identical Processes

In the previous section, we have discussed testing for monotonic trends in identical processes. Heterogeneity in the rate functions or gap time distributions of processes is common in applications (Rigdon and Basu, 2000), and ignoring heterogeneity may lead to wrong conclusions about the trend in recurrent event processes (Cox and Lewis, 1966; Kvaløy and Linqvist, 2003; and Lawless et al., 2012). Therefore, it is important to extend the trend tests of the previous section to deal with the heterogeneity. In this section, we therefore discuss tests for monotonic trends in nonidentical recurrent event processes. The tests in this section are extensions of the tests given in the previous section. Therefore, we briefly introduce the methodology. Our main goal is to give the forms of the tests used in the following chapters.

In the remainder part of this section, we assume that there are m independent processes are under observation over the observation window $(0, \tau_i]$, $i = 1, \ldots, m$. The process $\{N_i(t), t \geq 0\}$, $i = 1, \ldots, m$, has the associated intensity function $\lambda_i(t \mid H_i(t))$, which may be different from process to process. We assume all process are exposed to the same type of trend; that is, either monotonically increasing or decreasing. However, methods can be extended in a straightforward manner if there is a need for modeling different type of monotonic trends.

3.4.1 Tests Based on Poisson Processes

In Section 3.3.1, we discussed general procedures to develop tests for monotonic trends in recurrent event processes pertaining to Poisson processes. Then, we focused on two tests: (i) The Laplace test in Section 3.3.1.1, and (ii) the power law process trend test in Section 3.3.1.2. In this section, we assume the processes are Poisson with rate functions $\rho_i(t)$, $t > 0$, for $i = 1, \ldots, m$, but we do not require the rate functions are the same for all processes.

We start with the Laplace test. Let the ith model, $i = 1, \ldots, m$, under the null hypothesis of no trend be a HPP with the rate function

$$\rho_i(t) = \alpha_i, \qquad t > 0, \tag{3.61}$$

where $\alpha_i > 0$ are unknown parameters. We define the model with a monotonic trend as a NHPP with the rate function

$$\rho_i(t) = \alpha_i \exp\{\beta g(t)\}, \qquad t \geq 0, \tag{3.62}$$

where β is a real-valued parameter representing the monotonic trend and $g(.)$ is a specified function. As we discussed at the beginning of this section, we assume that the trend has a similar shape for all processes so that we use β in the above trend model. If there is an indication of different trends, then β can be replaced with β_i in the above model.

A test for the absence of trend in nonidentical process is then defined by the null hypothesis $H_0 : \beta = 0$. In this case, the α_i are nuisance parameters, and conditioning on the observed number of events over the observation windows $(0,\, \tau_i]$, $i = 1, \ldots, m$; that is, n_i, $i = 1, \ldots, m$, we obtain the conditional likelihood function proportional to

$$L_c(\beta) = \prod_{i=1}^{m} \prod_{j=1}^{n_i} \left\{ \frac{e^{\beta g(t_{ij})}}{\int_0^{\tau_i} e^{\beta g(u)} du} \right\}. \tag{3.63}$$

This is exactly the same conditional likelihood function given in (3.5). Therefore, by replacing $g(t) = t$, where $t > 0$, in (3.62) and following the steps explained in Section 3.3.1.1, we obtain the Laplace test statistic for monotonic trends in nonidentical processes as follows.

$$Z_{LT} = \frac{\sum_{i=1}^{m} \{\sum_{j=1}^{n_i} t_{ij} - n_i \tau_i / 2\}}{\{\sum_{i=1}^{m} n_i \tau_i^2 / 12\}^{1/2}}. \tag{3.64}$$

The Laplace test statistic Z_{LT} given in (3.64) is exactly the same with the Laplace test statistic given in (3.10) in Section 3.3.1.1. Therefore, the asymptotic distribution of Z_{LT} in (3.64) follows from the asymptotic distribution of Z_{LT} given in Section 3.3.1.1. It should be noted that we obtained this result because we condition on the value of the sufficient statistic $N_i(\tau_i) = n_i$ of the nuisance parameter α_i (Cox and Lewis, 1966).

Next, we consider testing for monotonic trends in nonidentical power law processes. We discussed the trend tests based on identical power law processes in Section 3.3.1.2. In this section, we assume the ith trend-free model, $i = 1, \ldots, m$, is a HPP with the rate function

$$\rho_i(t) = \frac{1}{\theta_i}, \qquad t > 0, \tag{3.65}$$

where $\theta_i > 0$ for $i = 1, \ldots, m$. The corresponding monotonic trend model is a power law process with the rate function

$$\rho_i(t) = \frac{\beta}{\theta_i}\left(\frac{t}{\theta_i}\right)^{\beta-1}, \qquad t > 0, \tag{3.66}$$

where $\theta_i > 0$ for $i = 1, \ldots, m$, and $\beta > 0$. Then, as we developed in Section 3.3.1.2, the likelihood function for β and θ, where $\theta = (\theta_1, \ldots, \theta_m)$, is given by

$$L(\beta, \theta) = \prod_{i=1}^{m}\left\{\prod_{j=1}^{n_i}\frac{\beta}{\theta_i}\left(\frac{t_{ij}}{\theta_i}\right)^{\beta-1}\right\}\exp\left\{-\left(\frac{\tau_i}{\theta_i}\right)^{\beta}\right\}, \tag{3.67}$$

which gives the log likelihood function $l(\beta, \theta) = \log L(\beta, \theta)$ as follows.

$$l(\beta, \theta) = \sum_{i=1}^{m}\left\{n_i \log\beta - n_i\beta\log\theta_i + (\beta - 1)\sum_{j=1}^{n_i}\log t_{ij} - \left(\frac{\tau_i}{\theta_i}\right)^{\beta}\right\}. \tag{3.68}$$

The value of θ that maximizes $l(1, \theta)$ in (3.68) is given by $\tilde{\theta} = (\tilde{\theta}_1, \ldots, \tilde{\theta}_m)$. Therefore, the score function can be developed by $U_\theta(\beta, \theta) = \frac{\partial l(\beta,\theta)}{\partial\theta} = \sum_{i=1}^{m} U_{\theta i}(\beta, \theta)$ and $U_\beta(\beta, \theta) = \frac{\partial l(\beta,\theta)}{\partial\beta} = \sum_{i=1}^{m} U_{\beta i}(\beta, \theta)$, where

$$U_{\theta i}(\beta, \theta) = -\frac{\beta n_i}{\theta_i} + \frac{\beta\tau_i^{\beta}}{\theta_i^{\beta+1}}, \tag{3.69}$$

$$U_{\beta i}(\beta,\theta)=\frac{n_i}{\beta}-n_i\log\theta_i+\sum_{j=1}^{n_i}\log t_{ij}-(\frac{\tau_i}{\theta_i})^{\beta}\log(\frac{\tau_i}{\theta_i}). \tag{3.70}$$

To solve $U_\theta(\beta,\theta)=0$, and define $\tilde{\theta}=(\tilde{\theta_1},\cdots,\tilde{\theta_m})$. We can obtain

$$\tilde{\theta}_i=\frac{\tau_i}{n_i}, \qquad i=1,\ldots,m. \tag{3.71}$$

Then, the function $U_\beta(1,\tilde{\theta}_i)$ is

$$U_\beta(1,\tilde{\theta})=n.+\sum_{i=1}^{m}\sum_{j=1}^{n_i}\log t_{ij}-\sum_{i=1}^{m}n_i\log\tau_i. \tag{3.72}$$

From (3.19), we can get

$$\widehat{Var}\{U_\beta(1,\tilde{\theta})\}=I_{\beta\beta}(1,\tilde{\theta})-I_{\beta\theta}(1,\tilde{\theta})I_{\theta\theta}^{-1}(1,\tilde{\theta})I_{\theta\beta}(1,\tilde{\theta}), \tag{3.73}$$

where $I_{\beta\beta}(\beta,\theta)=-\frac{\partial^2 l(\beta,\theta)}{\partial\beta^2}$, $I_{\beta\theta}(\beta,\theta)=I_{\theta,\beta}(\beta,\theta)=-\frac{\partial^2 l(\beta,\theta)}{\partial\beta\theta}$ and $I_{\theta\theta}(\beta,\theta)=-\frac{\partial^2 l(\beta,\theta)}{\partial\theta^2}$. That is,

$$I_{\beta\beta}(\beta,\theta)=\sum_{i=1}^{m}\left\{\frac{n_i}{\beta}+(\frac{\tau_i}{\theta_i})^{\beta}\log^2(\frac{\tau_i}{\theta_i})\right\}, \tag{3.74}$$

$$I_{\theta\theta}(\beta,\theta)=\sum_{i=1}^{m}\frac{-\beta n_i}{\theta_i^2}+\sum_{i=1}^{m}\frac{\beta\tau_i^{\beta}}{\theta_i^{\beta+2}}(\beta+1), \tag{3.75}$$

and

$$I_{\beta\theta}(\beta,\theta)=-\sum_{i=1}^{m}\left\{\frac{\tau_i^{\beta}+\beta\tau_i^{\beta}\log(\frac{\tau_i}{\theta_i})-n_i\theta_i^{\beta}}{\theta_i^{\beta+1}}\right\}. \tag{3.76}$$

Plugging $\beta=1$ and $\theta=\tilde{\theta}$, we obtain

$$I_{\beta\beta}(1,\tilde{\theta})=n.+\sum_{i=1}^{m}(\frac{\tau_i}{\tilde{\theta}_i})\log^2(\frac{\tau_i}{\tilde{\theta}_i}), \tag{3.77}$$

$$I_{\theta\theta}(1,\tilde{\theta}) = \sum_{i=1}^{m} \frac{-n_i}{\tilde{\theta}_i^2} + \sum_{i=1}^{m} \frac{2\tau_i}{\tilde{\theta}_i^3}, \tag{3.78}$$

and

$$I_{\beta\theta}(1,\tilde{\theta}) = -\sum_{i=1}^{m} \left\{ \frac{\tau_i \log(\frac{\tau_i}{\tilde{\theta}_i})}{\tilde{\theta}_i^2} \right\}. \tag{3.79}$$

We therefore obtain the PLP test statistic for testing $H_0 : \beta = 1$ in the model (3.66)

$$Z_{PLP} = \frac{U_\beta(1,\tilde{\theta}_i)}{\sqrt{\widehat{Var}\{U_\beta(1,\tilde{\theta}_i)\}}}. \tag{3.80}$$

3.4.2 Tests Based on Renewal Processes

In Section 3.3.2, we discussed monotonic trend tests based on identical renewal processes, where the gap times W_{ij} independent and identically distributed (i.i.d.) random variables for all $i = 1, \ldots, m$ and $j = 1, \ldots, n_i$. We then introduced two tests (the rank test for trend in Section 3.3.2.1 and the Lewis-Robinson test in Section 3.3.2.2). In this section, we allow the gap times W_{ij}, $j = 1, \ldots, m$, be i.i.d. for each i ($i = 1, \ldots, m$), but may not be identically distributed for all i. Trend tests for nonidentical renewal processes can be then expressed as H_0 : For each $i, W_{ij}, j = 1, \ldots, n_i$ are i.i.d.

As discussed by Lawless et al. (2012), the rank test given in Section 3.3.2.1 and the Lewis-Robinson test given in Section 3.3.2.2 are combined tests developed separately for the trend for each process under observation. Therefore, these tests can be directly applied for testing monotonic trends in the nonidentical case as well. Therefore, the rank test statistic Z_R given in (3.31) and the Lewis-Robinson test statistic Z_{LR} given in (3.34) and its adjusted version Z_{ALR} given in (3.36) are used in the following chapters. The asymptotic properties of these test statistics discussed in the previous sections are the same in the nonidentical case.

We would like to note that, since rank test introduced in Section 3.3.2 is based on the rank of observations, it can be used for the nonidentical cases as well (Cox and Lewis, 1966, Chapter 3). A similar discussion is also true for the Lewis-Robinson test. Lawless et al. (2012) noted that if exponential ordered scores S_{ij} in (3.30) are replaced with the actual gap times W_{ij}, we can obtain the term $\sum_{j=1}^{n_i-1} t_{ij} - (n_i - 1)t_{in_i}/2$. Hence, (3.30) and (3.33) are related to rank statistics. As we discussed above, the Lewis-Robinson and the adjusted Lewis-Robinson tests can be used for testing the monotonic trends in nonidentical processes.

It also worth mentioning that these tests are developed when the observed number of events are prespecified for each process (Type 2 censoring), instead of the presprecification of the τ_i (Type 1 censoring). In many applications, Type 1 censoring scheme is applied, but the researchers still use these test statistics. Aalen and Husebye (1991) showed that the bias may occur with this practice. Also, this issue was investigated by Lawless et al. (2012) through Monte Carlo simulations. They showed that the Lewis-Robinson test may be affected by the violation of this assumption in some situations, but the rank test for monotonic trends still provides good results comparing with the Lewis-Robinson test. The reason behind this result is that the rank test is based on exchangeable rank statistics, which is not the case for the Lewis-Robinson test (Lawless et al., 2012). In the next chapter, we investigate the power of these tests (Z_R, Z_{LR} and Z_{ALR}) under Type 1 censoring scheme because this type of observation scheme is common in applications and more suitable for the data types considered in this practicum.

3.4.3 Robust Trend Tests Based on Rate Functions

The robust trend tests based on the marginal characteristics of the recurrent event processes have been discussed in Section 3.3.3. In this section, we extend those procedures to non-identical processes settings. It should be noted that the generalized

Laplace test given in Section 3.3.3.2 can be used in non-identical processes settings as well. However, the generalized PLP test statistic can be applied after replacing $\tilde{\theta}$ with (3.71) in non-identical processes settings. This is explained as follows.

As we discussed in Section 3.3.3.2, the variance of $U_\beta(1,\tilde{\theta})$ can be estimated by $\sum_{i=1}^m U_{\beta i}(1,\tilde{\theta})^2$. Therefore, the generalized PLP test statistic in non-identical processes settings can be written as

$$Z_{GPLP} = \frac{\sum_{i=1}^m U_{\beta i}(1,\tilde{\theta}_i)}{\sqrt{\sum_{i=1}^m U_{\beta i}(1,\tilde{\theta}_i)^2}}, \tag{3.81}$$

where

$$U_{\beta i}(\beta,\theta) = \frac{n_i}{\beta} - n_i \log\theta_i + \sum_{j=1}^{n_i} \log t_{ij} - (\frac{\tau_i}{\theta_i})^\beta \log(\frac{\tau_i}{\theta_i}), \tag{3.82}$$

and

$$\tilde{\theta}_i = \frac{\tau_i}{n_i}, \qquad i = 1,\ldots,m. \tag{3.83}$$

This gives

$$U_\beta(1,\tilde{\theta}) = n. + \sum_{i=1}^m \sum_{j=1}^{n_i} \log t_{ij} - \sum_{i=1}^m n_i \log\tau_i. \tag{3.84}$$

The asymptotic distribution of the test statistic Z_{GPLP} is the same with the asymptotic distribution of the generalized PLP test derived in Section 3.3.3.2.

Chapter 4

Comparison of Trend Tests

In this chapter, we first explain the simulation procedures and then summarize the results of extensive simulation studies conducted to assess the adequacy of the asymptotic normal approximations and to investigate the power of the tests considered in the previous chapter. In Section 4.1, we enumerate the tests used in simulations, explain the simulation procedures and introduce the models from which the data generated under the null and alternative hypotheses. We also give the steps of the simulations in each case considered. We present the results of simulations conducted to assess the normal approximations for the tests first in identical processes and then in nonidentical processes with normal quantile-quantile plots as well as in tables in Appendix B.1 and Appendix B.2, respectively. Appendix C.1 and Appendix C.2 include the tables of power based on simulations for identical and nonidentical cases, respectively. In Section 4.2, we give a summary of the results obtained in through simulation studies.

4.1 Introduction

We generated $B = 10{,}000$ samples under the null models for various scenarios with combinations of (m, τ), where m is the number of processes under observation and τ

is the end of observation period for these processes. Our first goal is to investigate the adequacy of the normal approximation for the test statistics under the null hypothesis. Normal quantile-quantile (Q-Q) plots and tables were used to assess the accuracy of $N(0,1)$ approximations. The tables and plots are given in Appendix B. In the tables, we presented the empirical pth quantile $\hat{Q}_p$ of the test statistics estimated under 10,000 generated samples and the estimates of the probability that the test statistic is greater than the pth quantile Q_p of the standard normal distribution. We used $p = 0.975$ and 0.995 so that in the tables $\hat{P}(1.96)$ and $\hat{P}(2.576)$ represent the estimate of $P(Z > Q_p) = 1 - p$ for a test statistic Z when $Q_p = 1.96$ or 2.576, respectively. The power of the tests was found from 1,000 simulation runs. We used the empirical 0.05 quantiles of test statistics obtained from the 10,000 simulation runs under various (m, τ) combinations, as well as theoretical size (Type I error rate) 0.05.

We considered the following trend test statistics:

1. Laplace test statistic Z_{LT} given in (3.10).

2. Power law process (PLP) test statistic Z_{PLP} given in (3.28).

3. Linear rank test statistic Z_R given in (3.31).

4. Lewis-Robinson test statistic Z_{LR} given in (3.34).

5. Adjusted Lewis-Robinson test statistic Z_{ALR} given in (3.36).

6. Generalized Laplace test Z_{GLT} given in (3.48).

7. Generalized PLP test statistic Z_{GPLP} given in (3.60).

The details of these test statistics can be found in the previous chapter. We considered three cases (Case (a), Case (b) and Case (c)) to generate the data from trend-free models, and generated the simulated failure data from the following models.

(a) H_0: Process i is a HPP with rate α_i, $i = 1, \ldots, m$.

(b) H_0: Process i is a RP with the gap times following a gamma distribution with scale a_i and shape b (i.e., $W_{ij} \sim G(a_i, b)$, $j = 1, 2, \ldots$) for $i = 1, \ldots, m$.

(c) H_0: Process i has intensity function $\alpha_i \exp(\beta Z_i(t))$, where $Z_i(t) = I(N_i(t^-) > 0)I(B_i(t) \leq \Delta)$, for $i = 1, \ldots, m$. Here, $B_i(t) = t - T_{N_i(t^-)}$ gives the elapsed time since the last event.

In Case (a), we generated the data from a HPP with the rate functions $\alpha_i = 0.5 + (i-1)/(m-1)$ for $i = 1, \ldots, m$. In Case (b), $a_i = (\alpha_i b)^{-1}$ and $b = 0.75$. We consider $\Delta = 0.05$ and $\exp(\beta) = 5$ in Case (c). The model in Case (c), $\lambda_i(t \,|\, H_i(t)) = \alpha_i \exp(\beta Z_i(t))$, is called the carryover effect model. It is actually a delayed RP, where the first gap time W_{i1} has an exponential distribution with the rate parameter α_i and the remaining gap times W_{ij} (j=2, 3, ...) has a mixture type with the hazard function $h_i(w) = \alpha_i e^{\beta} I(w \leq \Delta) + \alpha_i I(w > \Delta)$ for $i = 1, \ldots, m$. This model has no trend, but incorporates carryover effects and shows some kind of clustering of events together over time. Furthermore, since it is a delayed RP, the gap times are identically distributed. In the scenarios of the simulations, we took $\tau = 5$, 10, 20 and $m = 10$, 20, 50 throughout this chapter. It should be noted that these cases were also used by Cigsar (2010) and Lawless et al. (2012). Therefore, we also compared our results with theirs.

We considered two types of settings; (1) identical (homogeneous) processes, and (2) non-identical (heterogeneous) processes. In the identical processes setting, all processes have the same intensity function; that is, $\alpha_i = \alpha$ ($i = 1, \ldots, m$), where α is a positive constant. We took $\alpha = 1$, which is the average of α_i ranging form 0.5 to 1.5 in the nonidentical processes settings. For nonidentical processes, the parameters are different for each process. In this setting, we applied Case (a), Case (b) and Case

(c) given above. The steps of the data generation procedures for each case is given later in this section.

We considered three models (Case (d), Case (e) and Case(f)) with monotonic trends to generate data with trend. We obtained the power of the tests, and then made a comparison between them. Once again, we followed the studies by Cigsar (2010) and Lawless et al. (2012), and we used the following models under trend alternatives:

(d) Process i is a NHPP with rate function $\rho_i(t) = \alpha_i^\star \exp(\gamma t)$ for $i = 1, \ldots, m$.

(e) Process i is a Markov RP in which the gap times W_{ij} $(i = 1, \ldots, m; j = 1, \ldots, n_i)$ are gamma distributed random variables with the scale and shape parameters $a_i^{'} \exp(\gamma j)$ and b, respectively.

(f) Process i has intensity function $\lambda_i(t \mid H_i(t)) = \alpha_i^\star \exp(\gamma t + \beta Z_i(t))$ $(i = 1, \ldots, m)$.

When $\gamma = 0$, the null models considered in Case (a), Case (b), and Case (c) correspond to the alternative models given in Case (d), Case (e) and Case (f), respectively. Therefore, a test for the absence of monotonic trends can be based on the hypotheses

$$H_0 : \gamma = 0 \qquad vs \qquad H_1 : \gamma \neq 0. \tag{4.1}$$

In Case (d) and Case (f), $\alpha_i^\star$ were assumed as the expected total number of events in Case (a) and Case (b), respectively. Similarly, in Case (e), we used $a_i^{'} = (\alpha_i^{'} b)^{-1}$. The values of γ, the parameter that represents the monotonic trend, was decided by $R = \exp(\gamma\tau)$. We considered two values of R; $R = 1.5$ or $R = 2$. The data generation steps for these cases are explained in Sections 4.1.4, 4.1.5 and 4.1.6.

We presented the results of the power studies in tables in Appendix C. We showed the results under the identical processes setting in Appendix C.1 and under the non-identical processes setting in Appendix C.2. As discussed before, we defined $\hat{Q}_p$ as the empirical pth quantile based on 10,000 samples. $\hat{P}(\cdot)$ is the proportion of the values of a test statistic in 10,000 samples which exceeded the standard normal pth quantile ($p = 0.975$ or 0.995). In the power tables, $\hat{\Omega}(1)$ gives the proportion of the values of test statistics greater than $\hat{Q}_{0.975}$ or less than $\hat{Q}_{0.025}$. Similarly, $\hat{\Omega}(2)$ gives the proportion of the values of test statistics greater than $\hat{Q}_{0.995}$ or less than $\hat{Q}_{0.005}$. Also, $\Omega(1)$ is defined as the proportion of the values of test statistics less than -1.96 or greater than -1.96 ($0.025th$ and $0.975th$ quantiles of the standard normal distribution, respectively). Similarly $\Omega(2)$ is defined as the proportion of the values of test statistics less than -2.576 or greater than 2.576 ($0.005th$ and $0.995th$ quantiles of the standard normal distribution, respectively).

4.1.1 Simulation Procedure for Case (a)

1. Set $i = 1$.

2. Set $j = 1$ and $t_{i0} = 0$.

3. Generate $U_{ij} \sim U(0, 1)$.

4. Use the transformation $E_{ij} = -log(U_{ij})$.

5. Obtain $W_{ij} = \frac{E_{ij}}{\alpha_i}$, where α_i is the rate function and $i = 1, \ldots, m$.

6. Calculate the event time $T_{ij} = t_{i,j-1} + W_{ij}$.

7. If $T_{ij} \leq \tau$, let $t_{ij} = T_{ij}$ and set $j = j+1$, and return to the third step. Otherwise, observed event times for the ith process are $t_{i1}, \ldots, t_{in_i}$, where $n_i = j - 1$, and go to Step 8.

8. Set $i = i + 1$. If $i \leq m$, go to Step 2. Otherwise, stop.

4.1.2 Simulation Procedure for Case (b)

1. Set $i = 1$.

2. Set $j = 1$ and $t_{i0} = 0$.

3. Generate W_{ij} from $G(a_i, b)$, where $a_i = (\alpha_i b)^{-1}$ and $b = 0.75$, $i = 1, \ldots, m$.

4. Calculate the event time $T_{ij} = t_{i,j-1} + W_{ij}$.

5. If $T_{ij} \leq \tau$, let $t_{ij} = T_{ij}$ and set $j = j+1$, and return to the third step. Otherwise, observed event times for the ith process are $t_{i1}, \ldots, t_{in_i}$, where $n_i = j - 1$, and go to Step 6.

6. Set $i = i + 1$. If $i \leq m$, go to Step 2. Otherwise, stop.

4.1.3 Simulation Procedure for Case (c)

1. Set $i = 1$.

2. Generate $U_{i1} \sim U(0, 1)$, and set $t_{i0} = 0$.

3. Use the transformation $E_{i1} = -log(U_{i1})$.

4. Obtain $W_{i1} = \frac{E_{i1}}{\alpha_i}$. Then, $T_{i1} = t_{i0} + W_{i1}$.

5. If $T_{i1} \leq \tau$, let $t_{i1} = T_{i1}$ and set $j = 2$, and go to Step 6. Otherwise, $N_i(\tau) = 0$ and jump to Step 10.

6. Generate E_{ij} from $Exp\,(1)$ distribution.

7. If $E_{ij} \leq \alpha_i e^{\beta} \Delta$, then $W_{ij} = \frac{E_{ij}}{\alpha_i e^{\beta}}$. Otherwise, $W_{ij} = \frac{E_{ij}}{\alpha_i} - \Delta e^{\beta} + \Delta$.

8. Calculate the event time $T_{ij} = t_{i,j-1} + W_{ij}$.

9. If $T_{ij} \leq \tau$, let $t_{ij} = T_{ij}$ and set $j = j + 1$, and return to Step 6. Otherwise, observed event times for the ith process are $t_{i1}, \ldots, t_{in_i}$, where $n_i = j - 1$, and go to Step 10.

10. Set $i = i + 1$. If $i \leq m$, go to Step 2. Otherwise, stop.

4.1.4 Simulation Procedure for Case (d)

1. Set $i = 1$.

2. Set $j = 1$, and $t_{i0} = 0$.

3. Set $\alpha_i' = \frac{\gamma\alpha_i\tau}{e^{\gamma\tau}-1} = \gamma\alpha_i\tau$, where $\gamma = \frac{log(R)}{\tau}$ and $R = e^{\gamma\tau}$.

4. Generate $E_{ij} \sim Exp(1)$ distribution. If $j = 1$, let $W_{i1} = \log\left(\frac{\gamma E_{i1}}{\alpha_i'} + 1\right)/\gamma$. Otherwise, let $W_{ij} = \log\left(\frac{\gamma E_{ij}}{\alpha_i'} + e^{\gamma t_{i,j-1}}\right)/\gamma - t_{i,j-1}$.

5. Calculate the event time $T_{ij} = t_{i,j-1} + W_{ij}$.

6. If $T_{ij} \leq \tau$, let $t_{ij} = T_{ij}$ and set $j = j+1$, and go to Step 4. Otherwise, observed event times for the ith process are $t_{i1}, \ldots, t_{in_i}$, where $n_i = j - 1$, and go to Step 7.

7. Set $i = i + 1$. If $i \leq m$, go to Step 2. Otherwise, stop.

4.1.5 Simulation Procedure for Case (e)

1. Set $i = 1$.

2. Set $j = 1$ and $t_{i0} = 0$.

3. Generate W_{ij} come from $G(a_i', b)$, where $a_i = (\alpha_i' b)^{-1}$ and $b = 0.75$.

4. Calculate the event time $T_{ij} = t_{i,j-1} + W_{ij}$.

5. If $T_{ij} \leq \tau$, let $t_{ij} = T_{ij}$ and set $j = j+1$, and return to the third step. Otherwise, observed event times for the ith process are $t_{i1}, \ldots, t_{in_i}$, where $n_i = j - 1$, and go to Step 6.

6. Set $i = i + 1$. If $i \leq m$, go to Step 2. Otherwise, stop.

4.1.6 Simulation Procedure for Case (f)

1. Set $i = 1$.

2. Generate $U_{i1} \sim U(0, 1)$, and set $t_{i0} = 0$.

3. Use the transformation $E_{i1} = -\log(U_{i1})$.

4. Obtain $W_{i1} = \log\left(\frac{E_{i1}}{\alpha_i'} + 1\right)/\gamma$, where $\alpha_i' = \gamma\alpha_i\tau/(e^{\gamma} - 1)$. Let $T_{i1} = t_{i0} + W_{i1}$.

5. If $T_{i1} \leq \tau$, let $t_{i1} = T_{i1}$ and set $j = 2$, and go to Step 6. Otherwise, $N_i(\tau) = 0$ and jump to Step 10.

6. Generate E_{ij} from $Exp(1)$ distribution.

7. Set $d = \frac{\alpha_i' e^{\beta}}{\gamma}\left\{e^{\gamma(t_{i,j-1}+\Delta)} - e^{\gamma t_{i,j-1}}\right\}$, where $\alpha_i' = \gamma\alpha_i\tau/(e^{\gamma j} - 1)$.

8. If $E_{ij} \leq d$, then $W_{ij} = \left\{\log\left(\frac{\gamma E_{ij}}{\alpha_i'} + e^{\gamma t_{i,j-1}}\right)/\gamma\right\} - t_{i,j-1}$. Otherwise, $W_{ij} = \left\{\log\left[\frac{\gamma E_{ij}}{\alpha_i'} - e^{\beta}\left(e^{\gamma(t_{i,j-1}+\Delta)} - e^{\gamma(t_{i,j-1}+\Delta)}\right)\right]/\gamma\right\} - t_{i,j-1}$.

9. Calculate the event time $T_{ij} = t_{i,j-1} + W_{ij}$.

10. If $T_{ij} \leq \tau$, let $t_{ij} = T_{ij}$ and set $j = j + 1$, and return to Step 6. Otherwise, observed event times for the ith process are $t_{i1}, \ldots, t_{in_i}$, where $n_i = j - 1$, and go to Step 11.

