Towards the Orlicz-Brunn-Minkowski theory for geominimal surface areas and capacity

by

© **Han Hong**

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Memorial University of Newfoundland

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Abstract

This thesis is dedicated to study Orlicz-Petty bodies, the p-capacitary Orlicz-Brunn-Minkowski theory and the general p-affine capacity as well as isocapacitary inequalities.

In the second chapter, the homogeneous Orlicz affine and geominimal surface areas are defined and their basic properties are established including homogeneity, affine invariance and continuity. Some related affine isoperimetric inequalities are proved. Similar results for the nonhomogeneous ones are proved as well. In the third chapter, we develop the p-capacitary Orlicz-Brunn-Minkowski theory by combining the p-capacity for $p \in (1, n)$ with the Orlicz addition of convex domains. In particular, Orlicz-Brunn-Minkowski type and Orlicz-Minkowski type inequalities are proved. In the last chapter, the general p-affine capacity for $p \in [1, n)$ is defined and its properties are discussed. Furthermore, the newly proposed general p-affine capacity is compared with many classical geometric quantities, e.g., the volume, the p-variational capacity and the p-integral affine surface area. Consequently, several sharp geometric inequalities for the general p-affine capacity are obtained. Theses inequalities extend and strengthen many well-known (affine) isoperimetric and (affine) isocapacitary inequalities.

Key words: Orlicz affine and geominimal surface areas, Orlicz-Brunn-Minkowski theory, Orlicz-Petty bodies, L_p projection body, p-capacity, p-affine capacity, p-integral affine surface area, isocapacitary inequalities, L_p affine isoperimetric inequalities, L_p affine Sobolev inequalities.

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Contents

Chapter 1

Backgrounds and Introduction

This chapter is dedicated to provide an overview of our main results and some backgrounds related to our topics. Refer to [20, 70] for more details and motivations.

1.1 Backgrounds

1.1.1 Basic facts about convex geometry

We now introduce the basic well-known facts and standard notations needed in this thesis. For more details and more concepts in convex geometry, please see [19, 26, 70].

A convex and compact subset $K \subset \mathbb{R}^n$ with nonempty interior is called a convex body in $\mathbb{R}^n$. By $\mathscr{K}$ we mean the set of all convex bodies containing the origin o and by $\mathscr{K}_0$ the set of all convex bodies with the origin in their interiors. A convex body K is said to be origin-symmetric if $K = -K$ where $-K = \{x \in \mathbb{R}^n : -x \in K\}$. Let $\mathscr{K}_e$ denote the set of all origin-symmetric convex bodies in $\mathbb{R}^n$. The volume of K is denoted by $|K|$ and the volume radius of K is denoted by $\mathrm{vrad}(K)$. By B_2^n and S^{n-1}, we mean the Euclidean unit ball and the unit sphere in $\mathbb{R}^n$ respectively. The volume of B_2^n will be often written by ω_n and the natural spherical measure on S^{n-1} is written by σ. Consequently, $\mathrm{vrad}(K) = (|K|/\omega_n)^{1/n}$. It is well known that

$$\omega_n = \frac{\pi^{n/2}}{\Gamma(1 + n/2)},$$

where $\Gamma(\cdot)$ is the Gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt.$$

Beta function $B(\cdot,\cdot)$ is closely related to Gamma function, and it has the form

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt.$$

It is easily checked that

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The standard notation $GL(n)$ stands for the set of all invertible linear transforms on $\mathbb{R}^n$. For $A \in GL(n)$, we use $\det A$ to denote the determinant of A. Let $SL(n) = \{A : \ A \in GL(n) \text{ and } \det A = \pm 1\}$. By A^t and A^{-t} we mean the transpose of A and the inverse of A^t respectively. For a set $E \in \mathbb{R}^n$, define *conv*(E) the convex hull of E, to be the smallest convex set containing E.

Each convex body $K \in \mathscr{K}$ has a continuous support function $h_K : S^{n-1} \to [0,\infty)$ defined by $h_K(u) = \max_{x\in K}\langle x, u\rangle$ for $u \in S^{n-1}$, where $\langle\cdot,\cdot\rangle$ denotes the usual inner product. Note that h_K for $K \in \mathscr{K}$ is nonnegative on S^{n-1}, but it is strictly positive on S^{n-1} if $K \in \mathscr{K}_0$. The support function $h_K : S^{n-1} \to (0,\infty)$ of a convex body $K \in \mathscr{K}_0$ can be extended to $\mathbb{R}^n \setminus \{o\}$ as follows: $h_K(x) = rh_K(u)$ for any $x \in \mathbb{R}\setminus\{o\}$ with $x = ru$. It can be easily checked that the extended function $h_K : \mathbb{R}^n\setminus\{o\} \to (0,\infty)$ has the positive homogeneity of degree 1 and is also subadditive: $h_K(x+y) \le h_K(x) + h_K(y)$ for all $x, y \in \mathbb{R}^n \setminus \{o\}$. Conversely, if a function $h : \mathbb{R}^n \setminus \{o\} \to (0,\infty)$ has the positive homogeneity of degree 1 and is also subadditive, then h must be a support function of a convex body $K \in \mathscr{K}_0$ [70].

One can define a probability measure $\widetilde{V}_K$ on each $K \in \mathscr{K}$ by

$$d\widetilde{V}_K(u) = \frac{h_K(u)\,dS_K(u)}{n|K|} \quad \text{for } \ u \in S^{n-1},$$

where S_K is the surface area measure of K. It is well known that S_K satisfies

$$\int_{S^{n-1}} u\,dS_K(u) = o \ \text{ and } \ \int_{S^{n-1}} |\langle u, v\rangle|\,dS_K(u) > 0 \text{ for each } v \in S^{n-1}. \qquad (1.1.1)$$

The first formula of (1.1.1) asserts that S_K has its centroid at the origin and the second one states that S_K is not concentrated on any great subsphere. Let $\nu_K(x)$

denote a unit outer normal vector of $x \in \partial K$. For each $f \in C(S^{n-1})$, where $C(S^{n-1})$ denotes the set of all continuous functions defined on S^{n-1}, one has

$$\int_{S^{n-1}} f(u)\, dS_K(u) = \int_{\partial K} f(\nu_K(x))\, d\mathscr{H}^{n-1}(x). \tag{1.1.2}$$

The dilation of K is of form $sK = \{sx : x \in K\}$ for $s > 0$. Clearly, $h_{sK}(u) = s \cdot h_K(u)$ for all $u \in S^{n-1}$. Moreover, sK and K share the same probability measure $d\widetilde{V}_K(u)$. Two convex bodies K and L are said to be dilates of each other if $K = sL$ for some constant $s > 0$.

A compact set $M \subset \mathbb{R}^n$ is said to be a star body (with respect to the origin o) if the line segment jointing o and x is contained in M, for all $x \in M$. For each star body M, one can define the radial function ρ_M of M as follows: for all $x \in \mathbb{R}^n \setminus \{o\}$,

$$\rho_M(x) = \max\{\lambda \geq 0 : \ \lambda x \in M\}.$$

The star body M is said to be a Lipschitz star body if the boundary of M is Lipschitz.

Denote by $\mathscr{S}_0$ the set of star bodies about the origin in $\mathbb{R}^n$ and clearly $\mathscr{K}_0 \subset \mathscr{S}_0$. The volume of $L \in \mathscr{S}_0$ can be calculated by

$$|L| = \frac{1}{n} \int_{S^{n-1}} \rho_L^n(u)\, d\sigma(u) \quad \text{and} \quad |K^\circ| = \frac{1}{n} \int_{S^{n-1}} \frac{1}{h_K^n(u)}\, d\sigma(u). \tag{1.1.3}$$

Hereafter, $K^\circ \in \mathscr{K}_0$ is the polar body of $K \in \mathscr{K}_0$; and the support function h_{K° and the radial function ρ_{K° are given by

$$h_{K^\circ}(u) = \frac{1}{\rho_K(u)} \quad \text{and} \quad \rho_{K^\circ}(u) = \frac{1}{h_K(u)}, \quad \text{for all } \ u \in S^{n-1}.$$

Alternatively, K° can be defined by

$$K^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \ \text{ for all } y \in K\}.$$

The bipolar theorem states that $(K^\circ)^\circ = K$ if $K \in \mathscr{K}_0$.

Let $\mathscr{K}_c \subset \mathscr{K}_0$ be the set of convex bodies with their centroids at origin; that is, $\int_K x\, dx = 0$ if $K \in \mathscr{K}_c$. We say $K \in \mathscr{K}_0$ has the Santaló point at the origin if $K^\circ \in \mathscr{K}_c$. Denote by $\mathscr{K}_s \subset \mathscr{K}_0$ the set of convex bodies with their Santaló points at the origin, and let $\widetilde{\mathscr{K}} = \mathscr{K}_s \cup \mathscr{K}_c$. The set $\widetilde{\mathscr{K}}$ is important in the famous Blaschke-Santaló inequality: for $K \in \widetilde{\mathscr{K}}$, one has

$$|K| \cdot |K^\circ| \leq \omega_n^2$$

with equality if and only if K is an origin-symmetric ellipsoid (i.e., $K = A(B_2^n)$ for some $A \in GL(n)$).

On the set $\mathscr{K}$, we consider the topology generated by the Hausdorff distance $d_H(\cdot,\cdot)$. For $K, K' \in \mathscr{K}$, define $d_H(K, K')$ by

$$d_H(K, K') = \|h_K - h_{K'}\|_\infty = \sup_{u \in S^{n-1}} |h_K(u) - h_{K'}(u)|.$$

A sequence $\{K_i\}_{i\geq 1} \subset \mathscr{K}$ is said to be convergent to a convex body K_0 if $d_H(K_i, K_0) \to 0$ as $i \to \infty$. Note that if $K_i \to K_0$ in the Hausdorff distance, then S_{K_i} is weakly convergent to S_{K_0}. That is, for all $f \in C(S^{n-1})$, one has

$$\lim_{i\to\infty} \int_{S^{n-1}} f(u)\, dS_{K_i}(u) = \int_{S^{n-1}} f(u)\, dS_{K_0}(u).$$

We will use a modified form of the above limit: if $\{f_i\}_{i\geq 1} \subset C(S^{n-1})$ is uniformly convergent to $f_0 \in C(S^{n-1})$ and $\{K_i\}_{i\geq 1} \subset \mathscr{K}$ converges to $K_0 \in \mathscr{K}$ in the Hausdorff distance, then

$$\lim_{i\to\infty} \int_{S^{n-1}} f_i(u)\ dS_{K_i}(u) = \int_{S^{n-1}} f_0(u)\ dS_{K_0}(u). \tag{1.1.4}$$

The Blaschke selection theorem is a powerful tool in convex geometry (see e.g., [26, 70]) and will be often used in this thesis. It reads: *every bounded sequence of convex bodies has a subsequence that converges to a compact convex subset of* $\mathbb{R}^n$.

The following result, proved by Lutwak [49], is essential for our main results.

Lemma 1.1.1. *Let $\{K_i\}_{i\geq 1} \subset \mathscr{K}_0$ be a convergent sequence with limit K_0, i.e., $K_i \to K_0$ in the Hausdorff distance. If the sequence $\{|K_i^\circ|\}_{i\geq 1}$ is bounded, then $K_0 \in \mathscr{K}_0$.*

Associated to each $f \in C^+(S^{n-1})$, the set of positive functions in $C(S^{n-1})$, one can define a convex body $K_f \in \mathscr{K}_0$ by

$$K_f = \bigcap_{u \in S^{n-1}} \Big\{x \in \mathbb{R}^n : \langle x, u\rangle \leq f(u)\Big\}.$$

The convex body K_f is called the Aleksandrov body (or Aleksandrov domain in the case of capacity) associated to $f \in C^+(S^{n-1})$. The Aleksandrov body provides a powerful tool in convex geometry and plays crucial roles in this thesis. Here we list some important properties for the Aleksandrov body which will be used in later context. First of all, if $f \in C^+(S^{n-1})$ is the support function of a convex body $K \in \mathscr{K}_0$, then $K = K_f$. Secondly, for $f \in C^+(S^{n-1})$, $h_{K_f}(u) \leq f(u)$ for all $u \in S^{n-1}$, and $h_{K_f}(u) = f(u)$ almost everywhere with respect to $S(K_f, \cdot)$, the surface area measure of K_f defined on S^{n-1}. Furthermore, the convergence of $\{K_{f_m}\}_{m\geq 1}$ in the Hausdorff metric is guaranteed by the convergence of $\{f_m\}_{m\geq 1}$. This is the Aleksandrov's convergence lemma [1]: if the sequence $f_1, f_2, \cdots \in C^+(S^{n-1})$ converges to $f \in C^+(S^{n-1})$ uniformly, then $K_{f_1}, K_{f_2}, \cdots \in \mathscr{K}_0$ converges to $K_f \in \mathscr{K}_0$ with respect to the Hausdorff metric.

1.1.2 Orlicz addition and Orlicz-Brunn-Minkowski theory

Let $m \geq 1$ be an integer number. Denote by Φ_m the set of convex functions $\varphi : [0,\infty)^m \to [0,\infty)$ that are increasing in each variable, and satisfy $\varphi(o) = 0$ and $\varphi(e_j) = 1$ for $j = 1, \ldots, m$. The Orlicz L_φ sum of $K_1, \cdots, K_m \in \mathscr{K}_0$ [22] is the convex body $+_\varphi(K_1, \ldots, K_m)$ whose support function $h_{+_\varphi(K_1,\ldots,K_m)}$ is defined by the unique positive solution of the following equation:

$$\varphi\left(\frac{h_{K_1}(u)}{\lambda}, \ldots, \frac{h_{K_m}(u)}{\lambda}\right) = 1, \quad \text{for} \quad u \in S^{n-1}.$$

That is, for each fixed $u \in S^{n-1}$,

$$\varphi\left(\frac{h_{K_1}(u)}{h_{+_\varphi(K_1,\ldots,K_m)}(u)}, \ldots, \frac{h_{K_m}(u)}{h_{+_\varphi(K_1,\ldots,K_m)}(u)}\right) = 1.$$

The fact that $\varphi \in \Phi_m$ is increasing in each variable implies that, for $j = 1, \cdots, m$,

$$K_j \subset +_\varphi(K_1, \ldots, K_m). \tag{1.1.5}$$

It is easily checked that if K_i for all $1 < i \leq m$ are dilates of K_1, then $+_\varphi(K_1, \ldots, K_m)$ is dilate of K_1 as well. The related Orlicz-Brunn-Minkowski inequality has the following form:

$$\varphi\left(\frac{|K_1|^{1/n}}{| +_\varphi (K_1, \ldots, K_m)|^{1/n}}, \ldots, \frac{|K_m|^{1/n}}{| +_\varphi (K_1, \ldots, K_m)|^{1/n}}\right) \leq 1. \tag{1.1.6}$$

The classical Brunn-Minkowski and the L_q Brunn-Minkowski inequalities are associated to

$$\varphi(x_1, \cdots, x_m) = \sum_{i=1}^{m} x_i \in \Phi_m$$

and $\varphi(x_1, \cdots, x_m) = \sum_{i=1}^{m} x_i^q \in \Phi_m$ with $q > 1$, respectively. In these cases, the L_q sum of $K_1, \cdots, K_m$ for $q \geq 1$ is the convex body $K_1 +_q \cdots +_q K_m$ whose support function is formulated by

$$h^q_{K_1 +_q \cdots +_q K_m} = h^q_{K_1} + \cdots + h^q_{K_m}.$$

When $q = 1$, we often write $K_1 + \cdots + K_m$ instead of $K_1 +_1 \cdots +_1 K_m$.

Consider the convex body $K +_{\varphi,\varepsilon} L \in \mathscr{K}_0$ whose support function is given by, for $u \in S^{n-1}$,

$$1 = \varphi_1\left(\frac{h_K(u)}{h_{K+_{\varphi,\varepsilon}L}(u)}\right) + \varepsilon\varphi_2\left(\frac{h_L(u)}{h_{K+_{\varphi,\varepsilon}L}(u)}\right), \tag{1.1.7}$$

where $\varepsilon > 0$, $K, L \in \mathscr{K}_0$, and $\varphi_1, \varphi_2 \in \Phi_1$. If $(\varphi_1)'_l(1)$, the left derivative of φ_1 at $t = 1$, exists and is positive, then the L_{φ_2} mixed volume of $K, L \in \mathscr{K}_0$ can be defined by [22, 80]

$$V_{\varphi_2}(K, L) = \frac{(\varphi_1)'_l(1)}{n} \cdot \frac{d}{d\varepsilon}|K +_{\varphi,\varepsilon} L|\bigg|_{\varepsilon=0^+} = \frac{1}{n}\int_{S^{n-1}} \varphi_2\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u)\,dS(K, u). \tag{1.1.8}$$

Together with the Orlicz-Brunn-Minkowski inequality (1.1.6), one gets the following fundamental Orlicz-Minkowski inequality: if $\varphi \in \Phi_1$, then for all $K, L \in \mathscr{K}_0$,

$$V_\varphi(K, L) \geq |K| \cdot \varphi\left(\left(\frac{|L|}{|K|}\right)^{1/n}\right),$$

with equality, if in addition φ is strictly convex, if and only if K and L are dilates of each other. The classical Minkowski and the L_q Minkowski inequalities are associated with $\varphi = t$ and $\varphi = t^q$ for $q > 1$ respectively.

Formula (1.1.8) was proved in [22, 80] with assumptions $\varphi_1, \varphi_2 \in \Phi_1$ (i.e., convex and increasing functions); however, it can be extended to more general increasing or decreasing functions (see Chapter 2). To this end, we work on the following classes of nonnegative continuous functions:

$$\mathscr{I} = \{\phi : [0, \infty) \to [0, \infty) \text{ such that } \phi \text{ is strictly increasing with } \phi(1) = 1, \; \phi(0) = 0, \phi(\infty) = \infty\}$$

$$\mathscr{D} = \{\phi : (0,\infty) \to (0,\infty) \text{ such that } \phi \text{ is strictly decreasing with } \phi(1) = 1,$$
$$\phi(0) = \infty, \phi(\infty) = 0\}$$

where for simplicity we let $\phi(0) = \lim_{t\to 0^+}\phi(t)$ and $\phi(\infty) = \lim_{t\to\infty}\phi(t)$. Note that all results may still hold if the normalization on $\phi(0), \phi(1)$ and $\phi(\infty)$ are replaced by other quantities. The linear Orlicz addition of h_K and h_L in formula (1.1.7) can be defined in the same way for either $\varphi_1, \varphi_2 \in \mathscr{I}$ or $\varphi_1, \varphi_2 \in \mathscr{D}$. Namely, for either $\varphi_1, \varphi_2 \in \mathscr{I}$ or $\varphi_1, \varphi_2 \in \mathscr{D}$, and for $\varepsilon > 0$, define $f_\varepsilon : S^{n-1} \to (0,\infty)$ the linear Orlicz addition of h_K and h_L by, for $u \in S^{n-1}$,

$$\varphi_1\left(\frac{h_K(u)}{f_\varepsilon(u)}\right) + \varepsilon\varphi_2\left(\frac{h_L(u)}{f_\varepsilon(u)}\right) = 1. \tag{1.1.9}$$

See [36] for more details. In general, f_ε may not be the support function of a convex body; however f_ε is the support function of $K +_{\varphi,\varepsilon} L$ when $\varphi_1, \varphi_2 \in \Phi_1$. It is easily checked that $f_\varepsilon \in C^+(S^{n-1})$ for all $\varepsilon > 0$. Moreover, $h_K \le f_\varepsilon$ if $\varphi_1, \varphi_2 \in \mathscr{I}$ and $h_K \ge f_\varepsilon$ if $\varphi_1, \varphi_2 \in \mathscr{D}$. Denote by K_ε the Aleksandrov body associated to f_ε. The following result in Chapter 2 extends formula (1.1.8) to not necessarily convex functions φ_1 and φ_2: if $K, L \in \mathscr{K}_0$ and $\varphi_1, \varphi_2 \in \mathscr{I}$ such that $(\varphi_1)'_l(1)$ exists and is positive, then

$$V_{\varphi_2}(K,L) = \frac{(\varphi_1)'_l(1)}{n} \cdot \frac{d}{d\varepsilon}|K_\varepsilon|\bigg|_{\varepsilon=0^+} = \frac{1}{n}\int_{S^{n-1}} \varphi_2\bigg(\frac{h_L(u)}{h_K(u)}\bigg) h_K(u)\,dS(K,u), \tag{1.1.10}$$

while if $\varphi_1, \varphi_2 \in \mathscr{D}$ such that $(\varphi_1)'_r(1)$, the right derivative of φ_1 at $t = 1$, exists and is nonzero, then (1.1.10) holds with $(\varphi_1)'_l(1)$ replaced by $(\varphi_1)'_r(1)$.

1.1.3 The p-Capacity

Throughout this thesis, the standard notation $C_c^\infty(\mathbb{R}^n)$ or C_c^∞denotes the set of all infinitely differentiable functions with compact support in $\mathbb{R}^n$ and ∇f denotes the gradient of f. Let $n \ge 2$ be an integer and $p \in (1,n)$. The p-capacity of a compact subset $E \subset \mathbb{R}^n$, denoted by $C_p(E)$, is defined by

$$C_p(E) = \inf\left\{\int_{\mathbb{R}^n} \|\nabla f\|^p\,dx : f \in C_c^\infty(\mathbb{R}^n) \ \text{ such that } \ f \ge 1 \ \text{ on } \ E\right\}.$$

If $O \subset \mathbb{R}^n$ is an open set, then the p-capacity of O is defined by

$$C_p(O) = \sup \left\{ C_p(E) : \quad E \subset O \ \text{ and } E \text{ is a compact set in } \mathbb{R}^n \right\}.$$

For general bounded measurable subset $F \subset \mathbb{R}^n$, the p-capacity of F is then defined by

$$C_p(F) = \inf \left\{ C_p(O) : \quad F \subset O \ \text{ and } O \text{ is an open set in } \mathbb{R}^n \right\}.$$

The p-capacity is monotone, that is, if $A \subset B$ are two measurable subsets of $\mathbb{R}^n$, then $C_p(A) \leq C_p(B)$. It is translation invariant: $C_p(F + x_0) = C_p(F)$ for all $x_0 \in \mathbb{R}^n$ and measurable subset $F \subset \mathbb{R}^n$. Its homogeneous degree is $n - p$, i.e., for all $\lambda > 0$,

$$C_p(\lambda A) = \lambda^{n-p} C_p(A). \tag{1.1.11}$$

For $K \in \mathscr{K}_0$, let $int(K)$ denote its interior. It follows from the monotonicity of the p-capacity that $C_p(int(K)) \leq C_p(K)$. On the other hand, for all $\varepsilon > 0$, one sees that

$$K \subset (1 + \varepsilon) \cdot int(K).$$

It follows from the homogeneity and the monotonicity of the p-capacity that

$$C_p(K) \leq (1 + \varepsilon)^{n-p} \cdot C_p(int(K)).$$

Hence $C_p(int(K)) = C_p(K)$ for all $K \in \mathscr{K}_0$ by letting $\varepsilon \to 0^+$. Please see [17] for more properties.

Following the convention in the literature of p-capacity, in later context we will work on convex domains containing the origin, i.e., all open subsets $\Omega \subset \mathbb{R}^n$ whose closure $\overline{\Omega} \in \mathscr{K}_0$. For convenience, we use $\mathscr{C}_0$ to denote the set of all open convex domains containing the origin. Moreover, geometric notations for $\Omega \in \mathscr{C}_0$, such as the support function and the surface area measure, are considered to be the ones for its closure, for instance,

$$h_\Omega(u) = \sup_{x \in \Omega} \langle x, u \rangle = h_{\overline{\Omega}}(u) \quad \text{for} \quad u \in S^{n-1}.$$

There exists the p-capacitary measure of $\Omega \in \mathscr{C}_0$, denoted by $\mu_p(\Omega, \cdot)$, on S^{n-1} such that for any Borel set $\Sigma \subset S^{n-1}$ (see e.g., [39, 40, 41]),

$$\mu_p(\Omega, \Sigma) = \int_{\nu_\Omega^{-1}(\Sigma)} \|\nabla U_\Omega\|^p \, d\mathscr{H}^{n-1}, \tag{1.1.12}$$

where $\nu_\Omega^{-1}: S^{n-1} \to \partial\Omega$ is the inverse Gauss map (i.e., $\nu_\Omega^{-1}(u)$ contains all points $x \in \partial\Omega$ such that u is an unit outer normal vector of x) and U_Ω is the p-equilibrium potential of Ω. Note that U_Ω is the unique solution to the boundary value problem of the following p-Laplace equation

$$\begin{cases} \operatorname{div}(\|\nabla U\|^{p-2}\nabla U) = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ U = 1 & \text{on } \partial\Omega, \\ \lim_{\|x\|\to\infty} U(x) = 0. \end{cases}$$

With the help of the p-capacitary measure, the Poincaré p-capacity formula [15] gives

$$C_p(\Omega) = \frac{p-1}{n-p}\int_{S^{n-1}} h_\Omega(u)\, d\mu_p(\Omega, u).$$

Lemma 4.1 in [15] asserts that $\mu_p(\Omega_m, \cdot)$ converges to $\mu_p(\Omega, \cdot)$ weakly on S^{n-1} and hence $C_p(\Omega_m)$ converges to $C_p(\Omega)$, if Ω_m converges to Ω in the Hausdorff metric.

The beautiful Hadamard variational formula for $C_p(\cdot)$ was provided in [15]: for two convex domains $\Omega, \Omega_1 \in \mathscr{C}_0$, one has

$$\frac{1}{n-p} \cdot \left.\frac{dC_p(\Omega + \varepsilon\Omega_1)}{d\varepsilon}\right|_{\varepsilon=0} = \frac{p-1}{n-p}\int_{S^{n-1}} h_{\Omega_1}(u)\, d\mu_p(\Omega, u) =: C_p(\Omega, \Omega_1), \quad (1.1.13)$$

where $C_p(\Omega, \Omega_1)$ is called the mixed p-capacity of Ω and Ω_1. By (1.2.26) and (1.1.13), one gets the p-capacitary Minkowski inequality

$$C_p(\Omega, \Omega_1)^{n-p} \geq C_p(\Omega)^{n-p-1} C_p(\Omega_1), \quad (1.1.14)$$

with equality if and only if Ω and Ω_1 are homothetic [15]. It is also well known that the centroid of $\mu_p(\Omega, \cdot)$ is o, that is,

$$\int_{S^{n-1}} u\, d\mu_p(\Omega, u) = o.$$

Moreover, the support of $\mu_p(\Omega, \cdot)$ is not contained in any closed hemisphere, i.e., there exists a constant $c > 0$ (see e.g., [86, Theorem 1]) such that

$$\int_{S^{n-1}} \langle \theta, u\rangle_+\, d\mu_p(\Omega, u) > c \qquad \text{for each} \quad \theta \in S^{n-1}, \quad (1.1.15)$$

where a_+ denotes $\max\{a, 0\}$ for all $a \in \mathbb{R}$.

For $f \in C^+(S^{n-1})$, denote by Ω_f the Aleksandrov domain associated to f (i.e., the interior of the Aleksandrov body associated to f). For $\Omega \in \mathscr{C}_0$ and $f \in C^+(S^{n-1})$, define the mixed p-capacity of Ω and f by

$$C_p(\Omega, f) = \frac{p-1}{n-p} \int_{S^{n-1}} f(u)\, d\mu_p(\Omega, u). \tag{1.1.16}$$

Clearly $C_p(\Omega, h_L) = C_p(\Omega, L)$ and $C_p(\Omega, h_\Omega) = C_p(\Omega)$ for all $\Omega, L \in \mathscr{C}_0$. Moreover,

$$C_p(\Omega_f) = C_p(\Omega_f, f) \tag{1.1.17}$$

holds for any $f \in C^+(S^{n-1})$.

1.1.4 Sobolev space and Level sets

For $1 \le p < \infty$ and $f \in C_c^\infty$, consider the norm

$$\|f\|_{1,p} = \|f\|_p + \|\nabla f\|_p = \left(\int_{\mathbb{R}^n} |f|^p\, dx\right)^{1/p} + \left(\int_{\mathbb{R}^n} |\nabla f|^p\, dx\right)^{1/p}.$$

We also use $\|f\|_\infty$ to denote the maximal value (or supremum) of $|f|$. The closure of C_c^∞ under the norm $\|\cdot\|_{1,p}$ is denoted by $W_0^{1,p}$. Note that the Sobolev space $W_0^{1,p}$ is a Banach space and consists of all real valued L_p functions on $\mathbb{R}^n$ with weak L_p partial derivatives (see e.g. [17] for more details about the Sobolev space). Hereafter, when $f \in W_0^{1,p}$ is not smooth enough, ∇f means the weak partial gradient. By $\nabla_z f$ we mean the inner product of z and ∇f, namely $\nabla_z f = z \cdot \nabla f$. When $u \in S^{n-1}$, $\nabla_u f$ is just the directional derivative of f along the direction u. Clearly $\nabla_z f$ is linear in $z \in \mathbb{R}^n$.

For a subset $E \subset \mathbb{R}^n$, $\mathbf{1}_E$ denotes the indicator function of E, that is, $\mathbf{1}_E(x) = 1$ if $x \in E$ and otherwise 0. Let $|x| = \sqrt{x \cdot x}$ be the Euclidean norm of $x \in \mathbb{R}^n$. The distance from a point $x \in \mathbb{R}^n$ to a subset $E \subset \mathbb{R}^n$, denoted by $\operatorname{dist}(x, E)$, is defined by

$$\operatorname{dist}(x, E) = \inf\{|x-y| :\ y \in E\}.$$

Note that if $x \in \bar{E}$, the closure of E, then $\operatorname{dist}(x, E) = 0$.

For any real number $t > 0$, define the level set $[f]_t$ of $f \in C_c^\infty$ by

$$[f]_t = \{x \in \mathbb{R}^n : |f(x)| \ge t\}. \tag{1.1.18}$$

For all $t \in (0, \|f\|_\infty)$, $[f]_t$ is a compact set. Sard's theorem implies that, for almost every $t \in (0, \|f\|_\infty)$, the smooth $(n-1)$ submanifold

$$\partial[f]_t = \{x \in \mathbb{R}^n : |f(x)| = t\}$$

has nonzero normal vector $\nabla f(x)$ for all $x \in \partial[f]_t$. Denoted by $\nu(x) = -\nabla f(x)/|\nabla f(x)|$ and

$$\{\nu(x) : x \in \partial[f]_t\} = S^{n-1}.$$

An often used formula in our proofs is the well-known Federer's coarea formula (see [18], p.289): suppose that Ω is an open set in $\mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function, then

$$\int_{f^{-1}(\Omega)\bigcap\{|\nabla f|>0\}} g(x)\ dx = \int_\Omega \int_{f^{-1}(t)} \frac{g(x)}{|\nabla f(x)|}\ d\mathscr{H}^{n-1}(x)\ dt, \tag{1.1.19}$$

for any measurable function $g : \mathbb{R}^n \to [0, \infty)$.

Denote by $\mathbb{R}^*$ the subset of $\mathbb{R}$ that contains nonnegative real numbers. Let $\varphi_\tau : \mathbb{R} \to \mathbb{R}^*$ be the function given by formula (1.2.32), that is, for $\tau \in [-1, 1]$ and $t \in \mathbb{R}$,

$$\varphi_\tau(t) = \Big(\frac{1+\tau}{2}\Big)^{1/p} t_+ + \Big(\frac{1-\tau}{2}\Big)^{1/p} t_-. \tag{1.1.20}$$

It is easily checked that φ_τ has positive homogeneous of degree 1 and subadditive, i.e.

$$\varphi_\tau(\lambda t) = \lambda\varphi_\tau(t) \quad \text{for} \quad \lambda \geq 0 \ \text{ and } \ \varphi_\tau(t_1 + t_2) \leq \varphi_\tau(t_1) + \varphi_\tau(t_1). \tag{1.1.21}$$

Special cases, which are commonly used, are $\varphi_0(t) = 2^{-1/p}|t|$, $\varphi_1(t) = t_+$ and $\varphi_{-1}(t) = t_-$. We would like to mention that the function $\psi_\eta : \mathbb{R} \to \mathbb{R}^*$ for each $\eta \in [-1, 1]$ given by

$$\psi_\eta(t) = |t| + \eta t$$

is also commonly used in convex geometry (see e.g. [29, 42]). However, if we let

$$\tau = \frac{(1+\eta)^p - (1-\eta)^p}{(1+\eta)^p + (1-\eta)^p},$$

then $\psi_\eta^p = \big((1+\eta)^p + (1-\eta)^p\big) \cdot \varphi_\tau^p$. In later context, the theory for the general p-affine capacity will be developed only based on the function φ_τ because it is more convenient to prove the convexity or concavity of the general p-affine capacity with φ_τ.

We shall need the following result (see, e.g., [24, Lemma 1.3.1 (ii)]), which is crucial in the computation of involved integral on S^{n-1}.

Lemma 1.1.2. *If $v \in S^{n-1}$ and Φ is a bounded Lebesgue integrable function on $[-1,1]$, then $\Phi(u\cdot v)$, considered as a function of $u \in S^{n-1}$, is integrable with respect to the normalized spherical measure du. Moreover,*

$$\int_{S^{n-1}} \Phi(u\cdot v)\,du = \frac{(n-1)\omega_{n-1}}{n\omega_n}\int_{-1}^{1}\Phi(t)(1-t^2)^{\frac{n-3}{2}}\,dt.$$

It can be easily checked that for $p>0$

$$\begin{aligned}\int_{-1}^{1} t_+^p(1-t^2)^{\frac{n-3}{2}}\,dt &= \int_{-1}^{1} t_-^p(1-t^2)^{\frac{n-3}{2}}\,dt\\ &= \int_0^1 t^p(1-t^2)^{\frac{n-3}{2}}\,dt\\ &= \frac{1}{2}\cdot\int_0^1 t^{\frac{p+1}{2}-1}(1-t)^{\frac{n-1}{2}-1}\,dt\\ &= \frac{1}{2}\cdot B\Big(\frac{p+1}{2},\frac{n-1}{2}\Big).\end{aligned}$$

In particular, if $\Phi=\varphi_\tau^p$, it follows from (1.2.32) and Lemma 1.1.2 that, for $p>0$ and for any $u\in S^{n-1}$,

$$\begin{aligned}\int_{S^{n-1}}[\varphi_\tau(u\cdot v)]^p\,du &= \frac{(n-1)\omega_{n-1}}{n\omega_n}\int_{-1}^{1}\Big[\Big(\frac{1+\tau}{2}\Big)t_+^p+\Big(\frac{1-\tau}{2}\Big)t_-^p\Big](1-t^2)^{\frac{n-3}{2}}\,dt\\ &= \frac{(n-1)\omega_{n-1}}{2n\omega_n}\cdot B\Big(\frac{p+1}{2},\frac{n-1}{2}\Big)\qquad (:=A(n,p)).\end{aligned}\tag{1.1.22}$$

1.2 Introduction and overview of the main results

The theory of convex geometry was greatly enriched by the combination of two notions: the volume and the linear Orlicz addition of convex bodies [22, 80]. This new theory, usually called the Orlicz-Brunn-Minkowski theory for convex bodies, started from the works of Lutwak, Yang and Zhang [53, 54], and received considerable attention (see e.g., [6, 7, 8, 13, 27, 28, 87, 88, 98, 99]). The linear Orlicz addition of convex bodies was proposed by Gardner, Hug and Weil [22] (independently Xi, Jin and Leng [80]). Let $\varphi_i : [0, \infty) \to [0, \infty)$, $i = 1, 2$, be convex functions such that φ_i is strictly increasing with $\varphi_i(1) = 1$, $\varphi_i(0) = 0$ and $\lim_{t\to\infty} \varphi_i(t) = \infty$. Let S^{n-1} be the unit sphere in $\mathbb{R}^n$ and $h_K : S^{n-1} \to (0, \infty)$ denote the support function of convex body K (i.e., a compact convex subset with nonempty interior). For any given $\varepsilon > 0$ and two convex bodies K and L with the origin in their interiors, the linear Orlicz addition $K +_{\varphi,\varepsilon} L$ is determined by its support function $h_{K+_{\varphi,\varepsilon}L}$, the unique solution of

$$\varphi_1\Big(\frac{h_K(u)}{\lambda}\Big) + \varepsilon\varphi_2\Big(\frac{h_L(u)}{\lambda}\Big) = 1 \quad \text{for} \quad u \in S^{n-1}.$$

Denote by $|K +_{\varphi,\varepsilon} L|$ the volume of $K +_{\varphi,\varepsilon} L$. If $(\varphi_1)'_l(1)$, the left derivative of φ_1 at $t = 1$, exists and is positive, then

$$\frac{(\varphi_1)'_l(1)}{n} \cdot \frac{d}{d\varepsilon}|K +_{\varphi,\varepsilon} L|\bigg|_{\varepsilon=0^+} = \frac{1}{n}\int_{S^{n-1}} \varphi_2\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u)dS_K(u), \tag{1.2.23}$$

where S_K is the surface area measure of K (see Subsection 1.1.1 for more details). That is, formula (1.2.23) provides a geometric interpretation of $V_\phi(K, L)$ for ϕ being convex and strictly increasing. Here, for any continuous function $\phi : (0, \infty) \to (0, \infty)$, $V_\phi(K, L)$ denotes the nonhomogeneous Orlicz L_ϕ mixed volume of K and L:

$$V_\phi(K, L) = \frac{1}{n}\int_{S^{n-1}} \phi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u)dS_K(u). \tag{1.2.24}$$

To the best of our knowledge, there are no geometric interpretations of $V_\phi(K, L)$ for non-convex functions ϕ (even for $\phi(t) = t^p$ with $p < 1$) in literature; and such geometric interpretations will be provided in Subsection 2.3.1 in this thesis. Note that formula (1.2.23) is essential for the Orlicz-Minkowski inequality and many other objects, such as the Orlicz affine and geominimal surface areas [90].

Introduced by Blaschke in 1923 [4], the classical affine surface area was thought to be one of the core concepts in the Brunn-Minkowski theory of convex bodies due to its important applications in, such as, approximation of convex bodies by polytopes [25, 46, 72] and valuation theory [2, 3, 44]. Since the groundbreaking paper by Lutwak [49], considerable progress has been made on the theory of the L_p affine surface areas (see e.g., [37, 45, 59, 60, 64, 71, 77, 78, 79]). Like the classical affine surface area, the

L_p affine surface areas play fundamental roles in applications and provide powerful tools in convex geometry. Note that the L_p affine surface areas are affine invariant valuations with homogeneity.

In the Orlicz-Brunn-Minkowski theory for convex bodies, a central task is to find the "right" definitions for the Orlicz affine surface areas. Here, we will discuss two different approaches by Ludwig [43] and Ye [90]. Based on an integral formula, Ludwig proposed the general affine surface areas [43]. Ludwig's definitions work perfectly in studying properties such as valuation [43], the characterization of valuation [28, 43] and the monotonicity under the Steiner symmetrization [88]. In order to define the Orlicz geominimal surface areas, new ideas are needed because geominimal surface areas do not have convenient integral expression like their affine relatives. Ye provided a unified approach to define the Orlicz affine and geominimal surface areas [90] based on the Orlicz L_ϕ mixed volume $V_\phi(\cdot,\cdot)$ defined in formula (1.2.24). In fact, the approach in [90] is related to an optimization problem for the f-divergence [36] and could be used to define other versions of Orlicz affine and geominimal surface areas [11, 91, 92].

