Dynamics of Some Partial Differential Equation Models Arising in Fluid Mechanics and Biology

by

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Abstract

In this thesis, we study the dynamics of some partial differential models arising in fluid mechanics and biology. First, we analyze a long-wave model for a liquid thin film on an inclined periodic substrate that is valid at a near-critical Reynolds number. The existence and the uniqueness, as well as the asymptotic formula, of a periodic steady-state are derived. Floquet-Bloch theory and asymptotic analysis are carried out to study the stability in a weighted functional space. The generalized Burgers equation is another fluid model that we consider. After transforming the problem into a constant coefficients problem, a shooting method is used to prove the existence of separable solutions. The total number of them is given and the uniqueness of the positive solution is proved. The stability of the small-amplitude positive steady-state is provided using the bifurcation analysis. Dynamics of a two-species competition model with diffusion is studied in the last part. The minimal wave speed selection mechanism (linear vs. nonlinear) is investigated. Hosono conjectured that there is a critical value of the birth rate so that the speed selection changes only at this value. We prove a modified version of this conjecture and establish some new results for the linear and the nonlinear speed selection. The local and the global stability, using the comparison principle together with the squeezing technique, of the traveling wavefront are studied in a weighted functional space. Some open problems and future works are presented.

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To my parents Salman and Feryal

with love, gratitude, and admiration

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Chapter 1

Introduction

Partial differential equation models arise mainly in the formulation of the physical, chemical, and biological laws. Most of these models are nonlinear and have a complicated structure. This makes the solution formula not always valid, even numerically. However, the existence of the solution, or the so-called steady-state solution, can be proved for some models. Also, the long-time behavior of the solution can be found. Generally, such information leads to further studies of the problem to obtain more explanation and significant results.

This thesis is concerned with the steady-state (stationary or equilibrium) solutions to some partial differential equations and their stability. A steady-state is a time-independent solution, i.e., there is no change with respect to time in the functions which describe the behavior of the system. The stability of the steady-state is the behavior of the solution under perturbations of the initial condition. If the solution, after a long enough time period, converges to the steady-state, then we say that the steady-state is stable. Otherwise, it is unstable. Usually, when the perturbation is sufficiently close to the steady-state, the stability analysis becomes easier. In this case, if we get the required convergence, the steady-state is said to be locally stable.

For arbitrary initial value, we have global stability.

In this thesis, we study the dynamics of three partial differential equation models arising in fluid mechanics and biology. In the following sections, we introduce these models and give a brief introduction for our research works on each model.

1.1 A Liquid Thin Film on a Periodic Wall

In the first research work of this thesis, we consider a flow of a thin film over an inclined periodic wall under gravity. This problem has been of great interest to scientific researchers, as it arises in considerable applications for many topics, for example see [1, 6, 40, 74, 75, 92, 93]. In 1955, Yih [100] investigated flow over a vertical plane. By numerical computations, the instability of the flow is proved for a large value of a Reynolds number (R), which is given in terms of the liquid density and the liquid viscosity. Based on Yih's formulation, Binjamin 1957 [5] proved that the steady flow is unstable for all finite Reynolds numbers. Yih 1963 [101] considered flow on an inclined flat wall. When the wall is inclined at an angle θ to the horizontal line, he proved that there exists a critical value

$$R_c = \frac{5}{4}\cot(\theta)$$

so that the steady-state solution to the equation of motion is stable if $R \leq R_c$, and unstable if $R > R_c$, see also the earlier articles [45, 55]. It is easy to see that $R_c = 0$ for vertical inclinations. This means that the flow is unstable for all values of R, or simply critical Reynolds number does not exist. In the last thirty years a large number of works considered the problem with a flat wall , e.g. [2, 3, 23, 38, 66, 69, 102] and the reference therein.

The surface between the liquid and the air responds to the wall topography shape when it becomes uneven. The flow on an uneven wall has been investigated in many previous works, e.g. [8,16,36,68,71,83,87,95]. Some of previous numerical, experimental, and analytical results will be discussed in Chapter 2. To study the problem, we consider a long wave model that is valid at a near-critical Reynolds number. Assume that the flow is in the x-direction and let $h(x,t)$ be the film thickness at location x and time t. The equation of motion is given by (see [83])

$$h_t + \frac{d}{dx}\left[\frac{2}{3}h^3 + \frac{8R}{15}h^6 h_x - \frac{2\cot(\theta)}{3}h^3(h+s)_x + \frac{1}{3C}h^3(h+s)_{xxx}\right] = 0.$$

Here,

$$R = \frac{gh_0^3\sin(\theta)}{2\nu^2} \quad \text{and} \quad C = \frac{\rho g h_0^2 \sin(\theta)}{2\gamma}$$

are Reynolds and capillary numbers, respectively, where g is gravity, h_0 is the average film thickness, ν is the liquid kinematic viscosity, ρ is the liquid density, and γ is the surface tension. The model is derived in [82–84] based on the Navier-Stokes equation.

In Chapter 2, we construct an iteration scheme in terms of an integral form of the original steady-state problem. The uniform convergence of the scheme is proved so that we can derive the existence and the uniqueness, as well as the asymptotic formula, of the periodic solutions. The analysis is split into three cases based on the formulation of the integral form. We re-write the equation into a new form so that we can combine the different cases in a single case. By the method of abstract contraction mapping, we prove the existence and the uniqueness of the steady-state in a particular functional space. Using the Floquet-Bloch theory and asymptotic method, we establish several analytic results on the stability of the periodic steady-state solution in a weighted functional space.

1.2 The Generalized Burgers Equation

A physical model that describes fluid turbulences is the convection-diffusion Burgers equation

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = \delta \nabla^2 \mathbf{u},$$

where $\mathbf{u}$ is the flow velocity, t is time, and δ is the kinematic liquid viscosity. This equation is introduced for the first time by Burgers [10], and it can be derived from the Navier-Stokes equation for Newtonian incompressible fluid

$$\rho \left(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{F},$$

where ρ is the density, p is the pressure, μ is the dynamic liquid viscosity, and $\mathbf{F}$ is an external force. If we assume no pressure or external forces and use the relation $\delta = \mu/\rho$, then the Burgers equation follows. In one space dimension, $0 \leq x \leq l$, the equation becomes

$$u_t + uu_x = \delta u_{xx}.$$

The exact solution to the Burgers equation in one dimension with the boundary-initial conditions

$$u(0,t) = u(l,t) = 0,$$
$$u(x,0) = u_0(x),$$

can be obtained by using the Hopf-Cole transformation (see [27])

$$v(x,t) = \exp\left(-\frac{1}{2\delta} \int_0^x u(\xi,t) d\xi \right).$$

This transforms the equation into heat equation form $v_t = \delta v_{xx}$. If the initial function is given by

$$u_0(x) = u_0 \sin\left(\frac{\pi x}{l}\right),$$

then the exact solution has the formula

$$u(x,t) = \frac{4\pi\delta}{l}\left\{\frac{\sum_{n=1}^{\infty}\exp\left(-\frac{n^2\pi^2\delta t}{l^2}\right) n \sin\left(\frac{n\pi x}{l}\right)}{1 + 2\sum_{n=1}^{\infty}\exp\left(-\frac{n^2\pi^2\delta t}{l^2}\right)\cos\left(\frac{n\pi x}{l}\right)}\right\}.$$

See a similar formula (8.7.8) in [12] and also formula (2.114) in [65].

The generalized Burgers equation with time-dependent viscosity in the form

$$u_t + uu_x = f(t)u_{xx},$$

has been considered in some previous works [24,72,73]. In Chapter 3, we consider the last equation, a time-dependent viscose equation, with

$$f(t) = \frac{\delta}{(t+1)^M},$$

for constant M, subject to the initial-boundary conditions

$$u(0,t) = u(l,t) = 0,\ t \in \mathbb{R}^+,$$
$$u(x,0) = u_0(x),\ x \in [0,l].$$

This model is investigated in [72,73]. We study the dynamics of separable solutions to the equation. We first incorporate a transformation to reduce the separable solutions into steady-states of a nonlinear partial differential equation with constant coefficients. By developing a shooting method, the existence of steady-state solutions is proved and their number is given by an explicit formula. The uniqueness of the positive solution is also verified. The weakly nonlinear bifurcation-analysis is conducted and the stability of the small-amplitude positive solution is provided.

1.3 Traveling Waves to a Two-species Competition Model

A traveling wave solution to a partial differential equation model is a wave-shaped function that travels in a special domain with a constant speed $c \geq 0$. At any time the shape will be the same. This kind of solution has been extensively investigated in the last few years, e.g. [17, 33, 44, 50, 59, 64, 88, 89, 98]. To have a mathematical understanding of this kind of solution, we present the work of Fisher [19] and KPP [41] on the reaction-diffusion scalar equation (Fisher-KPP equation)

$$\begin{cases} u_t = u_{xx} + f(u), \\ u(x,0) = u_0(x), \end{cases}$$

where $u(x,t)$ is a function of a special variable x and time t. Here $f(u)$ is a nonlinear function which is positive inside the interval $(0,1)$ and satisfies

$$f(0) = f(1) = 0, \quad f'(0) > 0, \quad \text{and} \quad f'(1) < 0.$$

A simple common example of the nonlinearity in the equation is the Fisher function $f(u) = (1-u)u$.

A traveling wave solution connecting 1 to 0 and spreads with speed $c \geq 0$ is a solution in the form

$$u(x,t) = U(z), \quad z = x - ct.$$

Here U is called the wavefront, z is the wave variable, and c is the wave speed. From the Fisher-KPP equation, $U(z)$ satisfies the ordinary differential equation

$$U_{zz} + cU_z + f(U) = 0, \tag{1.3.1}$$

subject to

$$U(-\infty) = 1, \quad U(+\infty) = 0. \tag{1.3.2}$$

To be applicable in physics, chemistry, and biology the wave profile $U(z)$ has to be bounded and non-negative in the domain for which we are concerned. By the linearization and the phase plane analysis, a positive monotone solution $U(z)$ to (1.3.1)-(1.3.2) exists with

$$c \geq c_0 = 2\sqrt{f'(0)}.$$

Define $c_{\min}$ as the minimal wave speed so that solution to (1.3.1)-(1.3.2) exists. Indeed, $c_{\min}$ is greater than or equal to c_0. These two cases are said to be nonlinear and linear speed selection, respectively. It is known that when $f(u)$ is bounded by its linearization about 0, i.e., satisfies the inequality

$$f(u) \leq f'(0)u, \tag{1.3.3}$$

a traveling wave exists for any $c \geq c_0$, see e.g. [4, 41]. In fact, the existence of a traveling wave which spreads with the same speed of the corresponding linear system can also be obtained by the upper-lower solution method similar to that in [103], that is, the linear speed selection is realized. Lucia *et al* [48] completely studied the problem of speed selection to the Fisher-KPP equation, where sufficient conditions for the linear and the nonlinear selection mechanisms were obtained. We include here some of their results.

Theorem 1.3.1. [48, Theorems 5.1-5.2].

(i) *If* $2\int_0^u f(s)ds \leq f'(0)u^2$, *then the linear speed selection is realized.*

(ii) *If* $2f'(0) \leq \int_0^1 f(s)ds$, *then the nonlinear speed selection is realized.*

Observe that the condition in (i) is the generalization of the condition (1.3.3). In general, determination of the speed selection mechanisms is not trivial and depends on the nonlinearity of the equations, especially for systems of equations.

For the stability of the traveling wave $U(x-ct)$ to (1.3.1)-(1.3.2), we recall the work of Moet [53]. Let the condition (1.3.3) be satisfied and $u(x,t)$ be the solution of (1.3.1) which is perturbed initially from $U(x-ct)$. Hence,

$$u(x,t) = U(x-ct) + v(x,t;c),$$

for some function $v(x,t;c)$. The partial differential equation for v in the (z,t)-coordinates is given by

$$\begin{cases} v_t = v_{zz} + cv_z + f(U+v) - f(U), \\ v(z,0) = v_0(z) := u_0(z) - U(z). \end{cases}$$

Introduce a weight function $w(z) = \exp\left(\frac{c}{2}z\right)$ and a weighted functional space

$$L^p_w(\mathbb{R}) = \{v(z) : w(z)v(z) \in L^p(\mathbb{R}), p \geq 1\}$$

with the norm defined by

$$\|v\|_w = \left(\int_{-\infty}^{\infty} w(z)|v(z)|^p dz\right)^{\frac{1}{p}}.$$

Assume $0 \le U(z) + v_0(z) \le 1$, for all $z \in \mathbb{R}$, and $v_0(z) \in L^p_w(\mathbb{R})$, for some $p \ge 1$. Let $\bar{v}(z,t) = w(z)v(z,t)$, then the equation for $\bar{v}$ is given by

$$\begin{cases} \bar{v}_t = \bar{v}_{zz} + F(\bar{v}), \\ \bar{v}(z,0) = \bar{v}_0(z) := w(z)v_0(z), \end{cases}$$

where

$$F(\bar{v}) = -\frac{c^2}{4}\bar{v} + wf\left(U + \frac{\bar{v}}{w}\right) - wf(U).$$

Define $p_1(z,t)$ and $p_2(z,t)$ as solutions of the differential equation

$$p_t = p_{zz} + F(p),$$

with the initial conditions $p(z,0) = \min\{\bar{v}_0(z), 0\}$ and $p(z,0) = \max\{\bar{v}_0(z), 0\}$, respectively. By comparison, we have

$$p_1(z,t) \le \bar{v}(z,t) \le p_2(z,t), \ \forall (z,t) \in \mathbb{R} \times \mathbb{R}^+.$$

By the condition (1.3.3), it is easy to get

$$F(\bar{v}) \le -\left(\frac{c^2}{4} - f'(0)\right)\bar{v}.$$

Using this fact in the $p-$equation, Moet proved that $p_1(z,t)$ and $p_2(z,t)$ tend to zero as $t \to \infty$. By the squeezing technique, this is true for $v(z,t)$, which gives the stability of $U(z)$ in the weighted space L^p_w.

We choose to work on the speed selection problem and the stability of the traveling wave solution to a two-species competition model in a Lotka-Volterra type. Consider the system

$$\begin{cases} \phi_t = d_1\phi_{xx} + r_1\phi(1 - b_1\phi - a_1\psi), \\ \psi_t = d_2\psi_{xx} + r_2\psi(1 - a_2\phi - b_2\psi), \end{cases} \tag{1.3.4}$$

with the initial data

$$\phi(x,0) = \phi_0(x) \geq 0, \qquad \psi(x,0) = \psi_0(x) \geq 0, \quad \forall x \in \mathbb{R},$$

where $\phi(x,t)$ and $\psi(x,t)$ are the population densities of the species at time t and location x. Here d_i, r_i, a_i, and b_i, for $i = 1, 2$, are non-negative biological parameters.

Equilibrium solutions and their stability, with $d_i = 0$, can be determined by the standard linearization in terms of the parameters. We consider a traveling wave solution to the diffusive system (1.3.4) that connects a stable equilibrium to an unstable one. This is called a monostable traveling wave and equivalent to assuming

$$\frac{a_1}{b_2} < 1 \;\text{ and }\; \frac{a_2}{b_1} > 1,$$

with considering a traveling wave which connects

$$(1/b_1, 0) \;\text{ and }\; (0, 1/b_2).$$

In contrast, when a traveling wave connects two stable equilibria it is called a bistable case, e.g. [20] for scaler equations, [33] for systems, and [44] for equations with delay.

For the speed selection mechanism of the monostable traveling waves to (1.3.4), a conjecture by Hosono [30] states that there exists a positive constant r_c so that the wave speed is linearly selected when

$$(r_1, r_2, a_1, a_2, b_1, b_2) \in \left\{ \frac{a_1 a_2}{b_1 b_2} \leq 1 \right\} \cup \left\{ \frac{a_1 a_2}{b_1 b_2} > 1 \text{ and } \frac{r_2}{r_1} \leq r_c \right\},$$

and nonlinearly selected otherwise. This conjecture attracted the attention of researchers since it was raised in 1998, see [32, 33, 42]. Lewis *et al* [42] proved a part of this conjecture when $\frac{d_2}{d_1} \leq 2$ and gave a lower bound for the critical value r_c. Huang [32] claimed that the result in [42] proves the Hosono's conjecture for the case when $\frac{d_2}{d_1} \leq 2$. We study the problem of the speed selection in the case when $d_2 = 0$ in Chapter 4. After transforming the partial differential equations into a cooperative system, the problem is investigated for the new system. We show that the result in [42] does not give the value of r_c, for the case when $\frac{d_2}{d_1} \leq 2$, in the Hosono's conjecture and the conjecture itself is not completely true. We successfully prove a modified version of the conjecture. Estimation of the critical value is given and some new conditions for linear or nonlinear selection are established. The previous results are presented in detail and compared with ours.

In Chapter 5, we study the local and the global stability of the traveling wavefront to the diffusive competition model (1.3.4) in a weighted functional space. For the global stability, comparison principle together with the squeezing technique, as discussed above for the Fisher-KPP equation, are applied to derive the main results.

The speed selection problem for the full system when $d_1, d_2 > 0$ is still challenging and will be discussed in Chapter 6, the future work.

Chapter 2

Steady-states to a Thin Film on an Inclined Periodic Substrate

Results in Section 2.2 of this chapter have been published in the *Canadian Mathematical Bulletin*[1]. The other part has been accepted for publication in the *Asymptotic Analysis* journal[2].

2.1 Introduction

In literature, many studies initially dealt with the problem of a viscous liquid falling down an inclined wall with a flat surface, where the steady-state solution and its stability characteristics were discussed numerically or theoretically. A change of flatness in the wall surface is more reasonable in practice and this definitely affects the liquid surface behavior. Flow over an inclined corrugated topography has a long history in literature studies. Tougou [78], by using asymptotic analysis, derived an approximate

[1] Alhasanat, A. and Ou, C. H. Periodic steady-state solutions of a liquid film model via a classical method, CMB, 2017, `http://dx.doi.org/10.4153/CMB-2017-035-5`.

[2] Alhasanat, A. and Ou, C. H. Existence and stability of the steady-state to a thin film on an inclined periodic substrate under gravity, Asym. Anal., 2017.

system up to first-order accuracy of the model based on the continuity equation and the Navier-Stokes equation to describe the liquid flow over an uneven wall. The unevenness factor was included in the system and was also addressed in the stability analysis to show its significant impact compared to the flat wall case. Wang [86] applied the perturbation theory to study the flow at low Reynolds numbers on a three-dimensional uneven plate with small amplitude compared to the liquid depth. He investigated the combined effect of the plate wavelength, the inclination angle, and the surface tension on the flow behavior of the liquid surface. Based on the analysis, he found that the liquid surface shares the same period of the plate, while the amplitude and flow rate have more complicated dependency. Pozrikidis [61] investigated creeping flow along a periodic solid wall with arbitrary geometrical shapes including smooth boundaries and corners. The mathematical model was formulated by using the boundary-integral method for the Stokes flow. Detailed numerical calculations for the flow along a sinusoidal wall were performed. The results were compared to previous studies, with an excellent agreement with the asymptotic analysis in [86] for small amplitude wall, but no agreement was also observed in the case for low flow rate. Shetty and Cerro [71] investigated a flow on a wall with semicircles shape. A nonlinear equation of the motion based on the linear momentum balance equation was derived. For small average film thickness (compared to the wall amplitude and wavelength), they found that the film thickness agrees with the Nusselt solution for flow over a flat surface. In the frame of Stokes equation and the continuity equation, flow over a sinusoidal wall with small amplitude (compared to the film thickness) was studied numerically by Bontozoglou and Papapolymerou [8]. For a wide range of Reynolds numbers and a fixed inclination angle, they successfully calculated the resonance phenomenon. Trifonov in [79] investigated a flow down a vertical wall. It was shown that the flow is controlled by the forces of surface tension for small Reynolds number, and by inertia forces for large Reynolds number. For fixed wall amplitude and wavelength, behavior of the liquid surface was studied numerically. Comparison with experimental data was carried out. Kalliadasis *et al* in [35] studied the motion of a thin viscous film flowing over a topographical feature (trench or mound) under the action of an external body force. They applied the lubrication theory to derive a nonlinear partial differential equation of the liquid motion. By solving this equation numerically, it has been shown that the dynamics of the film is governed by the feature depth, feature width, and the capillary scale. Bontozoglou [7] studied flow along large amplitude periodic wall. A numerical method was applied to extend the resonance observed in [8]. Wierschem and Aksel [94] studied the linear stability of a liquid film falling down an inclined wavy wall with long wavelength compared to the film thickness. They found that the critical Reynolds number for instability is greater than that on the flat wall. Further in [96], Wierschem *et al* extended the analysis in [94] by including a missing term to the model. Perturbation theory was carried out to analyze the film flow. Away from the singularity, they found a good agreement between experimental results and the perturbation analysis. They also applied the Floquet theory to study the linear stability. Trifonov in [80] followed the spirit in [79] to study the steady-state solution of the flow and its stability on an inclined wavy wall. Numerical method that allows to describe more complicated regimes of the flow without asymptotic approximation was applied to find the steady-state and show the effect of the parameter values on the stability.

Recently, Tesuilko and Blyth [82] studied the effect of inertia on a film flowing on an uneven wall in the presence of an electric field. They investigated the flow on a wall with small-amplitude sinusoidal corrugations, and derived a nonlinear equation for a thin-film flow (see equation (36) in [82]). This result included the special case derived in [84] (eq. 3.25) for a flow over a flat wall. Tseluiko *et al* [83] worked on the long-wave model derived in [82], assuming that the flow variation as well as the variation in the wall shape in the flow direction are subtle. Ignoring the electric effects, they solved the steady-state problem numerically, and applied the Floquet-Bloch theory to work on the spectrum problem numerically.

As can be seen above, most of previous works dealt with the problem numerically or experimentally. Analytic studies, which give more general results and deep understanding, were not widely carried out. The purpose of this work is to study the problem with a mathematical rigor. Consider a liquid film flow over a periodic wavy wall inclined at an angle θ to the horizontal line. Introduce the (x, y)-coordinates so that x^+-axis represents the flow direction. Let $y = s(x)$ be the periodic function that describes the wall topography. See Figure 2.1.

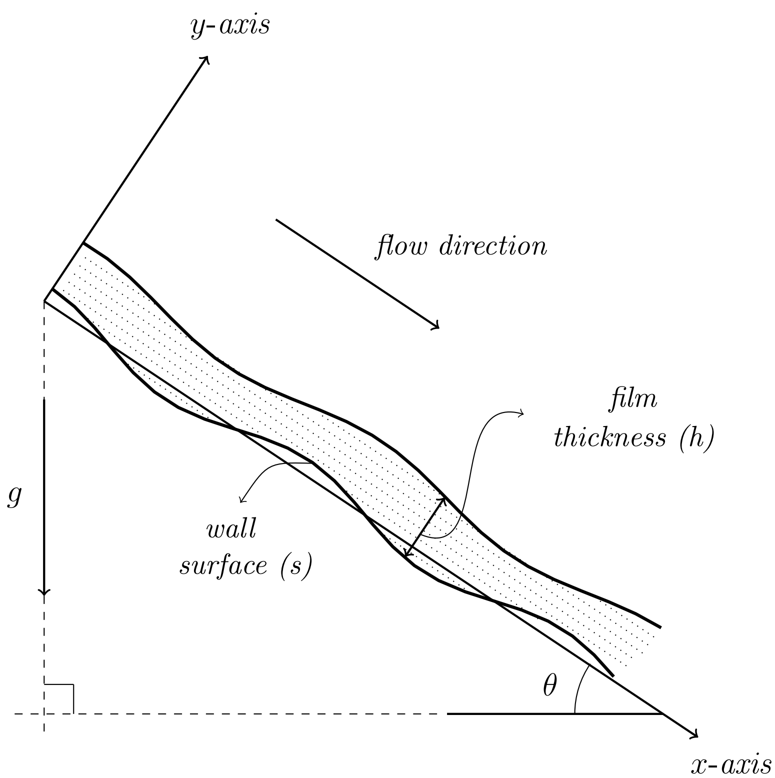


Figure 2.1: The representation of a thin film on a periodic uneven wall.

The flow is governed by the partial differential equation (see [82–84])

$$h_t + q_x = 0, \tag{2.1.1}$$

where $h(x,t)$ is the dimensionless film thickness at time t and location x, and $q(x,t)$ is the flux rate given by

$$q = \frac{2}{3}h^3 + \frac{8R}{15}h^6 h_x - \frac{2\cot(\theta)}{3}h^3(h+s)_x + \frac{1}{3C}h^3(h+s)_{xxx}. \tag{2.1.2}$$

Here, R and C are the Reynolds and capillary numbers, respectively, which are given in terms of the liquid density, the liquid viscosity, and the wall friction. Equation (2.1.1) represents the conservation of mass. The first and third terms in (2.1.2) are due to the $x-$ and $y-$component of gravity, respectively, the second term represents the inertia effects, and the fourth term is due to the surface tension (see [83]).

Throughout this chapter, we assume that the wall surface shape $s(x)$ satisfies

$$|s'(x)| \leq a_1\epsilon \quad \text{and} \quad |s'''(x)| \leq a_2\epsilon, \tag{2.1.3}$$

for small positive number ϵ, and constants a_1, a_2. Actually, this assumption also arose in [83], where both a sinusoidal wall with $s(x) = A\cos\left(\frac{\pi x}{l}\right)$ and a rectangular wall with $s(x) = A\tanh(\cos(\frac{\pi x}{l})/d)$ were considered. Here A is the amplitude, l is the period, and d is a constant such that the smaller the value of d the steeper the wall is. They assumed that A/l is small so that the condition (2.1.3) holds true. However, the analysis in the present work is valid for any pattern subject to this condition.

We should mention that the rigorous proof for the existence of periodic steady state in [82] and [83] is left open, to the best of our knowledge. Our new contribution is proving the existence of periodic steady-states to the partial differential equations analytically. Based on the asymptotic solution formula, we obtain the stability condition of the periodic solution via a perturbation argument in a weighted functional space. Previously this was only carried out numerically in [83].

We study the existence of the steady-states first via a classical method. We give the details in three cases in terms of integral equations. The result not only provides the existence and the uniqueness of a periodic solution, but also gives a generalized asymptotic formula. As can be seen in [25], by "classical methods in differential equations", we mean finite dimensional methods, derived from what is called "classical analysis". Whereas modern applied analysis is commonly used to cast differential equation problems (including boundary value problems) into infinite dimensional settings so that degree theory or infinite-dimensional fixed point theorems can be applied to prove the existence of solutions, "classical analysis", in handling the same problems, often provides more information than the abstract approaches. In particular, the "classical analysis" methods used are more likely to be constructive in some sense and so can form the basis of numerical methods. They are sometimes more global, for instance giving estimates of the size of a small parameter. By applying the technique of modern functional analysis, we can also prove the existence of the steady-state in a unified abstract method.

The rest of the chapter is organized as follows. In Section 2.2, we give the detailed prove of the existence and the uniqueness. By this analysis we derive the asymptotic formula of the steady-state solution. In Section 2.3, we show how to use the contraction mapping method to obtain the existence and the uniqueness in a simple fashion. Linear stability is analytically investigated in Section 2.4, where Floquet-Bloch theory is used to find the stability criteria. Conclusions and summary are presented in Section 2.5.

2.2 Steady-state and its Asymptotic Formula

In this section, we prove the existence and find the asymptotic formula of a periodic steady-state solution, $h(x,t) = h_0(x)$, to (2.1.1)-(2.1.2) via a classical method. By (2.1.2), this is equivalent to find $h_0(x)$ that solves the ordinary differential equation $q_x(x,t) = q'(x) = 0$ or $q(x) = q_0$ for a constant q_0 that is related to the flow flux of the model. For convenience and without loss of generality, we choose $q_0 = 2/3$. Therefore, the steady-state $h_0(x)$ from equation (2.1.2) satisfies

$$\frac{2}{3}h_0^3 + \frac{8R}{15}h_0^6 h_0' - \frac{2\cot(\theta)}{3}h_0^3(h_0+s)' + \frac{1}{3C}h_0^3(h_0+s)''' = \frac{2}{3}, \tag{2.2.1}$$

where prime denotes the derivative d/dx. When condition (2.1.3) holds, $h_0(x) = 1$ is an approximation solution to (2.2.1) (for any q_0, the approximation is $h_0(x) = \sqrt[3]{3q_0/2}$). This suggests that $h_0(x) = 1 + w(x)$ is the exact steady-state solution to (2.2.1), for some periodic small-amplitude function $w(x) \neq -1$. Substitute it into equation (2.2.1) and multiply the equation by $3C/h_0^3$ to get

$$\begin{aligned}\frac{8RC}{5}(3w+3w^2+w^3)w' + \left(\frac{8RC}{5} - 2C\cot(\theta)\right)w' - 2C\cot(\theta)s' + w''' + s''' \\ = 2C\left[\frac{1}{(1+w)^3} - 1\right].\end{aligned}$$

By collecting the linear terms, the latter equation is equivalent to

$$w''' + \left(\frac{8RC}{5} - 2C\cot(\theta)\right)w' + 6Cw = F(s', s''', w, w'), \tag{2.2.2}$$

where

$$F(s', s''', w, w')(x) = 2C\cot(\theta)s'(x) - s'''(x) + \frac{2Cw^2(x)}{(1+w(x))^3}\left[6 + 8w(x) + 3w^2(x)\right] - \frac{8RC}{5}\left[3w(x) + 3w^2(x) + w^3(x)\right] w'(x).$$

Define

$$a := \frac{8RC}{5} - 2C\cot(\theta) \quad \text{and} \quad b := 6C. \tag{2.2.3}$$

The homogeneous part of the non-homogeneous equation (2.2.2) becomes

$$w''' + aw' + bw = 0. \tag{2.2.4}$$

To find the fundamental set of solutions for the third-order homogeneous equation (2.2.4), which has the characteristic equation

$$r^3 + ar + b = 0, \tag{2.2.5}$$

we need the following lemma, which we will use in the stability analysis as well.

