On the polar Orlicz-Minkowski problems and the p-capacitary Orlicz-Petty bodies

by

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Abstract

This thesis deals with the polar Orlicz-Minkowski problems and the p-capacitary Orlicz-Petty bodies. The polar Orlicz-Minkowski problems are introduced and the solvability of such problems is discussed under different conditions. In particular, under certain condition on φ, the existence of a solution is proved for a nonzero finite measure μ on S^{n-1} which is not concentrated on any hemisphere of S^{n-1}. The existence of the p-capacitary Orlicz-Petty bodies is also established. The Orlicz and L_q geominimal capacities with respect to $\mathscr{K}_0$ and $\mathscr{S}_0$ are proposed and their properties, such as invariance under orthogonal matrices, isoperimetric type inequalities and cyclic type inequalities are provided as well.

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Chapter 1

Introduction

Let $\mathscr{K}_0$ denote the set of all convex bodies in $\mathbb{R}^n$ with the origin o in their interiors, i.e., $K \in \mathscr{K}_0$ is a convex compact subset of $\mathbb{R}^n$ such that $o \in \text{int}K$, the interior of K. The mixed volume is one of the central concepts in the Brunn-Minkowski theory of convex bodies. It comes naturally from the combination of Minkowski sum and volume in $\mathbb{R}^n$ and specifically, the mixed volume of $K, L \in \mathscr{K}_0$, denoted by $V_1(K, L)$, is the variation of the volume of K with respect to L, i.e.,

$$nV_1(K, L) = \lim_{\epsilon \to 0^+} \frac{V(K + \epsilon L) - V(K)}{\epsilon}.$$

The mixed volume of $K, L \in \mathscr{K}_0$ (see e.g. [20, 57]), can also be formulated by

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L(u) dS(K, u),$$

where $S(K, \cdot)$ is the surface area measure of K (see e.g. [1, 16]) and h_L is the support function of L defined on S^{n-1}, the unit sphere of $\mathbb{R}^n$ (see Chapter 2 for more details on the notations). The mixed volume plays fundamental roles in many important objects in convex geometry. For instance, the classical Minkowski inequality (see e.g.

[17, 20, 57]) states that, for all $K, L \in \mathscr{K}_0$,

$$V_1(K,L) \geq |K|^{\frac{n-1}{n}}|L|^{\frac{1}{n}} \tag{1.1}$$

with equality if and only if K and L are homothetic of each other, i.e. there exist a constant $\lambda > 0$ and $x_0 \in \mathbb{R}^n$ such that $K = \lambda L + x_0$. Hereafter $|K|$ refers to the volume of $K \in \mathscr{K}_0$ and $\omega_n = |B_2^n|$ denotes the volume of the unit ball B_2^n in $\mathbb{R}^n$. In fact, the classical Minkowski inequality (1.1) implies that: for any $K \in \mathscr{K}_0$ given, the following optimization problem

$$\inf\left\{\frac{1}{n}\int_{S^{n-1}} h_L(u)dS(K,u):\ L \in \mathscr{K}_0 \ \text{ and } \ |L| = \omega_n\right\} \tag{1.2}$$

has a unique solution (up to a translation).

The classical Minkowski problem is a fundamental problem in convex geometry and it has inspired a lot of problems with a similar nature, such as L_p Minkowski and Orlicz Minkowski problems. The classical Minkowski problem asks the necessary and sufficient conditions for a Borel measure μ on S^{n-1} such that there exists a convex body $K \in \mathscr{K}_0$ with $dS(K,\cdot) = d\mu$. It has been proved in, e.g. [52, 53], that there exists a unique convex body K (up to a translation) such that $dS(K,\cdot) = d\mu$, if μ is not concentrated on any hemisphere of S^{n-1} and has the centroid at the origin, i.e.,

$$\int_{S^{n-1}} \langle \xi, u\rangle_+ d\mu(u) > 0 \text{ for any } \xi \in S^{n-1} \ \text{ and } \ \int_{S^{n-1}} u d\mu(u) = o,$$

where $\langle \xi, u\rangle_+ = \max\{\langle \xi, u\rangle, 0\}$. To solve the classical Minkowski problem is equivalent to find an optimizer for the following optimization problem:

$$\inf\left\{\frac{1}{n}\int_{S^{n-1}} h_L d\mu:\ L \in \mathscr{K}_0 \ \text{ and } \ |L| = \omega_n\right\}. \tag{1.3}$$

Note that (1.3) is obtained by replacing the surface area measure $S(K,\cdot)$ by μ in (1.2). Another important concept in convex geometry closely related to the mixed volume is the classical geominimal surface area, which is denoted by $G(\cdot)$ and is defined by [56]: for any $K\in\mathscr{K}_0$,

$$G(K)=\inf\left\{\int_{S^{n-1}}h_L(u)dS(K,u):\ L\in\mathscr{K}_0\ \text{ and }\ |L^\circ|=\omega_n\right\},\tag{1.4}$$

where L° denotes the polar body of L, i.e., $L^\circ=\{y\in\mathbb{R}^n:\langle x,y\rangle\leq 1 \text{ for any } x\in L\}$. It has been proved in [56] that there exists a (unique) convex body T_1K, which is called the Petty body of K, such that

$$G(K)=nV_1(K,T_1K)=\int_{S^{n-1}}h_{T_1K}(u)dS(K,u)\quad\text{and}\quad|(T_1K)^\circ|=\omega_n.$$

In view of (1.3) and (1.4), one sees that the main difference is to replace L in (1.3) by L° in (1.4). Consequently, one may call the optimization problem in (1.4) the polar Minkowski problem. We would like to mention that both (1.3) and (1.4) are the key ingredients in the development of the Brunn-Minkowski theory of convex bodies. Indeed, a variation problem of (1.4) [42]

$$\inf\left\{V_1(K,L^\circ)=\frac{1}{n}\int_{S^{n-1}}\rho_L^{-1}(u)dS(K,u):L\in\mathscr{S}_0\ \text{ and }\ |L|=\omega_n\right\},$$

where $\rho_L:S^{n-1}\to(0,\infty)$ is the radial function of a star body $L\in\mathscr{S}_0$, provides an equivalent formula for the classical affine area [4] and plays arguably more important roles in the Brunn-Minkowski theory of convex bodies, due to its applications in, such as, the theory of valuation (see e.g. [2, 3, 38]) and the approximation of convex bodies by polytopes (see e.g. [19, 40, 59]).

For $p\in\mathbb{R}$ but $p\neq 0,1$, the L_p mixed volume of $K,L\in\mathscr{K}_0$ (see e.g. [43, 68]) can be defined as

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} \Big(\frac{h_L(u)}{h_K(u)}\Big)^p h_K(u) dS(K, u). \tag{1.5}$$

The L_p mixed volume plays fundamental roles in the development of the L_p Brunn-Minkowski theory of convex bodies (see e.g. [6, 7, 22, 44, 45, 48, 51, 59]). Analogous to (1.1), one has the L_p Minkowski inequality [43]: for all $p > 1$ and all $K, L \in \mathscr{K}_0$,

$$V_p(K, L) \geq |K|^{\frac{n-p}{n}} |L|^{\frac{p}{n}} \tag{1.6}$$

with equality if and only if K and L are dilates of each other, i.e., $K = \lambda L$ for some $\lambda > 0$. Again the L_p Minkowski inequality (1.6) implies that: for any fixed $K \in \mathscr{K}_0$ and $p > 1$, the following optimization problem

$$\inf \left\{ \int_{S^{n-1}} \Big(\frac{h_L(u)}{h_K(u)}\Big)^p h_K(u) dS(K, u) : L \in \mathscr{K}_0 \ \text{ and } \ |L| = \omega_n \right\} \tag{1.7}$$

has a unique solution.

Related to (1.7) is the L_p Minkowski problem: for $p \in \mathbb{R}$ and $p \neq 0$, under what condition on a given nonzero finite measure μ defined on S^{n-1}, there is a convex body K (ideally with the origin o in its interior) such that $h_K^{p-1} d\mu = dS(K, \cdot)$? The L_p Minkowski problem is a popular problem in geometry and has attracted considerable attention (see e.g. [8, 9, 29, 32, 43, 47, 60, 76, 77, 78]). Solutions to the L_p Minkowski problems have fundamental applications in, for instance, establishing the L_p Sobolev type inequalities (see e.g. [10, 23, 46, 72]).

Replacing L by L° in (1.7), one can ask the following problem: for $p > 1$, find a convex body to solve

$$\inf\left\{\int_{S^{n-1}}\Big(\frac{h_L(u)}{h_K(u)}\Big)^p h_K(u)dS(K,u) : L\in\mathscr{K}_0 \ \text{ and } \ |L^\circ|=\omega_n\right\}. \tag{1.8}$$

In fact, Lutwak in [44] showed that there exists a unique convex body $T_pK\in\mathscr{K}_0$, the L_p Petty body of K, such that $|(T_pK)^\circ|=\omega_n$ and

$$nV_p(K,T_pK)=\inf\left\{\int_{S^{n-1}}\Big(\frac{h_L(u)}{h_K(u)}\Big)^p h_K(u)dS(K,u) : L\in\mathscr{K}_0 \ \text{ and } \ |L^\circ|=\omega_n\right\}.$$

The quantity $G_p(K)=nV_p(K,T_pK)$ for $p>1$ is called the L_p geominimal surface area of K [44]. Recently, Ye extended the L_p geominimal surface area of K to all $0\neq p\in\mathbb{R}$ [68] and Zhu, Hong and Ye showed the existence of the L_p Petty body for $p>0$ [74]. Replacing $\mathscr{K}_0$ by $\mathscr{S}_0$ in (1.7), one gets,

$$\Omega_p(K)=\inf\left\{\int_{S^{n-1}}\Big(\frac{1}{h_K(u)\rho_L(u)}\Big)^p h_K(u)dS(K,u) : L\in\mathscr{S}_0 \ \text{ and } \ |L|=\omega_n\right\}.$$

In literature, $\Omega_p(K)$ is called the L_p affine surface area and has equivalent convenient integral formulas (see e.g. [31, 44, 51, 58, 59]). It is well known that the L_p affine surface area has many applications in, such as, the valuation theory, the approximation of convex bodies by polytopes, the f-divergence of convex bodies and the L_p affine isoperimetric inequalities (see e.g. [19, 28, 33, 39, 40, 55, 59, 61, 62, 63, 73]).

Extension from the L_1 and L_p Brunn-Minkowski theories to the Orlicz theory is rather dedicated and involves nonhomogeneous functions. In view of (1.2)-(1.8), a major task is to get the "right" formula of the Orlicz mixed volume. In the Orlicz theory, there are at least 3 different ways to define the Orlicz mixed volume and each of them has their own advantages. These Orlicz mixed volumes are given as follows:

for $K, L \in \mathscr{K}_0$, and $\phi,\ \varphi : (0,\infty) \to (0,\infty)$ continuous functions,

$$V_\varphi(K,L) = \frac{1}{n}\int_{S^{n-1}} \varphi\Big(\frac{h_L(u)}{h_K(u)}\Big) h_K(u)dS(K,u), \tag{1.9}$$

$$V_{\varphi,\phi}(K,L) = \frac{1}{n}\int_{S^{n-1}} \frac{\varphi(h_L(u))}{\phi(h_K(u))}dS(K,u), \tag{1.10}$$

and if in addition $\varphi \in \mathscr{I}$,

$$\widehat{V}_\varphi(K,L) = \inf\left\{\lambda > 0 : \int_{S^{n-1}} \varphi\Big(\frac{n|K|\cdot h_L(u)}{\lambda \cdot h_K(u)}\Big) h_K(u)dS(K,u) \le n|K|\right\}, \tag{1.11}$$

where $\mathscr{I}$ refers to the set of continuous functions $\varphi : (0,\infty) \to (0,\infty)$ such that φ is strictly increasing, $\lim_{t\to 0^+}\varphi(t) = 0$, $\varphi(1) = 1$ and $\lim_{t\to\infty}\varphi(t) = \infty$. The L_p mixed volume defined by (1.5) is a special case of (1.9)-(1.11). For instance,

$$V_p(K,L) = V_\varphi(K,L) = V_{\varphi,\phi}(K,L) = n^{-p}\cdot |K|^{1-p}\big(\widehat{V}_\varphi(K,L)\big)^p$$

if $\varphi(t) = t^p$ and $\phi(t) = t^{p-1}$. Note that $V_\varphi(\cdot,\cdot)$ given by (1.9) and $V_{\varphi,\phi}(\cdot,\cdot)$ given by (1.10) can be obtained by the combination of volume and a family of linear Orlicz additions of K and L (see e.g. [18, 65, 74]). However, $\widehat{V}_\varphi(\cdot,\cdot)$ seems not have a geometric interpretation. On the other hand, both $V_\varphi(\cdot,\cdot)$ and $\widehat{V}_\varphi(\cdot,\cdot)$ have the Orlicz-Minkowski inequalities [18, 65], which extend the L_1 and L_p Minkowski inequalities. The Orlicz-Minkowski inequalities read: for $K, L \in \mathscr{K}_0$ and φ being a convex function, then

$$V_\varphi(K,L) \ge |K|\cdot\varphi\left(\left(\frac{|L|}{|K|}\right)^{\frac{1}{n}}\right), \tag{1.12}$$

and if $\varphi \in \mathscr{I}$,

$$\widehat{V}_{\varphi}(K,L) \geq n|K|^{\frac{n-1}{n}}|L|^{\frac{1}{n}}. \tag{1.13}$$

Equality holds if and only if K and L are dilates, if φ is also strictly convex. Also note that $\widehat{V}_{\varphi}(\cdot,\cdot)$ has homogeneity [74] but $V_{\varphi}(\cdot,\cdot)$ and $V_{\varphi,\phi}(\cdot,\cdot)$ do not have homogeneity.

The Orlicz-Minkowski inequalities (1.12) and (1.13) imply that, if $\varphi \in \mathscr{I}$ is strictly convex, the following problems

$$\inf\left\{\int_{S^{n-1}} \varphi\Big(\frac{h_L(u)}{h_K(u)}\Big) h_K(u)dS(K,u) : L \in \mathscr{K}_0 \ \text{ and } \ |L| = \omega_n\right\}$$

and
$$\inf\left\{\widehat{V}_{\varphi}(K,L) : L \in \mathscr{K}_0 \ \text{ and } \ |L| = \omega_n\right\}$$

have unique solutions. On the other hand, it seems attractable to pose the Minkowski type problems related to $V_{\varphi}(\cdot,\cdot)$ and $\widehat{V}_{\varphi}(\cdot,\cdot)$ similar to the L_p Minkowski problems. However, such Orlicz-Minkowski problems can be asked for the case related to $V_{\varphi,\phi}(\cdot,\cdot)$: under what condition on a finite measure μ on S^{n-1} and on a continuous function $\varphi : (0,\infty) \to (0,\infty)$, there exists a convex body K such that $dS(K,\cdot) = c\cdot\varphi(h_K)d\mu$ for some positive constant c? Solutions of this Orlicz-Minkowski problems can be found in [21, 30, 37].

Ye [69] and Zhu, Hong and Ye [74] investigated the following optimization problems and gave a detailed study of the (homogeneous and nonhomogeneous) Orlicz geominimal surface areas $G_{\varphi}^{orlicz}(\cdot)$ and $\widehat{G}_{\varphi}^{orlicz}(\cdot)$: under certain conditions on φ, define

$$G_{\varphi}^{orlicz}(K) = \inf\left\{\int_{S^{n-1}} \varphi\Big(\frac{h_L(u)}{h_K(u)}\Big) h_K(u)dS(K,u) : L \in \mathscr{K}_0 \ \text{ and } \ |L^{\circ}| = \omega_n\right\},$$

$$\widehat{G}_{\varphi}^{orlicz}(K) = \inf\left\{\widehat{V}_{\varphi}(K,L) : L \in \mathscr{K}_0 \text{ and } |L^{\circ}| = \omega_n\right\}.$$

Again, one can replace $\mathscr{K}_0$ by $\mathscr{S}_0$ to get (homogeneous and nonhomogeneous) Orlicz affine surface areas. In particular, Zhu, Hong and Ye [74] proved that, under certain conditions on φ, there exist convex bodies $T_{\varphi}K$ and $\widehat{T}_{\varphi}K$, called the Orlicz-Petty bodies of K, such that $|(T_{\varphi}K)^{\circ}| = |(\widehat{T}_{\varphi}K)^{\circ}| = \omega_n$,

$$G_{\varphi}^{orlicz}(K) = nV_{\varphi}(K, T_{\varphi}K) \text{ and } \widehat{G}_{\varphi}^{orlicz}(K) = \widehat{V}_{\varphi}(K, \widehat{T}_{\varphi}K).$$

One can also see [71] for a special case.

In Chapter 3, we will study the polar Orlicz-Minkowski problems, which are the Orlicz settings of the L_p polar Minkowski problems: under what conditions on φ and a finite nonzero measure μ defined on S^{n-1}, there exists a convex body $M \in \mathscr{K}_0$ such that $|M^{\circ}| = \omega_n$ and

$$\int_{S^{n-1}} \varphi(h_M(u))d\mu(u) = \inf\left\{\int_{S^{n-1}} \varphi(h_L(u))d\mu(u) : L \in \mathscr{K}_0 \text{ and } |L^{\circ}| = \omega_n\right\}.$$

Our main result in Chapter 3 is summarized in the following theorem. Let Ω be the set of all finite positive Borel measures on S^{n-1} that are not concentrated on any hemisphere of S^{n-1}.

Theorem 1.1. *Let $\mu \in \Omega$ and $\varphi \in \mathscr{I}$. Then there exists a convex body $M \in \mathscr{K}_0$ such that $|M^{\circ}| = \omega_n$ and*

$$\int_{S^{n-1}} \varphi(h_M(u))d\mu(u) = \inf\left\{\int_{S^{n-1}} \varphi(h_L(u))d\mu(u) : L \in \mathscr{K}_0 \text{ and } |L^{\circ}| = \omega_n\right\}.$$

Moreover, if $\varphi \in \mathscr{I}$ is convex, then M is the unique solution to the polar Orlicz-Minkowski problem.

In Chapter 4, we replace $V_\varphi(\cdot,\cdot)$ and $\widehat{V}_\varphi(\cdot,\cdot)$ in (1.9) and (1.11) by their p-capacitary counterparts, and study the existence and continuity of the p-capacitary Orlicz-Petty bodies. Here for $K, L \in \mathscr{K}_0$, $p \in (1,n)$ and $\varphi : (0,\infty) \to (0,\infty)$, the Orlicz mixed p-capacities of K and L are given by:

$$C_{p,\varphi}(K,L) = \frac{p-1}{n-p}\int_{S^{n-1}} \varphi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u)d\mu_p(K,u), \tag{1.14}$$

$$\int_{S^{n-1}} \varphi\left(\frac{C_p(K)\cdot h_L(u)}{\widehat{C}_{p,\varphi}(K,L)\cdot h_K(u)}\right) d\mu_p^*(K,u) = 1 \quad \text{for } \varphi \in \mathscr{I}, \tag{1.15}$$

where $\mu_p(K,\cdot)$ is the p-capacitary measure given by (2.10) and $\mu_p^*(K,\cdot)$ is the normalized p-capacitary measure on S^{n-1} given by (2.15). Note that $C_{p,\varphi}(\cdot,\cdot)$ can be obtained by the combination of the p-capacity and a family of linear Orlicz additions of K and L [26]. Here for a compact set $E \subseteq \mathbb{R}^n$, the p-capacity of E (see e.g. [14, 15]), is defined by

$$C_p(E) = \inf\left\{\int_{\mathbb{R}^n} |\nabla f(x)|^p dx : \ f \in C_c^\infty(\mathbb{R}^n) \text{ and } f(x) \geq 1 \text{ on } x \in E\right\},$$

where $C_c^\infty(\mathbb{R}^n)$ denotes the set of all infinitely differentiable functions on $\mathbb{R}^n$ with compact supports and ∇f denotes the gradient of $f \in C_c^\infty(\mathbb{R}^n)$. Again $\widehat{C}_{p,\varphi}(\cdot,\cdot)$ has homogeneity about K and L (see e.g. [26] or Corollary 4.1), i.e.,

$$\widehat{C}_{p,\varphi}(sK,tL) = s^{n-p-1}\cdot t\cdot \widehat{C}_{p,\varphi}(K,L), \quad \text{for any } s>0 \text{ and any } t>0.$$

Related to $C_{p,\varphi}(\cdot,\cdot)$ and $\widehat{C}_{p,\varphi}(\cdot,\cdot)$, there are the p-capacitary Orlicz-Minkowski inequalities: for $p \in (1,n)$, $K, L \in \mathscr{K}_0$ and $\varphi \in \mathscr{I}$ convex [26], one has,

$$C_{p,\varphi}(K,L) \geq C_p(K)\cdot\varphi\left(\left(\frac{C_p(L)}{C_p(K)}\right)^{\frac{1}{n-p}}\right) \text{ and } \widehat{C}_{p,\varphi}(K,L) \geq C_p(K)\left(\frac{C_p(L)}{C_p(K)}\right)^{\frac{1}{n-p}},$$

with equality if and only if K and L are dilates, if φ is strictly convex. From these inequalities, we can get the following facts: for $p \in (1, n)$ and $\varphi \in \mathscr{I}$ strictly convex,

$$\inf\left\{C_{p,\varphi}(K, L) : L \in \mathscr{K}_0 \text{ and } |L| = \omega_n\right\},$$
$$\inf\left\{\widehat{C}_{p,\varphi}(K, L) : L \in \mathscr{K}_0 \text{ and } |L| = \omega_n\right\}$$

have unique solutions. Analogous to the L_q and Orlicz Minkowski problems related to $S(K, \cdot)$, one can ask the p-capacitary L_q and Orlicz Minkowski problems (i.e., with $S(K, \cdot)$ replaced by $\mu_p(K, \cdot)$). These problems have received extensive attention, see [13, 25, 26, 34, 35, 79] for more details.

Replacing L by L°, we will study the following problems: for $p \in (1, n)$, find the optimizers to the following optimization problems:

$$\sup / \inf\left\{C_{p,\varphi}(K, L) : L \in \mathscr{K}_0 \text{ and } |L^\circ| = \omega_n\right\},$$
$$\sup / \inf\left\{\widehat{C}_{p,\varphi}(K, L) : L \in \mathscr{K}_0 \text{ and } |L^\circ| = \omega_n\right\}.$$

Our main result in Chapter 4 is stated as follows:

Theorem 1.2. *Let $K \in \mathscr{K}_0$ be a convex body and $\varphi \in \mathscr{I}$.*
(i) There exists a convex body $M \in \mathscr{K}_0$ such that $|M^\circ| = \omega_n$ and

$$C_{p,\varphi}(K, M) = \inf\left\{C_{p,\varphi}(K, L) : L \in \mathscr{K}_0 \textit{ and } |L^\circ| = \omega_n\right\}.$$

(ii) There exists a convex body $\widehat{M} \in \mathscr{K}_0$ such that $|\widehat{M}^\circ| = \omega_n$ and

$$\widehat{C}_{p,\varphi}(K, \widehat{M}) = \inf\left\{\widehat{C}_{p,\varphi}(K, L) : L \in \mathscr{K}_0 \textit{ and } |L^\circ| = \omega_n\right\}.$$

In addition, if $\varphi \in \mathscr{I}$ is convex, then both M and $\widehat{M}$ are unique.

The convex bodies M and $\widehat{M}$ in Theorem 1.2 are called the p-capacitary Orlicz-Petty bodies of K. The continuity of the p-capacitary Orlicz-Petty bodies is provided in Theorem 4.2. In Chapter 5, we propose the Orlicz geominimal p-capacities of K and provide a detailed study on their properties. For instance, we define the homogeneous Orlicz geominimal p-capacity of K with respect to $\mathscr{K}_0$ for $\varphi \in \mathscr{I}$ by $\widehat{\mathscr{G}}_{\varphi}^{orlicz}(K) = \widehat{C}_{p,\varphi}(K, \widehat{M})$, where $\widehat{M}$ is the convex body given in Theorem 1.2. Properties of $\widehat{\mathscr{G}}_{\varphi}^{orlicz}(K)$ are provided, such as the invariance under orthogonal matrices. In particular, we show the following inequality.

Theorem 1.3. *Let $K \in \mathscr{K}_0$ be a convex body with its Santaló point or centroid at the origin and B_K be an origin symmetric ball defined by $B_K = vrad(K)B_2^n$.*
(i) If $\varphi \in \mathscr{I}_0 \cup \mathscr{D}_0$, then

$$\frac{\widehat{\mathscr{A}}_{p,\varphi}^{orlicz}(K)}{\widehat{\mathscr{A}}_{p,\varphi}^{orlicz}(B_K)} \leq \frac{\widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(K)}{\widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(B_K)} \leq \frac{C_p(K)}{C_p(B_K)}.$$

Equality holds if K is an origin symmetric ball.
(ii) If $\varphi \in \mathscr{D}_1$, then there exists a universal constant $c > 0$ such that

$$\frac{\widehat{\mathscr{A}}_{p,\varphi}^{orlicz}(K)}{\widehat{\mathscr{A}}_{p,\varphi}^{orlicz}(B_K)} \geq \frac{\widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(K)}{\widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(B_K)} \geq c \cdot \frac{C_p(K)}{C_p(B_K)}.$$

Similarly, we could define $\mathscr{G}_{\varphi}^{orlicz}(K) = C_{p,\varphi}(K, M)$ where M is the convex body given in Theorem 1.2, and establish properties and inequalities similar to those for $\widehat{\mathscr{G}}_{\varphi}^{orlicz}(K)$. Special attention is paid on the case when $\varphi(t) = t^q$ for $-n \neq q \in \mathbb{R}$. Besides, we also investigate the p-capacitary Orlicz-Petty bodies of $\boldsymbol{K} = (K_1, \cdots, K_m)$, a vector of convex bodies, and establish analogous results for the L_q mixed geominimal p-capacity.

Chapter 2

Preliminaries and Notations

A subset $K \subseteq \mathbb{R}^n$ is said to be convex if $\lambda x + (1-\lambda)y \in K$ for any $\lambda \in [0,1]$ and $x, y \in K$. A convex body is a convex compact subset of $\mathbb{R}^n$ with nonempty interior. A convex body K is said to be origin-symmetric if $-x \in K$ for any $x \in K$. We use $\mathscr{K}$ and $\mathscr{K}_0 \subseteq \mathscr{K}$ to denote the set of all convex bodies and the set of all convex bodies with the origin in their interiors, respectively. The Minkowski sum of $K, L \in \mathscr{K}$, denoted by $K + L$, is defined by

$$K + L = \{x + y : x \in K, y \in L\}.$$

For $\lambda \in \mathbb{R}$, the scalar product of λ and K, denoted by λK, is defined by

$$\lambda K = \{\lambda x : x \in K\}.$$

For $K \in \mathscr{K}$, $|K|$ refers to the volume of K. In particular, ω_n represents the volume of the unit ball $B_2^n \subseteq \mathbb{R}^n$. For $K \in \mathscr{K}$, one can define the volume radius of K by

$$\mathrm{vrad}(K) = \left(\frac{|K|}{\omega_n}\right)^{\frac{1}{n}}.$$

For a $n \times n$ matrix ϕ, $\det \phi$ refers to the determinant of ϕ and ϕ^t refers to the transpose of ϕ. If $\det \phi \neq 0$, we use ϕ^{-1} to denote the inverse of ϕ. By an ellipsoid $\mathscr{E}$, we refer to a subset of $\mathbb{R}^n$ given by

$$\mathscr{E} = \phi B_2^n + x_0 = \{\phi x + x_0 : x \in B_2^n\},$$

where ϕ is an invertible $n \times n$ matrix on $\mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$ is some vector. Let $O(n)$ be the set of all $n \times n$ matrices such that $\phi\phi^t = \phi^t\phi = I_n$, where I_n is the identity matrix on $\mathbb{R}^n$.

The polar body K° of $K \in \mathscr{K}_0$ is defined by

$$K^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for any } y \in K\},$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$. By $K^{\circ\circ}$, we mean the polar body of K°, and it is well known that $K^{\circ\circ} = K$ if $K \in \mathscr{K}_0$ [57, Theorem 1.6.1]. For $K \in \mathscr{K}$ and $z \in \text{int}K$, the polar body of K with respect to z, denoted by K^z, is defined as

$$K^z = (K - z)^\circ + z.$$

It has been proved in [50] that there exists a unique point $z_0 \in \text{int}K$, such that,

$$|K^{z_0}| = \inf\{|K^z| : z \in \text{int}K\}.$$

This unique point $z_0 \in \text{int}K$, denoted by $s(K)$, is called the Santaló point of K. The famous Blaschke-Santaló inequality can be stated as follows: for any $K \in \mathscr{K}$,

$$|K| \cdot |K^{s(K)}| \leq \omega_n^2 \tag{2.1}$$

with equality if and only if K is an ellipsoid. On the other hand, the inverse Santaló inequality reads (see e.g. [5, 36, 54]): there exists a universal constant $c > 0$ such that for any $K \in \mathscr{K}$,

$$|K| \cdot |K^{s(K)}| \geq c^n \omega_n^2. \tag{2.2}$$

The support function of a nonempty convex compact $K \subseteq \mathbb{R}^n$, $h_K : S^{n-1} \to \mathbb{R}$, is defined by

$$h_K(u) = \max_{x \in K} \langle x, u \rangle \quad \text{for any } u \in S^{n-1},$$

where S^{n-1} is the unit sphere. It can be easily checked that, for any $\lambda \geq 0$ and $K, L \in \mathscr{K}$, $h_{\lambda K}(u) = \lambda h_K(u)$ and $h_{K+L}(u) = h_K(u) + h_L(u)$ for any $u \in S^{n-1}$. A subset $L \subseteq \mathbb{R}^n$ is called a star-shaped set about the origin if for any $x \in L$, the line segment from the origin o to x is contained in L. The radial function of a star-shaped set L about the origin o, $\rho_L : S^{n-1} \to [0, \infty)$, is defined by

$$\rho_L(u) = \max\{r \geq 0 : \ ru \in L\} \quad \text{for any } u \in S^{n-1}.$$

A star-shaped set $L \subseteq \mathbb{R}^n$ about the origin o is called a star body about the origin o if the radial function ρ_L is positive and continuous on S^{n-1}. Denote by $\mathscr{S}_0$ the set of all star bodies about the origin o. Obviously, $\mathscr{K}_0 \subseteq \mathscr{S}_0$. It can be proved in [57] that for any $K \in \mathscr{K}_0$,

$$\rho_{K^\circ}(u) = \frac{1}{h_K(u)} \quad \text{and} \quad h_{K^\circ}(u) = \frac{1}{\rho_K(u)} \quad \text{for any } u \in S^{n-1}. \tag{2.3}$$

For $K \in \mathscr{S}_0$, the following volume formula for $|K|$ holds:

$$|K| = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^n d\sigma(u), \tag{2.4}$$

where $\sigma(\cdot)$ is the spherical measure on S^{n-1}. Associated to each $K \in \mathscr{K}_0$, the surface area measure (a positive Borel measure) on S^{n-1} of K, denoted by $S(K,\cdot)$, is defined by: for any measurable subset $A \subseteq S^{n-1}$,

$$S(K,A) = \int_{\nu_K^{-1}(A)} d\mathscr{H}^{n-1},$$

where $\nu_K^{-1} : S^{n-1} \to \partial K$ is the inverse Gauss map [57] and $\mathscr{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure on ∂K. If the measure $S(K,\cdot)$ is absolutely continuous with respect to the spherical measure $\sigma(\cdot)$, by the Radon-Nikodym theorem, there is a function f_K: $S^{n-1} \to \mathbb{R}$, called the curvature function of K, such that, for any $u \in S^{n-1}$,

$$dS(K,u) = f_K(u)d\sigma(u).$$

By $\mathscr{F}_0^+$, we mean the subset of $\mathscr{K}_0$ defined by:

$$\mathscr{F}_0^+ = \{K \in \mathscr{K}_0 : f_K \text{ exists and is positively continuous on } S^{n-1}\}.$$

The Hausdorff distance between $K, L \in \mathscr{K}_0$, denoted by $d_H(K,L)$, is given by

$$d_H(K,L) = \max_{u \in S^{n-1}} |h_K(u) - h_L(u)|.$$

In fact, the Hausdorff distance can be defined on two compact sets [57, (1.60)], i.e., for compact sets $E, F \subseteq \mathbb{R}^n$,

$$d_H(E,F) = \min\{\lambda \geq 0 : E \subseteq F + \lambda B_2^n \text{ and } F \subseteq E + \lambda B_2^n\}.$$

For a sequence $\{K_i\}_{i=1}^{\infty} \subseteq \mathscr{K}_0$ and a convex compact set K, $K_i \to K$ as $i \to \infty$ with respect to the Hausdorff metric means that $d_H(K_i, K) \to 0$ as $i \to \infty$.

