Individual differences in conceptual and procedural knowledge: The case of Algebra

Felix Ayesu

A thesis submitted in partial fulfilment of the requirement of the degree of

Master of Science

Department of Psychology

Memorial University

October, 2017

Abstract

As children learn algebra, it is not certain whether they are learning both the procedural and conceptual aspects in a balanced way. Instead, students may be learning one of these types of knowledge more than the other. Previous research on children's understanding of fractions have used cluster analysis to demonstrate that there are some students who rely more on conceptual knowledge, some who rely more on procedural knowledge, and some who rely equally on both. Using cluster analysis, the current study found that there are individual differences in the understanding of algebra in a sample of 104 grade eight students. Four clusters were found representing students who do relatively poorly on conceptual and procedural knowledge, those who do well on both types of knowledge, those who are relatively better conceptual problem solvers, and those who are relatively better procedural problem solvers. Furthermore, both conceptual and procedural knowledge are significant contributors to students' overall performance in algebra, which suggests that both conceptual and procedural knowledge are important in algebraic learning.

Keywords: conceptual knowledge, procedural knowledge, individual differences, algebra, youth

Acknowledgements

My sincere gratitude goes first to Professor Darcy Hallett, who guided me expertly through my graduate education. His unwavering enthusiasm for my study has kept me engaged constantly with my research. My appreciation also extends to my lab colleagues. Cheryll Fitzpatrick’s mentoring and encouragement have been especially valuable, and her insights launched the greater part of this study. My profound gratitude also goes to my committee members (Dr. Brent Snook and Dr. Mary Courage) who have provided guidance to ensure the success of this study. Above all, I am highly indebted to my family for their support throughout these years.

Table of contents

List of tables

List of figures

Chapter 1

Introduction

In the field of numerical cognition, there is a long-standing debate about the developmental relation between conceptual and procedural knowledge. This decades-old debate has been informed by researchers and educator's beliefs about how children develop conceptual and procedural knowledge and what constitutes the best ways of teaching and learning these types of knowledge (Rittle-Johnson & Alibali, 2001). Specifically, the main bone of contention has been whether children should be taught to solve math problems by emphasizing procedures (known as procedural first) or by constructing rich connections between mathematical thoughts (known as conceptual first). The latter is taught through the use of an enhanced discovery-based learning, where children are guided to devise their problem-solving schemes given their understanding of the principles. On the other hand, students may be actively engaged to continually solve problems to bring forth the understanding of concepts needed to thrive in a domain (Bruner, 1961).

Older research on the relation between these knowledge types – conceptual and procedural – have yielded contradictory findings. Some group of researchers (e.g., Byrne & Wasik, 1991) have suggested that conceptual knowledge develops first and influences the invention of procedural knowledge. Further, they indicate that children may depend more heavily on their conceptual understanding than their procedural understanding when solving math problems. In contrast, another group of researchers (e.g., Fyre, Braisby, Lowe, Maroudas, & Nichools, 1989) have claimed that children first learn the procedures for solving given math problems, and that their conceptual understanding will develop with continued practice. Recent studies (eg., Rittle- Johnson & Alibali, 1999; Rittle-Johnson, Alibali & Siegler, 2001) on the other hand, have found a bi-directional relation between conceptual and procedural knowledge. They claimed that children could first develop the conceptual knowledge for a given math domain and this conceptual knowledge will then lead to the generation of procedural knowledge for solving questions in that domain. On the contrary, children can learn procedural knowledge first, and with continued practice of the problem-solving procedures, they will subsequently discern the principles underlying the given problem – a process they called iterative learning.

Although it may be true that children learn conceptual and procedural knowledge iteratively, Hallett and his colleagues have proposed there may also be individual differences, or clusters, in conceptual and procedural knowledge (Hallett, Nunes, & Bryant, 2010; Hallett, Nunes, Bryant, & Thorpe, 2012). In the area of fractions understanding, these researchers found clusters that suggest there may be some children who rely primarily on conceptual knowledge, other children who may rely more on procedural knowledge, and yet others who may rely on both types of knowledge relatively equally. In other words, not all children rely equally on conceptual or procedural knowledge and therefore may approach math problems differently.

To date, no study has specifically explored the existence of these individual differences in other domains, especially in algebra, a domain that has been found to be challenging for students. In this thesis, I will extend this research by investigating whether or not these same individual differences in fractions demonstrated by Hallett and his colleagues (2010; 2012) are evident in children's understanding of algebra. The existence of similar clusters in algebra learning would further support the notion that not all children combine conceptual knowledge and procedural knowledge in the same way. It would also highlight the importance of taking these individual differences into account when investigating mathematical learning.

What is Algebra and Why Does it Matter?

Algebra is a domain of mathematics that employs symbols and letters to represent unknown quantities, and devise mathematical statements that are used to make connections between objects or events that vary over time (Coolman, 2015). A critical look at algebra would reveal that it is a lot more like arithmetic, as it follows all the rules and operations (addition, multiplication, subtraction, and division) used in arithmetic. However, algebra introduces a new element called the 'unknown,' and the primary goal of algebra is to manipulate the known terms given in an equation to find the unknown.

Algebra is considered to be one of the most significant concepts and tools in mathematics. Early knowledge of algebra, for example, is regarded as a significant predictor of future career prospects and overall life outcome [National Mathematics Advisory Panel (NMAP), 2008]. It is considered an entryway math domain, as it arms learners with the necessary foundation to thrive later in more advanced courses in college, and lifelong careers. It imparts students the language of math and helps them to build resilient general problem solving and critical thinking proficiencies (Education Commission of the States, 1998; Silver, 1997; Star et al., 2014; U.S. Department of Education, 1999).

Research suggests that algebra knowledge provides students with the needed foundation and skills that facilitate their understanding of other abstract and complex areas such as the fields of Science, Technology, Engineering, and Mathematics (STEM), and hence, students who excel in algebra are more likely to take advantage of the growth of technology and the concomitant job opportunities (Rose & Betts, 2001; Russell, 2014). In effect, proficiency in algebra during one's early academic experiences (e.g., secondary and post-secondary) may provide opportunities for careers in STEM fields. For individuals who are motivated, a career in STEM fields can facilitate circumstances to make meaningful contributions to the global community in the long-run. This leads to the conclusion that educators should among other things, focus on improving students understanding and performance in algebra if they aim to help contribute to the enhancement of imminent human resource base of every nation.

It is widely agreed that knowledge of algebra facilitates children's abilities in other related math domains (Lucariello, Tine, & Ganley, 2014; Star, 2014). In the United States, Allenworth and Easton (2007), have analyzed data from Chicago Public Schools on students' performance and how that predicts their graduation rates. In their study, they gathered data from more than 24,000 students in Chicago on specific aspects of school course works during their first year in high school and analyzed how students' performance on those indicators affects their ability to graduate on time. They found that students who can pass Algebra I by the 8th grade are more likely to graduate on time (i.e., within the average four-year high school period) than their unsuccessful counterparts.

In a similar study, Silver, Saunders, and Karate (2008) collected data on more than 48,000 students from 163 schools who are part of the Los Angeles Unified School District to track their academic progress until they reached the 9th grade. The data analyzed by Silver et al. (2008) consisted of the school transcript records of students and their scores from other standardized test taken from the sixth grade through their expected graduation year. They found that taking Algebra I increased the probability of graduating high school on time by approximately 75%. Other studies have found that students who complete Algebra II have a greater possibility to enter into a post-secondary education and subsequently finish a bachelor's degree (Adelman, 1999).

As noted, the benefits of algebra transcend students' academic experience. Some researchers have gone to the extent of describing algebra as the present day civil right (Moses, 1993). For the past few decades, governments and actors in educational sectors have begun to recognize the benefits of algebra and have started taking measures aimed at improving teaching and learning of algebra. In the United States, for instance, students are required to enroll and complete Algebra I in the 8th grade before graduating to 9th grade and beyond in some School Districts (Loveless, 2008). Despite the aforementioned significance of algebra, it still remains one of the most difficult branches of math (NMAP, 2008). Before presenting these data, however, it is important to first describe the current research on algebra learning, as well as the previous research on conceptual and procedural knowledge of mathematics

Students Struggles with Algebra

The struggles students face with algebra are well documented (see Blume & Heckman, 1997). For example, in a study by the National Center for Educational Statistics of United States (2005), researchers reported that only 6.9% of 12-graders were able to score at or above a proficient level in algebra in that particular year's assessment. In the National Report Card, (2011), when grade 8 students were tasked to solve for n in a simple algebra problem (e.g., n + 18 = 28), only 59 % of the respondents accomplished the tasks correctly.

It is not surprising therefore that poor performance in algebra is ascribed as one of the major predictors of high school drop-outs. In a longitudinal study that tracked almost 49,000 California students to assess the factors that lead to school drop-outs, Silver, Saunders, & Karate, (2008) found that students' performance in algebra 1 at 9th grade was a significant predictor of their completion, controlling for all other variables. Specifically, the study found that the demographic background of students accounted for only 4% of why students drop out of school whereas students' academic experiences, including algebra, was considered an essential factor that accounted for a larger variability (Silver, Saunders, & Karate, 2008).

The challenges with algebraic comprehension are widespread (Schmidt, McKnight, Cogan, Jakwerth, & Houang, 1999; Sfard, 1991), and this has attracted the attention of researchers over the past decades. Researchers have sought to find the specifics of algebra that make it challenging for students at least at the basic level. Several reasons have been proposed as the possible basis to this quandary. It has been suggested that the difficulty with algebra may be a result of a cognitive gap; that learning algebra demands that students operate with symbolic representations of limited concrete objects, and must process these symbols to produce numerical solutions (Kieran, 1992). Children learning algebra are therefore exposed to many abstract concepts that they must deal with by using symbols, and they may be overwhelmed by such tasks. This is particularly true because hitherto being introduced to algebra, children have mostly had experience with arithmetic (Sfard & Linchevski, 1994, Livneh & Linchevski, 2007; Kieran, 1992).

In algebra, unlike arithmetic where final answers are obtained after operations, students must work with unknown variables, the relationships between these unknown variables, and express the relationships using symbols (Benejee & Subramanian, 2012; Fischbein & Barash, 1993). Furthermore, although both algebra and arithmetic domains involve written symbols and the understanding of how operations work, arithmetic does not operate at the same level of abstraction as algebra. Arithmetic requires limited numbers and computations (Sfard & Linchevski, 1994). Thus, such complex tasks as the notion of the equal sign representing equivalence, operational laws, and understanding groups of numbers and symbols as objects may not be necessary for arithmetic but are crucial for algebra (Knuth, McNeil, & Alibali, 2006). Arithmetic is relatively straightforward and requires correct procedures and operations. As a result, shifting from concrete manipulation of numbers as pertains to arithmetic problem solving to a complicated and an ideal algebraic environment sometimes becomes a herculean task for these nascent learners.

This experience may present an abrupt increase in the cognitive demands of these students, and the results are sometimes poor performance when solving algebra problems. Star and Newton (2009) explain that children may not have developed the cognitive resources required to meet the demands of the abstract concepts and procedures that come with algebra. Hence, it is likely that they tend to be overwhelmed when algebra is introduced.

The seeming abstraction of algebra may, however, be beneficial to children's cognitive development. This is because algebra requires the application of critical thinking abilities. As such, the constant use of the working memory in attempting algebra problems sharpens one’s brain faculty. Recently, Star et al. (2014) have suggested that although the nonconcrete reasoning nature of algebra makes it difficult for students, it also serves as a vital learning instrument and network by preparing kids to acquire more advanced concepts in high school and beyond.

Other researchers have asserted that the teaching methodology adopted by educators may be a factor to why many children find algebra learning challenging (National Research Council, 2001; Star & Rittle-Johnson, 2008). The basis of this claim is that the nature of algebra demands a reliable connection of the knowledge of concepts to the exact procedures to get the appropriate solutions to problems. It is, therefore, imperative that educators identify this connection and teach kids how to master it. This can be achieved only when the right teaching methods are defined and employed. Consequently, the inability of pedagogy to make this connection could make learning algebra a difficult task. So the question now is: What exemplifies algebra and what are the best ways to improve students learning of algebraic concepts?

Resolving the challenge with algebra

Identifying students' struggles in algebra and the reasons for such struggles is one thing but helping students to improve their performance is another. To improve students' performance, the National Mathematics Advisory Panel (NMAP; 2008) have recommended a curricular focus on enhancing sound foundations in three primary areas including mastery of conceptual and computational skills. Specifically, as stated by the panel, "to prepare students for algebra, the curriculum must simultaneously develop conceptual understanding, computational fluency, and problem-solving skills" (NMAP, 2008, p.18).