Lemma 2.2.1 (Cardano's Formula, see [34, formulas (50)-(51), chapter 4])**.** *The cubic algebraic equation (2.2.5) has the roots*

$$r_1 = \phi + \psi, \;\; r_2 = -\frac{1}{2}(\phi+\psi) + \frac{\sqrt{3}}{2}(\phi-\psi)i, \quad \textit{and} \;\; r_3 = -\frac{1}{2}(\phi+\psi) - \frac{\sqrt{3}}{2}(\phi-\psi)i,$$

where

$$\phi = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} \quad \textit{and} \quad \psi = \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}.$$

Moreover, let $\Delta = b^2/4 + a^3/27$. Then we have the following three cases:

- *If $\Delta = 0$, then (2.2.5) has three real roots, at least two of which are equal. Here* *when a and b are not equal to 0, the number of equal roots is exactly two.*

- *If $\Delta < 0$, then (2.2.5) has three real distinct roots.*
- *If $\Delta > 0$, then (2.2.5) has a real root and two conjugate complex roots.*

The three different possibilities in Lemma 2.2.1 divide our work into three subsections. In subsection 2.1, we will show the existence of the steady-state solution $h_0(x)$ to (2.2.1) by proving the existence of a periodic solution $w(x)$ to (2.2.2) when a and b, defined in (2.2.3), satisfy $\Delta = 0$. After that, we will use the same idea in subsections 2.2 and 2.3 to prove the existence when $\Delta < 0$ or $\Delta > 0$ is satisfied.

2.2.1 The Steady-state When $\Delta = 0$

In the case $b^2/4 + a^3/27 = 0$, a must be negative, that is, $R < \frac{5}{4}\cot(\theta) = R_c$, where R_c is the critical Reynolds number for the flat wall. In particular, $R = R_c - (15/8)\sqrt[3]{9/C}$. By applying Lemma 2.2.1, the characteristic equation (2.2.5) associated to the homogeneous equation (2.2.4) has a simple root $r = -2\alpha$, and a root of multiplicity 2, $r = \alpha$, where $\alpha = \sqrt[3]{3C}$. Then the fundamental set of solutions to the homogeneous equation (2.2.4) is

$$\{w_1, w_2, w_3\} = \{e^{-2\alpha x}, e^{\alpha x}, xe^{\alpha x}\},$$

with a constant Wronskian $W(w_1, w_2, w_3) = 9\alpha^2$. Using the variation-of-parameters method, the integral form of the non-homogeneous equation (2.2.2) becomes

$$w(x) = e^{-2\alpha x}\int_{-\infty}^{x} \frac{e^{2\alpha t}}{9\alpha^2}\,F(t)dt + e^{\alpha x}\int_{\infty}^{x} \frac{-(3\alpha t+1)e^{-\alpha t}}{9\alpha^2}\,F(t)dt + xe^{\alpha x}\int_{\infty}^{x} \frac{3\alpha e^{-\alpha t}}{9\alpha^2}\,F(t)dt,$$

which can be further written as

$$w(x) = \frac{1}{9\alpha^2}\int_{-\infty}^{x} e^{2\alpha(t-x)}F(t)dt + \frac{1}{3\alpha}\int_{x}^{\infty}(t-x)e^{-\alpha(t-x)}F(t)dt + \frac{1}{9\alpha^2}\int_{x}^{\infty} e^{-\alpha(t-x)}F(t)dt. \tag{2.2.6}$$

In order to construct a better iteration scheme for $w(x)$ in a simple functional space so that the estimate of the norm of the integral operator becomes affordable, we want to remove the derivative term w' in the right-hand side of (2.2.6) and rewrite it as a functional of $w(x)$ only. To do this, we substitute the formula $F(t)$ and integrate the w'-term by parts. The first term in the right-hand side of (2.2.6) becomes

$$\begin{aligned}
&\int_{-\infty}^{x} e^{2\alpha(t-x)}F(t)dt \\
&\quad = \int_{-\infty}^{x} e^{2\alpha(t-x)}\left\{2C\cot(\theta)s' - s''' + \frac{2Cw^2}{(1+w)^3}\left(6+8w+3w^2\right)\right\}dt \\
&\qquad - \frac{8RC}{5}\int_{-\infty}^{x} e^{2\alpha(t-x)}(3w+3w^2+w^3)w'dt \\
&\quad = \int_{-\infty}^{x} e^{2\alpha(t-x)}\left\{2C\cot(\theta)s' - s''' + \frac{2Cw^2}{(1+w)^3}\left(6+8w+3w^2\right)\right\}dt \\
&\qquad - \frac{2RC}{5}(w^4+4w^3+6w^2) + \frac{4RC\alpha}{5}\int_{-\infty}^{x} e^{2\alpha(t-x)}(w^4+4w^3+6w^2)dt.
\end{aligned}$$

Similarly for the second and the last term, we have

$$\begin{aligned}
&\int_{x}^{\infty}(t-x)e^{-\alpha(t-x)}F(t)dt \\
&\qquad = \int_{x}^{\infty}(t-x)e^{-\alpha(t-x)}\left\{2C\cot(\theta)s' - s''' + \frac{2Cw^2}{(1+w)^3}\left(6+8w+3w^2\right)\right\}dt \\
&\qquad + \frac{2RC}{5}\int_{x}^{\infty}(1-\alpha(t-x))e^{-\alpha(t-x)}(w^4+4w^3+6w^2)dt,
\end{aligned}$$

and

$$\begin{aligned}
&\int_x^\infty e^{-\alpha(t-x)} F(t)dt \\
&\quad = \int_x^\infty e^{-\alpha(t-x)} \left\{ 2C\cot(\theta)s' - s''' + \frac{2Cw^2}{(1+w)^3}\left(6+8w+3w^2\right) \right\} dt \\
&\qquad + \frac{2RC}{5}(w^4+4w^3+6w^2) - \frac{2RC\alpha}{5}\int_x^\infty e^{-\alpha(t-x)}(w^4+4w^3+6w^2)dt.
\end{aligned}$$

Now, we define functions G, H, and Q by

$$\begin{aligned}
G(s) &:= 2C\cot(\theta)s' - s''', \\
H(w) &:= 2C\frac{w^2}{(1+w)^3}(6+8w+3w^2), \\
Q(w) &:= \frac{2RC}{5}(w^4+4w^3+6w^2).
\end{aligned} \tag{2.2.7}$$

Then we re-write the integral equation (2.2.6) in the form

$$w(x) = T_0(G)(x) + T_1(H)(x) + T_2(Q)(x) := T(w)(x), \tag{2.2.8}$$

where

$$\begin{aligned}
T_0(G)(x) &= \frac{1}{9\alpha^2}\int_{-\infty}^x e^{2\alpha(t-x)}G(s(t))dt + \frac{1}{3\alpha}\int_x^\infty (t-x)e^{-\alpha(t-x)}G(s(t))dt \\
&\quad + \frac{1}{9\alpha^2}\int_x^\infty e^{-\alpha(t-x)}G(s(t))dt, \\
T_1(H)(x) &= \frac{1}{9\alpha^2}\int_{-\infty}^x e^{2\alpha(t-x)}H(w(t))dt + \frac{1}{3\alpha}\int_x^\infty (t-x)e^{-\alpha(t-x)}H(w(t))dt \\
&\quad + \frac{1}{9\alpha^2}\int_x^\infty e^{-\alpha(t-x)}H(w(t))dt, \\
T_2(Q)(x) &= \frac{2}{9\alpha}\int_{-\infty}^x e^{2\alpha(t-x)}Q(w(t))dt - \frac{1}{3}\int_x^\infty (t-x)e^{-\alpha(t-x)}Q(w(t))dt \\
&\quad + \frac{2}{9\alpha}\int_x^\infty e^{-\alpha(t-x)}Q(w(t))dt.
\end{aligned} \tag{2.2.9}$$

To find a periodic function $w(x)$ that satisfies equation (2.2.8), we define an iteration scheme with the initial periodic function $w_0(x)$ as

$$\begin{aligned} w_0(x) &= T_0(G)(x), \\ w_{n+1}(x) &= T(w_n)(x), \text{ for } n \geq 0. \end{aligned} \tag{2.2.10}$$

Obviously, the operator T maps a periodic function into a periodic function with the same prime period. We shall show that the series $\sum_{n=1}^{\infty}(w_n(x) - w_{n-1}(x))$ converges uniformly for x in $(-\infty, \infty)$. Then the required periodic solution $w(x)$ can be obtained by the limit

$$w(x) = \lim_{n\to\infty} w_n(x) = w_0(x) + \sum_{i=1}^{\infty}(w_i(x) - w_{i-1}(x)).$$

First of all, we want to estimate the initial function $w_0(x)$. Note that

$$|w_0(x)| \leq \|G(s(x))\| \left\{ \frac{1}{9\alpha^2}\left|\int_{-\infty}^{x} e^{2\alpha(t-x)}dt\right| + \frac{1}{3\alpha}\left|\int_{x}^{\infty}(t-x)e^{-\alpha(t-t)}dt\right| + \frac{1}{9\alpha^2}\left|\int_{x}^{\infty} e^{-\alpha(t-x)}dt\right| \right\},$$

which implies

$$|w_0(x)| \leq \frac{1}{2\alpha^3}\|G(s(x))\|,$$

where $\|\cdot\|$ is the maximum norm. This means that we can determine the bound of the periodic function $w_0(x)$ by the bound of $s(x)$, that is, for $s(x)$ satisfying inequalities in (2.1.3) and using the definition of $G(s)$, we have

$$|w_0(x)| \leq \|w_0(x)\| \leq B\epsilon < \frac{1}{2}, \tag{2.2.11}$$

where $B = \frac{1}{2\alpha^3}(2C\cot(\theta)a_1 + a_2)$, and ϵ is sufficiently small (less than ϵ_0 below).

Now we are ready to show the uniform convergence of the series $\sum_{n=1}^{\infty}(w_n - w_{n-1})$. To this end, we define the constants

$$M_1 := \sup_{|w|\le\frac{1}{2}} |H''(w)|, \qquad M_2 := \sup_{|w|\le\frac{1}{2}} |Q''(w)|, \tag{2.2.12}$$
$$M := \frac{1}{2\alpha^3}M_1 + \frac{2}{3\alpha^2}M_2, \qquad \beta := 2MB.$$

We shall show that there exists a constant ϵ_0 such that for $0 < \epsilon < \epsilon_0$, we have

$$|w_n - w_0| \le \beta\epsilon\|w_0\|, \quad n = 1, 2, 3, \ldots, \tag{2.2.13}$$

and

$$|w_n - w_{n-1}| \le (2\beta\epsilon)^n\|w_0\|, \quad n = 1, 2, 3, \ldots. \tag{2.2.14}$$

Indeed, for $n = 1$, we use the iteration definition (2.2.10) and (2.2.8) to have

$$|w_1 - w_0| = |T(w_0) - w_0| \le |T_1(H(w_0))| + |T_2(Q(w_0))|. \tag{2.2.15}$$

Using Taylor expansion, $Q(w) = Q''(\nu)w^2$ for $\nu \in (0, w)$ and $|w| < \frac{1}{2}$. This implies

$$\|Q(w_0)\| \le M_2\|w_0\|^2. \tag{2.2.16}$$

Similarly,

$$\|H(w_0)\| \le M_1\|w_0\|^2. \tag{2.2.17}$$

By using (2.2.9), (2.2.16), and (2.2.17) in (2.2.15) yields

$$|w_1 - w_0| \le M\|w_0\|^2. \tag{2.2.18}$$

Hence, from inequality (2.2.11), we have

$$|w_1 - w_0| \le MB\epsilon\|w_0\| \le \beta\epsilon\|w_0\|,$$

which proves that inequalities (2.2.13) and (2.2.14) hold for $n = 1$. To complete our argument, we assume, by induction, that inequalities (2.2.13) and (2.2.14) are true for $n = k$. This gives $|w_k| \le (1+\beta\epsilon)B\epsilon \le \frac{1}{2}$ as long as $\epsilon < \epsilon_0$ for a given small ϵ_0. We need to show that both of (2.2.13) and (2.2.14) hold true for $n = k+1$. Actually we have

$$\begin{aligned} |w_{k+1} - w_0| &= |T(w_k) - w_0| \\ &\le |T_1(H(w_k))| + |T_2(Q(w_k))| \\ &\le M\|w_k\|^2 && \text{similar to (2.2.18)} \\ &\le M(1+\beta\epsilon)^2\|w_0\|^2 && \text{from our assumption} \\ &\le BM(1+\beta\epsilon)^2\epsilon\|w_0\| && \text{using (2.2.11)} \\ &\le \beta\epsilon\|w_0\|. \end{aligned}$$

This implies that the inequality (2.2.13) is satisfied for all n. Here, we have assumed that ϵ is sufficiently small so that $(1+\beta\epsilon)^2 \le 2$ for $\epsilon < \epsilon_0$. For inequality (2.2.14), we have

$$\begin{aligned} |w_{k+1} - w_k| &= |T(w_k) - T(w_{k-1})| \\ &\le |T_1(H(w_k) - H(w_{k-1}))| + |T_2(Q(w_k) - Q(w_{k-1}))|. \end{aligned} \tag{2.2.19}$$

By the Mean Value Theorem, for $0 \le \theta \le 1$, we get

$$\begin{aligned} \|Q(w_k) - Q(w_{k-1})\| &\le \|Q'(\theta w_k + (1-\theta)w_{k-1})\| \cdot \|w_k - w_{k-1}\| \\ &= \|Q''(\nu)\| \cdot \|\theta w_k + (1-\theta)w_{k-1}\| \cdot \|w_k - w_{k-1}\| \qquad \text{for some } \nu \\ &\le M_2(1+\beta\epsilon)\|w_0\| \cdot \|w_k - w_{k-1}\|, \end{aligned}$$

and similarly,

$$\|H(w_k) - H(w_{k-1})\| \le M_1(1+\beta\epsilon)\|w_0\| \cdot \|w_k - w_{k-1}\|.$$

Hence, inequality (2.2.19) implies

$$\begin{aligned}|w_{k+1} - w_k| &\le M(1+\beta\epsilon)\|w_0\|\|w_k - w_{k-1}\| \\ &\le M(1+\beta\epsilon)(2\beta\epsilon)^k\|w_0\|^2 \\ &\le MB\epsilon(1+\beta\epsilon)(2\beta\epsilon)^k\|w_0\| \\ &\le (2\beta\epsilon)^{k+1}\|w_0\|,\end{aligned}$$

which proves that inequality (2.2.14) is true for all n. By the well-known Weierstrass M-test, series

$$w_0(x) + \sum_{n=1}^{\infty}(w_n(x) - w_{n-1}(x))$$

is uniformly convergent for $x \in (-\infty, \infty)$. Consequently, we have the following theorem.

Theorem 2.2.1. *Assume that a and b, defined in (2.2.3), satisfy $b^2/4 + a^3/27 = 0$. There exists a small ϵ_0 such that for $\epsilon < \epsilon_0$, (2.2.1) has a solution $h_0(x) = 1 + w(x)$, where $w(x)$ is a solution of the differential equation (2.2.2) with the asymptotic expansion*

$$w(x) = w_0(x) + \sum_{n=1}^{\infty}(w_n(x) - w_{n-1}(x)),$$

and $w_n(x), n = 0, 1, 2, \ldots,$ are defined in (2.2.10).

Remark 2.2.1. *Based on (2.2.13) and (2.2.14), Theorem 2.1 also provides a generalized asymptotic expansion to the periodic steady-state solution.*

2.2.2 The Steady-state When $\Delta < 0$

In this subsection, we shall study the existence of periodic steady-state in the case $b^2/4 + a^3/27 < 0$. The fundamental set of solutions to the homogeneous equation (2.2.4), in this case, is $\{w_1, w_2, w_3\} = \{e^{r_1 x}, e^{r_2 x}, e^{r_3 x}\}$, where r_1, r_2, and r_3 are the real distinct roots of the characteristic equation (2.2.5) defined in Lemma 2.2.1, with a constant Wronskian

$$\widehat{W} := W(w_1, w_2, w_3) = r_2 r_3 (r_3 - r_2) - r_1 r_3 (r_3 - r_1) + r_1 r_2 (r_2 - r_1).$$

Note that, when $\Delta < 0$, we have $r_1 < 0$ and $r_2, r_3 > 0$. Then using the variation-of-parameters method, we have the following integral form of the non-homogeneous differential equation (2.2.2):

$$w(x) = C_1 \int_{-\infty}^{x} e^{-r_1(t-x)} F(t) dt + C_2 \int_{x}^{\infty} e^{-r_2(t-x)} F(t) dt + C_3 \int_{x}^{\infty} e^{-r_3(t-x)} F(t) dt, \tag{2.2.20}$$

where

$$C_1 = \frac{r_3 - r_2}{\widehat{W}}, \quad C_2 = \frac{r_3 - r_1}{\widehat{W}} \quad , \text{and } C_3 = \frac{-(r_2 - r_1)}{\widehat{W}}.$$

Substitute $F(t)$ and integrate the w'-term by parts to have

$$\begin{aligned} &\int_{-\infty}^{x} e^{-r_1(t-x)} F(t) dt \\ &\quad = \int_{-\infty}^{x} e^{-r_1(t-x)} \left\{ 2C \cot(\theta) s' - s''' + \frac{2Cw^2}{(1+w)^3} \left(6 + 8w + 3w^2\right) \right\} dt \\ &\quad\quad - \frac{2RC}{5}(w^4 + 4w^3 + 6w^2) - \frac{2RCr_1}{5} \int_{-\infty}^{x} e^{-r_1(t-x)} (w^4 + 4w^3 + 6w^2) dt \end{aligned}$$

and

$$\begin{aligned}
&\int_x^\infty e^{-r_i(t-x)}F(t)dt \\
&\quad = \int_x^\infty e^{-r_i(t-x)}\left\{2C\cot(\theta)s' - s''' + \frac{2Cw^2}{(1+w)^3}\left(6+8w+3w^2\right)\right\}dt \\
&\quad\quad + \frac{2RC}{5}(w^4+4w^3+6w^2) - \frac{2RCr_i}{5}\int_x^\infty e^{-r_i(t-x)}(w^4+4w^3+6w^2)dt,
\end{aligned}$$

for $i = 2, 3$. In terms of $G(s)$, $H(w)$, and $Q(w)$ defined in (2.2.7), the integral equation (2.2.20) can be written in the form

$$w(x) = \widehat{T}_0(G)(x) + \widehat{T}_1(H)(x) + \widehat{T}_2(Q)(x) := \widehat{T}(w)(x), \tag{2.2.21}$$

where

$$\begin{aligned}
\widehat{T}_0(G)(x) &= C_1\int_{-\infty}^x e^{-r_1(t-x)}G(s(t))dt + \sum_{i=2}^3 C_i\int_x^\infty e^{-r_i(t-x)}G(s(t))dt, \\
\widehat{T}_1(H)(x) &= C_1\int_{-\infty}^x e^{-r_1(t-x)}H(w(t))dt + \sum_{i=2}^3 C_i\int_x^\infty e^{-r_i(t-x)}H(w(t))dt,
\end{aligned}$$

and

$$\widehat{T}_2(Q)(x) = -C_1r_1\int_{-\infty}^x e^{-r_1(t-x)}Q(w(t))dt - \sum_{i=2}^3 C_ir_i\int_x^\infty e^{-r_i(t-x)}Q(w(t))dt.$$

Similar to the previous subsection, we define an iteration scheme

$$\begin{aligned}
\widehat{w}_0(x) &= \widehat{T}_0(G)(x), \\
\widehat{w}_{n+1}(x) &= \widehat{T}(\widehat{w}_n)(x), \text{ for } n \geq 0,
\end{aligned} \tag{2.2.22}$$

and later use the following constants:

$$\begin{aligned}
\widehat{B} &:= (2C\cot(\theta)a_1 + a_2)\sum_{i=1}^{3}\left|\frac{C_i}{r_i}\right|, \\
\widehat{M} &:= M_1\sum_{i=1}^{3}\left|\frac{C_i}{r_i}\right| + M_2\sum_{i=1}^{3}|C_i|, \\
\widehat{\beta} &:= 2\widehat{M}\widehat{B},
\end{aligned}$$

where M_1 and M_2 are the same as those in (2.2.12). The operator $\widehat{T}$ maps periodic functions into periodic functions. Then we can apply the same technique used in the previous subsection to show that, there exists an $\epsilon_0 > 0$ such that for sufficiently small $\epsilon < \epsilon_0$, the inequalities

$$\begin{aligned}
|\widehat{w}_0| &\leq \widehat{B}\epsilon, \\
|\widehat{w}_n - \widehat{w}_0| &\leq \widehat{\beta}\epsilon\|\widehat{w}_0\|, \quad n = 1, 2, 3, \ldots,
\end{aligned}$$

and

$$|\widehat{w}_n - \widehat{w}_{n-1}| \leq (2\widehat{\beta}\epsilon)^n\|\widehat{w}_0\|, \quad n = 1, 2, 3, \ldots$$

hold. Hence, the Weierstrass M-test implies that series

$$\widehat{w}_0(x) + \sum_{n=1}^{\infty}(\widehat{w}_n(x) - \widehat{w}_{n-1}(x))$$

is uniformly convergent for $x \in (-\infty, \infty)$. Then the following result is valid:

Theorem 2.2.2. *Assume that a and b, defined in (2.2.3), satisfy $b^2/4 + a^3/27 < 0$. There exists a constant $\epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$, (2.2.1) has a periodic solution $h_0(x) = 1 + w(x)$, where $w(x)$ is a solution of the differential equation (2.2.2) with* *the asymptotic expansion*

$$w(x) = \widehat{w}_0(x) + \sum_{n=1}^{\infty} \left(\widehat{w}_n(x) - \widehat{w}_{n-1}(x) \right),$$

and $\widehat{w}_n(x), n = 0, 1, 2, \ldots,$ *are defined in (2.2.22).*

2.2.3 The Steady-state When $\Delta > 0$

When $\Delta > 0$, Lemma 2.2.1 implies that the characteristic equation (2.2.5), associated to the homogeneous equation (2.2.4), has a real root r and two complex conjugate roots $u \pm iv$, where r, u, and v can be defined in terms of ϕ and ψ in Lemma 2.2.1. The fundamental set of solutions is $\{w_1, w_2, w_3\} = \{e^{rx}, e^{ux}\cos(vx), e^{ux}\sin(vx)\}$, with a constant Wronskian

$$\overline{W} := W(w_1, w_2, w_3) = v(2r^2 + u^2 + v^2).$$

Note that, since $b > 0$, we have $r < 0$ and $u > 0$, with $r + 2u = 0$. Hence, the integral form of the differential equation (2.2.2), in this case, is

$$w(x) = e^{rx} \int_{-\infty}^{x} \frac{W_1(t)}{\overline{W}} F(t)dt + e^{ux}\cos(vx) \int_{\infty}^{x} \frac{W_2(t)}{\overline{W}} F(t)dt + e^{ux}\sin(vx) \int_{\infty}^{x} \frac{W_3(t)}{\overline{W}} F(t)dt,$$

where

$$W_1(t) = ve^{-rt}, \quad W_2(t) = -[(u-r)\sin(vt) + v\cos(vt)]e^{-ut},$$
$$W_3(t) = [(u-r)\cos(vt) - v\sin(vt)]e^{-ut}.$$

This integral form can be written as

$$w(x) = \frac{v}{\overline{\overline{W}}} \int_{-\infty}^{x} e^{-r(t-x)} F(t)dt + \int_{x}^{\infty} g(x,t) e^{-u(t-x)} F(t)dt, \tag{2.2.23}$$

where $g(x,t)$ is given by

$$g(x,t) = \frac{1}{\overline{\overline{W}}} \left[(u-r)\sin(v(t-x)) + v\cos(v(t-x)) \right].$$

We write the integrals in (2.2.23) as

$$\begin{aligned}
&\frac{v}{\overline{\overline{W}}} \int_{-\infty}^{x} e^{-r(t-x)} F(t)dt \\
&\quad= \frac{v}{\overline{\overline{W}}} \int_{-\infty}^{x} e^{-r(t-x)} \left\{ 2C\cot(\theta)s' - s''' + \frac{2Cw^2}{(1+w)^3}\left(6+8w+3w^2\right) \right\} dt \\
&\quad\quad - \frac{2RCv}{5\overline{W}}(w^4+4w^3+6w^2) - \frac{2RCrv}{5\overline{W}} \int_{-\infty}^{x} e^{-r(t-x)}(w^4+4w^3+6w^2)dt,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{x}^{\infty} g(x,t) e^{-u(t-x)} F(t)dt \\
&\quad= \int_{x}^{\infty} e^{-u(t-x)} g(x,t) \left\{ 2C\cot(\theta)s' - s''' + \frac{2Cw^2}{(1+w)^3}\left(6+8w+3w^2\right) \right\} dt \\
&\quad\quad + \frac{2RCv}{5\overline{W}}(w^4+4w^3+6w^2) \\
&\quad\quad + \frac{2RC}{5} \int_{x}^{\infty} [g_t(x,t) - ug(x,t)] e^{-u(t-x)}(w^4+4w^3+6w^2)dt.
\end{aligned}$$

From this, the formula of $w(x)$ in (2.2.23) can be expressed as

$$w(x) = \overline{T}_0(G)(x) + \overline{T}_1(H)(x) + \overline{T}_2(Q)(x) := \overline{T}(w)(x),$$

where $G(s), H(w)$, and $Q(w)$ are defined in (2.2.7), and

$$\begin{aligned}
\overline{T}_0(G)(x) &= \frac{v}{\overline{\overline{W}}}\int_{-\infty}^{x} e^{-r(t-x)}G(s(t))dt + \int_{x}^{\infty} g(x,t)e^{-u(t-x)}G(s(t))dt,\\
\overline{T}_1(H)(x) &= \frac{v}{\overline{\overline{W}}}\int_{-\infty}^{x} e^{-r(t-x)}H(w(t))dt + \int_{x}^{\infty} g(x,t)e^{-u(t-x)}H(w(t))dt,\\
\overline{T}_2(Q)(x) &= -\frac{vr}{\overline{\overline{W}}}\int_{-\infty}^{x} e^{-r(t-x)}Q(w(t))dt + \int_{x}^{\infty} [g_t(x,t) - ug(x,t)]e^{-u(t-x)}Q(w(t))dt.
\end{aligned}$$

Similar to the previous cases, we define an iteration scheme, for this case, as

$$\begin{aligned}
\overline{w}_0(x) &= \overline{T}_0(G)(x),\\
\overline{w}_{n+1}(x) &= \overline{T}(\overline{w}_n)(x), \text{ for } n \geq 0.
\end{aligned} \tag{2.2.24}$$

Then we can show that, there exists an $\epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$, the inequalities

$$|\overline{w}_0| \leq \overline{B}\epsilon,$$

$$|\overline{w}_n - \overline{w}_{n-1}| \leq \overline{\beta}\epsilon\|\overline{w}_0\|,$$

and

$$|\overline{w}_n - \overline{w}_{n-1}| \leq (2\overline{\beta}\epsilon)^n\|\overline{w}_0\|, \quad n = 1, 2, 3, \ldots,$$

hold, where

$$\begin{aligned}
\overline{B} &= (2C\cot(\theta)a_1 + a_2)\left\{\left|\frac{v}{r\overline{\overline{W}}}\right| + \frac{\|g\|}{|u|}\right\},\\
\overline{M} &:= M_1\left\{\left|\frac{v}{r\overline{\overline{W}}}\right| + \frac{\|g\|}{|u|}\right\} + M_2\left\{\left|\frac{v}{\overline{\overline{W}}}\right| + \frac{\|g_t\|}{|u|} + \|g\|\right\},
\end{aligned}$$

and

$$\overline{\beta} := 2\overline{MB},$$

with the same constants M_1 and M_2 defined in (2.2.12). Note that g and g_t are bounded and satisfy

$$\|g\| \leq \frac{1}{|\overline{W}|}(|u-r|+|v|), \quad \|g_t\| \leq \left|\frac{v}{\overline{W}}\right|(|u-r|+|v|).$$

Then, the uniform convergence of

$$\overline{w}_0(x) + \sum_{n=1}^{\infty} (\overline{w}_n(x) - \overline{w}_{n-1}(x))$$

is confirmed for $x \in (-\infty, \infty)$. Hence, we obtain

Theorem 2.2.3. *Assume that a and b, defined in (2.2.3), satisfy $b^2/4 + a^3/27 > 0$. There exists an $\epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$, (2.2.1) has a periodic solution $h_0(x) = 1 + w(x)$, where $w(x)$ is a solution of the differential equation (2.2.2) with the asymptotic expansion*

$$w(x) = \overline{w}_0(x) + \sum_{n=1}^{\infty} (\overline{w}_n(x) - \overline{w}_{n-1}(x)),$$

and $\overline{w}_n(x), n = 0, 1, 2, \ldots,$ are defined in (2.2.24).