Denote by $C(S^{n-1})$ the set of all continuous functions on S^{n-1}. Let $\{\mu_i\}_{i=1}^{\infty}$ be a sequence of measures on S^{n-1} and μ also be a measure on S^{n-1}. We say μ_i converges weakly to μ if for any $f \in C(S^{n-1})$,

$$\lim_{i\to\infty} \int_{S^{n-1}} f d\mu_i = \int_{S^{n-1}} f d\mu.$$

The following variation formula of weak convergence $\mu_i \to \mu$ is often used.

Lemma 2.1. *If a sequence of measures $\{\mu_i\}_{i=1}^{\infty}$ on S^{n-1} converges weakly to a finite measure μ on S^{n-1} and a sequence of functions $\{f_i\}_{i=1}^{\infty} \subseteq C(S^{n-1})$ converges uniformly to a function $f \in C(S^{n-1})$, then*

$$\lim_{i\to\infty} \int_{S^{n-1}} f_i d\mu_i = \int_{S^{n-1}} f d\mu.$$

Proof. The weak convergence of $\mu_i \to \mu$ gives that for any $f \in C(S^{n-1})$,

$$\left| \int_{S^{n-1}} f d\mu_i - \int_{S^{n-1}} f d\mu \right| \to 0. \tag{2.5}$$

In particular, when $f(u) = 1$ for any $u \in S^{n-1}$, one gets

$$\left| \int_{S^{n-1}} 1 d\mu_i - \int_{S^{n-1}} 1 d\mu \right| = \left| \mu_i(S^{n-1}) - \mu(S^{n-1}) \right| \to 0. \tag{2.6}$$

Together with $\mu(S^{n-1}) < \infty$, one gets, for any i big enough, say $i \geq N_0$,

$$\mu_i(S^{n-1}) \leq \mu(S^{n-1}) + 1. \tag{2.7}$$

On the other hand, the uniform convergence of $f_i \to f$ gives that

$$\max_{u\in S^{n-1}} \left| f_i(u) - f(u) \right| \to 0.$$

Combining this with (2.5), (2.6) and (2.7), one has, for any $i \geq N_0$,

$$\begin{aligned}
&\left| \int_{S^{n-1}} f_i d\mu_i - \int_{S^{n-1}} f d\mu \right| \\
&\leq \left| \int_{S^{n-1}} f_i d\mu_i - \int_{S^{n-1}} f d\mu_i \right| + \left| \int_{S^{n-1}} f d\mu_i - \int_{S^{n-1}} f d\mu \right| \\
&\leq \max_{u \in S^{n-1}} \big| f_i(u) - f(u) \big| \cdot \big(\mu(S^{n-1}) + 1 \big) + \left| \int_{S^{n-1}} f d\mu_i - \int_{S^{n-1}} f d\mu \right| \\
&\to 0, \quad \text{as} \quad i \to \infty.
\end{aligned}$$

This gives the desired result. $\square$

We shall also need the following lemmas. The first one is the famous Blaschke selection theorem [57].

Lemma 2.2. *If $\{M_i\}_{i=1}^{\infty}$ is a bounded sequence of convex compact sets in $\mathbb{R}^n$, then there exist a subsequence $\{M_{i_k}\}_{k=1}^{\infty} \subseteq \{M_i\}_{i=1}^{\infty}$ and a convex compact set M, such that, $M_{i_k} \to M$ as $k \to \infty$ with respect to the Hausdorff metric.*

The second one is due to Lutwak [44].

Lemma 2.3. *Let $\{K_i\}_{i=1}^{\infty} \subseteq \mathscr{K}_0$ and K be a convex compact set. If $K_i \to K$ as $i \to \infty$ with respect to the Hausdorff metric and the sequence $\{|K_i^{\circ}|\}_{i=1}^{\infty}$ is bounded, then $K \in \mathscr{K}_0$.*

Capacity was introduced to study the small subsets of $\mathbb{R}^n$, since it could give more precise measurement than the Lebesgue measure. It is also a very useful tool in nonlinear theories, such as p-potential theory, and now we provide some basic background for the p-capacity, and all results can be found in e.g. [14, 15]. By $\operatorname{supp}(f)$, we mean the support set of $f : \mathbb{R}^n \to \mathbb{R}$, i.e.,

$$\operatorname{supp}(f) = \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}.$$

Let $C_c^\infty(\mathbb{R}^n)$ be the set of all infinitely differentiable functions on $\mathbb{R}^n$ with compact supports. For a bounded open set O, $C_c^\infty(O)$ refers to the set of all infinitely differentiable functions on $\mathbb{R}^n$ with their supports contained in O. For a compact set E and a bounded open set O with $E \subseteq O$, let

$$\mathscr{R}(E) = \{f : f \in C_c^\infty(\mathbb{R}^n) \text{ and } f(x) \geq 1 \text{ on } x \in E\};$$

$$\mathscr{R}(E, O) = \{f : f \in C_c^\infty(O) \text{ and } f(x) \geq 1 \text{ on } x \in E\}.$$

For $x \in \mathbb{R}^n$, we use $|x|$ to mean the Euclidean norm of x. For a compact subset $E \subseteq \mathbb{R}^n$ and $1 \leq p < n$, define $C_p(E)$, the p-capacity of E, by

$$C_p(E) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla f(x)|^p dx : f \in \mathscr{R}(E) \right\}.$$

When $p = 1$ and $K \in \mathscr{K}_0$, the 1-capacity of K is just the surface area of K [66]. When $p = 2$, the 2-capacity $C_2(E)$ is just the classical electrostatic capacity of E.

If O is a bounded open subset of $\mathbb{R}^n$ with $E \subseteq O$, one could define $C_p(E, O)$, the p-capacity of E relative to O, by

$$C_p(E, O) = \inf \left\{ \int_O |\nabla f(x)|^p dx : f \in \mathscr{R}(E, O) \right\}.$$

Clearly, $C_p(E, O) \geq C_p(E)$. Moreover, $E \subseteq F$ implies that $C_p(E) \leq C_p(F)$, $C_p(E, O) \leq C_p(F, O)$ with $E \subseteq F \subseteq O$ and $C_p(E, O_1) \geq C_p(E, O_2)$ for two bounded open sets O_1 and O_2 such that $E \subseteq O_1 \subseteq O_2$.

Lemma 2.4. *[14] Let E be a compact set and $\{O_i\}_{i=1}^\infty$ be a sequence of bounded open* *sets such that* $E \subseteq O_1 \subseteq O_2 \subseteq O_3 \subseteq \cdots \subseteq O_i \subseteq O_{i+1} \cdots$ *and* $\cup_{i=1}^{\infty} O_i = \mathbb{R}^n$. *Then*

$$\lim_{i\to\infty} C_p(E, O_i) = C_p(E).$$

Proof. As mentioned above, one has $C_p(E, O_i) \geq C_p(E)$ for any $i \geq 1$ and hence

$$\liminf_{i\to\infty} C_p(E, O_i) \geq C_p(E). \tag{2.8}$$

On the other hand, since $\{O_i\}_{i=1}^{\infty}$ is increasing and $\cup_{i=1}^{\infty} O_i = \mathbb{R}^n$, then for any function $g \in \mathscr{R}(E)$, there exists an integer N_g such that $\mathrm{supp}(g) \subseteq O_i$ for any $i \geq N_g$ and thus $g \in C_c^{\infty}(O_i)$ for any $i \geq N_g$. By the definition of $C_p(E, O_i)$, one has

$$C_p(E, O_i) \leq \int_{O_i} |\nabla g(x)|^p dx = \int_{\mathbb{R}^n} |\nabla g(x)|^p dx \quad \text{for any } i \geq N_g,$$

and thus

$$\limsup_{i\to\infty} C_p(E, O_i) \leq \int_{\mathbb{R}^n} |\nabla g(x)|^p dx.$$

Taking the infimum over $g \in \mathscr{R}(E)$, one gets $\limsup_{i\to\infty} C_p(E, O_i) \leq C_p(E)$. This, together with (2.8), gives $\lim_{i\to\infty} C_p(E, O_i) = C_p(E)$. $\square$

Now we state some basic properties of the p-capacity, please see [15, Chapter 4] for details.

Lemma 2.5. *Let E be a compact set and $p \in [1, n)$.*

(i) For any $\lambda > 0$,

$$C_p(\lambda E) = \lambda^{n-p} C_p(E).$$

(ii) For any $x_0 \in \mathbb{R}^n$,

$$C_p(E + x_0) = C_p(E).$$

(iii) For any $\phi \in O(n)$,

$$C_p(\phi E) = C_p(E).$$

(iv) The functional $C_p(\cdot)$ *is continuous on* $\mathscr{K}_0$ *with respect to the Hausdorff metric.*

Proof. (i) For a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, let $f_\lambda(x) = f(\lambda x)$. Clearly, $f \in \mathscr{R}(\lambda E)$ if and only if $f_\lambda \in \mathscr{R}(E)$. By the definition of $C_p(\cdot)$, one has

$$\begin{aligned}
C_p(\lambda E) &= \inf\left\{\int_{\mathbb{R}^n} |\nabla f(x)|^p dx : \; f \in \mathscr{R}(\lambda E)\right\} \\
&= \inf\left\{\int_{\mathbb{R}^n} |\nabla f(x)|^p dx : \; f_\lambda \in \mathscr{R}(E)\right\} \\
&= \lambda^{n-p} \cdot \inf\left\{\int_{\mathbb{R}^n} |\nabla f_\lambda(y)|^p dy : \; f_\lambda \in \mathscr{R}(E)\right\} \\
&= \lambda^{n-p} \cdot C_p(E).
\end{aligned}$$

(ii) Similarly, we define $f_{x_0}(x) = f(x+x_0)$ for a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ and hence $f \in \mathscr{R}(E + x_0)$ if and only if $f_{x_0} \in \mathscr{R}(E)$. By the definition of $C_p(\cdot)$, one has

$$\begin{aligned}
C_p(E + x_0) &= \inf\left\{\int_{\mathbb{R}^n} |\nabla f(x)|^p dx : \; f \in \mathscr{R}(E + x_0)\right\} \\
&= \inf\left\{\int_{\mathbb{R}^n} |\nabla f_{x_0}(y)|^p dy : \; f_{x_0} \in \mathscr{R}(E)\right\} \\
&= C_p(E).
\end{aligned}$$

(iii) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function and $f_\phi(x) = f(\phi x)$ with $\phi \in O(n)$. Hence, $f \in \mathscr{R}(\phi E)$ if and only if $f_\phi \in \mathscr{R}(E)$. Moreover, if $x = \phi y$, then $|\nabla f(x)| =$

$|\nabla f_\phi(y)|$. From the definition of $C_p(\cdot)$, one has

$$\begin{aligned} C_p(\phi E) &= \inf\left\{\int_{\mathbb{R}^n} |\nabla f(x)|^p dx : \ f \in \mathscr{R}(\phi E)\right\} \\ &= \inf\left\{\int_{\mathbb{R}^n} |\nabla f_\phi(y)|^p dy : \ f_\phi \in \mathscr{R}(E)\right\} \\ &= C_p(E). \end{aligned}$$

(iv) First of all, for $K \in \mathscr{K}_0$, $C_p(K) > 0$ (see e.g. [14, 66]). For any $\epsilon > 0$, choose two positive constants $\lambda > 1$ and $\rho > 0$ such that $(\lambda^{n-p} - 1) \cdot \lambda^{n-p} \cdot C_p(K) < \epsilon$ and $\rho B_2^n \subseteq K$. It follows from [57, Lemma 1.8.18] that there exists a positive number $\delta > 0$ such that $\delta \leq \rho(\lambda - 1)$ and $\rho B_2^n \subseteq \widetilde{K}$ when $d_H(K, \widetilde{K}) < \delta$. Thus,

$$K \subseteq \widetilde{K} + \delta B_2^n \subseteq \widetilde{K} + (\lambda - 1)\rho B_2^n \subseteq \widetilde{K} + (\lambda - 1)\widetilde{K} = \lambda \widetilde{K}.$$

This, together with the monotonicity and homogeneity of $C_p(\cdot)$, implies that

$$C_p(K) \leq C_p(\lambda \widetilde{K}) = \lambda^{n-p} \cdot C_p(\widetilde{K}).$$

Similarly, one has $\widetilde{K} \subseteq \lambda K$ and $C_p(\widetilde{K}) \leq \lambda^{n-p} \cdot C_p(K)$. Hence

$$\begin{aligned} C_p(K) - C_p(\widetilde{K}) \leq (\lambda^{n-p} - 1) \cdot C_p(\widetilde{K}) \leq (\lambda^{n-p} - 1) \cdot \lambda^{n-p} \cdot C_p(K); \\ C_p(\widetilde{K}) - C_p(K) \leq (\lambda^{n-p} - 1) \cdot C_p(K) \leq (\lambda^{n-p} - 1) \cdot \lambda^{n-p} \cdot C_p(K). \end{aligned}$$

Thus, one gets

$$|C_p(K) - C_p(\widetilde{K})| \leq (\lambda^{n-p} - 1) \cdot \lambda^{n-p} \cdot C_p(K) < \epsilon.$$

$\square$

In later context, we only consider $1 < p < n$. By the p-Laplace equation, we mean the equation

$$\mathrm{div}(|\nabla U|^{p-2}\nabla U) = 0.$$

The p-Laplace equation has a fundamental solution $U_0(x) = |x|^{\frac{p-n}{p-1}}(x \neq o)$. For $K \in \mathscr{K}_0$, the p-capacitary function of K is a weak solution of the following p-Laplace equation with the boundary conditions:

$$\begin{cases} \mathrm{div}(|\nabla U|^{p-2}\nabla U) = 0 & \text{in } \mathbb{R}^n \setminus K, \\ U(x) = 1 & \text{on } \partial K, \\ \lim_{|x|\to\infty} U(x) = 0. \end{cases} \tag{2.9}$$

It has been proved that there exists a unique solution U_K to (2.9). Moreover,

$$C_p(K) = \int_{\mathbb{R}^n\setminus K} |\nabla U_K(x)|^p dx$$

and $U_K \in C(\mathbb{R}^n \setminus \mathrm{int}K) \cap C^\infty(\mathbb{R}^n \setminus K)$ (see more details in [14]). In particular, the p-capacitary function of B_2^n is the fundamental solution to the p-Laplace equation, i.e., $U_{B_2^n}(x) = U_0(x) = |x|^{\frac{p-n}{p-1}}(x \neq o)$.

Lemma 2.6. *Let $K \in \mathscr{K}_0$ and U_K be the p-capacitary function of K.*

(i) The p-capacitary function of λK, for any $\lambda > 0$, is

$$U_{\lambda K}(x) = U_K\big(x/\lambda\big).$$

(ii) The p-capacitary function of $K + x_0$, for any $x_0 \in \mathbb{R}^n$, is

$$U_{K+x_0}(x) = U_K(x - x_0).$$

(iii) The p-capacitary function of ϕK, for any $\phi \in O(n)$, is

$$U_{\phi K}(x) = U_K(\phi^t x).$$

Proof. The proofs of the assertions (i)-(iii) are similar, and we only provide the proof of (iii) which requires the most work. For convenience, let $U_\phi(x) = U_K(\phi^t x)$ for any $x \in \mathbb{R}^n$. Note that $U_K(x) = 1$ on ∂K and $\lim_{|x|\to\infty} U_K(x) = 0$. Along with $\phi \in O(n)$, one gets $U_\phi(x) = U_K(\phi^t x) = 1$ on $\partial(\phi K)$ and $\lim_{|x|\to\infty} U_\phi(x) = \lim_{|x|\to\infty} U_K(\phi^t x) = 0$. Moreover, for any $x \in \mathbb{R}^n \setminus \phi K$,

$$\operatorname{div}(|\nabla U_\phi|^{p-2}\nabla U_\phi)(x) = \operatorname{div}(|\nabla U_K|^{p-2}\nabla U_K)(\phi^t x).$$

As U_K is the p-capacitary function of K, for any $x \in \mathbb{R}^n \setminus \phi K$, one has

$$\operatorname{div}(|\nabla U_\phi|^{p-2}\nabla U_\phi)(x) = \operatorname{div}(|\nabla U_K|^{p-2}\nabla U_K)(\phi^t x) = 0.$$

Thus U_ϕ is the p-capacitary function of ϕK, i.e., $U_{\phi K}(x) = U_\phi(x) = U_K(\phi^t x)$ for any $x \in \mathbb{R}^n$. $\square$

Let $K \in \mathscr{K}_0$. Define $\mu_p(K, \cdot)$, the p-capacitary measure on S^{n-1}, by

$$\mu_p(K, A) = \int_{\nu_K^{-1}(A)} |\nabla U_K(x)|^p d\mathscr{H}^{n-1}, \text{ for any measurable subset } A \subseteq S^{n-1}, \quad (2.10)$$

where U_K is the p-capacitary function of K. For any $\lambda > 0$, by Lemma 2.6 and $\nu_{\lambda K}^{-1}(\cdot) = \lambda \cdot \nu_K^{-1}(\cdot)$ on S^{n-1}, one can easily get

$$\mu_p(\lambda K, \cdot) = \lambda^{n-p-1}\mu_p(K, \cdot) \text{ on } S^{n-1}.$$

Moreover, for any $K \in \mathscr{K}_0$,

$$d\mu_p(K,u) = |\nabla U_K(\nu_K^{-1}(u))|^p dS(K,u) \quad \text{for any } u \in S^{n-1}. \tag{2.11}$$

In particular,

$$d\mu_p(B_2^n,u) = \left(\frac{n-p}{p-1}\right)^p d\sigma(u) \quad \text{for any } u \in S^{n-1}. \tag{2.12}$$

The translation invariance of $C_p(\cdot)$ yields that for any $K \in \mathscr{K}_0$, the centroid of $\mu_p(K,\cdot)$ is at the origin, i.e.,

$$\int_{S^{n-1}} u d\mu_p(K,u) = o.$$

It is also well known that $\mu_p(K,\cdot)$ is not concentrated on any hemisphere of S^{n-1} [67, Theorem 1], i.e.,

$$\int_{S^{n-1}} \langle v,u\rangle_+ \, d\mu_p(K,u) > 0 \quad \text{for any } v \in S^{n-1}.$$

The p-capacity of $K \in \mathscr{K}_0$ can be calculated by the famous Poincaré formula:

$$C_p(K) = \frac{p-1}{n-p}\int_{S^{n-1}} h_K(u)\, d\mu_p(K,u). \tag{2.13}$$

This together with (2.12) implies that for any $p \in (1,n)$,

$$C_p(B_2^n) = \left(\frac{n-p}{p-1}\right)^{p-1} \cdot n \cdot \omega_n. \tag{2.14}$$

From (2.13), for any $K \in \mathscr{K}_0$, one can define a probability measure $\mu_p^*(K,\cdot)$ on S^{n-1}: for any $u \in S^{n-1}$,

$$d\mu_p^*(K,u) = \frac{p-1}{n-p}\cdot\frac{h_K(u)}{C_p(K)}\cdot d\mu_p(K,u). \tag{2.15}$$

The following lemma is [49, (8.9)], which compares the volume with the p-capacity relative to a bounded open set O.

Lemma 2.7. *If* $1 < p < n$, $K \in \mathscr{K}_0$ *and* O *is a bounded open subset of* $\mathbb{R}^n$ *containing* K, *then*

$$C_p(K, O) \geq n{\omega_n}^{p/n} \left(\frac{n-p}{p-1}\right)^{p-1} |K|^{(n-p)/n}.$$

By Lemmas 2.4 and 2.7, one has the following isocapacitary inequality:

$$C_p(K) \geq n{\omega_n}^{p/n} \left(\frac{n-p}{p-1}\right)^{p-1} |K|^{(n-p)/n} \tag{2.16}$$

holds for any $K \in \mathscr{K}_0$ with equality if and only if K is a ball.

The p-capacity and the volume belong to a large family of functionals defined on $\mathscr{K}_0$. Such a family of functionals will be called the variational functionals compatible with the mixed volume [27] (see also [11, 12]). We summarize its definition below.

Definition 2.1. *A variational functional* $\mathscr{V} : \mathscr{K}_0 \to (0, \infty)$ *is said to be compatible with the mixed volume if* $\mathscr{V}$ *satisfies*

(i) homogeneous, i.e., there exists a constant $\alpha \neq 0$ *such that* $\mathscr{V}(\lambda K) = \lambda^\alpha \mathscr{V}(K)$ *for any* $\lambda > 0$ *and any* $K \in \mathscr{K}_0$;

(ii) translation invariant, i.e., $\mathscr{V}(K + x) = \mathscr{V}(K)$ *for any* $x \in \mathbb{R}^n$ *and any* $K \in \mathscr{K}_0$;

(iii) monotone increasing, i.e., $\mathscr{V}^{\frac{1}{\alpha}}(K) \leq \mathscr{V}^{\frac{1}{\alpha}}(L)$ *if* $K \subseteq L$;

(iv) the Brunn-Minkowski inequality, i.e., for any $\lambda \in [0, 1]$ *and* $K, L \in \mathscr{K}_0$,

$$\mathscr{V}^{\frac{1}{\alpha}}(\lambda K + (1-\lambda)L) \geq \lambda \mathscr{V}^{\frac{1}{\alpha}}(K) + (1-\lambda)\mathscr{V}^{\frac{1}{\alpha}}(L)$$

with equality if and only if K *and* L *are homothetic to each other;*

(v) there exists a measure $S_{\mathscr{V}}(K, \cdot)$ *on* S^{n-1} *such that for any* $L \in \mathscr{K}_0$,

$$\frac{1}{\alpha} \cdot \lim_{\epsilon \to 0^+} \frac{\mathscr{V}(K + \epsilon \cdot L) - \mathscr{V}(K)}{\epsilon} = \int_{S^{n-1}} h_L(u) dS_{\mathscr{V}}(K, u),$$

and $S_{\mathscr{V}}(K,\cdot)$ is not concentrated on any hemisphere of S^{n-1}. Moreover, the convergence of $K_i \to K$ with respect to the Hausdorff metric implies that $S_{\mathscr{V}}(K_i,\cdot)$ converges weakly to $S_{\mathscr{V}}(K,\cdot)$.

Besides the p-capacity and the volume, there are many other functionals on $\mathscr{K}_0$ satisfying conditions in Definition 2.1, such as $\tau(K)$, the torsional rigidity of K, whose definition is given by (see [11]):

$$\frac{1}{\tau(K)} = \inf\left\{\frac{\int_K |\nabla u(x)|^2 dx}{(\int_K |u(x)|dx)^2} \ \text{ s.t. } u \in W_0^{1,2}(\text{int}K) \text{ and } \int_K |u(x)|dx > 0\right\},$$

where $W^{1,2}(\text{int}K)$ refers to the Sobolev space of the functions in $L^2(\text{int}K)$ whose first order weak derivatives belong to $L^2(\text{int}K)$, and $W_0^{1,2}(\text{int}K)$ denotes the closure of $C_c^\infty(\text{int}K)$ in the Sobolev space $W^{1,2}(\text{int}K)$. By the definition of torsional rigidity, one can easily get that $\tau(K) \le \tau(L)$ if $K \subseteq L$. Moreover, for any $K \in \mathscr{K}_0$,

$$\tau(\lambda K) = \lambda^{n+2}\tau(K) \ \text{ for any } \lambda > 0 \ \text{ and } \ \tau(K+x) = \tau(K) \ \text{ for any } x \in \mathbb{R}^n.$$

The torsional rigidity satisfies the Brunn-Minkowski inequality, i.e., for any $\lambda \in [0,1]$ and $K, L \in \mathscr{K}_0$,

$$\tau^{\frac{1}{n+2}}(\lambda K + (1-\lambda)L) \ge \lambda \tau^{\frac{1}{n+2}}(K) + (1-\lambda)\tau^{\frac{1}{n+2}}(L)$$

with equality if and only if K and L are homothetic to each other; please refer to [11] for more details. For any $K \in \mathscr{K}_0$, there exists a unique solution $U_{\tau,K} \in C^\infty(\text{int}K) \cap C(K)$ to the following boundary value equation:

$$\begin{cases} \text{div}(\nabla U_{\tau,K}) = -2 & \text{in int}K, \\ U_{\tau,K}(x) = 0 & \text{on } \partial K. \end{cases}$$

In particular, when $K = B_2^n$, $U_{\tau,B_2^n}(x) = \dfrac{1-|x|^2}{n}$. Similar to $\mu_p(K,\cdot)$, one can define $\mu_\tau(K,\cdot)$, a nonnegative Borel measure on S^{n-1}, as follows (see [12]): for any measurable subset $A \subseteq S^{n-1}$,

$$\mu_\tau(K,A) = \int_{\nu_K^{-1}(A)} |\nabla U_{\tau,K}(x)|^2 d\mathscr{H}^{n-1}.$$

Then it follows from [12, Corollary 1] and [12, Theorem 6] that the measure $\mu_\tau(K,\cdot)$ satisfies the condition (v) in Definition 2.1.

Chapter 3

The polar Orlicz-Minkowski problems

Let $\varphi : (0,\infty) \to (0,\infty)$ be a continuous function. In this chapter, we consider the following problems.

The polar Orlicz-Minkowski problems: *under what condition on a nonzero finite measure μ and a function $\varphi : (0,\infty) \to (0,\infty)$, there exists a convex body $K \in \mathscr{K}_0$ such that K is an optimizer of the following optimization problem:*

$$\inf / \sup \left\{ \int_{S^{n-1}} \varphi\big(h_L\big) d\mu : L \in \mathscr{K}_0 \textit{ and } |L^\circ| = \omega_n \right\}. \tag{3.1}$$

The following theorem asserts that the polar Orlicz-Minkowski problem (3.1) is solvable under the assumptions $\varphi \in \mathscr{I}$ and $\mu \in \Omega$. For convenience, if $\varphi \in \mathscr{I}$, let

$$\widehat{\mathscr{G}_\varphi}(\mu) = \inf \left\{ \int_{S^{n-1}} \varphi\big(h_L\big) d\mu : L \in \mathscr{K}_0 \text{ and } |L^\circ| = \omega_n \right\}. \tag{3.2}$$

Clearly,

$$\widehat{\mathscr{G}_\varphi}(\mu) \le \int_{S^{n-1}} \varphi\big(h_{B_2^n}\big) d\mu \le \mu(S^{n-1}) < \infty.$$

Moreover, due to $|(\mathrm{vrad}(L^\circ)L)^\circ| = \left|\dfrac{L^\circ}{\mathrm{vrad}(L^\circ)}\right| = \omega_n$ and $h_{\mathrm{vrad}(L^\circ)L} = \mathrm{vrad}(L^\circ)h_L$ for any $L \in \mathscr{K}_0$, one has

$$\widehat{\mathscr{G}}_\varphi(\mu) = \inf_{L\in\mathscr{K}_0} \left\{ \int_{S^{n-1}} \varphi\big(\mathrm{vrad}(L^\circ)h_L\big)d\mu \right\}.$$

Theorem 3.1. *Let $\mu \in \Omega$ and $\varphi \in \mathscr{I}$. Then there exists a convex body $M \in \mathscr{K}_0$ such that $|M^\circ| = \omega_n$ and*

$$\widehat{\mathscr{G}}_\varphi(\mu) = \int_{S^{n-1}} \varphi\big(h_M\big)d\mu.$$

Moreover, if $\varphi \in \mathscr{I}$ is convex, then M is the unique solution to the polar Orlicz-Minkowski problem (3.2).

Proof. Let $\{M_i\}_{i=1}^\infty \subseteq \mathscr{K}_0$ be a sequence of convex bodies such that $|M_i^\circ| = \omega_n$ for any $i \geq 1$ and

$$\int_{S^{n-1}} \varphi\big(h_{M_i}\big)d\mu \to \widehat{\mathscr{G}}_\varphi(\mu) < \infty. \tag{3.3}$$

Let $R_i = \rho_{M_i}(u_i) = \max_{u\in S^{n-1}}\{\rho_{M_i}(u)\}$. Obviously, $h_{M_i}(u) \geq R_i \cdot \langle u, u_i\rangle_+$ for any $u \in S^{n-1}$ and any $i \geq 1$. Since S^{n-1} is compact, we can assume $u_i \to v \in S^{n-1}$ as $i \to \infty$. The fact that μ is not concentrated on any hemisphere of S^{n-1}, together with the monotone convergence theorem, implies that

$$\lim_{j\to\infty} \int_{\{u\in S^{n-1}:\,\langle u,v\rangle_+ \geq \frac{1}{j}\}} \langle u, v\rangle_+ \, d\mu(u) = \int_{S^{n-1}} \langle u, v\rangle_+ \, d\mu(u) > 0.$$

Thus, there exists an integer $j_0 \geq 1$ such that

$$\int_{\{u\in S^{n-1}:\,\langle u,v\rangle_+ \geq \frac{1}{j_0}\}} \langle u, v\rangle_+ \, d\mu(u) > 0. \tag{3.4}$$

We now prove $\sup_{i\geq 1} R_i < \infty$. This will follow if we can get a contradiction by assuming $R_i \to \infty$ as $i \to \infty$ or, more precisely, some subsequence $R_{i_j} \to \infty$ as $j \to \infty$.

By the monotonicity of φ, Fatou's Lemma and the fact that $h_{M_i}(u) \geq R_i \cdot \langle u, u_i \rangle_+$, one has, for any positive constant $C > 0$,

$$\begin{aligned}
\widehat{\mathscr{G}}_\varphi(\mu) &= \lim_{i\to\infty} \int_{S^{n-1}} \varphi\big(h_{M_i}\big) d\mu \\
&\geq \liminf_{i\to\infty} \int_{S^{n-1}} \varphi\big(R_i \cdot \langle u, u_i \rangle_+\big) d\mu(u) \\
&\geq \liminf_{i\to\infty} \int_{S^{n-1}} \varphi\big(C \cdot \langle u, u_i \rangle_+\big) d\mu(u) \\
&\geq \int_{S^{n-1}} \liminf_{i\to\infty} \varphi\big(C \cdot \langle u, u_i \rangle_+\big) d\mu(u) \\
&= \int_{S^{n-1}} \varphi\big(C \cdot \langle u, v \rangle_+\big) d\mu(u) \\
&\geq \varphi\left(\frac{C}{j_0}\right) \cdot \int_{\{u\in S^{n-1} :\, \langle u,v\rangle_+ \geq \frac{1}{j_0}\}} \langle u, v \rangle_+ \, d\mu(u).
\end{aligned}$$

Letting $C \to \infty$, one gets $\widehat{\mathscr{G}}_\varphi(\mu) \geq \infty$, which is impossible. Hence $\sup_{i\geq 1} R_i < \infty$ and $\{M_i\}_{i=1}^\infty$ is bounded.