Note that the natural property of "homogeneity" is missing in the Orlicz affine surface areas in [43, 90]. To define the Orlicz affine surface areas with homogeneity is one of the main objects in this thesis; and it will be done in Section 2.1. As an example, we give the definition for $\phi \in \widehat{\Phi}_1$, where $\widehat{\Phi}_1$ is the set of functions $\phi : [0,\infty) \to [0,\infty)$ such that ϕ is strictly increasing with $\phi(0) = 0, \phi(1) = 1$, $\lim_{t\to\infty} \phi(t) = \infty$ and $\phi(t^{-1/n})$ being strictly convex on $(0,\infty)$. For convex body K and star body L with the origin in their interiors, define $\widehat{V}_\phi(K, L^\circ)$ for $\phi \in \widehat{\Phi}_1$ by

$$\widehat{V}_\phi(K, L^\circ) = \inf_{\lambda>0} \left\{ \int_{S^{n-1}} \phi\Big(\frac{n|K|}{\lambda \cdot \rho_L(u) \cdot h_K(u)}\Big) h_K(u)\, dS_K(u) \le n|K| \right\}$$

where ρ_L denotes the radial function of L. We now define $\widehat{\Omega}_\phi^{orlicz}(K)$ for $\phi \in \widehat{\Phi}_1$, the homogeneous Orlicz L_ϕ affine surface area of K, by the infimum of $\widehat{V}_\phi(K, L^\circ)$ where L runs over all star bodies with the origin in their interiors and $|L| = |B_2^n|$ (the volume of the Euclidean unit ball of $\mathbb{R}^n$). In Proposition 2.1.1, we show that $\widehat{\Omega}_\phi^{orlicz}(K)$ is invariant under the volume preserving linear maps and has homogeneous degree $(n-1)$. Moreover, the following affine isoperimetric inequality is established in Theorem 2.1.1: *if K has its centroid at the origin and $\phi \in \widehat{\Phi}_1$, then*

$$\frac{\widehat{\Omega}_{\phi}^{orlicz}(K)}{\widehat{\Omega}_{\phi}^{orlicz}(B_2^n)} \le \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-1}{n}}$$

with equality if and only if K is an origin-symmetric ellipsoid. Note that affine isoperimetric inequalities are fundamental in convex geometry; and these inequalities compare affine invariant functionals with the volume (see e.g., [12, 29, 55, 50, 53, 54, 79, 95]).

The Petty body and its L_p extensions for $p > 1$ were used to study the continuity of the classical geominimal surface area and its L_p counterparts [49, 66]. To prove the existence and uniqueness of the Orlicz-Petty bodies is one of the main goals of Section 2.2 in this thesis. In order to fulfill these goals, we first define $\widehat{G}_{\phi}^{orlicz}(K)$, the homogeneous Orlicz geominimal surface area of K, by the infimum of $\widehat{V}_{\phi}(K, L^{\circ})$ where L runs over all convex bodies with the origin in their interiors and $|L| = |B_2^n|$. The classical geominimal surface area, which corresponds to $\phi(t) = t$, was introduced by Petty [66] in order to study the affine isoperimetric problems [66, 67]. The classical geominimal surface area and its L_p extensions (corresponding to $\phi(t) = t^p$) for $p > 1$ by Lutwak [49] are continuous on the set of convex bodies in terms of the Hausdorff distance; while their affine relatives are only semicontinuous. The main ingredients to prove the continuity of the L_p geominimal surface area for $p \geq 1$ are the existence of the L_p Petty bodies and the uniform boundedness of the L_p Petty bodies of a convergent sequence of convex bodies (hence, the Blaschke selection theorem can be used). In Section 2.2, we will prove that $\widehat{G}_{\phi}^{orlicz}(\cdot)$ is also continuous for $\phi \in \widehat{\Phi}_1$. Note that $\phi(t) = t^p \in \widehat{\Phi}_1$ if $p \in (0, \infty)$. Consequently, the L_p geominimal surface area for $p \in (0, 1)$, proposed by the third author in [89], is also continuous. Our approach basically follows the steps in [49, 66]; however, our proof is more delicate and requires much more careful analysis due to the lack of convexity of ϕ (note that in $\widehat{\Phi}_1$, $\phi(t^{-1/n})$ is assumed to be convex, not ϕ itself). In particular, we prove the existence and uniqueness of the Orlicz-Petty bodies in Proposition 2.2.3. Our main result is Theorem 2.2.1: *if $\phi \in \widehat{\Phi}_1$, then the homogeneous L_ϕ Orlicz geominimal* *surface area is continuous on the set of convex bodies with respect to the Hausdorff distance.* The continuity of nonhomogeneous Orlicz geominimal surface areas [90] will be discussed in Subsection 2.3.2. The L_p Petty body for $p \in (-1, 0)$ is more involved and will be discussed in Section 2.4.

For a compact set $K \subset \mathbb{R}^n$, its p-variational capacity, denoted by $C_p(K)$, can be formulated by (see e.g. [17, 57, 58])

$$C_p(K) = \inf \Big\{ \int_{\mathbb{R}^n} |\nabla f|^p \, dx : \quad f \in C_c^\infty \text{ and } f \geq 1 \text{ on } K \Big\}$$

where ∇f denotes the gradient of f and C_c^∞ is the set of smooth functions with compact supports in $\mathbb{R}^n$. The p-variational capacity is an important geometric invariant which has close connection with the p-Laplacian partial differential equation and has important applications in many areas, e.g., analysis, mathematical physics and partial differential equations (see e.g., [17, 57, 58] and references therein).

The p-capacitary measure $\mu_p(\Omega, \cdot)$ of convex domain Ω can be derived from an integral related to the p-equilibrium potential of Ω. Note that the p-equilibrium potential of Ω is the solution of a p-Laplace equation with certain boundary conditions (see Subsection 1.1.3 for details). For a convex domain $\Omega \subset \mathbb{R}^n$, the Poincaré p-capacity formula [15] asserts that the p-capacity of Ω has the following form:

$$C_p(\Omega) = \frac{p-1}{n-p} \int_{S^{n-1}} h_\Omega(u) \, d\mu_p(\Omega, u). \tag{1.2.25}$$

Although the definition of the p-capacity involves rather complicate partial differential equations, formula (1.2.25) suggests that the p-capacity has high resemblance with the volume. For a convex domain $\Omega \subset \mathbb{R}^n$, its volume can be calculated by

$$|\Omega| = \frac{1}{n} \int_{S^{n-1}} h_\Omega(u) \, dS(\Omega, u)$$

with $S(\Omega, \cdot)$ the surface area measure of Ω defined on S^{n-1}. For instance, the p-capacitary Brunn-Minkowski inequality asserts that for all convex domains Ω and Ω_1, one has

$$C_p(\Omega + \Omega_1)^{\frac{1}{n-p}} \geq C_p(\Omega)^{\frac{1}{n-p}} + C_p(\Omega_1)^{\frac{1}{n-p}} \tag{1.2.26}$$

with equality if and only if Ω and Ω_1 are homothetic (see [5, 10, 16]). Hereafter

$$\Omega + \Omega_1 = \{x + y : x \in \Omega, y \in \Omega_1\}$$

denotes the Minkowski sum of Ω an Ω_1. Inequality (1.2.26) has the formula similar to the classical Brunn-Minkowski inequality regarding the volume:

$$|\Omega+\Omega_1|^{\frac{1}{n}} \geq |\Omega|^{\frac{1}{n}} + |\Omega_1|^{\frac{1}{n}}$$

with equality if and only if Ω and Ω_1 are homothetic (see e.g., [20, 70]). Moreover, the p-capacitary Minkowski inequality (1.1.14) shares the formula similar to its volume counterpart (see e.g., [15, 20, 70]).

Sections 3.1 and 3.2 in this thesis reveal another surprising similarity between the p-capacity and the volume regarding the Orlicz additions. We develop the p-capacitary Orlicz-Brunn-Minkowski theory based on the combination of the Orlicz additions and the p-capacity. In particular, we establish the p-capacitary Orlicz-Brunn-Minkowski inequality (see Theorem 3.2.1) and Orlicz-Minkowski inequality (see Theorem 3.1.2). The p-capacitary Orlicz-Minkowski inequality provides a tight lower bounded for $C_{p,\phi}(\Omega,\Omega_1)$, the Orlicz L_ϕ mixed p-capacity of $\Omega,\Omega_1 \in \mathscr{C}_0$ (the collection of all convex domains containing the origin), in terms of $C_p(\Omega)$ and $C_p(\Omega_1)$. In Theorem 3.1.1, we prove the p-capacitary Orlicz-Hadamard variational formula based on a linear Orlicz addition $\Omega,\Omega_1 \in \mathscr{C}_0$. This p-capacitary Orlicz-Hadamard variational formula gives a geometric interpretation of $C_{p,\phi}(\Omega,\Omega_1)$.

Many objects of interest and fundamental results in convex geometry are related to the L_p projection bodies. For $p \geq 1$, the L_p projection body of a convex body $K \subset \mathbb{R}^n$ with the origin in its interior is determined by its support function $h_{\Pi_p(K)} : S^{n-1} \to \mathbb{R}$, whose definition is formulated as follows: for any $\theta \in S^{n-1}$,

$$h_{\Pi_p(K)}(\theta) = \left(\int_{\partial K} \left(\frac{\theta \cdot \nu_K(x)}{2}\right)^p \cdot |x \cdot \nu_K(x)|^{1-p} \, d\mathscr{H}^{n-1}(x) \right)^{\frac{1}{p}}$$

with ν_K the unit outer normal vector of K at $x \in \partial K$ and $\mathscr{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure of ∂K, the boundary of K. Define $\Phi_p(K)$, the p-integral affine surface area of K, by

$$\Phi_p(K) = \left(\int_{S^{n-1}} \left[h_{\Pi_p(K)}(u)\right]^{-n} du \right)^{-\frac{p}{n}}$$

where du is the normalized spherical measure on the unit sphere S^{n-1}. Let B_2^n be the unit Euclidean ball in $\mathbb{R}^n$ and $|K|$ denote the volume of K. The following L_p affine isoperimetric inequality for the p-integral affine surface area holds [47, 50, 51, 65, 95]: for $p \geq 1$ and for K a convex body with the origin in its interior,

$$\left(\frac{\Phi_p(K)}{\Phi_p(B_2^n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{1}{n}} \tag{1.2.27}$$

with equality if and only if $K = TB_2^n$ if $p > 1$ and $K = TB_2^n + x_0$ if $p = 1$ for some invertible linear transform T on $\mathbb{R}^n$ and some $x_0 \in \mathbb{R}^n$. Note that inequality (1.2.27) is invariant under the volume preserving linear transforms and hence is stronger than the well-known L_p isoperimetric inequality [20, 48, 70]:

$$\left(\frac{S_p(K)}{S_p(B_2^n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{1}{n}} \tag{1.2.28}$$

with equality if and only if K is an Euclidean ball in $\mathbb{R}^n$ (if $p > 1$, the center needs to be at the origin). Here $S_p(K)$ is the p-surface area of K and can be formulated by

$$S_p(K) = \int_{\partial K} |x \cdot \nu_K(x)|^{1-p} \, d\mathscr{H}^{n-1}(x). \tag{1.2.29}$$

It is well known that inequality (1.2.28) can be strengthened by the isocapacitary inequality related to the p-variational capacity. The following inequality for the p-variational capacity holds [47, 57]: for $p \in [1, n)$ and for K being a Lipschitz star body with the origin in its interior,

$$\left(\frac{S_p(K)}{S_p(B_2^n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{C_p(K)}{C_p(B_2^n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{1}{n}}. \tag{1.2.30}$$

The p-variational capacity behaves rather similar to the p-surface area and is lack of the affine invariance. Very recently, Xiao [83] introduced an affine relative of the p-variational capacity and named it as the p-affine capacity. This new notion is denoted by $C_{p,0}(K)$ in this thesis and its definition is equivalent to: for $p \in [1, n)$ and for K a compact set in $\mathbb{R}^n$,

$$C_{p,0}(K) = \inf\left\{\mathscr{H}_p(f) : \ f \in C_c^\infty \text{ and } f \geq 1 \text{ on K}\right\}$$

where $\mathscr{H}_p(f)$ is the p-affine energy of f:

$$\mathscr{H}_p(f) = \left(\int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \frac{|\langle u \cdot \nabla f\rangle|^p}{2} dx\right)^{-\frac{n}{p}} du\right)^{-\frac{p}{n}}.$$

The following affine isocapacitary inequality was also established in [83, Theorems 3.2 and 3.5] and [82, Theorems 1.3' and 1.4']: for $p \in [1, n)$ and for K an origin-symmetric convex body, one has

$$\left(\frac{\Phi_p(K)}{\Phi_p(B_2^n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{C_{p,0}(K)}{C_{p,0}(B_2^n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{1}{n}}. \tag{1.2.31}$$

The second inequality of (1.2.31) indeed also holds for any compact set $K \subset \mathbb{R}^n$. Again inequality (1.2.31) is invariant under the volume preserving linear transforms and hence is stronger than inequality (1.2.30). Moreover, inequality (1.2.31) can be viewed as the affine relative of inequality (1.2.30). See e.g., [75, 84, 85] for more works related to affine capacities. We would like to mention that the p-affine energy is the key ingredient in many fundamental analytical inequalities, see e.g., [14, 32, 52, 62, 73, 74, 81, 94].

It is our goal in Chapter 4 to study a concept more general than the p-affine capacity, and to establish stronger sharp geometric inequalities. Our motivations are results from recent series decent works, for example, the general L_p affine isoperimetric inequalities and asymmetric affine L_p Sobolev inequalities by Haberl and Schuster [29, 30], asymmetric affine Pólya-Szegö principle by Haberl, Schuster and Xiao [31] and Minkowski valuations by Ludwig [42]. The key in [29] is to replace h_{Π_p} by its asymmetric counterpart $h_{\Pi_{p,\tau}(K)} : S^{n-1} \to \mathbb{R}$: for $p \geq 1$, for $\tau \in [-1, 1]$ and for K a convex body with the origin in its interior,

$$\left[h_{\Pi_{p,\tau}(K)}(\theta)\right]^p = \int_{\partial K} \left[\varphi_\tau(\theta \cdot \nu_K(x))\right]^p \cdot |x \cdot \nu_K(x)|^{1-p} \, d\mathscr{H}^{n-1}(x)$$

for $\theta \in S^{n-1}$, where

$$\left[\varphi_\tau(t)\right]^p = \left(\frac{1+\tau}{2}\right) t_+^p + \left(\frac{1-\tau}{2}\right) t_-^p \tag{1.2.32}$$

with $t_+ = \max\{0, t\}$ and $t_- = \max\{0, -t\}$ for any $t \in \mathbb{R}$. We point out that this extension is a key step from the L_p Brunn-Minkowski theory of convex bodies to the Orlicz theory and its dual (see e.g., [22, 23, 45, 53, 54, 80, 97]). Similarly, the key in [30, 31] is to replace the p-affine energy function $\mathscr{H}_p(f)$ by its asymmetric counterpart: for $p \in [1, n)$ and for $\tau \in [-1, 1]$,

$$\mathscr{H}_{p,\tau}(f) = \left(\int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \left[\varphi_\tau(\nabla_u f) \right]^p dx \right)^{-\frac{n}{p}} du \right)^{-\frac{p}{n}}.$$

When $\tau = 0$, $\mathscr{H}_{p,\tau}(f)$ goes back to the p-affine energy $\mathscr{H}_p(f)$. It is worth to mention that to deal with $\mathscr{H}_{p,\tau}(f)$ is much more challenge than $\mathscr{H}_p(f)$, mainly because the L_p convexification of level sets of a smooth function f in the latter case always contains the origin in their interiors but in the former may not contains the origin in their interiors. These asymmetric extensions have also been widely used to study affine Sobolev type inequalities, the affine Pólya-Szegö principle as well as many other affine isoperimetric inequalities, see e.g., [56, 62, 63, 75, 76].

In Section 4.1, we provide several equivalent definitions for the general p-affine capacity, which will be denoted by $C_{p,\tau}(\cdot)$. One of them reads: for any $p \in [1, n)$, for any $\tau \in [-1, 1]$ and for any compact set $K \subset \mathbb{R}^n$,

$$C_{p,\tau}(K) = \inf \left\{ \mathscr{H}_{p,\tau}(f) : \ f \in C_c^\infty \text{ and } f \geq 1 \text{ on } K \right\}.$$

Basic properties for the general p-affine capacity, such as, monotonicity, affine invariance, translation invariance, homogeneity and the continuity from above, are established in Section 4.2. Similarly, the general p-integral affine surface area of a Lipschitz star body K is defined in Subsection 4.3.3 by: for any $p \in [1, n)$ and for any $\tau \in [-1, 1]$,

$$\Phi_{p,\tau}(K) = \left(\int_{S^{n-1}} h_{\Pi_{p,\tau}(K)}(u)^{-n} \, du \right)^{-\frac{p}{n}}.$$

Note that when $\tau = 0$, then $\Phi_{p,0}(K) = \Phi_p(K)$. The sharp geometric inequalities for the general p-affine capacity are established in Section 4.3. Roughly speaking, for K a convex body containing the origin in its interior, these sharp geometric inequalities can be summarized as follows: for all $p \in [1, n)$ and for all $0 \leq \tau \leq \eta \leq 1$, then

$$\begin{array}{ccccccc} \left(\frac{|K|}{|B_2^n|} \right)^{\frac{1}{n}} & \leq & \left(\frac{C_{p,\eta}(K)}{C_{p,\eta}(B_2^n)} \right)^{\frac{1}{n-p}} & \leq & \left(\frac{\Phi_{p,\eta}(K)}{\Phi_{p,\eta}(B_2^n)} \right)^{\frac{1}{n-p}} & \leq & \left(\frac{S_p(K)}{S_p(B_2^n)} \right)^{\frac{1}{n-p}} \\ & & \leq & & \leq & & \\ \left(\frac{|K|}{|B_2^n|} \right)^{\frac{1}{n}} & \leq & \left(\frac{C_{p,\tau}(K)}{C_{p,\tau}(B_2^n)} \right)^{\frac{1}{n-p}} & \leq & \left(\frac{\Phi_{p,\tau}(K)}{\Phi_{p,\tau}(B_2^n)} \right)^{\frac{1}{n-p}} & \leq & \left(\frac{S_p(K)}{S_p(B_2^n)} \right)^{\frac{1}{n-p}}. \end{array} \tag{1.2.33}$$

Inequality (1.2.31) turns out to be a special (and indeed the maximal) case of the above chain of inequalities. Hence, (1.2.33) extends and strengthens many well-known (affine) isoperimetric and (affine) isocapacitary inequalities, such as, [29, Theorem 1] by Haberl and Schuster, [47, inequality (13)] by Ludwig, Xiao and Zhang, and [83, Theorems 3.2 and 3.5] by Xiao. Moreover, we also prove that, for all $p \in [1, n)$ and for all $\tau \in [-1, 1]$,

$$\left(\frac{S_p(K)}{S_p(B_2^n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{C_p(K)}{C_p(B_2^n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{C_{p,\tau}(K)}{C_{p,\tau}(B_2^n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{1}{n}} \tag{1.2.34}$$

which extends and strengthens, e.g., inequality (1.2.30), [47, (12)] by Ludwig, Xiao and Zhang, and [83, Remark 2.7] by Xiao. Note that inequalities (1.2.33) and (1.2.34) work for more general compact sets than convex bodies, and we will explain the details in Section 4.3.

Chapter 2

The Orlicz-Petty bodies

This chapter is based on paper [96] collaborated with Deping Ye and Baocheng Zhu, which has been published online by *International Mathematics Research Notices*. It is dedicated to the Orlicz-Petty bodies. We first propose the homogeneous Orlicz affine and geominimal surface areas, and establish their basic properties such as homogeneity, affine invariance and affine isoperimetric inequalities. We also prove that the homogeneous geominimal surface areas are continuous, under certain conditions, on the set of convex bodies in terms of the Hausdorff distance. Our proofs rely on the existence of the Orlicz-Petty bodies and the uniform boundedness of the Orlicz-Petty bodies of a convergent sequence of convex bodies. Similar results for the nonhomogeneous Orlicz geominimal surface areas are proved as well.

2.1 The homogeneous Orlicz affine and geominimal surface areas

This section is dedicated to Orlicz affine and geominimal surface areas with homogeneity. Let $\mathscr{I}$ denote the set of continuous functions $\phi : [0,\infty) \to [0,\infty)$ which are strictly increasing with $\phi(1) = 1$, $\phi(0) = 0$ and $\phi(\infty) = \lim_{t\to\infty} \phi(t) = \infty$. Similarly, $\mathscr{D}$ denotes the set of continuous functions $\phi : (0,\infty) \to (0,\infty)$ which are strictly decreasing with $\phi(1) = 1$, $\phi(0) = \lim_{t\to 0} \phi(t) = \infty$ and $\phi(\infty) = \lim_{t\to\infty} \phi(t) = 0$.

Note that the conditions on $\phi(0), \phi(1)$ and $\phi(\infty)$ are mainly for convenience; results may still hold for more general strictly increasing or decreasing functions.

The Orlicz L_ϕ mixed volume of convex bodies K and L, $V_\phi(K, L)$, given in formula (1.2.24) does not have homogeneity in general. In order to define the homogeneous Orlicz affine and geominimal surface areas, a homogeneous Orlicz L_ϕ mixed volume of convex bodies K and L, denoted by $\widehat{V}_\phi(K, L)$, is needed.

Definition 2.1.1. *For $K, L \in \mathscr{K}_0$ and $\phi \in \mathscr{I}$, define $\widehat{V}_\phi(K, L)$ by*

$$\widehat{V}_\phi(K, L) = \inf_{\lambda>0} \Big\{ \int_{S^{n-1}} \phi\Big(\frac{n|K| \cdot h_L(u)}{\lambda \cdot h_K(u)}\Big)\, d\widetilde{V}_K(u) \leq 1 \Big\}. \tag{2.1.1}$$

While if $\phi \in \mathscr{D}$, $\widehat{V}_\phi(K, L)$ is defined as above with "≤ 1" replaced by "≥ 1".

Clearly $\widehat{V}_\phi(K, L) > 0$ for $K, L \in \mathscr{K}_0$. Definition 2.1.1 is motivated by formula (10.5) in [22] with a slight modification; namely, an extra term $n|K|$ has been added in the numerator of the variable inside ϕ. This extra term $n|K|$ is added in order to get, as $\phi(1) = 1$,

$$\widehat{V}_\phi(K, K) = n|K|. \tag{2.1.2}$$

Formula (2.1.1) coincides with formula (10.5) in [22] if $\phi \in \mathscr{I}$ is convex.

The following corollary states the homogeneity of $\widehat{V}_\phi(K, L)$, which has been made to be the same as the classical mixed volume $V_1(K, L)$.

Corollary 2.1.1. *Let $s, t > 0$ be constants. For $K, L \in \mathscr{K}_0$, one has, for $\phi \in \mathscr{I} \cup \mathscr{D}$,*

$$\widehat{V}_\phi(sK, tL) = s^{n-1} t \cdot \widehat{V}_\phi(K, L). \tag{2.1.3}$$

Proof. For $\phi \in \mathscr{I}$, one has, by letting $\eta = s^{n-1} t \lambda$,

$$\begin{aligned} \widehat{V}_\phi(sK, tL) &= \inf_{\eta>0} \Big\{ \int_{S^{n-1}} \phi\Big(\frac{t \cdot n|K| \cdot h_L(u)}{\eta \cdot s^{1-n} \cdot h_K(u)}\Big)\, d\widetilde{V}_K(u) \leq 1 \Big\} \\ &= s^{n-1} t \cdot \inf_{\lambda>0} \Big\{ \int_{S^{n-1}} \phi\Big(\frac{n|K| \cdot h_L(u)}{\lambda \cdot h_K(u)}\Big)\, d\widetilde{V}_K(u) \leq 1 \Big\}. \end{aligned}$$

That is, $\widehat{V}_\phi(sK, tL) = s^{n-1} t \cdot \widehat{V}_\phi(K, L)$. In particular, if $s = 1$ and $t > 0$, then

$$\widehat{V}_\phi(K, tL) = t \cdot \widehat{V}_\phi(K, L);$$

while if $t = 1$ and $s > 0$, then

$$\widehat{V}_\phi(sK, L) = s^{n-1} \cdot \widehat{V}_\phi(K, L).$$

The case for $\phi \in \mathscr{D}$ follows along the same way. $\square$

Let the function $G : (0, \infty) \to (0, \infty)$ be given by

$$G(\lambda) = \int_{S^{n-1}} \phi\Big(\frac{n|K| \cdot h_L(u)}{\lambda \cdot h_K(u)}\Big)\, d\widetilde{V}_K(u).$$

For $\phi \in \mathscr{I}$, the function G is strictly decreasing on λ with

$$\lim_{\lambda \to 0} G(\lambda) = \lim_{t \to \infty} \phi(t) \quad \text{and} \quad \lim_{\lambda \to \infty} G(\lambda) = \lim_{t \to 0} \phi(t).$$

As an example, we show that $\lim_{\lambda \to 0} G(\lambda) = \lim_{t \to \infty} \phi(t)$. To this end, as $\phi \in \mathscr{I}$ is strictly increasing, we have

$$\begin{aligned} G(\lambda) &= \int_{S^{n-1}} \phi\Big(\frac{n|K| \cdot h_L(u)}{\lambda \cdot h_K(u)}\Big)\, d\widetilde{V}_K(u) \\ &\geq \int_{S^{n-1}} \phi\bigg(\frac{n|K| \cdot \min_{u \in S^{n-1}} h_L(u)}{\lambda \cdot \max_{u \in S^{n-1}} h_K(u)}\bigg)\, d\widetilde{V}_K(u) \\ &= \phi\bigg(\frac{n|K| \cdot \min_{u \in S^{n-1}} h_L(u)}{\lambda \cdot \max_{u \in S^{n-1}} h_K(u)}\bigg). \end{aligned}$$

This yields

$$\lim_{\lambda \to 0} G(\lambda) \geq \lim_{\lambda \to 0} \phi\bigg(\frac{n|K| \cdot \min_{u \in S^{n-1}} h_L(u)}{\lambda \cdot \max_{u \in S^{n-1}} h_K(u)}\bigg) = \lim_{t \to \infty} \phi(t).$$

Similarly, one has

$$\lim_{\lambda \to 0} G(\lambda) \leq \lim_{\lambda \to 0} \phi\bigg(\frac{n|K| \cdot \max_{u \in S^{n-1}} h_L(u)}{\lambda \cdot \min_{u \in S^{n-1}} h_K(u)}\bigg) = \lim_{t \to \infty} \phi(t),$$

and the desired result follows. On the other hand, the function G for $\phi \in \mathscr{D}$ is strictly increasing on λ with

$$\lim_{\lambda \to 0} G(\lambda) = \lim_{t \to \infty} \phi(t) \quad \text{and} \quad \lim_{\lambda \to \infty} G(\lambda) = \lim_{t \to 0} \phi(t).$$

Together with $\phi(1) = 1$, we have proved the following corollary.

Corollary 2.1.2. *Let $\phi \in \mathscr{I} \cup \mathscr{D}$ and $K, L \in \mathscr{K}_0$. Then $\widehat{V}_\phi(K,L) > 0$, and $\lambda_0 = \widehat{V}_\phi(K,L)$ if and only if*

$$G(\lambda_0) = \int_{S^{n-1}} \phi\Big(\frac{n|K| \cdot h_L(u)}{\lambda_0 \cdot h_K(u)}\Big)\, d\widetilde{V}_K(u) = 1.$$

For $\phi(t) = t^p$, one writes $\widehat{V}_p(K,L)$ instead of $\widehat{V}_\phi(K,L)$. A simple calculation shows that

$$\begin{aligned} \widehat{V}_p(K,L) &= n|K| \cdot \bigg[\int_{S^{n-1}} \Big(\frac{h_L(u)}{h_K(u)}\Big)^p d\widetilde{V}_K(u)\bigg]^{1/p} \\ &= (n|K|)^{1-\frac{1}{p}} \cdot \big(nV_p(K,L)\big)^{1/p}, \end{aligned}$$

where $V_p(K,L)$ is the L_p mixed volume of K and L for $p \in \mathbb{R}$ [49, 89], i.e.,

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L(u)^p h_K(u)^{1-p}\, dS_K(u).$$

If $\phi \in \mathscr{I}$ is convex, the Orlicz-Minkowski inequality holds [22]: for $K, L \in \mathscr{K}_0$, one has

$$\widehat{V}_\phi(K,L) \geq n \cdot |K|^{\frac{n-1}{n}} |L|^{\frac{1}{n}}. \tag{2.1.4}$$

If in addition ϕ is strictly convex, equality holds if and only if K and L are dilates to each other. In particular, the classical Minkowski inequality is related to $\phi(t) = t$: for $K, L \in \mathscr{K}_0$, one has

$$V_1(K,L)^n \geq |K|^{n-1}|L|, \tag{2.1.5}$$

with equality if and only if K and L are homothetic to each other (i.e., there exist a constant $s > 0$ and a vector $a \in \mathbb{R}^n$ such that $K = sL + a$).

In order to define the homogeneous Orlicz affine surface areas, we need to define $\widehat{V}_\phi(K, L^\circ)$ for $L \in \mathscr{S}_0$. The definition is similar to Definition 2.1.1 but with h_{L° replaced by $1/\rho_L$. That is, for $\phi \in \mathscr{I} \cup \mathscr{D}$, $\widehat{V}_\phi(K, L^\circ)$ for $K \in \mathscr{K}_0$ and $L \in \mathscr{S}_0$ is defined by the constant λ_0 such that

$$\int_{S^{n-1}} \phi\Big(\frac{n|K|}{\lambda_0 \cdot \rho_L(u) \cdot h_K(u)}\Big)\, d\widetilde{V}_K(u) = 1. \tag{2.1.6}$$

Of course, $\widehat{V}_\phi(K, L^\circ)$ for $K, L \in \mathscr{K}_0$ given by formula (2.1.6) coincides with the one given by formula (2.1.1). Note that $\widehat{V}_\phi(K, L^\circ)$ for $K \in \mathscr{K}_0$ and $L \in \mathscr{S}_0$ is also homogeneous as stated in Corollary 2.1.1.

For function $\phi \in \mathscr{I} \cup \mathscr{D}$, let $F(t) = \phi(t^{-1/n})$ and hence $\phi(t) = F(t^{-n})$. The relations between ϕ and F have been discussed in [90]. For example, a): ϕ and F have opposite monotonicity, that is, if one is strictly decreasing (increasing), then the other one will be strictly increasing (decreasing); b): if one is convex and increasing, then the other one is convex and decreasing. As mentioned in [90], to define Orlicz affine and geominimal surface areas, one needs to consider the convexity and concavity of F instead of the convexity and concavity of ϕ itself. Let

$$\begin{aligned}\widehat{\Phi}_1 &= \big\{\phi : \phi \in \mathscr{I} \text{ and } F \text{ is strictly convex}\big\};\\ \widehat{\Phi}_2 &= \{\phi : \phi \in \mathscr{D} \text{ and } F \text{ is strictly concave}\}.\end{aligned}$$

We often use $\widehat{\Phi}$ for $\widehat{\Phi}_1 \cup \widehat{\Phi}_2$. Sample functions in $\widehat{\Phi}$ are: t^p with $p \in (-n, 0) \cup (0, \infty)$. Similarly, let

$$\widehat{\Psi} = \{\phi : \phi \in \mathscr{D} \text{ and } F \text{ is strictly convex}\}.$$

Note that if $\phi \in \mathscr{I}$ such that F is strictly concave, then ϕ is a constant. We are not interested in this case. The set $\widehat{\Psi}$ contains functions such as t^p with $p \in (-\infty, -n)$.

Definition 2.1.2. *Let $K \in \mathscr{K}_0$. The homogeneous Orlicz L_ϕ affine surface area of K, denoted by $\widehat{\Omega}^{orlicz}_\phi(K)$, is defined by*

$$\widehat{\Omega}^{orlicz}_\phi(K) = \inf_{L \in \mathscr{S}_0} \Big\{\widehat{V}_\phi(K, \mathrm{vrad}(L)L^\circ)\Big\} \quad \text{for} \quad \phi \in \widehat{\Phi}; \tag{2.1.7}$$

$$\widehat{\Omega}^{orlicz}_\phi(K) = \sup_{L \in \mathscr{S}_0} \Big\{\widehat{V}_\phi(K, \mathrm{vrad}(L)L^\circ)\Big\} \quad \text{for} \quad \phi \in \widehat{\Psi}. \tag{2.1.8}$$

The homogeneous Orlicz L_ϕ geominimal surface area of K, denoted by $\widehat{G}^{orlicz}_\phi(K)$, is defined similarly with $\mathscr{S}_0$ replaced by $\mathscr{K}_0$.

Clearly $\widehat{\Omega}^{orlicz}_\phi(K) \le \widehat{G}^{orlicz}_\phi(K)$ if $\phi \in \widehat{\Phi}$ and $\widehat{\Omega}^{orlicz}_\phi(K) \ge \widehat{G}^{orlicz}_\phi(K)$ if $\phi \in \widehat{\Psi}$. For $\phi(t) = t^p$, one writes $\widehat{\Omega}^{orlicz}_p(K)$ instead of $\widehat{\Omega}^{orlicz}_\phi(K)$. In particular, for $-n \neq p \in \mathbb{R}$,

$$\widehat{\Omega}^{orlicz}_p(K) = (n\omega_n)^{-1/n} \cdot \big(as_p(K)\big)^{\frac{n+p}{np}} \cdot (n|K|)^{1-\frac{1}{p}},$$

where $as_p(K)$ is the L_p affine surface area of K (see e.g., [49, 89]):

$$\begin{aligned} as_p(K) &= \inf_{L \in \mathscr{S}_0} \Big\{nV_p(K, L^\circ)^{\frac{n}{n+p}}\, |L|^{\frac{p}{n+p}}\Big\}, \quad p \ge 0;\\ as_p(K) &= \sup_{L \in \mathscr{S}_0} \Big\{nV_p(K, L^\circ)^{\frac{n}{n+p}}\, |L|^{\frac{p}{n+p}}\Big\}, \quad -n \neq p < 0.\end{aligned}$$

Similarly, for $-n \neq p \in \mathbb{R}$,

$$\widehat{G}_p^{orlicz}(K) = (n\omega_n)^{-1/n} \cdot \big(\tilde{G}_p(K)\big)^{\frac{n+p}{np}} \cdot (n|K|)^{1-\frac{1}{p}},$$

where $\tilde{G}_p(K)$ is the L_p geominimal surface area [49, 89]:

$$\begin{aligned} \tilde{G}_p(K) &= \inf_{L \in \mathscr{K}_0} \Big\{ nV_p(K, L^\circ)^{\frac{n}{n+p}} \ |L|^{\frac{p}{n+p}} \Big\}, \quad p \geq 0; \\ \tilde{G}_p(K) &= \sup_{L \in \mathscr{K}_0} \Big\{ nV_p(K, L^\circ)^{\frac{n}{n+p}} \ |L|^{\frac{p}{n+p}} \Big\}, \quad -n \neq p < 0. \end{aligned}$$

When $K = B_2^n$, both $\widehat{\Omega}_\phi^{orlicz}(B_2^n)$ and $\widehat{G}_\phi^{orlicz}(B_2^n)$ can be calculated precisely.

Corollary 2.1.3. *For $\phi \in \widehat{\Phi} \cup \widehat{\Psi}$, one has*

$$\widehat{\Omega}_\phi^{orlicz}(B_2^n) = \widehat{G}_\phi^{orlicz}(B_2^n) = n\omega_n. \tag{2.1.9}$$

Proof. We only prove $\widehat{\Omega}_\phi^{orlicz}(B_2^n) = n\omega_n$ with $\phi \in \widehat{\Phi}_1$, and the other cases follow along the same lines. As $\phi \in \widehat{\Phi}_1$, one sees that ϕ is strictly increasing and $F(t) = \phi(t^{-1/n})$ is strictly convex. First of all, by formulas (2.1.2) and (2.1.7), one has

$$\widehat{\Omega}_\phi^{orlicz}(B_2^n) \leq \widehat{V}_\phi(B_2^n, B_2^n) = n\omega_n. \tag{2.1.10}$$

From Corollary 2.1.2 and Jensen's inequality, the fact that F is strictly convex yields

$$\begin{aligned} 1 &= \int_{S^{n-1}} \phi\bigg(\frac{n\omega_n \cdot \mathrm{vrad}(L)}{\widehat{V}_\phi(B_2^n, \mathrm{vrad}(L)L^\circ) \cdot \rho_L(u)} \bigg) \cdot \frac{1}{n\omega_n}\, d\sigma(u) \\ &\geq F\bigg(\int_{S^{n-1}} \frac{[\widehat{V}_\phi(B_2^n, \mathrm{vrad}(L)L^\circ)]^n \cdot \rho_L^n(u)}{[n\omega_n \cdot \mathrm{vrad}(L)]^n \cdot n\omega_n}\, d\sigma(u) \bigg) \\ &= \phi\bigg(\frac{n\omega_n}{\widehat{V}_\phi(B_2^n, \mathrm{vrad}(L)L^\circ)} \bigg). \end{aligned}$$

As $\phi(1) = 1$ and ϕ is strictly increasing, one gets, for all $L \in \mathscr{S}_0$,

$$\widehat{V}_\phi(B_2^n, \mathrm{vrad}(L)L^\circ) \geq n\omega_n.$$

The desired equality follows by taking the infimum over $L \in \mathscr{S}_0$ and by formula (2.1.10). $\square$

Proposition 2.1.1. *Let $K \in \mathscr{K}_0$ and $A \in GL(n)$. For $\phi \in \widehat{\Phi} \cup \widehat{\Psi}$, one has*

$$\widehat{\Omega}_{\phi}^{orlicz}(AK) = |\det A|^{\frac{n-1}{n}} \cdot \widehat{\Omega}_{\phi}^{orlicz}(K) \quad \text{and} \quad \widehat{G}_{\phi}^{orlicz}(AK) = |\det A|^{\frac{n-1}{n}} \cdot \widehat{G}_{\phi}^{orlicz}(K).$$

Proof. Let $A \in GL(n)$. For $v \in S^{n-1}$, let $u = u(v) = \frac{A^t v}{|A^t v|}$. By the definitions of support and radial functions, one can easily check that

$$h_{AK}(v) = |A^t v| \cdot h_K(u) \quad \text{and} \quad \rho_{A^{-t}L}(v)|A^t v| = \rho_L(u).$$

Consequently, $h_{AK}(v)\rho_{A^{-t}L}(v) = h_K(u)\rho_L(u)$ and

$$\begin{aligned} \widehat{V}_{\phi}(AK, (A^{-t}L)^{\circ}) &= \inf_{\lambda>0} \left\{ \int_{S^{n-1}} \phi\Big(\frac{n|AK|}{\lambda \cdot h_{AK}(v)\rho_{A^{-t}L}(v)}\Big)\, d\widetilde{V}_{AK}(v) \le 1 \right\} \\ &= \inf_{\lambda>0} \left\{ \int_{S^{n-1}} \phi\Big(\frac{|\det A| \cdot n|K|}{\lambda \cdot h_K(u)\rho_L(u)}\Big)\, d\widetilde{V}_K(u) \le 1 \right\} \\ &= |\det A| \cdot \inf_{\eta>0} \left\{ \int_{S^{n-1}} \phi\Big(\frac{n|K|}{\eta \cdot h_K(u)\rho_L(u)}\Big)\, d\widetilde{V}_K(u) \le 1 \right\}, \end{aligned}$$

where $\lambda = |\det A| \cdot \eta$. Consequently,

$$\widehat{V}_{\phi}(AK, (A^{-t}L)^{\circ}) = |\det A| \cdot \widehat{V}_{\phi}(K, L^{\circ}). \tag{2.1.11}$$

Combining with equation (2.1.3), one gets, for $\phi \in \widehat{\Phi}$,

$$\begin{aligned} \widehat{V}_{\phi}(AK, \mathrm{vrad}(A^{-t}L) \cdot (A^{-t}L)^{\circ}) &= |\det A| \cdot |\det A^t|^{-1/n} \cdot \widehat{V}_{\phi}(K, \mathrm{vrad}(L)L^{\circ}) \\ &= |\det A|^{\frac{n-1}{n}} \cdot \widehat{V}_{\phi}(K, \mathrm{vrad}(L)L^{\circ}). \end{aligned}$$

The desired result follows immediately by taking the infimum over $L \in \mathscr{S}_0$. Other cases follow along the same lines. $\square$

Proposition 2.1.1 implies that both $\widehat{\Omega}_{\phi}^{orlicz}(\cdot)$ and $\widehat{G}_{\phi}^{orlicz}(\cdot)$ are invariant under the volume preserving linear transforms on $\mathbb{R}^n$. That is, for all $A \in SL(n)$ and $K \in \mathscr{K}_0$,

$$\widehat{\Omega}_{\phi}^{orlicz}(AK) = \widehat{\Omega}_{\phi}^{orlicz}(K) \quad \text{and} \quad \widehat{G}_{\phi}^{orlicz}(AK) = \widehat{G}_{\phi}^{orlicz}(K).$$

In particular, $\widehat{\Omega}_{\phi}^{orlicz}(\lambda K) = \lambda^{n-1} \cdot \widehat{\Omega}_{\phi}^{orlicz}(K)$ and $\widehat{G}_{\phi}^{orlicz}(\lambda K) = \lambda^{n-1} \cdot \widehat{G}_{\phi}^{orlicz}(K)$ for $\lambda > 0$ a constant. This means that both $\widehat{\Omega}_{\phi}^{orlicz}(\cdot)$ and $\widehat{G}_{\phi}^{orlicz}(\cdot)$ have homogeneity.

