Extended symmetry analysis of isothermal no-slip drift flux model

by
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Abstract

We carry out extended symmetry analysis of the hydrodynamic-type system of differential equations modeling an isothermal no-slip drift flux. The maximal Lie invariance algebra of this system is proved to be infinite-dimensional. We also find its complete point symmetry group, including discrete symmetries, using the megaideal-based version of the algebraic method. Optimal lists of one- and two-dimensional subalgebras of the above algebra are constructed to obtain group-invariant solutions. Applying the generalized hodograph method and linearizing the essential subsystem, we represent the general solution of the system under study in terms of solutions of the Klein–Gordon equation. Amongst first-order generalized symmetries, we single out genuinely generalized ones and relate them to Lie symmetries of the essential subsystem. Moreover, we construct infinite families of recursion operators, conservation laws and Hamiltonian structures of the entire system.

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Introduction

In the 1870s Sophus Lie started developing a theory for integrating ordinary differential equations, which should have become a counterpart of Abel's theory of solvability of algebraic equations. Lie noticed that incoherent special methods for solving unrelated types of differential equations could be unified within the framework of the general procedure based on the invariance of differential equations under the action of continuous groups of symmetries, which nowadays are widely known as Lie groups. Loosely speaking, such a group is constituted by point transformations of independent and dependent variables, that map solutions of a system of differential equations to solutions of the same system. The invariance condition of systems of differential equations under the action of point transformations generally leads to a nonlinear system of partial differential equations, which is hard to solve. It is much simpler to work with the infinitesimal counterparts of the invariance condition, which is equivalent to a linear system of partial differential equations, specifying the associated Lie algebras of infinitesimal generators of one-parameter point symmetry groups. Subalgebras of the maximal Lie invariance algebra of a system of differential equations can be used to construct so-called group-invariant solutions. Although for some partial differential equations the closed expression of the general solution can be found, e.g., the linear first-order equation or the d'Alembert solution to the heat equation, we are frequently forced to be satisfied by particular, however physically important solutions. Besides, knowing the symmetry transformations of a differential equation one can generate its new solutions from known ones.

One can also generalize symmetries letting them to act non-geometrically, that is, allowing for the vector field components to depend on derivatives of dependent variables as well. The generalized vector field satisfying the invariance criterion for the system of differential equations is called a generalized symmetry of this system. The remarkable theorem by Emmy Noether states that every symmetry, geometric or generalized, of a Lagrangian induces a conservation law of the corresponding Euler–Lagrange equations, and in fact every conservation law appears in this manner. In turn, conservation laws are of interest in both physics and mathematics. The existence of infinite family of linearly independent generalized symmetries of the system of differential equations is regarded as its "complete integrability". This family is often found by computing recursion operators of the system that map symmetries of the system to symmetries of the same system.

A number of specific symmetry methods naturally arise for Hamiltonian systems of differential equations. For example, distinguished functions of a Hamiltonian operator are conservation laws of the corresponding Hamiltonian system of differential equations. Nonetheless, the infinite-dimensional Hamiltonian formalism for partial differential equations differs from the one known in classical mechanics. In the case of partial differential equations one needs to concentrate on the Poisson bracket instead of canonical coordinates. Given a system of differential equations one can define a differential operator, called the Fréchet derivative of a differential function, the kernel of which coincides with the set of characteristics of generalized evolutionary symmetries of the system. In turn, the kernel of the formally adjoint to the operator is the set of cosymmetries, including the characteristics of the conservation laws of the system (but not necessarily coincides). From the symmetry point of view, Hamiltonian operators map cosymmetries of the corresponding Hamiltonian systems to their generalized symmetries.

The work is based on the preprint paper [43]. In the thesis work we add the additional theoretical chapter and Sections 2.7 and 2.9–2.12. Its structure is as follows.

In the first chapter we give some theoretical foundations on Lie groups and Lie algebras (Section 1.1), variational calculus (Section 1.5) and Riemannian geometry (Section 1.7), following the classical textbooks [9, 28, 33, 34, 60], and present the theory of symmetry methods based on this ground. Closely following the specialized textbooks [6, 41, 42, 44] on the topic of the thesis, we describe an action of symmetry groups on differential equations (Section 1.2), an algebraic method of finding maximal point symmetry groups (Section 1.3), provide the theoretical foundation of group-invariant solutions (Section 1.4) and Hamiltonian systems of partial differential equations (Section 1.6). We omit proofs of assertions, but each of them in this chapter is provided by reference, where the proof can be found.

In the second chapter we apply the described symmetry methods to a specific hydrodynamic-type system $\mathcal{S}$ of differential equations, modeling an isothermal no-slip drift flux model. This chapter is organized as follows. Section 2.1 is devoted to the problem statement and deriving simplified forms of the system $\mathcal{S}$. In Section 2.2 we compute the maximal Lie invariance algebra $\mathfrak{g}$ of the system $\mathcal{S}$. The computation of the complete point symmetry group of this system is presented in Section 2.3. One- and two-dimensional subalgebras of the algebra $\mathfrak{g}$ are classified in Section 2.4. Section 2.5 contains results on symmetry reductions of the system $\mathcal{S}$, which are based on the optimal lists of inequivalent subalgebras. In Section 2.6 we study Lie symmetries of the essential subsystem $\mathcal{S}_0$ of the system $\mathcal{S}$ and linearize $\mathcal{S}_0$ by the hodograph transformation. Using this, in Section 2.7 we construct the general solution of the entire system $\mathcal{S}$. Alternatively, in Section 2.8 we employ the generalized hodograph transformation. Generalized symmetries, conservation laws and Hamiltonian structures of the system $\mathcal{S}$ are studied in Sections 2.9, 2.10 and 2.11, respectively. Section 2.12 is devoted to the computation of recursions operators.

In the final chapter we discuss results of the thesis 3.1 as well as outline possible directions of the future work on the study of isothermal no-slip drift flux model 3.2.

Co-authorship statement

The thesis work is based on the preprint [43]. The investigation of the isothermal no-slip flux model was proposed by Roman O. Popovych, the idea of expanding the research to generalized symmetries and Hamiltonian structures is due to A. Sergyeyev, all practical aspects of the research and manuscript preparation were carried out by the principal author, S. Opanasenko. A. Bihlo supervised the entire research and provided several corrections.

Chapter 1

Theoretical foundation

1.1 Lie groups of transformations and associated Lie algebras

Here we present the basic objects of differential geometry, which are widely used in the sequel. The main references in this section are [28, 34, 41].

The underlying notion is of manifold, which generalizes the notion of Euclidean space.

Definition 1.1. An m-dimensional differential manifold is a set M and a family of injective maps $\Phi_i \colon U_i \to M$ of open subsets $U_i \subset \mathbb{R}^m$ such that

1) $\bigcup\limits_i \Phi_i(U_i) = M$;

2) if $\Phi_i(U_i) \bigcap \Phi_j(U_j) = W \neq \emptyset$, then both $\Phi_i^{-1}(W)$ and $\Phi_j^{-1}(W)$ are open in $\mathbb{R}^m$ and the composition $\Phi_j^{-1} \circ \Phi_i \colon U_i \to U_j$ is differentiable;

3) $\{(U_i, \Phi_i)\}$ is maximal relative to 1) and 2).

The family $\{(U_i, \Phi_i)\}$ is called a differentiable structure on M, each pair (U_i, Φ_i) is called a chart.

Definition 1.2. Let M and N be differentiable manifolds of dimension m and n, respectively. A map $f \colon M \to N$ is differentiable at $p \in M$ if given a chart

$$\Psi \colon V \to N$$

such that $f(p) \in N \subset \mathbb{R}^n$, then there exists a chart

$$\Phi \colon U \to M$$

with $U \subset \mathbb{R}^m$, $p \in \Phi(U)$ and $f \circ \Phi(U) \subset \Psi(V) \subset N$ such that

$$\Psi^{-1} \circ f \circ \Phi \colon \mathbb{R}^m \to \mathbb{R}^n$$

is differentiable at $\Phi^{-1}(p) \in \mathbb{R}^m$.

In the sequel the maps between manifolds are assumed to be smooth.

Likewise functions of one variable can be approximated by their tangent lines to their graph, studying a manifold can be replaced by studying its tangent space, which is rigorously formulated in the following definitions.

Definition 1.3. Let $p \in M$, $\gamma\colon (-\varepsilon, \varepsilon) \to M$ be a smooth curve with $\gamma(0) = p$ and $\mathrm{C}^\infty(M)$ denote the set of smooth functions on M. Then the tangent vector to γ at p is the map $\mathrm{v}\colon \mathrm{C}^\infty(M) \to \mathbb{R}$ defined by $\mathrm{v}(g) = \frac{\mathrm{d}}{\mathrm{d}t}(g \circ \gamma)\big|_{t=0}$ for any $g \in \mathrm{C}^\infty(M)$. The set of tangent vectors at p is denoted by T_pM. It is a vector space of dimension m.

In a chart (U, Φ) such that $\Phi(0, \dots, 0) = p$, a given $g \in \mathrm{C}^\infty(M)$ can be expressed as $g \circ \Phi\colon U \to \mathbb{R}$. Similarly, in the same chart the curve γ is expressed as

$$\Phi^{-1} \circ \gamma\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^m, \quad \Phi^{-1} \circ \gamma(t) = (x^1(t), \dots, x^m(t)).$$

Then

$$\mathrm{v}(g) = \frac{\mathrm{d}(g \circ \gamma)}{\mathrm{d}t}\bigg|_{t=0} = \frac{\mathrm{d}(g \circ \Phi \circ \Phi^{-1} \circ \gamma)}{\mathrm{d}t}\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} g\left(x^1(t), \dots, x^m(t)\right)\bigg|_{t=0} =$$
$$= \sum_{i=1}^{m} \frac{\partial g}{\partial x^i} \frac{\mathrm{d}x^i}{\mathrm{d}t}\bigg|_{t=0} = \left(\sum_{i=1}^{m} \dot{x}^i(0)\partial_i\right) g\bigg|_{t=0}.$$

Hereafter in the first chapter $\dot{x} = \mathrm{d}x/\mathrm{d}t$ and $\partial_i = \partial/\partial x^i$.

Besides, this definition justifies the fact that we can think of elements of T_pM as independent of any curve γ.

Given a map between two manifolds, the relation between the corresponding tangent spaces is characterized in terms of the pushforward of vector fields.

Definition 1.4. Given a map $f\colon M \to N$ between differentiable manifolds M and N, a point $p \in M$, for each $\mathrm{v} \in T_pM$ consider the curve $\alpha\colon (-t, t) \to M$ with $\alpha(0) = p$ and $\alpha'(0) = \mathrm{v}$. Let the curve $\beta\colon (-\varepsilon, \varepsilon) \to N$ be given by $\beta = f \circ \alpha$, then the pushforward of v, denoted by $f_*\mathrm{v} \in T_{f(p)}N$, is defined by

$$f_*\mathrm{v} = \frac{\mathrm{d}}{\mathrm{d}t}\beta(t)\bigg|_{t=0}.$$

The pushforward $f_*\colon T_pM \to T_{f(p)}N$ is a linear map.

Definition 1.5. A map $f\colon M \to N$ is said to be an immersion if f_* is injective as a linear map for each $p \in M$. If, in addition, the map f is a homeomorphism onto its image $f(M) \subset N$ with $f(M)$ having the subspace topology induced from N, the map f is said to be an embedding and the image $f(M)$ is called an embedded (regular) submanifold.

Definition 1.6. A map $f\colon M \to N$ is said to be a diffeomorphism if it is bijective and its inverse f^{-1} is a smooth map.

Using introduced geometric objects, we can define the main object in symmetry analysis of differential equations, namely the notion of Lie group, a particular case of which is the symmetry group of a system of differential equations. We assume the reader is familiar with basic notions of group theory.

Definition 1.7. An s-parameter Lie group G is a group carrying the structure of an s-dimensional smooth manifold, with both the group operation $m\colon G\times G \to G$, defined as $m(g,h) = gh$ for all $g,h \in G$, and the inversion $i\colon G \to G$, defined as $i(g) = g^{-1}$ for all $g \in G$, being smooth as maps between manifolds.

Since any Lie group is a group, we can further define different group-theoretic objects, related to the notion thereof.

Definition 1.8. A smooth map $\phi\colon G \to H$ between two Lie groups G and H, preserving the group operations,

$$\phi(g_1 \cdot g_2) = \phi(g_1) \circ \phi(g_2)$$

for all $g_1, g_2 \in G$, is called a Lie group homomorphism. Here $\cdot$ is the group operation in G and $\circ$ is the group operation in H.

Definition 1.9. Given a Lie group homomorphism $\phi\colon \tilde{H} \to G$, a Lie subgroup H of a Lie group G is the submanifold $H = \phi(\tilde{H})$.

Proposition 1.10. *(see [60]) Given a Lie group G, a closed subgroup H is a regular submanifold of G and thus a Lie subgroup of the group G. Vice versa, any embedded Lie subgroup of G is a closed subgroup.*

In practical applications, one often encounters not with global Lie groups but with local ones, containing only elements sufficiently close to the identity element. The advantage of local Lie groups is that both group operations and taking the inverse element can be determined in terms of local coordinate expressions.

Definition 1.11. Given connected open subsets V_0 and V such that $0 \in V_0 \subset V \subset \mathbb{R}^r$, and smooth maps $m\colon V \times V \to \mathbb{R}^r$ and $i\colon V_0 \to V$, an s-parameter local Lie group is defined by sets V_0 and V, the group multiplication m and group inversion i if the multiplication is associative, that is, if $x, y, z \in V$ as well as if $m(x,y)$ and $m(y,z)$ are in V, then $m(x, m(y,z)) = m(m(x,y),z)$; there exists the identity element, that is, for all $x \in V$, $m(0,x) = m(x,0) = x$; and for each $x \in V_0$ there exists the inverse $i(x)$, that is, $m(x,i(x)) = 0 = m(i(x),x)$.

In practice, Lie groups most often encountered are groups of transformations on some manifold M. For instance, the Lie group $\mathrm{GL}(n)$ determines the group of invertible linear transformations on $\mathbb{R}^n$ and the group $\mathrm{SO}(3)$ is the group of rotations in the three-dimensional space $\mathbb{R}^3$. Nevertheless, we are interested also in nonlinear transformations, with the action of some of them being defined only locally, that is, neither for all elements of the group nor for all points on the manifold. These groups are determined by the following definition.

Definition 1.12. A local group of transformations acting on a smooth manifold M is given by a local Lie group G with unit element e, an open subset $\mathcal{U}$, with

$$\{e\} \times M \subset \mathcal{U} \subset G \times M,$$

which is the domain of the group action, and a smooth map

$$\Psi\colon \mathcal{U} \to M$$

with the following properties: if $(h,x) \in \mathcal{U}$, $(g,\Psi(h,x)) \in \mathcal{U}$ and also $(g \cdot h, x) \in \mathcal{U}$, then $\Psi(g,\Psi(h,x)) = \Psi(g \cdot h, x)$; for all $x \in M$, $\Psi(e,x) = x$; if $(g,x) \in \mathcal{U}$, then $(g^{-1},\Psi(g,x)) \in \mathcal{U}$ and $(g^{-1},\Psi(g,x)) = x$.

Note if $\mathcal{U} = G \times M$, then the Lie group of transformations becomes global.

Now having at our disposal the notion of Lie groups we want to define the associated notion of Lie algebras, which allows us to linearize problems and thus to switch the consideration from differential geometry to linear algebra. We start with introducing the essential notion of vector fields which eventually leads us to the desired goal.

Definition 1.13. A vector field v is a map $\mathrm{v}\colon M \to TM$ assigning an element of T_pM to each $p \in M$. Here TM is the tangent bundle of the manifold M, $TM = \bigcup_{x \in M} T_xM$.

Definition 1.14. Let v be a vector field on a manifold M and $p \in M$. An integral curve of v through p, whose tangent vector at each its point coincides with the value of the vector field v at this point.

Given a smooth vector field v and a point $p \in M$, there exists a neighbourhood $U \subset M$ containing p and a smooth map $\phi\colon (-\varepsilon,\varepsilon) \times U \to M$ such that the curve $\phi(t,q)$ is the unique integral curve passing through q with

$$\frac{\mathrm{d}\phi(t,q)}{\mathrm{d}t} = \mathrm{v}(\phi(t,q)); \quad \phi(0,q) = q. \tag{1.1}$$

We call this curve the flow generated by a vector field v, denote $\phi(t,q) = \phi_t(q)$, and think of it as a diffeomorphism from U to M. Note that $\phi_0(q) = q$, $\phi_t \circ \phi_s(q) = \phi_{t+s}(q)$, $\phi_t^{-1}(q) = \phi_{-t}(q)$. Thus, the flow generated by a vector field has a group structure. Besides, the flow coincides with a local group action of the Lie group $\mathbb{R}$ on the manifold M, and called a one-parameter group of transformations. The vector field v is then called the infinitesimal generator of the action. The computation of the flow or one-parameter group generated by a given vector field v is equivalent to solving the system (1.1) of ordinary differential equations and is also referred to as an exponentiation of the vector field. Thus the suggestive notation

$$\exp(\varepsilon \mathrm{v})x = \phi_\varepsilon(x)$$

is fully justified.

Proposition 1.15. *(see [34]) For any two vector fields* v_1 *and* v_2 *there exists a unique vector field* $[\mathrm{v}_1,\mathrm{v}_2]$ *called the commutator of the vector fields* v_1 *and* v_2*, defined for any smooth function* $g \in \mathrm{C}^\infty(M)$ *as*

$$[\mathrm{v}_1,\mathrm{v}_2](g) = \mathrm{v}_1(\mathrm{v}_2(g)) - \mathrm{v}_2(\mathrm{v}_1(g)).$$

Consider a Lie group G with the group operation $R^g\colon G \to G$ being right multiplication $R^g(h) = hg$. It is clearly a diffeomorphism with the inverse $(R^g)^{-1} = R^{g^{-1}}$. We call a vector field v on G right-invariant if

$$R^g_*(\mathrm{v}|_h) = \mathrm{v}|_{R^g h} = \mathrm{v}|_{hg}$$

for all g and h in G. Right-invariant vector fields form a vector space as any linear combination of two right-invariant vector fields v and w is right-invariant as well: for any $a \in \mathbb{R}$

$$R^g_*(a\mathrm{v}|_h + \mathrm{w}|_h) = a\mathrm{v}|_{R^g h} + \mathrm{w}|_{R^g h} = a\mathrm{v}|_{hg} + \mathrm{w}|_{hg}.$$

Besides, they form an algebra $\mathfrak{g}$, called Lie algebra, the general definition of which is given below. Note that given the value of a right-invariant vector field at the identity element e, one can uniquely define it for any $g \in G$ as

$$\mathrm{v}|_g = \mathrm{v}|_{eg} = R^g_* \mathrm{v}|_e.$$

Similarly, any tangent vector to G at e uniquely determines a right-invariant vector field as

$$R^g_*(\mathrm{v}|_h) = R^g_*(R^h_*(\mathrm{v}|_e)) = R^g_* \circ R^h_*(\mathrm{v}|_e) = (R^g \circ R^h)_*(\mathrm{v}|_e) = R^{hg}_*(\mathrm{v}|_e) = \mathrm{v}|_{hg}.$$

This implies that one can identify the algebra $\mathfrak{g}$ of right-invariant vector fields with the tangent space of G at the identity element e,

$$\mathfrak{g} \cong T_e G.$$

In particular, this means that the dimensions of a Lie group and the corresponding Lie algebra are equal.

Definition 1.16. A Lie algebra $\mathfrak{g}$ is a vector space over the field $\mathbb{R}$ closed under a bilinear map $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the Lie bracket or Lie multiplication, for each $\mathrm{u}, \mathrm{v}, \mathrm{w} \in \mathfrak{g}$ satisfying the following properties:

- *Skew-symmetry*: $[\mathrm{v}, \mathrm{v}] = 0$;
- *Jacobi identity*: $[\mathrm{u}, [\mathrm{v}, \mathrm{w}]] + [\mathrm{w}, [\mathrm{u}, \mathrm{v}]] + [\mathrm{v}, [\mathrm{w}, \mathrm{u}]] = 0$.

We define a subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ as a vector subspace $\mathfrak{h} \subset \mathfrak{g}$ which is closed under the Lie bracket, that is, $[\mathrm{v}, \mathrm{w}] \in \mathfrak{h}$ whenever $\mathrm{v}, \mathrm{w} \in \mathfrak{h}$. Furthermore, each one-dimensional subspace of $\mathfrak{g}$ gives rise to a one-parameter subgroup of the associated Lie group G. This is done via the procedure of exponentiation. In general, one defines the exponential map $\exp\colon \mathfrak{g} \to G$ by setting $\varepsilon = 1$ in the one-parameter subgroup generated by $\mathrm{v} \in \mathfrak{g}$:

$$\exp(\mathrm{v}) = \exp(\mathrm{v})e.$$

It is known that $(\exp_*)_0 = \mathrm{Id}$, so by the inverse function theorem, exp determines a local diffeomorphism from $\mathfrak{g}$ onto a neighbourhood of the identity element in G. Based on this ground, the correspondence between Lie subalgebras and subgroups is generalized as follows.

Theorem 1.17. *(see [60]) Given a Lie group G and the associated Lie algebra $\mathfrak{g}$, for each $s \leq \dim G$ there is a one-to-one correspondence between s-parameter connected subgroups of G and the s-dimensional subalgebras of $\mathfrak{g}$.*

Now we can explain the origin of another important object in the symmetry analysis of differential equations – the maximal Lie invariance algebra.

Theorem 1.18. *Let $G \subset \mathbb{R}^n$ be a local Lie group defined as in Definition 1.11. Then the vector fields*

$$\mathrm{v}_k = \sum_{i=1}^{n} \xi^i_k(x)\partial_i, \quad k = 1, ..., n, \ \textit{where} \quad \xi^i_k(x) = \frac{\partial m^i}{\partial x^k}(0, x)$$

span the Lie algebra $\mathfrak{g}$ of right-invariant vector fields on G.

Assuming a Lie algebra $\mathfrak{g}$ to be finite-dimensional, there exists a basis $\mathrm{v}_1, \dots, \mathrm{v}_n$ of $\mathfrak{g}$. Since $\mathfrak{g}$ is closed under the Lie bracket, there exist uniquely defined constants c^k_{ij}, $i, j, k = 1, ..., n$, called the structure constants of $\mathfrak{g}$, such that

$$[\mathrm{v}_i, \mathrm{v}_j] = \sum_{k=1}^{n} c^k_{ij} \mathrm{v}_k, \quad i, j = 1, \dots, n.$$

Given an action of a local group of transformations G on a manifold M, one can define on M an infinitesimal action of the Lie algebra $\mathfrak{g}$ of the Lie group G. The vector fields forming the Lie algebra $\mathfrak{g}$ are known as infinitesimal generators of G and their flows coincide with corresponding one-parameters subgroups of G. The explicit formula is as follows

$$\psi(\mathrm{v})|_x = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \Psi(\exp(\varepsilon \mathrm{v}), x),$$

and thus $\psi(\mathrm{v})|_x$ is the vector field generating the flow which is induced by the action of the one-parameter subgroup $\exp(\varepsilon \mathrm{v})$ of G on M.

Theorem 1.19. *(see [41]) Given vector fields $\mathrm{w}_1, \dots, \mathrm{w}_n$ on a manifold M, with*

$$[\mathrm{w}_i, \mathrm{w}_j] = \sum_{k=1}^{n} c^k_{ij} \mathrm{w}_k, \quad i, j = 1, \dots, n,$$

for some constants c^k_{ij}, there exists a Lie group G with an associated Lie algebra having c^k_{ij} as structure constants with respect to some basis $\mathrm{v}_1, \dots, \mathrm{v}_n$, such that a local group action of G on M is defined by $\psi(\mathrm{v}_i) = \mathrm{w}_i$ for $i = 1, \dots, n$, where ψ is an infinitesimal action from $\mathfrak{g}$ on M.

1.2 Symmetries of differential equations

This section is the most essential in the first chapter. Starting with the general definition of a symmetry group of algebraic equations, we adjust it to (systems of) differential equations, making symmetry transformations map graphs of solutions of differential equations in the jet space one into another. Based on this ground, we derive the invariance criterion for differential equations, which allows one to find the maximal Lie invariance algebra of the system of differential equations, being the infinitesimal counterpart of the corresponding point symmetry group. Further we generalize symmetries of differential equations, allowing them to act non-geometrically, and develop the similar theory as for geometric symmetries. The main references here are [41, 42].

Consider a system $\mathcal{L}$ of algebraic equations with n independent variables $x = (x^1, \dots, x^n)$ and m dependent variables $u = (u^1, \dots, u^m)$. Suppose that the solutions of the system are defined by vector-functions $f = (f^1, \dots, f^m)$, that is, $u^i = f^i(x^1, \dots, x^n)$, where $i = 1, \dots, m$. Denote by $X = \{(x^1, \dots, x^n)\} \subset \mathbb{R}^n$ and $U = \{(u^1, \dots, u^m)\} \subset \mathbb{R}^m$ the spaces of independent and dependent variables, respectively.

We start with the naïve idea of the symmetry group.

Definition 1.20. A symmetry group of the system $\mathcal{L}$ is a local Lie group G of transformations acting on some open subset of the space of independent and dependent variables $M = X \times U$ by mapping solutions of the system $\mathcal{L}$ into solutions of the system $\mathcal{L}$. Each such transformation is called a symmetry of the system.

The geometric interpretation of this definition is as follows. One identifies a solution f of the system with its graph $\Gamma_f = \{(x, f(x)) \colon x \in \Omega\} \subset X \times U$, where $\Omega \subset X$ is the domain of f. Then acting by the element g from the local Lie group of transformations on Γ_f

$$g \cdot \Gamma_f = \{(\tilde{x}, \tilde{u}) = g \cdot (x, u) : (x, u) \in \Gamma_f\},$$

one finds another solution $\tilde{f}$ of the system, called the transformation of f by the group element g.

This interpretation allows us to give a rigorous definition of a symmetry group of a system of algebraic equations.

Definition 1.21. A symmetry group of the system $\mathcal{L}$ of algebraic equations is a local group of transformations G acting on an open subset of the space of independent and dependent variables for the system with the property that whenever $u = f(x)$ is a solution of $\mathcal{L}$ and $g \cdot f$ is defined for $g \in G$, then $u = g \cdot f(x)$ is also a solution of the system.

Dealing with symmetries of differential equations, we are interested not only in transformations of independent and dependent variables, but in transformations of derivatives of dependent variables as well. This can be formalized using the notion of jet spaces.

Given a smooth real-valued function $f(x) = f(x^1, \dots, x^n)$ of n independent variables, there are

$$n_k = \begin{pmatrix} n+k-1 \\ k \end{pmatrix}$$

different kth order partial derivatives of f. Hereafter $J = (j_1, \dots, j_k)$ denotes an unordered k-tuple of integers and

$$\partial_J = \frac{\partial^k}{\partial x^{j_1} \cdots \partial x^{j_k}}$$

is the corresponding derivative of order $\#J = k$. For a given smooth function $f\colon X \to U$ with $X = \mathbb{R}^n$, $U = \mathbb{R}^m$, there exist mn_k different kth order partial derivatives $u^\alpha_J = \partial_J f^\alpha(x)$ of components of f at a given point x. The total number of partial derivatives of all orders from 0 to l is then

$$m^{(l)} := m + mn_1 + \cdots + mn_l = m \begin{pmatrix} l+n \\ l \end{pmatrix}.$$

Thus, one can define $U^{(l)}$ to be an Euclidean space of dimension $m^{(l)}$, with its coordinates being all possible partial derivatives of u of order from 0 (just u) up to l. More generally, we have the following definition.

Definition 1.22. The l-jet space $J^l = X \times U^{(l)}$ of the underlying space $X \times U$ is an Euclidean space of dimension $n+m^{(l)}$, whose coordinates represent the independent variables, the dependent variables and the derivatives of the dependent variables up to order l.

Remark 1.23. Though some differential equations are defined only on some open subsets of the underlying space $X \times U$, we will avoid this technical remark as much as possible.

Now we should adapt the notion of a solution of the system to jet spaces. This is done via the prolongation of the function to the space $U^{(l)}$. Given a smooth function $u = f(x)$, such that $f\colon X \to U$, we define its lth prolongation $u^{(l)} = \mathrm{pr}^{(l)} f(x)\colon X \to U^{(l)}$ as

$$u^\alpha_J = \partial_J f^\alpha(x).$$

Thus $\mathrm{pr}^{(l)} f(x)$ is a vector-function whose coordinates represent the values of f and all its derivatives up to order l at the given point x.