By Lemmas 2.2 and 2.3, and $|M_i^\circ| = \omega_n$ for any $i \geq 1$, there exists a subsequence of $\{M_i\}_{i=1}^\infty$ which converges to some convex body $M \in \mathscr{K}_0$ with $|M^\circ| = \omega_n$. Without loss of generality, we assume $M_i \to M$ as $i \to \infty$. Thus there exist two positive constants r_0 and R_0 such that for any $i \geq 1$ and any $u \in S^{n-1}$,

$$r_0 \leq h_{M_i}(u), h_M(u) \leq R_0.$$

From the fact that φ is continuous on $[r_0, R_0]$ and the dominated convergence theorem, one has

$$\int_{S^{n-1}} \varphi\big(h_{M_i}\big) d\mu \to \int_{S^{n-1}} \varphi\big(h_M\big) d\mu.$$

Together with (3.3), one has

$$\widehat{\mathscr{G}}_\varphi(\mu) = \int_{S^{n-1}} \varphi\big(h_M\big)\, d\mu.$$

In other words, we prove that $M \in \mathscr{K}_0$ such that $|M^\circ| = \omega_n$ and

$$\widehat{\mathscr{G}}_\varphi(\mu) = \int_{S^{n-1}} \varphi\big(h_M\big)\, d\mu,$$

hence M is a solution to the polar Orlicz-Minkowski problem (3.2).

For the uniqueness, let M_1 and M_2 be two convex bodies such that $|M_1^\circ| = |M_2^\circ| = \omega_n$ and

$$\widehat{\mathscr{G}}_\varphi(\mu) = \int_{S^{n-1}} \varphi\big(h_{M_1}\big)\, d\mu = \int_{S^{n-1}} \varphi\big(h_{M_2}\big)\, d\mu.$$

Let $M_0 = \dfrac{M_1 + M_2}{2}$. Clearly, due to the fact that t^{-n} is strictly convex, (2.3) and (2.4), $\mathrm{vrad}(M_0^\circ) \leq 1$ with $\mathrm{vrad}(M_0^\circ) = 1$ if and only if $M_1 = M_2$. By the strict monotonicity of φ and the fact that φ is convex, one has

$$\begin{aligned}
\widehat{\mathscr{G}}_\varphi(\mu) &\leq \int_{S^{n-1}} \varphi\big(\mathrm{vrad}(M_0^\circ) \cdot h_{M_0}\big)\, d\mu \\
&\leq \int_{S^{n-1}} \varphi\big(h_{M_0}\big)\, d\mu \\
&= \int_{S^{n-1}} \varphi\left(\frac{h_{M_1} + h_{M_2}}{2}\right) d\mu \\
&\leq \int_{S^{n-1}} \frac{\varphi\big(h_{M_1}\big) + \varphi\big(h_{M_2}\big)}{2}\, d\mu \\
&= \widehat{\mathscr{G}}_\varphi(\mu).
\end{aligned}$$

This implies $\mathrm{vrad}(M_0^\circ) = 1$ and hence $M_1 = M_2$. $\square$

The following proposition states that the solutions to the polar Orlicz-Minkowski problem (3.2) for discrete measures must be polytopes.

Proposition 3.1. *Let $\varphi \in \mathscr{I}$ and $\mu \in \Omega$ be a discrete measure on S^{n-1} whose support*

$\{u_1, u_2, \cdots, u_m\} \subseteq S^{n-1}$ is not concentrated on any hemisphere of S^{n-1}. If $M \in \mathscr{K}_0$ is a solution of the polar Orlicz-Minkowski problem (3.2) for μ, then M is a polytope with $u_1, u_2, \cdots u_m$ being the unit normal vectors of its faces.

Proof. Let P be a polytope with $u_1, u_2, \cdots u_m$ being the unit normal vectors of its faces and circumscribing M. Thus, $h_P(u_i) = h_M(u_i)$ $(1 \le i \le m)$, $P^\circ \subseteq M^\circ$ and $\text{vrad}(P^\circ) \le \text{vrad}(M^\circ) = 1$. It follows from the fact that φ is strictly increasing that

$$\begin{aligned}
\inf_{L \in \mathscr{K}_0} \left\{ \int_{S^{n-1}} \varphi\big(\text{vrad}(L^\circ) h_L\big) d\mu \right\} &\le \int_{S^{n-1}} \varphi\big(\text{vrad}(P^\circ) h_P\big) d\mu \\
&\le \int_{S^{n-1}} \varphi\big(h_P\big) d\mu \\
&= \sum_{i=1}^{m} \varphi\left(h_P(u_i)\right) \cdot \mu(\{u_i\}) \\
&= \sum_{i=1}^{m} \varphi\left(h_M(u_i)\right) \cdot \mu(\{u_i\}) \\
&= \int_{S^{n-1}} \varphi\big(h_M\big) d\mu \\
&= \inf_{L \in \mathscr{K}_0} \left\{ \int_{S^{n-1}} \varphi\big(\text{vrad}(L^\circ) h_L\big) d\mu \right\}.
\end{aligned}$$

This shows that $\text{vrad}(P^\circ) = \text{vrad}(M^\circ) = 1$ and hence $M = P$. □

Note that if $\varphi \in \mathscr{I}$ is not convex, then the solution to the polar Orlicz-Minkowski problem (3.2) may not be unique. We use $\mathscr{M}_\varphi(\mu)$ for the set of all convex bodies satisfying the polar Orlicz-Minkowski problem (3.2) for $\mu \in \Omega$. When $\varphi \in \mathscr{I}$ is convex, $\mathscr{M}_\varphi(\mu)$ contains only one convex body.

The following theorem states the continuity of $\widehat{\mathscr{G}}_\varphi(\cdot)$ and $\mathscr{M}_\varphi(\cdot)$.

Theorem 3.2. *Let $\{\mu_i\}_{i=1}^\infty \subseteq \Omega$ and $\mu \in \Omega$ be such that μ_i converges weakly to μ as $i \to \infty$.*

(i) If $\varphi \in \mathscr{I}$, then $\widehat{\mathscr{G}}_\varphi(\mu_i) \to \widehat{\mathscr{G}}_\varphi(\mu)$.

(ii) If $\varphi \in \mathscr{I}$ is convex, then $\mathscr{M}_\varphi(\mu_i) \to \mathscr{M}_\varphi(\mu)$.

Proof. Let $M \in \mathscr{M}_\varphi(\mu)$ and $M_i \in \mathscr{M}_\varphi(\mu_i)$ be convex bodies such that $|M^\circ| = |M_i^\circ| = \omega_n$ for any $i \geq 1$,

$$\widehat{\mathscr{G}}_\varphi(\mu) = \int_{S^{n-1}} \varphi\big(h_M\big) d\mu \quad \text{and} \quad \widehat{\mathscr{G}}_\varphi(\mu_i) = \int_{S^{n-1}} \varphi\big(h_{M_i}\big) d\mu_i.$$

The weak convergence of $\mu_i \to \mu$ yields

$$\begin{aligned}
\widehat{\mathscr{G}}_\varphi(\mu) &= \int_{S^{n-1}} \varphi\big(h_M\big) d\mu \\
&= \lim_{i\to\infty} \int_{S^{n-1}} \varphi\big(h_M\big) d\mu_i \\
&= \limsup_{i\to\infty} \int_{S^{n-1}} \varphi\big(h_M\big) d\mu_i \\
&\geq \limsup_{i\to\infty} \widehat{\mathscr{G}}_\varphi(\mu_i).
\end{aligned} \tag{3.5}$$

Let R_i, u_i and v be as in the proof of Theorem 3.1, i.e., for any $u \in S^{n-1}$,

$$R_i = \rho_{M_i}(u_i) = \max_{u \in S^{n-1}} \{\rho_{M_i}(u)\}, \quad u_i \to v \quad \text{and} \quad h_{M_i}(u) \geq R_i \cdot \langle u, u_i \rangle_+.$$

Assume $\sup_{i\geq 1} R_i = \infty$, and without loss of generality, let $R_i \to \infty$. Since μ is not concentrated on any hemisphere of S^{n-1}, there exists an integer j_0 such that (3.4) holds. By the weak convergence of $\mu_i \to \mu$, (3.5) and Lemma 2.1, one gets, for any positive constant $C > 0$,

$$\begin{aligned}
\widehat{\mathscr{G}}_\varphi(\mu) &= \int_{S^{n-1}} \varphi\big(h_M\big) d\mu \\
&= \lim_{i\to\infty} \int_{S^{n-1}} \varphi\big(h_M\big) d\mu_i \\
&\geq \liminf_{i\to\infty} \int_{S^{n-1}} \varphi\big(h_{M_i}\big) d\mu_i
\end{aligned}$$

$$
\begin{aligned}
&\geq \liminf_{i\to\infty} \int_{S^{n-1}} \varphi\big(R_i \cdot \langle u, u_i\rangle_+\big) d\mu_i(u) \\
&\geq \liminf_{i\to\infty} \int_{S^{n-1}} \varphi\big(C \cdot \langle u, u_i\rangle_+\big) d\mu_i(u) \\
&= \int_{S^{n-1}} \varphi\big(C \cdot \langle u, v\rangle_+\big) d\mu(u) \\
&\geq \varphi\left(\frac{C}{j_0}\right) \cdot \int_{\{u\in S^{n-1}\colon \langle u,v\rangle_+ \geq \frac{1}{j_0}\}} \langle u, v\rangle_+ \, d\mu(u).
\end{aligned}
$$

This yields a contradiction $\widehat{\mathscr{G}}_\varphi(\mu) \geq \infty$ if we let $C \to \infty$. Therefore $\sup_{i\geq 1} R_i < \infty$ and hence $\{M_i\}_{i=1}^\infty$ is bounded.

Let $\{M_{i_k}\}_{k=1}^\infty$ be any subsequence of $\{M_i\}_{i=1}^\infty$. By the boundedness of $\{M_{i_k}\}_{k=1}^\infty$, Lemmas 2.2 and 2.3, and $|M_{i_k}^\circ| = \omega_n$ for any $k \geq 1$, one can find a subsequence $\{M_{i_{k_j}}\}_{j=1}^\infty$ of $\{M_{i_k}\}_{k=1}^\infty$ and a convex body $M' \in \mathscr{K}_0$ such that $M_{i_{k_j}} \to M'$ as $j \to \infty$ and $|(M')^\circ| = \omega_n$. Moreover, $\varphi(h_{M_{i_{k_j}}}) \to \varphi(h_{M'})$ uniformly on S^{n-1}.

(i) Let $\{\mu_{i_k}\}_{k=1}^\infty \subseteq \{\mu_i\}_{i=1}^\infty$ be a subsequence such that

$$
\lim_{k\to\infty} \widehat{\mathscr{G}}_\varphi(\mu_{i_k}) = \liminf_{i\to\infty} \widehat{\mathscr{G}}_\varphi(\mu_i).
$$

By the argument above, there exist a subsequence $\{M_{i_{k_j}}\}_{j=1}^\infty$ of $\{M_{i_k}\}_{k=1}^\infty$ and a convex body $M' \in \mathscr{K}_0$ such that $M_{i_{k_j}} \to M'$ as $j \to \infty$ and $|(M')^\circ| = \omega_n$. Thus, by Lemma 2.1, one has

$$
\begin{aligned}
\liminf_{i\to\infty} \widehat{\mathscr{G}}_\varphi(\mu_i) &= \lim_{j\to\infty} \widehat{\mathscr{G}}_\varphi(\mu_{i_{k_j}}) \\
&= \lim_{j\to\infty} \int_{S^{n-1}} \varphi\big(h_{M_{i_{k_j}}}\big) d\mu_{i_{k_j}} \\
&= \int_{S^{n-1}} \varphi\big(h_{M'}\big) d\mu \\
&\geq \widehat{\mathscr{G}}_\varphi(\mu).
\end{aligned}
$$

Together with (3.5), one has $\widehat{\mathscr{G}}_\varphi(\mu_i) \to \widehat{\mathscr{G}}_\varphi(\mu)$ as $i \to \infty$.

(ii) Let $\{M_{i_k}\}_{k=1}^{\infty}$ be any subsequence of $\{M_i\}_{i=1}^{\infty}$. The weak convergence of $\mu_{i_k} \to \mu$, along with part(i) above, implies

$$\widehat{\mathscr{G}}_{\varphi}(\mu) = \lim_{k\to\infty} \widehat{\mathscr{G}}_{\varphi}(\mu_{i_k}) = \lim_{k\to\infty} \int_{S^{n-1}} \varphi\big(h_{M_{i_k}}\big) d\mu_{i_k}.$$

Again, $\{M_{i_k}\}_{k=1}^{\infty}$ is uniformly bounded with $|M_{i_k}^{\circ}| = \omega_n$ for any $k \geq 1$. There exist a subsequence $\{M_{i_{k_j}}\}_{j=1}^{\infty}$ of $\{M_{i_k}\}_{k=1}^{\infty}$ and a convex body $M' \in \mathscr{K}_0$ such that $M_{i_{k_j}} \to M'$ as $j \to \infty$ and $|(M')^{\circ}| = \omega_n$. By Lemma 2.1, one has

$$\widehat{\mathscr{G}}_{\varphi}(\mu) = \lim_{j\to\infty} \widehat{\mathscr{G}}_{\varphi}(\mu_{i_{k_j}}) = \lim_{j\to\infty} \int_{S^{n-1}} \varphi\big(h_{M_{i_{k_j}}}\big) d\mu_{i_{k_j}} = \int_{S^{n-1}} \varphi\big(h_{M'}\big) d\mu.$$

The uniqueness in Theorem 3.1 yields $M = M'$. Consequently, $M_{i_{k_j}} \to M$ as $j \to \infty$. In summary, we prove that any subset of $\{M_i\}_{i=1}^{\infty}$ has a subsequent convergent to M, and then $M_i \to M$ as $i \to \infty$ with respect to the Hausdorff metric. $\square$

Let $\mathscr{D}$ be the set of continuous functions $\varphi : (0, \infty) \to (0, \infty)$ such that φ is strictly decreasing, $\lim_{t\to 0^+} \varphi(t) = \infty, \varphi(1) = 1$ and $\lim_{t\to\infty} \varphi(t) = 0$. The following proposition states that the polar Orlicz-Minkowski problem (3.1) might not be solvable in general for cases other than (3.2).

Proposition 3.2. *Let $\mu = \sum_{i=1}^{m} \lambda_i \delta_{u_i}$ with $\lambda_i > 0$ for any $1 \leq i \leq m$ be a given nonzero finite discrete measure whose support $\{u_1, u_2, \cdots, u_m\}$ is not concentrated on any hemisphere of S^{n-1}.*

(i) If $\varphi \in \mathscr{D}$ and the first coordinates of $u_1, u_2, \cdots u_m$ are all nonzero, then

$$\inf \left\{ \int_{S^{n-1}} \varphi\big(h_L\big) d\mu : L \in \mathscr{K}_0 \ \text{and} \ |L^{\circ}| = \omega_n \right\} = 0.$$

(ii) If $\varphi \in \mathscr{I} \cup \mathscr{D}$, then

$$\sup\left\{\int_{S^{n-1}}\varphi\big(h_L\big)d\mu : L\in\mathscr{K}_0 \ \mathit{and}\ |L^\circ|=\omega_n\right\}=\infty.$$

Proof. (i) Let $\alpha=\min_{1\le i\le m}\{|(u_i)_1|\}$ and $\alpha>0$ by assumption. For any $\epsilon>0$, let

$$\phi_\epsilon=\operatorname{diag}(1,1,\cdots,1,\epsilon^{-n}) \ \text{ and } \ L_\epsilon=\epsilon\cdot\phi_\epsilon B_2^n.$$

Thus $(L_\epsilon)^\circ=(\epsilon\cdot\phi_\epsilon^t)^{-1}B_2^n$ and $|L_\epsilon^\circ|=\omega_n$. Moreover, for $1\le i\le m$, one has

$$|\phi_\epsilon u_i|=\sqrt{(u_i)_1^2+(u_i)_2^2+\cdots+\epsilon^{-2n}(u_i)_n^2}\ge|(u_i)_1|\ge\alpha,$$

and

$$h_{L_\epsilon}(u_i)=\max_{v_1\in L_\epsilon}\langle v_1,u_i\rangle=\max_{v_2\in B_2^n}\langle\epsilon\cdot\phi_\epsilon v_2,u_i\rangle=\epsilon\cdot\max_{v_2\in B_2^n}\langle v_2,\phi_\epsilon u_i\rangle=\epsilon\cdot|\phi_\epsilon u_i|\ge\epsilon\cdot\alpha.$$

It follows from the fact φ is strictly decreasing that

$$\begin{aligned}\int_{S^{n-1}}\varphi\big(h_{L_\epsilon}\big)d\mu&=\sum_{i=1}^m\varphi\left(h_{L_\epsilon}(u_i)\right)\cdot\mu(\{u_i\})\\&\le\sum_{i=1}^m\varphi\left(\epsilon\cdot\alpha\right)\cdot\mu(\{u_i\})\\&=\varphi\left(\epsilon\cdot\alpha\right)\cdot\mu(S^{n-1}).\end{aligned}$$

Note that $\varphi(\epsilon)\to0$ as $\epsilon\to\infty$, and then

$$\inf\left\{\int_{S^{n-1}}\varphi\big(h_L\big)d\mu : L\in\mathscr{K}_0 \text{ and } |L^\circ|=\omega_n\right\}\le\varphi\left(\epsilon\cdot\alpha\right)\cdot\mu(S^{n-1})\to0 \ \text{ as } \epsilon\to\infty.$$

(ii) Without loss of generality, we assume $\mu(\{u_1\})>0$. By the Gram-Schmidt process, one could get an orthogonal matrix $T\in O(n)$ with its first column vector being u_1. For any $\epsilon > 0$, let

$$\phi_\epsilon = T \cdot \mathrm{diag}(\epsilon, \epsilon^{-1}, 1, 1, \cdots, 1) \cdot T^t \quad \text{and} \quad L_\epsilon = \phi_\epsilon B_2^n.$$

Then $|L_\epsilon^\circ| = \omega_n$, and

$$h_{L_\epsilon}(u_1) = \max_{v_1 \in L_\epsilon} \langle v_1, u_1 \rangle = \max_{v_2 \in B_2^n} \langle \phi_\epsilon v_2, u_1 \rangle = \max_{v_2 \in B_2^n} \langle v_2, \phi_\epsilon u_1 \rangle = \max_{v_2 \in B_2^n} \langle v_2, \epsilon u_1 \rangle = \epsilon.$$

Thus, one has

$$\int_{S^{n-1}} \varphi\big(h_{L_\epsilon}\big) d\mu = \sum_{i=1}^m \varphi\left(h_{L_\epsilon}(u_i)\right) \cdot \mu(\{u_i\}) \geq \varphi\left(h_{L_\epsilon}(u_1)\right) \cdot \mu(\{u_1\}) = \varphi\left(\epsilon\right) \cdot \mu(\{u_1\}).$$

For $\varphi \in \mathscr{I}$, letting $\epsilon \to \infty$, one gets

$$\sup\left\{ \int_{S^{n-1}} \varphi\big(h_L\big) d\mu : L \in \mathscr{K}_0 \text{ and } |L^\circ| = \omega_n \right\} = \infty,$$

while the desired result for $\varphi \in \mathscr{D}$ is obtained if we let $\epsilon \to 0$. $\square$

Let $\phi : (0, \infty) \to (0, \infty)$ be a continuous function and $K \in \mathscr{K}_0$, then $d\mu = \dfrac{1}{\phi(h_K)} dS(K, \cdot) \in \Omega$ is a nonzero finite measure on S^{n-1} which is not concentrated on any hemisphere of S^{n-1}. Theorem 3.1 yields that if $\varphi \in \mathscr{I}$, there exists a convex body $M \in \mathscr{K}_0$ such that $|M^\circ| = \omega_n$ and

$$nV_{\varphi,\phi}(K, M) = \int_{S^{n-1}} \frac{\varphi(h_M(u))}{\phi(h_K(u))} dS(K, u) = \inf\left\{ nV_{\varphi,\phi}(K, L) : L \in \mathscr{K}_0 \text{ and } |L^\circ| = \omega_n \right\}.$$

This is the polar analogue of the Orlicz-Minkowski problems studied in [21, 30, 37]. In particular, if $\varphi(t) = t^p$ and $\phi(t) = t^{p-1}$ for $p > 0$, it goes back to (1.4) and (1.8): the existence of the L_p Petty bodies.

Other examples include the measure $S_{\mathscr{V}}$ induced by the variational functional $\mathscr{V}$. For example, let $d\mu = \frac{1}{\phi(h_K)} d\mu_\tau(K,\cdot)$, which is not concentrated on any hemisphere of S^{n-1}. Theorem 3.1 implies the existence of a convex body $M \in \mathscr{K}_0$ such that $|M^\circ| = \omega_n$ and

$$\int_{S^{n-1}} \frac{\varphi(h_M(u))}{\phi(h_K(u))} d\mu_\tau(K,u) = \inf_{L\in\mathscr{K}_0} \left\{ \int_{S^{n-1}} \frac{\varphi(h_L(u))}{\phi(h_K(u))} d\mu_\tau(K,u) : |L^\circ| = \omega_n \right\}.$$

In particular, if $\varphi(t) = t^p$ and $\phi(t) = t^{p-1}$ for $p \geq 1$, one gets, similar to (1.4) and (1.8), the L_p torsional Petty body: i.e., $M \in \mathscr{K}_0$ such that $|M^\circ| = \omega_n$ and

$$\begin{aligned}\mu_{\tau,p}(K,M) &= \int_{S^{n-1}} \Big(\frac{h_M(u)}{h_K(u)}\Big)^p h_K(u) d\mu_\tau(K,u)\\ &= \inf\left\{ \int_{S^{n-1}} \Big(\frac{h_L(u)}{h_K(u)}\Big)^p h_K(u) d\mu_\tau(K,u) : L \in \mathscr{K}_0 \text{ and } |L^\circ| = \omega_n \right\}.\end{aligned}$$

Of course, one can also let $d\mu = \frac{1}{\phi(h_K)} d\mu_p(K,\cdot)$ and gets the similar results for the p-capacitary measure.

It is well known that $\mu \in \Omega$, i.e., μ is not concentrated on any hemisphere, is the minimal requirement for solutions to various Minkowski problems. For instance, for $p > 1$, it has been proved in [32] that if $\mu \in \Omega$, there exists a convex body K containing o (note that K may not be in $\mathscr{K}_0$ unless $p > n$) such that $|K| \cdot h_K^{p-1} d\mu = dS(K,\cdot)$. In this case, especially, if $K \in \mathscr{K}_0$, one can link μ to a convex body. However, it is not clear whether, in general, there exists a convex body $K \in \mathscr{K}_0$ such that $d\mu = c \cdot h_K^{1-p} dS(K,\cdot)$ for $p < 1$, see special cases in [21, 75, 76, 77, 78]. In other words, $\mu \in \Omega$, although closely related to convex bodies, is in fact more general than the measures generated from convex bodies. Consequently, the polar Orlicz-Minkowski problem is much more general than (1.4) and (1.8), and their direct extensions involving convex bodies.

Now let us discuss some dissimilarities between the Minkowski and the polar Minkowski problems. First of all, the solutions are always convex bodies in $\mathscr{K}_0$ for the polar Minkowski problem (3.2), while this may not be true for Minkowski problems as mentioned above. Secondly, as showed in Proposition 3.2, the solutions to the polar Minkowski problems for discrete measures usually do not exist, except in the case (3.2). However, as showed in a series of works, e.g. [77, 78], the solutions to the L_p Minkowski problems for discrete measures could be well-existed for all $p<0$. Finally, it seems intractable to find a direct relation between $\mu\in\Omega$ and the solutions to the polar Minkowski problems, while such a relation usually can be established as long as the solutions exist for the related Minkowski problems.

One can define $\|f\|_{L_\varphi(\mu)}$ as follows: for $\mu\in\Omega$ and $f: S^{n-1}\to\mathbb{R}$,

$$\|f\|_{L_\varphi(\mu)}=\inf\left\{\lambda>0:\int_{S^{n-1}}\varphi\Big(\frac{f}{\lambda}\Big)d\mu\leq\mu(S^{n-1})\right\}\ \text{ for }\ \varphi\in\mathscr{I},$$

$$\|f\|_{L_\varphi(\mu)}=\inf\left\{\lambda>0:\int_{S^{n-1}}\varphi\Big(\frac{f}{\lambda}\Big)d\mu\geq\mu(S^{n-1})\right\}\ \text{ for }\ \varphi\in\mathscr{D},$$

which have the positive homogeneity of degree 1, that is, $\|t\cdot f\|_{L_\varphi(\mu)}=t\cdot\|f\|_{L_\varphi(\mu)}$ for any $t>0$. One can easily check that for $L\in\mathscr{K}_0$, $\|h_L\|_{L_\varphi(\mu)}>0$ and

$$\int_{S^{n-1}}\varphi\Big(\frac{h_L}{\|h_L\|_{L_\varphi(\mu)}}\Big)d\mu=\mu(S^{n-1}).$$

Let $\{L_i\}_{i=1}^\infty\subseteq\mathscr{K}_0$ and $L\in\mathscr{K}_0$ be such that $L_i\to L$ with respect to the Hausdorff metric, then $\|h_{L_i}\|_{L_\varphi(\mu)}\to\|h_L\|_{L_\varphi(\mu)}$. This can be proved along the same lines as part (ii) of Proposition 4.1 in Chapter 4. Moreover, if $\varphi\in\mathscr{I}$, $\mu_i\to\mu$ weakly and there exists a positive constant $C>0$, such that, $\|h_{L_i}\|_{L_\varphi(\mu_i)}\leq C$ for any $i\geq1$, then $\{L_i\}_{i=1}^\infty$ is uniformly bounded; this can be proved along the same lines as Proposition 4.2.

For $\|h_L\|_{L_\varphi(\mu)}$, we can also ask the related polar Orlicz-Minkowski problem: Under what condition on φ and μ, there exists a convex body $M \in \mathscr{K}_0$ such that M is an optimizer of

$$\inf\big\{\|h_L\|_{L_\varphi(\mu)} : L \in \mathscr{K}_0 \text{ and } |L^\circ| = \omega_n\big\}, \tag{3.6}$$

$$\sup\big\{\|h_L\|_{L_\varphi(\mu)} : L \in \mathscr{K}_0 \text{ and } |L^\circ| = \omega_n\big\}. \tag{3.7}$$

The following theorem states that problem (3.6) is solvable for $\varphi \in \mathscr{I}$ and $\mu \in \Omega$. The proof follows along the same lines as Theorem 3.1 and will be omitted.

Theorem 3.3. *Let $\mu \in \Omega$ and $\varphi \in \mathscr{I}$. Then there exists a convex body $\widehat{M} \in \mathscr{K}_0$ such that $|\widehat{M}^\circ| = \omega_n$ and*

$$\|h_{\widehat{M}}\|_{L_\varphi(\mu)} = \inf\big\{\|h_L\|_{L_\varphi(\mu)} : L \in \mathscr{K}_0 \textit{ and } |L^\circ| = \omega_n\big\}.$$

Moreover, if $\varphi \in \mathscr{I}$ is convex, then $\widehat{M}$ is the unique solution to the polar Orlicz-Minkowski problem (3.6)*.*

Moreover, one can get arguments similar to Theorem 3.2. When μ is a discrete measure, part (i) of the following proposition states that the solutions to problem (3.6) for $\varphi \in \mathscr{I}$ are polytopes. However, part (ii), (iii) and (iv) show that the polar Orlicz-Minkowski problems (3.6) and (3.7) might not be solvable in general.

Proposition 3.3. *Let $\mu = \sum_{i=1}^m \lambda_i \delta_{u_i}$ with $\lambda_i > 0$ for any $1 \le i \le m$ be a given nonzero finite discrete measure whose support $\{u_1, u_2, \cdots, u_m\}$ is not concentrated on any hemisphere of S^{n-1}.*

(i) If $\varphi \in \mathscr{I}$ and $\widehat{M} \in \mathscr{K}_0$ is a solution to the polar Orlicz-Minkowski problem (3.6)*, then $\widehat{M}$ is a polytope with $u_1, u_2, \cdots u_m$ being the unit normal vectors of its faces.*

(ii) If $\varphi \in \mathscr{I}$*, then*

$$\sup\left\{\|h_L\|_{L_\varphi(\mu)} : L \in \mathscr{K}_0 \text{ and } |L^\circ| = \omega_n\right\} = \infty.$$

(iii) If $\varphi \in \mathscr{D}$*, then*

$$\inf\left\{\|h_L\|_{L_\varphi(\mu)} : L \in \mathscr{K}_0 \text{ and } |L^\circ| = \omega_n\right\} = 0.$$

(iv) If $\varphi \in \mathscr{D}$ *and the first coordinates of* $u_1, u_2, \cdots u_m$ *are all nonzeros, then*

$$\sup\left\{\|h_L\|_{L_\varphi(\mu)} : L \in \mathscr{K}_0 \text{ and } |L^\circ| = \omega_n\right\} = \infty.$$

Chapter 4

The p-capacitary Orlicz-Petty bodies

As indicated in Chapter 3, if $\varphi(t) = t^q$ and $d\mu = \frac{p-1}{n-p} h_K^{1-q} d\mu_p(K, \cdot)$, then Theorem 3.1 implies the existence of a convex body $M \in \mathscr{K}_0$, which will be called the p-capacitary L_q Petty body, such that $|M^\circ| = \omega_n$ and

$$\begin{aligned} C_{p,q}(K, M) &= \frac{p-1}{n-p} \int_{S^{n-1}} \left(\frac{h_M(u)}{h_K(u)} \right)^q h_K(u) d\mu_p(K, u) \\ &= \inf \left\{ \frac{p-1}{n-p} \int_{S^{n-1}} \left(\frac{h_L(u)}{h_K(u)} \right)^q h_K(u) d\mu_p(K, u) : L \in \mathscr{K}_0 \text{ and } |L^\circ| = \omega_n \right\}. \end{aligned}$$

This motivates our interest in studying the p-capacitary Orlicz-Petty bodies.

4.1 The nonhomogeneous and homogeneous Orlicz mixed p-capacities

For $\varphi \in \mathscr{I} \cup \mathscr{D}$, the nonhomogeneous L_φ Orlicz mixed p-capacity $C_{p,\varphi}(\cdot, \cdot)$ in (1.14) is introduced in [26]. When $\varphi(t) = t$, the mixed p-capacity was provided in [13].

Definition 4.1. *Let $\varphi \in \mathscr{I} \cup \mathscr{D}$, $p \in (1,n)$ and $K, L \in \mathscr{K}_0$. Define $C_{p,\varphi}(K,L)$, the L_φ Orlicz mixed p-capacity of K and L, by*

$$C_{p,\varphi}(K,L) = \frac{p-1}{n-p}\int_{S^{n-1}} \varphi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u) d\mu_p(K,u). \tag{4.1}$$

If $L \in \mathscr{S}_0$, we use $C_{p,\varphi}(K,L^\circ)$ for

$$C_{p,\varphi}(K,L^\circ) = \frac{p-1}{n-p}\int_{S^{n-1}} \varphi\left(\frac{1}{\rho_L(u)\cdot h_K(u)}\right) h_K(u) d\mu_p(K,u). \tag{4.2}$$

Note that φ in Definition 4.1 can be any continuous functions. However, the monotonicity of φ is crucial in later context so we only focus on $\varphi \in \mathscr{I} \cup \mathscr{D}$. We would like to mention that Hong, Ye and Zhang in [26] provided a geometric interpretation of the Orlicz mixed p-capacity of $K, L \in \mathscr{K}_0$. When $L \in \mathscr{S}_0$ is a convex body, then $C_{p,\varphi}(K,L^\circ)$ coincides with the one given by (4.1). Clearly, $C_{p,\varphi}(K,K) = C_p(K)$ for $\varphi \in \mathscr{I} \cup \mathscr{D}$. Moreover, for any $r > 0$,

$$C_{p,\varphi}(rB_2^n, B_2^n) = r^{n-p}\cdot\varphi\left(\frac{1}{r}\right)\cdot C_p(B_2^n);$$
$$C_{p,\varphi}(B_2^n, rB_2^n) = \varphi(r)\cdot C_p(B_2^n).$$

These imply that $C_{p,\varphi}(\cdot\,,\cdot\,)$ is nonhomogeneous on K and L, if φ is not a homogeneous function. The homogeneous analogue [26] is defined as follows.