To ascertain how these knowledge types, contribute to improving algebra performance, it is imperative that the relationships between them be explored. Knowing these relationships will enable researchers, educators and other players in the teaching and learning setting to identify the extent to which the acquisition of one type of knowledge affects the development of the other, and devise the right pedagogical strategy to achieve those ends. This is particularly important in algebra considering that to be a successful algebra problem solver one needs to successfully employ all the steps involved in solving a given question while simultaneously understanding why each step is indispensable. Failure to build a strong connection between how and why steps work in algebra will most likely lead to an undesired result. That is, students may struggle when learning other higher-level math, which requires the application of algebraic understanding (NMAP, 2008; Star, 2014; Wang, 2015).

One can make the conclusion that conceptual and procedural understanding in algebra are essential factors that determine how students perform in algebra. What then represents these two types of understanding? Are they separate from each other? Moreover, how do these two knowledge types differ or relate to one another? The following sections discuss conceptual and procedural knowledge and how researchers in numerical cognition have defined and described the relationships that exist among them.

Conceptual Knowledge

There is no universally accepted definition of conceptual knowledge (see Crooks & Alibali, 2014). Conceptual knowledge is the ability to discern mathematical concepts and operations, and the relations that exist between them (Kilpatrick, Swafford, & Findell, 2001). Learning conceptual knowledge may be associated with declarative memory since children have to learn facts, rules, operations, and symbols and match all these ranges of information, and apply it to make meaning of the given math domain (Mastin, 2010). Some researchers have argued that the definitions of conceptual knowledge are often vague and therefore using such definitions as the basis for the measurement of conceptual knowledge makes it difficult. Crooks and Alibali (2014) conducted a review of the literature investigating conceptual knowledge in mathematical cognition, and they concluded that there are two categories of conceptual knowledge.

The first kind of conceptual knowledge is the understanding of the principles that guides a given math domain. In this first case, Crooks and Alibali (2014) claimed that individuals learning math could possess such conceptual understanding without being able to link it with any given procedures. The authors suggested that this aspect of conceptual knowledge may include rules, definitions, and aspects of any domain structure. For example, children's ability to discern the meaning of the equal sign as being a representative of equivalence which is not tied to any procedure or any particular domain (e.g., algebra, fractions). They further stressed that general conceptual knowledge could take the form of categorization and grouping of formulae and symbols applicable in a general math sense. For example, a child must be able to understand that the equal sign (=), greater-than (>) and less than (<) symbols all belong to the same family because they are all used to demonstrate the relations existing between the left and the right sides of these symbols (i.e., they symbolize a connection). Crooks and Alibali (2014), contended that conceptualizing conceptual knowledge this way enables children to have a better understanding of the idea that different symbols and formulae belong to the different or same classes, and as such, children are more likely to recognize and select the appropriate procedures for solving questions on such problems.

The second kind of conceptual knowledge is the ability to form connections between the various procedures involved in solving any given problem (Crooks & Alibali, 2014). Baroody, Feil, and Johnson (2007) emphasized that conceptual knowledge should not be just an understanding of the principles that govern a particular math field, but that conceptual knowledge should be, in part, an understanding of the rich connections underlining particular action sequences or procedures. This leads to the conclusion that the ability of children to understand why certain steps must precede others, and the role of such steps in solving a problem, is an integral part of conceptual understanding.

In the case of algebra, one must understand that when a number crosses the equal sign, its sign changes. That is, when a positive number is moved to the other side of the equation (i.e., across the equal sign), it changes to a negative number. One must understand how the second step, where the number changes from positive to negative in the above instance, is related to the previous step when it was positive. That is, to have conceptual knowledge means to understand that this change of sign is not just magical, but a result of the process where that original number is negated on one side of the equal sign with its opposite, and therefore its opposite must also be added to the other end of the equal sign so that the overall relation remains equal. Another key aspect of conceptual knowledge of algebra is the ability to understand equivalence. Equivalence is the idea that both the right and the left-hand side of the equal sign or an equation are equal, and that whatever is done to the right side must be done to the left side. For example, suppose a student is asked to compare the answers to these two equations without doing the calculation: 214x + 214 = 428 and 214x + 214 + 6 = 428 +6. A student who possesses a conceptual understanding of algebra in general, and the equal sign more specifically, will understand that the answers to both equations will be the same because the equal sign represents a balance or equivalence and hence adding the same number to both sides of the sign will cancel out, and the result from equation two will be unaffected by the 6 added.

To be a competent math problem solver, one needs to combine their knowledge of the concepts with their ability to execute the right procedures to solve a given math problem. Knowledge of procedures is, therefore, another significant knowledge that students must possess.

Procedural Knowledge

Procedural knowledge refers to the ability to summon the right steps or sequences of actions to accomplish a goal (Hallett, Bryant and Nunes, 2010; Rittle-Johnson, 2015). The phrase "computational skills" is sometimes used interchangeably to refer to procedural knowledge (Byrne & Wasik, 1991). Some researchers have suggested that procedural knowledge can be acquired through rote learning and may not necessarily require an understanding when learning (Hallett. et al, 2012; Rittle-Johnson, 2015). It is possible that children can execute the steps involved in solving some given math problems successfully without having an understanding of why those steps worked (Fyre, Braisby, Lowe, Maroudas, & Nicholls, 1989).

However, the best math problem solvers appear to possess the skills to 'know how' to apply the rules about a particular problem and execute the procedures concurrently (Hallett et al., 2012). Studies have demonstrated that although children can learn the steps involved in solving a particular math problem without necessarily understanding why such procedures work, it is also possible to learn these procedures with meaning. That is, procedural knowledge may be linked to conceptual knowledge and that generally, children's knowledge of procedures is enhanced when they can link conceptual knowledge to procedural knowledge (Hiebert & LeFevre, 1986).

Rittle-Johnson and Alibali (2001) have suggested that procedures learned in one domain may not necessarily apply in a different field. Thus, procedural knowledge is often not transferable. For example, adding a zero to a number when multiplying by 10 (e.g., 2 x 10 = 20) does not work in situations where a decimal figure is multiplied by 10 (e.g., 0.134 x 10 ≠ 0.1340 but rather 1.34). A child who transfers the procedural knowledge used to solve questions that require the use of this 'add zero' method to solving questions that need the 'move the decimal point to the right or left' procedure is indeed bound to deviate on the solution. Nevertheless, procedural knowledge can involve the ability to adapt known methods to new problems (Rittle-Johnson & Star, 2007). For example, understanding the many ways of using the Lowest Common Multiple (LCM) principles when solving math fraction problems can also be adapted when solving algebra problems.

Hiebert and Lefevre (1986) claimed that procedural knowledge might be divided into two broad categories. One type of procedural knowledge would represent the ability to use symbols and legal rules (proper syntax). For example, a student who possesses this aspect of procedural ability will be able to discern an acceptable expression as f(x) = 2x-1 from an unacceptable one f3 = x/(). This does not include the capacity to know how to perform calculations or interpretations of the expressions, but rather the ability to discern what is right from wrong in using math symbols. The other aspect of procedural knowledge according to Hiebert and Lefevre (1986) relates to algorithms, which are step-by-step procedures. This involves knowledge about the actual sequence of actions that will eventually lead to the final answer to the problem. In carrying out these steps, the action to be done next is determined by the previous step, and each step is relatively distinct from the other. For example, in algebra, a student who has procedural understanding would be able to know that to solve a question such as $\frac{1}{4}(3y + 3) = 4$, they would first have to remove any fractions by multiplying through the equation by the LCM (which in this case would be 4), expand to eliminate the bracket, and group like terms before finally simplifying and solving for the answer.

The Relationship Between Conceptual and Procedural Knowledge

Many researchers have claimed that conceptual knowledge is differentiated from procedural knowledge (Byrne & Wasik, 1991; Fyre et al.,1989; Fyre, Fuson, & Hall 1983; Rittle-Johnson & Alibali, 2001; Hallett. et al., 2010, 2012; Peck & Jencks, 1981). Although in many spheres of learning children must learn both the fundamental concepts and correct procedures for solving problems to be proficient problem solvers, there are instances where children can possess the ability to execute the right procedures successfully to solving a problem without necessarily having the capacity to understand the concepts behind those procedures. The quest to describe the relationships between knowledge concepts and procedural understanding has led to virtually an impasse and a longstanding debate among numerical cognition researchers (Byrne & Wasik, 1991; Peck

& Jencks, 1981; Fyre et al.,1989; Fyre et al., 1983; Rittle-Johnson & Alibali, 2001, Hallett et al, 2010; 2012).

Historically, the age-old debate has been largely informed by researchers and educator's beliefs about which of these two types of knowledge children first develop and whether building one type of knowledge will influence the acquisition of the other knowledge type. While one group of researchers (e.g., Byrne & Wasik, 1991) argued that conceptual knowledge precedes procedural knowledge, another group of researchers (e.g., Fyre et al., 1989) have claimed vice-versa. The third group of researchers (Rittle-Johnson & Alibali, 2001) have contended that it is possible to build conceptual knowledge first before procedural knowledge and the reverse is true. An emerging school of thought (e.g., Hallett et al., 2010, 2012) is that there are individual differences. A review of studies that have found evidence in support of each of these perspectives is provided below.

Concept First?

The first group of researchers (e.g., Byrne & Wasik, 1991; Gelman & Bailargeon, 1983; Gelman & Gallistel, 1978; Gelman & Meck, 1983; 1986) have suggested that children, implicitly through the environment they are initially exposed to, be it at home or school, develop conceptual knowledge first, and this later influences the generation of procedural knowledge. This perspective is what Rittle-Johnson and Alibali (2001), described as the concept first viewpoint.

Gelman and Meck (1983) adopted an error detection task to explore 3 and 4- year olds understanding of counting. Gelman and Meck claimed that young children possess an implicit understanding of the five counting principles. These counting principles are: the stable order principle (that counting is done by following a single sequence of counting words. Example, one can only start counting from one, two, etc., or starting with the last number and counting downwards), the one on one counting principle (that each member of a target objects is counted once and with only one counting word), the cardinality principle (that the last count represents the numerosity of the set), the abstraction principle (that any given set of objects can be counted), and finally the order irrelevance (that counting can be done in any order; that is, it is not always necessary to start counting from the first object).

To investigate their claim, three separate studies were conducted by Gelman and Meck (1983). In all three studies, children were exposed to a puppet counting objects and were asked to judge whether the puppet was correct or wrong. In the first study, children's understanding of the one-on-one principle was explored. Children watched a puppet count two rows of red and blue objects. In half of the counts, the puppet made errors (i.e., skipped one item or double counted one item). On the other half, the puppet made accurate counts. These groups of objects were of sizes: 6, 8, 12 and 20. Each object size had two trials or wrong counts, and two trials of accurate counts. For the second study, children were asked to judge whether a puppet violated the stable order principle of counting. For a count to be considered wrong, the puppet skipped number words (eg., counted 1, 2, 4,…) or counted two objects using the same word (eg., 1, 2, 2, …). The third study examined children's understanding of cardinality. Here, children watched a puppet counting, after counting, the puppet mentioned a number which supposedly representing the size or quantity of the objects counted. In half of the trials, the puppet was wrong, and was correct on half of the counts.

In all these three studies, Gelman and Meck (1983) found that children were able to detect when any of the counting principles tested were violated The authors concluded that children as young as 3 and 4-year olds possess the ability to understand the principles underlying a counting procedure, although they are unable to demonstrate this ability. This means that children develop conceptual understanding before they finally learn to engage in the actual counting procedures.

In the area of fractions understanding, Byrne and Wasik (1991) conducted two studies among fourth, fifth, and sixth-graders to determine the relationship between conceptual and procedural knowledge. In Study 1, knowledge of concepts was assessed regarding the ability of children to understand equivalence and ordering of fractions using items designed to reflect three aspects of fractions, namely, picture symbol, simple morphism, and the order of items. Procedural knowledge was assessed using items designed to reflect two kinds of problems: multiplication and addition (e.g., using the LCM principle to add) of simple fractions. These problems required the use of computational skills and knowledge of algorithms.

Byrne and Wasik (1991) found that approximately 81% of children score above chance on all the conceptual items and that 93% of the children did not score above chance on the LCM. Since the LCM assessed procedural knowledge, the authors concluded that the children possessed more conceptual understanding than procedural knowledge. A similar result was found for the multiplication items as well. In Study 2, Byrne and Wasik classified participants as being significantly above chance or not on each of the conceptual and procedural scales. They found that almost all participants were above chance on conceptual scales before they were above chance on procedural scales.