11. Set $i = i + 1$. If $i \leq m$, go to Step 2. Otherwise, stop.

4.2 Summary of Results

In this chapter, we first discuss the validity of normal approximations for the seven test statistics discussed in Chapter 3. We used normal Q-Q plots based on 10,000 simulation samples of test statistics. We also provided tables to have a better understanding of approximations in the tails of the standard normal distribution. The tables include the empirical pth quantiles based on 10,000 samples and the proportion of the values of test statistics in 10,000 samples which exceeded standard normal pth quantiles for $p = 0.975$ and 0.995. The normal Q-Q plots and tables can be found in Appendix B for two settings; (i) identical processes in Appendix B.1 and (ii) non-identical processes in Appendix B.2. We next considered the power of the tests through simulations under various scenarios. The results of the power studies are presented in Appendix C, once again, for two settings; (i) identical processes in Appendix C.1 and (ii) non-identical processes in Appendix C.1. The details of the simulation procedures are given in Section 4.1. We summarize these results in this section. For all simulations, we considered (m, τ) combinations where $m = 10$, 20, 50 and $\tau = 5$, 10, 20. The test statistics considered are

1. Laplace test statistic Z_{LT} given in (3.10),
2. Power law process (PLP) test statistic Z_{PLP} given in (3.28),
3. Linear rank test statistic Z_R given in (3.31),
4. Lewis-Robinson test statistic Z_{LR} given in (3.34),
5. Adjusted Lewis-Robinson test statistic Z_{ALR} given in (3.36),
6. Generalized Laplace test Z_{GLT} given in (3.48),
7. Generalized PLP test statistic Z_{GPLP} given in (3.60).

We first summarize the results for normal approximations in identical processes setting when the trend-free model is a HPP (i.e., Case (a) in Section 4.1). The normal Q-Q plots resemble a straight line for the test statistics Z_{LT}, Z_R, Z_{LR}, and Z_{ALR} under all scenarios with (m, τ) combinations. This result indicates that the standard normal approximation is adequate for these statistics under the scenarios considered in this study. For the power law process test statistic Z_{PLP}, the normal Q-Q plots (Figure B.4, Figure B.5, and Figure B.6) are little off at the extreme tails when $m = 10$ and $\tau = 5$ and 10. As m or τ increases, the normal Q-Q plots form roughly straight lines, which indicates that the normal approximations are adequate for these scenarios. As for the robust test statistic Z_{GL}, the normal Q-Q plots (Figure B.16, Figure B.17 and Figure B.18) show that the normal approximations are off when $m = 10$, and $\tau = 5$, 10, or 20. When $m = 20$, and $\tau = 5$, 10, or 20, the normal approximations are adequate. With m increasing from 20 to 50 and $\tau = 5$, 10, or 20, normal approximations are better. These results show that the normal approximations can be used for the generalized Laplace test when there are moderate to large number of processes (e.g., $m > 10$) are under observation. A similar conclusion to ours was also given by Lawless et al. (2012). We obtained similar results for the other robust test statistic Z_{GPLP} as well. However, in this case, as m increases, the convergence is a little slower comparing with the convergence of Z_{GL}. For example, when $m = 10$ and 20, $\tau = 5$, 10, or 20, the normal Q-Q plots (Figure B.19, Figure B.20 and Figure B.21) reveal that the normal approximations are not adequate. When m increases from 20 to 50, the normal Q-Q plots of generalized power law process test statistic Z_{GPLP} are roughly a straight line; that is, the normal approximations are good when m is large enough, for example, $m > 20$. All of these results are also supported by the results given in Table B.1. We also obtained similar results under the non-identical processes settings. The normal Q-Q plots for the non-identical processes in Case (a) are given in

Figures B.64–B.84. Furthermore, extreme tail approximations of the standard normal distribution in Case (a) and identical processes setting can be found in Table B.4 in Appendix B.

We would like to note that our careful investigation of the normal Q-Q plots and tables given in Appendix B showed similar results for Case (b) and Case (c) as well. Therefore, in summary, the normal Q-Q plots and tables given in Appendix B indicate that the normal approximations are adequate for $Z_{LT}, Z_R, Z_{LR}, Z_{ALR}$ test statistics for all scenarios and cases considered in our study when identical or non-identical processes are under observation. Therefore, the standard normal approximation can be used to find p-values for the tests based on these tests statistics when m and τ are as small as 10 and 5, respectively. For the power law process test statistic Z_{PLP}, the standard normal approximation is off in the extreme tails when m and τ are small. However, the convergence is fast and p-values can be obtained by using the standard normal distribution when m and τ are moderately large, say m and τ are larger than 10 and 5, respectively. The normal approximations for the robust trend test statistics Z_{GL} and Z_{GPLP} are not adequate with small m values in all cases and settings. The normal Q-Q plots become roughly a straight line as m increases. The convergence is, however, slower for the Z_{GPLP} test statistic. Our simulation results showed that the standard normal approximation can be used when $m \geq 20$ for the Z_{GL} test statistic and when $m > 20$ for the Z_{GPLP} test statistic. When the normal approximations are not adequate, the p-values can be estimated by using a bootstrapping procedure (Davison and Hinkley, 1997). It should be noted that the generalized Laplace test Z_{GL} and the generalized power law process test Z_{GPLP} cannot be applied when a single process is under observation (i.e., $m = 1$).

The tables given in Appendix B include the values of $\hat{P}(1.96)$ and $\hat{P}(2.576)$, the proportion of the absolute values of the seven test statistics in 10,000 samples exceeded the standard normal pth quantiles when $p = 0.975$ and 0.995. When τ is fixed and m increases, we obtained similar conclusions based on normal Q-Q plots. This can be summarized as follows. In Case (a), the null model (i.e., the trend-free model) is an HPP. In this case from Table B.1, $\hat{P}(1.96)$ was close to its nominal value 0.05 for the robust tests (Z_{GL} and Z_{GPLP}) in almost all scenarios, except when $m = 10$ for the generalized Laplace test. We obtained good results for $\hat{P}(1.96)$ from all other tests as well, but the Lewis-Robinson test (Z_{LR}) when $\tau = 5$ and 10. In these scenarios, the expected numbers of events per process are 5 and 10, respectively. As discussed by Lawless et al. (2012), the Lewis-Robinson may not perform well when the number of observations per process is small, and the adjusted Lewis-Robinson test is recommended in such cases. Table B.1 shows that $\hat{P}(1.96)$ is close to its nominal value 0.05 for all values of τ in the simulations when Z_{ALR} test statistic is used. We also obtained similar conclusions in the non-identical processes setting in Table B.4.

In Case (b), we generated samples from a renewal process. As expected, the values of $\hat{P}(1.96)$ in Table B.2 are close to the nominal value 0.05 of a two-sided test for the tests based on renewal processes; rank test (Z_R), Lewis-Robinson test (Z_{LR}) and adjusted Lewis-Robinson test (Z_{ALR}). Among these tests, rank test performed the best in terms of approximating the nominal value 0.05, and the adjusted Lewis-Robinson performed much better than the Lewis-Robinson test. The tests based on Poisson processes and robust trend tests were not adequate to approximate the extreme tails of the standard normal distribution. Among these test, the generalized Laplace test (Z_{GL}) performed the best.Table B.5 shows the results for the non-identical processes setting. The conclusions are similar.

For Case (c), we generated the samples from a delayed renewal process. The only difference between renewal processes and the delayed renewal processes is that the first gap times of the processes follow a different distribution than the remaining gap times.

The results given in Table B.3 show that the values of $\hat{P}(1.96)$ are not close to 0.05 for the tests based renewal processes as well as Poisson processes. However, robust trend tests performed much better than the other tests in terms of approximation of $P(1.96)$ in this case. We also obtained similar results given in Table B.6 for the non-identical processes setting.

We next discuss the power of the seven tests considered in simulation studies under monotonically increasing trend alternatives given in Section 4.1 (i.e., Case (d), Case (e) and Case (f)). Once again, we conducted power studies for various scenarios of (m, τ) combinations (m = 10, 20, 50 and τ = 5, 10, 20) under two settings; (i) identical processes and (ii) non-identical processes. We took two values of R ($R = 1.5$ and 2), where $R = \exp(\gamma\tau)$ is the relative risk of a failure occurrence under the trend model at the end of the observation periods (i.e., when $t = \tau$) comparing with the trend-free process (i.e., $\gamma = 0$). Note that, when $\gamma = 0$, the alternative models given in Cases (d), (e) and (f) reduces to their corresponding trend-free null models given in Cases (a), (b) and (c), respectively. We presented the results in tables in Appendix C; first, in identical processes setting in Appendix C.1, and then, in non-identical processes setting in Appendix C.2. In the remaining part of this section, we summarize these results.

Table C.1 shows the proportions of rejection of the null hypothesis $H_0 : \gamma = 0$ against the alternative hypothesis $H_1 : \gamma \neq 0$ for the Case (d) in identical processes setting when R = 2. In this case, we generated 1,000 samples from a NHPP with a trend component. As discussed before, we obtained the proportions based on the quantiles of the standard normal distribution (denoted by $\Omega(\cdot)$) as well as the empirical pth quantile of the test statistics from 10,000 simulation runs corresponding to the matching null hypothesis Case (a) (denoted by $\hat{\Omega}(\cdot)$). We considered power of the tests with size 0.05 and 0.01 (Type 1 errors). In the tables in Appendix C, $\Omega(1)$ and

$\hat{\Omega}(1)$ show the proportions corresponding to the size 0.05, and $\Omega(2)$ and $\hat{\Omega}(2)$ show the proportions corresponding to the size 0.01. In this section, we only compare $\hat{\Omega}(1)$ values to discuss the performance of test statistics so that the adequacy of the standard normal approximations is not an issue in the power comparisons. We obtained similar conclusions when $\hat{\Omega}(2)$ is used. In Case (d), the Laplace test (Z_{LT}) is overall the most powerful test among the seven tests. The other tests performed well in terms of power include the generalized Laplace test (Z_{GL}) and the power law process (Z_{PLP}). Other tests performed poor in this case. As m increases for a fixed τ, the power increases for all tests, but the increase is more significant with the robust tests. Therefore, we conclude that Z_{LT}, Z_{GL} and Z_{PLP} tests have good power in Case (d) for testing the monotonic trend alternatives even when τ is small ($\tau = 5$) and $m > 10$. For moderate τ and m values such as $\tau = 10$ and $m = 20$, Z_{LT} and Z_{GL} performed better than the other tests. Furthermore, for large τ and m values, all tests provide good power for testing the trend alternative. The results given in Table C.2 lead to similar conclusions. Also, Table C.7 and Table C.8 show the proportions in non-identical processes setting when $R = 2$ and 1.5, respectively. We obtained similar conclusion.

In Case (e), we generated samples from a Markov renewal process, in which the gap times W_{ij} have gamma distribution with an increasing scale parameter $a_i' \exp(\gamma t)$ and a constant shape parameter b. Note that the scale parameter increases as j increases for the ith process, $i = 1, \ldots, m$. We presented the proportions of rejection of $H_0 : \gamma = 0$ in Table C.3 when $R = 2$ in the identical processes setting. In this case, the generalized Laplace test Z_{GL} has the highest power overall among the tests included in this study. However, the overall power of Z_{GL}, Z_{PLP}, Z_{LT} and Z_{GPLP} tests are close. In this case, the overall power of the tests based on the renewal processes (i.e., Z_R, Z_{LR} and Z_{ALR}) are less than the other tests. This is an interesting result because the trend-free model in this case is a renewal process, and we may expect to observe that the tests based on renewal processes perform better than the tests based on HPPs in terms of power. However, it should be noted that the Z_R, Z_{LR} and Z_{ALR} tests are developed under the Type 2 censoring mechanism. In our simulation studies, we applied a Type 1 censoring mechanism, which is more common in applications (see Section 1.2 and Section 3.3.2). This might result in a little lower power in these tests. Lawless et al. (2012) also obtained similar results in a power study comparing the Z_{GL}, Z_R and Z_{ALR} tests. They found that overall the most powerful test was the Z_{GL} test. Table C.4 shows the power results when $R = 1.5$ in identical processes setting. The conclusions are similar to those found when $R = 2$. Furthermore, in non-identical processes settings, we presented the power results in Table C.9 and Table C.10 when $R = 2$ and $R = 1.5$, respectively. We obtained similar results with the identical processes setting. Therefore, we conclude that, when there is a monotonically increasing trend in the gap times of a recurrent event process given in Case (e), the robust generalized Laplace test Z_{GL} was the most powerful in the scenarios and settings considered in this study. However, its power was close to Z_{PLP}, Z_{LT} and Z_{GPLP} tests. Based on our simulations, we do not recommend to use of the rank test Z_R, and the Lewis-Robinson test Z_{LR} and the adjusted Lewis-Robinson test Z_{ALR}.

In Case (f), we generated the sample from a carryover effect model with a monotonically increasing time trend. Under the null hypothesis of no trend (i.e., $H_0 : \gamma = 0$), the model becomes a carryover effect model given in Case (c). Carryover effect is an effect that may result in cluster of events together over time after each event occurrence in the same process. In essence, the carryover effect model given in Case (c) is a delayed renewal process, where the first gap time follows an exponential distribution with rate parameter $\alpha_i^\star$ and the remaining gap times have a mixture type of distribution with a substantial mass given over Δ time period after each event occurrences.

In this case, Table C.5 shows the power results for the seven test statistics under identical processes setting when R=2. The results indicate that the generalized Laplace test Z_{GL} is the overall most powerful test. The Laplace test Z_{LT} follows the Z_{GL}. The overall power of the Z_{PLP} and Z_{GPLP} are similar and lower than the overall power of Z_{LT} and Z_{GL} tests. The tests based on renewal processes (Z_R, Z_{LR} and Z_{ALR}) are once again provided low overall power results. In this case, we recommend the use of the generalized Laplace test for testing the monotonic trends in recurrent even processes. For $R = 1.5$, we presented the power results in Table C.6. We obtained similar conclusions. Also, Table C.11 and Table C.12 show the power of the tests in non-identical processes setting when $R = 2$ and $R = 1.5$, respectively. Once again, we observed that the generalized Laplace test Z_{GL} is overall the most powerful test among the trend tests included in our study.

In summary, we recommend the robust tests statistics Z_{GLT} and Z_{GPLP} when m is moderately large and there are relatively few events per process. The tests Z_R, Z_{LT}, and Z_{LR} can be considered when m is small (10 or less) and there are moderate number of events per process. We found that robust trend tests perform well in all scenarios considered in this practicum.

Chapter 5

Applications

In this chapter, we discuss two data sets from industry. These data sets have been discussed by other authors in the literature. Our goal in this chapter is to illustrate the methods discussed in the previous chapters.

5.1 Example 1: Hydraulic Systems of LHD Machines

Kumar and Klefsjö (1992) provided failure data of hydraulic systems of six load-haul-dump (LHD) machines working on a mine in Sweden. This data set is discussed in Section 1.3. In this section, we apply the trend tests considered in the previous chapter to test whether there is a monotonic trend in the data or not. This data set was analyzed by many other authors as well, including Kumar and Klefsjö (1992), Baker (1996) and Lawless et al. (2012). It includes the gap times between successive failures in six LHD machines. The data set is given in Appendix A.1. The reliability study considered by Kumar and Klefsjö (1992) originally includes failure times of a fleet of LHD machines, but as they mentioned, the data collection and putting them

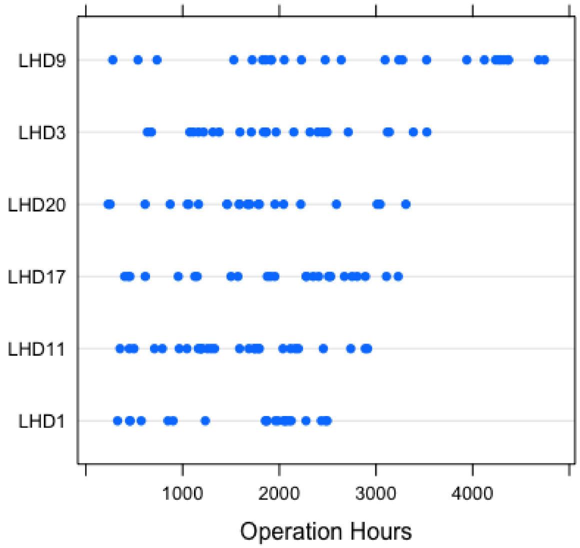

Figure 5.1: Dot plots of the failure times of the hydraulic systems in LHD machines

in a proper format was a tedious process. Therefore, they only reported the failure times of six LHD machines with the most correct failure time information. These machines was categorized as old (LHD1 and LHD 3), medium old (LHD 9 and LHD 11) and new (LHD 17 and LHD 20). The data were collected over a two-year period. The gap times between successive failure were reported in operating hours of LHD machines. Other details about this data set can be found in Kumar and Klefsjö (1992, Section 4). For the analysis purposes, we consider that the observation of each LHD machine starts at time $t = 0$. As denoted in Section 1.3.1, the end-of-followup times of LHD 1, LHD 3, LHD 9, LHD 11, LHD 17 and LHD 20 are, respectively, 2496, 3526, 4743, 2913, 3230 and 3309 hours and the observed numbers of failures over these observation periods are 23, 25, 27, 28, 26 and 23, respectively.

We start our analysis with simple plots to reveal the presence and forms of the trends in each LHD machine separately. Figure 5.1 shows the dot plots of the LHD machines. We note that the gap times between successive failures are decreasing in

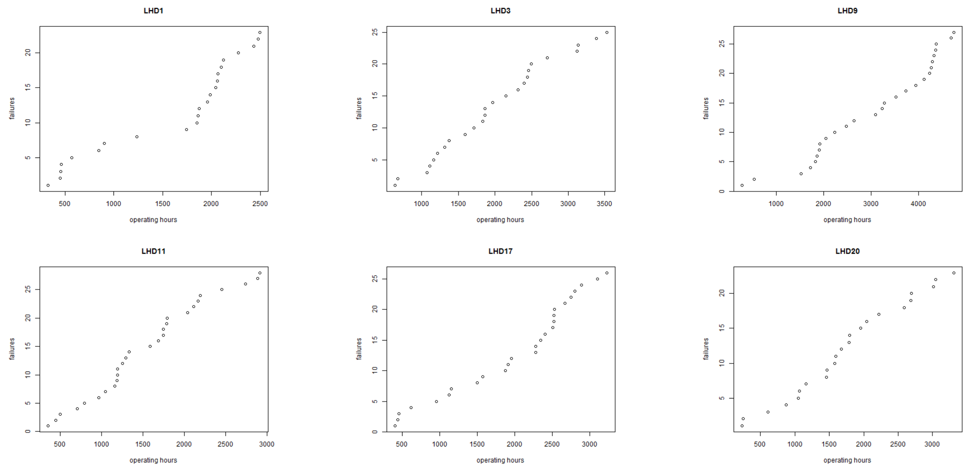

Figure 5.2: Plots of cumulative number of failures versus operting hours for LHD machines

LHD 1, LHD 9 and LHD 17 as the operating time t increases. To further investigate these patterns, we next present the Nelson-Aalen plots (cumulative number of failures versus cumulative operating time in hours) in Figure 5.2. The concave up shapes in these plots indicate an increasing monotonic trend in LHD 1, LHD 9 and LHD 17 machines. The Nelson-Aalen plots of LHD 3, LHD 11 and LHD 20 machines resemble an approximate linear line, indicating the absence of a time trend.

We next consider trend tests that can be used for a single processes. Therefore, we calculate the Laplace test statistic Z_{LT}, rank statistic Z_R, the Lewis-Robinson statistic Z_{LR}, the adjusted Lewis-Robinson statistic Z_{ALR} and the power law process test statistic Z_{PLP}. The values are given in Table 5.1 along with their corresponding standard normal distribution based p-values in parenthesis. It should be noted that, when $m = 1$, simulation studies conducted by other authors showed that the tests statistics Z_{LT}, Z_R, Z_{LR}, Z_{ALR} converge to the standard normal distribution very fast as the observed number of event increases. Therefore, we obtained p-values by using the standard normal distribution. We also conducted a simulation study (results are not shown here) to discuss the validity of the normality assumption for each LHD

	Z_{LT}	Z_R	Z_{LR}	Z_{ALR}	Z_{PLP}
LHD 1	2.294(0.022)	-1.218(0.223)	1.613(0.107)	1.579(0.114)	1.850(0.064)
LHD 3	1.075(0.282)	-0.321(0.748)	0.699(0.484)	0.686(0.493)	1.650(0.099)
LHD 9	2.409(0.016)	-1.681(0.093)	1.888(0.059)	1.854(0.064)	2.054(0.040)
LHD 11	0.565(0.572)	-0.441(0.659)	0.255(0.799)	0.251(0.802)	1.272(0.204)
LHD 17	1.855(0.064)	-1.655(0.098)	1.561(0.119)	1.531(0.126)	1.763(0.078)
LHD 20	0.455(0.650)	0.183(0.855)	0.114(0.909)	0.112(0.911)	0.864(0.387)

Table 5.1: The values of trend test statistics and corresponding two-sided p-values for the single LHD machines.

machine. Our results showed that the standard normal approximations are quite accurate for all these test statistics but Z_{PLP}. The results in Table 5.1 shows that, for single LHD machines, the absence of monotonic trend in LHD 1 machine is rejected by only Z_{LT} test at 0.05 level of significance (p-value $= 0.022$). All tests have p-values greater than 0.05 for testing the absence of monotonic trend in LHD 3. We note that the Laplace test and the power law process test indicate a significant trend in LHD 9 at 0.05 level of significance, but trend is significant in all tests at 0.10 level. All tests show that the monotonic trend is not significant in LHD 11, LHD 17 and LHD 20 machines at 0.01 level. This is also true when the significance level is 0.05, but only Laplace test show significant monotonic trend in LHD 17 at this level.

We next consider the trend tests for the combined data set, which includes the failure times of LHD 1, LHD 3, LHD 9, LHD 11, LHD 17 and LHD 20 machines together. In this case, $m = 6$, and we apply all seven tests including the robust tests Z_{GL} and Z_{GPLP}. We obtain the p-values based on the standard normal approximations. It should be noted that, since the number of processes is small ($m = 6$), the normal approximation may be off at the extreme tails for the robust tests. Our simulation studies also indicate that the standard normal approximation for the power law process tests may not be accurate for this data set. However, p-values can be estimated by using a parametric bootstrap procedure as well. Table 5.2 shows the values of the test statistics and their corresponding p-values within parenthesis. The Generalized Laplace test, Laplace test, rank test, Lewis-Robinson test, adjusted Lewis-Robinson

	Z_{GL}	Z_{LT}	Z_R	Z_{LR}	Z_{ALR}	Z_{PLP}	Z_{GPLP}
LHD	2.017(0.0437)	3.567(0.0004)	-2.145(0.0320)	2.502(0.0120)	2.454(0.0140)	-0.522(0.6020)	-2.443(0.01470)

Table 5.2: The values of trend test statistics and corresponding two-sided p-values for the combined LHD machines.

test and Generalized PLP test reject the absence of a monotonic trend in the combined data set at 0.05 level of significance. The p-values of Z_{PLP} is quite big. We would like to underline again that these p-values are based on standard normal distribution, and the standard normal approximation for these tests may not be accurate when $m = 6$.

5.2 Example 2: Failures of Air-Conditioning Equipment

Proschan (1963) gave a data set of air-conditioning equipment failures on thirteen aircrafts. We discussed this data set briefly in Section Cox and Lewis (1966) used this data set to illustrate the Laplace test and rank test for testing the monotonic trends in the rate of occurrence of failures. Lawless and Thiagarajah (1996) analyzed the trend in Aircraft 6 and Aircraft 7. We selected four aircrafts (Aircrafts 2, 3, 6 and 7), and applied the tests considered in the previous chapters. Table A.2 in Appendix A shows the gap times between successive failures in these four aircrafts in operation hours. In the original data set, the end-of-followup times were not given. Therefore, we take the last failure occurrence times as the end-of-followup time for the observation. Therefore, the observation periods lasted 33074, 30853, 33847 and 27373 hours for Aircrafts 2, 3, 6 and 7, respectively. Over these time periods, there were 23, 27, 29 and 30 failures of air-conditioning equipment observed in Aircrafts 2, 3, 6 and 7, respectively.

We start our analysis with simple plots. Figure 5.3 shows the dot plots of the

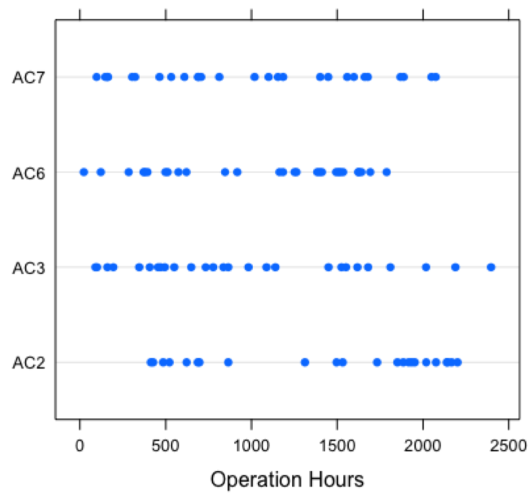

Figure 5.3: Dot plots of failure times of air conditioning equipment in aircrafts

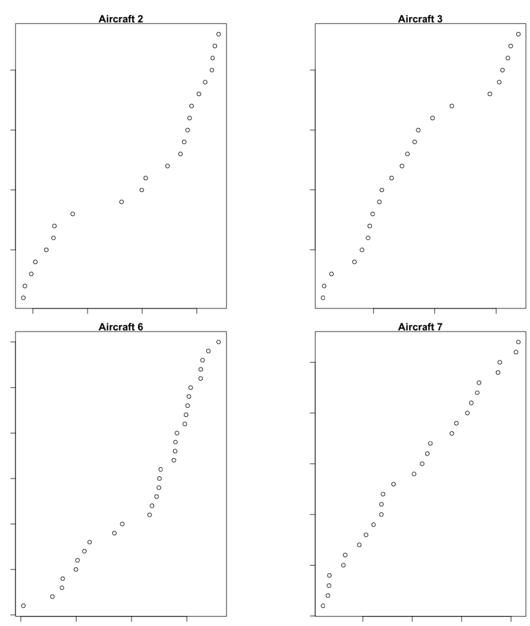

Figure 5.4: Plots of cumulative number of failures versus operating hours for aircrafts

air-conditioning equipment failures in four aircrafts. The dot plots indicate a mild increasing number of failures in Aircraft 2 and Aircraft 6. It appears that there are some clustering of failures in Aircraft 3, but there is no indication of a monotonic trend in Aircrafts 3 and 7. We also observe a similar pattern in the Nelseon-Aalen plots of the failure times given in Figure 5.4.

Next, we use the formal tests to test the absence of monotonic trend in the combined data set. The values of tests statistics and corresponding p-values based on the standard normal distribution are given as follows. The values of Z_{LR} and Z_{ALR} test statistics are 0.612 and 0.599, respectively, and the corresponding p-values are 0.541 and 0.549, respectively. Also, the value of the rank test statistic Z_R is -0.375, and the corresponding p-value is 0.708. Therefore, the tests based on renewal processes under the trend-free model do not reject the absence of a monotonic trend in the combined data set. Similarly, the values of Z_{LT} and Z_{PLP} test statistics are, respectively, 1.543 with a p-value of 0.436 and 0.123 with a p-value of 0.873. This result shows that the trend tests based on the HPP under the trend-free model do not reject the absence of a monotonic trend in the combined data set as well. Finally, we also obtain the values of the robust trend test statistics Z_{GL} and Z_{GPLP}. The value of the generalized Laplace test statistic Z_{GL} is 0.874 with a p-value of 0.382 and the value of the generalized PLP test statistic Z_{GPLP} is -1.987 and the corresponding p-value is 0.0469. Therefore, all trend tests considered in this chapter do not reject the absence of a monotonic trend in the combined data set.

Chapter 6

Conclusion and Future Work

Detection of trends is crucial in the analysis of recurrent event data. In this practicum, we discussed concept of trends and compared seven trend tests in terms of their power to detect monotonic trends. These tests include five classical tests, which are the Laplace test Z_{LT}, power law process (PLP) test Z_{PLP}, linear rank test Z_R, the Lewis-Robinson test Z_{LR} and the adjusted Lewis-Robinson test Z_{ALR}, as well as the two robust tests, the generalized Laplace test Z_{GLT} and the generalized PLP test Z_{GPLP}.

The classical tests are developed under certain model assumptions, and may lead to wrong conclusions about the presence of trends if these assumptions do not hold. Robust tests usually hold good power if the model assumptions are mildly violated. Therefore, they are important alternatives to the classical trend tests, and can be applied as routine checks for trends in multiple recurrent event processes.

In Section 4.2, we have discussed the validity of normal approximation for the formal test statistics. The normal approximation of Z_{LT}, Z_R, Z_{LP} and Z_{ALR} tests is valid, but it is off in the extreme tails for Z_{PLP} when m is small. As m approaches to infinity when τ is fixed or as τ approaches infinity when m is fixed, the normal approximation holds for all test statistics considered in this practicum. Therefore,

p-values may be calculated by using this approximation for sufficiently large m or τ values. In finite sample size (m or τ) settings, we found in our simulations that the normal approximation is not valid in the extreme tails for the robust test statistics Z_{GLT} and Z_{GPLP}. In these settings, p-values can be calculated via bootstrapping.

In the power studies, we found that robust test statistics Z_{GLT} and Z_{GPLP} performed well for prespecied observation periods τ and fixed number of processes m when m is moderately large. When m is small Z_R, Z_{LT} and Z_{LR} tests can be considered.

The main assumption for all simulations is that only monotonic trends may occur. Although generalized PLP test statistic Z_{GPLP} and generalized Laplace test statistics Z_{GLT} can not deal when a single process ($m = 1$) is under observation, they can be used when multiple processes are under observation ($m > 1$). Strictly speaking, we cannot conclude that which test statistic should be applied for all applications. Another important point is that the use of normal approximation to calculate p-values should be carefully applied for the robust test statistics. We hope that our simulation results given in Chapter 4 can be a reference for finite sample size settings.

In this practicum, we only discussed monotonic time trends as the only factor in event occurrence. More discussions should be considered in the future for the case of many factors affecting the intensity functions of the processes. For example, cluster is one of such important factors in analyzing recurrent event processes. In Chapter 5, Figure 5.1 and Figure 5.3 have showed some forms of cluster of events together over time. Cigsar (2010) discussed several carryover effect models to deal with some forms of event clustering in recurrent event processes. We will consider the power of these tests when carryover effects are present in the data as a future work. Non-monotonic trends can also appear in some applications. When non-monotonic trends are present, we recommend graphical checks and more elaborate modeling approach to detect such trends. This will be considered as a future work.