Definition 4.2. *Let $\varphi \in \mathscr{I} \cup \mathscr{D}$, $p \in (1,n)$, and $K, L \in \mathscr{K}_0$. Define $\widehat{C}_{p,\varphi}(K,L)$, the homogeneous L_φ Orlicz mixed p-capacity of K and L, by*

$$\int_{S^{n-1}} \varphi\left(\frac{C_p(K)\cdot h_L(u)}{\widehat{C}_{p,\varphi}(K,L)\cdot h_K(u)}\right) d\mu_p^*(K,u) = 1, \tag{4.3}$$

where $\mu_p^*(K,\cdot)$ is the probability measure on S^{n-1} associated with $K \in \mathscr{K}_0$ given in

(2.15).

If $L \in \mathscr{S}_0$, then we use $\widehat{C}_{p,\varphi}(K, L^\circ)$ for

$$\int_{S^{n-1}} \varphi\left(\frac{C_p(K)}{\widehat{C}_{p,\varphi}(K, L^\circ) \cdot \rho_L(u) \cdot h_K(u)}\right) d\mu_p^*(K, u) = 1. \tag{4.4}$$

In fact, it can be easily checked that the following function

$$G(\eta) = \int_{S^{n-1}} \varphi\left(\frac{C_p(K) \cdot h_L(u)}{\eta \cdot h_K(u)}\right) d\mu_p^*(K, u)$$

is continuous, strictly monotonic on $(0, \infty)$ and the range of $G(\eta)$ is $(0, \infty)$. These imply that $\widehat{C}_{p,\varphi}(K, L)$ is well-defined. Thus, for any $K, L \in \mathscr{K}_0$, $\widehat{C}_{p,\varphi}(K, L) > 0$. In addition, as $\varphi(1) = 1$ and $\mu_p^*(K, \cdot)$ is a probability measure on S^{n-1}, then for any $K \in \mathscr{K}_0$, $\widehat{C}_{p,\varphi}(K, K) = C_p(K)$. Similar arguments hold for $\widehat{C}_{p,\varphi}(K, L^\circ)$. The following result for the homogeneity of $\widehat{C}_{p,\varphi}(\cdot, \cdot)$ follows immediately from (4.3) and (4.4).

Corollary 4.1. *Let $K, L \in \mathscr{K}_0$ and $s, t > 0$. If $\varphi \in \mathscr{I} \cup \mathscr{D}$, then*

$$\widehat{C}_{p,\varphi}(sK, tL) = s^{n-p-1} \cdot t \cdot \widehat{C}_{p,\varphi}(K, L).$$

When $L \in \mathscr{S}_0$, then

$$\widehat{C}_{p,\varphi}(sK, (tL)^\circ) = s^{n-p-1} \cdot t^{-1} \cdot \widehat{C}_{p,\varphi}(K, L^\circ).$$

The following proposition deals with the continuity of $C_{p,\varphi}(\cdot, \cdot)$ and $\widehat{C}_{p,\varphi}(\cdot, \cdot)$.

Proposition 4.1. *Let $\{K_i\}_{i=1}^\infty \subseteq \mathscr{K}_0$ and $\{L_i\}_{i=1}^\infty \subseteq \mathscr{K}_0$ be two sequences of convex bodies such that $K_i \to K \in \mathscr{K}_0$ and $L_i \to L \in \mathscr{K}_0$ as $i \to \infty$. If $\varphi \in \mathscr{I} \cup \mathscr{D}$, then*

$$C_{p,\varphi}(K_i, L_i) \to C_{p,\varphi}(K, L) \quad \textit{and} \quad \widehat{C}_{p,\varphi}(K_i, L_i) \to \widehat{C}_{p,\varphi}(K, L) \quad \textit{as} \;\; i \to \infty.$$

Proof. As $L_i \to L$, then $h_{L_i} \to h_L$ uniformly on S^{n-1}. Similarly, the convergence of $K_i \to K$ implies that $h_{K_i} \to h_K$ uniformly on S^{n-1}, $C_p(K_i) \to C_p(K)$ and $\mu_p(K_i, \cdot)$ converges weakly to $\mu_p(K, \cdot)$ (see [13]). In addition, there exist two constants $r, R > 0$, such that, for any $i \geq 1$

$$r \cdot B_2^n \subseteq K_i, K, L_i, L \subseteq R \cdot B_2^n, \tag{4.5}$$

and hence for any $i \geq 1$ and $u \in S^{n-1}$,

$$\frac{r}{R} \leq \frac{h_{L_i}(u)}{h_{K_i}(u)}, \frac{h_L(u)}{h_K(u)} \leq \frac{R}{r}. \tag{4.6}$$

Since φ is continuous on the interval $\left[\frac{r}{R}, \frac{R}{r}\right]$, then

$$\varphi\left(\frac{h_{L_i}(u)}{h_{K_i}(u)}\right) \to \varphi\left(\frac{h_L(u)}{h_K(u)}\right) \text{ uniformly on } S^{n-1}.$$

Together with Lemma 2.1, one gets

$$\frac{p-1}{n-p}\int_{S^{n-1}} \varphi\left(\frac{h_{L_i}(u)}{h_{K_i}(u)}\right) h_{K_i}(u) d\mu_p(K_i, u) \to \frac{p-1}{n-p}\int_{S^{n-1}} \varphi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u) d\mu_p(K, u),$$

and hence $C_{p,\varphi}(K_i, L_i) \to C_{p,\varphi}(K, L)$ as $i \to \infty$.

For the case $\widehat{C}_{p,\varphi}(\cdot, \cdot)$, we only prove the argument for $\varphi \in \mathscr{I}$, and the case $\varphi \in \mathscr{D}$ follows along the same argument. It follows from the monotonicity of $C_p(\cdot)$ and φ, (4.5) and (4.6) that

$$1 = \int_{S^{n-1}} \varphi\left(\frac{C_p(K_i) \cdot h_{L_i}(u)}{\widehat{C}_{p,\varphi}(K_i, L_i) \cdot h_{K_i}(u)}\right) d\mu_p^*(K_i, u) \leq \varphi\left(\frac{C_p(R \cdot B_2^n) \cdot R}{\widehat{C}_{p,\varphi}(K_i, L_i) \cdot r}\right);$$

$$1 = \int_{S^{n-1}} \varphi\left(\frac{C_p(K_i) \cdot h_{L_i}(u)}{\widehat{C}_{p,\varphi}(K_i, L_i) \cdot h_{K_i}(u)}\right) d\mu_p^*(K_i, u) \geq \varphi\left(\frac{C_p(r \cdot B_2^n) \cdot r}{\widehat{C}_{p,\varphi}(K_i, L_i) \cdot R}\right).$$

Combining with the fact that $\varphi(1) = 1$, one gets, for any $i \geq 1$,

$$0 < \frac{C_p(r \cdot B_2^n) \cdot r}{R} \leq \widehat{C}_{p,\varphi}(K_i, L_i) \leq \frac{C_p(R \cdot B_2^n) \cdot R}{r} < \infty.$$

Let

$$S = \limsup_{i\to\infty} \widehat{C}_{p,\varphi}(K_i, L_i) < \infty \quad \text{and} \quad I = \liminf_{i\to\infty} \widehat{C}_{p,\varphi}(K_i, L_i) > 0.$$

Thus there exists a subsequence $\left\{\widehat{C}_{p,\varphi}(K_{i_k}, L_{i_k})\right\}_{k=1}^{\infty}$ such that

$$\frac{k}{k+1} S < \widehat{C}_{p,\varphi}(K_{i_k}, L_{i_k}) \;\text{ for any } k \geq 1 \text{ and } \lim_{k\to\infty} \widehat{C}_{p,\varphi}(K_{i_k}, L_{i_k}) = S.$$

These along with the fact that φ is increasing and Lemma 2.1 yield

$$\begin{aligned}
1 &= \lim_{k\to\infty} \int_{S^{n-1}} \varphi\left(\frac{C_p(K_{i_k}) \cdot h_{L_{i_k}}(u)}{\widehat{C}_{p,\varphi}(K_{i_k}, L_{i_k}) \cdot h_{K_{i_k}}(u)}\right) d\mu_p^*(K_{i_k}, u) \\
&\leq \lim_{k\to\infty} \int_{S^{n-1}} \varphi\left(\frac{(k+1) \cdot C_p(K_{i_k}) \cdot h_{L_{i_k}}(u)}{k \cdot S \cdot h_{K_{i_k}}(u)}\right) d\mu_p^*(K_{i_k}, u) \\
&= \int_{S^{n-1}} \varphi\left(\frac{C_p(K) \cdot h_L(u)}{S \cdot h_K(u)}\right) d\mu_p^*(K, u).
\end{aligned}$$

Similarly, there exists a sequence $\left\{\widehat{C}_{p,\varphi}(K_{i_l}, L_{i_l})\right\}_{l=1}^{\infty}$ such that

$$\frac{l+1}{l} I > \widehat{C}_{p,\varphi}(K_{i_l}, L_{i_l}) \;\text{ for any } l \geq 1 \text{ and } \lim_{l\to\infty} \widehat{C}_{p,\varphi}(K_{i_l}, L_{i_l}) = I.$$

Hence

$$1 = \lim_{l\to\infty} \int_{S^{n-1}} \varphi\left(\frac{C_p(K_{i_l}) \cdot h_{L_{i_l}}(u)}{\widehat{C}_{p,\varphi}(K_{i_l}, L_{i_l}) \cdot h_{K_{i_l}}(u)}\right) d\mu_p^*(K_{i_l}, u)$$

$$\geq \lim_{l\to\infty}\int_{S^{n-1}}\varphi\left(\frac{l\cdot C_p(K_{i_l})\cdot h_{L_{i_l}}(u)}{(l+1)\cdot I\cdot h_{K_{i_l}}(u)}\right)d\mu_p^*(K_{i_l},u)$$
$$= \int_{S^{n-1}}\varphi\left(\frac{C_p(K)\cdot h_L(u)}{I\cdot h_K(u)}\right)d\mu_p^*(K,u).$$

Together with Definition 4.2, one gets:

$$\limsup_{i\ \to\infty}\widehat{C}_{p,\varphi}(K_i,L_i)\leq\widehat{C}_{p,\varphi}(K,L)\leq\liminf_{i\to\infty}\widehat{C}_{p,\varphi}(K_i,L_i),$$

and hence $\widehat{C}_{p,\varphi}(K_i,L_i)\to\widehat{C}_{p,\varphi}(K,L)$ as $i\to\infty$ as desired. □

The following proposition is needed.

Proposition 4.2. *Let $\{K_i\}_{i=1}^\infty\subseteq\mathscr{K}_0$ and $K\in\mathscr{K}_0$ be such that $K_i\to K$ as $i\to\infty$. Let $\{M_i\}_{i=1}^\infty\subseteq\mathscr{K}_0$ and $\varphi\in\mathscr{I}$ be such that $\{C_{p,\varphi}(K_i,M_i)\}_{i=1}^\infty$ or $\{\widehat{C}_{p,\varphi}(K_i,M_i)\}_{i=1}^\infty$ is bounded. Then $\{M_i\}_{i=1}^\infty$ is uniformly bounded.*

Proof. As $K_i\to K$, then $h_{K_i}\to h_K$ uniformly on S^{n-1}, $C_p(K_i)\to C_p(K)$ and $\mu_p(K_i,\cdot)$ converges weakly to $\mu_p(K,\cdot)$. Again, one can find $r_0,R_0>0$ such that for any $i\geq 1$ and any $u\in S^{n-1}$,

$$r_0\leq h_K(u),\ h_{K_i}(u)\leq R_0.$$

Let $R_i=\rho_{M_i}(u_i)=\max_{u\in S^{n-1}}\{\rho_{M_i}(u)\}$. Thus $h_{M_i}(u)\geq R_i\cdot\langle u,u_i\rangle_+$ for any $u\in S^{n-1}$. As S^{n-1} is compact, without loss of generality, let $u_i\to v\in S^{n-1}$ as $i\to\infty$. Note that $\mu_p(K,\cdot)$ is not concentrated on any hemisphere of S^{n-1}. Hence,

$$0<\int_{S^{n-1}}\langle u,v\rangle_+\,d\mu_p(K,u)=\lim_{j\to\infty}\int_{\{u\in S^{n-1}:\,\langle u,v\rangle_+\geq\frac{1}{j}\}}\langle u,v\rangle_+\,d\mu_p(K,u).\qquad(4.7)$$

Thus there exists an integer $j_0\in N$ such that

$$\int_{\{u\in S^{n-1}:\,\langle u,v\rangle_+\geq\frac{1}{j_0}\}}\langle u,v\rangle_+\,d\mu_p(K,u)>0.$$

Suppose that M_i is not bounded uniformly, i.e., $\sup_{i\geq 1} R_i = \infty$. Without loss of generality, assume $R_i \to \infty$ as $i \to \infty$.

Firstly, we consider the case that $\{\widehat{C}_{p,\varphi}(K_i, M_i)\}_{i=1}^{\infty}$ is bounded. Then there exists a constant $B > 0$ such that $B \geq \widehat{C}_{p,\varphi}(K_i, M_i)$ for any $i \geq 1$. By (4.7), the monotonicity of φ and Lemma 2.1, for any constant $C > 0$, one has

$$
\begin{aligned}
1 &= \lim_{i\to\infty} \int_{S^{n-1}} \varphi\left(\frac{C_p(K_i)\cdot h_{M_i}(u)}{\widehat{C}_{p,\varphi}(K_i, M_i)\cdot h_{K_i}(u)}\right) d\mu_p^*(K_i, u) \\
&\geq \liminf_{i\to\infty} \int_{S^{n-1}} \varphi\left(\frac{C_p(K_i)\cdot R_i \cdot \langle u, u_i\rangle_+}{B\cdot R_0}\right) d\mu_p^*(K_i, u) \\
&\geq \liminf_{i\to\infty} \int_{S^{n-1}} \varphi\left(\frac{C_p(K_i)\cdot C \cdot \langle u, u_i\rangle_+}{B\cdot R_0}\right) d\mu_p^*(K_i, u) \\
&= \int_{S^{n-1}} \liminf_{i\to\infty} \varphi\left(\frac{C_p(K_i)\cdot C \cdot \langle u, u_i\rangle_+}{B\cdot R_0}\right) d\mu_p^*(K, u) \\
&= \int_{S^{n-1}} \varphi\left(\frac{C_p(K)\cdot C \cdot \langle u, v\rangle_+}{B\cdot R_0}\right) d\mu_p^*(K, u) \\
&= \int_{S^{n-1}} \varphi\left(\frac{C_p(K)\cdot C \cdot \langle u, v\rangle_+}{B\cdot R_0}\right) \frac{(p-1)\cdot h_K(u)}{(n-p)\cdot C_p(K)} d\mu_p(K, u) \\
&\geq \varphi\left(\frac{C_p(K)\cdot C}{B\cdot R_0\cdot j_0}\right) \frac{(p-1)\cdot r_0}{(n-p)\cdot C_p(K)} \cdot \int_{\{u\in S^{n-1}:\, \langle u,v\rangle_+ \geq \frac{1}{j_0}\}} \langle u, v\rangle_+ \, d\mu_p(K, u).
\end{aligned}
$$

A contradiction $1 \geq \infty$ is obtained if we let $C \to \infty$ and hence $\sup_{i\geq 1} R_i < \infty$.

Similarly, if $C_{p,\varphi}(K_i, M_i)\}_{i=1}^{\infty}$ is bounded, then there exists a positive constant $B > 0$ such that $B \geq C_{p,\varphi}(K_i, M_i)$ for any $i \geq 1$. Thus, for any given constant $C > 0$, one has

$$
\begin{aligned}
B &\geq \liminf_{i\to\infty} \frac{p-1}{n-p} \int_{S^{n-1}} \varphi\left(\frac{h_{M_i}(u)}{h_{K_i}(u)}\right) h_{K_i}(u) d\mu_p(K_i, u) \\
&\geq \liminf_{i\to\infty} \frac{p-1}{n-p} \int_{S^{n-1}} \varphi\left(\frac{C\cdot \langle u, u_i\rangle_+}{R_0}\right) h_{K_i}(u) d\mu_p(K_i, u) \\
&= \frac{p-1}{n-p} \int_{S^{n-1}} \varphi\left(\frac{C\cdot \langle u, v\rangle_+}{R_0}\right) h_K(u) d\mu_p(K, u)
\end{aligned}
$$

$$\geq \frac{p-1}{n-p} \cdot r_0 \cdot \varphi\left(\frac{C}{R_0 \cdot j_0}\right) \cdot \int_{\{u \in S^{n-1}:\, \langle u,v\rangle_+ \geq \frac{1}{j_0}\}} \langle u, v\rangle_+ \, d\mu_p(K, u).$$

A contradiction $B \geq \infty$ is obtained if we let $C \to \infty$ and hence $\sup_{i\geq 1} R_i < \infty$. $\square$

4.2 The p-capacitary Orlicz-Petty bodies

In this section, we will investigate the existence, uniqueness and continuity of the p-capacitary Orlicz-Petty bodies. Like the polar Orlicz-Minkowski problems in Chapter 3, we are interested in the following optimization problems for the homogeneous/nonhomoheneous L_φ Orlicz mixed p-capacity:

$$\sup / \inf \Big\{ C_{p,\varphi}(K, L) : L \in \mathscr{K}_0 \text{ and } |L^\circ| = \omega_n \Big\}; \tag{4.8}$$

$$\sup / \inf \Big\{ \widehat{C}_{p,\varphi}(K, L) : L \in \mathscr{K}_0 \text{ and } |L^\circ| = \omega_n \Big\}. \tag{4.9}$$

Our main result is the following theorem which establishes the solvability of (4.8) and (4.9) under certain conditions.

Theorem 4.1. *Let $K \in \mathscr{K}_0$ be a convex body and $\varphi \in \mathscr{I}$.*

(i) There exists a convex body $M \in \mathscr{K}_0$ such that $|M^\circ| = \omega_n$ and

$$C_{p,\varphi}(K, M) = \inf \Big\{ C_{p,\varphi}(K, L) : L \in \mathscr{K}_0 \textit{ and } |L^\circ| = \omega_n \Big\}.$$

(ii) There exists a convex body $\widehat{M} \in \mathscr{K}_0$ such that $|\widehat{M}^\circ| = \omega_n$ and

$$\widehat{C}_{p,\varphi}(K, \widehat{M}) = \inf \Big\{ \widehat{C}_{p,\varphi}(K, L) : L \in \mathscr{K}_0 \textit{ and } |L^\circ| = \omega_n \Big\}.$$

In addition, if $\varphi \in \mathscr{I}$ is convex, then both M and $\widehat{M}$ are unique.

Proof. For convenience, let

$$\mathscr{G}^{orlicz}_{p,\varphi}(K) = \inf\Big\{C_{p,\varphi}(K,L) : L \in \mathscr{K}_0 \text{ and } |L^\circ| = \omega_n\Big\}; \tag{4.10}$$

$$\widehat{\mathscr{G}}^{orlicz}_{p,\varphi}(K) = \inf\Big\{\widehat{C}_{p,\varphi}(K,L) : L \in \mathscr{K}_0 \text{ and } |L^\circ| = \omega_n\Big\}. \tag{4.11}$$

(i) Note that $\mathscr{G}^{orlicz}_{p,\varphi}(K) \le C_{p,\varphi}(K, B_2^n) < \infty$, due to (2.13) and Definition 4.1. Let $\{M_i\}_{i=1}^{\infty} \subseteq \mathscr{K}_0$ be an optimal sequence such that

$$C_{p,\varphi}(K, M_i) \to \mathscr{G}^{orlicz}_{p,\varphi}(K) \ \text{ and } \ |M_i^\circ| = \omega_n \text{ for any } i \ge 1.$$

By Proposition 4.2, one gets that $\{M_i\}_{i=1}^{\infty}$ is uniformly bounded. By Lemmas 2.2 and 2.3, and $|M_i^\circ| = \omega_n$ for any $i \ge 1$, one can find a subsequence $\{M_{i_k}\}_{k=1}^{\infty}$ of $\{M_i\}_{i=1}^{\infty}$ and $M \in \mathscr{K}_0$ such that $M_{i_k} \to M$ as $k \to \infty$ and $|M^\circ| = \omega_n$. Thus

$$\mathscr{G}^{orlicz}_{p,\varphi}(K) = \lim_{i\to\infty} C_{p,\varphi}(K, M_i) = \lim_{k\to\infty} C_{p,\varphi}(K, M_{i_k}) = C_{p,\varphi}(K, M).$$

The last identity is due to Proposition 4.1. So M is a solution to problem (4.8).

(ii) Following along the same lines, one gets a convex body $\widehat{M} \in \mathscr{K}_0$ such that $|\widehat{M}^\circ| = \omega_n$ and

$$\widehat{C}_{p,\varphi}(K, \widehat{M}) = \widehat{\mathscr{G}}^{orlicz}_{p,\varphi}(K) = \inf\Big\{\widehat{C}_{p,\varphi}(K,L) : L \in \mathscr{K}_0 \text{ and } |L^\circ| = \omega_n\Big\}.$$

Now we prove the uniqueness of M. Let M_1 and M_2 be two convex bodies such that $|M_1^\circ| = |M_2^\circ| = \omega_n$ and $\mathscr{G}^{orlicz}_{p,\varphi}(K) = C_{p,\varphi}(K, M_1) = C_{p,\varphi}(K, M_2)$. Let $M_0 = \dfrac{M_1 + M_2}{2}$ and $\mathrm{vrad}(M_0^\circ) \le 1$ with equality if and only if $M_1 = M_2$. The fact that φ is convex and strictly increasing implies

$$
\begin{aligned}
\mathscr{G}_{p,\varphi}^{orlicz}(K) &\leq C_{p,\varphi}(K, \mathrm{vrad}(M_0^\circ)\cdot M_0) \\
&= \frac{p-1}{n-p}\int_{S^{n-1}} \varphi\left(\frac{\mathrm{vrad}(M_0^\circ)\cdot h_{M_0}(u)}{h_K(u)}\right) h_K(u)d\mu_p(K,u) \\
&\leq \frac{p-1}{n-p}\int_{S^{n-1}} \varphi\left(\frac{h_{M_0}(u)}{h_K(u)}\right) h_K(u)d\mu_p(K,u) \\
&\leq \frac{p-1}{n-p}\int_{S^{n-1}} \left[\frac{1}{2}\varphi\left(\frac{h_{M_1}(u)}{h_K(u)}\right) h_K(u) + \frac{1}{2}\varphi\left(\frac{h_{M_2}(u)}{h_K(u)}\right) h_K(u)\right] d\mu_p(K,u) \\
&= \frac{C_{p,\varphi}(K,M_1) + C_{p,\varphi}(K,M_2)}{2} \\
&= \mathscr{G}_{p,\varphi}^{orlicz}(K).
\end{aligned}
$$

This implies $\mathrm{vrad}(M_0^\circ) = 1$ and hence $M_1 = M_2$.

For the uniqueness of $\widehat{M}$, let $\widehat{M_1}$ and $\widehat{M_2}$ be two convex bodies such that $|\widehat{M_1}^\circ| = |\widehat{M_2}^\circ| = \omega_n$ and $\widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(K) = \widehat{C}_{p,\varphi}(K, \widehat{M_1}) = \widehat{C}_{p,\varphi}(K, \widehat{M_2})$. Let $\widehat{M_0} = \dfrac{\widehat{M_1} + \widehat{M_2}}{2}$ and $\mathrm{vrad}(\widehat{M_0}^\circ) \leq 1$ with equality if and only if $\widehat{M_1} = \widehat{M_2}$. By the convexity of φ and the fact that $\widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(K) = \widehat{C}_{p,\varphi}(K, \widehat{M_1}) = \widehat{C}_{p,\varphi}(K, \widehat{M_2})$, one has

$$
\begin{aligned}
1 &= \int_{S^{n-1}} \frac{1}{2}\left[\varphi\left(\frac{C_p(K)\cdot h_{\widehat{M_1}}(u)}{\widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(K)\cdot h_K(u)}\right) + \varphi\left(\frac{C_p(K)\cdot h_{\widehat{M_2}}(u)}{\widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(K)\cdot h_K(u)}\right)\right] d\mu_p^*(K,u) \\
&\geq \int_{S^{n-1}} \varphi\left(\frac{C_p(K)\cdot \left(h_{\widehat{M_1}}(u) + h_{\widehat{M_2}}(u)\right)}{2\cdot \widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(K)\cdot h_K(u)}\right) d\mu_p^*(K,u) \\
&= \int_{S^{n-1}} \varphi\left(\frac{C_p(K)\cdot h_{\widehat{M_0}}(u)}{\widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(K)\cdot h_K(u)}\right) d\mu_p^*(K,u).
\end{aligned}
$$

By Definition 4.2 and monotonicity of φ, one obtains $\widehat{C}_{p,\varphi}(K, \widehat{M_0}) \leq \widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(K)$. Combining this with (4.11) and Corollary 4.1, one has

$$
\widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(K) \leq \widehat{C}_{p,\varphi}(K, \mathrm{vrad}(\widehat{M_0}^\circ)\cdot \widehat{M_0})
$$

$$
\begin{aligned}
&= \mathrm{vrad}(\widehat{M_0}^{\circ}) \cdot \widehat{C}_{p,\varphi}(K, \widehat{M_0}) \\
&\le \widehat{C}_{\varphi}(K, \widehat{M_0}) \\
&\le \widehat{\mathscr{G}}^{orlicz}_{p,\varphi}(K).
\end{aligned}
$$

This yields $\mathrm{vrad}(\widehat{M_0}^{\circ}) = 1$ and hence $M_1 = M_2$. □

Theorem 4.1 motivates the following definition of the p-capacitary Orlicz-Petty bodies.

Definition 4.3. *Let $K \in \mathscr{K}_0$ and $\varphi \in \mathscr{I}$. Define the set $\mathscr{T}_{p,\varphi}(K)$ to be the collection of all convex bodies M such that $|M^{\circ}| = \omega_n$ and*

$$C_{p,\varphi}(K, M) = \inf \Big\{ C_{p,\varphi}(K, L) : L \in \mathscr{K}_0 \text{ and } |L^{\circ}| = \omega_n \Big\}.$$

Similarly, let the set $\widehat{\mathscr{T}_{p,\varphi}}(K)$ be the collection of all convex bodies $\widehat{M}$ such that $|\widehat{M}^{\circ}| = \omega_n$ and

$$C_{p,\varphi}(K, \widehat{M}) = \inf \Big\{ \widehat{C}_{p,\varphi}(K, L) : L \in \mathscr{K}_0 \text{ and } |L^{\circ}| = \omega_n \Big\}.$$

A convex body $M \in \mathscr{T}_{p,\varphi}(K)$ is called a nonhomogeneous p-capacitary Orlicz-Petty body, and a convex body $\widehat{M} \in \widehat{\mathscr{T}_{p,\varphi}}(K)$ is called a homogeneous p-capacitary Orlicz-Petty body. Note when $\varphi \in \mathscr{I}$, both $\mathscr{T}_{p,\varphi}(K)$ and $\widehat{\mathscr{T}_{p,\varphi}}(K)$ are nonempty. When $\varphi \in \mathscr{I}$ is convex, $\mathscr{T}_{p,\varphi}(K)$ and $\widehat{\mathscr{T}_{p,\varphi}}(K)$ contain only one element. Again, if K is a polytope, then its p-capacity Orlicz-Petty bodies must be polytopes as well. That is the next proposition.

Proposition 4.3. *If $K \in \mathscr{K}_0$ is a polytope and $\varphi \in \mathscr{I}$, then $\mathscr{T}_{p,\varphi}(K)$ and $\widehat{\mathscr{T}_{p,\varphi}}(K)$ only contain polytopes with faces parallel to those of K.*

Proof. Let $M \in \widehat{\mathscr{T}_{p,\varphi}}(K)$ and P be a polytope with faces parallel to those of K and circumscribing M. As K is a polytope, its surface area measure $S(K, \cdot)$ must be concentrated on a finite subset $\{u_1, u_2, \cdots, u_m\} \subseteq S^{n-1}$. This, together with (2.10), implies that the p-capacitary measure $\mu_p(K, \cdot)$ is concentrated on $\{u_1, u_2, \cdots, u_m\}$ [26]. Moreover, $h_P(u_i) = h_M(u_i)$ $(1 \le i \le m)$. Thus, one has

$$\begin{aligned} 1 &= \int_{S^{n-1}} \varphi\left(\frac{C_p(K)\cdot h_P(u)}{\widehat{C}_{p,\varphi}(K,P)\cdot h_K(u)}\right) d\mu_p^*(K,u) \\ &= \frac{p-1}{n-p}\cdot\frac{1}{C_p(K)}\sum_{i=1}^m \varphi\left(\frac{C_p(K)\cdot h_P(u_i)}{\widehat{C}_{p,\varphi}(K,P)\cdot h_K(u_i)}\right)\cdot h_K(u_i)\cdot \mu_p(K,\{u_i\}) \\ &= \frac{p-1}{n-p}\cdot\frac{1}{C_p(K)}\sum_{i=1}^m \varphi\left(\frac{C_p(K)\cdot h_M(u_i)}{\widehat{C}_{p,\varphi}(K,P)\cdot h_K(u_i)}\right)\cdot h_K(u_i)\cdot \mu_p(K,\{u_i\}) \\ &= \int_{S^{n-1}} \varphi\left(\frac{C_p(K)\cdot h_M(u)}{\widehat{C}_{p,\varphi}(K,P)\cdot h_K(u)}\right) d\mu_p^*(K,u). \end{aligned}$$

This yields $\widehat{C}_{p,\varphi}(K,P) = \widehat{C}_{p,\varphi}(K,M)$. On the other hand, by (4.11) and Corollary 4.1, one gets

$$\widehat{C}_{p,\varphi}(K,P) = \widehat{C}_{p,\varphi}(K,M) = \widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(K) \le \mathrm{vrad}(P^\circ)\cdot \widehat{C}_{p,\varphi}(K,P).$$

This implies $\mathrm{vrad}(P^\circ) \ge 1$. Since P circumscribes M, then $P^\circ \subseteq M^\circ$ and $\mathrm{vrad}(P^\circ) \le \mathrm{vrad}(M^\circ) = 1$. Hence $|P^\circ| = |M^\circ|$ and then $M = P$.

Employing the same argument, one can prove that each $M \in \mathscr{T}_{p,\varphi}(K)$ is a polytope with faces parallel to those of K. $\square$

Theorem 4.1 implies that if $\varphi \in \mathscr{I}$ is convex, then both $\mathscr{T}_{p,\varphi}(K)$ and $\widehat{\mathscr{T}_{p,\varphi}}(K)$ only contain one element. Consequently, $\mathscr{T}_{p,\varphi} : \mathscr{K}_0 \to \mathscr{K}_0$ and $\widehat{\mathscr{T}_{p,\varphi}} : \mathscr{K}_0 \to \mathscr{K}_0$ define two operators on $\mathscr{K}_0$. The following theorem deals with the continuity of $\mathscr{T}_{p,\varphi}(\cdot)$, $\widehat{\mathscr{T}_{p,\varphi}}(\cdot)$, $\mathscr{G}_{p,\varphi}^{orlicz}(\cdot)$ and $\widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(\cdot)$.