2.3 The Existence of the Steady-state by an Abstract Method

We are wondering if we can study the existence and the uniqueness of the solution $w(x)$ to the ordinary differential equation (2.2.2) in a unified method. Actually, this can be done by writing the ordinary differential equation in a new form so that the characteristic equation corresponding to the homogeneous part has three distinct real roots, that is, a unique solution expression. This allows to write the problem in a fixed point problem form. Then we can use the contraction mapping theorem together with the fixed point theorem to get the desired result in a simple fashion. The new form of (2.2.2) is

$$w''' - 3w' + w = -(a+3)w' - (b-1)w + F. \tag{2.3.1}$$

Similar to the integral formula (2.2.20), the characteristic equation $r^3 - 3r + 1 = 0$ corresponding to the homogeneous part of (2.3.1) has three distinct real roots, $\rho_1 < 0, 0 < \rho_2 < 1$, and $\rho_3 > 1$. Then $w''' - 3w' + w = 0$ has the fundamental set of solutions $\{w_1, w_2, w_3\} = \{e^{\rho_1 x}, e^{\rho_2 x}, e^{\rho_3 x}\}$, with

$$W := \text{Wronskain}(w_1, w_2, w_3) = \rho_2\rho_3(\rho_3 - \rho_2) + \rho_1\rho_3(\rho_1 - \rho_3) + \rho_1\rho_2(\rho_2 - \rho_1).$$

Hence, one can get the integral form of the non-homogeneous differential equation (2.3.1) as follows

$$\begin{aligned} w(x) = & B_1 \int_{-\infty}^{x} e^{\rho_1(x-t)}\{-(a+3)w'(t) - (b-1)w(t) + F(t)\}dt \\ & + B_2 \int_{x}^{\infty} e^{\rho_2(x-t)}\{-(a+3)w'(t) - (b-1)w(t) + F(t)\}dt \\ & + B_3 \int_{x}^{\infty} e^{\rho_3(x-t)}\{-(a+3)w'(t) - (b-1)w(t) + F(t)\}dt, \end{aligned}$$

where

$$B_1 = \frac{\rho_3 - \rho_2}{W}, \quad B_2 = \frac{\rho_3 - \rho_1}{W}, \quad \text{and} \quad B_3 = \frac{-(\rho_2 - \rho_1)}{W}.$$

We write this integral form as

$$L(w) = N(w), \tag{2.3.2}$$

where the linear operator $L(w)$ and the nonlinear operator $N(w)$ are defined by

$$\begin{aligned} L(w)(x) =& w(x) - B_1 \int_{-\infty}^{x} e^{\rho_1(x-t)}\{-(a+3)w'(t) - (b-1)w(t)\}dt \\ &- B_2 \int_{x}^{\infty} e^{\rho_2(x-t)}\{-(a+3)w'(t) - (b-1)w(t)\}dt \\ &- B_3 \int_{x}^{\infty} e^{\rho_3(x-t)}\{-(a+3)w'(t) - (b-1)w(t)\}dt, \end{aligned} \tag{2.3.3}$$

and

$$N(w)(x) = B_1 \int_{-\infty}^{x} e^{\rho_1(x-t)}F(t)dt + B_2 \int_{x}^{\infty} e^{\rho_2(x-t)}F(t)dt + B_3 \int_{x}^{\infty} e^{\rho_3(x-t)}F(t)dt.$$

Note that if $w \in C_p^1[0,l]$, then $L(w) \in C_p^1[0,l]$, where $C_p^1[0,l]$ is the space of all $l-$periodic functions with continuous derivatives, with the norm defined by

$$\|\phi\| = \|\phi\|_\infty + \|\phi'\|_\infty.$$

Similar to that in (2.2.21), we substitute the formula $F(t)$ and integrate the w'-term by parts so that the integral operator $N(w)$ become a functional of $w(x)$ only. By this and using the same functions G, H, and Q, defined in (2.2.7), the nonlinear operator N can be written in the form

$$N = N_0(s) + N_1(w),$$

where $N_0(s)$ and the remainder part N_1 are given by

$$\begin{aligned} N_0(s)(x) =& B_1 \int_{-\infty}^{x} e^{\rho_1(x-t)} G(s(t))dt + \sum_{i=2}^{3} B_i \int_{x}^{\infty} e^{\rho_i(x-t)} G(s(t))dt, \\ N_1(w)(x) =& B_1 \int_{-\infty}^{x} e^{\rho_1(x-t)} H(w(t))dt + \sum_{i=2}^{3} B_i \int_{x}^{\infty} e^{\rho_i(x-t)} H(w(t))dt \\ &- B_1\rho_1 \int_{-\infty}^{x} e^{\rho_1(x-t)} Q(w(t))dt - \sum_{i=2}^{3} B_i\rho_i \int_{x}^{\infty} e^{\rho_i(x-t)} Q(w(t))dt. \end{aligned}$$

The following lemmas give the estimations of integrals in $N(w)$.

Lemma 2.3.1. *We have* $\|N_0(s)\| \leq O(\epsilon)$.

Proof. In veiw of definition of $N_0(G)$, we need to prove that

$$\left| B_1 \int_{-\infty}^{x} e^{\rho_1(x-t)}dt + B_2 \int_{x}^{\infty} e^{\rho_2(x-t)}dt + B_3 \int_{x}^{\infty} e^{\rho_3(x-t)}dt \right| = O(1),$$

which is readily satisfied. Since $s(x)$ satisfies relation (2.1.3), we then have $\|N_0(G)\| \leq O(\epsilon)$. ∎

Lemma 2.3.2. *For each δ, there is a σ such that*

$$\|N_1(\psi) - N_1(\phi)\| \leq \delta \|\psi - \phi\| \tag{2.3.4}$$

uniformly for all $\psi, \phi \in C_p^1[0, l]$ with $\|\psi\| \leq \sigma < 1/2, \|\phi\| \leq \sigma < 1/2$.

Proof. From definition of $H(w)$ and $Q(w)$ in (2.2.7), we have $\|H(w)\| = O(\|w\|^2)$ and $\|Q(w)\| = O(\|w\|^2)$, for $\|w\| \leq \sigma \leq 1/2$. Then estimation (2.3.4) follows. ∎

Now we state and prove the main result of this section.

Theorem 2.3.1. *There exists a constant $\epsilon_0 > 0$ such that for small $\epsilon < \epsilon_0$, the non-homogeneous equation (2.2.2) has a unique periodic solution $w \in C_p^1[0, l]$.*

Proof. We write the proof in four steps:

Step 1. Define an operator $\chi : Y \in C_p^3[0,l] \to C_p^3[0,l]$ by

$$\chi(Y)(x) = Y'''(x) + aY'(x) + bY(x).$$

Then the adjoint operator of $\chi(Y) = 0$ is given by $\chi^*(Z) = 0$, where χ^* is defined by

$$\chi^*(Z)(x) = -Z'''(x) - aZ'(x) + bZ(x).$$

It is obvious that $\chi^*(Z) = 0$ has only zero periodic solution. By Fredholm theory (see Lemma 4.2 in [60]), $\chi(Y) = f$ has a unique solution for any $f \in C_p^1[0,l]$.

Step 2. We define a linear operator $L : C_p^1[0,l] \to C_p^1[0,l]$ by (2.3.3), and prove that it is onto, that is, for any $\bar{f} \in C_p^1[0,l]$, equation

$$\begin{aligned} w(x) =& B_1 \int_{-\infty}^{x} e^{\rho_1(x-t)}\{-(a+3)w'(t) - (b-1)w(t)\}dt \\ &- B_2 \int_{x}^{\infty} e^{\rho_2(x-t)}\{-(a+3)w'(t) - (b-1)w(t)\}dt \\ &- B_3 \int_{x}^{\infty} e^{\rho_3(x-t)}\{-(a+3)w'(t) - (b-1)w(t)\}dt = \bar{f}(x) \end{aligned}$$

has a solution $w \in C_p^1[0,l]$. Indeed, assume that $u = w - \bar{f}$ and substitute it to get

$$\begin{aligned}
&u(x) - B_1 \int_{-\infty}^{x} e^{\rho_1(x-t)}\{-(a+3)u'(t) - (b-1)u(t)\}dt \\
&- B_2 \int_{x}^{\infty} e^{\rho_2(x-t)}\{-(a+3)u'(t) - (b-1)u(t)\}dt \\
&- B_3 \int_{x}^{\infty} e^{\rho_3(x-t)}\{-(a+3)u'(t) - (b-1)u(t)\}dt \\
=&B_1 \int_{-\infty}^{x} e^{\rho_1(x-t)}\{-(a+3)\bar{f}'(t) - (b-1)\bar{f}(t)\}dt \\
&+ B_2 \int_{x}^{\infty} e^{\rho_2(x-t)}\{-(a+3)\bar{f}'(t) - (b-1)\bar{f}(t)\}dt \\
&+ B_3 \int_{x}^{\infty} e^{\rho_3(x-t)}\{-(a+3)\bar{f}'(t) - (b-1)\bar{f}(t)\}dt,
\end{aligned}$$

which is equivalent to

$$u'''(x) + au'(x) + bu(x) = -(a+3)\bar{f}'(x) - (b-1)\bar{f}(x) := \hat{f}(x).$$

By step 1 and since $\hat{f} \in C_p^1[0,l]$ then $u \in C_p^1[0,l]$, which implies that $w = u + \bar{f} \in C_p^1[0,l]$.

Step 3. We claim that L is a one-to-one operator. Indeed, if $L(w_1) = L(w_2)$ for periodic functions $w_1, w_2 \in C_p^1[0,l]$, then $L(w_1 - w_2) = 0$. Since $\chi(w) = 0$ has only zero periodic solution, then so is $L(w) = 0$, which gives that $w_1 = w_2$. By the Banach Inverse Operator Theorem [49, pp. 149], $L^{-1} : C_p^1[0,l] \to C_p^1[0,l]$ is a linear bounded operator.

Step 4. Since $\|L^{-1}\|$ is independent of ϵ, it follows from lemmas 2.3.1 and 2.3.2 that, there exists a constant $\epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$, we have $\sigma = \sigma(\epsilon) > 0, \delta = \delta(\epsilon) > 0$, and $0 < \nu(\epsilon) < 1$ satisfying, for $w, \phi, \psi \in B(\sigma)$,

$$\|L^{-1}N(w)\| \leq \frac{1}{3}(\|w\| + \sigma),$$

and

$$\|L^{-1}N(\phi) - L^{-1}N(\psi)\| \le \nu\|\phi - \psi\|,$$

where $B(\sigma)$ is a ball in $C_p^1[0,l]$ with radius σ and center at origin. Consequently, $L^{-1}N$ is a contractive mapping for $w \in B(\sigma)$ and, by the Contractive Fixed Point Theorem (e.g. [11, pp. 177]), equation (2.3.2) has a unique periodic solution in the ball $B(\sigma)$ in $C_p^1[0,l]$, which is the desired result. ∎

2.4 Stability Analysis

In this section, we study the linear stability of the steady-state solution $h_0(x)$, founded in Sections 2.2-2.3. For this purpose, we add a small perturbation to $h_0(x)$, and study the behavior of the solution when t becomes very large. We say that $h_0(x)$ is stable if this perturbation decays when $t \to \infty$, and unstable if it grows when $t \to \infty$. This perturbation is written in the form $\delta_1\phi(x)e^{\lambda t}$, where $\delta_1 \ll 1, \phi(x) \in L^2(\mathbb{R})$, and λ is a parameter. Thus, we write

$$h(x,t) = h_0(x) + \delta_1\phi(x)e^{\lambda t}. \tag{2.4.1}$$

By this ansatz, if any value of λ lies in the right-half complex plane, then $h_0(x)$ is unstable, while if all λ lie in the left-half complex plane, then the perturbation term $\delta_1\phi(x)e^{\lambda t}$ decays exponentially and $h_0(x)$ is stable.

Substituting (2.4.1) into the problem (2.1.1) and linearizing the equation give

$$\mathcal{L}\phi = -\lambda\phi, \tag{2.4.2}$$

where the differential operator $\mathcal{L}$ is defined by

$$\mathcal{L}\phi = \frac{d}{dx}\left[2h_0^2\phi + \frac{8R}{15}\left(h_0^6\phi' + 6h_0^5\phi h_0'\right)\right.$$
$$\left. - \frac{2\cot(\theta)}{3}\left(h_0^3\phi' + 3h_0^2\phi(h_0' + s')\right) + \frac{1}{3C}\left(h_0^3\phi''' + 3h_0^2\phi(h_0''' + s''')\right)\right].$$

To study the stability analytically, we introduce a weighted functional space $\mathbb{L}^2_\eta$,

$$L^2_\eta(\mathbb{R}) = \{u(x) : e^{\eta x}u(x) \text{ is in } L^2(\mathbb{R})\},$$

with the norm defined by

$$\|u(x)\|^2_\eta = \int_{-\infty}^{\infty} |e^{\eta x}u(x)|^2\, dx,$$

where η is a real number. Then, we consider $\mathcal{L}$ on the new space $L^2_\eta(\mathbb{R})$ and find its spectrum. Since ϕ in the space is not periodic and all coefficients in equation (2.4.2) are periodic, we incorporate the Floquet-Bloch theory. For this purpose, we assume

$$\phi(x) = e^{(ik-\eta)x}g(x),$$

where $g(x)$ is an l-periodic function and $k \in [-\pi/l, \pi/l]$ is called the Bloch wave-number. Equation (2.4.2) reads

$$\mathcal{L}^k_\eta g := e^{-(ik-\eta)x}\mathcal{L}(e^{(ik-\eta)x}g) = -\lambda g. \tag{2.4.3}$$

The spectrum of $\mathcal{L}$ is the union of all point spectra of $\mathcal{L}^k_\eta$ when k varies in the interval from $-\pi/l$ to π/l.

Now, we use an asymptotic approach to find leading terms of λ. We know that

$h_0(x), s'(x)$, and $s'''(x)$ can be written in the forms

$$\begin{aligned} h_0(x) &= 1 + \epsilon h_{0,1}(x) + \epsilon^2 h_{0,2}(x) + \dots, \\ s'(x) &= \epsilon s_1'(x) + \epsilon^2 s_2'(x) + \dots, \\ s'''(x) &= \epsilon^3 s_3'''(x) + \dots . \end{aligned} \tag{2.4.4}$$

Then we set λ and g as

$$\begin{aligned} \lambda &= \lambda_0 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \dots, \\ g &= g_0 + \epsilon g_1 + \epsilon^2 g_2 + \dots, \end{aligned} \tag{2.4.5}$$

and determine the sign of the real part of λ_0. Substitute (2.4.4) and (2.4.5) into (2.4.3) and equate $O(1)$ terms to get the following differential equation

$$\begin{aligned} &-3C\lambda_0 g_0(x) \\ &\quad=[(ik-\eta)^4 g_0(x) + 4(ik-\eta)^3 g_0'(x) + 6(ik-\eta)^2 g_0''(x) + 4(ik-\eta) g_0'''(x) + g_0^{(4)}(x)] \\ &\qquad + a[(ik-\eta)^2 g_0(x) + 2(ik-\eta) g_0'(x) + g_0''(x)] + b[(ik-\eta) g_0(x) + g_0'(x)], \end{aligned} \tag{2.4.6}$$

where a and b are defined in (2.2.3), with the periodic boundary conditions

$$g_0^{(m)}(0) = g_0^{(m)}(l), \quad m = 0, 1, 2, 3. \tag{2.4.7}$$

$g_0^{(m)}(x)$ denotes the mth derivative of g_0 at x with $g_0^{(0)}$ is g_0 itself. Periodic solutions to the BVP (2.4.6)-(2.4.7) can be expressed in the form $g_0(x) = e^{rx}$, where $r \in \mathbb{C}$ is a purely imaginary number having the form $r = \omega i$ for the real number $\omega = \frac{2n\pi}{l}, n = 0, 1, 2, \dots$. Hence, by letting $\lambda_0 = \lambda_{0,R} + i\lambda_{0,I}$, we find from (2.4.6) that $\lambda_{0,R}$ and $\lambda_{0,I}$

satisfy

$$-3C\lambda_{0,R}(q) = q^4 - (6\eta^2 + a)q^2 + \eta^4 + a\eta^2 - b\eta, \tag{2.4.8}$$

$$-3C\lambda_{0,I}(q) = 4\eta q^3 + (b - 2a\eta - 4\eta^3)q, \tag{2.4.9}$$

where $q = (k + \omega) \in [-\pi/l, \infty)$. When k varies from $-\pi/l$ to π/l, and $\omega \in \{0, 2\pi/l, 4\pi/l, ...\}$, we determine the maximal real part of the spectrum, i.e, $\lambda_{0,R}(p)$ for some value of $p \in [-\pi/l, \infty)$, so that $\lambda_{0,R}(p) \geq \lambda_{0,R}(q)$ for all $q \in [-\pi/l, \infty)$. Since this maximization depends on the sign of $6\eta^2 + a$, we shall study the stability in terms of the following two cases.

2.4.1 Stability of the Periodic Steady-state When $a \leq -6\eta^2$

For $a \leq -6\eta^2$, the right-hand side of (2.4.8) has its minimum when $q = 0$, where $k = 0$ and $n = 0$ are the values satisfying this minimization, which in turn means that the maximal real part of the spectrum is $\lambda_0(0)$. Hence, we have the following theorem.

Theorem 2.4.1. *Assume $a \leq -6\eta^2$ and $\eta^4 + a\eta^2 - b\eta \neq 0$.*

(i) If $b^2/4 + a^3/27 \leq 0$, then the steady-state solution $h_0(x)$ is stable when $\eta \in (-\infty, \eta_3) \cup (\eta_2, 0) \cup (\eta_1, \infty)$, and unstable when $\eta \in (\eta_3, \eta_2) \cup (0, \eta_1)$, where η_i, for $i = 1, 2, 3$, are solutions of $\eta^3 + a\eta - b = 0$, and defined in Lemma 2.2.1 (equal to r_i by replacing $-b$ with b).

(ii) If $b^2/4 + a^3/27 > 0$, then the steady-state solution $h_0(x)$ is stable when $\eta \in (-\infty, 0) \cup (\eta_1, \infty)$, and unstable when $\eta \in (0, \eta_1)$.

Proof. The maximal real part of the spectrum, $\lambda_{0,R}(0)$, satisfies

$$-3C\lambda_{0,R}(0) = \eta^4 + a\eta^2 - b\eta := F_1(\eta, a, b).$$

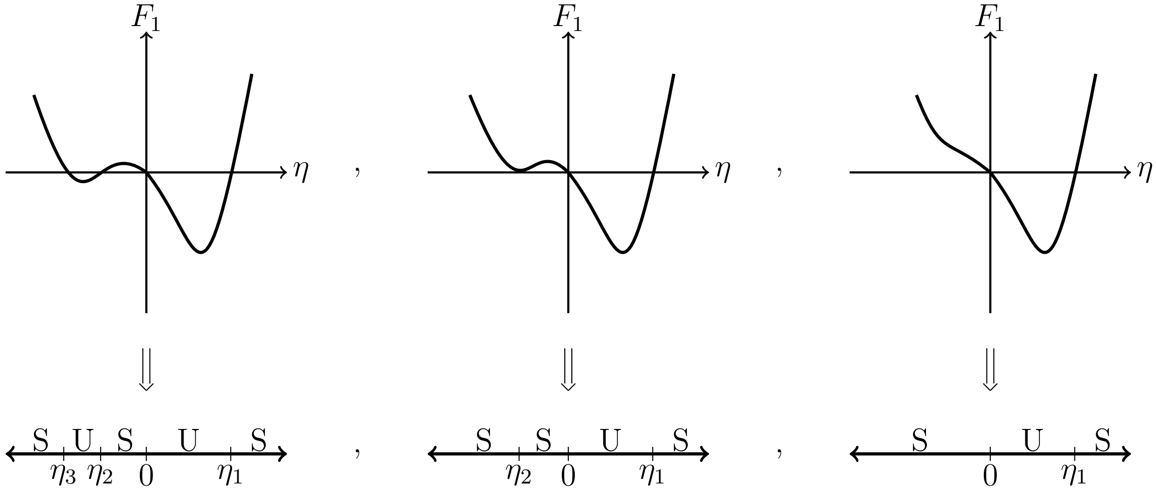


Figure 2.2: The sign of $F_1(\eta, a, b)$ and the stability/instability intervals provided in Theorem 2.4.1, where S: stable and U: unstable.

Therefore, we use Lemma 2.2.1 to solve $F_1(\eta, a, b) = 0$:

(i) if $b^2/4 + a^3/27 \leq 0$, then there exist four real solutions to $F_1(\eta, a, b) = 0$ for η, which are $\eta_1 > 0, \eta_0 = 0, \eta_2, \eta_3 < 0$ (we may have $\eta_2 = \eta_3$). Noting that $F_1(0^+, a, b) < 0$ and $F_1(0^-, a, b) > 0$, we find that $F_1(\eta, a, b) > 0$ for $\eta \in (-\infty, \eta_3) \cup (\eta_2, 0) \cup (\eta_1, \infty)$, and $F_1(\eta, a, b) < 0$ for $\eta \in (\eta_3, \eta_2) \cup (0, \eta_1)$. This implies that $\lambda_{0,R}(0) < 0$ and $h_0(x)$ is a stable steady-state solution, for $\eta \in (-\infty, \eta_3) \cup (\eta_2, 0) \cup (\eta_1, \infty)$. Also, $\lambda_{0,R}(0) > 0$ and $h_0(x)$ is unstable, for $\eta \in (\eta_3, \eta_2) \cup (0, \eta_1)$.

(ii) if $b^2/4 + a^3/27 > 0$, then there exist two real solutions $\eta = \eta_0$ and $\eta = \eta_1$ to $F_1(\eta, a, b) = 0$ with $\eta_1 > 0$ and $\eta_0 = 0$. The periodic steady-state is stable for $\eta \in (-\infty, 0) \cup (\eta_1, \infty)$, and unstable for $\eta \in (0, \eta_1)$.

Figure 2.2 shows the sign of F_1 around its zeros and the stability intervals. ∎

So far, we have studied the stability when $a \leq -6\eta^2$ for all possible values of η except for any value when $F_1 = 0$, where, from (2.4.8) and (2.4.9), the value of $\lambda_0(0)$

is 0. Hence, we need to find one correction term $\lambda_1(0)$ to determine the stability for these special cases.

By substituting (2.4.4)-(2.4.5) into (2.4.3) and using $k = 0, \omega = 0$ ($g_0(x) = 1$), and $F_1 = 0$, we get the BVP

$$\begin{aligned} \mathcal{L}_1 g_1 &= -3C\left[\lambda_1(0) + e^{\eta x}\frac{d}{dx}\left(e^{-\eta x}U_1(x)\right)\right], \\ g_1^{(m)}(0) &= g_1^{(m)}(l), \quad m = 0, 1, 2, 3, \end{aligned} \tag{2.4.10}$$

where

$$\begin{aligned} \mathcal{L}_1 g_1 &= g_1^{(4)} - 4\eta g_1''' + (6\eta^2 + a)g_1'' - (4\eta^3 + 2a\eta)g_1', \\ U_1(x) &= 4h_{0,1} + \frac{16R}{5}(-\eta h_{0,1} + h_{0,1}') - 2\cot(\theta)(-\eta h_{0,1} + h_{0,1}' + s_1') + \frac{1}{C}(h_{0,1}''' - \eta^3 h_{0,1}). \end{aligned}$$

To find the formula for λ_1, we define an adjoint problem from (2.4.10) by

$$\begin{aligned} \mathcal{L}_1^* u :=& u^{(4)} + 4\eta u''' + (6\eta^2 + a)u'' + (4\eta^3 + 2a\eta)u' = 0, \\ & u^{(m)}(0) = u^{(m)}(l), \quad m = 0, 1, 2, 3, \end{aligned}$$

where any constant solves this adjoint problem. Thus we take $u = 1$. Multiplying (2.4.10) by $u = 1$ and integrating from 0 to l give

$$\int_0^l 1 \cdot \mathcal{L}_1 g_1 dx = 0 \implies \lambda_1(0) = \frac{1}{l}\int_0^l \eta U_1(x)dx.$$

Here, we have made use of the technique of integration by parts. To simplify the last integral, let

$$h_{0,1}(x) = \alpha_0 + \sum_{m=1}^{\infty} \alpha_m \cos\left(\frac{2m\pi x}{l}\right) + \sum_{m=0}^{\infty} \beta_m \sin\left(\frac{2m\pi x}{l}\right),$$
$$s_1'(x) = \gamma_0 + \sum_{m=1}^{\infty} \gamma_m \cos\left(\frac{2m\pi x}{l}\right) + \sum_{m=0}^{\infty} \zeta_m \sin\left(\frac{2m\pi x}{l}\right)$$

be the Fourier series of $h_{0,1}(x)$ and $s_1'(x)$, where

$$\alpha_0 = \frac{1}{l}\int_0^l h_{0,1}(x)dx, \qquad \gamma_0 = \frac{1}{l}\int_0^l s_1'(x)dx,$$
$$\alpha_m = \frac{2}{l}\int_0^l h_{0,1}(x)\cos\left(\frac{2m\pi x}{l}\right)dx, \qquad \gamma_m = \frac{2}{l}\int_0^l s_1'(x)\cos\left(\frac{2m\pi x}{l}\right)dx,$$
$$\beta_m = \frac{2}{l}\int_0^l h_{0,1}(x)\sin\left(\frac{2m\pi x}{l}\right)dx, \qquad \zeta_m = \frac{2}{l}\int_0^l s_1'(x)\sin\left(\frac{2m\pi x}{l}\right)dx,$$

i.e., α_0 and γ_0 are the periodic constant part of $h_{0,1}(x)$ and $s_1'(x)$, respectively. Recall that, from (2.2.2), $h_{0,1}(x)$ and $s_1'(x)$ satisfy

$$h_{0,1}''' + ah_{0,1}' + bh_{0,1} = 2C\cot(\theta)s_1'.$$

Integrating from 0 to l gives $b\alpha_0 = 2C\cot(\theta)\gamma_0$ or $\gamma_0 = \frac{3}{\cot(\theta)}\alpha_0$. Then the formula for $\lambda_1(0)$ is given by

$$\lambda_1(0) = \eta\alpha_0\left(2\eta\cot(\theta) - 2 - \frac{16R}{5}\eta - \frac{1}{C}\eta^3\right). \tag{2.4.11}$$

$\lambda_1(0)$ depends on the shape of the wall surface topography $s(x)$, so that when (2.4.11) is not equal to zero, the stability can be determined by $\lambda_1(0)$; otherwise same steps can be repeated to find the formula for $\lambda_2(0)$, which depends on $s(x)$ as well. Note that we get $\lambda(0) = 0$, i.e., $\lambda_i(0) = 0$ for all $i = 1, 2, 3, \ldots$ when $\eta = 0$. This means the neutral stability in this case. We summarize these results in the following theorem.

Theorem 2.4.2. *Assume $a \leq -6\eta^2$.*

(i) When $\eta = 0$, we have $\lambda(0) = 0$ and the steady-state $h_0(x)$ is neutrally stable.

(ii) When $\eta = \eta_1, \eta_2$, or η_3, the steady-state solution $h_0(x)$ is stable if $\lambda_1(0) < 0$, and unstable if $\lambda_1(0) > 0$, where $\lambda_1(0)$ is defined in (2.4.11).

2.4.2 Stability of the Periodic Steady-state When $a > -6\eta^2$

It is easy to verify that, when $6\eta^2 + a$ is positive, the right-hand side of (2.4.8) has its minimum when

$$q = p, \quad \text{where } \; p^2 = \frac{6\eta^2 + a}{2}.$$

Note that values of k and ω can be determined uniquely to satisfy this minimization. Then the maximal spectrum, in this case, becomes $\lambda_0(p)$. Substituting this into (2.4.8)-(2.4.9) yields

$$\begin{aligned} -3C\lambda_{0,R}(p) &= -8\eta^4 - 2a\eta^2 - b\eta - \frac{a^2}{4} := F_2(\eta, a, b), \\ -3C\lambda_{0,I}(p) &= \pm(b + 8\eta^3)\sqrt{\frac{6\eta^2 + a}{2}} := \mp 3C\mu. \end{aligned}$$

To find the stability conditions, we define

$$\begin{aligned} &\eta_- := -\sqrt[3]{\frac{b}{4}}, \qquad \eta_+ := -\sqrt[3]{\frac{b}{5}}, \qquad a_- := -4\eta^2 - \sqrt{-4\eta(4\eta^3 + b)}, \quad \text{and} \\ &a_+ := -4\eta^2 + \sqrt{-4\eta(4\eta^3 + b)}, \end{aligned} \tag{2.4.12}$$

and present our main result in the following theorem.

Theorem 2.4.3. *When $a > -6\eta^2$, we have the following cases:*

(i) If η is given in the interval $\eta \leq \eta_-$ or $\eta \geq 0$, then the steady-state $h_0(x)$ is unstable for all $a \in (-6\eta^2, \infty) - \{a_\pm\}$.

(ii) If η is given in the interval $\eta_- < \eta < 0$, then the steady-state $h_0(x)$ is stable when $a \in (\max\{a_-, -6\eta^2\}, a_+)$, and unstable otherwise.

Figure 2.3 summarizes the regions in Theorem 2.4.3.

