Mixed Volumes and Anisotropic Potentials

by

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Department of Mathematics and Statistics
Memorial University

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Abstract

This PhD-thesis, comprizing five chapters deals with some topics combining potential theory and convex geometry through studying the mixed volumes and the anisotropic potentials, whence their applications in information theory and elliptic PDEs.

Chapter 1 is the introduction and overview for the whole dissertation. Chapter 2 studies a mixed volume induced by the anisotropic Riesz-potential including its reverse Minkowski-type inequality. It turns out that such a mixed volume is equal to the anisotropic Cordes-Nirenberg-capacity. Two restrictions on the Lorentz spaces are characterized. Besides, we also prove a Minkowski-type inequality and a log-Minkowski type inequality as well as its reverse form.

Chapter 3 investigates a mixed volume from the anisotropic potential with natural logarithm as a better complement to the end point case of the mixed volumes from the anisotropic Riesz-potential. An optimal polynomial log-inequality is not only discovered but also applicable to produce a polynomial dual for the conjectured fundamental log-Minkowski inequality in convex geometry analysis. Moreover, the star body with respect to the origin is characterized in terms of anisotropic potentials over the Euclidean spaces.

Chapter 4 establishes an interpretation of a functional type of mixed volume, the f-divergence via the Orlicz addition of measures. Fundamental inequalities, such as a dual functional Orlicz-Brunn-Minkowski inequality, are established. We also investigate an optimization problem for the f-divergence.

Chapter 5 characterizes the embeddings of associate Morrey spaces to Cordes-Nirenberg spaces and Cordes-Nirenberg spaces to Morrey spaces and hence produces the embedding chain. The trace of Riesz-Cordes-Nirenberg potentials, i.e., the boundedness of the Riesz operator mapping Cordes-Nirenberg spaces to the Radon measure based Campanato space, is also established with both sufficient and necessary conditions. Consequently, the regularity of an elliptic equation living on the Cordes-Nirenberg spaces can be characterized by means of the Campanato spaces.

To my dear parents and sister

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Statement of contribution

This dissertation studies several mixed volumes as geometric extensions of the Newton gravitational potential. Isovolumetric inequalities are established, which are widely applied to convex geometry analysis, function spaces and PDE. As a functional mixed volume, this thesis studies f-divergence by the Orlicz addition for measures. Applications in information theory are explored.

This dissertation combines the following five papers:

S. Hou, J. Xiao and D. Ye, A mixed volume from the anisotropic Riesz-potential. Submitted.

S. Hou and J. Xiao, A mixed volumetry for the anisotropic logarithmic potential. J. Geom. Anal. (2017). DOI: 10.1007/s12220-017-9895-z.

S. Hou and J. Xiao, Convex body via gravitational potential. Expo. Math. (2017). (Accepted)

S. Hou and D. Ye, Orlicz addition for measures and an optimization problem for the f-divergence. Submitted.

S. Hou and J. Xiao, Cordes-Nirenberg's embedding and restricting with application to an elliptic equation. Submitted.

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List of symbols

$\mathbb{N}$	natural numbers
$\mathbb{R}^n$	Euclidean space with dimension $n \in \mathbb{N}$
o	the origin point of $\mathbb{R}^n$
$B(x, r)$	the closed Euclidean ball with center x and radius $r > 0$
S^{n-1}	the unit sphere of $\mathbb{R}^n$
E^c	the complement of the Lebesgue measurable set $E \subseteq \mathbb{R}^n$
$int(E)$	the interior of the Lebesgue measurable set $E \subseteq \mathbb{R}^n$
$\overline{E}$	the closure of the Lebesgue measurable set $E \subseteq \mathbb{R}^n$
$V(E)$	the n-dimensional volume of Lebesgue measurable set $E \subseteq \mathbb{R}^n$
C_0^∞	the set of all smooth functions with compact support in $\mathbb{R}^n$
$X \lesssim Y$	$\exists C > 0$ not related with $X, Y \in \mathbb{R}$ such that $X \leq CY$
$X \gtrsim Y$	$\exists C > 0$ not related with $X, Y \in \mathbb{R}$ such that $X \geq CY$
$X \approx Y$	$\exists C_1, C_2 > 0$ not related with $X, Y \in \mathbb{R}$ such that $C_1 Y \leq X \leq C_2 Y$.

Chapter 1

Introduction and overview

Our starting point is the well-known Newton gravitational potential $B(o,r) \subseteq \mathbb{R}^3$ with unit mass density (see e.g. [56]):

$$\frac{1}{4\pi}\int_{B(o,r)} \frac{dy}{|x-y|} = \begin{cases} \frac{r^2}{2} - \frac{|x|^2}{6}, & \text{if } x \in B(o,r); \\ \frac{r^3}{3|x|}, & \text{if } x \in \mathbb{R}^3 \setminus B(o,r). \end{cases}$$

Clearly, we have

$$\begin{aligned} \sup_{x\in\mathbb{R}^3}\int_{B(o,r)} \frac{dy}{|x-y|} &= \int_{B(o,r)} \frac{dy}{|y|} \\ &= 2\pi r^2 \\ &= 2\pi \left(\frac{V\big(B(o,r)\big)}{V\big(B_1(o)\big)}\right)^{\frac{2}{3}} \\ &= \frac{3}{2} V\big(B(o,r)\big)^{\frac{2}{3}} V\big(B_1(o)\big)^{\frac{1}{3}}. \end{aligned} \tag{1.1}$$

Such a simple but important computation leads to the following question: *Is it possible to extend* (1.1) *to any n-dimensional space $(\mathbb{R}^n, \|\cdot\|)$, where $\|\cdot\|$ is a norm defined on $\mathbb{R}^n$?*

To settle this question, as a geometrical understanding of the maximal gravitational potential, we introduce a mixed volume induced by the anisotropic Riesz-potential in Section 2.1 and establish a reverse Minkowski-type inequality in Section 2.2, which implies a integral presentation to the lower bound of Mahler volume. Metric properties including inner and outer regularity are studied in Section 2.3.

It turns out that such a mixed volume is equal to the anisotropic Cordes-Nirenberg-capacity and has connections with the anisotropic Cordes-Nirenberg space in Section 2.4. In Section 2.5, two restrictions on the Lorentz spaces in terms of the anisotropic Cordes-Nirenberg-capacity are characterized. Moreover, we also prove a Minkowski-type inequality and a log-Minkowski type inequality as well as its reverse form respectively in Section 2.6 and 2.7.

In Chapter 3, we study the mixed volume $V_{\log,m}(E,K)$ from anisotropic potential with natural logarithm, as a better complement to the end point case of the mixed volumes $V_\alpha(E,K)$ developed in Chapter 2. Note that $V_{\log,m}(E,K)$ is of independent interest in engineering and mathematical physics [23, 35] under the circumstances.

If m is an even number then $V_{\log,m}(E,K) = +\infty$ (see Remark 3.1.2(ii)). The principal theorem of this chapter is the optimal polynomial inequality for $V_{\log,m}(E,K)$ established in Section 3.2 for m being an odd number. In the immediate sequel in Section 3.3, we prove the dual polynomial log-Minkowski inequality for two star bodies K and L in convex geometry analysis, which may be regarded as the polynomial dual for the fundamental log-Minkowski inequality for mixed volumes of two convex bodies conjectured by Böröczky-Lutwak-Yang-Zhang in [15] where it is only proved for $n = 2$. More precisely, the dual polynomial log-Minkowski inequality under $m = 1$ reduces to the known dual log-Minkowski inequality developed by Gardner-Hug-Weil-Ye in [30] and by Wang-Liu in [66].

We believe that there exists a corresponding dual polynomial log-Brunn-Minkowski inequality, which together with the dual polynomial log-Minkowski inequality, will establish the dual polynomial log-Brunn-Minkowski theory. This is a generalized endpoint case of the significant dual Brunn-Minkowski theory developed by Zhu-Zhou-Xu in [74] and Gardner-Hug-Weil-Ye in [30].

Moreover, we characterize the star body with respect to the origin in terms of anisotropic Riesz-potentials in Chapter 2 and logarithmic potentials in this chapter for all $\mathbb{R}^{n\geq 2}$ in Section 3.4.

In Chapter 4, we explore the connection between the Orlicz addition for measures and the f-divergence, which is a functional type of mixed volume. Let Ω be a nonempty set and μ be a measure on Ω. Assume that P and Q are two measures on Ω whose density functions p and q, respectively, with respect to μ are positive on Ω. That is, $p, q > 0$ such that

$$P(\Omega) = \int_\Omega p\, d\mu < \infty \quad \text{and} \quad Q(\Omega) = \int_\Omega q\, d\mu < \infty.$$

For a real valued function f, the f-divergence of P and Q, denoted by $D_f(P, Q)$, was introduced independently by Csiszár [21], Morimoto [50], and Ali and Silvey [8]). It can be formulated by

$$D_f(P, Q) = \int_\Omega f\left(\frac{p}{q}\right) q\, d\mu. \tag{1.2}$$

The f-divergence is an extension of the classical L_p distance of measures and contains many widely-used distances for measures as its special cases, e.g., Bhattcharyya distance [14], Hellinger, Kullback-Leibler divergence [36], Renyi distance [7], χ^2-distance [26] and total variation distance [42]. The f-divergence can be used to distinguish two measures and plays fundamental roles in topics such as image analysis, information theory, pattern matching and statistical learning (see [11, 20, 34, 42, 53]), where the measure of difference between measures is required.

The Brunn-Minkowsi inequality is arguably one of the most important inequalities in geometry. It can be used to prove, for instance, the celebrated Minkowski's and isoperimetric inequalities. See the excellent survey [28] by Gardner for more details. On the other hand, the dual Brunn-Minkowski inequality and dual Minkowski inequality are crucial for the solutions of the famous Busemann-Petty problem (see e.g., [27, 31, 48, 73]). These inequalities have been extended to the Orlicz theory in [29, 30, 67, 74].

Chapter 4 is dedicated to provide a basic theory for the dual functional Orlicz-Brunn-Minkowski theory and an interpertation for the f-divergence. In particular, we define the Orlicz addition of functions in Section 4.1 and further Orlicz addition of measures together with the dual functional Orlicz-Brunn-Minkowski inequality in Section 4.2, which is proved to be equivalent to Jensen's inequality under certain case in Section 4.4. Moreover, the f-divergence is proved to be the first order variation of the total mass of a measure obtained by a linear Orlicz addition of two measures in Section 4.3. Further connections between the f-divergence and geometry are provided. In Section 4.5, we investigate an optimization problem for the f-divergence, and define the dual functional Orlicz affine and geominimal surface areas for measures. Related functional affine isoperimetric inequalities for the dual functional Orlicz affine and geominimal surface areas for measures are established.

In Chapter 5, we apply the theory of the mixed volume from the anisotropic Riesz-potential in Chapter 2 to function spaces and PDE. As is well-known, the Morrey space $L^{q,\lambda}$ is a very useful tool for handling the regularity for solutions of some partial differential equations of basic importance (see [32, 63, 51, 64, 3, 62]). As one of generalizations of the Morrey space, the Campanato space (covering BMO and the Hölder space C^{γ}; see [54, 68]) was introduced in [17]. Moreover, the associate Morrey space $H^{s,\kappa}$ was explored in [5, 57, 69, 12]. Simultaneously, it is known that Cordes-Nirenberg space $CN^{p,\tilde{\alpha}}$ (regarded also as a variant of the Morrey space - see [41]), together with $L^{q,\lambda}$, plays a significant role in studying the local behaviour of solutions to some nonlinear elliptic equations (see [40, 41, 3]). When $p = 2$, these spaces are particularly important, since the techniques suggested by Cordes cannot be applied to the Morrey space (see [19]); see Section 5.1 for definitions of these function spaces.

In Section 5.2, we first establish the Cordes-Nirenberg embedding $CN^{p,\widetilde{\alpha}} \subseteq L^{p,\lambda}$ with both sufficient and necessary conditions. Then the associate Morrey spaces embedding $H^{s,k} \subseteq CN^{p,\widetilde{\alpha}}$ is completely studied for all $s \in [1, \infty]$, whence producing the embedding chains $H^{s,k} \subseteq CN^{p,\widetilde{\alpha}} \subseteq L^{q,\lambda}$.

On the other hand, the restricting/tracing of the Riesz-type potentials, i.e., the boundedness of the Riesz operator I_α on these function spaces, has been widely investigated within analysis, geometry and so on. For example, the famous Galiardo-Nirenberg-Sobolev's inequality can be implied by the boundedness of the Riesz operator from one Morrey space to another (see [1, Theorem 3.2]). The Radon measure based p-Laplace equation can be settled through the boundedness of the Riesz operator from the Morrey space to the Radon-measure-based Campanato space (see e.g. [6, 44, 45, 43, 69]). However, there has been still no restricting result on the Cordes-Nirenberg potential space $I_\alpha(CN^{p,\tilde{\alpha}})$. So, as a development of Theorem 5.2.1 Section 5.3 deals with this problem through restricting the Morrey potential space $I_\alpha(L^{p,\lambda})$ to the Radon measure μ-based Campanato space $\mathcal{L}^{q,\eta}_\mu$.

As an further application, we study the solution and its regularity of a widely studied Dirichlet problem of some elliptic equations with symmetric L^∞-coefficients through restricting some related function spaces on a bounded open set $\Omega \subsetneq \mathbb{R}^n$.

Chapter 2

A mixed volume from the anisotropic Riesz-potential

2.1 The first definition

Let us agree on some conventions. A set $K \subsetneq \mathbb{R}^n$ is star-shaped with respect to the origin if the intersection of every line through origin with K is a compact line segment. The radial function is defined by

$$\rho_K(x) = \max\{\lambda \geq 0 : \lambda x \in K\} \quad \text{for} \quad x \in \mathbb{R}^n \setminus o,$$

where o denotes the origin of $\mathbb{R}^n$. If ρ_K is positive and continuous, K is called a star body with respect to the origin. In this paper, we always assume K is a star body with respect to the origin unless otherwise stated.

The Minkowski functional of K, $\|\cdot\|_K$ is defined by:

$$\|x\|_K = \inf\{\lambda > 0 : x \in \lambda K\} \quad \& \quad \lambda K = \{\lambda y : y \in K\} \;\; \forall \;\; x \in \mathbb{R}^n. \tag{2.1}$$

Note that Minkowski functional is usually defined for convex bodies (see [58]). In this chapter, we extend the definition to star bodies. It is easy to check that $\rho_K^{-1}(x) = \|x\|_K$ and $\|\cdot\|_{B(o,r)} = |\cdot|$, where $|\cdot|$ is the Euclidean norm. For these and more information on convex geometry analysis, we refer to [30] and [58].

Below is a quasi-triangle inequality for $\|\cdot\|_K$.