Theorem 4.2. *Let $\varphi \in \mathscr{I}$ and $\{K_i\}_{i=1}^\infty \subseteq \mathscr{K}_0$ be a sequence converging to $K \in \mathscr{K}_0$. Then* *(i)* $\mathscr{G}_{p,\varphi}^{orlicz}(K_i) \to \mathscr{G}_{p,\varphi}^{orlicz}(K)$ *and* $\widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(K_i) \to \widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(K)$ *as* $i \to \infty$,

(ii) if, in addition, $\varphi \in \mathscr{I}$ *is convex,* $\mathscr{T}_{p,\varphi}(K_i) \to \mathscr{T}_{p,\varphi}(K)$ *and* $\widehat{\mathscr{T}_{p,\varphi}}(K_i) \to \widehat{\mathscr{T}_{p,\varphi}}(K)$ *as* $i \to \infty$.

Proof. (i) First of all, we prove $\mathscr{G}_{p,\varphi}^{orlicz}(K_i) \to \mathscr{G}_{p,\varphi}^{orlicz}(K)$ as $i \to \infty$. Let $M \in \mathscr{T}_{p,\varphi}(K)$ and $M_i \in \mathscr{T}_{p,\varphi}(K_i)$ for each $i \geq 1$. By part (i) of Proposition 4.1 and (4.10), one has

$$\begin{aligned} \mathscr{G}_{p,\varphi}^{orlicz}(K) &= C_{p,\varphi}(K, M) \\ &= \lim_{i\to\infty} C_{p,\varphi}(K_i, M) \\ &= \limsup_{i\to\infty} C_{p,\varphi}(K_i, M) \\ &\geq \limsup_{i\to\infty} \mathscr{G}_{p,\varphi}^{orlicz}(K_i). \end{aligned} \tag{4.12}$$

This implies that $\{\mathscr{G}_{p,\varphi}^{orlicz}(K_i)\}_{i=1}^{\infty}$ is bounded. It follows from Proposition 4.2 and $\mathscr{G}_{p,\varphi}^{orlicz}(K_i) = C_{p,\varphi}(K_i, M_i)$ for each $i \geq 1$ that $\{M_i\}_{i=1}^{\infty}$ is uniformly bounded. Let $\{K_{i_k}\}_{k=1}^{\infty} \subseteq \{K_i\}_{i=1}^{\infty}$ be a subsequence such that

$$\lim_{k\to\infty} \mathscr{G}_{p,\varphi}^{orlicz}(K_{i_k}) = \liminf_{i\to\infty} \mathscr{G}_{p,\varphi}^{orlicz}(K_i).$$

By the boundedness of $\{M_{i_k}\}_{k=1}^{\infty}$, Lemmas 2.2 and 2.3, and $|M_{i_k}^{\circ}| = \omega_n$ for any $k \geq 1$, there exist a subsequence $\{M_{i_{k_j}}\}_{j=1}^{\infty}$ of $\{M_{i_k}\}_{k=1}^{\infty}$ and $M' \in \mathscr{K}_0$ such that $M_{i_{k_j}} \to M'$ as $j \to \infty$ and $|(M')^{\circ}| = \omega_n$. Thus, Proposition 4.1 yields

$$\begin{aligned} \liminf_{i\to\infty} \mathscr{G}_{p,\varphi}^{orlicz}(K_i) &= \lim_{j\to\infty} \mathscr{G}_{p,\varphi}^{orlicz}(K_{i_{k_j}}) \\ &= \lim_{j\to\infty} C_{p,\varphi}(K_{i_{k_j}}, M_{i_{k_j}}) \\ &= C_{p,\varphi}(K, M') \\ &\geq \mathscr{G}_{p,\varphi}^{orlicz}(K). \end{aligned} \tag{4.13}$$

From (4.12) and (4.13), one concludes that

$$\mathscr{G}^{orlicz}_{p,\varphi}(K)=\lim_{i\to\infty}\mathscr{G}^{orlicz}_{p,\varphi}(K_i). \tag{4.14}$$

The assertion $\widehat{\mathscr{G}}^{orlicz}_{p,\varphi}(K_i)\to\widehat{\mathscr{G}}^{orlicz}_{p,\varphi}(K)$ can be proved in a similar manner.

(ii) Next we prove $\mathscr{T}_{p,\varphi}(K_i)\to\mathscr{T}_{p,\varphi}(K)$ when $\varphi\in\mathscr{I}$ is convex. In this case, by Theorem 4.1, $\mathscr{T}_{p,\varphi}(K)$ and $\mathscr{T}_{p,\varphi}(K_i)$ contain only one element which will be denoted by M and M_i for each $i\geq 1$. Let $\{M_{i_k}\}_{k=1}^{\infty}$ be any subsequence of $\{M_i\}_{i=1}^{\infty}$. By the convergence of $K_{i_k}\to K\in\mathscr{K}_0$ and (4.14), one has

$$\mathscr{G}^{orlicz}_{p,\varphi}(K)=\lim_{k\to\infty}\mathscr{G}^{orlicz}_{p,\varphi}(K_{i_k})=\lim_{k\to\infty}C_{p,\varphi}(K_{i_k},M_{i_k}). \tag{4.15}$$

Consequently, $\{C_{p,\varphi}(K_{i_k},M_{i_k})\}_{k=1}^{\infty}$ is uniformly bounded and then $\{M_{i_k}\}_{k=1}^{\infty}$ is bounded, due to Proposition 4.2. From Lemmas 2.2 and 2.3, and $|M_{i_k}^{\circ}|=\omega_n$ for any $k\geq 1$, there exist a subsequence $\{M_{i_{k_j}}\}_{j=1}^{\infty}$ of $\{M_{i_k}\}_{k=1}^{\infty}$ and a convex body $M'\in\mathscr{K}_0$ such that $M_{i_{k_j}}\to M'$ and $|(M')^{\circ}|=\omega_n$.

By part (i) of Proposition 4.1 and (4.15), one has

$$\mathscr{G}^{orlicz}_{p,\varphi}(K)=\lim_{j\to\infty}\mathscr{G}^{orlicz}_{p,\varphi}(K_{i_{k_j}})=\lim_{j\to\infty}C_{p,\varphi}(K_{i_{k_j}},M_{i_{k_j}})=C_{p,\varphi}(K,M').$$

Therefore $M=M'$, due to the uniqueness if $\varphi\in\mathscr{I}$ is convex. In other words, we have proved that every subsequence of $\{M_i\}_{i=1}^{\infty}$ has a convergent subsequence with limit of M. Thus $M_i\to M$ as $i\to\infty$ with respect to the Hausdorff metric.

Along the same lines, one can prove $\widehat{\mathscr{T}_{p,\varphi}}(K_i)\to\widehat{\mathscr{T}_{p,\varphi}}(K)$ as $i\to\infty$ under the condition that $\varphi\in\mathscr{I}$ is convex. $\square$

The following proposition can be proved by the techniques same as the proofs of Proposition 3.2. From this proposition, one sees that problems (4.8) and (4.9) may not be solvable in general except the case Theorem 4.1.

Proposition 4.4. *Let $K \in \mathscr{K}_0$ be a polytope and $S(K,\cdot)$ be its surface area measure on S^{n-1} which is concentrated on a finite subset $\{u_1, u_2, \cdots, u_m\} \subseteq S^{n-1}$.*
(i) If $\varphi \in \mathscr{I}$, then

$$\sup\left\{C_{p,\varphi}(K,L) : L \in \mathscr{K}_0 \text{ and } |L^\circ| = \omega_n\right\} = \infty.$$

$$\sup\left\{\widehat{C}_{p,\varphi}(K,L) : L \in \mathscr{K}_0 \text{ and } |L^\circ| = \omega_n\right\} = \infty.$$

(ii) If $\varphi \in \mathscr{D}$, then

$$\inf\left\{\widehat{C}_{p,\varphi}(K,L) : L \in \mathscr{K}_0 \text{ and } |L^\circ| = \omega_n\right\} = 0,$$

$$\sup\left\{C_{p,\varphi}(K,L) : L \in \mathscr{K}_0 \text{ and } |L^\circ| = \omega_n\right\} = \infty.$$

(iii) If $\varphi \in \mathscr{D}$ and some fixed jth ($1 \le j \le n$) coordinates of $u_1, u_2, \cdots, u_m$ are nonzero, then

$$\inf\left\{C_{p,\varphi}(K,L) : L \in \mathscr{K}_0 \text{ and } |L^\circ| = \omega_n\right\} = 0,$$

$$\sup\left\{\widehat{C}_{p,\varphi}(K,L) : L \in \mathscr{K}_0 \text{ and } |L^\circ| = \omega_n\right\} = \infty.$$

In fact, we can replace $|L^\circ|$ in problems (4.8) and (4.9) by $C_p(L^\circ)$ and consider the following optimization problems:

$$\sup / \inf\left\{\widehat{C}_{p,\varphi}(K,L) : L \in \mathscr{K}_0 \text{ and } C_p(L^\circ) = C_p(B_2^n)\right\}; \tag{4.16}$$

$$\sup / \inf\left\{C_{p,\varphi}(K,L) : L \in \mathscr{K}_0 \text{ and } C_p(L^\circ) = C_p(B_2^n)\right\}. \tag{4.17}$$

The following result can be obtained.

Theorem 4.3. *Let $K \in \mathscr{K}_0$ and $\varphi \in \mathscr{I}$.*

(i) There exists a convex body $\widehat{M} \in \mathscr{K}_0$ *such that* $C_p(\widehat{M}^\circ) = C_p(B_2^n)$ *and*

$$\widehat{\mathscr{H}_{p,\varphi}^{orlicz}}(K) = \inf\left\{\widehat{C}_{p,\varphi}(K,L) : L \in \mathscr{K}_0 \text{ and } C_p(L^\circ) = C_p(B_2^n)\right\} = \widehat{C}_{p,\varphi}(K,\widehat{M}).$$

Moreover, if $\{K_i\}_{i=1}^\infty \subseteq \mathscr{K}_0$ *satisfies* $K_i \to K$, *then* $\widehat{\mathscr{H}_{p,\varphi}^{orlicz}}(K_i) \to \widehat{\mathscr{H}_{p,\varphi}^{orlicz}}(K)$.

(ii) There exists a convex body $M \in \mathscr{K}_0$ *such that* $C_p(M^\circ) = C_p(B_2^n)$ *and*

$$\mathscr{H}_{p,\varphi}^{orlicz}(K) = \inf\left\{C_{p,\varphi}(K,L) : L \in \mathscr{K}_0 \text{ and } C_p(L^\circ) = C_p(B_2^n)\right\} = C_{p,\varphi}(K,M).$$

Moreover, if $\{K_i\}_{i=1}^\infty \subseteq \mathscr{K}_0$ *satisfies* $K_i \to K$, *then* $\mathscr{H}_{p,\varphi}^{orlicz}(K_i) \to \mathscr{H}_{p,\varphi}^{orlicz}(K)$.

Proof. (i) Let $\{M_i\}_{i=1}^\infty \subseteq \mathscr{K}_0$ be a sequence of convex bodies such that

$$\widehat{C}_{p,\varphi}(K,M_i) \to \widehat{\mathscr{H}_{p,\varphi}^{orlicz}}(K) \quad \text{and} \quad C_p({M_i}^\circ) = C_p(B_2^n) \text{ for any } i \geq 1.$$

As $\widehat{\mathscr{H}_{p,\varphi}^{orlicz}}(K) \leq \widehat{C}_{p,\varphi}(K,B_2^n) < \infty$, then $\{\widehat{C}_{p,\varphi}(K,M_i)\}_{i=1}^\infty$ is bounded. Proposition 4.2 implies that $\{M_i\}_{i=1}^\infty$ is uniformly bounded. By Lemma 2.2, there exist a subsequence $\{M_{i_k}\}_{k=1}^\infty$ of $\{M_i\}_{i=1}^\infty$ and a convex compact set $\widehat{M}$ such that $M_{i_k} \to \widehat{M}$ as $k \to \infty$. As $C_p(M_{i_k}^\circ) = C_p(B_2^n)$ for any $k \geq 1$, it follows from Lemma 2.3 and (2.16) that $\{|M_{i_k}^\circ|\}_{i=1}^\infty$ is bounded and then $\widehat{M} \in \mathscr{K}_0$. By Proposition 4.1 and Lemma 2.5 (iv), one has $C_p(\widehat{M}^\circ) = \lim_{k\to\infty} C_p(M_{i_k}^\circ) = C_p(B_2^n)$ and

$$\widehat{\mathscr{H}_{p,\varphi}^{orlicz}}(K) = \lim_{i\to\infty} \widehat{C}_{p,\varphi}(K,M_i) = \lim_{k\to\infty} \widehat{C}_{p,\varphi}(K,M_{i_k}) = \widehat{C}_{p,\varphi}(K,\widehat{M}).$$

Thus, $\widehat{M}$ is a solution to problem (4.16).

Let $\{\widehat{M_i}\}_{i=1}^{\infty}$ be a sequence of convex bodies such that

$$\widehat{\mathscr{H}}_{p,\varphi}^{orlicz}(K_i) = \widehat{C}_{p,\varphi}(K_i, \widehat{M_i}) \ \text{ and } \ C_p(\widehat{M_i}^{\circ}) = C_p(B_2^n) \ \text{ for any } i \geq 1.$$

By Proposition 4.1, one has

$$\begin{aligned}
\widehat{\mathscr{H}}_{p,\varphi}^{orlicz}(K) &= \widehat{C}_{p,\varphi}(K, \widehat{M}) \\
&= \lim_{i\to\infty} \widehat{C}_{p,\varphi}(K_i, \widehat{M}) \\
&= \limsup_{i\to\infty} \widehat{C}_{p,\varphi}(K_i, \widehat{M}) \\
&\geq \limsup_{i\to\infty} \widehat{\mathscr{H}}_{p,\varphi}^{orlicz}(K_i). \qquad (4.18)
\end{aligned}$$

This, together with Proposition 4.2, implies that $\{\widehat{M_i}\}_{i=1}^{\infty}$ is uniformly bounded. Let $\{K_{i_k}\}_{k=1}^{\infty} \subseteq \{K_i\}_{i=1}^{\infty}$ be a subsequence such that

$$\lim_{k\to\infty} \widehat{\mathscr{H}}_{p,\varphi}^{orlicz}(K_{i_k}) = \liminf_{i\to\infty} \widehat{\mathscr{H}}_{p,\varphi}^{orlicz}(K_i).$$

By the boundedness of $\{\widehat{M_{i_k}}\}_{k=1}^{\infty}$, Lemmas 2.2 and 2.3, together with (2.16) and $C_p(M_{i_k}^{\circ}) = C_p(B_2^n)$ for any $k \geq 1$, one can find a subsequence $\{\widehat{M_{i_{k_j}}}\}_{j=1}^{\infty}$ of $\{\widehat{M_{i_k}}\}_{k=1}^{\infty}$ and $\widehat{M_0} \in \mathscr{K}_0$ such that $\widehat{M_{i_{k_j}}} \to \widehat{M_0}$ as $j \to \infty$ and $C_p\big(\widehat{M_0}^{\circ}\big) = C_p(B_2^n)$. Thus, by Proposition 4.1 again, one gets

$$\begin{aligned}
\liminf_{i\to\infty} \widehat{\mathscr{H}}_{p,\varphi}^{orlicz}(K_i) &= \lim_{k\to\infty} \widehat{\mathscr{H}}_{p,\varphi}^{orlicz}(K_{i_k}) \\
&= \lim_{j\to\infty} \widehat{C}_{p,\varphi}(K_{i_{k_j}}, \widehat{M_{i_{k_j}}}) \\
&= \widehat{C}_{p,\varphi}(K, \widehat{M_0}) \\
&\geq \widehat{\mathscr{H}}_{p,\varphi}^{orlicz}(K).
\end{aligned}$$

Combining this with (4.18), one has $\widehat{\mathscr{H}}^{orlicz}_{p,\varphi}(K_i) \to \widehat{\mathscr{H}}^{orlicz}_{p,\varphi}(K)$. The case for (ii) follows along the same lines. $\square$

The p-capacitary measure $\mu_p(K,\cdot)$ in problems (4.8) and (4.9) could be replaced by the measure $S_{\mathscr{V}}(K,\cdot)$. In fact, in [27], Hong, Ye and Zhu proposed the following L_φ Orlicz mixed $\mathscr{V}$-measure of K and L:

$$\mathscr{V}_\varphi(K,L) = \int_{S^{n-1}} \varphi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u) dS_{\mathscr{V}}(K,u),$$

where $\varphi \in \mathscr{I} \cup \mathscr{D}$ and $K, L \in \mathscr{K}_0$. For $\varphi \in \mathscr{I} \cup \mathscr{D}$, one can define $\widehat{\mathscr{V}}_\varphi(K,L)$ by

$$\int_{S^{n-1}} \varphi\left(\frac{\mathscr{V}(K)\cdot h_L(u)}{\widehat{\mathscr{V}}_\varphi(K,L)\cdot h_K(u)}\right) h_K(u) dS_{\mathscr{V}}(K,u) = \mathscr{V}(K).$$

The following theorem can be proved similar to Theorem 4.1.

Theorem 4.4. *Let $K \in \mathscr{K}_0$ and $\varphi \in \mathscr{I}$. There exists a convex body $M \in \mathscr{K}_0$ such that $|M^\circ| = \omega_n$ and*

$$\mathscr{V}_\varphi(K,M) = \inf\left\{\mathscr{V}_\varphi(K,L) : L \in \mathscr{K}_0 \text{ and } |L^\circ| = \omega_n\right\}.$$

Moreover, if $\varphi \in \mathscr{I}$ is convex, then M is unique.

Similarly, there exists a convex body $\widehat{M} \in \mathscr{K}_0$ such that $|\widehat{M}^\circ| = \omega_n$ and

$$\widehat{\mathscr{V}}_\varphi(K,\widehat{M}) = \inf\left\{\widehat{\mathscr{V}}_\varphi(K,L) : L \in \mathscr{K}_0 \text{ and } |L^\circ| = \omega_n\right\}.$$

If $\varphi \in \mathscr{I}$ is convex, then $\widehat{M}$ is unique.

Besides, results in Proposition 4.3, Theorem 4.2 and Proposition 4.4 can be obtained for the case of variational functionals. We leave the details for readers.

4.3 The p-capacitary Orlicz-Petty bodies for multiple convex bodies

Let m be a positive integer and $\mathscr{Q}_0$ be a nonempty subset of $\mathscr{S}_0$. In the following, denote the cartesian product $\underbrace{\mathscr{Q}_0 \times \cdots \times \mathscr{Q}_0}_{m}$ by $(\mathscr{Q}_0)^m$. By $\boldsymbol{L} = (L_1, L_2, \cdots, L_m) \in (\mathscr{Q}_0)^m$, we mean that, for any $1 \le i \le m$, $L_i \in \mathscr{Q}_0$. Let $\boldsymbol{L}^\circ$ refer to the vector $(L_1^\circ, L_2^\circ, \cdots, L_m^\circ)$. Let $\boldsymbol{K_i} = (K_{i1}, K_{i2}, \cdots, K_{im})$ for any $i \ge 1$ and $\boldsymbol{K} = (K_1, K_2, \cdots, K_m)$. By $\boldsymbol{K_i} \to \boldsymbol{K}$ as $i \to \infty$ we mean that, for any $1 \le j \le m$, $K_{ij} \to K_j$ as $i \to \infty$. By $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots, \varphi_m) \in (\mathscr{I})^m$, we mean that each $\varphi_i \in \mathscr{I}$ for $i = 1, 2, \cdots, m$. Similarly, $\boldsymbol{\varphi} \in (\mathscr{D})^m$ means $\varphi_i \in \mathscr{D}$ for $i = 1, 2, \cdots, m$.

Definition 4.4. *Let $\boldsymbol{\varphi} \in \mathscr{I}^m$ or $\boldsymbol{\varphi} \in \mathscr{D}^m$, $\boldsymbol{K} \in (\mathscr{F}_0^+)^m$ and $\boldsymbol{L} \in (\mathscr{K}_0)^m$. The $L_{\boldsymbol{\varphi}}$ Orlicz mixed p-capacity of $\boldsymbol{K}$ and $\boldsymbol{L}$, denoted by $\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, \boldsymbol{L})$, is defined by*

$$\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, \boldsymbol{L}) = \frac{p-1}{n-p} \int_{S^{n-1}} \left(\prod_{i=1}^{m} \varphi_i \left(\frac{h_{L_i}(u)}{h_{K_i}(u)} \right) f_{K_i}^*(u) \right)^{\frac{1}{m}} d\sigma(u),$$

where $f_{K_i}^*(u) = h_{K_i}(u) \cdot |\nabla U_{K_i}(\nu_{K_i}^{-1}(u))|^p \cdot f_{K_i}(u)$, $\nu_{K_i}^{-1} : S^{n-1} \to \partial K_i$ is the inverse Gauss map of K_i, $f_{K_i}(u)$ is the curvature function of K_i and $U_{K_i}(u)$ is the p-capacitary function of K_i for any $1 \le i \le m$. If $\boldsymbol{L} \in (\mathscr{S}_0)^m$, then define $\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, \boldsymbol{L}^\circ)$ by

$$\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, \boldsymbol{L}^\circ) = \frac{p-1}{n-p} \int_{S^{n-1}} \left(\prod_{i=1}^{m} \varphi_i \left(\frac{1}{\rho_{L_i}(u) h_{K_i}(u)} \right) f_{K_i}^*(u) \right)^{\frac{1}{m}} d\sigma(u).$$

The continuity of $\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\cdot, \cdot)$ is stated as follows.

Proposition 4.5. *Let $\{\boldsymbol{K_i}\}_{i=1}^\infty \subseteq (\mathscr{F}_0^+)^m$ and $\{\boldsymbol{L_i}\}_{i=1}^\infty \subseteq (\mathscr{K}_0)^m$ be such that $\boldsymbol{K_i} \to \boldsymbol{K} \in (\mathscr{F}_0^+)^m$ and $\boldsymbol{L_i} \to \boldsymbol{L} \in (\mathscr{K}_0)^m$ as $i \to \infty$. If $\boldsymbol{\varphi} \in \mathscr{I}^m$ or $\in \mathscr{D}^m$ and $(\prod_{j=1}^m f_{K_{ij}})^{\frac{1}{m}}$ converges uniformly to $(\prod_{j=1}^m f_{K_j})^{\frac{1}{m}}$ on S^{n-1}, then $\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K_i}, \boldsymbol{L_i}) \to \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, \boldsymbol{L})$ as $i \to \infty$.*

Proof. For any $u \in S^{n-1}$, any $i \geq 1$ and any $1 \leq k \leq m$, let

$$\begin{aligned}
a_i(u) &= \left(\prod_{j=1}^{m} \varphi_j\left(\frac{h_{L_{ij}}(u)}{h_{K_{ij}}(u)}\right) \cdot h_{K_{ij}}(u) \cdot f_{K_{ij}}(u) \right)^{\frac{1}{m}}, \\
b_{i,k}(u) &= \left(\prod_{j=1}^{k} |\nabla U_{K_{ij}}\big({\nu_{K_{ij}}}^{-1}(u)\big)|^p \right)^{\frac{1}{m}}, \\
a(u) &= \left(\prod_{j=1}^{m} \varphi_j\left(\frac{h_{L_j}(u)}{h_{K_j}(u)}\right) \cdot h_{K_j}(u) \cdot f_{K_j}(u) \right)^{\frac{1}{m}}, \\
b_k(u) &= \left(\prod_{j=1}^{k} |\nabla U_{K_j}\big({\nu_{K_j}}^{-1}(u)\big)|^p \right)^{\frac{1}{m}}.
\end{aligned}$$

The convergences of $\boldsymbol{K_i} \to \boldsymbol{K}$ and $\boldsymbol{L_i} \to \boldsymbol{L}$ imply that, for any $1 \leq j \leq m$, $h_{K_{ij}} \to h_{K_j}$ and $h_{L_{ij}} \to h_{L_j}$ uniformly on S^{n-1}. As in the proof of Proposition 4.1, one sees

$$\left[\prod_{j=1}^{m} \varphi_j\left(\frac{h_{L_{ij}}(u)}{h_{K_{ij}}(u)}\right) \cdot h_{K_{ij}}(u) \right]^{\frac{1}{m}} \to \left[\prod_{j=1}^{m} \varphi_j\left(\frac{h_{L_j}(u)}{h_{K_j}(u)}\right) \cdot h_{K_j}(u) \right]^{\frac{1}{m}} \text{ uniformly on } \ S^{n-1}.$$

Together with the assumption that $(\prod_{j=1}^m f_{K_{ij}})^{\frac{1}{m}} \to (\prod_{j=1}^m f_{K_j})^{\frac{1}{m}}$ uniformly on S^{n-1}, one gets $a_i(u) \to a(u)$ uniformly on S^{n-1} and hence there exists a positive constant C_1, such that, $|a_i(u)| \leq C_1$ for any $i \geq 1$ and any $u \in S^{n-1}$. By [13, Lemmas 2.10 and 4.6], one has, for any $1 \leq j \leq m$,

$$\int_{S^{n-1}} \Big| \big|\nabla U_{K_{ij}}\big({\nu_{K_{ij}}}^{-1}(u)\big)\big|^p - \big|\nabla U_{K_j}\big({\nu_{K_j}}^{-1}(u)\big)\big|^p \Big| d\sigma(u) \to 0. \tag{4.19}$$

Moreover, there exist two positive constants C_2 (only dependent on $\boldsymbol{K}, n$ and p) and i_0 such that when $i \geq i_0$, for any $1 \leq j \leq m$,

$$\int_{S^{n-1}} |\nabla U_{K_{ij}}\big({\nu_{K_{ij}}}^{-1}(u)\big)|^p d\sigma(u) \leq C_2 \ \text{ and } \int_{S^{n-1}} |\nabla U_{K_j}\big({\nu_{K_j}}^{-1}(u)\big)|^p d\sigma(u) \leq C_2. \tag{4.20}$$

Note that $a_i(u) \cdot b_{i,m}(u) - a(u) \cdot b_m(u) = (a_i(u) - a(u)) \cdot b_m(u) + a_i(u) \cdot (b_{i,m}(u) - b_m(u))$. Hence, to prove $\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K_i}, \boldsymbol{L_i}) \to \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, \boldsymbol{L})$, it is enough to prove

$$\int_{S^{n-1}} (a_i(u) - a(u)) \cdot b_m(u) d\sigma(u) \to 0; \tag{4.21}$$

$$\int_{S^{n-1}} a_i(u) \cdot (b_{i,m}(u) - b_m(u)) d\sigma(u) \to 0. \tag{4.22}$$

By the uniform convergence of $a_i(u) \to a(u)$, together with (4.20) and Hölder inequality [24], one can easily get (4.21). As

$$\begin{aligned} & a_i(u) \cdot (b_{i,m}(u) - b_m(u)) \\ = {} & a_i(u) \cdot b_{i,m-1}(u) \Big(|\nabla U_{K_{im}}\big(\nu_{K_{im}}{}^{-1}(u)\big)|^{\frac{p}{m}} - |\nabla U_{K_m}\big(\nu_{K_m}{}^{-1}(u)\big)|^{\frac{p}{m}} \Big) \\ & + a_i(u) \cdot (b_{i,m-1}(u) - b_{m-1}(u)) |\nabla U_{K_m}\big(\nu_{K_m}{}^{-1}(u)\big)|^{\frac{p}{m}}, \end{aligned}$$

by the triangle inequality, $|a_i(u)| \le C_1$, inequality $|\sqrt[m]{a} - \sqrt[m]{b}| \le \sqrt[m]{|a-b|}$ for $a, b \ge 0$, Hölder inequality [24], and (4.19)-(4.20), one gets, for any $i \ge i_0$,

$$\begin{aligned} & \Big| \int_{S^{n-1}} a_i(u) \cdot (b_{i,m}(u) - b_m(u)) \big) d\sigma(u) \Big| \\ \le {} & C_1 \cdot C_2^{\frac{m-1}{m}} \cdot \bigg(\int_{S^{n-1}} \Big| \big|\nabla U_{K_{im}}\big(\nu_{K_{im}}{}^{-1}(u)\big)\big|^p - \big|\nabla U_{K_m}\big(\nu_{K_m}{}^{-1}(u)\big)\big|^p \Big| d\sigma(u) \bigg)^{\frac{1}{m}} \\ & + \Big| \int_{S^{n-1}} a_i(u) \cdot (b_{i,m-1}(u) - b_{m-1}(u)) |\nabla U_{K_m}\big(\nu_{K_m}{}^{-1}(u)\big)|^{\frac{p}{m}} d\sigma(u) \Big|. \end{aligned}$$

Repeating the process above, one gets

$$\begin{aligned} & \Big| \int_{S^{n-1}} a_i(u) \cdot (b_{i,m}(u) - b_m(u)) \big) d\sigma(u) \Big| \\ \le & \sum_{j=1}^{m} C_1 \cdot C_2^{\frac{m-1}{m}} \cdot \bigg(\int_{S^{n-1}} \Big| \big|\nabla U_{K_{ij}}\big(\nu_{K_{ij}}{}^{-1}(u)\big)\big|^p - \big|\nabla U_{K_j}\big(\nu_{K_j}{}^{-1}(u)\big)\big|^p \Big| d\sigma(u) \bigg)^{\frac{1}{m}} \to 0. \end{aligned}$$

Hence, (4.22) is also true and then $\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K_i}, \boldsymbol{L_i}) \to \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, \boldsymbol{L})$ as $i \to \infty$. $\square$

Similar to Theorem 4.1, the following theorem shows the existence of the p-capacitary Orlicz-Petty bodies for multiple convex bodies.

Theorem 4.5. *Let $\boldsymbol{K} \in (\mathscr{F}_0^+)^m$ and $\boldsymbol{\varphi} \in \mathscr{I}^m$. There exists a convex body $M \in \mathscr{K}_0$ such that $|M^\circ| = \omega_n$ and*

$$\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, M, \cdots, M) = \inf \Big\{ \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, L, \cdots, L) : L \in \mathscr{K}_0 \ \ \textit{and} \ \ |L^\circ| = \omega_n \Big\}.$$

Proof. For convenience, let

$$\boldsymbol{\mathscr{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}) = \inf \Big\{ \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, L, \cdots, L) : L \in \mathscr{K}_0 \ \ \text{and} \ \ |L^\circ| = \omega_n \Big\}.$$

Clearly, $\boldsymbol{\mathscr{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}) < \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, B_2^n, \cdots, B_2^n) < \infty$. Let $\{M_i\}_{i=1}^\infty \subseteq \mathscr{K}_0$ be a sequence of convex bodies such that

$$\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, M_i, \cdots, M_i) \to \boldsymbol{\mathscr{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}) \ \ \text{and} \ \ |M_i^\circ| = \omega_n \ \ \text{for any} \ \ i \geq 1.$$

As $\boldsymbol{K} \in (\mathscr{F}_0^+)^m$, there exist two positive constants $R_0 > 0$ and $C_1 > 0$, such that, $h_{K_j}(u) \leq R_0$ and $f_{K_j}(u) \cdot h_{K_j}(u) \geq C_1$ for any $1 \leq j \leq m$ and any $u \in S^{n-1}$. By [13, Lemma 2.18], there is a positive constant C_2, such that, $|\nabla U_{K_j}\big(\nu_{K_j}^{-1}(u)\big)|^p \geq C_2$ almost everywhere on S^{n-1} for any $1 \leq j \leq m$.