They concluded that conceptual knowledge was necessary but not essentially sufficient for understanding procedural items, like finding the LCM successfully. In conclusion, such evidence supports the claim that conceptual knowledge precedes procedural knowledge.

Procedure First?

In contrast, another group of researchers (e.g., Briars & Sieglar, 1984; Fyre et al., 1983; Fyre et al., 1989) have claimed that children first learn the procedures for solving given math problems and their conceptual understanding develops later with continued practice. This view has come to be labeled the “procedure first” perspective by Rittle-Johnson and Alibali (2001).

In a test of this view, Peck and Jencks (1981) found, from interviews with several sixth graders about their understanding of fractions, that less than 10% of the children had a good conceptual understanding of fractions. Although children demonstrated a limited conceptual ability, approximately 35% of these children were able to employ correct procedures for solving fractions. Other studies have found results that corroborate Peck and Jencks’ findings. For example, in a study that examined the type of strategies that children employ when solving fractions and the errors they usually commit, it was found that some children can perform addition computations with fraction correctly but are unable to explain why the procedures they used work (Kerslake, 1986)

In the realm of counting, Fyre et al. (1989) found that 4-year-olds who count accurately often have a limited understanding of the cardinal goal of counting. There are five core principles that one needs to know to be considered a competent counter. Previous studies (Gelman & Bailargeon, 1983; Gelman & Gallistel, 1978; Gelman &

Meck, 1983) found that children are born with the inherent understanding of these counting principles. In Experiment 1, Fyre. et al., (1989) investigated the assertion that young children have an understanding of cardinality of counting by asking children three cardinality questions; “are there X here?”, “give me X” and “how many objects are here?” (X represents the number of objects). In half of their experiment, these questions were posed before the child was allowed to count. They found that, children performed better on the “how many” questions than when they were asked “are there X,” and also performed better on the “are there X” than the “give me X”. In all of these trials, Fyre. et al., found that children performance were poor when they were asked to answer the questions (i.e., are there X?, give me X and how many are there?) before they were allowed to count.

To have a better understanding of children’s understanding of counting, Fyre. et al., (1989) included all the counting principles in Experiment 2. For Experiment 2, children watched the experimenter do the counting and were required to judge whether the experimenter was correct or incorrect on following the proper counting procedure. Also, children were asked whether the last count given by the experimenter was correct or incorrect. For half of the trials, the experimenter made a counting mistake, and for the other half, the correct counting procedure was followed. Wrong counting procedures violated at least one of the other four counting principles (i.e., one to one, stable order, abstraction and order irrelevance). Thus, counting procedures were assessed using the child’s ability detect whether the experimenter had violated each of the counting principles except cardinality. On cardinality, the experiment on half of the trials gave a number less than or greater than the actual numerosity while on the other half, the experimenter gave the correct figure representing the number of objects counted. Children were asked to judge whether the counting procedure used was incorrect or correct and whether the number mentioned at the end of the count as representing the cardinality of the objects counted was correct or incorrect.

Fyre et al. (1989) found that children mostly made accurate judgments on the counting procedures, except the order irrelevance counting principle. That is, children were able to judge when most counting procedures were violated. However, children were much less successful at judging when cardinality rules were violated, and there were also less successful at identifying when experimenters could count in a non-standard order (e.g., from the middle rather than from left to right). These errors on cardinality and the order irrelevance principle suggest a lack of conceptual understanding of what counting does. These findings support the perspective that children build procedural knowledge before conceptual knowledge.

Bidirectional Learning

The third group of researchers holds the view that conceptual and procedural knowledge learning is an iterative process. According to this group of researchers, drawing a sharp contrast between conceptual and procedural learning could be misleading and contradictory to the way that children build an understanding of mathematics (see Rittle-Johnson & Alibali, 2001; Rittle-Johnson, 2016). The basic tenet of this view is that knowledge is not a discrete quantity, and, as such, it is hard to determine at which point one knowledge type begins and how to completely separate it that knowledge from the other knowledge type. Proponents of this perspective, therefore, claim that the relationship between conceptual and procedural knowledge is bi-directional. This means that children can first develop the conceptual knowledge and this conceptual knowledge will then lead to the generation of procedural knowledge for solving questions in that domain. On the other hand, procedural knowledge can emerge first and, subsequently, with continued practice of the problem-solving sequence, learners will discern the principles underlying a given problem.

Rittle-Johnson and Alibali (1999) examined the relations between children's conceptual understanding of mathematical equivalence and their ability to execute procedures in solving equivalent problems among fourth and fifth graders. To examine how each knowledge type affects the other, the authors asserted that it would be important to investigate whether an instruction on conceptual knowledge would influence students' procedural knowledge and vice versa. To achieve this, students were first given general equivalence questions to solve. Equivalence problems are ones that present mathematical calculation in non-standard ways to see if children understand the meaning of the equal sign. For example, children would be asked to fill in the blank for the answer 6 + 5 = 2 + __. Students who performed well on this task were categorized as the equivalence group, while those who performed poorly were put into the nonequivalence group. The nonequivalence group was further randomized into three groups, two instructional and one control: a procedural instruction group, conceptual instruction group, and a non-instructional control group. A conceptually designed instruction was provided for those in the procedural group, and procedural instruction is given to the conceptual group. Participants were then reassessed at post-test to determine if their conceptual and procedural knowledge scores were affected. The pretest and post-test items had been designed to reflect students' knowledge in three components of equivalence: the meaning of two quantities being equal, the sense of the equal sign, and the idea that equation has two sides. The procedural task was also designed such that children could answer whether a given procedure for solving the equivalence task was right or wrong.

When Rittle-Johnson and Alibali (1999) controlled for differences in procedural knowledge at pretest, the results showed that students who received conceptual instruction at pretest not only increased their conceptual understanding but also improved in correctly applying procedures for solving those problems at posttest. Similarly, when the researchers controlled for differences in conceptual knowledge at pretest students who received procedural instruction at pretest improved in generating the correct problem-solving procedures, and also improved in their conceptual understanding at posttest. They concluded that procedural knowledge could lead to conceptual understanding and that conceptual knowledge can also lead to procedural understanding.

A follow-up study by Rittle-Johnson and Alibali (2001) sampled fifth and sixth graders learning decimal fractions, using similar methods as Rittle-Johnson and Alibali (1999). They explored if the iterative relationship would be found in decimal fractions. Children were assessed at pretest and were grouped into a conceptual or procedural instruction. Conceptual knowledge and procedural knowledge were assessed at pretest and instructions provided afterward. Rittle-Johnson and her colleague then assessed the kids' procedural knowledge on number line problems. They also administered a conceptual knowledge of general fraction and decimals. Overall, they found that children's fundamental conceptual knowledge at pretest predicted their procedural understanding at both pretest and post-test. Also, gains in procedural knowledge at post-test predicted conceptual knowledge at post-test. The authors concluded that children could start to learn conceptual knowledge first, and this could help improve their knowledge of procedures. On the other-hand, children can learn to use procedures first, and with further practices, they procedural knowledge could be enhanced.

Individual Differences

Although the iterative model is one way to reconcile the contradictory findings supporting “concepts-first” and those supporting “procedures-first," Hallett and colleagues (2010, 2012) proposed that individual differences in conceptual and procedural knowledge may also explain these findings. In other words, it could be the case that some children rely more on concepts and some children rely more on procedures. Individual differences in conceptual and procedural knowledge have been found in many mathematical domains such as addition and subtraction (e.g., Canobi, 2005; Canobi 2004), multiplication (e.g., Mabbott & Bisanz, 2003), fractions (e.g., Hallett et al., 2010; Hallett et al., 2012), and arithmetic (e.g., Gilmore & Bryant, 2006). In the past, the methods and procedures used have varied, but general findings have been similar – children have different profiles of conceptual and procedural knowledge that usually show up as different clusters.

Gilmore and Bryant (2006) tested fourth-grade children on inversion problems and standard problems to determine if there may be individual differences in their conceptual understanding and computational skills. Inversion problems refer to problems that can be very easily solved if one understands the inverse relationship between addition and subtraction. For example, the problem 89 + 123 – 123 can be very easily solved without calculation because the last two terms negate each other (i.e., an indication of conceptual understanding), whereas the problem 123 +123 -89 actually requires calculation even though it uses the same numbers (i.e., an indication of procedural skill). Gilmore and Bryant (2006) presented children with four-term problems that were either inverse problems (e.g., a + b – b = a) or control problems (e.g., a + b – c = d).

A cluster analysis was conducted on the children's accuracy scores on the inversion problems, the control problems, and an arithmetic reasoning test. Three different clusters, each of roughly the same size, were found. In one of the groups, children tended to have high scores on the inverse problems, the control problems, and the arithmetic test, and they labeled this high-ability group. The second group had low scores on all three measures; they suggested that this group included children with generally lower ability. However, the third group tended to have high scores on the inverse problems but low scores on the control problems and the arithmetic test. These were children who had a good understanding of the inverse principle but did poorly in computation skills. Gilmore and Bryant concluded that children in the 'high ability' and 'low ability' groups showed conceptual understanding that was in-line with their arithmetical skill, and the third group of children had more advanced conceptual understanding than their arithmetical skill. This showed individual differences in how children performed on tasks that had been grouped taking into consideration the type of knowledge children used in solving them.

Canobi (2004) assessed 6 to 8-year-old children's understanding of the part-whole principle and their ability to use the right strategy to solve such problems. In their study, problem-solving tasks were administered to assess the procedures employed by kids when performing addition and subtraction of the part-whole equations. For example, children saw a problem in the form a + b =? appear on a computer screen; they were asked by an interviewer after they have solved the problem to self-report which of the two numbers they first counted. For the conceptual tasks, the kids had been invited to judge whether a task performed by a puppet was correct or incorrect. The strategies used were coded into retrieval, decomposition, counting on, counting all, and fingers. Retrieval was when children self-reported that they knew the answer because they remember it, for example, if the student said: "I just knew it because I remember it." Decomposition was when children reported that they selected a particular answer because they remember how a related problem was solved (e.g., "I know that 5 + 5 equals 10, so 5 + 6 must be 11"). For answers to be coded into counting all, children must have counted, say for 2 + 3, "I counted 1, 2, 3, 4, and 5, while children who started counting on a term; say for 2 + 3, they started counting from 2, 3, 4, etc. Lastly, when children used their fingers to show the answer without counting, it was coded as fingers. For subtraction, answers were coded as counting up when they counted in an increasing order starting with the smallest number. When children report that counted backward starting from the smaller term, (e.g., starting from 6, 5, etc. for a 7 – 5 counting), they were coded as using a counting-down approach. A final method of counting was where fingers are used to model the arithmetic. To measure conceptual understanding, the researchers relied on children's ability to use solve part-whole relations to solve problems, their ability to judge correctly whether an approach employed by a puppet is right or not, and their ability to justify their answers.

Using cluster analysis, Canobi (2004) found individual differences in children's problem solving abilities. For procedural knowledge, they found three separate clusters representing the different strategies employed by children to solve addition and subtraction of part-whole problems. The first cluster represents those children who employed all the different methods of counting (i.e., counting-on, counting-all, retrieval, and finger), and they called this cluster the flexible counters. The second cluster represents those kids who applied only counting on and counting-all, and they labeled this efficient group counters. A final cluster was those who did not use any of the methods explained above, and the called this group the inefficient group. On conceptual knowledge, three clusters were identified as well. The first cluster was those kids who performed better on part-whole tasks; the second cluster were those who had a greater degree of understanding of the commutativity of addition and subtraction and a final group which represented those who were good at the judgement and commutative tasks. They concluded that there are individual differences in the conceptual and procedural knowledge for understanding subtraction and addition.

Canobi's (2004) approach to finding individual differences in conceptual and procedural knowledge, however, examined conceptual and procedural knowledge separately. It did not consider how these individual differences could be part of a larger cluster structure that included both types of knowledge. A better way of doing this would be to put these knowledge types together in a cluster analysis and explore how children combine these two knowledge types.