Proposition 2.1.1. *If*

$$\begin{cases} r_K = \sup\{\tilde{r} \geq 0 : \tilde{r}B(o,1) \subseteq K\}; \\ R_K = \inf\{\tilde{r} \geq 0 : K \subseteq \tilde{r}B(o,1)\}, \end{cases} \tag{2.2}$$

then

$$\|x+y\|_K \leq \frac{R_K}{r_K}\left(\|x\|_K + \|y\|_K\right) \quad \forall \quad x, y \in \mathbb{R}^n.$$

Proof. Since ρ_K is positive and continuous, we have

$$0 < r_K \leq R_K < +\infty \quad \& \quad r_K B(o,1) \subseteq K \subseteq R_K B(o,1).$$

Then, by the definition of Minkowski functional in (2.1), it follows that if $\forall x \in \mathbb{R}^n$ then

$$\begin{cases} \|x\|_K = \inf\{\lambda > 0 : x \in \lambda K\} \leq \inf\{\lambda > 0 : x \in \lambda r_K B(o,1)\} = \|x\|_{r_K B(o,1)}; \\ \|x\|_K = \inf\{\lambda > 0 : x \in \lambda K\} \geq \inf\{\lambda > 0 : x \in \lambda R_K B(o,1)\} = \|x\|_{R_K B(o,1)}, \end{cases}$$

and

$$\begin{cases} \|x\|_{r_K B(o,1)} = \inf\{\lambda > 0 : x \in \lambda r_K B(o,1)\} = \frac{1}{r_K}\inf\{\lambda > 0 : x \in \lambda B(o,1)\} = \frac{1}{r_K}|x|; \\ \|x\|_{R_K B(o,1)} = \inf\{\lambda > 0 : x \in \lambda R_K B(o,1)\} = \frac{1}{R_K}\inf\{\lambda > 0 : x \in \lambda B(o,1)\} = \frac{1}{R_K}|x|, \end{cases}$$

whence implying

$$\frac{|x|}{R_K} \le \|x\|_K \le \frac{|x|}{r_K}. \tag{2.3}$$

From this, it follows that if $\forall x, y \in \mathbb{R}^n$ then

$$\|x+y\|_K \le {r_K}^{-1}|x+y| \le {r_K}^{-1}(|x|+|y|) \le \frac{R_K}{r_K}\left(\|x\|_K + \|y\|_K\right).$$

□

Denote by

$$B_r^K(y) = \{x \in \mathbb{R}^n : \|x-y\|_K \le r\}$$

the K-ball centred at y with radius r. It follows that

$$V(\{x : \|x\|_K \le t\}) = t^n V(K) \ \ \forall \ \ t > 0.$$

For another measurable set $\widetilde{K}$, we say that $\widetilde{K}, K$ are dilates provided that $\exists \lambda > 0$ obeying $\widetilde{K} = \lambda K$, while $\widetilde{K}, K$ are homothetic if $\exists \lambda > 0$ and $y \in \mathbb{R}^n$ obeying $\widetilde{K} = \lambda K + y$.

A set L is said to be a convex body if L is a convex compact subset in $\mathbb{R}^n$ with nonempty interior. If the origin is in the interior of the convex body L, one can define its polar body L° as

$$L^\circ = \{x \in \mathbb{R}^n : \langle x, y\rangle \le 1, \quad \forall y \in L\}.$$

Clearly L° is a convex body with the origin in the interior. Moreover, if L is origin-symmetric, then its polar L° is clearly origin-symmetric as well. When K is a convex body with the origin in the interior, it is easy to check the triangle inequality

$$\|x+y\|_K \le \|x\|_K + \|y\|_K \quad \forall \quad x, y \in \mathbb{R}^n.$$

If K is additionally origin-symmetric, then $\|\cdot\|_K$ is a norm on $\mathbb{R}^n$. More background about convex geometry can be found in [58].

Let $0 \le \alpha < n$ and $E \subset \mathbb{R}^n$ be a bounded measurable set. We define the anisotropic α-Riesz-potential of E at y with respect to K by

$$I_\alpha(E, K; y) = \int_E \frac{dx}{\|x-y\|_K^\alpha}.$$

Now we can define $V_\alpha(E, K)$, the mixed volume induced by the anisotropic α-Riesz-potential $I_\alpha(E, K; \cdot)$.

Definition 2.1.2. *Let $0 \le \alpha < n$ and $E \subset \mathbb{R}^n$ be a bounded measurable set. Define*

$$V_\alpha(E, K) = \begin{cases} \sup_{y \in \mathbb{R}^n} I_\alpha(E, K; y), & \text{if } 0 < \alpha < n; \\ V(E), & \text{if } \alpha = 0. \end{cases}$$

If E is bounded, one has

$$\lim_{\|y\|_K \to \infty} I_\alpha(E, K; y) = 0. \tag{2.4}$$

In fact, as E is bounded, there is a constant $R > 0$ such that $|x| \le R$ for all $x \in E$. On the other hand, if $|y| > 2R$, one has $|x| \le |y|/2$ and

$$\|x-y\|_K \ge \frac{1}{R_K}|x-y| \ge \frac{1}{R_K}(|y| - |x|) \ge \frac{|y|}{2R_K} > \frac{R}{R_K}$$

by (2.3). This further implies that, for $0<\alpha<n$,

$$I_\alpha(E,K;y)=\int_E \frac{dx}{\|x-y\|_K^\alpha}\le \int_E \left(\frac{R_K}{R}\right)^\alpha dx \le \left(\frac{R_K}{R}\right)^\alpha V(E).$$

Therefore, for $0<\alpha<n$,

$$0\le \lim_{\|y\|_K\to\infty} I_\alpha(E,K;y)\le \lim_{R\to\infty}\left(\frac{R_K}{R}\right)^\alpha V(E)=0.$$

Moreover, $I_\alpha(E,K;y)$ is continuous on $y\in\mathbb{R}^n$ (see Lemma 2.3.1). Consequently, there is a point $y_0\in\mathbb{R}^n$, such that, for $0<\alpha<n$,

$$V_\alpha(E,K)=\sup_{y\in\mathbb{R}^n} I_\alpha(E,K;y)=I_\alpha(E,K;y_0)=\int_E \frac{dx}{\|x-y_0\|_K^\alpha}.$$

To see this, let $V_\alpha(E,K)>0$ (as otherwise it is trivial). Then, there exists $y_1\in\mathbb{R}^n$ such that $I_\alpha(E,K;y_1)>0$. By formula (2.4), one can find $R_0>0$ (depending on α), such that

$$0\le I_\alpha(E,K;y)<I_\alpha(E,K;y_1)/2,\qquad \forall y\in \big(B^K_{R_0}(0)\big)^c.$$

In other words, the supremum of $I_\alpha(E,K;y)$ *cannot* be obtained in $\big(B^K_{R_0}(0)\big)^c$. On the other hand, the function $I_\alpha(E,K;y)$ is continuous in $B^K_{R_0}(0)$, a compact set in $\mathbb{R}^n$. Hence, for $0<\alpha<n$, there is y_0 (depending on α) in $B^K_{R_0}(0)$, such that

$$V_\alpha(E,K)=\sup_{y\in B^K_{R_0}(0)} I_\alpha(E,K;y)=I_\alpha(E,K;y_0)=\int_E \frac{dx}{\|x-y_0\|_K^\alpha}.$$

For $0<\alpha<n$, denote by $\mathcal{M}_\alpha$ the set of all $y\in\mathbb{R}^n$ such that $V_\alpha(E,K)=I_\alpha(E,K;y)$. Clearly, $\mathcal{M}_\alpha\subseteq B^K_{R_0}(0)$.

Note that $(n-\alpha)V_\alpha(K,K)=nV(K)$ for $\alpha\in[0,n)$, a consequence following from the forthcoming Theorem 2.2.1 (i). In the literature, several anisotropic norms and perimeters have been introduced and investigated (see e.g., [9, 18, 24, 33, 47, 46, 70] and their references). The basic idea behind those anisotropic norms and perimeters is to substitute the Euclidean norm $|\cdot|$ by the Minkowski norm $\|\cdot\|_K$. This naturally brings convex geometry into consideration and greatly enhances the already existing connections between analysis and convex geometry.

2.2 A reverse Minkowski-type inequality

The Minkowski inequality is one of the most important inequalities in convex geometry with many applications (see e.g. [58]). For two convex bodies $L, M \subset \mathbb{R}^n$, the Minkowski inequality asserts that the mixed volume

$$V(L, M) = \lim_{\epsilon \to 0} \frac{V(L + \epsilon M) - V(L)}{n\epsilon} \quad \text{with} \quad L + \epsilon M = \{x + \epsilon y : \ x \in L \ \& \ y \in M\},$$

is bounded from below by $V(L)^{1-1/n} V(M)^{1/n}$, i.e.,

$$V(L, M) \geq V(L)^{\frac{n-1}{n}} V(M)^{\frac{1}{n}} \tag{2.5}$$

with equality if and only if L and M are homothetic to each other. We now establish a reverse Minkowski-type inequality for the mixed volume $V_\alpha(E, K)$, which actually provides a solution to the question mentioned above (right below formula (1.1)). Note that the characterization for equality in Theorem 2.2.1 gives, for $0 < \alpha < n$,

$$\sup_{y \in \mathbb{R}^n} \int_{B_r^K(0)} \frac{dx}{\|x - y\|_K^\alpha} = \frac{n}{n - \alpha} V(B_r^K(0))^{\frac{n-\alpha}{n}} V(B_1^K(0))^{\frac{\alpha}{n}}.$$

In particular, if $K = B(o,1)$, then for $0 < \alpha < n$,

$$\sup_{y\in\mathbb{R}^n}\int_{B(o,r)}\frac{dx}{|x-y|^\alpha} = \frac{n}{n-\alpha}V(B(o,r))^{\frac{n-\alpha}{n}}V(B(o,1))^{\frac{\alpha}{n}},$$

which is an extension of formula (1.1) to all n. A Minkowski-type inequality similar to inequality (2.5) for $\widetilde{V}_\alpha(E,K)$ will be proved in Section 2.6.

Theorem 2.2.1. *The following reverse Minkowski-type inequalities hold.*

(i) For a bounded measurable set $E \subset \mathbb{R}^n$ and for $0 \le \alpha < n$, one has

$$V_\alpha(E,K) \le \frac{n}{n-\alpha}V(E)^{\frac{n-\alpha}{n}}V(K)^{\frac{\alpha}{n}}. \tag{2.6}$$

Equality holds trivially if $\alpha = 0$ or $V(E) = 0$. For $\alpha \in (0,n)$ and bounded measurable set E with $V(E) > 0$, equality holds if and only if E is almost a $K-$ball; namely, there is $y \in \mathbb{R}^n$, such that

$$V(E^c \cap B_r^K(y)) = V\Big(\big(B_r^K(y)\big)^c \cap E\Big) = 0, \quad \text{with} \quad r = \left(\frac{V(E)}{V(K)}\right)^{\frac{1}{n}}.$$

(ii) For star body $L \subset \mathbb{R}^n$ with respect to the origin and $0 < \alpha < n$, one has

$$V_\alpha(L,K) \le \frac{n}{n-\alpha}V(L)^{\frac{n-\alpha}{n}}V(K)^{\frac{\alpha}{n}}, \tag{2.7}$$

with equality if and only if K and L are homothetic to each other.

Additionally, if K is a convex body with origin in its interior, then

$$\frac{(V_{n/2}(K^\circ,K))^2}{4} \le V(K^\circ)V(K) \tag{2.8}$$

with equality if and only if K is an Euclidean ball.

Proof. (i) The desired inequality (2.6) holds trivially if $V(E)=0$ or $\alpha=0$. We only need to consider the case $\alpha\in(0,n)$ and $0<V(E)<\infty$. Let $y\in\mathbb{R}^n$ be fixed and $B_r^K(y)$ be the $K-$ball with center y and radius

$$r=\left(\frac{V(E)}{V(K)}\right)^{\frac{1}{n}}>0.$$

Note that $V(\{x:\|x\|_K\le t\})=t^nV(K)$ for all $t>0$. Thus $V(B_r^K(y))=V(E)$, which further implies

$$V(E^c\cap B_r^K(y))=V\Big(\big(B_r^K(y)\big)^c\cap E\Big). \tag{2.9}$$

Moreover, the following integral can be calculated by Fubini's Theorem:

$$\begin{aligned}\int_{B_r^K(y)}\frac{dx}{\|x-y\|_K^\alpha}&=\int_{\{x:\|x-y\|_K\le r\}}\left(\int_{\|x-y\|_K}^\infty \alpha t^{-\alpha-1}\,dt\right)dx\\&=\int_0^r\alpha t^{-\alpha-1}\left(\int_{\{x:\|x-y\|_K\le t\}}dx\right)dt\\&\quad+\int_r^\infty\alpha t^{-\alpha-1}\left(\int_{\{x:\|x-y\|_K\le r\}}dx\right)dt\\&=V(K)\int_0^r\alpha t^{-\alpha+n-1}\,dt+r^nV(K)\int_r^\infty\alpha t^{-\alpha-1}\,dt\\&=\frac{\alpha}{n-\alpha}r^{n-\alpha}V(K)+r^{n-\alpha}V(K)\\&=\frac{n}{n-\alpha}V(E)^{\frac{n-\alpha}{n}}V(K)^{\frac{\alpha}{n}}.\end{aligned}\tag{2.10}$$

Formula (2.9) together with the fact

$$\begin{cases}\|x-y\|_K\le r, & \forall x\in E^c\cap B_r^K(y);\\ \|x-y\|_K>r, & \forall x\in\big(B_r^K(y)\big)^c\cap E,\end{cases}$$

implies

$$\int_{E^c\cap B_r^K(y)} \frac{dx}{\|x-y\|_K^\alpha} \geq \frac{V(B_r^K(y)\cap E^c)}{r^\alpha} = \frac{V\Big(\big(B_r^K(y)\big)^c\cap E\Big)}{r^\alpha} \geq \int_{\left(B_r^K(y)\right)^c\cap E} \frac{dx}{\|x-y\|_K^\alpha}. \tag{2.11}$$

Consequently, one has

$$\begin{aligned}\int_E \frac{dx}{\|x-y\|_K^\alpha} &= \int_{E\cap B_r^K(y)} \frac{dx}{\|x-y\|_K^\alpha} + \int_{E\cap\left(B_r^K(y)\right)^c} \frac{dx}{\|x-y\|_K^\alpha}\\ &\leq \int_{E\cap B_r^K(y)} \frac{dx}{\|x-y\|_K^\alpha} + \int_{E^c\cap B_r^K(y)} \frac{dx}{\|x-y\|_K^\alpha}\\ &= \int_{B_r^K(y)} \frac{dx}{\|x-y\|_K^\alpha}.\end{aligned}$$

By formula (2.10), one has, for $0<\alpha<n$,

$$V_\alpha(E,K) = \sup_{y\in\mathbb{R}^n}\int_E \frac{dx}{\|x-y\|_K^\alpha} \leq \sup_{y\in\mathbb{R}^n}\int_{B_r^K(y)} \frac{dx}{\|x-y\|_K^\alpha} = \frac{n}{n-\alpha}V(E)^{\frac{n-\alpha}{n}}V(K)^{\frac{\alpha}{n}}.$$

To check the equality situation of inequality (2.6), let us make the following consideration. On the one hand, if E is almost a $K-$ball, that is, there is $y_1\in\mathbb{R}^n$ and $r_0>0$, such that,

$$V(E^c\cap B_{r_0}^K(y_1)) = V\Big(\big(B_{r_0}^K(y_1)\big)^c\cap E\Big) = 0,$$

then the equality in (2.11) holds:

$$\int_{E^c\cap B_{r_0}^K(y_1)} \frac{dx}{\|x-y_1\|_K^\alpha} = \int_{\left(B_{r_0}^K(y_1)\right)^c\cap E} \frac{dx}{\|x-y_1\|_K^\alpha} = 0,$$

which, together with formula (2.10) and $r_0 = \left(\frac{V(E)}{V(K)}\right)^{1/n}$, implies that,

$$\int_E \frac{dx}{\|x-y_1\|_K^\alpha} = \int_{B_{r_0}^K(y_1)} \frac{dx}{\|x-y_1\|_K^\alpha} = \frac{n}{n-\alpha} V(E)^{\frac{n-\alpha}{n}} V(K)^{\frac{\alpha}{n}}.$$

Consequently, the equality in (2.6) holds.

On the other hand, assume that E is not a K-ball with center at $y \in \mathcal{M}_\alpha$ (indeed we can assume $y \in \mathcal{M}_\alpha$ due to translation invariance, see Theorem 2.3.2), where $\mathcal{M}_\alpha$ is as above. Then, for $r = \left(\frac{V(E)}{V(K)}\right)^{1/n} > 0$ and for $y \in \mathcal{M}_\alpha$, one has,

$$V(E^c \cap B_r^K(y)) \neq 0 \;\; \text{and} \;\; V(B_r^K(y)^c \cap E) \neq 0.$$

Consequently, inequality (2.11) is strict and *cannot* have equality. Namely,

$$\int_{E^c \cap B_r^K(y)} \frac{dx}{\|x-y\|_K^\alpha} > \int_{\left(B_r^K(y)\right)^c \cap E} \frac{dx}{\|x-y\|_K^\alpha}.$$

Thus, equality in inequality (2.6) *cannot* hold, because for $y \in \mathcal{M}_\alpha$,

$$V_\alpha(E,K) = \int_E \frac{dx}{\|x-y\|_K^\alpha} < \int_{B_r^K(y)} \frac{dx}{\|x-y\|_K^\alpha} = \frac{n}{n-\alpha} V(E)^{\frac{n-\alpha}{n}} V(K)^{\frac{\alpha}{n}}.$$

In conclusion, to have equality in the inequality (2.6), E must be almost a K-ball.