To formulate this geometrically, given a function $u = f(x)$ whose graph lies in $X \times U$, its lth prolongation $\mathrm{pr}^{(l)} f(x)$ is a function whose graph lies in a jet space J^l.

Finally, we can determine a geometric interpretation of differential equations.

Consider a system Δ of lth order differential equations in n independent and m dependent variables

$$\Delta_\nu(x, u^{(l)}) = 0, \quad \nu = 1, 2, \ldots, r.$$

Assuming functions $\Delta(x, u^{(l)}) = (\Delta_1(x, u^{(l)}), \ldots, \Delta_r(x, u^{(l)}))$ to be smooth in their arguments, one can consider Δ as a smooth map from the jet space J^l to some r-dimensional Euclidean space,

$$\Delta\colon J^l \to \mathbb{R}^r.$$

An alternative geometric definition of a system Δ is then a subvariety

$$\mathcal{L}_\Delta = \{\big(x, u^{(l)}\big) : \Delta(x, u^{(l)}) = 0\} \subset J^l, \tag{1.2}$$

of a jet space J^l, that is, the subset of the l-jet space, where the map Δ vanishes. Similarly, a function $u = f(x)$ is called a solution of a system Δ if the graph of its prolongation $\mathrm{pr}^{(l)} f(x)$ lies within the subvariety $\mathcal{L}_\Delta$ of the jet space J^l, that is,

$$\Gamma_f^{(l)} \equiv \{(x, \mathrm{pr}^{(l)} f(x))\} \subset \mathcal{L}_\Delta.$$

Thus we have reformulated the notion of a domain of elements of a symmetry group. The next goal is to define an action of this group, which is done by prolongation of the group action from the space $X \times U$ of independent and dependent variables to the jet space J^l.

Given a local group of transformations G acting on the space $X \times U$ of independent and dependent variables, one can define the lth prolongation of G denoted by $\mathrm{pr}^{(l)} G$, which is in fact the induced local action of G on the l-jet space $U^{(l)}$ transforming the derivatives of functions $u = f(x)$ into the corresponding derivatives of the transformed function $\tilde{u} = \tilde{f}(\tilde{x})$.

More rigorously, given a point $(x_0, u_0^{(l)})$ in $J^{(l)}$, one chooses any smooth function $u = f(x)$ defined in a neighbourhood of x_0 such that $u_0^{(l)} = \mathrm{pr}^{(l)} f(x_0)$.

By virtue of the local action of G, one can guarantee that the transformed function $g \cdot f$ is well defined in a neighbourhood of the point $(\tilde{x}_0, \tilde{u}_0) = g \cdot (x_0, u_0)$, where $u_0 = f(x_0)$ is conventionally the zeroth order derivative of u, or in other words, the zeroth component of $u_0^{(l)}$. Evaluating the derivatives of the transformed function $g \cdot f$ at $\tilde{x}_0$ we can determine the action of the prolonged group transformation $\mathrm{pr}^{(l)} g$ on the point $(x_0, u_0^{(l)})$,

$$\mathrm{pr}^{(l)} g \cdot (x_0, u_0^{(l)}) = \left(\tilde{x}_0, \mathrm{pr}^{(l)}(g \cdot f)(\tilde{x}_0)\right).$$

It is worth noting that this definition of $\mathrm{pr}^{(l)} g \cdot (x_0, u_0^{(l)})$ depends only on the derivatives of f at x_0 up to order l and thus the chain rule guarantees an independence on the choice of representative function f for $(x_0, u_0^{(l)})$. Hence the prolonged group action is well defined.

Taking into account the local action of the group of transformations, we can restrict ourselves to groups acting on local subsets of the space $X \times U$. So we can determine the symmetry group of the system of differential equations as follows.

Proposition 1.24. *(see [41]) Let M be an open subset of $X \times U$ and $\Delta(x, u^{(l)}) = 0$ an lth order system of differential equations defined over M, with corresponding subvariety $\mathcal{L}_\Delta \subset J^{(l)}$ defined by* (1.2). *Let a local group of transformations G act on M so that its prolongation leaves $\mathcal{L}_\Delta$ invariant. Then G is a symmetry group of the system of differential equations.*

Taking advantage of Proposition 1.18, we can also determine the prolongation of the corresponding infinitesimal generators. In fact, these are infinitesimal generators of the prolonged group action.

Definition 1.25. Let M be an open subset of the space $X \times U$ of independent and dependent variables and v a vector field on M with corresponding one-parameter group $\exp(\varepsilon \mathrm{v})$. The lth prolongation $\mathrm{pr}^{(l)} \mathrm{v}$ of a vector field v is a vector field on the jet space J^l, defined as the infinitesimal generator of the corresponding prolonged one-parameter group $\mathrm{pr}^{(l)}[\exp(\varepsilon \mathrm{v})]$,

$$\mathrm{pr}^{(l)}\mathrm{v}|_{(x,u^{(l)})} = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \mathrm{pr}^{(l)}[\exp(\varepsilon \mathrm{v})](x, u^{(l)})$$

for any $(x, u^{(l)}) \in J^l$.

Having at our disposal all these tools, we can derive the infinitesimal condition for a group G to be a symmetry group of a given system of differential equations. Nonetheless we need the technical condition of total nondegeneracy for the system of differential equations to proceed further.

The system Δ of differential equations is said to be of maximal rank if its $r \times (n + mn^{(l)})$-dimensional Jacobian matrix

$$\mathrm{J}_\Delta(x, u^{(l)}) = \left(\frac{\partial \Delta_\nu}{\partial x^i}, \frac{\partial \Delta_\nu}{\partial u^\alpha_J}\right)$$

is of rank r whenever $(x, u^{(l)}) \in \mathcal{L}_\Delta$.

A system Δ is called locally solvable at the point $(x_0, u_0^{(l)}) \in \mathcal{L}_\Delta$ if for any x in a neighbourhood of x_0 there exists a smooth solution $u = f(x)$ of the system satisfying $u_0^{(l)} = \mathrm{pr}^{(l)} f(x_0)$. The system is locally solvable if it is locally solvable at every point of $\mathcal{L}_\Delta$.

Differentiating the equations in Δ_ν for all $\nu = 1, \ldots, r$ in all possible ways up to order k one obtains the kth prolongation of the system Δ, which is a system of differential equations itself of order $(l + k)$.

Definition 1.26. A system of differential equations is called totally nondegenerate if it and all its prolongations are both of maximal rank and locally solvable.

An important example of a system of partial differential equations which is totally nondegenerate is that in Kovalevskaya form, that is, of the form

$$\frac{\partial^l u^\alpha}{\partial t^l} = \Lambda_\alpha(t, x, u^{(l)}), \quad \alpha = 1, \ldots, r,$$

where the functions Λ_α's are analytic functions of their arguments and the derivatives $\partial^l u^\alpha / \partial t^l$ do not arise on the right hand side.

Theorem 1.27. *(see [41]) The system of differential equations in Kovalevskaya form is totally nondegenerate.*

Corollary 1.28. *The system* (2.3) *which is under consideration in this thesis is totally nondegenerate.*

Proof. The time derivative in evolution systems of differential equations occurs only in the left hand side, so they are clearly in Kovalevskaya form. □

Given a totally nondegenerate system of differential equations, one can determine its symmetry group via the following invariance criterion.

Theorem 1.29. *(see [44]) Let Δ be a totally nondegenerate system of differential equations over $M \subset X \times U$. If G is a local group of transformations acting on M, and*

$$\mathrm{pr}^{(l)}\mathrm{v}[\Delta] = 0 \quad \textit{for} \quad (x, u^{(l)}) \in \mathcal{L}_\Delta$$

and every infinitesimal generator v *of G, then G is a symmetry group of the system.*

In view of this theorem the principal and only task remaining for us is to find an explicit formula for the prolongation of a vector field. In spite of the complexity of the prolonged group action, the calculation of prolonged vector fields is relatively simple. As was said before, this justifies the transition from the consideration of Lie groups to that of Lie algebras. The cornerstone of most of the computations is the notion of the total derivative.

Let $\mathcal{A}$ be the space (actually, algebra) of differential functions, that is, smooth functions of the independent variables x, the dependent variables u, as well as a finite number of derivatives of the dependent variables. If the order of derivatives of u on which the differential function P depends is not essential, then we denote it by $P[u]$. By $\mathcal{A}^r$ we denote the vector space of r-tuples of differential functions, $P[u] = (P^1[u], \ldots, P^r[u])$, where each $P^i \in \mathcal{A}$.

Definition 1.30. Let $P(x, u^{(l)})$ be a differential function. Its total derivative with respect to x^i is the differential function $\mathrm{D}_i P(x, u^{(l+1)})$ such that

$$\mathrm{D}_i P(x, \mathrm{pr}^{(l+1)} f(x)) = \partial_i \Big(P(x, \mathrm{pr}^{(l)} f(x)) \Big)$$

for any smooth function f.

Using the straightforward chain rule argument one defines the general formula for determining the action of the total derivative D_i,

$$\mathrm{D}_i P = \partial_i P + \sum_{\alpha=1}^{m} \sum_J u^\alpha_{J,i} \frac{\partial P}{\partial u^\alpha_J,}$$

where J, i is the multi-index $(j_1, \ldots, j_k, i)$.

The computation of the prolongation of vector fields is carried out in the following way.

Theorem 1.31. *(see [41]) Let*

$$\mathrm{v} = \sum_{i=1}^{n} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{m} \eta_\alpha(x,u) \frac{\partial}{\partial u^\alpha}$$

be a vector field defined on an open subset $M \subset X \times U$. Its lth prolongation $\mathrm{pr}^{(l)}\mathrm{v}$ *is the vector field*

$$\mathrm{pr}^{(l)}\mathrm{v} = \mathrm{v} + \sum_{\alpha=1}^{m} \sum_J \eta^J_\alpha(x, u^{(l)}) \frac{\partial}{\partial u^\alpha_J}$$

defined in the jet space J^l, the multi-indices $J = (j_i, \ldots, j_k)$ run through all possible indices with $1 \leqslant j_k \leqslant n$ and $1 \leqslant k \leqslant l$. The components η_α^J of $\mathrm{pr}^{(l)}\mathrm{v}$ are determined as

$$\eta_\alpha^J(x, u^{(l)}) = \mathrm{D}_J \left(\eta_\alpha - \sum_{i=1}^{n} \xi^i u_i^\alpha \right) + \sum_{i=1}^{n} \xi^i u_{J,i}^\alpha. \tag{1.3}$$

Similar as the infinitesimal generators of the symmetry group, their prolongations also form a Lie algebra.

Theorem 1.32. *(see [42]) Let Δ be a totally nondegenerate system of differential equations over an open subset $M \subset X \times U$. The vector space of infinitesimal symmetries of this system forms a Lie algebra of vector fields on M. If this Lie algebra is finite-dimensional, the symmetry group of the system is a local Lie group of transformations acting on M.*

A vector field

$$\mathrm{v} = \sum_{i=1}^{n} \xi_i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{m} \phi_\alpha(x, u) \frac{\partial}{\partial u^\alpha}$$

defined on some open subset M of the space of independent and dependent variables $X \times U$ has a geometric sense, generating a one-parameter transformation acting pointwise on $X \times U$. Letting vector-field components depend on derivatives of dependent variables, this sense is evidently being lost. Nonetheless, this idea has another important interpretation. It provides a connection with conservation laws, which are of significant importance in both physics and mathematics. We call such vector fields generalized and discuss them in the rest of the section.

Definition 1.33. A generalized vector field is a formal expression of the form

$$\mathrm{v} = \sum_{i=1}^{n} \xi_i[u] \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{m} \eta_\alpha[u] \frac{\partial}{\partial u^\alpha} \tag{1.4}$$

where ξ_i's and ϕ_α's are smooth differential functions.

Just as for ordinary geometric vector fields, we can define the prolongation of a generalized vector field

$$\mathrm{pr}^{(l)}\mathrm{v} = \mathrm{v} + \sum_{\alpha=1}^{m} \sum_{J} \eta_\alpha^J[u] \frac{\partial}{\partial u_J^\alpha}$$

whose coefficients are determined by the prolongation formula (1.3).

Also the similar invariance criterion holds for generalized vector fields.

Theorem 1.34. *(see [41]) A generalized vector field* v *is a generalized infinitesimal symmetry of a system Δ of differential equations if and only if*

$$\mathrm{pr}^{(l)}\, \mathrm{v}[\Delta] = 0$$

for any solution $u = f(x)$ of the system Δ.

Among all the generalized vector fields defined by (1.4), those for which the coefficients $\xi^i[u]$ vanish play a distinguished role.

Definition 1.35. An m-tuple $Q[u] = (Q_1[u], \ldots, Q_m[u]) \in \mathcal{A}^m$ is called the characteristic of the evolutionary generalized vector field

$$\mathrm{v} = \sum_{\alpha=1}^{m} Q_\alpha[u] \frac{\partial}{\partial u^\alpha}.$$

Note that the l-prolongation of an evolutionary vector field is a vector field of the form

$$\mathrm{pr}^{(l)}\, \mathrm{v}_Q = \sum_{\alpha,J} \mathrm{D}_J Q_\alpha[u] \frac{\partial}{\partial u^\alpha_J}.$$

Any vector field v, geometric or generalized, has the associated evolutionary representative v_Q with the characteristic Q defined by

$$Q_\alpha = \eta_\alpha - \sum_{i=1}^{n} \xi^i u^\alpha_i, \quad \alpha = 1, \ldots, m. \tag{1.5}$$

Thus, every geometric vector field has the evolutionary representative with characteristic depending on at most first-order derivatives. At the same time, not every first-order evolutionary vector field has a geometric counterpart. This is the case only when its characteristic is of the specific form (1.5), with ξ^i and η^α not depending on derivatives of u.

Theorem 1.36. *(see [41]) A generalized vector field* v *is a symmetry of a system of differential equations if and only if its evolutionary representative* v_Q *is.*

This property makes evolutionary vector fields distinguished. The generalized vector field is called trivial if its characteristic vanishes on solutions of the system Δ. Two generalized symmetries are called equivalent if they differ by a trivial one. This gives rise to equivalence relation on the space of generalized symmetries of the system. In particular, the geometric symmetry and its evolutionary counterpart are equivalent.

The remaining question concerns the group of transformations corresponding to a generalized vector field. As was already emphasized, the geometric interpretation is lost. Besides, there are also problems with prolongations of group actions on finite jet spaces as prolongations of generalized vector fields depend on still higher derivatives. Therefore, the best way an action of the group $\exp(\varepsilon \mathrm{v})$ on a space of smooth functions can be defined is as follows. Replacing the generalized vector field v by its evolutionary representative v_Q, one considers the system of evolution equations

$$\frac{\partial u}{\partial \varepsilon} = Q(x, u^{(q)}), \tag{1.6}$$

where Q is the characteristic of v_Q. Under the assumption of existence and uniqueness of the solution of the Cauchy problem for system (1.6) with the initial condition $u(x, 0) = f(x)$ at least in some small interval of ε and for the function f from some appropriate functional space, the solution defines the flow on the related functional space.

$$[\exp(\varepsilon \mathrm{v}_Q)f](x) \equiv u(x,\varepsilon).$$

Supposing some technical assumption including the totally nondegeneracy condition of the system Δ and uniqueness of the solution of the above Cauchy problem, one can state the following theorem.

Proposition 1.37. *(see [41]) The evolutionary vector field* v_Q *is a generalized symmetry of a system* Δ *of differential equations if and only if the corresponding flow* $\exp(\varepsilon \mathrm{v}_Q)$ *transforms solutions of the system* Δ *to solutions of the same system.*

Definition 1.38. A (local) recursion operator for a system Δ of differential equations is a linear operator $\mathfrak{R}\colon \mathcal{A}^m \to \mathcal{A}^m$ mapping a generalized symmetry v_Q of the system Δ to the generalized symmetry $\mathrm{v}_{\tilde Q}$ of the same system by the rule $\tilde Q = \mathfrak{R}Q$, where Q and $\tilde Q$ are characteristics of the corresponding vector fields.

1.3 Methods of finding complete point symmetry groups

Applying the Lie infinitesimal method to a system of differential equations $\mathcal{L}$ with further generating finite transformations allows one to construct the Lie symmetry group of $\mathcal{L}$, which consists of continuous symmetry transformations of the system $\mathcal{L}$ and is the identity component of the complete point symmetry group of this system. At the same time, discrete symmetry transformations are also of interest for applications. If the Lie symmetry group of $\mathcal{L}$ is known, which is always can be assumed to be the case, finding discrete symmetry transformations of $\mathcal{L}$ is equivalent to the construction of the complete point symmetry group of $\mathcal{L}$. There are two methods for computing complete point symmetry groups of systems of differential equations in the literature, the direct method [47, 65, 66] and the algebraic method [3, 4, 26, 27, 49]. Although the latter in general gives only a part of restrictions on the form of point symmetry transformations and thus also involves computations within the framework of the direct method on the final step of the corresponding procedure, it is usually advantageous since, in contrast to using the direct method alone, computations are not so cumbersome.

The direct method is based on the definition of a point symmetry transformation and is the most universal. The application of this method results in a system of PDEs as determining equations which are, in general, nonlinear highly-coupled and thus difficult to solve. Therefore several methods were elaborated to improve the direct method, one of which is the algebraic one.

The algebraic method for finding the complete point symmetry group of a system of differential equations was suggested by Hydon [26, 27]. The underlying idea is that each point symmetry transformation $\mathcal{T}$ of a system of differential equations $\mathcal{L}$ induces an automorphism of the maximal Lie invariance algebra $\mathfrak{g}$

of $\mathcal{L}$. If the algebra $\mathfrak{g}$ is finite dimensional with $\dim\mathfrak{g}=n$, then the above means that

$$\mathcal{T}_*e_i=\sum_{j=1}^n a_{ij}e_j,\quad i=1,\dots,n,$$

where $\mathcal{T}_*$ is the pushforward of vector fields induced by $\mathcal{T}$, and $(a_{ij})_{i,j=1}^n$ is the matrix of an automorphism of $\mathfrak{g}$ in the chosen basis $(e_1,\dots,e_n)$. Computing the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{g}$, one obtains a restricted form of the matrix $(a_{ij})_{i,j=1}^n$. Under the supposition $\mathfrak{g}\neq\{0\}$, expanding the above condition for $\mathcal{T}_*$ gives constraints for the transformation $\mathcal{T}$. These constraints are to be used within the direct method afterwards.

Nevertheless, the computation of the entire automorphism group $\mathrm{Aut}(\mathfrak{g})$ can be challenging, especially if the maximal Lie invariance algebra $\mathfrak{g}$ is infinite-dimensional. This is why another version of the algebraic method, which involves the notion of megaideals [48] (also known as fully characteristics ideals [25]), was developed in [3, 4, 49]. A megaideal $\mathfrak{i}$ of a Lie algebra $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ that is invariant under any automorphism of $\mathfrak{g}$. Providing $\mathfrak{g}$ is finite-dimensional with $\dim\mathfrak{g}=n$ and possesses a megaideal $\mathfrak{i}$ spanned by the first k, $k<n$, basis elements, i.e., $\mathfrak{i}=\langle e_1,\dots,e_k\rangle$, the matrix $(a_{ij})_{i,j=1}^n$ of any automorphism of $\mathfrak{g}$ is of block structure with $a_{ij}=0$ for $i>k$ and $j\leqslant k$. In other words, the megaideal hierarchy of a finite-dimensional Lie algebra is directly related to the block structure of matrices of its automorphisms. This observation can be also used for infinite-dimensional Lie algebras with finite-dimensional megaideals.

Simple tools for constructing megaideals were presented in [4, 48, 49]. First of all, both the improper subalgebras of a Lie algebra $\mathfrak{g}$ (the zero subalgebra and $\mathfrak{g}$ itself) are (improper) megaideals of $\mathfrak{g}$. Moreover, sums, intersections and Lie products of megaideals are megaideals, megaideals of megaideals $\mathfrak{g}$ are megaideals of $\mathfrak{g}$, all elements of the derived series, ascending and descending central series of $\mathfrak{g}$, in particular, the center and the derivatives of $\mathfrak{g}$, as well as the radical and the nilradical, that is, the maximal solvable and nilpotent ideals of $\mathfrak{g}$, are its megaideals. Recall that the nth derivative $\mathfrak{g}^{(n)}$ of the Lie algebra $\mathfrak{g}$ is defined by $\mathfrak{g}^{(n)}=[\mathfrak{g}^{(n-1)},\mathfrak{g}^{(n-1)}]$ for $n\geqslant 1$ with $\mathfrak{g}^{(0)}=\mathfrak{g}$ and the nth power $\mathfrak{g}^n$ of the algebra $\mathfrak{g}$ is $\mathfrak{g}^n=[\mathfrak{g}^{n-1},\mathfrak{g}]$ for $n\geqslant 2$ with $\mathfrak{g}^1=\mathfrak{g}$. One more, less obvious, way of obtaining new megaideals from known ones is given by the following assertion.

Proposition 1.39. *(see [3]) If $\mathfrak{i}_0$, $\mathfrak{i}_1$ and $\mathfrak{i}_2$ are megaideals of $\mathfrak{g}$, then the set*

$$\mathfrak{s}=\{z\in\mathfrak{i}_0\colon [\langle z\rangle,\mathfrak{i}_1]\subseteq\mathfrak{i}_2\}$$

is also a megaideal of $\mathfrak{g}$.

Within the megaideal-based version of the algebraic method, one use the condition $\mathcal{T}_*\mathfrak{i}=\mathfrak{i}$ for each megaideal $\mathfrak{i}$ from a certain set of known megaideals of the maximal Lie invariance algebra $\mathfrak{g}$ of the system $\mathcal{L}$. Megaideals that are sums of other megaideals gives constraints that are consequences of constraints that are derived from the consideration of the summands. This is why such decomposed megaideals should be neglected in the course of the computation. After deriving constraints for a point symmetry transformation $\mathcal{T}$ implied by the megaideal invariance, one finishes the computation of the complete point symmetry group with the direct method.

1.4 Group-invariant solutions

In this section we consider how one can use the Lie symmetry group of a system of differential equations to find its solutions. The main references here are [44] and [41].

Let G be the Lie symmetry group of a system of differential equations Δ of order l in n independent variables. For the system of ODEs, its s-parameter solvable subgroup H allows one to construct ansatzes reducing the order of the system by s. For the system of PDEs an s-parameter subgroup allows one to reduce the number of independent variables by s upon the transversality condition for Lie algebra $\mathfrak{g}$ of Lie group G, that is, the rank of matrix composed by components of basis vector fields of the algebra $\mathfrak{g}$ is equal to the rank of matrix composed of the components relating to the corresponding independent variables. The solutions of the reduced system Δ/H are called H-invariant solutions of the initial system Δ. Since typically the number of possible subgroups is infinite, it is necessary to construct an optimal list of subgroups in the sense that any subgroup can be obtained from the unique one of the list. The classification of subgroups is obtained by using the adjoint representation of the group on the corresponding Lie algebra. To proceed, one has to require the regularity of the action of the group in the following sense.

Definition 1.40. An orbit of a local transformation group G action on a manifold M is a minimal nonempty G-invariant subset of M. The group G acts regularly if all the orbits are of the same dimension as submanifolds of M (semi-regular action) and each point $x \in M$ has an arbitrarily small neighborhood whose intersection with each orbit is a connected subset of M.

Assuming the action of the group G to be regular, one can state the basic result on an optimal list of its subgroups.

Proposition 1.41. *(see [44]) Let G be the symmetry group of a system of differential equations Δ, $H < G$ an s-parameter subgroup and g an arbitrary element of G. Given an H-invariant solution $u = f(x)$ of the system Δ, the transformed function $u = \tilde{f}(x) = g \cdot f(x)$ is an $\tilde{H}$-invariant solution, where $\tilde{H} = gHg^{-1}$.*

Therefore, we need to study the conjugacy map $h \mapsto ghg^{-1}$ on a Lie group more thoroughly.

Let G be a Lie group. The group conjugation $K^g(h) = ghg^{-1}$ of an element $h \in G$ by any $g \in G$ defines a diffeomorphism on G since it is a bijection of g, the group operation is a smooth map by the definition of Lie group and the composition of smooth maps is smooth. The set of group conjugation maps is naturally endowed with a group structure since

$$K^g \circ K^{g'} = K^{gg'} \text{ and } K^e = \mathrm{Id}_G,$$

and thus K^g defines a global group action of G on itself. Note that each conjugacy map K^g is a group homomorphism as

$$K^g(hh') = ghh'g^{-1} = ghg^{-1}gh'g^{-1} = K^g(h)K^g(h').$$

Finally, the pushforward $K^g_*\colon T_hG \to T_{K^g_h}G$ preserves the right-invariance of vector fields, thus giving rise to a linear map on the Lie algebra $\mathfrak{g}$ of G, called the adjoint representation of G,

$$\mathrm{Ad}(g)\mathrm{v} = K^g_*\mathrm{v}, \quad \mathrm{v} \in \mathfrak{g}.$$

If the one-parameter subgroup $H < G$ is generated by a vector field $\mathrm{v} \in \mathfrak{g}$, $H = \{\exp(\varepsilon\mathrm{v})\colon \varepsilon \in \mathbb{R}\}$, then the conjugate one-parameter subgroup $K^g(H) = gHg^{-1}$ corresponds to the vector field $\mathrm{Ad}(g)\mathrm{v}$ in view of the one-to-one correspondence between one-parameter subgroups and one-parameter subalgebras. This argument is also generalized to higher dimensional subgroups and can be stated as follows.

Theorem 1.42. *(see [44]) Let H and $\tilde{H}$ be connected, s-dimensional Lie subgroups of the Lie group G, generated by vector fields of the Lie subalgebras $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$, respectively, of the Lie algebra $\mathfrak{g}$ of G. Then $\tilde{H} = gHg^{-1}$ are conjugate subgroups if and only if $\tilde{\mathfrak{h}} = \mathrm{Ad}(g)\mathfrak{h}$.*

Two different algorithms of computing adjoint representations of the Lie group are in use today.

One way to compute the adjoint representation of a Lie group G on its Lie algebra $\mathfrak{g}$ is to use its infinitesimal generators. Each vector field $\mathrm{v} \in \mathfrak{g}$ generates the one-parameter subgroup $\{\exp(\varepsilon\mathrm{v})\}$ of the group G. This is why the vector field $\mathrm{ad}\,\mathrm{v}$ generates the corresponding one-parameter group of adjoint actions, where

$$\mathrm{ad}\,\mathrm{v}|_{\mathrm{w}} \equiv \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \mathrm{Ad}(\exp(\varepsilon\mathrm{v}))\mathrm{w} = [\mathrm{w}, \mathrm{v}], \quad \mathrm{w} \in \mathfrak{g}.$$

Knowing the infinitesimal adjoint action $\mathrm{ad}\,\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ on itself, one can obtain the corresponding adjoint representation $\mathrm{Ad}\,G$ of the underlying Lie group by the standard procedure of exponentiation of vector fields by solving the Cauchy problem

$$\frac{\mathrm{dw}}{\mathrm{d}\varepsilon} = \mathrm{ad}\,\mathrm{v}|_{\mathrm{w}}, \quad \mathrm{w}(0) = \mathrm{w}_0,$$

with solution

$$\mathrm{w}(\varepsilon) = \mathrm{Ad}(\exp(\varepsilon\mathrm{v}))\mathrm{w}_0$$

or, alternatively, by computing the Lie series

$$\mathrm{Ad}(\exp(\varepsilon\mathrm{v}))\mathrm{w}_0 = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!}(\mathrm{ad}\,\mathrm{v})^i(\mathrm{w}_0).$$

The second method [10, 61] requires calculating the actions of pushforwards of the transformations from the symmetry group on generating vector fields of the algebra. This method works particularly well for infinite-dimensional Lie algebras, for which the Lie series method can be cumbersome. A further advantage of the second method is that once the complete point symmetry group of a system of differential equations has been constructed, the finite form of admitted point symmetries is already known, which is all that is required in this method.

Definition 1.43. An optimal list of s-parameter subgroups of a given Lie group G is a list such that every s-parameter subgroup of G is conjugated to a unique subgroup in the list. An list of s-dimensional subalgebras of the corresponding Lie algebra $\mathfrak{g}$ is called optimal if every s-dimensional subalgebra of $\mathfrak{g}$ is $\mathrm{Ad}\, G$-equivalent to the unique subalgebra in the list.