In the realm of fractions, Hallett and colleagues (2010) investigated whether there are individual differences in how children use conceptual and procedural knowledge. In their study, fourth- and fifth-grade students were drawn from schools across England and were presented different kinds of fraction questions which were later coded to reflect conceptual and procedural knowledge. Items defined as conceptual were those that involved an understanding of equivalence (that two fractions with different numbers can be equal), a comparison between two quantities (judging which fractions are larger), or the realization that fractions can refer to different wholes (a third of something can be greater than a half of something smaller). For example, children were asked to order 1/4, 1/2, 1/100, and 1/3 in increasing order of magnitude. On the other hand, procedural items were those that were judged to be primarily solved by applying an algorithm or procedure taught in school, and that can be implemented without checking for meaning outside the proceedings. Because there was a significant correlation between the two knowledge types, they adopted a procedure where procedural scores were regressed against the conceptual scores while also regressing the conceptual scores against the procedural scores.

In using this regression procedure, Hallett and his colleagues were able to have residuals that were representative of scores of the procedural scale independent of the conceptual scores and conceptual scores of the conceptual scale that were also independent of the procedural scores. Furthermore, these scores were illustrative of relative abilities and not absolute competencies in conceptual or procedural knowledge. These residualized scores were then cluster analyzed to look for different profiles of conceptual and procedural knowledge. Hallett and his colleagues found five different groups: Children who performed highly on the conceptual scales relative to what their procedural score would have predicted (higher conceptual-lower procedural); those who performed high on procedural items relative to what their conceptual scores would have predicted (higher procedural-lower conceptual; students who performed highly on both types of knowledge (higher group); and, lastly, two groups of students who performed poorly overall, but one that did especially poorly on procedural items (lower procedural) and one that did especially poorly on conceptual items (lower conceptual). These results added further credence to what past studies have concluded – that individual differences may exist in the way children acquire and use conceptual and procedural knowledge.

In a follow-up study, Hallett and colleagues (2012) attempted to examine if the same results found among grades four and five would be replicated among older grades. Again, they provided the participants with measures that had been coded into conceptual knowledge (e.g., questions that asked them to order fractions from smallest to largest) and those that reflected procedural understanding (e.g., asking students to solve 2/5 + 3/10) and then interviewed them to explain how they arrived at the solution. Conceptual and procedural scores were again residualized, and separate analyses were conducted for each grade. The result of the cluster analysis using these residualized scores replicated the previous findings in Hallett et al. (2010) among the grade six students but not in the grade eight students. That is, they found similar four clusters among the Grade 6 students representing those who were more procedural, more conceptual, high on both, and low on both. However, for the Grade 8 students, only two clusters were evident, representing children who are dominantly conceptual (more conceptual) and those who are dominantly procedural (more procedural). Hallett et al concluded that it is possible that the differences in the number of clusters may be a result of the fact that as students get older, they tend to specialize in each of these types of knowledge. The results give further evidence that there are individual differences in how children rely on conceptual and procedural knowledge when solving math problems.

The Current Study

While Hallett and colleagues (2010, 2012) demonstrated that individual differences might exist in the conceptual and procedural understanding of fractions, it has yet to be determined if the same or similar pattern can be found in other math domains. Moreover, as asserted by Star (2016), "notably absent are the studies of the development of conceptual and procedural in algebra" (p. 2). As a consequence, the current study involves an investigation into whether or not the individual differences (i.e., clusters) found in fractions by Hallett and his colleagues (2010, 2012) are evident in children's understanding of algebra.

Exploring this relationship is very essential, especially in the field of algebra. In algebra, one needs to be very proficient in applying procedures without compromising on conceptual abilities to be a successful problem solver. This study facilitates a comparison of the extent to which children use different conceptual-procedural strategies in algebra than in other math domains. The existence of similar clusters in algebra learning would further support the notion that not all children combine conceptual and procedural understanding in the same way, but that there are individual variations. Thus, if the same profiles found in fractions (see Hallett et al., 2010, 2012) are seen with algebra, it will strengthen the claim that children do not follow a universal developmental pattern in learning conceptual and procedural understanding.

Research questions

Three major research questions are examined.

1. Do children learn and use conceptual and procedural knowledge differently in algebra?

2. If there are individual differences in conceptual and procedural knowledge, do these clusters differ in their overall understanding of algebra?

3. Does conceptual and procedural knowledge contribute significant variance independently on kids' overall algebra performance?

Chapter 2

Method

Participants

A total of 124 Grade 8 students were recruited from a junior high school in Newfoundland, Canada. Twenty students were excluded from the analyses because they did not complete at least three tasks that were used in this study. Of the 104 that were included in the analysis, 65 were boys, and 39 were girls (M_{age} = 13.660, *SD* = 0.451).

Design and Measures

This was a correlational study that used a within-participant design. Four separate measures were administered to participants. These are the conceptual measure, procedural measure, the Chelsea Diagnostic Mathematics Algebra Test, and the Ravens Standard of Progression Test. The Conceptual and procedural measure was adopted from Rittle-Johnson and Star (2007). Below are details of each measure and how they were scored.

Conceptual and Procedural knowledge assessment (Rittle-Johnson & Star, 2007). The conceptual and procedural knowledge assessment was designed to measure conceptual and procedural knowledge, as well as assess whether or not comparing solutions in algebra influences students understanding and performance. This assessment consists of 29 questions (12 multiple-choice, 17 open-ended). Nine of the questions are procedural knowledge items which require the use of procedural understanding to solve, eight questions test flexibility (e.g., judging the best first step in a solution), and 12 questions test conceptual knowledge (e.g., understanding of equivalence). The test was recently used by Star et al., 2014 with a little modification and yielded overall internal consistency (conceptual and procedural items *a*= .89 and *a*= .77, respectively) and *a*= .76

on the procedural flexibility items. The flexibility items were removed from the measure because it was not the core focus of this research and it shortened the duration of the test. This brought the total number of questions to 16 although the conceptual items have aspects that require further explanations.

The nine procedural knowledge items were algebraic equations where the student was asked to solve for x. One point was awarded for each right step that was shown. Scores obtained were calculated based on the overall percentage of the correct steps for a total possible score of 40. The conceptual knowledge questions were designed to test children's understanding of different aspects of algebra, including the meaning of the equal sign (e.g., that adding the same thing to both sides of an equation do not change the validity of the equation, or the solution for x), the combination of like terms (e.g., what is m + m + m + m the same as?), the nature of unknowns (e.g., how many possible values are there for k in the expression k + 6), and how children could conceptually understand procedures in order to be able to pick out which next steps in a problem would be valid (and potentially more than one of them could be; see Parts II and III in Appendix B).

On the conceptual knowledge assessment, students received one point for correctly answering each of the nine objective questions. Of these nine multiple choice items, six had more than one answer correct, and participants were asked to select all options that were correct. A point was given for each correct answer selected. Also, students were asked to explain their reasoning on three items, and these explanations were scored for one point each. The scores from these explanations were added to students' conceptual knowledge totals. Thus, a conceptual knowledge score was calculated as a percentage of total possible points. On the conceptual measure, students could earn a maximum of 20 points. The results were coded by the researcher and his supervisor based on the operational definition of conceptual understanding in this study and the guide provided by the authors of the measure. The conceptual measure and procedural measures had Cronbach alpha's of = 0.60 and = 0.81, respectively. See Appendix B for the full measure.

Raven's Standard Progressive Matrices (Raven, Raven, & Court, 1998). The Raven's Progressive Matrices is a non-verbal test to measure students overall cognitive abilities. This standardized measure consists of geometric analogy problems in which a matrix of geometric figures is presented with one entry missing, and the correct missing entry must be selected from a set of answer choices. This measure has been designed to be and has been separately validated as, an index of general conceptual ability or fluid intelligence (Raven et al., 1998). Participants were asked to complete a subset of 32 items, based on age-norms, ranging from C3 to E10.

The Chelsea Diagnostic Mathematics Algebra Test [Hart & National Foundation for Education Research (NFER), 1984]. The Chelsea Diagnostic Mathematics Algebra Test was used as a measure of students' general algebra understanding. This test has been designed to test children's level of understanding across a broad range of typical secondary school algebra tasks. These include substitution, simplifying expressions and constructing, and interpreting and solving equations. It contains 23 questions and focusses on the different ways in which children use and interpret letters in generalized arithmetic. It has been shown to be an appropriate test for children in their second, third, and fourth year in high school. Specifically, it can be used for students aged 12 years or older.

Procedure

Students were given a permission form for their parents to sign and return. Only those students with signed, returned forms participated in the study. Before distributing the assessment measure, students were reminded that despite receiving permission from their parents to participate, it was still their decision as to whether they would like to participate. They were also told that they could withdraw from the study at any time and it will not impact their grades. The administration of the tests was counter-balanced. One group of students (*n* = 39 in the final analysis) had the Raven's test first before the conceptual and procedural measure and the general algebra measure, while another group of students (*n* = 65 in the final analysis) had the conceptual and procedural measures taken first before the general algebra measure and Raven's test. All measures were group administered. Instructions were read out loud, and the experimenter completed two examples with the students on the Raven's and the Chelsea Diagnostic. After each student had finished all measures, the students were debriefed and thanked for participating in the research and were given $10 in compensation.

Analyses

This thesis adopted the method of analysis used by Hallett et al. (2010, 2012). However, as explained in more detail below, the analyses were slightly modified by adding a third variable into the cluster analysis, which helped to better classify the data into clusters.

Residualized Conceptual and Procedural Scores. Although the measures of conceptual and procedural knowledge were designed to include items that participants would be expected to mostly employ conceptual or procedural understanding, respectively, to solve, previous studies have shown a significant correlation between these two measures. To have scores that were separate from each other, conceptual scores were regressed against procedural scores while procedural scores were also regressed against procedural scores (see Fig. 1, Panels A & B), and these residuals were then standardized. Regressing procedural scores against conceptual scores generated residuals which represent conceptual scores independent of procedural ability while regressing conceptual scores against procedural scores created residuals which represents procedural scores that were independent of conceptual scores.

It is important to note that these residuals are relative scores and not absolute scores. Thus, a score represented students' performance on one type of knowledge relative to how the other type of knowledge would predict they would do. For example, a participant who received a score of 2 on conceptual residual did not score 2 standard deviations above the mean, but instead, 2 standard errors above their conceptual predicted from their procedural score. Likewise, a procedural score of -1 would not necessarily mean that that student is below the mean on their procedural skill, just that their procedural skill is lower (by 1 standard error) than what we would expect that score to be, given their conceptual score. That is what makes these scores relative instead of absolute.

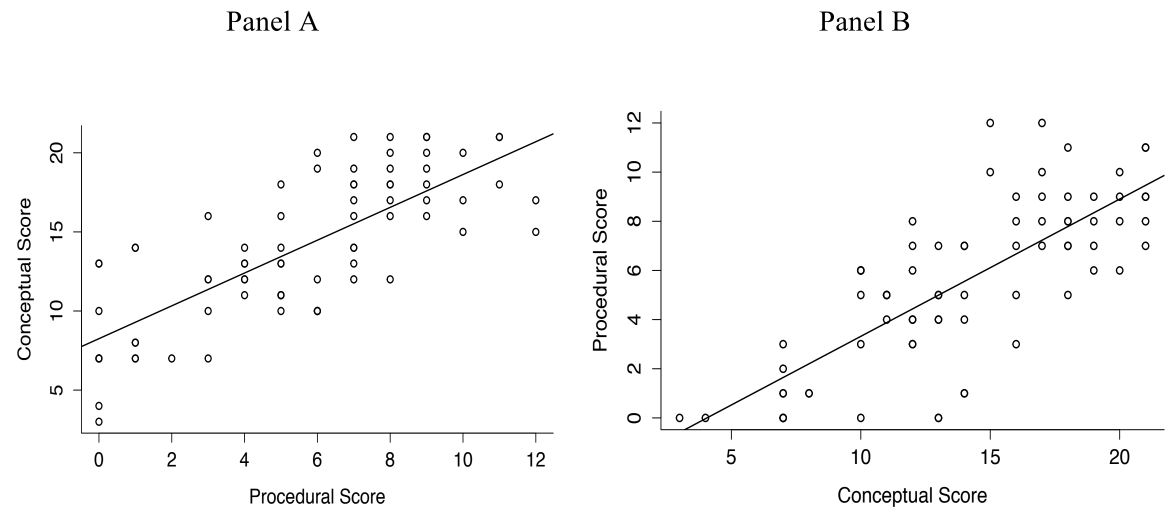


Figure 1. The above figures show how residualized conceptual and procedural scores were obtained. In Panel A, conceptual raw scores are entered into the regression analysis as the criterion while procedural scores are entered as the predictor. In Panel B, the procedural raw score is entered in the regression analysis as the dependent variable (criterion) while conceptual raw scores are entered as the predictors. The results are two separate residualized scores. This was adopted from Hallett et al., (2010, 2012).