(ii) Inequality (2.7) follows immediately from inequality (2.6). Note that $V(L) > 0$ and $0 < \alpha < n$. Thus, equality holds in inequality (2.7) if and only if L is almost a K-ball. Then there is $y \in \mathbb{R}^n$, such that, for $r = \left(\frac{V(L)}{V(K)}\right)^{1/n} > 0$

$$V(L^c \cap B_r^K(y)) = V\left(\left(B_r^K(y)\right)^c \cap L\right) = 0.$$

Definition of star body with respect to the origin shows that $L = B_r^K(y) = y + rK$, and hence K and L are homothetic to each other.

If K is a convex body with origin in its interior, inequality (2.8) follows immediately from inequality (2.7) if we let $L = K^\circ$ and $\alpha = n/2$. Equality holds if and only if K and K° are dilated to each other; namely, $K^\circ = aK$ for some constant $a > 0$. Consequently, $\langle x, ax\rangle \leq 1$ for all $x \in K$, which is equivalent to $K \subseteq a^{-1/2}B(o,1)$ or $a^{1/2}K \subseteq B(o,1)$. This further implies that $B(o,1) \subseteq a^{-1/2}K^\circ = a^{1/2}K$, and hence $K = a^{-1/2}B(o,1)$. $\square$

Remark 2.2.2. *If K is a convex body with origin in its interior, note that $V(K^\circ)V(K)$ is known as the Mahler volume product of K and its polar body K°. The well-known Blaschke-Santaló inequality states that, for all origin-symmetric convex body K in $\mathbb{R}^n$,*

$$V(K^\circ)V(K) \leq [V(B(o,1))]^2,$$

with equality if and only if K is an ellipsoid (i.e., $TB(o,1)$ for some invertible linear transform T defined on $\mathbb{R}^n$). Regarding the lower bound of $V(K^\circ)V(K)$, the famous Mahler conjecture asks whether

$$V(K^\circ)V(K) \geq \frac{4^n}{n!}$$

holds for all origin-symmetric convex body K in $\mathbb{R}^n$. Inequality (2.8) provides a lower bound for $V(K^\circ)V(K)$ and may be useful in improving well-known results for the isomorphic solutions of the Mahler conjecture: there is a universal constant $c > 0$ (independent of n and K), such that

$$V(K^\circ)V(K) \geq c^n[V(B(o,1))]^2$$

holds for all origin-symmetric convex body K in $\mathbb{R}^n$ (see [16, 37, 52]).

2.3 Metric properties

We first prove the continuity of the potential functions.

Lemma 2.3.1. *Let E be a bounded measurable set in $\mathbb{R}^n$ and $\alpha \in (0, n)$. Then anisotropic α-Riesz-potential $I_\alpha(E, K; \cdot)$ is continuous on $\mathbb{R}^n$.*

Proof. $\forall y \in \mathbb{R}^n$, since E is bounded, $\exists R > 0$ such that

$$x - y \in \frac{R}{2} B(o, 1) \quad \forall \quad x \in E.$$

If $z \in \mathbb{R}^n$ and $|z - y| \leq \frac{R}{2}$, then

$$|x - z| \leq |x - y| + |y - z| \leq \frac{R}{2} + \frac{R}{2} = R \quad \forall \quad x \in E, \tag{2.12}$$

and hence $x - z \in RB(o, 1)$.

Note that $\rho_K^{-1}(\cdot) = \|\cdot\|_K$ and ρ_K is continuous on $\mathbb{R}^n \setminus o$ and positive since K is a star body with respect to the origin, then $\|\cdot\|_K$ is continuous on $\mathbb{R}^n \setminus o$. Moreover, it is easy to check that

$$\lim_{x \to o} \|x\|_K = 0 \quad \& \quad \|o\|_K = 0$$

from the definition of Minkowski functional in (2.1). Hence $\|\cdot\|_K$ is continuous on $\mathbb{R}^n$ and uniformly continuous on $RB(o, 1)$, since $RB(o, 1)$ is compact.

Note that

$$\lim_{t \to 0^+} \frac{2nV(K)}{n - \alpha} t^{n-\alpha} = 0,$$

Consequently, $\forall \varepsilon > 0$, $\exists t_1 > 0$ such that if $0 < t \leq t_1$ then

$$\frac{4nV(K)}{n - \alpha} t^{n-\alpha} < \varepsilon. \tag{2.13}$$

Let

$$a \in \left(0, \frac{r_K t_1}{2R_K}\right]. \tag{2.14}$$

Since $\|\cdot\|_K$ is uniformly continuous on $RB(o,1)$ and (2.12) is valid, there is $\delta > 0$ such that if

$$|y - z| < \min\left\{\delta, \frac{r_K^2 t_1}{2R_K}, \frac{R}{2}\right\} \tag{2.15}$$

then

$$\left|\|x-z\|_K^\alpha - \|x-y\|_K^\alpha\right| < \frac{\varepsilon a^{2\alpha}}{2V(E)} \quad \forall \quad x \in E. \tag{2.16}$$

As a consequence, it follows that

$$\begin{aligned}
&|I_\alpha(E,K;y) - I_\alpha(E,K;z)| \\
&= \left|\int_E \frac{1}{\|x-y\|_K^\alpha}\,dx - \int_E \frac{1}{\|x-z\|_K^\alpha}\,dx\right| \\
&\le \left|\int_{E\cap(B_a^K(y)\cup B_a^K(z))} \frac{1}{\|x-y\|_K^\alpha}\,dx - \int_{E\cap(B_a^K(y)\cup B_a^K(z))} \frac{1}{\|x-z\|_K^\alpha}\,dx\right| \\
&\quad + \left|\int_{E\setminus(B_a^K(y)\cup B_a^K(z))} \frac{1}{\|x-y\|_K^\alpha}\,dx - \int_{E\setminus(B_a^K(y)\cup B_a^K(z))} \frac{1}{\|x-z\|_K^\alpha}\,dx\right| \\
&:= I_1 + I_2.
\end{aligned}$$

Part 1: For I_1, by (2.14), (2.15) and Proposition 2.1.1, we have the following two situations.

Situation 1: $x \in E \cap B_a^K(y)$. This yields

$$\|x-y\|_K \le a \le \frac{r_K t_1}{2R_K} < t_1.$$

Situation 2: $x \in E \cap B_a^K(z)$. This yields

$$\|x-y\|_K \le \frac{R_K}{r_K}(\|x-z\|_K + \|z-y\|_K) \le \frac{R_K}{r_K}a + \frac{R_K}{r_K^2}|z-y| < t_1.$$

Similarly, we have

$$\begin{cases} \|x-z\|_K < t_1 & \text{as } x \in E \cap B_a^K(z); \\ \|x-z\|_K < t_1 & \text{as } x \in E \cap B_a^K(y), \end{cases}$$

whence getting that if

$$x \in \big(E \cap B_a^K(y)\big) \cup \big(E \cap B_a^K(z)\big) = E \cap \big(B_a^K(y) \cup B_a^K(z)\big)$$

then

$$\begin{cases} \|x-y\|_K < t_1; \\ \|x-z\|_K < t_1. \end{cases}$$

Hence, by similar methods as in (2.10) and (2.13), we have

$$\begin{aligned} I_1 &\le \int_{E\cap(B_a^K(y)\cup B_a^K(z))} \frac{1}{\|x-y\|_K^\alpha}\,dx + \int_{E\cap(B_a^K(y)\cup B_a^K(z))} \frac{1}{\|x-z\|_K^\alpha}\,dx \\ &\le \int_{B_{t_1}^K(y)} \frac{1}{\|x-y\|_K^\alpha}\,dx + \int_{B_{t_1}^K(z)} \frac{1}{\|x-z\|_K^\alpha}\,dx \\ &\le \frac{2nV(K)}{n-\alpha} t_1^{n-\alpha} \\ &< \frac{\varepsilon}{2}. \end{aligned} \tag{2.17}$$

Part 2: For I_2, by (2.16), we have

$$\begin{aligned} I_2 &\le \int_{E\setminus(B_a^K(y)\cup B_a^K(z))} \frac{\big|\|x-y\|_K^\alpha - \|x-z\|_K^\alpha\big|}{(\|x-y\|_K\|x-z\|_K)^\alpha}\,dx \\ &\le \frac{1}{a^{2\alpha}} \int_{E\setminus(B_a^K(y)\cup B_a^K(z))} \big|\|x-y\|_K^\alpha - \|x-z\|_K^\alpha\big|\,dx \end{aligned}$$

$$< \frac{\varepsilon}{2V(E)} \int_{E \setminus (B_\alpha^K(y) \cup B_\alpha^K(z))} dx$$
$$\leq \frac{\varepsilon}{2}.$$

This, together with (2.17), implies

$$|I_\alpha(E, K; y) - I_\alpha(E, K; z)| \leq I_1 + I_2 < \varepsilon,$$

thereby ensuring that $I_\alpha(E, K; \cdot)$ is continuous in y. Because $y \in \mathbb{R}^n$ is arbitrary, $I_\alpha(E, K; \cdot)$ is continuous on $\mathbb{R}^n$. □

The following theorem establishes the fundamental metric properties of the newly defined mixed volume $V_\alpha(E, K)$.

Theorem 2.3.2. *Let $E, E_1, E_2, E_3, \cdots \subset \mathbb{R}^n$ be bounded measurable sets and $\alpha \in [0, n)$. The set-function $E \mapsto V_\alpha(E, K)$ is nonnegative and has the following metric properties.*

(i) Homogeneity and translation invariance: $\forall r, s > 0$ and $\forall x_0 \in \mathbb{R}^n$, one has,

$$V_\alpha(sE, rK) = s^{n-\alpha} r^\alpha V_\alpha(E, K) \quad \& \quad V_\alpha(x_0 + E, K) = V_\alpha(E, K),$$

where $x_0 + E = \{x_0 + y : y \in E\}$.

(ii) Monotonicity: if $E_1 \subseteq E_2$, then $V_\alpha(E_1, K) \leq V_\alpha(E_2, K)$. On the other hand, if $K_1 \subseteq K_2$, then $V_\alpha(E, K_1) \leq V_\alpha(E, K_2)$.

(iii) Sub-additivity: $V_\alpha(E_1 \cup E_2, K) \leq V_\alpha(E_1, K) + V_\alpha(E_2, K)$.

(iv) Downward-monotone-convergence: if $\{E_j\}_{j=1}^\infty$ is decreasing, i.e., $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$, then

$$\lim_{j \to \infty} V_\alpha(E_j, K) = V_\alpha(\cap_{j=1}^\infty E_j, K).$$

(v) Upward-monotone-convergence: if $\{E_j\}_{j=1}^{\infty}$ *is increasing, i.e.,* $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$, *and* $\cup_{j=1}^{\infty} E_j$ *is bounded, then*

$$\lim_{j\to\infty} V_\alpha(E_j, K) = V_\alpha(\cup_{j=1}^{\infty} E_j, K).$$

(vi) Interpolation: if $0 \le \alpha < \beta < \gamma < n$, *then*

$$\big[V_\beta(E,K)\big]^{\gamma-\alpha} \le \big[V_\alpha(E,K)\big]^{\gamma-\beta} \, \big[V_\gamma(E,K)\big]^{\beta-\alpha}.$$

In particular,

$$\beta \mapsto \left[\frac{V_\beta(E,K)}{V(E)}\right]^{\frac{1}{\beta}}$$

is an increasing function on $(0, n)$.

Proof. (i) The formula $V_\alpha(E, rK) = r^\alpha V_\alpha(E,K)$ follows immediately from

$$\|x-y\|_{rK} = r^{-1}\|x-y\|_K, \quad \forall x, y \in \mathbb{R}^n.$$

The formula $V_\alpha(rE, K) = r^{n-\alpha} V_\alpha(E,K)$ follows from Definition 2.1.2 and the following calculation:

$$\begin{aligned}
V_\alpha(rE, K) &= \sup_{y\in\mathbb{R}^n} \int_{rE} \frac{1}{\|x-y\|_K^\alpha}\, dx \\
&= \sup_{y\in\mathbb{R}^n} \int_{rE} \frac{1}{\|\frac{x}{r} - \frac{y}{r}\|_K^\alpha} r^{n-\alpha}\, dx/r \\
&= \sup_{y\in\mathbb{R}^n} \int_{E} \frac{1}{\|x-y\|_K^\alpha} r^{n-\alpha}\, dx \\
&= r^{n-\alpha} V_\alpha(E,K).
\end{aligned}$$

Combining the above two formulas, one can easily get the desired homogeneous result:

$$V_\alpha(sE, rK) = r^\alpha V_\alpha(sE, K) = s^{n-\alpha} r^\alpha V_\alpha(E, K).$$

Now let us prove the translation invariance. For $x_0 \in \mathbb{R}^n$, one has,

$$\begin{aligned} V_\alpha(x_0 + E, K) &= \sup_{y \in \mathbb{R}^n} \int_{x_0+E} \frac{1}{\|x-y\|_K^\alpha}\, dx \\ &= \sup_{y \in \mathbb{R}^n} \int_E \frac{1}{\|z+x_0-y\|_K^\alpha}\, dz \\ &= \sup_{w \in \mathbb{R}^n} \int_E \frac{1}{\|z-w\|_K^\alpha}\, dz \\ &= V_\alpha(E, K), \end{aligned}$$

where we have let $x = x_0 + z$ and $y = w + x_0$.