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Appendix A

Data Sets

Table A.1: Table of the time between successive failures of LHD machines

Failure Number	Time between successive failures of hydraulic system					
	LHD 1	LHD 3	LHD 9	LHD 11	LHD 17	LHD 20
1	327	637	278	353	401	231
2	125	40	261	6	36	20
3	7	397	990	49	18	361
4	6	36	191	211	159	260
5	107	54	107	82	341	176
6	277	53	32	175	171	16
7	54	97	51	79	24	101
8	332	63	10	117	350	293
9	510	216	132	26	72	5
10	110	118	176	4	303	119
11	10	125	247	5	34	9
12	9	25	165	60	45	80
13	85	4	454	39	324	112
14	27	101	142	35	2	10
15	59	184	38	258	70	162
16	16	167	249	97	57	90
17	8	81	212	59	103	176
18	34	46	204	3	11	370
19	21	18	182	37	5	90
20	152	32	116	8	3	15
21	158	219	30	245	144	315
22	44	405	24	79	80	32
23	18	20	32	49	53	266
24		248	38	31	84	
25		140	10	259	218	
26			311	283	122	
27			61	150		
28				24		

Table A.2: Table of the intervals in operating hours between successive failures of airconditioning equipment in 13 Boeing 720 aircraft

Failure Number	Operating hours between successive failures of airconditioning equipment			
	Aircraft 2	Aircraft 3	Aircraft 6	Aircraft 7
1	413	90	23	97
2	14	10	261	51
3	58	60	87	11
4	37	186	7	4
5	110	61	120	141
6	65	49	14	18
7	9	14	62	142
8	169	24	47	68
9	447	56	225	77
10	184	20	71	80
11	36	79	246	1
12	201	84	21	16
13	118	44	42	106
14	34	59	20	206
15	31	29	5	82
16	18	118	12	54
17	18	25	120	31
18	67	156	11	216
19	57	310	3	46
20	62	76	14	111
21	7	26	71	39
22	22	44	11	63
23	34	23	14	18
24		62	11	191
25		130	16	18
26		208	90	163
27		70	1	24
28		101	16	
29		208	52	
30			95	

Appendix B

Normal Approximation

B.1 Normal Approximation with Identical processes

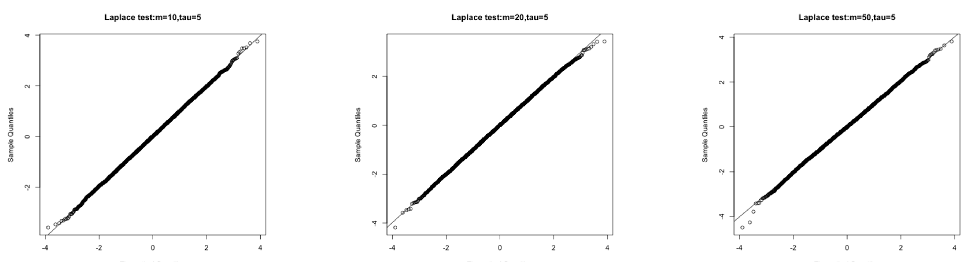

Figure B.1: Normal Q-Q plots of 10,000 simulated values of Laplace test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: : Case (a) with $\alpha_i = 1$.

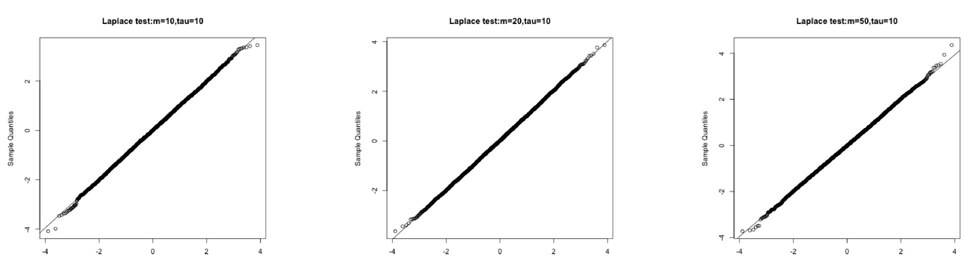

Figure B.2: Normal Q-Q plots of 10,000 simulated values of Laplace test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with $\alpha_i = 1$.

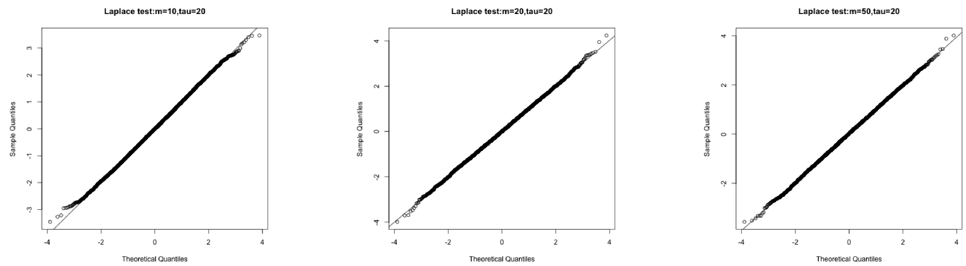

Figure B.3: Normal Q-Q plots of 10,000 simulated values of Laplace test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with $\alpha_i = 1$.

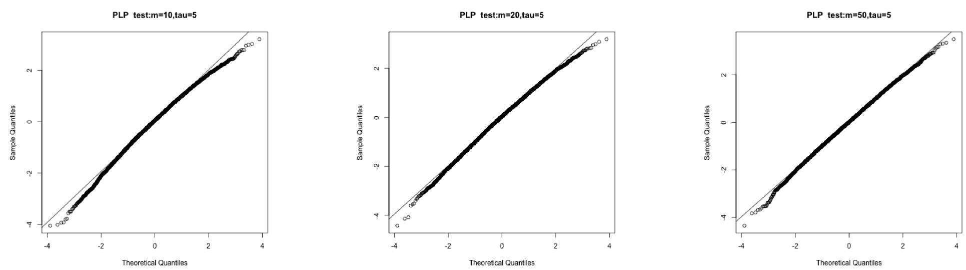

Figure B.4: Normal Q-Q plots of 10,000 simulated values of PLP test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with $\alpha_i = 1$.

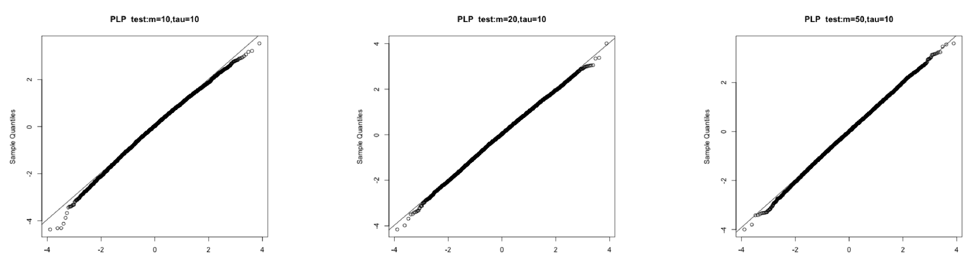

Figure B.5: Normal Q-Q plots of 10,000 simulated values of PLP test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$:Case (a) with $\alpha_i = 1$.

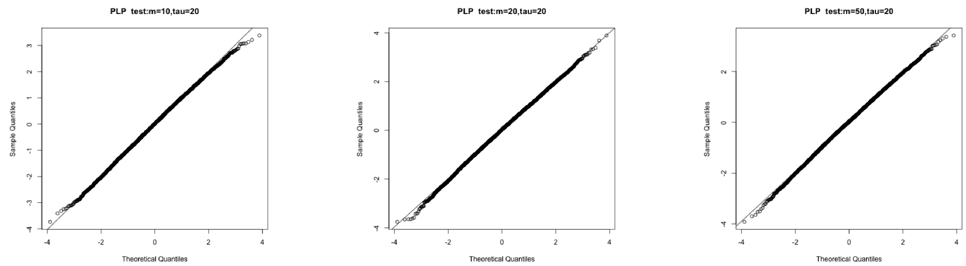

Figure B.6: Normal Q-Q plots of 10,000 simulated values of PLP test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with $\alpha_i = 1$.

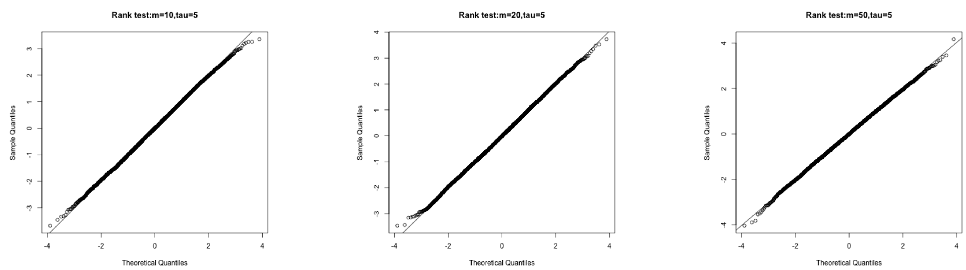

Figure B.7: Normal Q-Q plots of 10,000 simulated values of Rank test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with $\alpha_i = 1$.

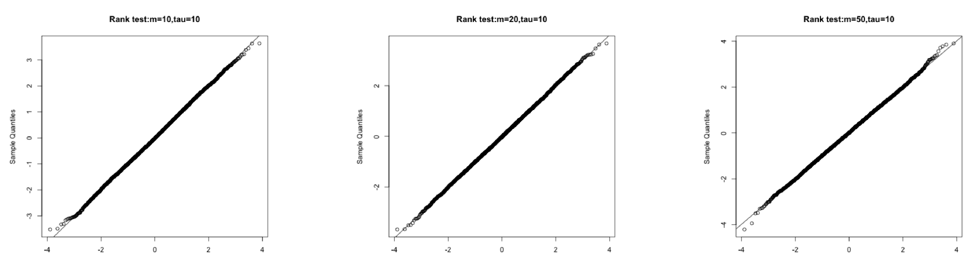

Figure B.8: Normal Q-Q plots of 10,000 simulated values of Rank test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with $\alpha_i = 1$.

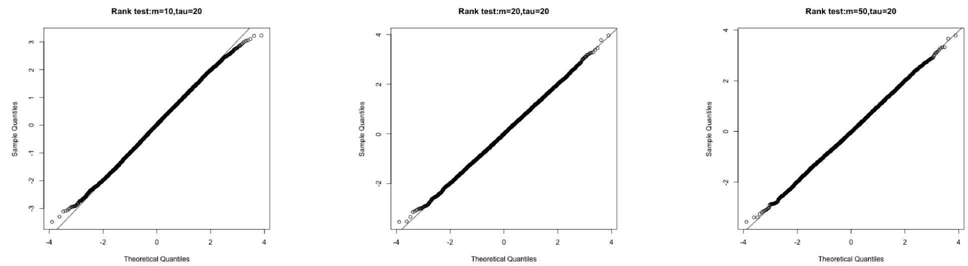

Figure B.9: Normal Q-Q plots of 10,000 simulated values of Rank test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with $\alpha_i = 1$.

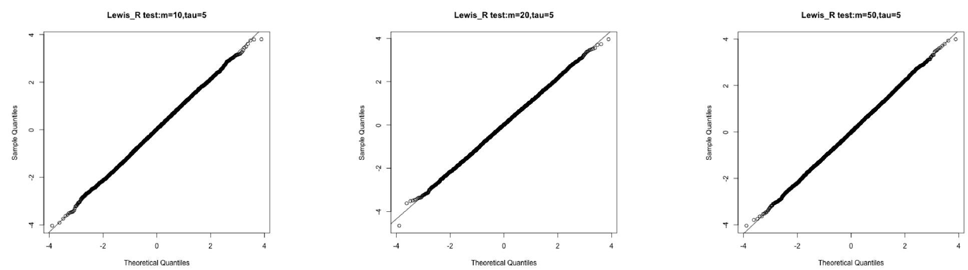

Figure B.10: Normal Q-Q plots of 10,000 simulated values of Lewis-Robinson test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$:Case (a) with $\alpha_i = 1$.

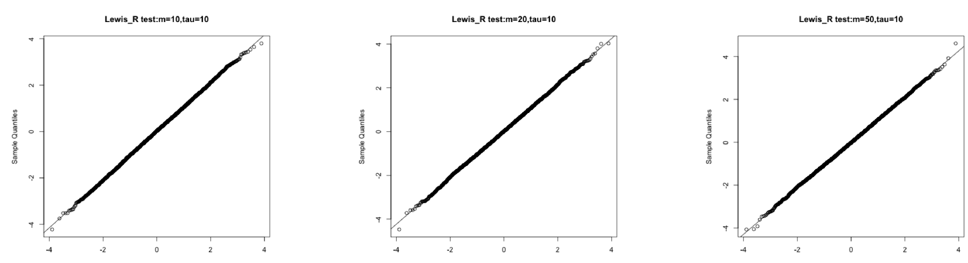

Figure B.11: Normal Q-Q plots of 10,000 simulated values of Lewis-Robinson test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with $\alpha_i = 1$.

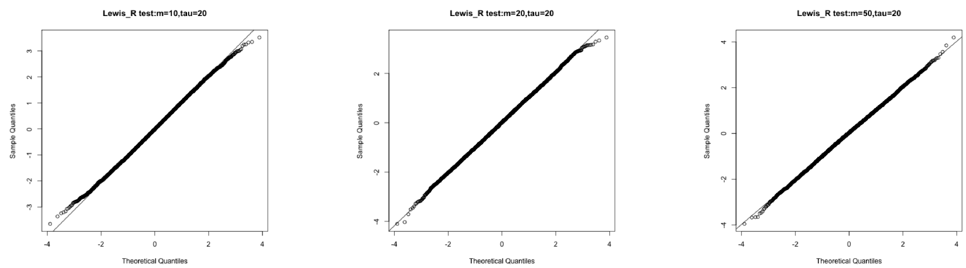

Figure B.12: Normal Q-Q plots of 10,000 simulated values of Lewis-Robinson test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with $\alpha_i = 1$.

Figure B.13: Normal Q-Q plots of 10,000 simulated values of Lewis -Robinson test with adjustment when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with $\alpha_i = 1$.

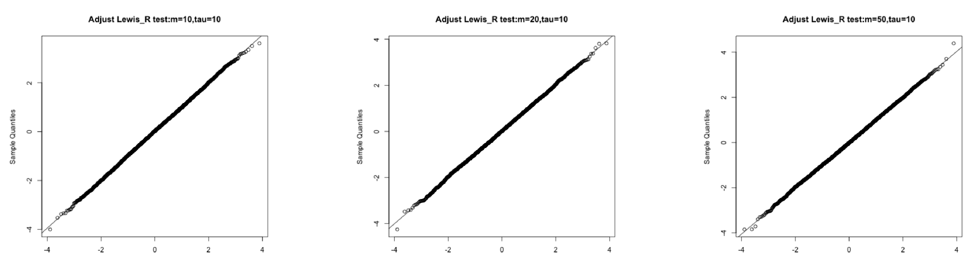

Figure B.14: Normal Q-Q plots of 10,000 simulated values of Lewis -Robinson test with adjustment when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with $\alpha_i = 1$.

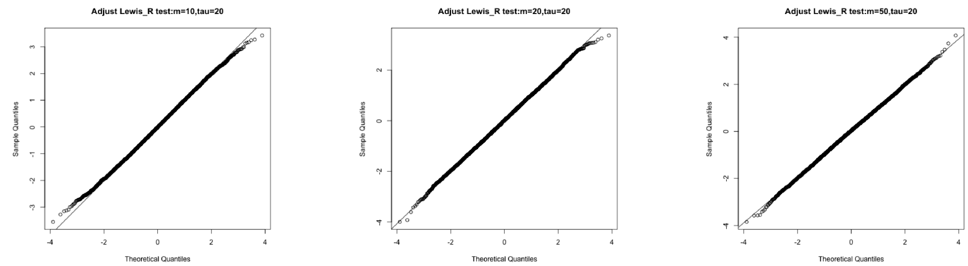

Figure B.15: Normal Q-Q plots of 10,000 simulated values of Lewis -Robinson test with adjustment when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with $\alpha_i = 1$.

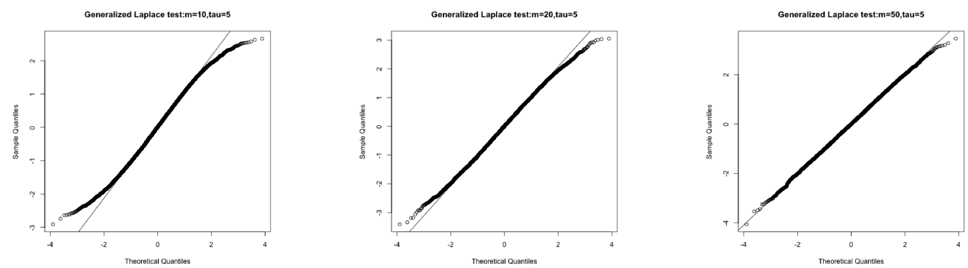

Figure B.16: Normal Q-Q plots of 10,000 simulated values of generalized Laplace test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with $\alpha_i = 1$.

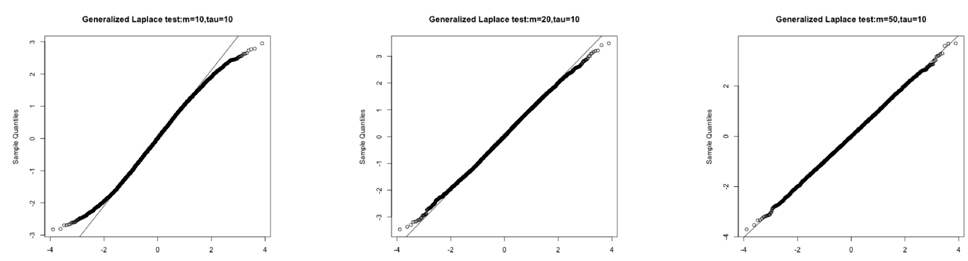

Figure B.17: Normal Q-Q plots of 10,000 simulated values of generalized Laplace test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with $\alpha_i = 1$.

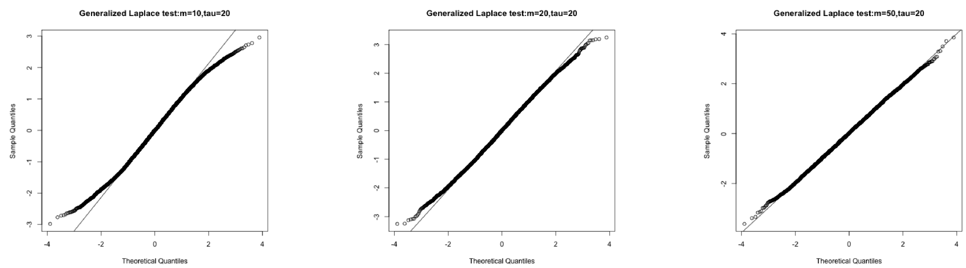

Figure B.18: Normal Q-Q plots of 10,000 simulated values of generalized Laplace test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with $\alpha_i = 1$.

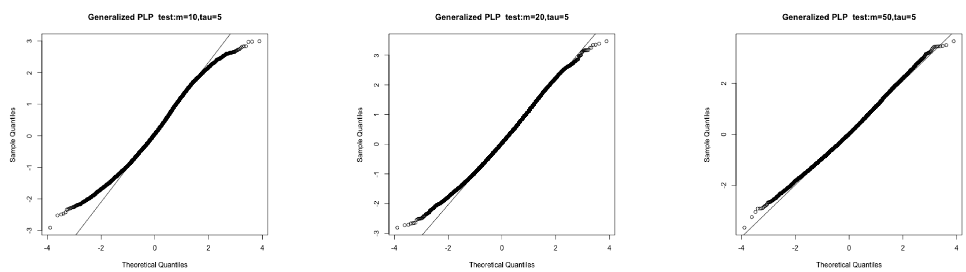

Figure B.19: Normal Q-Q plots of 10,000 simulated values of generalized PLP test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with $\alpha_i = 1$.

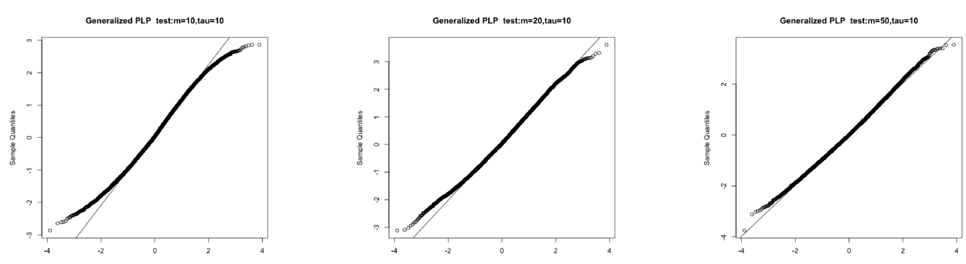

Figure B.20: Normal Q-Q plots of 10,000 simulated values of generalized PLP test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with $\alpha_i = 1$.

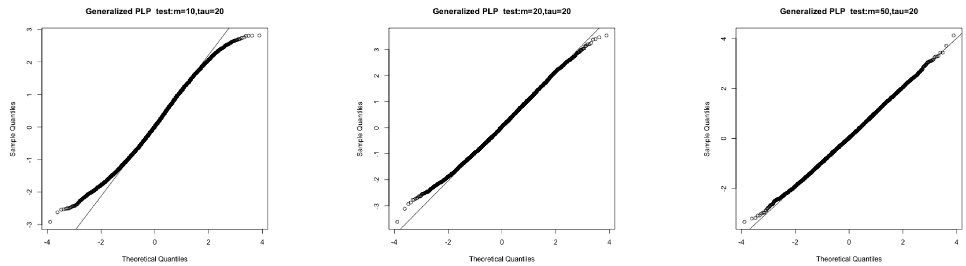

Figure B.21: Normal Q-Q plots of 10,000 simulated values of generalized PLP test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with $\alpha_i = 1$.

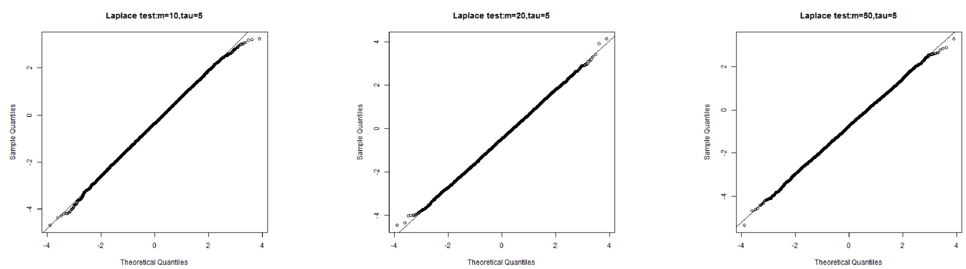

Figure B.22: Normal Q-Q plots of 10,000 simulated values of Laplace test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with $\alpha_i = 1$.

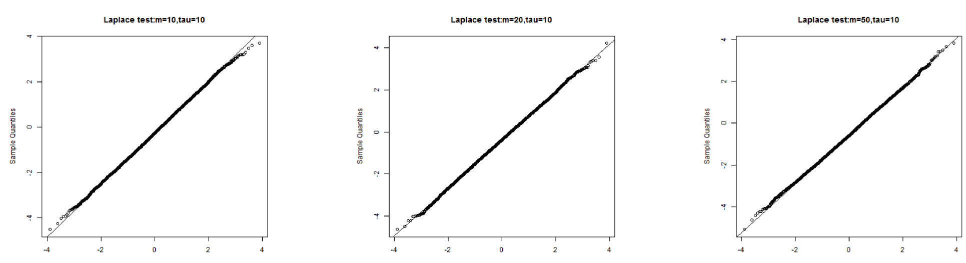

Figure B.23: Normal Q-Q plots of 10,000 simulated values of Laplace test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with $\alpha_i = 1$.

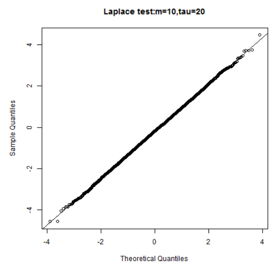

Figure B.24: Normal Q-Q plots of 10,000 simulated values of Laplace test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with $\alpha_i = 1$.

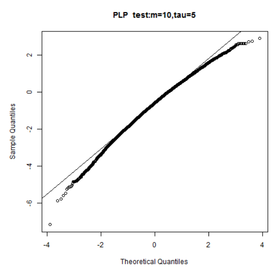

Figure B.25: Normal Q-Q plots of 10,000 simulated values of PLP test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with $\alpha_i = 1$.

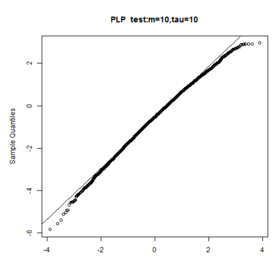

Figure B.26: Normal Q-Q plots of 10,000 simulated values of PLP test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with $\alpha_i = 1$.

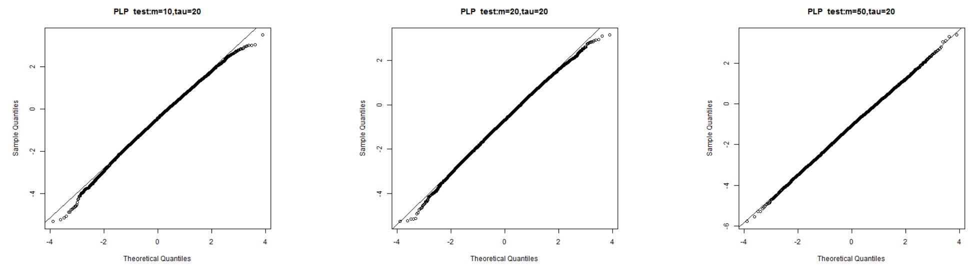

Figure B.27: Normal Q-Q plots of 10,000 simulated values of PLP test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with $\alpha_i = 1$.

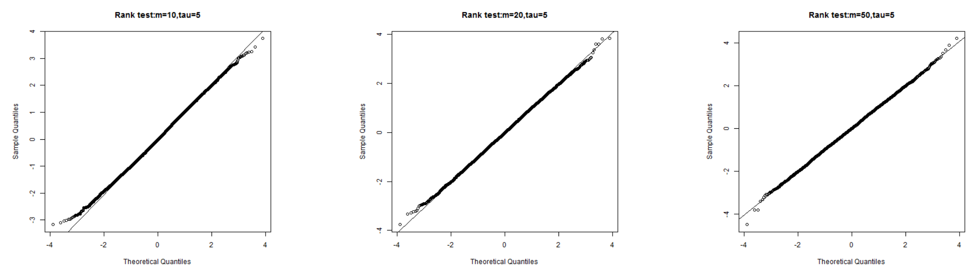

Figure B.28: Normal Q-Q plots of 10,000 simulated values of Rank test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with $\alpha_i = 1$.

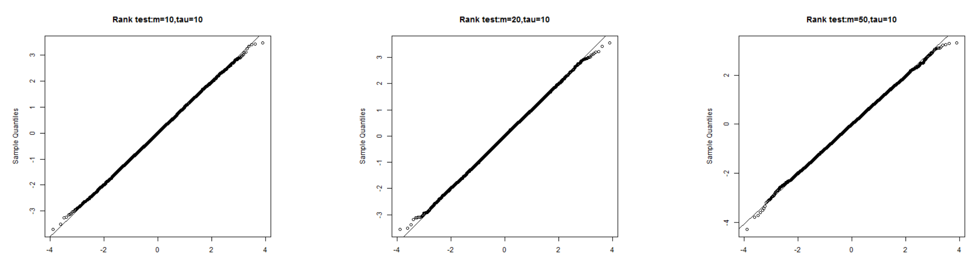

Figure B.29: Normal Q-Q plots of 10,000 simulated values of Rank test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with $\alpha_i = 1$.

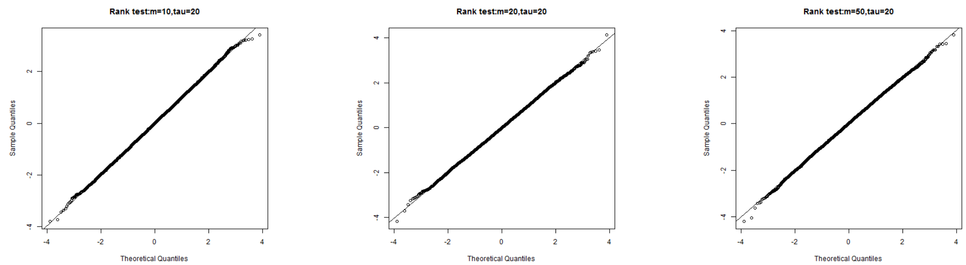

Figure B.30: Normal Q-Q plots of 10,000 simulated values of Rank test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with $\alpha_i = 1$.

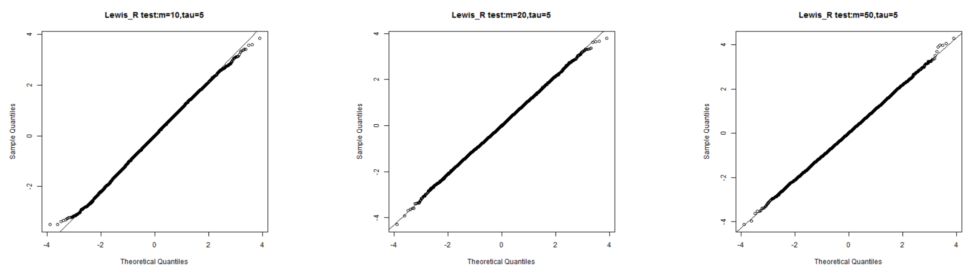

Figure B.31: Normal Q-Q plots of 10,000 simulated values of Lewis-Robinson test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with $\alpha_i = 1$.

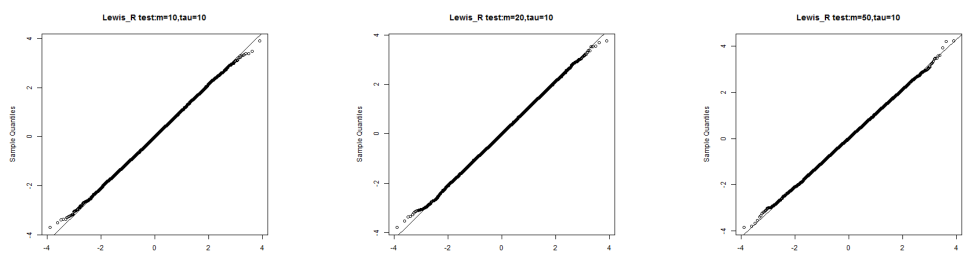

Figure B.32: Normal Q-Q plots of 10,000 simulated values of Lewis-Robinson test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with $\alpha_i = 1$.