Improved Way of Doing Cluster Analysis. Creating residualized scores in the manner explained above means that there is a possibility for an individual who has a very weak score on one variable to be wrongly classified as possessing that knowledge. This is because the nature of regression is that predicted scores can be negative even if it is not actually possible to receive a negative score, especially if there are a substantial number of students who score near zero on the predictor variable. In these data, 21 of the 104 participants scored a zero on the procedural measure. With data having this signature, a participant who had a very low score on the procedural measure, even if that score was zero, could end up with a positive procedural residual if their score on the conceptual measure was low enough that it predicted a negative score on the procedural measure. This would make it look like a student with zero on the procedural measure, probably erroneously, was relatively strong on procedural skill. This means the cluster solutions obtained may not be a true representation of the actual situation on the ground.

To rectify this problem, two options were considered. The first option was to exclude the students who scored zero on the conceptual and the procedural measures. Of the final 104 students included in this analysis, 21 scored zero on those two measures and as such their data would have be excluded if this option was adopted. A second option would be to add a third variable representing overall ability to control for the potential clustering errors. The addition of this general ability measure would help separate out low overall ability people from others, which would include those who have zero on the procedural measure but appear to have a relative strength in procedural knowledge. A preliminary analysis compared the results of the cluster analysis when these students were deleted to when a third variable was added. This analysis found that the number of clusters and the cluster pattern were essentially the same. For this reason, the present analysis used the third variable control, as this included the most subjects.

To obtain a third variable, we created a measure that was an equally weighted and standardized combination of the conceptual and procedural scores. This variable was a representation of the absolute and actual scores obtained by students and was used as a proxy measure of their general strength in algebra. This variable was named the overall score. This was then included in the cluster analysis to control for students' general ability, which would have the effect of preventing children with low ability with relative scores that might not reflect their relative strength (because of the boundary limits of the measure) from being grouped with children whose relative strength in procedural knowledge is likely more real.

Chapter 3

Results

Before proceeding to cluster analyze the data, correlations were run on all of the assessments (Table 1). There was a significant positive correlation between the Chelsea Diagnostic test and all other measures. The Raven's Matrices, however, did not correlate with the Conceptual and Procedural Knowledge measures, which was not expected. However, there was a significant positive correlation between conceptual and procedural knowledge.

Table 1

Pearson correlations between Conceptual and Procedural measures, Chelsea Diagnostics Test and the Ravens Matrices (N= 92)

Measures	*1*	*2*	*3*	*4*
1. Concepts	-			
2. Procedures	0.244*	-		
3. Chelsea	0.388**	0.342**	-	
4. Ravens	0.130	0.091	0.294**	-

Note. Concepts is the conceptual part of the Conceptual and Procedural Knowledge task, Procedures is the procedural part for the Conceptual and Procedural Knowledge task, Chelsea is the Chelsea Diagnostic Test for General Knowledge of Algebra, Raven's is the Raven's Standard Progressive Matrices task. *$p < 0.05$, **$p < 0.01$

Cluster Analysis

The first thing to determine was if there are individual differences in conceptual and procedural knowledge of algebra. Two separate residual scores were created from regressing one variable against the other as described above in the Method. Furthermore, an overall score was created by combining the conceptual and procedural scores, again, as outlined in the Method. These three variables were then cluster analyzed together.

To obtain the ideal cluster solution, this thesis adopted the procedure used by Hallett and his colleagues (2010, 2012). In this method, a rate (denoted by C*(g)*) is calculated based on the amount of variance between and within the clusters. Thus, the good cluster solution would have a larger variance between the clusters but a relatively small variance within each cluster. A better cluster solution, therefore, is one that has the bigger calculated C*(g)* statistic. Furthermore, cluster solutions that had splinter clusters were to be ignored. Table 2 displays the C*(g)* statistic for each solution from 2 to 10 clusters. Although cluster solutions 7, 8, 9, and 10 have higher C*(g)* scores, they all have at least one group that have very low number of members (2). Because of the potential for such small groups to act as outliers, these cluster solutions were not considered – a practice consistent with others who have done cluster analysis in mathematical cognition (see Gilmore and Bryan, 2008). For this reason, the best cluster solution was the four-cluster solution.

Research question 1: Do individual differences exist in conceptual and procedural knowledge?

To answer the question of whether there are individual differences, we relied on the cluster solutions explained above. As mentioned earlier, the analysis found four different clusters representing the different ways in which Grade 8 students combine conceptual and procedural knowledge. The first group of students (33.65%), belonged to the cluster designated the lower cluster. For these students, their performance was below expectation on both conceptual and procedural knowledge.

Table 2

C(g) statistics showing the number of cluster solution

Solution	C*(g)*
10	107.73421
9	96.22245
8	91.34717
7	90.08904
6	82.13914
5	79.45711
4	86.77680
3	66.63228
2	55.50238

C(g) is the rate of variability between clusters divided by the variability within each cluster. Generally, the higher the C(g), the better the cluster. Although cluster solutions 10, 9, 8, and 7 had higher C(g) statistical rate, they have too many splinter clusters that have very few members. In this study, cluster 4 was adopted as the best cluster solution although it has a lower C(g) than cluster solutions 10, 9, 8, and 7. This was because cluster 4 have relatively comparable number of members.

The last cluster was called the higher cluster. This represented a group of students (12.50%), who performed above expectation on both their conceptual and procedural scores. The two middle groups represented students (20.19%) who performed better on conceptual scores than their procedural would have predicted, and students (33.65%), who performed better on procedural scores than their conceptual scores would have predicted. These groups were called more conceptual and more procedural, respectively (see Fig. 2).

As would be expected, students who did relatively poorly on both conceptual and procedural knowledge were those with lower scores on the overall score, and the students who had relatively better scores on both conceptual and procedural knowledge were the ones with better performance on the overall variable. The more conceptual group had relatively higher performance on the overall score than the more procedural group.

Research question 2: Do these clusters differ on general algebra performance?

To give meaning to the clusters, we compared these clusters on the Chelsea Diagnostic test. This was to examine whether the cluster to which one belongs affects how they will perform on a general algebra ability. To achieve this, a One-way ANOVA was conducted with Cluster as the grouping variable and performance on the Chelsea Diagnostic test as the dependent variable. Overall, there was a significant difference in performance between these clusters, $F_{(3, 99)} = 8.515$, $p < 0.001$, $\eta^2 = 0.212$. Tukey's post hoc test showed that the Lower group ($M = 23.53$), differed significantly from the Higher group ($M = 35.538$, $p < 0.001$), the more procedural group ($M = 29.882$, $p = 0.015$), and the more conceptual group, ($M = 32.700$, $p = 0.001$)[1].

The more procedural group, ($M = 29.882$), did not differ significantly from the more conceptual group, ($M = 32.700$, $p = 0.636$), and the Higher group, ($M = 35.538$, $p = 0.174$). Finally, the more conceptual group, $M = 32.700$, also did not differ significantly from the Higher group, ($M = 35.538$, $p = 0.790$). The analysis shows that the differences that exist among these clusters at the general level are driven by the Lower cluster.

[1] *p*-values are reported exactly to three decimal places. When p values were too small to be represented to three decimal places, $p < .001$ was used.

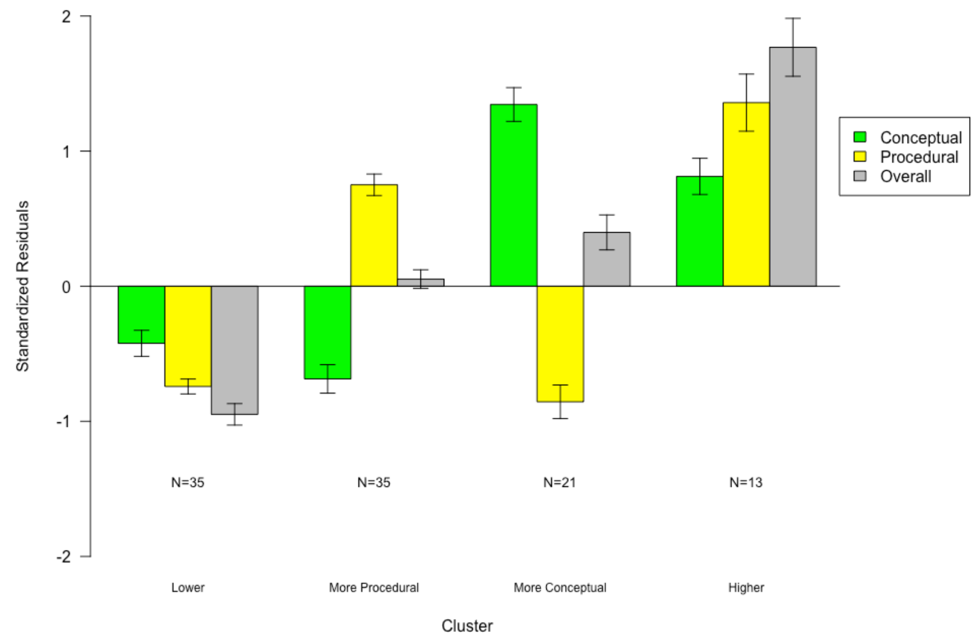


Figure 2. Cluster solution for residualized conceptual and procedural scores for grades 8 students on Procedural and Conceptual Knowledge of Algebra measure.

The inability to detect a statistically significant group differences between the more conceptual and the more procedural clusters could be another way of arguing that both conceptual and procedural knowledge are equally beneficial in learning algebra. To test for this specifically, would require analyzing to determine the amount of variance in the Chelsea task explained by each of these knowledge types. This brings up the third research question examined in this study.

Research question 3: Do conceptual and procedural knowledge independently predict general performance in algebra?

The next question explored in this study was whether conceptual or procedural knowledge was more strongly related to one's general performance in algebra, assessed using the Chelsea Diagnostic Algebra Test. Using hierarchical regression analysis, scores on the Chelsea Diagnostic test were entered as the criterion variable, Gender, Order, and Raven's scores were entered into the first block, and conceptual and procedural scores were entered in the second block. Gender, Order, and Raven's predicted a significant amount of variance in scores on the general algebra performance (see Table 3), but only scores on the Raven's added a unique amount of variance in general algebra performance. When conceptual and procedural scores were added in the second block, the model continued to account for a significant amount of variance in the general algebra scores, $\Delta R2$ = 0.204, $F_{(2, 87)} = 13.397$, p <.001. Both conceptual and procedural scores independently predicted general algebra performance and scores on the Raven's remained a unique predictor of the variable in general algebra scores.

To test for the amount of variance uniquely explained by conceptual and procedural knowledge, a three block regression analysis was conducted. It was found that conceptual knowledge uniquely predicted 13.400%, p <0.001 of the variance controlling for Order, Gender and the general cognitive ability of the students. On the other hand, procedural knowledge exclusively accounted for 7.007%, p = 0.003 of the variance in the general algebra measure. This is quite surprising, especially because both conceptual and procedural knowledge were highly correlated. This indicates that both conceptual and procedural knowledge independently explained a significant amount of variance above and beyond the order in which a test was presented and gender.

Table 3

Predictors of General Algebra performance in a hierarchical regression indicating variance explained by each block, B values, and ß values (SE) (N = 93)

	Predictors	R^2	*B*	*ß(SE)*
Block 1				
	Gender		-2.562	-0.134(1.898)
	Order		-0.333	-0.020(1.642)
	Raven's		0.705*	0.333(0.210)
	Total R^2	0.133**		
Block 2				
	Gender		-2.928	-0.153(1.685)
	Order		-1.016	-0.062(1.459)
	Raven's		0.577*	0.273(0.188)
	Conceptual		0.802**	0.317(0.229)
	Procedural		0.303*	0.271(0.100)
	Total R^2	0.337**		
Total R^2		0.0.470**		

Note. R^2 is the amount of variance accounted for in general algebra scores; *B* is the unstandardized coefficient, *ß* is the standardized coefficient, and SE is the standard error of the *ß* coefficient *$p < 0.01$, **$p < .001$

Chapter 4
Discussion

The current study examined three main research questions. The first was whether there are individual differences in the way in which students combine conceptual and procedural knowledge when solving algebra questions, as has been found in other math domains (see Canobi 2003, 2004; Hallett et al., 2010, 2012). The second issue was whether these clusters would differ in overall algebra ability. Finally, this study explored whether conceptual and procedural knowledge would independently predict significant variance in students' general performance of algebra.