In order to understand why subgroups of the Lie group of transformations induce group-invariant solutions of the system of differential equations one needs to consider the underlying geometric interpretation of the construction. The answer lies in the notion of a quotient manifold which is introduced in the following.

Let G be a local group of transformations acting on a smooth manifold M. Assume two points x and y on M to be equivalent if there exists an element $g \in G$ such that $y = g \cdot x$, that is, x and y lie in the same orbit of G. This determines an equivalence relation of elements of the manifold M. Denote the set of equivalence classes by M/G, which are orbits of G, and introduce the projection map $\pi\colon M \to M/G$ assigning each element of M its orbit, or, in other words, some canonical representative of the orbit that the element belongs to. One can define a topology on the quotient space requiring the image $\pi[U]$ of the open subset $U \subset M$ to be open in M/G. This topology can be very complicated, and therefore the requirement that the group G acts regularly on the manifold M comes into play. This allows us to endow the quotient space M/G with the structure of a smooth manifold. If the dimension of the manifold M is equal to m and the group G is an s-parameter group, then the quotient manifold M/G is of dimension $m - s$. The existence of a quotient manifold is stated in the following theorem.

Theorem 1.44. *(see [41]) Suppose G is a local group of transformations regularly acting on an m-dimensional manifold M with s-dimensional orbits. Then there exists a smooth $(m-s)$-dimensional quotient manifold M/G and a smooth map $\pi\colon M \to M/G$ such that the points x and y lie in the same orbit of G in M if and only if $\pi(x) = \pi(y)$. Moreover, for the Lie algebra $\mathfrak{g}$ of infinitesimal generators of the action of G the linear map*

$$\pi_*\colon T_xM \to T_{\pi(x)}(M/G)$$

is surjective with kernel $\mathfrak{g}_x = \{\mathrm{v}|_x\colon \mathrm{v} \in \mathfrak{g}\}$.

Finally, after introducing the notion of group-invariance and stating the basic theorem we can formulate an algorithm of the construction of group-invariant solutions.

Let G_x be a subset of a group G such that $g \cdot x$ is defined. A subset $\mathfrak{I} \subset M$ is called locally G-invariant if for every $x \in \mathfrak{I}$ there is a neighbourhood of the identity in G, $\tilde{G}_x \subset G_x$ such that $g \cdot x \in \mathfrak{I}$ for all $g \in \tilde{G}_x$. A smooth function F is called locally G-invariant if its graph is locally G-invariant.

Evidently, invariants are defined non-uniquely, as any smooth function of invariants is again an invariant. This is why we need an auxiliary notion of functional dependence of invariants, after which the basic result on the number of invariants is stated.

Definition 1.45. Smooth functions $\zeta^1(x), \dots, \zeta^k(x)$ defined on a manifold M are called functionally dependent if for each $x \in M$ there is a neighbourhood U of x and a smooth function $F(z^1, \dots, z^k)$, not identically equal to zero on any open subset of $\mathbb{R}^k$ such that $F(\zeta^1(x), \dots, \zeta^k(x)) = 0$ for all $x \in U$; the functions are functionally independent if they are not functionally dependent on any open subset $U \subset M$.

The number of local invariants of a regular group action is determined by the dimension of the group orbits and the basic theorem is asserted as follows.

Theorem 1.46. *(see [44]) Let G be a Lie group of transformations, acting semi-regularly with s-dimensional orbits on the m-dimensional manifold M. In a neighbourhood of any point $x_0 \in M$ there exist precisely $m - s$ functionally independent local invariants $\zeta^1(x), \dots, \zeta^{m-s}(x)$.*

We consider the construction of group-invariant solutions of the system of PDEs. Let H be an s-parameter subgroup of the Lie symmetry group G of the system Δ of partial differential equations. As the point symmetry group of the system under consideration in this thesis admit only fiber-preserving transformations, that is, transformations of the form

$$(\tilde{x}, \tilde{u}) = g \cdot (x, u) = (\Xi_g(x), \Upsilon_g(x, u)),$$

we describe the algorithm of reduction only in this case.

According to Theorem 1.46 there exist $n - s$ functionally independent local invariants $y^1 = \zeta^1(x), \dots, y^{n-s} = \zeta^{n-s}(x)$ of the group H action. A complete set of functionally independent invariants for H on M consists of $n+m-s$ invariants. As local invariants are invariants of the full group action, we can additionally find m invariants of the action of H on M. Denote them by $v^1 = \eta^1(x, u), \dots, v^m = \eta^m(x, u)$, so the relation between old independent and dependent variables x and u and new ones y and v, respectively, can be written as

$$y = \zeta(x), \quad v = \eta(x, u).$$

Moreover, one can find the correspondence between new dependent and independent variables, $v = h(y)$, and then between H-invariant functions $u = f(x)$ and $v = h(y)$. In the new variables the number of independent variables in the reduced system Δ/H of partial differential equations (which we are interested in) is always less by s than in initial one provided the group of transformations H is a symmetry group of the system. This leads to a system of equations of the form

$$(\Delta/H)_\nu(y, v^{(l-s)}) = 0, \quad \nu = 1, \dots, r.$$

Every solution of Δ/H is a G-invariant solution of Δ. Besides, each G-invariant solution is constructed in this manner.

1.5 Variational and Fréchet derivatives

As it has already been mentioned, there is a correspondence between symmetries of a Lagrangian and conservation laws of the corresponding Euler–Lagrange equations. To understand the idea of conservation laws it is necessary to introduce some notions of variational calculus. The main reference here is [41].

Given an open, connected subset $\Omega \subset X$ with smooth boundary $\partial\Omega$ and a smooth function $L = L(x, u^{(n)})$ of its arguments, a variational problem is stated as follows: to find the extremals of a functional

$$\mathfrak{L}[u] = \int_\Omega L(x, u^{(n)}(x))\mathrm{d}x$$

in some class of functions $u = f(x)$ defined over Ω. In this notation, we call the integrand L the Lagrangian of the functional $\mathfrak{L}$. Like the smooth real-valued functions, the functional takes its extremals where its derivative, called the variational derivative, vanishes.

Definition 1.47. The variational derivative of a variational problem $\mathfrak{L}$ is the unique m-tuple

$$\delta\mathfrak{L}[u] = (\delta_1\mathfrak{L}[u], \dots, \delta_m\mathfrak{L}[u])$$

with the property that

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \mathfrak{L}[f + \varepsilon\eta] = \int_\Omega \delta\mathfrak{L}[f(x)] \cdot \eta(x)\mathrm{d}x.$$

Here a smooth function $\eta(x) = (\eta^1(x), \dots, \eta^m(x))$ with compact support in Ω is chosen such that the function $f + \varepsilon\eta$ satisfies the boundary conditions imposed for the given variational problem. The components $\delta_\alpha\mathfrak{L} = \delta\mathfrak{L}/\delta u^\alpha$ are called the variational derivatives of $\mathfrak{L}$ with respect to u^α.

The general formula to find the variational derivative is obtained by applying the integration by parts, and is given as follows

$$\delta_\alpha\mathfrak{L} = \sum_J (-\mathrm{D})_J \frac{\partial L}{\partial u^\alpha_J}.$$

Here $J = (j_1, \dots, j_k)$ is a unordered multi-index,

$$u^\alpha_J = \frac{\partial^k u^\alpha}{\partial x^{j_1} \cdots \partial x^{j_k}}$$

for $\alpha = 1, \dots, m$ and the operator

$$\mathrm{E}_\alpha = \sum_J (-\mathrm{D})_J \frac{\partial}{\partial u^\alpha_J},$$

is called the αth Euler operator. Thus the variational derivative of $\mathfrak{L}[u]$ is found by applying the Euler operator to the Lagrangian: $\delta\mathfrak{L}[u] = \mathrm{E}(L)$, where $\mathrm{E}(L) = (\mathrm{E}_1(L), \dots, \mathrm{E}_m(L))$.

Another notion of the calculus of variations that is essential for the study of symmetries of differential equations is of Fréchet derivative. It arises in the study of generalized symmetries, conservation laws and Hamiltonian structures of systems of differential equations.

Definition 1.48. The Fréchet derivative of a differential function $P[u] \in \mathcal{A}^r$ is called the differential operator $\mathbf{D}_P\colon \mathcal{A}^m \to \mathcal{A}^r$ defined for any $Q \in \mathcal{A}^m$ so that

$$\mathbf{D}_P(Q) = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} P\big[u + \varepsilon Q[u]\big].$$

The Fréchet derivative of an r-tuple $P = (P_1, \ldots, P_r)$ is presented by the $m \times r$-matrix differential operator with entries

$$(\mathbf{D}_P)_{\mu\nu} = \sum_J \frac{\partial P_\mu}{\partial u^\nu_J}\mathrm{D}_J,$$

where $\mu = 1, \ldots, r$, $\nu = 1, \ldots, m$ and the sum is running over all possible unordered multi-indices J.

It turns out that the infinitesimal invariance criterion of systems of differential equations can be reformulated in terms of Fréchet derivative, which is based on the following proposition.

Proposition 1.49. *(see [7]) If $P \in \mathcal{A}^r$ and $Q \in \mathcal{A}^m$, then*

$$\mathbf{D}_P(Q) = \mathrm{pr}\,\mathrm{v}_Q(P).$$

In the study of conservation laws of systems of differential equations one need to consider formally adjoint operators to Fréchet derivatives.

Definition 1.50. Given a differential operator $\mathcal{D} = \sum\limits_J P_J[u]\mathrm{D}_J$, its (formally) adjoint is the differential operator $\mathcal{D}^*$ such that

$$\int_\Omega P \cdot \mathcal{D}Q \mathrm{d}x = \int_\Omega Q \cdot \mathcal{D}^* P \mathrm{d}x$$

for every pair of differential functions P and $Q \in \mathcal{A}$ and appropriate domain Ω.

Given a differential operator $\mathcal{D}$ as in the above definition, its adjoint is determined by the action on a differential function $Q \in \mathcal{A}$ as follows

$$\mathcal{D}^* Q = \sum_J (-D)_J(P_J Q). \tag{1.7}$$

Similarly, a matrix differential operator $\mathcal{D}\colon \mathcal{A}^p \to \mathcal{A}^q$ with entries $\mathcal{D}_{\mu\nu}$ has as adjoint the operator $\mathcal{D}^*\colon \mathcal{A}^q \to \mathcal{A}^p$ with entries $\mathcal{D}^*_{\mu\nu} = (\mathcal{D}_{\nu\mu})^*$.

Definition 1.51. An operator $\mathcal{D}$ is self-adjoint if $\mathcal{D}^* = \mathcal{D}$, it is skew-adjoint if $\mathcal{D}^* = -\mathcal{D}$.

Finally, the adjoint operator $\mathbf{D}^*_P\colon \mathcal{A}^r \to \mathcal{A}^m$ of the Fréchet derivative of the differential function $P \in \mathcal{A}^r$ has entries

$$(\mathbf{D}^*_P)_{\nu\mu} = \sum_J (-\mathrm{D})_J \cdot \frac{\partial P_\mu}{\partial u^\nu_J},$$

where $\mu = 1, \ldots, r$ and $\nu = 1, \ldots, m$.

1.6 Hamiltonian systems of evolution equations

Although the theory of finite-dimensional Hamiltonian systems is known for a long time, the infinite-dimensional generalization to evolution differential equations started to be studied only recently. Unlike finite-dimensional systems, one cannot rely on Darboux's theorem guaranteeing the local existence of canonical coordinates. The main object to investigate in this section is the Poisson bracket of functionals [41].

Consider the algebra $\mathcal{A}$ of differential functions over $M = X \times U$. Each differential function $P \in \mathcal{A}$ determines the functional $\int_M P \mathrm{d}x$. We define the space $\mathcal{F}$ of functionals as the set of equivalence classes on the algebra $\mathcal{A}$ under the equivalence relation

$$\tilde{P} \sim P \text{ iff } \tilde{P} = P + \mathrm{Div}\ Q \text{ for some } Q \in \mathcal{A}^n.$$

Definition 1.52. A Poisson bracket of functionals on a smooth manifold M is an operation that assigns a functional $\{\mathcal{P}, \mathcal{Q}\}$ on M to each pair $\mathcal{P}, \mathcal{Q} \in \mathcal{F}$, with the basic properties

(a) Bilinearity:

$$\{a\mathcal{P} + \mathcal{Q}, \mathcal{R}\} = a\{\mathcal{P}, \mathcal{R}\} + \{\mathcal{Q}, \mathcal{R}\},$$
$$\{\mathcal{P}, a\mathcal{Q} + \mathcal{R}\} = a\{\mathcal{P}, \mathcal{Q}\} + \{\mathcal{P}, \mathcal{R}\};$$

(a) Skew-symmetry:

$$\{\mathcal{P}, \mathcal{Q}\} = -\{\mathcal{Q}, \mathcal{P}\};$$

(b) Jacobi identity:

$$\{\{\mathcal{P}, \mathcal{Q}\}, \mathcal{R}\} + \{\{\mathcal{Q}, \mathcal{R}\}, \mathcal{P}\} + \{\{\mathcal{R}, \mathcal{P}\}, \mathcal{Q}\} = 0,$$

for any $a \in \mathbb{R}$ and $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in \mathcal{F}$.

A manifold M with a Poisson bracket is called a Poisson manifold, the bracket defines a Poisson structure on M.

Consider a linear operator $\mathfrak{D} \colon \mathcal{A}^m \to \mathcal{A}^m$ on the space of m-tuples of differential functions and associate to it the bracket

$$\{\mathcal{P}, \mathcal{Q}\} = \int \delta\mathcal{P} \cdot \mathfrak{D}\delta\mathcal{Q}\mathrm{d}x,$$

where $\cdot$ stands for an inner product in $\mathbb{R}^m$.

Definition 1.53. A linear operator $\mathfrak{D}$ is called Hamiltonian if its associated bracket is Poisson.

The equilibrium solutions of the equations of nondissipative continuum mechanics are usually found by minimizing an appropriate variational integral. Therefore, smooth solutions satisfy the Euler-Lagrange equations for the relevant functional and thus one works in the Lagrangian framework discussed above. Nevertheless, this approach is not natural for the full dynamical problem described by a system of evolution equations and the Hamiltonian formulation of systems of evolution equations comes into the scene.

Having the definition of the Poisson bracket of functionals we can introduce the Hamiltonian formalism of systems of evolution equations of the form

$$u_t = K[u],$$

where K is a differential function depending on u and its spatial derivatives. We call the system Hamiltonian if it can be written as

$$u_t = \mathfrak{D}\delta\mathcal{H}$$

for some $\mathcal{H} \in \mathcal{F}$ called the Hamiltonian of the system.

Thus to verify that a differential operator is Hamiltonian, one must check the corresponding properties of the induced bracket.

Proposition 1.54. *(see [41]) Let $\mathfrak{D}$ be an $m \times m$ matrix differential operator in total derivatives with coefficients being differential functions. Then the associated bracket on the space of functional is skew-symmetric if and only if the differential operator $\mathfrak{D}$ is formally skew-adjoint, $\mathfrak{D}^* = -\mathfrak{D}$.*

See also [41] for the criterion for the bracket associated with a linear differential operator in total derivatives to satisfy the Jacobi identity.

1.7 Riemannian manifolds

In this section we give the necessary theoretic foundation on Riemannian geometry, following [9, 33].

Denote by T_p^*M the dual vector space to T_pM, called the cotangent space of M at p.

Definition 1.55. A tensor of type (r, s) (r-order contravariant, s-order covariant) at p is a multilinear map

$$T\colon \underbrace{T_p^*M \times \cdots \times T_p^*M}_{r \text{ times}} \times \underbrace{T_pM \times \cdots \times T_pM}_{s \text{ times}} \to \mathbb{R}.$$

One of the important tensors in geometry is the Riemannian metric which generalizes the notion of an inner product from inner product spaces to manifolds.

Definition 1.56. A Riemannian metric g at $p \in M$ is a $(0, 2)$-tensor with the properties

(a) $g(\mathrm{u}, \mathrm{v}) = g(\mathrm{v}, \mathrm{u})$ for any $\mathrm{u}, \mathrm{v} \in T_pM$;

(b) $g(\mathrm{u}, \mathrm{u}) \geqslant 0$ for any $\mathrm{u} \in T_pM$ with $g(\mathrm{u}, \mathrm{u}) = 0$ if and only if $\mathrm{u} = 0$.

A Riemannian metric is smooth if and only if $g(\mathrm{u}, \mathrm{v})$ is smooth on M for any smooth vector fields u and v.

A metric is pseudo-Riemannian if it is not positive-definite but only nondegenerate.

Definition 1.57. A Riemannian manifold (M, g) is a differential manifold M endowed with a smooth Riemannian metric.

Given a basis $(x^1, \dots, x^n)$ on the manifold M, the associated basis on the tangent space T_pM is $(\partial_{x^1}, \dots, \partial_{x^n})$ and one can determine the Riemannian metric coordinate-wise $g_{ij} = g(\partial_{x^i}, \partial_{x^j})$ for any $i, j = 1, \dots, n$.

A choice of a Riemannian metric on a manifold M uniquely determines a certain affine connection on M. Roughly speaking, this allows one to differentiate vector fields on M.

Definition 1.58. A connection (covariant derivative) on M is a smooth map

$$\nabla \colon T_pM \times T_pM \to T_pM$$

defined as $\nabla_{\mathrm{u}}\mathrm{v} = \nabla(\mathrm{u}, \mathrm{v})$ such that

(a) $\nabla_{f\mathrm{u}+\mathrm{v}}\mathrm{w} = f\nabla_{\mathrm{u}}\mathrm{w} + \nabla_{\mathrm{v}}\mathrm{w}$;

(b) $\nabla_{\mathrm{u}}(\mathrm{v} + \mathrm{w}) = \nabla_{\mathrm{u}}\mathrm{v} + \nabla_{\mathrm{u}}\mathrm{w}$;

(c) $\nabla_{\mathrm{u}}(f\mathrm{v}) = f\nabla_{\mathrm{u}}\mathrm{v} + \mathrm{u}(f)\mathrm{v}$

for any $\mathrm{u}, \mathrm{v}, \mathrm{w} \in T_pM$ and a smooth function f.

In a local chart $\nabla_{\partial_{x^i}}\partial_{x^j} = \Gamma^k_{ji}\partial_{x^k}$, where the Einstein summation is applied (that is, summation over pairs of indices, one of which is a superscript, one of which is a subscript) and Γ^k_{ji} are connection components (Christoffel symbols), defined as

$$\Gamma^k_{ji} = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ji}).$$

Here ∂_i stands for differentiation with respect to x^i and g^{ij} is the second-order contravariant tensor, whose components constitute the inverse to the matrix of the (pseudo-)Riemannian metric. Since the metric is nondegenerate, the associated matrix is invertible.

Remark 1.59. The definition of a connection can also be extended to general tensor fields. In general, the covariant derivative ∇T of an (r, s)-type tensor is an $(r, s+1)$-type tensor with coordinates

$$\nabla_m T^{j_1 \dots j_s}_{i_1 \dots i_r} = \partial_m T^{j_1 \dots j_s}_{i_1 \dots i_r} + \sum_{k=1}^{s} T^{j_1 \dots l \dots j_s}_{i_1 \dots i_r}\Gamma^{j_k}_{ml} - \sum_{k=1}^{r} T^{j_1 \dots j_s}_{i_1 \dots l \dots i_r}\Gamma^{l}_{mi_k}. \tag{1.8}$$

Definition 1.60. A connection ∇ is symmetric if and only if $\nabla_{\mathrm{u}}\mathrm{v} - \nabla_{\mathrm{v}}\mathrm{u} = [\mathrm{u}, \mathrm{v}]$. In a coordinate basis, the coordinates of a symmetric connection satisfy $\Gamma^k_{ij} = \Gamma^k_{ji}$.

Theorem 1.61. *(see [33]) A Riemannian manifold (M, g) admits a unique symmetric connection ∇ such that $\nabla g \equiv 0$, which is called the Levi-Civita connection, and defined by*

$$\begin{aligned} g(\nabla_{\mathrm{u}}\mathrm{v}, \mathrm{w}) = &\frac{1}{2}(\mathrm{u}g(\mathrm{v}, \mathrm{w}) + \mathrm{v}g(\mathrm{u}, \mathrm{w}) - \mathrm{w}g(\mathrm{u}, \mathrm{v}) \\ &+ g([\mathrm{u}, \mathrm{v}], \mathrm{w}) + g([\mathrm{w}, \mathrm{u}], \mathrm{v}) - g([\mathrm{v}, \mathrm{w}], \mathrm{u})). \end{aligned}$$

For every Riemannian manifold one can define objects which are invariant under isometries. All of them are expressed in terms of the curvature. Roughly speaking, the curvature measures the amount in which a Riemannian manifold deviates from being Euclidean.

Definition 1.62. The Riemann curvature tensor is a $(1,3)$-tensor field defined as

$$R(\mathrm{u},\mathrm{v})\mathrm{w} = \nabla_{\mathrm{u}}\nabla_{\mathrm{v}}\mathrm{w} - \nabla_{\mathrm{w}}\nabla_{\mathrm{u}}\mathrm{w} - \nabla_{[\mathrm{u},\mathrm{v}]}\mathrm{w},$$

where u, v and w are smooth vector fields on M.

If the Riemann curvature tensor vanishes on (M,g), then a manifold is said to be flat. For example, an Euclidean space is flat. In local coordinates, the components of the Riemann curvature tensor are defined as follows

$$R(\partial_i,\partial_j)\partial_k = (\partial_i\Gamma^l_{kj} - \partial_j\Gamma^l_{ik} + \Gamma^m_{kj}\Gamma^l_{mi} - \Gamma^m_{ki}\Gamma^l_{mj})\partial_l = R^l{}_{kij}\partial_l.$$

Although the Riemann curvature tensor contains n^4 components, it can be completely determined by $n^2(n^2-1)/12$ components using the symmetries of the curvature tensor such as $R^i{}_{jkl} = -R^i{}_{jlk}$, $R_{ijkl} = -R_{jikl}$ (where $R(\mathrm{u},\mathrm{v},\mathrm{w},\mathrm{x}) = g(\mathrm{x},R(\mathrm{u},\mathrm{v})\mathrm{w})$), algebraic, differential and contracted Bianchi identities etc. In particular, for two-dimensional manifolds the Riemann curvature tensor is determined by the single component known as the Gauß curvature, while for three-dimensional manifolds it is determined by the six components $R^i{}_{jij}$ for $i \neq j$.

Chapter 2

Application to a certain hydrodynamic-type system

2.1 Introduction

Solving problems in physics one often faces systems of first-order quasilinear differential equations, that is, the systems which are linear in the first derivatives of the dependent variables but whose coefficients are in general functions of the dependent variables. Such systems are common for acoustics, fluid mechanics, gas and shock dynamics [69] and for the case of two independent variables have a general form

$$\sum_{j=1}^{n} A^{ij}\frac{\partial u^j}{\partial t} + \sum_{j=1}^{n} B^{ij}\frac{\partial u^j}{\partial x} + C^i = 0, \quad i = 1, \ldots, n, \tag{2.1}$$

where the $n \times n$ matrices A, B and the n-component vector C are the functions of independent and dependent variables (t, x) and $(u^1, \ldots, u^n)$ but not of the derivatives of the latter. Furthermore, nowadays such systems are a subject of intense research, see for example [5, 8, 22, 24, 45, 54] and references therein.

An important class of such systems is given by (evolutionary translation-invariant) systems of hydrodynamic type, for which (A^{ij}) is the $n \times n$ unit matrix, the vector (C^i) vanishes, and the matrix (B^{ij}) depends only on dependent variables. If a hydrodynamic-type system can be diagonalized by the change of transformations $r = r(u)$, that is, $B^{ij}(r) = 0$ for $i \neq j$, then the new dependent variables are called its Riemann invariants. Note that this is not the common case for systems with more than two dependent variables. The eigenvalues V^i of the matrix (B^{ij}) for a hydrodynamic-type system are commonly referred to as the characteristic speeds of the latter. A hydrodynamic-type system is called genuinely nonlinear if $\partial V^i/\partial r^i \neq 0$ for all i, and linearly degenerate if all of these inequalities fail.

In the theory of hydrodynamic-type systems there exists a criterion of integrability using the generalized hodograph transformation [63, 64]. It states that a diagonalizable strictly hyperbolic (the strict hyperbolicity means that all characteristic velocities V^i are real and distinct) hydrodynamic-type system is integrable in the sense presented below if and only if the condition

$$\partial_i \frac{\partial_j V^k}{V^j - V^k} = \partial_j \frac{\partial_i V^k}{V^i - V^k}$$

holds for all $i \neq j \neq k$. Such hydrodynamic-type systems are called *semi-Hamiltonian* [63, 64] (see e.g. [13, p. 60] and [14] for further details). In general, generalized hodograph transformation allows one to locally represent the general solution (except the solutions for which $r^i_x = 0$ for some i) in the form

$$x - V^i(r)t = W^i(r), \tag{2.2}$$

where $V = (V^1, V^2, V^3)$ is the tuple of characteristic speeds of the system (2.5), $r = (r^1, r^2, r^3)$ is the tuple of its Riemann invariants and $W = (W^1, W^2, W^3)$ is the general solution of the system

$$\frac{W^i_j}{W^j - W^i} = \frac{V^i_j}{V^j - V^i}, \quad i \neq j,$$

with the nondegeneracy condition $\det(V^i_j t + W^i_j) \neq 0$, where the indices i and j runs from 1 to n, which guarantees that the ansatz (2.2) is locally solvable with respect to r.

The problem of two phase flow phenomena is of great importance in physics thanks to its applications in several rapidly developing branches such as nuclear power and chemical industries [15, 70]. In particular, the accurate prediction of void fraction in the sub-channel under two phase flow is of fundamental significance. Unfortunately, this problem is quite challenging and therefore various simplified models of two phase flow phenomena were developed. One of these is the drift flux model introduced in [72], which allows to describe the mixing motion rather than the individual phases. It was thoroughly studied in [16, 17, 18], where several sub-models of the drift flux model were found, and, in particular, the concept of the slip function was introduced. The model with no-slip condition

$$\rho^1_t + u\rho^1_x + u_x\rho^1 = 0, \tag{2.3a}$$
$$\rho^2_t + u\rho^2_x + u_x\rho^2 = 0, \tag{2.3b}$$
$$(\rho^1 + \rho^2)(u_t + uu_x) + a^2(\rho^1_x + \rho^2_x) = 0, \tag{2.3c}$$

where $u = u(t, x)$ is the common velocity in both phases, $\rho^i = \rho^i(t, x)$, $i = 1, 2$ are densities of liquids (or liquid and gas) and a is a constant depending on both phases, was considered in [2]. In [52] an attempt at performing the group analysis for this system was made. Unfortunately, the said work contains a number of inaccuracies, including an incorrect computation of the maximal Lie invariance algebra as well as some mistakes in the classification of its one-dimensional and two-dimensional subalgebras. Also, the system (2.3) has many nice additional properties as it is of hydrodynamic type. Thus, the goal of the present paper is to revisit and to extend the group analysis of (2.3).