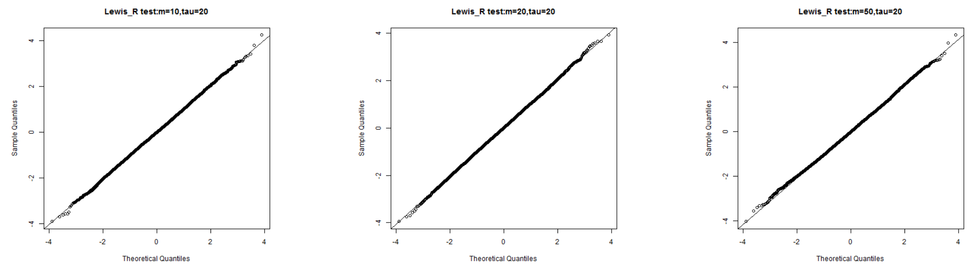

Figure B.33: Normal Q-Q plots of 10,000 simulated values of Lewis-Robinson test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with $\alpha_i = 1$.

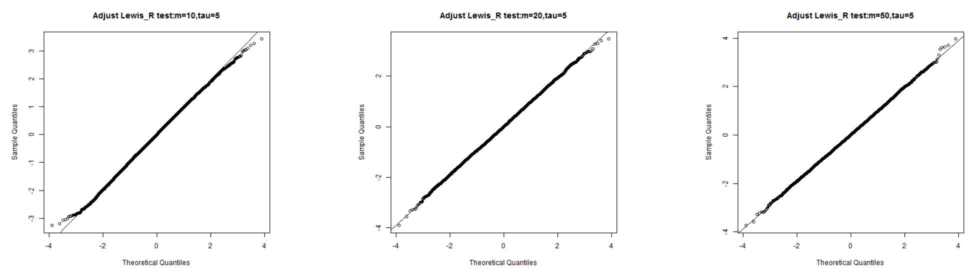

Figure B.34: Normal Q-Q plots of 10,000 simulated values of Lewis -Robinson test with adjustment when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with $\alpha_i = 1$.

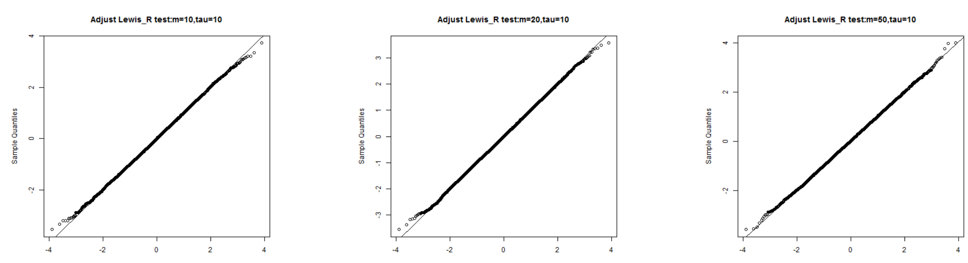

Figure B.35: Normal Q-Q plots of 10,000 simulated values of Lewis -Robinson test with adjustment when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with $\alpha_i = 1$.

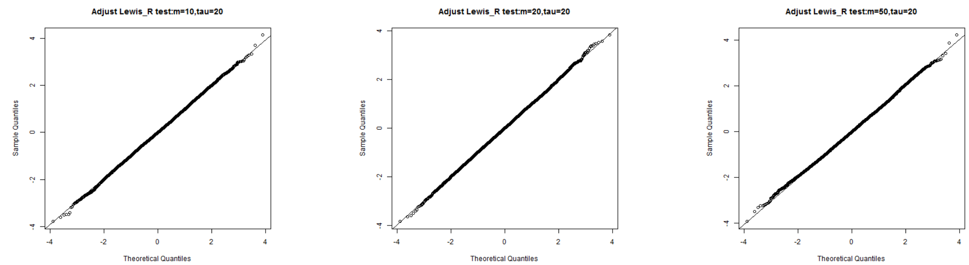

Figure B.36: Normal Q-Q plots of 10,000 simulated values of Lewis -Robinson test with adjustment when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with $\alpha_i = 1$.

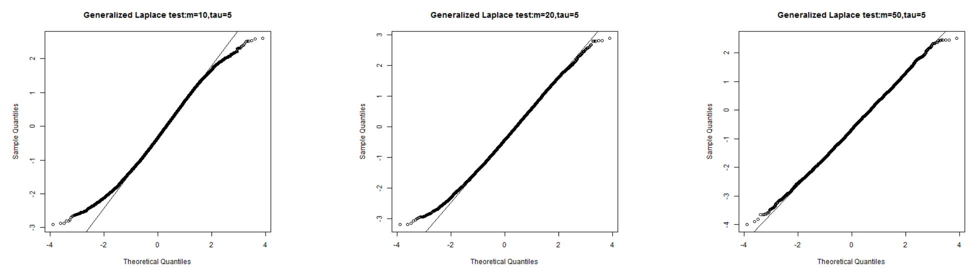

Figure B.37: Normal Q-Q plots of 10,000 simulated values of generalized Laplace test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with $\alpha_i = 1$.

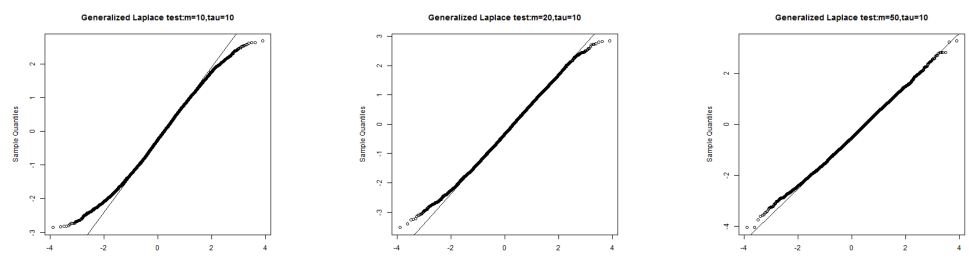

Figure B.38: Normal Q-Q plots of 10,000 simulated values of generalized Laplace test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with $\alpha_i = 1$.

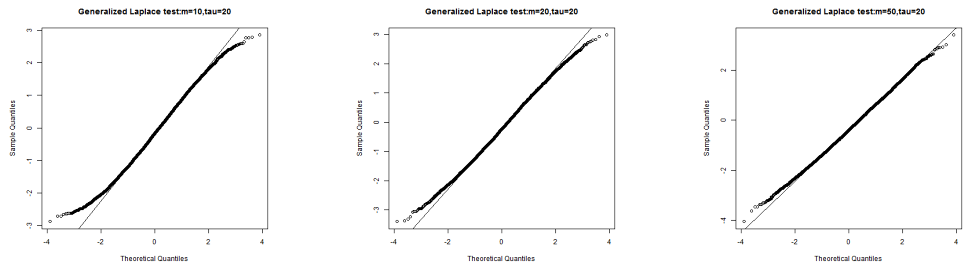

Figure B.39: Normal Q-Q plots of 10,000 simulated values of generalized Laplace test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with $\alpha_i = 1$.

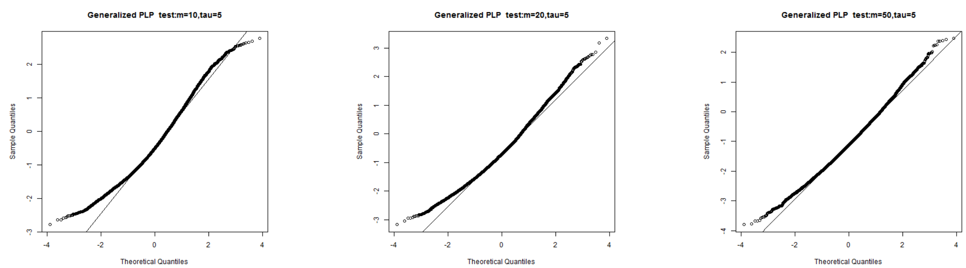

Figure B.40: Normal Q-Q plots of 10,000 simulated values of generalized PLP test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with $\alpha_i = 1$.

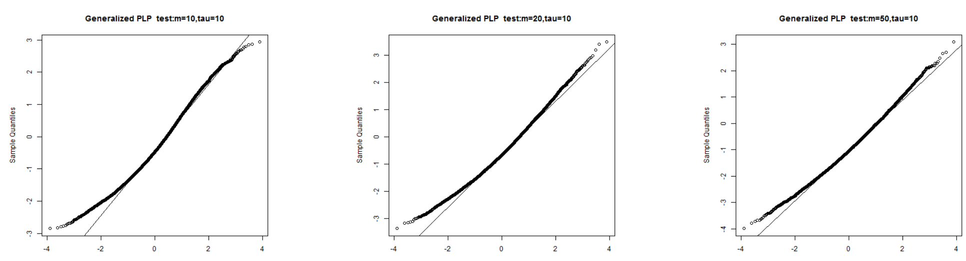

Figure B.41: Normal Q-Q plots of 10,000 simulated values of generalized PLP test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with $\alpha_i = 1$.

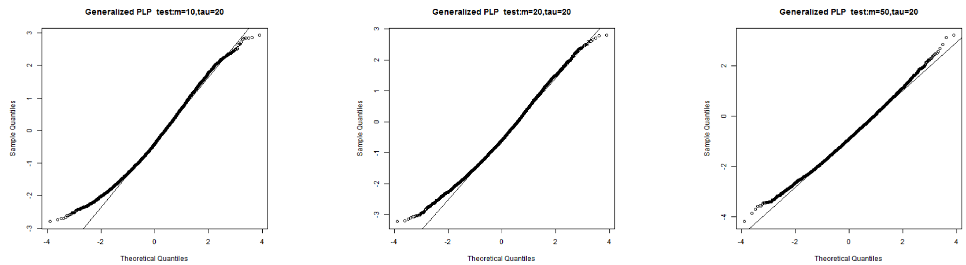

Figure B.42: Normal Q-Q plots of 10,000 simulated values of generalized PLP test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with $\alpha_i = 1$.

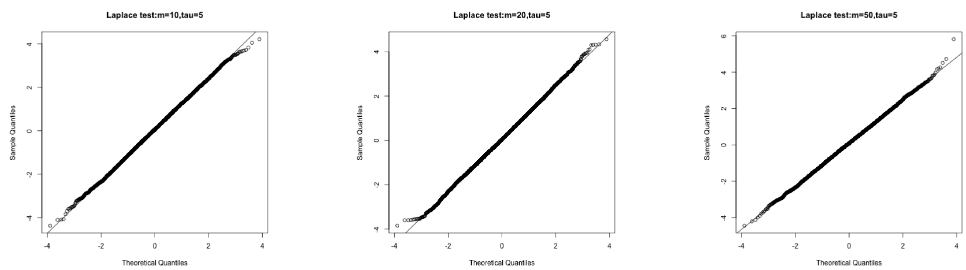

Figure B.43: Normal Q-Q plots of 10,000 simulated values of Laplace test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with $\alpha_i = 1$.

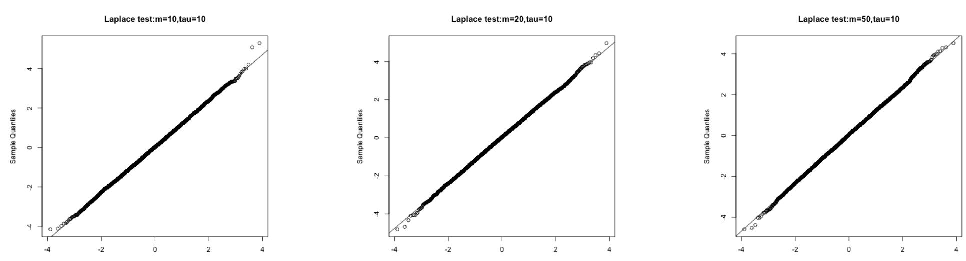

Figure B.44: Normal Q-Q plots of 10,000 simulated values of Laplace test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with $\alpha_i = 1$.

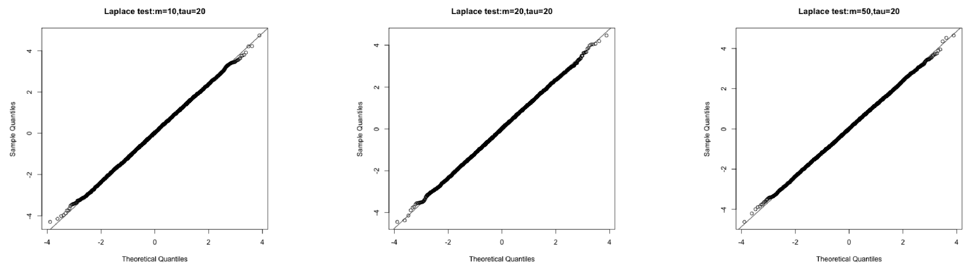

Figure B.45: Normal Q-Q plots of 10,000 simulated values of Laplace test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with $\alpha_i = 1$.

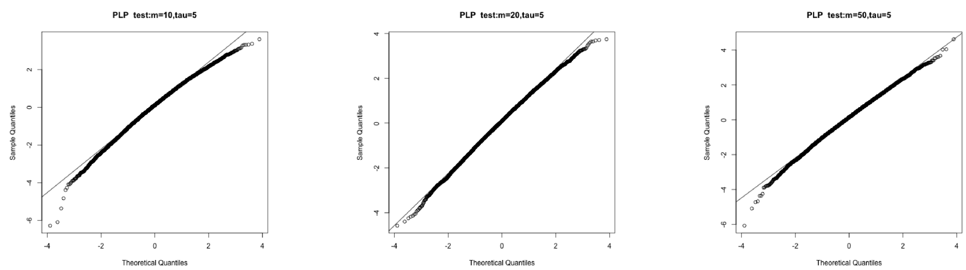

Figure B.46: Normal Q-Q plots of 10,000 simulated values of PLP test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with $\alpha_i = 1$.

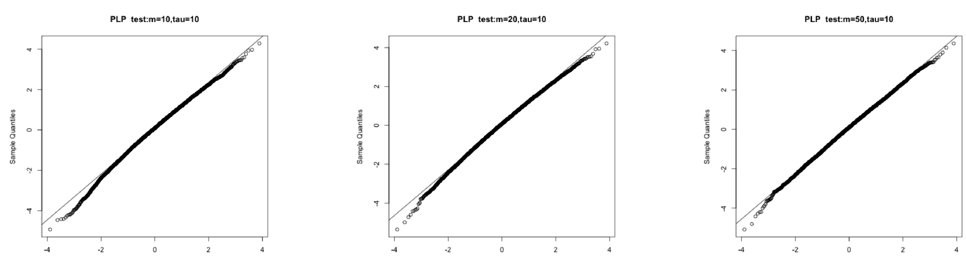

Figure B.47: Normal Q-Q plots of 10,000 simulated values of PLP test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with $\alpha_i = 1$.

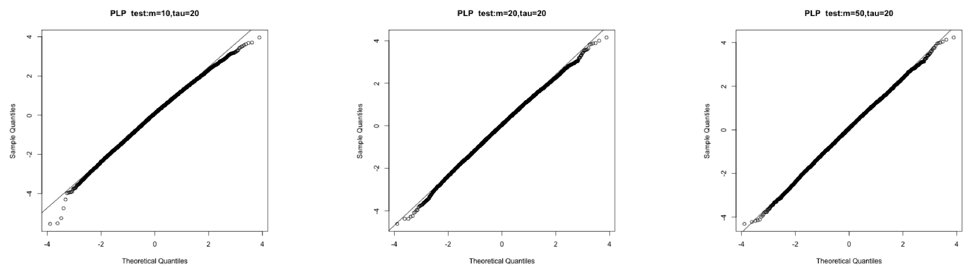

Figure B.48: Normal Q-Q plots of 10,000 simulated values of PLP test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with $\alpha_i = 1$.

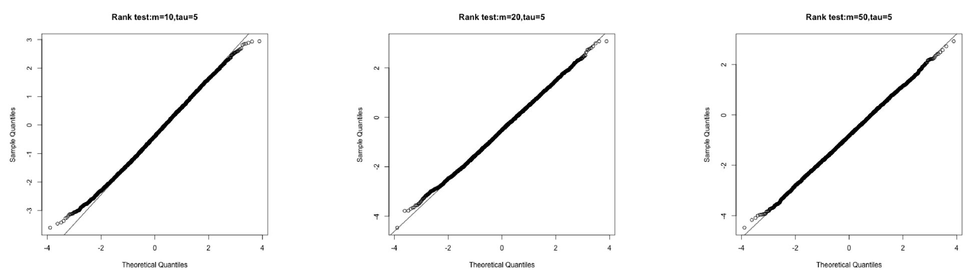

Figure B.49: Normal Q-Q plots of 10,000 simulated values of Rank test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with $\alpha_i = 1$.

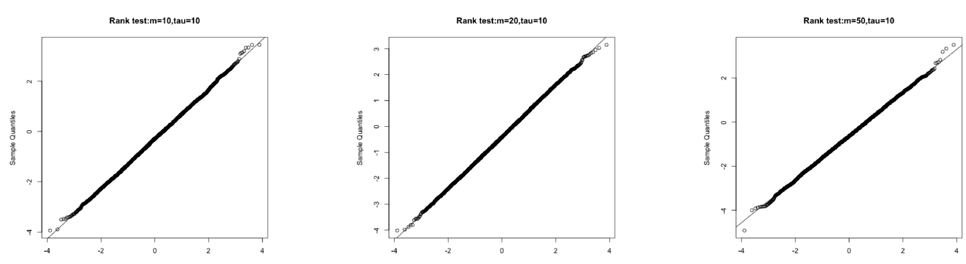

Figure B.50: Normal Q-Q plots of 10,000 simulated values of Rank test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with $\alpha_i = 1$.

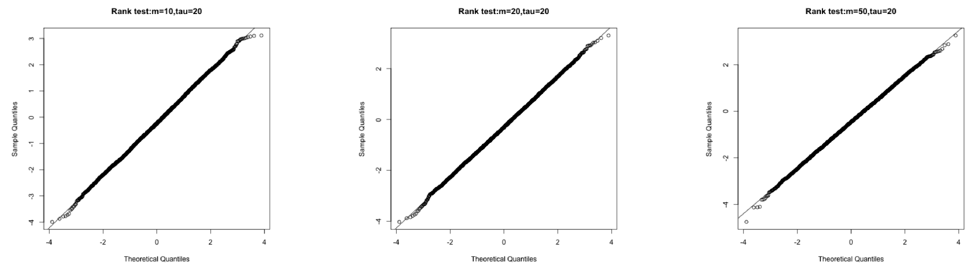

Figure B.51: Normal Q-Q plots of 10,000 simulated values of Rank test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with $\alpha_i = 1$.

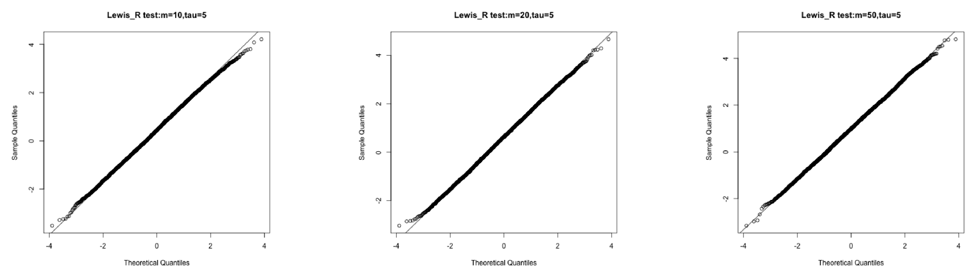

Figure B.52: Normal Q-Q plots of 10,000 simulated values of Lewis-Robinson test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with $\alpha_i = 1$.

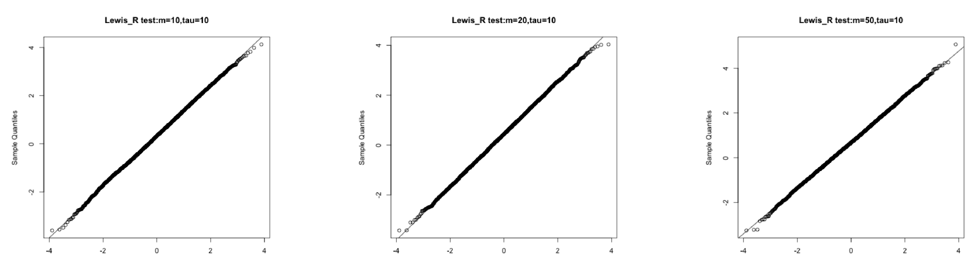

Figure B.53: Normal Q-Q plots of 10,000 simulated values of Lewis-Robinson test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with $\alpha_i = 1$.

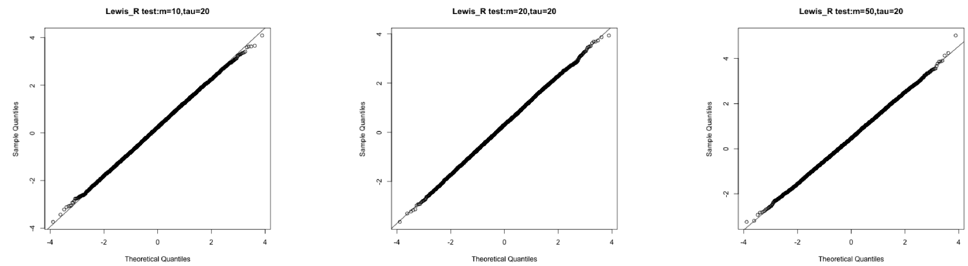

Figure B.54: Normal Q-Q plots of 10,000 simulated values of Lewis-Robinson test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with $\alpha_i = 1$.

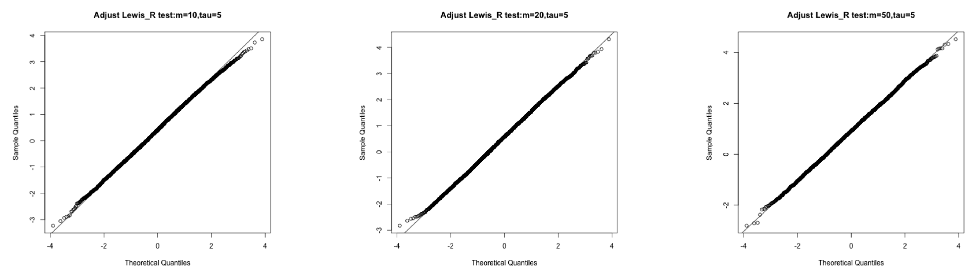

Figure B.55: Normal Q-Q plots of 10,000 simulated values of Lewis -Robinson test with adjustment when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with $\alpha_i = 1$.

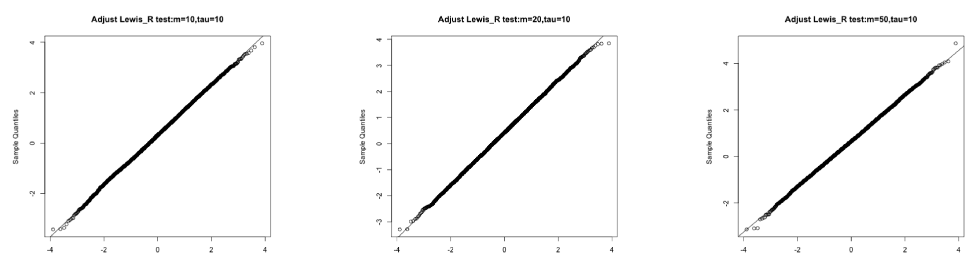

Figure B.56: Normal Q-Q plots of 10,000 simulated values of Lewis -Robinson test with adjustment when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with $\alpha_i = 1$.

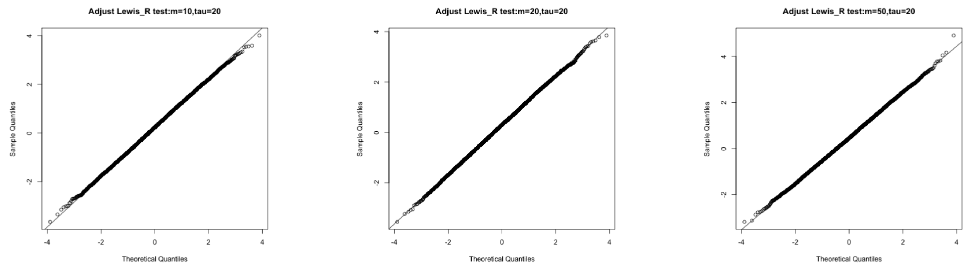

Figure B.57: Normal Q-Q plots of 10,000 simulated values of Lewis -Robinson test with adjustment when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with $\alpha_i = 1$.

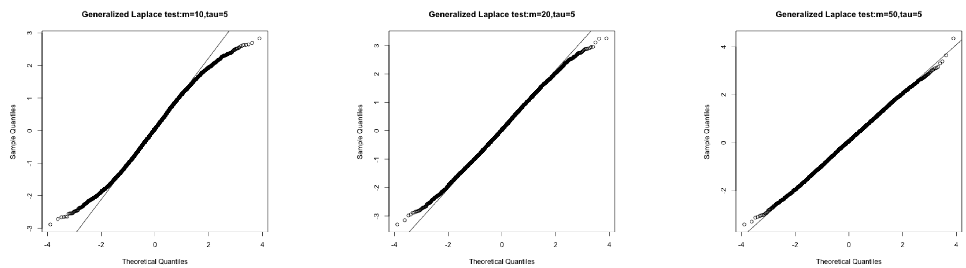

Figure B.58: Normal Q-Q plots of 10,000 simulated values of generalized Laplace test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with $\alpha_i = 1$.

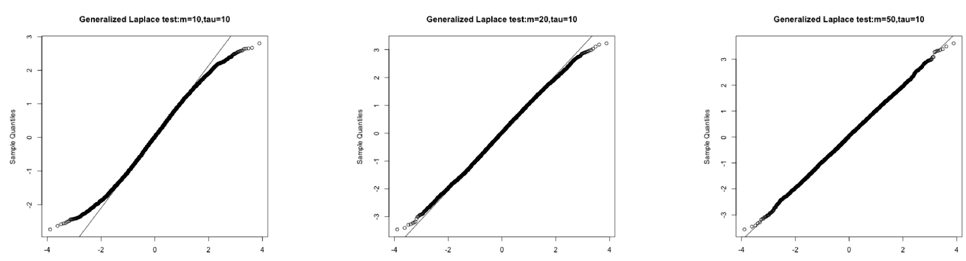

Figure B.59: Normal Q-Q plots of 10,000 simulated values of generalized Laplace test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with $\alpha_i = 1$.

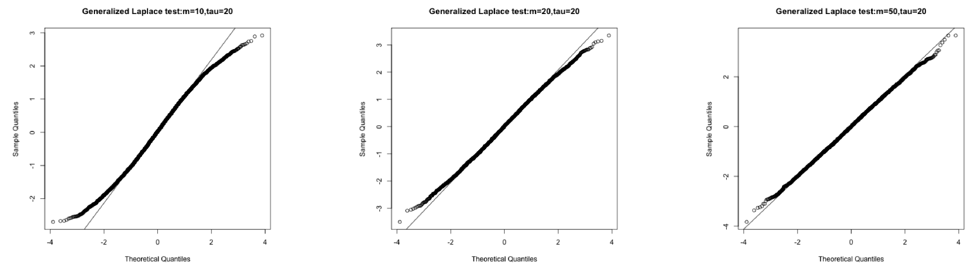

Figure B.60: Normal Q-Q plots of 10,000 simulated values of generalized Laplace test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with $\alpha_i = 1$.

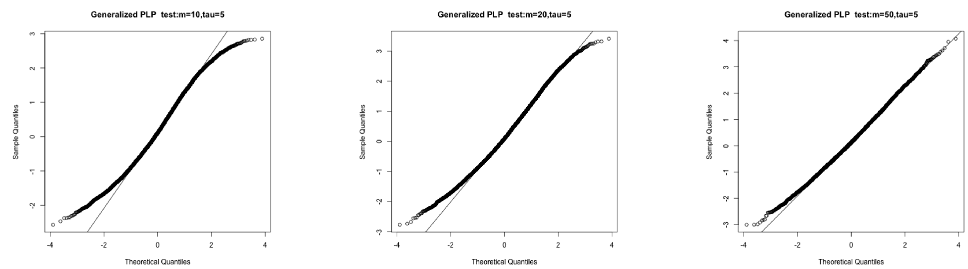

Figure B.61: Normal Q-Q plots of 10,000 simulated values of generalized PLP test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with $\alpha_i = 1$.

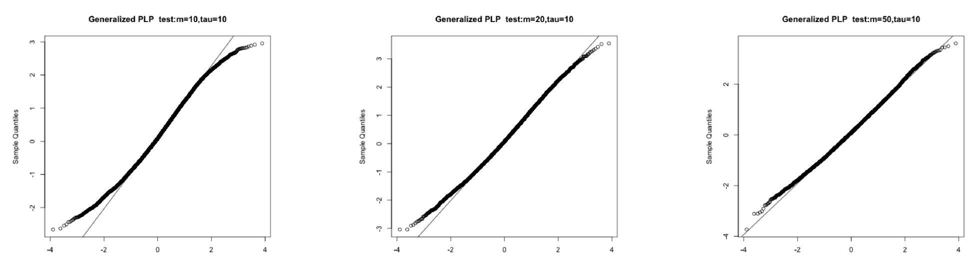

Figure B.62: Normal Q-Q plots of 10,000 simulated values of generalized PLP test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with $\alpha_i = 1$.