Generally, the results from this study have further supported the claim that there are individual differences in conceptual and procedural knowledge. As previously discussed, there are three main perspectives on what researchers believe to be the relationship between conceptual and procedural knowledge. These are the concepts first, procedural first, and the bi-directional outlook. If the claim by the proponents of the concepts first approach were true, then it would have been expected that there will be only one cluster that would be made up of only students who are relatively good at conceptual knowledge. On the other-hand, if the claim by the procedures first perspective is true, then it would be expected that we would find only one cluster that would represent students who are relatively good with procedural knowledge. Finally, if the view by the bi-directional perspective is true, then we would expect no clusters, as children would switch back and forth between learning conceptual knowledge and procedural knowledge in such a way that they would not develop a strong reliance on one over the other.

The findings of the current study, however, demonstrate that students may choose different ways of learning and use conceptual and procedural knowledge when solving algebra problems. This is evidenced in the four different cluster solutions obtained. The more concept cluster represents the perspective that claimed that children develop conceptual understanding first and later build upon this to derive the knowledge of procedures needed to solve a given math problem. The more procedural group represents the procedures first perspective which claims that procedural knowledge precedes conceptual knowledge. The iterative viewpoint might be represented in both the higher group and the lower group.

The findings from this study offer one way to reconcile all the various perspectives that exist in the literature. The four-cluster solution found in this study paralleled that observed by Hallett et al. (2010, 2012). In Hallett and colleague's study (2012), although four cluster solutions were found for the grade 6 students, only 2 cluster solutions were found for grades 8 students. One can speculate that the differences in the cluster solution may be a result of poor performance of the participants in their study. It is possible that the cluster solutions in this study returned to the four cluster solutions found by Hallett et al., (2010, 2012) with their grades 6 because of the third variable included in the current study. The grades 8 data in Hallett et al. (2012) has recently been re-analyzed by adding a similar third variable representing overall ability, and a 3-cluster solution became the optimal one (see Appendix A). By adding a third variable to the cluster analysis, the cluster solution found in this study more closely parallels the one found by Hallett and his colleagues (2012), as the three clusters seem to parallel the more conceptual, more procedural and higher clusters from this study.

Although individual differences in conceptual and procedural knowledge have been found in other math domains such as fractions (Hallett et al., 2010, 2012), addition and subtraction (Canobi, 2004), this is the first time conceptual and procedural knowledge in algebra has been investigated using this approach. Further, using residualized scores enabled the current study to create independent scores. It can, therefore, be concluded that there are individual differences in the way students combine their conceptual and procedural knowledge when solving algebra problems and individual differences are not restricted to only one math domain, and it also cuts across different age groups. Further, our inability to find significant group differences between the more procedural and the more conceptual group means that children who are good at using their conceptual knowledge of algebra perform equally well as those who perform best on procedural knowledge. Although we did not find significant group differences between the more procedural, more conceptual groups, and the higher group, this could be a result of differences in the sample sizes between the groups, as the pattern of means would suggest that the Higher group might outperform all other groups.

In addition to demonstrating individual differences in conceptual and procedural knowledge, the next research question that was investigated in this study was whether both conceptual and procedural independently predict variance in the overall algebra measure. The results confirm that conceptual knowledge uniquely explained 13.400% of the variance in the overall scores for algebra while procedural scores uniquely explained 7.00% of the variance in the general algebra scores. In a previous study, Hallett and colleagues (2012) similarly found that both conceptual and procedural knowledge were independently predictive of both grades 6s and 8s knowledge of fractions. The results from this study have further confirmed this observation. Children rely on both conceptual and procedural knowledge when they have to solve algebra questions. Educators must, therefore, make conscious efforts to ensure that children have the opportunity to learn both knowledge types. Thus, one type of knowledge must not be given preferential treatment or sacrificed in the classroom at the expense of the other.

Not much research has been done with the direct intention of investigating whether teaching concepts first or procedures first affects the way kids perform in algebra. However, generally, the latent reason for the longstanding debate on whether children develop conceptual knowledge first or procedural knowledge first is to argue for kids to be taught the concepts that underlay certain procedures before they are made to use the procedures to solve any given question. On the contrary, the group of researchers who believe that procedures precede concepts would presumably advocate for children to be taught to use procedures first so that they build an understanding of the principles underneath those procedures with time. One can argue that the purpose of these debates is to devise the best approach to learning math in generally irrespective of the domains.

The result of this study, however, has discounted the argument from both ends. In other words, the findings from this study could help to begin to bring closure to the debate. That is, this study has found that children's performance is dictated by first, both their level of conceptual understanding and their ability to execute the right procedures to solve problems. Secondly, learning conceptual or procedural knowledge does not have a significant bearing on the performance of the each other.

Limitations

Some issues may have affected the result of this study. The first is sample selection. Although the study had a sizeable sample, all of the participants were sampled from the same school. This hinders the ability to generalize the findings in this study, although the school was located in an area with a wide range of incomes. Also, even though the order in which the conceptual and procedural knowledge items were presented was counterbalanced, both of these measures were in the same booklet. It is possible that although kids had one task first, they flipped to the other task. For example, students who received a conceptual and procedural measure that has conceptual knowledge items first may have flipped to the procedural knowledge section. Consequently, although they would be classified as those who received conceptual items first, in reality, they may have answered procedural knowledge tasks first. A better way would be to present a timed version of each of these tasks separately. This would restrict students to answering the tasks in the order in which they are presented. Finally, this is a correlation study, and as such, it was difficult to infer a causal relationship.

On a separate note, the inability of the Ravens to correlate with the conceptual and procedural measures was somewhat surprising given our expectation that the student's general cognitive ability should relate to their performance. At the same time, the Raven's was correlated with the Chelsea Diagnostic Test of Algebra, which is to be expected, but it is interesting that this general measure of Algebra is related to the Raven's while the conceptual and procedural measures of Algebra are not. Furthermore, the regression done for Research Question 3 above found that the Raven's was independently predictive of scores on the Chelsea Diagnostic Test even after controlling for conceptual and procedural knowledge. In their work on fractions, Hallett and his colleagues (2012) performed a similar regression and found that the Raven's was not independently predictive of general fraction ability after controlling for their measures of conceptual and procedural knowledge of fractions. This would suggest that these measures of conceptual and procedural knowledge of algebra, developed by Rittle-Johnson & Star (2007), may be ones that measure conceptual and procedural knowledge independent from academic ability. This result, however, is still unexpected, and suggests that further research should be done to ensure these tasks are measuring the constructs that they were designed to measure.

Future Work

Future work could explore these individual differences with older grades. For instance, grades 10s or older could be considered. This would help to ascertain whether these individual differences are true across higher grades. Also, future research should investigate these individual differences in other math domains such as calculus, proportions, percentage calculations, and the like. This would help to explore the universality of these individual differences. Future work should consider a longitudinal method where the conceptual and procedural knowledge developments are tracked across developmental time periods. This will help to draw a better conclusion about the relationship between these two knowledge types.

Implications

The implications of this study are of two fold. The first implication of this study is in the classroom setting. The results of this study have shown that there are children who are better at absorbing the principles and concepts of algebra than employing the procedures to solve given problems. On the contrary, there are also some children who are better at employing procedures to solve algebra questions than understanding the concepts. In other words, some kids can learn to use algebra procedures without necessarily understanding why those procedures work, while other children could possess the understanding of the concepts behind algebra problems but have difficulties in using the right procedures to solve problems in the domain.

As implied by the results, the kids who performed best are the ones who are good at both conceptual and procedural knowledge. If this is the case, then educators can devise means of assessing the relative strength of kids in these two knowledge types. Thus, kids who lack conceptual knowledge could be identified, and the needed intervention could be attention provided to help them catch up. On the contrary, educators could identify procedural deficient kids and provide the necessary intervention to help improve their knowledge in that regard.

This study has further strengthened the argument that research on conceptual and procedural knowledge must involve an investigation of individual differences. It can, therefore, be argued that researchers must learn to adopt the investigation of individual differences when exploring conceptual and procedural knowledge. Exploring individual differences enables researchers to understand conceptual and procedural knowledge holistically and in a much more detail. This could help resolve future debates such as that which has been witnessed among numerical cognition researchers on the developmental relationship between conceptual and procedural knowledge.

Conclusion

This study is the first to explore the relationship between conceptual and procedural knowledge in algebra. Although several studies have in the past examined this relationship in other mathematical domains, this was the first attempt at exploring those relations with algebra by considering the possibility of individual differences. Additionally, the findings from this study have further strengthened the argument made by Hallett and his colleagues that children learn conceptual and procedural knowledge differently. This has contributed to resolving the long-standing debates on the developmental relation between conceptual and procedural knowledge. The study did not discount the three competing sides of the debate, i.e., the concept first, the procedures first, and the bidirectional approach. It does, however, offer a better approach to interpreting the results in so far as examining conceptual and procedural knowledge are concerned.

References

Achieve (2007). Closing the expectations gap: *An annual 50-state progress report on the alignment of high school policies with the demands of college and work*. Retrieved July 1, 2011, from http://www.achieve.org/node/844.

Mathematics teaching in the middle school (Vol. 2) Reston, VA: National Council of Teachers of Mathematics.

Allensworth, E., & Easton, J. Q. (2007). What Matters for Staying On-track and Graduating in Chicago Public High Schools: A Close Look at Course Grades, Failures, and Attendance in the Freshman Year. Chicago, IL: Consortium on Chicago School Research

Anderson, J. R., & Fincham, J. M. (1994). Acquisition of procedural skills from examples. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 20*, 1322–1340.

Baroody, A. J., & Ginsburg, H. P. (1983). The effects of instruction on children's understanding

Briars, D., & Siegier, R. (1984). A featural analysis of preschoolers' counting knowledge. *Develop- mental Psychology, 20*, 607-618

Brown, J.S. & VanLehn, K. (1980). Repair theory: A generative theory of bugs in procedural skills. Cognitive Science, 4, 379-426.

Byrnes, J. P. (1992). The conceptual basis of procedural learning. *Cognitive Development, 7,* 235–237.

Byrnes, J. P., & Wasik, B. A. (1991). The role of conceptual knowledge in mathematical procedural learning. *Developmental Psychology, 27*, 777–786. doi: 10.1037//0012-1649.27.5.777.

C.K. Gilmore, P. Bryant Individual differences in children's understanding of inversion and arithmetical skill. British Journal of Educational Psychology, 76 (2006), pp. 309–331

Canobi, K. H. (2005). Children's profiles of addition and subtraction understanding. *Journal of Experimental Child Psychology, 92*, 220–246.

Canobi, K. H., Reeve, R. A., & Pattison, P. E. (2003). Patterns of knowledge in children's addition. *Developmental Psychology, 39*, 521–534.

Chen, X. (2009*). Students Who Study Science, Technology, Engineering, and Mathematics (STEM) in Postsecondary Education* (NCES 2009-161).

Coolman, R, 2015 What Is Algebra?" https://www.livescience.com/50258-algebra.html

Cooper, T.J., Boulton-Lewis, G.M., Atweh, B., Pillay, H., Wilss, L., & Mutch, S. (1997). The transition from arithmetic to algebra: Initial understanding of equals, operations and variable. In the *Proceedings of the 2pt Conference of the International Group for the Psychology of Mathematics Education,* Lahti, Finland, 2, 89-96.

Fischbein, E., & Barash, A. (1993). Algorithmic models and their misuse in solving algebraic problems. *Proceedings of PME 17*, Tsukuba, Japan, Vol. I, pp. 162-172.

Fuson, K., & Hall, J. (1983). The acquisition of early number word meanings; A conceptual analysis and review. In H. Ginsburg (Ed.), *The develop- mental of mathematical thinking* (pp. 49-107). New York: Academic Press.

Fyre, D., Braisby, N., Lowe, J., Maroudas, C., and Nicholls, J. (1989). Young children's understanding of counting & and Cardinality. Child Development 60, 1158 -1171.

Gelman, R., & Baillargeon, R. (1983). A review of some Piagetian concepts. In J.H. Flavell& E.M. Markman (Eds.), Cognitive development, Vol. 3 of P.H. Mussen (General Ed.), Handbook of child psychology. New York: Wiley.

Gilmore, C. K., & Bryant, P. (2008). Can children construct inverse relations in arithmetic? Evidence for individual differences in the development of conceptual understanding and computational skill. British Journal of Developmental Psychology, 26, 301–316.

Gelman, R., & Meck, E. (1983). Preschooler's counting: Principles before skill. *Cognition, 13,* 343–359.

Gelman, R., & Meck, E. (1986). The notion of principle: The case of counting. In J. Hiebert (Ed.), *The relationship between procedural and conceptual competence* (pp. 29–57). Hillsdale, NJ: Erlbaum.