To this end it is convenient to simplify the initial model (2.3) by an appropriate change of variables. First of all, scaling simultaneously x and u we can set $a = 1$

provided that a is positive, which is justified from the physical point of view. Introducing the new dependent variables $v = \ln(\rho^1 + \rho^2)$ and $w = \rho^1/\rho^2$ instead of ρ^1 and ρ^2, we rewrite the system (2.3) as the system $\mathcal{S}$ which reads

$$u_t + uu_x + v_x = 0, \tag{2.4a}$$
$$v_t + uv_x + u_x = 0, \tag{2.4b}$$
$$w_t + uw_x = 0. \tag{2.4c}$$

The system (2.4) is obviously of the hydrodynamic type. Furthermore, diagonalizing the matrix B of the representation (2.1) for the system (2.4) we map the system (2.4) to the system

$$r^1_t + \left(\frac{r^1 + r^2}{2} + 1\right) r^1_x = 0, \tag{2.5a}$$
$$r^2_t + \left(\frac{r^1 + r^2}{2} - 1\right) r^2_x = 0, \tag{2.5b}$$
$$r^3_t + \frac{r^1 + r^2}{2} r^3_x = 0 \tag{2.5c}$$

using the change of dependent variables $r^1 = u + v$, $r^2 = u - v$, $r^3 = w$. Therefore, r^1, r^2 and r^3 are Riemann invariants of the system (2.4), and Riemann invariants of the initial system (2.3) are

$$r^1 = u + \ln(\rho^1 + \rho^2), \quad r^2 = u - \ln(\rho^1 + \rho^2), \quad r^3 = \frac{\rho^1}{\rho^2},$$

and thus the expressions for the initial dependent variables (u, ρ^1, ρ^2) in terms of the Riemann invariants are

$$u = \frac{r^1 + r^2}{2}, \quad \rho^1 = \frac{r^3}{r^3 + 1} \exp\frac{r^1 - r^2}{2}, \quad \rho^2 = \frac{1}{r^3 + 1} \exp\frac{r^1 - r^2}{2}.$$

We also readily see that

$$V^1 = \frac{r^1 + r^2}{2} + 1, \quad V^2 = \frac{r^1 + r^2}{2} - 1, \quad V^3 = \frac{r^1 + r^2}{2} \tag{2.6}$$

are characteristic speeds of the system (2.5). Besides, the system (2.5) is not genuinely nonlinear as $\partial_{r^3} V^3 = 0$, although it is strictly hyperbolic and diagonalizable, so generalized hodograph transformation can be utilized. Nevertheless, it also makes sense to extend the scope and to study this system within the framework of group analysis. Note that the subsystem of the first two equations of (2.5) coincides with the diagonalized form of the system describing one-dimensional isentropic gas flows with constant sound speed [53, Section 2.2.7, Eq. (16)].

Throughout the text we switch among the forms (2.4) and (2.5). It is often more convenient to use diagonalized form for computation, although many results are more concisely expressed in terms of the variables u, v and w. Here and in what follows the subscript i denotes the derivative with respect to the Riemann invariant r^i.

2.2 Lie symmetries

In order to compute the maximal Lie invariance algebra of a given system of differential equations we invoke the infinitesimal method (see Theorem 1.29).

The infinitesimal generators of one-parameter Lie symmetry groups for the system $\mathcal{S}$ are defined as $Q = \tau\partial_t + \xi\partial_x + \eta\partial_u + \theta\partial_v + \kappa\partial_w$, where the components τ, ξ, η, θ and κ depend on t, x, u, v, and w. The infinitesimal invariance criterion requires that

$$Q^{(1)}\left(u_t + uu_x + v_x\right)|_{\mathcal{S}} = 0, \quad Q^{(1)}\left(v_t + u_x + uv_x\right)|_{\mathcal{S}} = 0, \quad Q^{(1)}\left(w_t + uw_x\right)|_{\mathcal{S}} = 0.$$

The first prolongation $Q^{(1)}$ of the vector field Q is given by

$$Q^{(1)} = Q + \eta^{(1,0)}\partial_{u_t} + \eta^{(0,1)}\partial_{u_x} + \theta^{(1,0)}\partial_{v_t} + \theta^{(0,1)}\partial_{v_x} + \kappa^{(1,0)}\partial_{w_t} + \kappa^{(0,1)}\partial_{w_x},$$

where the components $\eta^{(1,0)}$, $\eta^{(0,1)}$, $\theta^{(1,0)}$, $\theta^{(0,1)}$, $\kappa^{(1,0)}$ and $\kappa^{(0,1)}$ of the prolonged vector field $Q^{(1)}$ are readily derived from the general prolongation formula (see Theorem 1.31),

$$\begin{aligned}
\eta^{\alpha} &= \mathrm{D}^{\alpha}\left(\eta - \tau u_t - \xi u_x\right) + \tau u_{\alpha+\delta_1} + \xi u_{\alpha+\delta_2},\\
\theta^{\alpha} &= \mathrm{D}^{\alpha}\left(\theta - \tau v_t - \xi v_x\right) + \tau v_{\alpha+\delta_1} + \xi v_{\alpha+\delta_2},\\
\kappa^{\alpha} &= \mathrm{D}^{\alpha}\left(\kappa - \tau w_t - \xi w_x\right) + \tau w_{\alpha+\delta_1} + \xi w_{\alpha+\delta_2}.
\end{aligned}$$

Here $\alpha = (\alpha_1, \alpha_2)$ is a multi-index, $\delta_1 = (1,0)$, $\delta_2 = (0,1)$, $\mathrm{D}^{\alpha} = \mathrm{D}_t^{\alpha_1}\mathrm{D}_x^{\alpha_2}$, D_t and D_x are the total derivative operators with respect to t and x, respectively,

$$\begin{aligned}
\mathrm{D}_t &= \partial_t + \sum_{\alpha}(u_{\alpha+\delta_1}\partial_{u_\alpha} + v_{\alpha+\delta_1}\partial_{v_\alpha} + w_{\alpha+\delta_1}\partial_{w_\alpha}),\\
\mathrm{D}_x &= \partial_x + \sum_{\alpha}(u_{\alpha+\delta_2}\partial_{u_\alpha} + v_{\alpha+\delta_2}\partial_{v_\alpha} + w_{\alpha+\delta_2}\partial_{w_\alpha})
\end{aligned}$$

with $u_\alpha = \partial^{\alpha_1+\alpha_2}u/\partial t^{\alpha_1}\partial x^{\alpha_2}$, etc.

Thus the infinitesimal invariance criterion implies that

$$\begin{aligned}
&\eta^{(1,0)} + u\eta^{(0,1)} + \eta u_x + \theta^{(0,1)} = 0,\\
&\theta^{(1,0)} + u\theta^{(0,1)} + \eta v_x + \eta^{(0,1)} = 0,\\
&\kappa^{(1,0)} + u\kappa^{(0,1)} + \eta w_x = 0
\end{aligned}$$

when substituting $u_t = -uu_x - v_x$, $v_t = -uv_x - u_x$ and $w_t = -uw_x$. Splitting with respect to the parametric derivatives u_x, v_x and w_x results in the system of determining equations for the components of Lie symmetry vector fields,

$$\begin{aligned}
&\kappa_t = \kappa_x = \kappa_u = \kappa_v = \tau_x = \tau_u = \tau_v = \tau_w = \eta_u = \eta_v = \eta_w = \theta_u = \theta_v = \theta_w = 0,\\
&\xi_x = \tau_t, \quad \eta = \xi_t, \quad \eta_t + u\eta_x + \theta_x = 0, \quad \theta_t + u\theta_x + \eta_x = 0.
\end{aligned}$$

The general solution of this system is

$$\tau = \xi^1 t + \tau^0, \quad \xi = \eta^0 t + \xi^1 x + \xi^0, \quad \eta = \eta^0, \quad \theta = \theta^0, \quad \kappa = \kappa(w),$$

where τ^0, ξ^0, ξ^1, η^0, η^1 and θ^0 are arbitrary constants and κ runs through the set of smooth functions of its argument. This proves the following theorem.

Theorem 2.1. *The maximal Lie invariance algebra $\mathfrak{g}$ of the system $\mathcal{S}$ is infinite-dimensional and spanned by the vector fields*

$$\begin{aligned}&\mathcal{D} = t\partial_t + x\partial_x, \quad \mathcal{G} = t\partial_x + \partial_u, \quad \mathcal{P}^t = \partial_t,\\ &\mathcal{P}^x = \partial_x, \quad \mathcal{P}^v = \partial_v, \quad \mathcal{W}(\kappa) = \kappa(w)\partial_w,\end{aligned} \tag{2.7}$$

where κ runs through the set of smooth functions of w.

Remark 2.2. The systems (2.3) and (2.4) are connected by the point transformation, where t, x and u are not changed, and

$$\rho^1 = \frac{w}{w+1}e^v, \quad \rho^2 = \frac{e^v}{w+1}.$$

This transformation pushforwards the algebra $\mathfrak{g}$ to the maximal Lie invariance algebra $\tilde{\mathfrak{g}}$ of the initial system (2.3), where $\tilde{\mathfrak{g}} = \langle \tilde{\mathcal{D}}, \tilde{\mathcal{G}}, \tilde{\mathcal{P}}^t, \tilde{\mathcal{P}}^x, \tilde{\mathcal{P}}^v, \tilde{\mathcal{W}}(\tilde{\kappa})\rangle$, where $\tilde{\mathcal{D}}$, $\tilde{\mathcal{G}}$, $\tilde{\mathcal{P}}^t$, $\tilde{\mathcal{P}}^x$ are formally of the same form as their counterparts in the algebra $\mathfrak{g}$,

$$\tilde{\mathcal{P}}^v = \rho^1\partial_{\rho^1} + \rho^2\partial_{\rho^2}, \quad \tilde{\mathcal{W}}(\tilde{\kappa}) = \tilde{\kappa}\left(\frac{\rho^1}{\rho^2}\right)\rho^2(\partial_{\rho^1} - \partial_{\rho^2}),$$

and $\tilde{\kappa}$ runs through the set of the smooth functions of its argument. Note that the infinite-dimensional part $\langle \tilde{\mathcal{W}}(\tilde{\kappa})\rangle$ of the algebra $\tilde{\mathfrak{g}}$ was missed in [52].

Remark 2.3. Analogously, the maximal Lie invariance algebra of the system (2.5) is spanned by vector fields

$$\begin{aligned}&\hat{\mathcal{D}} = t\partial_t + x\partial_x, \quad \hat{\mathcal{G}} = t\partial_x + \partial_{r^1} + \partial_{r^2}, \quad \hat{\mathcal{P}}^t = \partial_t, \quad \hat{\mathcal{P}}^x = \partial_x,\\ &\hat{\mathcal{P}}^v = \partial_{r^1} - \partial_{r^2}, \quad \hat{\mathcal{W}}(\kappa) = \kappa(r^3)\partial_{r^3},\end{aligned}$$

where κ runs through the set of smooth functions of r^3.

2.3 Complete point symmetry group

We compute the complete point symmetry group of the system (2.4) using the megaideal-based version of the algebraic method, described in Section 1.3.

The nonzero commutation relations among generating elements (2.7) of the maximal Lie invariance algebra $\mathfrak{g}$ of the system $\mathcal{S}$ are exhausted by

$$\begin{aligned}&[\mathcal{D}, \mathcal{P}^t] = -\mathcal{P}^t, \quad [\mathcal{D}, \mathcal{P}^x] = -\mathcal{P}^x, \quad [\mathcal{G}, \mathcal{P}^t] = -\mathcal{P}^x,\\ &[\mathcal{W}(\kappa^1), \mathcal{W}(\kappa^2)] = \mathcal{W}(\kappa^1\kappa^2_w - \kappa^2\kappa^1_w).\end{aligned}$$

Therefore, the algebra $\mathfrak{g}$ is the direct sum of its finite-dimensional and infinite-dimensional parts, $\mathfrak{g} = \langle \mathcal{D}, \mathcal{G}, \mathcal{P}^t, \mathcal{P}^x, \mathcal{P}^v\rangle \oplus \langle \mathcal{W}(\kappa)\rangle$. Moreover, the finite-dimensional part can be split into a direct sum as well, $\mathfrak{g} = \langle \mathcal{D}, \mathcal{G}, \mathcal{P}^t, \mathcal{P}^x\rangle \oplus \langle \mathcal{P}^v\rangle \oplus \langle \mathcal{W}(\kappa)\rangle$.

We now construct a list of megaideals of the algebra $\mathfrak{g}$. Firstly, the derivatives of $\mathfrak{g}$ are megaideals of $\mathfrak{g}$, so $\mathfrak{g}' = \langle \mathcal{P}^t, \mathcal{P}^x, \mathcal{W}(\kappa)\rangle$ and $\mathfrak{g}'' = \langle \mathcal{W}(\kappa)\rangle$ are megaideals, with $\mathfrak{g}^{(i)} = \mathfrak{g}''$ for $i \geqslant 2$. The center $\mathcal{Z}(\mathfrak{g}) = \langle \mathcal{P}^v\rangle$ of the algebra $\mathfrak{g}$ is also its megaideal.

Lemma 2.4. *The radical* $\mathfrak{r}$ *of* $\mathfrak{g}$ *coincides with the finite-dimensional part of* $\mathfrak{g}$,

$$\mathfrak{r} = \langle \mathcal{D}, \mathcal{G}, \mathcal{P}^t, \mathcal{P}^x, \mathcal{P}^v \rangle.$$

Proof. We temporarily denote by $\mathfrak{s} = \langle \mathcal{D}, \mathcal{G}, \mathcal{P}^t, \mathcal{P}^x, \mathcal{P}^v \rangle$ the finite-dimensional part of $\mathfrak{g}$. The subspace $\mathfrak{s}$ is an ideal of $\mathfrak{g}$, which is solvable since $\mathfrak{s}'' = \{0\}$. Therefore, it is contained in the radical $\mathfrak{r}$ of $\mathfrak{g}$, $\mathfrak{s} \subseteq \mathfrak{r}$. If an ideal of $\mathfrak{g}$ contains a vector field $\mathcal{W}(\kappa^0)$ for a nonvanishing value $\kappa^0 = \kappa^0(w)$ of the parameter function κ, then it contains the entire infinite-dimensional part $\langle \mathcal{W}(\kappa) \rangle$ and hence it is not solvable. This means that $\mathfrak{r} \cap \langle \mathcal{W}(\kappa) \rangle = \{0\}$. Therefore, $\mathfrak{r} = \mathfrak{s}$. □

Thus, we found a Levi decomposition of the infinite-dimensional algebra $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{g}''$, where $\mathfrak{r}$ is the (finite-dimensional) radical and $\mathfrak{g}''$ is an (infinite-dimensional) simple subalgebra, which is also an (mega)ideal of $\mathfrak{g}$ but contains no proper subideals.

Lemma 2.5. *The nilradical* $\mathfrak{n}$ *of* $\mathfrak{g}$ *is spanned by the vector fields* $\mathcal{G}$, $\mathcal{P}^t$, $\mathcal{P}^x$ *and* $\mathcal{P}^v$,

$$\mathfrak{n} = \langle \mathcal{G}, \mathcal{P}^t, \mathcal{P}^x, \mathcal{P}^v \rangle.$$

Proof. The nilradical of $\mathfrak{g}$ is contained in the radical $\mathfrak{r}$ of $\mathfrak{g}$. We temporarily denote by $\mathfrak{s}$ the span of the vector fields $\mathcal{G}$, $\mathcal{P}^t$, $\mathcal{P}^x$ and $\mathcal{P}^v$, $\mathfrak{s} = \langle \mathcal{G}, \mathcal{P}^t, \mathcal{P}^x, \mathcal{P}^v \rangle$. The subspace $\mathfrak{s}$ is an ideal of $\mathfrak{g}$, and it is nilpotent since $\mathfrak{s}^2 = \{0\}$. Moreover, $\mathfrak{s}$ is the maximal nilpotent ideal of $\mathfrak{g}$ since the only subspace of $\mathfrak{r}$ properly containing $\mathfrak{s}$ is the radical $\mathfrak{r}$ itself, which is not nilpotent. Thus, $\mathfrak{n} = \mathfrak{s}$. □

Corollary 2.6. *The derivatives* $\mathfrak{r}' = \langle \mathcal{P}^t, \mathcal{P}^x \rangle$ *and* $\mathfrak{n}' = \langle \mathcal{P}^x \rangle$ *of the radical* $\mathfrak{r}$ *and the nilradical* $\mathfrak{n}$ *of* $\mathfrak{g}$*, respectively, are megaideals of* $\mathfrak{g}$.

Corollary 2.7. *The ideal* $\mathfrak{m}_1 = \langle \mathcal{G}, \mathcal{P}^x, \mathcal{P}^v \rangle$ *of the algebra* $\mathfrak{g}$ *is its megaideal.*

Proof. This is a simple consequence of Proposition 1.39 for $\mathfrak{i}_0 = \mathfrak{r}$, $\mathfrak{i}_1 = \mathfrak{r}$ and $\mathfrak{i}_2 = \mathfrak{n}'$. □

The nilradical $\mathfrak{n}$ is not essential for the megaideal-based version of the algebraic method since it is the sum of other megaideals of $\mathfrak{g}$, $\mathfrak{n} = \mathfrak{m}_1 + \mathfrak{r}'$. As a result, for finding the complete point symmetry group of the system (2.4) with the megaideal-based version of the algebraic method we use the following list of megaideals of the algebra $\mathfrak{g}$:

$$\langle \mathcal{D}, \mathcal{G}, \mathcal{P}^t, \mathcal{P}^x, \mathcal{P}^v \rangle, \quad \langle \mathcal{G}, \mathcal{P}^x, \mathcal{P}^v \rangle, \quad \langle \mathcal{P}^t, \mathcal{P}^x \rangle, \quad \langle \mathcal{P}^x \rangle, \quad \langle \mathcal{P}^v \rangle, \quad \langle \mathcal{W}(\kappa) \rangle. \tag{2.8}$$

Theorem 2.8. *The complete point symmetry group* G *of the modified no-slip isothermal drift flux model* (2.4) *consists of the transformations*

$$\begin{aligned} &\tilde{t} = T^1 t + T^0, \quad \tilde{x} = T^1 x + T^1 U^0 t + X^0, \\ &\tilde{u} = u + U^0, \quad \tilde{v} = v + V^0, \quad \tilde{w} = W(w), \end{aligned} \tag{2.9}$$

where T^0, T^1, X^0, U^0 *and* V^0 *are arbitrary constants with* $T^1 \neq 0$ *and* W *runs through the set of smooth functions of* w *with* $W_w \neq 0$.

Proof. The general form of a point symmetry transformation for the system (2.4) is

$$\mathcal{T}\colon\ (\tilde t,\tilde x,\tilde u,\tilde v,\tilde w)=(T,X,U,V,W),$$

where T, X, U, V and W are functions of t, x, u, v and w with non-vanishing Jacobian. To obtain constrains for a point symmetry transformation $\mathcal{T}$, we push-forward each of the Lie symmetry generators (2.7) by this transformation and use the invariance, with respect to the pushforward $\mathcal{T}_*$, of the minimal megaideal from the list (2.8) that contains the taken Lie symmetry generator. This leads to the following conditions:

$$\begin{aligned}
&\mathcal{T}_*\mathcal{D}=a_{11}(\tilde t\partial_{\tilde t}+\tilde x\partial_{\tilde x})+a_{21}(\tilde t\partial_{\tilde x}+\partial_{\tilde u})+a_{31}\partial_{\tilde t}+a_{41}\partial_{\tilde x}+a_{51}\partial_{\tilde v},\\
&\mathcal{T}_*\mathcal{G}=a_{22}(\tilde t\partial_{\tilde x}+\partial_{\tilde u})+a_{42}\partial_{\tilde x}+a_{52}\partial_{\tilde v},\\
&\mathcal{T}_*\mathcal{P}^t=T_t\partial_{\tilde t}+X_t\partial_{\tilde x}+U_t\partial_{\tilde u}+V_t\partial_{\tilde v}+W_t\partial_{\tilde w}=a_{33}\partial_{\tilde t}+a_{43}\partial_{\tilde x}\\
&\mathcal{T}_*\mathcal{P}^x=T_x\partial_{\tilde t}+X_x\partial_{\tilde x}+U_x\partial_{\tilde u}+V_x\partial_{\tilde v}+W_x\partial_{\tilde w}=a_{44}\partial_{\tilde x},\\
&\mathcal{T}_*\mathcal{P}^v=T_v\partial_{\tilde t}+X_v\partial_{\tilde x}+U_v\partial_{\tilde u}+V_v\partial_{\tilde v}+W_v\partial_{\tilde w}=a_{55}\partial_{\tilde v},\\
&\mathcal{T}_*\mathcal{W}(\kappa)=\kappa(T_w\partial_{\tilde t}+X_w\partial_{\tilde x}+U_w\partial_{\tilde u}+V_w\partial_{\tilde v}+W_w\partial_{\tilde w})=\tilde\kappa^\kappa\partial_{\tilde w},
\end{aligned}\tag{2.10}$$

where all a's are constants and $\tilde\kappa$ is a smooth function of $\tilde w$ depending on the parameter function $\kappa=\kappa(w)$.

We collect components of vector fields in the last four conditions of (2.10) and then take into account the obtained equations when expending and componentwise splitting the first two conditions. As a result, we derive the following system:

$$\begin{aligned}
&T_t=a_{33},\quad X_t=a_{43},\quad U_t=V_t=W_t=0;\quad T_x=U_x=V_x=W_x=0;\\
&X_x=a_{44};\quad T_v=X_v=U_v=W_v=0,\quad V_v=a_{55};\\
&T_w=X_w=U_w=V_w=0;\quad \kappa W_w=\tilde\kappa^\kappa(W);\\
&tT_t=a_{11}T+a_{31},\quad tX_t+xX_x=a_{11}X+a_{41},\quad a_{21}=a_{51}=0,\\
&T_u=W_u=0,\quad tX_x+X_u=a_{22}T+a_{42},\quad U_u=a_{22},\quad V_u=a_{52}.
\end{aligned}\tag{2.11}$$

Equations of this system are partitioned into groups according to their source conditions in (2.10). The general solutions of the system (2.11) is given by

$$\begin{aligned}
&T=a_{33}t-a_{31},\quad X=a_{22}a_{33}x+a_{43}t-a_{41},\\
&U=a_{22}u+U^0,\quad V=a_{55}v+a_{52}u+V^0,\quad W=W(w),
\end{aligned}\tag{2.12}$$

where U^0 and V^0 are arbitrary constants, and additionally $a_{22}a_{33}a_{55}\neq0$, $a_{11}=1$, $a_{44}=a_{22}a_{33}$ and $a_{42}=a_{31}a_{22}$.

Now the direct method of computing complete point symmetry groups should be applied. To this end, we apply a transformation of the form (2.12) to the system $\mathcal{S}$. To do this, the derivative operators with respect to the new independent variables, $\partial_{\tilde t}$ and $\partial_{\tilde x}$, have to be determined,

$$\partial_{\tilde t}=\frac{1}{a_{33}}\left(\partial_t-\frac{a_{43}}{a_{22}a_{33}}\partial_x\right),\quad \partial_{\tilde x}=\frac{1}{a_{22}a_{33}}\partial_x.$$

Using these derivative operators to express the derivatives of the new variables, and enforcing the symmetry condition requires that $a_{43} = a_{33}U^0$, $a_{22} = a_{55} = 1$, $a_{52} = 0$.

Re-denoting parameter constants completes theorem's proof. □

Corollary 2.9. *The modified no-slip isothermal drift flux model* (2.4) *possesses two independent (up to combining with each other and with continuous symmetries) discrete point symmetries given by the reflections*

$$(t, x, u, v, w) \to (-t, -x, u, v, w) \quad \textit{and} \quad (t, x, u, v, w) \to (t, x, u, v, -w).$$

Remark 2.10. As has been already mentioned in Section 1.3, the automorphism-based version of the algebraic method for finding the complete point symmetry group is based on knowing the automorphism group of the corresponding maximal invariance algebra. For the system $\mathcal{S}$, this algebra is infinite-dimensional, and therefore the computation of its automorphism group is complicated. However, it might be useful to look for the automorphism group $\mathrm{Aut}(\mathfrak{r})$ of the radical $\mathfrak{r}$ which coincides, in view of Lemma 2.4, with the finite-dimensional part of the algebra $\mathfrak{g}$. Note that the algebra $\mathfrak{r}$ is isomorphic to the algebra $A^0_{4,8} \oplus A_1$ from the classification list of five-dimensional real Lie algebras presented in [36, 37]. In the basis $(\mathcal{D}, \mathcal{G}, \mathcal{P}^t, \mathcal{P}^x, \mathcal{P}^v)$, the automorphism group $\mathrm{Aut}(\mathfrak{r})$ group can be identified with the matrix group constituted by the matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & b_{22} & 0 & 0 & 0 \\ b_{31} & 0 & b_{33} & 0 & 0 \\ b_{41} & b_{31}b_{22} & b_{43} & b_{22}b_{33} & 0 \\ b_{51} & b_{52} & 0 & 0 & b_{55} \end{pmatrix} \qquad \text{with} \qquad b_{22}b_{33}b_{55} \neq 0.$$

The knowledge of $\mathrm{Aut}(\mathfrak{r})$ allows us to set constraints on the constants a's in the conditions (2.10) before analyzing these conditions,

$$a_{11} = 1, \quad a_{22}a_{33}a_{55} \neq 0, \quad a_{21} = 0, \quad a_{44} = a_{22}a_{33}, \quad a_{42} = a_{31}a_{22}.$$

The structure of automorphism matrices obviously implies that there is one more megaideal of $\mathfrak{r}$ and, therefore, of $\mathfrak{g}$, $\mathfrak{m}_2 = \langle \mathcal{D}, \mathcal{P}^t, \mathcal{P}^x, \mathcal{P}^v \rangle$. This completes the description of megaideals of the algebra $\mathfrak{g}$. More specifically, the megaideals of $\mathfrak{g}$ are exhausted by the essential megaideals $\mathfrak{n}'$, $\mathcal{Z}(\mathfrak{g})$, $\mathfrak{r}'$, $\mathfrak{m}_1$, $\mathfrak{m}_2$ and $\mathfrak{g}''$ and their sums. The megaideal $\mathfrak{m}_2$ cannot be found using means presented in Section 1.3. The presence of this megaideal explains the first of the above constraints on the constants a's whilst the rest of these constraints cannot be obtained using the megaideal-based version of algebraic method, being, at the same time, a direct consequence of its automorphism-based counterpart applied to $\mathfrak{r}$. In fact, this discussion shows a possibility for combining the automorphism- and megaideal-based versions of algebraic method for finding the complete point symmetry groups of systems of differential equations.

Remark 2.11. There is one constraint for the constants a's, $a_{51} = 0$, among those derived from the conditions (2.10) that cannot be obtained from the structure of automorphism matrices of the radical $\mathfrak{r}$. This means that there exist automorphisms of the entire algebra $\mathfrak{g}$ that are not induced by point transformations of (t, x, u, v, w).

2.4 Classification of subalgebras

In order to efficiently carry out group-invariant reductions for finding exact solutions of partial differential equations, it is necessary to determine an optimal list of inequivalent (under the action of the adjoint representation of a Lie group on the associate Lie algebra) subalgebras of the admitted maximal Lie invariance algebra $\mathfrak{g}$ of the system under consideration (see Section 1.4). Such an optimal list is derived as follows: one simplifies the set of linear independent admitted infinitesimal generators in the most general form by actions of adjoint representations of the one-parameters point transformations on corresponding Lie invariance algebra, recombining basis operators, taking into account the requirement of being closed under Lie multiplication and collecting all arising inequivalent subalgebras in this list.

To determine adjoint representations of one-parameter subgroups of a group G on the maximal Lie invariance algebra $\mathfrak{g}$ we compute the actions of pushforwards of the transformations from the group G on generating vector fields of the algebra $\mathfrak{g}$. Any transformation $\mathcal{T}$ from G can be presented as a composition

$$\mathcal{T} = \mathcal{D}(T^1)\mathcal{P}(T^0)\mathcal{P}(X^0)\mathcal{P}(V^0)\mathcal{G}(U^0)\mathcal{W}(W),$$

where

$$\begin{aligned}
&\mathcal{P}^t(T^0)\colon (\tilde t,\tilde x,\tilde u,\tilde v,\tilde w) = (t+T^0,x,u,v,w),\\
&\mathcal{P}^x(X^0)\colon (\tilde t,\tilde x,\tilde u,\tilde v,\tilde w) = (t,x+X^0,u,v,w),\\
&\mathcal{P}^v(V^0)\colon (\tilde t,\tilde x,\tilde u,\tilde v,\tilde w) = (t,x,u,v+V^0,w),\\
&\mathcal{D}(T^1)\colon (\tilde t,\tilde x,\tilde u,\tilde v,\tilde w) = (T^1t,T^1x,u,v,w),\\
&\mathcal{G}(U^0)\colon (\tilde t,\tilde x,\tilde u,\tilde v,\tilde w) = (t,x+U^0t,u+U^0,v,w),\\
&\mathcal{W}(W)\colon (\tilde t,\tilde x,\tilde u,\tilde v,\tilde w) = (t,x,u,v,W(w)),
\end{aligned}$$

where, as in Theorem 2.8, T^0, T^1, X^0, U^0 and V^0 are constants and $W(w)$ runs through the set of smooth functions of w with $T^1W_w\neq 0$. The nonidentical actions of pushforwards of these transformations on generating elements of $\mathfrak{g}$ are exhausted by the following:

$$\begin{aligned}
&\mathcal{P}^t_*(T^0)\mathcal{D} = \mathcal{D} - T^0\mathcal{P}^t, \quad \mathcal{P}^t_*(T^0)\mathcal{G} = \mathcal{G} - T^0\mathcal{P}^x,\\
&\mathcal{P}^x_*(X^0)\mathcal{D} = \mathcal{D} - X^0\mathcal{P}^x, \quad \mathcal{D}_*(T^1)\mathcal{P}^t = T^1\mathcal{P}^t,\\
&\mathcal{D}_*(T^1)\mathcal{P}^x = T^1\mathcal{P}^x, \quad \mathcal{G}_*(U^0)\mathcal{P}^t = \mathcal{P}^t + U^0\mathcal{P}^x,\\
&\mathcal{W}_*(W)\mathcal{W}(\kappa) = \mathcal{W}(\tilde\kappa),
\end{aligned} \tag{2.13}$$

where κ is an arbitrary smooth functions of w and $\tilde\kappa$ is a function of $\tilde w$ is related to κ by $\tilde\kappa(W(w)) = W_w(w)\kappa(w)$, and the other parameters are as in Theorem 2.8.