(ii) If $E_1 \subseteq E_2$, then for all $y \in \mathbb{R}^n$,

$$\int_{E_1} \frac{dx}{\|x-y\|_K^\alpha} \leq \int_{E_2} \frac{dx}{\|x-y\|_K^\alpha}.$$

Hence

$$V_\alpha(E_1, K) = \sup_{y \in \mathbb{R}^n} \int_{E_1} \frac{dx}{\|x-y\|_K^\alpha} \leq \sup_{y \in \mathbb{R}^n} \int_{E_2} \frac{dx}{\|x-y\|_K^\alpha} = V_\alpha(E_2, K).$$

On the other hand, if $K_1 \subseteq K_2$, one can check that $\|x\|_{K_1} \geq \|x\|_{K_2}$ for all $x \in \mathbb{R}^n$. Hence,

$$V_\alpha(E, K_1) = \sup_{y \in \mathbb{R}^n} \int_E \frac{dx}{\|x-y\|_{K_1}^\alpha} \leq \sup_{y \in \mathbb{R}^n} \int_E \frac{dx}{\|x-y\|_{K_2}^\alpha} = V_\alpha(E, K_2).$$

(iii) By Definition 2.1.2, one has

$$\begin{aligned} V_\alpha(E_1 \cup E_2, K) &= \sup_{y\in\mathbb{R}^n} \int_{E_1\cup E_2} \frac{dx}{\|x-y\|_K^\alpha} \\ &\le \sup_{y\in\mathbb{R}^n} \int_{E_1} \frac{dx}{\|x-y\|_K^\alpha} + \sup_{y\in\mathbb{R}^n} \int_{E_2} \frac{dx}{\|x-y\|_K^\alpha} \\ &= V_\alpha(E_1, K) + V_\alpha(E_2, K). \end{aligned}$$

(iv) If $\{E_j\}_{j=1}^\infty$ is a decreasing sequence of measurable sets with E_1 bounded, then

$$\lim_{j\to\infty} V(E_j) = V(\cap_{j=1}^\infty E_j).$$

Hence, $\forall \varepsilon > 0$, $\exists i_0 \in \mathbb{N}$ such that

$$V(E_{i_0} \setminus \cap_{j=1}^\infty E_j) \le \left[\frac{(n-\alpha)\varepsilon}{nV(K)^{\frac{\alpha}{n}}}\right]^{\frac{n}{n-\alpha}}.$$

Thus, by Theorem 2.2.1 and Properties (ii)-(iii) above, one has

$$\begin{aligned} \lim_{j\to\infty} V_\alpha(E_j, K) &\le V_\alpha(E_{i_0}, K) \\ &\le V_\alpha(\cap_{j=1}^\infty E_j, K) + V_\alpha\big(E_{i_0} \setminus \cap_{j=1}^\infty E_j, K\big) \\ &\le V_\alpha(\cap_{j=1}^\infty E_j, K) + \frac{n}{n-\alpha} V(E_{i_0} \setminus \cap_{j=1}^\infty E_j)^{\frac{n-\alpha}{n}} V(K)^{\frac{\alpha}{n}} \\ &\le V_\alpha(\cap_{j=1}^\infty E_j, K) + \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ in the above inequality, one has,

$$\lim_{j\to\infty} V_\alpha(E_j, K) \le V_\alpha(\cap_{j=1}^\infty E_j, K).$$

For the other side, one has, for all $j \in \mathbb{N}$,

$$V_\alpha(\cap_{j=1}^{\infty} E_j, K) \leq V_\alpha(E_j, K).$$

Letting $j \to \infty$ gives

$$V_\alpha(\cap_{j=1}^{\infty} E_j, K) \leq \lim_{j\to\infty} V_\alpha(E_j, K),$$

which leads to the desired equality:

$$\lim_{j\to\infty} V_\alpha(E_j, K) = V_\alpha(\cap_{j=1}^{\infty} E_j, K).$$

(v) Let $\{E_j\}_{j=1}^{\infty}$ be increasing such that $\cup_{j=1}^{\infty} E_j$ is bounded. The monotonicity in (ii) implies that

$$V_\alpha(E_k, K) \leq \lim_{i\to\infty} V_\alpha(E_i, K) \leq V_\alpha(\cup_{j=1}^{\infty} E_j, K), \quad \forall k \in \mathbb{N}.$$

On the other hand, $\forall \varepsilon > 0$, $\exists i_0 \in \mathbb{N}$ such that

$$V(\cup_{j=1}^{\infty} E_j \setminus E_{i_0}) \leq \left[\frac{(n-\alpha)\varepsilon}{nV(K)^{\frac{\alpha}{n}}}\right]^{\frac{n}{n-\alpha}}.$$

By Theorem 2.2.1 and (ii)-(iii) above, one has,

$$\begin{aligned} V_\alpha(\cup_{j=1}^{\infty} E_j, K) &\leq V_\alpha(E_{i_0}, K) + V_\alpha\big(\cup_{j=1}^{\infty} E_j \setminus E_{i_0}, K\big) \\ &\leq \lim_{i\to\infty} V_\alpha(E_i, K) + \frac{n}{n-\alpha} V(\cup_{j=1}^{\infty} E_j \setminus E_{i_0})^{\frac{n-\alpha}{n}} V(K)^{\frac{\alpha}{n}} \\ &\leq \lim_{i\to\infty} V_\alpha(E_i, K) + \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$, one has,

$$V_\alpha(\cup_{j=1}^{\infty} E_j, K) \leq \lim_{j\to\infty} V_\alpha(E_j, K),$$

which leads to the desired equality:

$$\lim_{j\to\infty} V_\alpha(E_j, K) = V_\alpha(\cup_{j=1}^{\infty} E_j, K).$$

(vi) Under the assumption on α, β, γ one has $0 < \frac{\beta-\alpha}{\gamma-\alpha} < 1$. By Hölder's inequality, it follows that

$$\begin{aligned}
V_\beta(E, K) &= \sup_{y\in\mathbb{R}^n} \int_E \frac{dx}{\|x-y\|_K^\beta} \\
&= \sup_{y\in\mathbb{R}^n} \int_E \left(\frac{1}{\|x-y\|_K^\alpha}\right)^{\frac{\gamma-\beta}{\gamma-\alpha}} \left(\frac{1}{\|x-y\|_K^\gamma}\right)^{\frac{\beta-\alpha}{\gamma-\alpha}} dx \\
&\leq \left(\sup_{y\in\mathbb{R}^n} \int_E \frac{dx}{\|x-y\|_K^\alpha}\right)^{\frac{\gamma-\beta}{\gamma-\alpha}} \left(\sup_{y\in\mathbb{R}^n} \int_E \frac{dx}{\|x-y\|_K^\gamma}\right)^{\frac{\beta-\alpha}{\gamma-\alpha}} \\
&= \big(V_\alpha(E, K)\big)^{\frac{\gamma-\beta}{\gamma-\alpha}} \big(V_\gamma(E, K)\big)^{\frac{\beta-\alpha}{\gamma-\alpha}}.
\end{aligned}$$

The desired inequality follows by taking power $\gamma - \alpha$ from both sides.

Of course, a rearrangement of the interpolation inequality with $\alpha = 0$ derives the desired monotonicity right away. □

From Theorem 2.3.2, one sees that $V_\alpha(\cdot, K)$ has many properties similar to the Lebesgue measure. This can be further strengthened by the following regularity for $V_\alpha(\cdot, K)$. Denote by $G\Delta E$ the symmetric difference set of two sets E and G in $\mathbb{R}^n$.

Theorem 2.3.3. *Let $0 \leq \alpha < n$ and $E \subset \mathbb{R}^n$ be a bounded measurable set.*

(i) If $G \subset \mathbb{R}^n$ *is bounded and measurable with* $V(G\Delta E) = 0$*, then*

$$V_\alpha(G, K) = V_\alpha(E, K).$$

(ii) If $G \subset \mathbb{R}^n$ *is bounded and measurable with* $E \subseteq G$ *and* $V(G \setminus E) = 0$*, then*

$$V_\alpha(G, K) = V_\alpha(E, K).$$

In particular, $V_\alpha(\overline{E}, K) = V_\alpha(E, K)$ *if* $V(\overline{E} \setminus E) = 0$*.*

(iii) The mixed volume $V_\alpha(\cdot, K)$ *is outer regular: for all bounded measurable set* $E \subset \mathbb{R}^n$*, one has,*

$$V_\alpha(E, K) = \inf_{\text{open } O \supseteq E} V_\alpha(O, K).$$

The mixed volume $V_\alpha(\cdot, K)$ *is also inner regular: for all bounded measurable set* E*,*

$$V_\alpha(E, K) = \sup_{\text{compact } L \subseteq E} V_\alpha(L, K).$$

Proof. (i) The monotonicity and sub-additivity in Theorem 2.3.2 imply that

$$V_\alpha(G, K) \leq V_\alpha(G \cap E, K) + V_\alpha(G \setminus E, K) \leq V_\alpha(E, K) + V_\alpha(G\Delta E, K).$$

Similarly, one has,

$$V_\alpha(E, K) \leq V_\alpha(G, K) + V_\alpha(G\Delta E, K).$$

Suppose that $V(G\Delta E) = 0$. By Theorem 2.2.1, one has,

$$0 \leq \big|V_\alpha(G, K) - V_\alpha(E, K)\big| \leq V_\alpha(G\Delta E, K) \leq \frac{n}{n-\alpha} V(G\Delta E)^{\frac{n-\alpha}{n}} V(K)^{\frac{\alpha}{n}} = 0,$$

which implies $V_\alpha(G,K) = V_\alpha(E,K)$ as desired.

(ii) This is an immediate consequence of (i).

(iii) First of all, monotonicity in Theorem 2.3.2 implies that $V_\alpha(E,K) \le V_\alpha(O,K)$ for all open set O with $E \subseteq O$. Taking the infimum over O, one gets

$$V_\alpha(E,K) \le \inf_{\text{open } O \supseteq E} V_\alpha(O,K).$$

On the other hand, similar to the calculation in (i), one has, for all open sets O such that $E \subseteq O$,

$$0 \le V_\alpha(O,K) - V_\alpha(E,K) \le \frac{n}{n-\alpha} V(O \setminus E)^{\frac{n-\alpha}{n}} V(K)^{\frac{\alpha}{n}}.$$

As E is measurable, for any $\varepsilon > 0$, one can select an open set O_ε such that $E \subseteq O_\varepsilon$ and

$$V(O_\varepsilon \setminus E) < \left[\frac{(n-\alpha)\varepsilon}{nV(K)^{\frac{\alpha}{n}}}\right]^{\frac{n}{n-\alpha}}.$$

This in turns implies

$$V_\alpha(O_\varepsilon, K) < V_\alpha(E,K) + \varepsilon,$$

and consequently $V_\alpha(E,K) = \inf_{\text{open } O \supseteq E} V_\alpha(O,K)$ as desired.

For the inner regularity, the monotonicity in Theorem 2.3.2 implies that

$$V_\alpha(E,K) \ge V_\alpha(L,K)$$

for all compact set L with $L \subseteq E$. This implies

$$V_\alpha(E,K) \ge \sup_{\text{compact } L \subseteq E} V_\alpha(L,K).$$

On the other hand, similar to the calculation in (i), one has, for all compact sets L such that $L \subseteq E$,

$$0 \leq V_\alpha(E, K) - V_\alpha(L, K) \leq \frac{n}{n-\alpha} V(E \setminus L)^{\frac{n-\alpha}{n}} V(K)^{\frac{\alpha}{n}}.$$

As E is measurable, for any $\varepsilon > 0$, one can select a compact set L_ε such that $L_\varepsilon \subseteq E$ and

$$V(E \setminus L_\varepsilon) < \left[\frac{(n-\alpha)\varepsilon}{nV(K)^{\frac{\alpha}{n}}}\right]^{\frac{n}{n-\alpha}}.$$

This in turns implies

$$V_\alpha(L_\varepsilon, K) > V_\alpha(E, K) - \varepsilon,$$

and consequently $V_\alpha(E, K) = \sup_{\text{compact } L \subseteq E} V_\alpha(L, K)$ as desired. □

2.4 The anisotropic Cordes-Nirenberg-capacity and the second definition

In this section, we will provide another definition for $V_\alpha(E, K)$. To this end, we first introduce the anisotropic Cordes-Nirenberg space, which is closely related to the weighted Morrey's space, see e.g., [41, 55].

Definition 2.4.1. *Let $(p, \widetilde{\alpha}) \in [1, \infty) \times [0, n)$. The anisotropic Cordes-Nirenberg space $CN_K^{p,\widetilde{\alpha}}$ consists of all Lebesgue measurable functions f such that*

$$\|f\|_{CN^{p,\widetilde{\alpha}}} = \sup_{y \in \mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|f(x)|^p}{\|x-y\|_K^{\widetilde{\alpha}}} \, dx \right)^{\frac{1}{p}} < \infty.$$

Denote $CN^{p,\widetilde{\alpha}}$ by $CN^{p,\widetilde{\alpha}}_{B_1(o)}$. Note that

$$V_\alpha(E,K)=\|\chi_E\|_{CN^{1,\widetilde{\alpha}}_K},$$

where χ_E is the characteristic function of E (i.e., $\chi_E(x)=1$ for $x\in E$ and $\chi_E(x)=0$ for $x\in E^c$).

We now define the anisotropic Cordes-Nirenberg-capacity $\mathrm{cap}(\cdot;CN^{1,\widetilde{\alpha}}_K)$ induced by the norm $\|\cdot\|_{CN^{1,\widetilde{\alpha}}_K}$.

Definition 2.4.2. *Let $\alpha\in[0,n)$ and K be an origin-symmetric convex body.*

(i) The anisotropic Cordes-Nirenberg-capacity of a compact set $L\subset\mathbb{R}^n$ is defined by:

$$\mathrm{cap}(L;CN^{1,\widetilde{\alpha}}_K)=\inf\left\{\|f\|_{CN^{1,\widetilde{\alpha}}_K}:\quad f\in C_0^\infty\quad\&\quad f\geq\chi_L\right\}.$$

(ii) The anisotropic Cordes-Nirenberg-capacity of any open set $O\subset\mathbb{R}^n$ is defined by:

$$\mathrm{cap}(O;CN^{1,\widetilde{\alpha}}_K)=\sup_{\text{compact }L\subseteq O}\mathrm{cap}(L;CN^{1,\widetilde{\alpha}}_K).$$

(iii) The anisotropic Cordes-Nirenberg-capacity of an arbitrary measurable set $E\subset\mathbb{R}^n$ is defined by:

$$\mathrm{cap}(E;CN^{1,\widetilde{\alpha}}_K)=\inf_{\text{open }O\supseteq E}\mathrm{cap}(O;CN^{1,\widetilde{\alpha}}_K).$$

The following theorem proves that $V_\alpha(E,K)$ is equal to the anisotropic Cordes-Nirenberg-capacity, which is usually not true for general capacities.

Theorem 2.4.3. *If $E\subset\mathbb{R}^n$ is a bounded measurable set and $0\leq\alpha<n$, then*

$$V_\alpha(E,K)=\mathrm{cap}(E;CN^{1,\widetilde{\alpha}}_K).$$

Proof. According to Definition 2.4.2 and Theorem 2.3.3 (iii), it is enough to prove the theorem for compact sets.

Let L be a compact set in $\mathbb{R}^n$. For $f \in C_0^\infty$ with $f \geq \chi_L$, one has,

$$V_\alpha(L,K) = \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\chi_L(x)}{\|x-y\|_K^\alpha}\,dx \leq \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)|}{\|x-y\|_K^\alpha}\,dx.$$

Taking the infimum over all $f \in C_0^\infty$ with $f \geq \chi_L$, one gets

$$V_\alpha(L,K) \leq \mathrm{cap}(L; CN_K^{1,\widetilde{\alpha}}).$$

On the other hand, $\forall \varepsilon > 0$, there is an open set $O_\varepsilon \supsetneq L$ such that

$$0 < V(O_\varepsilon \setminus L) \leq \left[\frac{(n-\alpha)\varepsilon}{nV(K)^{\frac{\alpha}{n}}}\right]^{\frac{n}{n-\alpha}}.$$

Moreover, one can find a function g, such that, $g = 1$ on L, $g \in C_0^\infty$, $0 \leq g \leq 1$, and the support of g (denoted by $\mathrm{supp}(g)$) is contained in O_ε. Note that $g \geq \chi_L$. From Theorem 2.2.1, it follows that

$$\begin{aligned}
\mathrm{cap}(L; CN_K^{1,\widetilde{\alpha}}) &\leq \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|g(x)|}{\|x-y\|_K^\alpha}\,dx \\
&\leq \sup_{y\in\mathbb{R}^n} \int_{L} \frac{|g(x)|}{\|x-y\|_K^\alpha}\,dx + \sup_{y\in\mathbb{R}^n} \int_{L^c} \frac{|g(x)|}{\|x-y\|_K^\alpha}\,dx \\
&\leq \sup_{y\in\mathbb{R}^n} \int_{L} \frac{dx}{\|x-y\|_K^\alpha} + \sup_{y\in\mathbb{R}^n} \int_{O_\varepsilon\setminus L} \frac{dx}{\|x-y\|_K^\alpha} \\
&\leq V_\alpha(L,K) + \frac{n}{n-\alpha} V(O_\varepsilon \setminus L)^{\frac{n-\alpha}{n}} V(K)^{\frac{\alpha}{n}} \\
&\leq V_\alpha(L,K) + \varepsilon.
\end{aligned}$$

This in turn yields

$$\mathrm{cap}(L; CN_K^{1,\widetilde{\alpha}}) \leq V_\alpha(L,K)$$

by letting $\varepsilon \to 0$. Hence, one has $\mathrm{cap}(L; CN_K^{1,\widetilde{\alpha}}) = V_\alpha(L, K)$, as desired. □

2.5 Two restrictions on the Lorentz spaces

To begin with, we have the following restriction result.