Gelman, R., & Tucker, M. (1975). Further investigations of the young child's conception of number. *Child Development, 46*,167–175.

Gelman, R., Meck, E., & Merkin, S. (1986). Young 25 children's numerical competence. *Cognitive Development, 1*(1), 1–29.

Goldin-Meadow, S., Alibali, M. W., & Church, R. B. 1993. Transitions in Concept Acquisition: Using the Hand to Read the Mind. *Psychological Review*, 100(2):279-297.

Hallett, D., Nunes, T., & Bryant, P. (2010). Individual differences in conceptual and procedural knowledge when learning fractions. *Journal of Educational Psychology, 102,* 395–406. doi: 10.1037/a0017486

Hallett, D., Nunes, T., Bryant, P., & Thorpe, C. M. (2012). Individual differences in conceptual and procedural fraction understanding: the role of abilities and school experience. Journal of Experimental Child Psychology, 113, 469–486. doi:10.1016/j. jecp.2012.07.009.

Hart, K. M. (Kathleen M.) & National Foundation for Educational Research in England and Wales & Chelsea College. CSMS Mathematics Team & Chelsea College. Concepts in Secondary Mathematics and Science Programme (1984). *Chelsea diagnostic mathematics tests*. NFER-Nelson, Windsor, Berks

Hiebert, J. (1986). Conceptual and Procedural Knowledge: *The Case of Mathematics*. Hillsdale,

Hiebert, J. & Lefevre, P. (1986). *Conceptual and Procedural Knowledge in Mathematics: An Introductory Analysis (pp. 1–27)*. Hillsdale, NJ: Erlbaum.

K.H. Canobi (2004) *Individual differences in children's addition and subtraction knowledge* Cognitive Development, 19 (2004), pp. 81–93

Kerslake, D. (1986). Fractions: Children's strategies and errors: A report of the strategies and errors in Secondary Mathematics Project. Windsor, UK: NFER–Nelson.

Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.), *Handbook*

Kieren, T. E. (1993). *Rational and fractional numbers: From quotient fields to recursive understanding*. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 49–84). Hillsdale, NJ: Erlbaum.

Kilpatrick, J., Swafford, J. O., & Findell, B. (2001). Adding it up: Helping Children Learn Mathematics. Washington, DC: National Academy Press.

knowledge, procedural knowledge, and procedural flexibility in two samples differing in prior knowledge. *Developmental Psychology, 47* (6), 1525–1538. doi: doi:10.1037/a0024997.

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 37*(4), 297-312.

LeFevre, J.-A., Smith-Chant, B. L., Fast, L., et al. (2006). What counts as knowing? The development of conceptual and procedural knowledge of counting from kindergarten through grade 2. *Journal of Experimental Child Psychology, 93,* 285–303. doi: 10.1016/j.jecp.2005.11.002.

Livneh, D., & Linchevski, L. (2007). Algebrification of arithmetic: Developing algebraic structure sense in the context of arithmetic. In J. Woo, H. Lew, K. Park, & D. Seo (Eds), Proceedings of the 31st conference of the psychology of mathematics education (Vol. 3, pp. 217–225). Seoul, Korea: PME

Loveless, T. (2008). The misplaced math student: Lost in eighth-grade algebra. Washington DC: Brown Center on Education Policy at Brookings

Lucariello, J., Tine, M., & Ganley, C. M. (2014). A formative assessment of students' algebraic variable misconceptions. *Journal of Mathematical Behavior, 33*, 30-4

Mabbott, D. J., & Bisanz, J. (2003). Developmental change and individual differences in children’s multiplication. Child Development, 74, 1091–1107.

Mastin, L. (Ed.). (2010). Declarative (Explicit) and Procedural (Implicit) Memory. Retrieved October 30, 2016, from http://www.human-memory.net/types_declarative.html

McNeil, N. M. & Alibali, M. W. (2000). Learning mathematics from procedural instruction: externally imposed goals influence what is learned. *Journal of Educational Psychology, 92*, 734–744. doi: 10.1037//0022–0663.92.4.734.

McNeil, N. M., Weinberg, A., Hattikudur, S., Stephens, A. C., Asquith, P., Knuth, E. J, ., et al. (2010). A is for apple: Mnemonic symbols hinder the interpretation of algebraic expressions. Journal of Educational Psychology, 102(3), 625–634.

National Center for Educational Statistics (2000, August). *The NAEP 1999 Long-Term Trend Mathematics Summary Data Tables for Age 17 Student*. Accessed at http://nces.ed.gov/naep3/tables/Ltt1999/NTM31031.PDF.

National Center for Educational Statistics. (2005). The nations report card: Mathematics. Accessed from http://nces.ed.gov/nationsreportcard/

National Council of Teachers of Mathematics (2006). Curriculum and Evaluation Standards for school mathematics. Reston, VA. Author.

National Mathematics Advisory Panel (2008). Foundations for success: The final report of the national mathematics advisory panel. Washington, DC: Department of Education.

NCTM (2000). *Principles and standards for school mathematics*. Reston, Va: NCTM. Assessing knowledge of mathematical equivalence p. 41

NJ: Erlbaum.

of research on mathematics teaching and learning (pp. 390-419). New York: Simon &

of the "Equals" Sign. *Elementary School Journal, 84*(2), 199-212.

Peck, D. M., & Jencks, S. M. (1981). Conceptual issues in the teaching and learning of fractions. Journal for Research in Mathematics Education, 12, 339–348.

Posner, G. J., Strike, K. A., Hewson, P. W., & Gertzog, W. A. (1982). Accommodation of a scientific conception: Toward a theory of conceptual change. Science Education, 66, 211–227. http://dx.doi.org/10.1002/sce.3730660207

Raven, J., Raven, J. C., & Court, J. H. (1998). Manual for Raven's progressive matrices and vocabulary scales, Section 3: Standard progressive matrices. San Antonio, TX: Harcourt Assessment.

Resnick, L. B. (1982). Syntax and semantics in learning to subtract. In T. P. Carpenter, J. M. Moser, & T. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 136–155). Hillsdale, NJ: Erlbaum.

review of 25 years of research (Santa Barbara: California Dropout Research Project, University of California, 2008), http://cdrp.ucsb.edu/Dropouts/Pubs_ Reports.Htm#15

Rittle-Johnson, B. (2006). Promoting transfer: effects of self-explanation and direct instruction. *Child Development, 77,* 1–15. doi: 10.1111/j.1467-8624.2006.00852.x.

Rittle-Johnson, B. & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: does one lead to the other? *Journal of Educational Psychology, 91*, 175–189. doi: 10.1037//0022-0663.91.1.175.

Rittle-Johnson, B. & Koedinger, K. R. (2009). Iterating between lessons concepts and procedures can improve mathematics knowledge. *British Journal of Educational Psychology, 79,* 483–500. doi: doi:10.1348/000709908X398106.

Rittle-Johnson, B. & Siegler, R. S. (1998). The relation between conceptual and procedural knowledge in learning mathematics: a review. In C. Donlan (Ed.), *The Development of Mathematical Skills* (pp. 75–110). London: Psychology Press.

Rittle-Johnson, B. & Star, J. R. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology, 99,* 561–574. doi: 10.1037/0022–0663.99.3.561.

Rittle-Johnson, B. & Star, J. R. (2009). Compared with what? The effects of different comparisons on conceptual knowledge and procedural flexibility for equation solving. *Journal of Educational Psychology, 101,* 529–544. doi: 10.1037/a0014224.

Rittle-Johnson, B., Schneider, M., & Star, J. R. (2015). Not a One-Way Street: Bidirectional Relations Between Procedural and Conceptual Knowledge of Mathematics. Educational Psychology Review, *27* (4) 587-597.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing the conceptual understanding and procedural skill in mathematics: an iterative process. *Journal of Educational Psychology, 93,* 346–362. doi: 10.1037//0022–0663.93.2.346.

Rittle-Johnson, B., Star, J. R., & Durkin, K. (2009). The importance of prior knowledge when comparing examples: influences on conceptual and procedural knowledge of equation solving. *Journal of Educational Psychology, 101*, 836–852. doi: 10.1037/a0016026.

Rittle-Johnson, B., Star, J. R., & Durkin, K. (2011). Developing procedural flexibility: are novices prepared to learn from comparing procedures? *British Journal of Educational Psychology, 82* (3), 436–455.doi: 10.1111/j.2044–8279.2011.02037. x.

Rose, H., & Betts, J. R. (2001). *Math matters: The links between high school curriculum, college graduation, and earnings*. San Francisco: Public Policy Institute of California.

Rosnick, P. (1981). Some misconceptions concerning the concept of a variable. Mathematics Teacher, 74, 418–420.

Rosnick, P., & Clements, J. (1980). Learning without understanding: The effect of tutoring strategies on algebra misconceptions. *Journal of Mathematical Behavior,* 3(1), 3-27.

Russell W. Rumberger and Sun Ah Lim, *why students drop out of school: A*

Schmidt, W. H., McKnight, C., Cogan, L. S., Jakwerth, P. M., & Houang, R. T. (1999). *Facing the consequences: Using TIMSS for a closer look at U.S. mathematics and science education*. Boston: Kluwer

Schneider, M., Rittle-Johnson, B., & Star, J. R. (2011). Relations between conceptual

Schuster.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. Educational studies in Mathematics, 22(1), 1–36

Sfard, A. & Linchevski, L. (1994). The gains and the pitfalls of reification - the case of algebra. *Educational Studies in Mathematics,* 26, 191-228.

Silver, E. A. (1997). Algebra for all – Increasing students' access to algebraic ideas. Not just

Star, J. R. (2005). Reconceptualizing procedural knowledge. *Journal for Research in Mathematics Education, 36,* 404–411.

Star, J. R. & Newton, K. J. (2009). The nature and development of expert are strategy flexibility for solving equations. *ZDM Mathematics Education, 41,* 557–567. doi: 10.1007/s11858–009-01855.

Star, J. R. & Rittle-Johnson, B. (2008). Flexibility in problem-solving: the case of equation solving. *Learning and Instruction, 18*, 565–579. doi: 10.1016/j.learninstruc.2007.09.018.

Star, J. R. & Rittle-Johnson, B. (2009). It pays to compare: an experimental study on computational estimation. *Journal of Experimental Child Psychology, 101,* 408–426. doi: 10.1016/j.jecp.2008.11.00.

Star, J. R., & Stylianides, G. J. (2013). Procedural and conceptual knowledge: Exploring the gap between knowledge type and knowledge quality. Canadian Journal of Science, Mathematics, and Technology Education, 13, 169–181. doi:10.1080/14926156.2013.784828.

Star, J. R., Pollack, C., Durkin, K., Rittle-Johnson, B., Lynch, K., Newton, K., & Gogolen, C.

(2014, May 28). Learning from the comparison in algebra. *Contemporary Educational Psychology*, 41-54. doi:10.1016

U.S. Department of Education, National Center for Education Statistics. (2001). ECLS-K, *Base Year Public-Use Data File, Kindergarten Class of 1998–99: Data Files and Electronic Code Book (Child, Teacher, School Files), and User's Manual* (NCES 2001-029rev). Washington, DC: Author.

VanLehn, K. (1990). *Mind bugs*. Cambridge, MA: MIT Press.

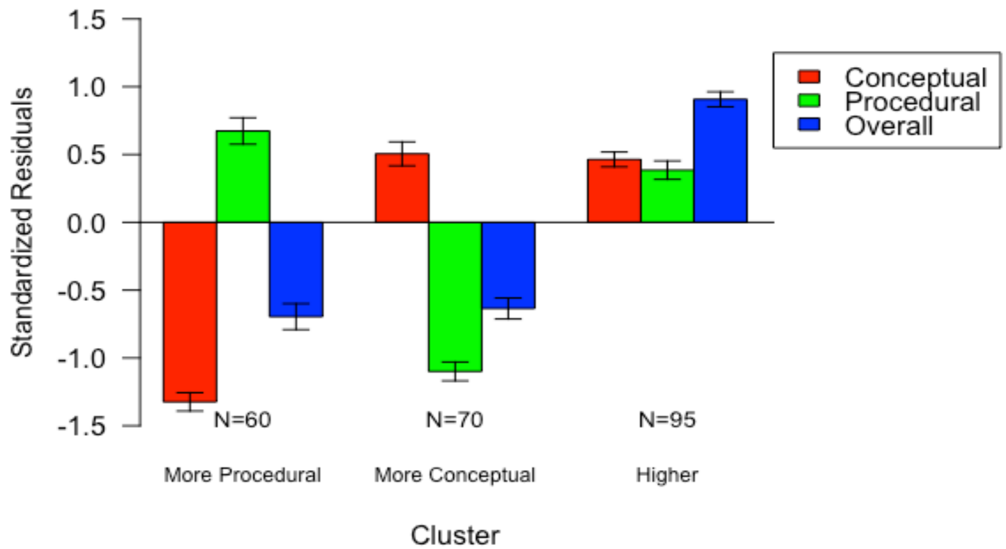
Weibell, C. J. (2011). *Principles of learning: 7 principles to guide personalized, student-centered learning in the technology-enhanced, blended learning environment.* Retrieved July 4, 2011 from [https://principlesoflearning.wordpress.com]

Wittrock, M. C. 1989. Generative processes of comprehension. *Educational Psychologist*, *24*, 345–376.