Theorem 2.12. *An optimal list of one-dimensional subalgebras of the maximal Lie invariance algebra* $\mathfrak{g}$ *is exhausted by the subalgebras*

$$\begin{aligned}
&\langle \mathcal{D} + a\mathcal{G} + b\mathcal{P}^v + \mathcal{W}(\delta_1)\rangle, \quad \langle \mathcal{G} + \delta_2\mathcal{P}^t + b\mathcal{P}^v + \mathcal{W}(\delta_1)\rangle,\\
&\langle \mathcal{P}^t + \delta_2\mathcal{P}^v + \mathcal{W}(\delta_1)\rangle, \quad \langle \mathcal{P}^x + \delta_2\mathcal{P}^v + \mathcal{W}(\delta_1)\rangle,\\
&\langle \mathcal{P}^v + \mathcal{W}(\delta_1)\rangle, \quad \langle \mathcal{W}(1)\rangle,
\end{aligned} \tag{2.14}$$

where $\delta_1, \delta_2 \in \{0, 1\}$, *and* $a, b \in \mathbb{R}$ *are arbitrary constants.*

Proof. Starting with the most general form of a generating vector field of a one-dimensional subalgebra of the algebra $\mathfrak{g}$,

$$\mathrm{v} = a_1\mathcal{D} + a_2\mathcal{G} + a_3\mathcal{P}^t + a_4\mathcal{P}^x + a_5\mathcal{P}^v + \mathcal{W}(\kappa),$$

where $a_i \in \mathbb{R}$, $i = 1, \ldots, 5$, and the function $\kappa(w)$ runs through the set of smooth functions of w, we simplify it by using the actions of pushforwards (2.13). In all cases when $\kappa \neq 0$, the parameter function κ can be set to 1 by applying the pushforward $\mathcal{W}_*(\kappa^{-1})$ (see the last equation in the system (2.13) and explanations after it). In other words, we can always suppose $\kappa(w) = \delta_1$. Moreover, the operator $\mathcal{P}^v$ cannot be cancelled by application of any pushforward (i.e. one cannot set $a_5 = 0$), and hence is presented in all subalgebras of the maximal Lie invariance algebra $\mathfrak{g}$.

The further consideration is carried out recursively: regarding the leading coefficient of the vector field as nonzero, we simplify this vector field and eliminate this coefficient afterwards.

If $a_1 \neq 0$, then v can be scaled to set $a_1 = 1$. Using successively the pushforwards $\mathcal{P}^t_*$ and $\mathcal{P}^x_*$ we can set $a_3 = a_4 = 0$. This yields the first subalgebra of the list (2.14).

If $a_1 = 0$ and $a_2 \neq 0$ we can set $a_2 = 1$. After applying $\mathcal{P}^t_*$, $\mathcal{D}_*$ and renaming the second subalgebra is obtained.

If $a_1 = a_2 = 0$ and $a_3 \neq 0$ we can scale $a_3 = 1$. If $a_5 \neq 0$, we can set $a_5 = a_3 = 1$ by acting $\mathcal{D}_*$; otherwise we act by $\mathcal{G}_*$ and derive the third subalgebra.

If $a_1 = a_2 = a_3 = 0$ and $a_4 \neq 0$ then we can set $a_4 = 1$ and upon applying the pushforward action $\mathcal{D}_*$ we derive the fourth subalgebra.

The fifth subalgebra is obtained when $a_i = 0$ for $i = 1, \ldots, 4$ and $a_5 \neq 0$, in which case we can scale $a_5 = 1$.

The last subalgebra follows from having $a_i = 0$ for $i = 1, \ldots, 5$. In this case κ has to be non-vanishing. □

Theorem 2.13. *An optimal list of two-dimensional subalgebras of the maximal* *algebra of invariance* $\mathfrak{g}$ *is given by*

$$
\begin{aligned}
&\langle \mathcal{D} + a\mathcal{P}^v + \mathcal{W}(\delta_1), \mathcal{G} + b\mathcal{P}^v + \mathcal{W}(\delta_2)\rangle, \quad \langle \mathcal{D} + a\mathcal{P}^v + \mathcal{W}(\delta_5), \mathcal{P}^t\rangle,\\
&\langle \mathcal{D} + a\mathcal{P}^v + \mathcal{W}(w), \mathcal{P}^t + \mathcal{W}(1)\rangle, \quad \langle \mathcal{D} + a\mathcal{G} + b\mathcal{P}^v + \mathcal{W}(\delta_5), \mathcal{P}^x\rangle,\\
&\langle \mathcal{D} + a\mathcal{G} + b\mathcal{P}^v + \mathcal{W}(w), \mathcal{P}^x + \mathcal{W}(1)\rangle,\\
&\langle \mathcal{D} + a\mathcal{G} + \mathcal{W}(\delta_1), \mathcal{P}^v + \mathcal{W}(\delta_2)\rangle,\\
&\langle \mathcal{G} + \delta_3\mathcal{P}^t + a\mathcal{P}^v + \mathcal{W}(\delta_1), \mathcal{P}^x + \delta_4\mathcal{P}^v + \mathcal{W}(\delta_2)\rangle,\\
&\langle \mathcal{G} + \delta_5\mathcal{P}^t + \mathcal{W}(\delta_1), \mathcal{P}^v + \mathcal{W}(\delta_2)\rangle,\\
&\langle \mathcal{P}^t + \delta_3\mathcal{P}^v + \mathcal{W}(\delta_1), \mathcal{P}^x + \delta_4\mathcal{P}^v + \mathcal{W}(\delta_2)\rangle,\\
&\langle \mathcal{P}^t + \mathcal{W}(\delta_1), \mathcal{P}^v + \mathcal{W}(\delta_2)\rangle, \quad \langle \mathcal{P}^x + \mathcal{W}(\delta_1), \mathcal{P}^v + \mathcal{W}(\delta_2)\rangle,\\
&\langle \mathcal{D} + a\mathcal{G} + b\mathcal{P}^v + c\mathcal{W}(w), \mathcal{W}(1)\rangle,\\
&\langle \mathcal{G} + \delta_5\mathcal{P}^t + b\mathcal{P}^v + c\mathcal{W}(w), \mathcal{W}(1)\rangle,\\
&\langle \mathcal{P}^t + \delta_1\mathcal{P}^v + \delta_2\mathcal{W}(w), \mathcal{W}(1)\rangle,\\
&\langle \mathcal{P}^x + \delta_1\mathcal{P}^v + \delta_2\mathcal{W}(w), \mathcal{W}(1)\rangle,\\
&\langle \mathcal{P}^v + c\mathcal{W}(w), \mathcal{W}(1)\rangle, \quad \langle \mathcal{W}(w), \mathcal{W}(1)\rangle,
\end{aligned}
\tag{2.15}
$$

if in the pairs (δ_1, δ_2) *and* (δ_3, δ_4) *there is at least one nonzero coefficient, it can be set 1;* $\delta_5 \in \{0, 1\}$*, and* a*,* b *and* c *are arbitrary real numbers.*

Proof. We start with two vector fields v_1 and v_2 of the most general form spanning the two-dimensional subalgebra $\langle \mathrm{v}_1, \mathrm{v}_2\rangle$, where

$$
\begin{aligned}
\mathrm{v}_1 &= a_1\mathcal{D} + a_2\mathcal{G} + a_3\mathcal{P}^t + a_4\mathcal{P}^x + a_5\mathcal{P}^v + \mathcal{W}(\kappa^1),\\
\mathrm{v}_2 &= b_1\mathcal{D} + b_2\mathcal{G} + b_3\mathcal{P}^t + b_4\mathcal{P}^x + b_5\mathcal{P}^v + \mathcal{W}(\kappa^2).
\end{aligned}
$$

For the sake of efficient classification it is convenient to introduce the matrix

$$
A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \end{pmatrix}
$$

as well as 2×2 matrices $A_{i,j}$ composed by i and j columns of A. As was shown earlier the algebra $\mathfrak{g}$ is split into a direct sum and therefore the elements $\mathcal{P}^v$, $\mathcal{W}(\kappa)$ are present in every subalgebra and cannot be cancelled by any pushforward action. Nevertheless, we can always set $\kappa^i = 1$ for nonzero κ^i by the action of pushforward $\mathcal{W}_*((\kappa^i)^{-1})$, here $i \in \{1, 2\}$. Then due to the fact that the commutator of vector fields v_1 and v_2 must be in the span of v_1 and v_2, for $j \in \{1, 2\}$, $j \neq i$, $\kappa^j = cw + \delta_2$. This condition gives additional restrictions on the form of all subalgebras. The problem of classification of subalgebras is split into two different cases, namely: $\operatorname{rk} A = 2$ and $\operatorname{rk} A < 2$. Hereafter I is a 2×2 unit matrix.

I. $\mathbf{rk}\, \boldsymbol{A} = \mathbf{2}$**.**

1) $\operatorname{rk} A_{1,2} = 2$. We can set $A_{1,2} = I$, then by the action of pushforwards $\mathcal{P}^t_*(a_3)$ and $\mathcal{P}^x_*(a_4)$ we can cancel operators $\mathcal{P}^t$ and $\mathcal{P}^x$ in the vector field v_1. Using the commutativity condition yields the first subalgebra in the list (2.15).

2) $\operatorname{rk} A_{1,2} < 2$, $\operatorname{rk} A_{1,3} = 2$. In this case we can set $A_{1,3} = I$ and the first condition requires that $b_2 = 0$. Then by the action of the pushforwards $\mathcal{P}^x_*(a_4)$ and $\mathcal{G}_*(b_4)$ we can eliminate a_4 and $b_4 - a_2a_4$ and by virtue of the closure condition of

Lie algebras $a_2 = b_5 = 0$ and $\kappa^2\kappa^1_w - \kappa^2_w\kappa^1 = \kappa^2$. Note that here we use the closure condition before the action of pushforward $\mathcal{W}_*$. This leads to the second and third subalgebras in the list (2.15), depending on whether κ^2 vanish or not (in the latter the pushforward $\mathcal{W}_*$ acts one more time).

3) $\operatorname{rk} A_{1,2} < 2$, $\operatorname{rk} A_{1,3} < 2$, $\operatorname{rk} A_{1,4} = 2$. Immediately $A_{1,4} = I$ can be set. In order to avoid repeating the cases $b_2 = b_3 = 0$ as well. Then we can cancel a_3 by the action of pushforward $\mathcal{P}^t_*(-a_3)$, but at the same time nonzero coefficient of $\mathcal{P}^x$ in the vector field v_1 appear, which we cancel by action of pushforward $\mathcal{P}^x_*$. Moreover, commutativity conditions causes the same situation as in the previous case, leading to the fourth and fifth subalgebras in the list (2.15).

4) $\operatorname{rk} A_{1,2} < 2$, $\operatorname{rk} A_{1,3} < 2$, $\operatorname{rk} A_{1,4} < 2$, $\operatorname{rk} A_{1,5} = 2$. Readily $A_{1,5} = I$ and $b_2 = b_3 = b_4 = 0$ due to the same reason as before. Push-forwards $\mathcal{P}^t_*$ and $\mathcal{P}^x_*$ allow us to cancel coefficients a_3 and a_4. Recombining the basis operators produce the sixth subalgebra in the list (2.15).

In the following cases we can assume that $a_1 = b_1 = 0$.

5) $\operatorname{rk} A_{2,3} = 2$. $[\mathrm{v}_1, \mathrm{v}_2] = \mathcal{P}^x$ and hence $\langle \mathrm{v}_1, \mathrm{v}_2 \rangle$ is not a subalgebra.

6) $\operatorname{rk} A_{2,3} < 2$, $\operatorname{rk} A_{2,4} = 2$. Readily $A_{2,4} = I$ by recombining basis elements. The first condition requires that $b_3 = 0$. Again recombining basis elements leads to $b_4 = 0$ and one of a_3, b_5 can be scaled to 1 (if nonzero) using the action of pushforward $\mathcal{D}_*(a_3)$ for a_3 and additional recombining basis elements for b_5.

7) $\operatorname{rk} A_{2,3} < 2$, $\operatorname{rk} A_{2,4} < 2$, $\operatorname{rk} A_{2,5} = 2$. Set $A_{2,5} = I$ and $b_2 = b_3 = b_4 = 0$ to avoid repeating cases. Further consideration is similar to the previous case, with the last straw being recombining basis operators.

We can now additionally assume that $a_2 = b_2 = 0$.

8) $\operatorname{rk} A_{3,4} = 2$. Then $A_{3,4} = I$, and one of a_5 and b_5 can be scaled to 1 acting with $\mathcal{D}_*(1/a_5)$ or $\mathcal{D}_*(1/b_5)$ and recombining basis elements.

9) $\operatorname{rk} A_{3,4} < 2$, $\operatorname{rk} A_{3,5} = 2$. Instantly $A_{3,5} = I$ and $b_4 = 0$. Recombining basis operators and commutation conditions yield the subalgebra.

Now set $a_3 = b_3 = 0$.

10) $\operatorname{rk} A_{4,5} = 2$, hence $A_{4,5} = I$ and again Recombining basis operators and commutation conditions yield the subalgebra.

If we now set $a_4 = b_4 = 0$, then by recombining basis operators we are going to leave class $\operatorname{rk} A = 2$, therefore we consider this case in the following subsection of the proof.

II. rk $\boldsymbol{A = 1}$. In this case b_i can be set 0 for $i = 1, 2, \dots, 5$ and as $\kappa^2 \neq 0$, by the action of pushforward $\mathcal{W}_*$ $\kappa^2 = 1$. Commutation condition additionally gives $\kappa^1 = cw$ for arbitrary number c, which by action of pushforward $\mathcal{W}_*$ can be reduced to 0 or 1. Then the list of classified subalgebras is split into the cases of vanishing the leading coefficients of the vector field v_1. Since the proof is similar to the classification of one-dimensional subalgebras, we omit it.

This produces the final form of the list (2.15) as presented above. □

2.5 Reductions

In this section we use the optimal lists of one- and two-dimensional subalgebras of algebra $\mathfrak{g}$ from Theorems 2.12 and 2.13, respectively, to obtain associated with them Lie reductions. Lie reductions corresponding to one-dimensional subalgebras derive systems of PDEs with respect to independent variables t and x to the system of ODE with respect to new independent variable ω defined in each case separately. Given a symmetry operator of a system of differential equations, its characteristic determines a first-order linear PDE, solving which defines a reduction ansatz for the initial system. Moreover, we will try to use enhanced ansatzes in order to obtain the simplest forms of reduced equations. Only first four subalgebras from the Theorem 2.12 are applicable for Lie reductions since for the rest of subalgebras the transversality condition does not hold.

It is known that reduction ansatzes corresponding to two-dimensional subalgebras derive system of PDEs with two independent variables to algebraical systems whose solutions are just constant. This is why such solutions are not of significant importance. Nevertheless, we are going to construct one of these solutions for reduced system as an example. Since the procedure of reduction and finding invariant solution is similar for the first six subalgebras from Theorem 2.13 we present only those corresponding to the third subalgebra from the aforementioned theorem. And again, not every subalgebra therein is appropriate for Lie reduction, though with the help of some of them it is possible to construct partially invariant solutions [44] by virtue of the fact that the system $\mathcal{S}$ is partially coupled (see Section 2.6). For this purpose subalgebras with the basis operator $\mathcal{W}(1)$ fit, for instance $\langle \mathcal{D} + a\mathcal{G} + \mathcal{P}^v, \mathcal{W}(1)\rangle$. Beside, using all the rest subalgebras, except for the last $\langle \mathcal{P}^v, \mathcal{W}(1)\rangle$, non-invariant solution can be calculated. For example, we take the subalgebra $\langle \mathcal{P}^t + \mathcal{W}(1), \mathcal{P}^v\rangle$. Within this section we assume ϕ_0, χ_0 and ψ_0 to be arbitrary constants.

1. $\boldsymbol{\langle \mathcal{D} + a\mathcal{G} + b\mathcal{P}^v + \mathcal{W}(\delta_1)\rangle}$. An ansatz is $u = \phi(\omega) + \omega + a\ln|t|$, $v = \chi(\omega) + b\ln|t|$ and $w = \psi(\omega) + \delta_1 \ln|t|$, where $\omega = x/t - a\ln|t| + a$. The corresponding reduced system of ODEs is as follows

$$a + \chi' + (\phi' + 1)\phi^1 = 0, \quad b + 1 + \phi' + \phi\chi' = 0, \quad \delta_1 + \phi\psi' = 0,$$

As will be seen later, it is convenient to consider two special cases, namely:

$$\phi = 0 \quad \text{and} \quad \phi^2 + a\phi - 1 - b = 0 \quad \text{provided } b \neq -1.$$

Then in the first case $\chi(\omega) = -a\omega + \chi_0$, ψ is an arbitrary function of ω, as well as $b = -1$ and $\delta_1 = 0$ are the only possible values. In the second case $\phi = \phi_0$, where ϕ_0 is a root of the aforementioned algebraic equation, $\chi = -(a + \phi_0)\omega + \chi_0$, $\psi = -\delta_1\omega/\phi_0 + \psi_0$ and the additional constrain $a = b + 1$ holds. Note that ϕ_0 here is a nonzero. In the general case the reduced system results in

$$\phi'\left(1 - \phi^2\right) + b + 1 - a\phi - \phi^2 = 0 \text{ and hence } \omega = \int \frac{(1 - \phi^2)d\phi}{\phi^2 + a\phi - 1 - b}.$$

Then readily $\chi' = (b\phi^1 - a)/(1 - \phi^2)$. Now we apply the hodograph transformation, making ϕ the independent variable and ω dependent. In such a case derivatives with respect to ω one need to recompute following the rule: $d\omega = \frac{d\phi}{\omega_\phi}$.

$$\chi = \int \frac{b\phi - a}{\phi^2 + a\phi - 1 - b} d\phi, \quad \psi = -\int \frac{\delta_1}{\phi} \frac{1 - \phi^2}{\phi^2 + a\phi - 1 - b} d\phi.$$

All three integrals are the integrals of rational functions and thus can be easily computed. Since the expression of the anti-derivative depends significantly on the values of parameters, we omit the explicit expression here.

2. $\langle \boldsymbol{\mathcal{G}} + \boldsymbol{\delta_2 \mathcal{P}^t} + \boldsymbol{b\mathcal{P}^v} + \boldsymbol{\mathcal{W}(\delta_1)}\rangle$. Let us consider separately cases $\delta_2 = 1$ and $\delta_2 = 0$. In the first case the reduction ansatz is $u = \phi(\omega) + t$, $v = \chi(\omega) + bt$ and $w = \psi(\omega) + \delta_1 t$, where $\omega = x - t^2/2$. Then the corresponding reduced system of ODEs is as follows

$$1 + \phi\phi' + \chi' = 0, \quad b + \phi' + \phi\chi' = 0, \quad \delta_1 + \phi\psi' = 0.$$

Likewise the previous subalgebra, we single out the special case $\phi = b$. In this case readily $\chi = -\omega + \chi_0$ and $\psi = \delta_1\omega/b + \psi_0$. In the general case expressing χ' from the second equation and substituting it into the first one obtains $b + \phi' - \phi(1 + \phi\phi') = 0$ and hence $\phi' = (\phi - b)/(1 - \phi^2)$, therefore

$$\omega = -\phi^2/2 - b\phi - (b^2 - 1)\ln|\phi - b| + \phi_0.$$

Since we have the explicit dependence of independent variable on dependent, it is convenient to change their roles, having made the hodograph transformation. Then immediately

$$\chi = b\phi + (b^2 - 1)\ln|\phi - b| + \chi_0,$$
$$\psi = \delta_1\phi + \frac{\delta_1}{b}\ln|\phi| + \frac{\delta_1(b^2 - 1)}{b}\ln|\phi - b| + \psi_0.$$

In the second case, when $\delta_2 = 0$, the reduction ansatz is $u = x/t + \phi(\omega)$, $v = bx/t + \chi(\omega)$ and $w = \delta_1 x/t + \psi(\omega)$, where the new independent variable $\omega = t$. The reduced system of ODEs is as follows

$$\omega\phi' + \phi + b = 0, \quad \omega\chi' + 1 + b\phi = 0, \quad \omega\psi' + \delta_1\phi = 0.$$

Then immediately $\phi = \phi_0/\omega - b$, $\chi = (b^2 - 1)\ln|\omega| + \phi_0 b/\omega + \chi_0$ and $\psi = \delta_1 b\ln|\omega| + \phi_0/\omega + \psi_0$.

3. $\langle \boldsymbol{\mathcal{P}^t} + \boldsymbol{\delta_2\mathcal{P}^v} + \boldsymbol{\mathcal{W}(\delta_1)}\rangle$. In this case the reduction ansatz is $u = \phi(\omega)$, $v = \chi(\omega) + \delta_2 t$ and $w = \psi(\omega) + \delta_1 t$, where $\omega = x$. Then the corresponding reduced system of ODEs is as follows

$$\phi\phi' + \chi' = 0, \quad \delta_2 + \phi' + \phi\chi' = 0, \quad \delta_1 + \phi\psi' = 0.$$

Expressing χ' from the first equation and substituting it into the second one obtains the following $\delta_2 + \phi' = \phi^2\phi'$, resulting in $\phi^3/3 - \phi = \delta_2\omega + \phi_0$.

If $\delta_2 = 0$, then ϕ is just a constant satisfying the algebraic equation and hence χ is another constant and ψ is linear function with respect to ω providing $\phi \neq 0$, otherwise δ_1 vanishes and $\psi(\omega)$ is an arbitrary smooth function of its argument.

If $\delta_2 \neq 0$, then we take advantage of the procedure applied above, changing the roles of ω and ϕ. In such a case

$$\chi = \phi^2/2 + \chi_0, \quad \psi = \delta_1(\phi^2/2 - \ln|\phi|) + \psi_0.$$

4. $\langle \boldsymbol{\mathcal{P}^x + \delta_2\mathcal{P}^v + \mathcal{W}(\delta_1)} \rangle$. In this case the reduction ansatz is $u = \phi(\omega)$, $v = \chi(\omega) + \delta_2 x$ and $w = \psi(\omega) + \delta_1 x$, where $\omega = t$. Then the corresponding reduced system of ODEs is as follows

$$\phi' + \delta_2 = 0, \quad \chi' + \delta_2\phi = 0, \quad \psi' + \delta_1\phi = 0.$$

Then immediately $\phi = -\delta_2\omega + \phi_0$, $\chi = \delta_2^2\omega^2/2 - \phi_0\delta_2\omega + \chi_0$ and $\psi = \delta_1\delta_2\omega^2/2 - \phi_0\delta_1\omega + \psi_0$.

5. $\langle \boldsymbol{\mathcal{D} + a\mathcal{G} + b\mathcal{P}^v + \mathcal{W}(\delta_1), \mathcal{P}^x} \rangle$. Successively using both basis operators we find the reduction ansatz corresponding to the subalgebra, i.e., the second basis element produces $u = \phi(t)$, $v = \chi(t)$ and $w = \psi(t)$ and then using this constrain the first one yields $\phi = a\ln|t| + \phi_0$, $\chi = b\ln|t| + \chi_0$, $\psi = \delta_1\ln|t| + \psi_0$. Substituting these values into $\mathcal{S}$ produces $\phi' = \chi' = \psi' = 0$, hence $a = b = \delta_1 = 0$ and $\phi = \phi_0$, $\chi = \chi_0$ and $\psi = \psi_0$.

6. $\langle \boldsymbol{\mathcal{D} + a\mathcal{G} + b\mathcal{P}^v, \mathcal{W}(1)} \rangle$. The form of the second basis operator $\mathcal{W}(1)$ implies that the subalgebra can not be used for Lie reductions. Though it can be used for finding partially invariant solutions, that is, using the first basis operator the Lie reduction can be found for the essential subsystem $\mathcal{S}_0$ and then the third equation which is the first-order linear partial differential, can be solved separately (see Section 2.6 for details). The operator $\mathcal{D} + a\mathcal{G} + \mathcal{P}^v$ is the partial case of already considered subalgebra for which the reduction ansatz and the exact solutions are known. Then after substituting the value u into the equation (2.4c) we obtain the linear first-order partial differential equation for w. Likewise the first considered subalgebra we consider three possible cases.

1) $\phi = 0$, $\chi = -a(x/t - a\ln|t| + a) + \chi_0$, then the function w satisfies the equation

$$w_t + \left(\frac{x}{t} + a\right) w_x = 0,$$

general solution of which is $w = \psi(x - t + at\ln|t|)$, where ψ runs through the set of smooth functions of its argument;

2) $\phi = \phi_0$, where ϕ_0 is a root of $(\phi)^2 + a\phi - 1 - b = 0$ and $b \neq -1$, $\chi = a\omega/\phi_0 + \chi_0$. Then w is the general solution $\psi(x - t + (\phi_0 + a)t\ln|t|)$ of the equation $w_t + (x/t + \phi_0 + a)w_x = 0$, where ψ runs through the set of smooth functions of its argument;

3) again it is convenient to make hodograph transformation with ϕ and ω being the new independent and dependent variables, respectively. Then readily

$$\omega = \int \frac{(1 - (\phi)^2)d\phi}{(\phi)^2 + a\phi - 1 - b}, \quad \chi = \int \frac{b\phi - a}{(\phi)^2 + a\phi - 1 - b} d\phi,$$

and plugging these expressions in the third equation and transferring to the new independent variables t and ϕ, one can find w as a general solution of the first-order linear partial differential equation

$$w_t + \frac{(\phi + \omega(\phi) + a\ln|t|)}{\omega_\phi t} w_\phi = 0.$$

7. $\langle \boldsymbol{\mathcal{P}^t + \mathcal{W}(1), \mathcal{P}^v} \rangle$. The reduction ansatz constructed with the aim of the first basis element yields $u = \phi(x)$, $v = \chi(x)$ and $w = t + \psi(x)$. Then substituting this into the equation (2.4c) one finds $u = -1/\psi'$. Note that ψ' can not vanish since then the equation (2.4c) is inconsistent. The equations (2.4a)–(2.4b) in turn readily produce $\psi = x + \psi_0$. Hence the corresponding solution is

$$\phi = \pm 1, \quad \chi = \mp 1, \quad \psi = t + x + \psi_0.$$

2.6 Group analysis of the essential subsystem

It is worth noticing that the system of equations (2.4a)–(2.4b) (which we denote for brevity as $\mathcal{S}_0$) does not depend on the variable w, and thus, the system $\mathcal{S}$ is partially coupled. That is why we can solve the system $\mathcal{S}_0$ in the first place and then substitute the obtained value u into the equation (2.4c). Also it is sufficient to find particular solution of the equation (2.4c) and then obtain the family of solutions acting on it by vector field $\mathcal{W}(w)$. Moreover, the system $\mathcal{S}_0$ is simpler because it has less dependent variables, besides it keeps the homogeneous property of the system $\mathcal{S}$ and has wider Lie symmetry algebra.