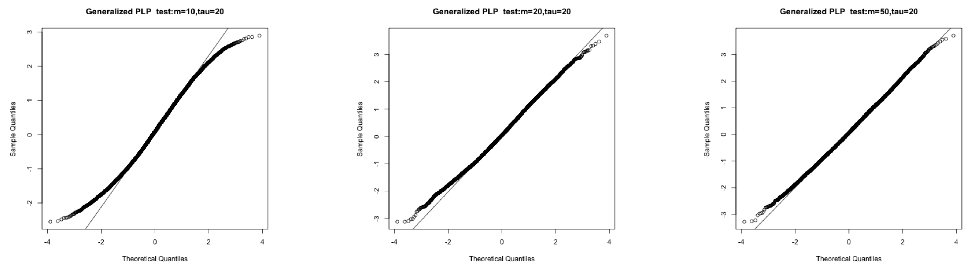

Figure B.63: Normal Q-Q plots of 10,000 simulated values of generalized PLP test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with $\alpha_i = 1$.

	τ	m	$\hat{Q}_{0.975}$	$\hat{Q}_{0.995}$	$\hat{P}(1.96)$	$\hat{P}(2.576)$		$\hat{Q}_{0.975}$	$\hat{Q}_{0.995}$	$\hat{P}(1.96)$	$\hat{P}(2.576)$
	5	10	1.844	2.263	0.038	0.002		1.902	2.496	0.046	0.009
	5	20	1.922	2.501	0.045	0.007		1.981	2.599	0.050	0.010
	5	50	1.931	2.488	0.049	0.009		1.938	2.551	0.050	0.011
	10	10	1.916	2.373	0.043	0.003		1.956	2.606	0.050	0.011
Z_{GL}	10	20	1.930	2.459	0.045	0.006	Z_{LT}	1.962	2.603	0.050	0.010
	10	50	1.959	2.565	0.049	0.009		1.979	2.608	0.051	0.012
	20	10	1.888	2.297	0.040	0.001		1.938	2.567	0.047	0.008
	20	20	1.960	2.475	0.049	0.007		1.949	2.498	0.049	0.009
	20	50	1.897	2.505	0.048	0.009		1.915	2.534	0.049	0.010
	5	10	1.939	2.475	0.049	0.008		2.089	2.668	0.071	0.015
	5	20	1.920	2.497	0.047	0.008		2.121	2.752	0.071	0.017
	5	50	1.924	2.600	0.049	0.010		2.132	2.736	0.072	0.017
	10	10	1.965	2.545	0.048	0.010		2.095	2.721	0.065	0.014
Z_R	10	20	1.974	2.583	0.051	0.010	Z_{LR}	2.062	2.694	0.063	0.015
	10	50	2.010	2.715	0.055	0.013		2.039	2.725	0.065	0.016
	20	10	1.962	2.582	0.047	0.009		1.988	2.596	0.052	0.011
	20	20	1.984	2.573	0.051	0.009		2.030	2.589	0.057	0.011
	20	50	1.980	2.597	0.047	0.010		1.991	2.550	0.056	0.011
	5	10	1.884	2.410	0.044	0.005		1.806	2.200	0.047	0.010
	5	20	1.915	2.463	0.044	0.008		1.860	2.437	0.048	0.009
	5	50	1.930	2.493	0.045	0.008		1.906	2.508	0.047	0.011
	10	10	1.979	2.588	0.053	0.010		1.851	2.399	0.051	0.012
Z_{ALR}	10	20	1.957	2.555	0.048	0.010		1.905	2.524	0.049	0.012
	10	50	1.942	2.581	0.052	0.011	Z_{GPLP}	1.919	2.570	0.050	0.012
	20	10	1.940	2.530	0.046	0.009		1.866	2.439	0.047	0.010
	20	20	1.975	2.525	0.050	0.009		1.910	2.503	0.049	0.009
	20	50	1.941	2.484	0.050	0.009		1.880	2.444	0.047	0.009
	5	10	2.133	2.525	0.050	0.004					
	5	20	2.183	2.798	0.052	0.010					
	5	50	2.135	2.775	0.053	0.011					
	10	10	2.041	2.473	0.044	0.003					
Z_{PLP}	10	20	2.127	2.713	0.049	0.008					
	10	50	2.060	2.797	0.050	0.011					
	20	10	2.026	2.462	0.043	0.003					
	20	20	2.066	2.621	0.047	0.008					
	20	50	2.006	2.606	0.045	0.009					

Table B.1: The empirical pth quantiles and p values for case (a) with identical processes

	τ	m	$\hat{Q}_{0.975}$	$\hat{Q}_{0.995}$	$\hat{P}(1.96)$	$\hat{P}(2.576)$		$\hat{Q}_{0.975}$	$\hat{Q}_{0.995}$	$\hat{P}(1.96)$	$\hat{P}(2.576)$
Z_{GL}	5	10	1.636	2.040	0.048	0.002	Z_{LT}	1.844	2.480	0.095	0.028
	5	20	1.571	2.095	0.064	0.011		1.758	2.410	0.110	0.035
	5	50	1.236	1.845	0.097	0.021		1.387	2.092	0.152	0.051
	10	10	1.734	2.173	0.047	0.003		1.928	2.631	0.091	0.026
	10	20	1.627	2.243	0.059	0.009		1.829	2.604	0.104	0.034
	10	50	1.469	2.071	0.076	0.016		1.641	2.310	0.129	0.043
	20	10	1.779	2.296	0.046	0.002		2.047	2.779	0.091	0.028
	20	20	1.712	2.202	0.051	0.007		1.948	2.672	0.092	0.029
	20	50	1.590	2.253	0.066	0.013		1.821	2.539	0.112	0.035
Z_R	5	10	1.959	2.563	0.046	0.008	Z_{LR}	2.063	2.664	0.070	0.018
	5	20	1.939	2.524	0.051	0.009		2.112	2.777	0.066	0.016
	5	50	1.948	2.553	0.050	0.010		2.154	2.818	0.070	0.016
	10	10	1.930	2.502	0.049	0.008		2.078	2.686	0.061	0.015
	10	20	1.966	2.575	0.050	0.010		2.092	2.730	0.064	0.016
	10	50	1.904	2.481	0.047	0.008		2.086	2.698	0.065	0.014
	20	10	1.960	2.579	0.049	0.010		2.015	2.607	0.055	0.013
	20	20	1.964	2.516	0.050	0.010		2.013	2.712	0.056	0.013
	20	50	1.947	2.483	0.051	0.010		2.064	2.700	0.058	0.012
Z_{ALR}	5	10	1.867	2.413	0.044	0.007	Z_{GPLP}	1.753	2.248	0.043	0.001
	5	20	1.904	2.513	0.041	0.007		1.394	2.130	0.066	0.006
	5	50	1.949	2.545	0.045	0.008		0.865	1.472	0.169	0.040
	10	10	1.973	2.545	0.051	0.009		1.699	2.248	0.046	0.003
	10	20	1.981	2.592	0.051	0.011		1.450	2.110	0.071	0.009
	10	50	1.974	2.565	0.050	0.009		0.990	1.686	0.154	0.036
	20	10	1.957	2.537	0.049	0.010		1.713	2.250	0.044	0.002
	20	20	1.964	2.644	0.049	0.011		1.479	2.024	0.064	0.008
	20	50	2.005	2.634	0.051	0.010		1.089	1.798	0.139	0.028
Z_{PLP}	5	10	1.565	2.120	0.153	0.067					
	5	20	1.346	1.946	0.191	0.091					
	5	50	0.881	1.458	0.321	0.172					
	10	10	1.641	2.327	0.137	0.056					
	10	20	1.451	2.199	0.183	0.077					
	10	50	1.053	1.714	0.281	0.143					
	20	10	1.743	2.419	0.118	0.044					
	20	20	1.552	2.124	0.150	0.058					
	20	50	1.184	1.922	0.238	0.112					

Table B.2: The empirical pth quantiles and p values for case (b) with identical processes

	τ	m	$\hat{Q}_{0.975}$	$\hat{Q}_{0.995}$	$\hat{P}(1.96)$	$\hat{P}(2.576)$		$\hat{Q}_{0.975}$	$\hat{Q}_{0.995}$	$\hat{P}(1.96)$	$\hat{P}(2.576)$
Z_{GL}	5	10	1.914	2.303	0.039	0.001	Z_{LT}	2.355	3.115	0.101	0.029
	5	20	1.996	2.529	0.049	0.007		2.453	3.144	0.098	0.033
	5	50	2.018	2.578	0.051	0.008		2.434	3.153	0.102	0.033
	10	10	1.875	2.260	0.037	0.001		2.322	3.063	0.091	0.029
	10	20	1.944	2.508	0.047	0.009		2.366	3.063	0.104	0.030
	10	50	1.926	2.571	0.046	0.009		2.304	3.163	0.095	0.028
	20	10	1.914	2.308	0.041	0.002		2.318	3.075	0.099	0.030
	20	20	1.902	2.406	0.043	0.006		2.302	2.953	0.093	0.027
	20	50	1.945	2.505	0.049	0.009		2.332	3.052	0.098	0.031
Z_R	5	10	1.575	2.158	0.063	0.012	Z_{LR}	2.485	3.079	0.091	0.022
	5	20	1.427	2.026	0.087	0.019		2.727	3.275	0.117	0.035
	5	50	1.107	1.665	0.133	0.040		3.092	3.691	0.182	0.066
	10	10	1.592	2.273	0.060	0.013		2.375	2.986	0.074	0.019
	10	20	1.561	2.161	0.070	0.017		2.526	3.101	0.084	0.023
	10	50	1.293	1.912	0.097	0.027		2.731	3.311	0.113	0.036
	20	10	1.756	2.358	0.053	0.011		2.214	2.822	0.061	0.014
	20	20	1.649	2.244	0.056	0.013		2.243	2.761	0.061	0.013
	20	50	1.479	2.059	0.073	0.018		2.480	3.038	0.081	0.020
Z_{ALR}	5	10	2.274	2.799	0.059	0.012	Z_{GPLP}	2.165	2.541	0.052	0.004
	5	20	2.472	3.008	0.086	0.020		2.329	2.858	0.061	0.012
	5	50	2.840	3.421	0.140	0.044		2.253	2.856	0.059	0.011
	10	10	2.286	2.854	0.062	0.014		2.125	2.515	0.049	0.004
	10	20	2.413	2.963	0.070	0.016		2.186	2.717	0.055	0.010
	10	50	2.613	3.168	0.096	0.028		2.165	2.797	0.050	0.012
	20	10	2.169	2.767	0.056	0.011		2.087	2.506	0.046	0.003
	20	20	2.197	2.707	0.056	0.010		2.050	2.672	0.047	0.009
	20	50	2.423	2.968	0.075	0.018		2.094	2.722	0.052	0.010
Z_{PLP}	5	10	2.119	2.674	0.088	0.023					
	5	20	2.254	2.845	0.093	0.027					
	5	50	2.318	2.955	0.096	0.026					
	10	10	2.208	2.787	0.090	0.027					
	10	20	2.259	2.879	0.099	0.030					
	10	50	2.295	2.975	0.091	0.028					
	20	10	2.262	2.852	0.094	0.028					
	20	20	2.218	2.870	0.090	0.027					
	20	50	2.313	2.964	0.095	0.031					

Table B.3: The empirical pth quantiles and p values for case (c) with identical processes

B.2 Normal Approximation with Non-identical processes

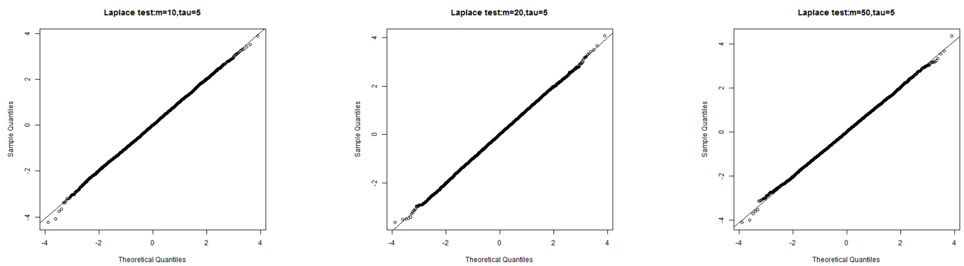

Figure B.64: Normal Q-Q plots of 10,000 simulated values of Laplace test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with non-identical processes.

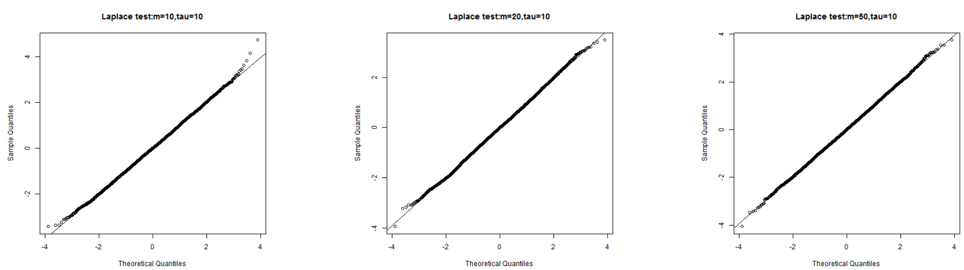

Figure B.65: Normal Q-Q plots of 10,000 simulated values of Laplace test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with non-identical processes.

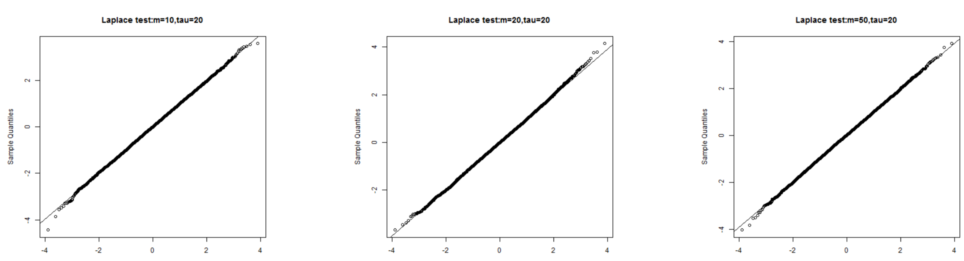

Figure B.66: Normal Q-Q plots of 10,000 simulated values of Laplace test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with non-identical processes.

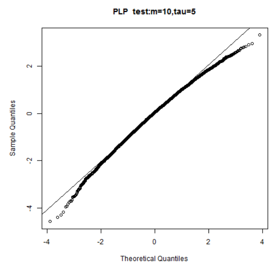

Figure B.67: Normal Q-Q plots of 10,000 simulated values of PLP test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with non-identical processes.

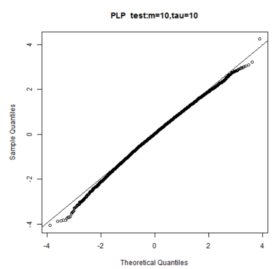

Figure B.68: Normal Q-Q plots of 10,000 simulated values of PLP test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with non-identical processes.

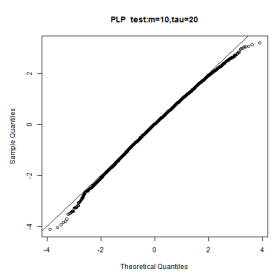

Figure B.69: Normal Q-Q plots of 10,000 simulated values of PLP test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with non-identical processes.

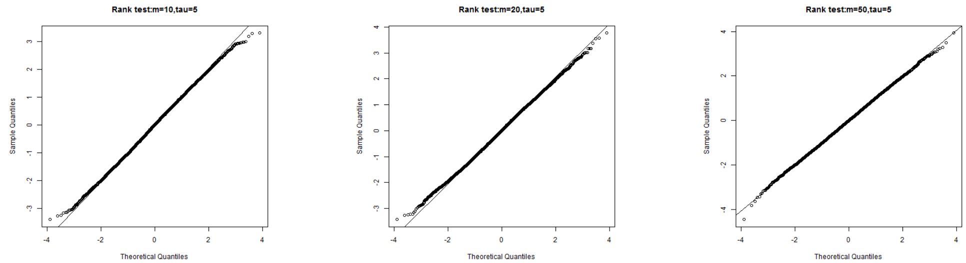

Figure B.70: Normal Q-Q plots of 10,000 simulated values of Rank test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with non-identical processes.

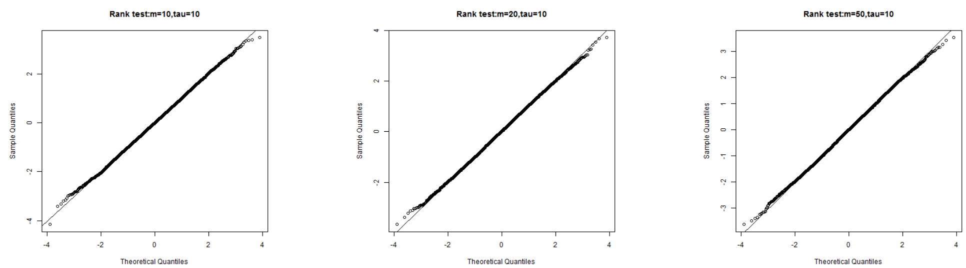

Figure B.71: Normal Q-Q plots of 10,000 simulated values of Rank test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with non-identical processes.

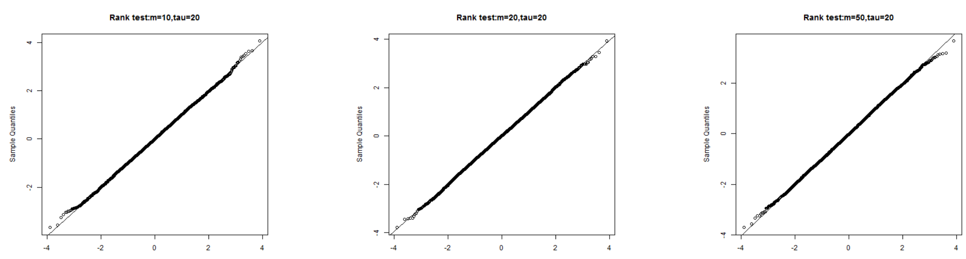

Figure B.72: Normal Q-Q plots of 10,000 simulated values of Rank test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with non-identical processes.

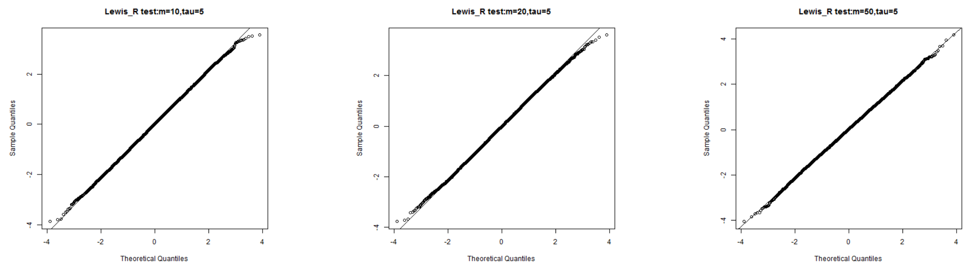

Figure B.73: Normal Q-Q plots of 10,000 simulated values of Lewis-Robinson test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with nonidentical processes.

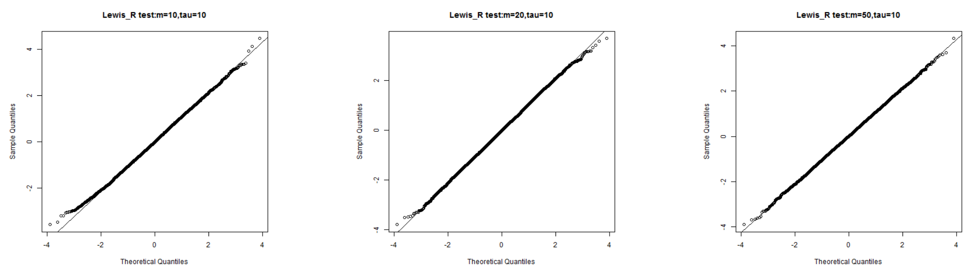

Figure B.74: Normal Q-Q plots of 10,000 simulated values of Lewis-Robinson test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with non-identical processes.

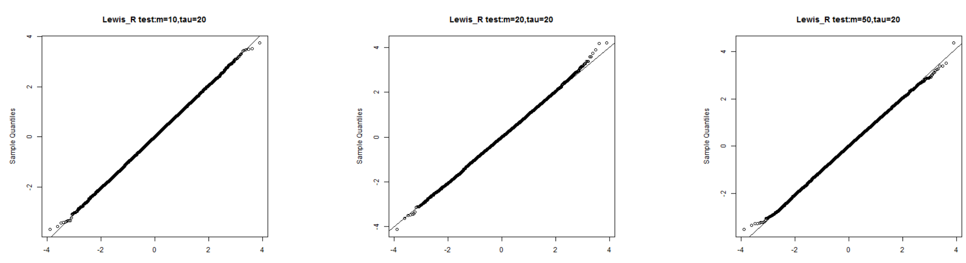

Figure B.75: Normal Q-Q plots of 10,000 simulated values of Lewis-Robinson test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with non-identical processes.

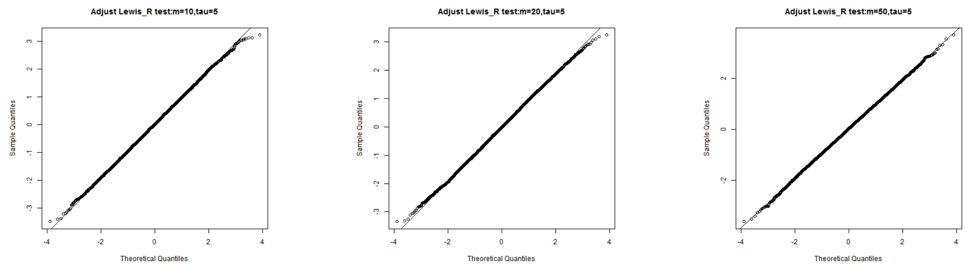

Figure B.76: Normal Q-Q plots of 10,000 simulated values of Lewis -Robinson test with adjustment when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with non-identical processes.

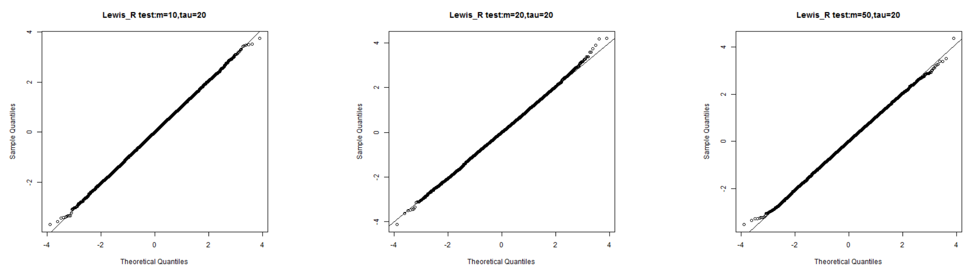

Figure B.77: Normal Q-Q plots of 10,000 simulated values of Lewis -Robinson test with adjustment when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with non-identical processes.

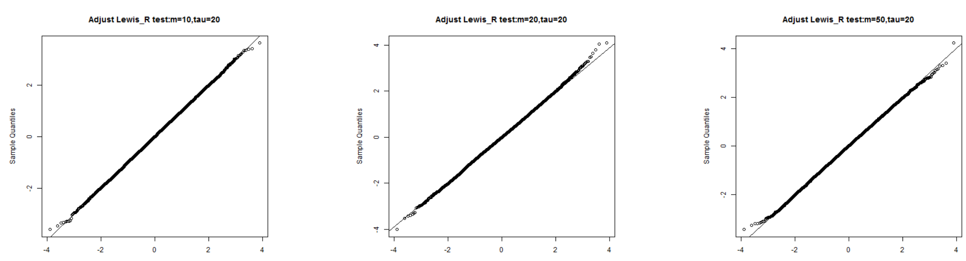

Figure B.78: Normal Q-Q plots of 10,000 simulated values of Lewis -Robinson test with adjustment when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with non-identical processes.

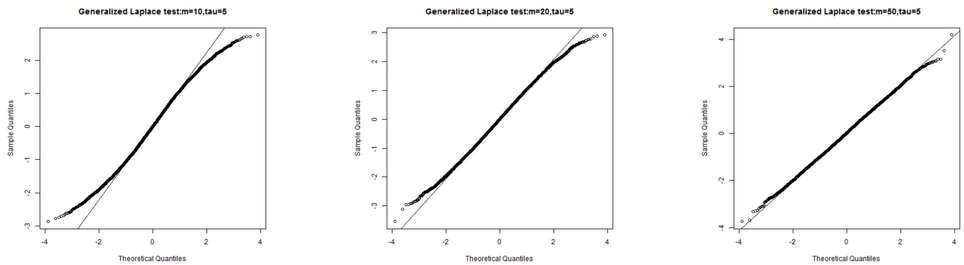

Figure B.79: Normal Q-Q plots of 10,000 simulated values of generalized Laplace test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with non-identical processes.

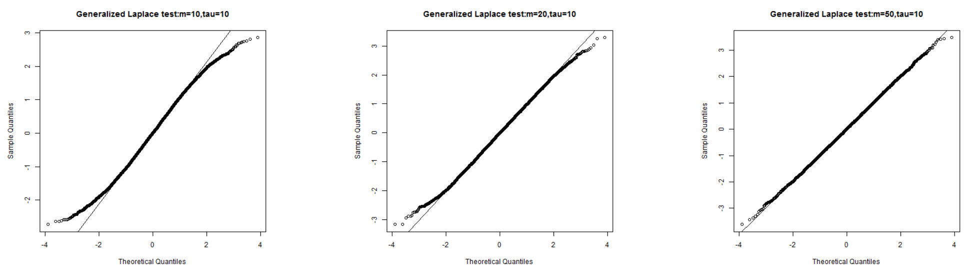

Figure B.80: Normal Q-Q plots of 10,000 simulated values of generalized Laplace test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with non-identical processes.

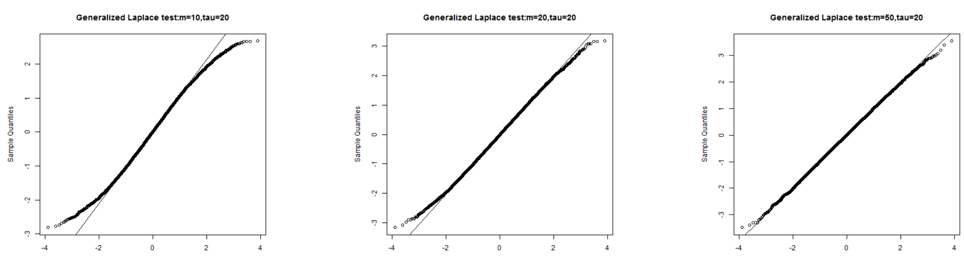

Figure B.81: Normal Q-Q plots of 10,000 simulated values of generalized Laplace test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with non-identical processes.

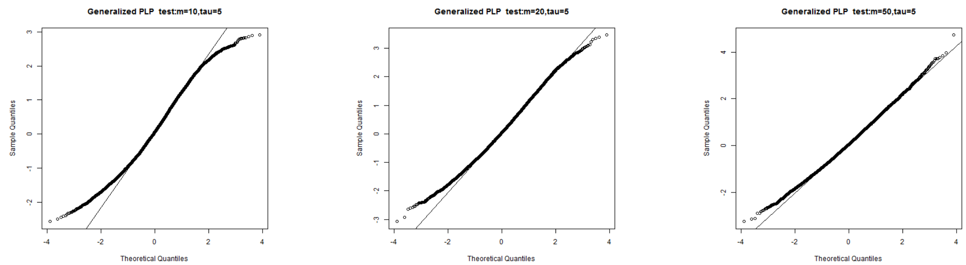

Figure B.82: Normal Q-Q plots of 10,000 simulated values of generalized PLP test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with non-identical processes.

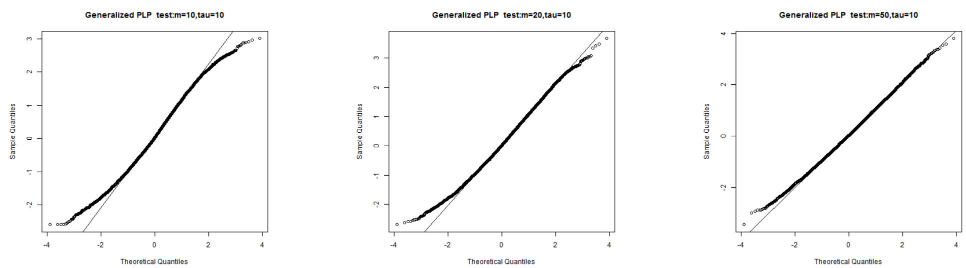

Figure B.83: Normal Q-Q plots of 10,000 simulated values of generalized PLP test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with non-identical processes.

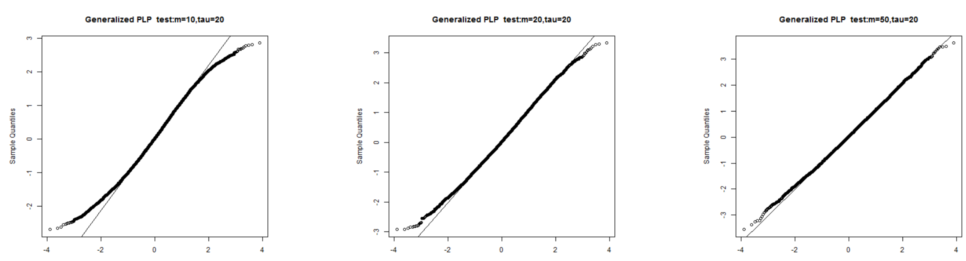

Figure B.84: Normal Q-Q plots of 10,000 simulated values of generalized PLP test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (a) with non-identical processes.

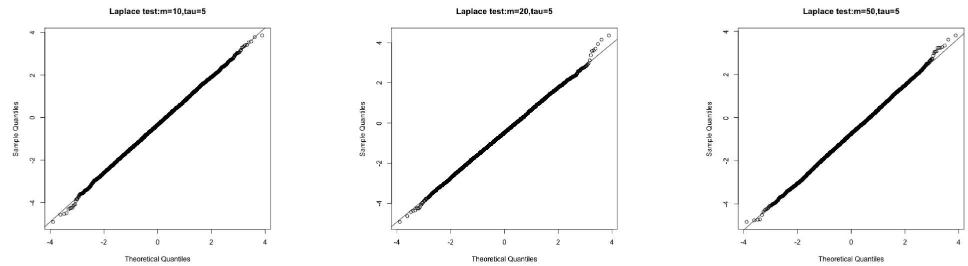

Figure B.85: Normal Q-Q plots of 10,000 simulated values of Laplace test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with non-identical processes.