Appendix

Appendix A

Cluster solutions from Hallett et, (2012), grade 8 students with a third variable



Appendix B
CONCEPTUAL AND PROCEDURAL ALGEBRA MEASURE

First Name & Last Initial ________________________________
Date ________________________________

Class Period: 1 2 4 5 6
______

This exercise will help us learn how you think about algebra. Please do your best to complete all the questions.
If you don't know an answer, you may guess or write "I don't know".
Please don't leave any questions blank – we want to know how much you had time to try.
If you make a mistake, please gently erase it.

Each section is timed. If you finish a section early, you may go ahead to the next section. You may not go back, even if you have extra time later. Once you finish a page, please move to the next page and do not look back.

Thank you for doing your best work on this exercise.

Part I. Answer the questions below. {10 mins}

1) What does the equal sign mean?
Can the equal sign mean anything else? If yes, what?

____
____
____
____
____
____
____
____

2) If m is a positive number, which of these is equivalent to (the same as) m + m + m + m? Circle your answer.

a. m + 4
b. 4m
c. m^4
d. 4(m + 1)

3) Here are two equations.

Equation #1: 213x + 476 = 984

Equation #2: 213x + 476 + 4 = 984 + 4

Without doing the math, what can you say about the answers to this two equations? Circle a, b, or c below.

a. The answer to Equation #1 is same as the answer to Equation #2
b. The answer to Equation #1 is different from the answer to Equation #2
c. I can't tell without doing the math

Explain your reasoning:

____
____
____
____
____
____
____

____

____

4) If k can be replaced by any number, how many different values can the expression k + 6 have?

a. None
b. One
c. Six
d. Seven
e. Infinitely many

5) Without solving each equation, which of the following equations are equivalent to (will have the same answer as) the equation: $32(x - 12) = 96$. Circle all that apply

a. $32x – 12 = 96$
b. $x – 12 = 96 – 32$
c. $16x – 16 \cdot 12 = 48$
d. $16x – 6 = 48$
e. $\frac{32(x − 12)}{32} = \frac{96}{32}$

Explain why you chose those ones:

____
____
____
____
____
____
____
____
____

Part II. On each of the problems in this section, you will see one line in a student's solution
to an equation. Listed below each line are four possible next steps. For each equation, figure
out ALL possible steps that the student could do in the NEXT step. For each problem, circle
ALL steps that are possible for the student to do in the NEXT step. {3mins}

6) $2(x + 1) + 4 = 12$

a. Combine like terms
b. Distribute across parentheses
c. Add or subtract the same number on both sides
d. Multiply or divide by the same number on both sides

7) $15(x + 3) + 5(x + 3) = 10(x + 3)$

a. Combine like terms
b. Distribute across parentheses
c. Add or subtract the same number on both sides
d. Multiply or divide by the same number on both sides

Part III. Below are the beginning steps of how a student tried to solve several equations.
Look at the way that this student started each equation and answer the questions below. {8mins}

8) $3(x + 2) = 12$
$x + 2 = 4$
a. In the part of the solution shown above, what step did the student use to get
from the first line to the second line? Circle your answer below.

a. Combine like terms
b. Distribute across parentheses
c. Add or subtract the same number on both sides
d. Multiply or divide by the same number on both sides

b. Looking at the problem shown above, do you think that this way of starting
to do this problem is a good idea? An ok step to make? Circle your answer below and explain your reasoning.
(a) Very good way (b) Ok to do, but not a very
good way
(c) Not OK to do

Explain your reasoning:

____
____
____
____
____
____
____
____
____
____
____

9) $13(x + 25) + 38 = 15(x + 25)$
$38 = 2(x + 25)$

a. In the part of the solution shown above, what step did the student use to get
from the first line to the second line? Circle your answer below.

a. Combine like terms
b. Distribute across parentheses
c. Add or subtract the same number on both sides
d. Multiply or divide by the same number on both sides

b. Looking at the problem shown above, do you think that this way of starting
to do this problem is a good idea? An ok step to make? Circle your answer below and explain your reasoning.

(a) Very good way
(b) Ok to do, but not a very good way
(c) Not OK to do

Explain your reasoning:

____
____
____

Part IV. Solve the following 8 equations. Show all your work and steps you used to arrive at your answer. {20mins)

12) $-1/4(x - 3) = 10$

13) $-3(b + 2) + 9(b + 2) + 4(b + 2) = -30$

14) $5(y - 12) = 3(y - 12) + 20$

15) $5(2x + 3) + 4x + 2 = 7(x + 1) + 20 + 2x$

16) $0.25(t + 3) = 0.5$

17) $-3(x + 5 + 3x) - 5(x + 5 + 3x) = 24$

18) $-2(y + 1) + 7(y + 1) - 3(y + 1) + 5(2y - 6) = 2(y + 1) + 6(y + 1)$

19) $3(x + 1) + 2x + 7 – 3(x + 1) – 7 = 4x + 12 – 4x – 12 + 10$

Appendix C *Student's knowledge of algebra*

Researcher: Felix Ayesu, Master of Science Candidate
Email: fayesu@mun.ca, Phone: 709-690-9452

Supervisor: Darcy Hallett, Associate Professor of Psychology
Email: darcy@mun.ca
Phone: 709-864-4871, Fax: 709-864-2340

Greetings,

Your child has been invited to take part in a research project investigating children's knowledge and understanding of algebra problems.

This form is part of the process of informed consent. It will provide you with information regarding what the research is about and what your child's participation will involve. It also describes your child's right to withdraw from the study. To decide whether you wish to have your child participate in this research study, you should understand enough about it risks and benefits to be able to make an informed decision. Please take the time to read this carefully and to understand the information given to you. Please contact the researcher, Felix Ayesu if you have any questions about the study.

It is you and your child's decision whether to take part in this research. If you or your child chooses not to participate in this study, or if you or your child decides to withdraw from the research once it has started, there will be no adverse consequences for your child now or in the future.

Introduction:

I am a Masters' student in the Psychology Department at Memorial University of Newfoundland conducting data collection for a research project in your child's school. The published results of this study will be publicly available at the QEII library, but these results will be aggregated across groups of students and will not include any identifying information about your child. The purpose of this study is to understand better how students approach algebra problems. If we can better understand the different ways that students perform these problems, we may be able to devise better ways to teach them.

What will your child do in this study?

If your child participates in this study, he or she will complete three paper-and-pencil tasks in the classroom or any designated room on school grounds. The first, called Raven's Progressive Matrices, asks participants to choose missing symbols that fit the various patterns presented to them. The second is a general algebra measure, which will be a lot like a standard algebra test. The third measure will be another algebra test, but this one will ask questions that are more focused on particular ways of understanding or doing algebra.

Length of time:

This study will occur over two sessions (two different days) in total, and the entire study will take your child 2 hours of regular course instructional time.

Withdrawal from the study:

Your child is free to withdraw from the study with no consequences. If your child decides to no longer be involved in the study, he or she can inform the researcher during or after data collection. You or your child will cease to be able to withdraw from the study once the school year has ended (June 2017). If you or your child decides, before June 2017, to withdraw from the study, please contact my supervisor or myself using the above-listed contact information. In the case of withdrawal, the related data will be disposed of before data analysis.

Possible benefits:

This will give your child extra practice with algebra problems, which has been reported as an area in math that children find challenging. Also, your child will have an opportunity to experience scientific research firsthand and contribute to the advancement of the field.

Compensation:

As a gesture of thanks for their time spent participating in this study, your child will be given $10. If they complete only one of the data collections sessions and then decide to withdraw, they will receive $5.

Possible risks:

Conceivable risks are test and math anxiety, but these risks are no higher than it would be participating in a regular math class. Students will be reminded that this is not a test and will not count towards their grades. If your child experiences any anxiety during the study, they will be reminded that they can withdraw without consequence. Those who do withdraw due to anxiety will be brought back to their classroom (if the test is administered outside of the child's classroom) or will be sent to any room deemed appropriate by the heads of the school (if the test is being administered in the classroom); those experiencing extreme anxiety (e.g., crying) will be brought to the school's guidance counselor.

Confidentiality:

Results for each child are kept strictly confidential. If the results are published in a scientific journal, it will be two or three years after the end of the study. Summaries of information about different groups of participants will be given. There will be no permanent record kept that your child participated in this study.

Your child's data will be stored at Memorial University of Newfoundland's Research Centre for the Development of Mathematical Cognition. Data will be stored in a secured area in this locked laboratory in which only those associated with the study have access. Electronic data will also be stored on a computer that is password protected in the locked facility. Data will be kept for a minimum of five years, as required by Memorial University policy on Integrity in Scholarly Research.

Anonymity:

Anonymity cannot be maintained in this research project. Your child will participate in a group setting. Therefore, their peers will likely know that your child is participating. Complete anonymity will be maintained in the published findings.

Recording of Data:

Data will be recorded primarily using written responses. There will be no interview sessions and hence no audio information will be recorded.

Storage of Data:

All hard copy data will be stored in a locked filing cabinet, and electronic data will be stored on a password-protected computer in a locked laboratory at Memorial University's Research Centre for the Development of Mathematical Cognition, where only lab members have keys. All electronic data will be accessible by only the lead researcher. All data will be kept for a minimum of five years, as required by Memorial University's policy on Integrity in Scholarly Research.

Questions:

You are welcome to ask questions at any time before, during, or after your child's participation in this research. If you would like more information about this study, please contact Felix Ayesu at fayesu@mun.ca or Darcy Hallett at darcy@mun.ca.

The proposal for this research has been reviewed by the Interdisciplinary Committee on Ethics in Human Research and found to be in compliance with Memorial University's ethics policy. If you have ethical concerns about the research, such as the way you have been treated or your rights as a participant, you may contact the Chairperson of the ICEHR at icehr@mun.ca or by telephone at 709-864-2861.

Felix Ayesu, Master of Science Candidate
Email: fayesu@mun.ca, Phone: 709-690-9452

Supervisor: Darcy Hallett, Associate Professor of Psychology
Email: darcy@mun.ca
Phone: 709-864-4871, Fax: 709-864-2340

You and your child have agreed to take part in a study through Memorial University that is designed to investigate children's knowledge and understanding of Algebra problems.

Children will be visited by researchers in their school and will be asked to complete a series of tasks.

Participation in this study is not a requirement of your child's school or the teacher, and hence, will have no effect on your child's school grades

If you have any additional questions that are not answered by the information sheet, please contact Darcy Hallett, or Felix Ayesu through the contact details listed above.

The proposal for this research has been reviewed by the Interdisciplinary Committee on Ethics in Human Research and found to be in compliance with Memorial University's ethics policy. If you have ethical concerns about the research, such as the way you have been treated or your rights as a participant, you may contact the Chairperson of the ICEHR at icehr@mun.ca or by telephone at 709-864-2861.

Please fill out the form below to indicate whether or not you would like your child to participate.

Your child is also required to complete this form to indicate whether they would like to participate.

Your signature on this form means that:

Parents:

- You have read the information about the research.
- You understand what the study is about and what your level and your child's level of involvement are.
- You understand that any data collected from your child up to the point of your withdrawal will be discarded.
- You understand that you are consenting to the use of the data provided by your child on the demographics sheet (date of birth, gender, and grade) accompanying this consent form. This information will be used ONLY for the purpose of analysis and nothing else.

Student:

- You have read the information about the research.
- You understand what the study is about and what your level of involvement are.
- You understand that any data collected from you up to the point of your withdrawal will be discarded.
- You understand that you are consenting to participate in this study.

Your signature:

I have read and understood what this study is about and appreciate the risks and benefits. I have had adequate time to think about this and I agree to participate voluntarily, and I understand that I may end my or my child's participation at any time.

Signature of Guardian Date

Signature of Student Date