For any $i \geq 1$, let $R_i = \rho_{M_i}(u_i) = \max_{u \in S^{n-1}}\{\rho_{M_i}(u)\}$ and $h_{M_i}(u) \geq R_i \cdot \langle u, u_i \rangle_+$ for any $u \in S^{n-1}$. Again, suppose that u_i converges to $v \in S^{n-1}$. Since the spherical measure $\sigma(\cdot)$ is not concentrated on any hemisphere of S^{n-1}, there exists an integer j_0 such that

$$\int_{\{u \in S^{n-1} :\, \langle u, v \rangle_+ \geq \frac{1}{j_0}\}} \langle u, v \rangle_+ \, d\sigma(u) > 0.$$

Assume that M_i is not bounded uniformly, i.e., $\sup_{i\geq 1} R_i = \infty$. Without loss of generality, let $R_i \to \infty$ as $i \to \infty$. Thus, for any positive constant $C > 0$,

$$\begin{aligned}
\boldsymbol{\mathscr{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}) &= \lim_{i\to\infty} \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, M_i, M_i, \cdots, M_i) \\
&= \liminf_{i\to\infty} \frac{p-1}{n-p} \int_{S^{n-1}} \left(\prod_{j=1}^{m} \varphi_j \left(\frac{h_{M_i}(u)}{h_{K_j}(u)} \right) f^*_{K_j}(u) \right)^{\frac{1}{m}} d\sigma(u) \\
&\geq \frac{C_1 \cdot C_2 \cdot (p-1)}{n-p} \cdot \liminf_{i\to\infty} \int_{S^{n-1}} \left[\prod_{j=1}^{m} \varphi_j \left(\frac{R_i \cdot \langle u, u_i \rangle_+}{R_0} \right) \right]^{\frac{1}{m}} d\sigma(u) \\
&\geq \frac{C_1 \cdot C_2 \cdot (p-1)}{n-p} \cdot \liminf_{i\to\infty} \int_{S^{n-1}} \left[\prod_{j=1}^{m} \varphi_j \left(\frac{C \cdot \langle u, u_i \rangle_+}{R_0} \right) \right]^{\frac{1}{m}} d\sigma(u) \\
&= \frac{C_1 \cdot C_2 \cdot (p-1)}{n-p} \cdot \int_{S^{n-1}} \left[\prod_{j=1}^{m} \varphi_j \left(\frac{C \cdot \langle u, v \rangle_+}{R_0} \right) \right]^{\frac{1}{m}} d\sigma(u) \\
&\geq \frac{C_1 \cdot C_2 \cdot (p-1)}{n-p} \cdot \left[\prod_{j=1}^{m} \varphi_j \left(\frac{C}{R_0 \cdot j_0} \right) \right]^{\frac{1}{m}} \\
&\quad \times \int_{\{u \in S^{n-1} :\, \langle u, v \rangle_+ \geq \frac{1}{j_0}\}} \langle u, v \rangle_+ \, d\sigma(u). \qquad (4.23)
\end{aligned}$$

Letting $C \to \infty$, one gets a contradiction $\boldsymbol{\mathscr{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}) \geq \infty$. Thus, $\sup_{i\geq 1} R_i < \infty$ and $\{M_i\}_{i=1}^{\infty}$ is bounded. By Lemmas 2.2 and 2.3, and $|M_i^\circ| = \omega_n$ for any $i \geq 1$, one gets a convergent subsequence of $\{M_i\}_{i=1}^{\infty}$ which converges to some convex body $M \in \mathscr{K}_0$ with $|M^\circ| = \omega_n$. Without loss of generality, let $M_i \to M$ as $i \to \infty$. Thus, by Proposition 4.5, one has

$$\boldsymbol{\mathscr{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}) = \lim_{i\to\infty} \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, M_i, \cdots, M_i) = \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, M, \cdots, M), \qquad (4.24)$$

as desired. □

The convex body $M \in \mathscr{K}_0$ in (4.24) can be called a p-capacitary Orlicz-Petty bodies of $\boldsymbol{K}$, and if $\boldsymbol{\varphi} \in (\mathscr{I})^m$, such a convex body M exists for $\boldsymbol{K} \in (\mathscr{F}_0^+)^m$. The following theorem deals with the continuity of the functional $\boldsymbol{\mathscr{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\cdot)$ on $(\mathscr{F}^+_0)^m$ for the case $\boldsymbol{\varphi} \in \mathscr{I}^m$.

Theorem 4.6. *Let* $\{\boldsymbol{K_i}\}_{i=1}^\infty \subseteq (\mathscr{F}^+_0)^m$ *and* $\boldsymbol{K} \in (\mathscr{F}^+_0)^m$ *be such that* $\boldsymbol{K_i} \to \boldsymbol{K}$ *as* $i \to \infty$ *and* $\boldsymbol{\varphi} \in \mathscr{I}^m$. *If* $(\prod_{j=1}^m f_{K_{ij}})^{\frac{1}{m}}$ *converges uniformly to* $(\prod_{j=1}^m f_{K_j})^{\frac{1}{m}}$ *on* S^{n-1}, *then* $\boldsymbol{\mathscr{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K_i}) \to \boldsymbol{\mathscr{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K})$ *as* $i \to \infty$.

Proof. Let $M \in \mathscr{K}_0$ and $M_i \in \mathscr{K}_0$ be such that for any $i \geq 1$, $|M^\circ| = |M_i^\circ| = \omega_n$

$$\boldsymbol{\mathscr{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}) = \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, M, \cdots, M) \quad \text{and} \quad \boldsymbol{\mathscr{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K_i}) = \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K_i}, M_i, \cdots, M_i).$$

Similar to the proof of (4.12), Proposition 4.5 yields

$$\boldsymbol{\mathscr{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}) \;\; \geq \;\; \limsup_{i\to\infty} \boldsymbol{\mathscr{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K_i}). \tag{4.25}$$

By [13, (4.19)], there exist two positive constants C_3 (only dependent on $\boldsymbol{K}, n$ and p) and i_0, such that, $|\nabla U_{K_{ij}}\big(\nu^{-1}_{K_{ij}}(u)\big)|^p \geq C_3$ and $|\nabla U_{K_j}\big(\nu^{-1}_{K_j}(u)\big)|^p \geq C_3$ almost everywhere on S^{n-1} for any $i \geq i_0$ and $1 \leq j \leq m$. With a modification of (4.23), one gets that $\{M_i\}_{i=1}^\infty$ is bounded. Let $\{\boldsymbol{K_{i_k}}\}_{k=1}^\infty \subseteq \{\boldsymbol{K_i}\}_{i=1}^\infty$ be a subsequence, such that,

$$\lim_{k\to\infty} \boldsymbol{\mathscr{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K_{i_k}}) = \liminf_{i\to\infty} \boldsymbol{\mathscr{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K_i}).$$

It follows from the boundedness of $\{M_{i_k}\}_{k=1}^\infty$, Lemmas 2.2 and 2.3, and $|M^\circ_{i_k}| = \omega_n$ for any $k \geq 1$ that there exist a subsequence $\{M_{i_{k_j}}\}_{j=1}^\infty$ of $\{M_{i_k}\}_{k=1}^\infty$ and $M' \in \mathscr{K}_0$ such that $M_{i_{k_j}} \to M'$ as $j \to \infty$ and $|(M')^\circ| = \omega_n$. By Proposition 4.5, one has

$$\begin{aligned}
\liminf_{i\to\infty} \boldsymbol{\mathscr{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K_i}) &= \lim_{j\to\infty} \boldsymbol{\mathscr{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K_{i_{k_j}}}) \\
&= \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, M', \cdots, M')
\end{aligned}$$

$$\geq \boldsymbol{\mathscr{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}).$$

Together with (4.25), one gets $\boldsymbol{\mathscr{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}) = \lim_{i\to\infty} \boldsymbol{\mathscr{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K_i})$ as desired. $\square$

The arguments in Theorem 4.5 and Theorem 4.6 may still work if we replace the measure $\mu_p(K,\cdot)$ by $S_{\mathscr{V}}(K,\cdot)$, for instance, if $S_{\mathscr{V}} = S(K,\cdot)$, and we leave the details for readers.

Chapter 5

The Orlicz and L_q geominimal p-capacities

In this chapter, the Orlicz and L_q geominimal p-capacities and their properties are provided. In particular, we establish isoperimetric type inequalities related to these newly proposed geominimal p-capacities.

5.1 The Orlicz geominimal p-capacity

In this section, we provide a detailed study of the Orlicz geominimal p-capacities. Let

$$\mathscr{I}_0 = \mathscr{I} \cap \{\varphi : (0,\infty) \to (0,\infty) \,|\, \varphi(t^{-1/n}) \text{ is strictly convex on } (0,\infty)\};$$

$$\mathscr{D}_0 = \mathscr{D} \cap \{\varphi : (0,\infty) \to (0,\infty) \,|\, \varphi(t^{-1/n}) \text{ is strictly concave on } (0,\infty)\};$$

$$\mathscr{D}_1 = \mathscr{D} \cap \{\varphi : (0,\infty) \to (0,\infty) \,|\, \varphi(t^{-1/n}) \text{ is strictly convex on } (0,\infty)\}.$$

Let $\mathscr{Q}_0 \subseteq \mathscr{S}_0$ be a nonempty subset of $\mathscr{S}_0$.

Definition 5.1. *For $K \in \mathscr{K}_0$, define $\mathscr{G}^{orlicz}_{p,\varphi}(K, \mathscr{Q}_0)$, the nonhomogeneous Orlicz geominimal p-capacity of K with respect to $\mathscr{Q}_0$, as follows:*

$$\mathscr{G}^{orlicz}_{p,\varphi}(K, \mathscr{Q}_0) = \inf_{L \in \mathscr{Q}_0} \Big\{ C_{p,\varphi}(K, vrad(L)\, L^\circ) \Big\} \ \textit{for } \varphi \in \mathscr{I} \cup \mathscr{D}_1,$$
$$\mathscr{G}^{orlicz}_{p,\varphi}(K, \mathscr{Q}_0) = \sup_{L \in \mathscr{Q}_0} \Big\{ C_{p,\varphi}(K, vrad(L)\, L^\circ) \Big\} \ \textit{for } \varphi \in \mathscr{D}_0.$$

Similarly, the homogeneous Orlicz geominimal p-capacity with respect to $\mathscr{Q}_0$, denoted by $\widehat{\mathscr{G}}^{orlicz}_{p,\varphi}(K, \mathscr{Q}_0)$, can be defined with $C_{p,\varphi}(\cdot,\cdot)$ replaced by $\widehat{C}_{p,\varphi}(\cdot,\cdot)$ and $\mathscr{D}_1$ switching with $\mathscr{D}_0$.

Two special cases are important and we will focus on their properties in later context. The first one is the case when $\mathscr{Q}_0 = \mathscr{K}_0$, and we use $\mathscr{G}^{orlicz}_{p,\varphi}(K)$ and $\widehat{\mathscr{G}}^{orlicz}_{p,\varphi}(K)$ to denote $\mathscr{G}^{orlicz}_{p,\varphi}(K, \mathscr{K}_0)$ and $\widehat{\mathscr{G}}^{orlicz}_{p,\varphi}(K, \mathscr{K}_0)$. The second case is $\mathscr{Q}_0 = \mathscr{S}_0$, and we use $\mathscr{A}^{orlicz}_{p,\varphi}(K)$ and $\widehat{\mathscr{A}}^{orlicz}_{p,\varphi}(K)$ for $\mathscr{G}^{orlicz}_{p,\varphi}(K, \mathscr{S}_0)$ and $\widehat{\mathscr{G}}^{orlicz}_{p,\varphi}(K, \mathscr{S}_0)$. As $\mathscr{K}_0 \subseteq \mathscr{S}_0$, then

$$\mathscr{A}^{orlicz}_{p,\varphi}(K) \le \mathscr{G}^{orlicz}_{p,\varphi}(K) \ \text{ for } \varphi \in \mathscr{I} \cup \mathscr{D}_1 \ \text{ and } \ \mathscr{A}^{orlicz}_{p,\varphi}(K) \ge \mathscr{G}^{orlicz}_{p,\varphi}(K) \ \text{ for } \varphi \in \mathscr{D}_0;$$
$$\widehat{\mathscr{A}}^{orlicz}_{p,\varphi}(K) \le \widehat{\mathscr{G}}^{orlicz}_{p,\varphi}(K) \ \text{ for } \varphi \in \mathscr{I} \cup \mathscr{D}_0 \ \text{ and } \ \widehat{\mathscr{A}}^{orlicz}_{p,\varphi}(K) \ge \widehat{\mathscr{G}}^{orlicz}_{p,\varphi}(K) \ \text{ for } \varphi \in \mathscr{D}_1.$$

By Corollary 4.1, one can easily get, for any $\lambda > 0$,

$$\widehat{\mathscr{G}}^{orlicz}_{p,\varphi}(\lambda K) = \lambda^{n-p-1} \widehat{\mathscr{G}}^{orlicz}_{p,\varphi}(K) \ \text{ and } \ \widehat{\mathscr{A}}^{orlicz}_{p,\varphi}(\lambda K) = \lambda^{n-p-1} \widehat{\mathscr{A}}^{orlicz}_{p,\varphi}(K).$$

The following results state that all the quantities above are $O(n)$-invariant. Moreover, when $\varphi \in \mathscr{I}$, it follows from Theorem 4.1 that $\mathscr{G}^{orlicz}_{p,\varphi}(K) = C_{p,\varphi}(K, M)$ for $M \in \mathscr{T}_{p,\varphi}(K)$ and $\widehat{\mathscr{G}}^{orlicz}_{p,\varphi}(K) = \widehat{C}_{p,\varphi}(K, \widehat{M})$ for $\widehat{M} \in \widehat{\mathscr{T}}_{p,\varphi}(K)$.

Corollary 5.1. *If $\varphi \in \mathscr{I} \cup \mathscr{D}_0 \cup \mathscr{D}_1$, then for any $\phi \in O(n)$ and for any $K \in \mathscr{K}_0$,*

$$\mathscr{G}_{p,\varphi}^{orlicz}(\phi K) = \mathscr{G}_{p,\varphi}^{orlicz}(K) \textit{ and } \widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(\phi K) = \widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(K);$$

$$\mathscr{A}_{p,\varphi}^{orlicz}(\phi K) = \mathscr{A}_{p,\varphi}^{orlicz}(K) \textit{ and } \widehat{\mathscr{A}}_{p,\varphi}^{orlicz}(\phi K) = \widehat{\mathscr{A}}_{p,\varphi}^{orlicz}(K).$$

Proof. Here we only prove the equality of $\mathscr{G}_{p,\varphi}^{orlicz}(\phi K) = \mathscr{G}_{p,\varphi}^{orlicz}(K)$, and the other cases can be proved along a similar argument. Let $L \in \mathscr{K}_0$. Since $\phi \in O(n)$, then $|\phi L| = |L|$ and $\mathrm{vrad}(\phi L) = \mathrm{vrad}(L)$. Moreover, by Lemma 2.6 and (2.11), one has, for any $u \in S^{n-1}$,

$$\begin{aligned} d\mu_p(\phi K, u) &= |\nabla U_{\phi K}\big(\nu_{\phi K}^{-1}(u)\big)|^p dS(\phi K, u) \\ &= |\nabla U_K\big(\phi^t \cdot \phi \cdot \nu_K^{-1}(\phi^t u)\big) \cdot \phi^t|^p dS(K, \phi^t u) \\ &= |\nabla U_K\big(\nu_K^{-1}(\phi^t u)\big)|^p dS(K, \phi^t u) \\ &= d\mu_p(K, \phi^t u), \end{aligned} \tag{5.1}$$

where ϕ^t is the transpose of ϕ. For $u \in S^{n-1}$ and $\phi \in O(n)$, let $v = \phi^t u$. By (5.1) and $(\phi L)^\circ = \phi L^\circ$, one gets

$$\begin{aligned} C_{p,\varphi}(\phi K, \mathrm{vrad}(\phi L)(\phi L)^\circ) &= C_{p,\varphi}(\phi K, \mathrm{vrad}(L)\phi L^\circ) \\ &= \frac{p-1}{n-p}\int_{S^{n-1}} \varphi\left(\frac{h_{\mathrm{vrad}(L)\phi L^\circ}(u)}{h_{\phi K}(u)}\right) h_{\phi K}(u) d\mu_p(\phi K, u) \\ &= \frac{p-1}{n-p}\int_{S^{n-1}} \varphi\left(\frac{h_{\mathrm{vrad}(L)L^\circ}(\phi^t u)}{h_K(\phi^t u)}\right) h_K(\phi^t u) d\mu_p(K, \phi^t u) \\ &= \frac{p-1}{n-p}\int_{S^{n-1}} \varphi\left(\frac{h_{\mathrm{vrad}(L)L^\circ}(v)}{h_K(v)}\right) h_K(v) d\mu_p(K, v) \\ &= C_{p,\varphi}(K, \mathrm{vrad}(L)L^\circ). \end{aligned}$$

This, together with Definition 5.1, implies that if $\varphi \in \mathscr{I} \cup \mathscr{D}_1$,

$$\begin{aligned}\mathscr{G}^{orlicz}_{p,\varphi}(\phi K) &= \inf_{\phi L \in \mathscr{K}_0} \Big\{ C_{p,\varphi}(K, \mathrm{vrad}(\phi L)\,(\phi L)^\circ) \Big\} \\ &= \inf_{L \in \mathscr{K}_0} \Big\{ C_{p,\varphi}(K, \mathrm{vrad}(L)\, L^\circ) \Big\} \\ &= \mathscr{G}^{orlicz}_{p,\varphi}(K).\end{aligned}$$

Replacing "inf" by "sup", one gets $\mathscr{G}^{orlicz}_{p,\varphi}(\phi K) = \mathscr{G}^{orlicz}_{p,\varphi}(K)$ when $\varphi \in \mathscr{D}_0$. $\square$

In general, it is not easy to calculate $\mathscr{G}^{orlicz}_{p,\varphi}(\cdot)$, $\widehat{\mathscr{G}}^{orlicz}_{p,\varphi}(\cdot)$, $\mathscr{A}^{orlicz}_{p,\varphi}(\cdot)$ and $\widehat{\mathscr{A}}^{orlicz}_{p,\varphi}(\cdot)$. However, when $K = rB_2^n$ for some $r > 0$, we are able to calculate their precise values.

Proposition 5.1. *Let $\varphi \in \mathscr{I}_0 \cup \mathscr{D}_0 \cup \mathscr{D}_1$ and $r > 0$. Then*

$$\mathscr{A}^{orlicz}_{p,\varphi}(rB_2^n) = \mathscr{G}^{orlicz}_{p,\varphi}(rB_2^n) = \varphi\Big(\frac{1}{r}\Big) \cdot C_p(rB_2^n) \tag{5.2}$$

$$\widehat{\mathscr{A}}^{orlicz}_{p,\varphi}(B_2^n) = \widehat{\mathscr{G}}^{orlicz}_{p,\varphi}(B_2^n) = C_p(B_2^n). \tag{5.3}$$

Proof. The proofs of (5.2) and (5.3) are similar, and we only prove (5.3). For any $L \in \mathscr{S}_0$, let $\tilde{L} = \dfrac{L}{\mathrm{vrad}(L)}$. Thus $|\tilde{L}| = \omega_n$ and $\mathrm{vrad}(\tilde{L}) = 1$. If $\varphi \in \mathscr{I}_0$, with the help of (2.12), (2.14) and Jensen's inequality for the convex function $\varphi(t^{-\frac{1}{n}})$, one has

$$\begin{aligned}1 &= \int_{S^{n-1}} \varphi\left(\frac{C_p(B_2^n)}{\widehat{C}_{p,\varphi}\big(B_2^n, \tilde{L}^\circ\big) \cdot \rho_{\tilde{L}}(u) \cdot h_{B_2^n}(u)} \right) d\mu_p^*(B_2^n, u) \\ &= \int_{S^{n-1}} \varphi\left(\frac{C_p(B_2^n)}{\widehat{C}_{p,\varphi}\big(B_2^n, \tilde{L}^\circ\big) \cdot \rho_{\tilde{L}}(u)} \right) \frac{d\sigma(u)}{n\omega_n} \\ &\geq \varphi\left(\left(\int_{S^{n-1}} \left(\frac{C_p(B_2^n)}{\widehat{C}_{p,\varphi}\big(B_2^n, \tilde{L}^\circ\big) \cdot \rho_{\tilde{L}}(u)} \right)^{-n} \frac{d\sigma(u)}{n\omega_n} \right)^{-\frac{1}{n}} \right) \\ &= \varphi\left(\frac{C_p(B_2^n)}{\widehat{C}_{p,\varphi}(B_2^n, \tilde{L}^\circ)} \right).\end{aligned}$$

Since φ is increasing and $\varphi(1) = 1$, one gets

$$C_p(B_2^n) \leq \widehat{C}_{p,\varphi}(B_2^n, \tilde{L}^\circ) = \widehat{C}_{p,\varphi}(B_2^n, \mathrm{vrad}(L)L^\circ).$$

Taking the infimum over $L \in \mathscr{S}_0$ and by Definition 5.1, one has

$$C_p(B_2^n) \leq \widehat{\mathscr{A}}_{p,\varphi}^{orlicz}(B_2^n) \leq \widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(B_2^n) = \inf_{L \in \mathscr{K}_0} \left\{ \widehat{C}_{p,\varphi}(B_2^n, \mathrm{vrad}(L)\, L^\circ) \right\} \leq C_p(B_2^n)$$

and hence $C_p(B_2^n) = \widehat{\mathscr{A}}_{p,\varphi}^{orlicz}(B_2^n) = \widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(B_2^n)$. The results for $\varphi \in \mathscr{D}_0 \cup \mathscr{D}_1$ follow from a similar argument. $\square$

The isoperimetric type inequalities for $\widehat{\mathscr{A}}_{p,\varphi}^{orlicz}(\cdot)$, $\widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(\cdot)$, $\mathscr{A}_{p,\varphi}^{orlicz}(\cdot)$ and $\mathscr{G}_{p,\varphi}^{orlicz}(\cdot)$ are established in the following theorems.

Theorem 5.1. *Let $K \in \mathscr{K}_0$ be a convex body with its Santaló point or centroid at the origin and B_K be an origin symmetric ball defined by $B_K = vrad(K)B_2^n$.*
(i) If $\varphi \in \mathscr{I}_0 \cup \mathscr{D}_0$, then

$$\frac{\widehat{\mathscr{A}}_{p,\varphi}^{orlicz}(K)}{\widehat{\mathscr{A}}_{p,\varphi}^{orlicz}(B_K)} \leq \frac{\widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(K)}{\widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(B_K)} \leq \frac{C_p(K)}{C_p(B_K)}.$$

Equality holds if K is an origin symmetric ball.
(ii) If $\varphi \in \mathscr{D}_1$, then there exists a universal constant $c > 0$ such that

$$\frac{\widehat{\mathscr{A}}_{p,\varphi}^{orlicz}(K)}{\widehat{\mathscr{A}}_{p,\varphi}^{orlicz}(B_K)} \geq \frac{\widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(K)}{\widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(B_K)} \geq c \cdot \frac{C_p(K)}{C_p(B_K)}.$$

Proof. (i) Let $\varphi \in \mathscr{I}_0 \cup \mathscr{D}_0$. It follows from the homogeneity of $\widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(\cdot)$, $\widehat{\mathscr{A}}_{p,\varphi}^{orlicz}(\cdot)$

and $C_p(\cdot)$, and Proposition 5.1 that

$$\widehat{\mathscr{A}}_{p,\varphi}^{orlicz}(B_K) = \widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(B_K) = \frac{C_p(B_K)}{\mathrm{vrad}(K)}. \tag{5.4}$$

By Definition 5.1 and Corollary 4.1, one has,

$$\widehat{\mathscr{A}}_{p,\varphi}^{orlicz}(K) \leq \widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(K) \leq \widehat{C}_{p,\varphi}(K, \mathrm{vrad}(K^\circ)\, K) = \mathrm{vrad}(K^\circ) \cdot C_p(K).$$

Together with (5.4) and the Blaschke-Santaló inequality (2.1), one has

$$\frac{\widehat{\mathscr{A}}_{p,\varphi}^{orlicz}(K)}{\widehat{\mathscr{A}}_{p,\varphi}^{orlicz}(B_K)} \leq \frac{\widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(K)}{\widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(B_K)} \leq \frac{C_p(K)}{C_p(B_K)}.$$

If K is an origin symmetric ball, say $K = rB_2^n$ for some $r > 0$, one can easily get $K = B_K$ and thus equality in part (i) holds.

(ii) If $\varphi \in \mathscr{D}_1$, by a similar argument and the inverse Santaló inequality (2.2), one has

$$\frac{\widehat{\mathscr{A}}_{p,\varphi}^{orlicz}(K)}{\widehat{\mathscr{A}}_{p,\varphi}^{orlicz}(B_K)} \geq \frac{\widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(K)}{\widehat{\mathscr{G}}_{p,\varphi}^{orlicz}(B_K)} \geq \frac{\mathrm{vrad}(K) \cdot \mathrm{vrad}(K^\circ) \cdot C_p(K)}{C_p(B_K)} \geq c \cdot \frac{C_p(K)}{C_p(B_K)}.$$

□

Along the same lines, one can get the similar results for $\mathscr{G}_{p,\varphi}^{orlicz}(K)$ and $\mathscr{A}_{p,\varphi}^{orlicz}(K)$.

Theorem 5.2. *Let $K \in \mathscr{K}_0$ be a convex body with its Santaló point or centroid at the origin and $B_K = vrad(K)B_2^n$.*
(i) If $\varphi \in \mathscr{I}_0 \cup \mathscr{D}_1$, then

$$\frac{\mathscr{A}_{p,\varphi}^{orlicz}(K)}{\mathscr{A}_{p,\varphi}^{orlicz}\big((B_{K^\circ})^\circ\big)} \leq \frac{\mathscr{G}_{p,\varphi}^{orlicz}(K)}{\mathscr{G}_{p,\varphi}^{orlicz}\big((B_{K^\circ})^\circ\big)} \leq \frac{C_p(K)}{C_p\big((B_{K^\circ})^\circ\big)}.$$

Moreover, if $\varphi \in \mathscr{I}_0$, then

$$\frac{\mathscr{A}^{orlicz}_{p,\varphi}(K)}{\mathscr{A}^{orlicz}_{p,\varphi}(B_K)} \leq \frac{\mathscr{G}^{orlicz}_{p,\varphi}(K)}{\mathscr{G}^{orlicz}_{p,\varphi}(B_K)} \leq \frac{C_p(K)}{C_p(B_K)}.$$

Equality holds if K is an origin symmetric ball.

(ii) If $\varphi \in \mathscr{D}_0$, then

$$\frac{\mathscr{A}^{orlicz}_{p,\varphi}(K)}{\mathscr{A}^{orlicz}_{p,\varphi}(B_K)} \geq \frac{\mathscr{G}^{orlicz}_{p,\varphi}(K)}{\mathscr{G}^{orlicz}_{p,\varphi}(B_K)} \geq \frac{C_p(K)}{C_p(B_K)}.$$

Equality holds if K is an origin symmetric ball.

Proof. (i) It follows from Definition 5.1 that

$$\mathscr{A}^{orlicz}_{p,\varphi}(K) \leq \mathscr{G}^{orlicz}_{p,\varphi}(K) \leq C_{p,\varphi}(K, \mathrm{vrad}(K^\circ)\, K) = \varphi(\mathrm{vrad}(K^\circ)) \cdot C_p(K). \tag{5.5}$$

Note that $(B_{K^\circ})^\circ = \big(\mathrm{vrad}(K^\circ)B_2^n\big)^\circ = \dfrac{1}{\mathrm{vrad}(K^\circ)}B_2^n$. By (5.2) in Proposition 5.1, one has

$$\mathscr{A}^{orlicz}_{p,\varphi}(B_K) = \mathscr{G}^{orlicz}_{p,\varphi}(B_K) = \varphi\Big(\frac{1}{\mathrm{vrad}(K)}\Big) \cdot C_p(B_K); \tag{5.6}$$

$$\mathscr{A}^{orlicz}_{p,\varphi}\big((B_{K^\circ})^\circ\big) = \mathscr{G}^{orlicz}_{p,\varphi}\big((B_{K^\circ})^\circ\big) = \varphi(\mathrm{vrad}(K^\circ)) \cdot C_p\big((B_{K^\circ})^\circ\big). \tag{5.7}$$

The desired result follows from (5.5) and (5.7).

If $\varphi \in \mathscr{I}_0$, by (5.5) and the Blaschke-Santaló inequality (2.1), one has

$$\mathscr{A}^{orlicz}_{p,\varphi}(K) \leq \mathscr{G}^{orlicz}_{p,\varphi}(K) \leq \varphi(\mathrm{vrad}(K^\circ)) \cdot C_p(K) \leq \varphi\Big(\frac{1}{\mathrm{vrad}(K)}\Big) \cdot C_p(K).$$

This along with (5.6) yields

$$\frac{\mathscr{A}_{p,\varphi}^{orlicz}(K)}{\mathscr{A}_{p,\varphi}^{orlicz}(B_K)} \leq \frac{\mathscr{G}_{p,\varphi}^{orlicz}(K)}{\mathscr{G}_{p,\varphi}^{orlicz}(B_K)} \leq \frac{C_p(K)}{C_p(B_K)}.$$

If K is an origin symmetric ball, it can be easily checked that the equality holds. The case (ii) follows from the same lines as the proof of the case $\varphi \in \mathscr{I}_0$. $\square$

Again, one can replace the p-capacitary measure $\mu_p(K, \cdot)$ by the more general measure $S_{\mathscr{V}}(K, \cdot)$. For instance, if $\varphi \in \mathscr{I}_0 \cup \mathscr{D}_0 \cup \mathscr{D}_1$, then

$$\inf_{L \in \mathscr{K}_0} \left\{ \frac{1}{n+2} \int_{S^{n-1}} \varphi\left(\frac{h_L(u)}{h_{B_2^n}(u)}\right) h_{B_2^n}(u) d\mu_\tau(B_2^n, u) : |L^\circ| = \omega_n \right\} = \tau(B_2^n).$$

Results for the case $S_{\mathscr{V}}(K, \cdot)$ can be obtained in the same ways.