An immediate consequence of formula (2.1.9) and Proposition 2.1.1 is: for $\phi \in \widehat{\Phi} \cup \widehat{\Psi}$ and for the ellipsoid $\mathscr{E} = AB_2^n$ with $A \in GL(n)$,

$$\widehat{\Omega}_\phi^{orlicz}(\mathscr{E}) = \widehat{G}_\phi^{orlicz}(\mathscr{E}) = |\det A|^{\frac{n-1}{n}} \cdot n\omega_n.$$

We can prove the following affine isoperimetric inequalities for the homogeneous Orlicz L_ϕ affine and geominimal surface areas.

Theorem 2.1.1. *Let $K \in \widetilde{\mathscr{K}}$ be a convex body with its centroid or Santaló point at the origin.*

(i) For $\phi \in \widehat{\Phi}$, one has

$$\frac{\widehat{\Omega}_\phi^{orlicz}(K)}{\widehat{\Omega}_\phi^{orlicz}(B_2^n)} \le \frac{\widehat{G}_\phi^{orlicz}(K)}{\widehat{G}_\phi^{orlicz}(B_2^n)} \le \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-1}{n}}$$

with equality if and only if K is an origin-symmetric ellipsoid.

(ii) For $\phi \in \widehat{\Psi}$, there is a universal constant $c > 0$ such that

$$\frac{\widehat{\Omega}_\phi^{orlicz}(K)}{\widehat{\Omega}_\phi^{orlicz}(B_2^n)} \ge \frac{\widehat{G}_\phi^{orlicz}(K)}{\widehat{G}_\phi^{orlicz}(B_2^n)} \ge c \cdot \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-1}{n}}.$$

Remark. Theorem 2.1.1 asserts that among all convex bodies in $\widetilde{\mathscr{K}}$ with fixed volume, the homogeneous Orlicz L_ϕ affine and geominimal surface areas for $\phi \in \widehat{\Phi}$ attain their maximum at origin-symmetric ellipsoids. The L_p affine isoperimetric inequalities for the L_p affine and geominimal surface areas are special cases of Theorem 2.1.1 with $\phi(t) = t^p$ (see e.g., [49, 66, 67, 79, 89]).

Proof. Formulas (2.1.2) and (2.1.3) together with Definition 2.1.2 imply that for all $\phi \in \widehat{\Phi}$ and $K \in \mathscr{K}_0$,

$$\widehat{\Omega}_\phi^{orlicz}(K) \le \widehat{G}_\phi^{orlicz}(K) \le \widehat{V}_\phi(K, \mathrm{vrad}(K^\circ)K) = n|K| \cdot \mathrm{vrad}(K^\circ). \qquad (2.1.12)$$

If $K \in \widetilde{\mathscr{K}}$, the Blaschke-Santaló inequality further implies, for all $\phi \in \widehat{\Phi}$,

$$\widehat{\Omega}_\phi^{orlicz}(K) \le \widehat{G}_\phi^{orlicz}(K) \le n|K|^{\frac{n-1}{n}} \cdot \omega_n^{1/n}$$

with equality if and only if K is an origin-symmetric ellipsoid (i.e., those make the equality hold in the Blaschke-Santaló inequality). Dividing both sides by $\widehat{\Omega}_{\phi}^{orlicz}(B_2^n) = n\omega_n$, one gets the desired inequality in (i).

Similarly, for all $\phi \in \widehat{\Psi}$ and for all $K \in \mathscr{K}_0$,

$$\widehat{\Omega}_{\phi}^{orlicz}(K) \geq \widehat{G}_{\phi}^{orlicz}(K) \geq n|K| \cdot \operatorname{vrad}(K^{\circ}). \tag{2.1.13}$$

Dividing both sides by $\widehat{\Omega}_{\phi}^{orlicz}(B_2^n) = n\omega_n$, one gets

$$\frac{\widehat{\Omega}_{\phi}^{orlicz}(K)}{\widehat{\Omega}_{\phi}^{orlicz}(B_2^n)} \geq \frac{\widehat{G}_{\phi}^{orlicz}(K)}{\widehat{G}_{\phi}^{orlicz}(B_2^n)} \geq c \cdot \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-1}{n}},$$

where the inverse Santaló inequality [9] has been used: there is a universal constant $c > 0$ such that for all $K \in \widetilde{\mathscr{K}}$,

$$|K| \cdot |K^{\circ}| \geq c^n \omega_n^2. \tag{2.1.14}$$

See [38, 61] for estimates of the constant c. □

The following Santaló type inequalities follow immediately from Theorem 2.1.1 and the Blaschke-Santaló inequality.

Theorem 2.1.2. *Let $K \in \widetilde{\mathscr{K}}$ be a convex body with its centroid or Santaló point at the origin.*

(i) For $\phi \in \widehat{\Phi}$, one has

$$\frac{\widehat{\Omega}_{\phi}^{orlicz}(K) \cdot \widehat{\Omega}_{\phi}^{orlicz}(K^{\circ})}{[\widehat{\Omega}_{\phi}^{orlicz}(B_2^n)]^2} \leq \frac{\widehat{G}_{\phi}^{orlicz}(K) \cdot \widehat{G}_{\phi}^{orlicz}(K^{\circ})}{[\widehat{G}_{\phi}^{orlicz}(B_2^n)]^2} \leq 1.$$

Equality holds if and only if K is an origin-symmetric ellipsoid.

(ii) For $\phi \in \widehat{\Psi}$, there is a universal constant $c > 0$ such that

$$\frac{\widehat{\Omega}_{\phi}^{orlicz}(K) \cdot \widehat{\Omega}_{\phi}^{orlicz}(K^{\circ})}{[\widehat{\Omega}_{\phi}^{orlicz}(B_2^n)]^2} \geq \frac{\widehat{G}_{\phi}^{orlicz}(K) \cdot \widehat{G}_{\phi}^{orlicz}(K^{\circ})}{[\widehat{G}_{\phi}^{orlicz}(B_2^n)]^2} \geq c^{n+1}.$$

A finer calculation could lead to stronger arguments than Theorem 2.1.1, where the conditions on the centroid or the Santaló point of K can be removed. That is, $\widetilde{\mathscr{K}}$ in Theorem 2.1.1 can be replaced by $\mathscr{K}_0$. See similar results in [88, 89, 90, 95].

Corollary 2.1.4. *Let $K \in \mathscr{K}_0$. If either $\phi \in \widehat{\Phi}_1$ is concave or $\phi \in \widehat{\Phi}_2$ is convex, then*

$$\frac{\widehat{\Omega}_\phi^{orlicz}(K)}{\widehat{\Omega}_\phi^{orlicz}(B_2^n)} \le \frac{\widehat{G}_\phi^{orlicz}(K)}{\widehat{G}_\phi^{orlicz}(B_2^n)} \le \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-1}{n}}.$$

In addition, if either $\phi \in \widehat{\Phi}_1$ is strictly concave or $\phi \in \widehat{\Phi}_2$ is strictly convex, equality holds if and only if K is an origin-symmetric ellipsoid.

To prove this corollary, one needs the following cyclic inequality. For convenience, let $H = \phi \circ \psi^{-1}$, where ψ^{-1}, the inverse of ψ, always exists if $\psi \in \widehat{\Phi} \cup \widehat{\Psi}$.

Theorem 2.1.3. *Let $K \in \mathscr{K}_0$. Assume one of the following conditions holds: a) $\phi \in \widehat{\Phi}$ and $\psi \in \widehat{\Psi}$; b) H is convex with $\phi \in \widehat{\Phi}_2$ and $\psi \in \widehat{\Phi}_1$; c) H is concave with $\phi, \psi \in \widehat{\Phi}_1$; d) H is convex with either $\phi, \psi \in \widehat{\Phi}_2$ or $\phi, \psi \in \widehat{\Psi}$. Then*

$$\widehat{\Omega}_\phi^{orlicz}(K) \le \widehat{\Omega}_\psi^{orlicz}(K) \quad \text{and} \quad \widehat{G}_\phi^{orlicz}(K) \le \widehat{G}_\psi^{orlicz}(K).$$

Proof. The case for condition a) follows immediately from formulas (2.1.12) and (2.1.13). We only prove the case for condition b), and the other cases follow along the same fashion. Assume that condition b) holds and then H is convex. Corollary 2.1.2 and Jensen's inequality imply that

$$\begin{aligned} 1 &= \int_{S^{n-1}} \phi\Big(\frac{n|K|}{\widehat{V}_\phi(K, L^\circ) \cdot \rho_L(u) \cdot h_K(u)}\Big)\, d\widetilde{V}_K(u) \\ &= \int_{S^{n-1}} H\bigg(\psi\Big(\frac{n|K|}{\widehat{V}_\phi(K, L^\circ) \cdot \rho_L(u) \cdot h_K(u)}\Big)\bigg)\, d\widetilde{V}_K(u) \\ &\ge H\bigg(\int_{S^{n-1}} \psi\Big(\frac{n|K|}{\widehat{V}_\phi(K, L^\circ) \cdot \rho_L(u) \cdot h_K(u)}\Big)\, d\widetilde{V}_K(u)\bigg). \end{aligned}$$

Together with Corollary 2.1.2 and the facts that H is decreasing and $H(1) = 1$, one has

$$\int_{S^{n-1}} \psi\Big(\frac{n|K|}{\widehat{V}_\psi(K, L^\circ) \cdot \rho_L(u) \cdot h_K(u)}\Big)\, d\widetilde{V}_K(u) \le \int_{S^{n-1}} \psi\Big(\frac{n|K|}{\widehat{V}_\phi(K, L^\circ) \cdot \rho_L(u) \cdot h_K(u)}\Big)\, d\widetilde{V}_K(u).$$

Note that $\psi \in \mathscr{I}$ (increasing). It follows from above that $\widehat{V}_\phi(K, L^\circ) \le \widehat{V}_\psi(K, L^\circ)$. Together with Corollary 2.1.1 and Definition 2.1.2, one gets the desired result. □

Proof of Corollary 2.1.4. Let $\psi(t) = t$ and $\phi \in \widehat{\Phi}_2$ be convex. Then $H = \phi$ satisfies condition b) in Theorem 2.1.3 and thus $\widehat{\Omega}_\phi^{orlicz}(K) \leq \widehat{\Omega}_1^{orlicz}(K)$. Note that $\widehat{\Omega}_1^{orlicz}(K)$ is essentially the classical geominimal surface area and is translation invariant. That is, for any $z_0 \in \mathbb{R}^n$, $\widehat{\Omega}_1^{orlicz}(K - z_0) = \widehat{\Omega}_1^{orlicz}(K)$. In particular, one selects z_0 to be the point in $\mathbb{R}^n$ such that $K - z_0 \in \widetilde{\mathscr{K}}$ (i.e., z_0 is either the centroid or the Santaló point of K). Theorem 2.1.1 implies that

$$\frac{\widehat{\Omega}_\phi^{orlicz}(K)}{\widehat{\Omega}_\phi^{orlicz}(B_2^n)} \leq \frac{\widehat{\Omega}_1^{orlicz}(K - z_0)}{\widehat{\Omega}_1^{orlicz}(B_2^n)} \leq \left(\frac{|K - z_0|}{|B_2^n|}\right)^{\frac{n-1}{n}} = \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-1}{n}}.$$

To characterize the equality, due to the homogeneity of $\widehat{\Omega}_\phi^{orlicz}(\cdot)$, it is enough to prove that if ϕ is in addition strictly convex, $\widehat{\Omega}_\phi^{orlicz}(K) = \widehat{\Omega}_\phi^{orlicz}(B_2^n)$ if and only if K is an origin-symmetric ellipsoid with $|K| = \omega_n$. First of all, if K is an origin-symmetric ellipsoid with $|K| = \omega_n$, then $\widehat{\Omega}_\phi^{orlicz}(K) = \widehat{\Omega}_\phi^{orlicz}(B_2^n)$ follows from Corollary 2.1.3 and Proposition 2.1.1. On the other hand, by Theorem 2.1.2, $\widehat{\Omega}_\phi^{orlicz}(K) = \widehat{\Omega}_\phi^{orlicz}(B_2^n)$ holds only if $K - z_0$ is an origin-symmetric ellipsoid with $|K| = \omega_n$. By Proposition 2.1.1, it is enough to claim $K = B_2^n + z_0$ with $z_0 = o$. Corollary 2.1.3 and Definition 2.1.2 yield

$$n\omega_n = \widehat{\Omega}_\phi^{orlicz}(B_2^n) = \widehat{\Omega}_\phi^{orlicz}(K) = \widehat{\Omega}_\phi^{orlicz}(B_2^n + z_0) \leq \widehat{V}_\phi(B_2^n + z_0, B_2^n).$$

Note that $\phi \in \widehat{\Phi}$ is convex and decreasing. Combining with Corollary 2.1.2, one has

$$\begin{aligned} 1 &= \int_{S^{n-1}} \phi\left(\frac{n\omega_n}{\widehat{V}_\phi(B_2^n + z_0, B_2^n) \cdot h_{B_2^n+z_0}(u)}\right) \cdot \frac{h_{B_2^n+z_0}(u)}{n\omega_n} \cdot d\sigma(u) \\ &\geq \phi\left(\int_{S^{n-1}} \frac{d\sigma(u)}{\widehat{V}_\phi(B_2^n + z_0, B_2^n)}\right) \geq 1. \end{aligned}$$

As ϕ is strictly convex, equality holds if and only if $h_{B_2^n+z_0}(u)$ is a constant on S^{n-1}. This yields $z_0 = o$ as desired.

The case for $\phi \in \widehat{\Phi}_1$ being concave (with characterization for equality) follows along the same lines. $\square$

2.2 The Orlicz-Petty bodies and the continuity

This section concentrates on the continuity of the homogeneous Orlicz geominimal surface areas. In Subsection 2.2.1, we first show that the homogeneous Orlicz geominimal surface areas are semicontinuous on $\mathscr{K}_0$ with respect to the Hausdorff distance. The existence and uniqueness of the Orlicz-Petty bodies, under certain conditions, will be proved in Subsection 2.2.2. Our main result on the continuity will be given in Subsection 2.2.3.

2.2.1 Semicontinuity of the homogeneous Orlicz geominimal surface areas

Let us first establish the semicontinuity of the homogeneous Orlicz geominimal surface areas. Recall that for $\phi \in \widehat{\Phi}$ and for $K \in \mathscr{K}_0$,

$$\widehat{G}_{\phi}^{orlicz}(K) = \inf_{L \in \mathscr{K}_0} \{\widehat{V}_{\phi}(K, \mathrm{vrad}(L)L^{\circ})\}.$$

It is often more convenient, by the bipolar theorem (i.e., $(L^{\circ})^{\circ} = L$ for $L \in \mathscr{K}_0$) and Corollary 2.1.1, to formulate $\widehat{G}_{\phi}^{orlicz}(K)$ for $\phi \in \widehat{\Phi}$ by

$$\widehat{G}_{\phi}^{orlicz}(K) = \inf\{\widehat{V}_{\phi}(K, L) : \ L \in \mathscr{K}_0 \quad \text{with} \quad |L^{\circ}| = \omega_n\}. \tag{2.2.15}$$

Similarly, for $\phi \in \widehat{\Psi}$,

$$\widehat{G}_{\phi}^{orlicz}(K) = \sup\{\widehat{V}_{\phi}(K, L) : \ L \in \mathscr{K}_0 \quad \text{with} \quad |L^{\circ}| = \omega_n\}. \tag{2.2.16}$$

Denote by r_K and R_K the inner and outer radii of convex body $K \in \mathscr{K}_0$, respectively. That is,

$$r_K = \min\{h_K(u) : \ u \in S^{n-1}\} \quad \text{and} \quad R_K = \max\{h_K(u) : \ u \in S^{n-1}\}.$$

Lemma 2.2.1. *Let $K, L \in \mathscr{K}_0$. For $\phi \in \mathscr{I} \cup \mathscr{D}$, one has*

$$\frac{n\omega_n \cdot r_K^n \cdot r_L}{R_K} \le \widehat{V}_{\phi}(K, L) \le \frac{n\omega_n \cdot R_K^n \cdot R_L}{r_K}.$$

Proof. For $\phi \in \mathscr{I}$, let $\lambda = \widehat{V}_\phi(K,L)$. By Corollary 2.1.2 and the fact that ϕ is increasing on $(0,\infty)$, one has

$$\begin{aligned} 1 &= \int_{S^{n-1}} \phi\left(\frac{n|K|\cdot h_L(u)}{\lambda\cdot h_K(u)}\right) d\widetilde{V}_K(u) \\ &\leq \int_{S^{n-1}} \phi\left(\frac{n|K|\cdot R_L}{\lambda\cdot r_K}\right) d\widetilde{V}_K(u) \\ &\leq \phi\left(\frac{n\omega_n\cdot R_K^n\cdot R_L}{\lambda\cdot r_K}\right). \end{aligned}$$

Moreover, as $\phi(1)=1$, one gets

$$\widehat{V}_\phi(K,L) = \lambda \leq \frac{n\omega_n\cdot R_K^n\cdot R_L}{r_K}.$$

For the lower bound,

$$\begin{aligned} 1 &= \int_{S^{n-1}} \phi\left(\frac{n|K|\cdot h_L(u)}{\lambda\cdot h_K(u)}\right) d\widetilde{V}_K(u) \\ &\geq \int_{S^{n-1}} \phi\left(\frac{n|K|\cdot r_L}{\lambda\cdot R_K}\right) d\widetilde{V}_K(u) \\ &\geq \phi\left(\frac{n\omega_n\cdot r_K^n\cdot r_L}{\lambda\cdot R_K}\right). \end{aligned}$$

As ϕ is increasing on $(0,\infty)$ and $\phi(1)=1$, one gets

$$\widehat{V}_\phi(K,L) \geq \frac{n\omega_n\cdot r_K^n\cdot r_L}{R_K}.$$

The case for $\phi \in \mathscr{D}$ follows along the same lines. □

We will often need the following result.

Lemma 2.2.2. *Let $\varphi : I \to \mathbb{R}$ be a uniformly continuous function on an interval $I \subset \mathbb{R}$. Let $\{f_i\}_{i\geq 0}$ be a sequence of functions such that $f_i : E \to I$ for all $i \geq 0$ and $f_i \to f_0$ uniformly on E as $i \to \infty$. Then $\varphi(f_i) \to \varphi(f_0)$ uniformly on E as $i \to \infty$.*

Proof. For any $\epsilon > 0$. As φ is uniformly continuous, there exists $\delta(\epsilon) > 0$ such that $|\varphi(x) - \varphi(y)| < \epsilon$ for all $x, y \in I$ with $|x-y| < \delta(\epsilon)$. On the other hand, as $f_i \to f_0$ uniformly on E, there exists an integer $N_0(\epsilon) := N(\delta(\epsilon)) > 0$ such that $|f_i(z) - f_0(z)| < \delta(\epsilon)$ for all $i > N_0(\epsilon)$ and all $z \in E$. Hence, $|\varphi(f_i(z)) - \varphi(f_0(z))| < \epsilon$ for all $i > N_0(\epsilon)$ and all $z \in E$. That is, $\varphi(f_i) \to \varphi(f_0)$ uniformly on E. □

Proposition 2.2.1. *Let $\{K_i\}_{i\geq 1}$ and $\{L_i\}_{i\geq 1}$ be two sequences of convex bodies in $\mathscr{K}_0$ such that $K_i \to K \in \mathscr{K}_0$ and $L_i \to L \in \mathscr{K}_0$. For $\phi \in \mathscr{I} \cup \mathscr{D}$, one has $\widehat{V}_\phi(K_i, L_i) \to \widehat{V}_\phi(K, L)$.*

Proof. As $K_i \to K \in \mathscr{K}_0$, one can find constants $c_K, C_K > 0$, such that, for all $i \geq 1$,

$$c_K B_2^n \subset K_i, K \subset C_K B_2^n. \tag{2.2.17}$$

Similarly, one can find constants $c_L, C_L > 0$, such that, for all $i \geq 1$,

$$c_L B_2^n \subset L_i, L \subset C_L B_2^n. \tag{2.2.18}$$

For simplicity, let $\lambda_i = \widehat{V}_\phi(K_i, L_i)$. Lemma 2.2.1 yields, for all $i \geq 1$,

$$\frac{n\omega_n \cdot c_K^n \cdot c_L}{C_K} \leq \lambda_i \leq \frac{n\omega_n \cdot C_K^n \cdot C_L}{c_K}, \tag{2.2.19}$$

and thus the sequence $\{\lambda_i\}_{i\geq 1}$ is bounded from both sides. Let f_i and f be given by

$$f_i(u) = \frac{n|K_i| \cdot h_{L_i}(u)}{\lambda_i \cdot h_{K_i}(u)} \quad \text{and} \quad f(u) = \frac{n|K| \cdot h_L(u)}{\lambda_0 \cdot h_K(u)} \quad \text{for} \;\; u \in S^{n-1}.$$

On the one hand, suppose that $\{\lambda_{i_k}\}_{k\geq 1}$ is a convergent subsequence of $\{\lambda_i\}_{i\geq 1}$ with limit λ_0. That is, $\lim_{k\to\infty} \lambda_{i_k} = \lambda_0$ and then $0 < \lambda_0 < \infty$. Note that $K_i \to K \in \mathscr{K}_0$ yields $h_{K_i} \to h_K$ uniformly on S^{n-1}. Similarly, $h_{L_i} \to h_L$ uniformly on S^{n-1}. Together with (2.2.17) and (2.2.18), one sees that $f_{i_k} \to f$ uniformly on S^{n-1}. Moreover, the ranges of f_{i_k}, f are all in the interval

$$I = \left[\frac{c_L}{C_L} \cdot \left(\frac{c_K}{C_K}\right)^{n+1}, \;\; \frac{C_L}{c_L} \cdot \left(\frac{C_K}{c_K}\right)^{n+1}\right].$$

Note that the interval $I \subsetneq (0, \infty)$ is a compact set. Hence $\phi \in \mathscr{I} \cup \mathscr{D}$ restricted on I is uniformly continuous. Lemma 2.2.2 implies that $\phi(f_{i_k}) \to \phi(f)$ uniformly on S^{n-1}. Moreover, as both $\{\phi(f_{i_k})\}_{k\geq 1}$ and $\{h_{K_i}\}_{i\geq 1}$ are uniformly bounded on S^{n-1}, one sees that $\phi(f_{i_k})h_{K_{i_k}} \to \phi(f)h_K$ uniformly on S^{n-1}. Formula (1.1.4) then yields

$$\begin{aligned} 1 &= \lim_{k\to\infty}\int_{S^{n-1}} \phi\left(\frac{n|K_{i_k}|\cdot h_{L_{i_k}}(u)}{\lambda_{i_k}\cdot h_{K_{i_k}}(u)}\right) d\widetilde{V}_{K_{i_k}}(u) \\ &= \lim_{k\to\infty}\int_{S^{n-1}} \frac{\phi\left(f_{i_k}(u)\right) h_{K_{i_k}}(u)}{n|K_{i_k}|}\, dS_{K_{i_k}}(u) \\ &= \int_{S^{n-1}} \frac{\phi\left(f(u)\right) h_K(u)}{n|K|}\, dS_K(u) \\ &= \int_{S^{n-1}} \phi\left(\frac{n|K|\cdot h_L(u)}{\lambda_0\cdot h_K(u)}\right) d\widetilde{V}_K(u). \end{aligned}$$

Therefore $\lambda_0 = \widehat{V}_\phi(K,L)$ and $\lim_{k\to\infty}\widehat{V}_\phi(K_{i_k},L_{i_k}) = \widehat{V}_\phi(K,L)$. We have proved that if a subsequence of $\{\widehat{V}_\phi(K_i,L_i)\}_{i\geq 1}$ is convergent, then its limit must be $\widehat{V}_\phi(K,L)$.

To conclude Proposition 2.2.1, it is enough to claim that the sequence $\{\widehat{V}_\phi(K_i,L_i)\}_{i\geq 1}$ is indeed convergent. Suppose that $\{\widehat{V}_\phi(K_i,L_i)\}_{i\geq 1}$ is not convergent. One has two convergent subsequences whose limits exist by (2.2.19) and are different. This contradicts with the arguments in the previous paragraph, and hence the sequence $\{\widehat{V}_\phi(K_i,L_i)\}_{i\geq 1}$ is convergent. $\square$

The following result states that the homogeneous Orlicz geominimal surface areas are semicontinuous. For the homogeneous Orlicz affine surface areas, similar semi-continuous arguments also hold.

Proposition 2.2.2. *For $\phi\in\widehat{\Phi}$, the functional $\widehat{G}^{orlicz}_\phi(\cdot)$ is upper semicontinuous on $\mathscr{K}_0$ with respect to the Hausdorff distance. That is, for any convergent sequence $\{K_i\}_{i\geq 1}\subset\mathscr{K}_0$ whose limit is $K_0\in\mathscr{K}_0$, then*

$$\widehat{G}^{orlicz}_\phi(K_0) \geq \limsup_{i\to\infty}\widehat{G}^{orlicz}_\phi(K_i).$$

While for $\phi\in\widehat{\Psi}$, the functional $\widehat{G}^{orlicz}_\phi(\cdot)$ is lower semicontinuous on $\mathscr{K}_0$: for any $K_i\to K_0$, then

$$\widehat{G}^{orlicz}_\phi(K_0) \leq \liminf_{i\to\infty}\widehat{G}^{orlicz}_\phi(K_i).$$

Proof. Let $\phi\in\widehat{\Phi}$. For any given $\epsilon>0$, by formula (2.2.15), there exists a convex body $L_\epsilon\in\mathscr{K}_0$, such that $|L_\epsilon^\circ| = \omega_n$ and

$$\widehat{G}^{orlicz}_\phi(K_0)+\epsilon > \widehat{V}_\phi(K_0,L_\epsilon) \geq \widehat{G}^{orlicz}_\phi(K_0).$$

By Proposition 2.2.1, one has

$$\widehat{G}_\phi^{orlicz}(K_0)+\epsilon > \widehat{V}_\phi(K_0,L_\epsilon)=\lim_{i\to\infty}\widehat{V}_\phi(K_i,L_\epsilon)=\limsup_{i\to\infty}\widehat{V}_\phi(K_i,L_\epsilon)\geq \limsup_{i\to\infty}\widehat{G}_\phi^{orlicz}(K_i).$$

The desired result follows by letting $\epsilon\to 0$. The case for $\phi\in\widehat{\Psi}$ can be proved along the same lines. $\square$

2.2.2 The Orlicz-Petty bodies: existence and basic properties

In this subsection, we will prove the existence of the Orlicz-Petty bodies under the condition $\phi\in\widehat{\Phi}_1$. The following lemma is needed for our goal. Recall that $a_+=\max\{a,0\}$ and hence $a_+=\frac{a+|a|}{2}$ for $a\in\mathbb{R}$.

Lemma 2.2.3. *Let $K\in\mathscr{K}_0$ and $\phi\in\widehat{\Phi}_1$. For fixed $v\in S^{n-1}$, define $G_v:(0,\infty)\to(0,\infty)$ by*

$$G_v(\eta)=\int_{S^{n-1}}\phi\left(\frac{n|K|\cdot\langle u,v\rangle_+}{\eta\cdot h_K(u)}\right)d\widetilde{V}_K(u).$$

Then G_v is strictly decreasing, and

$$\lim_{\eta\to 0}G_v(\eta)=\lim_{t\to\infty}\phi(t)=\infty \quad\text{and}\quad \lim_{\eta\to\infty}G_v(\eta)=\lim_{t\to 0}\phi(t)=0.$$

Proof. Since $K\in\mathscr{K}_0$, (1.1.1) implies that there exists a constant $c_1>0$ such that for all $v\in S^{n-1}$,

$$\int_{S^{n-1}}\langle u,v\rangle_+ dS_K(u)\geq c_1.$$

For any given $v\in S^{n-1}$, let $\Sigma_j(v)=\{u\in S^{n-1}:\langle u,v\rangle_+>\frac{1}{j}\}$ for all integers $j\geq 1$. It is obvious that $\Sigma_j(v)\subset\Sigma_{j+1}(v)$ for all $j\geq 1$ and $\cup_{j=1}^\infty\Sigma_j(v)=\{u\in S^{n-1}:\langle u,v\rangle_+>0\}$. Hence,

$$\lim_{j\to\infty}\int_{\Sigma_j(v)}\langle u,v\rangle_+\,dS_K(u)=\int_{\cup_{j=1}^\infty\Sigma_j(v)}\langle u,v\rangle_+\,dS_K(u)=\int_{S^{n-1}}\langle u,v\rangle_+\,dS_K(u)\geq c_1.$$

Then, there exists an integer $j_0\geq 1$ (depending on $v\in S^{n-1}$) such that

$$\frac{c_1}{2}\leq\int_{\Sigma_{j_0}(v)}\langle u,v\rangle_+\,dS_K(u)\leq\int_{\Sigma_{j_0}(v)}dS_K(u). \tag{2.2.20}$$

Assume that $\phi \in \widehat{\Phi}_1$ and then ϕ is strictly increasing. Let $0 < \eta_1 < \eta_2 < \infty$. For all $u \in \Sigma_{j_0}(v)$, one has

$$\phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\eta_2 \cdot h_K(u)}\right) < \phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\eta_1 \cdot h_K(u)}\right),$$

and by (2.2.20),

$$\int_{\Sigma_{j_0}(v)} \phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\eta_2 \cdot h_K(u)}\right) d\widetilde{V}_K(u) < \int_{\Sigma_{j_0}(v)} \phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\eta_1 \cdot h_K(u)}\right) d\widetilde{V}_K(u).$$

The desired monotone argument (i.e., G_v is strictly decreasing) follows immediately from

$$G_v(\eta) = \int_{\Sigma_{j_0}(v)} \phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\eta \cdot h_K(u)}\right) d\widetilde{V}_K(u) + \int_{S^{n-1}\setminus\Sigma_{j_0}(v)} \phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\eta \cdot h_K(u)}\right) d\widetilde{V}_K(u).$$

Now let us prove that

$$\lim_{\eta\to 0} G_v(\eta) = \lim_{t\to\infty} \phi(t) = \infty \quad \text{and} \quad \lim_{\eta\to\infty} G_v(\eta) = \lim_{t\to 0} \phi(t) = 0.$$

To this end, as $\phi \in \widehat{\Phi}_1$ is increasing,

$$G_v(\eta) = \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\eta \cdot h_K(u)}\right) d\widetilde{V}_K(u) \le \int_{S^{n-1}} \phi\left(\frac{n|K|}{\eta \cdot r_K}\right) d\widetilde{V}_K(u) = \phi\left(\frac{n|K|}{\eta \cdot r_K}\right).$$

By letting $t = \frac{n|K|}{\eta \cdot r_K}$, one has $0 \le \lim_{\eta\to\infty} G_v(\eta) \le \lim_{t\to 0} \phi(t) = 0$ and thus we have $\lim_{\eta\to\infty} G_v(\eta) = 0$. On the other hand,

$$\begin{aligned}
G_v(\eta) &= \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\eta \cdot h_K(u)}\right) d\widetilde{V}_K(u) \\
&\ge \int_{\Sigma_{j_0}(v)} \phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\eta \cdot h_K(u)}\right) d\widetilde{V}_K(u) \\
&\ge \int_{\Sigma_{j_0}(v)} \phi\left(\frac{n|K|}{\eta \cdot j_0 \cdot R_K}\right) \cdot \frac{r_K}{n|K|} \cdot dS_K(u) \\
&\ge \phi\left(\frac{n|K|}{\eta \cdot j_0 \cdot R_K}\right) \cdot \frac{r_K}{n|K|} \cdot \frac{c_1}{2}. \qquad (2.2.21)
\end{aligned}$$

The desired result $\lim_{\eta\to 0} G_v(\eta) = \lim_{t\to\infty} \phi(t) = \infty$ follows by taking $\eta \to 0$. $\square$

A direct consequence of Lemma 2.2.3 is that if $\phi \in \widehat{\Phi}_1$ and $v \in S^{n-1}$, then there is a unique $\eta_0 \in (0, \infty)$ such that

$$G_v(\eta_0) = \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\eta_0 \cdot h_K(u)}\right) d\widetilde{V}_K(u) = 1.$$

Such a unique η_0 can be defined as the homogeneous Orlicz L_ϕ mixed volume of $K \in \mathscr{K}_0$ and the line segment $[0, v] = \{tv : t \in [o, 1]\}$, namely, $\eta_0 = \widehat{V}_\phi(K, [o, v])$ and

$$\int_{S^{n-1}} \phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\widehat{V}_\phi(K, [0, v]) \cdot h_K(u)}\right) d\widetilde{V}_K(u) = 1. \tag{2.2.22}$$

Proposition 2.2.3. *Let $K \in \mathscr{K}_0$ and $\phi \in \widehat{\Phi}_1$. There exists a convex body $M \in \mathscr{K}_0$ such that*

$$\widehat{G}_\phi^{orlicz}(K) = \widehat{V}_\phi(K, M) \quad \text{and} \quad |M^\circ| = \omega_n.$$

If in addition ϕ is convex, such a convex body M is unique.

Proof. Formula (2.2.15) implies that for $\phi \in \widehat{\Phi}_1$, there exists a sequence $\{M_i\}_{i\ge 1} \subset \mathscr{K}_0$ such that $\widehat{V}_\phi(K, M_i) \to \widehat{G}_\phi^{orlicz}(K)$ as $i \to \infty$, $|M_i^\circ| = \omega_n$ and $\widehat{V}_\phi(K, M_i) \le 2\widehat{V}_\phi(K, B_2^n)$ for all $i \ge 1$. For each fixed $i \ge 1$, let

$$R_i = \rho_{M_i}(u_i) = \max\{\rho_{M_i}(u) : u \in S^{n-1}\}.$$

This yields $\{\lambda u_i : 0 \le \lambda \le R_i\} \subset M_i$ and hence for all $u \in S^{n-1}$,

$$h_{M_i}(u) \ge R_i \cdot \frac{|\langle u, u_i\rangle| + \langle u, u_i\rangle}{2} = R_i \cdot \langle u, u_i\rangle_+.$$

Let $\phi \in \widehat{\Phi}_1$ and $\eta_i = \widehat{V}_\phi(K, [o, u_i]) \in (0, \infty)$ for $i \ge 1$. Recall that formula (2.2.22) states

$$1 = \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot \langle u, u_i\rangle_+}{\eta_i \cdot h_K(u)}\right) d\widetilde{V}_K(u).$$

By Corollary 2.1.2 and the fact that ϕ is increasing, we have

$$\begin{aligned} 1 &= \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot h_{M_i}(u)}{\widehat{V}_\phi(K, M_i) \cdot h_K(u)}\right) d\widetilde{V}_K(u) \\ &\ge \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot R_i \cdot \langle u, u_i\rangle_+}{2\widehat{V}_\phi(K, B_2^n) \cdot h_K(u)}\right) d\widetilde{V}_K(u). \end{aligned}$$

This further leads to, for all $i \geq 1$,

$$R_i \leq \frac{2\widehat{V}_\phi(K, B_2^n)}{\eta_i}.$$

Next, we prove that $\inf_{i\geq 1} \eta_i > 0$. We will use the method of contradiction and assume that $\inf_{i\geq 1} \eta_i = 0$. Consequently, there is a subsequence of $\{\eta_i\}_{i\geq 1}$ (still denoted by $\{\eta_i\}_{i\geq 1}$), such that, $\eta_i \to 0$ as $i \to \infty$. Due to the compactness of S^{n-1}, one can also have a convergent subsequence of $\{u_i\}_{i\geq 1}$ (again denoted by $\{u_i\}_{i\geq 1}$) whose limit is $v \in S^{n-1}$. In summary, we have two sequences $\{u_i\}_{i\geq 1}$ and $\{\eta_i\}_{i\geq 1}$ such that $u_i \to v$ and $\eta_i \to 0$ as $i \to \infty$. It is easily checked that $\langle u, u_i\rangle_+ \to \langle u, v\rangle_+$ uniformly on S^{n-1} by the triangle inequality. For any given $\varepsilon > 0$, Corollary 2.1.2, Fatou's lemma and formula (2.2.20) imply

$$\begin{aligned}
1 &= \lim_{i\to\infty} \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot \langle u, u_i\rangle_+}{\eta_i \cdot h_K(u)}\right) d\widetilde{V}_K(u) \\
&\geq \liminf_{i\to\infty} \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot \langle u, u_i\rangle_+}{(\eta_i+\varepsilon) \cdot h_K(u)}\right) d\widetilde{V}_K(u) \\
&\geq \int_{S^{n-1}} \liminf_{i\to\infty} \phi\left(\frac{n|K| \cdot \langle u, u_i\rangle_+}{(\eta_i+\varepsilon) \cdot h_K(u)}\right) d\widetilde{V}_K(u) \\
&= \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot \langle u, v\rangle_+}{\varepsilon \cdot h_K(u)}\right) d\widetilde{V}_K(u) \\
&= G_v(\varepsilon).
\end{aligned}$$

It follows from Lemma 2.2.3 that $\lim_{\varepsilon\to 0^+} G_v(\varepsilon) = \infty$, which leads to a contradiction (i.e., $1 \geq \infty$). Therefore, $\inf_{i\geq 1} \eta_i > 0$ and

$$\sup_{i\geq 1} R_i \leq \frac{2\widehat{V}_\phi(K, B_2^n)}{\inf_{i\geq 1} \eta_i} < \infty.$$

This concludes that the sequence $\{M_i\}_{i\geq 1} \subset \mathscr{K}_0$ is uniformly bounded.

The Blaschke selection theorem yields that there exists a convergent subsequence of $\{M_i\}_{i\geq 1}$ (still denoted by $\{M_i\}_{i\geq 1}$) and a convex body $M \in \mathscr{K}$ such that $M_i \to M$ as $i \to \infty$. Since $|M_i^\circ| = \omega_n$ for all $i \geq 1$, Lemma 1.1.1 implies $M \in \mathscr{K}_0$. Moreover, $|M^\circ| = \omega_n$ because $|M_i^\circ| = \omega_n$ for all $i \geq 1$ and $M_i \to M$ (hence, $M_i^\circ \to M^\circ$). It follows from Proposition 2.2.1 that

$$\widehat{V}_\phi(K, M_i) \to \widehat{V}_\phi(K, M) \quad \text{and} \quad |M^\circ| = \omega_n.$$

By the uniqueness of the limit, one gets

$$\widehat{G}_\phi^{orlicz}(K)=\widehat{V}_\phi(K,M)\quad\text{and}\quad|M^\circ|=\omega_n.$$

This concludes the existence of the Orlicz-Petty bodies.

If $\phi\in\widehat{\Phi}_1$ is also convex, the uniqueness of M can be proved as follows. Suppose that $M_1,M_2\in\mathscr{K}_0$ such that $|M_1^\circ|=|M_2^\circ|=\omega_n$ and

$$\widehat{V}_\phi(K,M_1)=\inf_{L\in\mathscr{K}_0}\{\widehat{V}_\phi(K,\mathrm{vrad}(L^\circ)L)\}=\widehat{V}_\phi(K,M_2).$$

Define $M\in\mathscr{K}_0$ by $M=\frac{M_1+M_2}{2}$. That is, $h_M=\frac{h_{M_1}+h_{M_2}}{2}$. By formula (1.1.3), it can be checked that $|M^\circ|\le\omega_n$ (hence $\mathrm{vrad}(M^\circ)\le 1$) with equality if and only if $M_1=M_2$. In fact, the function t^{-n} is strictly convex, and hence

$$\begin{aligned}|M^\circ| &= \frac{1}{n}\int_{S^{n-1}}h_M(u)^{-n}\,d\sigma(u)\\ &= \frac{1}{n}\int_{S^{n-1}}\left(\frac{h_{M_1}(u)+h_{M_2}(u)}{2}\right)^{-n}d\sigma(u)\\ &\le \frac{1}{n}\int_{S^{n-1}}\frac{h_{M_1}(u)^{-n}+h_{M_2}(u)^{-n}}{2}\,d\sigma(u)\\ &= \frac{|M_1^\circ|+|M_2^\circ|}{2}=\omega_n,\end{aligned}\tag{2.2.23}$$

with equality if and only if $h_{M_1}=h_{M_2}$ on S^{n-1}, i.e., $M_1=M_2$.