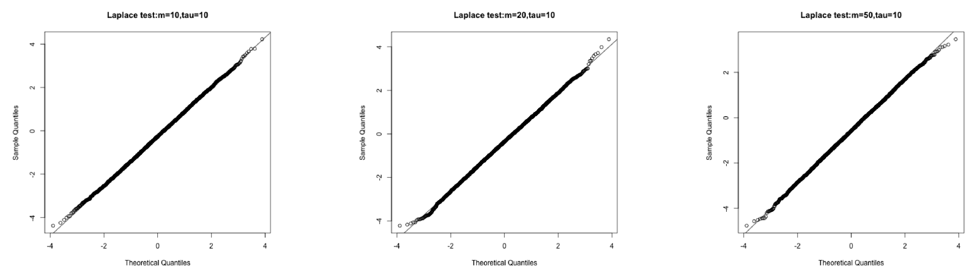

Figure B.86: Normal Q-Q plots of 10,000 simulated values of Laplace test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with non-identical processes.

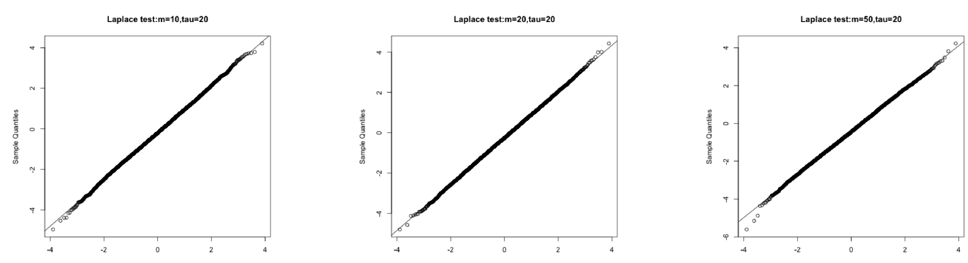

Figure B.87: Normal Q-Q plots of 10,000 simulated values of Laplace test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with non-identical processes.

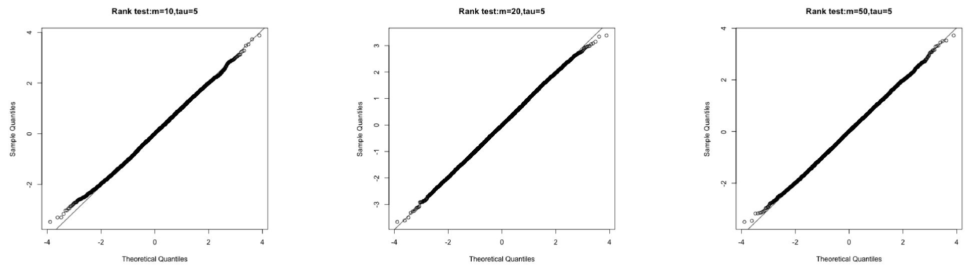

Figure B.88: Normal Q-Q plots of 10,000 simulated values of Rank test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with non-identical processes.

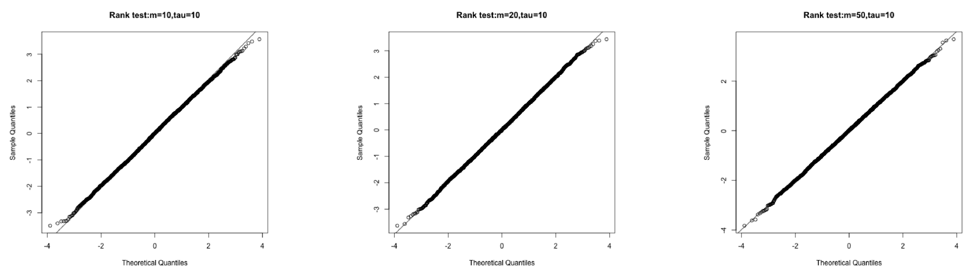

Figure B.89: Normal Q-Q plots of 10,000 simulated values of Rank test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with non-identical processes.

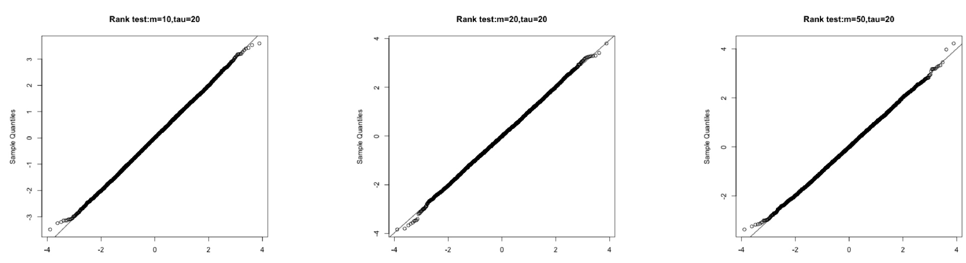

Figure B.90: Normal Q-Q plots of 10,000 simulated values of Rank test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with non-identical processes.

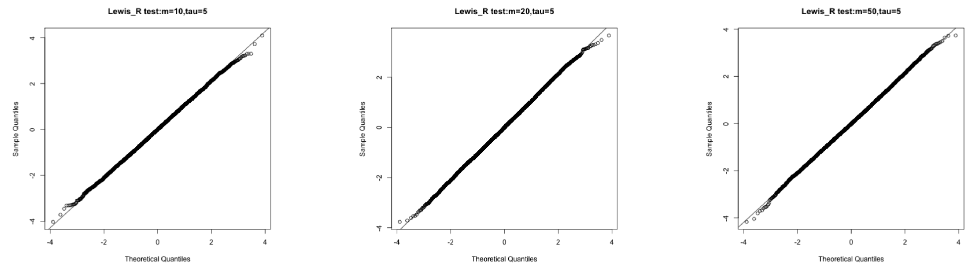

Figure B.91: Normal Q-Q plots of 10,000 simulated values of Lewis-Robinson test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with non-identical processes.

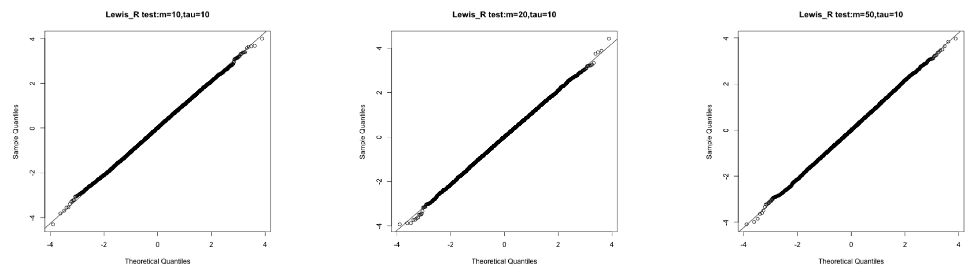

Figure B.92: Normal Q-Q plots of 10,000 simulated values of Lewis-Robinson test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with non-identical processes.

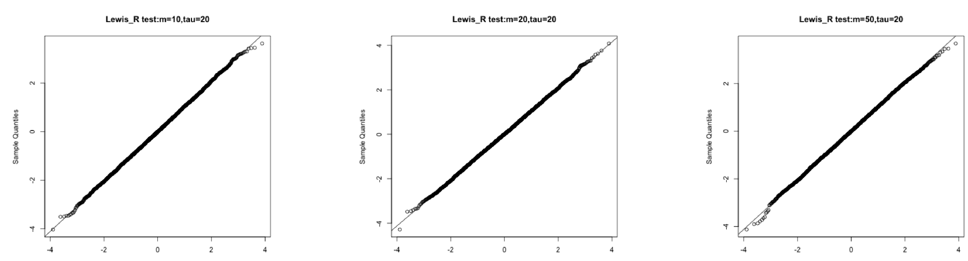

Figure B.93: Normal Q-Q plots of 10,000 simulated values of Lewis-Robinson test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with non-identical processes.

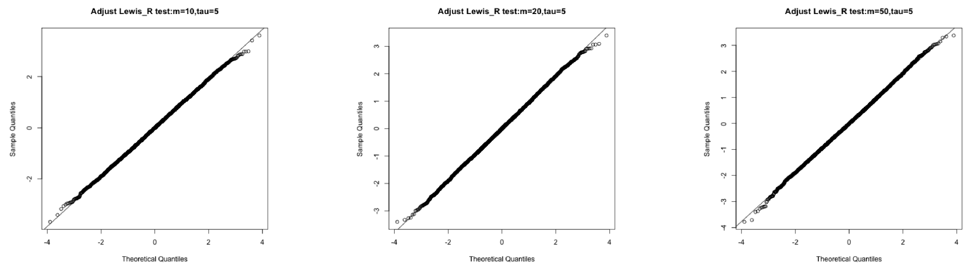

Figure B.94: Normal Q-Q plots of 10,000 simulated values of Lewis -Robinson test with adjustment when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with non-identical processes.

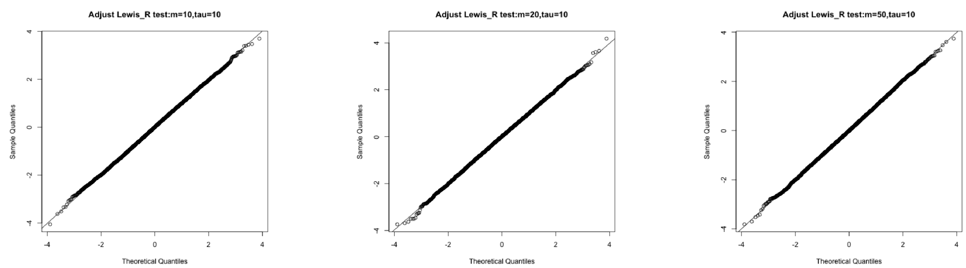

Figure B.95: Normal Q-Q plots of 10,000 simulated values of Lewis -Robinson test with adjustment when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with non-identical processes.

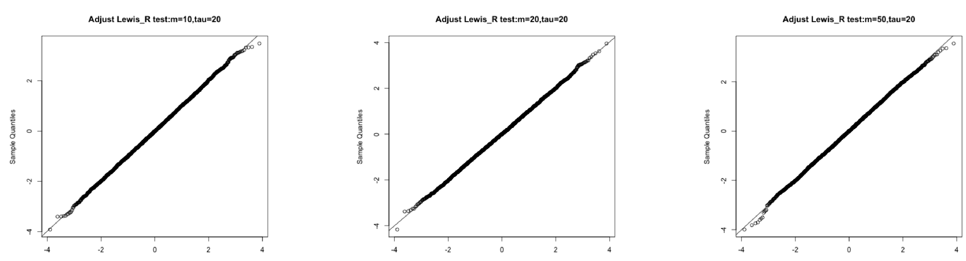

Figure B.96: Normal Q-Q plots of 10,000 simulated values of Lewis -Robinson test with adjustment when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with non-identical processes.

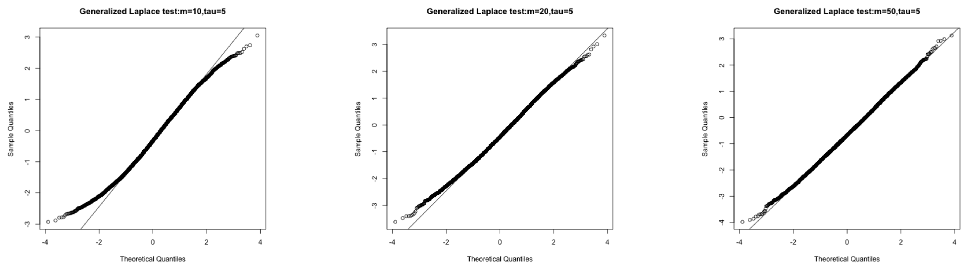

Figure B.97: Normal Q-Q plots of 10,000 simulated values of generalized Laplace test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with non-identical processes.

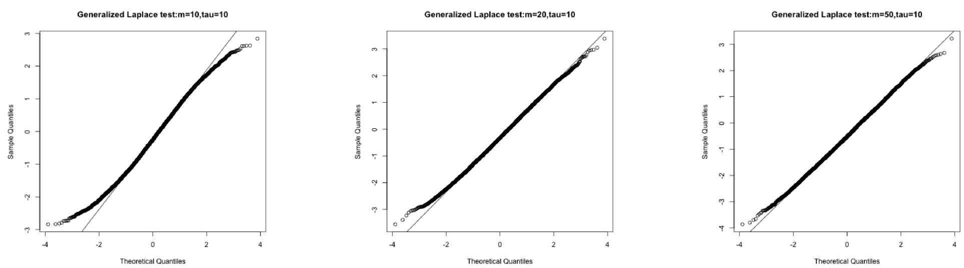

Figure B.98: Normal Q-Q plots of 10,000 simulated values of generalized Laplace test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with non-identical processes.

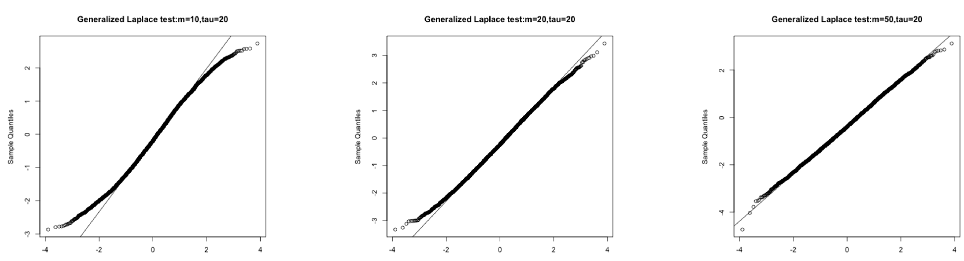

Figure B.99: Normal Q-Q plots of 10,000 simulated values of generalized Laplace test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with non-identical processes.

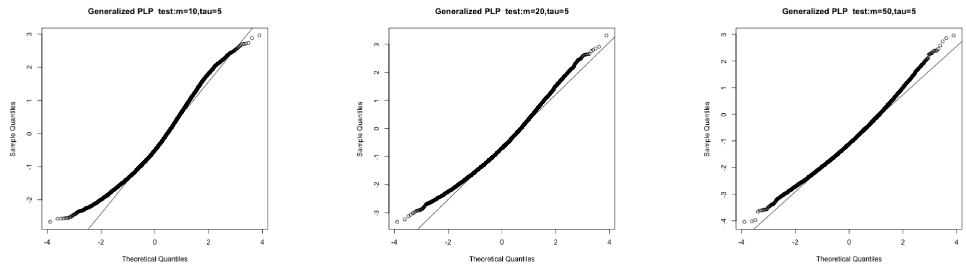

Figure B.100: Normal Q-Q plots of 10,000 simulated values of generalized PLP test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with non-identical processes.

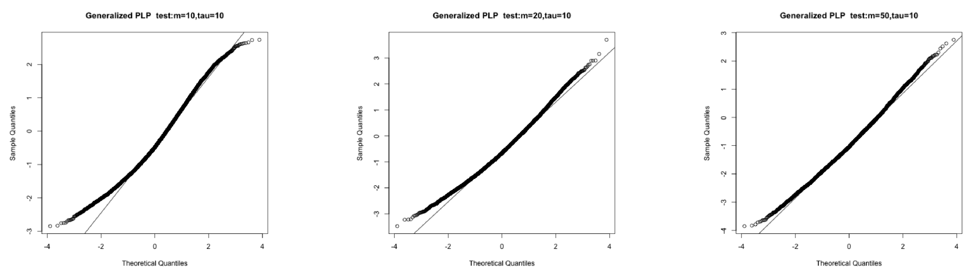

Figure B.101: Normal Q-Q plots of 10,000 simulated values of generalized PLP test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with non-identical processes.

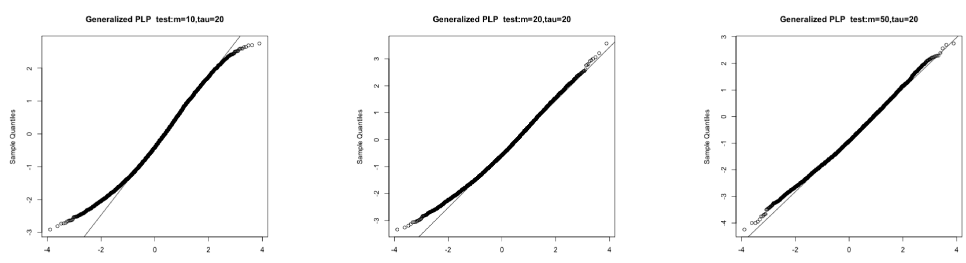

Figure B.102: Normal Q-Q plots of 10,000 simulated values of generalized PLP test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with non-identical processes.

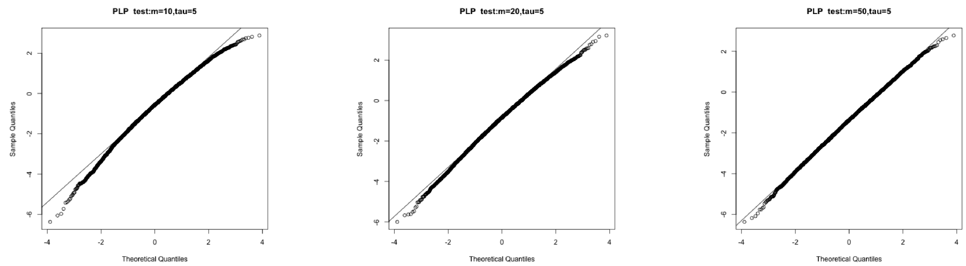

Figure B.103: Normal Q-Q plots of 10,000 simulated values of PLP test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with non-identical processes.

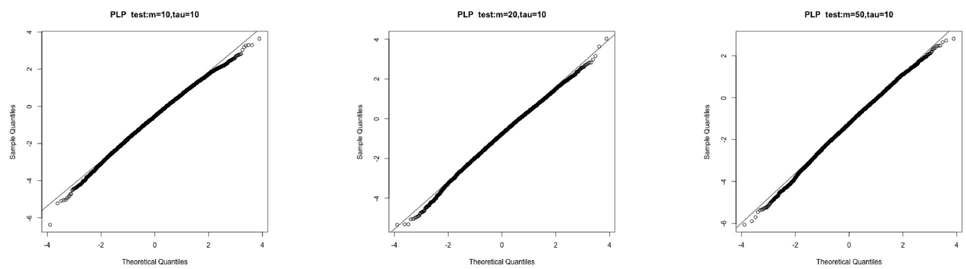

Figure B.104: Normal Q-Q plots of 10,000 simulated values of PLP test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with non-identical processes.

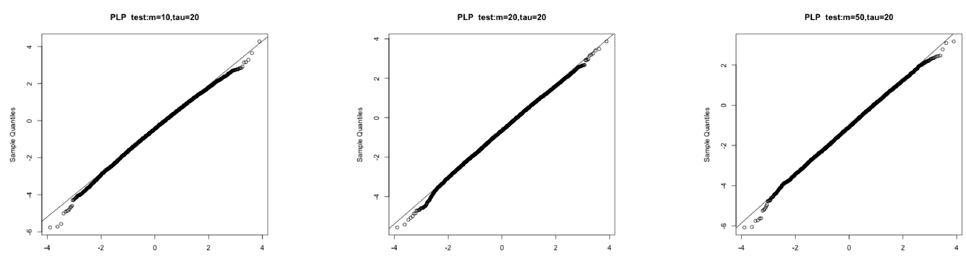

Figure B.105: Normal Q-Q plots of 10,000 simulated values of PLP test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (b) with non-identical processes.

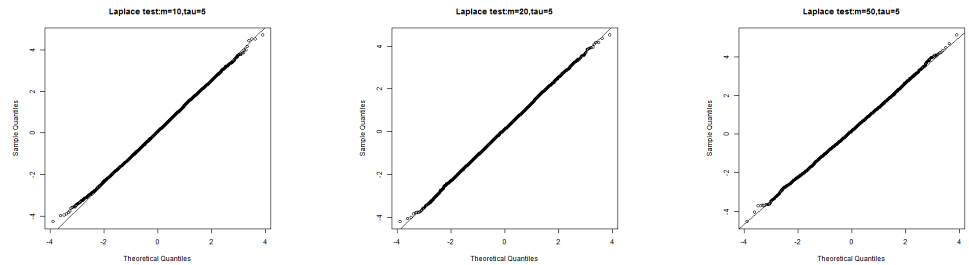

Figure B.106: Normal Q-Q plots of 10,000 simulated values of Laplace test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with non-identical processes.

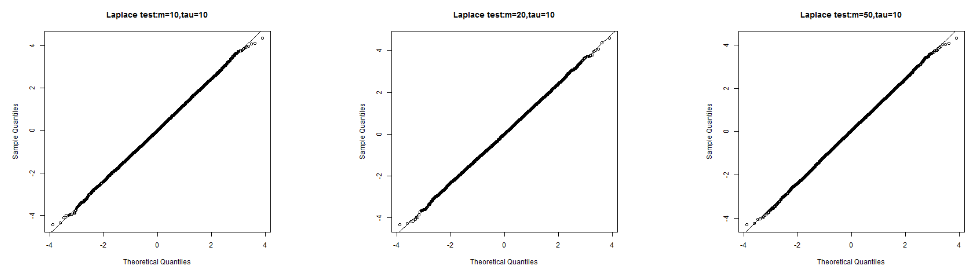

Figure B.107: Normal Q-Q plots of 10,000 simulated values of Laplace test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with non-identical processes.

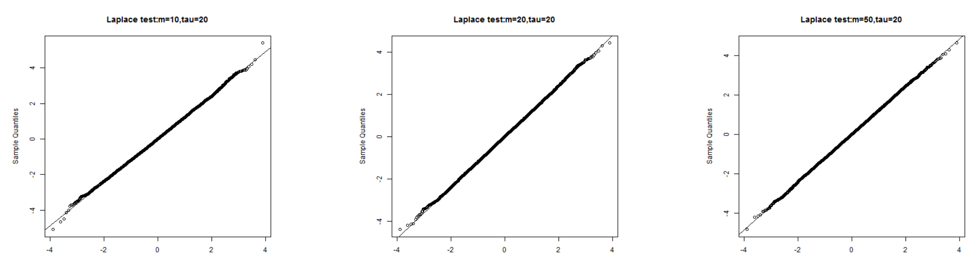

Figure B.108: Normal Q-Q plots of 10,000 simulated values of Laplace test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with non-identical processes.

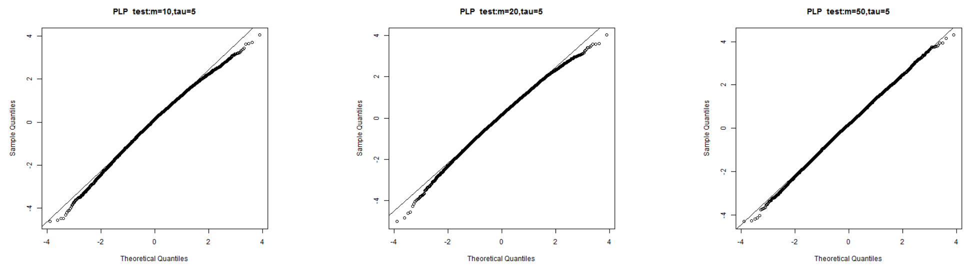

Figure B.109: Normal Q-Q plots of 10,000 simulated values of PLP test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with non-identical processes.

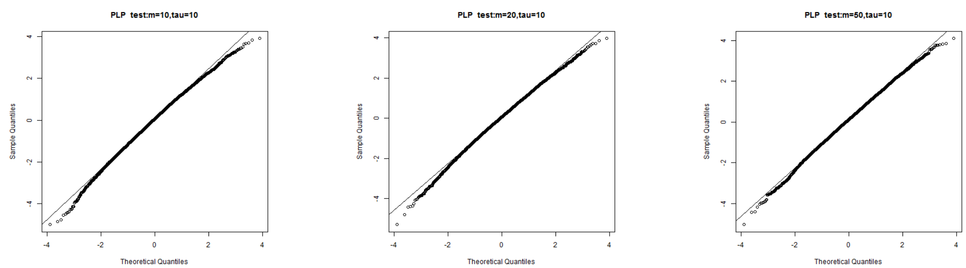

Figure B.110: Normal Q-Q plots of 10,000 simulated values of PLP test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with non-identical processes.

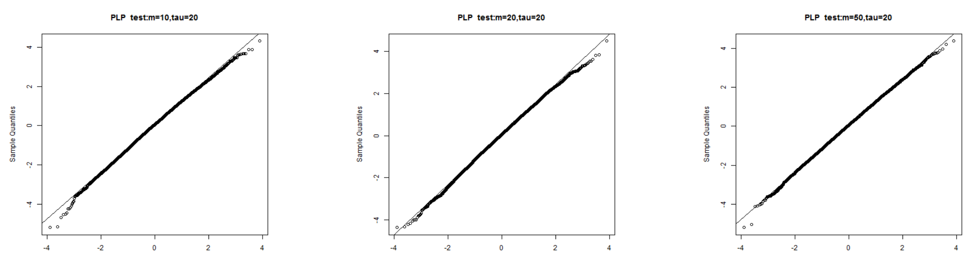

Figure B.111: Normal Q-Q plots of 10,000 simulated values of PLP test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with non-identical processes.

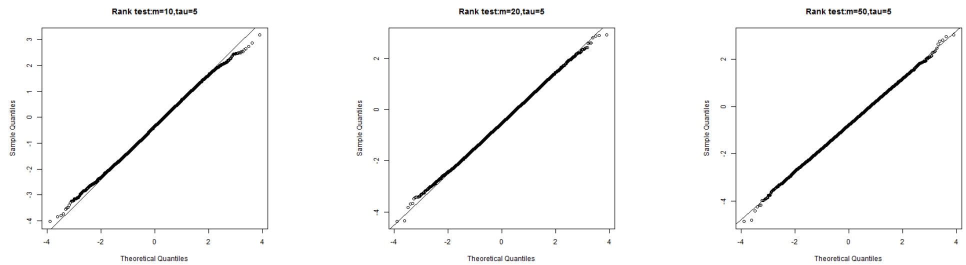

Figure B.112: Normal Q-Q plots of 10,000 simulated values of Rank test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with non-identical processes.

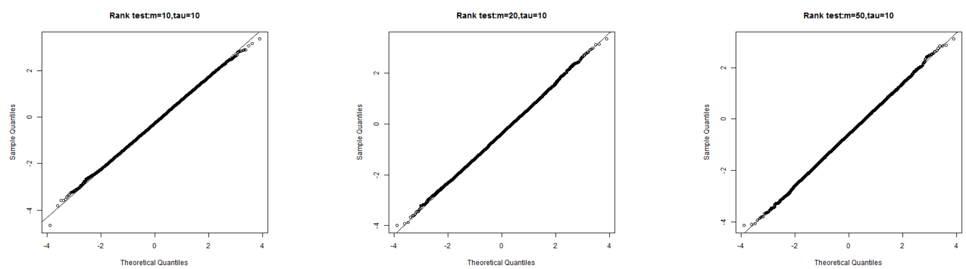

Figure B.113: Normal Q-Q plots of 10,000 simulated values of Rank test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with non-identical processes.

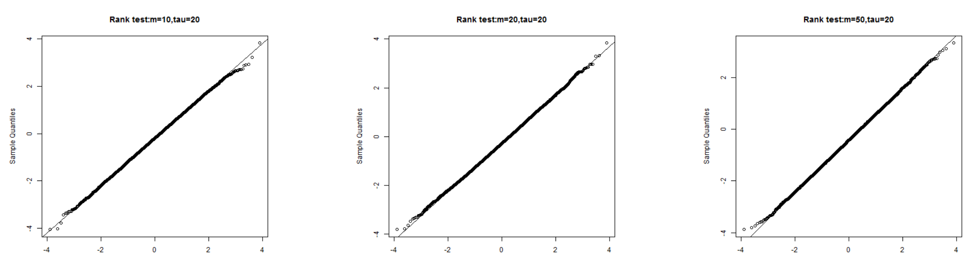

Figure B.114: Normal Q-Q plots of 10,000 simulated values of Rank test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with non-identical processes.

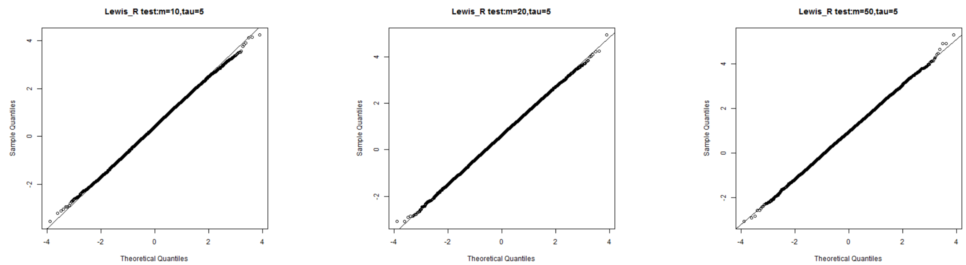

Figure B.115: Normal Q-Q plots of 10,000 simulated values of Lewis-Robinson test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with non-identical processes.

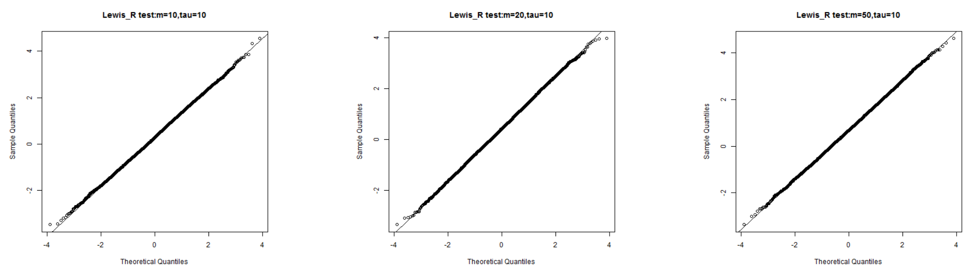

Figure B.116: Normal Q-Q plots of 10,000 simulated values of Lewis-Robinson test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with non-identical processes.