Theorem 2.5.1. *Let μ be a Radon measure on $\mathbb{R}^n$ and $0 < p < \infty$. The following two inequalities are equivalent (with the same constants $c_{\alpha,p} > 0$).*

(i) The analytic inequality: there is a constant $c_{\alpha,p} > 0$ such that

$$\|f\|_{L^p_\mu} \le c_{\alpha,p} \left(\int_0^\infty \Big(V_\alpha\big(\{x \in \mathbb{R}^n : |f(x)| \ge t\}, K\big) \Big)^p \, dt^p \right)^{\frac{1}{p}}, \tag{2.18}$$

for any $f = g$ a.e., where $g \in C_0^\infty$.

(ii) The anisotropic isoperimetric inequality: there is a constant $c_{\alpha,p} > 0$ such that

$$\big(\mu(\overline{O})\big)^{\frac{1}{p}} \le c_{\alpha,p} \, V_\alpha(\overline{O}, K) \tag{2.19}$$

for all bounded open set $O \subset \mathbb{R}^n$.

Proof. (ii) ⇒ (i). Suppose that inequality (2.19) holds. Note that, for any $f = g$ a.e., where $g \in C_0^\infty$, $\forall t > 0$, the set $O_t(g) = \{x \in \mathbb{R}^n : \ |g(x)| > t\}$ is a bounded open domain. By (2.19), Fubini's theorem and Theorem 2.3.3 (i), we get the desired inequality (2.18) as follows:

$$\begin{aligned}
\|f\|_{L^p_\mu} = \|g\|_{L^p_\mu} &= \left(\int_{\mathbb{R}^n} |g(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}^n} \left[\int_0^{|g(x)|} p t^{p-1} \, dt \right] d\mu(x) \right)^{\frac{1}{p}} \\
&= \left(\int_0^\infty \left[\int_{O_t(g)} p t^{p-1} \, d\mu(x) \right] dt \right)^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_0^\infty \mu\big(O_t(g)\big)\, dt^p\right)^{\frac{1}{p}}. \\
&\le \left(\int_0^\infty \mu\big(\overline{O_t(g)}\big)\, dt^p\right)^{\frac{1}{p}} \\
&\le c_{\alpha,p}\left(\int_0^\infty \Big(V_\alpha\big(\overline{O_t(g)}, K\big)\Big)^p\, dt^p\right)^{\frac{1}{p}} \\
&= c_{\alpha,p}\left(\int_0^\infty \Big(V_\alpha\big(\overline{O_t(f)}, K\big)\Big)^p\, dt^p\right)^{\frac{1}{p}}.
\end{aligned}
$$

(i) $\Rightarrow$ (ii). Suppose that inequality (2.18) holds. For any bounded open set $O \subset \mathbb{R}^n$ and $0 < \epsilon < 1$, let

$$
f_\epsilon(x) = \begin{cases} 1 - \epsilon^{-1}\mathrm{dist}(x, \overline{O}), & \text{if } \ \mathrm{dist}(x, \overline{O}) < \epsilon, \\ 0, & \text{if } \ \mathrm{dist}(x, \overline{O}) \ge \epsilon, \end{cases}
$$

where $\mathrm{dist}(x, E)$ denotes the Euclidean distance of a point x to a set E. Then $\exists \tilde{g} \in C_0^\infty$ such that $f_\epsilon = \tilde{g}$ a.e. and hence inequality (2.18) holds for f_ϵ. Moreover, one can check, by the dominated convergence theorem, that

$$
\big(\mu(\overline{O})\big)^{\frac{1}{p}} = \lim_{\epsilon \to 0^+} \|f_\epsilon\|_{L^p_\mu}. \tag{2.20}
$$

Let $O_\epsilon = \{x \in \mathbb{R}^n : \ \mathrm{dist}(x, \overline{O}) < \epsilon\}$. Inequality (2.18) implies that for all $\epsilon \in (0, 1)$,

$$
\begin{aligned}
\|f_\epsilon\|_{L^p_\mu} &\le c_{\alpha,p}\left(\int_0^\infty \Big(V_\alpha\big(\overline{O_t(f_\epsilon)}, K\big)\Big)^p\, dt^p\right)^{\frac{1}{p}} \\
&= c_{\alpha,p}\left(\int_0^1 \big(V_\alpha(\overline{O_t(f_\epsilon)}, K)\big)^p\, dt^p\right)^{\frac{1}{p}} \\
&\le c_{\alpha,p} V_\alpha(\overline{O_\epsilon}, K),
\end{aligned}
$$

where the last inequality is due to the monotonicity in Theorem 2.3.2 (ii) and $\overline{O_t(f_\epsilon)} \subseteq \overline{O_\epsilon}$. Theorem 2.3.2 (iv) and formula (2.20) imply inequality (2.19), if we let $\epsilon \to$

0^+. $\square$

Coming-up-next is a result on how $CN_K^{1,\widetilde{\alpha}}$ is contained in the weak Lebesgue space $L_\mu^{p,\infty}$ on $\mathbb{R}^n$. Here, the weak Lebesgue space $L_\mu^{p,\infty}$ is defined as

$$L_\mu^{p,\infty} = \left\{f : \text{measurable function with } \|f\|_{L_\mu^{p,\infty}} < \infty\right\},$$

where for all measurable function f,

$$\|f\|_{L_\mu^{p,\infty}} = \sup_{t>0} t\mu(O_t(f))^{\frac{1}{p}}$$

with $O_t(f) = \{x \in \mathbb{R}^n : \ |f(x)| > t\}$. Below is an embedding of $CN_K^{1,\widetilde{\alpha}}$ into $L_\mu^{p,\infty}$.

Theorem 2.5.2. *Let μ be a Radon measure on $\mathbb{R}^n$ and $0 < p < \infty$. The following two inequalities are equivalent (with the same constants $c_{\widetilde{\alpha},p} > 0$).*

(i) The analytic inequality: there is a constant $c_{\widetilde{\alpha},p} > 0$, such that,

$$\|f\|_{L_\mu^{p,\infty}} \leq c_{\widetilde{\alpha},p}\|f\|_{CN_K^{1,\widetilde{\alpha}}}$$

for any $f = g$ a.e., where $g \in C_0^\infty$.

(ii) The anisotropic isoperimetric inequality: there is a constant $c_{\alpha,p} > 0$, such that,

$$\big(\mu(\overline{O})\big)^{\frac{1}{p}} \leq c_{\alpha,p} V_\alpha(\overline{O}, K) \tag{2.21}$$

for all bounded open set $O \subset \mathbb{R}^n$.

Proof. (ii) $\Rightarrow$ (i). Let $f = g$ a.e., where $g \in C_0^\infty$. Assume that (ii) holds true. Then, it follows that

$$\begin{aligned}
\|f\|_{L^{p,\infty}_\mu} = \|g\|_{L^{p,\infty}_\mu} &\le \sup_{t>0} t\mu(\overline{O_t(g)})^{\frac{1}{p}} \\
&\le c_{\alpha,p} \sup_{t>0} tV_\alpha(\overline{O_t(g)}, K) \\
&= c_{\alpha,p} \sup_{t>0} \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{t\cdot \chi_{\overline{O_t(g)}}(x)}{\|x-y\|_K^\alpha}\,dx \\
&\le c_{\alpha,p} \sup_{t>0} \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|g(x)|\cdot \chi_{\overline{O_t(g)}}(x)}{\|x-y\|_K^\alpha}\,dx \\
&\le c_{\alpha,p} \sup_{t>0} \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|g(x)|}{\|x-y\|_K^\alpha}\,dx \\
&\le c_{\alpha,p}\|g\|_{CN_K^{1,\tilde{\alpha}}} \\
&= c_{\alpha,p}\|f\|_{CN_K^{1,\tilde{\alpha}}}.
\end{aligned}$$

(i) $\Rightarrow$ (ii). Let $O \subset \mathbb{R}^n$ be a bounded open set and $0 < \epsilon < 1$. As in Theorem 2.5.1, we let

$$f_\epsilon(x) = \begin{cases} 1 - \epsilon^{-1}\text{dist}(x, \overline{O}), & \text{if } \text{dist}(x, \overline{O}) < \epsilon, \\ 0, & \text{if } \text{dist}(x, \overline{O}) \ge \epsilon. \end{cases}$$

Then $\exists \tilde{g} \in C_0^\infty$ such that $f_\epsilon = \tilde{g}$ a.e.

By Fubini's theorem and Theorem 2.4.3, it follows that

$$\begin{aligned}
(1-\epsilon)\cdot\left(\mu(\overline{O})\right)^{\frac{1}{p}} \le \|f_\epsilon\|_{L^{p,\infty}_\mu} &\le c_{\alpha,p}\|f_\epsilon\|_{CN_K^{1,\tilde{\alpha}}} \\
&= c_{\alpha,p} \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f_\epsilon(x)|}{\|x-y\|_K^\alpha}\,dx \\
&= c_{\alpha,p} \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\int_0^{|f_\varepsilon(x)|} \frac{1}{\|x-y\|_K^\alpha}\,dt \right) dx \\
&= c_{\alpha,p} \sup_{y\in\mathbb{R}^n} \int_0^\infty \left(\int_{O_t(f_\epsilon)} \frac{dx}{\|x-y\|_K^\alpha} \right) dt \\
&\le c_{\alpha,p} \int_0^\infty V_\alpha(O_t(f_\epsilon), K)\,dt
\end{aligned}$$

$$= c_{\alpha,p} \int_0^1 V_\alpha(O_t(f_\epsilon), K)\, dt$$
$$\leq c_{\alpha,p} V_\alpha(\overline{O_\epsilon}, K),$$

where $O_\epsilon = \{x \in \mathbb{R}^n : \ \operatorname{dist}(x, \overline{O}) < \epsilon\}$ and the last inequality follows from the monotonicity in Theorem 2.3.2 (ii) and $O_t(f_\epsilon) \subseteq \overline{O_\epsilon}$. By taking $\epsilon \to 0^+$, inequality (2.21) follows from Theorem 2.3.2 (iv). □

2.6 A Minkowski type inequality

Notice that Theorem 2.2.1 depends essentially on the hypothesis $\alpha \in [0, n)$. A natural question to ask is: *can we obtain appropriate results for $\alpha \geq n$?* In this section, we will establish a Minkowski-type inequality for $\alpha > n$ which provide a solution to this question for $\alpha > n$. In Section 2.7, a log-Minkowski inequality as well as a reverse log-Minkowski inequality for $\alpha = n$ will be proved.

Observe that for $\alpha \geq n$, the function $I_\alpha(E, K; y)$ may not be well-defined and so is $V_\alpha(E, K)$. We will modify the function $I_\alpha(E, K; y)$ as

$$\tilde{I}_\alpha(E, K; y) = \int_{E^c} \frac{dx}{\|x - y\|_K^\alpha}$$

and define the analogous mixed volume $\widetilde{V}_\alpha(E, K)$ induced by $\tilde{I}_\alpha(E, K; y)$ as

$$\widetilde{V}_\alpha(E, K) = \inf_{y \in \mathbb{R}^n} \int_{E^c} \frac{dx}{\|x - y\|_K^\alpha}.$$

Similar result to Theorem 2.3.2 (such as translation invariance) can be established for $\widetilde{V}_\alpha(E, K)$, and we leave this to the readers.

Note that $\widetilde{V}_\alpha(E, K) < \infty$ if E has nonempty interior. To see this, let y_0 be an interior point of E. Then there is $r_0 > 0$ such that $\|x - y_0\|_K \geq r_0$ for all $x \in E^c$.

Therefore, $E^c \subseteq \left(B^K_{r_0}(y_0)\right)^c$ and

$$(2.22)\qquad \widetilde{V}_\alpha(E,K) \le \int_{E^c} \frac{dx}{\|x-y_0\|_K^\alpha} \le \int_{\left(B^K_{r_0}(y_0)\right)^c} \frac{dx}{\|x-y_0\|_K^\alpha} = \frac{n}{\alpha-n}\, r_0^{n-\alpha}\, V(K),$$

as calculated in formula (2.25).

We say that a measurable set $E \subset \mathbb{R}^n$ with $int(E) \neq \emptyset$ is regular if for all (small enough) $\epsilon > 0$ and $y \notin int(E)$, one has

$$V(B^n_\epsilon(y) \cap E^c) \approx \epsilon^n.$$

A simple argument by separation shows that if E is a convex body, then E is regular. Note that, such a regularity condition may also be called type (A) condition (see e.g., [32]) and is common in analysis, especially in the study of partial differential equations.

When E is regular, one has, for all $y \notin int(E)$ and for all (small enough) $\epsilon > 0$,

$$\tilde{I}_\alpha(E,K;y) = \int_{E^c} \frac{dx}{\|x-y\|_K^\alpha} \ge \int_{E^c \cap B^K_\epsilon(y)} \frac{dx}{\|x-y\|_K^\alpha} \ge \int_{E^c \cap B^K_\epsilon(y)} \epsilon^{-\alpha}\, dx \gtrsim \epsilon^{n-\alpha},$$

where we have used the equivalence between $\|\cdot\|_K$ and $|\cdot|$ in the last inequality. Letting $\epsilon \to 0$, one gets, for $\alpha > n$ and for all $y \notin int(E)$,

$$(2.23)\qquad \tilde{I}_\alpha(E,K;y) = \infty.$$

We now prove the following Minkowski type inequality.

Theorem 2.6.1. *Let $\alpha > n$ be a constant. For all bounded measurable set $E \subset \mathbb{R}^n$,* *one has*

$$\widetilde{V}_\alpha(E,K) \geq \frac{n}{\alpha-n} V(E)^{\frac{n-\alpha}{n}} V(K)^{\frac{\alpha}{n}}. \tag{2.24}$$

If in addition E is regular and has nonempty interior, equality holds in inequality (2.24) if and only if E is almost a $K-$ball.

In particular, if E is a star body with respect to the origin, then inequality (2.24) holds with equality if and only if K and E are homothetic to each other.

Proof. The proof of Theorem 2.6.1 is similar to that for Theorem 2.2.1 and is essentially identical to that for Theorem 3 in [70] (which is corresponding to $\alpha \in (n, n+1)$). Here we include a brief proof for completeness.

It is enough to consider $0 < V(E) < \infty$. Let $r = \left(\frac{V(E)}{V(K)}\right)^{1/n} > 0$. For any fixed $y \in \mathbb{R}^n$, one has, $V(E^c \cap B_r^K(y)) = V\Big(\big(B_r^K(y)\big)^c \cap E\Big)$. Note that $\|x-y\|_K \leq r$ for $x \in E^c \cap B_r^K(y)$ and $\|x-y\|_K > r$ for $x \in \big(B_r^K(y)\big)^c \cap E$. Thus,

$$\begin{aligned}
\int_{E^c} \frac{dx}{\|x-y\|_K^\alpha} &= \int_{E^c \cap B_r^K(y)} \frac{dx}{\|x-y\|_K^\alpha} + \int_{E^c \cap \left(B_r^K(y)\right)^c} \frac{dx}{\|x-y\|_K^\alpha} \\
&\geq \int_{\left(B_r^K(y)\right)^c \cap E} \frac{dx}{\|x-y\|_K^\alpha} + \int_{E^c \cap \left(B_r^K(y)\right)^c} \frac{dx}{\|x-y\|_K^\alpha} \\
&= \int_{\left(B_r^K(y)\right)^c} \frac{dx}{\|x-y\|_K^\alpha},
\end{aligned}$$

where the last integral can be calculated by Fubini's theorem as follows:

$$\begin{aligned}
\int_{\left(B_r^K(y)\right)^c} \frac{dx}{\|x-y\|_K^\alpha} &= \int_{\{x:\|x-y\|_K > r\}} \left(\int_{\|x-y\|_K}^\infty \alpha t^{-\alpha-1}\, dt\right) dx \\
&= \int_r^\infty \alpha t^{-\alpha-1} \left(\int_{\{x: r<\|x-y\|_K \leq t\}} dx\right) dt \\
&= V(K) \int_r^\infty \alpha t^{-\alpha-1} \left(t^n - r^n\right) dt
\end{aligned} \tag{2.25}$$

$$= \frac{n}{\alpha - n} V(E)^{\frac{n-\alpha}{n}} V(K)^{\frac{\alpha}{n}}.$$

Hence, we can conclude that, for all $y \in \mathbb{R}^n$,

$$\int_{E^c} \frac{dx}{\|x-y\|_K^\alpha} \geq \frac{n}{\alpha - n} V(E)^{\frac{n-\alpha}{n}} V(K)^{\frac{\alpha}{n}}.$$

The desired inequality follows by taking the infimum over all $y \in \mathbb{R}^n$.