For convenience, let $\lambda=\widehat{V}_\phi(K,M_1)=\widehat{V}_\phi(K,M_2)$. The fact that ϕ is convex imply

$$\begin{aligned}\int_{S^{n-1}}\phi\left(\frac{n|K|\cdot h_M(u)}{\lambda\cdot h_K(u)}\right)d\widetilde{V}_K(u) &= \int_{S^{n-1}}\phi\left(\frac{n|K|\cdot(h_{M_1}(u)+h_{M_2}(u))}{2\cdot\lambda\cdot h_K(u)}\right)d\widetilde{V}_K(u)\\ &\le \frac{1}{2}\int_{S^{n-1}}\left[\phi\left(\frac{n|K|\cdot h_{M_1}(u)}{\lambda\cdot h_K(u)}\right)\right.\\ &+ \left.\phi\left(\frac{n|K|\cdot h_{M_2}(u)}{\lambda\cdot h_K(u)}\right)\right]d\widetilde{V}_K(u)\\ &= 1.\end{aligned}$$

Hence, $\widehat{V}_\phi(K,M)\le\lambda$ which follows from the facts that ϕ is strictly increasing and

$$\int_{S^{n-1}}\phi\left(\frac{n|K|\cdot h_M(u)}{\lambda\cdot h_K(u)}\right)d\widetilde{V}_K(u)\le 1=\int_{S^{n-1}}\phi\left(\frac{n|K|\cdot h_M(u)}{\widehat{V}_\phi(K,M)\cdot h_K(u)}\right)d\widetilde{V}_K(u).$$

Assume that $M_1 \neq M_2$, then $\text{vrad}(M^\circ) < 1$. Note that $\widehat{V}_\phi(K, M) > 0$. Together with Corollary 2.1.1, one can check that

$$\widehat{V}_\phi(K, \text{vrad}(M^\circ)M) < \widehat{V}_\phi(K, M) \leq \widehat{V}_\phi(K, M_1).$$

This contradicts with the minimality of M_1. Therefore, $M_1 = M_2$ and the uniqueness follows. $\square$

Definition 2.2.1. *Let $K \in \mathscr{K}_0$ and $\phi \in \widehat{\Phi}_1$. A convex body M is said to be an L_ϕ Orlicz-Petty body of K, if $M \in \mathscr{K}_0$ satisfies*

$$\widehat{G}_\phi^{orlicz}(K) = \widehat{V}_\phi(K, M) \quad \text{and} \quad |M^\circ| = \omega_n.$$

Denote by $\widehat{T}_\phi K$ the set of all L_ϕ Orlicz-Petty bodies of K.

Clearly, if $\phi \in \widehat{\Phi}_1$, the set $\widehat{T}_\phi K$ is nonempty and may contain more than one convex body. If in addition $\phi \in \widehat{\Phi}_1$ is convex, $\widehat{T}_\phi K$ must contain only one convex body; and in this case $\widehat{T}_\phi K$ is called the L_ϕ Orlicz-Petty body of K. Moreover, the set $\widehat{T}_\phi K$ is $SL(n)$-invariant. In fact, for $A \in SL(n)$ and all $M \in \widehat{T}_\phi K$, by Proposition 2.1.1 and formula (2.1.11), one sees

$$\widehat{G}_\phi^{orlicz}(AK) = \widehat{G}_\phi^{orlicz}(K) = \widehat{V}_\phi(K, M) = \widehat{V}_\phi(AK, AM).$$

It follows from $|(AM)^\circ| = \omega_n$ that $AM \in \widehat{T}_\phi(AK)$ and thus $A(\widehat{T}_\phi K) \subset \widehat{T}_\phi(AK)$. Replacing K by AK and A by its inverse, one also gets $\widehat{T}_\phi(AK) \subset A(\widehat{T}_\phi K)$ and thus $\widehat{T}_\phi(AK) = A(\widehat{T}_\phi K)$.

On the other hand, $\widehat{T}_\phi(\lambda K) = \widehat{T}_\phi K$ for all $\lambda > 0$. To this end, for $M \in \widehat{T}_\phi K$, one has $|M^\circ| = \omega_n$ and $\widehat{G}_\phi^{orlicz}(K) = \widehat{V}_\phi(K, M)$. This leads to, by Corollary 2.1.1 and Proposition 2.1.1,

$$\widehat{G}_\phi^{orlicz}(\lambda K) = \lambda^{n-1}\widehat{G}_\phi^{orlicz}(K) = \lambda^{n-1}\widehat{V}_\phi(K, M) = \widehat{V}_\phi(\lambda K, M).$$

Thus, $M \in \widehat{T}_\phi(\lambda K)$ and then $\widehat{T}_\phi K \subset \widehat{T}_\phi(\lambda K)$. Similarly, $\widehat{T}_\phi(\lambda K) \subset \widehat{T}_\phi K$ and thus $\widehat{T}_\phi(\lambda K) = \widehat{T}_\phi K$.

When $\phi \in \widehat{\Phi}_1$ is convex, the L_ϕ Orlicz-Petty body $\widehat{T}_\phi K$ satisfies the following inequality: for all $K \in \mathscr{K}_0$, one has

$$|\widehat{T}_\phi K| \cdot |(\widehat{T}_\phi K)^\circ| \leq |K| \cdot |K^\circ|. \tag{2.2.24}$$

In fact, it follows from (2.1.12) that for $K \in \mathscr{K}_0$, $\widehat{G}_\phi^{orlicz}(K) \leq n|K| \cdot \mathrm{vrad}(K^\circ)$. Definition 2.2.1 and the Orlicz-Minkowski inequality (2.1.4) imply that

$$\widehat{G}_\phi^{orlicz}(K) = \widehat{V}_\phi(K, \widehat{T}_\phi K) \geq n \cdot |K|^{\frac{n-1}{n}} |\widehat{T}_\phi K|^{\frac{1}{n}}.$$

The desired inequality (2.2.24) is then a simple consequence of the combination of the two inequalities above and $|(\widehat{T}_\phi K)^\circ| = \omega_n$. Note that it is an open problem (i.e., the famous Mahler conjecture) to find the minimum of $|K| \cdot |K^\circ|$ among all convex bodies $K \in \widetilde{\mathscr{K}}$. The inverse Santaló inequality (2.1.14) provides an isomorphic solution to the Mahler conjecture. We think that the L_ϕ Orlicz-Petty body $\widehat{T}_\phi K$ and inequality (2.2.24) may be useful in attacking the Mahler conjecture.

The following proposition states that an L_ϕ Orlicz-Petty body of a polytope is again a polytope.

Proposition 2.2.4. *Let $K \in \mathscr{K}_0$ be a polytope and $\phi \in \widehat{\Phi}_1$. If $M \in \widehat{T}_\phi K$, then M is a polytope with faces parallel to those of K.*

Proof. Let K be a polytope whose surface area measure S_K is focused on a finite set $\{u_1, \cdots, u_m\} \subset S^{n-1}$. Let $M \in \widehat{T}_\phi K$ be an L_ϕ Orlicz-Petty body of K. Denote by P the polytope whose faces are parallel to those of K and P circumscribes M.

Note that S_K is concentrated on $\{u_1, \cdots, u_m\}$ and $h_P(u_i) = h_M(u_i)$ for all $1 \leq i \leq m$. Let $\lambda = \widehat{V}_\phi(K, P)$. Then

$$\begin{aligned}
1 &= \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot h_P(u)}{\lambda \cdot h_K(u)}\right) d\widetilde{V}_K(u) \\
&= \frac{1}{n|K|} \cdot \sum_{i=1}^m \phi\left(\frac{n|K| \cdot h_P(u_i)}{\lambda \cdot h_K(u_i)}\right) h_K(u_i) S_K(u_i) \\
&= \frac{1}{n|K|} \cdot \sum_{i=1}^m \phi\left(\frac{n|K| \cdot h_M(u_i)}{\lambda \cdot h_K(u_i)}\right) h_K(u_i) S_K(u_i) \\
&= \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot h_M(u)}{\lambda \cdot h_K(u)}\right) d\widetilde{V}_K(u).
\end{aligned}$$

Consequently, $\lambda = \widehat{V}_\phi(K,P) = \widehat{V}_\phi(K,M)$.

As P circumscribes M, then $P^\circ \subset M^\circ$ and $|P^\circ| \leq |M^\circ| = \omega_n$ with equality if and only if $M = P$. Formula (2.1.7) and Corollary 2.1.1 yield that for $\phi \in \widehat{\Phi}_1$,

$$\widehat{G}_\phi^{orlicz}(K) \leq \widehat{V}_\phi(K, \mathrm{vrad}(P^\circ)P) \leq \widehat{V}_\phi(K,M) = \widehat{G}_\phi^{orlicz}(K).$$

This requires in particular $|P^\circ| = |M^\circ| = \omega_n$. Hence $M = P$ is a polytope whose faces are parallel to those of K. $\square$

Proposition 2.2.5. *Let $K \in \mathscr{K}_0$ and $r_K, R_K > 0$ be such that $r_K B_2^n \subset K \subset R_K B_2^n$. For $\phi \in \widehat{\Phi}_1$ and $M \in \widehat{T}_\phi K$, there exists an integer $j_0 > 1$ such that, for all $u \in S^{n-1}$,*

$$h_M(u) \leq \frac{j_0 \cdot R_K^{n+1}}{r_K^{n+1}} \cdot \phi^{-1}\left(\frac{2n\omega_n \cdot R_K^n}{c_1 \cdot r_K}\right),$$

where $c_1 > 0$ is the constant in (2.2.20).

Proof. Let $M \in \widehat{T}_\phi K$. First of all, the minimality of M gives that

$$\widehat{V}_\phi(K,M) \leq \widehat{V}_\phi(K,B_2^n) \leq \frac{n\omega_n \cdot R_K^n}{r_K},$$

where the second inequality follows from Lemma 2.2.1. Let $\lambda = \widehat{V}_\phi(K,M)$ and $R(M) = \rho_M(v) = \max\{\rho_M(u) : u \in S^{n-1}\}$. A calculation similar to (2.2.21) leads to

$$\begin{aligned} 1 &= \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot h_M(u)}{\lambda \cdot h_K(u)}\right) d\widetilde{V}_K(u) \\ &\geq \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot R(M) \cdot \langle u,v\rangle_+}{\lambda \cdot R_K}\right) \frac{r_K}{n|K|} dS_K(u) \\ &\geq \int_{\Sigma_{j_0}(v)} \phi\left(\frac{n|K| \cdot R(M)}{\lambda \cdot j_0 \cdot R_K}\right) \frac{r_K}{n|K|} dS_K(u) \\ &\geq \phi\left(\frac{n|K| \cdot R(M)}{\lambda \cdot j_0 \cdot R_K}\right) \frac{r_K \cdot c_1}{2n|K|}. \end{aligned}$$

By the facts that $\phi(1) = 1$ and ϕ is increasing, one has

$$R(M) \leq \frac{\lambda \cdot j_0 \cdot R_K}{n|K|} \cdot \phi^{-1}\left(\frac{2n|K|}{c_1 \cdot r_K}\right) \leq \frac{j_0 \cdot R_K^{n+1}}{r_K^{n+1}} \cdot \phi^{-1}\left(\frac{2n\omega_n \cdot R_K^n}{c_1 \cdot r_K}\right).$$

This completes the proof. $\square$

2.2.3 Continuity of the homogeneous Orlicz geominimal surface areas

This subsection is dedicated to prove the continuity of the homogeneous Orlicz geominimal surface areas under the condition $\phi \in \widehat{\Phi}_1$. The following uniform boundedness argument is needed.

Lemma 2.2.4. *Let $\{K_\alpha\}_{\alpha\in\Lambda} \subset \mathscr{K}_0$ be a family of convex bodies satisfying the uniformly bounded property: there exist constants $r, R > 0$ such that $rB_2^n \subset K_\alpha \subset RB_2^n$ for all $\alpha \in \Lambda$. For $\phi \in \widehat{\Phi}_1$ and for any $M_\alpha \in \widehat{T}_\phi(K_\alpha)$, there exist constants $r', R' > 0$ such that*

$$r'B_2^n \subset M_\alpha \subset R'B_2^n \quad \text{for all} \quad \alpha \in \Lambda.$$

Proof. We only need to prove the case that $\{K_\alpha\}_{\alpha\in\Lambda}$ contains infinite many different convex bodies, as otherwise the argument is trivial.

Let $M_\alpha \in \widehat{T}_\phi(K_\alpha)$. First, we prove the existence of R' by contradiction. To this end, we assume that there is no constant R' such that $M_\alpha \subset R'B_2^n$ for all $\alpha \in \Lambda$. In other words, there is a sequence of $\{M_\alpha\}_{\alpha\in\Lambda}$, denoted by $\{M_i\}_{i\geq 1}$, such that $R(M_i) \to \infty$. Hereafter, for all $i \geq 1$,

$$R(M_i) = \rho_{M_i}(u_i) = \max\{\rho_{M_i}(u) : u \in S^{n-1}\}.$$

Similar to the proof of Proposition 2.2.3, one can find a subsequence, which will not be relabeled, such that, $u_i \to v \in S^{n-1}$ (due to the compactness of S^{n-1}), $R(M_i) \to \infty$ and $K_i \to K$ (by the Blaschke selection theorem due to the uniform boundedness of $\{K_\alpha\}_{\alpha\in\Lambda}$) as $i \to \infty$.

It follows from Proposition 2.2.1 that $\widehat{V}_\phi(K_i, B_2^n) \to \widehat{V}_\phi(K, B_2^n)$ as $i \to \infty$. This implies the boundedness of the sequence $\{\widehat{V}_\phi(K_i, B_2^n)\}_{i\geq 1}$ and hence

$$\lambda_i = \frac{\widehat{V}_\phi(K_i, B_2^n)}{R(M_i)} \to 0 \quad \text{as} \quad i \to \infty.$$

Let $\varepsilon > 0$ be given. The triangle inequality yields the uniform convergence of $\langle u, u_i\rangle_+ \to \langle u, v\rangle_+$ on S^{n-1} as $i \to \infty$. Moreover, as $K_i \to K$, one sees $rB_2^n \subset$

$K \subset RB_2^n$ and

$$0 \le \frac{n|K_i| \cdot \langle u, u_i \rangle_+}{(\lambda_i + \varepsilon) \cdot h_{K_i}(u)} \le \frac{n\omega_n \cdot R^n}{\varepsilon \cdot r} \quad \text{and} \quad 0 \le \frac{n|K| \cdot \langle u, v \rangle_+}{\varepsilon \cdot h_K(u)} \le \frac{n\omega_n \cdot R^n}{\varepsilon \cdot r}.$$

A simple calculation yields that

$$\frac{n|K_i| \cdot \langle u, u_i \rangle_+}{(\lambda_i + \varepsilon) \cdot h_{K_i}(u)} \to \frac{n|K| \cdot \langle u, v \rangle_+}{\varepsilon \cdot h_K(u)} \qquad \text{uniformly on } S^{n-1} \text{ as } i \to \infty.$$

Let $I = [0, n\omega_n R^n \varepsilon^{-1} r^{-1}]$ and then $\phi \in \widehat{\Phi}_1$ is uniformly continuous on I. By Lemma 2.2.2, one gets

$$\phi\left(\frac{n|K_i| \cdot \langle u, u_i \rangle_+}{(\lambda_i + \varepsilon) \cdot h_{K_i}(u)}\right) \to \phi\left(\frac{n|K| \cdot \langle u, v \rangle_+}{\varepsilon \cdot h_K(u)}\right) \quad \text{uniformly on } S^{n-1} \text{ as } i \to \infty. \tag{2.2.25}$$

Note that $\phi \in \widehat{\Phi}_1$ is increasing. By Corollary 2.1.2, (1.1.4), (2.2.20) and (2.2.25), a calculation similar to (2.2.21) leads to, for any given $\varepsilon > 0$,

$$\begin{aligned}
1 &= \lim_{i\to\infty} \int_{S^{n-1}} \phi\left(\frac{n|K_i| \cdot h_{M_i}(u)}{\widehat{V}_\phi(K_i, M_i) \cdot h_{K_i}(u)}\right) d\widetilde{V}_{K_i}(u) \\
&\ge \lim_{i\to\infty} \int_{S^{n-1}} \phi\left(\frac{n|K_i| \cdot R(M_i) \cdot \langle u, u_i \rangle_+}{\widehat{V}_\phi(K_i, B_2^n) \cdot h_{K_i}(u)}\right) d\widetilde{V}_{K_i}(u) \\
&\ge \lim_{i\to\infty} \int_{S^{n-1}} \phi\left(\frac{n|K_i| \cdot \langle u, u_i \rangle_+}{(\lambda_i + \varepsilon) \cdot h_{K_i}(u)}\right) d\widetilde{V}_{K_i}(u) \\
&= \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot \langle u, v \rangle_+}{\varepsilon \cdot h_K(u)}\right) d\widetilde{V}_K(u) \\
&= G_v(\varepsilon).
\end{aligned}$$

It follows from Lemma 2.2.3 that $\lim_{\varepsilon\to 0^+} G_v(\varepsilon) = \infty$, which leads to a contradiction (i.e., $1 \ge \infty$). Thus $R(M_i) \to \infty$ is impossible. This concludes the existence of R' such that $M_\alpha \subset R'B_2^n$ for all $\alpha \in \Lambda$. In other words, $\{M_\alpha\}_{\alpha\in\Lambda} \subset \mathscr{K}_0$ is uniformly bounded.

Next, we show the existence of $r' > 0$ such that $r'B_2^n \subset M_\alpha$ for all $\alpha \in \Lambda$. Assume that there is no such a constant $r' > 0$. In other words, there is a sequence $\{M_j\}_{j\ge1}$ such that $w_j \to w \in S^{n-1}$ (due to the compactness of S^{n-1}) and $r_j \to 0$ as $j \to \infty$, where

$$r_j = h_{M_j}(w_j) = \min\{h_{M_j}(u) : u \in S^{n-1}\}.$$

Note that the sequence $\{M_j\}_{j\geq 1}\subset \mathscr{K}_0$ is uniformly bounded (as proved above). The Blaschke selection theorem, Lemma 1.1.1 and $|M_j^\circ|=\omega_n$ for all $j\geq 1$ imply that there exists a subsequence of $\{M_j\}_{j\geq 1}$, which will not be relabeled, and a convex body $M\in\mathscr{K}_0$, such that, $M_j\to M$ as $j\to\infty$. That is,

$$\lim_{j\to\infty}\sup_{u\in S^{n-1}}|h_{M_j}(u)-h_M(u)|=0.$$

This further implies, as $w_j\to w$,

$$h_M(w)=\lim_{j\to\infty}h_{M_j}(w_j)=\lim_{j\to\infty}r_j=0.$$

This contradicts with the positivity of the support function of M. Hence, there is a constant $r'>0$ such that $r'B_2^n\subset M_\alpha$ for all $\alpha\in\Lambda$. $\square$

Now let us prove our main result which states that the homogeneous Orlicz geominimal surface areas are continuous on $\mathscr{K}_0$ with respect to the Hausdorff distance.

Theorem 2.2.1. *For $\phi\in\widehat{\Phi}_1$, the functional $\widehat{G}_\phi^{orilcz}(\cdot)$ on $\mathscr{K}_0$ is continuous with respect to the Hausdorff distance. In particular, the L_p geominimal surface surface area for $p\in(0,\infty)$ is continuous on $\mathscr{K}_0$ with respect to the Hausdorff distance.*

Proof. The upper semicontinuity has been proved in Proposition 2.2.2. To get the continuity, it is enough to prove that the homogeneous Orlicz geominimal surface areas are lower semicontinuous on $\mathscr{K}_0$. To this end, let $\{K_i\}_{i\geq 1}\subset\mathscr{K}_0$ be a convergent sequence whose limit is $K_0\in\mathscr{K}_0$. Let $M_i\in\widehat{T}_\phi(K_i)$ for $i\geq 1$. Clearly, $\{K_i\}_{i\geq 0}$ satisfies the uniformly bounded condition in Lemma 2.2.4, which implies the uniform boundedness of the sequence $\{M_i\}_{i\geq 1}$.

Let $l=\liminf_{i\to\infty}\widehat{G}_\phi^{orlicz}(K_i)$. Consequently, one can find a subsequence $\{K_{i_k}\}_{k\geq 1}$ such that $l=\lim_{k\to\infty}\widehat{G}_\phi^{orlicz}(K_{i_k})$. By the Blaschke selection theorem and Lemma 1.1.1, there exists a subsequence of $\{M_{i_k}\}_{k\geq 1}$ (still denoted by $\{M_{i_k}\}_{k\geq 1}$) and a body $M\in\mathscr{K}_0$, such that, $M_{i_k}\to M$ as $k\to\infty$ and $|M^\circ|=\omega_n$. Proposition 2.2.1 then yields

$$\widehat{G}_\phi^{orlicz}(K_{i_k})=\widehat{V}_\phi(K_{i_k},M_{i_k})\to\widehat{V}_\phi(K_0,M)\quad\text{as}\quad k\to\infty.$$

It follows from (2.2.15) that

$$\widehat{G}_{\phi}^{orlicz}(K_0) \le \widehat{V}_{\phi}(K_0, M) = \lim_{k\to\infty} \widehat{G}_{\phi}^{orlicz}(K_{i_k}) = \liminf_{i\to\infty} \widehat{G}_{\phi}^{orlicz}(K_i).$$

This completes the proof. □

Proposition 2.2.3 states that if $\phi \in \widehat{\Phi}_1$ is convex, the L_ϕ Orlicz-Petty body is unique. In this case, $\widehat{T}_\phi K$ contains only one element. Consequently, $\widehat{T}_\phi : \mathscr{K}_0 \to \mathscr{K}_0$ defines an operator. The following result states that the operator $\widehat{T}_\phi$ is continuous.

Proposition 2.2.6. *Let $\phi \in \widehat{\Phi}_1$ be convex. Then $\widehat{T}_\phi : \mathscr{K}_0 \mapsto \mathscr{K}_0$ is continuous with respect to the Hausdorff distance.*

Proof. It is enough to prove that $\{\widehat{T}_\phi K_i\}_{i\ge1} \subset \mathscr{K}_0$ is convergent to $\widehat{T}_\phi K_0 \in \mathscr{K}_0$ for every convergent sequence $\{K_i\}_{i\ge1} \subset \mathscr{K}_0$ with limit $K_0 \in \mathscr{K}_0$, in particular, every subsequence of $\{\widehat{T}_\phi K_i\}_{i\ge1}$ has a convergent subsequence whose limit is $\widehat{T}_\phi K_0$.

Let $\{K_{i_k}\}_{k\ge1}$ be any subsequence of $\{K_i\}_{i\ge1}$. Of course, $K_{i_k} \to K_0$ as $k \to \infty$ and $\{\widehat{T}_\phi K_{i_k}\}_{k\ge1}$ is uniformly bounded by Lemma 2.2.4. Following the Blaschke selection theorem, one can find a subsequence of $\{\widehat{T}_\phi K_{i_k}\}_{k\ge1}$, which will not be relabeled, and $M \in \mathscr{K}_0$ such that $\widehat{T}_\phi K_{i_k} \to M$ as $k \to \infty$ and $|M^\circ| = \omega_n$. By Proposition 2.2.1, one has

$$\widehat{G}_{\phi}^{orlicz}(K_{i_k}) = \widehat{V}_{\phi}(K_{i_k}, \widehat{T}_\phi K_{i_k}) \to \widehat{V}_{\phi}(K_0, M) \quad \text{as} \quad k \to \infty.$$

By Theorem 2.2.1, one has

$$\widehat{G}_{\phi}^{orlicz}(K_{i_k}) \to \widehat{G}_{\phi}^{orlicz}(K_0) = \widehat{V}_{\phi}(K_0, \widehat{T}_\phi K_0) \quad \text{as} \quad k \to \infty.$$

Hence, $\widehat{V}_{\phi}(K_0, \widehat{T}_\phi K_0) = \widehat{V}_{\phi}(K_0, M)$ and then $\widehat{T}_\phi K_0 = M$ by the uniqueness of the L_ϕ Orlicz-Petty body for $\phi \in \widehat{\Phi}_1$ being convex. □

2.3 The nonhomogeneous Orlicz geominimal surface areas

In this section, we will briefly discuss the continuity of the nonhomogeneous Orlicz geominimal surface areas defined in [90]. In particular, we prove the existence, uniqueness and affine invariance for the L_φ Orlicz-Petty bodies in Subsection 2.3.2. In Subsection 2.3.1, we provide a geometric interpretation for the nonhomogeneous Orlicz L_φ mixed volume with $\varphi \in \mathscr{I} \cup \mathscr{D}$ (in particular, for $\varphi(t) = t^p$ with $p < 1$).

2.3.1 The geometric interpretation for the nonhomogeneous Orlicz L_φ mixed volume

For any continuous function $\varphi : (0, \infty) \to (0, \infty)$, $V_\varphi(K, L)$ denotes the nonhomogeneous Orlicz L_φ mixed volume of K and L. It has the following integral expression:

$$V_\varphi(K, L) = \frac{1}{n} \int_{S^{n-1}} \varphi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u) dS_K(u). \tag{2.3.26}$$

We can use the following examples to see that $V_\varphi(\cdot, \cdot)$ is not homogeneous:

$$V_\varphi(rB_2^n, B_2^n) = \varphi(1/r) \cdot r^n \cdot \omega_n \quad \text{and} \quad V_\varphi(B_2^n, rB_2^n) = \varphi(r) \cdot \omega_n.$$

The geometric interpretation of $V_\varphi(\cdot, \cdot)$ for convex $\varphi \in \mathscr{I}$ was given in [22, 80]. However, there are no geometric interpretations of $V_\varphi(\cdot, \cdot)$ for non-convex functions φ (even if $\varphi(t) = t^p$ for $p < 1$). In this subsection, we will provide such a geometric interpretation for all $\varphi \in \mathscr{I} \cup \mathscr{D}$.

Recall that $C^+(S^{n-1})$ is the set of all positive continuous functions on S^{n-1} and K_f is the Aleksandrov body associated with $f \in C^+(S^{n-1})$, by

$$K_f = \cap_{u \in S^{n-1}} H^-(u, f(u)),$$

where $H^-(u, \alpha)$ is the half space with normal vector u and constant $\alpha > 0$:

$$H^-(u, \alpha) = \{x \in \mathbb{R}^n : \langle x, u \rangle \le \alpha\}.$$

This implies that

$$K_f = \{x \in \mathbb{R}^n : \langle x, u \rangle \le f(u) \quad \text{for all} \quad u \in S^{n-1}\}.$$

Equivalently, K_f is the (unique) maximal element (with respect to set inclusion) of the set

$$\{K \in \mathscr{K}_0 : h_K(u) \le f(u) \quad \text{for all} \quad u \in S^{n-1}\}.$$

When $f = h_L$ for some convex body $L \in \mathscr{K}_0$, one sees $K_f = L$.

For $K \in \mathscr{K}_0$ and $f \in C^+(S^{n-1})$, the L_1 mixed volume of K and f, denoted by $V_1(K, f)$, can be formulated by

$$V_1(K, f) = \frac{1}{n} \int_{S^{n-1}} f(u) dS_K(u).$$

When f is the support function of a convex body L, then $V_1(K, f)$ is just the usual L_1 mixed volume of K and L (i.e., $\varphi(t) = t$ in formula (2.3.26)). In particular, $V_1(K, h_K) = |K|$ for all $K \in \mathscr{K}_0$. Lemma 3.1 in [48] states that

$$|K_f| = V_1(K_f, f). \tag{2.3.27}$$

In order to prove the geometric interpretation for $V_\varphi(\cdot, \cdot)$, the linear Orlicz addition of functions [36] is needed. A special case is given below.

Definition 2.3.1. *Assume that either $\varphi_1, \varphi_2 \in \mathscr{I}$ or $\varphi_1, \varphi_2 \in \mathscr{D}$. For $\varepsilon > 0$, define $p_1 +_{\varphi,\varepsilon} p_2$, the linear Orlicz addition of positive functions p_1, p_2 (on whatever common domain), by*

$$\varphi_1 \left(\frac{p_1(x)}{(p_1 +_{\varphi,\varepsilon} p_2)(x)} \right) + \varepsilon \varphi_2 \left(\frac{p_2(x)}{(p_1 +_{\varphi,\varepsilon} p_2)(x)} \right) = 1.$$

For our context, $p_1 = h_K$ and $p_2 = h_L$ where $K, L \in \mathscr{K}_0$ are two convex bodies. Namely we let $f_\varepsilon = h_K +_{\varphi,\varepsilon} h_L$ and then for any $u \in S^{n-1}$,

$$\varphi_1 \left(\frac{h_K(u)}{f_\varepsilon(u)} \right) + \varepsilon \varphi_2 \left(\frac{h_L(u)}{f_\varepsilon(u)} \right) = 1. \tag{2.3.28}$$

When $\varphi_1, \varphi_2 \in \mathscr{I}$ are convex functions, $f_\varepsilon = h_K +_{\varphi,\varepsilon} h_L$ is the support function of a convex body (see [22, 80]). Clearly, $f_\varepsilon \in C^+(S^{n-1})$ determines an Aleksandrov body K_{f_ε}, which will be written as K_ε for simplicity. Moreover, $h_K \le f_\varepsilon$ if $\varphi_1, \varphi_2 \in \mathscr{I}$ and $h_K \ge f_\varepsilon$ if $\varphi_1, \varphi_2 \in \mathscr{D}$.

Let $(\varphi_1)'_l(1)$ and $(\varphi_1)'_r(1)$ stand for the left and the right derivatives of φ_1 at $t = 1$, respectively, if they exist. From the proof of Theorem 9 in [36], one sees that $f_\varepsilon \to h_K$ uniformly on S^{n-1} as $\varepsilon \to 0^+$. Following similar arguments in [22, 23, 36, 97], we can prove the following result.

Lemma 2.3.1. *Let $K, L \in \mathscr{K}_0$ and $\varphi_1, \varphi_2 \in \mathscr{I}$ be such that $(\varphi_1)'_l(1)$ exists and is positive. Then*

$$(\varphi_1)'_l(1) \lim_{\varepsilon\to 0^+} \frac{f_\varepsilon(u) - h_K(u)}{\varepsilon} = h_K(u) \cdot \varphi_2\left(\frac{h_L(u)}{h_K(u)}\right) \quad \text{uniformly on } S^{n-1}. \tag{2.3.29}$$

For $\varphi_1, \varphi_2 \in \mathscr{D}$, (2.3.29) holds with $(\varphi_1)'_l(1)$ replaced by $(\varphi_1)'_r(1)$.

Proof. Let $\varphi_1, \varphi_2 \in \mathscr{I}$. Note that $f_\varepsilon \downarrow h_K$ uniformly on S^{n-1} as $\varepsilon \downarrow 0^+$. Then, for all $u \in S^{n-1}$,

$$\begin{aligned}
(\varphi_1)'_l(1) &= \lim_{\varepsilon\to 0^+} f_\varepsilon(u) \cdot \frac{1 - \varphi_1\big(\frac{h_K(u)}{f_\varepsilon(u)}\big)}{f_\varepsilon(u) - h_K(u)} \\
&= \lim_{\varepsilon\to 0^+} f_\varepsilon(u) \cdot \varphi_2\bigg(\frac{h_L(u)}{f_\varepsilon(u)}\bigg) \cdot \frac{\varepsilon}{f_\varepsilon(u) - h_K(u)} \\
&= h_K(u) \cdot \varphi_2\left(\frac{h_L(u)}{h_K(u)}\right) \cdot \lim_{\varepsilon\to 0^+} \frac{\varepsilon}{f_\varepsilon(u) - h_K(u)}.
\end{aligned}$$

Rewrite the above limit as follows:

$$(\varphi_1)'_l(1) \cdot \lim_{\varepsilon\to 0^+} \frac{f_\varepsilon(u) - h_K(u)}{\varepsilon} = \lim_{\varepsilon\to 0^+} f_\varepsilon(u) \cdot \lim_{\varepsilon\to 0^+} \varphi_2\bigg(\frac{h_L(u)}{f_\varepsilon(u)}\bigg) = h_K(u) \cdot \varphi_2\left(\frac{h_L(u)}{h_K(u)}\right).$$

Moreover, the convergence is uniform because both $\{f_\varepsilon(u)\}_{\varepsilon>0}$ and $\big\{\varphi_2\big(\frac{h_L(u)}{f_\varepsilon(u)}\big)\big\}_{\varepsilon>0}$ are uniformly convergent and uniformly bounded on S^{n-1}.

If $\varphi_1, \varphi_2 \in \mathscr{D}$ such that $(\varphi_1)'_r(1)$ exists and is nonzero, the proof goes along the same manner. $\square$

The geometric interpretation for the nonhomogeneous Orlicz L_φ mixed volume with $\varphi \in \mathscr{I} \cup \mathscr{D}$ is given in the following theorem.

Theorem 2.3.1. *Let $K, L \in \mathscr{K}_0$ and $\varphi_1, \varphi_2 \in \mathscr{I}$ be such that $(\varphi_1)'_l(1)$ exists and is positive. Then,*

$$V_{\varphi_2}(K, L) = \frac{(\varphi_1)'_l(1)}{n} \lim_{\varepsilon\to 0^+} \frac{|K_\varepsilon| - |K|}{\varepsilon}. \tag{2.3.30}$$

For $\varphi_1, \varphi_2 \in \mathscr{D}$, (2.3.30) holds with $(\varphi_1)'_l(1)$ replaced by $(\varphi_1)'_r(1)$.

Proof. The uniform convergence of f_ε on S^{n-1} implies that K_ε converges to K in the Hausdorff distance as $\varepsilon \to 0^+$. In particular $|K_\varepsilon| \to |K|$ as $\varepsilon \to 0^+$ and S_{K_ε} converges to S_K weakly on S^{n-1}. It follows from (1.1.4), (2.3.27), the Minkowski inequality (2.1.5) and Lemma 2.3.1 that

$$\begin{aligned}
\liminf_{\varepsilon\to 0^+} |K_\varepsilon|^{\frac{n-1}{n}} \cdot \frac{|K_\varepsilon|^{\frac{1}{n}} - |K|^{\frac{1}{n}}}{\varepsilon} &\geq \liminf_{\varepsilon\to 0^+} \frac{|K_\varepsilon| - V_1(K_\varepsilon, K)}{\varepsilon} \\
&= \liminf_{\varepsilon\to 0^+} \frac{V_1(K_\varepsilon, f_\varepsilon) - V_1(K_\varepsilon, h_K)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{n} \int_{S^{n-1}} \frac{f_\varepsilon(u) - h_K(u)}{\varepsilon} dS_{K_\varepsilon}(u) \\
&= \frac{1}{(\varphi_1)'_l(1)} V_{\varphi_2}(K, L).
\end{aligned}$$

Similarly, due to $h_{K_\varepsilon} \leq f_\varepsilon$,

$$\begin{aligned}
|K|^{\frac{n-1}{n}} \cdot \limsup_{\varepsilon\to 0^+} \frac{|K_\varepsilon|^{\frac{1}{n}} - |K|^{\frac{1}{n}}}{\varepsilon} &\leq \limsup_{\varepsilon\to 0^+} \frac{V_1(K, K_\varepsilon) - |K|}{\varepsilon} \\
&\leq \limsup_{\varepsilon\to 0^+} \frac{V_1(K, f_\varepsilon) - V_1(K, h_K)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{n} \int_{S^{n-1}} \frac{f_\varepsilon(u) - h_K(u)}{\varepsilon} dS_K(u) \\
&= \frac{1}{(\varphi_1)'_l(1)} V_{\varphi_2}(K, L).
\end{aligned}$$

Combing the inequalities above, one has

$$(\varphi_1)'_l(1) \cdot |K|^{\frac{n-1}{n}} \cdot \lim_{\varepsilon\to 0^+} \frac{|K_\varepsilon|^{\frac{1}{n}} - |K|^{\frac{1}{n}}}{\varepsilon} = V_{\varphi_2}(K, L).$$

Let $g(\varepsilon) = |K_\varepsilon|^{\frac{1}{n}}$ and $g(0) = |K|^{\frac{1}{n}}$. Then

$$\begin{aligned}
\frac{(\varphi_1)'_l(1)}{n} \cdot \lim_{\varepsilon\to 0^+} \frac{|K_\varepsilon| - |K|}{\varepsilon} &= \frac{(\varphi_1)'_l(1)}{n} \cdot \lim_{\varepsilon\to 0^+} \frac{g(\varepsilon)^n - g(0)^n}{\varepsilon} \\
&= (\varphi_1)'_l(1) \cdot g(0)^{n-1} \lim_{\varepsilon\to 0^+} \frac{g(\varepsilon) - g(0)}{\varepsilon} \\
&= V_{\varphi_2}(K, L).
\end{aligned}$$

The result for $\varphi_1, \varphi_2 \in \mathscr{D}$ follows along the same lines. □

Let $\varphi_1(t) = \varphi_2(t) = t^p$ for $0 \neq p \in \mathbb{R}$. Then formula (2.3.28) gives the L_p addition of h_K and h_L:

$$f_{p,\varepsilon}(u) = \big[h_K(u)^p + \varepsilon h_L(u)^p\big]^{1/p} \quad \text{for } \ u \in S^{n-1}.$$

Then the L_p mixed volume of K and L [48, 89] is the first order variation at $\varepsilon = 0$ of the volume of $K_{f_{p,\varepsilon}}$, the Aleksandrov body associated to $f_{p,\varepsilon}$:

$$V_p(K, L) = \frac{p}{n} \cdot \lim_{\varepsilon \to 0^+} \frac{|K_{f_{p,\varepsilon}}| - |K|}{\varepsilon}.$$

2.3.2 The Orlicz-Petty bodies and the continuity of nonhomogeneous Orlicz geominimal surface areas

In this subsection, we establish the continuity of the nonhomogeneous Orlicz geominimal surface areas, whose proof is similar to that in Section 2.2. For completeness, we still include the proof with emphasis on the modification.

The nonhomogeneous Orlicz geominimal surface areas can be defined as follows.