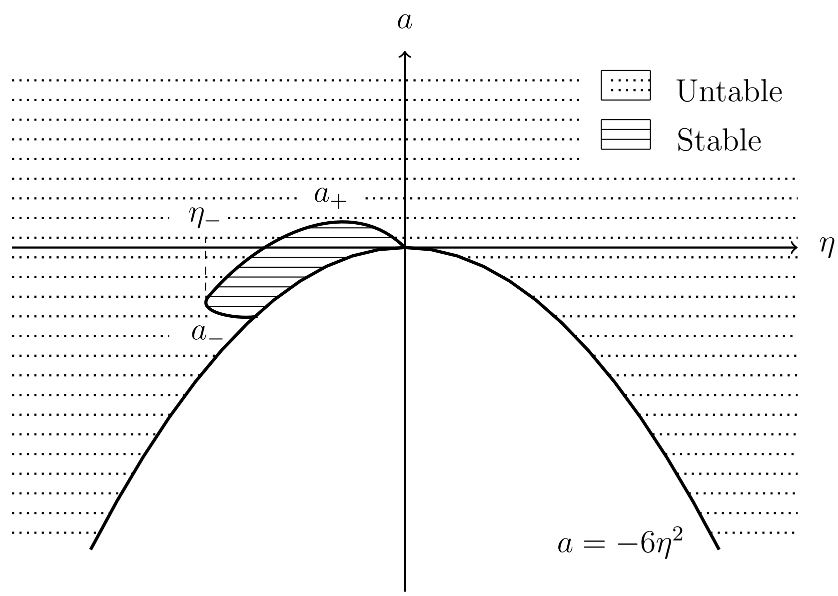


Figure 2.3: Stability/instability regions for the film flow provided in Theorem 2.4.3.

Proof. Equation $F_2(\eta, a, b) = 0$ is equivalent to

$$a^2 + 8\eta^2 a + 4b\eta + 32\eta^4 = 0, \tag{2.4.13}$$

which has solutions at $a = a_\pm$, where $a_\pm$ are defined in (2.4.12).

For $\eta \leq \eta_-$ or $\eta \geq 0$, there is no real solutions to (2.4.13), that is, $F_2(\eta, a, b)$ does not change its sign from negativity, then $\lambda_{0,R}(p) > 0$ and the steady-state solution is unstable. For $\eta = 0$ or η_-, we need to exclude $a_\pm$. This gives the proof of part (i).

For $\eta_- < \eta < 0$, $F_2(\eta, a, b)$ changes its sign at $a = a\pm$ and

$$F_2(\eta, a, b) \begin{cases} > 0 & \text{when } a \in (a_-, a_+), \\ < 0 & \text{when } a \in (-\infty, a_-) \cup (a_+, \infty). \end{cases}$$

To complete our proof, we connect these regions with the main condition $a > -6\eta^2$. When $\eta_- < \eta < 0$, we have $a_+ > -6\eta^2$, but the branch $a = a_-$ intersects $a = -6\eta^2$ only at $\eta = \eta_+ > \eta_-$, where $a_- > -6\eta^2$ when $\eta < \eta_+$. Then, case (ii) can be easily verified. ∎

At $a = a_\pm$, we have $\lambda_0(p) = \pm i\mu$. Since the real part of the leading term of $\lambda(p)$ is zero, we need to find the formula for $\lambda_1(p)$. From substituting (2.4.4)-(2.4.5) into (2.4.3), we have

$$\begin{aligned} \mathcal{L}_2 g_1 &= -3C\left[\lambda_1(p)g_0(x) + e^{-(ik-\eta)x}\frac{d}{dx}\left(e^{(ip-\eta)x}U_2(x)\right)\right], \\ g_1^{(m)}(0) &= g_1^{(m)}(l), \quad m = 0, 1, 2, 3, \end{aligned} \tag{2.4.14}$$

with

$$\begin{aligned} \mathcal{L}_2 g_1 &= g_1^{(4)} + A_1 g_1''' + A_2 g_1'' + A_3 g_1' + A_4 g_1, \\ U_2(x) &= 4h_{0,1} + \frac{16R}{5}\left[(ip-\eta)h_{0,1} + h_{0,1}'\right] - 2\cot(\theta)\left[(ip-\eta)h_{0,1} + h_{0,1}' + s_1'\right] \\ &\quad + \frac{1}{C}\left[(ip-\eta)^3 h_{0,1} + h_{0,1}'''\right], \end{aligned}$$

where $A_i, i = 1, \ldots, 4$ are given by

$$\begin{aligned} A_1 &= 4(ik-\eta), \qquad A_2 = 6(ik-\eta)^2, \\ A_3 &= 4(ik-\eta)^3 + 2a(ik-\eta), \\ A_4 &= (ik-\eta)^4 + a(ik-\eta)^2 + b(ik-\eta) + 3C\lambda_0(p). \end{aligned}$$

In view of (2.4.6)-(2.4.7), $g_0 = e^{i\omega x}$ satisfies $\mathcal{L}_2 g_0 = 0$. Then it is easy to verify that $u = e^{-i\omega x}$ solves the adjoint problem of (2.4.14), i.e.,

$$\mathcal{L}_2^* u := u^{(4)} - A_1 u''' + A_2 u'' - A_3 u' + A_4 u = 0,$$
$$u^{(m)}(0) = u^{(m)}(l), \quad m = 0, 1, 2, 3.$$

Similar to the previous subsection, we multiply (2.4.14) by $u = e^{-i\omega x}$ and integrate from 0 to L. Then we substitute the value of p and use the relation $\gamma_0 = \frac{3}{\cot(\theta)}\alpha_0$ to find $\lambda_1(p)$ as

$$\lambda_1(p) = \alpha_0 \left\{ 2\eta - \left(\frac{8R}{5} - \cot(\theta) \right) (4\eta^2 + a) - \frac{1}{4C}(a^2 - 32\eta^4) \right\}. \qquad (2.4.15)$$

Therefore, we have the result.

Theorem 2.4.4. *When $a > -6\eta^2$ and $a = a_\pm$, the steady-state $h_0(x)$ is stable if $\lambda_1(p) < 0$, and unstable if $\lambda_1(p) > 0$, where $\lambda_1(p)$ is defined in (2.4.15).*

2.5 Conclusions and Summary

We analytically studied the flow of a thin film over an inclined periodic wavy wall governed by a long-wave model. The existence of periodic steady-state solution was proved rigorously and its stability was analyzed by a perturbation analysis.

For the existence and the uniqueness of the steady-state solution, the variation-of-parameter method was used to write the steady-state problem in an integral form. We have started by constructing an iteration scheme in terms of the integral forms to find periodic solutions in the form $h_0(x) = 1 + w(x)$, where $w(x)$ is solution to the non-homogeneous equation (2.2.2). Three distinct cases have been handled depending on the values of Reynolds number (R), the capillary number (C), and the inclination angle (θ). For each case, we proved the result and found an asymptotic formula for $w(x)$. To work in a unified case, we chose to re-write equation (2.2.2) in the form (2.3.1). Then the existence and the uniqueness were proved by incorporating an abstract Banach contractive theorem.

For the stability, by using the Floquet-Bloch theory and the method of perturbation analysis, we obtained the stability of the steady-state solutions in a weighted functional space L_η. This study has been split into two different cases depending on a relation between η, the real parameter defined in the weighted space, and the values of R, C, and θ, particularly, the value of

$$a = \frac{8RC}{5} - 2C\cot(\theta).$$

For each case, stability conditions were successfully determined, see Theorems 2.4.1-2.4.4.

Chapter 3

Separable Solutions to the Generalized Burgers Equation and Their Stability

3.1 Introduction

Recently there have been extensive interests in the study of the generalized Burgers equation with time dependent viscosity

$$u_t + uu_x = \frac{\delta}{(t+1)^M} u_{xx},\ 0 \leq x \leq l,\ t > 0, \tag{3.1.1}$$

subject to

$$u(0,t) = u(l,t) = 0,\ t > 0, \tag{3.1.2}$$

$$u(x,0) = u_0(x),\ x \in [0,l], \tag{3.1.3}$$

where $M \geq 0$, $\delta > 0$ and $l > 0$ are constants. Here, $u_0(x)$ is a continuous function on $[0, l]$ satisfying $u_0(0) = 0$ and $u_0(l) = 0$. For the importance of this equation in nonlinear acoustics, we refer to the references [14, 15].

Srinivasarao and Satyanarayana [73] studied the large time asymptotics of the solutions to (3.1.1)-(3.1.3) by developing the method of separation variables that used to be valid to linear equations. They balanced the dominated contribution terms and obtained the large time behavior of (3.1.1)-(3.1.3) for different values of M. When $0 \leq M < 1$, and $M = 1$ with $\delta > l^2/\pi^2$, they showed that the term uu_x in (3.1.1) can be ignored and the large time behavior to (3.1.1)-(3.1.3) can be approximated by the linear partial differential equation

$$u_t = \frac{\delta}{(t+1)^M} u_{xx}, \ 0 < x < l, \ t > 0,$$

subject to (3.1.2)-(3.1.3). Indeed, they obtained

$$u(t,x) \sim A_1 \ \exp\left(-\frac{\delta\pi^2(t+1)^{1-M}}{l^2(1-M)}\right) \ \sin\left(\frac{\pi x}{l}\right) \ \text{as } t \to \infty, \ \text{for } 0 \leq M < 1,$$

and

$$u(x,t) \sim A_2 \ (t+1)^{-\frac{\delta\pi^2}{l^2}} \ \sin\left(\frac{\pi x}{l}\right) \ \text{as } t \to \infty, \ \text{for } M = 1, \delta > \frac{l^2}{\pi^2},$$

where A_1 and A_2 are constants that can be determined from the initial functions. The other case when $M > 1$ was also studied in [73] and they found that the solution behaves like

$$u(x,t) = \frac{x}{t+1}$$

for x near to zero.

The most difficult case to study is the critical case when $M = 1$ with $\delta < l^2/\pi^2$.

This was investigated in 2015 by Srinivasarao and Nath [72]. The existence of positive separable solution was proved, and they numerically claimed that the positive separable solution, in the form

$$u(x,t) = \frac{v(x,t)}{t+1}, \tag{3.1.4}$$

describes the large time behavior of the original problem, where $v(x)$ is the positive steady-state solution to the problem (3.1.5) below.

The purpose of this work is further to study (3.1.1)-(3.1.3) in the last case when $M = 1$. We incorporate the transformation (3.1.4) and the time rescaling

$$\tau = \ln(1+t)$$

to reduce the original problem into

$$\begin{cases} v_\tau = v - vv_x + \delta v_{xx}, \\ v(0,\tau) = 0, v(l,\tau) = 0, \\ v(x,0) = v_0(x) = u_0(x). \end{cases} \tag{3.1.5}$$

Since the existence of a steady-state solution $v(x,\tau) = v(x)$ to the partial differential equation (3.1.5) gives the existence of a separable solution to the problem (3.1.1)-(3.1.3) in the form (3.1.4), we shall focus on the existence and stability of the steady-state solutions to (3.1.5). Also, the stability of $v(x)$ in (3.1.5) implies that the separable solution to the problem is stable. By developing the shooting arguments in [57, 58], we not only obtain the existence and the uniqueness of the positive solution, but also provide the existence of sign-changed steady-state solutions. More interestingly, we also estimate the number of the total solutions of the problem in terms of the parameter values. Compared to [72], our method is new and seems easy to follow. Our results greatly extend their studies. Furthermore, using a perturbative bifurcation analysis, we obtain the asymptotic formula for the small-amplitude positive solution when the parameter δ is near its first bifurcation location $\delta = l^2/\pi^2$. Based on this asymptotic formula, we also find that this solution is stable by finding the principal eigenvalue to the eigenvalue problem corresponding to the linearized equation. We should mention that when δ is sufficiently small, stability of the steady-state to (3.1.5) has been studied by Sun and Ward [76] by estimating the principal eigenvalue.

The rest of this chapter is as follows. We derive the partial differential equation (3.1.5), and obtain the stability of the trivial solution in Section 3.2. In Section 3.3, we use a shooting method to prove the existence of the non-constant steady-state solutions. Then we derive the exact number of all solutions depending on the viscosity parameter δ and the space bound l. Using bifurcation analysis, linear stability of small-amplitude positive steady-state solution is investigated in Section 3.4. Conclusions and summary are presented in Section 3.5.

3.2 Time Rescaling and Stability of the Trivial Solution

Consider the generalized Burgers equation (3.1.1) with $M = 1$,

$$u_t + uu_x = \frac{\delta}{t+1} u_{xx}, \qquad 0 \le x \le l,\ t > 0, \tag{3.2.1}$$

subject to the initial-boundary conditions (3.1.2)-(3.1.3).

As mentioned before, we remove the time dependent coefficient $1/(t+1)$ by introducing a special transformation. To do this, for some function $v(x,t)$, assume that the solution $u(x,t)$ to (3.2.1), with the initial-boundary conditions (3.1.2)-(3.1.3), has the form (3.1.4). Substitute it into (3.2.1) to get

$$(1+t)v_t = v - vv_x + \delta v_{xx}.$$

From (3.1.2), the boundary conditions become

$$v(0,t) = 0, v(l,t) = 0.$$

Rescale time as $\tau = \ln(1+t)$. By finding the derivative v_τ and simplifying the latter partial differential equation, we obtain

$$\begin{cases} v_\tau = v - vv_x + \delta v_{xx}, \\ v(0,\tau) = 0, \quad v(l,\tau) = 0. \end{cases} \tag{3.2.2}$$

Note that when we study the large time behavior as $\tau \to \infty$ we also have $t \to \infty$.

For the local stability of the trivial solution (the zero solution) to (3.2.2) we use the standard linear analysis, that is, we let

$$v(x,\tau) = \sigma w(x) e^{\lambda\tau},$$

where $\sigma \ll 1$, $w(x)$ is a non-zero continuously differentiable function, and λ is a parameter. We study the behavior of the small perturbation $\sigma w(x)e^{\lambda\tau}$, which can be determined by finding the sign of the parameter λ. Substituting it into (3.2.2) and taking the linear terms give

$$\begin{cases} \delta w'' + (1-\lambda)w = 0, \\ w(0) = 0, \quad w(l) = 0. \end{cases} \tag{3.2.3}$$

The non-trivial solution to the boundary value problem (3.2.3) is given by

$$w(x) = A\sin\left(\sqrt{\frac{1-\lambda}{\delta}}x\right),$$

for some constant A. By the boundary conditions, we have $\sqrt{\frac{1-\lambda}{\delta}}l = n\pi$, $n = 1, 2, 3, ...$, which implies that

$$\lambda_n = 1 - \frac{\delta n^2\pi^2}{l^2}.$$

Since the principal eigenvalue $\lambda_1 = 1 - \frac{\delta\pi^2}{l^2}$ has the same sign of $\frac{l^2}{\pi^2} - \delta$, we have the following result.

Theorem 3.2.1. *The trivial solution of (3.2.2), and hence of (3.2.1), is locally stable when $\delta > l^2/\pi^2$, and unstable when $\delta < l^2/\pi^2$.*

To prove the global stability of the trivial solution when $\delta > l^2/\pi^2$, we use an energy argument (see Logan [47]). Define the energy function

$$E(\tau) = \int_0^l v^2(x,\tau)dx.$$

By differentiating both sides with respect to τ and using the differential equation in (3.2.2), we get

$$E'(\tau) = 2\int_0^l [v^2 - v^2 v_x + \delta v v_{xx}]dx.$$

The last two terms can be simplified by using integration by parts. Together with the boundary conditions in system (3.2.2), this leads to

$$E'(\tau) = 2\int_0^l v^2 dx - 2\delta \int_0^l v_x^2 dx.$$

Making use of the Poincaré inequality (see e.g. [47]) yields

$$\begin{aligned} E'(\tau) &\leq 2\int_0^l v^2 dx - 2\frac{\pi^2}{l^2}\delta \int_0^l v^2 dx \\ &= 2\big(1 - \frac{\delta\pi^2}{l^2}\big)E(\tau), \end{aligned}$$

which gives that

$$E(\tau) \leq E(0)\exp\left[2\tau\left(1 - \frac{\delta\pi^2}{l^2}\right)\right]. \tag{3.2.4}$$

Then, $E(\tau) \to 0$ as $\tau \to \infty$ when $\delta > l^2/\pi^2$. Hence,

Theorem 3.2.2. *When $\delta > l^2/\pi^2$, the trivial solution of (3.2.2), and hence of (3.2.1), is globally asymptotically stable.*

In the next section, we proceed to study the existence of the non-trivial steady-state to (3.2.2). We will see that the problem has only the trivial solution when $\delta \geq l^2/\pi^2$. The global stability result for the case when $\delta = l^2/\pi^2$, can be obtained from (3.2.4) and the uniqueness of the trivial solution.

3.3 The Existence and the Number of Steady-states

We study here the existence of the non-trivial solutions to the steady-state problem corresponding to (3.2.2), namely

$$\begin{cases} \delta v'' - vv' + v = 0, \\ v(0) = 0, \;\; v(l) = 0, \end{cases} \tag{3.3.1}$$

where prime denotes the derivative d/dx. To do this, we develop the shooting method in [57, 58] and first consider the initial value problem

$$\begin{cases} \delta v'' - vv' + v = 0, \\ v(0) = 0, v'(0) = k \end{cases} \tag{3.3.2}$$

where k is a constant. We want to study the behavior of the solution $v(x, k)$ to (3.3.2) and seek possible values of k so that the second boundary condition in (3.3.1) is satisfied, i.e., $v(l, k) = 0$.

3.3.1 Pre-analysis

We start by analyzing the solution $v(x, k)$ to (3.3.2). The following lemma gives the possible values of k so that the problem has a non-trivial solution.

Lemma 3.3.1. *For the solution $v(x, k)$ of the initial-value problem (3.3.2), we have*

(i) $v(x) = 0$ and $v(x) = x$ are two solutions to (3.3.2) when $k = 0$ and $k = 1$, respectively.

(ii) $v(x, k) > x$,when exists, for all $k > 1$.

(iii) If $0 < k < 1$, then $v(x,k)$ is periodic and has infinitely many zeros at x-axis.

Proof. It is easy to verify part (i) by direct substitution.

For part (ii), re-write the differential equation from (3.3.2) in the form

$$\frac{\delta v''}{v'-1} = v,$$

and integrate both sides from 0 to x to get the equation

$$v'(x,k) = 1 + (k-1)\exp\left(\frac{1}{\delta}\int_0^x v(s)\,ds\right).$$

For $k > 1$, we have $v'(x,k) > 1$, for all $x \in [0,\infty)$, and it follows that $v(x,k) > x$.

To prove part (iii), define $p(x) = \frac{dv(x)}{dx}$ for all $x \in [0,\infty)$. Then

$$\frac{dp}{dv} = \frac{dp/dx}{dv/dx} = \frac{v''}{p},$$

which gives, by substituting the formula of v'' from the equation in (3.3.1),

$$\frac{dp}{dv} = \frac{v(p-1)}{\delta p} \implies \frac{\delta p}{(p-1)}\,dp = v\,dv.$$

Integrating both sides yields

$$\delta(p + \ln(1-p)) = \frac{v^2}{2} + c. \tag{3.3.3}$$

Define $Q : (-\infty, 1) \times \mathbb{R} \to \mathbb{R}$ by

$$Q(p,v) = \delta(p + \ln(1-p)) - \frac{v^2}{2}.$$

Then the contour of Q shows that every solution is periodic for $0 < k < 1$, see Figure

3.1. The proof is complete. ∎

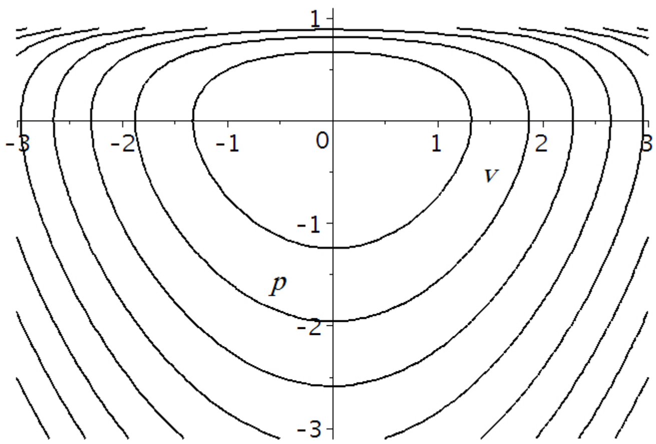


Figure 3.1: Contour of $Q(p, v) = \delta(p + \ln(1 - p)) - v^2/2$, with $\delta = 2$.

Remark 3.3.1. *From Figure 3.1, the curve of Q is symmetrical about the variable v. This leads to the following observation about the symmetry of the solution $v(x, k)$:*

(i) Assume that $v(x, k)$ is a solution to (3.3.2) with $0 < k < 1$ and x_2 is its first zero after $x = 0$. Then we have another solution $\hat{v}(x, \hat{k})$ that is a shift of half period of $v(x, k)$, i.e., we have $\hat{v}'(0) = v'(x_2) < 0$ and $\hat{v}'(x_2) = k = v'(0)$.

(ii) Solution $v(x, k)$ to (3.3.2) is oscillatory about the line $v = 0$, and every arch above $v = 0$ is followed by an arch below $v = 0$ and vise versa, which are symmetrical. See Figure 3.2.

In view of Lemma 3.3.1, solutions to the boundary value problem (3.3.1) only exist when $|k| < 1$. We consider $0 < k < 1$ first. Let x_2 be the first positive value of x such that $v(x, k) = 0$, and a be the maximum value of $v(x, k)$ in the interval $[0, x_2]$, which occurs at $x = x_1$, i.e.,

$$\max_{0<x<x_2} v(x, k) = v(x_1, k) = a.$$

Denote the length of intervals $[0, x_1]$ and $[x_1, x_2]$ by T_1 and T_2, respectively, see Figure 3.3 for details.

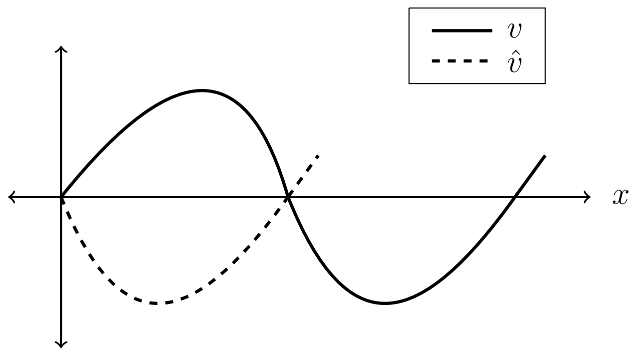


Figure 3.2: Oscillation of the solution $v(x,k)$ to (3.3.2) and its associated function $\hat{v}(x,\hat{k})$ defined in Remark 3.3.1.

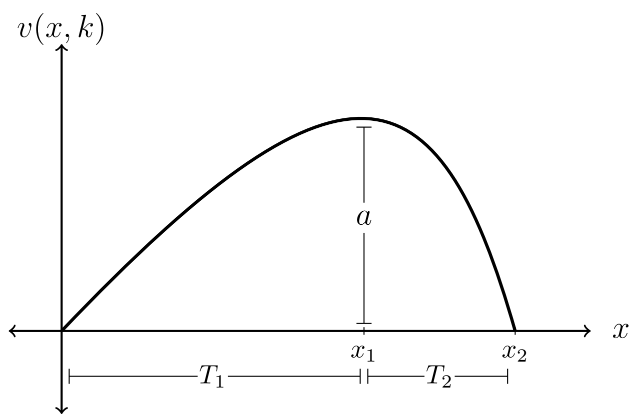


Figure 3.3: The base length, in terms of T_1 and T_2, and the maximum value of the solution $v(x,k)$ to (3.3.2).

We want to write T_1 and T_2 as functions of a. This leads to find a value of $a > 0$ (or $0 < k < 1$) so that $v(k, l) = 0$. To do this, define

$$y = q(p) := -\delta(p + \ln(1 - p)). \tag{3.3.4}$$

Note that $q(p)$ is defined and continuous on $(-\infty, 1)$, with

$$q'(p) \begin{cases} < 0, & p \in (-\infty, 0), \\ = 0, & p = 0, \\ > 0, & p \in (0, 1). \end{cases}$$

This means that $y = q(p)$ is not one-to-one, that is, to find p in terms of y (> 0) we need to split $q(p)$ into two branches as

$$q(p) = \begin{cases} q_0(p), & p \in (0, 1), \\ q_1(p), & p \in (-\infty, 0). \end{cases}$$

Let $p_+(y) = q_0^{-1}(y)$ and $p_-(y) = q_1^{-1}(y)$. From $dv/dx = p$, we have

$$\int_0^a \frac{dv}{p_+(y)} = \int_0^{x_1} dx = T_1,$$

By letting $v = at$, we obtain that T_1 has the form

$$T_1(a) = \int_0^1 \frac{adt}{p_+(y)} = \int_0^1 \frac{adt}{p_+\left(\frac{a^2}{2}(1 - t^2)\right)},$$

where

$$y = c - \frac{v^2}{2} = \frac{a^2}{2}(1 - t^2)$$

follows from the definition of y in (3.3.4) and by using the point $x = x_1$ in (3.3.3).

Similarly, we can find T_2 as a function of a to be

$$T_2(a) = \int_{x_1}^{x_2} dx = \int_0^1 -\frac{adt}{p_-(y)} = \int_0^1 -\frac{adt}{p_-\left(\frac{a^2}{2}(1-t^2)\right)}.$$

Lemma 3.3.2. *$T_1(a)$ is increasing and $T_2(a)$ is decreasing for all $a > 0$.*

Proof. To prove $T_1(a)$ is increasing, it is enough to show that $\frac{p_+(y)}{a}$ is decreasing in a. Actually we have

$$\begin{aligned}\frac{d}{da}\left(\frac{p_+(y)}{a}\right) &= \frac{1}{a^2}\left(a\frac{dp_+(y)}{dy}\frac{dy}{da} - p_+(y)\right) \\ &= \frac{-1}{a^2 p}\left(2(1-p)(p+\ln(1-p)) + p^2\right), \quad p \in (0,1).\end{aligned}$$

Here, we have made use of

$$\begin{aligned}a\frac{dy}{da} &= \frac{a}{2}\,\frac{d}{da}\left[a^2(1-t^2)\right] \\ &= a^2(1-t^2) \\ &= 2y \\ &= -2\delta(p+\ln(1-p))\end{aligned}$$

and, by implicit differentiation of (3.3.4),

$$\frac{dp_+}{dy} = \frac{1-p}{\delta p}, \quad \text{for} \quad p \in (0,1).$$

It is easy to verify that $g(p) = 2(1-p)(p+\ln(1-p)) + p^2$ satisfies $g'(p) > 0$ for all $p \in (-\infty, 1) - \{0\}$, $g(0) = 0$, and $g'(0) = 0$. This means that

$$g(p)\begin{cases} > 0 & \text{for } p \in (0,1), \\ < 0 & \text{for } p \in (-\infty, 0). \end{cases}$$

Then $\frac{d}{da}\left(\frac{p_+}{a}\right) < 0$ and $T_1(a)$ is increasing with respect to a.

Similarly,

$$\begin{aligned}\frac{d}{da}\left(\frac{dp_-}{a}\right) &= \frac{1}{a^2 p}\left(2(1-p)(p+\ln(1-p))+p^2\right), \quad p \in (-\infty, 0)\\ &> 0.\end{aligned}$$

Then $T_2(a)$ is decreasing with respect to a, which completes the proof. ∎

Lemma 3.3.3. *The following is true:*

$$\max_{a>0} T_1(a) = \min_{a>0} T_2(a) = \frac{\sqrt{\delta}\pi}{2}.$$

Proof. From the above lemma we conclude that $\max_{a>0} T_1(a) = \lim_{a\to 0^+} T_1(a)$. To find this limit, we use Taylor expansion to the left-hand side of (3.3.3) when $a \to 0^+$ ($p \to 0^+$) to get

$$\delta\left(p - p - \frac{p^2}{2} + O(p^3)\right) = \frac{a^2}{2}(t^2-1).$$

By simplifying this relation, we obtain the behavior

$$p \sim a\sqrt{\frac{1-t^2}{\delta}}, \quad \text{as } a \to 0.$$

Then

$$\lim_{a\to 0^+} T_1(a) = \lim_{a\to 0^+} \int_0^1 \sqrt{\frac{\delta}{1-t^2}}dt = \frac{\sqrt{\delta}\pi}{2}.$$

Similarly, we can show that

$$\min_{a>0} T_2(a) = \frac{\sqrt{\delta}\pi}{2}.$$

This completes the proof. ∎

Lemma 3.3.4. *$T(a) = T_1(a) + T_2(a)$ is increasing for all $a > 0$. Moreover,*

$$\min_{a>0} T(a) = \sqrt{\delta}\pi.$$

Proof. We have the following formula of $T(a)$:

$$T(a) = \int_0^1 a\left(\frac{1}{p_+} - \frac{1}{p_-}\right) dt.$$

Define $F(a) = a\left(\frac{1}{p_+} - \frac{1}{p_-}\right)$, then

$$\frac{dF}{da} = \frac{1}{(p_+)^2}\left(p_+ - 2y\frac{1-p_+}{\delta p_+}\right) - \frac{1}{(p_-)^2}\left(p_- - 2y\frac{1-p_-}{\delta p_-}\right).$$

Let

$$G_\pm = \frac{1}{(p_\pm)^2}\left(p_\pm - 2y\frac{1-p_\pm}{\delta p_\pm}\right).$$

Then we have

$$\frac{dG_\pm}{dy} = \frac{1-p_\pm}{(p_\pm)^5}\left(-3(p_\pm)^2 + 6y - 4yp_\pm\right).$$

Since $(-3p^2 + 6y - 4yp) = p^2 - 6p + (4p-6)\ln(1-p) := h(p)$ is defined and continuous on $(-\infty, 1)$ with $h(0) = 0, h'(0) = 0$, and $h''(p) = \frac{2p^2}{(1-p)^2} > 0$, for all $p \in (-\infty, 1) - \{0\}$, we then obtain $h(p) > 0$ for all $p \in (-\infty, 1) - \{0\}$. See graph of $h(p)$ in Figure 3.4. From the definition of p_+ and p_-, we have

$$\frac{1-p_+}{(p_+)^5} > 0 \text{ and } \frac{1-p_-}{(p_-)^5} < 0.$$

It follows that

$$\frac{dG_+}{da} > 0 \text{ and } \frac{dG_-}{da} < 0,$$

that is, G_+ is increasing and G_- is decreasing for all $a > 0$.