5.2 The L_q geominimal p-capacity

In this section, we let $\varphi(t) = t^q$ and consider the L_q geominimal p-capacity of K with respect to $\mathscr{K}_0$ and $\mathscr{S}_0$. Let

$$C_{p,q}(K, L) = \frac{p-1}{n-p} \int_{S^{n-1}} \left(\frac{h_L(u)}{h_K(u)}\right)^q h_K(u) d\mu_p(K, u) \quad \text{for } L \in \mathscr{K}_0;$$
$$C_{p,q}(K, L^\circ) = \frac{p-1}{n-p} \int_{S^{n-1}} \left(\frac{1}{\rho_L(u) \cdot h_K(u)}\right)^q h_K(u) d\mu_p(K, u) \quad \text{for } L \in \mathscr{S}_0.$$

Definition 5.2. *Let $-n \neq q \in \mathbb{R}$ and $K \in \mathscr{K}_0$. Define $\mathscr{G}_{p,q}(K)$, the L_q geominimal p-capacity with respect to $\mathscr{K}_0$, by*

$$\mathscr{G}_{p,q}(K) = \inf_{L \in \mathscr{K}_0} \left\{ \big(C_{p,q}(K, L)\big)^{\frac{n}{n+q}} \cdot |L^\circ|^{\frac{q}{n+q}} \right\}, \quad q \geq 0, \tag{5.8}$$

$$\mathscr{G}_{p,q}(K) = \sup_{L \in \mathscr{K}_0} \left\{ \big(C_{p,q}(K, L)\big)^{\frac{n}{n+q}} \cdot |L^\circ|^{\frac{q}{n+q}} \right\}, \quad -n \neq q < 0; \tag{5.9}$$

and define $\mathscr{A}_{p,q}(K)$, the L_q geominimal p-capacity with respect to $\mathscr{S}_0$, by

$$\mathscr{A}_{p,q}(K) = \inf_{L\in\mathscr{S}_0} \Big\{ \big(C_{p,q}(K,L^\circ)\big)^{\frac{n}{n+q}} \cdot |L|^{\frac{q}{n+q}} \Big\}, \quad q \geq 0, \tag{5.10}$$

$$\mathscr{A}_{p,q}(K) = \sup_{L\in\mathscr{S}_0} \Big\{ \big(C_{p,q}(K,L^\circ)\big)^{\frac{n}{n+q}} \cdot |L|^{\frac{q}{n+q}} \Big\}, \quad -n \neq q < 0. \tag{5.11}$$

Clearly, $\mathscr{G}_{p,0}(K) = \mathscr{A}_{p,0}(K) = C_p(K)$ for any $K \in \mathscr{K}_0$. Moreover, it can be easily checked that for $\varphi(t) = t^q$ $(q \neq -n)$ and any $K \in \mathscr{K}_0$,

$$\begin{aligned} &\mathscr{G}_{p,q}(\lambda K) = \lambda^{\frac{n(n-p-q)}{n+q}} \mathscr{G}_{p,q}(K) \quad \text{and} \quad \mathscr{A}_{p,q}(\lambda K) = \lambda^{\frac{n(n-p-q)}{n+q}} \mathscr{A}_{p,q}(K) \quad \text{for any } \lambda > 0; \\ &\mathscr{G}_{p,q}(\phi K) = \mathscr{G}_{p,q}(K) \qquad\qquad \text{and} \quad \mathscr{A}_{p,q}(\phi K) = \mathscr{A}_{p,q}(K) \qquad\qquad \text{for any } \phi \in O(n). \end{aligned}$$

Moreover, if $q \neq 0, -n$, then with $\varphi(t) = t^q$, one has

$$\widehat{\mathscr{G}}^{orlicz}_{p,\varphi}(K) = \frac{C_p(K)^{1-\frac{1}{q}}}{\omega_n^{1/n}} \cdot \big(\mathscr{G}_{p,q}(K)\big)^{\frac{n+q}{nq}}; \tag{5.12}$$

$$\widehat{\mathscr{A}}^{orlicz}_{p,\varphi}(K) = \frac{C_p(K)^{1-\frac{1}{q}}}{\omega_n^{1/n}} \cdot \big(\mathscr{A}_{p,q}(K)\big)^{\frac{n+q}{nq}}. \tag{5.13}$$

Remark 5.1. *By Proposition 5.1 and (5.8), for any $-n \neq q \in \mathbb{R}$,*

$$\mathscr{G}_{p,q}(B_2^n) = \mathscr{A}_{p,q}(B_2^n) = \big(C_p(B_2^n)\big)^{\frac{n}{n+q}} \cdot |B_2^n|^{\frac{q}{n+q}} = \big(C_{p,q}(B_2^n, B_2^n)\big)^{\frac{n}{n+q}} \cdot |B_2^n|^{\frac{q}{n+q}}.$$

The following corollary provides a convenient formula to calculate $\mathscr{A}_{p,q}(K)$ for $q \neq -n$. For $K \in \mathscr{F}_0^+$, let

$$f_{\mu_p,q}(K,u) = h_K^{1-q}(u) \cdot |\nabla U_K\big(\nu_K^{-1}(u)\big)|^p \cdot f_K(u),$$

where U_K is the p-capacitary function of K, f_K is the curvature function of K and

$\nu_K^{-1} : S^{n-1} \to \partial K$ is the inverse Gauss map. For $-n \neq q \in \mathbb{R}$, let

$$\xi_{\mu_p,q} = \big\{K \in \mathscr{F}_0^+ : \exists Q \in \mathscr{S}_0 \ \text{ s.t. } f_{\mu_p,q}(K,u) = \big(\rho_Q(u)\big)^{n+q} \text{ for any } u \in S^{n-1}\big\}.$$

Clearly, $B_2^n \in \xi_{\mu_p,q}$ as one can let $Q_0 = \left(\dfrac{n-p}{p-1}\right)^{p/(n+q)} \cdot B_2^n \in \mathscr{S}_0$ and thus for any $u \in S^{n-1}$,

$$f_{\mu_p,q}(B_2^n, u) = \left(\frac{n-p}{p-1}\right)^p = \big(\rho_{Q_0}(u)\big)^{n+q}.$$

Corollary 5.2. *If $K \in \xi_{\mu_p,q}$, then for $-n \neq q \in \mathbb{R}$,*

$$\mathscr{A}_{p,q}(K) = \left(\frac{1}{n}\right)^{\frac{q}{n+q}} \cdot \left(\frac{p-1}{n-p}\right)^{\frac{n}{n+q}} \cdot \int_{S^{n-1}} f_{\mu_p,q}(K,u)^{\frac{n}{n+q}} d\sigma(u). \tag{5.14}$$

Proof. Let $L \in \mathscr{S}_0$. It can be easily checked that (5.14) is true for $q = 0$, i.e.,

$$\mathscr{A}_{p,0}(K) = \frac{p-1}{n-p} \cdot \int_{S^{n-1}} h_K(u) \cdot d\mu_p(K,u) = C_p(K).$$

If $q > 0$, by Hölder inequality, one has

$$\begin{aligned}
&\left(\frac{1}{n}\right)^{\frac{q}{n+q}} \cdot \left(\frac{p-1}{n-p}\right)^{\frac{n}{n+q}} \cdot \int_{S^{n-1}} f_{\mu_p,q}(K,u)^{\frac{n}{n+q}} d\sigma(u) \\
&= \left(\frac{1}{n}\right)^{\frac{q}{n+q}} \cdot \left(\frac{p-1}{n-p}\right)^{\frac{n}{n+q}} \cdot \int_{S^{n-1}} \left[\rho_L^{-q}(u) f_{\mu_p,q}(K,u) \rho_L^q(u)\right]^{\frac{n}{n+q}} d\sigma(u) \\
&\leq \left(\frac{p-1}{n-p} \cdot \int_{S^{n-1}} \rho_L^{-q}(u) f_{\mu_p,q}(K,u) d\sigma(u)\right)^{\frac{n}{n+q}} \left(\frac{1}{n}\int_{S^{n-1}} \rho_L^n(u) d\sigma(u)\right)^{\frac{q}{n+q}} \\
&= C_{p,q}(K, L^\circ)^{\frac{n}{n+q}} \cdot |L|^{\frac{q}{n+q}}.
\end{aligned}$$

Take the infimum over $L \in \mathscr{S}_0$ and thus

$$\left(\frac{1}{n}\right)^{\frac{q}{n+q}} \cdot \left(\frac{p-1}{n-p}\right)^{\frac{n}{n+q}} \cdot \int_{S^{n-1}} f_{\mu_p,q}(K,u)^{\frac{n}{n+q}} d\sigma(u) \leq \mathscr{A}_{p,q}(K). \tag{5.15}$$

On the other hand, since $K \in \xi_{\mu_p,q}$, there exists a star body $Q \in \mathscr{S}_0$ such that

$$\rho_Q(u) = \Big(f_{\mu_p,q}(K,u)\Big)^{\frac{1}{n+q}} \quad \text{for any } u \in S^{n-1}.$$

Then

$$\Big(\frac{1}{n}\Big)^{\frac{q}{n+q}} \cdot \Big(\frac{p-1}{n-p}\Big)^{\frac{n}{n+q}} \cdot \int_{S^{n-1}} f_{\mu_p,q}(K,u)^{\frac{n}{n+q}} d\sigma(u) = C_{p,q}(K,Q^\circ)^{\frac{n}{n+q}} \cdot |Q|^{\frac{q}{n+q}} \geq \mathscr{A}_{p,q}(K).$$

This together with (5.15) yields

$$\mathscr{A}_{p,q}(K) = \Big(\frac{1}{n}\Big)^{\frac{q}{n+q}} \cdot \Big(\frac{p-1}{n-p}\Big)^{\frac{n}{n+q}} \cdot \int_{S^{n-1}} f_{\mu_p,q}(K,u)^{\frac{n}{n+q}} d\sigma(u).$$

Along the same lines, one can prove (5.14) when $-n \neq q < 0$. $\square$

Remark 5.2. *Motivated by the definition of the p-curvature image of $K \in \mathscr{F}_0^+$ in [44, 68], for any $K \in \xi_{\mu_p,q}$ and $-n \neq q \in \mathbb{R}$, we can define $\Lambda_{\mu_p,q}K \in \mathscr{S}_0$, the p-capacitary q-curvature image of K, by*

$$f_{\mu_p,q}(K,u) = \frac{n-p}{n(p-1)|\Lambda_{\mu_p,q}K|} \cdot \big(\rho_{\Lambda_{\mu_p,q}K}(u)\big)^{n+q} \quad \textit{for any } u \in S^{n-1}.$$

By the proof of Corollary 5.2, one also gets

$$\mathscr{A}_{p,q}(K) = \Big(C_{p,q}\big(K,(\Lambda_{\mu_p,q}K)^\circ\big)\Big)^{\frac{n}{n+q}} \cdot |\Lambda_{\mu_p,q}K|^{\frac{q}{n+q}} = |\Lambda_{\mu_p,q}K|^{\frac{q}{n+q}}.$$

For $-n \neq q \in \mathbb{R}$, let

$$\nu_{\mu_p,q} = \Big\{K \in \mathscr{F}_0^+ : \exists Q \in \mathscr{K}_0 \ \text{ s.t. } f_{\mu_p,q}(K,u) = \big(\rho_Q(u)\big)^{n+q} \text{ for any } u \in S^{n-1}\Big\}.$$

Clearly, $\nu_{\mu_p,q} \subseteq \xi_{\mu_p,q}$ and $B_2^n \in \nu_{\mu_p,q}$, which yields $\nu_{\mu_p,q} \neq \emptyset$. The following results provide a convenient formula to calculate $\mathscr{G}_{p,q}(K)$ when $K \in \nu_{\mu_p,q}$.

Proposition 5.2. *If $-n \neq q \in \mathbb{R}$ and $K \in \nu_{\mu_p,q}$, then $\mathscr{G}_{p,q}(K) = \mathscr{A}_{p,q}(K)$.*

Proof. First of all, we prove that $\Lambda_{\mu_p,q}K \in \mathscr{K}_0$ if $K \in \nu_{\mu_p,q}$. As $K \in \nu_{\mu_p,q}$, there is a convex body $Q \in \mathscr{K}_0$ such that $f_{\mu_p,q}(K,u) = \big(\rho_Q(u)\big)^{n+q}$ for any $u \in S^{n-1}$. Together with Remark 5.2, one gets, for any $u \in S^{n-1}$,

$$\frac{n-p}{n(p-1)|\Lambda_{\mu_p,q}K|} \cdot \big(\rho_{\Lambda_{\mu_p,q}K}(u)\big)^{n+q} = \big(\rho_Q(u)\big)^{n+q},$$

and hence

$$\Lambda_{\mu_p,q}K = \left(\frac{n(p-1)|\Lambda_{\mu_p,q}K|}{n-p}\right)^{\frac{1}{n+q}} Q \in \mathscr{K}_0.$$

Next we shall prove $\mathscr{G}_{p,q}(K) = \mathscr{A}_{p,q}(K)$. The case $q = 0$ is trivial as $\mathscr{G}_{p,0}(K) = \mathscr{A}_{p,0}(K) = C_p(K)$.

If $q > 0$, by (5.8) and (5.10), one gets $\mathscr{G}_{p,q}(K) \geq \mathscr{A}_{p,q}(K)$. On the other hand, by Remark 5.2, $\Lambda_{\mu_p,q}K \in \mathscr{K}_0$ and Definition 5.2, one has

$$\mathscr{A}_{p,q}(K) = \big(C_{p,q}\big(K, (\Lambda_{\mu_p,q}K)^\circ\big)\big)^{\frac{n}{n+q}} \cdot |\Lambda_{(\mu_p,q)}K|^{\frac{q}{n+q}} \geq \mathscr{G}_{p,q}(K).$$

These imply $\mathscr{G}_{p,q}(K) = \mathscr{A}_{p,q}(K)$.

If $-n \neq q < 0$, similarly, employing (5.9) and (5.11), Remark 5.2, $\Lambda_{\mu_p,q}K \in \mathscr{K}_0$ and Definition 5.2, one gets

$$\mathscr{G}_{p,q}(K) \leq \mathscr{A}_{p,q}(K) = \big(C_{p,q}(K, \big(\Lambda_{\mu_p,q}K\big)^\circ\big)^{\frac{n}{n+q}} \cdot |\Lambda_{\mu_p,q}K|^{\frac{q}{n+q}} \leq \mathscr{G}_{p,q}(K).$$

Thus $\mathscr{G}_{p,q}(K) = \mathscr{A}_{p,q}(K)$ when $-n \neq q < 0$. $\square$

The following isoperimetric type inequalities for $\mathscr{G}_{p,q}(K)$ and $\mathscr{A}_{p,q}(K)$ can be easily obtained from Theorem 5.1, Theorem 5.2, (5.12) and (5.13), and $\mathscr{G}_{p,0}(K) = \mathscr{A}_{p,0}(K) =$

$C_p(K)$ for any $K \in \mathscr{K}_0$.

Proposition 5.3. *Let $K \in \mathscr{K}_0$ be a convex body with its Santaló point or centroid at the origin and $B_K = vrad(K)B_2^n$.*

(i) For $q \geq 0$,

$$\frac{\mathscr{A}_{p,q}(K)}{\mathscr{A}_{p,q}(B_K)} \leq \frac{\mathscr{G}_{p,q}(K)}{\mathscr{G}_{p,q}(B_K)} \leq \left(\frac{C_p(K)}{C_p(B_K)}\right)^{\frac{n}{n+q}}.$$

(ii) For $-n < q < 0$,

$$\frac{\mathscr{A}_{p,q}(K)}{\mathscr{A}_{p,q}(B_K)} \geq \frac{\mathscr{G}_{p,q}(K)}{\mathscr{G}_{p,q}(B_K)} \geq \left(\frac{C_p(K)}{C_p(B_K)}\right)^{\frac{n}{n+q}}.$$

(iii) For $q < -n$, there exists a universal constant $c > 0$ such that

$$\frac{\mathscr{A}_{p,q}(K)}{\mathscr{A}_{p,q}(B_K)} \geq \frac{\mathscr{G}_{p,q}(K)}{\mathscr{G}_{p,q}(B_K)} \geq c^{\frac{nq}{n+q}}\left(\frac{C_p(K)}{C_p(B_K)}\right)^{\frac{n}{n+q}}.$$

The cyclic inequality for $\mathscr{G}_{p,r}(K)$ is given by the following theorem.

Theorem 5.3. *Let $K \in \mathscr{K}_0$.*

(i) If $-n < t < 0 < r < s$ or $-n < s < 0 < r < t$, then

$$\mathscr{G}_{p,r}(K) \leq \big(\mathscr{G}_{p,t}(K)\big)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} \cdot \big(\mathscr{G}_{p,s}(K)\big)^{\frac{(r-t)(n+s)}{(s-t)(n+r)}}.$$

(ii) If $-n < t < r < s < 0$ or $-n < s < r < t < 0$, then

$$\mathscr{G}_{p,r}(K) \leq \big(\mathscr{G}_{p,t}(K)\big)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} \cdot \big(\mathscr{G}_{p,s}(K)\big)^{\frac{(r-t)(n+s)}{(s-t)(n+r)}}.$$

(iii) If $t < r < -n < s < 0$ or $s < r < -n < t < 0$, then

$$\mathscr{G}_{p,r}(K) \geq \big(\mathscr{G}_{p,t}(K)\big)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} \cdot \big(\mathscr{G}_{p,s}(K)\big)^{\frac{(r-t)(n+s)}{(s-t)(n+r)}}.$$

Proof. Let $K, L \in \mathscr{K}_0$ and s, r, t be three real numbers such that $0 < \dfrac{t-r}{t-s} < 1$. By Hölder inequality, one has

$$\begin{aligned}
&C_{p,r}(K,L)\\
&= \frac{p-1}{n-p}\int_{S^{n-1}} h_L^r(u)\cdot h_K^{1-r}(u)\,d\mu_p(K,u)\\
&\le \frac{p-1}{n-p}\Big(\int_{S^{n-1}} h_L^t(u)\cdot h_K^{1-t}(u)\,d\mu_p(K,u)\Big)^{\frac{r-s}{t-s}}\Big(\int_{S^{n-1}} h_L^s(u)\cdot h_K^{1-s}(u)\,d\mu_p(K,u)\Big)^{\frac{r-t}{s-t}}\\
&= \big(C_{p,t}(K,L)\big)^{\frac{r-s}{t-s}}\cdot\big(C_{p,s}(K,L)\big)^{\frac{r-t}{s-t}}. \qquad (5.16)
\end{aligned}$$

(i) Assume that $-n < t < 0 < r < s$. Then $0 < \dfrac{t-r}{t-s} < 1$, $\dfrac{n}{n+r} > 0$, $\dfrac{(r-s)(n+t)}{(t-s)(n+r)} > 0$ and $\dfrac{(r-t)(n+s)}{(s-t)(n+r)} > 0$. Together with (5.16) and Definition 5.2, one has

$$\begin{aligned}
&\mathscr{G}_{p,r}(K)\\
&= \inf_{L\in\mathscr{K}_0}\Big\{\big(C_{p,r}(K,L)\big)^{\frac{n}{n+r}}\cdot|L^\circ|^{\frac{r}{n+r}}\Big\}\\
&\le \inf_{L\in\mathscr{K}_0}\Bigg\{\Big[\big(C_{p,t}(K,L)\big)^{\frac{n}{n+t}}\cdot|L^\circ|^{\frac{t}{n+t}}\Big]^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}\cdot\Big[\big(C_{p,s}(K,L)\big)^{\frac{n}{n+s}}\cdot|L^\circ|^{\frac{s}{n+s}}\Big]^{\frac{(r-t)(n+s)}{(s-t)(n+r)}}\Bigg\}\\
&\le \sup_{L\in\mathscr{K}_0}\Big\{\big(C_{p,t}(K,L)\big)^{\frac{n}{n+t}}\cdot|L^\circ|^{\frac{t}{n+t}}\Big\}^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}\cdot\inf_{L\in\mathscr{K}_0}\Big\{\big(C_{p,s}(K,L)\big)^{\frac{n}{n+s}}\cdot|L^\circ|^{\frac{s}{n+s}}\Big\}^{\frac{(r-t)(n+s)}{(s-t)(n+r)}}\\
&= \big(\mathscr{G}_{p,t}(K)\big)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}\cdot\big(\mathscr{G}_{p,s}(K)\big)^{\frac{(r-t)(n+s)}{(s-t)(n+r)}}.
\end{aligned}$$

By switching the roles of s and t, one gets the case $-n < s < 0 < r < t$.

(ii) It's enough to prove the case $-n < t < r < s < 0$, since the case $-n < s < r < t < 0$ can be proved by switching the roles of s and t. In this case, one has $0 < \dfrac{t-r}{t-s} < 1$, $\dfrac{n}{n+r} > 0$, $\dfrac{(r-s)(n+t)}{(t-s)(n+r)} > 0$ and $\dfrac{(r-t)(n+s)}{(s-t)(n+r)} > 0$. Together with

(5.16) and Definition 5.2, one has

$$
\begin{aligned}
&\mathscr{G}_{p,r}(K)\\
&= \sup_{L\in\mathscr{K}_0}\left\{\big(C_{p,r}(K,L)\big)^{\frac{n}{n+r}}\cdot|L^\circ|^{\frac{r}{n+r}}\right\}\\
&\le \sup_{L\in\mathscr{K}_0}\left\{\left[\big(C_{p,t}(K,L)\big)^{\frac{r-s}{t-s}}\cdot\big(C_{p,s}(K,L)\big)^{\frac{r-t}{s-t}}\right]^{\frac{n}{n+r}}\cdot|L^\circ|^{\frac{r}{n+r}}\right\}\\
&\le \sup_{L\in\mathscr{K}_0}\left\{\big(C_{p,t}(K,L)\big)^{\frac{n}{n+t}}\cdot|L^\circ|^{\frac{t}{n+t}}\right\}^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}\cdot\sup_{L\in\mathscr{K}_0}\left\{\big(C_{p,s}(K,L)\big)^{\frac{n}{n+s}}\cdot|L^\circ|^{\frac{s}{n+s}}\right\}^{\frac{(r-t)(n+s)}{(s-t)(n+r)}}\\
&= \big(\mathscr{G}_{p,t}(K)\big)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}\cdot\big(\mathscr{G}_{p,s}(K)\big)^{\frac{(r-t)(n+s)}{(s-t)(n+r)}}.
\end{aligned}
$$

(iii) Let $t<r<-n<s<0$. Thus $0<\dfrac{t-r}{t-s}<1$, $\dfrac{n}{n+r}<0$, $\dfrac{(r-s)(n+t)}{(t-s)(n+r)}>0$ and $\dfrac{(r-t)(n+s)}{(s-t)(n+r)}<0$. Together with (5.16) and Definition 5.2, one has

$$
\begin{aligned}
&\mathscr{G}_{p,r}(K)\\
&= \sup_{L\in\mathscr{K}_0}\left\{\big(C_{p,r}(K,L)\big)^{\frac{n}{n+r}}\cdot|L^\circ|^{\frac{r}{n+r}}\right\}\\
&\ge \sup_{L\in\mathscr{K}_0}\left\{\left[\big(C_{p,t}(K,L)\big)^{\frac{r-s}{t-s}}\cdot\big(C_{p,s}(K,L)\big)^{\frac{r-t}{s-t}}\right]^{\frac{n}{n+r}}\cdot|L^\circ|^{\frac{r}{n+r}}\right\}\\
&\ge \sup_{L\in\mathscr{K}_0}\left\{\big(C_{p,t}(K,L)\big)^{\frac{n}{n+t}}\cdot|L^\circ|^{\frac{t}{n+t}}\right\}^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}\cdot\sup_{L\in\mathscr{K}_0}\left\{\big(C_{p,s}(K,L)\big)^{\frac{n}{n+s}}\cdot|L^\circ|^{\frac{s}{n+s}}\right\}^{\frac{(r-t)(n+s)}{(s-t)(n+r)}}\\
&= \big(\mathscr{G}_{p,t}(K)\big)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}\cdot\big(\mathscr{G}_{p,s}(K)\big)^{\frac{(r-t)(n+s)}{(s-t)(n+r)}}.
\end{aligned}
$$

The case $s<r<-n<t<0$ follows by switching the roles of s and t. $\square$

The following results regarding the monotonicity of $\mathscr{G}_{p,s}(K)$ on $s\in\mathbb{R}$ can be obtained.

Theorem 5.4. *Let $K\in\mathscr{K}_0$ and $t,s\neq 0$.*

(i) If $-n < s < t$ *or* $s < t < -n$*, then*

$$\left(\frac{\mathscr{G}_{p,s}(K)}{C_p(K)}\right)^{\frac{n+s}{s}} \leq \left(\frac{\mathscr{G}_{p,t}(K)}{C_p(K)}\right)^{\frac{n+t}{t}}. \tag{5.17}$$

(ii) If $s < -n < t$*, then*

$$\left(\frac{\mathscr{G}_{p,s}(K)}{C_p(K)}\right)^{\frac{n+s}{s}} \geq \left(\frac{\mathscr{G}_{p,t}(K)}{C_p(K)}\right)^{\frac{n+t}{t}}. \tag{5.18}$$

Proof. (i) Clearly, the condition $-n < s < t$ or $s < t < -n$ consists of four different possibilities: $-n < 0 < s < t$, $-n < s < 0 < t$, $-n < s < t < 0$ and $s < t < -n$. Here we choose to prove the case $-n < 0 < s < t$. The other cases can be proved by employing the corresponding parts of Theorem 5.3.

Indeed, Theorem 5.3 holds if we replace t, r and s in part (i) by 0, s and t, that is,

$$\mathscr{G}_{p,s}(K) \leq \big(C_p(K)\big)^{\frac{n(t-s)}{t(n+s)}} \cdot \big(\mathscr{G}_{p,t}(K)\big)^{\frac{s(n+t)}{t(n+s)}}.$$

Dividing by $C_p(K)$ from both sides above, one has

$$\frac{\mathscr{G}_{p,s}(K)}{C_p(K)} \leq \big(C_p(K)\big)^{\frac{n(t-s)}{t(n+s)}-1} \cdot \big(\mathscr{G}_{p,t}(K)\big)^{\frac{s(n+t)}{t(n+s)}} = \left(\frac{\mathscr{G}_{p,t}(K)}{C_p(K)}\right)^{\frac{s(n+t)}{t(n+s)}}.$$

The desired inequality (5.17) follows immediately after taking the power of $\dfrac{n+s}{s}$ from both sides.

(ii) Clearly, $s < -n < t$ contains two cases: $s < -n < 0 < t$ and $s < -n < t < 0$. We shall prove the case $s < -n < 0 < t$. The case $s < -n < t < 0$ follows along the same lines.

Again, Theorem 5.3 holds if t, r and s in part (iii) are replaced by s, 0 and t, i.e.,

$$C_p(K) = \mathscr{G}_{p,0}(K) \geq \big(\mathscr{G}_{p,s}(K)\big)^{\frac{-t(n+s)}{n(s-t)}} \cdot \big(\mathscr{G}_{p,t}(K)\big)^{\frac{-s(n+t)}{n(t-s)}}.$$

Dividing by $C_p(K)$, one has

$$\left(\frac{\mathscr{G}_{p,s}(K)}{C_p(K)}\right)^{\frac{-t(n+s)}{n(t-s)}} \geq \left(\frac{\mathscr{G}_{p,t}(K)}{C_p(K)}\right)^{\frac{-s(n+t)}{n(t-s)}},$$

and inequality (5.18) follows immediately after taking the power of $\dfrac{n(t-s)}{-ts}$ from both sides. $\square$

5.3 The mixed L_q geominimal p-capacity

The mixed L_q and Orlicz affine and geominimal surface areas were investigated in [44, 64, 68, 70], which extended the L_q and Orlicz affine and geominimal surface areas to multiple convex bodies. As proved in Section 4.3, one can define the mixed Orlicz geominimal p-capacity for multiple convex bodies as well.

Definition 5.3. *Let $\boldsymbol{K} \in (\mathscr{F}_0^+)^m$.*

(i) If $\boldsymbol{\varphi} \in \mathscr{I}^m$ or $\boldsymbol{\varphi} \in \mathscr{D}_1^m$, define $\boldsymbol{\mathscr{G}}_{p,\boldsymbol{\varphi}}^{orlicz}(\boldsymbol{K})$, the mixed Orlicz geominimal p-capacity with respect to $\mathscr{K}_0$, by

$$\boldsymbol{\mathscr{G}}_{p,\boldsymbol{\varphi}}^{orlicz}(\boldsymbol{K}) = \inf\Big\{\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, \underbrace{L, \cdots, L}_{m}) : L \in \mathscr{K}_0 \quad and \quad |L^\circ| = \omega_n\Big\}.$$

(ii) If $\boldsymbol{\varphi} \in \mathscr{D}_0^m$, define $\boldsymbol{\mathscr{G}}_{p,\boldsymbol{\varphi}}^{orlicz}(\boldsymbol{K})$, the mixed Orlicz geominimal p-capacity with respect to $\mathscr{K}_0$, by

$$\boldsymbol{\mathscr{G}}_{p,\boldsymbol{\varphi}}^{orlicz}(\boldsymbol{K}) = \sup\Big\{\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, \underbrace{L, \cdots, L}_{m}) : L \in \mathscr{K}_0 \quad and \quad |L^\circ| = \omega_n\Big\}.$$

Let $\boldsymbol{L} = (L_1, L_2, \cdots, L_m) \in (\mathscr{S}_0)^m$. Define the dual mixed volume of L by [41]

$$\tilde{V}(\boldsymbol{L}) = \tilde{V}(L_1, L_2, \cdots, L_m) = \frac{1}{n} \int_{S^{n-1}} \Big(\prod_{i=1}^{m} \rho_{L_i}(u) \Big)^{\frac{n}{m}} d\sigma(u).$$

Clearly, for any $L \in \mathscr{S}_0$, $\tilde{V}(\underbrace{L, L, \cdots, L}_{m}) = |L|$. Moreover, by Hölder inequality [24], one has

$$\tilde{V}(\boldsymbol{L}) \leq \prod_{i=1}^{m} |L_i|^{\frac{1}{m}} \quad \text{for any} \quad \boldsymbol{L} = (L_1, L_2, \cdots L_m) \in (\mathscr{S}_0)^m,$$

and equality holds if and only if L_i $(1 \leq i \leq m)$ are dilates of each other. For $\phi \in O(n)$ and $\boldsymbol{L} = (L_1, L_2, \cdots, L_m) \in (\mathscr{S}_0)^m$, define $\phi\boldsymbol{L}$ by $\phi\boldsymbol{L} = (\phi L_1, \phi L_2, \cdots, \phi L_m)$. It can be checked that $\tilde{V}(\phi\boldsymbol{L}) = \tilde{V}(\boldsymbol{L})$. When $\varphi_i(t) = t^q$ for any $1 \leq i \leq m$, $\boldsymbol{C}_{p,q}(\boldsymbol{K}, \boldsymbol{L})$, the Orlicz mixed p-capacity of $\boldsymbol{K}$ and $\boldsymbol{L}$, is given by

$$\boldsymbol{C}_{p,q}(\boldsymbol{K}, \boldsymbol{L}) = \frac{p-1}{n-p} \int_{S^{n-1}} \bigg(\prod_{i=1}^{m} \big(h_{L_i}(u)\big)^q f_{\mu_p,q}(K_i, u) \bigg)^{\frac{1}{m}} d\sigma(u).$$

If $\boldsymbol{L} \in (\mathscr{S}_0)^m$, we let

$$\boldsymbol{C}_{p,q}(\boldsymbol{K}, \boldsymbol{L}^\circ) = \frac{p-1}{n-p} \int_{S^{n-1}} \bigg(\prod_{i=1}^{m} \big(\rho_{L_i}(u)\big)^{-q} f_{\mu_p,q}(K_i, u) \bigg)^{\frac{1}{m}} d\sigma(u).$$

Let $\mathscr{Q}_0$ be a nonempty subset of $\mathscr{S}_0$.

Definition 5.4. *Let $\boldsymbol{K} = (K_1, K_2, \cdots, K_m) \in (\mathscr{F}_0^+)^m$ and $-n \neq q \in \mathbb{R}$.*

(i) For $q \geq 0$, the mixed L_q geominimal p-capacity with respect to $\mathscr{Q}_0$, is defined by

$$\boldsymbol{\mathscr{G}}_{p,q}(\boldsymbol{K}, \mathscr{Q}_0) = \inf_{L \in \mathscr{Q}_0} \Big\{ \big(\boldsymbol{C}_{p,q}(\boldsymbol{K}, \underbrace{L^\circ, \cdots, L^\circ}_{m})\big)^{\frac{n}{n+q}} \cdot |L|^{\frac{q}{n+q}} \Big\}.$$

(ii) For $-n \neq q < 0$, the mixed L_q geominimal p-capacity with respect to $\mathscr{Q}_0$, is defined

by

$$\boldsymbol{\mathscr{G}}_{p,q}(\boldsymbol{K}, \mathscr{Q}_0) = \sup_{L\in\mathscr{Q}_0} \Big\{ \big(\boldsymbol{C}_{p,q}(\boldsymbol{K}, \underbrace{L^\circ, \cdots, L^\circ}_{m})\big)^{\frac{n}{n+q}} \cdot |L|^{\frac{q}{n+q}} \Big\}.$$

There are many ways to extend/modify Definition 5.4 and to define different mixed L_q geominimal p-capacities. For instance, one can replace $|L|^{\frac{q}{n+q}}$ by $\prod_{i=1}^m |L_i|^{\frac{q}{m(n+q)}}$ or $\tilde{V}(\boldsymbol{L})^{\frac{q}{n+q}}$. However, their properties are similar to these for $\boldsymbol{\mathscr{G}}_{p,q}(\cdot)$ defined in Definition 5.4 and hence will not be discussed here.