Definition 2.3.2. *Let $K \in \mathscr{K}_0$ be a convex body with the origin in its interior.*
(i) For $\varphi \in \widehat{\Phi}_1 \cup \widehat{\Psi}$, define the nonhomogeneous Orlicz L_φ geominimal surface area of K by

$$G_\varphi^{orlicz}(K) = \inf_{L \in \mathscr{K}_0} \{nV_\varphi(K, \mathrm{vrad}(L^\circ)L)\} = \inf\{nV_\varphi(K, L) : L \in \mathscr{K}_0 \text{ with } |L^\circ| = \omega_n\}.$$

(ii) For $\varphi \in \widehat{\Phi}_2$, define the nonhomogeneous Orlicz L_φ geominimal surface area of K by

$$G_\varphi^{orlicz}(K) = \sup_{L \in \mathscr{K}_0} \{nV_\varphi(K, \mathrm{vrad}(L^\circ)L)\} = \sup\{nV_\varphi(K, L) : L \in \mathscr{K}_0 \text{ with } |L^\circ| = \omega_n\}.$$

Note that the nonhomogeneous Orlicz L_φ geominimal surface area can be defined for more general functions than $\varphi \in \mathscr{I} \cup \mathscr{D}$ (see more details in [90]). However, from Section 2.2, one sees that the monotonicity of φ is crucial to establish continuity of Orlicz geominimal surface areas. Hence, in this section, we only consider $\varphi \in \widehat{\Phi} \cup \widehat{\Psi}$. We can use the following example to see that $G_\varphi^{orlicz}(\cdot)$ is not homogeneous (see Corollary 3.1 in [90]):

$$G_\varphi^{orlicz}(rB_2^n) = \varphi(1/r) \cdot r^n \cdot n\omega_n.$$

Proposition 2.3.1. *Let $\{K_i\}_{i \geq 1} \subset \mathscr{K}_0$ and $\{L_i\}_{i \geq 1} \subset \mathscr{K}_0$ be such that $K_i \to K \in \mathscr{K}_0$ and $L_i \to L \in \mathscr{K}_0$ as $i \to \infty$. For $\varphi \in \widehat{\Phi} \cup \widehat{\Psi}$, one has $V_\varphi(K_i, L_i) \to V_\varphi(K, L)$ as $i \to \infty$.*

Proof. As $K_i \to K \in \mathscr{K}_0$ and $L_i \to L \in \mathscr{K}_0$, one can find constants $r, R > 0$ such that these bodies contain rB_2^n and are contained in RB_2^n. Moreover, $h_{K_i} \to h_K$ and $h_{L_i} \to h_L$ uniformly on S^{n-1}. Together with Lemma 2.2.2 (where we can let $I = [r/R, R/r]$), one has

$$\varphi\left(\frac{h_{L_i}(u)}{h_{K_i}(u)}\right) h_{K_i}(u) \to \varphi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u) \quad \text{uniformly on} \quad S^{n-1}.$$

Formula (1.1.4) then implies

$$\int_{S^{n-1}} \varphi\left(\frac{h_{L_i}(u)}{h_{K_i}(u)}\right) h_{K_i}(u) dS_{K_i}(u) \to \int_{S^{n-1}} \varphi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u) dS_K(u).$$

This completes the proof. □

Similar to Proposition 2.2.2, the nonhomogeneous Orlicz L_φ geominimal surface area is upper (lower, respectively) semicontinuous on $\mathscr{K}_0$ with respect to the Hausdorff distance for $\varphi \in \widehat{\Phi}_1 \cup \widehat{\Psi}$ (for $\varphi \in \widehat{\Phi}_2$, respectively).

The following proposition states that the Orlicz-Petty bodies exist. See [93] for special results when $\varphi \in \mathscr{I}$ is convex (in this case, $\varphi \in \widehat{\Phi}_1$).

Proposition 2.3.2. *Let $K \in \mathscr{K}_0$ and $\varphi \in \widehat{\Phi}_1$. There exists a convex body $M \in \mathscr{K}_0$ such that*

$$G_\varphi^{orlicz}(K) = nV_\varphi(K, M) \quad \text{and} \quad |M^\circ| = \omega_n.$$

If in addition φ is convex, such a convex body is unique.

Proof. Let $\varphi \in \widehat{\Phi}_1$. It follows from the definition of $G_\varphi^{orlicz}(K)$ that there exists a sequence $\{M_i\}_{i\geq 1} \subset \mathscr{K}_0$ such that $nV_\varphi(K, M_i) \to G_\varphi^{orlicz}(K)$, $|M_i^\circ| = \omega_n$ and $2V_\varphi(K, B_2^n) \geq V_\varphi(K, M_i)$ for all $i \geq 1$. Let $R_i = \rho_{M_i}(u_i) = \max\{\rho_{M_i}(u) : u \in S^{n-1}\}$ and assume that $\sup_{i\geq 1} R_i = \infty$. Without loss of generality, let $R_i \to \infty$ and $u_i \to v$ (due to the compactness of S^{n-1}) as $i \to \infty$. As before, $h_{M_i}(u) \geq R_i \cdot \langle u, u_i \rangle_+$ for all $u \in S^{n-1}$.

Let $D > 0$ be given. By Definition 2.3.2, Fatou's lemma, continuity of φ, (2.2.20) and the fact that φ is increasing, one has

$$\begin{aligned} 2V_\varphi(K, B_2^n) &\geq \lim_{i\to\infty} \frac{1}{n} \int_{S^{n-1}} \varphi\left(\frac{h_{M_i}(u)}{h_K(u)}\right) h_K(u) dS_K(u) \\ &\geq \liminf_{i\to\infty} \frac{1}{n} \int_{S^{n-1}} \varphi\left(\frac{R_i \cdot \langle u, u_i\rangle_+}{R_K}\right) r_K dS_K(u) \\ &\geq \liminf_{i\to\infty} \frac{1}{n} \int_{S^{n-1}} \varphi\left(\frac{D \cdot \langle u, u_i\rangle_+}{R_K}\right) r_K dS_K(u) \\ &\geq \frac{1}{n} \int_{S^{n-1}} \varphi\left(\frac{D \cdot \langle u, v\rangle_+}{R_K}\right) r_K dS_K(u) \\ &\geq \varphi\left(\frac{D}{j_0 \cdot R_K}\right) \cdot \frac{r_K}{n} \cdot \frac{c_1}{2}. \end{aligned}$$

A contradiction (i.e., $V_\varphi(K, B_2^n) > \infty$) is obtained by letting $D \to \infty$ and the fact that $\lim_{t\to\infty} \phi(t) = \infty$ (as $\varphi \in \widehat{\Phi}_1$ is increasing and unbounded). That is, $\{M_i\}_{i\geq 1}$ is uniformly bounded, and a convergent subsequence of $\{M_i\}_{i\geq 1}$, which will not be relabeled, can be found due to the Blaschke selection theorem. Let M be the limit of $\{M_i\}_{i\geq 1}$ and then $M \in \mathscr{K}_0$ due to Lemma 1.1.1. Moreover, $|M_i^\circ| = \omega_n$ for all $i \geq 1$ implies $|M^\circ| = \omega_n$. It follows from Proposition 2.3.1 that M is the desired body such that $G_\varphi^{orlicz}(K) = nV_\varphi(K, M)$ and $|M^\circ| = \omega_n$.

For uniqueness, let $M_1, M_2 \in \mathscr{K}_0$ be such that $|M_1^\circ| = |M_2^\circ| = \omega_n$ and

$$V_\varphi(K, M_1) = \inf_{L\in\mathscr{K}_0} \{V_\varphi(K, \mathrm{vrad}(L^\circ)L)\} = V_\varphi(K, M_2).$$

Let $M = \frac{M_1+M_2}{2}$. Then $\mathrm{vrad}(M^\circ) \leq 1$ with equality if and only if $M_1 = M_2$ (see inequality (2.2.23)). The fact that φ is convex yields that $V_\varphi(K, M) \leq V_\varphi(K, M_1)$. Therefore, if $M_1 \neq M_2$ (hence $\mathrm{vrad}(M^\circ) < 1$), the fact that φ is strictly increasing implies that

$$nV_\varphi(K, \mathrm{vrad}(M^\circ)M) < nV_\varphi(K, M) \leq nV_\varphi(K, M_1) = nV_\varphi(K, \mathrm{vrad}(M_1^\circ)M_1).$$

This contradicts with the minimality of M_1 and hence the uniqueness follows. □

Definition 2.3.3. *Let $K \in \mathscr{K}_0$ and $\varphi \in \widehat{\Phi}_1$. A convex body $M \in \mathscr{K}_0$ is said to be an L_φ Orlicz-Petty body of K, if $M \in \mathscr{K}_0$ satisfies*

$$G_\varphi^{orlicz}(K) = nV_\varphi(K, M) \quad \text{and} \quad |M^\circ| = \omega_n.$$

Denote by $T_\varphi K$ the set of all L_φ Orlicz-Petty bodies of K.

Let $\varphi \in \widehat{\Phi}_1$. The set $T_\varphi K$ has many properties same as those for $\widehat{T}_\phi K$. For instance, $T_\varphi K$ is $SL(n)$-invariant: $T_\varphi(AK) = A(T_\varphi K)$ for all $A \in SL(n)$. Moreover, if K is a polytope, then any convex body in $T_\varphi K$ must be a polytope with faces parallel to those of K. If in addition φ is convex, $|T_\varphi K| \cdot |(T_\varphi K)^\circ| \le |K| \cdot |K^\circ|$.

The continuity of the nonhomogeneous Orlicz L_φ geominimal surface areas is proved in the following theorem. See [93] for special results when $\varphi \in \mathscr{I}$ is convex (in this case, $\varphi \in \widehat{\Phi}_1$).

Theorem 2.3.2. *If $\varphi \in \widehat{\Phi}_1$, then the functional $G_\varphi^{orilcz}(\cdot)$ on $\mathscr{K}_0$ is continuous with respect to the Hausdorff distance.*

Proof. Let $\varphi \in \widehat{\Phi}_1$. The upper semicontinuity has been stated after Proposition 2.3.1. To conclude the continuity, it is enough to prove the lower semicontinuity.

To this end, we need the following statement: if $K_i \to K$ as $i \to \infty$ with $K_i, K \in \mathscr{K}_0$ for all $i \ge 1$, there exists a constant $R' > 0$ such that $M_i \subset R' B_2^n$ for all (given) $M_i \in T_\varphi K_i$, $i \ge 1$. The proof basically follows the idea in Lemma 2.2.4. In fact, assume that there is no constant R' such that $M_i \subset R' B_2^n$ for $i \ge 1$. Let $R_i = \rho_{M_i}(u_i) = \max\{\rho_{M_i}(u) : u \in S^{n-1}\}$. It follows from the Blaschke selection theorem and the compactness of S^{n-1} that there is a subsequence of $\{K_i\}_{i\ge1}$, which will not be relabeled, such that, $R_i \to \infty$ and $u_i \to v$ as $i \to \infty$. For any given $\varepsilon > 0$, one has

$$\begin{aligned}
V_\varphi(K, B_2^n) &= \lim_{i\to\infty} V_\varphi(K_i, B_2^n) \\
&\ge \lim_{i\to\infty} \frac{1}{n} \int_{S^{n-1}} \varphi\left(\frac{h_{M_i}(u)}{h_{K_i}(u)}\right) h_{K_i}(u) dS_{K_i}(u) \\
&\ge \lim_{i\to\infty} \frac{1}{n} \int_{S^{n-1}} \varphi\left(\frac{\langle u, u_i\rangle_+}{(R_i^{-1} + \varepsilon) \cdot R}\right) r dS_{K_i}(u) \\
&= \frac{1}{n} \int_{S^{n-1}} \varphi\left(\frac{\langle u, v\rangle_+}{\varepsilon \cdot R}\right) r dS_K(u) \\
&\ge \frac{1}{n} \int_{\Sigma_{j_0}(v)} \varphi\left(\frac{1}{\varepsilon \cdot j_0 \cdot R}\right) r dS_K(u) \\
&= \varphi\left(\frac{1}{\varepsilon \cdot j_0 \cdot R}\right) \cdot \frac{r}{n} \cdot \frac{c_1}{2}
\end{aligned}$$

where $r, R > 0$ are constants such that $rB_2^n \subset K_i, K \subset RB_2^n$ for all $i \geq 1$. A contradiction (i.e., $V_\varphi(K, B_2^n) \geq \infty$) is obtained by taking $\varepsilon \to 0^+$ and the fact that $\lim_{t\to\infty} \varphi(t) = \infty$.

Now let us prove the lower semicontinuity of $G_\varphi^{orilcz}(\cdot)$ and the continuity then follows. Let $l = \liminf_{i\to\infty} G_\varphi^{orlicz}(K_i)$. There is a subsequence of $\{K_i\}_{i\geq 1}$, say $\{K_{i_k}\}_{k\geq 1}$, such that we have $l = \lim_{k\to\infty} G_\varphi^{orlicz}(K_{i_k})$. From the arguments in the previous paragraph, one sees that $\{M_{i_k}\}_{k\geq 1}$ is uniformly bounded. The Blaschke selection theorem and Lemma 1.1.1 imply that there exists a subsequence of $\{M_{i_k}\}_{k\geq 1}$, which will not be relabeled, and a convex body $M \in \mathscr{K}_0$ such that $M_{i_k} \to M$ as $k \to \infty$ and $|M^\circ| = \omega_n$. Proposition 2.3.1 yields

$$G_\varphi^{orlicz}(K_{i_k}) = nV_\varphi(K_{i_k}, M_{i_k}) \to nV_\varphi(K, M) \geq G_\varphi^{orlicz}(K) \quad \text{as} \quad k \to \infty.$$

Hence, $\liminf_{i\to\infty} G_\varphi^{orlicz}(K_i) \geq G_\varphi^{orlicz}(K)$ and this completes the proof. $\square$

Similar to Proposition 2.2.6, we can prove that if $\varphi \in \widehat{\Phi}_1$ is convex, then $T_\varphi : \mathscr{K}_0 \mapsto \mathscr{K}_0$ is continuous with respect to the Hausdorff distance.

2.4 The Orlicz geominimal surface areas with respect to $\mathscr{K}_e$ and the related Orlicz-Petty bodies

In Sections 2.2 and 2.3, we prove the existence of the Orlicz-Petty bodies and the continuity for the Orlicz geominimal surface areas under the condition $\phi \in \widehat{\Phi}_1$. For $\phi \in \widehat{\Phi}_2 \cup \widehat{\Psi}$, our method fails. In fact, when $\phi \in \widehat{\Phi}_2$, we can prove the following result.

Proposition 2.4.1. *Let $\phi, \varphi \in \widehat{\Phi}_2$ and $K \in \mathscr{K}_0$ be a polytope. Then*

$$\widehat{G}_\phi^{orlicz}(K) = 0 \quad \text{and} \quad G_\varphi^{orlicz}(K) = \infty.$$

Proof. Let $\phi \in \widehat{\Phi}_2$ and $K \in \mathscr{K}_0$ be a polytope. Then the surface area measure of K is concentrated on finite directions, say $\{u_1, \cdots, u_m\} \subset S^{n-1}$. As $\widehat{G}_\phi^{orlicz}(K)$ is $SL(n)$ invariant, we can assume that, without loss of generality, $S_K(u_1) > 0$ and $u_1 = e_1$ with $\{e_1, \cdots, e_n\}$ the canonical orthonormal basis of $\mathbb{R}^n$.

Let $\epsilon > 0$ and $A_\epsilon = \text{diag}(\epsilon, b_2, \cdots, b_n)$ with constants $b_2, \cdots, b_n > 0$ such that $b_2 \cdots b_n = 1/\epsilon$. Clearly $\det A_\epsilon = 1$ and then $A_\epsilon \in SL(n)$. Let $L_\epsilon = A_\epsilon K \in \mathscr{K}_0$ and $\lambda_\epsilon = \widehat{V}_\phi(K, L_\epsilon)$. Then, $h_{L_\epsilon}(e_1) = \epsilon \cdot h_K(e_1)$ for all $\epsilon > 0$ and

$$\begin{aligned} 1 &= \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot h_{L_\epsilon}(u)}{\lambda_\epsilon \cdot h_K(u)}\right) d\widetilde{V}_K(u) \\ &= \frac{1}{n|K|} \cdot \sum_{i=1}^{m} \phi\left(\frac{n|K| \cdot h_{L_\epsilon}(u_i)}{\lambda_\epsilon \cdot h_K(u_i)}\right) h_K(u_i) S_K(u_i) \\ &\geq \frac{1}{n|K|} \cdot \phi\left(\frac{n|K| \cdot h_{L_\epsilon}(e_1)}{\lambda_\epsilon \cdot h_K(e_1)}\right) h_K(e_1) S_K(e_1) \\ &= \frac{1}{n|K|} \cdot \phi\left(\frac{n|K| \cdot \epsilon}{\lambda_\epsilon}\right) h_K(e_1) S_K(e_1). \end{aligned}$$

Assume that $\inf_{\epsilon>0} \lambda_\epsilon > 0$. There exists a constant $c > 0$ such that $\lambda_\epsilon > c$ for all $\epsilon > 0$. The above inequality and the fact that $\phi \in \widehat{\Phi}_2$ is decreasing imply

$$1 \geq \frac{1}{n|K|} \cdot \phi\left(\frac{n|K| \cdot \epsilon}{\lambda_\epsilon}\right) h_K(e_1) S_K(e_1) \geq \frac{1}{n|K|} \cdot \phi\left(\frac{n|K| \cdot \epsilon}{c}\right) h_K(e_1) S_K(e_1).$$

Recall that $\lim_{t\to 0} \phi(t) = \infty$ as $\phi \in \widehat{\Phi}_2 \subset \mathscr{D}$. A contradiction (i.e., $1 \geq \infty$) is obtained if we let $\epsilon \to 0^+$. This means that

$$\inf_{\epsilon>0} \lambda_\epsilon = \inf_{\epsilon>0} \widehat{V}_\phi(K, L_\epsilon) = 0.$$

On the other hand, $\text{vrad}(L_\epsilon^\circ) = \text{vrad}(K^\circ)$ for all $\epsilon > 0$. This yields that

$$0 \leq \widehat{G}_\phi^{orlicz}(K) = \inf_{L \in \mathscr{K}_0} \{\widehat{V}_\phi(K, \text{vrad}(L^\circ) L)\} \leq \inf_{\epsilon>0} \{\widehat{V}_\phi(K, \text{vrad}(L_\epsilon^\circ) L_\epsilon)\} = 0.$$

For the nonhomogeneous Orlicz L_φ geominimal surface area, the proof follows along the same lines. In fact, for all $\epsilon > 0$,

$$\begin{aligned} V_\varphi(K, \text{vrad}(L_\epsilon^\circ) L_\epsilon) &= \frac{1}{n} \int_{S^{n-1}} \varphi\left(\frac{\text{vrad}(K^\circ) h_{L_\epsilon}(u)}{h_K(u)}\right) h_K(u)\, dS_K(u) \\ &\geq \frac{1}{n} \cdot \varphi(\text{vrad}(K^\circ) \cdot \epsilon) \cdot h_K(e_1) \cdot S_K(e_1). \end{aligned}$$

and the desired result follows

$$G_\varphi^{orlicz}(K) = \sup_{L \in \mathscr{K}_0} \{n V_\varphi(K, \text{vrad}(L^\circ) L)\} \geq \sup_{\epsilon>0} \{n V_\varphi(K, \text{vrad}(L_\epsilon^\circ) L_\epsilon)\} = \infty.$$

This completes the proof. $\square$

An immediate consequence of Proposition 2.4.1 is that for $\phi \in \widehat{\Phi}_2$, the homogeneous Orlicz L_ϕ geominimal surface area is not continuous but only upper semicontinuous on $\mathscr{K}_0$ with respect to the Hausdorff distance. To this end, let $K = B_2^n$. One can find a sequence of polytopes $\{P_i\}_{i\geq 1}$ such that $P_i \to B_2^n$ as $i \to \infty$ with respect to the Hausdorff distance. However, one cannot expect to have $\widehat{G}_\phi^{orlicz}(P_i) \to \widehat{G}_\phi^{orlicz}(B_2^n)$ as $i \to \infty$, since $\widehat{G}_\phi^{orlicz}(P_i) = 0$ for all $i \geq 1$ and $\widehat{G}_\phi^{orlicz}(B_2^n) = n\omega_n > 0$. Moreover, if $\phi \in \widehat{\Phi}_2$ and K is a polytope, the Orlicz-Petty bodies for K do not exist (i.e., $\widehat{T}_\phi K = \emptyset$). This is because $\widehat{G}_\phi^{orlicz}(K) = 0$, but $\widehat{V}_\phi(K, M) > 0$ for $M \in \widehat{T}_\phi K \subset \mathscr{K}_0$ if $\widehat{T}_\phi K \neq \emptyset$. Similarly, the nonhomogeneous Orlicz L_φ geominimal surface area is not continuous but only lower semicontinuous on $\mathscr{K}_0$ with respect to the Hausdorff distance as $G_\varphi^{orlicz}(P_i) = \infty$ for all $i \geq 1$. Moreover, if $\varphi \in \widehat{\Phi}_2$ and K is a polytope, the Orlicz-Petty bodies for K do not exist.

Our method to prove the existence of the Orlicz-Petty bodies in Sections 2.2 and 2.3 heavily relies on the value of the Orlicz mixed volumes of K and line segments $[o, v] = \{tv : t \in [0, 1]\}$ for $v \in S^{n-1}$ (for instance $\widehat{V}_\phi(K, [o, v])$ in Section 2.2). However, $\widehat{V}_\phi(K, [o, v])$ are always 0 for all $v \in S^{n-1}$ if $\phi \in \mathscr{D}$. It seems impossible to prove the existence of the Orlicz-Petty bodies for $\phi \in \mathscr{D}$ and for general (even with enough smoothness) convex bodies $K \in \mathscr{K}_0$.

When $\phi(t) = t^p$ for $p \in (-1, 0)$, one can calculate that, for all $v \in S^{n-1}$ (see e.g., [95]),

$$\int_{S^{n-1}} |\langle u, v\rangle|^p d\sigma(u) = C_{n,p}, \tag{2.4.31}$$

where $C_{n,p} > 0$ is a finite constant depending on n and p. Note that the integrand includes $|\langle u, v\rangle|$ rather than $\langle u, v\rangle_+$. This suggests that our method in Sections 2.2 and 2.3 may still work for smooth enough $K \in \mathscr{K}_0$ and a modified Orlicz geominimal surface area.

Our modified Orlicz geominimal surface area is given by the following definition. Recall that $\mathscr{K}_e$ is the set of all origin-symmetric convex bodies.

Definition 2.4.1. *Let $K \in \mathscr{K}_0$ and $\phi \in \widehat{\Phi}$. The homogeneous Orlicz L_ϕ geominimal surface area of K with respect to $\mathscr{K}_e$ is defined by*

$$\widehat{G}_\phi^{orlicz}(K, \mathscr{K}_e) = \inf\{\widehat{V}_\phi(K, L) : \ L \in \mathscr{K}_e \quad \text{with} \quad |L^\circ| = \omega_n\}. \tag{2.4.32}$$

While if $\phi \in \widehat{\Psi}$, $\widehat{G}_{\phi}^{orlicz}(\cdot, \mathscr{K}_e)$ can be defined similarly with "$\inf$" replaced by "$\sup$".

Properties for $\widehat{G}_{\phi}^{orlicz}(\cdot, \mathscr{K}_e)$, such as affine invariance, homogeneity, affine isoperimetric inequalities (requiring $K \in \mathscr{K}_e$), and continuity if $\phi \in \widehat{\Phi}_1$, are the same as those for $\widehat{G}_{\phi}^{orlicz}(\cdot)$ proved in Sections 2.1 and 2.2. The details are left for readers.

In the rest of this section, we will prove the existence of the Orlicz-Petty bodies and the "continuity" of $\widehat{G}_{\phi}^{orlicz}(\cdot, \mathscr{K}_e)$ for certain $\phi \in \widehat{\Phi}_2$. We will work on convex bodies $K \in C_+^2$. A convex body K is said to be in C_+^2 if K has C^2 boundary and positive curvature function f_K. Hereafter, the curvature function of K is the function $f_K : S^{n-1} \to (0, \infty)$ such that

$$f_K(u) = \frac{dS_K(u)}{d\sigma(u)} \quad \text{for} \quad u \in S^{n-1}.$$

Let $\phi \in \widehat{\Phi}_2$ be such that for all $x \in \mathbb{R}^n$,

$$\int_{S^{n-1}} \phi(|\langle u, x\rangle|)\, d\sigma(u) < \infty \quad \text{and} \quad \lim_{|x|\to\infty} \int_{S^{n-1}} \phi(|\langle u, x\rangle|)\, d\sigma(u) = 0. \qquad (2.4.33)$$

Note that $\phi(t) = t^p$ for $p \in (-1, 0)$ satisfies the condition (2.4.33) due to formula (2.4.31). Moreover, (2.4.33) is equivalent to, for all $s > 0$,

$$\int_{S^{n-1}} \phi(s \cdot |\langle u, e_1\rangle|)\, d\sigma(u) < \infty \quad \text{and} \quad \lim_{s\to\infty} \int_{S^{n-1}} \phi(s \cdot |\langle u, e_1\rangle|)\, d\sigma(u) = 0.$$

Proposition 2.4.2. *Let $K \in C_+^2$ and $\phi \in \widehat{\Phi}_2$ satisfy (2.4.33). Then there exists $M \in \mathscr{K}_e$ such that*

$$\widehat{G}_{\phi}^{orlicz}(K, \mathscr{K}_e) = \widehat{V}_{\phi}(K, M) \quad \text{and} \quad |M^{\circ}| = \omega_n.$$

Proof. Let $K \in C_+^2$. Its curvature function f_K is continuous on S^{n-1} and hence has maximum which will be denoted by $F_K < \infty$. By (2.4.32), for $\phi \in \widehat{\Phi}_2$, there exists a sequence $\{M_i\}_{i\ge1} \subset \mathscr{K}_e$ such that $\widehat{V}_{\phi}(K, M_i) \to \widehat{G}_{\phi}^{orlicz}(K, \mathscr{K}_e)$ as $i \to \infty$, $2\widehat{V}_{\phi}(K, B_2^n) \ge \widehat{V}_{\phi}(K, M_i)$ and $|M_i^{\circ}| = \omega_n$ for all $i \ge 1$. Again let $R_i = \rho_{M_i}(u_i) = \max\{\rho_{M_i}(u) : u \in S^{n-1}\}$. Then $h_{M_i}(u) \ge R_i \cdot |\langle u, u_i\rangle|$ for all $u \in S^{n-1}$ and all $i \ge 1$. Corollary 2.1.2, together with (2.4.33) and the fact that $\phi \in \widehat{\Phi}_2$ is decreasing, implies that, for all $i \geq 1$,

$$\begin{aligned}
1 &= \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot h_{M_i}(u)}{\widehat{V}_\phi(K, M_i) \cdot h_K(u)}\right) d\widetilde{V}_K(u) \\
&\leq \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot R_i \cdot |\langle u, u_i\rangle|}{2\widehat{V}_\phi(K, B_2^n) \cdot h_K(u)}\right) \cdot \frac{h_K(u) f_K(u)}{n|K|}\, d\sigma(u) \\
&\leq \int_{S^{n-1}} \phi\left(\frac{n|K| \cdot R_i \cdot |\langle u, u_i\rangle|}{2\widehat{V}_\phi(K, B_2^n) \cdot R_K}\right) \cdot \frac{R_K F_K}{n|K|}\, d\sigma(u) < \infty.
\end{aligned}$$

Assume that $\sup_{i\geq 1} R_i = \infty$. Without loss of generality, let $\lim_{i\geq 1} R_i = \infty$ and

$$x_i = \frac{n|K| \cdot R_i \cdot u_i}{2\widehat{V}_\phi(K, B_2^n) \cdot R_K}.$$

Then $\lim_{i\to\infty} |x_i| = \infty$. It follows from (2.4.33) that

$$1 \leq \frac{R_K F_K}{n|K|} \cdot \lim_{i\to\infty} \int_{S^{n-1}} \phi\left(|\langle u, x_i\rangle|\right) d\sigma(u) = 0.$$

This is a contradiction and hence $\sup_{i\geq 1} R_i < \infty$. In other words, the sequence $\{M_i\}_{i\geq 1}$ is uniformly bounded. By the Blaschke selection theorem, there exists a convergent subsequence of $\{M_i\}_{i\geq 1}$ (still denoted by $\{M_i\}_{i\geq 1}$) and a convex body $M \in \mathscr{K}$ such that $M_i \to M$ as $i \to \infty$. As $|M_i^\circ| = \omega_n$ for all $i \geq 1$, Lemma 1.1.1 gives $M \in \mathscr{K}_e$ and $|M^\circ| = \omega_n$. Proposition 2.2.1 concludes that M is the desired body. $\square$

Definition 2.4.2. *Let $K \in C_+^2$ and $\phi \in \widehat{\Phi}_2$ satisfy (2.4.33). A convex body $M \in \mathscr{K}_e$ is said to be an L_ϕ Orlicz-Petty body of K with respect to $\mathscr{K}_e$, if $M \in \mathscr{K}_e$ satisfies*

$$\widehat{G}_\phi^{orlicz}(K, \mathscr{K}_e) = \widehat{V}_\phi(K, M) \quad \text{and} \quad |M^\circ| = \omega_n.$$

Denote by $\widehat{T}_\phi(K, \mathscr{K}_e)$ the set of all such bodies.

Theorem 2.4.1. *Let $\phi \in \widehat{\Phi}_2$ satisfy (2.4.33). Assume that $\{K_i\}_{i\geq 0} \subset C_+^2$ such that $K_i \to K_0$ as $i \to \infty$ and $\{f_{K_i}\}_{i\geq 1}$ is uniformly bounded on S^{n-1}. Then*

$$\lim_{i\to\infty} \widehat{G}_\phi^{orlicz}(K_i, \mathscr{K}_e) = \widehat{G}_\phi^{orlicz}(K_0, \mathscr{K}_e).$$

Proof. As $K_i \to K_0$, there exist $r, R > 0$ such that $rB_2^n \subset K_i \subset RB_2^n$ for all $i \geq 0$. We claim that there is a finite constant $R' > 0$ such that $M_i \subset R'B_2^n$ for all (given) $M_i \in \widehat{T}_\phi(K_i, \mathscr{K}_e)$, $i \geq 1$. Suppose that there is no such finite constant. Without loss of generality, assume that $\lim_{i\to\infty} R_i = \infty$ and $u_i \to v$ (due to the compactness of S^{n-1}) as $i \to \infty$, where again

$$R_i = \rho_{M_i}(u_i) = \max\{\rho_{M_i}(u) : u \in S^{n-1}\}.$$

As before, $h_{M_i}(u) \geq R_i \cdot |\langle u, u_i\rangle|$ for all $u \in S^{n-1}$ and $i \geq 1$. Corollary 2.1.2, together with (2.4.33) and the fact that $\phi \in \widehat{\Phi}_2$ is decreasing, implies that, for all $i \geq 1$,

$$\begin{aligned} 1 &= \int_{S^{n-1}} \phi\left(\frac{n|K_i| \cdot h_{M_i}(u)}{\widehat{V}_\phi(K_i, M_i) \cdot h_{K_i}(u)}\right) d\widetilde{V}_{K_i}(u) \\ &\leq \int_{S^{n-1}} \phi\left(\frac{n|K_i| \cdot R_i \cdot |\langle u, u_i\rangle|}{\widehat{V}_\phi(K_i, B_2^n) \cdot h_{K_i}(u)}\right) \cdot \frac{h_{K_i}(u) f_{K_i}(u)}{n|K_i|}\, d\sigma(u) \\ &\leq \int_{S^{n-1}} \phi\left(\frac{r^{n+1} \cdot R_i \cdot |\langle u, u_i\rangle|}{R^{n+1}}\right) \cdot \frac{R \cdot F_0}{n\omega_n \cdot r^n}\, d\sigma(u), \end{aligned}$$

where the last inequality follows from Lemma 2.2.1 and F_0 is the uniform bound of $\{f_{K_i}\}_{i\geq 1}$ on S^{n-1} (i.e., $F_0 = \sup_{i\geq 1} \sup_{u\in S^{n-1}} f_{K_i}(u)$). As in the proof of Proposition 2.4.2, one gets

$$1 \leq \lim_{i\to\infty} \int_{S^{n-1}} \phi\left(\frac{r^{n+1} \cdot R_i \cdot |\langle u, u_i\rangle|}{R^{n+1}}\right) \cdot \frac{R \cdot F_0}{n\omega_n \cdot r^n}\, d\sigma(u) = 0,$$

which is a contradiction. Hence there is a finite constant $R' > 0$ such that $M_i \subset R'B_2^n$ for all (given) $M_i \in \widehat{T}_\phi(K_i, \mathscr{K}_e)$, $i \geq 1$. In other words, $\{M_i\}_{i\geq 1}$ is uniformly bounded.

Let $l = \liminf_{i\to\infty} \widehat{G}_\phi^{orlicz}(K_i, \mathscr{K}_e)$. Clearly, one can find a subsequence $\{K_{i_k}\}_{k\geq 1}$ such that $l = \lim_{k\to\infty} \widehat{G}_\phi^{orlicz}(K_{i_k}, \mathscr{K}_e)$. By the Blaschke selection theorem and Lemma 1.1.1, there exists a subsequence of $\{M_{i_k}\}_{k\geq 1}$ (still denoted by $\{M_{i_k}\}_{k\geq 1}$) and a body $M \in \mathscr{K}_e$, such that, $M_{i_k} \to M$ as $k \to \infty$ and $|M^\circ| = \omega_n$. Proposition 2.2.1 then yields

$$\widehat{G}_\phi^{orlicz}(K_{i_k}, \mathscr{K}_e) = \widehat{V}_\phi(K_{i_k}, M_{i_k}) \to \widehat{V}_\phi(K_0, M) \quad \text{as} \quad k \to \infty.$$

By (2.4.32), one has

$$\widehat{G}_\phi^{orlicz}(K_0, \mathscr{K}_e) \leq \widehat{V}_\phi(K_0, M) = \lim_{k\to\infty} \widehat{G}_\phi^{orlicz}(K_{i_k}, \mathscr{K}_e) = \liminf_{i\to\infty} \widehat{G}_\phi^{orlicz}(K_i, \mathscr{K}_e).$$

On the other hand, for any given $\epsilon > 0$, by (2.4.32) and Proposition 2.2.1, there exists a convex body $L_\epsilon \in \mathscr{K}_e$ such that $|L_\epsilon^\circ| = \omega_n$ and

$$\widehat{G}_\phi^{orlicz}(K_0, \mathscr{K}_e) + \epsilon > \widehat{V}_\phi(K_0, L_\epsilon) = \limsup_{i\to\infty} \widehat{V}_\phi(K_i, L_\epsilon) \geq \limsup_{i\to\infty} \widehat{G}_\phi^{orlicz}(K_i, \mathscr{K}_e).$$

By letting $\epsilon \to 0$, one gets $\widehat{G}_\phi^{orlicz}(K_0, \mathscr{K}_e) \geq \limsup_{i\to\infty} \widehat{G}_\phi^{orlicz}(K_i, \mathscr{K}_e)$ and the desired limit follows. $\square$

Let $K \in \mathscr{K}_0$ and $\varphi \in \widehat{\Phi}_1 \cup \widehat{\Psi}$. The nonhomogeneous Orlicz L_φ geominimal surface area of K with respect to $\mathscr{K}_e$ can be defined by

$$G_\varphi^{orlicz}(K, \mathscr{K}_e) = \inf\{nV_\varphi(K, L) : \ L \in \mathscr{K}_e \quad \text{with} \quad |L^\circ| = \omega_n\}.$$

While if $\varphi \in \widehat{\Phi}_2$, $G_\varphi^{orlicz}(\cdot, \mathscr{K}_e)$ can be defined similarly with " inf" replaced by " sup". Analogous results to Proposition 2.4.2 and Theorem 2.4.1 can be proved for $G_\varphi^{orlicz}(\cdot, \mathscr{K}_e)$ if $\varphi \in \widehat{\Phi}_2$ satisfies (2.4.33). We leave the details for readers.

Chapter 3

The Orlicz Brunn-Minkowski theory for p-capacity

This chapter is based on paper [35] collaborated with Deping Ye and Ning Zhang. In this chapter, combining the p-capacity for $p \in (1, n)$ with the Orlicz addition of convex domains, we develop the p-capacitary Orlicz-Brunn-Minkowski theory. In particular, the Orlicz L_ϕ mixed p-capacity of two convex domains is introduced and its geometric interpretation was obtained by the p-capacitary Orlicz-Hadamard variational formula. The p-capacitary Orlicz-Brunn-Minkowski and Orlicz-Minkowski inequalities are established, and the equivalence of these two inequalities are discussed as well.

3.1 The Orlicz L_ϕ mixed p-capacity and related Orlicz-Minkowski inequality

This section is dedicated to prove the p-capacitary Orlicz-Hadamard variational formula and establish the p-capacitary Orlicz-Minkowski inequality. Let $\phi : (0, \infty) \to (0, \infty)$ be a continuous function. We now define the Orlicz L_ϕ mixed p-capacity. The mixed p-capacity defined in (1.1.13) is related to $\phi = t$.

Definition 3.1.1. *Let $\Omega, \Omega_1 \in \mathscr{C}_0$ be two convex domains. Define $C_{p,\phi}(\Omega, \Omega_1)$, the*

Orlicz L_ϕ mixed p-capacity of Ω and Ω_1, by

$$C_{p,\phi}(\Omega,\Omega_1)=\frac{p-1}{n-p}\int_{S^{n-1}}\phi\left(\frac{h_{\Omega_1}(u)}{h_\Omega(u)}\right)h_\Omega(u)\,d\mu_p(\Omega,u). \tag{3.1.1}$$

When Ω and Ω_1 are dilates of each other, say $\Omega_1=\lambda\Omega$ for some $\lambda>0$, one has

$$C_{p,\phi}(\Omega,\lambda\Omega)=\phi\left(\lambda\right)C_p(\Omega). \tag{3.1.2}$$

Let φ_1 and φ_2 be either both in $\mathscr{I}$ or both in $\mathscr{D}$. For $\varepsilon>0$, let g_ε be defined as in (1.1.9). That is, for $\Omega,\Omega_1\in\mathscr{C}_0$ and for $u\in S^{n-1}$,

$$\varphi_1\left(\frac{h_\Omega(u)}{g_\varepsilon(u)}\right)+\varepsilon\varphi_2\left(\frac{h_{\Omega_1}(u)}{g_\varepsilon(u)}\right)=1.$$

Clearly $g_\varepsilon\in C^+(S^{n-1})$. Denote by $\Omega_\varepsilon\in\mathscr{C}_0$ the Aleksandrov domain associated to g_ε.

From Lemma 2.3.1, one sees that g_ε converges to h_Ω uniformly on S^{n-1}. According to the Aleksandrov convergence lemma, Ω_ε converges to Ω in the Hausdorff metric. We are now ready to establish the geometric interpretation for the Orlicz L_ϕ mixed p-capacity. Formula (1.1.13) is the special case when $\varphi_1=\varphi_2=t$.

Theorem 3.1.1. *Let $\Omega,\Omega_1\in\mathscr{C}_0$ be two convex domains. Suppose $\varphi_1,\varphi_2\in\mathscr{I}$ such that $(\varphi_1)'_l(1)$ exists and is nonzero. Then*

$$C_{p,\varphi_2}(\Omega,\Omega_1)=\frac{(\varphi_1)'_l(1)}{n-p}\cdot\lim_{\varepsilon\to0^+}\frac{C_p(\Omega_\varepsilon)-C_p(\Omega)}{\varepsilon}.$$

With $(\varphi_1)'_l(1)$ replaced by $(\varphi_1)'_r(1)$ if $(\varphi_1)'_r(1)$ exists and is nonzero, one gets the analogous result for $\varphi_1,\varphi_2\in\mathscr{D}$.