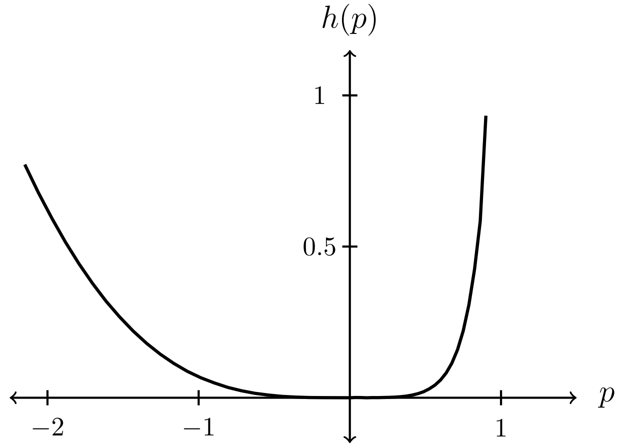


Figure 3.4: Graph of $h(p) = (4p - 6)\ln(1 - p) + p^2 - 6p$

To complete our proof, we need to compare $\inf G_+$ and $\sup G_-$. When $a \to 0^+$ we have $p_+ \to 0^+$. Using (3.3.4) and L'Hopital's rule, we can show that $\lim_{a \to 0^+} G_+ = \frac{1}{3}$ $(= \inf G_+)$. Similarly $\lim_{a \to 0^+} G_- = \frac{1}{3}$ $(= \sup G_-)$. Then we conclude that $G_+ > G_-$, for all $a > 0$. Hence $F(a)$ is increasing and so is $T(a)$. Its minimum value can be easily computed as

$$\begin{aligned} \min_{a>0} T(a) &= \lim_{a \to 0^+} T(a) \\ &= \lim_{a \to 0^+} T_1(a) + \lim_{a \to 0^+} T_2(a) \\ &= \sqrt{\delta}\pi. \end{aligned}$$

The proof is complete. ∎

3.3.2 Main Result

We are now ready to prove the existence of solutions to the boundary value problem (3.3.1), and count the number of them.

Theorem 3.3.1. *Define the number*

$$N(\delta) = \left\lceil \frac{l}{\sqrt{\delta}\pi} \right\rceil, \quad \text{for } \delta > 0,$$

where $\lceil x \rceil$ *denotes the smallest integer number greater than or equal to* x*. Then the problem (3.3.1) has exactly* $2N(\delta) - 1$ *solutions.*

Proof. We are looking for the value of k (or $a > 0$) so that the non-trivial solution $v(x, k)$ to the initial value problem (3.3.2) satisfies

$$v(l, k) = 0.$$

By Remark 3.3.1 (ii), this equation with some peaks can be expressed into the form

$$m(T_1 + T_2)(a) = l, \tag{3.3.5}$$

where m is a non-negative integer representing the number of intervals with the length $T_1 + T_2$.

If $m \geq N(\delta)$, then by Lemma 3.3.4, we have

$$m(T_1 + T_2)(a) > N(\delta)(T_1 + T_2)(0^+) \geq l, \ \forall a > 0,$$

which means that (3.3.5) fails to hold. Hence, there is no non-trivial solution to the boundary value problem (3.3.1) in this case.

If $0 \leq m < N(\delta)$, then

$$m(T_1 + T_2)(0^+) \leq (N(\delta) - 1)(T_1 + T_2)(0^+) < l.$$

In view of the continuity and the monotonicity of $(T_1 + T_2)(a)$, as well as the limit

$$\lim_{a\to\infty} m(T_1 + T_2)(a) = \infty,$$

the Intermediate Value Theorem implies that there exists exactly one point $a_0 > 0$ satisfying $m(T_1 + T_2)(a_0) = l$, which means that there exists one solution to the problem (3.3.1) with m arches of base-length $T_1 + T_2$.

Now, we can count the number of solutions when m changes from 0 to $N(\delta) - 1$:

(1) If $m = 0$, then there is no solution with base length $T_1 + T_2$, that is, (3.3.1) has the unique trivial solution.

(2) For $0 < m < N(\delta)$, there exists one solution with m arches of base-length T_1+T_2 and $k > 0$. By Remark 3.3.1 (i), a symmetrical solution of base-length $T_1 + T_2$ with m arches and a negative slope at $x = 0$ exists. Thus when m changes from 1 to $N(\delta) - 1$, we have two solutions for each m. Hence, the number of total solutions in this case is given by

$$\sum_{i=1}^{N(\delta)-1} 2 = 2N(\delta) - 2.$$

From (1) and (2), the number of total solutions including the trivial solution is $2N(\delta) - 1$, which completes our proof. ∎

The above result gives the existence and the number of solutions based on the values of l and δ. For the uniqueness, we have the following result.

Theorem 3.3.2. *The boundary value problem (3.3.1) has a unique solution, which is $v(x) = 0$ if $\delta \geq l^2/\pi^2$, and a unique positive solution if $\delta < l^2/\pi^2$.*

Proof. If $\delta \geq l^2/\pi^2$, then $N(\delta) = 1$, where $N(\delta)$ is defined in Theorem 3.3.1. Hence, (3.3.1) has a unique trivial solution. In the case $\delta < l^2/\pi^2$ we have $N(\delta) \geq 2$, that is, (3.3.1) has a positive solution with one arch of base-length $T_1 + T_2$. All other non-trivial solutions are not positive in $(0, l)$. This completes the proof. ∎

From the above theorem and using (3.2.4), we have the following result.

Theorem 3.3.3. *The trivial solution is a global asymptotically stable steady-state solution to (3.2.2) when* $\delta = l^2/\pi^2$.

3.4 Bifurcation Analysis

In this section we study the stability of small-amplitude positive steady-state solution to (3.2.2) by estimating the sign of the principal eigenvalue via a perturbative approach.

3.4.1 Weakly Nonlinear Analysis

Assume that $\bar{v}$ is the positive steady-state solution bifurcated from the trivial solution. For the bifurcation analysis, assume that δ and the positive solution $\bar{v}$ have the forms, for $\epsilon > 0$,

$$\delta = \delta_0 + \epsilon\delta_1 + \epsilon^2\delta_2 + \cdots = \sum_{i=0}^{\infty} \epsilon^i \delta_i, \tag{3.4.1}$$

$$\bar{v} = v_0 + \epsilon v_1 + \epsilon^2 v_2 + \cdots = \sum_{i=0}^{\infty} \epsilon^i v_i. \tag{3.4.2}$$

In this analysis, δ is the bifurcation parameter, with $\delta_0 = l^2/\pi^2$ being the first bifurcation location, where the corresponding small amplitude positive steady-state solution $\bar{v}$ can be bifurcated from the trivial solution $v_0 = 0$. Substitute (3.4.2) and (3.4.1) into (3.3.1) and equate $O(\epsilon)$, $O(\epsilon^2)$, and $O(\epsilon^3)$ to get the following boundary value problems

$$\begin{cases} \dfrac{l^2}{\pi^2} v_1'' + v_1 = 0, \\ v_1(0) = 0,\ v_1(l) = 0, \end{cases} \tag{3.4.3}$$

$$\begin{cases} \dfrac{l^2}{\pi^2} v_2'' + v_2 = v_1' v_1 - \delta_1 v_1'', \\ v_2(0) = 0,\ v_2(l) = 0, \end{cases} \tag{3.4.4}$$

and

$$\begin{cases} \dfrac{l^2}{\pi^2} v_3'' + v_3 = (v_1 v_2)' - \delta_1 v_2'' - \delta_2 v_1'', \\ v_3(0) = 0,\ v_3(l) = 0, \end{cases} \tag{3.4.5}$$

Solving the boundary value problem (3.4.3) gives the unique (up to constant multiple) positive solution

$$v_1(x) = \sin\left(\frac{\pi x}{l}\right). \tag{3.4.6}$$

Consider the adjoint system obtained from the left-hand side of (3.4.4)

$$\begin{cases} \dfrac{l^2}{\pi^2} z'' + z = 0, \\ z(0) = 0\ , z(l) = 0, \end{cases}$$

which gives a solution

$$z(x) = \sin\left(\frac{\pi x}{l}\right).$$

Multiplying both sides of the differential equation in (3.4.4) by $z(x)$ and integrating from 0 to l, give the solvability condition

$$\int_0^l (v_1' v_1 - \delta_1 v_1'') z \, dx = 0.$$

By direct computations, we find $\delta_1 = 0$. Then (3.4.4) can be simplified as

$$\begin{cases} \dfrac{l^2}{\pi^2} v_2'' + v_2 = v_1' v_1, \\ v_2(0) = 0, \;\; v_2(l) = 0. \end{cases} \tag{3.4.7}$$

Plugging v_1 from (3.4.6) into (3.4.7) gives a non-homogeneous term in the form $v_1 v_1' = \frac{\pi}{2l} \sin\left(\frac{2\pi x}{l}\right)$. Use the method of undetermined coefficients to find a particular solution to (3.4.7) as

$$v_2(x) = \sin\left(\frac{\pi x}{l}\right) - \frac{\pi}{6l} \sin\left(\frac{2\pi x}{l}\right). \tag{3.4.8}$$

In order to find δ_2, we use the solvability condition of (3.4.5) with $\delta_1 = 0$. Hence, the formula of δ_2 is given by

$$\delta_2 = \frac{\int_0^l (v_1 v_2)' \sin\left(\frac{\pi x}{l}\right) dx}{\int_0^l v_1'' \sin\left(\frac{\pi x}{l}\right) dx}.$$

By using (3.4.6) and (3.4.8), we find

$$\int_0^l (v_1 v_2)' \sin\left(\frac{\pi x}{l}\right) dx = \frac{\pi^2}{24l}, \text{ and } \int_0^l v_1'' \sin\left(\frac{\pi x}{l}\right) dx = \frac{-\pi^2}{2l}.$$

Then

$$\delta_2 = -\frac{1}{12}. \tag{3.4.9}$$

3.4.2 Stability of Small-amplitude Steady-states

Now, we are in a position to estimate the principal eigenvalue and show the stability of the small-amplitude steady-state $\bar{v}$. We write the solution of (3.2.2) in the form

$$v(x) = \bar{v}(x) + \sigma_1 w_1(x) e^{\lambda \tau},$$

where $\sigma_1 \ll 1$, $w_1(x)$ is a continuously differentiable function, and λ is a parameter. Expand

$$\begin{aligned}\bar{v} =& \epsilon v_1 + \epsilon^2 v_2 + \dots \\ w =& \sin\left(\frac{\pi x}{l}\right) + \epsilon w_1 + \epsilon^2 w_2 + \dots \\ \lambda =& \epsilon\lambda_1 + \epsilon^2 \lambda_2 + \dots \\ \delta =& \frac{l^2}{\pi^2} + \epsilon^2 \delta_2 + \dots,\end{aligned}$$

where v_1, v_2, and δ_2 are given in (3.4.6),(3.4.8), and (3.4.9), respectively. Proceed to substitute them into (3.2.2), linearize this equation, and compute the expansion of λ. We have the following system in the order of $O(\epsilon)$:

$$\begin{cases} \dfrac{l^2}{\pi^2} w_1'' + w_1 = \lambda_1 w_0 + (v_1 w_0)', \\ \quad w_1(0) = 0,\ w_1(l) = 0, \end{cases} \tag{3.4.10}$$

where $w_0 = \sin(\pi x/l)$. To obtain the formula of λ_1, we use the solvability condition

$$\int_0^l [\lambda_1 w_0 + (v_1 w_0)'] \sin\left(\frac{\pi x}{l}\right) dx = 0$$

of the system (3.4.10). Then

$$\lambda_1 = \frac{-\int_0^l (v_1 w_0)' \sin\left(\frac{\pi x}{l}\right) dx}{\int_0^l w_0 \sin\left(\frac{\pi x}{l}\right) dx},$$

which gives $\lambda_1 = 0$. In this case, we need to find λ_2 to determine the stability. We simplify (3.4.10) to be

$$\begin{cases} \dfrac{l^2}{\pi^2} w_1'' + w_1 = (v_1 w_0)', \\ w_1(0) = 0, w_1(l) = 0, \end{cases}$$

and use the method of undetermined coefficients to find

$$w_1 = \sin\left(\frac{\pi x}{l}\right) - \frac{\pi}{3l}\sin\left(\frac{2\pi x}{l}\right).$$

Then, at the order of $O(\epsilon^2)$, we have the boundary value problem

$$\begin{cases} \frac{l^2}{\pi^2} w_2'' + w_2 = \lambda_2 w_0 + (v_1 w_1)' + (v_2 w_0)' - \delta_2 w_0'', \\ w_2(0) = 0,\ w_2(l) = 0, \end{cases}$$

where the solvability condition gives

$$\lambda_2 = \frac{\int_0^l [-(v_1 w_1)' - (v_2 w_0)' + \delta_2 w_0''] \sin\left(\frac{\pi x}{l}\right) dx}{\int_0^l w_0 \sin\left(\frac{\pi x}{l}\right) dx}.$$

A computation gives

$$\lambda_2 = -\frac{\pi^2}{6l^2} < 0.$$

Hence, the following theorem is valid.

Theorem 3.4.1. *When $\delta < l^2/\pi^2$, the small-amplitude positive steady-state solution to (3.2.2) is stable.*

Figure 3.5 shows the bifurcation diagram near δ_0.

3.5 Conclusions and Summary

In this chapter, we have investigated the dynamics of the separable solutions to the generalized Burgers equation (3.2.1), subject to (3.1.2)-(3.1.3), by transforming the generalized Burgers equation into a new constant-coefficient equation (3.2.2) and by analyzing the steady-state solutions to the new equation. A shooting method was used to prove the existence of steady-state solutions to (3.2.2) and we find the number

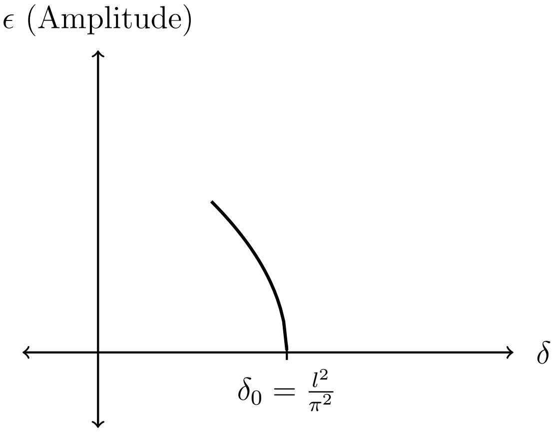


Figure 3.5: Bifurcation diagram of the positive steady-state to the generalized Burgers equation in $\epsilon\delta$-space.

of solutions depending on the viscosity parameter δ and the space bound l. This number of solutions gave a full understanding of what the solutions look like and how to determine the uniqueness of them.

We have shown the stability of trivial solution for the generalized Burgers equation (3.2.1) when $\delta > l^2/\pi^2$, which agrees with results in [72], and we proved that the trivial solution is stable for the critical case $\delta = l^2/\pi^2$. Using the bifurcation analysis, we have given the asymptotic formula for the positive small-amplitude steady-state solution of (3.2.2), and showed its stability when $\delta < l^2/\pi^2$. Stability of the large-amplitude positive steady-state solution is still challenging, and we will consider it in the future.

Chapter 4

The Minimal Wave Speed Selection to the Competition Model

4.1 Introduction

Consider the diffusive Lotka-Volterra competition model

$$\begin{cases} \phi_t = d_1\phi_{xx} + r_1\phi(1 - b_1\phi - a_1\psi), \\ \psi_t = d_2\psi_{xx} + r_2\psi(1 - a_2\phi - b_2\psi), \end{cases}$$

with the initial data

$$\phi(x,0) = \phi_0(x) \geq 0, \qquad \psi(x,0) = \psi_0(x) \geq 0, \quad \forall x \in \mathbb{R}.$$

Here $\phi(x,t)$ and $\psi(x,t)$ are the population densities of the first and the second species at time t and location x, respectively; d_1 and d_2 are the diffusion coefficients; r_1 and r_2 are the net birth rates; a_1 and a_2 are the competition coefficients; $1/b_1$ and $1/b_2$ are the carrying capacities of two species. All these parameters are assumed to be non-negative. Biologically, the model is used to study the logistic growth of two species population under competition. Originally, Okubo *et al* [56] used this model to describe the interaction between the externally introduced gray squirrels and the indigenous red squirrels in Britain.

Non-dimensionalizing the problem by

$$\begin{aligned}
&\sqrt{r_1/d_1}\; x \to x, \qquad r_1 t \to t, \\
&b_1\phi(x,t) = \tilde{\phi}(x,t), \qquad b_2\psi(x,t) = \tilde{\psi}(x,t), \\
&d = \frac{d_2}{d_1}, \qquad r = \frac{r_2}{r_1}, \qquad \frac{a_1}{b_2} \to a_1, \qquad \frac{a_2}{b_1} \to a_2,
\end{aligned}$$

gives a new system

$$\begin{cases} \tilde{\phi}_t = \tilde{\phi}_{xx} + \tilde{\phi}(1 - \tilde{\phi} - a_1\tilde{\psi}), \\ \tilde{\psi}_t = d\tilde{\psi}_{xx} + r\tilde{\psi}(1 - a_2\tilde{\phi} - \tilde{\psi}). \end{cases}$$

A change of variable $u = \tilde{\phi}$ and $v = 1 - \tilde{\psi}$ transforms the above model into a cooperative system

$$\begin{cases} u_t = u_{xx} + u(1 - a_1 - u + a_1 v), \\ v_t = dv_{xx} + r(1 - v)(a_2 u - v), \end{cases} \tag{4.1.1}$$

with the initial data

$$u(x,0) = u_0(x) = b_1\phi_0(x), \qquad v(x,0) = v_0(x) = 1 - b_2\psi_0(x), \quad \forall x \in \mathbb{R}.$$

Throughout this chapter, we assume that a_1 and a_2 satisfy the condition

$$0 < a_1 < 1 < a_2 \tag{4.1.2}$$

that arose in many previous studies. In [56], (4.1.2) means that the gray squirrels out-competes the reds. For biological interpretation of this condition, see also [28–30, 43, 91].

The cooperative system (4.1.1), under the condition (4.1.2), has three equilibria in the region $\{(u,v)|0 \le u \le 1, 0 \le v \le 1\}$, which are $e_0 = (0,0), e_1 = (1,1)$, and $e_2 = (0,1)$. When $a_1a_2 \ne 1$, another equilibrium exists with

$$e_4 = \left(\frac{1-a_1}{1-a_1a_2}, \frac{a_2(1-a_1)}{1-a_1a_2} \right).$$

It is in the first quadrant and satisfies $e_4 \gg (1,1)$ when $a_1a_2 < 1$; otherwise when $a_1a_2 > 1$, it is in the third quadrant and has negative components.

It is easy to see that e_0 is an unstable and e_1 is a stable equilibrium to the following ordinary differential equations system

$$\begin{cases} u' = u(1-a_1-u+a_1v), \\ v' = r(1-v)(a_2u-v). \end{cases}$$

Let

$$u(x,t) = U(z),\ v(x,t) = V(z), \quad z = x - ct,$$

be the traveling wave solution to the system (4.1.1), with speed $c \ge 0$, that connects e_1 and e_0, that is,

$$(U,V)(-\infty) = e_1, \quad (U,V)(\infty) = e_0.$$

Substituting this into the system (4.1.1) leads to an ordinary differential system

$$\begin{cases} -cU' = U'' + U(1-a_1-U+a_1V), \\ -cV' = dV'' + r(1-V)(a_2U-V), \\ (U,V)(-\infty) = e_1, \quad (U,V)(\infty) = e_0, \end{cases} \tag{4.1.3}$$

where prime denotes the derivative with respect to z. Results in [37,43,46,85] proved that there exists a constant $c_{\min} \geq 0$ so that the system has a traveling wave solution if and only if $c \geq c_{\min}$. In other words, $c_{\min}$ can be expressed as

$$c_{\min} = \inf\{c : (4.1.3) \text{ has a non-negative solution } (U, V)(z)\}$$

Standard linearization analysis near the equilibrium point e_0 shows that the necessary condition for the existence of a traveling wave solution is

$$c \geq c_0 = 2\sqrt{1 - a_1}.$$

The value of c_0 is the minimal wave speed for the linear system with non-negative traveling wave solutions. Based on the relation between the two speed values $c_{\min}$ and c_0, we have the following definition.

Definition 1. If $c_{\min} = c_0$, then we say that the minimal wave speed is linearly selected; otherwise, if $c_{\min} > c_0$, we say that the minimal wave speed is nonlinearly selected.

The problem of speed selection (linear and nonlinear) has been of a great interest in biological and mathematical studies, see e.g. [28–30, 32, 33, 42, 43, 48, 63, 64, 89, 91]. In literature, the linear speed selection for the system (4.1.1) was studied in [18, 29, 32, 42, 43, 54, 56]. Particularly, in [29], it was proved that the linear speed selection is realized if

$$d = 0 \quad \text{and} \quad (a_1 a_2 - 1)r \leq 2(1 - a_1). \tag{4.1.4}$$

Lewis *et al* [42] applied the results in [91] and proved that the minimal wave speed for (4.1.3) is linearly selected when the condition

$$d \leq 2 \quad \text{and} \quad (a_1 a_2 - 1)r \leq (2-d)(1-a_1) \tag{4.1.5}$$

holds. By extending the above result, Huang [32] proved that, by constructing an upper and a lower solution, the linear speed selection is realized without the restriction $d \leq 2$ but with the condition

$$\frac{(2-d)(1-a_1)+r}{ra_2} \geq \max\left\{a_1, \frac{d-2}{2|d-1|}\right\}. \tag{4.1.6}$$

These two conditions ((4.1.5) and (4.1.6)) are equivalent when $d \leq 2$, and are similar to (4.1.4) when $d = 0$. For the special case when $d = r = 1$ and $a_1 + a_2 = 2$, the system of equations can be reduced to a single equation in Fisher-KPP type and the minimal wave speed can be found as $c_{\min} = c_0$, e.g. [30].

We should mention that, in 1998, Hosono [30] studied the speed selection problem numerically and found that the wave speed is not always linearly selected. Based on his numerical simulations, he raised the following conjecture.

Hosono's conjecture. *If $a_1 a_2 \leq 1$, then $c_{\min} = c_0$ for all $r > 0$. If $a_1 a_2 > 1$, then there exits a positive number r_c such that $c_{\min} = c_0$ for $0 < r \leq r_c$, and $c_{\min} > c_0$ for $r > r_c$.*

This conjecture has been outstanding for almost twenty years and it is still open now. The purpose of this chapter is to work on the Hosono's conjecture for the special case when $d = 0$ in (4.1.3). We find that the conjecture is not completely correct, since the critical number r_c could be infinite even though $a_1 a_2 > 1$ is true. Therefore we provide a modified version of this conjecture and prove it rigorously. Our main result is the following theorem.

Theorem 4.1.1. *Suppose $d = 0$ in (4.1.1). There exists r_c, $0 \leq r_c \leq \infty$, such that*

(1) If $r \leq r_c$, the minimal wave speed is linearly selected.

(2) If $r > r_c$, the minimal wave speed is nonlinearly selected.

We also give some estimates of r_c. This successfully leads to some explicit and new conditions for both linear and nonlinear speed selection mechanism. In [32], Huang strongly believes that the condition (4.1.5) is necessary and sufficient for the linear speed selection. Our results are against this claim.

We should emphasize that we will use the upper-lower solution method coupled with the comparison principle to prove our result. The method originates from Weinberger [90] and Diekmann [13] with two classical constructions of upper and lower solutions that have been extensively applied in the research of traveling wave solutions. We will construct a new and smooth upper solution in the linear selection and a new lower solution in the nonlinear selection mechanism. We find that these new types of solutions approximate more accurately to the actual traveling waves, and this not only improves previous explicit results on the linear selection, but also provides some new results on the nonlinear selection that was thought to be very difficult in study.

The rest of the chapter is organized as follows. We study the asymptotic behavior of the traveling wave solution to (4.1.1), when $d = 0$, in Section 4.2. By applying the upper-lower solution method, we study the speed selection mechanisms and prove the modified Hosono's conjecture, Theorem 4.1.1, in Section 4.3. In Section 4.4, we estimate the critical value r_c and give explicit conditions for the speed selection. Conclusions are presented in Section 4.5. Section 4.6 is an appendix where the upper-lower solution technique is illustrated to our model.

4.2 The Asymptotic Behavior of the Wave Profiles

By letting $d = 0$ in (4.1.3), we get

$$\begin{cases} U'' + cU' + U(1 - a_1 - U + a_1 V) = 0, \\ cV' + r(1 - V)(a_2 U - V) = 0, \\ (U, V)(-\infty) = e_1, \quad (U, V)(\infty) = e_0. \end{cases} \tag{4.2.1}$$

We assume here that the traveling wave solution to (4.2.1) exists and want to find its asymptotic behavior. To this end, suppose that it exists and satisfies

$$(U, V)(z) \sim (\zeta_1 e^{-\mu z}, \zeta_2 e^{-\mu z}) \quad \text{as} \quad z \to \infty,$$

for some positive ζ_1, ζ_2, and μ. Substitute this into (4.2.1) and linearize the equation to get the algebraic system

$$A(\mu)\zeta = \mathbf{0},$$

where $\zeta = (\zeta_1 \;\; \zeta_2)^T$, $\mathbf{0}$ is the zero vector, and $A(\mu)$ is a 2×2 matrix given by

$$A(\mu) = \begin{pmatrix} \mu^2 - c\mu + 1 - a_1 & 0 \\ ra_2 & -c\mu - r \end{pmatrix}.$$

This algebraic equation has a non-trivial solution ζ if and only if $\det(A(\mu)) = 0$, that is,

$$\left[\mu^2 - c\mu + 1 - a_1\right][c\mu + r] = 0,$$

which implies that

$$\mu = \mu_1(c) = \frac{c - \sqrt{c^2 - 4(1 - a_1)}}{2} \quad \text{or} \quad \mu = \mu_2(c) = \frac{c + \sqrt{c^2 - 4(1 - a_1)}}{2}. \tag{4.2.2}$$

As such, if the traveling wave solution is non-negative, we require

$$c \geq 2\sqrt{1-a_1} := c_0.$$

Here c_0 is called the linear speed of the system.

For $c > c_0$, it gives $\mu_1 < \mu_2$, and the eigenvector of the matrix $A(\mu)$ corresponding to $\mu_i, i = 1, 2$ is the strongly positive vector

$$\zeta = (\zeta_1 \quad \zeta_2)^T = (c\mu_i + r \quad ra_2)^T, i = 1, 2. \tag{4.2.3}$$

Hence, as $z \to \infty$,

$$\begin{pmatrix} U(z) \\ V(z) \end{pmatrix} = C_1 \begin{pmatrix} c\mu_1 + r \\ ra_2 \end{pmatrix} e^{-\mu_1 z} + C_2 \begin{pmatrix} c\mu_2 + r \\ ra_2 \end{pmatrix} e^{-\mu_2 z}$$

for some constants C_1 and C_2.

4.3 The Speed Selection Mechanism

In this section we will study the speed selection of (4.2.1). The method used is the upper-lower solution pair coupled with the comparison technique, see the Appendix in this chapter for details. Due to $d = 0$, V in the second nonlinear equation can be solved explicitly in terms of U. Indeed, define first

$$y(z) = \frac{V(z)}{1 - V(z)} \quad \text{and} \quad \mu(z) = \exp\left(\frac{r}{c}\int_0^z (a_2 U(t) - 1) dt\right).$$

From the second equation in (4.2.1), the differential equation of $y(z)$ is given by

$$y' + \frac{r}{c}(a_2U - 1)y = -\frac{r}{c}a_2U,$$

with the boundary conditions

$$y(-\infty) = \infty, \quad y(\infty) = 0.$$

Multiplying both sides by $\mu(z)$ and integrating over $[z, \infty)$ give the formula of $y(z)$,

$$y(z) = \frac{ra_2}{c\mu(z)} \int_z^\infty \mu(s)U(s)ds.$$

This gives a formula for $V(z)$ as

$$V(z) = \frac{y(z)}{1 + y(z)} = \frac{ra_2 \int_z^\infty \mu(s)U(s)ds}{c\mu(z) + ra_2 \int_z^\infty \mu(s)U(s)ds} := H(U)(z). \tag{4.3.1}$$

By using this formula, (4.2.1) reduces to a non-local equation

$$\begin{cases} L_1(U, V) := U'' + cU' + U(1 - a_1 - U + a_1V) = 0, \\ U(-\infty) = 1, \ U(\infty) = 0, \end{cases} \tag{4.3.2}$$

where V is given in (4.3.1).