To find the Lie symmetries of system $\mathcal{S}_0$, as in Section 2.2 we use the infinitesimal Criterion 1.29, with all the notations given therein. Define the generators of one-parameter point symmetry group as $Q = \tau\partial_t + \xi\partial_x + \eta\partial_u + \theta\partial_v$, where the coefficients τ, ξ, η and θ depend on t, x, u, v. Then the infinitesimal invariance criterion reads

$$Q^{(1)}\left(u_t + uu_x + v_x\right)|_{\mathcal{S}_0} = 0, \qquad Q^{(1)}\left(v_t + uv_x + u_x\right)|_{\mathcal{S}_0} = 0.$$

The first prolongation $Q^{(1)}$ of the vector field Q is given by $Q^{(1)} = Q + \eta^{(1,0)}\partial_{u_t} + \eta^{(0,1)}\partial_{u_x} + \theta^{(1,0)}\partial_{v_t} + \theta^{(0,1)}\partial_{v_x}$, where the coefficients $\eta^{(1,0)}$, $\eta^{(0,1)}$, $\theta^{(1,0)}$ and $\theta^{(0,1)}$ in the prolonged vector field $Q^{(1)}$ are obtained from the general prolongation formula, see Theorem 1.31. Then the infinitesimal invariance criterion on solutions of the system $\mathcal{S}_0$ reads

$$\eta^{(1,0)} + u\eta^{(0,1)} + \eta u_x + \theta^{(0,1)} = 0,$$
$$\theta^{(1,0)} + u\theta^{(0,1)} + \eta v_x + \eta^{(0,1)} = 0.$$

After splitting these equations with respect to the independent variables and their derivatives the following system of determining equations is produced

$$\begin{aligned}
&\theta_t + u\theta_x + \eta_x = 0, \qquad \eta_t + u\eta_x + \theta_x = 0,\\
&\eta_v = \theta_u, \qquad \eta_u = \theta_v,\\
&\eta - \xi_t = (u^2-1)\tau_x, \quad 2u\tau_x + \tau_t - \xi_x = 0,\\
&u\tau_u - \tau_v - \xi_u = 0, \qquad u\tau_v - \tau_u - \xi_v = 0.
\end{aligned} \tag{2.16}$$

Cross-differentiating the first pair of equations with respect to u and v, respectively, and taking into account the second pair of equations one successively yields $\theta_t = \theta_x = \eta_t = \eta_x = 0$. The second pair after cross-differentiation takes the form $\eta_{uu} + \eta_{vv} = 0$ and $\theta_{uu} + \theta_{vv} = 0$ implying

$$\theta = \theta^1(u+v) + \theta^2(u-v) \quad \text{and} \quad \eta = \theta^1(u+v) - \theta^2(u-v),$$

where θ^1 and θ^2 are smooth functions of their arguments. Cross-differentiation of the third pair of (2.16) of equations by the operators ∂_t, ∂_x and $(u^2-1)\partial_x$, $2u\partial_x+\partial_t$ gives rise to the equations

$$(u^2-1)\tau_{xx}+2u\tau_{tx}+\tau_{tt}=0 \quad \text{and} \quad (u^2-1)\xi_{xx}+2u\xi_{tx}+\xi_{tt}=0,$$

respectively, which general solutions are

$$\tau=\tau^1(\omega^1,u,v)+\tau^2(\omega^2,u,v),$$
$$\xi=\xi^1(\omega^1,u,v)+\xi^2(\omega^2,u,v),$$

where $\omega^1=(u+1)t-x$, $\omega^2=(u-1)t-x$ and τ^1, τ^2, ξ^1, ξ^2 are arbitrary functions of their arguments. So we can recompute the derivatives in the fourth pair of equations (2.16). Having differentiated the first of them independently twice with respect to these new variables one derives $\tau^i_{iii}=\xi^i_{iii}=0$, where ∂_i means ∂_{ω^i} for $i=1,2$. At same time this procedure for the equation $\xi_x=2u\tau_x+\tau_t$ produces $(u-1)\tau^1_{11}=\xi^1_{11}$ and $(u+1)\tau^2_{22}=\xi^2_{22}$. Thus, for $i=1,2$,

$$\tau^i=\tau^{i2}(\omega^i)^2+\tau^{i1}\omega^i+\tau^{i0},$$
$$\xi^i=(u+(-1)^i)\xi^{i2}(\omega^i)^2+\xi^{i1}\omega^i+\xi^{i0}.$$

Plugging these values into the fourth pair of equations (2.16) and splitting with respect to powers of ω^i one immediately finds that the leading coefficients τ^{i2}, and hence ξ^{i2} vanish.

The complete list of conditions derived from splitting the fourth pair of equations in (2.16) with respect to the new variables ω^1,ω^2 and rewriting of the third pair of equations in (2.16) is as follows:

$$\xi^{11}_u+\frac{1}{2}(\xi^{11}+\xi^{21})=u\tau^{11}_u-\tau^{11}_v+\frac{u}{2}(\tau^{11}+\tau^{21}), \tag{2.17a}$$
$$\xi^{21}_u-\frac{1}{2}(\xi^{11}+\xi^{21})=u\tau^{21}_u-\tau^{21}_v-\frac{u}{2}(\tau^{11}+\tau^{21}), \tag{2.17b}$$
$$\xi^{11}_v=u\tau^{11}_v-\tau^{11}_u-\frac{1}{2}(\tau^{11}+\tau^{21}), \tag{2.17c}$$
$$\xi^{21}_v=u\tau^{21}_v-\tau^{21}_u+\frac{1}{2}(\tau^{11}+\tau^{21}), \tag{2.17d}$$
$$\xi^{11}+\xi^{21}=(u-1)\tau^{11}+(u+1)\tau^{21}, \tag{2.17e}$$
$$\eta-\xi^{11}(u-1)-\xi^{21}(u+1)=(u^2-1)(-\tau^{11}-\tau^{21}), \tag{2.17f}$$
$$\begin{aligned}\tau^{10}_{uu}&=\tau^{10}_{vv}+\tau^{10}_v, \quad \xi^{10}_{uu}=\xi^{10}_{vv}+\xi^{10}_v,\\ \tau^{20}_{uu}&=\tau^{20}_{vv}+\tau^{20}_v, \quad \xi^{20}_{uu}=\xi^{20}_{vv}+\xi^{20}_v.\end{aligned} \tag{2.17g}$$

Firstly, denote $\tau^0=\tau^{10}+\tau^{20}$ and $\xi^0=\xi^{10}+\xi^{20}$. These functions satisfy the telegraph equations $\tau^0_{vv}+\tau^0_v=\tau^0_{uu}$ and $\xi^0_{vv}+\xi^0_v=\xi^0_{uu}$, respectively. Then combining (2.17e) with pairs (2.17a)–(2.17b) and (2.17c)–(2.17d) one immediately has $(\tau^{11}+\tau^{21})_v=(\tau^{11}-\tau^{21})_u$ and $(\tau^{11}+\tau^{21})_v=(\tau^{11}-\tau^{21})_u-(\tau^{11}+\tau^{21})$. Moreover, bringing τ^{11} and τ^{21} to the other sides of the respective equations and cross-differentiating them with respect to pairs of operators $\partial_u-\partial_v$, $\mathrm{id}-\partial_u+\partial_v$ and

$\partial_u+\partial_v$, $\mathrm{id}+\partial_u+\partial_v$, where id is an identity transformation, one finds that τ^{i1} satisfies the telegraph equation as well for $i=1,2$. Differentiating (2.17f) successively with respect to operators ∂_{u+v} and ∂_{u-v} one obtains $(\tau^{11}+\tau^{21})_v=0$ and hence $(\tau^{11}-\tau^{21})_{uv}=0$. This means that τ^{i1} can be presented as follows: $\tau^{11}=U^1(u)+V(v)$ and $\tau^{21}=U^2(u)-V(v)$. Taking into account that τ^{i1} satisfy the telegraph equation one derives that U^1 and U^2 are quadratic whilst V is linear with the leading coefficients being $\frac{V_1}{2}$, $-\frac{V_1}{2}$ and V_1, respectively. Using all the previous conditions one finally has $\tau^{11}=\frac{V_1}{2}u^2+U_{11}u+U_{10}+V_1v+V_0$ and $\tau^{21}=-\frac{V_1}{2}u^2+U_{21}u+U_{20}-V_1v-V_0$, with $U_{11}+U_{21}=2V_1$ and $U_{10}+U_{20}=U_{11}-U_{21}$. Plugging this into (2.17a)–(2.17b) one yields $\xi^{11}=-\frac{1}{2}(2U_{11}+U_{10}+U_{20})v-V_1uv+c^{11}(u)$ and $\xi^{21}=\frac{1}{2}(-2U_{21}+U_{10}+U_{20})v+V_1uv+c^{21}(u)$. On this stage splitting with respect to v gives $V_1=0$, and further splitting is impossible due to the undetermined functions $c^{11}(u)$ and $c^{21}(u)$. Thus, $U_{21}=-U_{11}$, $U_{10}+U_{20}=2U_{11}$ and

$$\begin{aligned}&\tau^{11}=U_{11}u+U_{10}+V_0,\quad \tau^{21}=-U_{11}u+U_{20}-V_0,\\&\xi^{11}=-2U_{11}v+c^{11}(u)\text{ and }\xi^{21}=2U_{11}v+c^{21}(u).\end{aligned}$$

Substituting this into (2.17c) and (2.17d) one is able to determine $c^{11}=U_{11}u^2-c_0u+\tilde{c}$ and $c^{21}=-U_{11}u^2+c_0u+2c_0-\tilde{c}$. Furthermore, splitting the equation (2.17f) gives the last relating condition $2c_0=U_{20}-U_{10}-2V_0$. Knowing τ and ξ now we can find η from (2.17f) and afterwards obtain θ. The form of coefficients of Q is the following:

$$\begin{aligned}&\tau=(4U_{11}u-2c_0)t-2U_{11}x+\tau^0(u,v),\\&\xi=(-4U_{11}v+2U_{11}u^2+2(\tilde{c}-c_0))t-2c^0x+\xi^0(u,v),\\&\eta=-4U_{11}v+2U_{11}+2(\tilde{c}-c_0),\\&\theta=-4U_{11}u+U_0.\end{aligned}$$

This proves the theorem.

Theorem 2.14. *The maximal Lie invariance algebra $\mathfrak{g}_0$ of the system $\mathcal{S}_0$ is spanned by the vector fields*

$$\begin{aligned}&\breve{\mathcal{D}}=t\partial_t+x\partial_x,\quad \breve{\mathcal{G}}=t\partial_x+\partial_u,\quad \breve{\mathcal{P}}(\tau^0,\xi^0)=\tau^0(u,v)\partial_t+\xi^0(u,v)\partial_x,\\&\breve{\mathcal{P}}^v=\partial_v,\quad \breve{\mathcal{J}}=\left(\tfrac{1}{2}x-tu\right)\partial_t+t\left(v-\tfrac{1}{2}u^2+\tfrac{1}{2}\right)\partial_x+v\partial_u+u\partial_v,\end{aligned}\tag{2.18}$$

where (τ^0,ξ^0) runs through the solution set of the system

$$u\tau^0_u-\tau^0_v=\xi^0_u,\quad u\tau^0_v-\tau^0_u=\xi^0_v.$$

In terms of Riemann invariants, the system for (τ^0,ξ^0) takes the form

$$\xi^0_1=V^2\tau^0_1,\quad \xi^0_2=V^1\tau^0_2.$$

Remark 2.15. It is obvious that for any κ the system $\mathcal{S}_0$ cannot admit a counterpart of the Lie symmetry vector field $\mathcal{W}(\kappa)$ of the system $\mathcal{S}$. The algebra $\mathfrak{g}/\langle\mathcal{W}(\kappa)\rangle$ is isomorphic to the proper subalgebra $\langle\breve{\mathcal{D}},\breve{\mathcal{G}},\breve{\mathcal{P}}(1,0),\breve{\mathcal{P}}(0,1),\breve{\mathcal{P}}^v\rangle$ of $\mathfrak{g}_0$, where the spanning vector fields correspond to $\mathcal{D}$, $\mathcal{G}$, $\mathcal{P}^t$, $\mathcal{P}^x$, $\mathcal{P}^v$, respectively. Moreover, since the functions τ^0 and ξ^0 are interrelated, it is impossible to span the operator $\mathcal{P}(\tau^0,\xi^0)$ by two operators of one function. However, one can find the solutions $(\tau^0,\xi^0)=(1,0)$ and $(\tau^0,\xi^0)=(0,1)$ that correspond the translation operators $\mathcal{P}^t$ and $\mathcal{P}^x$.

By virtue of the structure of the algebra $\mathfrak{g}_0$, the system $\mathcal{S}_0$ can be linearized by the hodograph transformation; cf. [32]. In general, every (1+1)-dimensional system of hydrodynamic type with two dependent variables can be linearized by the two-dimensional hodograph transformation [53]. A hodograph transformation exchanges the roles of dependent and independent variables. For the system $\mathcal{S}_0$, the pair of independent and dependent variables are to be exchanged, i.e., we set $y=u$, $z=v$, $p=t$, $q=x$, where y and z are the new independent variables and p and q are the new dependent variables. Then differentiating the equality $p(u,v)=t$, $q(u,v)=x$ with respect to t and x using the chain rule, we obtain the system of linear algebraic equations for the first derivatives of (u,v)

$$\begin{aligned}&u_tp_z+u_xq_z=0,\quad u_tp_y+u_xq_y=1,\\&v_tp_z+v_xq_z=1,\quad v_tp_y+v_xq_y=0.\end{aligned}$$

When assuming the nondegeneracy condition $\Delta:=p_yq_z-p_zq_y\neq0$, which is equivalent to $u_tv_x-u_xv_t\neq0$, the solution of this system is

$$u_t=\frac{q_z}{\Delta},\quad u_x=-\frac{p_z}{\Delta},\quad v_t=-\frac{q_y}{\Delta},\quad v_x=\frac{p_y}{\Delta}. \tag{2.19}$$

In the new variables, the system $\mathcal{S}_0$ takes the form

$$q_z-yp_z+p_y=0,\quad -q_y+yp_y-p_z=0. \tag{2.20}$$

As expected, the system (2.20) coincides with the system for (τ^0,ξ^0) from Theorem 2.14. The cross-differentiation with respect to y and z leads to a differential consequence of the system (2.20), which is the telegraph equation for single p,

$$p_{zz}+p_z=p_{yy}. \tag{2.21}$$

In other words, the system (2.20) is a potential system for the equation (2.21). The substitution $p=e^{-z/2}\tilde p$ reduces (2.21) to the Klein–Gordon equation

$$\tilde p_{yy}=\tilde p_{zz}-\frac{\tilde p}{4}. \tag{2.22}$$

The maximal Lie invariance algebra $\mathfrak{g}_{\rm KG}$ of the Klein–Gordon equation (2.22) is well known [20]. It is spanned by the vector fields

$$\hat{\mathcal{P}}^y=\partial_y,\quad \hat{\mathcal{P}}^z=\partial_z,\quad \hat{\mathcal{J}}=y\partial_z+z\partial_y,\quad \hat{\mathcal{D}}=\tilde p\partial_{\tilde p},\quad \hat{\mathcal{P}}(\psi)=\psi(y,z)\partial_{\tilde p},$$

where ψ runs through the solution set of (2.22). The maximal Lie invariance algebra of the telegraph equation (2.21) is spanned by the pushforwards of these vector fields with the transformation $p=e^{-z/2}\tilde p$, where the variables y and z are not changed,

$$\partial_y,\quad \partial_z-\frac12p\partial_p,\quad y\partial_z+z\partial_y-\frac12yp\partial_p,\quad p\partial_p,\quad e^{-z/2}\psi(y,z)\partial_p.$$

Although the maximal Lie invariance algebras $\mathfrak{g}_0$ and $\mathfrak{g}_{\rm KG}$ of the system $\mathcal{S}_0$ and the Klein–Gordon equation (2.22) look similar and there seems to be an obvious relation between generating elements, $\breve{\mathcal{D}} \sim \hat{\mathcal{D}}$, $\breve{\mathcal{G}} \sim \hat{\mathcal{P}}^y$, $\breve{\mathcal{P}}(\tau^0, \xi^0) \sim \hat{\mathcal{P}}(e^{z/2}\tau^0)$, $\breve{\mathcal{P}}^v \sim \hat{\mathcal{P}}^z + \frac{1}{2}\hat{\mathcal{D}}$, $\breve{\mathcal{J}} \sim \hat{\mathcal{J}}$, these algebras are in fact not isomorphic. This can be explained with the following arguments. Since the systems $\mathcal{S}_0$ and (2.20) are related by the hodograph transformation, the maximal Lie invariance algebra $\tilde{\mathfrak{g}}_0$ of the system (2.20) is the pushforward of $\mathfrak{g}_0$ by this transformation. Hence the algebras $\mathfrak{g}_0$ and $\tilde{\mathfrak{g}}_0$ coincide up to re-denoting variables. The transition from the system (2.20) to the equation (2.21) needs the projection $(y, z, p, q) \to (y, z, p)$. At the same time, the vector field $\breve{\mathcal{P}}(0, 1)$ is projected to the zero vector field. Moreover, the vector field $\breve{\mathcal{J}}$ is not projectable, and thus the commutators $[\breve{\mathcal{J}}, \breve{\mathcal{P}}(\tau^0, \xi^0)]$ and $[\hat{\mathcal{J}}, \hat{\mathcal{P}}(e^{z/2}\tau^0)]$ are not related to each other in the above way. This is why there is no direct relation between the set of Lie solutions of the systems $\mathcal{S}_0$ and that of the equation (2.22) (resp. of the equation (2.21)).

To summarize, an alternative to investigate the system $\mathcal{S}$ is to consider the Klein–Gordon equation (2.22), and using the change of variables (2.19) obtain the solution of the system $\mathcal{S}_0$, in general, in implicit form. The next step is to substitute the obtained u into the equation (2.4c) in order to find its particular solution. Then by virtue of the fact that any smooth function of the derived particular solution is a solution of the equation (2.4c) as well, we can derive the families of solutions. In particular, the equation (2.4c) in the new variables takes the form

$$(q_z - yp_z)w_y - (q_y - yp_y)w_z = 0,$$

or, taking into account equation (2.20),

$$p_z w_z - p_y w_y = 0.$$

The fact that this equation is the linear PDE of first order, general solution of which is easily obtainable, justifies this alternative approach.

2.7 Solution through linearization of the essential subsystem

Using the facts that the system $\mathcal{S}$ is partially coupled and the subsystem $\mathcal{S}_0$ can be linearized, we construct an implicit representation of the general solution of the system $\mathcal{S}$ in terms of the general solution of the telegraph equation (2.21) or, equivalently, of the Klein–Gordon equation (2.22). Consider the potential system $p_y = \Upsilon_z + \Upsilon$, $p_z = \Upsilon_y$ of the equation (2.21). The "pseudopotential" $\Upsilon = \Upsilon(y, z)$ is in fact related to a usual potential of the equation (2.21): the function $e^z\Upsilon$ is the potential associated with the conservation-law characteristic e^z of the equation (2.21). Since the equation $p_z = \Upsilon_y$ is in conserved form, we can introduce the second-level potential $\Psi = \Psi(y, z)$ defined by the system $\Psi_y = p$, $\Psi_z = \Upsilon$. The equation $p_y = \Upsilon_z + \Upsilon$ implies that the potential Ψ satisfies the same telegraph equation (2.21) as p, $\Psi_{zz} + \Psi_z = \Psi_{yy}$. Substituting the expression $p = \Psi_y$

into the system (2.20) and using the equation for Ψ, we obtain the integrable system $(q-y\Psi_y+\Psi_z+\Psi)_y=0$, $(q-y\Psi_y+\Psi_z+\Psi)_z=0$ for q. The potential Ψ is defined up to a constant summand. Hence we can assume that $q=y\Psi_y-\Psi_z-\Psi$. Then $\Delta:=p_yq_z-p_zq_y=(\Psi_{yy})^2-(\Psi_{yz})^2$, and the nondegeneracy condition $\Delta\neq 0$ reduces, on the solution set of the equation for Ψ, to the condition $\Psi_{yy}\neq\Psi_{yz}$ or, equivalently, $\Psi_{yy}\neq-\Psi_{yz}$.[1] To find the component of the general solution for w, we write the equation (2.4c) in the new variables, $(q_z-yp_z)w_y-(q_y-yp_y)w_z=0$, or $p_yw_y-p_zw_z=0$ when taking into account the system (2.20). We substitute the expression $p=\Psi_y$ into the last equation multiplied by e^z and take into account the equation for Ψ. As a result, we derive the equation $(e^z\Psi_z)_zw_y-(e^z\Psi_z)_yw_z=0$ meaning, in view of $e^z\Psi_z\neq\mathrm{const}$ that w is a function of $e^z\Psi_z$ only. Re-writing the expressions obtained for p, q and w in terms of the initial variables, we construct the following implicit representation of all solution of the system $\mathcal S$ with $u_tv_x-u_xv_t\neq 0$: $t=\Psi_u$, $x=u\Psi_u-\Psi_v-\Psi$, $w=W(e^v\Psi_v)$, where the function $\Psi=\Psi(u,v)$ runs through the solution set of the equation $\Psi_{vv}+\Psi_v=\Psi_{uu}$, and W is an arbitrary function of its argument $e^v\Psi_v$. This representation can also be interpreted as parametric.

The condition $u_tv_x-u_xv_t=0$ for singular solutions of the system $\mathcal S$ is equivalent to the constraint $v_x^2=u_x^2$, i.e. $v_x=\varepsilon u_x$ with $\varepsilon=\pm1$. Then the subsystem $\mathcal S_0$ also implies $v_t=\varepsilon u_t$. Therefore, $v=\varepsilon u+c$, where c is an arbitrary constant, and the subsystem $\mathcal S_0$ reduces to the single equation $u_t+uu_x+\varepsilon u_x=0$. Suppose that $u\neq\mathrm{const}$, which is equivalent to the condition $u_x\neq 0$. We carry out the one-dimensional hodograph transformation with $s=t$ and $y=u$ being the new independent variables and $q=x$ and w being the new dependent variables. Deriving the expressions for first derivatives of u and w in the hodograph variables,

$$u_x=\frac{1}{q_y},\quad u_t=-\frac{q_s}{q_y},\quad w_x=\frac{w_y}{q_y},\quad w_t=w_s-\frac{q_s}{q_y}w_y,$$

we represent the equations of the reduced system $\mathcal S$, $u_t+uu_x+\varepsilon u_x=0$ and $w_t+uw_x=0$, in these variables, which read $q_s=y+\varepsilon$ and $q_yw_s=\varepsilon w_y$. The equation $q_s=y+\varepsilon$ integrates to $q=(y+\varepsilon)s+\varphi(y)$, where φ is an arbitrary function of y. The general solution of the equation $p_yw_s=\varepsilon w_y$ with respect to w is an arbitrary function of a first integral I of the ODE $\mathrm ds/\mathrm dy=\varepsilon(s+\varphi_y)$. To derive a nice expression for this integral, instead of φ we introduce a new arbitrary function $\Theta=\Theta(y)$ such that $\varphi=e^{\varepsilon y}\Theta_y$, which gives $I=e^{-\varepsilon y}s-\varepsilon\Theta_y-\Theta$. Returning to the initial variable, we obtain $x-ut=e^{\varepsilon u}\Theta_u$, $v=\varepsilon u+c$ and $w=W(I)$, where c is an arbitrary constant, $\varepsilon=\pm1$, $\Theta=\Theta(u)$ is an arbitrary function of u and W is an arbitrary function of $I=e^{-\varepsilon u}t-\varepsilon\Theta_u-\Theta$.

If $u=\mathrm{const}$, then also $v=\mathrm{const}$, and the system $\mathcal S$ reduces to the equation (2.4c) whose general solution in this case takes the form $w=W(x-ut)$ with W being an arbitrary function of its argument $x-ut$. We refer to this case as ultra-singular.

We collect all the three cases for solutions of the system $\mathcal S$ into a single assertion.

[1] The overdetermined system $\Psi_{zz}+\Psi_z=\Psi_{yy}$, $\Psi_{yy}=\varepsilon\Psi_{yz}$ with $\varepsilon\in\{-1,1\}$ implies that $\Psi_{yyy}=\Psi_{yzz}+\Psi_{yz}=\varepsilon\Psi_{yyz}+\varepsilon\Psi_{yy}=\Psi_{yyy}+\varepsilon\Psi_{yy}$ and thus $\Psi_{yy}=\Psi_{yz}=0$ resulting also to $\Psi_{yy}=-\varepsilon\Psi_{yz}$.

Theorem 2.16. *Any solution of the system $\mathcal{S}$ (locally) belongs to one of the following families; below W is an arbitrary function of its argument.*

1. The regular family, where $u_t v_x - u_x v_t \neq 0$ (the general solution):

$$t = \Psi_u, \quad x = u\Psi_u - \Psi_v - \Psi, \quad w = W(e^v \Psi_v). \tag{2.23}$$

Here the function $\Psi = \Psi(u,v)$ runs through the solutions of the equation $\Psi_{vv} + \Psi_v = \Psi_{uu}$ with $\Psi_{uu} \neq \Psi_{uv}$.

2. The singular family, where $u_t v_x - u_x v_t = 0$ but u and v are not constants:

$$x - ut = e^{\varepsilon u}\Theta_u, \quad v = \varepsilon u + c, \quad w = W(e^{-\varepsilon u} t - \varepsilon\Theta_u - \Theta).$$

Here c is an arbitrary constant, $\varepsilon = \pm 1$, and $\Theta = \Theta(u)$ is an arbitrary function of u.

3. The ultra-singular family, where u and v are arbitrary constants and $w = W(x - ut)$.

In other words, the regular, singular and ultra-singular families of solutions of the system $\mathcal{S}$ are associated with solutions of the subsystem $\mathcal{S}_0$ with rank 2, 1 and 0, respectively; cf. [23].

We demonstrate the results with the following examples.

Example 2.17. According to [46], $\tilde{p} = C_1 \exp \frac{ky + \lambda z}{2\sqrt{\lambda^2 - k^2}}$, where $C_1 > 0$, λ and k are real numbers with $\lambda^2 > k^2$, is a travelling-wave solution of the Klein–Gordon equation. Then immediately

$$p = C_1 \exp \frac{ky + (\lambda - \sqrt{\lambda^2 - k^2})z}{2\sqrt{\lambda^2 - k^2}} =: A$$

and

$$p_z = \frac{\lambda - \sqrt{\lambda^2 - k^2}}{2\sqrt{\lambda^2 - k^2}} A, \quad p_y = \frac{k}{2\sqrt{\lambda^2 - k^2}} A,$$
$$q_z = \frac{(\lambda - \sqrt{\lambda^2 - k^2})y - k}{2\sqrt{\lambda^2 - k^2}} A, \quad q_y = \frac{ky - \lambda + \sqrt{\lambda^2 - k^2}}{2\sqrt{\lambda^2 - k^2}} A,$$

resulting in

$$u = \frac{x - C_0}{t} + \frac{k}{\lambda - \sqrt{\lambda^2 - k^2}},$$
$$v = \frac{2\sqrt{\lambda^2 - k^2}}{\lambda - \sqrt{\lambda^2 - k^2}} \left(\ln|t| - \frac{k}{2\sqrt{\lambda^2 - k^2}} \left(\frac{x - C_0}{t} \right) \right) + C_1.$$

With the help of the symmetries of the Klein–Gordon equation, we can set $C_1 = 0$ and $C_0 = 0$, whilst due to the arbitrariness of λ and k we can also set $\lambda = 5$ and $k = 4$, leading to the simplified form of the solutions written above

$$u = \frac{x}{t} + 2, \quad v = 3\ln|t| - 2\frac{x}{t}.$$

Then the particular solution of (2.4c) is $w = x/t - 2\ln|t|$, and any smooth function $\varphi(w)$ of ω is a solution of (2.4c) as well.