Proof. The proof of this theorem is similar to analogous results in [15, 22, 27, 80] and Theorem 2.3.1. A brief proof is included here for completeness. As $\Omega_\varepsilon\to\Omega$ in the Hausdorff metric, $\mu_p(\Omega_\varepsilon,\cdot)\to\mu_p(\Omega,\cdot)$ weakly on S^{n-1} due to Lemma 4.1 in [15]. Moreover, if $h_\varepsilon\to h$ uniformly on S^{n-1}, then

$$\lim_{\varepsilon\to0^+}\int_{S^{n-1}}h_\varepsilon(u)\,d\mu_p(\Omega_\varepsilon,u)=\int_{S^{n-1}}h(u)\,d\mu_p(\Omega,u).$$

In particular, it follows from (1.1.16) and Lemma 2.3.1 that

$$\begin{aligned}
& (\varphi_1)_l'(1)\cdot \lim_{\varepsilon\to 0^+}\frac{C_p(\Omega_\varepsilon, g_\varepsilon)-C_p(\Omega_\varepsilon, h_\Omega)}{\varepsilon}\\
= \;& (\varphi_1)_l'(1)\cdot \lim_{\varepsilon\to 0^+}\frac{p-1}{n-p}\int_{S^{n-1}}\frac{g_\varepsilon(u)-h_\Omega(u)}{\varepsilon}d\mu_p(\Omega_\varepsilon, u)\\
= \;& \frac{p-1}{n-p}\int_{S^{n-1}} h_\Omega(u)\varphi_2\left(\frac{h_{\Omega_1}(u)}{h_\Omega(u)}\right)d\mu_p(\Omega, u)\\
= \;& C_{p,\varphi_2}(\Omega,\Omega_1).
\end{aligned}$$

Inequality (1.1.14), formula (1.1.17), and the continuity of p-capacity yield that

$$\begin{aligned}
C_{p,\varphi_2}(\Omega,\Omega_1) &= (\varphi_1)_l'(1)\cdot \liminf_{\varepsilon\to 0^+}\frac{C_p(\Omega_\varepsilon)-C_p(\Omega_\varepsilon,\Omega)}{\varepsilon}\\
&\le (\varphi_1)_l'(1)\cdot \liminf_{\varepsilon\to 0^+}\left[C_p(\Omega_\varepsilon)^{\frac{n-p-1}{n-p}}\cdot\frac{C_p(\Omega_\varepsilon)^{\frac{1}{n-p}}-C_p(\Omega)^{\frac{1}{n-p}}}{\varepsilon}\right]\\
&= (\varphi_1)_l'(1)\cdot C_p(\Omega)^{\frac{n-p-1}{n-p}}\cdot \liminf_{\varepsilon\to 0^+}\frac{C_p(\Omega_\varepsilon)^{\frac{1}{n-p}}-C_p(\Omega)^{\frac{1}{n-p}}}{\varepsilon}.
\end{aligned}$$

Similarly, as $h_{\Omega_\varepsilon}\le g_\varepsilon$ and $C_p(\Omega)=C_p(\Omega,h_\Omega)$, one has

$$\begin{aligned}
C_{p,\varphi_2}(\Omega,\Omega_1) &= (\varphi_1)_l'(1)\cdot \lim_{\varepsilon\to 0^+}\frac{p-1}{n-p}\int_{S^{n-1}}\frac{g_\varepsilon(u)-h_\Omega(u)}{\varepsilon}d\mu_p(\Omega, u)\\
&\ge (\varphi_1)_l'(1)\cdot \limsup_{\varepsilon\to 0^+}\frac{C_p(\Omega,\Omega_\varepsilon)-C_p(\Omega)}{\varepsilon}\\
&\ge (\varphi_1)_l'(1)\cdot C_p(\Omega)^{\frac{n-p-1}{n-p}}\cdot \limsup_{\varepsilon\to 0^+}\frac{C_p(\Omega_\varepsilon)^{\frac{1}{n-p}}-C_p(\Omega)^{\frac{1}{n-p}}}{\varepsilon}.
\end{aligned}$$

This concludes that

$$\begin{aligned}
C_{p,\varphi_2}(\Omega,\Omega_1) &= (\varphi_1)_l'(1)\cdot C_p(\Omega)^{\frac{n-p-1}{n-p}}\cdot \lim_{\varepsilon\to 0^+}\frac{C_p(\Omega_\varepsilon)^{\frac{1}{n-p}}-C_p(\Omega)^{\frac{1}{n-p}}}{\varepsilon}\\
&= \frac{(\varphi_1)_l'(1)}{n-p}\cdot \lim_{\varepsilon\to 0^+}\frac{C_p(\Omega_\varepsilon)-C_p(\Omega)}{\varepsilon},
\end{aligned}$$

where the second equality follows from a standard argument by the chain rule. □

Let $p\in(1,n)$ and $q\neq 0$ be real numbers. For $\Omega,\Omega_1\in\mathscr{C}_0$, define $C_{p,q}(\Omega,\Omega_1)$, the L_q mixed p-capacity of Ω and Ω_1, by

$$C_{p,q}(\Omega,\Omega_1)=\frac{p-1}{n-p}\int_{S^{n-1}}\left[h_{\Omega_1}(u)\right]^q d\mu_{p,q}(\Omega,u), \tag{3.1.3}$$

where $\mu_{p,q}(\Omega, \cdot)$ denotes the L_q p-capacitary measure of Ω:

$$d\mu_{p,q}(\Omega, \cdot) = h_\Omega^{1-q}\, d\mu_p(\Omega, \cdot).$$

For $\varepsilon > 0$, let $h_{q,\varepsilon} = \left[h_\Omega^q + \varepsilon h_{\Omega_1}^q\right]^{1/q}$ and $\Omega_{h_{q,\varepsilon}}$ be the Aleksandrov domain associated to $h_{q,\varepsilon}$. By letting $\varphi_1 = \varphi_2 = t^q$ for $q \neq 0$ in Theorem 3.1.1, one gets the geometric interpretation for $C_{p,q}(\cdot, \cdot)$.

Corollary 3.1.1. *Let $\Omega, \Omega_1 \in \mathscr{C}_0$ and $p \in (1, n)$. For all $0 \neq q \in \mathbb{R}$, one has*

$$C_{p,q}(\Omega, \Omega_1) = \frac{q}{n-p} \cdot \lim_{\varepsilon \to 0^+} \frac{C_p(\Omega_{h_{q,\varepsilon}}) - C_p(\Omega)}{\varepsilon}.$$

Regarding the Orlicz L_ϕ mixed p-capacity, one has the following p-capacitary Orlicz-Minkowski inequality. When $\phi = t$, one recovers the p-capacitary Minkowski inequality (1.1.14).

Theorem 3.1.2. *Let $\Omega, \Omega_1 \in \mathscr{C}_0$ and $p \in (1, n)$. Suppose that $\phi : [0, \infty) \to [0, \infty)$ is increasing and convex. Then*

$$C_{p,\phi}(\Omega, \Omega_1) \geq C_p(\Omega) \cdot \phi\bigg(\left(\frac{C_p(\Omega_1)}{C_p(\Omega)}\right)^{\frac{1}{n-p}}\bigg).$$

If in addition ϕ is strictly convex, equality holds if and only if Ω and Ω_1 are dilates of each other.

Proof. It follows from Jensen's inequality (see [23]), $C_p(\Omega) > 0$ and the convexity of ϕ that

$$\begin{aligned} C_{p,\phi}(\Omega, \Omega_1) &= \frac{p-1}{n-p} \int_{S^{n-1}} \phi\left(\frac{h_{\Omega_1}(u)}{h_\Omega(u)}\right) h_\Omega(u)\, d\mu_p(\Omega, u) \\ &\geq C_p(\Omega) \cdot \phi\left(\int_{S^{n-1}} \frac{p-1}{n-p} \cdot \frac{h_{\Omega_1}(u)}{C_p(\Omega)}\, d\mu_p(\Omega, u)\right) \\ &= C_p(\Omega) \cdot \phi\left(\frac{C_{p,1}(\Omega, \Omega_1)}{C_p(\Omega)}\right) \\ &\geq C_p(\Omega) \cdot \phi\bigg(\left(\frac{C_p(\Omega_1)}{C_p(\Omega)}\right)^{\frac{1}{n-p}}\bigg) \end{aligned} \tag{3.1.4}$$

where the last inequality follows from (1.1.14) and the fact that ϕ is increasing.

From (1.1.11) and (3.1.2), if Ω and Ω_1 are dilates of each other, then clearly

$$C_{p,\phi}(\Omega,\Omega_1) = C_p(\Omega)\cdot\phi\bigg(\Big(\frac{C_p(\Omega_1)}{C_p(\Omega)}\Big)^{\frac{1}{n-p}}\bigg).$$

On the other hand, if ϕ is strictly convex, equality holds in (3.1.4) only if equalities hold in both the first and the second inequalities of (3.1.4). For the second one, Ω and Ω_1 are homothetic to each other. That is, there exists $r > 0$ and $x \in \mathbb{R}^n$, such that $\Omega_1 = r\Omega + x$ and hence for all $u \in S^{n-1}$,

$$h_{\Omega_1}(u) = r \cdot h_\Omega(u) + \langle x, u\rangle.$$

As ϕ is strictly convex, the characterization of equality in Jensen's inequality implies that

$$\frac{h_{\Omega_1}(v)}{h_\Omega(v)} = \int_{S^{n-1}} \frac{p-1}{n-p}\cdot\frac{h_{\Omega_1}(u)}{C_p(\Omega)}\,d\mu_p(\Omega,u)$$

for $\mu_p(\Omega,\cdot)$-almost all $v \in S^{n-1}$. This together with the fact that $\mu_p(\Omega,\cdot)$ has its centroid at the origin yield $\langle x, v\rangle = 0$ for $\mu_p(\Omega,\cdot)$-almost all $v \in S^{n-1}$. As the support of $\mu_p(\Omega,\cdot)$ is not contained in a closed hemisphere, one has $x = o$. That is, Ω and Ω_1 are dilates of each other. $\square$

An application of the above p-capacitary Orlicz-Minkowski inequality is stated below.

Theorem 3.1.3. *Let $\phi \in \Phi_1$ be strictly increasing and strictly convex. Assume that $\Omega, \widetilde{\Omega} \in \mathscr{C}_0$ are two convex domains. Then $\Omega = \widetilde{\Omega}$ if the following equality holds for all $\Omega_1 \in \mathscr{C}_0$:*

$$\frac{C_{p,\phi}(\Omega,\Omega_1)}{C_p(\Omega)} = \frac{C_{p,\phi}(\widetilde{\Omega},\Omega_1)}{C_p(\widetilde{\Omega})}. \tag{3.1.5}$$

Moreover, $\Omega = \widetilde{\Omega}$ also holds if, for any $\Omega_1 \in \mathscr{C}_0$,

$$C_{p,\phi}(\Omega_1,\Omega) = C_{p,\phi}(\Omega_1,\widetilde{\Omega}). \tag{3.1.6}$$

Proof. It follows from equality (3.1.5) and the p-capacitary Orlicz-Minkowski inequality that

$$1 = \frac{C_{p,\phi}(\Omega,\Omega)}{C_p(\Omega)} = \frac{C_{p,\phi}(\widetilde{\Omega},\Omega)}{C_p(\widetilde{\Omega})} \geq \phi\bigg(\Big(\frac{C_p(\Omega)}{C_p(\widetilde{\Omega})}\Big)^{\frac{1}{n-p}}\bigg). \tag{3.1.7}$$

The fact that ϕ is strictly increasing with $\phi(1) = 1$ and $n - p > 0$ yield $C_p(\widetilde{\Omega}) \geq C_p(\Omega)$. Similarly, $C_p(\widetilde{\Omega}) \leq C_p(\Omega)$ and then $C_p(\widetilde{\Omega}) = C_p(\Omega)$. Hence, equality holds in inequality (3.1.7). This can happen only if Ω and $\widetilde{\Omega}$ are dilates of each other, due to Theorem 3.1.2 and the fact that ϕ is strictly convex. Combining with the above proved fact $C_p(\widetilde{\Omega}) = C_p(\Omega)$, one gets $\Omega = \widetilde{\Omega}$.

Follows along the same lines, $\Omega = \widetilde{\Omega}$ if equality (3.1.6) holds for any $\Omega_1 \in \mathscr{C}_0$. $\square$

Note that $\phi = t^q$ for $q > 1$ is a strictly convex and strictly increasing function. Theorem 3.1.2 yields the p-capacitary L_q Minkowski inequality: for $\Omega, \Omega_1 \in \mathscr{C}_0$, one has

$$C_{p,q}(\Omega, \Omega_1) \geq \left[C_p(\Omega)\right]^{\frac{n-p-q}{n-p}} \cdot \left[C_p(\Omega_1)\right]^{\frac{q}{n-p}}$$

with equality if and only if Ω and Ω_1 are dilates of each other.

Corollary 3.1.2. *Let $p \in (0, n)$ and $q > 1$. If $\Omega, \widetilde{\Omega} \in \mathscr{C}_0$ such that*

$$\mu_{p,q}(\Omega, \cdot) = \mu_{p,q}(\widetilde{\Omega}, \cdot),$$

then $\Omega = \widetilde{\Omega}$ if $q \neq n - p$, and Ω is dilate of $\widetilde{\Omega}$ if $q = n - p$.

Proof. Firstly let $q > 1$ and $q \neq n - p$. As $\mu_{p,q}(\Omega, \cdot) = \mu_{p,q}(\widetilde{\Omega}, \cdot)$, it follows form (3.1.3) that, for all $\Omega_1 \in \mathscr{C}_0$,

$$C_{p,q}(\Omega, \Omega_1) = C_{p,q}(\widetilde{\Omega}, \Omega_1). \tag{3.1.8}$$

By letting $\Omega_1 = \widetilde{\Omega}$, one has,

$$C_{p,q}(\Omega, \widetilde{\Omega}) = C_p(\widetilde{\Omega}) \geq \left[C_p(\Omega)\right]^{\frac{n-p-q}{n-p}} \cdot \left[C_p(\widetilde{\Omega})\right]^{\frac{q}{n-p}}.$$

This yields $C_p(\Omega) \geq C_p(\widetilde{\Omega})$ if $q > n-p$ and $C_p(\Omega) \leq C_p(\widetilde{\Omega})$ if $q < n-p$. Similarly, by letting $\Omega_1 = \Omega$, one has $C_p(\Omega) \leq C_p(\widetilde{\Omega})$ if $q > n - p$ and $C_p(\Omega) \geq C_p(\widetilde{\Omega})$ if $q < n - p$. In any cases, $C_p(\Omega) = C_p(\widetilde{\Omega})$. Together with (3.1.8), Theorem 3.1.3 yields the desired argument $\Omega = \widetilde{\Omega}$.

Now assume that $q = n - p > 1$. Then (3.1.8) yields

$$C_{p,q}(\Omega, \widetilde{\Omega}) = C_p(\widetilde{\Omega}) \geq \left[C_p(\Omega)\right]^{\frac{n-p-q}{n-p}} \cdot \left[C_p(\widetilde{\Omega})\right]^{\frac{q}{n-p}} = C_p(\widetilde{\Omega}).$$

It follows from Theorem 3.1.2 that Ω and $\widetilde{\Omega}$ are dilates of each other. $\square$

It is worth to mention that $C_{p,\phi}(\cdot,\cdot)$ is not homogeneous if ϕ is not a homogeneous function; this can be seen from formula (3.1.2). When $\phi \in \mathscr{I}$, we can define $\widehat{C}_{p,\phi}(\Omega,\Omega_1)$, the homogeneous Orlicz L_ϕ mixed p-capacity of $\Omega,\Omega_1 \in \mathscr{C}_0$, by

$$\widehat{C}_{p,\phi}(\Omega,\Omega_1) = \inf\left\{\eta > 0 : \frac{p-1}{n-p}\int_{S^{n-1}} \phi\left(\frac{h_{\Omega_1}(u)}{\eta \cdot h_\Omega(u)}\right) h_\Omega(u)\, d\mu_p(\Omega,u) \le C_p(\Omega)\right\},$$

while $\widehat{C}_{p,\phi}(\Omega,\Omega_1)$ for $\phi \in \mathscr{D}$ is defined as above with "$\le$" replaced by "$\ge$". If $\phi = t^q$ for $q \neq 0$,

$$\widehat{C}_{p,\phi}(\Omega,\Omega_1) = \left(\frac{C_{p,q}(\Omega,\Omega_1)}{C_p(\Omega)}\right)^{1/q}.$$

For all $\eta > 0$ and for $\phi \in \mathscr{I}$, let

$$g(\eta) = \frac{p-1}{n-p}\int_{S^{n-1}} \phi\left(\frac{h_{\Omega_1}(u)}{\eta \cdot h_\Omega(u)}\right) h_\Omega(u)\, d\mu_p(\Omega,u).$$

The fact that ϕ is monotone increasing yields

$$\phi\left(\frac{\min_{u\in S^{n-1}} h_{\Omega_1}(u)}{\eta \cdot \max_{u\in S^{n-1}} h_\Omega(u)}\right) \le \frac{g(\eta)}{C_p(\Omega)} \le \phi\left(\frac{\max_{u\in S^{n-1}} h_{\Omega_1}(u)}{\eta \cdot \min_{u\in S^{n-1}} h_\Omega(u)}\right).$$

Hence $\lim_{\eta\to 0^+} g(\eta) = \infty$ and $\lim_{\eta\to\infty} g(\eta) = 0$. It is also easily checked that g is strictly decreasing. This concludes that if $\phi \in \mathscr{I}$,

$$\frac{p-1}{n-p}\int_{S^{n-1}} \phi\left(\frac{h_{\Omega_1}(u)}{\widehat{C}_{p,\phi}(\Omega,\Omega_1) \cdot h_\Omega(u)}\right) h_\Omega(u)\, d\mu_p(\Omega,u) = C_p(\Omega). \tag{3.1.9}$$

Following along the same lines, formula (3.1.9) also holds for $\phi \in \mathscr{D}$.

The p-capacitary Orlicz-Minkowski inequality for $\widehat{C}_{p,\phi}(\cdot,\cdot)$ is stated in the following result.

Corollary 3.1.3. *Let $\phi \in \mathscr{I}$ be convex. For all $\Omega,\Omega_1 \in \mathscr{C}_0$, one has,*

$$\widehat{C}_{p,\phi}(\Omega,\Omega_1) \ge \left(\frac{C_p(\Omega_1)}{C_p(\Omega)}\right)^{\frac{1}{n-p}}. \tag{3.1.10}$$

If in addition ϕ is strictly convex, equality holds if and only if Ω and Ω_1 are dilates of each other.

Proof. It follows from formula (3.1.9) and Jensen's inequality that

$$\begin{aligned}
1 &= \int_{S^{n-1}} \phi\bigg(\frac{h_{\Omega_1}(u)}{\widehat{C}_{p,\phi}(\Omega,\Omega_1)\cdot h_\Omega(u)}\bigg)\cdot\frac{p-1}{n-p}\cdot\frac{h_\Omega(u)}{C_p(\Omega)}\,d\mu_p(\Omega,u)\\
&\geq \phi\bigg(\int_{S^{n-1}}\frac{h_{\Omega_1}(u)}{\widehat{C}_{p,\phi}(\Omega,\Omega_1)}\cdot\frac{p-1}{n-p}\cdot\frac{1}{C_p(\Omega)}\,d\mu_p(\Omega,u)\bigg)\\
&= \phi\bigg(\frac{C_p(\Omega,\Omega_1)}{\widehat{C}_{p,\phi}(\Omega,\Omega_1)\cdot C_p(\Omega)}\bigg).
\end{aligned}$$

As $\phi(1)=1$ and ϕ is monotone increasing, one has

$$\widehat{C}_{p,\phi}(\Omega,\Omega_1)\geq\frac{C_p(\Omega,\Omega_1)}{C_p(\Omega)}\geq\bigg(\frac{C_p(\Omega_1)}{C_p(\Omega)}\bigg)^{\frac{1}{n-p}},$$

where the second inequality follows from (1.1.14).

It is easily checked that equality holds in (3.1.10) if Ω_1 is dilate of Ω. Now assume that in addition ϕ is strictly convex and equality holds in (3.1.10). Then equality must hold in (1.1.14) and hence Ω is homothetic to Ω_1. Following along the same lines in the proof of Theorem 3.1.2, one obtains that Ω is dilate of Ω_1. $\square$

3.2 The p-capacitary Orlicz-Brunn-Minkowski inequality

This section aims to establish the p-capacitary Orlicz-Brunn-Minkowski inequality (i.e., Theorem 3.2.1). We also show that the p-capacitary Orlicz-Brunn-Minkowski inequality is equivalent to the p-capacitary Orlicz-Minkowski inequality (i.e., Theorem 3.1.2) in some sense. Let $m\geq 2$. Recall that the support function of $+_\varphi(\Omega_1,\ldots,\Omega_m)$ satisfies the following equation: for any $u\in S^{n-1}$,

$$\varphi\bigg(\frac{h_{\Omega_1}(u)}{h_{+_\varphi(\Omega_1,\ldots,\Omega_m)}(u)},\ldots,\frac{h_{\Omega_m}(u)}{h_{+_\varphi(\Omega_1,\ldots,\Omega_m)}(u)}\bigg)=1. \tag{3.2.11}$$

Theorem 3.2.1. *Suppose that $\Omega_1,\cdots,\Omega_m\in\mathscr{C}_0$ are convex domains. For all $\varphi\in\Phi_m$, one has*

$$1\geq\varphi\Bigg(\bigg(\frac{C_p(\Omega_1)}{C_p(+_\varphi(\Omega_1,\cdots,\Omega_m))}\bigg)^{\frac{1}{n-p}},\cdots,\bigg(\frac{C_p(\Omega_m)}{C_p(+_\varphi(\Omega_1,\cdots,\Omega_m))}\bigg)^{\frac{1}{n-p}}\Bigg). \tag{3.2.12}$$

If in addition φ is strictly convex, equality holds if and only if Ω_i are dilates of Ω_1 for all $i = 2, 3, \cdots, m$.

Proof. Let $\varphi \in \Phi_m$ and $\Omega_1, \cdots, \Omega_m \in \mathscr{C}_0$. Recall that $\Omega_1 \subset +_\varphi(\Omega_1, \ldots, \Omega_m)$ (see (1.1.5)). The fact that the p-capacity is monotone increasing yields

$$C_p(+_\varphi(\Omega_1, \ldots, \Omega_m)) \geq C_p(\Omega_1) > 0.$$

Define a probability measure on S^{n-1} by

$$d\omega_{p,\varphi}(u) = \frac{p-1}{n-p} \cdot \frac{1}{C_p(+_\varphi(\Omega_1, \ldots, \Omega_m))} \cdot h_{+_\varphi(\Omega_1,\ldots,\Omega_m)}(u)\, d\mu_p(+_\varphi(\Omega_1, \ldots, \Omega_m), u).$$

It follows from formulas (3.1.1) and (3.2.11), and Jensen's inequality (see [23, Proposition 2.2]) that

$$\begin{aligned} 1 &= \int_{S^{n-1}} \varphi\left(\frac{h_{\Omega_1}(u)}{h_{+_\varphi(\Omega_1,\cdots,\Omega_m)}(u)}, \cdots, \frac{h_{\Omega_m}(u)}{h_{+_\varphi(\Omega_1,\cdots,\Omega_m)}(u)}\right) d\omega_{p,\varphi}(u) \\ &\geq \varphi\left(\int_{S^{n-1}} \frac{h_{\Omega_1}(u)}{h_{+_\varphi(\Omega_1,\cdots,\Omega_m)}(u)}\, d\omega_{p,\varphi}(u), \cdots, \int_{S^{n-1}} \frac{h_{\Omega_m}(u)}{h_{+_\varphi(\Omega_1,\cdots,\Omega_m)}(u)}\, d\omega_{p,\varphi}(u)\right) \\ &= \varphi\left(\frac{C_{p,1}(+_\varphi(\Omega_1, \cdots, \Omega_m), \Omega_1)}{C_p(+_\varphi(\Omega_1, \cdots, \Omega_m))}, \cdots, \frac{C_{p,1}(+_\varphi(\Omega_1, \cdots, \Omega_m), \Omega_m)}{C_p(+_\varphi(\Omega_1, \cdots, \Omega_m))}\right) \\ &\geq \varphi\left(\left(\frac{C_p(\Omega_1)}{C_p(+_\varphi(\Omega_1, \cdots, \Omega_m))}\right)^{\frac{1}{n-p}}, \cdots, \left(\frac{C_p(\Omega_m)}{C_p(+_\varphi(\Omega_1, \cdots, \Omega_m))}\right)^{\frac{1}{n-p}}\right), \end{aligned}$$

where the last inequality follows from inequality (1.1.14).

Let us now characterize the conditions for equality. In fact, if Ω_i are dilates of Ω_1 for all $1 < i \leq m$, then $+_\varphi(\Omega_1, \cdots, \Omega_m)$ is also dilate of Ω_1 and the equality clearly holds. Now suppose that $\varphi \in \Phi_m$ is strictly convex. Equality must hold for Jensen's inequality and hence there exists a vector $z_0 \in \mathbb{R}^m$ (see [23, Proposition 2.2]) such that

$$\left(\frac{h_{\Omega_1}(u)}{h_{+_\varphi(\Omega_1,\cdots,\Omega_m)}(u)}, \cdots, \frac{h_{\Omega_m}(u)}{h_{+_\varphi(\Omega_1,\cdots,\Omega_m)}(u)}\right) = z_0$$

for $\omega_{p,\varphi}$-almost all $u \in S^{n-1}$. Moreover, as $\varphi \in \Phi_m$ is strictly increasing on each component, one must have

$$\frac{C_{p,1}(+_\varphi(\Omega_1, \cdots, \Omega_m), \Omega_j)}{C_p(+_\varphi(\Omega_1, \cdots, \Omega_m))} = \left(\frac{C_p(\Omega_j)}{C_p(+_\varphi(\Omega_1, \cdots, \Omega_m))}\right)^{\frac{1}{n-p}}$$

for all $j = 1, 2, \cdots, m$. The characterization of equality for (1.1.14) yields that Ω_j for $j = 1, \cdots, m$ are all homothetic to $+_\varphi(\Omega_1, \cdots, \Omega_m)$. Following the argument similar to that of Theorem 3.1.2, one can conclude that Ω_i for all $j = 1, \cdots, m$ are dilates of $+_\varphi(\Omega_1, \cdots, \Omega_m)$, as desired. $\square$

If $\varphi(x) = \sum_{i=1}^m x_i$ for $x \in [0, \infty)^m$, then $\varphi \in \Phi_m$ and inequality (3.2.12) becomes the classical p-capacitary Brunn-Minkowski inequality (see inequality (1.2.26)): for $\Omega_1, \cdots, \Omega_m \in \mathscr{C}_0$, one has

$$C_p(\Omega_1 + \cdots + \Omega_m)^{\frac{1}{n-p}} \geq C_p(\Omega_1)^{\frac{1}{n-p}} + \cdots + C_p(\Omega_m)^{\frac{1}{n-p}}. \tag{3.2.13}$$

From the proof of Theorem 3.2.1, one sees that equality holds if and only if Ω_i is homothetic to Ω_j for all $1 \leq i < j \leq m$. When $\varphi(x) = \sum_{i=1}^m x_j^q \in \Phi_m$ for $q > 1$, one gets the p-capacitary L_q-Brunn-Minkowski inequality: for $\Omega_1, \cdots, \Omega_m \in \mathscr{C}_0$, one has

$$C_p(\Omega_1 +_q \cdots +_q \Omega_m)^{\frac{q}{n-p}} \geq C_p(\Omega_1)^{\frac{q}{n-p}} + \cdots + C_p(\Omega_m)^{\frac{q}{n-p}}.$$

As $\varphi(x) = \sum_{i=1}^m x_j^q$ for $q > 1$ is strictly convex, equality holds if and only if Ω_i is dilate of Ω_j for all $1 \leq i < j \leq m$. This has been proved by Zou and Xiong in [100] with a different approach.

Now let us consider the linear Orlicz addition of $\Omega_1, \cdots, \Omega_m \in \mathscr{C}_0$. This is related to

$$\varphi(x) = \alpha_1\varphi_1(x_1) + \cdots + \alpha_m\varphi_m(x_m), \quad x = (x_1, \cdots, x_m) \in (0, \infty)^m, \tag{3.2.14}$$

where $\alpha_j > 0$ are constants and $\varphi_j \in \Phi_1$ for all $j = 1, \cdots, m$. Clearly $\varphi \in \Phi_m$ and the p-capacitary Orlicz-Brunn-Minkowski inequality in Theorem 3.2.1 can be rewritten as the following form.

Theorem 3.2.2. *Let φ be given in (3.2.14) with $\alpha_j > 0$ constants and $\varphi_j \in \Phi_1$ for $j = 1, \cdots, m$. For $\Omega_1, \cdots, \Omega_m \in \mathscr{C}_0$, one has*

$$1 \geq \sum_{j=1}^m \alpha_j\varphi_j\left(\left(\frac{C_p(\Omega_j)}{C_p(+_\varphi(\Omega_1, \cdots, \Omega_m))}\right)^{\frac{1}{n-p}}\right). \tag{3.2.15}$$

In fact, inequality (3.2.15) is equivalent to, in some sense, the p-capacitary Orlicz-Minkowski inequality in Theorem 3.1.2. Let $m = 2$, $\varphi_1, \varphi_2 \in \Phi_1$, $\Omega, \widetilde{\Omega} \in \mathscr{C}_0$, $\alpha_1 = 1$ and $\alpha_2 = \varepsilon > 0$. In this case, the linear Orlicz addition of Ω and $\widetilde{\Omega}$ is denoted by $\Omega +_{\varphi,\varepsilon} \widetilde{\Omega}$, whose support function is given by, for $u \in S^{n-1}$,

$$\varphi_1\bigg(\frac{h_\Omega(u)}{h_{\Omega+_{\varphi,\varepsilon}\widetilde{\Omega}}(u)}\bigg) + \varepsilon\varphi_2\bigg(\frac{h_{\widetilde{\Omega}}(u)}{h_{\Omega+_{\varphi,\varepsilon}\widetilde{\Omega}}(u)}\bigg) = 1.$$

The p-capacitary Orlicz-Brunn-Minkowski inequality in Theorem 3.2.2 becomes

$$1 \geq \varphi_1\bigg(\bigg(\frac{C_p(\Omega)}{C_p(\Omega +_{\varphi,\varepsilon} \widetilde{\Omega})}\bigg)^{\frac{1}{n-p}}\bigg) + \varepsilon\varphi_2\bigg(\bigg(\frac{C_p(\widetilde{\Omega})}{C_p(\Omega +_{\varphi,\varepsilon} \widetilde{\Omega})}\bigg)^{\frac{1}{n-p}}\bigg),$$

for all $\varepsilon > 0$. It is equivalent to

$$1 - \varphi_1^{-1}\bigg(1 - \varepsilon\varphi_2\bigg(\bigg(\frac{C_p(\widetilde{\Omega})}{C_p(\Omega +_{\varphi,\varepsilon} \widetilde{\Omega})}\bigg)^{\frac{1}{n-p}}\bigg)\bigg) \leq 1 - \bigg(\frac{C_p(\Omega)}{C_p(\Omega +_{\varphi,\varepsilon} \widetilde{\Omega})}\bigg)^{\frac{1}{n-p}}. \qquad (3.2.16)$$

For convenience, let $z(\varepsilon)$ be

$$z(\varepsilon) = \varphi_1^{-1}\bigg(1 - \varepsilon\varphi_2\bigg(\bigg(\frac{C_p(\widetilde{\Omega})}{C_p(\Omega +_{\varphi,\varepsilon} \widetilde{\Omega})}\bigg)^{\frac{1}{n-p}}\bigg)\bigg).$$

Then $z(\varepsilon) \to 1^-$ as $\varepsilon \to 0^+$ and

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \frac{1 - z(\varepsilon)}{\varepsilon} &= \lim_{\varepsilon\to 0^+} \frac{1 - z(\varepsilon)}{1 - \varphi_1(z(\varepsilon))} \cdot \lim_{\varepsilon\to 0^+} \varphi_2\bigg(\bigg(\frac{C_p(\widetilde{\Omega})}{C_p(\Omega +_{\varphi,\varepsilon} \widetilde{\Omega})}\bigg)^{\frac{1}{n-p}}\bigg) \\ &= \frac{1}{(\varphi_1)_l'(1)} \cdot \varphi_2\bigg(\bigg(\frac{C_p(\widetilde{\Omega})}{C_p(\Omega)}\bigg)^{\frac{1}{n-p}}\bigg),\end{aligned}$$

where $(\varphi_1)_l'(1)$ is assumed to exist and to be nonzero. Together with inequality (3.2.16), one gets

$$(\varphi_1)_l'(1) \cdot \lim_{\varepsilon\to 0^+} \frac{1 - \Big(\frac{C_p(\Omega)}{C_p(\Omega+_{\varphi,\varepsilon}\widetilde{\Omega})}\Big)^{\frac{1}{n-p}}}{\varepsilon} \geq \varphi_2\bigg(\bigg(\frac{C_p(\widetilde{\Omega})}{C_p(\Omega)}\bigg)^{\frac{1}{n-p}}\bigg).$$

This together with Theorem 3.1.1 further imply the p-capacitary Orlicz-Minkowski inequality:

$$
\begin{aligned}
(n-p)\cdot C_{p,\varphi_2}(\Omega,\widetilde{\Omega}) &= (\varphi_1)'_l(1)\cdot\lim_{\varepsilon\to 0^+} C_p(\Omega+_{\varphi,\varepsilon}\widetilde{\Omega})\cdot\lim_{\varepsilon\to 0^+}\frac{1-\frac{C_p(\Omega)}{C_p(\Omega+_{\varphi,\varepsilon}\widetilde{\Omega})}}{\varepsilon}\\
&= (\varphi_1)'_l(1)\cdot(n-p)\cdot C_p(\Omega)\cdot\lim_{\varepsilon\to 0^+}\frac{1-\Big(\frac{C_p(\Omega)}{C_p(\Omega+_{\varphi,\varepsilon}\widetilde{\Omega})}\Big)^{\frac{1}{n-p}}}{\varepsilon}\\
&\geq (n-p)\cdot C_p(\Omega)\cdot\varphi_2\bigg(\bigg(\frac{C_p(\widetilde{\Omega})}{C_p(\Omega)}\bigg)^{\frac{1}{n-p}}\bigg).
\end{aligned}
$$

On the other hand, assume that the p-capacitary Orlicz-Minkowski inequality in Theorem 3.1.2 holds. In particular, for $\varphi_1,\varphi_2\in\Phi_1$ and for $\Omega,\widetilde{\Omega}\in\mathscr{C}_0$,

$$
\begin{aligned}
\frac{C_{p,\varphi_1}(+_\varphi(\Omega,\widetilde{\Omega}),\Omega)}{C_p(+_\varphi(\Omega,\widetilde{\Omega}))} &\geq \varphi_1\bigg(\bigg(\frac{C_p(\Omega)}{C_p(+_\varphi(\Omega,\widetilde{\Omega}))}\bigg)^{\frac{1}{n-p}}\bigg),\\
\frac{C_{p,\varphi_2}(+_\varphi(\Omega,\widetilde{\Omega}),\widetilde{\Omega})}{C_p(+_\varphi(\Omega,\widetilde{\Omega}))} &\geq \varphi_2\bigg(\bigg(\frac{C_p(\widetilde{\Omega})}{C_p(+_\varphi(\Omega,\widetilde{\Omega}))}\bigg)^{\frac{1}{n-p}}\bigg),
\end{aligned}
$$

where $\varphi=\alpha_1\varphi_1+\alpha_2\varphi_2$ with $\alpha_1,\alpha_2>0$ and $\varphi_1,\varphi_2\in\Phi_1$, and $+_\varphi(\Omega,\widetilde{\Omega})$ is the convex domain whose support function $h_{+_\varphi(\Omega,\widetilde{\Omega})}$ is given by

$$
1=\alpha_1\varphi_1\bigg(\frac{h_\Omega(u)}{h_{+_\varphi(\Omega,\widetilde{\Omega})}(u)}\bigg)+\alpha_2\varphi_2\bigg(\frac{h_{\widetilde{\Omega}}(u)}{h_{+_\varphi(\Omega,\widetilde{\Omega})}(u)}\bigg),\quad \text{for}\ \ u\in S^{n-1}.
$$

This together with (3.1.1) lead to inequality (3.2.15) with $m=2$:

$$
\begin{aligned}
1 &= \frac{p-1}{n-p}\cdot\int_{S^{n-1}}\bigg[\alpha_1\varphi_1\bigg(\frac{h_\Omega(u)}{h_{+_\varphi(\Omega,\widetilde{\Omega})}(u)}\bigg)+\alpha_2\varphi_2\bigg(\frac{h_{\widetilde{\Omega}}(u)}{h_{+_\varphi(\Omega,\widetilde{\Omega})}(u)}\bigg)\bigg]\\
&\quad\cdot\ \frac{h_{+_\varphi(\Omega,\widetilde{\Omega})}(u)}{C_p(+_\varphi(\Omega,\widetilde{\Omega}))}\cdot d\mu_p(+_\varphi(\Omega,\widetilde{\Omega}),u)\\
&= \alpha_1\cdot\frac{C_{p,\varphi_1}(+_\varphi(\Omega,\widetilde{\Omega}),\Omega)}{C_p(+_\varphi(\Omega,\widetilde{\Omega}))}+\alpha_2\cdot\frac{C_{p,\varphi_2}(+_\varphi(\Omega,\widetilde{\Omega}),\widetilde{\Omega})}{C_p(+_\varphi(\Omega,\widetilde{\Omega}))}\\
&\geq \alpha_1\cdot\varphi_1\bigg(\bigg(\frac{C_p(\Omega)}{C_p(+_\varphi(\Omega,\widetilde{\Omega}))}\bigg)^{\frac{1}{n-p}}\bigg)+\alpha_2\cdot\varphi_2\bigg(\bigg(\frac{C_p(\widetilde{\Omega})}{C_p(+_\varphi(\Omega,\widetilde{\Omega}))}\bigg)^{\frac{1}{n-p}}\bigg).
\end{aligned}
$$

The M-addition of convex domains are closely related to the Orlicz addition. For an arbitrary subset $M\subset\mathbb{R}^m$, the M-addition of $\Omega_1,\cdots,\Omega_m\in\mathscr{C}_0$, denoted by $\oplus_M(\Omega_1,\cdots,\Omega_n)$, is defined by (see e.g. [21, 22, 69, 68])

$$
\oplus_M(\Omega_1,\cdots,\Omega_m)=\bigg\{\sum_{j=1}^m a_jx^j:\ \ x^j\in\Omega_j\ \ \text{and}\ \ (a_1,\cdots,a_m)\in M\bigg\}.
$$

It is equivalent to the following more convenient formula:

$$\oplus_M(\Omega_1, \cdots, \Omega_m) = \cup\big\{a_1\Omega_1 + \cdots + a_m\Omega_m : \quad (a_1, a_2, \cdots, a_m) \in M\big\}, \qquad (3.2.17)$$

where $a_1\Omega_1 + \cdots + a_m\Omega_m$ is the Minkowski addition of $a_j\Omega_j = \{a_jx^j : x^j \in \Omega_j\}$ for $j = 1, 2, \cdots, m$. Note that if M is compact, then $\oplus_M(\Omega_1, \cdots, \Omega_m)$ is again a convex domain. In general, the M-addition is different from the Orlicz addition. However, when M is a 1-unconditional convex body in $\mathbb{R}^m$ that contains $\{e_1, \cdots, e_m\}$ in its boundary, then the M-addition coincides with the Orlicz L_φ addition for some $\varphi \in \Phi_m$. More properties and historical remarks for the M-addition, such as convexity, $GL(n)$ covariant, homogeneity and monotonicity, can be founded in [21, 22, 69, 68].

Lemma 3.2.1. *If $M \subset \mathbb{R}^m$ is compact and $\Omega_1, \cdots, \Omega_m \in \mathscr{C}_0$, then for any $a = (a_1, \cdots, a_m) \in M$,*

$$C_p\big(\oplus_M (\Omega_1, \cdots, \Omega_m)\big)^{\frac{1}{n-p}} \geq \sum_{i=1}^{m} \left[|a_i| \cdot C_p(\Omega_i)^{\frac{1}{n-p}}\right]. \qquad (3.2.18)$$

If equality holds in (3.2.18) for some $a \in M$ with $a_j \neq 0$ for all $j = 1, 2, \cdots, m$, then Ω_i is homothetic to Ω_j for all $1 \leq i < j \leq m$.