Remark 4.3.1. *V is a continuous function of c. When $c \to c_0$, V tends to*

$$V_{c_0}(z) = \frac{ra_2 \int_z^\infty \mu(s)U(s)ds}{c_0\mu(z) + ra_2 \int_z^\infty \mu(s)U(s)ds}.$$

For any $c > c_0$, we proceed to construct an upper solution to the equation (4.3.2), which in turn, with the exact formula of $V(z)$, is an upper solution to the two-equation system (4.2.1). Define a continuous monotone function $\overline{U}(z)$ as

$$\overline{U} = \frac{\bar{k}}{1 + Ae^{\mu_1 z}} \quad \text{and let} \quad \overline{V} = H(\overline{U}), \tag{4.3.3}$$

for some constants $\bar{k} \geq 1$ and $A > 0$, where μ_1 is defined in (4.2.2). Finding the derivatives $\overline{U}'(z)$ and $\overline{U}''(z)$, and substituting them into (4.3.2) yield

$$L_1(\overline{U},\overline{V}) = \overline{U}\left(1 - \frac{\overline{U}}{\bar{k}}\right)\left\{\left(\mu_1^2 - c\mu_1 + 1 - a_1\right) + \frac{\overline{U}}{\bar{k}}\left(-2\mu_1^2 + a_1 \frac{\overline{V} - \overline{U}\left(\frac{a_1 - 1 + \bar{k}}{a_1 \bar{k}}\right)}{\left(1 - \frac{\overline{U}}{\bar{k}}\right)\frac{\overline{U}}{\bar{k}}}\right)\right\}. \tag{4.3.4}$$

Let $c = c_0 + \epsilon_1$, where ϵ_1 is a sufficiently small positive number. Take also $\bar{k} = 1 + \epsilon_1$. The formula of $\mu_1(c)$ gives $\mu_1 = \sqrt{1 - a_1} + \delta_1(\epsilon_1)$, with $\delta_1(\epsilon_1) \to 0$ as $\epsilon_1 \to 0$. By using Lemma 4.6.1 in the Appendix, it is easy to see that, for $\epsilon_1 \ll 1$, the pair of functions $(\overline{U}(z), \overline{V}(z))$ is an upper solution to (4.2.1) when

$$-2(1 - a_1) + a_1 Y_1(z) < 0, \quad \text{where} \quad Y_1(z) = \frac{\overline{V} - \overline{U}}{(1 - \overline{U})\overline{U}}. \tag{4.3.5}$$

In the following lemmas we want to prove the boundedness of $Y_1(z)$ and its monotonicity with respect to the parameter r.

Lemma 4.3.1. *The function $Y_1(z)$ is bounded above for all $z \in \mathbb{R}$.*

Proof. Since $Y_1(z)$ is continuous in $\mathbb{R}$, it is enough to show that $\lim\limits_{z \to \pm\infty} Y_1(z) < \infty$. Note that, as $z \to -\infty$, we have

$$\mu(z) \sim D_1 \exp\left(\frac{r}{c}(a_2 - 1)z\right), \qquad D_1 = \exp\left(\int_0^{-\infty} \frac{ra_2}{c}(\overline{U}(s) - 1)ds\right),$$
$$y(z) \sim D_1^{-1} D_2 \exp\left(-\frac{r}{c}(a_2 - 1)z\right), \qquad D_2 = \exp\left(\int_{-\infty}^{\infty} \frac{ra_2}{c}\mu(s)\overline{U}(s)ds\right),$$

$$\overline{V}(z) \sim 1 - D_1 D_2^{-1} \exp\left(\frac{r}{c}(a_2 - 1)z\right),$$

$$\overline{U}(z) \sim 1 - A\exp(\mu_1 z).$$

This gives

$$\lim_{z\to-\infty} Y_1(z) = \lim_{z\to-\infty} \frac{Ae^{\mu_1 z} - D_1 D_2^{-1} e^{\frac{r}{c}(a_2-1)z}}{Ae^{\mu_1 z}}$$
$$= \begin{cases} 1 & \text{, when } r(a_2-1) > c\mu_1 \\ D_3 & \text{, when } r(a_2-1) = c\mu_1 \\ -\infty & \text{, when } r(a_2-1) < c\mu_1 \end{cases}$$

where $D_3 = 1 - D_1 D_2^{-1} A^{-1} < 1$.

For the limit when $z \to \infty$, we also have

$$\lim_{z\to\infty} \frac{\overline{V} - \overline{U}}{(1-\overline{U})\overline{U}} = \lim_{z\to\infty}\left(\frac{y(z)}{\overline{U}(z)} - 1\right) = \lim_{z\to\infty} \frac{ra_2 \int_z^\infty \mu(s)\overline{U}(s)ds}{c\mu(z)\overline{U}(z)} - 1.$$

By making use of L'Hopital's rule, it follows that

$$\lim_{z\to\infty} \frac{\overline{V} - \overline{U}}{(1-\overline{U})\overline{U}} = \frac{r(a_2-1) - c\mu_1}{r + c\mu_1}.$$

This implies that $Y_1(z)$ is bounded above. ∎

Lemma 4.3.2. *The function $Y_1(z)$ is non-decreasing with respect to r.*

Proof. Since $\overline{U}(z)$ is independent of r, it is enough to show that $\overline{V}(z)$ is non-decreasing with respect to r. We prove this in the following two steps:

Step 1. We prove here $a_2\overline{U}(z) \geq \overline{V}(z)$, $\forall z \in \mathbb{R}$. Note that $0 \leq \overline{V}(z) \leq 1$ with $\overline{V}(-\infty) = 1$ and $\overline{V}(\infty) = 0$. On the other hand, we have $a_2\overline{U}(-\infty) = a_2\bar{k} > 1$ and $\overline{U}'(z) < 0$, $\forall z \in \mathbb{R}$. From these facts, there exists a z^* so that $a_2\overline{U}(z^*) = 1$

and $a_2\overline{U}(z) > \overline{V}(z)$, $\forall z < z^*$. Assume by contradiction there exists a first point $\bar{z}$, $z^* < \bar{z} < \infty$, so that $a_2\overline{U}(\bar{z}) < \overline{V}(\bar{z})$. From the formula of $\overline{V}'(z)$,

$$\overline{V}'(z) = -\frac{r}{c}(1 - \overline{V}(z))(a_2\overline{U}(z) - \overline{V}(z)),$$

$\overline{V}(z)$ is increasing in the right neighborhood of $\bar{z}$, that is, for small $\delta > 0$, $\overline{V}(\bar{z} + \delta) > \overline{V}(\bar{z})$. But since $\overline{U}(z)$ is a decreasing function, $\overline{V}(\bar{z}) > a_2\overline{U}(\bar{z}) \geq a_2\overline{U}(\bar{z} + \delta)$ and $\overline{V}(\bar{z} + \delta) > a_2\overline{U}(\bar{z} + \delta)$. This implies that $\overline{V}(z)$ is greater than $a_2\overline{U}(z)$ and, hence by the differential equation, increasing for all $z > \bar{z}$, which contradicts the fact that $\overline{V}(\infty) = 0$.

Step 2. Let $\tau = z/r$ and $(\overline{U},\overline{V})(z) = (\tilde{U}, \tilde{V})(\tau)$. Substituting into the $\overline{V}'(z)$ formula gives

$$\tilde{V}_\tau = -\frac{1}{c}(1 - \tilde{V})(a_2\tilde{U} - \tilde{V}).$$

From step 1, $\tilde{V}(\tau)$ is a non-increasing function in τ. Since τ is decreasing in r, then $\tilde{V}(\tau)$ (hence $\overline{V}(z)$) is a non-decreasing function in r. The lemma is proved. ∎

By the above lemmas, we can define

$$r_- = \sup\{\ r \geq 0 \mid \text{the inequality (4.3.5) holds for } c = c_0 \text{ and all } z \in \mathbb{R}\}. \qquad (4.3.6)$$

Hence, the following lemma is true.

Lemma 4.3.3. *For $c = c_0 + \epsilon_1$ and $r \leq r_-$, where ϵ_1 is a sufficiently small positive number and r_- is defined in (4.3.6), the pair of functions $(\overline{U}(z),\overline{V}(z))$, defined in (4.3.3), is an upper solution to the system (4.2.1) with $(\overline{U},\overline{V})(-\infty) = (\bar{k}, 1)$ and $(\overline{U},\overline{V})(\infty) = (0, 0)$.*

To show the existence of traveling waves $(U, V)(z)$, we want to use Theorem 4.6.1 in the Appendix. To this end, we need to construct a lower solution to the system

(4.2.1) when c is near c_0. Define a continuous function $\underline{U}(z)$ as

$$\underline{U}(z) = \begin{cases} \zeta_1 e^{-\mu_1 z}(1 - Me^{-\epsilon_2 z}) & , z \geq z_1, \\ 0 & , z < z_1, \end{cases}$$

where $0 < \epsilon_2 \ll 1$, M is a positive constant to be determined, $z_1 = \frac{1}{\epsilon_2} \log M$, and ζ_1 is defined in (4.2.3). Let $\underline{V}(z) = H(\underline{U})(z)$. We can obtain the following lemma.

Lemma 4.3.4. *When $c = c_0 + \epsilon_1$, the pair of functions $(\underline{U}(z), \underline{V}(z))$ is a lower solution to the system (4.2.1).*

Proof. Since $\underline{V}(z)$ is the exact solution to the V-equation when $U(z) = \underline{U}(z)$. This automatically gives

$$c\underline{V}' + r(1 - \underline{V})(a_2 \underline{U} - \underline{V}) = 0, \quad \forall z \in \mathbb{R}.$$

For the U-equation, when $z \leq z_1$, we have

$$\underline{U}'' + c\underline{U}' + \underline{U}(1 - a_1 - \underline{U} + a_1 \underline{V}) = 0.$$

When $z > z_1$, it follows that

$$\begin{aligned} L_1(\underline{U}, \underline{V}) =& \underline{U}'' + c\underline{U}' + \underline{U}(1 - a_1 - \underline{U} + a_1 \underline{V}) \\ =& \zeta_1 e^{-\mu z} \left\{ \mu_1^2 - c\mu_1 + 1 - a_1 \right\} - M\zeta_1 e^{-(\mu + \epsilon_2) z} \left\{ (\mu_1 + \epsilon_2)^2 - c(\mu_1 + \epsilon_2) + 1 - a_1 \right\} \\ & - \zeta_1^2 e^{-2\mu_1 z} \left(1 - Me^{-\epsilon_2 z}\right)^2 + a_1 \zeta_1 \underline{V} e^{-\mu_1 z} \left(1 - Me^{-\epsilon_2 z}\right). \end{aligned}$$

In view of definition of μ_1, the first term vanishes and, for sufficiently small ϵ_2, the second term is positive. We choose M sufficiently large so that $z_1 > 0$ and the second term dominates the third one. The last term is positive. Hence, $L_1(\underline{U}, \underline{V}) \geq 0$. ∎

Now, we are ready to state our result for the linear speed selection.

Theorem 4.3.1. *The linear speed selection of the system (4.2.1) is realized when $r \leq r_-$.*

Proof. When $r < r_-$, by using $(\overline{U},\overline{V})(z)$ and $(\underline{U},\underline{V})(z)$ in Theorem 4.6.1, we conclude that the system (4.2.1) has a traveling wave solution $(U,V)(x-ct)$ with $(U,V)(-\infty) = (1,1)$ and $(U,V)(\infty) = (0,0)$ for $c = c_0+\epsilon_1$, $0 < \epsilon_1 \ll 1$. When r_- is finite and $r = r_-$, a limiting argument can show the linear selection of the wave speed. This completes the proof. ∎

Remark 4.3.2. *We can use the exponential function $(\zeta_1,\zeta_2)e^{-\mu_1 z}$ as an upper solution to the system (4.2.1). This gives that the linear selection is realized when*

$$r \leq r_0 := \begin{cases} \infty, & a_1a_2 \leq 1, \\ \dfrac{2(1-a_1)}{a_1a_2-1}, & a_1a_2 > 1, \end{cases}$$

which agrees with the condition (4.1.4). This is also found in [62]. We will see that our choice of upper solution (4.3.3) gives some better and new results.

To see the novel contribution of our upper solution to the linear selection, we will show that the condition (4.1.5) is not necessary for the linear speed selection when $d = 0$. Indeed, the following remark shows that $r_- > r_0$ when $a_1a_2 > 1$.

Remark 4.3.3. *We give a counterexample with $r_- > r_0$ to show the non-necessity of the condition (4.1.5). Let $d = 0, a_1 = 0.5, a_2 = 3, r = 4, \bar{k} = 1.001, c = c_0 + 0.001,$ and $A = 1$. Then $r_0 = 2$,*

$$\overline{U}(z) = \frac{1.001}{1+e^{0.6310z}}, \qquad \mu(z) = \exp\left(2.8264\int_0^z (3\,\overline{U}(z)-1)dt\right),$$
$$y(z) = \frac{8.4793}{\mu(z)}\int_z^\infty \mu(s)\overline{U}(s)ds, \qquad \overline{V}(z) = \frac{y(z)}{1+y(z)},$$

and

$$-2(1-a_1)+a_1\frac{\overline{V}-\overline{U}}{(1-\overline{U})\overline{U}}=-1+0.5Y_1(z):=Y_0(z).$$

Using MATLAB, we plot the graph of $Y_0(z)$. Figure 4.1 shows that $Y_0(z)<0$ for all $z\in\mathbb{R}$. This implies that the wave speed is linearly selected for $r<4$. The result is better than previous one that only gives the linear selection for $r\in(-\infty,2]$. In other words, we have

$$r_0=2<r_-.$$

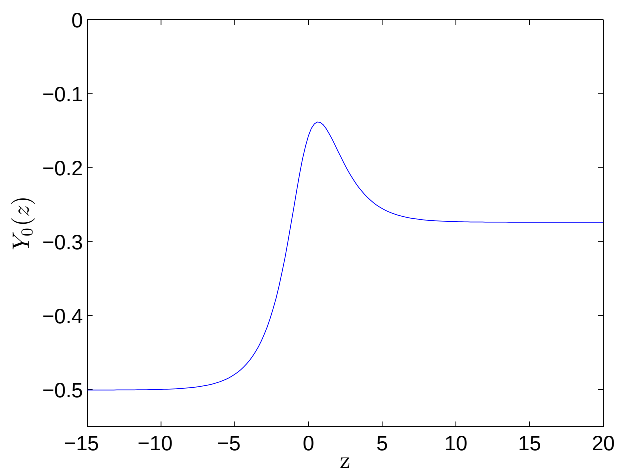


Figure 4.1: Graph of $Y_0(z)$ defined in Remark 4.3.3.

A natural question to ask is whether the speed selection mechanism changes to nonlinear selection at some value of $r\geq r_-$. To answer this question, we will prove the existence of a threshold value of r, in the sense that when r increases, the speed selection changes from linear to nonlinear when r crosses this threshold value. For this purpose, we first prove the following comparison lemma.

Lemma 4.3.5. *For the system (4.2.1), if the wave speed is linearly selected when $r=r_\beta$, for some $r_\beta>0$, then it is linearly selected for all $r<r_\beta$.*

Proof. Let $(U_\beta, V_\beta)(z)$ be the solution to the system (4.2.1) when $r = r_\beta$, that is,

$$\begin{cases} U_\beta'' + cU_\beta' + U_\beta(1 - a_1 - U_\beta + a_1 V_\beta) = 0, \\ cV_\beta' + r_\beta(1 - V_\beta)(a_2 U_\beta - V_\beta) = 0, \\ (U_\beta, V_\beta)(-\infty) = e_1, \quad (U_\beta, V_\beta)(\infty) = e_0. \end{cases} \tag{4.3.7}$$

We want to show that $(U_\beta, V_\beta)(z)$ is an upper solution to the system with $r < r_\beta$, i.e.,

$$\begin{cases} U_\beta'' + cU_\beta' + U_\beta(1 - a_1 - U_\beta + a_1 V_\beta) \le 0, \\ cV_\beta' + r(1 - V_\beta)(a_2 U_\beta - V_\beta) \le 0. \end{cases}$$

The first inequality is naturally satisfied from (4.3.7). For the second inequality, add and subtract $r_\beta(1 - V_\beta)(a_2 U_\beta - V_\beta)$ to the left-hand side to get

$$\begin{aligned} & cV_\beta' + r(1 - V_\beta)(a_2 U_\beta - V_\beta) \\ & \qquad = cV_\beta' + r_\beta(1 - V_\beta)(a_2 U_\beta - V_\beta) + (r - r_\beta)(1 - V_\beta)(a_2 U_\beta - V_\beta) \\ & \qquad = (r - r_\beta)(1 - V_\beta)(a_2 U_\beta - V_\beta) \\ & \qquad \le 0. \end{aligned}$$

Here, we have used the fact that $a_2 U_\beta(z) \ge V_\beta(z)$, $\forall z \in \mathbb{R}$, which can be proved similarly to the proof of Lemma 4.3.2. Using the upper solution $(U_\beta, V_\beta)(z)$ and the lower solution defined in Lemma 4.3.4, we conclude that the wave speed is linearly selected for $r < r_\beta$. ∎

Form this lemma, we define a critical value of r as

$$r_c = \sup\{\, r \mid \text{The linear speed selection of the system (4.2.1) is realized}\}. \tag{4.3.8}$$

Clearly $0 \le r_c \le \infty$ and the following result holds true.

Theorem 4.3.2. *The minimal wave speed of the system (4.2.1) is linearly selected for all $r \le r_c$, and nonlinearly selected for $r > r_c$.*

Remark 4.3.4. *This theorem is the main result Theorem 1.1 which we emphasize in the Introduction section of this chapter. In the above theorem, if $r_c = 0$ then the interval $0 < r \le r_c$ is empty. This means that the nonlinear selection is realized for all r. Similarly, by $r_c = \infty$ we mean that the linear selection is realized for all r.*

From the result in Theorem 4.3.1, it is obvious to see that r_- is a lower bound of r_c, that is $r_- \le r_c$. To give an upper bound to the value of r_c, we proceed to find a value of r so that the nonlinear speed selection is realized when r is greater than this value.

Lemma 4.3.6. *For $c_1 > c_0$, assume that there exists a lower monotonic solution $(\underline{U}, \underline{V})$ to (4.2.1), with $(0,0) \le (\underline{U}, \underline{V}) < (1,1)$, satisfying $(\underline{U}, \underline{V})(\xi) \sim (\underline{\zeta}_1, \underline{\zeta}_2)e^{-\mu_2 \xi}$ for some $(\underline{\zeta}_1, \underline{\zeta}_2) > (0,0)$ as $\xi \to \infty$, where μ_2 is defined in (4.2.2) and $\xi = x - c_1 t$, i.e., $(\underline{U}, \underline{V})(\xi)$ have the faster decay rate near infinity. Then no traveling wave solution to (4.2.1) exists with speed $c \in [c_0, c_1)$.*

Proof. By the assumption, it follows that $(\underline{U}, \underline{V})(x - c_1 t)$ is a lower solution to the following partial differential equation

$$\begin{cases} u_t = u_{xx} + u(1 - a_1 - u + a_1 v), \\ v_t = r(1 - v)(a_2 u - v), \end{cases} \tag{4.3.9}$$

with the initial conditions

$$u(x,0) = \underline{U}(x) \quad \text{and} \quad v(x,0) = \underline{V}(x).$$

Assume to the contrary, for some $c \in [c_0, c_1)$, there exists a monotonic traveling wave solution $(U, V)(x - ct)$ to the system (4.3.9), with the initial condition

$$u(x, 0) = U(x) \ \text{ and } \ v(x, 0) = V(x).$$

The asymptotic behavior of this solution near $\pm\infty$ can be easily found, see e.g., Section 4.2. By a simple computation, we can always assume (by shifting if necessay) $(\underline{U}, \underline{V})(x) \leq (U, V)(x)$ for all $x \in (-\infty, \infty)$. Since $(\underline{U}, \underline{V})(x - c_1 t)$ is a lower solution to the system (4.3.9) and by comparison, we have

$$\begin{aligned} \underline{U}(x - c_1 t) &\leq U(x - ct), \\ \underline{V}(x - c_1 t) &\leq V(x - ct), \end{aligned} \tag{4.3.10}$$

for all $(x, t) \in (\mathbb{R}, \mathbb{R}^+)$. On the other hand, fix $\xi = x - c_1 t$. Then $\underline{U}(\xi) > 0$ is fixed, and we have

$$U(x - ct) = U(\xi + (c_1 - c)t) \sim U(\infty) = 0 \text{ as } t \to \infty.$$

By (4.3.10), this implies that $\underline{U}(\xi) \leq 0$, which is a contradiction. The proof is complete. ∎

By this lemma, we will find an upper bound of r_c by a suitable choice of a lower solution. Define

$$\underline{U}_1 = \frac{\underline{k}}{1 + Be^{\mu_2 z}} \ \text{ and } \ \underline{V}_1 = H(\underline{U}_1), \tag{4.3.11}$$

for some constant B and $0 < \underline{k} < 1$. Similar as previous analysis we find

$$L_1(\underline{U}_1, \underline{V}_1) = \underline{U}_1 \left(1 - \frac{\underline{U}_1}{\underline{k}}\right) \left\{ \left(\mu_2^2 - c\mu_2 + 1 - a_1\right) + \frac{\underline{U}_1}{\underline{k}} \left(-2\mu_2^2 + a_1 \frac{\underline{V}_1 - \underline{U}_1 \left(\frac{a_1 - 1 + \underline{k}}{a_1 \underline{k}}\right)}{\left(1 - \frac{\underline{U}_1}{\underline{k}}\right) \frac{\underline{U}_1}{\underline{k}}} \right) \right\}. \tag{4.3.12}$$

The pair of functions $(\underline{U}_1(z), \underline{V}_1(z))$ is a lower solution to (4.2.1) when

$$-2\mu_2^2 + a_1 Y_2(z) > 0, \ \ z \in (-\infty, \infty), \tag{4.3.13}$$

where

$$Y_2(z) = \frac{\underline{V}_1 - \underline{U}_1 \left(\frac{a_1 - 1 + \underline{k}}{a_1 \underline{k}}\right)}{\left(1 - \frac{\underline{U}_1}{\underline{k}}\right) \frac{\underline{U}_1}{\underline{k}}}.$$

It is easy to find $\lim_{z \to -\infty} Y_2(z) = \infty$, for $0 < \underline{k} < 1$. The same argument as that in the proof of Lemma 4.3.1 can yield that $\lim_{z \to \infty} Y_2(z)$ is finite. Hence, the minimum value of $Y_2(z)$ is defined. In view of the monotonicity of $Y_1(z)$ with respect to r in Lemma 4.3.2, the result is true for $Y_2(z)$ as well. Then we can define

$$r_+ = \inf\{\ r \geq 0 \mid \text{The inequality (4.3.13) holds for some } c > c_0\}. \tag{4.3.14}$$

Hence, $(\underline{U}_1, \underline{V}_1)(z)$ is a lower solution to (4.2.1) when $r \geq r_+$. Then the following result is valid.

Theorem 4.3.3. *The nonlinear speed selection of the system (4.2.1) is realized when* $r \geq r_+$.

By the above analysis, we have a general estimation of r_c, defined in (4.3.8), as

$$r_- \leq r_c < r_+.$$

We can use formulas of r_- and r_+ defined in (4.3.6) and (4.3.14), respectively, to estimate the value of r_c. This analysis will lead to new results and cover some previous results. It will be done in the next section.

4.4 Estimation of r_c

The extreme values of $Y_1(z)$ and $Y_2(z)$ cannot be easily found. For this reason, we will estimate the upper and the lower solutions in the V-equation instead of using the exact formula. This will lead to some new and explicit results on the speed selection.

Theorem 4.4.1. *When $a_1a_2 \leq 2(1-a_1)$, the minimal wave speed of the system (4.2.1) is linearly selected for all $r \geq 0$, that is, $r_c = \infty$.*

Proof. In (4.3.5), let

$$\overline{V}(z) = \min\{1, a_2\overline{U}\} = \begin{cases} 1, & z \leq z_2, \\ a_2\overline{U}(z), & z > z_2, \end{cases}$$

where z_2 satisfies $a_2\overline{U}(z_2) = 1$. This function is an upper solution to the V-equation. Indeed, when $z \leq z_2$, $c\overline{V}' + r(1-\overline{V})(a_2\overline{U} - \overline{V}) = 0$, and when $z > z_2$, we have

$$c\overline{V}' + r(1-\overline{V})(a_2\overline{U} - \overline{V}) = -a_2c\mu_1\overline{U}(1-\overline{U}) \leq 0.$$

Same formulas as those in (4.3.4)-(4.3.5) hold true, and an estimate of $Y_1(z)$ is giving by

$$Y_1(z) = \begin{cases} \dfrac{1}{\overline{\overline{U}}} \leq a_2, & \text{when } z \leq z_2, \\ \dfrac{a_2 - 1}{1-\overline{U}} \leq a_2, & \text{when } z > z_2. \end{cases}$$

Then $-2(1-a_1) + a_1Y_1(z) \leq -2(1-a_1) + a_1a_2 \leq 0$. From Theorem 4.3.1, the result is true. ∎

From Remark 4.3.2, $a_1a_2 \leq 1$ implies that $r_c = \infty$. We combine this and the above theorem to have the following corollary.

Corollary 4.4.1. *The condition* $a_1a_2 \leq \max\{1, 2(1-a_1)\}$ *implies the linear speed selection for (4.2.1).*

By an another choice of the upper solution, we have the following theorem.

Theorem 4.4.2. *When* $a_1 \leq 2/3$ *and* $a_1a_2 > 2(1-a_1)$*, the minimal wave speed of the system (4.2.1) is linearly selected for all*

$$r \leq \frac{4(1-a_1)^2}{a_1a_2 - 2(1-a_1)}, \quad \textit{that is,} \quad r_c > \frac{4(1-a_1)^2}{a_1a_2 - 2(1-a_1)}.$$

Proof. Here we choose $\overline{V}(z)$ as

$$\overline{V}(z) = \min\left\{1, \frac{2(1-a_1)}{a_1}\overline{U}\right\} = \begin{cases} 1, & z \leq z_3, \\ \dfrac{2(1-a_1)}{a_1}\overline{U}(z), & z > z_3, \end{cases}$$

so that z_3 satisfies $2(1-a_1)\overline{U}(z_3) = a_1$. When $z \leq z_3$, we have $c\overline{V}' + r(1-\overline{V})(a_2\overline{U} - \overline{V}) = 0$, and when $z > z_3$, we have

$$\begin{aligned} c\overline{V}' &+ r(1-\overline{V})(a_2\overline{U} - \overline{V}) \\ &= -\frac{2c(1-a_1)}{a_1}\left\{-\mu_1\overline{U}(1-\overline{U})\right\} + r\left(1 - \frac{2(1-a_1)}{a_1}\overline{U}\right)\left(a_2\overline{U} - \frac{2(1-a_1)}{a_1}\overline{U}\right). \end{aligned}$$

Since $a_1 \leq 2/3$, the inequality $1 - \frac{2(1-a_1)}{a_1}\overline{U} \leq 1 - \overline{U}$ is true. Hence,

$$\begin{aligned} c\overline{V}' &+ r(1-\overline{V})(a_2\overline{U} - \overline{V}) \\ &\leq \frac{2(1-a_1)}{a_1}\overline{U}(1-\overline{U})\left\{-c\mu_1 + r\left(\frac{a_1a_2}{2(1-a_1)} - 1\right)\right\} \\ &\leq 0, \end{aligned}$$

when

$$r < \frac{4(1-a_1)^2}{a_1a_2 - 2(1-a_1)} \quad \text{and} \quad c = c_0 + \epsilon_1,$$

for small ϵ_1. Also, we have $Y_1(z) \leq \dfrac{2(1-a_1)}{a_1}$. Then $-2(1-a_1) + a_1Y_1(z) \leq 0$. By Theorem 4.3.1, the proof is complete. ∎

Again, from Remark 4.3.2, we have seen that, when $a_1a_2 > 1$,

$$r_c \geq \frac{2(1-a_1)}{a_1a_2 - 1}.$$

Define $M =: \max\{1, 2(1-a_1)\}$. If $a_1 \leq 1/2 < 2/3$, then $M = 2(1-a_1)$. In this case, we have showed that, for $a_1a_2 > M$,

$$r_c \geq \frac{4(1-a_1)^2}{a_1a_2 - 2(1-a_1)} = \frac{2M(1-a_1)}{a_1a_2 - M}.$$

This implies the following extension to the condition (4.1.5) with $d = 0$.