Now let us check the equality situation of inequality (2.24). Assume that E is regular and has nonempty interior. On the one hand, if E is almost a $K-$ball, there is $y_0 \in \mathbb{R}^n$ and $r_0 > 0$, such that

$$V(E \setminus B_{r_0}^K(y_0)) = V(B_{r_0}^K(y_0) \setminus E) = 0.$$

Hence, we have

$$\int_{E^c \cap B_{r_0}^K(y_0)} \frac{dx}{\|x-y_0\|_K^\alpha} = \int_{\left(B_{r_0}^K(y_0)\right)^c \cap E} \frac{dx}{\|x-y_0\|_K^\alpha} = 0.$$

This further implies that,

$$\int_{E^c} \frac{dx}{\|x-y_0\|_K^\alpha} = \int_{\left(B_{r_0}^K(y_0)\right)^c} \frac{dx}{\|x-y_0\|_K^\alpha} = \frac{n}{\alpha - n} V(E)^{\frac{n-\alpha}{n}} V(K)^{\frac{\alpha}{n}}.$$

Consequently, the equality in inequality (2.24) holds.

On the other hand, we can prove that $\tilde{I}_\alpha(E, K; y)$ is continuous in $int(E)$ by a similar way as in Lemma 2.3.1 (we omit the details here). Note that $\widetilde{V}_\alpha(E, K) < \infty$ by inequality (2.22) as $int(E) \neq \emptyset$. Moreover, there is a sequence $\{y_m\}_{m \geq 1} \subset \mathbb{R}^n$ such that

$$\widetilde{V}_\alpha(E, K) = \lim_{m \to \infty} \tilde{I}_\alpha(E, K; y_m) < \infty.$$

Formula (2.23) implies that $y_m \in int(E) \subset \overline{E}$ for all $m > m_0$ with m_0 some fixed integer. As E is bounded, one can find a convergent subsequence of y_m (without loss of generality, denote by y_m this subsequence), such that $y_m \to y_0 \in int(E)$ and

$$\widetilde{V}_\alpha(E,K) = \lim_{m\to\infty} \tilde{I}_\alpha(E,K;y_m) = \tilde{I}_\alpha(E,K;y_0) < \infty.$$

For $\alpha > n$, denote by $\mathcal{N}_\alpha$ the set of all $y \in \mathbb{R}^n$ such that $\widetilde{V}_\alpha(E,K) = \tilde{I}_\alpha(E,K;y)$, and hence $\mathcal{N}_\alpha \subset int(E)$. If E is not a K-ball, then for $r = \left(\frac{V(E)}{V(K)}\right)^{1/n} > 0$ and for all $y \in \mathbb{R}^n$, one has,

$$V(E^c \cap B_r^K(y)) \neq 0 \;\text{ and }\; V(B_r^K(y)^c \cap E) \neq 0.$$

Without loss of generality (due to translation invariance), one can assume $y \in \mathcal{N}_\alpha$. Hence,

$$\int_{E^c \cap B_r^K(y)} \frac{dx}{\|x-y\|_K^\alpha} > \int_{\left(B_r^K(y)\right)^c \cap E} \frac{dx}{\|x-y\|_K^\alpha}.$$

Thus, equality in inequality (2.24) *cannot* hold, because for $y \in \mathcal{N}_\alpha$,

$$\widetilde{V}_\alpha(E,K) = \int_{E^c} \frac{dx}{\|x-y\|_K^\alpha} > \int_{\left(B_r^K(y)\right)^c} \frac{dx}{\|x-y\|_K^\alpha} = \frac{n}{\alpha-n} V(E)^{\frac{n-\alpha}{n}} V(K)^{\frac{\alpha}{n}}.$$

In conclusion, to have equality in the inequality (2.24), E must be almost a K-ball.

The case for E being a star body with respect to origin follows easily from the above argument and we omit the details here. □

Remark 2.6.2. *The anisotropic fractional α-perimeter of E with respect to origin-symmetric convex body K [46] is defined by, for $\alpha \in (n, n+1)$,*

$$P_\alpha(E,K) = \int_E \int_{E^c} \frac{1}{\|x-y\|_K^\alpha}\, dxdy.$$

It has been proved in [70] that, for $\alpha \in (n, n+1)$,

$$P_\alpha(E,K) \geq \frac{n}{\alpha-n} V(K)^{\frac{\alpha}{n}} V(E)^{\frac{2n-\alpha}{n}}. \tag{2.26}$$

It is clear that

$$P_\alpha(E,K) \geq V(E) \cdot \widetilde{V}_\alpha(E,K) \geq \frac{n}{\alpha-n} V(K)^{\frac{\alpha}{n}} V(E)^{\frac{2n-\alpha}{n}}.$$

Therefore, to have equality in inequality (2.26) for E *being a convex body, one must have,*

$$\widetilde{V}_\alpha(E,K) = \frac{n}{\alpha-n} V(E)^{\frac{n-\alpha}{n}} V(K)^{\frac{\alpha}{n}}.$$

In other words, E *being homothetic to* K *is the only possibility to have equality in inequality (2.26). A more detailed discussion on the sharpness of inequality (2.26) can be found in [70].*

2.7 Two log-Minkowski type inequalities

As promised, we now establish a reverse log-Minkowski inequality for $\alpha = n$. To deal with the case $\alpha = n$, we must bring the logarithm into play. In fact, we have the following reverse log-Minkowski inequality.

Theorem 2.7.1. *Let* $E \subset \mathbb{R}^n$ *be a bounded measurable set. For all* $A > 0$, *one has*

$$\begin{aligned} &\sup_{y\in\mathbb{R}^n} \int_{E\cap\{x\in\mathbb{R}^n : \|x-y\|_K \geq A\}} \frac{dx}{\|x-y\|_K^n} \\ &\quad \leq \sup_{y\in\mathbb{R}^n} V(K) \log\left(\frac{V\big(E\cap\{x\in\mathbb{R}^n : \|x-y\|_K \geq A\}\big)}{A^n V(K)} + 1\right). \end{aligned} \tag{2.27}$$

Proof. It is enough to verify the following inequality: for all $A > 0$ and all fixed

$y \in \mathbb{R}^n$, one has,

$$\int_{E\cap\{x\in\mathbb{R}^n:\|x-y\|_K\geq A\}} \frac{1}{\|x-y\|_K^n}\,dx \leq V(K)\log\left(\frac{V\big(E\cap\{x\in\mathbb{R}^n:\|x-y\|_K\geq A\}\big)}{A^nV(K)}+1\right). \tag{2.28}$$

We only consider $0<V(E)<\infty$, as otherwise inequality (2.28) holds trivially if $V(E)=0$. The calculation is similar to that in the proof of Theorem 2.2.1, so we will keep our calculation short with concentration on the main modification. Let $A>0$ and $y\in\mathbb{R}^n$ be fixed. For simplicity, we let

$$E_y = E\cap\{x\in\mathbb{R}^n : \|x-y\|_K\geq A\}.$$

Let $R_r(y)$ be the $K-$annulus centered at y with inner radius A and outer radius r_y:

$$R_r(y)=\{x\in\mathbb{R}^n : A\leq\|x-y\|_K\leq r_y\},$$

where $r_y=\left(\frac{V(E_y)}{V(K)}+A^n\right)^{\frac{1}{n}}$. A simple calculation shows that $V(R_r(y))=V(E_y)$. This in turn implies

$$V(E_y^c\cap R_r(y)) = V\big((R_r(y))^c\cap E_y\big).$$

Together with the facts $\|x-y\|_K\leq r_y$ if $x\in E_y^c\cap R_r(y)$ and $\|x-y\|_K>r_y$ if $x\in R_r(y)^c\cap E_y$, one gets

$$\int_{E_y^c\cap R_r(y)}\frac{dx}{\|x-y\|_K^n}\geq\frac{V(E_y^c\cap R_r(y))}{r_y^n}=\frac{V(R_r(y)^c\cap E_y)}{r_y^n}\geq\int_{R_r(y)^c\cap E_y}\frac{dx}{\|x-y\|_K^n}.$$

This further implies

$$\int_{E_y} \frac{dx}{\|x-y\|_K^n} \le \int_{R_r(y)} \frac{dx}{\|x-y\|_K^n}.$$

The last integral can be calculated by Fubini's Theorem as follows:

$$\begin{aligned}
(2.29)\qquad \int_{R_r(y)} \frac{dx}{\|x-y\|_K^n} &= \int_{\{x: A\le \|x-y\|_K \le r_y\}} \left(\int_{\|x-y\|_K}^{\infty} nt^{-n-1}\,dt\right) dx \\
&= \int_A^{r_y} nt^{-n-1} \left(\int_{\{x: A\le \|x-y\|_K \le t\}} dx\right) dt \\
&\quad + \int_{r_y}^{\infty} nt^{-n-1} \left(\int_{\{x: A\le \|x-y\|_K \le r_y\}} dx\right) dt \\
&= V(K)\cdot n \cdot \log\left(\frac{r_y}{A}\right) \\
&= V(K)\log\left(\frac{V\big(E\cap\{x\in\mathbb{R}^n : \|x-y\|_K \ge A\}\big)}{A^n V(K)} + 1\right),
\end{aligned}$$

which gives the desired inequality (2.28). Consequently, inequality (2.27) follows from inequality (2.28) by taking the supremum over $y \in \mathbb{R}^n$. □

Remark 2.7.2. *Since $E\cap\{x\in\mathbb{R}^n : \|x-y\|_K \ge A\} \subseteq E$, one can easily have*

$$\sup_{y\in\mathbb{R}^n} \int_{E\cap\{x\in\mathbb{R}^n:\|x-y\|_K\ge A\}} \frac{dx}{\|x-y\|_K^n} \le V(K)\log\left(\frac{V(E)}{A^nV(K)} + 1\right).$$

The conjecture of the log-Minkowski inequality can be stated as follows (see [15, p. 1976]): for origin-symmetric convex bodies K and L in $\mathbb{R}^n$, does the following inequality hold true

$$\int_{S^{n-1}} \log\left(\frac{h_L(u)}{h_K(u)}\right) d\overline{V}_K(u) \ge \log\left(\left(\frac{V(L)}{V(K)}\right)^{1/n}\right)?$$

Here h_K and h_L are the support functions of K and L:

$$h_K(u) = \max_{x\in K}\langle x, u\rangle, \quad \text{for } u \in S^{n-1},$$

$S(K,\cdot)$ is the surface area measure of K defined on S^{n-1} (the unit sphere of $\mathbb{R}^n$), and

$$d\overline{V}_K(u) = \frac{h_K(u)}{nV(K)}\, dS(K,u),$$

is the normalized cone measure of K. The conjecture has been confirmed only for $n = 2$, but it is still open in general. A dual log-Minkowski inequality has been established in [30] and [66] for all dimension n (see Corollary 3.3.2). Here, we will prove a log-Minkowski type inequality. For convenience, we let $\log(\infty) = \infty$.

Theorem 2.7.3. *Let $0 < B < \infty$ be a constant. For a bounded measurable set E with $V(E) > 0$, one has:*

$$\begin{aligned}(2.30)\qquad &\inf_{y\in\mathbb{R}^n} \int_{E^c\cap\{x\in\mathbb{R}^n:\|x-y\|_K\le B\}} \frac{dx}{\|x-y\|_K^n} \\ &\ge \inf_{y\in\mathbb{R}^n} V(K)\log\left(\frac{B^nV(K)}{V\big(E\cap\{x\in\mathbb{R}^n : \|x-y\|_K\le B\}\big)}\right).\end{aligned}$$

Proof. Let $0 < B < \infty$ be a constant and $y \in \mathbb{R}^n$ be a fixed point. For simplicity, we let

$$\widetilde{E}_y = E^c \cap \{x\in\mathbb{R}^n : \|x-y\|_K \le B\}.$$

Denote by $\widetilde{R}_\rho(y)$ the K-annulus center at y with inner radius ρ and outer radius B:

$$\widetilde{R}_\rho(y) = \{x\in\mathbb{R}^n : \rho \le \|x-y\|_K \le B\},$$

where the inner radius ρ is defined by

$$\rho^n = B^n - \frac{V(\widetilde{E}_y)}{V(K)}.$$

Note that we can rewrite ρ as

$$V(K)\rho^n = V\big(E \cap \{x \in \mathbb{R}^n : \|x-y\|_K \le B\}\big),$$

which follows from

$$\begin{aligned} V(K)B^n &= V\big(\{x \in \mathbb{R}^n : \|x-y\|_K \le B\}\big) \\ &= V\Big(\widetilde{E}_y \cup \big(E \cap \{x \in \mathbb{R}^n : \|x-y\|_K \le B\}\big)\Big) \\ &= V(\widetilde{E}_y) + V\Big(\big(E \cap \{x \in \mathbb{R}^n : \|x-y\|_K \le B\}\big)\Big). \end{aligned}$$

A simple calculation shows that $V(\widetilde{R}_\rho(y)) = V(\widetilde{E}_y)$ and $V\big(\widetilde{E}_y \backslash \widetilde{R}_\rho(y)\big) = V\big(\widetilde{R}_\rho(y) \backslash \widetilde{E}_y\big)$. These lead to, by a similar calculation to the formula (2.29), if $\rho > 0$,

$$\int_{\widetilde{E}_y} \frac{dx}{\|x-y\|_K^n} \ge \int_{\widetilde{R}_\rho(y)} \frac{dx}{\|x-y\|_K^n} = V(K) \cdot n \cdot \log\left(\frac{B}{\rho}\right)$$

and while if $\rho = 0$,

$$\int_{\widetilde{E}_y} \frac{dx}{\|x-y\|_K^n} \ge \int_{\widetilde{R}_\rho(y)} \frac{dx}{\|x-y\|_K^n} \ge \lim_{\eta \to 0} V(K) \cdot n \cdot \log\left(\frac{B}{\eta}\right) = \infty.$$

The desired inequality (2.30) holds if we take the infimum over $y \in \mathbb{R}^n$, that is,

$$\begin{aligned} &\inf_{y \in \mathbb{R}^n} \int_{E^c \cap \{x \in \mathbb{R}^n : \|x-y\|_K \le B\}} \frac{dx}{\|x-y\|_K^n} \\ &\ge \inf_{y \in \mathbb{R}^n} V(K) \log\left(\frac{B^n V(K)}{V\big(E \cap \{x \in \mathbb{R}^n : \|x-y\|_K \le B\}\big)}\right). \end{aligned}$$

□

Remark 2.7.4. *If E is a bounded measurable set with $V(E) > 0$, the right hand side of inequality (2.30) is bounded above. In fact, $\overline{E}$ is a bounded closed set, hence $\overline{E}$ is a compact set in $\mathbb{R}^n$. Therefore, there is a finite number of open covering $\{x \in \mathbb{R}^n : \|x - y_j\|_K < B\}$, $j = 1, 2, \cdots, m$, such that*

$$E \subseteq \overline{E} \subseteq \cup_{j=1}^{m} \{x \in \mathbb{R}^n : \|x - y_j\|_K < B\} \subseteq \cup_{j=1}^{m} \{x \in \mathbb{R}^n : \|x - y_j\|_K \leq B\}.$$

By sub-additivity of Lebesgue measure, one has,

$$0 < V(E) \leq \sum_{j=1}^{m} V(E \cap \{x \in \mathbb{R}^n : \|x - y_j\|_K \leq B\}).$$

Thus, there must have (at least) one $y_0 \in \mathbb{R}^n$ such that

$$V(E \cap \{x \in \mathbb{R}^n : \|x - y_0\|_K \leq B\}) > 0.$$

Consequently, one has,

$$\begin{aligned}
&\inf_{y \in \mathbb{R}^n} V(K) \log \left(\frac{B^n V(K)}{V\big(E \cap \{x \in \mathbb{R}^n : \|x - y\|_K \leq B\}\big)} \right) \\
&\leq V(K) \log \left(\frac{B^n V(K)}{V\big(E \cap \{x \in \mathbb{R}^n : \|x - y_0\|_K \leq B\}\big)} \right) < \infty.
\end{aligned}$$