Proof. Recall that the p-capacity is invariant under the affine isometry and has homogeneous degree $n - p$ (see [17]). Then for all $a \in \mathbb{R}$ and for all $\Omega \in \mathscr{C}_0$, one has

$$C_p(a\Omega) = |a|^{n-p}C_p(\Omega).$$

Note that $n - p > 0$. It follows from (3.2.13), (3.2.17) and the monotonicity of the p-capacity that, for all $a = (a_1, \cdots, a_m) \in M$,

$$C_p\big(\oplus_M (\Omega_1, \cdots, \Omega_m)\big)^{\frac{1}{n-p}} \geq C_p\big(a_1\Omega_1 + \cdots + a_m\Omega_m\big)^{\frac{1}{n-p}} \geq \sum_{i=1}^{m} \left[|a_i| \cdot C_p(\Omega_i)^{\frac{1}{n-p}}\right].$$

Assume that equality holds in (3.2.18) for some $a \in M$ with $a_j \neq 0$ for all $j = 1, 2, \cdots, m$. Then equality in (3.2.13) must hold and hence Ω_i is homothetic to Ω_j for all $1 \leq i < j \leq m$. $\square$

Let $e_j^{\perp} = \{x \in \mathbb{R}^m : \langle x, e_j \rangle = 0\}$ for all $j = 1, 2, \cdots, m$. For a nonzero vector $x \in \mathbb{R}^m$ and a convex set $E \subset \mathbb{R}^m$, define the support set of E with outer normal vector x to be the set

$$F(E, x) = \left\{ y \in \mathbb{R}^m : \langle x, y \rangle = \sup_{z \in E} \langle x, z \rangle \right\} \cap E.$$

Theorem 3.2.3. *Let $M \subset \mathbb{R}^m$ be a compact subset and $\Omega_1, \cdots, \Omega_m \in \mathscr{C}_0$. Then*

$$C_p\big(\oplus_M (\Omega_1, \cdots, \Omega_m)\big)^{\frac{1}{n-p}} \geq h_{conv(M)}\big(C_p(\Omega_1)^{\frac{1}{n-p}}, \cdots, C_p(\Omega_m)^{\frac{1}{n-p}}\big). \tag{3.2.19}$$

If $M \cap F(conv(M), x) \not\subset \cup_{j=1}^m e_j^{\perp}$ for all $x = (x_1, \cdots, x_m)$ with all $x_i > 0$ and equality holds in (3.2.19), then Ω_i is homothetic to Ω_j for all $1 \leq i < j \leq m$.

Proof. It is easily checked that $h_{conv(M)}(x) = \max_{y \in M} \langle x, y \rangle$ for all $x \in \mathbb{R}^m$. Following (3.2.18), one has, as all $C_p(\Omega_i) > 0$,

$$\begin{aligned}
& C_p\big(\oplus_M (\Omega_1, \cdots, \Omega_m)\big)^{\frac{1}{n-p}} \\
\geq \; & \max_{(a_1, \cdots, a_m) \in M} \Big\langle \big(|a_1|, \cdots, |a_m|\big), \big(C_p(\Omega_1)^{\frac{1}{n-p}}, \cdots, C_p(\Omega_m)^{\frac{1}{n-p}}\big) \Big\rangle \\
\geq \; & \max_{(a_1, \cdots, a_m) \in M} \Big\langle \big(a_1, \cdots, a_m\big), \big(C_p(\Omega_1)^{\frac{1}{n-p}}, \cdots, C_p(\Omega_m)^{\frac{1}{n-p}}\big) \Big\rangle \\
= \; & h_{conv(M)}\big(C_p(\Omega_1)^{\frac{1}{n-p}}, \cdots, C_p(\Omega_m)^{\frac{1}{n-p}}\big).
\end{aligned}$$

Now let us characterize the conditions for equality. Let

$$x_0 = \big(C_p(\Omega_1)^{\frac{1}{n-p}}, \cdots, C_p(\Omega_m)^{\frac{1}{n-p}}\big).$$

Assume that equality holds in (3.2.19). There exists a vector $a_0 \in M \cap F(conv(M), x_0)$ such that

$$\begin{aligned}
& C_p\big(\oplus_M (\Omega_1, \cdots, \Omega_m)\big)^{\frac{1}{n-p}} \\
= \; & \max_{(a_1, \cdots, a_m) \in M} \Big\langle \big(|a_1|, \cdots, |a_m|\big), \big(C_p(\Omega_1)^{\frac{1}{n-p}}, \cdots, C_p(\Omega_m)^{\frac{1}{n-p}}\big) \Big\rangle \\
= \; & \max_{(a_1, \cdots, a_m) \in M} \Big\langle \big(a_1, \cdots, a_m\big), \big(C_p(\Omega_1)^{\frac{1}{n-p}}, \cdots, C_p(\Omega_m)^{\frac{1}{n-p}}\big) \Big\rangle \\
= \; & h_{conv(M)}(x_0) = \langle a_0, x_0 \rangle.
\end{aligned}$$

Note that $M \cap F(conv(M), x_0) \not\subset \cup_{j=1}^m e_j^{\perp}$ and then all coordinates of a_0 must be strictly positive. As all coordinates of x_0 are strictly positive, it follows from the conditions of equality for (3.2.18) that Ω_i is homothetic to Ω_j for all $1 \leq i < j \leq m$. $\square$

Chapter 4

The general p-affine capacity and affine isocapacitary inequalities

This chapter is based on paper [34] collaborated with Deping Ye. In this chapter, we propose the notion of the general p-affine capacity and prove some basic properties for the general p-affine capacity, such as affine invariance and monotonicity. Moreover, the newly proposed general p-affine capacity is compared with several classical geometric quantities, e.g., the volume, the p-variational capacity and the p-integral affine surface area. Consequently, several sharp geometric inequalities for the general p-affine capacity are obtained. Theses inequalities extend and strengthen many well-known (affine) isoperimetric and (affine) isocapacitary inequalities.

4.1 The general p-affine capacity

In this section, the general p-affine capacity is proposed and several equivalent formulas for the general p-affine capacity are provided. Throughout, the general p-affine capacity of a compact set $K \subset \mathbb{R}^n$ will be denoted by $C_{p,\tau}(K)$. For convenience, let

$$\mathscr{E}(K) = \{f : f \in W_0^{1,p},\ f \geq \mathbf{1}_K\}.$$

For each $f \in W_0^{1,p}$, let $\nabla_u^+ f(x) = \max\{\nabla_u f(x), 0\}$, $\nabla_u^- f(x) = \max\{-\nabla_u f(x), 0\}$, and

$$\mathscr{H}_{p,\tau}(f) \;=\; \left(\int_{S^{n-1}} \|\varphi_\tau(\nabla_u f)\|_p^{-n}\, du\right)^{-\frac{p}{n}} \tag{4.1.1}$$

Definition 4.1.1. *Let K be a compact subset in $\mathbb{R}^n$ and the function φ_τ be as in (1.1.20). For $1 \le p < n$, define the general p-affine capacity of K by*

$$C_{p,\tau}(K) \;=\; \inf_{f \in \mathscr{E}(K)} \mathscr{H}_{p,\tau}(f).$$

Remark. For any compact set $K \subset \mathbb{R}^n$ and for any $\tau \in [-1,1]$, $C_{p,\tau}(K) < \infty$ if $p \in [1,n)$. According to the proofs of (4.2.4) and Theorem 4.2.1, the desired boundedness argument follows if $C_{p,\tau}(B_2^n) < \infty$ is verified. To this end, let $K = B_2^n$ and $\varepsilon > 0$. Consider

$$f_\varepsilon(x) = \begin{cases} 0, & \text{if } |x| \ge 1+\varepsilon, \\ 1 - \frac{|x|-1}{\varepsilon}, & \text{if } 1 < |x| < 1+\varepsilon, \\ 1, & \text{if } |x| \le 1. \end{cases}$$

It can be checked that $f_\varepsilon \in W_0^{1,p}$ and f_ε has its weak derivative to be

$$\nabla f_\varepsilon(x) = \begin{cases} 0, & \text{if } |x| \notin (1, 1+\varepsilon), \\ -\frac{x}{\varepsilon|x|}, & \text{if } |x| \in (1, 1+\varepsilon). \end{cases}$$

This further implies that, together with Fubini's theorem, (1.1.21) and (1.1.22),

$$\begin{aligned} \|\varphi_\tau(\nabla_u f_\varepsilon)\|_p^p &= \int_{\mathbb{R}^n} \big[\varphi_\tau(\nabla_u f_\varepsilon(x))\big]^p\, dx \\ &= \int_{\{x \in \mathbb{R}^n : 1 < |x| < 1+\varepsilon\}} \left[\varphi_\tau\Big(-\frac{u \cdot x}{\varepsilon|x|}\Big)\right]^p dx \\ &= \varepsilon^{-p} \int_1^{1+\varepsilon} r^{n-1}\, dr \cdot \int_{S^{n-1}} \big[\varphi_\tau(-u \cdot v)\big]^p\, d\sigma(v) \\ &= \frac{(1+\varepsilon)^n - 1}{\varepsilon^p} \cdot \omega_n \cdot A(n,p). \end{aligned}$$

It follows from (4.1.1) that

$$\mathscr{H}_{p,\tau}(f_\varepsilon) = \left(\int_{S^{n-1}} \|\varphi_\tau(\nabla_u f_\varepsilon)\|_p^{-n}\, du\right)^{-\frac{p}{n}} = \frac{(1+\varepsilon)^n - 1}{\varepsilon^p} \cdot \omega_n \cdot A(n,p).$$

By Definition 4.1.1, for $p \in [1, n)$,

$$C_{p,\tau}(B_2^n) \le \mathscr{H}_{p,\tau}(f_\varepsilon)\Big|_{\varepsilon=1} < 2^n \cdot \omega_n \cdot A(n,p) < \infty.$$

We would like to mention that the general p-affine capacity can be also defined for $p \in (0,1) \cup [n, \infty)$ along the same manner in Definition 4.1.1, however in these cases the general p-affine capacities are trivial. For instance, if $p \in (0,1)$,

$$C_{p,\tau}(B_2^n) \le \lim_{\varepsilon \to 0^+} \mathscr{H}_{p,\tau}(f_\varepsilon) = \lim_{\varepsilon \to 0^+} \frac{(1+\varepsilon)^n - 1}{\varepsilon^p} \cdot \omega_n \cdot A(n,p) = 0,$$

and hence, again due to the proofs of (4.2.4) and Theorem 4.2.1, $C_{p,\tau}(K) = 0$ for any compact set $K \subset \mathbb{R}^n$ and for any $\tau \in [-1, 1]$. The case for $p > n$ can be seen intuitively from the above estimate with $\varepsilon \to \infty$ instead, but more details for $p \ge n$ will be discussed in Theorem 4.3.1. The precise value of $C_{p,\tau}(B_2^n)$ will be provided in formulas (4.3.10) and (4.3.11). $\square$

As $\varphi_0(t) = 2^{-1/p}|t|$, one gets the p-affine capacity defined by Xiao in [82, 83]:

$$C_{p,0}(K) = \frac{1}{2} \inf_{f \in \mathscr{E}(K)} \left(\int_{S^{n-1}} \|\nabla_u f\|_p^{-n} \, du \right)^{-\frac{p}{n}}.$$

As $\varphi_1(\nabla_u f) = \nabla_u^+ f$, one has

$$C_{p,1}(K) = \inf_{f \in \mathscr{E}(K)} \left(\int_{S^{n-1}} \|\nabla_u^+ f\|_p^{-n} \, du \right)^{-\frac{p}{n}},$$

which will be called the asymmetric p-affine capacity and denoted by $C_{p,+}$ instead of $C_{p,1}$ for better intuition. Similarly, as $\varphi_{-1}(\nabla_u f) = \nabla_u^- f$, one can have the following p-affine capacity:

$$C_{p,-}(K) = \inf_{f \in \mathscr{E}(K)} \left(\int_{S^{n-1}} \|\nabla_u^- f\|_p^{-n} \, du \right)^{-\frac{p}{n}}.$$

The following theorem plays important roles in later context. For a compact set $K \subset \mathbb{R}^n$, let

$$\mathscr{F}(K) = \Big\{ f : f \in W_0^{1,p},\ 0 \le f \le 1 \text{ in } \mathbb{R}^n, \text{ and } f = 1 \text{ in a neighborhood of K} \Big\}.$$

Theorem 4.1.1. *Let $1 \le p < n$ and K be a compact set in $\mathbb{R}^n$. Then*

$$C_{p,\tau}(K) = \inf_{f \in \mathscr{F}(K)} \mathscr{H}_{p,\tau}(f).$$

Moreover, the general p-affine capacity is upper-semicontinuous: for any $\varepsilon > 0$, there exists an open set O_ε such that for any compact set F with $K \subset F \subset O_\varepsilon$,

$$C_{p,\tau}(F) \le C_{p,\tau}(K) + \varepsilon.$$

Proof. Our proof is based on the standard technique in [58] and is similar to that in [83, 85]. A short proof is included for completeness. Recall that $C_{p,\tau}(K) < \infty$. Due to $\mathscr{F}(K) \subset \mathscr{E}(K)$, one has

$$\inf_{f \in \mathscr{F}(K)} \mathscr{H}_{p,\tau}(f) \ge C_{p,\tau}(K).$$

On the other hand, for any $\varepsilon > 0$, let $f_\varepsilon \in \mathscr{E}(K)$ satisfy that

$$C_{p,\tau}(K) + \varepsilon \ge \mathscr{H}_{p,\tau}(f_\varepsilon).$$

For $i = 1, 2, \cdots$, there are functions $\phi_i \in C_c^\infty(\mathbb{R})$, such that, for all $t \in \mathbb{R}$,

$$0 \le \phi_i'(t) \le i^{-1} + 1,$$

$\phi_i = 0$ in a neighborhood of $(-\infty, 0]$, and $\phi_i = 1$ in a neighborhood of $[1, \infty)$. It follows from the chain rule in [17, Theorem 4 on p.129] and the homogeneity of φ_τ (see (1.1.21)) that, for all i, $\phi_i(f_\varepsilon) \in \mathscr{F}(K)$ and

$$\begin{aligned} \inf_{f \in \mathscr{F}(K)} \mathscr{H}_{p,\tau}(f) &\le \mathscr{H}_{p,\tau}(\phi_i(f_\varepsilon)) \\ &\le (1 + i^{-1})^p \cdot \mathscr{H}_{p,\tau}(f_\varepsilon) \\ &\le (1 + i^{-1})^p \cdot (C_{p,\tau}(K) + \varepsilon). \end{aligned}$$

Taking $i \to \infty$ first and then letting $\varepsilon \to 0$, one gets

$$\inf_{f \in \mathscr{F}(K)} \mathscr{H}_{p,\tau}(f) \le C_{p,\tau}(K)$$

and hence the following desired formula holds:

$$\inf_{f \in \mathscr{F}(K)} \mathscr{H}_{p,\tau}(f) = C_{p,\tau}(K).$$

Now let us prove the upper-semicontinuity. For any given $\varepsilon > 0$, let $g_\varepsilon \in \mathscr{F}(K)$ and O_ε be a neighborhood of K such that $g_\varepsilon = 1$ on O_ε and

$$C_{p,\tau}(K) + \varepsilon \geq \mathscr{H}_{p,\tau}(g_\varepsilon).$$

On the other hand, for any compact set F such that $K \subset F \subset O_\varepsilon$, one has $g_\varepsilon \in \mathscr{F}(F)$ and hence

$$\mathscr{H}_{p,\tau}(g_\varepsilon) \geq C_{p,\tau}(F),$$

by Definition 4.1.1. The desired inequality follows from the above two inequalities. □

Our next result regarding the definition of the general p-affine capacity for compact sets is to replace $\mathscr{E}(K)$ by the bigger set $\mathscr{D}(K)$:

$$\mathscr{D}(K) = \Big\{ f \in W_0^{1,p} \ \text{ such that } \ f \geq 1 \text{ on } K \Big\}.$$

Theorem 4.1.2. *Let $1 \leq p < n$ and K be a compact set in $\mathbb{R}^n$. Then*

$$C_{p,\tau}(K) \ = \inf_{f \in \mathscr{D}(K)} \mathscr{H}_{p,\tau}(f).$$

Proof. It follows from (1.1.20) and [33, Lemma 1.19] that, for any $f \in W_0^{1,p}$ and for any $u \in S^{n-1}$,

$$\varphi_\tau(\nabla_u f_+(x)) = \begin{cases} \varphi_\tau(\nabla_u f(x)), & \text{if } \ f(x) > 0, \\ 0, & \text{if } \ f(x) \leq 0. \end{cases}$$

Hence, for any $u \in S^{n-1}$ and all $x \in \mathbb{R}^n$, one has

$$\varphi_\tau(\nabla_u f_+(x)) \leq \varphi_\tau(\nabla_u f(x)).$$

This further implies that $\mathscr{H}_{p,\tau}(f_+) \leq \mathscr{H}_{p,\tau}(f)$ for any $f \in W_0^{1,p}$. Let $\{f_k\}_{k\geq 1} \subset \mathscr{D}(K)$ be such that

$$\lim_{k\to\infty} \mathscr{H}_{p,\tau}(f_k) = \inf_{f \in \mathscr{D}(K)} \mathscr{H}_{p,\tau}(f).$$

Then $\{f_{k,+}\}_{k\geq 1}$ is a sequence in $\mathscr{E}(K)$. Definition 4.1.1 yields

$$\lim_{k\to\infty} \mathscr{H}_{p,\tau}(f_k) \ \geq \ \limsup_{k\to\infty} \mathscr{H}_{p,\tau}(f_{k,+}) \geq \inf_{f \in \mathscr{E}(K)} \mathscr{H}_{p,\tau}(f) = C_{p,\tau}(K).$$

This concludes that

$$\inf_{f\in\mathscr{D}(K)} \mathscr{H}_{p,\tau}(f) \ge C_{p,\tau}(K).$$

On the other hand, as $\mathscr{E}(K)\subset\mathscr{D}(K)$, the following inequality holds trivially:

$$\inf_{f\in\mathscr{D}(K)} \mathscr{H}_{p,\tau}(f) \le C_{p,\tau}(K).$$

Combining the above two inequalities, one has $C_{p,\tau}(K)=\inf_{f\in\mathscr{D}(K)}\mathscr{H}_{p,\tau}(f)$. □

The following result asserts that $f\in W_0^{1,p}$ in Definition 4.1.1, Theorems 4.1.1 and 4.1.2 could be replaced by $f\in C_c^\infty$. The smoothness of functions is convenient in establishing many properties for the general p-affine capacity.

Theorem 4.1.3. *Let $p\in[1,n)$ and K be a compact set in $\mathbb{R}^n$. For any $\tau\in[-1,1]$, one has*

$$C_{p,\tau}(K)=\inf_{f\in C_c^\infty\cap\mathscr{D}(K)}\mathscr{H}_{p,\tau}(f)=\inf_{f\in C_c^\infty\cap\mathscr{E}(K)}\mathscr{H}_{p,\tau}(f)=\inf_{f\in C_c^\infty\cap\mathscr{F}(K)}\mathscr{H}_{p,\tau}(f). \quad (4.1.2)$$

Proof. Let $p\in[1,n)$. Let $f\in\mathscr{F}(K)$, i.e., $f\in W_0^{1,p}$ such that $0\le f\le 1$ in $\mathbb{R}^n$ and $f=1$ in U, a neighborhood of K. As $W_0^{1,p}$ is the closure of C_c^∞ under $\|\cdot\|_{1,p}$, there is a sequence $\{f_k\}_{k=1}^\infty\subset C_c^\infty$ such that $f_k\to f$ in $W_0^{1,p}$, i.e.,

$$\|f_k-f\|_p+\|\nabla f_k-\nabla f\|_p\to 0.$$

Without loss of generality, we can assume that $f_k\in C_c^\infty\cap\mathscr{D}(K)$ for all k. To see this, from the regularization technique (see, e.g., [33]), one can choose a cut off function $\kappa\in C^\infty$, such that, $0\le\kappa\le 1$ on $\mathbb{R}^n$, $\kappa=1$ on $\mathbb{R}^n\setminus U$, and $\kappa=0$ in a neighborhood (contained in U) of K. Let

$$g_k=1-(1-f_k)\kappa.$$

Clearly, $g_k\in C_c^\infty$, such that, $g_k=1$ in a neighborhood (contained in U) of K and $g_k=f_k$ on $\mathbb{R}^n\setminus U$. This implies $g_k\in C_c^\infty\cap\mathscr{D}(K)$ for all k. Moreover, $\|g_k-f\|_{1,p}\to 0$ and hence

$$\|g_k-f\|_p\to 0 \quad\text{and}\quad \|\nabla g_k-\nabla f\|_p\to 0.$$

Let $f_k \in C_c^\infty \cap \mathscr{D}(K)$ be such that $f_k \to f$ in $W_0^{1,p}$. It can be checked that, for any $u \in S^{n-1}$,

$$|\nabla_u^+ f_k - \nabla_u^+ f| \le |\nabla f_k - \nabla f| \quad \text{and} \quad |\nabla_u^- f_k - \nabla_u^- f| \le |\nabla f_k - \nabla f|.$$

This together with (1.1.20) yield, for any $\tau \in [-1,1]$ and for all $k \ge 1$,

$$\begin{aligned} \Big|\varphi_\tau(\nabla_u f_k) - \varphi_\tau(\nabla_u f)\Big| &= \Big|\Big(\frac{1+\tau}{2}\Big)^{1/p}\big[\nabla_u^+ f_k - \nabla_u^+ f\big] + \Big(\frac{1-\tau}{2}\Big)^{1/p}\big[\nabla_u^- f_k - \nabla_u^- f\big]\Big| \\ &\le \Big(\frac{1+\tau}{2}\Big)^{1/p}\Big|\nabla_u^+ f_k - \nabla_u^+ f\Big| + \Big(\frac{1-\tau}{2}\Big)^{1/p}\Big|\nabla_u^- f_k - \nabla_u^- f\Big| \\ &\le C(p,\tau)\cdot\big|\nabla f_k - \nabla f\big|, \end{aligned}$$

where we have let $C(p,\tau)$ be the constant

$$C(p,\tau) = \Big(\frac{1+\tau}{2}\Big)^{1/p} + \Big(\frac{1-\tau}{2}\Big)^{1/p}.$$

It follows from the triangle inequality that, for any $u \in S^{n-1}$, for any $\tau \in [-1,1]$ and for any $p \in [1,n)$,

$$\begin{aligned} \Big|\|\varphi_\tau(\nabla_u f_k)\|_p - \|\varphi_\tau(\nabla_u f)\|_p\Big| &\le \|\varphi_\tau(\nabla_u f_k) - \varphi_\tau(\nabla_u f)\|_p \\ &\le C(p,\tau)\cdot\|\nabla f_k - \nabla f\|_p. \end{aligned}$$

Consequently, for any $u \in S^{n-1}$, for any $\tau \in [-1,1]$ and for any $p \in [1,n)$, one has

$$\lim_{k\to\infty} \|\varphi_\tau(\nabla_u f_k)\|_p = \|\varphi_\tau(\nabla_u f)\|_p.$$

By Fatou's lemma, one has

$$\begin{aligned} \mathscr{H}_{p,\tau}(f) &= \Big(\int_{S^{n-1}} \|\varphi_\tau(\nabla_u f)\|_p^{-n}\,du\Big)^{-\frac{p}{n}} \\ &= \Big(\int_{S^{n-1}} \lim_{k\to\infty} \|\varphi_\tau(\nabla_u f_k)\|_p^{-n}\,du\Big)^{-\frac{p}{n}} \\ &\ge \Big(\liminf_{k\to\infty}\int_{S^{n-1}} \|\varphi_\tau(\nabla_u f_k)\|_p^{-n}\,du\Big)^{-\frac{p}{n}} \\ &= \limsup_{k\to\infty}\Big(\int_{S^{n-1}} \|\varphi_\tau(\nabla_u f_k)\|_p^{-n}\,du\Big)^{-\frac{p}{n}} \\ &= \limsup_{k\to\infty} \mathscr{H}_{p,\tau}(f) \\ &\ge \inf_{g\in C_c^\infty\cap\mathscr{D}(K)} \mathscr{H}_{p,\tau}(g). \end{aligned} \tag{4.1.3}$$

It follows from Theorem 4.1.1 that, by taking the infimum over $f \in \mathscr{F}(K)$,

$$C_{p,\tau}(K) \geq \inf_{C_c^\infty \cap \mathscr{D}(K)} \mathscr{H}_{p,\tau}(f).$$

It is easily checked that, due to $C_c^\infty \subset W_0^{1,p}$,

$$C_{p,\tau}(K) \leq \inf_{C_c^\infty \cap \mathscr{D}(K)} \mathscr{H}_{p,\tau}(f),$$

and hence equality holds, as desired.

The desired formula (4.1.2) follows, due to $\mathscr{F}(K) \subset \mathscr{E}(K) \subset \mathscr{D}(K)$, once the following inequality is proved:

$$\inf_{f \in C_c^\infty \cap \mathscr{F}(K)} \mathscr{H}_{p,\tau}(f) \leq \inf_{f \in C_c^\infty \cap \mathscr{D}(K)} \mathscr{H}_{p,\tau}(f) = C_{p,\tau}(K).$$

This inequality follows along the same lines as the proof of Theorem 4.1.1. In fact, for any $\varepsilon > 0$, let $f_\varepsilon \in \mathscr{D}(K) \cap C_c^\infty$ satisfy that

$$C_{p,\tau}(K) + \varepsilon \geq \mathscr{H}_{p,\tau}(f_\varepsilon).$$

Let $\phi_i \in C_c^\infty(\mathbb{R})$ be as in Theorem 4.1.1. Then, $\phi_i(f_\varepsilon) \in \mathscr{F}(K) \cap C_c^\infty$ and

$$\inf_{f \in \mathscr{F}(K) \cap C_c^\infty} \mathscr{H}_{p,\tau}(f) \leq (1 + i^{-1})^p \cdot (C_{p,\tau}(K) + \varepsilon).$$

Taking $i \to \infty$ first and then letting $\varepsilon \to 0$, one gets

$$\inf_{f \in \mathscr{F}(K) \cap C_c^\infty} \mathscr{H}_{p,\tau}(f) \leq C_{p,\tau}(K)$$

as desired. $\square$

It follows from (1.1.21) and $\nabla_y f = y \cdot \nabla f$ that, for all $\lambda > 0$ and $y \in \mathbb{R}^n \setminus \{o\}$,

$$\|\varphi_\tau(\nabla_{\lambda y} f)\|_p = \lambda \|\varphi_\tau(\nabla_y f)\|_p.$$

Moreover, for $p \in [1, n)$ and for any $y_1, y_2 \in \mathbb{R}^n \setminus \{o\}$, by the Minkowski's inequality, one has

$$\begin{aligned} \|\varphi_\tau(\nabla_{y_1+y_2} f)\|_p &\leq \|\varphi_\tau(\nabla_{y_1} f) + \varphi_\tau(\nabla_{y_2} f)\|_p \\ &\leq \|\varphi_\tau(\nabla_{y_1} f)\|_p + \|\varphi_\tau(\nabla_{y_2} f)\|_p. \end{aligned}$$

Hence, $\|\varphi_\tau(\nabla_y f)\|_p : \mathbb{R}^n \setminus \{o\} \to [0,\infty)$, as a function of $y \in \mathbb{R}^n \setminus \{o\}$, is sublinear. According to the proof of [62, Lemma 3.1] (or [30, Lemma 2]), if $f \in \mathscr{F}(K)$, then $\|\varphi_\tau(\nabla_u f)\|_p > 0$ and $\|\varphi_\tau(\nabla_y f)\|_p$ is the support function of a convex body in $\mathscr{K}_0$. Let $L_{f,\tau}$ be the convex body. An application of (1.1.3) yields (see also [62, (3.2)])

$$\begin{aligned}
\mathscr{H}_{p,\tau}(f) &= \left(\int_{S^{n-1}} \|\varphi_\tau(\nabla_u f)\|_p^{-n}\, du\right)^{-\frac{p}{n}} \\
&= \left(\int_{S^{n-1}} \left[h_{L_{f,\tau}}(u)\right]^{-n} du\right)^{-\frac{p}{n}} \\
&= \left(\frac{1}{n|B_2^n|}\int_{S^{n-1}} \left[\rho_{L_{f,\tau}^\circ}(u)\right]^{n} d\sigma(u)\right)^{-\frac{p}{n}} \\
&= \left(\frac{|L_{f,\tau}^\circ|}{|B_2^n|}\right)^{-\frac{p}{n}}.
\end{aligned}$$

Taking the infimum over $f \in \mathscr{F}(K)$, Theorem 4.1.1 implies that for any compact set $K \subset \mathbb{R}^n$, for any $\tau \in [-1,1]$ and for any $p \in [1,n)$,

$$C_{p,\tau}(K) = \inf_{f\in\mathscr{F}(K)} \mathscr{H}_{p,\tau}(f) = \inf_{f\in\mathscr{F}(K)} \left(\frac{|L_{f,\tau}^\circ|}{|B_2^n|}\right)^{-\frac{p}{n}}.$$

This provides a connection of the general p-affine capacity with the volume of convex bodies.

The general p-affine capacity of a general bounded measurable set $E \subset \mathbb{R}^n$ can be defined as well. In fact, for $O \subset \mathbb{R}^n$ a bounded open set,

$$C_{p,\tau}(O) = \sup\Big\{C_{p,\tau}(K) : \quad K \subset O \text{ and } K \text{ is compact}\Big\}.$$

Then the general p-affine capacity of a bounded measurable set $E \subset \mathbb{R}^n$ is formulated by

$$C_{p,\tau}(E) = \inf\Big\{C_{p,\tau}(O) : \quad E \subset O \text{ and } O \text{ is open}\Big\}.$$

In later context of this chapter, we only concentrate on the general p-affine capacity for compact sets. We would like to mention that many properties proved in Chapter 4.2, such as, monotonicity, affine invariance and homogeneity etc, for compact sets could work for general sets too.

4.2 Properties of the general p-affine capacity

This section aims to establish basic properties for the general p-affine capacity, such as, monotonicity, affine invariance, translation invariance, homogeneity and the continuity from above.

The following result provides the properties of $C_{p,\tau}(\cdot)$ as a function of $\tau \in [-1,1]$.

Corollary 4.2.1. *Let $p \in [1,n)$ and K be a compact set in $\mathbb{R}^n$. The following properties hold.*

i) For any $\tau \in [-1,1]$, one has

$$C_{p,\tau}(K) = C_{p,-\tau}(K).$$

ii) For any $\lambda \in [0,1]$ and for any $\tau, \gamma \in [-1,1]$, one has

$$C_{p,\lambda\tau+(1-\lambda)\gamma}(K) \geq \lambda \cdot C_{p,\tau}(K) + (1-\lambda) \cdot C_{p,\gamma}(K).$$

Proof. i) Let $v = -u$. Then for any $x \in \mathbb{R}^n$, one has

$$\nabla_u^+ f(x) = \nabla_v^- f(x) \quad \text{and} \quad \nabla_u^- f(x) = \nabla_v^+ f(x).$$

This leads to, as $du = dv$, for any $f \in \mathscr{E}(K)$,

$$\begin{aligned}
\mathscr{H}_{p,\tau}(f) &= \left(\int_{S^{n-1}} \|\varphi_\tau(\nabla_u f)\|_p^{-n} \, du \right)^{-\frac{p}{n}} \\
&= \left(\int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \left[\left(\frac{1+\tau}{2} \right) (\nabla_u^+ f(x))^p + \left(\frac{1-\tau}{2} \right) (\nabla_u^- f(x))^p \right] dx \right)^{-\frac{n}{p}} du \right)^{-\frac{p}{n}} \\
&= \left(\int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \left[\left(\frac{1+\tau}{2} \right) (\nabla_v^- f(x))^p + \left(\frac{1-\tau}{2} \right) (\nabla_v^+ f(x))^p \right] dx \right)^{-\frac{n}{p}} dv \right)^{-\frac{p}{n}} \\
&= \mathscr{H}_{p,-\tau}(f).
\end{aligned}$$

It follows from Definition 4.1.1 that, for any $\tau \in [-1,1]$, for any $p \in [1,n)$ and for any compact set $K \subset \mathbb{R}^n$,

$$C_{p,\tau}(K) = C_{p,-\tau}(K).$$

ii) For any $\lambda \in [0,1]$ and for any $\tau, \gamma \in [-1,1]$, it follows from (1.2.32) that, for any $t \in \mathbb{R}$,

$$\big[\varphi_{\lambda\tau+(1-\lambda)\gamma}(t)\big]^p = \lambda\big[\varphi_\tau(t)\big]^p + (1-\lambda)\big[\varphi_\gamma(t)\big]^p,$$

which implies

$$\int_{\mathbb{R}^n} \big[\varphi_{\lambda\tau+(1-\lambda)\gamma}(\nabla_u f(x))\big]^p\, dx = \lambda \int_{\mathbb{R}^n} \big[\varphi_\tau(\nabla_u f(x))\big]^p\, dx + (1-\lambda) \int_{\mathbb{R}^n} \big[\varphi_\gamma(\nabla_u f(x))\big]^p\, dx.$$

According to the proof of [62, Lemma 3.1] (or [30, Lemma 2]), $\|\varphi_\tau(\nabla_u f)\|_p > 0$ if $f \in \mathscr{F}(K)$. The reverse Minkowski inequality yields that for any $\lambda \in [0,1]$ and for any $\tau, \gamma \in [-1,1]$,

$$\begin{aligned} \lambda \left(\int_{S^{n-1}} \|\varphi_\tau(\nabla_u f)\|_p^{-n} du\right)^{-\frac{p}{n}} &+ (1-\lambda)\left(\int_{S^{n-1}} \|\varphi_\gamma(\nabla_u f)\|_p^{-n}\, du\right)^{-\frac{p}{n}} \\ &\le \left(\int_{S^{n-1}} \|\varphi_{\lambda\tau+(1-\lambda)\gamma}(\nabla_u f)\|_p^{-n} du\right)^{-\frac{p}{n}}. \end{aligned}$$

Taking the infimum over $f \in \mathscr{F}(K)$, by Theorem 4.1.1,

$$C_{p,\lambda\tau+(1-\lambda)\gamma}(K) \ge \lambda \cdot C_{p,\tau}(K) + (1-\lambda)\cdot C_{p,\gamma}(K)$$

holds for any $\lambda \in [0,1]$ and for any $\tau, \gamma \in [-1,1]$. $\square$

From Corollary 4.2.1, one sees that, for any $p \in [1,n)$ and for any compact set $K \subset \mathbb{R}^n$, $C_{p,\tau}(K) \le C_{p,\gamma}(K)$ holds if $-1 \le \tau < \gamma \le 0$, and $C_{p,\gamma}(K) \le C_{p,\tau}(K)$ holds if $0 \le \tau < \gamma \le 1$. In particular, for any $\tau \in [-1,1]$, one has

$$C_{p,+}(K) = C_{p,-}(K) \le C_{p,\tau}(K) = C_{p,-\tau}(K) \le C_{p,0}(K).$$

Given two compact sets $K \subset L$, one sees $\mathscr{E}(L) \subset \mathscr{E}(K)$ and hence the general p-affine capacity is monotone by Definition 4.1.1, namely,

$$C_{p,\tau}(K) \le C_{p,\tau}(L). \tag{4.2.4}$$

The general p-affine capacity is also translation invariant. To see this, let $a \in \mathbb{R}^n$ and consider the function $g(x) = f(x+a)$ for any $x \in \mathbb{R}^n$. It is easily checked that $f \in \mathscr{E}(K+a)$ if and only if $g \in \mathscr{E}(K)$. Moreover, $\nabla g(x) = \nabla f(x+a)$, and thus

$\mathscr{H}_{p,\tau}(g) = \mathscr{H}_{p,\tau}(f)$. Taking the infimum over $g \in \mathscr{E}(K)$ from both sides, by Definition 4.1.1, for any $a \in \mathbb{R}^n$ and for any compact set $K \subset \mathbb{R}^n$,

$$C_{p,\tau}(K + a) = C_{p,\tau}(K).$$

An interesting (and common for many capacities) fact for the general p-affine capacity is that

$$C_{p,\tau}(K) = C_{p,\tau}(\partial K)$$

for any compact set $K \subset \mathbb{R}^n$. To see this, let $\varepsilon > 0$ be given. There exists $f_\varepsilon \in \mathscr{E}(\partial K)$ such that

$$C_{p,\tau}(\partial K) + \varepsilon \geq \mathscr{H}_{p,\tau}(f_\varepsilon).$$

Let $g = \max\{f_\varepsilon, 1\}$ on K and $g = f_\varepsilon$ on $\mathbb{R}^n \setminus K$. It can be checked, along the manner same as the proof of Theorem 4.1.2, that $g \in \mathscr{E}(K)$ and

$$\int_{\mathbb{R}^n} \left[\varphi_\tau(\nabla_u g)\right]^p \, dx \leq \int_{\mathbb{R}^n} \left[\varphi_\tau(\nabla_u f_\varepsilon)\right]^p \, dx.$$

Consequently, due to Definition 4.1.1,

$$C_{p,\tau}(K) \leq \mathscr{H}_{p,\tau}(g) \leq \mathscr{H}_{p,\tau}(f_\varepsilon) < C_{p,\tau}(\partial K) + \varepsilon.$$

Letting $\varepsilon \to 0$, one gets

$$C_{p,\tau}(K) \leq C_{p,\tau}(\partial K).$$

The monotonicity of the general p-affine capacity yields that

$$C_{p,\tau}(\partial K) \leq C_{p,\tau}(K)$$

and hence $C_{p,\tau}(\partial K) = C_{p,\tau}(K)$ holds for all compact set $K \subset \mathbb{R}^n$.

Let $GL(n)$ be the group of all invertible linear transforms defined on $\mathbb{R}^n$. For $T \in GL(n)$, denote by T^t and $\det(T)$ the transpose of T and the determinant of T, respectively. The affine invariance of the general p-affine capacity is stated in the following theorem.

Theorem 4.2.1. *The general p-affine capacity has the affine invariance and homogeneity: for any $T \in GL(n)$ and for any compact set $K \subset \mathbb{R}^n$,*

$$C_{p,\tau}(TK) = |\det(T)|^{\frac{n-p}{n}} C_{p,\tau}(K).$$

In particular, the general p-affine capacity is affine invariant: for any $T \in GL(n)$ with $|\det(T)| = 1$,

$$C_{p,\tau}(TK) = C_{p,\tau}(K).$$

Moreover, the general p-affine capacity has positive homogeneity of degree $n-p$, i.e.,

$$C_{p,\tau}(\lambda K) = \lambda^{n-p} C_{p,\tau}(K)$$

for all $\lambda > 0$, where $\lambda K = \{\lambda x : x \in K\}$.

Proof. For $T \in GL(n)$ and $f \in \mathscr{E}(TK)$, one has $g = f \circ T \in \mathscr{E}(K)$. For simplicity, assume that $|\det(T)| = 1$. Thus, by $x = Ty$,

$$\int_{\mathbb{R}^n} \left[\varphi_\tau(\nabla_u g(y))\right]^p dy = \int_{\mathbb{R}^n} \left[\varphi_\tau(\nabla_u (f \circ T)(y))\right]^p dy = \int_{\mathbb{R}^n} \left[\varphi_\tau(\nabla_{Tu}(f(x))\right]^p dx,$$

where the second equality follows from the chain rule

$$\nabla g(y) = \nabla(f \circ T)(y) = T^t \nabla f(Ty).$$

By letting $v = Tu/|Tu|$, it follows from (1.1.21) that

$$\begin{aligned}
\int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \left[\varphi_\tau(\nabla_u g(y))\right]^p dy\right)^{-\frac{n}{p}} du &= \int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \left[\varphi_\tau(\nabla_{Tu}(f(x))\right]^p dx\right)^{-\frac{n}{p}} du \\
&= \int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \left[\varphi_\tau(\nabla_v(f(x))\right]^p dx\right)^{-\frac{n}{p}} |Tu|^{-n} du \\
&= \int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \left[\varphi_\tau(\nabla_v(f(x))\right]^p dx\right)^{-\frac{n}{p}} dv.
\end{aligned}$$

Consequently, $\mathscr{H}_{p,\tau}(g) = \mathscr{H}_{p,\tau}(f)$. Taking the infimum over $f \in \mathscr{E}(TK)$ from both sides, which is equivalent to taking the infimum over $g \in \mathscr{E}(K)$ from the left hand side, one gets the affine invariance: for all $T \in GL(n)$ with $|\det(T)| = 1$, then

$$C_{p,\tau}(TK) = C_{p,\tau}(K).$$

For the homogeneity, let $\lambda > 0$ be given. For any $f \in \mathscr{E}(\lambda K)$, one sees $g_\lambda \geq \mathbf{1}_K$ where $g_\lambda(x) = f(\lambda x)$ for all $x \in \mathbb{R}^n$. It is easily checked, by letting $y = \lambda x$, that

$$\int_{\mathbb{R}^n} \left[\varphi_\tau(\nabla_u g_\lambda(x))\right]^p dx = \lambda^{p-n} \int_{\mathbb{R}^n} \left[\varphi_\tau(\nabla_u f(y))\right]^p dy,$$

which further implies that $\mathscr{H}_{p,\tau}(f) = \lambda^{n-p}\mathscr{H}_{p,\tau}(g_\lambda)$. The desired formula $C_{p,\tau}(\lambda K) = \lambda^{n-p}C_{p,\tau}(K)$ follows immediately by Definition 4.1.1 and by taking the infimum over $f \in \mathscr{E}(\lambda K)$.