Corollary 4.4.2. *When $a_1a_2 > M$, the minimal wave speed of the system (4.2.1) is linearly selected for all*

$$r \leq \frac{2M(1-a_1)}{a_1a_2 - M}, \quad \textit{that is,} \quad r_c > \frac{2M(1-a_1)}{a_1a_2 - M}.$$

Theorem 4.4.3. *If there exists $\eta < 1$ so that $\eta \geq (2/a_1)\max\{1-a_1, 1/a_2\}$, then the minimal wave speed of the system (4.2.1) is nonlinearly selected for all*

$$r > \frac{2(1-a_1)\eta}{(1-\eta)^2}, \quad \textit{that is,} \quad r_c \leq \frac{2(1-a_1)\eta}{(1-\eta)^2}.$$

Proof. In (4.3.13) we choose $\underline{V}_1(z)$ as

$$\underline{V}_1(z) = \min\{\eta, \eta a_2 \underline{U}_1\} = \begin{cases} \eta, & z \le z_4, \\ \eta a_2 \underline{U}_1, & z > z_4, \end{cases}$$

where z_4 satisfies $a_2 \underline{U}_1(z_4) = 1$. When $z \le z_4$, since $a_2 \underline{U}(z) \ge 1$, we have

$$c\underline{V}_1' + r(1 - \underline{V}_1)(a_2 \underline{U}_1 - \underline{V}_1) = r(1-\eta)(a_2 \underline{U}_1 - \eta) \ge 0.$$

For the region $z > z_4$, we obtain

$$\begin{aligned} & c\underline{V}_1' + r(1 - \underline{V}_1)(a_2 \underline{U}_1 - \underline{V}_1) \\ & \quad = -\eta a_2 c \mu_2 \underline{U}_1 (1 - \underline{U}_1) + r(1 - \eta a_2 \underline{U}_1)(a_2 \underline{U}_1 - \eta a_2 \underline{U}_1) \\ & \quad \ge -\eta a_2 c \mu_2 \underline{U}_1 (1 - \underline{U}_1) + r(1-\eta)(1-\eta) a_2 \underline{U}_1 \\ & \quad \ge \eta a_2 \underline{U}_1 \left\{ -c\mu_2 + \frac{r}{\eta}(1-\eta)^2 \right\} \\ & \quad \ge 0, \end{aligned}$$

when $r > \frac{2(1-a_1)\eta}{(1-\eta)^2}$ and $c = c_0 + \epsilon_1$, for some small ϵ_1. On the other hand, since $\eta a_1 a_2 \ge 2$, we can fix the value of $\underline{k}$ in the formula of $\underline{U}(z)$ defined in (4.3.11) so that the following holds true

$$\frac{1 - a_1}{\eta a_1 a_2 - 1} \le \underline{k} \le 1 - a_1.$$

Obviously, the same formula as that is (4.3.12) is still true with the neww choice of $\underline{V}_1(z)$. By this choice and since $0 \le \underline{U} \le \underline{k} < 1$, we have

$$
\begin{aligned}
-2(1-a_1)+a_1Y_2(z) &= -2(1-a_1)+a_1\frac{\underline{V}_1-\underline{U}_1\left(\frac{a_1-1+\underline{k}}{a_1\underline{k}}\right)}{\left(1-\frac{\underline{U}_1}{\underline{k}}\right)\frac{\underline{U}_1}{\underline{k}}} \\
&\geq \begin{cases} -2(1-a_1)+a_1\eta-(a_1-1+\underline{k}), & z\leq z_4 \\ -2(1-a_1)+a_1a_2\eta\underline{k}-(a_1-1+\underline{k}), & z>z_4 \end{cases} \\
&\geq 0.
\end{aligned}
$$

Hence, by Lemma 4.3.6, we conclude that the minimal wave speed is nonlinearly selected. ∎

Remark 4.4.1. *We can include the case $a_1 = 0$ in the condition (4.1.2), where the speed selection can be studied directly. In this case, $c_0 = 2\sqrt{1-a_1} = 2$, and the system (4.2.1) reads*

$$
\begin{cases} U''+cU'+U(1-U)=0, \\ cV'+r(1-V)(a_2U-V)=0, \\ (U,V)(-\infty)=e_1, (U,V)(\infty)=e_0. \end{cases}
$$

The first equation is the well-known Fisher equation. It has a monotone solution for all $c \geq 2$. Using its solution in the formula $V = H(U)$ shows that the system has a solution for any $c \geq 2$. Hence, the minimal wave speed is linearly selected.

4.5 Conclusions and Summary

The speed selection mechanisms (linear and nonlinear) for traveling waves to a two-species Lotka-Volterra competition model (4.1.1) are investigated when $d = 0$ and $0 \leq a_1 < 1 < a_2$. New types of the upper-lower solutions are constructed. We prove a modified version of Hosono's conjecture, and provided some estimates of the critical value r_c.

The linear determinacy in the condition (4.1.5) with $d = 0$, has been extended to the condition

$$d = 0 \quad \text{and} \quad (a_1 a_2 - M)r \leq 2M(1 - a_1),$$

where $M = \max\{1, 2(1 - a_1)\}$. It extends the results in [29,32], when $d = 0$, as well. This together with a counterexample show that they are sufficient and not necessary for the linear speed selection. Our result also indicates that the wave speed is linearly selected when $a_1 a_2 > 1$ for all values of r provided an extra condition on a_1 and a_2 is satisfied. This shows the failure of Hosono's conjecture for the existence of finite r_c when $a_1 a_2 > 1$.

By our analysis, some new results on nonlinear speed selection are also established, see e.g. Theorem 4.4.3.

The speed selection mechanism when $d > 0$ is challenging and will be considered in our future work.

4.6 Appendix: Upper-lower Solution Method

A useful method to prove the existence of monotone traveling wave solution is the upper-lower solution technique originated in [13,90]. Here we illustrate the main idea. By transforming the system (4.2.1) to a system of integral equations, we can define a monotone iteration scheme in terms of the integral system. By construction an upper and a lower solutions to the system and using the iteration scheme, we can give the existence of traveling wave solutions.

Let α be sufficiently large positive number so that

$$\alpha U + U(1 - a_1 - U + V) := F(U, V)$$

and

$$\alpha V + r(1-V)(a_2U - V) := G(U,V)$$

are monotone in U and V, respectively. Equations in (4.2.1) are equivalent to

$$\begin{cases} U'' + cU' - \alpha U = -F(U,V), \\ cV' - \alpha V = -G(U,V). \end{cases} \tag{4.6.1}$$

Define constants $\lambda_1^{\pm}$ as

$$\lambda_1^- = \frac{-c-\sqrt{c^2+4\alpha}}{2} < 0 \quad \text{and} \quad \lambda_1^+ = \frac{-c+\sqrt{c^2+4\alpha}}{2} > 0.$$

By applying the variation-of-parameter method to the first equation in the system (4.6.1), and the first order theory of differential equations to the second equation, the system can be written in the form

$$\begin{cases} U(z) = T_1(U,V)(z), \\ V(z) = T_2(U,V)(z), \end{cases} \tag{4.6.2}$$

where

$$T_1(U,V)(z) = \frac{1}{\lambda_1^+ - \lambda_1^-}\left\{\int_{-\infty}^{z} e^{\lambda_1^-(z-s)}F(U,V)(s)ds + \int_z^{\infty} e^{\lambda_1^+(z-s)}F(U,V)(s)ds\right\},$$
$$T_2(U,V)(z) = \frac{1}{c}\int_z^{\infty} e^{\frac{\alpha}{c}(z-s)}G(U,V)(s)ds.$$

Definition 2. A pair of continuous functions $(U(z),V(z))$ is an upper (a lower) solution to the integral equations system (4.6.2) if

$$\begin{cases} U(z) \geq (\leq)\ T_1(U,V)(z), \\ V(z) \geq (\leq)\ T_2(U,V)(z). \end{cases}$$

Definition 3. A pair of continuous functions $(U(z), V(z))$ that are differentiable on $\mathbb{R}$ except at finite number of points $\{z_i, i = 1, \ldots, n\}$ is an upper (a lower) solution to the ordinary differential equations system (4.2.1) if

$$\begin{cases} U'' + cU' + U(1 - a_1 - U + a_1 V) \leq (\geq)\ 0, \\ cV' + r(1 - V)(a_2 U - V) \leq (\geq)\ 0, \end{cases}$$

for all $z \neq z_i, i = 1, \ldots, n$.

The relation between these two definitions is presented in the following lemma.

Lemma 4.6.1. *A continuous upper solution $(U, V)(z)$ to the system (4.2.1) which is differentiable on $\mathbb{R}$ except at finite number of points $\{z_i, i = 1, \ldots, n\}$ and satisfies $(U', V')(z_i^-) \geq (U', V')(z_i^+)$, for all $z = z_i, i = 1, \ldots, n$, is an upper solution to the integral equations system (4.6.2). A same result is true for the lower solution by reversing the inequality.*

Proof. We give the proof for the upper solution where the same argument can be applied for the lower solution. When the inequalities in Definition 3 are true, it is easy to verify that

$$U'' + cU' - \alpha U + F(U, V) \leq 0$$

$$V'' + cV' - \alpha V + G(U, V) \leq 0.$$

From the first inequality, we have

$$T_1(U,V)(z) = \frac{1}{\lambda_1^+ - \lambda_1^-}\left\{\int_{-\infty}^{z} e^{\lambda_1^-(z-s)}F(U,V)(s)ds + \int_{z}^{\infty} e^{\lambda_1^+(z-s)}F(U,V)(s)ds\right\}$$
$$\leq \frac{-1}{\lambda_1^+ - \lambda_1^-}\left\{\int_{-\infty}^{z} e^{\lambda_1^-(z-s)}(U'' + cU' - \alpha U)(s)ds + \int_{z}^{\infty} e^{\lambda_1^+(z-s)}(U'' + cU' - \alpha U)(s)ds\right\}.$$

Simple computations as that in [51, proof of Lemma 2.5] yield

$$T_1(U,V)(z) \leq U(z).$$

Similarly $T_2(U,V) \leq V(z)$. This implies that $(U,V)(z)$ is an upper solution to the system (4.6.2). ∎

The existence of an upper and a lower solution to the system (4.6.2) will give the existence of the actual traveling wave solution. Indeed, for our problem, we assume that the following hypothesis is true.

Hypothesis 1. *There exists a monotone non-increasing upper solution* $(\overline{U},\overline{V})(z)$ *and a non-zero lower solution* $(\underline{U},\underline{V})(z)$ *to the system (4.6.2) with the properties*

(1) $(\underline{U},\underline{V})(z) \leq (\overline{U},\overline{V})(z)$, *for all* $z \in \mathbb{R}$,

(2) $(\overline{U},\overline{V})(+\infty) = e_0, \quad (\overline{U},\overline{V})(-\infty) = (\bar{k}_1,\bar{k}_2)$,

(3) $(\underline{U},\underline{V})(+\infty) = e_0, \quad (\underline{U},\underline{V})(-\infty) = (\underline{k}_1,\underline{k}_2)$,

for $e_0 \leq (\underline{k}_1,\underline{k}_2) \leq e_1$ *and* $(\bar{k}_1,\bar{k}_2) \geq e_1 = (1,1)$ *so that no equilibrium solution to (4.2.1) exists in the set* $\{(U,V)|e_1 < (U,V) \leq (\bar{k}_1,\bar{k}_2)\}$.

From the integral system, we define an iteration scheme as

$$\begin{cases} (U_0, V_0) = (\overline{U}, \overline{V}), \\ U_{n+1} = T_1(U_n, V_n), \quad n = 0, 1, 2, \ldots, \\ V_{n+1} = T_2(U_n, V_n), \quad n = 0, 1, 2, \ldots, \end{cases} \tag{4.6.3}$$

and arrive to the following result by the upper-lower solution method, see e.g. [13].

Theorem 4.6.1. *If Hypothesis 1 holds, then the iteration (4.6.3) converges to a non-increasing function* $(U, V)(z)$*, which is a solution to the system (4.2.1) with* $(U, V)(-\infty) = e_1$ *and* $(U, V)(\infty) = e_0$*. Moreover,* $(\underline{U}, \underline{V})(z) \leq (U, V)(z) \leq (\overline{U}, \overline{V})(z)$ *for all* $z \in \mathbb{R}$*.*

Chapter 5

Stability of Traveling Waves to the Competition Model

5.1 Introduction

In this chapter, we are concerned with the stability of the traveling wave solution to the diffusive Lotka-Volterra competition model. Re-consider the non-dimensional cooperative system (4.1.1),

$$\begin{cases} u_t = u_{xx} + u(1 - a_1 - u + a_1 v), \\ v_t = dv_{xx} + r(1 - v)(a_2 u - v), \end{cases} \tag{5.1.1}$$

with

$$u(x, 0) = u_0(x) \geq 0, \qquad v(x, 0) = v_0(x) \geq 0, \quad \forall x \in \mathbb{R}.$$

We also assume the condition (4.1.2),

$$0 < a_1 < 1 < a_2, \tag{5.1.2}$$

and consider the monostable traveling wave solution, connecting $(1,1)$ to $(0,0)$, in the form

$$(u,v)(x,t) = (\overline{U},\overline{V})(z),$$

where $z = x - ct$ and $c \geq 0$. The wavefront $(\overline{U},\overline{V})(z)$ satisfies

$$\begin{cases} 0 = \overline{U}_{zz} + c\overline{U}_z + \overline{U}(1 - a_1 - \overline{U} + a_1\overline{V}), \\ 0 = d\overline{V}_{zz} + c\overline{V}_z + r(1 - \overline{V})(a_2\overline{U} - \overline{V}), \end{cases} \tag{5.1.3}$$

subject to

$$(\overline{U},\overline{V})(-\infty) = (1,1), \quad (\overline{U},\overline{V})(\infty) = (0,0).$$

We know that $(\overline{U},\overline{V})(x-ct)$ is a special pattern that only satisfies the equations in (5.1.1). For the stability of this pattern, we want to know how the solution of (5.1.1) tends to $(\overline{U},\overline{V})(x-ct)$ for given initial data $u_0(x)$ and $v_0(x)$. To this end, we use the (z,t)-coordinates and

$$(u,v)(x,t) = (U,V)(z,t),$$

to transform (5.1.1) into the partial differential model

$$\begin{cases} U_t = U_{zz} + cU_z + U(1 - a_1 - U + a_1V), \\ V_t = dV_{zz} + cV_z + r(1 - V)(a_2U - V), \end{cases} \tag{5.1.4}$$

subject to

$$U(z,0) = u_0(z), \qquad V(z,0) = v_0(z), \quad \forall z \in \mathbb{R}.$$

It is easy to see that $(\overline{U},\overline{V})(z)$ is the steady-state to the above new system.

The stability of traveling waves to a scalar partial differential equation has been well-studied, e.g., [21,22,31,39,52,53,67,70,81,97], the monograph [9,85], the survey paper [99]. As far as we know, most of previous works were concerned with a scalar equation, since the extension of this method to a general system is not trivial.

Or goal here is to systematically study the local and the global stability of the steady-state $(\overline{U},\overline{V})(z)$ to the system (5.1.4). Using the method of spectrum analysis in [26], we give the local stability. For the global stability, we construct an upper and a lower solution to the system (5.1.4), and prove their convergence to the traveling wave $(\overline{U},\overline{V})(z)$. In view of comparison together with the squeezing technique, we arrive at new results on the global stability of the traveling waves.

The rest of the chapter is organized as follows. The asymptotic behavior of the traveling waves are found in Section 5.2. In Section 5.3, we study the local stability of the steady-state by applying the standard linearization. The resulted spectrum problem is studied by the method in [26]. A suitable weighted functional space is chosen to proceed the analysis. In Section 5.4, beside the weighted functional space, the upper-lower solution method together with the squeezing technique are applied to derive the global stability results. Conclusions and summary are presented in Section 5.5.

5.2 The Asymptotic Behavior of the Steady-state

In this section, we will derive the exponential asymptotic behavior of the steady-state $(\overline{U},\overline{V})(z)$ of the model (5.1.3) as $z \to \infty$. Assume

$$(\overline{U},\overline{V})(z) \sim (\zeta_1 e^{-\mu z}, \zeta_2 e^{-\mu z}) \quad \text{as} \quad z \to \infty,$$

for positive constants ζ_1, ζ_2, and μ. By substituting this into (5.1.3) and linearizing the equations we have

$$A(\mu)\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}, \tag{5.2.1}$$

where $A(\mu)$ is given by

$$A(\mu)=\begin{pmatrix}\mu^2-c\mu+1-a_1 & 0\\ ra_2 & d\mu^2-c\mu-r\end{pmatrix}. \tag{5.2.2}$$

The system of algebraic equations (5.2.1) has a non-trivial solution if and only if $\det(A)=0$. This implies $\mu=\mu_{1,2,3}>0$, where

$$\mu_1(c)=\frac{c-\sqrt{c^2-4(1-a_1)}}{2},\quad \mu_2(c)=\frac{c+\sqrt{c^2-4(1-a_1)}}{2}, \tag{5.2.3}$$

and

$$\mu=\mu_3(c)=\frac{c+\sqrt{c^2+4dr}}{2d}. \tag{5.2.4}$$

For $c>c_0$, obviously $\mu_1<\mu_2$. When $0\le d\le 1$, we have also $\mu_2<\mu_3$ for all $c>c_0$, i.e., $e^{-\mu_1 z}$ dominates both of $e^{-\mu_2 z}$ and $e^{-\mu_3 z}$. In this case, the eigenvector of $A(\mu)$ corresponding to μ_i, for $i=1,2$, is the strongly positive vector $(\zeta_1(\mu_i)\ \ \zeta_2(\mu_i))^T$, where

$$\zeta_1(\mu_i)=-(d\mu_i^2-c\mu_i-r)\ \text{ and }\ \zeta_2(\mu_i)=ra_2. \tag{5.2.5}$$

It follows that

$$\begin{pmatrix}\overline{U}(z)\\ \overline{V}(z)\end{pmatrix}=C_1\begin{pmatrix}\zeta_1(\mu_1)\\ \zeta_2(\mu_1)\end{pmatrix}e^{-\mu_1 z}+C_2\begin{pmatrix}\zeta_1(\mu_2)\\ \zeta_2(\mu_2)\end{pmatrix}e^{-\mu_2 z},\quad \text{as } z\to\infty, \tag{5.2.6}$$

for $C_1 > 0$ or $C_1 = 0, C_2 > 0$. For the case when

$$1 < d < 2 + \frac{r}{1-a_1},$$

the same behavior in (5.2.6) is still true if $c_{\min} \le c \le \hat{c}$, where

$$\hat{c} = \sqrt{\frac{r+1-a_1}{d-1}} + (1-a_1)\sqrt{\frac{d-1}{r+1-a_1}}.$$

If $c > \hat{c}$, then $\mu_1 < \mu_3 < \mu_2$ and we have, as $z \to \infty$,

$$\begin{pmatrix} \overline{U}(z) \\ \overline{V}(z) \end{pmatrix} = C_1 \begin{pmatrix} \zeta_1(\mu_1) \\ \zeta_2(\mu_1) \end{pmatrix} e^{-\mu_1 z} + C_2 \begin{pmatrix} -\zeta_1(\mu_2) \\ -\zeta_2(\mu_2) \end{pmatrix} e^{-\mu_2 z} + C_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\mu_2 z}, \tag{5.2.7}$$

for $C_1 > 0$ or $C_1 = 0, C_{2,3} > 0$. Here, $(0 \;\; 1)^T$ is the eigenvector of $A(\mu)$ corresponding to μ_3, and note that $\zeta_1(\mu_2) < 0$ in this case. On the other hand, when

$$d > 2 + \frac{r}{1-a_1},$$

$(\overline{U},\overline{V})(z)$ behaves like (5.2.7) when $c > \hat{c}$. For the case when $c_0 < c < \hat{c}$, we have $\mu_3 < \mu_1 < \mu_2$. Hence, as $z \to \infty$,

$$\begin{pmatrix} \overline{U}(z) \\ \overline{V}(z) \end{pmatrix} = C_1 \begin{pmatrix} -\zeta_1(\mu_1) \\ -\zeta_2(\mu_1) \end{pmatrix} e^{-\mu_1 z} + C_2 \begin{pmatrix} -\zeta_1(\mu_2) \\ -\zeta_2(\mu_2) \end{pmatrix} e^{-\mu_2 z} + C_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\mu_3 z},$$

for $C_{1,3} > 0$, or $C_1 = 0,\; C_{2,3} > 0$.

Finally, we have the asymptotic behavior for the solution $\overline{U}(z)$ when the wave speed is greater than the minimal speed $c_{\min}$.

Theorem 5.2.1. *For $c > c^*$, the wavefront $\overline{U}$ has the following behavior*

$$\overline{U}(z) \sim C_1 e^{-\mu_1 z}, \text{ as } z \to \infty$$

for some $C_1 > 0$.

Proof. Assume that for some $c_1 > c^*$, the wavefront $\overline{U}$ has the following behavior

$$\overline{U}(z) \sim C_2 e^{-\mu_2 z}, \text{as } z \to \infty \tag{5.2.8}$$

for some $C_2 > 0$. Similar to the proof of Lemma 4.3.6, the result can be proved by contradiction. ∎

5.3 The Local Stability

To study the local stability, as usual, we add a small perturbation to the steady-state solution and study the behavior of this perturbation for large time period. If this perturbation decays, then we say that the steady-state is locally stable. For $\delta \ll 1$, and a parameter λ, let

$$U(z,t) = \overline{U}(z) + \delta\phi_1(z)e^{\lambda t},$$

$$V(z,t) = \overline{V}(z) + \delta\phi_2(z)e^{\lambda t}.$$

where ϕ_1 and ϕ_2 are two real functions. Substitute these formulas into (5.1.4) and linearize the system about $(\overline{U},\overline{V})$ to get the following spectrum problem

$$\lambda\Phi = \mathcal{L}\Phi := D\Phi'' + c\Phi' + J(z)\Phi, \tag{5.3.1}$$

where $\Phi = (\phi_1 \;\; \phi_2)^T$, $D = diag(1,d)$, and $J(z)$ is a 2×2 matrix given by

$$J(z) = \begin{pmatrix} 1 - a_1 - 2\overline{U} + a_1\overline{V} & a_1\overline{U} \\ ra_2(1-\overline{V}) & r(-1 - a_2\overline{U} + 2\overline{V}) \end{pmatrix}. \tag{5.3.2}$$

For Φ in a suitable space, we shall find sign of the maximal real part to the spectrum (λ) of the operator $\mathcal{L}$ to determine the local stability of the steady-state solution. To proceed, we introduce a weighted functional space L^p_w,

$$L^p_w = \{f(z) \;: w(z)f(z) \in L^p(\mathbb{R}), p \geq 1\}$$

with the norm

$$\|f(z)\|_{L^p_w} = \left(\int_{-\infty}^{\infty} w(z)|f(z)|^p dz\right)^{\frac{1}{p}},$$

where

$$w(z) = (1/w_1(z), \;\; 1/w_2(z)) \tag{5.3.3}$$

is the weight function with

$$w_1(z) = \begin{cases} e^{-\alpha(z-z_0)} & , z > z_0 \\ 1 & , z \leq z_0 \end{cases}, \quad w_2(z) = \begin{cases} e^{-\beta(z-z_0)} & , z > z_0 \\ 1 & , z \leq z_0 \end{cases}, \tag{5.3.4}$$

for some positive constants α, β and z_0 to be chosen. Here, $L^p(\mathbb{R})$, for $p \geq 1$, is the well-known Lebesgue space of the integrable functions defined on $\mathbb{R}$. Then we consider the operator $\mathcal{L}$ on this new space and find its spectrum. To do this, we write $\Phi(z)$ in the form

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} w_1\psi_1 \\ w_2\psi_2 \end{pmatrix}, \tag{5.3.5}$$

for L^p-functions ψ_1 and ψ_2. Substituting (5.3.5) into (5.3.1) gives a new spectrum problem in the weighted space L^p_w,

$$\lambda\Psi = \mathcal{L}_w\Psi := D\Psi'' + M(z)\Psi' + N(z)\Psi,$$

where $\Psi = (\psi_1 \;\; \psi_2)^T$, $M(z)$ and $N(z)$ are 2×2 matrices defined by

$$M(z) = \begin{pmatrix} c + 2\frac{w_1'}{w_1} & 0 \\ 0 & c + 2d\frac{w_2'}{w_2} \end{pmatrix} \tag{5.3.6}$$

and

$$N(z) = \begin{pmatrix} \frac{w_1''}{w_1} + c\frac{w_1'}{w_1} & 0 \\ 0 & d\frac{w_2''}{w_2} + c\frac{w_2'}{w_2} \end{pmatrix} + Y(z),$$

with the ik-element of the matrix $Y(z)$, y_{ik}, is given in terms of the ik-element of the matrix $J(z)$ as $y_{ik} = \frac{w_k}{w_i} j_{ik}$, that is,

$$N(z) = \begin{pmatrix} \frac{w_1''}{w_1} + c\frac{w_1'}{w_1} + 1 - a_1 - 2\overline{U} + a_1\overline{V} & a_1\overline{U}\frac{w_2}{w_1} \\ ra_2(1-\overline{V})\frac{w_1}{w_2} & d\frac{w_2''}{w_2} + c\frac{w_2'}{w_2} + r(-1 - a_2\overline{U} + 2\overline{V}) \end{pmatrix}. \tag{5.3.7}$$

The details to find the essential spectrum of the operator $\mathcal{L}_w$ can be finalized by using Theorem A.2 in [26] and are given below. After we choose the weight function so that the essential spectrum is on the left-half complex plane, we shall determine the sign of the maximal real part of the point spectrum in the weighted space.

First of all, to apply the method in [26], we need to choose α and β so that the matrix functions $M(z)$ and $N(z)$ are bounded, i.e.,

$$\lim_{z\to\infty} \overline{U}(z)\frac{w_2(z)}{w_1(z)} = A_1 \quad \text{and} \quad \lim_{z\to\infty}(1-\overline{V}(z))\frac{w_1(z)}{w_2(z)} = A_2,$$

for some constants A_1 and A_2. We choose

$$\alpha - \mu_1 < \beta \leq \alpha, \tag{5.3.8}$$

where μ_1 is defined in (5.2.3). This makes $A_1 = 0$ and

$$A_2 = \begin{cases} 0 & \text{when } \beta < \alpha, \\ 1 & \text{when } \beta = \alpha. \end{cases}$$

Now, we define

$$S_{\pm} := \{\lambda \mid \det(-\tau^2 D + i\tau M_{\pm} + N_{\pm} - \lambda I) = 0, -\infty < \tau < \infty\},$$

where $M_{\pm}$ and $N_{\pm}$ are the limits of $M(z)$ and $N(z)$ as $z \to \pm\infty$, respectively. Then the essential spectrum of the operator $\mathcal{L}_w$ is contained in the union of regions inside or on the curves S_+ and S_-, see [26, pp. 140]. By letting $z \to +\infty$, M_+ and N_+ are given as (taking condition (5.3.8) into account)

$$M_+ = \begin{pmatrix} c - 2\alpha & 0 \\ 0 & c - 2d\beta \end{pmatrix} \quad \text{and} \quad N_+ = \begin{pmatrix} \alpha^2 - c\alpha + 1 - a_1 & 0 \\ C & d\beta^2 - c\beta - r \end{pmatrix}.$$

The equation $\det(-\tau^2 D + i\tau M_+ + N_+ - \lambda I) = 0$ has two solutions $\lambda = \lambda_{1,2}$, where

$$\lambda_1 = -\tau^2 + i\tau(c - 2\alpha) + \alpha^2 - c\alpha + 1 - a_1,$$
$$\lambda_2 = -\tau^2 d + i\tau(c - 2d\beta) + d\beta^2 - c\beta - r.$$

This means that S_+ is the union of two parabolas in the complex plane which are symmetric about the real axis, namely

$$S_{+,1} = \{\lambda_1 \mid -\infty < \tau < \infty\} \quad \text{and} \quad S_{+,2} = \{\lambda_2 \mid -\infty < \tau < \infty\}.$$

The most right points of these curves are $\alpha^2 - c\alpha + 1 - a_1$ and $d\beta^2 - c\beta - r$, respectively, which are negative if

$$\alpha \in (\mu_1, \mu_2) \quad \text{and} \quad \beta \in (0, \mu_3), \tag{5.3.9}$$

where μ_1, μ_2, and μ_3 are defined in (5.2.3)-(5.2.4). Hence, when the above condition satisfies, $S_+ = S_{+,1} \cup S_{+,2}$ is on the left-half complex plane.

Similarly, we find S_- by solving the equation $\det(-\tau^2 D + i\tau M_- + N_- - \lambda I) = 0$, with

$$M_- = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \quad \text{and} \quad N_- = \begin{pmatrix} -1 & a_1 \\ 0 & r(1-a_2) \end{pmatrix}.$$

This gives two solutions $\lambda = \lambda_{3,4}$, where

$$\lambda_3 = -\tau^2 + i\tau c - 1,$$
$$\lambda_4 = -\tau^2 d + i\tau c + r(1-a_2).$$

From (5.1.2), $S_- = \{\lambda_3 \mid -\infty < \tau < \infty\} \cup \{\lambda_4 \mid -\infty < \tau < \infty\}$ is on the left-half complex plane.

The above analysis shows that the essential spectrum of $\mathcal{L}_w$ is on the left-half complex plane as long as conditions (5.3.8) and (5.3.9) are satisfied. In fact, there are many choices of α and β satisfying these conditions depending on μ_1, μ_2, and μ_3. We choose them by the following algorithm.

Algorithm 1. Two mechanisms are valid to choose α and β so that all conditions in (5.3.8) and (5.3.9) hold:

(1) If $\mu_1 < \mu_3$, then we choose $\beta = \alpha$ for any $\alpha \in (\mu_1, \min\{\mu_2, \mu_3\})$.

(2) If $\mu_1 \geq \mu_3$, then we choose $\epsilon < \beta < \mu_3$ and $\alpha = \mu_1 + \epsilon$, for small $\epsilon > 0$. In particular, we can choose $\beta = 2\epsilon$ and $\alpha = \mu_1 + \epsilon$, for $\epsilon < \min\{\mu_2 - \mu_1, \mu_3/2\}$.

For any $c > c^*$, we have from 5.2.1 that $\overline{U}(z) \sim C_1 e^{-\mu_1 z}$, $C_1 > 0$, as $z \to \infty$. Since $\lambda = 0$ is the principal eigenvalue to the operator $\mathcal{L}$ defined in (5.3.1) with the one-sign eigenvector $(\overline{U}',\overline{V}')(z)$. By the choice of the weighted functional space L^p_w, the one-sign eigenvector $(\overline{U}',\overline{V}')(z)$ is not inside. Hence, the eigenvalues of the operator $\mathcal{L}_w$ in L^p_w are all negative. Now we are in a position to state the local stability result.