Again, in the following we will focus on the case $\boldsymbol{\mathscr{G}}_{p,q}(\boldsymbol{K}) = \boldsymbol{\mathscr{G}}_{p,q}(\boldsymbol{K}, \mathscr{K}_0)$ and $\boldsymbol{\mathscr{A}}_{p,q}(\boldsymbol{K}) = \boldsymbol{\mathscr{G}}_{p,q}(\boldsymbol{K}, \mathscr{S}_0)$. Clearly, for any $K \in \mathscr{K}_0$,

$$\boldsymbol{\mathscr{G}}_{p,q}(\underbrace{K, \cdots, K}_{m}) = \mathscr{G}_{p,q}(K) \quad \text{and} \quad \boldsymbol{\mathscr{A}}_{p,q}(\underbrace{K, \cdots, K}_{m}) = \mathscr{A}_{p,q}(K).$$

Moreover, if $\varphi_i(t) = t^q$ $(1 \le i \le m, -n \ne q \in \mathbb{R})$, then, for any $\boldsymbol{K} = (K_1, \cdots, K_m) \in (\mathscr{F}_0^+)^m$,

$$\boldsymbol{\mathscr{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}) = \omega_n^{-\frac{q}{n}} \cdot \boldsymbol{\mathscr{G}}_{p,q}^{\frac{n+q}{n}}(\boldsymbol{K}).$$

The following proposition states that $\boldsymbol{\mathscr{G}}_{p,q}(\cdot)$ and $\boldsymbol{\mathscr{A}}_{p,q}(\cdot)$ are $O(n)$-invariant.

Proposition 5.4. *Let* $\boldsymbol{K} = (K_1, K_2, \cdots, K_m) \in (\mathscr{F}_0^+)^m$ *and* $-n \ne q \in \mathbb{R}$*. Then for any* $\phi \in O(n)$*, one has*

$$\boldsymbol{\mathscr{G}}_{p,q}(\phi\boldsymbol{K}) = \boldsymbol{\mathscr{G}}_{p,q}(\boldsymbol{K}) \quad \textit{and} \quad \boldsymbol{\mathscr{A}}_{p,q}(\phi\boldsymbol{K}) = \boldsymbol{\mathscr{A}}_{p,q}(\boldsymbol{K}).$$

Proof. We only prove $\boldsymbol{\mathscr{G}}_{p,q}(\phi\boldsymbol{K}) = \boldsymbol{\mathscr{G}}_{p,q}(\boldsymbol{K})$, and $\boldsymbol{\mathscr{A}}_{p,q}(\phi\boldsymbol{K}) = \boldsymbol{\mathscr{A}}_{p,q}(\boldsymbol{K})$ follows along a similar argument. For any $1 \le i \le m$ and any $u \in S^{n-1}$, let $v = \phi^t u$ and then

$$\begin{aligned} f_{\mu_p,q}(\phi K_i, u) &= h_{\phi K_i}^{1-q}(u) \cdot |\nabla U_{\phi K_i}\big(\nu_{\phi K_i}^{-1}(u)\big)|^p \cdot f_{\phi K_i}(u) \\ &= h_{K_i}^{1-q}(\phi^t u) \cdot |\nabla U_{K_i}\big(\nu_{K_i}^{-1}(\phi^t u)\big)|^p \cdot f_{K_i}(\phi^t u) \\ &= f_{\mu_p,q}(K_i, v). \end{aligned}$$

Hence for any $\boldsymbol{L} \in (\mathscr{S}_0)^m$,

$$\begin{aligned} \boldsymbol{C}_{p,q}(\phi\boldsymbol{K}, (\phi\boldsymbol{L})^\circ) &= \frac{p-1}{n-p}\int_{S^{n-1}} \left(\prod_{i=1}^m \big(\rho_{\phi L_i}(u)\big)^{-q} f_{\mu_p,q}(\phi K_i, u)\right)^{\frac{1}{m}} d\sigma(u) \\ &= \frac{p-1}{n-p}\int_{S^{n-1}} \left(\prod_{i=1}^m \big(\rho_{L_i}(\phi^t u)\big)^{-q} f_{\mu_p,q}(K_i, \phi^t u)\right)^{\frac{1}{m}} d\sigma(u) \\ &= \frac{p-1}{n-p}\int_{S^{n-1}} \left(\prod_{i=1}^m \big(\rho_{L_i}(v)\big)^{-q} f_{\mu_p,q}(K_i, v)\right)^{\frac{1}{m}} d\sigma(v) \\ &= \boldsymbol{C}_{p,q}(\boldsymbol{K}, \boldsymbol{L}^\circ). \end{aligned}$$

Together with $(\phi L)^\circ = \phi L^\circ$ and $|\phi L| = |L|$ for any $L \in \mathscr{S}_0$, one has, for $q \ge 0$,

$$\begin{aligned} \boldsymbol{\mathscr{G}}_{p,q}(\phi\boldsymbol{K}) &= \inf_{\phi L \in \mathscr{K}_0} \left\{\big(\boldsymbol{C}_{p,q}(\boldsymbol{\phi K}, (\phi L)^\circ, (\phi L)^\circ, \cdots, (\phi L)^\circ)\big)^{\frac{n}{n+q}} \cdot |\phi L|^{\frac{q}{n+q}}\right\} \\ &= \inf_{L \in \mathscr{K}_0} \left\{\big(\boldsymbol{C}_{p,q}(\boldsymbol{K}, L^\circ, L^\circ, \cdots, L^\circ)\big)^{\frac{n}{n+q}} \cdot |L|^{\frac{q}{n+q}}\right\} \\ &= \boldsymbol{\mathscr{G}}_{p,q}(\boldsymbol{K}). \end{aligned}$$

The case $-n \ne q < 0$ follows along the same lines. $\square$

For $\boldsymbol{\mathscr{A}}_{p,q}(\cdot)$, we have the following result.

Proposition 5.5. *Let* $\boldsymbol{K} = (K_1, K_2, \cdots, K_m) \in (\mathscr{F}_0^+)^m$.

(i) If $q \geq 0$, then

$$\boldsymbol{\mathscr{A}}_{p,q}(\boldsymbol{K}) = \inf_{\boldsymbol{L} \in (\mathscr{S}_0)^m} \left\{ \big(\boldsymbol{C}_{p,q}(\boldsymbol{K}, \boldsymbol{L}^\circ)\big)^{\frac{n}{n+q}} \cdot \prod_{i=1}^m |L_i|^{\frac{q}{m(n+q)}} \right\}$$
$$= \inf_{\boldsymbol{L} \in (\mathscr{S}_0)^m} \left\{ \big(\boldsymbol{C}_{p,q}(\boldsymbol{K}, \boldsymbol{L}^\circ)\big)^{\frac{n}{n+q}} \cdot \tilde{V}(\boldsymbol{L})^{\frac{q}{n+q}} \right\}.$$

(ii) If $-n < q < 0$, then

$$\boldsymbol{\mathscr{A}}_{p,q}(\boldsymbol{K}) = \sup_{\boldsymbol{L} \in (\mathscr{S}_0)^m} \left\{ \big(\boldsymbol{C}_{p,q}(\boldsymbol{K}, \boldsymbol{L}^\circ)\big)^{\frac{n}{n+q}} \cdot \prod_{i=1}^m |L_i|^{\frac{q}{m(n+q)}} \right\}$$
$$= \sup_{\boldsymbol{L} \in (\mathscr{S}_0)^m} \left\{ \big(\boldsymbol{C}_{p,q}(\boldsymbol{K}, \boldsymbol{L}^\circ)\big)^{\frac{n}{n+q}} \cdot \tilde{V}(\boldsymbol{L})^{\frac{q}{n+q}} \right\}.$$

(iii) If $q < -n$, then

$$\boldsymbol{\mathscr{A}}_{p,q}(\boldsymbol{K}) = \sup_{\boldsymbol{L} \in (\mathscr{S}_0)^m} \left\{ \big(\boldsymbol{C}_{p,q}(\boldsymbol{K}, \boldsymbol{L}^\circ)\big)^{\frac{n}{n+q}} \cdot \tilde{V}(\boldsymbol{L})^{\frac{q}{n+q}} \right\}.$$

Proof. For $\boldsymbol{L} = (L_1, L_2, \cdots, L_m) \in (\mathscr{S}_0)^m$, define a star body Q associated with $\boldsymbol{L}$ by

$$\rho_Q^m(u) = \prod_{i=1}^m \rho_{L_i}(u) \quad \text{for any } u \in S^{n-1}. \tag{5.19}$$

Thus it can be easily checked that $\tilde{V}(\boldsymbol{L}) = |Q|$ and $\boldsymbol{C}_{p,q}(\boldsymbol{K}, \boldsymbol{L}^\circ) = \boldsymbol{C}_{p,q}\big(\boldsymbol{K}, Q^\circ, \cdots, Q^\circ\big)$.

(i) The case $q = 0$ is trivial. If $q > 0$, as $\tilde{V}(\boldsymbol{L}) \leq \prod_{i=1}^m |L_i|^{1/m}$ for any $\boldsymbol{L} \in (\mathscr{S}_0)^m$ and $\dfrac{q}{n+q} > 0$, one has

$$\inf_{\boldsymbol{L} \in (\mathscr{S}_0)^m} \left\{ \big(\boldsymbol{C}_{p,q}(\boldsymbol{K}, \boldsymbol{L}^\circ)\big)^{\frac{n}{n+q}} \cdot \tilde{V}(\boldsymbol{L})^{\frac{q}{n+q}} \right\} \leq \inf_{\boldsymbol{L} \in (\mathscr{S}_0)^m} \left\{ \big(\boldsymbol{C}_{p,q}(\boldsymbol{K}, \boldsymbol{L}^\circ)\big)^{\frac{n}{n+q}} \cdot \prod_{i=1}^m |L_i|^{\frac{q}{m(n+q)}} \right\}$$
$$\leq \boldsymbol{\mathscr{A}}_{p,q}(\boldsymbol{K}). \tag{5.20}$$

On the other hand, let $\{\boldsymbol{L_i}\}_{i=1}^{\infty} \subseteq (\mathscr{S}_0)^m$ be a sequence such that

$$\inf_{\boldsymbol{L}\in(\mathscr{S}_0)^m} \left\{ \big(\boldsymbol{C}_{p,q}(\boldsymbol{K},\boldsymbol{L}^{\circ})\big)^{\frac{n}{n+q}} \cdot \tilde{V}(\boldsymbol{L})^{\frac{q}{n+q}} \right\} = \lim_{i\to\infty} \big(\boldsymbol{C}_{p,q}(\boldsymbol{K},\boldsymbol{L_i}^{\circ})\big)^{\frac{n}{n+q}} \cdot \tilde{V}(\boldsymbol{L_i})^{\frac{q}{n+q}}.$$

Letting Q_i be the star body associated with $\boldsymbol{L_i}$ as in (5.19), one has

$$\begin{aligned} \inf_{\boldsymbol{L}\in(\mathscr{S}_0)^m} \left\{ \big(\boldsymbol{C}_{p,q}(\boldsymbol{K},\boldsymbol{L}^{\circ})\big)^{\frac{n}{n+q}} \cdot \tilde{V}(\boldsymbol{L})^{\frac{q}{n+q}} \right\} &= \lim_{i\to\infty} \big(\boldsymbol{C}_{p,q}(\boldsymbol{K},\boldsymbol{L_i}^{\circ})\big)^{\frac{n}{n+q}} \cdot \tilde{V}(\boldsymbol{L_i})^{\frac{q}{n+q}} \\ &= \lim_{i\to\infty} \big(\boldsymbol{C}_{p,q}\big(\boldsymbol{K},Q_i^{\circ},\cdots,Q_i^{\circ}\big)\big)^{\frac{n}{n+q}} \cdot |Q_i|^{\frac{q}{n+q}} \\ &\geq \boldsymbol{\mathscr{A}}_{p,q}(\boldsymbol{K}). \end{aligned}$$

This, together with (5.20), gives the desired result.

(ii) For $q \in (-n,0)$, one has $\dfrac{q}{n+q} < 0$ and the desired argument follows along the lines in (i) with "inf" replaced by "sup" and "$\leq$" replaced by "$\geq$".

(iii) For $q < -n$, one has $\dfrac{q}{n+q} > 0$. Similar to (5.20), one has

$$\sup_{\boldsymbol{L}\in(\mathscr{S}_0)^m} \left\{ \big(\boldsymbol{C}_{p,q}(\boldsymbol{K},\boldsymbol{L}^{\circ})\big)^{\frac{n}{n+q}} \cdot \tilde{V}(\boldsymbol{L})^{\frac{q}{n+q}} \right\} \geq \boldsymbol{\mathscr{A}}_{p,q}(\boldsymbol{K}).$$

The result follows along the lines in (i) again with "inf" replaced by "sup" and "$\leq$" replaced by "$\geq$". $\square$

For $-n \neq q \in \mathbb{R}$, define $\boldsymbol{\xi}_{\mu_p,q}$, a subset of $(\mathscr{F}_0^+)^m$, by

$$\boldsymbol{\xi}_{\mu_p,q} = \left\{ \boldsymbol{K} : \exists Q \in \mathscr{S}_0 \ \text{ s.t. } \left(\prod_{i=1}^{m} f_{\mu_p,q}(K_i,u) \right)^{\frac{1}{m}} = \big(\rho_Q(u)\big)^{n+q} \text{ for any } u \in S^{n-1} \right\}.$$

One can easily check that $(B_2^n,\cdots,B_2^n) \in \boldsymbol{\xi}_{\mu_p,q}$, and hence $\boldsymbol{\xi}_{\mu_p,q} \neq \emptyset$. In general, it is difficult to get the precise value of $\boldsymbol{\mathscr{A}}_{p,q}(\boldsymbol{K})$. However, the following proposition provides a convenient formula to calculate $\boldsymbol{\mathscr{A}}_{p,q}(\boldsymbol{K})$ if $\boldsymbol{K} \in \boldsymbol{\xi}_{\mu_p,q}$.

Proposition 5.6. *Let* $\boldsymbol{K}=(K_1,K_2,\cdots,K_m)\in\boldsymbol{\xi}_{\mu_p,q}$. *Then, for any* $-n\neq q\in\mathbb{R}$,

$$\mathscr{A}_{p,q}(\boldsymbol{K})=\Big(\frac{1}{n}\Big)^{\frac{q}{n+q}}\cdot\Big(\frac{p-1}{n-p}\Big)^{\frac{n}{n+q}}\cdot\int_{S^{n-1}}\bigg(\prod_{i=1}^m f_{\mu_p,q}(K_i,u)\bigg)^{\frac{n}{m(n+q)}}d\sigma(u).$$

Proof. Let $L\in\mathscr{S}_0$.

(i) The case $q=0$ is trivial as

$$\mathscr{A}_{p,0}(\boldsymbol{K})=\frac{p-1}{n-p}\int_{S^{n-1}}\bigg(\prod_{i=1}^m f_{\mu_p,0}(K_i,u)\bigg)^{\frac{1}{m}}d\sigma(u).$$

(ii) $q>0$. By Hölder inequality [24], one has

$$\begin{aligned}
&\Big(\frac{1}{n}\Big)^{\frac{q}{n+q}}\cdot\Big(\frac{p-1}{n-p}\Big)^{\frac{n}{n+q}}\cdot\int_{S^{n-1}}\bigg(\prod_{i=1}^m f_{\mu_p,q}(K_i,u)\bigg)^{\frac{n}{m(n+q)}}d\sigma(u)\\
&=\Big(\frac{1}{n}\Big)^{\frac{q}{n+q}}\cdot\Big(\frac{p-1}{n-p}\Big)^{\frac{n}{n+q}}\cdot\int_{S^{n-1}}\bigg[\rho_L^{-q}(u)\bigg(\prod_{i=1}^m f_{\mu_p,q}(K_i,u)\bigg)^{\frac{1}{m}}\rho_L^q(u)\bigg]^{\frac{n}{n+q}}d\sigma(u)\\
&\le\bigg(\frac{p-1}{n-p}\int_{S^{n-1}}\rho_L^{-q}(u)\bigg(\prod_{i=1}^m f_{\mu_p,q}(K_i,u)\bigg)^{\frac{1}{m}}d\sigma(u)\bigg)^{\frac{n}{n+q}}\bigg(\frac{1}{n}\int_{S^{n-1}}\rho_L^n(u)d\sigma(u)\bigg)^{\frac{q}{n+q}}\\
&=\big(\boldsymbol{C}_{p,q}(\boldsymbol{K},\underbrace{L^\circ,L^\circ,\cdots,L^\circ}_{m})\big)^{\frac{n}{n+q}}\cdot|L|^{\frac{q}{n+q}}.
\end{aligned}$$

Taking the infimum over $L\in\mathscr{S}_0$, one gets

$$\Big(\frac{1}{n}\Big)^{\frac{q}{n+q}}\cdot\Big(\frac{p-1}{n-p}\Big)^{\frac{n}{n+q}}\cdot\int_{S^{n-1}}\bigg(\prod_{i=1}^m f_{\mu_p,q}(K_i,u)\bigg)^{\frac{n}{m(n+q)}}d\sigma(u)\le\mathscr{A}_{p,q}(\boldsymbol{K}). \tag{5.21}$$

On the other hand, due to the fact that $\boldsymbol{K}\in\boldsymbol{\xi}_{\mu_p,q}$, there exists a star body Q such that

$$\bigg(\prod_{i=1}^m f_{\mu_p,q}(K_i,u)\bigg)^{\frac{1}{m}}=\rho_Q^{n+q}(u)\ \text{ for any } u\in S^{n-1}.$$

Hence one gets

$$\begin{aligned}\mathscr{A}_{p,q}(\boldsymbol{K}) &\leq \big(\boldsymbol{C}_{p,q}(\boldsymbol{K}, Q^\circ, \cdots, Q^\circ)\big)^{\frac{n}{n+q}} \cdot |Q|^{\frac{q}{n+q}} \\ &= \Big(\frac{1}{n}\Big)^{\frac{q}{n+q}} \cdot \Big(\frac{p-1}{n-p}\Big)^{\frac{n}{n+q}} \cdot \int_{S^{n-1}} \Big(\prod_{i=1}^m f_{\mu_p,q}(K_i,u)\Big)^{\frac{n}{m(n+q)}} d\sigma(u).\end{aligned}$$

This, together with (5.21), gives the desired result.

(iii) $-n \neq q < 0$. Along the same lines, one can prove the case $-n \neq q < 0$. □

The following result can be obtained.

Corollary 5.3. *Let* $\boldsymbol{K} = (K_1, K_2, \cdots, K_m) \in \boldsymbol{\xi}_{\mu_p,q}$ *and* $-n \neq q \in \mathbb{R}$. *Then*

$$|\Lambda_{\mu_p,q}K_1|^n \cdots |\Lambda_{\mu_p,q}K_m|^n \cdot \mathscr{A}_{p,q}(\boldsymbol{K})^{m(n+q)} = \tilde{V}\big(\Lambda_{\mu_p,q}K_1, \cdots, \Lambda_{\mu_p,q}K_m\big)^{m(n+q)}.$$

Proof. By Remark 5.2 and Proposition 5.6, one has

$$\begin{aligned}&\tilde{V}\big(\Lambda_{\mu_p,q}K_1, \cdots, \Lambda_{\mu_p,q}K_m\big) \\ &= \frac{1}{n}\int_{S^{n-1}} \Big(\prod_{i=1}^m \rho_{\Lambda_{\mu_p,q}K_i}(u)\Big)^{\frac{n}{m}} d\sigma(u) \\ &= \frac{1}{n}\int_{S^{n-1}} \Big(\prod_{i=1}^m \frac{n(p-1)|\Lambda_{\mu_p,q}K_i|}{n-p} \cdot f_{\mu_p,q}(K_i,u)\Big)^{\frac{n}{m(n+q)}} d\sigma(u) \\ &= \frac{1}{n} \cdot \Big(\prod_{i=1}^m n|\Lambda_{\mu_p,q}K_i|\Big)^{\frac{n}{m(n+q)}} \cdot \Big(\frac{p-1}{n-p}\Big)^{\frac{n}{n+q}} \int_{S^{n-1}} \Big(\prod_{i=1}^m f_{\mu_p,q}(K_i,u)\Big)^{\frac{n}{m(n+q)}} d\sigma(u) \\ &= \Big(\prod_{i=1}^m |\Lambda_{\mu_p,q}K_i|\Big)^{\frac{n}{m(n+q)}} \cdot \mathscr{A}_{p,q}(\boldsymbol{K}).\end{aligned}$$

This yields the desired result. □

Let $-n \neq q \in \mathbb{R}$. We define $\boldsymbol{\nu}_{\mu_p,q}$, a subset of $(\mathscr{F}_0^+)^m$, as follows:

$$\boldsymbol{\nu}_{\mu_p,q} = \left\{ \boldsymbol{K} : \exists Q \in \mathscr{K}_0 \ \text{ s.t. } \left(\prod_{i=1}^m f_{\mu_p,q}(K_i, u) \right)^{\frac{1}{m}} = \big(\rho_Q(u)\big)^{n+q} \text{ for any } u \in S^{n-1} \right\}.$$

The following proposition provides a convenient formula to calculate $\boldsymbol{\mathscr{G}}_{p,q}(\boldsymbol{K})$ for $\boldsymbol{K} \in \boldsymbol{\nu}_{\mu_p,q}$. In particular,

$$\boldsymbol{\mathscr{G}}_{p,q}(B_2^n, \cdots, B_2^n) = \boldsymbol{\mathscr{A}}_{p,q}(B_2^n, \cdots, B_2^n) = \mathscr{A}_{p,q}(B_2^n) = \big(C_p(B_2^n)\big)^{\frac{n}{n+q}} \cdot |B_2^n|^{\frac{q}{n+q}}. \quad (5.22)$$

Proposition 5.7. *Let* $\boldsymbol{K} = (K_1, K_2, \cdots, K_m) \in \boldsymbol{\nu}_{\mu_p,q}$ *and* $-n \neq q \in \mathbb{R}$. *Then*

$$\boldsymbol{\mathscr{G}}_{p,q}(\boldsymbol{K}) = \boldsymbol{\mathscr{A}}_{p,q}(\boldsymbol{K}).$$

Proof. Due to $\boldsymbol{K} = (K_1, K_2, \cdots, K_m) \in \boldsymbol{\nu}_{\mu_p,q}$, we can define $L \in \mathscr{K}_0$ by its radial function:

$$\big(\rho_L(u)\big)^{n+q} = \left(\prod_{i=1}^m f_{\mu_p,q}(K_i, u) \right)^{\frac{1}{m}} \text{ for any } u \in S^{n-1}.$$

When $q = 0$, the desired formula follows trivially, i.e.,

$$\boldsymbol{\mathscr{G}}_{p,0}(\boldsymbol{K}) = \boldsymbol{\mathscr{A}}_{p,0}(\boldsymbol{K}) = \frac{p-1}{n-p} \int_{S^{n-1}} \left(\prod_{i=1}^m f_{\mu_p,0}(K_i, u) \right)^{\frac{1}{m}} d\sigma(u).$$

If $q > 0$, it follows from the proof of Proposition 5.6 and $L \in \mathscr{K}_0$ that

$$\boldsymbol{\mathscr{G}}_{p,q}(\boldsymbol{K}) \geq \boldsymbol{\mathscr{A}}_{p,q}(\boldsymbol{K}) = \big(\boldsymbol{C}_{p,q}(\boldsymbol{K}, L^\circ, \cdots, L^\circ)\big)^{\frac{n}{n+q}} \cdot |L|^{\frac{q}{n+q}} \geq \boldsymbol{\mathscr{G}}_{p,q}(\boldsymbol{K}).$$

Hence $\boldsymbol{\mathscr{G}}_{p,q}(\boldsymbol{K}) = \boldsymbol{\mathscr{A}}_{p,q}(\boldsymbol{K})$. The case $-n \neq q < 0$ follows from a similar argument. $\square$

Similar to Theorem 5.3, we have the following cyclic inequalities for $\boldsymbol{\mathscr{G}}_{p,q}(\cdot)$. Similar results hold for $\boldsymbol{\mathscr{A}}_{p,q}(\cdot)$.

Theorem 5.5. *Let $\boldsymbol{K} \in (\mathscr{F}_0^+)^m$.*

(i) If $-n < t < 0 < r < s$ or $-n < s < 0 < r < t$, then

$$\boldsymbol{\mathscr{G}}_{p,r}(\boldsymbol{K}) \leq \big(\boldsymbol{\mathscr{G}}_{p,t}(\boldsymbol{K})\big)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} \cdot \big(\boldsymbol{\mathscr{G}}_{p,s}(\boldsymbol{K})\big)^{\frac{(r-t)(n+s)}{(s-t)(n+r)}}.$$

(ii) If $-n < t < r < s < 0$ or $-n < s < r < t < 0$, then

$$\boldsymbol{\mathscr{G}}_{p,r}(\boldsymbol{K}) \leq \big(\boldsymbol{\mathscr{G}}_{p,t}(\boldsymbol{K})\big)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} \cdot \big(\boldsymbol{\mathscr{G}}_{p,s}(\boldsymbol{K})\big)^{\frac{(r-t)(n+s)}{(s-t)(n+r)}}.$$

(iii) If $t < r < -n < s < 0$ or $s < r < -n < t < 0$, then

$$\boldsymbol{\mathscr{G}}_{p,r}(\boldsymbol{K}) \geq \big(\boldsymbol{\mathscr{G}}_{p,t}(\boldsymbol{K})\big)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} \cdot \big(\boldsymbol{\mathscr{G}}_{p,s}(\boldsymbol{K})\big)^{\frac{(r-t)(n+s)}{(s-t)(n+r)}}.$$

Next we provide the Aleksandrov-Fenchel inequality for $\boldsymbol{\mathscr{G}}_{p,q}(\cdot)$. Similar results can be obtained for $\boldsymbol{\mathscr{A}}_{p,q}(\cdot)$.

Theorem 5.6. *Let $\boldsymbol{K} \in (\mathscr{F}_0^+)^m$. For $1 \leq j \leq m$ and $-n < q < 0$, one has*

$$\big(\boldsymbol{\mathscr{G}}_{p,q}(\boldsymbol{K})\big)^j \leq \prod_{i=1}^{j} \boldsymbol{\mathscr{G}}_{p,q}(K_1, K_2, \cdots, K_{m-j}; \underbrace{K_{m-j+i}, K_{m-j+i}, \cdots, K_{m-j+i}}_{j}).$$

Moreover, if $j = m$, one has

$$\big(\boldsymbol{\mathscr{G}}_{p,q}(\boldsymbol{K})\big)^m \leq \prod_{i=1}^{m} \mathscr{G}_{p,q}(K_i).$$

Proof. By Hölder inequality [24], it can be checked that

$$\big(\boldsymbol{C}_{p,q}(\boldsymbol{K}, L, \cdots, L)\big)^j \leq \prod_{i=1}^{j} \boldsymbol{C}_{p,q}(K_1, \cdots, K_{m-j}; \underbrace{K_{m-j+i}, \cdots, K_{m-j+i}}_{j}; \underbrace{L, \cdots, L}_{m}). \tag{5.23}$$

Together with Definition 5.4 and $\dfrac{n}{n+q} > 0$, one gets

$$\begin{aligned}
&\big(\boldsymbol{\mathscr{G}}_{p,q}(\boldsymbol{K})\big)^j \\
&= \sup_{L \in \mathscr{K}_0} \Big\{ \big(\boldsymbol{C}_{p,q}(\boldsymbol{K}, L, \cdots, L)\big)^{\frac{nj}{n+q}} \cdot |L^\circ|^{\frac{qj}{n+q}} \Big\} \\
&\leq \prod_{i=1}^{j} \sup_{L \in \mathscr{K}_0} \Big\{ \big(\boldsymbol{C}_{p,q}(K_1, \cdots, K_{m-j}; \underbrace{K_{m-j+i}, \cdots, K_{m-j+i}}_{j}; L, \cdots, L)\big)^{\frac{n}{n+q}} \cdot |L^\circ|^{\frac{q}{n+q}} \Big\} \\
&\leq \prod_{i=1}^{j} \boldsymbol{\mathscr{G}}_{p,q}(K_1, K_2, \cdots, K_{m-j}; \underbrace{K_{m-j+i}, K_{m-j+i}, \cdots, K_{m-j+i}}_{j}).
\end{aligned}$$

This proves the desired Aleksandrov-Fenchel inequality. $\square$

Corollary 5.4. *Let* $\boldsymbol{K} = (K_1, K_2, \cdots, K_m) \in (\mathscr{F}_0^+)^m$.

(i) If $q \geq 0$, *then*

$$\frac{\boldsymbol{\mathscr{G}}_{p,q}(K_1, K_2, \cdots, K_m)}{\boldsymbol{\mathscr{G}}_{p,q}(\underbrace{B_2^n, B_2^n, \cdots, B_2^n}_{m})} \leq \prod_{i=1}^{m} \left(\frac{C_{p,q}(K_i, B_2^n)}{C_{p,q}(B_2^n, B_2^n)} \right)^{\frac{n}{m(n+q)}}.$$

Equality holds if $K_i = r_i B_2^n$ *with* $r_i > 0$ *for any* $1 \leq i \leq m$ *and* $\prod_{i=1}^m r_1 = 1$.

(ii) If $q < -n$, *then*

$$\frac{\boldsymbol{\mathscr{G}}_{p,q}(K_1, K_2, \cdots, K_m)}{\boldsymbol{\mathscr{G}}_{p,q}(\underbrace{B_2^n, B_2^n, \cdots, B_2^n}_{m})} \geq \prod_{i=1}^{m} \left(\frac{C_{p,q}(K_i, B_2^n)}{C_{p,q}(B_2^n, B_2^n)} \right)^{\frac{n}{m(n+q)}}.$$

Equality holds if $K_i = r_i B_2^n$ *with* $r_i > 0$ *for any* $1 \le i \le m$ *and* $\prod_{i=1}^m r_1 = 1$.

Proof. (i) By Definition 5.4 and (5.23), one has

$$\begin{aligned}
\boldsymbol{\mathscr{G}}_{p,q}(\boldsymbol{K}) &= \inf_{L \in \mathscr{K}_0} \left\{ \left(\boldsymbol{C}_{p,q}(\boldsymbol{K}, L^\circ, \cdots, L^\circ) \right)^{\frac{n}{n+q}} \cdot |L|^{\frac{q}{n+q}} \right\} \\
&\le \left(\boldsymbol{C}_{p,q}(\boldsymbol{K}, B_2^n, \cdots, B_2^n) \right)^{\frac{n}{n+q}} \cdot |B_2^n|^{\frac{q}{n+q}} \\
&\le |B_2^n|^{\frac{q}{n+q}} \cdot \prod_{i=1}^m \left(C_{p,q}(K_i, B_2^n) \right)^{\frac{n}{m(n+q)}}.
\end{aligned}$$

This, along with (5.22), yields the desired result. If $K_i = r_i B_2^n$ for any $1 \le i \le m$ and $\prod_{i=1}^m r_i = 1$, it can be easily checked that

$$\frac{\boldsymbol{\mathscr{G}}_{p,q}(K_1, K_2, \cdots, K_m)}{\boldsymbol{\mathscr{G}}_{p,q}(\underbrace{B_2^n, B_2^n, \cdots, B_2^n}_{m})} = \prod_{i=1}^m \left(\frac{C_{p,q}(K_i, B_2^n)}{C_{p,q}(B_2^n, B_2^n)} \right)^{\frac{n}{m(n+q)}} = 1.$$

The assertion (ii) follows from the same lines. $\square$

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