Finally, we consider $T \in GL(n)$ be an invertible linear transform. Then

$$\widetilde{T} = |\det(T)|^{-1/n}T$$

has $|\det(\widetilde{T})| = 1$. Hence, the affine invariance and the homogeneity yield that, for all $T \in GL(n)$,

$$C_{p,\tau}(TK) = C_{p,\tau}(|\det(T)|^{1/n}\widetilde{T}K) = |\det(T)|^{\frac{n-p}{n}}C_{p,\tau}(\widetilde{T}K) = |\det(T)|^{\frac{n-p}{n}}C_{p,\tau}(K).$$

This concludes the proof. $\square$

The continuity from above for the general p-affine capacity is stated in the following theorem.

Theorem 4.2.2. *The general p-affine capacity is continuous from above: if $\{K_i\}_{i=1}^\infty$ is a decreasing sequence of compact sets, then*

$$C_{p,\tau}(\cap_{i=1}^\infty K_i) = \lim_{i\to\infty} C_{p,\tau}(K_i). \tag{4.2.5}$$

Proof. Recall that the general p-affine capacity of the compact set K_1 is finite. It follows from the monotonicity that, for all i,

$$C_{p,\tau}(K_{i+1}) \le C_{p,\tau}(K_i) \le C_{p,\tau}(K_1) < \infty,$$

and hence $\lim_{i\to\infty} C_{p,\tau}(K_i)$ exists and is finite. Moreover, the monotonicity of the general p-affine capacity also yields

$$C_{p,\tau}(\cap_{i=1}^\infty K_i) \le \lim_{i\to\infty} C_{p,\tau}(K_i).$$

The desired formula (4.2.5) follows if we prove the following inequality:

$$C_{p,\tau}(\cap_{i=1}^\infty K_i) \ge \lim_{i\to\infty} C_{p,\tau}(K_i).$$

First of all, the set $\cap_{i=1}^{\infty} K_i$ is clearly compact. By Definition 4.1.1 and Theorem 4.1.3, for any $\varepsilon > 0$, one can find a smooth function $f_\varepsilon \in \mathscr{E}(\cap_{i=1}^{\infty} K_i)$, such that, $f_\varepsilon \geq \mathbf{1}_{\cap_{i=1}^{\infty} K_i}$ and

$$C_{p,\tau}\big(\cap_{i=1}^{\infty} K_i\big) + \varepsilon \geq \mathscr{H}_{p,\tau}(f_\varepsilon).$$

Let $K_\varepsilon = \{x \in \mathbb{R}^n : f_\varepsilon(x) \geq 1-\varepsilon\}$. Then, $\frac{f_\varepsilon}{1-\varepsilon} \in \mathscr{E}(K_\varepsilon)$ and $K_i \subset K_\varepsilon$ for i big enough. Together with (1.1.21), Definition 4.1.1 and the monotonicity of the general p-affine capacity, one has

$$\lim_{i\to\infty} C_{p,\tau}(K_i) \leq C_{p,\tau}(K_\varepsilon) \leq (1-\varepsilon)^{-p}\mathscr{H}_{p,\tau}(f_\varepsilon) \leq \frac{C_{p,\tau}(\cap_{i=1}^{\infty} K_i) + \varepsilon}{(1-\varepsilon)^p}.$$

Taking $\varepsilon \to 0$, one gets the desired inequality

$$\lim_{i\to\infty} C_{p,\tau}(K_i) \leq C_{p,\tau}(\cap_{i=1}^{\infty} K_i)$$

and this concludes the proof. $\square$

Note that one cannot expect to have the subadditivity for the general p-affine capacity, even for $\tau = 0$; see [85] for the details. It is not clear whether the general p-affine capacity has the continuity from below.

4.3 Sharp geometric inequalities for the general p-affine capacity

This section aims to establish several sharp geometric inequalities for the general p-affine capacity. In particular, the general p-affine capacity is compared with the p-variational capacity, the general p-integral affine surface areas and the volume.

4.3.1 Comparison with the p-variational capacity

This subsection aims to compare the general p-affine capacity and the p-variational capacity. Recall that for $p \in [1, n)$ and a compact set $K \subset \mathbb{R}^n$, the p-variational capacity of K, denoted by $C_p(K)$, is formulated by

$$C_p(K) \quad = \inf_{f\in\mathscr{D}(K)} \int_{\mathbb{R}^n} |\nabla f|^p\, dx = \inf_{f\in\mathscr{D}(K)\cap C_c^\infty} \int_{\mathbb{R}^n} |\nabla f|^p\, dx.$$

Of course, the set $\mathscr{D}(K)$ in the above definition for the p-variational capacity could be replaced by $\mathscr{E}(K)$ and $\mathscr{F}(K)$ (see e.g., [17, 58]). The p-variational capacity is fundamental in many areas, such as, analysis, geometry and physics. It has many properties similar to those for the general p-affine capacity, such as, homogeneity, monotonicity; however the p-variational capacity does not have the affine invariance.

The comparison between the general p-affine capacity and the p-variational capacity is stated in the following theorem. The case $\tau = 0$ was discussed in [83, Remark 2.7] and [82, Theorem 1.5']. Let $A(n,p)$ be the constant given in (1.1.22).

Theorem 4.3.1. *Let $p \in [1,n)$ and $K \subset \mathbb{R}^n$ be a compact set. For any $\tau \in [-1,1]$, one has*

$$C_{p,\tau}(K) \leq A(n,p) \cdot C_p(K).$$

Proof. According to the proof of [62, Lemma 3.1] (or [30, Lemma 2]), $\|\varphi_\tau(\nabla_u f)\|_p > 0$ for any $f \in \mathscr{F}(K) \cap C_c^\infty$, for any $\tau \in [-1,1]$ and for any $u \in S^{n-1}$. By Jensen's inequality, Fubini's theorem, (1.1.21) and (1.1.22), one has, for any $f \in \mathscr{F}(K) \cap C_c^\infty$,

$$\begin{aligned}
\mathscr{H}_{p,\tau}(f) &= \left(\int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \left[\varphi_\tau(\nabla_u f)\right]^p dx\right)^{-\frac{n}{p}} du\right)^{-\frac{p}{n}} \\
&\leq \int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \left[\varphi_\tau(\nabla_u f)\right]^p dx\right) du \\
&= \int_{\mathbb{R}^n} \left(\int_{S^{n-1}} \left[\varphi_\tau(\nabla_u f)\right]^p du\right) dx \\
&= \left(\int_{S^{n-1}} \left[\varphi_\tau(u\cdot v)\right]^p du\right) \cdot \left(\int_{\mathbb{R}^n} |\nabla f|^p \, dx\right) \\
&= A(n,p) \cdot \int_{\mathbb{R}^n} |\nabla f|^p \, dx,
\end{aligned}$$

where $v \in S^{n-1}$ (depending on $x \in \mathbb{R}^n$) is given by

$$v = \frac{\nabla f(x)}{|\nabla f(x)|} \quad \text{on} \quad \{x \in \mathbb{R}^n : \nabla f \neq 0\}.$$

Taking the infimum over $f \in \mathscr{F}(K) \cap C_c^\infty$, one has, by Theorem 4.1.3 and the definition of the p-variational capacity,

$$\begin{aligned} C_{p,\tau}(K) &= \inf_{f \in \mathscr{F}(K) \cap C_c^\infty} \mathscr{H}_{p,\tau}(f) \\ &\le A(n,p) \cdot \inf_{f \in \mathscr{F}(K) \cap C_c^\infty} \int_{\mathbb{R}^n} |\nabla f|^p \, dx \\ &= A(n,p) \cdot C_p(K) \end{aligned}$$

holds for any $\tau \in [-1,1]$, for any $p \in [1,n)$ and for any compact set $K \subset \mathbb{R}^n$. $\square$

It is well known (see e.g., [58, (2.2.13) and (2.2.14)]) that

$$C_p(B_2^n) = n\omega_n \cdot \left(\frac{n-p}{p-1} \right)^{p-1} \tag{4.3.6}$$

for $p \in (1,n)$, $C_p(B_2^n) = 0$ for $p \ge n$, and $C_1(B_2^n) = \lim_{p \to 1^+} C_p(B_2^n) = n\omega_n$. Hence, for any $\tau \in [-1,1]$,

$$C_{p,\tau}(B_2^n) \le A(n,p) C_p(B_2^n) = A(n,p) \cdot n\omega_n \cdot \left(\frac{n-p}{p-1} \right)^{p-1} \tag{4.3.7}$$

holds for any $p \in (1,n)$, and

$$C_{1,\tau}(B_2^n) \le A(n,1) C_1(B_2^n) = A(n,1) \cdot n\omega_n. \tag{4.3.8}$$

Following along the same lines as the proof of Theorem 4.3.1, one has, for any $\tau \in [-1,1]$ and for any $p \ge n$,

$$0 \le C_{p,\tau}(B_2^n) \le A(n,p) C_p(B_2^n) = 0.$$

Again due to the proofs of (4.2.4) and Theorem 4.2.1, $C_{p,\tau}(K) = 0$ for any $\tau \in [-1,1]$, for any $p \ge n$ and for any compact set $K \subset \mathbb{R}^n$.

4.3.2 Affine isocapacitary inequalities

This subsection dedicates to establish the affine isocapacitary inequality which compares the general p-affine capacity with the volume. An ellipsoid is a convex body of form $TB_2^n + x_0$ for some $T \in GL(n)$ and $x_0 \in \mathbb{R}^n$.

Theorem 4.3.2. *Let $p \in [1, n)$. For any $\tau \in [-1, 1]$ and for any compact set $K \subset \mathbb{R}^n$, one has*

$$\left(\frac{C_{p,\tau}(K)}{C_{p,\tau}(B_2^n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{1}{n}}$$

with equality if K is an ellipsoid.

Proof. Let $p \in (1, n)$, $\tau \in [-1, 1]$ and $K \subset \mathbb{R}^n$ be a compact set. It follows from [30, inequality (5.8)] that for $f \in C_c^\infty \cap \mathscr{F}(K)$, $\|f\|_\infty = 1$ and

$$\left(\int_{S^{n-1}} \|\nabla_u^+ f\|_p^{-n}\, du\right)^{-\frac{p}{n}} \geq n^p \omega_n^{\frac{p}{n}} A(n,p) \int_0^1 \frac{|[f]_t|^{\frac{np-p}{n}}}{[-|[f]_t|']^{p-1}}\, dt,$$

where $|[f]_t|'$ is the derivative of $|[f]_t|$ with respect to t. Recall that for any real number $t > 0$ and for any $f \in C_c^\infty$,

$$[f]_t = \{x \in \mathbb{R}^n : |f(x)| \geq t\}.$$

Note that $|K| \leq |[f]_1| \leq |[f]_0|$. Together with Jensen's inequality, one has, for $p \in (1, n)$,

$$\begin{aligned}
\int_0^1 \frac{|[f]_t|^{\frac{np-p}{n}}}{\big[-|[f]_t|'\big]^{p-1}}\, dt &\geq \left(\int_0^1 |[f]_t|^{\frac{np-p}{n-np}} (-d|[f]_t|)\right)^{1-p} \\
&= \left(\frac{np-n}{n-p} \cdot |[f]_t|^{\frac{n-p}{n-np}}\Big|_0^1\right)^{1-p} \\
&\geq \left(\frac{np-n}{n-p}\right)^{1-p} |[f]_1|^{\frac{n-p}{n}} \\
&\geq \left(\frac{np-n}{n-p}\right)^{1-p} |K|^{\frac{n-p}{n}}.
\end{aligned}$$

Together with (4.3.6), Theorem 4.1.3 and Corollary 4.2.1, for any $p \in (1, n)$ and for any $\tau \in [-1, 1]$,

$$\begin{aligned}
C_{p,\tau}(K) &\geq C_{p,+}(K) \\
&= \inf_{f \in \mathscr{F}(K) \cap C_c^\infty} \left(\int_{S^{n-1}} \|\nabla_u^+ f\|_p^{-n}\, du\right)^{-\frac{p}{n}} \\
&\geq n\omega_n^{\frac{p}{n}} \cdot A(n,p) \cdot \left(\frac{n-p}{p-1}\right)^{p-1} |K|^{\frac{n-p}{n}} \\
&= A(n,p) \cdot C_p(B_2^n) \cdot \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n}}. \qquad (4.3.9)
\end{aligned}$$

Let $K = B_2^n$ in inequality (4.3.9). Then, for any $p \in (1, n)$ and for any $\tau \in [-1, 1]$,

$$C_{p,\tau}(B_2^n) \geq A(n,p) \cdot C_p(B_2^n).$$

Together with (4.3.7), one gets, for any $p \in (1, n)$ and for any $\tau \in [-1, 1]$,

$$C_{p,\tau}(B_2^n) = A(n,p) \cdot C_p(B_2^n) = A(n,p) \cdot n\omega_n \cdot \left(\frac{n-p}{p-1}\right)^{p-1}. \qquad (4.3.10)$$

Hence, inequality (4.3.9) can be rewritten as, for any $p \in (1, n)$, for any $\tau \in [-1, 1]$ and for any compact set $K \subset \mathbb{R}^n$,

$$\left(\frac{C_{p,\tau}(K)}{C_{p,\tau}(B_2^n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{1}{n}}.$$

Now let us consider the case $p = 1$. For $f \in C_c^\infty \cap \mathscr{F}(K)$, it can be checked, due to the dominated convergence theorem, that for any $u \in S^{n-1}$ and for any $\tau \in [-1, 1]$,

$$\lim_{p\to 1^+} \|\varphi_\tau(\nabla_u f)\|_p = \|\varphi_\tau(\nabla_u f)\|_1.$$

By Fatou's lemma, one has

$$\begin{aligned}
\left(\int_{S^{n-1}} \|\varphi_\tau(\nabla_u f)\|_1^{-n}\, du\right)^{-\frac{1}{n}} &= \left(\int_{S^{n-1}} \lim_{p\to 1^+} \|\varphi_\tau(\nabla_u f)\|_p^{-n}\, du\right)^{-\frac{1}{n}} \\
&\geq \left(\liminf_{p\to 1^+} \int_{S^{n-1}} \|\varphi_\tau(\nabla_u f)\|_p^{-n}\, du\right)^{-\frac{1}{n}} \\
&= \limsup_{p\to 1^+} \left(\int_{S^{n-1}} \|\varphi_\tau(\nabla_u f)\|_p^{-n}\, du\right)^{-\frac{p}{n}} \\
&\geq \limsup_{p\to 1^+} C_{p,\tau}(K).
\end{aligned}$$

It follows from Theorem 4.1.3, after taking the infimum over $f \in C_c^\infty \cap \mathscr{F}(K)$, that for any $\tau \in [-1, 1]$ and for any compact set $K \subset \mathbb{R}^n$,

$$C_{1,\tau}(K) \geq \limsup_{p\to 1^+} C_{p,\tau}(K).$$

In particular, by (4.3.8) and (4.3.10), one has

$$A(n,1) \cdot n\omega_n \geq C_{1,\tau}(B_2^n) \geq \limsup_{p\to 1^+} C_{p,\tau}(B_2^n) = A(n,1) \cdot n\omega_n.$$

This gives the precise value of $C_{1,\tau}(B_2^n)$:

$$C_{1,\tau}(B_2^n) = A(n,1) \cdot n\omega_n = \lim_{p \to 1^+} C_{p,\tau}(B_2^n), \tag{4.3.11}$$

and hence inequality (4.3.9) yields

$$\left(\frac{C_{1,\tau}(K)}{C_{1,\tau}(B_2^n)} \right)^{\frac{1}{n-1}} \geq \limsup_{p \to 1^+} \left(\frac{C_{p,\tau}(K)}{C_{p,\tau}(B_2^n)} \right)^{\frac{1}{n-p}} \geq \left(\frac{|K|}{|B_2^n|} \right)^{\frac{1}{n}}$$

for any $\tau \in [-1, 1]$ and for any compact set $K \subset \mathbb{R}^n$, as desired.

Due to the affine invariance and the translation invariance, it is trivial to see that equality holds if K is an ellipsoid. □

Theorem 4.3.2 asserts that the general p-affine capacity attains the minimum, among all compact sets with fixed volume, at ellipsoids. It also asserts that ellipsoids have the maximal volumes among all compact sets with fixed general p-affine capacity. When $\tau = 0$, one recovers the affine isocapacitary inequality for the p-affine capacity proved in [83, Theorem 3.2] and [82, Theorem 1.3']. Recall that the isocapacitary inequality for the p-variational capacity reads: for any $p \in [1, n)$ and any compact set $K \subset \mathbb{R}^n$,

$$\left(\frac{C_p(K)}{C_p(B_2^n)} \right)^{\frac{1}{n-p}} \geq \left(\frac{|K|}{|B_2^n|} \right)^{\frac{1}{n}}.$$

It follows from Theorem 4.3.1 that the affine isocapacitary inequality in Theorem 4.3.2 is stronger than the isocapacitary inequality for the p-variational capacity. That is, for any $p \in [1, n)$, for any $\tau \in [-1, 1]$ and for any compact set $K \subset \mathbb{R}^n$,

$$\left(\frac{C_p(K)}{C_p(B_2^n)} \right)^{\frac{1}{n-p}} \geq \left(\frac{C_{p,\tau}(K)}{C_{p,\tau}(B_2^n)} \right)^{\frac{1}{n-p}} \geq \left(\frac{|K|}{|B_2^n|} \right)^{\frac{1}{n}}.$$

Moreover, combining the above inequality with [47, (12)], when $K \subset \mathbb{R}^n$ is a Lipschitz star body with the origin in its interior, the following inequality holds: for any $p \in [1, n)$ and for any $\tau \in [-1, 1]$,

$$\left(\frac{S_p(K)}{S_p(B_2^n)} \right)^{\frac{1}{n-p}} \geq \left(\frac{C_p(K)}{C_p(B_2^n)} \right)^{\frac{1}{n-p}} \geq \left(\frac{C_{p,\tau}(K)}{C_{p,\tau}(B_2^n)} \right)^{\frac{1}{n-p}} \geq \left(\frac{|K|}{|B_2^n|} \right)^{\frac{1}{n}},$$

where $S_p(K)$ denotes the p-surface area of K given by formula (1.2.29).

4.3.3 Connection with the general p-integral affine surface area

In this subsection, we explore the relation between the general p-affine capacity and the general p-integral affine surface area. Throughout, denote by $\mathscr{L}_0$ the set of all Lipschitz star bodies (with respect to the origin o) containing o in their interiors. For a Lipschitz star body $K \in \mathscr{L}_0$, let $\nu_K(x)$ denote the unit outer normal vector of ∂K at x (sometimes may be abbreviated as $\nu(x)$). Let D_K, the core of K, be given by

$$D_K = \big\{tx : \ t > 0, \ x \in \partial K, \ |x \cdot \nu(x)| > 0\big\}.$$

According to [47, Lemma 5], for each Lipschitz star body $K \subset \mathbb{R}^n$, one has

$$\nu_K(x) = -\frac{\nabla \rho_K(x)}{|\nabla \rho_K(x)|} \quad \text{and} \quad \nabla \rho_K(x) = -\frac{\nu_K(x)}{x \cdot \nu_K(x)}$$

for almost all $x \in \partial K \cap D_K$.

For $p \geq 1$ and $\tau \in [-1, 1]$, define $\Pi_{p,\tau}(K)$, the general L_p projection body of $K \in \mathscr{L}_0$, to be the convex body with support function $h_{\Pi_{p,\tau}(K)}$; namely, for any $\theta \in S^{n-1}$,

$$h_{\Pi_{p,\tau}(K)}(\theta) = \bigg(\int_{\partial K} \big[\varphi_\tau(\theta \cdot \nu_K(x))\big]^p \cdot |x \cdot \nu_K(x)|^{1-p} \, d\mathscr{H}^{n-1}(x) \bigg)^{\frac{1}{p}}.$$

Note that $|x \cdot \nu_K(x)|^{-1} = |\nabla \rho_K(x)|$ is bounded on ∂K because $\rho_K(x)$ is Lipschitz continuous on ∂K, and hence $h_{\Pi_{p,\tau}(K)}$ is finite. The general L_1 projection body can be defined for more general sets in $\mathbb{R}^n$, such as compact domains (i.e., the closure of bounded open sets) with piecewise C^1 boundaries (or compact domains with finite perimeters). When $K \in \mathscr{K}_0$, formula (1.1.2) yields that, for any $\theta \in S^{n-1}$,

$$h_{\Pi_{p,\tau}(K)}(\theta) = \bigg(\int_{S^{n-1}} \big[\varphi_\tau(u \cdot \theta)\big]^p h_K(u)^{1-p} \, dS(K, u) \bigg)^{\frac{1}{p}}.$$

Denote by $v_{p,\tau}(K, \cdot) = h^p_{\Pi_{p,\tau}(K)}(\cdot)$ the general p-projection function of K. The general p-integral affine surface area of $K \in \mathscr{L}_0$ is defined by

$$\Phi_{p,\tau}(K) = \bigg(\int_{S^{n-1}} \big[v_{p,\tau}(K, u)\big]^{-\frac{n}{p}} \, du \bigg)^{-\frac{p}{n}} = \omega_n^{\frac{p}{n}} |\Pi^\circ_{p,\tau}(K)|^{-\frac{p}{n}}, \tag{4.3.12}$$

where du is the normalized spherical measure and $\Pi^\circ_{p,\tau}(K)$ is the polar body of $\Pi_{p,\tau}(K)$. When $\tau = 0$, one gets the p-integral affine surface area of $K \in \mathscr{L}_0$ in, e.g., [47, 95]. The case $\tau = 1$ defines the asymmetric p-integral affine surface area, denoted by $\Phi_{p,+}(K)$, of $K \in \mathscr{L}_0$. Similarly, one can also define $\Phi_{p,-}(K)$ if $\tau = -1$. When $K = B_2^n$, by (1.1.22), (4.3.10) and (4.3.11), for any $p \geq 1$ and for any $\tau \in [-1, 1]$,

$$\Phi_{p,\tau}(B_2^n) = \left(\frac{n-p}{p-1}\right)^{1-p} C_{p,\tau}(B_2^n). \tag{4.3.13}$$

It can be checked that for any $T \in GL(n)$,

$$\Phi_{p,\tau}(TK) = |\det T|^{\frac{n-p}{n}} \Phi_{p,\tau}(K).$$

Similar to the proof of Corollary 4.2.1, the following properties for the general p-integral affine surface area can be proved. One cannot expect that the general p-integral affine surface area has the translation invariance (unless $p = 1$, see following Proposition 4.3.1) and monotonicity.

Corollary 4.3.1. *Let $p \geq 1$ and $K \in \mathscr{L}_0$. The following statements hold:*

i) for any $\tau \in [-1, 1]$,

$$\Phi_{p,\tau}(K) = \Phi_{p,-\tau}(K);$$

ii) for any $\lambda \in [0, 1]$ and for any $\tau, \gamma \in [-1, 1]$,

$$\Phi_{p,\lambda\tau+(1-\lambda)\gamma}(K) \geq \lambda \cdot \Phi_{p,\tau}(K) + (1-\lambda) \cdot \Phi_{p,\gamma}(K);$$

iii) for any $\tau \in [-1, 1]$,

$$\Phi_{p,+}(K) = \Phi_{p,-}(K) \leq \Phi_{p,\tau}(K) \leq \Phi_{p,0}(K);$$

iv) if $-1 < \tau < \gamma \leq 0$, then

$$\Phi_{p,\tau}(K) \leq \Phi_{p,\gamma}(K)$$

and if $0 < \tau \leq \gamma < 1$, then

$$\Phi_{p,\gamma}(K) \leq \Phi_{p,\tau}(K).$$

By $\mathscr{C}_1$, we mean the set of all compact domains with piecewise C^1 boundaries. Again, for $M \in \mathscr{C}_1$, its outer unit normal vector is denoted by $\nu_M(x)$ for $x \in \partial M$. In the following proposition, we show that the general 1-affine capacity and the general 1-integral affine surface area are all equal to the 1-affine capacity (or equivalently, the 1-integral affine surface area) for any $M \in \mathscr{C}_1$.

Proposition 4.3.1. *Let $M \in \mathscr{C}_1$ be a compact domain with piecewise C^1 boundary. For any $\tau \in [-1, 1]$, one has*

$$C_{1,0}(M) = C_{1,\tau}(M) = \Phi_{1,\tau}(M) = \Phi_{1,0}(M).$$

Proof. We first prove $C_{1,0}(M) = C_{1,\tau}(M)$ for $M \in \mathscr{C}_1$; it follows immediately from Theorem 4.1.3 once $\|\varphi_\tau(\nabla_u f)\|_1 = \|\varphi_0(\nabla_u f)\|_1$ is established for any $f \in C_c^\infty \cap \mathscr{F}(M)$. To this end, for any $M_0 \in \mathscr{C}_1$ and for any $u \in S^{n-1}$,

$$\int_{\partial M_0} (u \cdot \nu_{M_0}(x))\, d\mathscr{H}^{n-1}(x) = 0 \quad \text{and} \quad \int_{\partial M_0} |u \cdot \nu_{M_0}(x)|\, d\mathscr{H}^{n-1}(x) > 0. \tag{4.3.14}$$

Note that (4.3.14) together with the Minkowski existence theorem leads to the powerful convexification technique, see e.g., [94, p.189-190]. For almost every $t \in (0, 1)$ with $f \in C_c^\infty \cap \mathscr{F}(M)$, it follows from the Sard's theorem, (1.1.19) and (4.3.14) that, for any $\tau \in [-1, 1]$,

$$\begin{aligned}
\|\varphi_\tau(\nabla_u f)\|_1 &= \int_{\mathbb{R}^n} \left(\frac{1}{2}|u \cdot \nabla f| + \frac{\tau}{2} u \cdot \nabla f\right) dx \\
&= \int_0^1 \int_{\partial[f]_t} \left(\frac{1}{2}|u \cdot \nu(x)| + \frac{\tau}{2} u \cdot \nu(x)\right) d\mathscr{H}^{n-1}(x)dt \\
&= \int_0^1 \int_{\partial[f]_t} \frac{|u \cdot \nu(x)|}{2}\, d\mathscr{H}^{n-1}(x)dt \\
&= \int_{\mathbb{R}^n} \frac{|u \cdot \nabla f|}{2}\, dx \\
&= \|\varphi_0(\nabla_u f)\|_1.
\end{aligned}$$

This concludes the proof of $C_{1,0}(M) = C_{1,\tau}(M)$ for $M \in \mathscr{C}_1$.

On the other hand, for any $\tau \in [-1,1]$,

$$\begin{aligned} v_{1,\tau}(M,\theta) &= \int_{\partial M} \varphi_\tau(\theta\cdot\nu_M(x))\, d\mathscr{H}^{n-1}(x) \\ &= \int_{\partial M} \left(\frac{|\theta\cdot\nu_M(x)|}{2} + \frac{\tau}{2}\big(\theta\cdot\nu_M(x)\big) \right) d\mathscr{H}^{n-1}(x) \\ &= \int_{\partial M} \frac{|\theta\cdot\nu_M(x)|}{2}\, d\mathscr{H}^{n-1}(x) \\ &= v_{0,\tau}(M,\theta), \end{aligned}$$

where the third equality follows again from (4.3.14). Consequently, for any $\tau \in [-1,1]$ and for any $M \in \mathscr{C}_1$,

$$\Phi_{1,\tau}(M) = \left(\int_{S^{n-1}} \big[v_{1,\tau}(K,u)\big]^{-n}\, du \right)^{-\frac{1}{n}} = \Phi_{1,0}(M).$$

Finally, let us prove that $C_{1,0}(M) = \Phi_{1,0}(M)$ holds for any $M \in \mathscr{C}_1$. For each function $f \in C_c^\infty \cap \mathscr{F}(M)$, it follows from (1.1.19), (1.1.21), and $M \subset [f]_t$ for any $t \in [0,1]$ that

$$\begin{aligned} \|\varphi_0(\nabla_u f)\|_1 &= \int_{\mathbb{R}^n} \varphi_0(\nabla_u f)\, dx \\ &= \frac{1}{2}\int_0^1 \int_{\partial[f]_t} |u\cdot\nu(x)|\ d\mathscr{H}^{n-1}(x)\ dt \\ &= \frac{1}{2}\int_0^1 \int_{\Pi_u[f]_t} \#([f]_t \cap (y+u\mathbb{R}))\, d\mathscr{H}^{n-1}(y)dt \\ &\geq \frac{1}{2}\int_0^1 \int_{\Pi_u M} \#(M \cap (y+u\mathbb{R}))\, d\mathscr{H}^{n-1}(y)dt \\ &= \frac{1}{2}\int_0^1 \int_{\partial M} |u\cdot\nu_M(x)|\ d\mathscr{H}^{n-1}(x)\ dt \\ &= v_{1,0}(M,u), \end{aligned}$$

where $\Pi_u K$ is the projection of $K \subset \mathbb{R}^n$ onto $u^\perp = \{x \in \mathbb{R}^n :\ x\cdot u = 0\}$ and $\#$ denotes the number of elements of a set (see e.g., [95]). Thus, for any $M \in \mathscr{C}_1$ and for any $f \in C_c^\infty \cap \mathscr{F}(M)$,

$$\left(\int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \varphi_0(\nabla_u f)\ dx \right)^{-n} du \right)^{-\frac{1}{n}} \geq \left(\int_{S^{n-1}} v_{1,0}(M,u)^{-n}\ du \right)^{-\frac{1}{n}} = \Phi_{1,0}(M).$$

Due to Theorem 4.1.3, by taking the infimum over $f \in C_c^\infty \cap \mathscr{F}(M)$, one gets, for any $M \in \mathscr{C}_1$,

$$C_{1,0}(M) \geq \Phi_{1,0}(M).$$

For the opposite direction, let $\varepsilon > 0$ be small enough and consider

$$f_\varepsilon(x) = \begin{cases} 0 & \text{if } \operatorname{dist}(x, K) \geq \varepsilon, \\ 1 - \frac{\operatorname{dist}(x,K)}{\varepsilon} & \text{if } \operatorname{dist}(x, K) < \varepsilon. \end{cases}$$

It has been proved in [94] that for any $u \in S^{n-1}$,

$$\lim_{\varepsilon \to 0} \|\varphi_0(\nabla_u f_\varepsilon)\|_1 = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \varphi_0(\nabla_u f_\varepsilon)\, dx = v_{1,0}(M, u).$$

Note that $f_\varepsilon \in \mathscr{F}(M)$ for any $\varepsilon > 0$ small enough. It follows from Theorem 4.1.1 that, for any $M \in \mathscr{C}_1$,

$$\begin{aligned}
C_{1,0}(M) &\leq \limsup_{\varepsilon \to 0} \left(\int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \varphi_0(\nabla_u f_\varepsilon)\ dx \right)^{-n} du \right)^{-\frac{1}{n}} \\
&= \left(\liminf_{\varepsilon \to 0} \int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \varphi_0(\nabla_u f_\varepsilon)\ dx \right)^{-n} du \right)^{-\frac{1}{n}} \\
&\leq \left(\int_{S^{n-1}} \lim_{\varepsilon \to 0} \left(\int_{\mathbb{R}^n} \varphi_0(\nabla_u f_\varepsilon)\ dx \right)^{-n} du \right)^{-\frac{1}{n}} \\
&= \left(\int_{S^{n-1}} \left(\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \varphi_0(\nabla_u f_\varepsilon)\ dx \right)^{-n} du \right)^{-\frac{1}{n}} \\
&= \left(\int_{S^{n-1}} (v_{1,0}(M, u))^{-n}\ du \right)^{-\frac{1}{n}} \\
&= \Phi_{1,0}(M),
\end{aligned}$$

where the second inequality is due to Fatou's lemma. This concludes the proof of

$$C_{1,0}(M) = \Phi_{1,0}(M)$$

for any $M \in \mathscr{C}_1$. $\square$

When M is an origin-symmetric convex body, the equality $C_{1,0}(M) = \Phi_{1,0}(M)$ was proved in [84, Theorem 2]; Proposition 4.3.1 extends it to all Lipschitz star bodies

$M \in \mathscr{L}_0$. The proof of $C_{1,0}(M) = C_{1,\tau}(M)$ basically relies on the smoothness (and the convexification) of $\partial[f]_t$ instead of the compact domain M itself; hence, the argument $C_{1,0}(M) = C_{1,\tau}(M)$ holds for any compact set $M \subset \mathbb{R}^n$ and for any $\tau \in [-1,1]$. The assumption $M \in \mathscr{C}_1$ is imposed here mainly in order to have $\Phi_{1,0}(M)$ well defined and finite. As commented in [95, p.247], the assumption $M \in \mathscr{C}_1$ could be relaxed to more general compact domains (such as compact domains with finite perimeters). Recall the affine isoperimetric inequality for the 1-integral affine surface area: for $M \in \mathscr{C}_1$,

$$\left(\frac{|M|}{|B_2^n|}\right)^{\frac{1}{n}} \le \left(\frac{\Phi_{1,0}(M)}{\Phi_{1,0}(B_2^n)}\right)^{\frac{1}{n-1}}$$

with equality if and only if M is an ellipsoid. Then Proposition 4.3.1 yields that for any $M \in \mathscr{C}_1$ and for any $\tau \in [-1,1]$,

$$\left(\frac{|M|}{|B_2^n|}\right)^{\frac{1}{n}} \le \left(\frac{\Phi_{1,\tau}(M)}{\Phi_{1,\tau}(B_2^n)}\right)^{\frac{1}{n-1}} = \left(\frac{C_{1,\tau}(M)}{C_{1,\tau}(B_2^n)}\right)^{\frac{1}{n-1}}$$

with equality if and only if M is an ellipsoid.

The following theorem compares the general p-affine capacity and the general p-integral affine surface area. We only concentrate on $p \in (1,n)$ as the case $p = 1$ has been discussed in Proposition 4.3.1. When $\tau = 0$ and K is an origin-symmetric convex body, it recovers [83, Theorem 3.5].

Theorem 4.3.3. *Let $K \in \mathscr{L}_0$ and $1 < p < n$. The following inequality*

$$\frac{C_{p,\tau}(K)}{C_{p,\tau}(B_2^n)} \le \frac{\Phi_{p,\tau}(K)}{\Phi_{p,\tau}(B_2^n)} \quad \text{for any} \quad \tau \in [-1,1]$$

holds with equality if K is an origin-symmetric ellipsoid.

Proof. Let $K \in \mathscr{L}_0$ and $p \in (1,n)$. Define the function g by: for $s > 0$,

$$g(s) = \min\Big\{1, \quad s^{\frac{n-p}{1-p}}\Big\}.$$

Let $f(x) = g\big(\frac{1}{\rho_K(x)}\big)$. Then $f(x) \ge \mathbf{1}_K$ and $\|f\|_\infty = 1$. From (1.1.18) and the fact that g is strictly decreasing on $s \in (1,\infty)$, it follows that, for all $t \in (0,1)$ with $t = g(s) = s^{\frac{n-p}{1-p}}$,

$$[f]_t = \{x \in \mathbb{R}^n : 1/\rho_K(x) \le s\}.$$

That is, $[f]_t = [f]_{g(s)} = sK$ for any $s > 1$. Together with [47, Lemma 6], for any $x \in \partial[f]_t$, there exists $z \in \partial K$ with $x = sz$ such that

$$|\nabla f(x)| = \frac{|g'(s)|}{|z \cdot \nu_K(z)|} \quad \text{and} \quad \nu_K(z) = \nu_{[f]_t}(x) = -\frac{\nabla f(x)}{|\nabla f(x)|}.$$

By (1.1.19), one has, for any $u \in S^{n-1}$,

$$\begin{aligned} \|\varphi_\tau(\nabla_u f)\|_p^p &= \int_0^1 \int_{\partial[f]_t} \left[\varphi_\tau(-u \cdot \nu_{[f]_t}(x))\right]^p \cdot |\nabla f(x)|^{p-1} \, d\mathscr{H}^{n-1}(x) \, dt \\ &= \int_1^\infty |g'(s)| \int_{\partial[f]_{g(s)}} \left[\varphi_\tau(-u \cdot \nu_{[f]_{g(s)}}(x))\right]^p \cdot |\nabla f(x)|^{p-1} \, d\mathscr{H}^{n-1}(x) \, ds \\ &= \int_1^\infty |g'(s)|^p s^{n-1} \int_{\partial K} \left[\varphi_\tau(-u \cdot \nu_K(z))\right]^p \cdot |z \cdot \nu_K(z)|^{1-p} \, d\mathscr{H}^{n-1}(z) \, ds \\ &= \left(\frac{n-p}{p-1}\right)^{p-1} \left(\int_{\partial K} \left[\varphi_\tau(-u \cdot \nu_K(z))\right]^p \cdot |z \cdot \nu_K(z)|^{1-p} \, d\mathscr{H}^{n-1}(z) \right) \\ &= \left(\frac{n-p}{p-1}\right)^{p-1} v_{p,\tau}(K, -u). \end{aligned}$$

It follows from (4.1.1) and (4.3.12) that

$$\begin{aligned} \mathscr{H}_{p,\tau}(f) &= \left(\int_{S^{n-1}} \|\varphi_\tau(\nabla_u f)\|_p^{-n} \, du \right)^{-\frac{p}{n}} \\ &= \left(\frac{n-p}{p-1}\right)^{p-1} \left(\int_{S^{n-1}} v_{p,\tau}(K, -u)^{-\frac{n}{p}} \, du \right)^{-\frac{p}{n}} \\ &= \left(\frac{n-p}{p-1}\right)^{p-1} \Phi_{p,\tau}(K). \end{aligned}$$

A standard limiting argument together with Definition 4.1.1 show that, for any $p \in (1, n)$, for any $\tau \in [-1, 1]$ and for any $K \in \mathscr{L}_0$,

$$C_{p,\tau}(K) \leq \left(\frac{n-p}{p-1}\right)^{p-1} \Phi_{p,\tau}(K).$$

By (4.3.13), the above inequality can be rewritten as

$$\frac{C_{p,\tau}(K)}{C_{p,\tau}(B_2^n)} \leq \frac{\Phi_{p,\tau}(K)}{\Phi_{p,\tau}(B_2^n)}.$$

Clearly equality holds in the above inequality if $K = B_2^n$. Due to the affine invariance of both $C_{p,\tau}(\cdot)$ and $\Phi_{p,\tau}(\cdot)$, equality holds in the above inequality if K is an origin-symmetric ellipsoid. $\square$

Together with [47, (13)], Corollary 4.3.1 and Theorem 4.3.2, for any $K \in \mathscr{L}_0$, for any $p \in (1, n)$ and for any $\tau \in [-1, 1]$, one has,

$$\left(\frac{S_p(K)}{S_p(B_2^n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{\Phi_{p,0}(K)}{\Phi_{p,0}(B_2^n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{\Phi_{p,\tau}(K)}{\Phi_{p,\tau}(B_2^n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{C_{p,\tau}(K)}{C_{p,\tau}(B_2^n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{1}{n}} \tag{4.3.15}$$

with equality if K is an origin-symmetric ellipsoid. Inequality (4.3.15) extends several known results in the literature. For example, inequality (4.3.15) strengthens the following (affine) isoperimetric inequality (see [47, inequality (13)]): for $\tau = 0$ and for any $K \in \mathscr{L}_0$,

$$\left(\frac{S_p(K)}{S_p(B_2^n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{\Phi_{p,0}(K)}{\Phi_{p,0}(B_2^n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{1}{n}}.$$

Moreover, inequality (4.3.15) holds for all $K \in \mathscr{K}_0 \subset \mathscr{L}_0$, and hence it extends the following affine isoperimetric inequality (4.3.16) for convex bodies to Lipschitz star bodies: for any $K \in \mathscr{K}_0$, for any $\tau \in [-1, 1]$ and for any $p \in (1, n)$,

$$\left(\frac{\Phi_{p,\tau}(K)}{\Phi_{p,\tau}(B_2^n)}\right)^{\frac{1}{n-p}} \geq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{1}{n}}, \tag{4.3.16}$$

which is an immediate consequence of the general L_p affine isoperimetric inequality for the general L_p projection body [29].

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