One also has, as $E \cap \{x \in \mathbb{R}^n : \|x - y\|_K \leq B\} \subseteq E$,

$$\inf_{y \in \mathbb{R}^n} \int_{E^c \cap \{x \in \mathbb{R}^n : \|x - y\|_K \leq B\}} \frac{dx}{\|x - y\|_K^n} \geq V(K) \log \left(\frac{B^n V(K)}{V(E)} \right).$$

Chapter 3

A mixed volume from the anisotropic logarithmic potential

3.1 Mixed volume from anisotropic log-potential

Note that the mixed volumes from the anisotropic Riesz potential in Chapter 2 cover all the cases $\alpha \in [0, +\infty)$. However, when $\alpha = 0$, $V_\alpha(E, K) = V(E)$, which is trivial. Their kernels have the following formula:

$$\lim_{\alpha \to 0} \|x - y\|_K^{-\alpha} = 1 \quad as \quad x \neq y. \tag{3.1}$$

Inspired by [13], we once again think the limitation for the kernel of $V_\alpha(E, K)$ in a derivative way instead of (3.1),

$$\frac{\partial}{\partial \alpha} \|x - y\|_K^{-\alpha} \bigg|_{\alpha=0} = \frac{\log \|x - y\|_K^{-1}}{\|x - y\|_K^{\alpha}} \bigg|_{\alpha=0} = \log \|x - y\|_K{}^{-1} \quad as \quad x \neq y, \tag{3.2}$$

which induces:

Definition 3.1.1. *For $m \in \mathbb{N}$ let $I_{\log,m}(E, K; y)$ be the anisotropic m-*log*-potential of*

E at y with respect to K,

$$I_{\log,m}(E,K;y) = \int_E \left(\log \frac{1}{\|x-y\|_K}\right)^m dx,$$

which generates the mixed volume of E and K from anisotropic potential with natural logarithm (written as log*),*

$$V_{\log,m}(E,K) = \sup_{y\in\mathbb{R}^n} I_{\log,m}(E,K;y).$$

Remark 3.1.2. *Three comments are in order.*

(i) *In the classical case,*

$$\Gamma(x) = \begin{cases} |x|^{2-n} & as \quad n \geq 3; \\ \log|x| & as \quad n = 2, \end{cases}$$

is harmonic on $\mathbb{R}^n \setminus o$, where the logarithmic function in $\mathbb{R}^2$ is a suitable replacement for these in higher dimensions.

(ii) $V_{\log,m}(E,K)$ *is defined for m being an integer since* $\log\|x-y\|_K^{-1}$ *may be negative.*

(iii) *If m is an even number, then* $V_{\log,m}(E,K) = +\infty$.

Actually, since E is bounded, $\sup_{x\in E}|x| < +\infty$. *For any $C > 0$, let*

$$|y| > \max\left\{2\sup_{x\in E}|x|, 2R_K e^{\left(\frac{C}{V(E)}\right)^{\frac{1}{m}}}\right\},$$

where R_K is in (2.2)*. From this, it follows that*

$$\|x-y\|_K \geq {R_K}^{-1}|x-y| \geq {R_K}^{-1}(|y|-|x|) > |y|(2R_K)^{-1} > e^{\left(\frac{C}{V(E)}\right)^{\frac{1}{m}}} > 1 \;\forall\; x \in E.$$

Hence, as m is even we have

$$\begin{aligned} I_{\log,m}(E,K;y) &= \int_E \left(\log \frac{1}{\|x-y\|_K}\right)^m \, dx \\ &= \int_E (\log \|x-y\|_K)^m \, dx \\ &> \int_E \left(\log e^{\left(\frac{C}{V(E)}\right)^{\frac{1}{m}}}\right)^m \, dx \\ &= C, \end{aligned}$$

which derives

$$V_{\log,m}(E,K) = +\infty \quad \text{owing to} \quad V_{\log,m}(E,K) = \sup_{y\in\mathbb{R}^n} I_{\log,m}(E,K;y).$$

Therefore, we will focus on the case when m is an odd number.

The volume $V_{\log,m}(E,K)$ enjoys the following metric properties.

Proposition 3.1.3. *Let m be an odd number.*

(i) Homogeneity: $V_{\log,1}(sE,sK) = s^n V_{\log,1}(E,K) \ \ \forall \ \ s > 0.$

(ii) Translation-invariance: $V_{\log,m}(x_0+E,K) = V_{\log,m}(E,K) \ \ \forall \ \ x_0 \in \mathbb{R}^n.$

(iii) Sub-additivity: if $F \subset \mathbb{R}^n$ is a bounded measurable set such that $E \cap F = \emptyset$, then $V_{\log,m}(E\cup F,K) \leq V_{\log,m}(E,K) + V_{\log,m}(F,K).$

Proof. (i) $\forall s > 0$, $\forall x, y \in \mathbb{R}^n$, by the definition of Minkowski functional in (2.1), it follows that

$$\begin{aligned} \|x-y\|_{sK} &= \inf\{\lambda > 0 : x-y \in \lambda sK\} \\ &= \frac{1}{s}\inf\{\lambda > 0 : x-y \in \lambda K\} \end{aligned}$$

$$= \frac{1}{s}\|x-y\|_K$$

and

$$\begin{aligned}\|sx-sy\|_K &= \inf\{\lambda>0 : sx-sy\in\lambda K\}\\ &= s\inf\{\lambda>0 : x-y\in\lambda K\}\\ &= s\|x-y\|_K.\end{aligned}$$

Hence

$$\begin{aligned}V_{\log,1}(E,sK) &= \sup_{y\in\mathbb{R}^n}\int_E \log\frac{1}{\|x-y\|_{sK}}\,dx\\ &= \sup_{y\in\mathbb{R}^n}\int_E \log\frac{s}{\|x-y\|_K}\,dx\\ &= \sup_{y\in\mathbb{R}^n}\int_E \log\frac{1}{\|x-y\|_K}\,dx + (\log s)V(E)\\ &= V_{\log,1}(E,K) + (\log s)V(E),\end{aligned} \tag{3.3}$$

and by changing the variables $x = s\tilde{x}$ and $y = s\tilde{y}$,

$$\begin{aligned}V_{\log,1}(sE,K) &= \sup_{y\in\mathbb{R}^n}\int_{sE} \log\frac{1}{\|x-y\|_K}\,dx\\ &= s^n\sup_{\tilde{y}\in\mathbb{R}^n}\int_E \log\frac{1}{\|s\tilde{x}-s\tilde{y}\|_K}\,d\tilde{x}\\ &= s^n\left(\sup_{\tilde{y}\in\mathbb{R}^n}\int_E \log\frac{1}{\|\tilde{x}-\tilde{y}\|_K}\,d\tilde{x} - (\log s)V(E)\right)\\ &= s^n\left(V_{\log,1}(E,K) - (\log s)V(E)\right).\end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4), we have

$$\begin{aligned} V_{\log,1}(sE, sK) &= s^n \left(V_{\log,1}(E, sK) - (\log s) V(E) \right) \\ &= s^n \left(V_{\log,1}(E, K) + (\log s) V(E) - (\log s) V(E) \right) \\ &= s^n V_{\log}(E, K). \end{aligned}$$

(ii) $\forall x_0 \in \mathbb{R}^n$, by changing the variables $x = x_0 + z$ and $y = w + x_0$, we have

$$\begin{aligned} V_{\log,m}(x_0 + E, K) &= \sup_{y \in \mathbb{R}^n} \int_{x_0+E} \left(\log \frac{1}{\|x - y\|_K} \right)^m dx \\ &= \sup_{y \in \mathbb{R}^n} \int_E \left(\log \frac{1}{\|z + x_0 - y\|_K} \right)^m dz \\ &= \sup_{w \in \mathbb{R}^n} \int_E \left(\log \frac{1}{\|z - w\|_K} \right)^m dz \\ &= V_{\log,m}(E, K). \end{aligned}$$

(iii) Since $E \cap F = \emptyset$, it follows that

$$\begin{aligned} &V_{\log,m}(E \cup F, K) \\ &= \sup_{y \in \mathbb{R}^n} \int_{E \cup F} \left(\log \frac{1}{\|x - y\|_K} \right)^m dx \\ &= \sup_{y \in \mathbb{R}^n} \left(\int_E \left(\log \frac{1}{\|x - y\|_K} \right)^m dx + \int_F \left(\log \frac{1}{\|x - y\|_K} \right)^m dx \right) \\ &\leq \sup_{y \in \mathbb{R}^n} \int_E \left(\log \frac{1}{\|x - y\|_K} \right)^m dx + \sup_{y \in \mathbb{R}^n} \int_F \left(\log \frac{1}{\|x - y\|_K} \right)^m dx \\ &= V_{\log,m}(E, K) + V_{\log,m}(F, K). \end{aligned}$$

$\square$

We need the following two more useful lemmas.

Lemma 3.1.4. *For all $m, n \in \mathbb{N}$, the anisotropic m-log-potential $I_{\log,m}(E, K; \cdot)$ is continuous on $\mathbb{R}^n$.*

Proof. $\forall y \in \mathbb{R}^n$, since E is bounded, $\exists R > 0$ such that

$$x - y \in \frac{R}{2}B(o,1) \quad \forall \quad x \in E.$$

If $z \in \mathbb{R}^n$ and $|z-y| \leq \frac{R}{2}$, then

$$|x-z| \leq |x-y| + |y-z| \leq \frac{R}{2} + \frac{R}{2} = R \quad \forall \quad x \in E, \tag{3.5}$$

and hence $x - z \in RB(o,1)$.

$\|\cdot\|_K$ is continuous on $\mathbb{R}^n$ (see Lemma 2.3.1) and uniformly continuous on $RB(o,1)$, since $RB(o,1)$ is compact.

The following two facts are useful:

$$\log(x+1) \leq x \quad \forall \quad x \geq -1, \tag{3.6}$$

and

$$b^m - c^m = (b-c)\sum_{i=1}^{m} b^{m-i}c^{i-1} \quad \forall \quad b, c \in \mathbb{R}, \tag{3.7}$$

where $\sum_{i=1}^{m} b^{m-i}c^{i-1} = 1$ when $m = 1$.

Note that

$$\lim_{t \to 0^+} \frac{2t^n V(K)}{n^m} \sum_{i=0}^{m} \frac{m!}{(m-i)!}\left(\log \frac{1}{t^n}\right)^{m-i} = 0.$$

Consequently, $\forall\varepsilon > 0$, $\exists t_1 \in (0,1]$ such that if $0 < t \leq t_1$ then

$$\frac{2t^n V(K)}{n^m} \sum_{i=0}^{m} \frac{m!}{(m-i)!} \left(\log \frac{1}{t^n}\right)^{m-i} < \varepsilon. \tag{3.8}$$

Let

$$a \in \left(0, \frac{r_K t_1}{2R_K}\right]. \tag{3.9}$$

Since $\|\cdot\|_K$ is uniformly continuous on $RB(o,1)$ and (3.5) is valid, there is $\delta > 0$ such that if

$$|y - z| < \min\left\{\delta, \frac{r_K^2 t_1}{2R_K}, \frac{R}{2}\right\} \tag{3.10}$$

then

$$|\|x - z\|_K - \|x - y\|_K| < \varepsilon H(a)^{-1} \quad \forall \quad x \in E, \tag{3.11}$$

where $H(a)$ is defined in (3.14).

As a consequence, it follows that

$$\begin{aligned}
&|I_{\log,m}(E,K;y) - I_{\log,m}(E,K;z)| \\
&\quad = \left| \int_E \left(\log \frac{1}{\|x-y\|_K}\right)^m dx - \int_E \left(\log \frac{1}{\|x-z\|_K}\right)^m dx \right| \\
&\quad \leq \left| \int_{E\cap(B_a^K(y)\cup B_a^K(z))} \frac{dx}{\left(\log \frac{1}{\|x-y\|_K}\right)^{-m}} - \int_{E\cap(B_a^K(y)\cup B_a^K(z))} \frac{dx}{\left(\log \frac{1}{\|x-z\|_K}\right)^{-m}} \right| \\
&\quad\quad + \left| \int_{E\setminus(B_a^K(y)\cup B_a^K(z))} \frac{dx}{\left(\log \frac{1}{\|x-y\|_K}\right)^{-m}} - \int_{E\setminus(B_a^K(y)\cup B_a^K(z))} \frac{dx}{\left(\log \frac{1}{\|x-z\|_K}\right)^{-m}} \right|
\end{aligned}$$

$$:= I_1 + I_2.$$

Part 1: For I_1, by (3.9), (2.3), (3.10) and Proposition 2.1.1, we have the following two situations.

Situation 1: $x \in E \cap B_a^K(y)$. This yields

$$\|x-y\|_K \leq a \leq \frac{r_K t_1}{2R_K} < t_1 \leq 1.$$

Situation 2: $x \in E \cap B_a^K(z)$. This yields

$$\|x-y\|_K \leq \frac{R_K}{r_K}(\|x-z\|_K + \|z-y\|_K) \leq \frac{R_K}{r_K}a + \frac{R_K}{r_K^2}|z-y| < t_1 \leq 1.$$

Similarly, we have

$$\begin{cases} \|x-z\|_K < t_1 \leq 1 \quad \text{as } x \in E \cap B_a^K(z); \\ \|x-z\|_K < t_1 \leq 1 \quad \text{as } x \in E \cap B_a^K(y), \end{cases}$$

whence getting that if

$$x \in \big(E \cap B_a^K(y)\big) \cup \big(E \cap B_a^K(z)\big) = E \cap \big(B_a^K(y) \cup B_a^K(z)\big)$$

then

$$\begin{cases} \log \|x-y\|_K^{-1} > 0; \\ \log \|x-z\|_K^{-1} > 0, \end{cases} \quad \& \quad \begin{cases} \|x-y\|_K < t_1; \\ \|x-z\|_K < t_1. \end{cases}$$

Hence, by (3.8) and similar methods as in (3.21) and (3.22), we have

$$
\begin{aligned}
(3.12)\quad I_1 &\le \left|\int_{E\cap(B_a^K(y)\cup B_a^K(z))} \frac{dx}{\left(\log\frac{1}{\|x-y\|_K}\right)^{-m}}\right| + \left|\int_{E\cap(B_a^K(y)\cup B_a^K(z))} \frac{dx}{\left(\log\frac{1}{\|x-z\|_K}\right)^{-m}}\right| \\
&= \int_{E\cap(B_a^K(y)\cup B_a^K(z))} \frac{dx}{\left(\log\frac{1}{\|x-y\|_K}\right)^{-m}} + \int_{E\cap(B_a^K(y)\cup B_a^K(z))} \frac{dx}{\left(\log\frac{1}{\|x-z\|_K}\right)^{-m}} \\
&\le \int_{B_{t_1}^K(y)} \left(\log\frac{1}{\|x-y\|_K}\right)^m dx + \int_{B_{t_1}^K(z)} \left(\log\frac{1}{\|x-z\|_K}\right)^m dx \\
&\le \frac{2t_1^n V(K)}{n^m}\sum_{i=0}^{m}\frac{m!}{(m-i)!}\left(\log\frac{1}{t_1^n}\right)^{m-i} \\
&\le \varepsilon.
\end{aligned}
$$