Theorem 5.3.1. *For any $c > c_{\min}$, the wavefront $(\overline{U},\overline{V})(z)$ is locally stable in the weighted functional space L^p_w with the weight function $w(z)$ defined in (5.3.3)-(5.3.4), where α and β in the formula of $w(z)$ are chosen by Algorithm 1.*

5.4 The Global Stability

We study here the global stability of the steady-state $(\overline{U},\overline{V})(z)$ in a special choice of the weighted functional space $L^p_w(\mathbb{R})$. Let $p = \infty$ and define the norm $\|f\|_{L^\infty_w} = \text{ess sup}_{z\in\mathbb{R}} |w(z)f(z)|$, for some weight function $w(z)$. Assume $\mu_1 < \mu_3$. By Algorithm 1, we choose $\alpha = \beta \in (\mu_1, \min\{\mu_2, \mu_3\})$. Specifically, let $\alpha = \beta = \mu_1 + \epsilon$, for small positive number ϵ. Also, we assume that the functions $\overline{U}(z)$ and $\overline{V}(z)$ satisfy the condition

$$\frac{\overline{V}(z)}{\overline{U}(z)} < \min\{a_2, 1/a_1\}, \quad \forall z \in (-\infty, +\infty). \tag{5.4.1}$$

Theorem 5.4.1. *Suppose $c > c_{\min}$, $\mu_1 < \mu_3$, and conditions (5.1.2)-(5.4.1) hold true.*

If the initial data $U(z,0) = U_0(z)$ and $V(z,0) = V_0(z)$ satisfy

$$(0,0) \le (U_0, V_0)(z) \le (1,1), \quad \forall z \in \mathbb{R},$$
$$\lim_{z\to -\infty} \inf (U_0, V_0)(z) > (0,0),$$

and

$$\left[U_0(z) - \overline{U}(z)\right] \in L^\infty_w(\mathbb{R}), \quad \left[V_0(z) - \overline{V}(z)\right] \in L^\infty_w(\mathbb{R}).$$

Then the solution $(U,V)(z,t)$ to (5.1.4) exists globally with

$$(0,0) \le (U,V)(z,t) \le (1,1), \quad \forall (z,t) \in \mathbb{R}\times\mathbb{R}^+,$$

and converges to the steady-state $(\overline{U},\overline{V})(z)$ exponentially in the sense of

$$\sup_{z\in\mathbb{R}} \left|U(z,t) - \overline{U}(z)\right| \le k e^{-\eta t}, \quad t > 0,$$
$$\sup_{z\in\mathbb{R}} \left|V(z,t) - \overline{V}(z)\right| \le k e^{-\eta t}, \quad t > 0,$$

for positive constants k and η.

To prove Theorem 5.4.1, we will find an upper and a lower solution to the partial differential equations system (5.1.4). For $z \in \mathbb{R}$, define

$$U_0^+(z) = \max\left\{U_0(z), \overline{U}(z)\right\}, \quad V_0^+(z) = \max\left\{V_0(z), \overline{V}(z)\right\},$$
$$U_0^-(z) = \min\left\{U_0(z), \overline{U}(z)\right\}, \quad V_0^-(z) = \min\left\{V_0(z), \overline{V}(z)\right\}.$$

It is easy to see that the following inequalities are true

$$\begin{aligned} &(0,0) \le (U_0^-, V_0^-)(z) \le (U_0, V_0)(z) \le (U_0^+, V_0^+)(z) \le (1,1), \\ &(0,0) \le (U_0^-, V_0^-)(z) \le (\overline{U},\overline{V})(z) \le (U_0^+, V_0^+)(z) \le (1,1). \end{aligned} \tag{5.4.2}$$

Denote $(U^+, V^+)(z,t)$ and $(U^-, V^-)(z,t)$ as the solutions to the system (5.1.4) with the initial data $(U_0^+, V_0^+)(z)$ and $(U_0^-, V_0^-)(z)$, respectively, that is,

$$\begin{cases} U_t^{\pm} = U_{zz}^{\pm} + cU_z^{\pm} + U^{\pm}(1 - a_1 - U^{\pm} + a_1 V^{\pm}), \\ V_t^{\pm} = dV_{zz}^{\pm} + cV_z^{\pm} + r(1 - V^{\pm})(a_2 U^{\pm} - V^{\pm}), \\ (U^{\pm}, V^{\pm})(z,0) = (U_0^{\pm}, V_0^{\pm})(z). \end{cases} \tag{5.4.3}$$

By the comparison principle, one gets

$$\begin{aligned} &(0,0) \le (U^-, V^-)(z,t) \le (U,V)(z,t) \le (U^+, V^+)(z,t) \le (1,1), \quad \forall (z,t) \in \mathbb{R} \times \mathbb{R}^+, \\ &(0,0) \le (U^-, V^-)(z,t) \le \ (\overline{U}, \overline{V})(z) \ \le (U^+, V^+)(z,t) \le (1,1), \quad \forall (z,t) \in \mathbb{R} \times \mathbb{R}^+. \end{aligned} \tag{5.4.4}$$

In the following lemmas we shall prove the convergence of $(U^+, V^+)(z,t)$ and $(U^-, V^-)(z,t)$ to the wavefront $(\overline{U}, \overline{V})(z)$. Then we apply the squeezing theorem to obtain the result in Theorem 5.4.1.

Lemma 5.4.1. *Under the conditions in Theorem 5.4.1, $(U^+, V^+)(z,t)$ converges to $(\overline{U}, \overline{V})(z)$.*

Proof. For $(z,t) \in \mathbb{R} \times \mathbb{R}^+$, define

$$P(z,t) = U^+(z,t) - \overline{U}(z) \ \text{ and } \ Q(z,t) = V^+(z,t) - \overline{V}(z).$$

These functions, P and Q, satisfy the initial value conditions

$$P(z,0) = U_0^+(z) - \overline{U}(z) \ \text{ and } \ Q(z,0) = V_0^+(z) - \overline{V}(z).$$

By (5.4.2) and (5.4.4), for all $z \in \mathbb{R}$ and $t \geq 0$, we have

$$(0,0) \leq (P,Q)(z,t) \leq (1,1).$$

By (5.1.3) and (5.4.3) and using condition (5.4.1), we can verify that P and Q satisfy

$$\begin{aligned} P_t &\leq P_{zz} + cP_z + (1-a_1)P + (P+\overline{U})(-P+a_1Q), \\ Q_t &\leq Q_{zz} + cQ_z + r(a_2P-Q) + r(Q+\overline{V})(-a_2P+Q). \end{aligned} \tag{5.4.5}$$

To study the stability in the weighted functional space L_w^∞, with $w(z)$ defined in (5.3.3), we first let

$$\begin{pmatrix} P \\ Q \end{pmatrix}(z,t) = e^{-\alpha(z-z_0)} \begin{pmatrix} \overline{P} \\ \overline{Q} \end{pmatrix}(z,t), \quad \text{for all } (z,t) \in \mathbb{R} \times \mathbb{R}^+,$$

where $\overline{P}$ and $\overline{Q}$ are functions in $L^\infty(\mathbb{R})$ and z_0 is the same used in the weight function $w(z)$. This gives

$$\begin{aligned} \begin{pmatrix} \overline{P} \\ \overline{Q} \end{pmatrix}_t &\leq D \begin{pmatrix} \overline{P} \\ \overline{Q} \end{pmatrix}_{zz} + M \begin{pmatrix} \overline{P} \\ \overline{Q} \end{pmatrix}_z + A(\alpha) \begin{pmatrix} \overline{P} \\ \overline{Q} \end{pmatrix} + \begin{pmatrix} (\overline{U}+e^{-\alpha z}\overline{P})(-\overline{P}+a_1\overline{Q}) \\ r(\overline{V}+e^{-\alpha z}\overline{Q})(-a_2\overline{P}+\overline{Q}) \end{pmatrix} \\ &:= \begin{pmatrix} \mathcal{L}_1(\overline{P},\overline{Q}) \\ \mathcal{L}_2(\overline{P},\overline{Q}) \end{pmatrix}, \end{aligned} \tag{5.4.6}$$

where $A(\alpha)$ is the same matrix defined in (5.2.2) and $M = diag(c-2\alpha, c-2d\alpha)$.

Define $\overline{P}_1(z,t)$ and $\overline{Q}_1(z,t)$ as

$$\overline{P}_1(z,t) = k_1\zeta_1 e^{-\eta_1 t} \ \text{ and } \ \overline{Q}_1(z,t) = k_1\zeta_2 e^{-\eta_1 t}, \ \ \forall (z,t) \in \mathbb{R} \times \mathbb{R}^+,$$

for some constants $k_1, \eta_1 > 0$ to be chosen and $(\zeta_1, \zeta_2) = (\zeta_1(\alpha), \zeta_2(\alpha))$ is the eigenvector of the matrix $A(\alpha)$ associated to the eigenvalue $\alpha^2 - c\alpha + 1 - a_1$. Simple computations give

$$\begin{aligned} \zeta_1(\alpha) &= (\alpha^2 - c\alpha + 1 - a_1) - (d\alpha^2 - c\alpha - r) \\ &= (\mu_1^2 + \epsilon)(1-d) + 1 - a_1 + r, \\ \zeta_2(\alpha) &= ra_2, \end{aligned}$$

which are positive for small ϵ and $\mu_1 < \mu_3$. Since the initial values $\overline{P}(z,0)$ and $\overline{Q}(z,0)$ are in the space L_w^∞, we can choose $k_1 \geq \max_{z\in\mathbb{R}}\{\overline{P}(z,0)/\zeta_1, \overline{Q}(z,0)/\zeta_2\}$. Direct computations and using condition (5.4.1) show that both of $\mathcal{L}_1(\overline{P}_1, \overline{Q}_1)$ and $\mathcal{L}_2(\overline{P}_1, \overline{Q}_1)$ are negative. This allows to choose a positive value to η_1 so that the inequality

$$\begin{pmatrix} \overline{P}_1 \\ \overline{Q}_1 \end{pmatrix}_t = -\eta_1 k_1 \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} e^{-\eta_1 t} \geq \begin{pmatrix} \mathcal{L}_1(\overline{P}_1, \overline{Q}_1) \\ \mathcal{L}_2(\overline{P}_1, \overline{Q}_1) \end{pmatrix}. \tag{5.4.7}$$

holds. Hence, since $(\overline{P}_1, \overline{Q}_1)(0,z) \geq (\overline{P}, \overline{Q})(0,z)$ and by comparison on unbounded domain, see e.g. [4, Proposition 2.1],

$$(P,Q)(z,t) = (\overline{P}, \overline{Q})e^{-\alpha(z-z_0)} \leq k_1(\zeta_1, \zeta_2)e^{-\alpha(z-z_0)-\eta_1 t}, \quad \forall (z,t) \in \mathbb{R} \times \mathbb{R}^+.$$

In particular, this is true when $z \in [z_0, \infty)$, for any fixed z_0.

Now, we introduce the weight function $w(z)$ defined in (5.3.3)-(5.3.4) with $\alpha = \beta = \mu_1 + \epsilon$. By the above analysis, we need to prove the convergence of $(P,Q)(z,t)$ to

$(0,0)$ for $z \in (-\infty, z_0]$. Note that the full system for $(P,Q)(z,t)$ can be expressed as

$$\begin{pmatrix} P \\ Q \end{pmatrix}_t = D \begin{pmatrix} P \\ Q \end{pmatrix}_{zz} + c \begin{pmatrix} P \\ Q \end{pmatrix}_z + J(z) \begin{pmatrix} P \\ Q \end{pmatrix} + \begin{pmatrix} (-P + a_1 Q)P \\ r(-a_2 P + Q)Q \end{pmatrix}. \quad (5.4.8)$$

Here, $J(z)$ is the same 2×2 matrix defined in (5.3.2). Let z_0 be chosen so that

$$J(z) \leq \begin{pmatrix} -1 + \epsilon_1 & a_1 + \epsilon_1 \\ \epsilon_1 & r(1 - a_2) + \epsilon_1 \end{pmatrix} := J_{\epsilon_1},$$

for some given small $\epsilon_1 > 0$, when $z \leq z_0$. This is equivalent to require that $(\overline{U}, \overline{V})(z)$ is close to $(1,1)$ for all $z \leq z_0$. Define $(\widehat{P}, \widehat{Q})(t)$ as the solution of the autonomous system

$$\begin{pmatrix} \widehat{P} \\ \widehat{Q} \end{pmatrix}_t = J_{\epsilon_1} \begin{pmatrix} \widehat{P} \\ \widehat{Q} \end{pmatrix} + \begin{pmatrix} (-\widehat{P} + a_1 \widehat{Q})\widehat{P} \\ r(-a_2 \widehat{P} + \widehat{Q})\widehat{Q} \end{pmatrix}, \quad (5.4.9)$$

with the initial data

$$\widehat{P}(0) \geq \overline{P}(z,0), \quad \widehat{Q}(0) \geq \overline{Q}(z,t), \ \forall z \in \mathbb{R}.$$

Then $(\widehat{P}, \widehat{Q})$ is an upper solution to the system (5.4.8).

Now we need to prove the convergence of $(\widehat{P}, \widehat{Q})(t)$ to $(0,0)$ as $t \to \infty$. The Jacobian matrix $J(0,0) = J_{\epsilon_1}$ of system (5.4.9) at the fixed point $(0,0)$ has two eigenvalues, $\hat{\lambda}_2 < \hat{\lambda}_1 < 0$. By the phase plane analysis, there exists $0 < \delta \leq 1$ so that the flow in the $\widehat{P}\widehat{Q}-$space converges to origin for any initial data $(\widehat{P}, \widehat{Q})(0)$ in the box $[0,1] \times [0, \delta]$. Hence, we conclude that

$$(\widehat{P}, \widehat{Q}) = \hat{k}_1 (\widehat{C}_1, \widehat{C}_2) e^{\hat{\lambda}_1 t} \ \text{ as } \ t \to \infty,$$

for positive constant $\hat{k}_1$ and $(\widehat{C}_1 \;\; \widehat{C}_2)^T$ is the eigenvector of J_{ϵ_1} corresponding to $\hat{\lambda}_1$. For the maximal possible choice of the constant δ so that we have the convergence result inside the box $[0,1] \times [0,\delta]$, see Remark 5.4.1 below.

We can choose $\hat{k}_1$ large and $\bar{\lambda}_1 = \min\{\eta_1, -\hat{\lambda}_1\}$ so that, at the boundary $z = z_0$, we have

$$(P,Q)(z_0,t) \leq k_1(\zeta_1,\zeta_2)e^{-\eta_1 t} \leq \hat{k}_1(\zeta_1,\zeta_2)e^{-\bar{\lambda}_1 t} = (\widehat{P},\widehat{Q})(z_0,t).$$

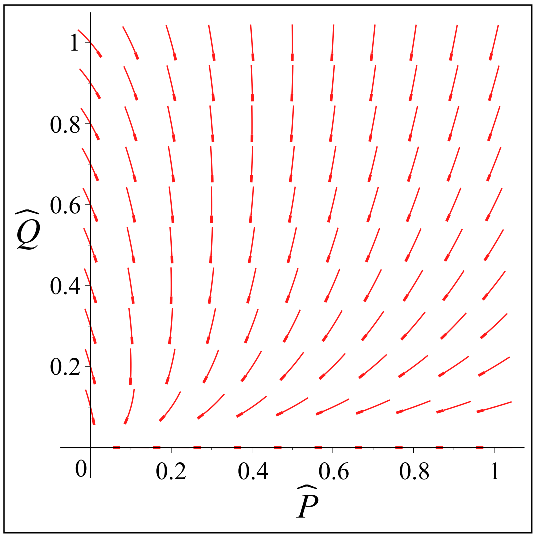
Hence, by comparison on the domain $(-\infty, z_0] \times [0,\infty)$, see e.g. [77, Lemma 3.2],

$$(P,Q)(z,t) \leq \hat{k}_1(\zeta_1,\zeta_2)e^{-\bar{\lambda}_1 t}, \quad \forall (z,t) \in (-\infty, z_0] \times \mathbb{R}^+.$$

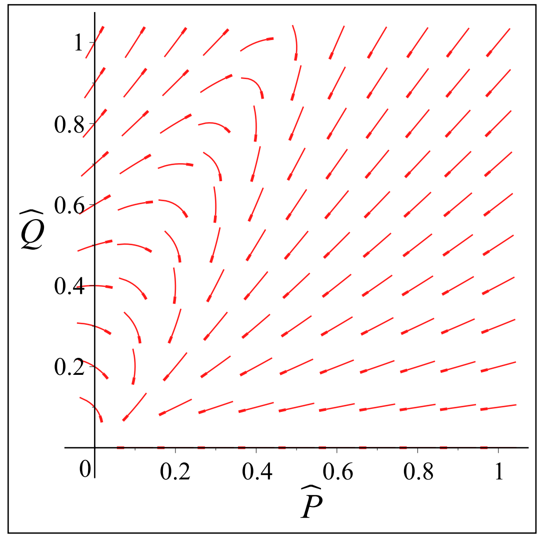
This completes the proof. ∎

Remark 5.4.1. The maximal possible value of the constant δ, which could be 1, depends on the location of the fourth fixed point to the system (5.4.9) near or inside the box $[0,1]\times[0,1]$. See Figure 5.1 for all possible different cases. In (a), the positive fixed point is far away from the box $[0,1] \times [0,1]$ and does not effect the flow. This happens when $a_2 > 2$. Hence we set $\delta = 1$. The second figure (b) shows the effect of the positive fixed point on the flow, which still outside the box. The maximal choice of δ for this case exists in the interval $(a_2 - 1 - \epsilon_1/r, 1)$. The number $a_2 - 1 - \epsilon_1/r$ is the positive $\widehat{Q}$-intercept of the nullcline $\widehat{Q}_t = 0$. A fixed point exists inside the box $[0,1] \times [0,1]$ in (c), where δ becomes close to the value $a_2 - 1 - \epsilon_1/r$.

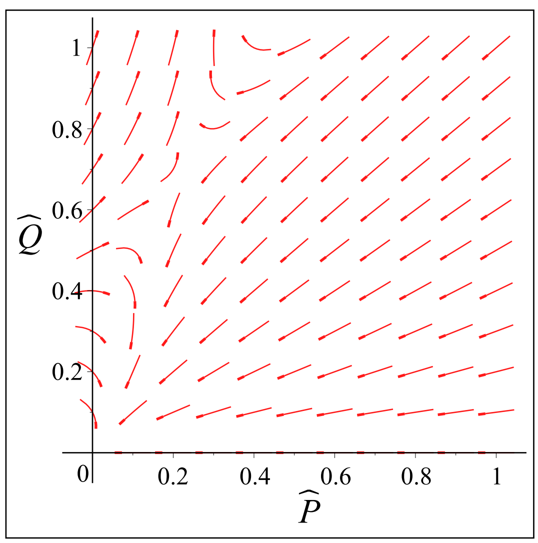
Lemma 5.4.2. *Under the conditions in Theorem 5.4.1, $(U^-,V^-)(z,t)$ converges to $(\overline{U},\overline{V})(z)$.*



(a) $a_1 = 0.5$ and $a_2 = 2.4$. We chose $\delta = 1$.



(b) $a_1 = 0.5$ and $a_2 = 1.4$. In this case, the maximal possible choice of δ is in $(0.3984, 1)$



(c) $a_1 = 0.3$ and $a_2 = 1.4$. Here, the maximal choice of δ becomes close to 0.3984.

Figure 5.1: The phase portrait of the system (5.4.9) when $\epsilon_1 = 0.003$ and $r = 1.875$.

Proof. For $(z,t) \in \mathbb{R} \times \mathbb{R}^+$, define

$$R(z,t) = \overline{U}(z) - U^-(z,t) \quad \text{and} \quad S(z,t) = \overline{V}(z) - V^(z,t).$$

These functions, R and S, satisfy the initial value conditions

$$R(z,0) = \overline{U}(z) - U_0^-(z) \quad \text{and} \quad S(z,0) = \overline{V}(z) - V_0^-(z).$$

From (5.4.2) and (5.4.4), for all $z \in \mathbb{R}$ and $t \geq 0$, we have

$$(0,0) \leq (R,S)(z,t) \leq (1,1).$$

From (5.1.3) and (5.4.3), R and S satisfy the system

$$\begin{pmatrix} R \\ S \end{pmatrix}_t = D \begin{pmatrix} R \\ S \end{pmatrix}_{zz} + c \begin{pmatrix} R \\ S \end{pmatrix}_z + J(z) \begin{pmatrix} R \\ S \end{pmatrix} - \begin{pmatrix} (-R + a_1 S)R \\ r(-a_2 R + S)S \end{pmatrix}, \quad (5.4.10)$$

with $J(z)$ defined in (5.3.2). By condition (5.4.1), we have

$$\begin{aligned} R_t &\leq R_{zz} + cR_z + (1-a_1)R + (R - \overline{U})(R - a_1 S), \\ S_t &\leq dS_{zz} + cS_z + r(a_2 R - S) + r(S - \overline{V})(a_2 R - S). \end{aligned} \quad (5.4.11)$$

Similar to the previous analysis in the proof of Lemma 5.4.1, and making a use of the facts $R < \overline{U}$ and $S < \overline{V}$, we can prove that there exist $\eta_2 > 0$ and

$$k_2 \geq e^{-\alpha(z-z_0)} \max_{z \in \mathbb{R}} \{R(z,0)/\zeta_1, S(z,0)/\zeta_2\}, \quad \forall (z,t) \in \mathbb{R} \times \mathbb{R}^+$$

so that

$$(R,S)(z,t) \leq k_2(\zeta_1,\zeta_2)e^{-\eta_2 t}, \quad \forall (z,t) \in \mathbb{R} \times \mathbb{R}^+.$$

For the choice of z_0 in proof of Lemma 5.4.1, we study the stability in the weighted space L_w^∞. To this end, define $(\widehat{R},\widehat{S})(t)$ as the solution of the system

$$\begin{pmatrix} \widehat{R} \\ \widehat{S} \end{pmatrix}_t = J_{\epsilon_1}\begin{pmatrix} \widehat{R} \\ \widehat{S} \end{pmatrix} - w_1 \begin{pmatrix} (-\widehat{R}+a_1\widehat{S})\widehat{R} \\ r(-a_2\widehat{R}+\widehat{S})\widehat{S} \end{pmatrix}, \tag{5.4.12}$$

with the initial data

$$\widehat{R}(0) \geq R(z,0), \quad \widehat{S}(0) \geq S(z,0), \ \forall z \in \mathbb{R}. \tag{5.4.13}$$

It is easy to see that $(\widehat{R},\widehat{S})$ is an upper solution to the system (5.4.10). The phase plane analysis shows that $(\widehat{R},\widehat{S})(t)$ converges to origin for any initial data in the region $[0,1]\times[0,1]$ except the point $(1,1)$. Similar to the previous lemma,

$$(R,S)(z,t) \leq \hat{k}_2(\zeta_1,\zeta_2)e^{-\bar{\lambda}_2 t}, \quad \forall (z,t) \in (-\infty,z_0]\times\mathbb{R}^+.$$

for some positive constants $\hat{k}_2$ and $\bar{\lambda}_2$. This completes the proof. ∎

Now, we are ready to give the proof of Theorem 5.4.1.

Proof of Theorem 5.4.1. From (5.4.4), for all $(z,t)\in\mathbb{R}\times\mathbb{R}^+$, we have

$$|R(z,t)| \leq |U(z,t)-\overline{U}(z)| \leq |P(z,t)|,$$
$$|S(z,t)| \leq |V(z,t)-\overline{V}(z)| \leq |Q(z,t)|.$$

By lemmas 5.4.1-5.4.2 and the squeezing theorem, it follows that there exist $k>0$

and $\eta > 0$ so that

$$|U(z,t) - \overline{U}(z)| \le ke^{-\eta t},$$
$$|V(z,t) - \overline{V}(z)| \le ke^{-\eta t},$$

for all $(z,t) \in \mathbb{R} \times \mathbb{R}^+$. This proves the desired result. ∎

Condition (5.4.1) is used in the previous analysis to construct the upper solutions in the proof of lemmas 5.4.1-5.4.2. It implies that, at $c = c_0$ and $z \to +\infty$,

$$\frac{\zeta_2(\mu_1)}{\zeta_1(\mu_1)} < \min\{a_2, 1/a_1\} \implies \begin{cases} d < 2, \\ (a_1 a_2 - 1)r < (2-d)(1-a_1). \end{cases}$$

This condition is the same derived in [42] for the linear speed selection. To see that the condition (5.4.1) can be realized for all $z \in \mathbb{R}$, we prove the following lemma.

Lemma 5.4.3. *$d = 0$ and $a_1 a_2 \le 1$ imply (5.4.1).*

Proof. Since $a_1 a_2 \le 1$, we only need to prove the inequality $\overline{V}(z) \le a_2 \overline{U}(z)$ for all $z \in \mathbb{R}$. Same argument as that in the proof of Lemma (4.3.2) completes the proof. ∎

5.5 Conclusions and Summary

The local and the global stability of traveling waves to the two-species Lotka-Volterra competition model (5.1.1) under the condition (5.1.2) are investigated. Using the linearization and the essential spectrum analysis in [26], we find that the traveling wavefront is stable in some weighted functional space, see Theorem 5.3.1. Many choices of the exponential weight functions are valid, see Algorithm 1.

Under some further condition, (5.4.1), we apply the upper-lower solution method to obtain a global stability result. Indeed, we prove that both the upper and the lower solutions tend to the wavefront. Our main results are presented in Theorem 5.4.1.

Chapter 6

Future Work

The minimal wave speed selection mechanisms of the traveling wave solution to the system

$$\begin{cases} U'' + cU' + U(1 - a_1 - U + a_1 V) = 0, \\ dV'' + cV' + r(1 - V)(a_2 U - V) = 0, \\ (U, V)(-\infty) = e_1, \quad (U, V)(\infty) = e_0, \end{cases} \tag{6.0.1}$$

which is the corresponding system to the non-dimensional competition model, has been studied in Chapter 4 for the special case when $d = 0$. The solution formula to the second equation, for a given monotone function $U(z)$, was given. We have used this formula to prove some properties of the functions regarding the boundedness and the monotonicity. Then we applied the upper-lower solution method to determine the speed selection mechanisms.

The speed selection problem becomes more challenging when $d > 0$, due to the invalidity of the solution formula $V(z)$ in terms of $U(z)$ for the second equation in (6.0.1). Also, $U(z)$ and $V(z)$ have different behaviors near infinity for some cases (see Theorem 6.0.2 below), which makes the construction of the upper and the lower solutions more complicated.

Indeed, as we can see in Chapter (5), if assume that the traveling wave solution to the full system (6.0.1) exists with the behavior

$$(U, V)(z) = (\xi_1, \xi_2)e^{-\mu z}, \quad \text{as } z \to \infty,$$

for some positive constants ξ_1, ξ_2, and μ, then the following theorem is true.

Theorem 6.0.1. *For $c \geq c_{\min}$, if*

$$0 \leq d \leq 2 + \frac{r}{1 - a_1}, \tag{6.0.2}$$

then $(U, V)(z)$ has the behavior

$$(U, V) \sim C_1(\xi_1, \xi_2)e^{-\mu_1 z}, \quad \text{as} \;\; z \to \infty, \tag{6.0.3}$$

where $\mu_1 = \frac{c - \sqrt{c^2 - 4(1 - a_1)}}{2}$, $\xi_1 = d\mu_1^2 - c\mu_1 + 1 - a_1$, $\xi_2 = ra_2$, and C_1 is a positive constant.

As mentioned in Chapter 4, Huang [32] proved that the linear speed selection is realized when

$$\frac{(2 - d)(1 - a_1) + r}{ra_2} \geq \max\left\{a_1, \frac{d - 2}{2|d - 1|}\right\},$$

which also extended the result of Lewis *et al* [42] when $0 \leq d \leq 2$. It is easy to see that the Huang's result contributes only when

$$2 < d \leq 2 + \frac{r}{1 - a_1},$$

i.e., the study of [42] and [32] consider the case in Theorem 6.0.1. Also, the nonlinear result by Huang and Han [33] requires

$$d = r < 2 + \frac{r}{1 - a_1}.$$

On the other hand, when condition (6.0.2) does not hold, we have the following result.

Theorem 6.0.2. *If* $d > 2 + \dfrac{r}{1 - a_1}$*, then there exists* $\hat{c} > c_0$ *so that* $(U, V)(z)$ *has the same behavior as that in (6.0.3) when* $c_{\min} \le c < \hat{c}$*, and has the behavior*

$$(U, V)(z) \sim (C_2 e^{-\mu_1 z}, C_3 e^{-\mu_3 z}), \ \text{as } z \to \infty,$$

when $c > \hat{c}$*. Here* C_2 *and* C_3 *are positive constants, and* $\mu_3 = \dfrac{c + \sqrt{c^2 + 4rd}}{2d}$*.*

As far as we know, the case in the above theorem has not been considered before.

To conclude, the speed selection problem of the system (6.0.1) for any value of $d > 0$ is quite interesting and challenging. This problem will be studied in another project in our future work, via the extension of our novel idea in the case when $d = 0$. Besides, for a short term plan, we expect to extend the method here for further study on the speed selection problem for general abstract monotone systems, including time-periodic and periodic habitat systems, as well as some non-monotone systems.

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