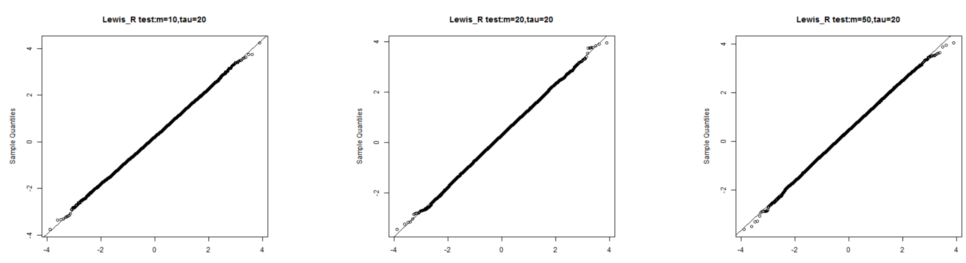

Figure B.117: Normal Q-Q plots of 10,000 simulated values of Lewis-Robinson test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with non-identical processes.

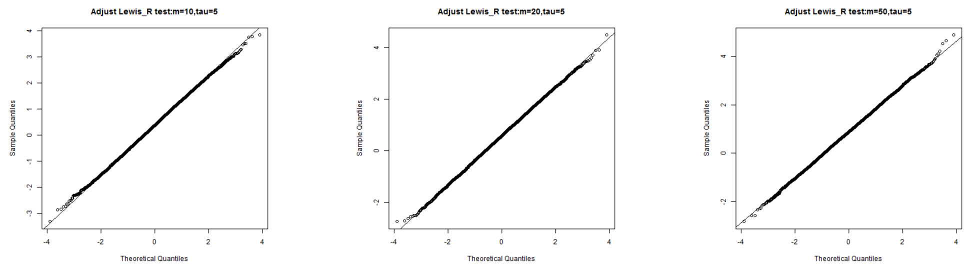

Figure B.118: Normal Q-Q plots of 10,000 simulated values of Lewis -Robinson test with adjustment when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with non-identical processes.

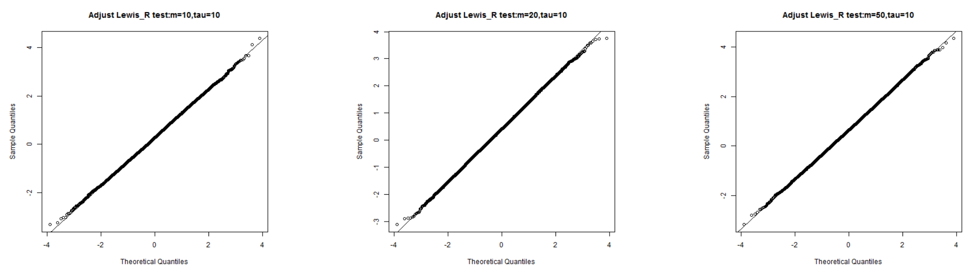

Figure B.119: Normal Q-Q plots of 10,000 simulated values of Lewis -Robinson test with adjustment when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with non-identical processes.

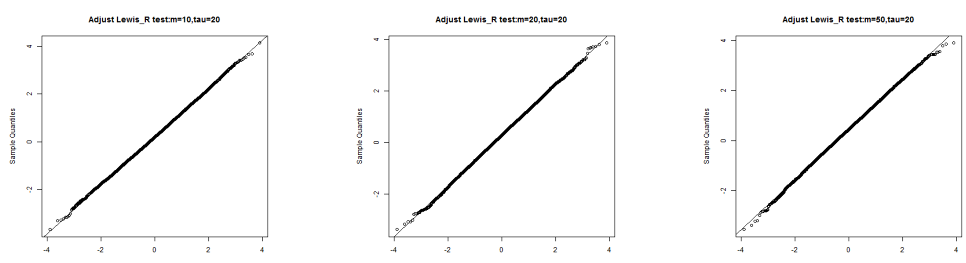

Figure B.120: Normal Q-Q plots of 10,000 simulated values of Lewis -Robinson test with adjustment when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with non-identical processes.

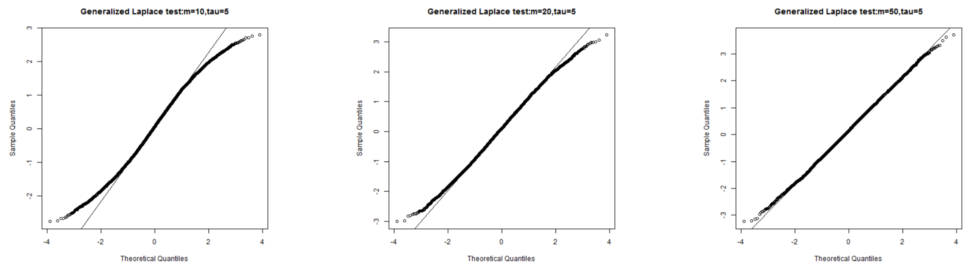

Figure B.121: Normal Q-Q plots of 10,000 simulated values of generalized Laplace test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with non-identical processes.

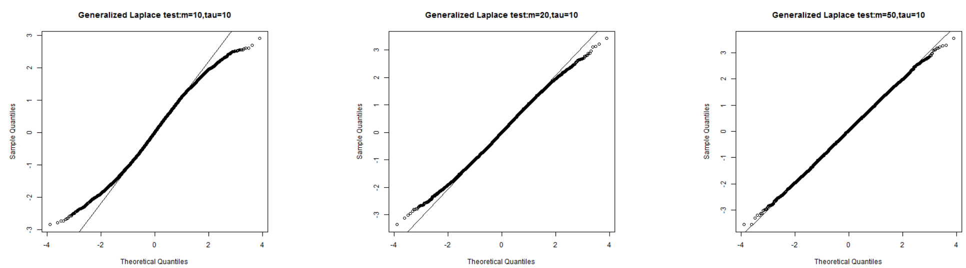

Figure B.122: Normal Q-Q plots of 10,000 simulated values of generalized Laplace test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with non-identical processes.

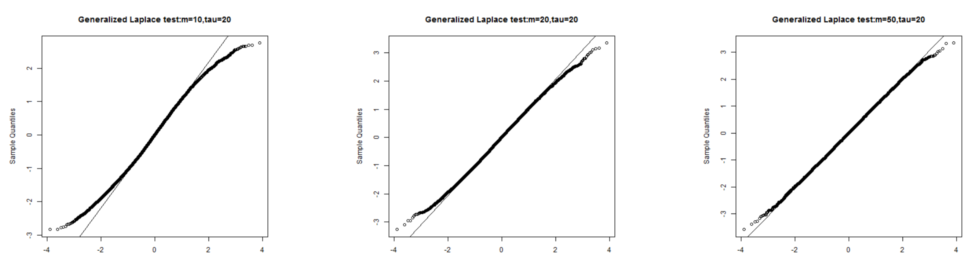

Figure B.123: Normal Q-Q plots of 10,000 simulated values of generalized Laplace test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with non-identical processes.

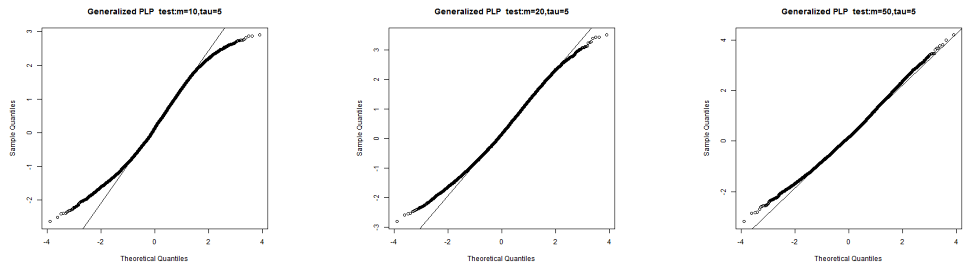

Figure B.124: Normal Q-Q plots of 10,000 simulated values of generalized PLP test when $\tau = 5$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with non-identical processes.

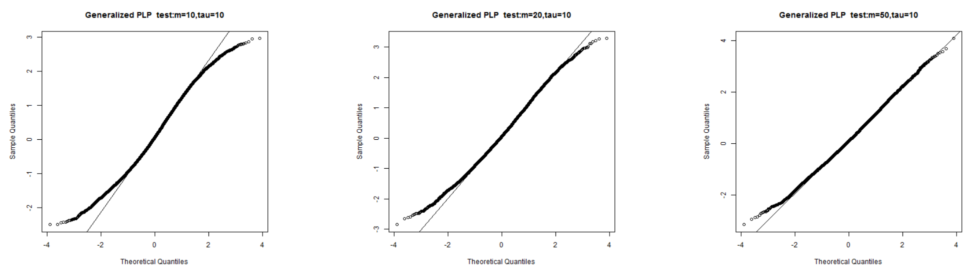

Figure B.125: Normal Q-Q plots of 10,000 simulated values of generalized PLP test when $\tau = 10$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with non-identical processes.

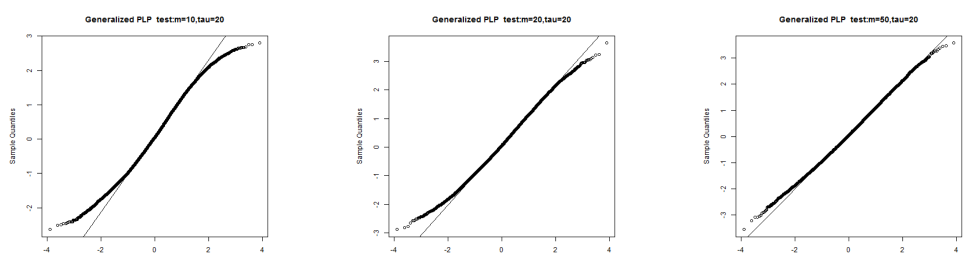

Figure B.126: Normal Q-Q plots of 10,000 simulated values of generalized PLP test when $\tau = 20$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$: Case (c) with non-identical processes.

	τ	m	$\hat{Q}_{0.975}$	$\hat{Q}_{0.995}$	$\hat{P}(1.96)$	$\hat{P}(2.576)$		$\hat{Q}_{0.975}$	$\hat{Q}_{0.995}$	$\hat{P}(1.96)$	$\hat{P}(2.576)$
	5	10	1.907	2.293	0.039	0.002		1.969	2.593	0.049	0.010
	5	20	1.922	2.399	0.046	0.005		1.950	2.489	0.050	0.010
	5	50	1.983	2.632	0.051	0.011		2.007	2.649	0.054	0.010
	10	10	1.914	2.308	0.041	0.002		1.978	2.613	0.052	0.010
Z_{GL}	10	20	1.921	2.419	0.047	0.004	Z_{LT}	1.930	2.572	0.050	0.009
	10	50	1.971	2.583	0.050	0.010		1.978	2.613	0.051	0.010
	20	10	1.881	2.308	0.042	0.002		1.931	2.531	0.046	0.009
	20	20	1.926	2.422	0.047	0.006		1.953	2.635	0.051	0.010
	20	50	1.908	2.497	0.048	0.008		1.921	2.551	0.050	0.010
	5	10	1.925	2.491	0.049	0.008		2.121	2.734	0.066	0.016
	5	20	1.921	2.539	0.047	0.007		2.036	2.670	0.064	0.013
	5	50	1.931	2.523	0.051	0.010		2.116	2.740	0.068	0.017
	10	10	1.992	2.576	0.053	0.009		2.071	2.727	0.065	0.013
Z_R	10	20	1.963	2.536	0.048	0.008	Z_{LR}	2.054	2.648	0.060	0.014
	10	50	1.942	2.454	0.048	0.007		2.081	2.695	0.064	0.013
	20	10	1.941	2.496	0.049	0.009		2.007	2.622	0.055	0.012
	20	20	1.985	2.567	0.052	0.011		1.992	2.686	0.057	0.012
	20	50	1.928	2.523	0.048	0.009		2.002	2.615	0.055	0.012
	5	10	1.931	2.437	0.043	0.007		2.148	2.508	0.053	0.003
	5	20	1.833	2.391	0.040	0.006		2.197	2.693	0.053	0.008
	5	50	1.903	2.476	0.043	0.008		2.164	2.817	0.053	0.012
	10	10	1.958	2.575	0.048	0.009		2.059	2.468	0.046	0.003
Z_{ALR}	10	20	1.919	2.489	0.048	0.009	Z_{GPLP}	2.104	2.626	0.046	0.007
	10	50	1.963	2.557	0.050	0.009		2.048	2.708	0.049	0.010
	20	10	1.951	2.551	0.048	0.010		2.016	2.372	0.042	0.002
	20	20	1.934	2.595	0.050	0.010		2.076	2.619	0.048	0.008
	20	50	1.941	2.526	0.049	0.010		2.041	2.618	0.048	0.009
	5	10	1.839	2.328	0.050	0.010					
	5	20	1.840	2.347	0.051	0.011					
	5	50	1.923	2.531	0.051	0.011					
	10	10	1.881	2.401	0.052	0.011					
Z_{PLP}	10	20	1.885	2.403	0.050	0.008					
	10	50	1.923	2.443	0.052	0.009					
	20	10	1.895	2.409	0.051	0.009					
	20	20	1.925	2.525	0.050	0.012					
	20	50	1.911	2.548	0.048	0.011					

Table B.4: The empirical pth quantiles and p values for case (a) with non-identical processes

	τ	m	$\hat{Q}_{0.975}$	$\hat{Q}_{0.995}$	$\hat{P}(1.96)$	$\hat{P}(2.576)$		$\hat{Q}_{0.975}$	$\hat{Q}_{0.995}$	$\hat{P}(1.96)$	$\hat{P}(2.576)$
	5	10	1.646	2.150	0.049	0.003		1.858	2.512	0.095	0.028
	5	20	1.548	2.067	0.061	0.009		1.717	2.321	0.108	0.034
	5	50	1.300	1.868	0.102	0.027		1.456	2.142	0.150	0.058
	10	10	1.695	2.166	0.045	0.002		1.942	2.602	0.094	0.027
Z_{GL}	10	20	1.627	2.133	0.058	0.009	Z_{LT}	1.834	2.536	0.101	0.030
	10	50	1.442	2.043	0.076	0.018		1.639	2.301	0.125	0.044
	20	10	1.758	2.208	0.041	0.002		2.055	2.751	0.091	0.029
	20	20	1.765	2.249	0.055	0.008		1.986	2.694	0.096	0.030
	20	50	1.548	2.103	0.059	0.013		1.785	2.408	0.105	0.033
	5	10	1.963	2.517	0.049	0.007		2.080	2.674	0.064	0.013
	5	20	1.921	2.494	0.048	0.009		2.076	2.676	0.064	0.015
	5	50	1.950	2.542	0.049	0.008		2.077	2.781	0.065	0.016
	10	10	1.928	2.533	0.047	0.009		2.040	2.627	0.063	0.013
Z_R	10	20	1.958	2.577	0.047	0.010	Z_{LR}	2.036	2.685	0.063	0.015
	10	50	1.962	2.606	0.051	0.010		2.115	2.744	0.068	0.015
	20	10	1.951	2.625	0.051	0.011		2.026	2.618	0.057	0.012
	20	20	1.957	2.613	0.052	0.011		2.012	2.676	0.058	0.013
	20	50	1.969	2.555	0.050	0.008		2.007	2.548	0.058	0.010
	5	10	1.868	2.422	0.040	0.006		1.767	2.254	0.041	0.001
	5	20	1.864	2.424	0.040	0.006		1.465	2.110	0.066	0.007
	5	50	1.872	2.524	0.039	0.008		0.948	1.706	0.162	0.036
	10	10	1.920	2.481	0.048	0.008		1.732	2.220	0.043	0.002
Z_{ALR}	10	20	1.924	2.528	0.049	0.010	Z_{GPLP}	1.440	2.120	0.068	0.009
	10	50	2.001	2.563	0.053	0.010		0.992	1.621	0.148	0.037
	20	10	1.962	2.548	0.050	0.010		1.694	2.238	0.045	0.002
	20	20	1.968	2.595	0.052	0.011		1.521	2.114	0.063	0.008
	20	50	1.951	2.476	0.051	0.008		1.123	1.765	0.125	0.029
	5	10	1.593	2.182	0.151	0.064					
	5	20	1.354	1.966	0.192	0.090					
	5	50	0.974	1.652	0.325	0.170					
	10	10	1.673	2.194	0.139	0.053					
Z_{PLP}	10	20	1.437	2.107	0.171	0.070					
	10	50	1.075	1.658	0.275	0.138					
	20	10	1.740	2.361	0.124	0.046					
	20	20	1.566	2.237	0.146	0.055					
	20	50	1.186	1.913	0.237	0.100					

Table B.5: The empirical pth quantiles and p values for case (b) with non-identical processes

	τ	m	$\hat{Q}_{0.975}$	$\hat{Q}_{0.995}$	$\hat{P}(1.96)$	$\hat{P}(2.576)$		$\hat{Q}_{0.975}$	$\hat{Q}_{0.995}$	$\hat{P}(1.96)$	$\hat{P}(2.576)$
	5	10	1.949	2.314	0.040	0.002		2.476	3.224	0.110	0.035
	5	20	2.013	2.448	0.046	0.005		2.491	3.248	0.110	0.034
	5	50	2.092	2.715	0.049	0.009		2.629	3.380	0.112	0.037
	10	10	1.917	2.302	0.038	0.001		2.382	3.151	0.103	0.032
Z_{GL}	10	20	1.918	2.376	0.042	0.005	Z_{LT}	2.346	3.096	0.100	0.030
	10	50	1.945	2.561	0.048	0.009		2.371	3.126	0.105	0.031
	20	10	1.910	2.290	0.041	0.003		2.350	3.240	0.104	0.032
	20	20	1.913	2.416	0.044	0.005		2.360	3.139	0.099	0.032
	20	50	1.988	2.535	0.051	0.009		2.394	3.120	0.108	0.035
	5	10	1.592	2.083	0.061	0.025		2.440	3.060	0.082	0.020
	5	20	1.406	1.965	0.078	0.016		2.670	3.435	0.107	0.031
	5	50	1.138	1.757	0.121	0.035		2.997	2.910	0.171	0.063
	10	10	1.677	2.259	0.057	0.010		2.354	2.836	0.072	0.017
Z_R	10	20	1.553	2.269	0.061	0.013	Z_{LR}	2.432	3.060	0.082	0.020
	10	50	1.338	1.975	0.095	0.025		2.781	3.435	0.118	0.037
	20	10	1.765	2.338	0.053	0.011		2.219	2.910	0.061	0.014
	20	20	1.662	2.317	0.055	0.012		2.295	2.836	0.065	0.014
	20	50	1.551	2.163	0.072	0.018		2.464	3.059	0.082	0.021
	5	10	2.218	3.019	0.055	0.010		2.178	2.513	0.055	0.004
	5	20	2.429	3.269	0.076	0.016		2.286	2.759	0.058	0.011
	5	50	2.740	3.620	0.130	0.037		2.326	2.987	0.061	0.015
	10	10	2.225	2.928	0.057	0.012		2.112	2.509	0.049	0.004
Z_{ALR}	10	20	2.318	3.060	0.066	0.014		2.117	2.593	0.058	0.006
	10	50	2.640	3.435	0.100	0.029	Z_{GPLP}	2.190	2.781	0.061	0.011
	20	10	2.170	2.910	0.054	0.012		2.068	2.435	0.047	0.002
	20	20	2.237	2.836	0.059	0.012		2.112	2.587	0.050	0.006
	20	50	2.408	3.059	0.075	0.018		2.096	2.737	0.051	0.010
	5	10	2.204	2.752	0.096	0.026					
	5	20	2.295	2.842	0.095	0.026					
	5	50	2.413	3.168	0.098	0.029					
	10	10	2.233	2.915	0.096	0.030					
Z_{PLP}	10	20	2.221	2.828	0.095	0.026					
	10	50	2.365	3.024	0.100	0.029					
	20	10	2.282	2.969	0.095	0.030					
	20	20	2.307	2.958	0.103	0.030					
	20	50	2.373	3.058	0.103	0.031					

Table B.6: The empirical pth quantiles and p values for case (c) with non-identical processes

Appendix C

Power Comparison of Trend Tests

C.1 Power of Tests with Identical Processes

	τ	m	$\hat{\Omega}(1)$	$\hat{\Omega}(2)$	$\Omega(1)$	$\Omega(2)$		$\hat{\Omega}(1)$	$\hat{\Omega}(2)$	$\Omega(1)$	$\Omega(2)$
	5	10	0.247	0.072	0.195	0.014		0.290	0.133	0.273	0.118
	5	20	0.464	0.188	0.441	0.164		0.493	0.262	0.504	0.270
	5	50	0.856	0.654	0.849	0.614		0.882	0.680	0.879	0.671
	10	10	0.419	0.127	0.396	0.046		0.523	0.265	0.521	0.278
Z_{GL}	10	20	0.777	0.509	0.770	0.431	Z_{LT}	0.813	0.592	0.814	0.606
	10	50	0.993	0.957	0.993	0.955		0.992	0.970	0.993	0.971
	20	10	0.728	0.408	0.685	0.171		0.829	0.630	0.820	0.625
	20	20	0.946	0.811	0.946	0.756		0.965	0.906	0.965	0.899
	20	50	1.000	1.000	1.000	1.000		1.000	1.000	1.000	1.000
	5	10	0.140	0.230	0.141	0.043		0.168	0.060	0.195	0.070
	5	20	0.275	0.379	0.274	0.099		0.285	0.125	0.340	0.158
	5	50	0.548	0.683	0.553	0.315		0.579	0.350	0.644	0.411
	10	10	0.357	0.474	0.341	0.131		0.362	0.144	0.411	0.184
Z_{R}	10	20	0.610	0.731	0.613	0.352	Z_{LR}	0.671	0.426	0.707	0.470
	10	50	0.947	0.974	0.948	0.842		0.971	0.870	0.974	0.901
	20	10	0.713	0.796	0.698	0.465		0.742	0.517	0.751	0.524
	20	20	0.932	0.963	0.932	0.801		0.935	0.841	0.948	0.845
	20	50	1.000	1.000	1.000	0.998		1.000	0.999	1.000	0.999
	5	10	0.171	0.059	0.145	0.040		0.159	0.039	0.215	0.027
	5	20	0.293	0.131	0.281	0.113		0.301	0.105	0.396	0.161
	5	50	0.587	0.351	0.572	0.320		0.698	0.464	0.752	0.526
	10	10	0.368	0.141	0.372	0.144		0.337	0.118	0.382	0.070
Z_{ALR}	10	20	0.671	0.429	0.671	0.422	Z_{GPLP}	0.582	0.309	0.650	0.368
	10	50	0.970	0.874	0.969	0.874		0.966	0.820	0.972	0.878
	20	10	0.741	0.519	0.733	0.506		0.547	0.258	0.585	0.176
	20	20	0.936	0.841	0.938	0.828		0.855	0.639	0.877	0.660
	20	50	1.000	0.999	1.000	0.999		0.999	0.996	0.999	0.997
	5	10	0.232	0.112	0.182	0.049					
	5	20	0.401	0.177	0.346	0.137					
	5	50	0.773	0.528	0.748	0.494					
	10	10	0.429	0.208	0.384	0.160					
Z_{PLP}	10	20	0.686	0.452	0.665	0.432					
	10	50	0.977	0.906	0.975	0.904					
	20	10	0.737	0.492	0.700	0.425					
	20	20	0.919	0.786	0.911	0.765					
	20	50	1.000	0.999	1.000	0.997					

Table C.1: Probability of rejection $H_0 : \gamma = 0$ under the case (d) with identical processes when R = 2

	τ	m	$\hat{\Omega}(1)$	$\hat{\Omega}(2)$	$\Omega(1)$	$\Omega(2)$		$\hat{\Omega}(1)$	$\hat{\Omega}(2)$	$\Omega(1)$	$\Omega(2)$
	5	10	0.133	0.051	0.107	0.005		0.156	0.050	0.143	0.036
	5	20	0.212	0.058	0.196	0.044		0.227	0.074	0.232	0.074
	5	50	0.430	0.210	0.420	0.184		0.433	0.227	0.425	0.218
	10	10	0.172	0.052	0.149	0.017		0.214	0.081	0.210	0.082
Z_{GL}	10	20	0.355	0.124	0.338	0.096	Z_{LT}	0.387	0.180	0.388	0.186
	10	50	0.734	0.460	0.734	0.454		0.747	0.510	0.753	0.523
	20	10	0.313	0.127	0.275	0.041		0.387	0.185	0.382	0.183
	20	20	0.584	0.291	0.584	0.245		0.655	0.449	0.651	0.412
	20	50	0.955	0.838	0.945	0.825		0.963	0.863	0.957	0.859
	5	10	0.082	0.147	0.082	0.020		0.097	0.037	0.112	0.037
	5	20	0.127	0.201	0.124	0.039		0.133	0.036	0.172	0.052
	5	50	0.244	0.353	0.248	0.089		0.246	0.104	0.313	0.144
	10	10	0.166	0.249	0.162	0.052		0.158	0.062	0.190	0.074
Z_R	10	20	0.247	0.357	0.251	0.108	Z_{LR}	0.280	0.123	0.311	0.139
	10	50	0.546	0.677	0.548	0.311		0.573	0.320	0.609	0.382
	20	10	0.300	0.393	0.292	0.133		0.332	0.153	0.341	0.155
	20	20	0.547	0.674	0.544	0.295		0.553	0.342	0.581	0.348
	20	50	0.896	0.930	0.886	0.740		0.916	0.793	0.920	0.786
	5	10	0.091	0.035	0.079	0.022		0.105	0.040	0.148	0.025
	5	20	0.134	0.044	0.119	0.030		0.144	0.030	0.202	0.052
	5	50	0.254	0.101	0.244	0.088		0.310	0.131	0.371	0.174
	10	10	0.163	0.062	0.166	0.057		0.147	0.051	0.176	0.029
Z_{ALR}	10	20	0.282	0.120	0.281	0.116	Z_{GPLP}	0.245	0.077	0.299	0.105
	10	50	0.574	0.322	0.566	0.323		0.568	0.292	0.605	0.370
	20	10	0.331	0.153	0.320	0.141		0.225	0.090	0.254	0.056
	20	20	0.555	0.341	0.559	0.320		0.423	0.200	0.471	0.213
	20	50	0.917	0.794	0.914	0.763		0.866	0.679	0.877	0.697
	5	10	0.125	0.057	0.095	0.017					
	5	20	0.180	0.054	0.157	0.031					
	5	50	0.348	0.161	0.333	0.145					
	10	10	0.187	0.070	0.152	0.044					
Z_{PLP}	10	20	0.316	0.120	0.286	0.104					
	10	50	0.616	0.356	0.600	0.355					
	20	10	0.307	0.131	0.282	0.097					
	20	20	0.534	0.276	0.512	0.250					
	20	50	0.900	0.763	0.892	0.715					

Table C.2: Probability of rejection H_0 : $\gamma = 0$ under the case (d) with identical processes when R = 1.5

	τ	m	$\hat{\Omega}(1)$	$\hat{\Omega}(2)$	$\Omega(1)$	$\Omega(2)$		$\hat{\Omega}(1)$	$\hat{\Omega}(2)$	$\Omega(1)$	$\Omega(2)$
	5	10	0.336	0.125	0.182	0.018		0.313	0.139	0.271	0.126
	5	20	0.648	0.368	0.429	0.152		0.585	0.349	0.487	0.265
	5	50	0.959	0.870	0.853	0.635		0.945	0.824	0.858	0.688
	10	10	0.547	0.259	0.378	0.066		0.498	0.244	0.493	0.254
Z_{GL}	10	20	0.863	0.689	0.772	0.414	Z_{LT}	0.847	0.619	0.820	0.607
	10	50	0.999	0.987	0.989	0.960		0.997	0.988	0.993	0.976
	20	10	0.778	0.472	0.657	0.157		0.763	0.505	0.780	0.586
	20	20	0.972	0.914	0.954	0.798		0.966	0.901	0.966	0.917
	20	50	1.000	1.000	1.000	1.000		1.000	1.000	1.000	1.000
	5	10	0.160	0.241	0.147	0.048		0.174	0.062	0.205	0.076
	5	20	0.246	0.371	0.245	0.088		0.282	0.125	0.320	0.147
	5	50	0.576	0.683	0.569	0.318		0.603	0.364	0.634	0.436
	10	10	0.342	0.457	0.339	0.132		0.368	0.183	0.406	0.192
Z_R	10	20	0.625	0.728	0.604	0.375	Z_{LR}	0.674	0.442	0.700	0.479
	10	50	0.943	0.969	0.944	0.814		0.955	0.856	0.970	0.896
	20	10	0.656	0.775	0.660	0.413		0.701	0.466	0.723	0.487
	20	20	0.913	0.953	0.922	0.830		0.940	0.854	0.943	0.869
	20	50	1.000	1.000	1.000	0.998		1.000	1.000	1.000	1.000
	5	10	0.180	0.061	0.156	0.036		0.306	0.124	0.218	0.039
	5	20	0.290	0.126	0.253	0.098		0.601	0.356	0.419	0.187
	5	50	0.609	0.366	0.586	0.351		0.950	0.831	0.754	0.536
	10	10	0.380	0.185	0.363	0.165		0.478	0.243	0.376	0.097
Z_{ALR}	10	20	0.682	0.453	0.670	0.434	Z_{GPLP}	0.832	0.584	0.648	0.366
	10	50	0.958	0.873	0.963	0.867		0.999	0.985	0.968	0.870
	20	10	0.707	0.469	0.708	0.455		0.708	0.396	0.570	0.153
	20	20	0.939	0.856	0.940	0.861		0.962	0.864	0.896	0.711
	20	50	1.000	1.000	1.000	1.000		1.000	1.000	0.999	0.995
	5	10	0.326	0.124	0.186	0.053					
	5	20	0.622	0.368	0.370	0.150					
	5	50	0.952	0.853	0.759	0.507					
	10	10	0.511	0.284	0.381	0.151					
Z_{PLP}	10	20	0.829	0.618	0.676	0.424					
	10	50	0.999	0.986	0.975	0.890					
	20	10	0.735	0.468	0.652	0.373					
	20	20	0.960	0.878	0.918	0.797					
	20	50	1.000	1.000	1.000	0.999					

Table C.3: Probability of rejection H_0 : $\gamma = 0$ under the case (e) with identical processes when R =2