Example 2.18. Using linearity of the Klein–Gordon equation (or the corresponding symmetry operator $\hat{\mathcal{P}}(\psi)$) we can construct more sophisticated solutions of the system $\mathcal{S}_0$. Taking the following solution of equation (2.22)

$$p = \exp\frac{2y+z}{3} + \exp\frac{5y+z}{24}$$

carrying out all the steps as in the previous example we finds that

$$t = \exp\frac{2u+v}{3} + \exp\frac{5u+v}{24},$$
$$x = u\left(\exp\frac{2u+v}{3} + \exp\frac{5u+v}{24}\right) - 2\exp\frac{2u+v}{3} - 5\exp\frac{5u+v}{24}$$

is a solution of the system $\mathcal{S}_0$ in a parameterized form. The equation (2.4c) in new variables takes the form

$$\left(16\exp\frac{2y+z}{3} + 5\exp\frac{5y+z}{24}\right)w_y - \left(8\exp\frac{2y+z}{3} + \exp\frac{5y+z}{24}\right)w_z = 0,$$

whose the general solution is $w = \kappa(I)$, where κ is an arbitrary function of I and $I = I(y,z)$ is a first integral of the ODE

$$\frac{dy}{16\exp\frac{2y+z}{3} + 5\exp\frac{5y+z}{24}} = -\frac{dz}{8\exp\frac{2y+z}{3} + \exp\frac{5y+z}{24}}.$$

In the new variables $\omega^1 = \exp\frac{2y+z}{3}$ and $\omega^2 = \exp\frac{5y+z}{24}$ the latter equation becomes the homogeneous ordinary differential equation

$$\frac{d\omega_2}{d\omega_1} = \frac{(\omega^2)^2 + 3\omega^1\omega^2}{3\omega^1\omega^2 + 8(\omega^1)^2},$$

which is derived by substitution $\omega = \dfrac{\omega^2}{\omega^1}$ to the separable ODE

$$\omega_1\frac{d\omega}{d\omega_1} = \frac{-2\omega^2 - 5\omega}{3\omega + 8},$$

with the first integral

$$\frac{\omega^{-16}(2\omega+5)}{(\omega^1)^{10}} = \text{const},$$

and hence equation (2.4c) has as the solution $w = \kappa(I)$, where κ is an arbitrary smooth function of its argument, and

$$I = 2\exp\frac{5y+25z}{24} + 5\exp\frac{2y+4z}{3}.$$

2.8 Solution with generalized hodograph transformation

Another approach to finding the general solution of the system (2.3) is to employ the generalized hodograph method (see [63] and Section 2.1) for the diagonalized form (2.5) of this system. In this way, one uses the semi-Hamiltonian property related to the diagonalized form (2.5) instead of the partial coupling, which emerges in both the simplified forms (2.4) and (2.5). For the system (2.5), the tuple of the parameter functions $W = (W^1, W^2, W^3)$ in the ansatz (2.2) runs through the solution set of the overdetermined linear systems of first-order partial differential equations

$$W^1_2 = W^2_1 = \frac{1}{4}(W^1 - W^2), \quad W^1_3 = W^2_3 = 0, \tag{2.24a}$$

$$W^3_1 = \frac{1}{2}(W^1 - W^3), \quad W^3_2 = \frac{1}{2}(W^3 - W^2). \tag{2.24b}$$

Introducing the function $\Lambda = \Lambda(r^1, r^2)$ defined by $\Lambda_1 = W^1$ and $\Lambda_2 = W^2$, we reduce the subsystem (2.24a) to the single equation

$$4\Lambda_{12} = \Lambda_1 - \Lambda_2. \tag{2.25}$$

Then we consider the subsystem (2.24b) as an overdetermined inhomogeneous linear system of first-order partial differential equations with respect to the function W^3. The general solution of this system is represented in the form

$$W^3 = F(r^3) \exp\frac{r^2 - r^1}{2} + \Phi(r^1, r^2)$$

where F is an arbitrary smooth function of r^3 and $\Phi = \Phi(r^1, r^2)$ is a particular solution of the subsystem (2.24b), i.e.,

$$\Phi_1 + \frac{1}{2}\Phi = \frac{1}{2}\Lambda_1, \quad \Phi_2 - \frac{1}{2}\Phi = -\frac{1}{2}\Lambda_2. \tag{2.26}$$

Therefore, the function Φ satisfies the same equation as the function Λ, $4\Phi_{12} = \Phi_1 - \Phi_2$. It is convenient to assume for Φ to be the principal parameter function and to express Λ in terms of Φ. Then the determinant $\det(V^i_j t + W^i_j)$, where the indices i and j run from 1 to 3, is equal to

$$(\Phi_{11} - \Phi_{22})t + 2(\Phi_{11} + \Phi_{12})(2\Phi_{12} + \Phi_2) - 2(\Phi_{22} + \Phi_{12})(2\Phi_{11} + \Phi_1).$$

It vanishes on solutions of equation for Φ if and only if $\Phi_{11} = \Phi_{22}$ and $\Phi_{11} + \Phi_{12} = 0$, which also gives $\Phi_{22} + \Phi_{12} = 0$. Nevertheless, the equation $\Phi_{11} = \Phi_{22}$ is a differential consequence of the overdetermined system $4\Phi_{12} = \Phi_1 - \Phi_2$, $\Phi_{11} + \Phi_{12} = 0$. The same result is true if we replace the second equation of this system by $\Phi_{22} + \Phi_{12} = 0$. This is why the nondegeneracy condition $\det(V^i_j t + W^i_j) \neq$ is equivalent to the single inequality $\Phi_{11} + \Phi_{12} \neq 0$ (resp. $\Phi_{22} + \Phi_{12} \neq 0$).

As a result, we construct the following implicit representation for the general solution of the system (2.5):

$$
\begin{aligned}
&x-\left(\frac{r^1+r^2}{2}+1\right)t=\Phi+2\Phi_1,\\
&x-\left(\frac{r^1+r^2}{2}-1\right)t=\Phi-2\Phi_2,\\
&x-\frac{r^1+r^2}{2}t=\Phi+F\exp\frac{r^2-r^1}{2},
\end{aligned}
\tag{2.27}
$$

where F is an arbitrary smooth function of r^3 with $F_{r^3}\neq 0$ and the function $\Phi=\Phi(r^1,r^2)$ runs through the set of solutions of the equation $4\Phi_{12}=\Phi_1-\Phi_2$ with $\Phi_{11}\neq-\Phi_{12}$ or, equivalently, $\Phi_{22}\neq-\Phi_{12}$. The equation for Φ is reduced by the transformation $\tilde\Phi=e^{(r^2-r^1)/4}\Phi$ to the Klein–Gordon equation $4\tilde\Phi_{12}=\tilde\Phi$.

The representation (2.27) can be derived from the particular case of the representation (2.23), where the parameter function W assumed to be nonconstant. It is just necessary to replace dependent variables in (2.23) by their expressions in terms of the Riemann invariants and re-denote Ψ and the inverse of W by $-\Phi$ and $-F$, respectively.

Families of solutions from Theorem 2.16 are nicely characterized in terms of conditions for Riemann invariants: none of (resp. exactly one of, resp. both) r^1 and r^2 are constants for the regular (resp. singular, resp. ultra-singular) family. To complete the set of solutions of the form (2.27) to the entire solution set of the system (2.5), we should add solutions, where (r^1,r^2) are defined by the first two equations of (2.27) and r^3 is a constant, and solutions obtained from the singular and ultra-singular solutions of Theorem 2.16 by the transition to Riemann invariants. The corresponding representations in terms of Riemann invariants are obvious.

2.9 Higher-order generalized symmetries

To investigate first-order generalized symmetries of the system (2.3) it is more convenient to take advantage of its diagonalized form (2.5) in terms of the Riemann invariants $r=(r^1,r^2,r^3)$. Also for the first-order generalized symmetries it is convenient to define vector fields in evolutionary forms

$$
Q=\eta^i(t,x,r,r_x)\partial_{r^i},
$$

that is equivalent to considering vector fields in the usual form, see Theorem 1.36

From now on repeating indices i and j mean summation for $i,j=1,2,3$, while the index k is fixed and takes the values $1,2,3$. For the vector field Q to be generalized symmetry of the system (2.5) invariance criterion 1.34 must hold and on the subvariety of solutions of the system $\mathcal{S}$ it reads

$$
\eta^k_t-\eta^k_{r^j}V^kr^j_x-\eta^k_{r^j_x}\left(\frac{r^1_x+r^2_x}{2}r^j_x+V^jr^j_{xx}\right)
$$

$$+\frac{\eta^1+\eta^2}{2}r^k_x + V^k(\eta^k_x + \eta^k_{r^j} r^j_x + \eta^k_{r^j_x} r^j_{xx}) = 0.$$

Splitting with respect to r^k_{xx} readily produces $\eta^k_{r^j_x} = 0$ for $j \neq k$, splitting with respect to the first derivatives yields systems on η^1 and η^2

$$\begin{aligned}
&\eta^k = \theta^k(t,x,r)r^k_x + \zeta^k(r^k),\\
&\theta^k_t + V^k\theta^k_x + \frac{\zeta^1+\zeta^2}{2} = 0
\end{aligned} \tag{2.28}$$

for $k = 1, 2$ with $\theta^1_{r^2} = \theta^2_{r^1} = (\theta^1 - \theta^2)/4$ and on η^3

$$\begin{aligned}
&r^3_x\theta^1 = r^3_x\eta^3_{r^3_x} + 2\eta^3_{r^1},\\
&r^3_x\theta^2 = r^3_x\eta^3_{r^3_x} - 2\eta^3_{r^2},\\
&\eta^3_t + V^3\eta^3_x + \frac{\zeta^1+\zeta^2}{2} = 0.
\end{aligned} \tag{2.29}$$

Cross-differentiating the second pair of equations of the system (2.28) with the auxiliary equation for it and successive direct plugging results into the initial system (2.28) produce

$$\begin{aligned}
&\zeta^1 = \sigma r^1 + \zeta^{10}, \quad \zeta^2 = -\sigma r^2 + \zeta^{20},\\
&\theta^1 = \left(\frac{\sigma}{2}\left(\frac{r^1+r^2}{2}+1\right) - 2\gamma\right)x\\
&\quad - \left(\frac{\sigma}{8}(r^1+r^2)^2 - \gamma(r^1+r^2) + \sigma r^1 - \frac{1}{2}(\sigma + \zeta^{10} + \zeta^{20} - 4\gamma)\right)t + \psi,\\
&\theta^2 = \left(\frac{\sigma}{2}\left(\frac{r^1+r^2}{2}-1\right) - 2\gamma\right)x\\
&\quad - \left(\frac{\sigma}{8}(r^1+r^2)^2 - \gamma(r^1+r^2) - \sigma r^1 - \frac{1}{2}(\sigma + \zeta^{10} + \zeta^{20} + 4\gamma)\right)t\\
&\quad + \psi - 4\psi_{r^2},
\end{aligned}$$

where $\psi(r^1, r^2)$ runs through the set of solutions of the equation

$$\psi_{r^1 r^2} = (\psi_{r^1} - \psi_{r^2})/4.$$

It simple to observe that the system (2.29) is the list of partial derivatives of the function η^3. Checking the compatibility condition we find that $\sigma = 0$ and $\eta^3 = -2\gamma r^3_x$. Successively integrating all three equations of the system (2.29) we find the final form of the coefficients of the generalized vector field Q

$$\begin{aligned}
&\eta^1 = \left(\beta + 2\beta_{r^1} + \gamma t(r^1+r^2) - \frac{\tilde{\zeta}^{10}}{2}t + 2\gamma t - 2\gamma x\right)r^1_x + \tilde{\zeta}^{10} - \tilde{\zeta}^{20},\\
&\eta^2 = \left(\beta - 2\beta_{r^2} + \gamma t(r^1+r^2) - \frac{\tilde{\zeta}^{10}}{2}t - 2\gamma t - 2\gamma x\right)r^2_x + \tilde{\zeta}^{20},
\end{aligned}$$

$$\eta^3 = \left(\beta + \gamma t(r^1 + r^2) - \frac{\tilde\zeta^{10}}{2}t - 2\gamma x\right) r^3_x + \alpha\left(r^3, r^3_x \exp\frac{r^2 - r^1}{2}\right),$$

where $\tilde\zeta^{10}$, $\tilde\zeta^{20}$ and γ are arbitrary constants, α is an arbitrary function of its arguments, the function $\beta(r^1, r^2)$ runs through the solution set of the equation $\beta_{r^1r^2} = (\beta_{r^1} - \beta_{r^2})/4$.

This proves the theorem.

Theorem 2.19. *The space of first-order generalized symmetries of the system $\mathcal{S}_D$ are spanned by first-order generalized vector fields in the evolutionary form*

$$\tilde{\mathcal{G}}_1 = \left(1 - \frac{t}{2}r^1_x\right)\partial_{r^1} - \frac{t}{2}r^2_x\partial_{r^2} - \frac{t}{2}r^3_x\partial_{r^3}, \quad \tilde{\mathcal{G}}_2 = \partial_{r^1} - \partial_{r^2},$$
$$\tilde{\mathcal{D}} = \big(t(r^1 + r^2 + 2) - 2x\big)\, r^1_x\partial_{r^1} + \big(t(r^1 + r^2 - 2) - 2x\big)\, r^2_x\partial_{r^2}$$
$$+(t(r^1 + r^2) - 2x)r^3_x\partial_{r^3}, \quad \tilde{\mathcal{W}} = \alpha\left(r^3, r^3_x \exp\frac{r^2 - r^1}{2}\right)\partial_{r^3},$$
$$\tilde{\mathcal{P}}(\beta) = (\beta + 2\beta_{r^1})r^1_x\partial_{r^1} + (\beta - 2\beta_{r^2})r^2_x\partial_{r^2} + \beta r^3_x\partial_{r^3},$$

where α is an arbitrary function of its arguments and $\beta(r^1, r^2)$ runs through the set of solutions of the equation $4\beta_{r^1r^2} = \beta_{r^1} - \beta_{r^2}$.

Remark 2.20. Amongst the generalized vector fields presented in Theorem 2.19, there are evolutionary forms of the vector fields of the algebra $\mathfrak{g}$ (see Remark 2.3) as well as genuine generalized symmetries. The operators $\tilde{\mathcal{D}}$, $\tilde{\mathcal{G}}_1$, $\tilde{\mathcal{G}}_2$, $\tilde{\mathcal{P}}(r^1 + r^2)$, $\tilde{\mathcal{P}}(1)$ and $\tilde{\mathcal{W}}(\alpha)$, depending on r^3 only, are evolutionary forms of the operators $\mathcal{D}$, $\mathcal{G}$, $\mathcal{P}^v$, $\mathcal{P}^t$, $\mathcal{P}^x$ and $\mathcal{W}(\alpha)$, respectively. If the generalized vector field $\tilde{\mathcal{W}}(\alpha)$ depends on α not being the function of r^3 only, then it is a genuine non–Lie symmetry. The appearance in generalized vector field $\tilde{\mathcal{P}}(\beta)$ with $\beta \notin \langle 1, r^1 + r^2\rangle$ can be explained by the observation that the vector field $\tilde{\mathcal{P}}(\beta)$ is the evolutionary form of the symmetry $\mathcal{P}(\tau^0, \xi^0)$ of the essential subsystem $\mathcal{S}_0$ with $\tau^0 = \beta_1 + \beta_2$ and $\xi^0 = (\beta_1 + \beta_2)((r^1 + r^2)/2 - 1) - \beta + 2\beta_2$. Therefore, the connection lost upon transition to the whole system is recovered in generalized symmetries. In contrast to $\mathcal{P}(\tau^0, \xi^0)$, the other personal symmetry operator $\mathcal{J}$ of the system $\mathcal{S}_0$ is connected with the parameter σ arising in the course of deriving of the theorem 2.19 and therefore has no counterpart amongst first-order generalized symmetries of the system $\mathcal{S}$.

Proposition 2.21. *The system* (2.5) *admits generalized symmetries of arbitrary high order of the form*

$$\tilde{\mathcal{W}}^k(\alpha) = \alpha(\omega^0, \dots, \omega^k)\partial_{r^3},$$

where $\omega^i = \mathcal{R}^i r^3$ for $i = 0, \dots, k$ with $\mathcal{R} = \exp\frac{r^2 - r^1}{2}\mathrm{D}_x$ and α runs through the set of smooth functions of its arguments.

Proof. The differential operator $\mathcal{R} = \exp\frac{r^2 - r^1}{2}\mathrm{D}_x$ commutes with the operator $\mathrm{D}_t + \frac{r^1 + r^2}{2}\mathrm{D}_x$ meaning that $\mathcal{R}r^3$ is the solution of the equation (2.5c) whenever r^3 is.

Moreover, the process can be iterated and so $\mathcal{R}^k$ has the same property. This observation as well as the fact that the equation (2.5c) is linear with respect to r^3 makes an allusion that $\mathcal{R}^i r^3 \partial_{r^3}$ is a generalized high-order symmetry of the system (2.5). Indeed, the invariance criterion holds for it. Furthermore, considering operator $\mathcal{W}(\alpha)$ as recursion operator, we can construct the family of generalized high-order symmetries $\tilde{\mathcal{W}}^k(\alpha) = \alpha(\omega^0, \dots, \omega^k)\partial_{r^3}$, where ω^i and α as in the statement. □

2.10 Conservation laws

Definition 2.22. Consider a system of differential equations $\Delta(x, u^{(l)}) = 0$. A conservation law is a divergence expression $\operatorname{Div} P = 0$ which vanishes for all solutions $u = f(x)$ of the given system. Here $P = (P^1[u], \dots, P^n[u])$ is a n-tuple of differential functions and $\operatorname{Div} P = \mathrm{D}_1 P^1 + \cdots + \mathrm{D}_n P^n$ is its total divergence.

In a dynamical problem, one of the independent variables is distinguished as the time t, the remaining variables $x = (x^1, \dots, x^n)$ being spatial variables. In this case a conservation law takes the form

$$\mathrm{D}_t\, \rho + \operatorname{Div} \sigma = 0,$$

in which Div is the total spatial divergence of σ and D_t is the total derivative operator with respect to t. The functions ρ and σ are called conserved density and flux respectively, the vector (ρ, σ) is called a conserved vector. For more theoretical background on conservation laws see [41, 71].

In this section we classify hydrodynamic conservation laws of system (2.4) by applying a modification of the most direct method, which was proposed in [50] and following [11] we classify first-order conservation laws with space-independent densities and show an existence of higher-order ones.

Note that a conservation law is called *hydrodynamic* if its components are functions of the dependent variables only. The general form of hydrodynamic conservation laws of the system $\mathcal{S}$ is

$$(\mathrm{D}_t \rho(u, v, w) + \mathrm{D}_x \sigma(u, v, w))\,|_{\mathcal{S}} = 0, \tag{2.30}$$

where D_t and D_x are the operators of total differentiation with respect to t and x, defined as in Section 2.2. The components ρ and σ of the conserved current (ρ, σ) are called the density and the flux of the conservation law. Expanding the total derivatives in (2.30) on the solutions of the system $\mathcal{S}$ and splitting the equations obtained with respect to space derivatives of the dependent variables one finds the system

$$-u\rho_u - \rho_v + \sigma_u = 0, \quad -\rho_u - u\rho_v + \sigma_v = 0, \quad -u\rho_w + \sigma_w = 0,$$

which can be integrated resulting in $\rho = (\rho^0(w) + \rho^1(u, v))e^v$ and $\sigma = u(\rho^0(w) + \rho^1(u, v))e^v + \sigma^0(u, v)$, where ρ_0 is an arbitrary function of its argument and ρ^1 and σ^0 satisfy the system of equations

$$\sigma^0_u = e^v \rho^1_v, \quad \sigma^0_v = e^v \rho^1_u.$$

Introducing the function $\Phi = \Phi(u,v)$ such that $\Phi_u = \sigma^0$ and $\Phi_v = \rho^1 e^v$ we can rewrite the elements of the conservation laws as

$$\rho = e^v\rho^0 + \Phi_v, \quad \sigma = ue^v\rho^0 + u\Phi_v + \Phi_u,$$

where the function $\Phi(u,v)$ satisfies the telegraph equation $\Phi_{uu} = \Phi_{vv} - \Phi_v$. Also, there are only two inequivalent values of the function $\rho^0(w)$ up to equivalence transformations of the symmetry group G, namely $\rho^0_1 = 1$ and $\rho^0_2 = w$. Nonetheless, it is easily seen that the generating element (roughly speaking, acting on the generating element by symmetries, one finds a conservation law of the system, see [31, 51] for details) for ρ^0_1 can be expressed via the solution e^v of the aforementioned telegraph equation.

This proves the following theorem.

Theorem 2.23. *The space of conserved currents of hydrodynamic conservation laws of the system $\mathcal{S}$ consists of the conserved currents of the general form (ρ,σ) with*

$$\rho = e^v\rho^0 + \Phi_v, \quad \sigma = ue^v\rho^0 + u\Phi_v + \Phi_u,$$

where $\rho^0(w)$ is an arbitrary smooth function of w and the function $\Phi(u,v)$ runs through the set of solutions of the telegraph equation

$$\Phi_{uu} = \Phi_{vv} - \Phi_v.$$

The generating elements of this space are exhausted by

$$(e^v w, ue^v w), \quad (\Phi_v, u\Phi_v + \Phi_u).$$

Corollary 2.24. *The generating elements of the space of conserved currents of hydrodynamic conservation laws of the system* (2.5) *are exhausted by conserved currents* DHC *and* EHC*, corresponding to equation* (2.5c) *("degenerate hydrodynamic current") and the essential subsystem ("essential hydrodynamic current"), respectively,*

$$\mathrm{DHC} = \left(r^3 \exp\frac{r^1 - r^2}{2},\ \frac{r^1 + r^2}{2} r^3 \exp\frac{r^1 - r^2}{2}\right),$$

$$\mathrm{EHC}(\Phi) = \left(\Phi_1 - \Phi_2,\ \frac{r^1 + r^2}{2}(\Phi_1 - \Phi_2) + \Phi_1 + \Phi_2\right),$$

where $\Phi(r^1, r^2)$ runs through the solution set of the equation

$$4\Phi_{12} = \Phi_2 - \Phi_1.$$

Remark 2.25. Every hydrodynamic conserved current of a system (2.5) can be obtained by the action of a symmetry transformation of the system on the span of the generating elements. For example,

$$\Big(W(r^3)\exp\frac{r^1 - r^2}{2} + \Phi_1 - \Phi_2,$$
$$\frac{r^1 + r^2}{2} W(r^3)\exp\frac{r^1 - r^2}{2} + \frac{r^1 + r^2}{2}(\Phi_1 - \Phi_2) + \Phi_1 + \Phi_2\Big)$$

is the conserved current of the system generated by the action of its symmetry transformation $\mathcal{W}(W)$ on the sum of generating elements.

Remark 2.26. Functions β and Φ parameterizing first-order generalized symmetries and hydrodynamic conservation laws of the system (2.5) satisfy adjoint differential equations. This can be explained by the fact that characteristics of conservations laws are cosymmetries of the system, cf. [67].

Remark 2.27. The existence of systems with an infinite number of independent hydrodynamic conservation laws has been known for a long time [1, 39, 40]. In general, for semi-Hamiltonian diagonalized systems they are parameterized by n functions of one variable [64], where n is a number of the dependent variables. For the system (2.4) two of those functions relate to solutions of the telegraph equation $\Phi_{uu} = \Phi_{vv} - \Phi_v$, while the third function is an arbitrary function ρ^0 of w.

Each semi-Hamiltonian system possesses first-order conservation laws [11, 59] (see also [55, 56]). To classify them, following Theorem 5.1 of the first of these papers, one should find smooth nonvanishing functions $G^i(r)$, $i = 1, 2, 3$ such that

$$\frac{V^i_{r^j}}{V^i - V^j} = -\frac{G^i_{r^j}}{G^i}, \quad i \neq j, \quad \text{with} \quad \sum_{i=1}^{2} (G^i)^2 f^i = \text{const}$$

for some smooth functions $f^1 = f^1(r^1)$, $f^2 = f^2(r^2)$.

For the system (2.5) with characteristic speeds V^i given by (2.6), we obtain

$$G^1 = g^1(r^1) \exp\left(-\frac{r^2}{4}\right), \quad G^2 = g^2(r^2) \exp\frac{r^1}{4}, \quad G^3 = g^3(r^3) \exp\frac{r^1 - r^2}{2}$$

Every conserved current of order not greater than one with time- and space-independent density can be presented as the span of the following currents

$$\begin{aligned} \mathrm{CC}_0 &= \left(\left(\frac{1}{r^1_x} - \frac{1}{r^2_x}\right) \exp\frac{r^1 - r^2}{2}, \left(\frac{V^1}{r^1_x} - \frac{V^2}{r^2_x}\right) \exp\frac{r^1 - r^2}{2}\right) \\ \mathrm{CC}_1(F^3) &= \left(\exp\frac{r^1 - r^2}{2} F^3\left(\omega^0, \omega^1\right), \frac{r^1 + r^2}{2} \exp\frac{r^1 - r^2}{2} F^3\left(\omega^0, \omega^1\right)\right), \end{aligned} \tag{2.31}$$

where F^3 runs through the set of smooth functions of its arguments. This is formulated as the following assertion.

Theorem 2.28. *The space of conserved currents of conservation laws of order not greater than one of the system* (2.5) *with time- and space-independent densities is spanned by the conserved currents* CC_0 *and* $\mathrm{CC}_1(F^3)$ *given by* (2.31).

Since the system $\mathcal{S}$ is not genuinely nonlinear, Theorem 5.2 of [11] has the direct consequence.

Proposition 2.29. *The system $\mathcal{S}$ possesses nontrivial conservation laws of arbitrarily high order.*

Remark 2.30. Alternatively to [11], presence of higher-order conservation laws of the system (2.5) can be explained by actions of generalized symmetries on conserved currents. Thus, a family of conserved currents $\mathrm{CC}_k(\alpha)$ of arbitrary order k is obtained by acting the generalized k-order symmetry $\tilde{\mathcal{W}}^k$ on a hydrodynamic conserved current DHC,

$$\mathrm{CC}_k(\alpha) = \tilde{\mathcal{W}}^k(\mathrm{DHC}) = \\ \left(\exp\frac{r^1 - r^2}{2}\alpha(\omega^0, \dots, \omega^k),\ \frac{r^1 + r^2}{2}\exp\frac{r^1 - r^2}{2}\alpha\Big(\omega^0, \dots, \omega^k\Big)\right),$$

where α runs through the set of smooth functions of its arguments. Note that this agrees with the first order conserved current $\mathrm{CC}_1(F^3)$ found in Theorem 2.28.

2.11 Hamiltonian structure of hydrodynamic type

Given a system $\mathcal{E}$ of evolution differential equations $u_t = K[u]$, where K is a differential function of independent variables and spatial derivatives (including ones of zero order) of the dependent variables u, an algorithm of finding its Hamiltonian structures is as follows:

- Using the right hand side of the system $\mathcal{E}$ one defines the Fréchet derivative $\mathbf{D}_K$ of K, see Theorem 1.49 (also known as the linearization operator [7]), and the adjoint to it differential operator (1.7), and writes down the corresponding linearized and adjoint to it systems of differential equations

  $$\mathbf{D}_K(\phi) = 0, \quad \mathbf{D}_K^*(\psi) = 0,$$

  for appropriate vector functions ϕ and ψ.

- Making an ansatz one finds the Noether operators $\mathfrak{N}$ (see [19]) which map solutions of the adjoint system into the solutions of the linearized system.

- Pick amongst the Noether operators the Hamiltonian ones $\mathfrak{H}$.To this end, one chooses at first skew-symmetric operators, and then those whose associated brackets satisfy the Jacobi identity.

- Making an ansatz, one finds the Hamiltonians $\mathcal{H}$ solving the system

  $$\mathfrak{H}\delta\mathcal{H} = K.$$

The packages of symbolic computation such as Jets [35] are of great help.

We find the Hamiltonian structure of the system (2.3), choosing for this purpose the diagonalized form (2.5) of the given system (2.3). We start with the system of equations which consists of the initial system of equations (2.32a), the linearized system (2.32b) and the adjoint system (2.32c), as before there is no summation with respect to k, here and below δ^i_j stands for the Kronecker symbol for any i and j

$$r^k_t + V^k r^k_x = 0, \tag{2.32a}$$

$$\mathrm{D}_t\eta^k + V^k\mathrm{D}_x\eta^k + \frac{\eta^1+\eta^2}{2} r^k_x = 0, \tag{2.32b}$$

$$\mathrm{D}_t\lambda^k + \mathrm{D}_x(V^k\lambda^k) - \frac{1}{2} r^l_x \lambda^l (\delta^1_k + \delta^2_k) = 0. \tag{2.32c}$$

We are looking for Noether operator $\mathfrak{N} = (\mathfrak{N}^{ij})$ for hydrodynamic system, with

$$\mathfrak{N}^{ij} = h^{ij}(r)\mathrm{D}_x + f^{ij}(r, r_x).$$

By definition of the Noether operator $\eta = \mathfrak{N}\lambda$, or in coordinate form

$$\eta^i = h^{ij}\mathrm{D}_x\lambda^j + f^{ij}\lambda^j,$$

which allows rewriting the subsystem (2.32b)–(2.32c) as the following equation

$$\begin{aligned}
&-h^{kj}_{r^l} V^l r^l_x \mathrm{D}_x\lambda^j + h^{kj}\mathrm{D}_x(-\mathrm{D}_x(V^j\lambda^j) + \frac{1}{2}(r^l_x\lambda^l)(\delta^1_j + \delta^2_j)) \\
&\quad -f^{kj}_{r^l_x}\left(V^l r^l_{xx} + \frac{r^1_x + r^2_x}{2} r^l_x\right)\lambda^j - f^{kj}_{r^l} V^l r^l_x \lambda^j \\
&\quad +f^{kj}\left(-\mathrm{D}_x(V^j\lambda^j) + \frac{r^l_x\lambda^l}{2}(\delta^1_j + \delta^2_j)\right) + V^k(h^{kj}_{r^l} r^l_x \mathrm{D}_x\lambda^j + h^{kj}\mathrm{D}^2_x\lambda^j \\
&\quad +f^{kj}_{r^l} r^l_x \lambda^j + f^{kj}_{r^l_x} r^l_{xx}\lambda^j + f^{kj}\mathrm{D}_x\lambda^j) \\
&\quad +\frac{r^k_x}{2}(h^{1j}\mathrm{D}_x\lambda^j + f^{1j}\lambda^j + h^{2j}\mathrm{D}_x\lambda^j + f^{2j}\lambda^j) = 0.
\end{aligned}$$

Splitting with respect to $\mathrm{D}^2_x\lambda^j$ immediately implies $h^{kj} = 0$ for all $j \neq k$, and splitting with respect to $\mathrm{D}_x\lambda^j$ yields

$$\begin{aligned}
&f^{21} = -f^{12} = \frac{1}{4}(h^{11}r^2_x + h^{22}r^1_x), \quad f^{31} = -f^{13} = \frac{1}{2}h^{11}r^3_x, \\
&f^{23} = -f^{32} = \frac{1}{2}h^{22}r^3_x, \quad h^{11}_{r^3} = 0, \quad h^{11}_{r^2} = \frac{1}{2}h^{11}, \\
&h^{22}_{r^3} = 0, \quad h^{22}_{r^1} = -\frac{1}{2}h^{22}, \quad h^{33}_{r^1} = -h^{33}, \quad h^{33}_{r^2} = h^{33}
\end{aligned}$$

for $k \neq j$ and $k = j$, respectively. Further splitting with respect to λ^j and derivatives of r^l is quite cumbersome and therefore we present only result

$$\begin{aligned}
\mathfrak{N} = \mathrm{diag}&\left\{-\exp\frac{r^2-r^1}{2}, \exp\frac{r^2-r^1}{2}, \Phi(r^3)e^{r^2-r^1}\right\}\mathrm{D}_x \\
&- \frac{1}{4}\exp\frac{r^2-r^1}{2}\begin{pmatrix} r^2_x - r^1_x & r^1_x - r^2_x & -2r^3_x \\ r^2_x - r^1_x & r^1_x - r^2_x & -2r^3_x \\ 2r^3_x & 2r^3_x & -4f^{33}\exp\frac{r^1-r^2}{2} \end{pmatrix},
\end{aligned} \tag{2.33}$$

where $f^{33} = \frac{1}{2}\Phi(r^3)e^{r^2-r^1}(r^2_x - r^1_x) + \Psi\left(r^3, r^3_x \exp\frac{r^2-r^1}{2}\right)\exp\frac{r^2-r^1}{2}$, and Φ and Ψ are arbitrary smooth functions of their arguments. To qualify as Hamiltonian operator the Noether operator must be skew-adjoint, i.e, $\mathfrak{N}^* = -\mathfrak{N}$, yielding

$$f^{33} = \frac{1}{2}\mathrm{D}_x h^{33} = \frac{1}{2}e^{r^2-r^1}(\Phi(r^3)(r^2_x - r^1_x) + \Phi_{r^3}(r^3)r^3_x). \tag{2.34}$$

Denote the Noether operator (2.33) with constraints (2.34) by $\mathfrak{H}$.

According to Theorem 5 in [30], skew-adjoint Noether operators of certain systems including the system under consideration $\mathcal{S}$ satisfy the Jacobi identity automatically. Nonetheless, for non-scalar first-order systems this is only a hypothesis, so this result should be verified (for systems of higher dimensions the theorem is proved in [21], while for scalar first-order systems it is not true in general [29]). The proof can be done straightforwardly, but we prefer to show this from the geometrical point of view. To this end, we introduce some theoretical background demonstrating an intimate connection between systems of hydrodynamic type and Riemannian geometry.

Definition 2.31. The (Poisson) bracket $\{\cdot,\cdot\}$ is said to be of hydrodynamic type if for any arbitrary functionals $\mathcal{I}$ and $\mathcal{J}$

$$\{\mathcal{I}, \mathcal{J}\} = \int \delta\mathcal{I} \cdot \mathfrak{A}\delta\mathcal{J}\mathrm{d}x,$$

where $\mathfrak{A} = (\mathfrak{A}^{ij}) = g^{ij}(r(x))\mathrm{D}_x + b^{ij}_k(r(x))r^k_x$ and the Einstein summation convention is used.

It is evident that the operator $\mathfrak{H}$ satisfies the conditions for being a hydrodynamic operator $\mathfrak{A}$ from this definition. The following theorem [12] is the cornerstone of the geometric interpretation of hydrodynamic systems.

Theorem 2.32. *1) Upon a local change of variables $u = u(w)$ not containing derivatives with respect to x, the coefficients g^{ij} in Definition 2.31 are transformed as a second-order contravariant tensor; if $\det(g^{ij}) \neq 0$, then the expression $b^{ij}_k = -g^{is}\Gamma^j_{sk}$ is transformed so that Γ^j_{sk} are Christoffel symbols of Levi-Civita connection;*

2) the bracket $\{\cdot,\cdot\}$ is skew-symmetric if and only if the tensor g^{ij} is symmetric and the connection Γ^i_{jk} agrees with the metric, $(\nabla_k g)^{ij} = 0$;

3) provided that $\det(g^{ij}) \neq 0$, the bracket $\{\cdot,\cdot\}$ satisfies the Jacobi identity if and only if the curvature tensor $R^a{}_{bcd}$ vanishes.

As was already mentioned the operator $\mathfrak{H}$ is of hydrodynamic type with the metric

$$g^{ij} = \begin{pmatrix} -\exp\frac{r^2-r^1}{2} & 0 & 0 \\ 0 & \exp\frac{r^2-r^1}{2} & 0 \\ 0 & 0 & \Phi(r^3)\exp\{r^2-r^1\} \end{pmatrix}$$

and coefficients b^{ij}_k

$$b^{ij}_1 = \frac{1}{4}\exp\frac{r^2-r^1}{2}\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2\Phi(r^3)\exp\frac{r^2-r^1}{2} \end{pmatrix},$$

$$b_2^{ij} = \frac{1}{4}\exp\frac{r^2-r^1}{2}\begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2\Phi(r^3)\exp\frac{r^2-r^1}{2} \end{pmatrix},$$

$$b_3^{ij} = \frac{1}{4}\exp\frac{r^2-r^1}{2}\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ -2 & -2 & 2\Phi_{r^3}(r^3)\exp\frac{r^2-r^1}{2} \end{pmatrix}.$$

Since the system (2.5) is semi-Hamiltonian the computation of non-diagonal elements Γ^i_{jk}, $j \neq k$, is quite straightforward [64]:

$$\Gamma^k_{ki} = \frac{\partial_i V^k}{V^i - V^k} \quad \text{if } i \neq k \quad \text{and} \quad \Gamma^i_{jk} = 0 \quad \text{if } i \neq j \neq k.$$

The diagonal elements are computed using the formula $\Gamma^j_{sk} = -g_{si}b_k^{ij}$, where g_{si} is a second order covariant tensor associated with g^{si}. Since the latter is diagonal, the elements of the former are just inverses of elements of g^{si}. The resulting Christoffel symbols are the following

$$\Gamma^1_{ij} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{2}\exp\frac{r^1-r^2}{2}/\Phi(r^3) \end{pmatrix},$$

$$\Gamma^2_{ij} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{2}\exp\frac{r^1-r^2}{2}/\Phi(r^3) \end{pmatrix},$$

$$\Gamma^3_{ij} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2}\Phi_{r^3}(r^3)/\Phi(r^3) \end{pmatrix}.$$

One can verify that the operator $\mathfrak{H}$ is skew-symmetric checking that the connection Γ^i_{jk} agrees with the metric g. To this end, we compute covariant derivatives $(\nabla_k g)^{ij}$ using the formula (1.8)

$$(\nabla_k g)^{ij} = \partial_k g^{ij} + g^{lj}\Gamma^i_{kl} + g^{il}\Gamma^j_{kl}.$$

The straightforward computation confirms that the operator $\mathfrak{H}$ is skew-adjoint. To verify the Jacobi identity one must check that the components of Riemann curvature tensor vanish. Due to its symmetries, we only need to verify that components ${R^i}_{jij}$ for $i \neq j$ vanish. Computing

$$R^i_{jij} = \partial_i\Gamma^i_{jj} - \partial_j\Gamma^i_{ij} + \Gamma^k_{ij}\Gamma^i_{kj} - \Gamma^k_{ji}\Gamma^i_{kj},$$

shows that the Noether operator $\mathfrak{H}$ satisfies the Jacobi identity and therefore proves that it is also a Hamiltonian operator.

From the geometric point of view, defining the diagonal flat metric is equivalent to defining orthogonal curvilinear system of coordinates in the flat pseudo-Euclidian space. We introduce complex Lamé coefficients

$$L_i = \sqrt{g_{ii}}, \tag{2.35}$$

and Darboux rotation coefficients

$$\beta_{ik}(r) = \frac{\partial_i L_k}{L_i} \text{ for } i \neq k \text{ and } \beta_{ii} = 0, \tag{2.36}$$

which we will need in the sequel.

Finally, we are looking for a Hamiltonian $\mathcal{H} = \int H(r)\mathrm{d}x$ satisfying the following system of equations

$$\mathfrak{H}\frac{\delta\mathcal{H}}{\delta r} = -\begin{pmatrix} V^1 r^1_x \\ V^2 r^2_x \\ V^3 r^3_x \end{pmatrix},$$

where $\delta\mathcal{H}/\delta r$ is the vector of variational derivatives with respect to Riemann invariants r^1, r^2 and r^3, which due to the fact that H is a function of r only coincides with the gradient of H, $\delta\mathcal{H}/\delta r = \nabla H$. Expanding this system we find

$$\left(-\mathrm{D}_x H_1 + \frac{r^2_x - r^1_x}{4}(H_2 - H_1) + \frac{r^3_x}{2}H_3\right)\exp\frac{r^2 - r^1}{2} = -V^1 r^1_x, \tag{2.37a}$$

$$\left(\mathrm{D}_x H_2 + \frac{r^2_x - r^1_x}{4}\left(H_2 - H_1 + \frac{r^3_x}{2}H_3\right)\right)\exp\frac{r^2 - r^1}{2} = -V^2 r^2_x, \tag{2.37b}$$

$$\begin{aligned} &-\frac{1}{2}r^3_x(H_1 + H_2)\exp\frac{r^2 - r^1}{2} + \mathrm{D}_x(H_3)\Phi e^{r^2 - r^1} \\ &\quad + \frac{1}{2}\left((r^2_x - r^1_x)\Phi + r^3_x \Phi_{r^3}\right) H_3 e^{r^2 - r^1} = -V^3 r^3_x. \end{aligned} \tag{2.37c}$$

Successively splitting with respect to derivatives r^1_x, r^2_x and r^3_x and using elementary algebraic transformations we find the final form of the function $H(r)$

$$H = \frac{1}{4}(r^1 + r^2)^2 \exp\frac{r^1 - r^2}{2} + C_0(r^1 + r^2) + (r^1 - r^2 + \theta(r^3))e^{r^1 - r^2}, \tag{2.38}$$

where C_0 is an arbitrary constant and $\theta(r^3)$ satisfies the equality

$$(|\Phi|^{1/2}\theta_{r^3})_{r^3} = C_0\frac{|\Phi|^{1/2}}{\Phi}.$$

It worth noting that provided $\theta_{r^3} \neq 0$ the last equality can be rewritten in the form

$$\Phi = \frac{2C_0\theta + C_1}{(\theta_{r^3})^2},$$

where C_1 is an arbitrary constant. On the other hand, the case $C_0 = \theta = 0$ produces the degenerate Hamiltonian

$$\mathcal{H} = \int\left(\left(\frac{r^1 + r^2}{2}\right)^2 + r^1 - r^2\right)\exp\frac{r^1 - r^2}{2}\mathrm{d}x.$$

Thus we proved the following theorem.

Theorem 2.33. *The system* (2.5) *admits an infinite family of Hamiltonian structures $r_t = \mathfrak{H}\delta\mathcal{H}$, where the Hamiltonian operators $\mathfrak{H}$ are defined by* (2.33)*, with Φ running through the set of arbitrary smooth functions of its arguments, Ψ and f^{33} being defined by* (2.34)*, and the family of Hamiltonians $\mathcal{H} = \int H\mathrm{d}x$ by* (2.38)*.*

2.12 Recursion operators

For hydrodynamic-type systems in [62] found were Teshukov's recursion operators which are first-order differential operators without pseudo-differential part. Nonetheless, due to criterion [58] such situation is possible when associated Darboux rotation coefficients (2.36) depend only on differences of Riemann invariants. The criterion fails for the system $\mathcal{S}$ as $\beta_{23} = -\frac{1}{4}\sqrt{\Phi(r^3)}\exp\{r^1 - r^2\}$. Thus, Teshukov's operators are not admitted. However the system (2.5) possesses generalized symmetries of arbitrary high order relating to the equation (2.5c) making an allusion about the form of the recursion operator. To avoid cumbersomeness of computations we make the following already simplified ansatz for recursion operator

$$\mathfrak{R}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & f^{33}(r, r_x) \end{pmatrix} \mathrm{D}_x + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ g^{31} & g^{32} & g^{33} \end{pmatrix},$$

where g^{31}, g^{32} and g^{33} are functions of r, r_x and r_{xx}. Let $\eta = (\eta^1, \eta^2, \eta^3)$ and $\lambda = (\lambda^1, \lambda^2, \lambda^3)$ be vector functions satisfying the system (2.32b), relating $\eta = \mathfrak{R}^1\lambda$. Coordinate-wise it is written as

$$\begin{aligned}
&f^{33}_{r^l}(-V^l r^l_x)\mathrm{D}_x\lambda^3 + f^{33}_{r^l_x}\left(-\frac{r^1_x + r^2_x}{2} r^l_x - V^l r^l_{xx}\right)\mathrm{D}_x\lambda^3 \\
&+ f^{33}\left(-\frac{r^1_x + r^2_x}{2}\mathrm{D}_x\lambda^3 - \frac{r^1 + r^2}{2}\mathrm{D}^2_x\lambda^3 - \frac{r^3_x}{2}(\mathrm{D}_x\lambda^1 + \mathrm{D}_x\lambda^2) - \frac{\lambda^1 + \lambda^2}{2} r^3_{xx}\right) \\
&+ \Bigg(g^{3j}_{r^l}(-V^l r^l_x) + g^{3j}_{r^l_x}\left(-\frac{r^1_x + r^2_x}{2} r^l_x - V^l r^l_{xx}\right) + g^{3j}_{r^l_{xx}}\Big(-\frac{r^1_{xx} + r^2_{xx}}{2} r^l_x \\
&-(r^1_x + r^2_x) r^l_{xx} - V^l r^l_{xxx}\Big)\Bigg)\lambda^j + g^{3j}\left(-V^j\mathrm{D}_x\lambda^j - \frac{\lambda^1 + \lambda^2}{2} r^j_x\right) \\
&+ \frac{r^1 + r^2}{2}\Big(f^{33}_{r^l} r^l_x \mathrm{D}_x\lambda^3 + f^{33}_{r^l_x} r^l_{xx}\mathrm{D}_x\lambda^3 + f^{33}\mathrm{D}^2_x\lambda^3 + g^{3j}_{r^l} r^l_x\lambda^j + g^{3j}_{r^l_x} r^l_{xx}\lambda^j \\
&+ g^{3j}_{r^l_{xx}} r^l_{xxx}\lambda^j + g^{3j}\mathrm{D}_x\lambda^j\Big) = 0.
\end{aligned}$$

Successively splitting this equation with respect to $\mathrm{D}_x\lambda^3$, $\mathrm{D}_x\lambda^2$, $\mathrm{D}_x\lambda^1$ and λ^2 one proves the theorem.

Theorem 2.34. *The system* (2.5) *admits the recursion operator* $\mathfrak{R}^1$ *of the form*

$$\begin{aligned}
\mathfrak{R}^1 = f(r^3)\exp\frac{r^2 - r^1}{2}\Bigg(&\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r^3_x \end{pmatrix}\mathrm{D}_x + \\
&\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -(r^3_x)^2 & (r^3_x)^2 & r^3_x(r^1_x - r^2_x) - r^3_{xx} \end{pmatrix}\Bigg),
\end{aligned}$$

where f runs through the set of smooth functions of its argument.

As Teshukov's conditions hold for the essential subsystem $\mathcal{S}_0$, it can be expected that making more general ansatz, one can find recursion operator generating generalized symmetries of the system (2.5) relating to the essential subsystem. Let η and λ as above and the recursion operator $\mathfrak{R}^2$ be the first-order pseudo-differential operator acting nonlocally as

$$\eta = \mathfrak{R}^2\lambda = A(r)\mathrm{D}_x\lambda + B(r, r_x)\lambda + C(r, r_x)Y, \tag{2.39}$$

where $A = (A^{ij})$ and $B = (B^{ij})$ are 3×3 matrices, C is a 3-component column and Y is a potential function that satisfies the system

$$\mathrm{D}_tY = -V^1\lambda^1 - V^2\lambda^2, \quad \mathrm{D}_xY = \lambda^1 + \lambda^2. \tag{2.40}$$

The right hand sides of the last system is nothing else than linearized components of the conserved current $(r^1 + r^2, (r^1 + r^2)^2/4 + r^1 - r^2)$ up to sign.

Taking into account that both η and λ satisfy (2.32b), we can write the equation (2.39) out coordinate-wise as

$$\begin{aligned}
&A^{kj}_{r^l}(-V^lr^l_x)\mathrm{D}_x\lambda^j + A^{kj}\mathrm{D}_x\left(-V^j\mathrm{D}_x\lambda^j - \frac{\lambda^1+\lambda^2}{2}r^j_x\right) + B^{kj}_{r^l}(-V^lr^l_x)\lambda^j \\
&+B^{kj}_{r^l_x}\left(-V^lr^l_{xx} - \frac{r^1+r^2}{2}r^l_x\right)\lambda^j + B^{kj}\left(-V^j\mathrm{D}_x\lambda^j - \frac{\lambda^1+\lambda^2}{2}r^j_x\right) \\
&+C^k_{r^l}(-V^lr^l_{xx})Y + C^k_{r^l_x}\left(-V^lr^l_{xx} - \frac{r^1_x+r^2_x}{2}r^l_x\right)Y + C^k(-V^1\lambda^1 - V^2\lambda^2) \\
&+V^k\Big(A^{kj}_{r^l}r^l_x\mathrm{D}_x\lambda^j + A^{kj}\mathrm{D}^2_x\lambda^j + B^{kj}_{r^l}r^l_{xx}\lambda^j + B^{kj}\mathrm{D}_x\lambda^j + C^k_{r^l}r^l_xY + C^k_{r^l_x}r^l_{xx}Y \\
&+ C^k(\lambda^1+\lambda^2)\Big) + \frac{r^k_x}{2}\left(A^{1j}\mathrm{D}_x\lambda^j + B^{1j}\lambda^j + C^1Y + A^{2j}\mathrm{D}_x\lambda^j + B^{2j}\lambda^j + C^2Y\right) \\
&= 0,
\end{aligned}$$

with summation with respect to repeatable indices j and l, and no summation over k. The following splitting with respect to $\mathrm{D}^2_x\lambda^j$, $\mathrm{D}_x\lambda^j$, λ^j and Y, respectively, yields

$$\begin{aligned}
&A^{kj}(V^k - V^j) = 0, \\
&A^{kj}_{r^l}(V^k - V^l)r^l_x - \frac{r^1_x(1+\delta^1_j) + r^2_x(1+\delta^2_j)}{2}A^{kj} + \\
&\qquad (V^k - V^j)B^{kj} + \frac{1}{2}r^k_x(A^{1j} + A^{2j}) = 0, \\
&-A^{kj}\frac{r^j_x}{2}(\delta^1_j + \delta^2_j) - B^{kj}\frac{r^k_x}{2}(\delta^1_j + \delta^2_j) + B^{kj}_{r^l}(V^k - V^l)r^l_x \\
&\qquad +B^{kj}_{r^l_x}\left((V^k - V^l)r^l_{xx} - \frac{r^1_x + r^2_x}{2}r^l_x\right) + C^k(V^k - V^j)(\delta^1_j + \delta^2_j) \\
&\qquad +\frac{r^k_x}{2}(B^{1j} + B^{2j}) = 0, \\
&C^k_{r^l}(V^k - V^l)r^l_x + C^k_{r^l_x}\left((V^k - V^l)r^l_{xx} - \frac{r^1_x + r^2_x}{2}r^3_x\right) + \frac{r^k_x}{2}(C^1 + C^2) = 0.
\end{aligned}$$

Splitting step by step these equations we prove the theorem.

Theorem 2.35. *The system* (2.5) *admits the formally pseudo-differential recursion operator* $\mathfrak{R}^2$ *acting as*

$$\eta = \mathfrak{R}^2 \lambda = B\lambda + CY,$$

where

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} r^1_x \\ r^2_x \\ r^3_x \end{pmatrix},$$

and Y *defined by* (2.40).

Chapter 3

Conclusion

3.1 Summary of results

In the thesis we have carried out extended symmetry analysis of the isothermal no-slip drift flux model given by the system (2.3). It turned out that the form (2.3) was not convenient for the study within the symmetry framework. In particular, the maximal Lie invariance algebra of the system (2.3) is difficult to compute even using specialized computer algebra packages, e.g. [68]. Therefore, transforming dependent variables we have represented the model in the form (2.4). This representation has allowed us to compute the maximal Lie invariance algebra $\mathfrak{g}$ of the initial system. This algebra is infinite-dimensional. Note that the infinite-dimensional part of $\mathfrak{g}$ spanned by $\{\tilde{\mathcal{W}}(\tilde{\kappa})\}$ was missed in [52].

Moreover, we have computed the complete point symmetry group of the system $\mathcal{S}$ using the combined algebraic method. Since the algebra $\mathfrak{g}$ is infinite-dimensional, the straightforward application of the automorphism-based method is not appropriate. For this reason we have chosen its megaideal-based counterpart. However, the fact that the finite-dimensional part of the algebra $\mathfrak{g}$ coincides with its radical $\mathfrak{r}$, which is a megaideal, allowed us to partially use the automorphism-based version of the algebraic method as well. To this end, we have computed the entire automorphism group of the radical $\mathfrak{r}$ of $\mathfrak{g}$. Finally, we have employed the constraints derived from the algebraic method to obtain the complete point symmetry group (2.9) of the isothermal no-slip drift flux model $\mathcal{S}$ applying the direct method.

Following the standard Lie reduction procedure [44], we have also obtained optimal lists of one- and two-dimensional subalgebras of the algebra $\mathfrak{g}$, which were used to find appropriate solution ansatzes. Substituting them into the initial system we found families of invariant and partially invariant solutions of the system $\mathcal{S}$.

Sections 2.6–2.7 and 2.8 provide alternative approaches to finding solutions of the system under study. The first of them is based on the fact that the subsystem $\mathcal{S}_0$ is partially decoupled from the rest of the system, so we could construct its solutions first and then solve the remaining equation (2.4c) in parametric form. This choice is justified by the fact that the system $\mathcal{S}_0$ admits larger Lie symmetry group than the Lie symmetry group of $\mathcal{S}$ and thus one can find more invariant or partially invariant solutions. Moreover, every two-dimensional hydrodynamic-type system can be linearized via the two-dimensional hodograph transformation with $(p,q) = (t,x)$, $(y,z) = (u,v)$ being the new dependent and independent variables, respectively. Here it is worth noticing that the equation (2.4c) in the new variables remains a first-order linear PDE in w which is easily solvable. The above change of variables had led to the potential system (2.20) of the telegraph equation (2.21). Using the further change of variables $\tilde{p} = p\exp(-z/2)$ we obtained the famous Klein–Gordon equation (2.22). As the latter equation is well studied, we could construct large classes of exact solutions, find p and solve equation (2.4c), which allows us to find solutions of the initial system $\mathcal{S}$ in parametric form.

The second approach of finding solutions of (2.4) is based on the semi-Hamiltonian property of (2.5), which has allowed us to employ the generalized hodograph transformation. Just as for the previous method, the problem of finding solutions of the system (2.4) was reduced to solving Klein–Gordon equation. Nonetheless, the solutions obtained via generalized hodograph transformations are included in the list of solutions obtaining via decoupling of the system (2.4).

We have computed the first-order generalized symmetries for the system (2.5), amongst which we observed both evolutionary forms of point symmetries and genuinely generalized symmetries. It is an interesting fact that some of genuinely generalized symmetries are nothing else but evolutionary forms of point symmetries of the essential subsystem. Moreover, the system $\mathcal{S}$ possesses genuinely generalized symmetries beyond those (see Remark 2.20). We have also found the recursion operators $\mathfrak{R}^1$ and $\mathfrak{R}^2$ first of which turned out not to have differential part unlike pseudo-differential part, while the second is the first-order differential operator and thus generates the higher-order generalized symmetries of the system (2.5).

We have also constructed the entire space of hydrodynamic conservation laws for the system (2.4) which is an infinite-dimensional. Note that the space of hydrodynamic conservation laws of a diagonalized semi-Hamiltonian system is parameterized by n functions of one variable, where n is a number of dependent variables. For the system (2.4) two of those functions are determined by solutions of the telegraph equation $\Phi_{uu} = \Phi_{vv} - \Phi_v$, while the third function is an arbitrary function of w, which can be reduced to 1 by actions of point symmetries unless it is non-zero. The first-order conservation laws with (t,x)-independent densities have been also classified. The existence of higher-order conservation laws has been established using the fact that system (2.5) was not genuinely nonlinear. Such conservation laws have been found using the actions of generalized symmetries of the system (2.5) of arbitrarily high order.

Furthermore, we have found the infinite family of Hamiltonian structures for the system (2.5). This family is parameterized by an arbitrary smooth function of r^3 and several constants. There is the degenerate Hamiltonian structure amongst constructed ones. Moreover, we have verified the assertion formulated in [30] stating that the Jacobi identity holds automatically for skew-adjoint Noether operators of the system of first-order evolution equations with more than one dependent variables, using the underlying geometric interpretation of hydrodynamic-type systems.

3.2 Future work

We found only generalized symmetries and conservation laws of the system $\mathcal{S}$ of order not greater than one as well as higher-order generalized symmetries relating to the linearly degenerate part of the system $\mathcal{S}$. In general, it is quite a common situation when a hydrodynamic-type system does not possess generalized symmetries of order greater than one. The key observation for studying higher-order symmetries is the fact that the essential subsystem $\mathcal{S}_0$ of the system $\mathcal{S}$ is linearized by the hodograph transformation to the Klein–Gordon equation for which all generalized symmetries of which are known [38, 57]. The further work is to figure out which of those symmetries are prolonged to the entire system $\mathcal{S}$ and how this prolongation is carried out.

The linearizability of the subsystem $\mathcal{S}_0$ can be applied to classification of all conservation laws of the system $\mathcal{S}$. The conjecture here is that the entire space of conservation laws of the system $\mathcal{S}$ is spanned by conservation laws from the family presented in Remark 2.30 and of the essential subsystem $\mathcal{S}_0$. Moreover, one does not need to prolong conservation laws of the essential subsystem $\mathcal{S}_0$ to conservation laws of the entire system $\mathcal{S}$, which makes the problem less difficult than that of finding generalized symmetries.

Applying the linearization of the subsystem $\mathcal{S}_0$ cannot guarantee the description of the complete set of local recursion operators of the system $\mathcal{S}$. Nonetheless, this can extend the set of computed operators. Besides, introducing the potential of the Klein–Gordon equation we can express it in terms of the essential subsystem $\mathcal{S}_0$ and find nonlocal recursion operators of the system $\mathcal{S}$.

Unfortunately, the partial coupling of the system $\mathcal{S}$ is not helpful for finding Hamiltonian structures of the system $\mathcal{S}$. Though we computed the infinite-dimensional family of hydrodynamic Hamiltonian structures, which is in fact, to the best of author's knowledge, the first such example in the literature, the existence of non-hydrodynamic Hamiltonian structures is also of interest.

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