	τ	m	$\hat{\Omega}(1)$	$\hat{\Omega}(2)$	$\Omega(1)$	$\Omega(2)$		$\hat{\Omega}(1)$	$\hat{\Omega}(2)$	$\Omega(1)$	$\Omega(2)$
	5	10	0.194	0.089	0.107	0.006		0.152	0.043	0.135	0.035
	5	20	0.336	0.154	0.200	0.046		0.284	0.107	0.224	0.078
	5	50	0.726	0.481	0.422	0.198		0.669	0.387	0.447	0.238
	10	10	0.257	0.106	0.180	0.020		0.241	0.074	0.233	0.083
Z_{GL}	10	20	0.466	0.198	0.315	0.081	Z_{LT}	0.412	0.149	0.360	0.158
	10	50	0.877	0.669	0.714	0.455		0.831	0.614	0.738	0.494
	20	10	0.383	0.120	0.294	0.038		0.346	0.147	0.372	0.197
	20	20	0.672	0.441	0.558	0.269		0.631	0.371	0.629	0.401
	20	50	0.974	0.898	0.946	0.812		0.963	0.865	0.958	0.856
	5	10	0.092	0.138	0.083	0.013		0.097	0.047	0.122	0.040
	5	20	0.129	0.207	0.132	0.036		0.120	0.045	0.153	0.058
	5	50	0.247	0.353	0.259	0.095		0.246	0.089	0.297	0.130
	10	10	0.168	0.258	0.168	0.046		0.157	0.050	0.196	0.060
Z_R	10	20	0.247	0.353	0.240	0.086	Z_{LR}	0.243	0.095	0.289	0.122
	10	50	0.534	0.661	0.534	0.306		0.535	0.331	0.580	0.371
	20	10	0.308	0.404	0.304	0.127		0.323	0.147	0.344	0.152
	20	20	0.517	0.648	0.516	0.305		0.542	0.290	0.568	0.345
	20	50	0.894	0.941	0.897	0.716		0.910	0.752	0.926	0.797
	5	10	0.094	0.044	0.081	0.024		0.196	0.069	0.140	0.022
	5	20	0.122	0.047	0.111	0.034		0.394	0.153	0.199	0.074
	5	50	0.248	0.093	0.246	0.084		0.775	0.582	0.395	0.187
	10	10	0.157	0.049	0.167	0.045		0.266	0.102	0.184	0.030
Z_{ALR}	10	20	0.252	0.097	0.255	0.098	Z_{GPLP}	0.511	0.245	0.300	0.105
	10	50	0.545	0.333	0.548	0.327		0.902	0.712	0.606	0.367
	20	10	0.328	0.148	0.328	0.134		0.369	0.140	0.253	0.044
	20	20	0.544	0.291	0.546	0.321		0.697	0.470	0.494	0.217
	20	50	0.913	0.751	0.921	0.771		0.978	0.906	0.870	0.696
	5	10	0.181	0.059	0.092	0.016					
	5	20	0.354	0.149	0.149	0.046					
	5	50	0.772	0.542	0.359	0.147					
	10	10	0.242	0.078	0.162	0.036					
Z_{PLP}	10	20	0.480	0.193	0.269	0.097					
	10	50	0.889	0.680	0.596	0.347					
	20	10	0.366	0.143	0.278	0.115					
	20	20	0.665	0.445	0.518	0.274					
	20	50	0.973	0.891	0.884	0.716					

Table C.4: Probability of rejection $H_0 : \gamma = 0$ under the case (e) with identical processes when R= 1.5

	τ	m	$\hat{\Omega}(1)$	$\hat{\Omega}(2)$	$\Omega(1)$	$\Omega(2)$		$\hat{\Omega}(1)$	$\hat{\Omega}(2)$	$\Omega(1)$	$\Omega(2)$
	5	10	0.215	0.070	0.192	0.019		0.159	0.045	0.263	0.116
	5	20	0.412	0.161	0.432	0.140		0.296	0.118	0.481	0.258
	5	50	0.837	0.638	0.852	0.638		0.750	0.486	0.868	0.696
	10	10	0.420	0.201	0.371	0.064		0.370	0.149	0.511	0.280
Z_{GL}	10	20	0.766	0.451	0.762	0.410	Z_{LT}	0.683	0.394	0.819	0.591
	10	50	0.997	0.956	0.997	0.955		0.994	0.907	0.997	0.975
	20	10	0.719	0.415	0.695	0.168		0.710	0.418	0.817	0.607
	20	20	0.972	0.867	0.968	0.799		0.948	0.844	0.977	0.917
	20	50	1.000	1.000	1.000	1.000		1.000	1.000	1.000	1.000
	5	10	0.086	0.136	0.140	0.037		0.094	0.045	0.184	0.075
	5	20	0.141	0.200	0.266	0.119		0.123	0.064	0.331	0.154
	5	50	0.263	0.377	0.569	0.329		0.274	0.114	0.667	0.443
	10	10	0.280	0.372	0.369	0.163		0.288	0.123	0.440	0.220
Z_R	10	20	0.432	0.568	0.612	0.348	Z_{LR}	0.464	0.256	0.684	0.448
	10	50	0.822	0.900	0.939	0.825		0.870	0.725	0.969	0.897
	20	10	0.640	0.760	0.728	0.462		0.684	0.443	0.770	0.533
	20	20	0.885	0.937	0.930	0.803		0.917	0.803	0.951	0.863
	20	50	0.999	1.000	1.000	0.999		1.000	0.998	1.000	0.999
	5	10	0.089	0.039	0.138	0.048		0.161	0.047	0.240	0.038
	5	20	0.126	0.056	0.282	0.110		0.262	0.106	0.422	0.181
	5	50	0.264	0.106	0.605	0.349		0.681	0.433	0.778	0.539
	10	10	0.278	0.123	0.402	0.187		0.286	0.094	0.360	0.073
Z_{ALR}	10	20	0.458	0.254	0.650	0.402	Z_{GPLP}	0.572	0.294	0.660	0.371
	10	50	0.866	0.721	0.965	0.873		0.949	0.828	0.966	0.887
	20	10	0.682	0.441	0.753	0.503		0.557	0.229	0.625	0.187
	20	20	0.917	0.798	0.946	0.835		0.884	0.619	0.906	0.673
	20	50	1.000	0.998	1.000	0.999		0.997	0.989	0.999	0.992
	5	10	0.148	0.037	0.192	0.054					
	5	20	0.247	0.085	0.366	0.146					
	5	50	0.631	0.356	0.763	0.506					
	10	10	0.298	0.100	0.395	0.150					
Z_{PLP}	10	20	0.536	0.297	0.677	0.397					
	10	50	0.946	0.801	0.978	0.905					
	20	10	0.578	0.312	0.700	0.427					
	20	20	0.880	0.692	0.927	0.786					
	20	50	0.998	0.992	1.000	0.996					

Table C.5: Probability of rejection $H_0 : \gamma = 0$ under the case (f) with identical processes when R =2

	τ	m	$\hat{\Omega}(1)$	$\hat{\Omega}(2)$	$\Omega(1)$	$\Omega(2)$		$\hat{\Omega}(1)$	$\hat{\Omega}(2)$	$\Omega(1)$	$\Omega(2)$
	5	10	0.130	0.046	0.119	0.014		0.15	0.056	0.243	0.122
	5	20	0.223	0.072	0.241	0.067		0.232	0.095	0.385	0.210
	5	50	0.456	0.235	0.486	0.235		0.473	0.25	0.63	0.435
	10	10	0.226	0.085	0.195	0.016		0.255	0.109	0.365	0.201
Z_{GL}	10	20	0.391	0.123	0.381	0.101	Z_{LT}	0.433	0.210	0.578	0.363
	10	50	0.808	0.532	0.797	0.528		0.821	0.564	0.888	0.747
	20	10	0.361	0.161	0.339	0.060		0.441	0.226	0.569	0.354
	20	20	0.706	0.445	0.677	0.330		0.752	0.556	0.839	0.674
	20	50	0.978	0.927	0.978	0.908		0.984	0.939	0.991	0.974
	5	10	0.091	0.155	0.162	0.047		0.103	0.048	0.225	0.080
	5	20	0.151	0.240	0.317	0.118		0.152	0.056	0.376	0.185
	5	50	0.258	0.381	0.605	0.332		0.260	0.113	0.673	0.436
	10	10	0.220	0.288	0.284	0.111		0.189	0.075	0.306	0.137
Z_R	10	20	0.301	0.437	0.484	0.221	Z_{LR}	0.288	0.117	0.543	0.274
	10	50	0.637	0.760	0.849	0.646		0.667	0.455	0.878	0.721
	20	10	0.363	0.485	0.440	0.230		0.371	0.197	0.489	0.257
	20	20	0.636	0.758	0.734	0.492		0.679	0.483	0.774	0.557
	20	50	0.954	0.980	0.987	0.933		0.959	0.885	0.988	0.953
	5	10	0.109	0.051	0.167	0.050		0.110	0.036	0.167	0.026
	5	20	0.163	0.058	0.320	0.142		0.150	0.052	0.262	0.103
	5	50	0.265	0.114	0.623	0.362		0.338	0.174	0.431	0.233
	10	10	0.183	0.075	0.277	0.115		0.168	0.055	0.22	0.038
Z_{ALR}	10	20	0.296	0.116	0.508	0.234	Z_{GPLP}	0.272	0.093	0.353	0.135
	10	50	0.670	0.458	0.865	0.684		0.652	0.407	0.722	0.501
	20	10	0.371	0.195	0.464	0.239		0.258	0.101	0.313	0.074
	20	20	0.678	0.480	0.769	0.527		0.545	0.280	0.584	0.312
	20	50	0.960	0.888	0.988	0.945		0.927	0.761	0.946	0.807
	5	10	0.136	0.056	0.172	0.066					
	5	20	0.206	0.085	0.298	0.124					
	5	50	0.390	0.201	0.502	0.304					
	10	10	0.212	0.079	0.299	0.118					
Z_{PLP}	10	20	0.350	0.147	0.472	0.229					
	10	50	0.712	0.486	0.801	0.621					
	20	10	0.355	0.172	0.465	0.250					
	20	20	0.642	0.411	0.741	0.514					
	20	50	0.947	0.862	0.971	0.912					

Table C.6: Probability of rejection $H_0 : \gamma = 0$ under the case (f) with identical processes when R = 1.5

C.2 Power of Tests with Nonidentical Processes

	τ	m	$\hat{\Omega}(1)$	$\hat{\Omega}(2)$	$\Omega(1)$	$\Omega(2)$		$\hat{\Omega}(1)$	$\hat{\Omega}(2)$	$\Omega(1)$	$\Omega(2)$
Z_{GL}	5	10	0.243	0.081	0.213	0.024	Z_{LT}	0.287	0.114	0.289	0.117
	5	20	0.452	0.218	0.426	0.142		0.496	0.282	0.493	0.252
	5	50	0.861	0.624	0.867	0.654		0.877	0.689	0.885	0.715
	10	10	0.398	0.165	0.363	0.053		0.485	0.253	0.493	0.263
	10	20	0.757	0.518	0.745	0.412		0.806	0.599	0.795	0.597
	10	50	0.989	0.940	0.989	0.942		0.993	0.953	0.994	0.955
	20	10	0.721	0.399	0.668	0.153		0.810	0.612	0.799	0.588
	20	20	0.970	0.870	0.964	0.796		0.975	0.916	0.975	0.929
	20	50	1.000	1.000	1.000	1.000		1.000	1.000	1.000	1.000
Z_{R}	5	10	0.161	0.247	0.162	0.037	Z_{LR}	0.158	0.071	0.200	0.074
	5	20	0.273	0.374	0.260	0.116		0.299	0.123	0.316	0.147
	5	50	0.566	0.699	0.573	0.315		0.598	0.348	0.646	0.409
	10	10	0.356	0.498	0.372	0.145		0.380	0.156	0.425	0.197
	10	20	0.640	0.727	0.626	0.380		0.650	0.427	0.680	0.459
	10	50	0.940	0.968	0.939	0.824		0.948	0.879	0.957	0.904
	20	10	0.703	0.791	0.702	0.442		0.738	0.483	0.756	0.505
	20	20	0.936	0.966	0.937	0.834		0.957	0.842	0.962	0.866
	20	50	0.999	1.000	0.999	0.999		1.000	1.000	1.000	1.000
Z_{ALR}	5	10	0.161	0.073	0.149	0.053	Z_{GPLP}	0.170	0.065	0.229	0.050
	5	20	0.305	0.130	0.257	0.087		0.280	0.122	0.379	0.156
	5	50	0.608	0.360	0.586	0.310		0.686	0.422	0.759	0.522
	10	10	0.385	0.159	0.385	0.158		0.320	0.136	0.367	0.090
	10	20	0.668	0.436	0.650	0.400		0.600	0.342	0.660	0.367
	10	50	0.948	0.883	0.948	0.879		0.949	0.823	0.963	0.851
	20	10	0.740	0.488	0.739	0.472		0.561	0.316	0.586	0.170
	20	20	0.960	0.846	0.956	0.852		0.882	0.665	0.907	0.681
	20	50	1.000	1.000	1.000	0.999		0.997	0.993	0.997	0.994
Z_{PLP}	5	10	0.240	0.104	0.200	0.045					
	5	20	0.394	0.189	0.343	0.123					
	5	50	0.766	0.538	0.760	0.510					
	10	10	0.408	0.213	0.383	0.149					
	10	20	0.699	0.488	0.675	0.145					
	10	50	0.960	0.899	0.958	0.868					
	20	10	0.685	0.493	0.657	0.417					
	20	20	0.940	0.806	0.936	0.791					
	20	50	0.999	0.997	0.999	0.997					

Table C.7: Probability of rejection $H_0 : \gamma = 0$ under the case (d) with non-identical processes when R =2

	τ	m	$\hat{\Omega}(1)$	$\hat{\Omega}(2)$	$\Omega(1)$	$\Omega(2)$		$\hat{\Omega}(1)$	$\hat{\Omega}(2)$	$\Omega(1)$	$\Omega(2)$
Z_{GL}	5	10	0.105	0.042	0.087	0.004	Z_{LT}	0.131	0.050	0.133	0.043
	5	20	0.200	0.071	0.187	0.034		0.216	0.093	0.214	0.078
	5	50	0.398	0.147	0.413	0.168		0.401	0.187	0.422	0.209
	10	10	0.163	0.056	0.145	0.017		0.198	0.080	0.202	0.082
	10	20	0.368	0.168	0.342	0.123		0.402	0.194	0.391	0.192
	10	50	0.691	0.430	0.695	0.431		0.715	0.478	0.725	0.493
	20	10	0.298	0.109	0.267	0.038		0.377	0.186	0.365	0.167
	20	20	0.621	0.345	0.600	0.275		0.655	0.396	0.654	0.410
	20	50	0.952	0.858	0.945	0.837		0.954	0.880	0.952	0.873
Z_{R}	5	10	0.083	0.138	0.083	0.020	Z_{LR}	0.095	0.040	0.125	0.041
	5	20	0.130	0.208	0.124	0.044		0.156	0.055	0.168	0.061
	5	50	0.220	0.334	0.224	0.083		0.234	0.094	0.283	0.117
	10	10	0.132	0.216	0.143	0.037		0.146	0.047	0.172	0.059
	10	20	0.284	0.388	0.273	0.112		0.285	0.132	0.324	0.153
	10	50	0.520	0.655	0.513	0.282		0.559	0.329	0.597	0.370
	20	10	0.298	0.394	0.296	0.124		0.317	0.125	0.325	0.142
	20	20	0.514	0.651	0.522	0.307		0.576	0.313	0.585	0.356
	20	50	0.888	0.926	0.888	0.742		0.910	0.798	0.919	0.814
Z_{ALR}	5	10	0.091	0.043	0.086	0.026	Z_{GPLP}	0.091	0.038	0.137	0.019
	5	20	0.159	0.061	0.133	0.035		0.136	0.051	0.197	0.066
	5	50	0.240	0.099	0.221	0.082		0.304	0.111	0.367	0.168
	10	10	0.148	0.048	0.148	0.044		0.133	0.036	0.163	0.015
	10	20	0.297	0.143	0.282	0.115		0.270	0.099	0.326	0.113
	10	50	0.568	0.330	0.568	0.326		0.557	0.293	0.588	0.357
	20	10	0.317	0.133	0.317	0.120		0.237	0.103	0.262	0.031
	20	20	0.581	0.325	0.564	0.333		0.462	0.206	0.521	0.219
	20	50	0.913	0.811	0.910	0.792		0.867	0.677	0.888	0.695
Z_{PLP}	5	10	0.112	0.043	0.097	0.021					
	5	20	0.179	0.073	0.152	0.046					
	5	50	0.332	0.137	0.324	0.127					
	10	10	0.159	0.061	0.141	0.036					
	10	20	0.320	0.147	0.293	0.106					
	10	50	0.599	0.387	0.583	0.325					
	20	10	0.306	0.134	0.273	0.108					
	20	20	0.539	0.283	0.527	0.273					
	20	50	0.902	0.743	0.893	0.735					

Table C.8: Probability of rejection $H_0 : \gamma = 0$ under the case (d) with non-identical processes when R = 1.5

	τ	m	$\hat{\Omega}(1)$	$\hat{\Omega}(2)$	$\Omega(1)$	$\Omega(2)$		$\hat{\Omega}(1)$	$\hat{\Omega}(2)$	$\Omega(1)$	$\Omega(2)$
	5	10	0.333	0.122	0.181	0.022		0.301	0.121	0.263	0.107
	5	20	0.649	0.384	0.479	0.171		0.607	0.317	0.532	0.278
	5	50	0.972	0.885	0.871	0.656		0.961	0.850	0.897	0.734
	10	10	0.564	0.278	0.367	0.036		0.532	0.316	0.517	0.287
Z_{GL}	10	20	0.840	0.605	0.729	0.391	Z_{LT}	0.820	0.620	0.788	0.586
	10	50	0.996	0.982	0.986	0.952		0.995	0.984	0.989	0.968
	20	10	0.774	0.429	0.636	0.124		0.771	0.533	0.799	0.591
	20	20	0.974	0.909	0.956	0.771		0.967	0.901	0.969	0.921
	20	50	1.000	1.000	1.000	1.000		1.000	1.000	1.000	1.000
	5	10	0.166	0.053	0.134	0.034		0.161	0.049	0.180	0.055
	5	20	0.293	0.117	0.267	0.100		0.289	0.119	0.328	0.141
	5	50	0.619	0.386	0.586	0.319		0.611	0.379	0.648	0.420
	10	10	0.377	0.198	0.365	0.154		0.367	0.183	0.399	0.202
Z_R	10	20	0.637	0.418	0.631	0.377	Z_{LR}	0.624	0.411	0.672	0.427
	10	50	0.945	0.865	0.947	0.863		0.941	0.855	0.957	0.880
	20	10	0.693	0.438	0.695	0.440		0.692	0.432	0.712	0.475
	20	20	0.940	0.831	0.941	0.830		0.941	0.828	0.942	0.851
	20	50	1.000	1.000	1.000	1.000		1.000	1.000	1.000	1.000
	5	10	0.180	0.061	0.156	0.036		0.281	0.093	0.214	0.025
	5	20	0.290	0.126	0.253	0.098		0.614	0.357	0.446	0.208
	5	50	0.609	0.366	0.586	0.351		0.964	0.872	0.772	0.543
	10	10	0.380	0.185	0.363	0.165		0.512	0.212	0.359	0.063
Z_{ALR}	10	20	0.682	0.453	0.670	0.434	Z_{GPLP}	0.812	0.582	0.656	0.340
	10	50	0.958	0.873	0.963	0.867		0.997	0.981	0.961	0.863
	20	10	0.707	0.469	0.708	0.455		0.690	0.390	0.566	0.118
	20	20	0.939	0.856	0.940	0.861		0.969	0.861	0.896	0.644
	20	50	1.000	1.000	1.000	1.000		1.000	1.000	0.999	0.997
	5	10	0.308	0.119	0.166	0.053					
	5	20	0.637	0.332	0.394	0.177					
	5	50	0.968	0.895	0.788	0.536					
	10	10	0.514	0.246	0.383	0.165					
Z_{PLP}	10	20	0.827	0.602	0.669	0.413					
	10	50	0.997	0.977	0.967	0.883					
	20	10	0.730	0.478	0.654	0.402					
	20	20	0.969	0.879	0.928	0.792					
	20	50	1.000	1.000	1.000	0.999					

Table C.9: Probability of rejection $H_0 : \gamma = 0$ under the case (e) with non-identical processes when R =2

	τ	m	$\hat{\Omega}(1)$	$\hat{\Omega}(2)$	$\Omega(1)$	$\Omega(2)$		$\hat{\Omega}(1)$	$\hat{\Omega}(2)$	$\Omega(1)$	$\Omega(2)$
	5	10	0.180	0.051	0.084	0.003		0.130	0.043	0.119	0.035
	5	20	0.311	0.129	0.186	0.033		0.264	0.095	0.212	0.071
	5	50	0.662	0.425	0.404	0.189		0.613	0.360	0.419	0.229
	10	10	0.270	0.105	0.150	0.009		0.223	0.078	0.216	0.068
Z_{GL}	10	20	0.510	0.247	0.337	0.101	Z_{LT}	0.429	0.218	0.386	0.194
	10	50	0.859	0.650	0.712	0.444		0.821	0.597	0.736	0.506
	20	10	0.389	0.139	0.277	0.033		0.339	0.146	0.380	0.187
	20	20	0.700	0.446	0.587	0.256		0.627	0.354	0.654	0.411
	20	50	0.976	0.907	0.945	0.823		0.963	0.876	0.952	0.873
	5	10	0.093	0.148	0.088	0.012		0.096	0.046	0.120	0.030
	5	20	0.137	0.213	0.145	0.041		0.133	0.054	0.170	0.064
	5	50	0.223	0.325	0.220	0.086		0.237	0.091	0.276	0.108
	10	10	0.161	0.244	0.148	0.040		0.147	0.056	0.173	0.054
Z_R	10	20	0.269	0.387	0.267	0.105	Z_{LR}	0.269	0.127	0.307	0.135
	10	50	0.544	0.663	0.544	0.297		0.549	0.299	0.597	0.356
	20	10	0.292	0.403	0.295	0.119		0.295	0.132	0.318	0.152
	20	20	0.513	0.650	0.509	0.280		0.555	0.309	0.580	0.339
	20	50	0.898	0.942	0.894	0.725		0.907	0.762	0.917	0.795
	5	10	0.100	0.047	0.083	0.011		0.184	0.057	0.130	0.011
	5	20	0.141	0.053	0.123	0.039		0.344	0.151	0.211	0.067
	5	50	0.243	0.094	0.207	0.073		0.760	0.480	0.362	0.171
	10	10	0.155	0.056	0.144	0.039		0.275	0.091	0.185	0.023
Z_{ALR}	10	20	0.273	0.127	0.272	0.107	Z_{GPLP}	0.513	0.229	0.290	0.112
	10	50	0.554	0.305	0.565	0.303		0.907	0.717	0.615	0.351
	20	10	0.297	0.135	0.299	0.135		0.378	0.153	0.263	0.037
	20	20	0.556	0.312	0.562	0.000		0.688	0.442	0.517	0.220
	20	50	0.909	0.767	0.908	0.773		0.983	0.893	0.869	0.670
	5	10	0.168	0.052	0.085	0.017					
	5	20	0.333	0.123	0.160	0.041					
	5	50	0.755	0.495	0.321	0.137					
	10	10	0.251	0.075	0.155	0.041					
Z_{PLP}	10	20	0.495	0.229	0.284	0.096					
	10	50	0.897	0.698	0.614	0.332					
	20	10	0.364	0.138	0.280	0.103					
	20	20	0.669	0.411	0.526	0.266					
	20	50	0.978	0.898	0.883	0.713					

Table C.10: Probability of rejection $H_0 : \gamma = 0$ under the case (e) with non-identical processes when R =1.5

	τ	m	$\hat{\Omega}(1)$	$\hat{\Omega}(2)$	$\Omega(1)$	$\Omega(2)$		$\hat{\Omega}(1)$	$\hat{\Omega}(2)$	$\Omega(1)$	$\Omega(2)$
	5	10	0.242	0.092	0.238	0.014		0.345	0.148	0.520	0.314
	5	20	0.500	0.265	0.537	0.191		0.575	0.350	0.740	0.550
	5	50	0.909	0.708	0.925	0.767		0.915	0.770	0.969	0.918
	10	10	0.469	0.192	0.431	0.047		0.623	0.386	0.756	0.554
Z_{GL}	10	20	0.834	0.589	0.815	0.463	Z_{LT}	0.882	0.702	0.936	0.844
	10	50	0.999	0.984	0.998	0.984		0.999	0.994	1.000	0.998
	20	10	0.776	0.432	0.742	0.145		0.892	0.694	0.936	0.856
	20	20	0.983	0.925	0.979	0.869		0.993	0.966	0.996	0.992
	20	50	1.000	1.000	1.000	1.000		1.000	1.000	1.000	1.000
	5	10	0.198	0.311	0.321	0.124		0.208	0.080	0.374	0.174
	5	20	0.371	0.468	0.536	0.310		0.326	0.151	0.588	0.357
	5	50	0.719	0.807	0.906	0.754		0.687	0.470	0.936	0.816
	10	10	0.464	0.584	0.558	0.315		0.470	0.255	0.615	0.372
Z_{R}	10	20	0.710	0.799	0.811	0.614	Z_{LR}	0.749	0.498	0.863	0.704
	10	50	0.974	0.991	0.995	0.974		0.990	0.930	0.999	0.993
	20	10	0.773	0.861	0.833	0.621		0.804	0.554	0.873	0.686
	20	20	0.978	0.986	0.986	0.930		0.976	0.927	0.989	0.954
	20	50	1.000	1.000	1.000	1.000		1.000	1.000	1.000	1.000
	5	10	0.212	0.079	0.311	0.116		0.187	0.055	0.297	0.039
	5	20	0.339	0.160	0.538	0.291		0.358	0.166	0.527	0.238
	5	50	0.704	0.492	0.920	0.757		0.790	0.543	0.871	0.718
	10	10	0.479	0.254	0.580	0.329		0.363	0.112	0.453	0.079
Z_{ALR}	10	20	0.753	0.510	0.847	0.657	Z_{GPLP}	0.680	0.445	0.753	0.451
	10	50	0.990	0.937	0.999	0.991		0.981	0.916	0.988	0.945
	20	10	0.805	0.563	0.863	0.664		0.608	0.293	0.684	0.158
	20	20	0.978	0.928	0.989	0.951		0.938	0.773	0.963	0.779
	20	50	1.000	1.000	1.000	1.000		0.999	0.997	0.999	0.998
	5	10	0.296	0.135	0.391	0.186					
	5	20	0.498	0.298	0.629	0.390					
	5	50	0.851	0.661	0.926	0.818					
	10	10	0.529	0.286	0.640	0.414					
Z_{PLP}	10	20	0.802	0.616	0.863	0.703					
	10	50	0.990	0.968	0.996	0.986					
	20	10	0.787	0.572	0.856	0.703					
	20	20	0.980	0.921	0.990	0.964					
	20	50	1.000	1.000	1.000	1.000					

Table C.11: Probability of rejection $H_0 : \gamma = 0$ under the case (f) with non-identical processes when R =2

	τ	m	$\hat{\Omega}(1)$	$\hat{\Omega}(2)$	$\Omega(1)$	$\Omega(2)$		$\hat{\Omega}(1)$	$\hat{\Omega}(2)$	$\Omega(1)$	$\Omega(2)$
	5	10	0.096	0.034	0.095	0.004		0.142	0.052	0.259	0.125
	5	20	0.208	0.083	0.231	0.050		0.235	0.093	0.405	0.213
	5	50	0.514	0.229	0.573	0.281		0.510	0.288	0.713	0.520
	10	10	0.189	0.065	0.173	0.011		0.273	0.102	0.401	0.219
Z_{GL}	10	20	0.374	0.163	0.349	0.100	Z_{LT}	0.449	0.208	0.562	0.359
	10	50	0.804	0.557	0.799	0.546		0.821	0.608	0.895	0.770
	20	10	0.334	0.131	0.302	0.029		0.452	0.211	0.586	0.377
	20	20	0.712	0.432	0.689	0.336		0.769	0.515	0.867	0.699
	20	50	0.975	0.906	0.975	0.901		0.981	0.942	0.993	0.974
	5	10	0.088	0.164	0.168	0.047		0.090	0.036	0.208	0.070
	5	20	0.152	0.244	0.305	0.111		0.151	0.050	0.364	0.167
	5	50	0.325	0.435	0.587	0.366		0.298	0.134	0.670	0.439
	10	10	0.170	0.263	0.243	0.091		0.176	0.071	0.281	0.127
Z_{R}	10	20	0.318	0.416	0.423	0.209	Z_{LR}	0.302	0.140	0.488	0.254
	10	50	0.619	0.739	0.831	0.616		0.616	0.375	0.871	0.687
	20	10	0.345	0.465	0.421	0.217		0.358	0.145	0.457	0.236
	20	20	0.658	0.764	0.740	0.492		0.648	0.439	0.764	0.551
	20	50	0.952	0.977	0.986	0.925		0.965	0.867	0.990	0.952
	5	10	0.092	0.036	0.157	0.043		0.080	0.027	0.146	0.010
	5	20	0.158	0.049	0.301	0.119		0.148	0.060	0.251	0.087
	5	50	0.306	0.150	0.615	0.362		0.363	0.155	0.504	0.281
	10	10	0.184	0.072	0.253	0.094		0.150	0.042	0.195	0.027
Z_{ALR}	10	20	0.303	0.144	0.453	0.214	Z_{GPLP}	0.288	0.121	0.351	0.128
	10	50	0.623	0.377	0.853	0.649		0.641	0.388	0.727	0.481
	20	10	0.361	0.153	0.440	0.209		0.248	0.070	0.309	0.308
	20	20	0.652	0.440	0.754	0.530		0.548	0.305	0.633	0.308
	20	50	0.965	0.868	0.990	0.947		0.925	0.765	0.936	0.835
	5	10	0.124	0.052	0.191	0.066					
	5	20	0.201	0.088	0.316	0.133					
	5	50	0.437	0.197	0.598	0.394					
	10	10	0.223	0.072	0.313	0.136					
Z_{PLP}	10	20	0.372	0.177	0.458	0.257					
	10	50	0.709	0.485	0.810	0.649					
	20	10	0.353	0.154	0.480	0.264					
	20	20	0.665	0.419	0.760	0.563					
	20	50	0.942	0.853	0.965	0.922					

Table C.12: Probability of rejection $H_0 : \gamma = 0$ under the case (f) with non-identical processes when R =1.5