Part 2: For I_2, note that $\forall x \in E$ ensures

$$
\begin{cases}
\frac{\|x-z\|_K-\|x-y\|_K}{\|x-y\|_K} \ge \frac{-\|x-y\|_K}{\|x-y\|_K} = -1; \\
\frac{\|x-y\|_K-\|x-z\|_K}{\|x-z\|_K} \ge \frac{-\|x-z\|_K}{\|x-z\|_K} = -1,
\end{cases}
$$

which, together with (3.6), implies

$$
\begin{cases}
\log\frac{1}{\|x-y\|_K} - \log\frac{1}{\|x-z\|_K} = \log\left(\frac{\|x-z\|_K-\|x-y\|_K}{\|x-y\|_K}+1\right) \le \frac{\|x-z\|_K-\|x-y\|_K}{\|x-y\|_K}; \\
\log\frac{1}{\|x-z\|_K} - \log\frac{1}{\|x-y\|_K} = \log\left(\frac{\|x-y\|_K-\|x-z\|_K}{\|x-z\|_K}+1\right) \le \frac{\|x-y\|_K-\|x-z\|_K}{\|x-z\|_K}.
\end{cases}
$$

Then

$$
\begin{aligned}
&\left|\log\frac{1}{\|x-y\|_K} - \log\frac{1}{\|x-z\|_K}\right| \\
&\quad \le \max\left\{\frac{\|x-z\|_K-\|x-y\|_K}{\|x-y\|_K}, \frac{\|x-y\|_K-\|x-z\|_K}{\|x-z\|_K}\right\} \\
&\quad \le \big|\|x-z\|_K - \|x-y\|_K\big|\left(\frac{1}{\|x-y\|_K}+\frac{1}{\|x-z\|_K}\right)
\end{aligned}
$$

$$\leq \frac{\varepsilon}{H(a)}\left(\frac{1}{\|x-y\|_K}+\frac{1}{\|x-z\|_K}\right),$$

which, together with (3.7), (3.11) and Hölder's inequality, implies

$$\begin{aligned}
(3.13)\quad I_2 &\leq \int_{E\setminus(B_a^K(y)\cup B_a^K(z))}\left|\log\frac{1}{\|x-y\|_K}-\log\frac{1}{\|x-z\|_K}\right|\sum_{i=1}^{m}\frac{\left|\log\frac{1}{\|x-y\|_K}\right|^{m-i}}{\left|\log\frac{1}{\|x-z\|_K}\right|^{1-i}}\,dx\\
&\leq \frac{\varepsilon}{H(a)}\int_{E\setminus(B_a^K(y)\cup B_a^K(z))}\left(\frac{1}{\|x-y\|_K}+\frac{1}{\|x-z\|_K}\right)\sum_{i=1}^{m}\frac{\left|\log\frac{1}{\|x-y\|_K}\right|^{m-i}}{\left|\log\frac{1}{\|x-z\|_K}\right|^{1-i}}\,dx\\
&\leq \frac{2\varepsilon}{aH(a)}\int_{E\setminus(B_a^K(y)\cup B_a^K(z))}\sum_{i=1}^{m}\left|\log\frac{1}{\|x-y\|_K}\right|^{m-i}\left|\log\frac{1}{\|x-z\|_K}\right|^{i-1}\,dx\\
&\lesssim \frac{J(a)\varepsilon}{H(a)},
\end{aligned}$$

where

$$J(a)=\frac{2}{a}\sum_{i=1}^{m}\frac{\left(\int_{E\setminus(B_a^K(y)\cup B_a^K(z))}\left|\log\frac{1}{\|x-y\|_K}\right|^{2(m-i)}\,dx\right)^{\frac{1}{2}}}{\left(\int_{E\setminus(B_a^K(y)\cup B_a^K(z))}\left|\log\frac{1}{\|x-z\|_K}\right|^{2(i-1)}\,dx\right)^{-\frac{1}{2}}},$$

$H(a)$ is defined in (3.14) and $U\lesssim V$ denotes that there is a constant $c>0$ such that $U\leq cV$.

Note that

$$\begin{cases} a<\|x-y\|_K\leq \frac{R}{2} & \text{as } x\in E\setminus B_a^K(y);\\ a<\|x-z\|_K\leq R & \text{as } x\in E\setminus B_a^K(z),\end{cases}$$

then $\forall k\in\mathbb{N}$,

$$\begin{aligned}
\int_{E\setminus(B_a^K(y)\cup B_a^K(z))}\left|\log\frac{1}{\|x-y\|_K}\right|^k\,dx &\leq \int_{E\setminus B_a^K(y)}\left|\log\frac{1}{\|x-y\|_K}\right|^k\,dx\\
&\leq V(E)\left(\max\left\{\log a,\log\frac{R}{2}\right\}\right)^k
\end{aligned}$$

and

$$\int_{E\setminus(B_a^K(y)\cup B_a^K(z))}\left|\log\frac{1}{\|x-y\|_K}\right|^k dx \le \int_{E\setminus B_a^K(z)}\left|\log\frac{1}{\|x-y\|_K}\right|^k dx$$
$$\le V(E)\left(\max\{\log a,\log R\}\right)^k,$$

which imply

$$J(a) \le \frac{2V(E)^m}{a}\sum_{i=1}^{m}\max\{\log a,\log R\}^{i-1}\max\left\{\log a,\log\frac{R}{2}\right\}^{m-i} \tag{3.14}$$
$$= H(a),$$

whence $I_2 \lesssim \varepsilon$ (via (3.13)). This, together with (3.12), implies

$$|I_{\log,m}(E,K;y)-I_{\log,m}(E,K;z)| \le I_1+I_2 \lesssim \varepsilon,$$

thereby ensuring that $I_{\log,m}(E,K;\cdot)$ is continuous in y. Because $y\in\mathbb{R}^n$ is arbitrary, $I_{\log,m}(E,K;\cdot)$ is continuous on $\mathbb{R}^n$. □

Lemma 3.1.5. *Let m be an odd number. The supremum in*

$$V_{\log,m}(E,K)=\sup_{y\in\mathbb{R}^n} I_{\log,m}(E,K;y)$$

is achieved at some $y\in\mathbb{R}^n$.

Proof. We first conclude that

$$\lim_{|y|\to+\infty} I_{\log,m}(E,K;y) = -\infty. \tag{3.15}$$

Actually, if E is bounded, then $\sup_{x\in E}|x| < +\infty$. For $C_1 < 0$ let

$$|y| \geq \max\left\{2R_K e^{-\left(\frac{C_1}{V(E)}\right)^{\frac{1}{m}}}, 2\sup_{x\in E}|x|\right\},$$

where R_K is in (2.2). Then, by (2.3), we have

$$\begin{aligned} I_{\log,m}(E,K;y) &= \int_E \left(\log\frac{1}{\|x-y\|_K}\right)^m dx \\ &\leq \int_E \left(\log\frac{R_K}{|x-y|}\right)^m dx \\ &\leq \int_E \left(\log\frac{R_K}{|y|-|x|}\right)^m dx \\ &\leq \int_E \left(\log\frac{2R_K}{|y|}\right)^m dx \\ &\leq C_1, \end{aligned}$$

thereby reaching (3.15).

Next we conclude that $I_{\log,m}(E,K;y)$ is not always equal to $-\infty$. Actually, if $|y_0| \geq \sup_{x\in E}|x|$ and r_K is in (2.2), then

$$\begin{aligned} I_{\log,m}(E,K;y_0) &= \int_E \left(\log\frac{1}{\|x-y_0\|_K}\right)^m dx \\ &\geq \int_E \left(\log\frac{r_K}{|x-y_0|}\right)^m dx \\ &\geq \int_E \left(\log\frac{r_K}{|y_0|+|x|}\right)^m dx \\ &\geq \int_E \left(\log\frac{r_K}{2|y_0|}\right)^m dx \\ &= \left(\log\frac{r_K}{2|y_0|}\right)^m V(E) \\ &> -\infty. \end{aligned} \tag{3.16}$$

Because of (3.15) & (3.16), there exists $C_2 \geq 0$ such that

$$|y| > C_2 \;\; \& \;\; I_{\log,m}(E,K;y) < \left(\log \frac{r_K}{2|y_0|}\right)^m V(E)$$

implies

$$y_0 \in G = \{y \in \mathbb{R}^n : \; \|y\|_K \leq C_2\}.$$

By Lemma 3.1.4, $I_{\log,m}(E,K;y)$ is continuous for y, and $I_{\log,m}(E,K;y)$ can attain its maximum at the point y_1 in the compact set G, and

$$\begin{aligned} I_{\log,m}(E,K;y_1) &= \sup_{y\in G} I_{\log,m}(E,K;y) \\ &\geq I_{\log,m}(E,K;y_0) \\ &\geq \left(\log \frac{r_K}{2|y_0|}\right)^m V(E) \\ &\geq \sup_{y\in G^c} I_{\log,m}(E,K;y), \end{aligned}$$

which means

$$I_{\log,m}(E,K;y_1) = \sup_{y\in\mathbb{R}^n} I_{\log,m}(E,K;y).$$

□

3.2 An optimal polynomial log-inequality

Now we are ready to establish the optimal polynomial log-inequality for $V_{\log,m}(E,K)$, where m is an odd number.

Theorem 3.2.1. *If m is an odd number, then*

$$V_{\log,m}(E,K) \leq \begin{cases} \frac{V(E)}{n^m}\sum_{i=0}^m \frac{m!}{(m-i)!}\left(\log\frac{V(K)}{V(E)}\right)^{m-i} & as \quad V(E)>0; \\ 0 & as \quad V(E)=0. \end{cases} \tag{3.17}$$

Equality in (3.17) *holds if and only if E is almost a $K-$ball, namely, there is $y \in \mathbb{R}^n$ such that*

$$V\left(E^c \cap B_r^K(y)\right) = V\left(\left(B_r^K(y)\right)^c \cap E\right) = 0 \quad \text{with} \quad r = \left(\frac{V(E)}{V(K)}\right)^{n^{-1}}.$$

Proof. Note that if $V(E)=0$ then $V_{\log,m}(E,K)=0$. So, it suffices to consider $V(E)>0$.

First of all, let $V(E)>0$, $y \in \mathbb{R}^n$ be fixed, and $B_r^K(y)$ be the $K-$ball with center y and radius

$$r = \left(\frac{V(E)}{V(K)}\right)^{1/n} > 0.$$

Thanks to

$$V(\{x : \|x-y\|_K \leq r\}) = r^n V(K) \Rightarrow V\left(B_r^K(y)\right) = V(E),$$

it follows that

$$\begin{aligned} V\left(E^c \cap B_r^K(y)\right) &= V\left(B_r^K(y) \setminus E\right) \\ &= V\left(B_r^K(y)\right) - V\left(B_r^K(y) \cap E\right) \\ &= V(E) - V\left(B_r^K(y) \cap E\right) \\ &= V\left(E \setminus B_r^K(y)\right) \\ &= V\left(\left(B_r^K(y)\right)^c \cap E\right). \end{aligned}$$

This, together with

$$\begin{cases} \|x-y\|_K \le r & \forall \quad x \in E^c \cap B_r^K(y); \\ \|x-y\|_K > r & \forall \quad x \in \left(B_r^K(y)\right)^c \cap E, \end{cases}$$

implies

$$\begin{aligned} \int_{E^c \cap B_r^K(y)} \left(\log \frac{1}{\|x-y\|_K}\right)^m dx &\ge \left(\log \frac{1}{r}\right)^m V\left(B_r^K(y) \cap E^c\right) \\ &= \left(\log \frac{1}{r}\right)^m V\left(\left(B_r^K(y)\right)^c \cap E\right) \\ &\ge \int_{(B_r^K(y))^c \cap E} \left(\log \frac{1}{\|x-y\|_K}\right)^m dx. \end{aligned} \tag{3.18}$$

Consequently,

$$\begin{aligned} & I_{\log,m}(E,K;y) \\ &= \int_E \left(\log \frac{1}{\|x-y\|_K}\right)^m dx \\ &= \int_{E \cap B_r^K(y)} \left(\log \frac{1}{\|x-y\|_K}\right)^m dx + \int_{E \cap \left(B_r^K(y)\right)^c} \left(\log \frac{1}{\|x-y\|_K}\right)^m dx \\ &\le \int_{E \cap B_r^K(y)} \left(\log \frac{1}{\|x-y\|_K}\right)^m dx + \int_{E^c \cap B_r^K(y)} \left(\log \frac{1}{\|x-y\|_K}\right)^m dx \\ &= \int_{B_r^K(y)} \left(\log \frac{1}{\|x-y\|_K}\right)^m dx. \end{aligned} \tag{3.19}$$

For convenience, we first compute the following integral directly for $r = 0$, $s \in \mathbb{N} \cup \{0\}$, or by integration by parts for r times for $r \ge 1$, $r, s \in \mathbb{N} \cup \{0\}$:

$$\int (\log t)^r t^s \, dt = \sum_{i=1}^{r+1} \frac{(-1)^{i-1} r! (\log t)^{r+1-i} t^{s+1}}{(s+1)^i (r+1-i)!} + C, \tag{3.20}$$

where C is a constant.

Case 1: If $r = \left(\frac{V(E)}{V(K)}\right)^{1/n} > 1$, then from Fubini's theorem and (3.20), it follows that

$$
\begin{aligned}
(3.21)\qquad & \int_{B_r^K(y)} \left(\log \frac{1}{\|x-y\|_K}\right)^m dx \\
&= (-1)^m m \int_{\{x:\|x-y\|_K \le r\}} \int_1^{\|x-y\|_K} \frac{(\log t)^{m-1}}{t}\, dt\, dx \\
&= (-1)^{m+1} m \int_{\{x:\|x-y\|_K \le 1\}} \int_{\|x-y\|_K}^1 \frac{(\log t)^{m-1}}{t}\, dt\, dx \\
&\quad + (-1)^m m \int_{\{x:1\le\|x-y\|_K \le r\}} \int_1^{\|x-y\|_K} \frac{(\log t)^{m-1}}{t}\, dt\, dx \\
&= (-1)^{m+1} m \int_0^1 \frac{(\log t)^{m-1}}{t} \int_{\{x:\|x-y\|_K \le t\}} dx\, dt \\
&\quad + (-1)^m m \int_1^r \frac{(\log t)^{m-1}}{t} \int_{\{x:t\le\|x-y\|_K \le r\}} dx\, dt \\
&= (-1)^{m+1} m V(K) \int_0^1 t^{n-1} (\log t)^{m-1}\, dt \\
&\quad + (-1)^m m V(K) \int_1^r \frac{(\log t)^{m-1}}{t} (r^n - t^n)\, dt \\
& \frac{V(E)}{n^m} \sum_{i=0}^m \frac{m!}{(m-i)!} \left(\log \frac{V(K)}{V(E)}\right)^{m-i}.
\end{aligned}
$$

Case 2: If $0 < r = \left(\frac{V(E)}{V(K)}\right)^{1/n} \le 1$, then from Fubini's theorem and (3.20) again, it follows that

$$
\begin{aligned}
(3.22)\qquad & \int_{B_r^K(y)} \left(\log \frac{1}{\|x-y\|_K}\right)^m dx \\
&= (-1)^{m+1} m \int_{\{x:\|x-y\|_K \le r\}} \int_{\|x-y\|_K}^1 \frac{(\log t)^{m-1}}{t}\, dt\, dx \\
&= (-1)^{m+1} m \int_{\{x:\|x-y\|_K \le r\}} \int_{\|x-y\|_K}^r \frac{(\log t)^{m-1}}{t}\, dt\, dx \\
&\quad + (-1)^{m+1} m \int_{\{x:\|x-y\|_K \le r\}} \int_r^1 \frac{(\log t)^{m-1}}{t}\, dt\, dx
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^{m+1} m \int_0^r \frac{(\log t)^{m-1}}{t} \int_{\{x:\|x-y\|_K \le t\}} dx\, dt \\
&\quad + (-1)^{m+1} m \int_r^1 \frac{(\log t)^{m-1}}{t} \int_{\{x:\|x-y\|_K \le r\}} dx\, dt \\
&= (-1)^{m+1} m V(K) \int_0^r (\log t)^{m-1} t^{n-1}\, dt \\
&\quad + (-1)^{m+1} m r^n V(K) \int_r^1 \frac{(\log t)^{m-1}}{t}\, dt \\
&\frac{V(E)}{n^m} \sum_{i=0}^m \frac{m!}{(m-i)!} \left(\log \frac{V(K)}{V(E)}\right)^{m-i}.
\end{aligned}
$$

Hence, by formula (3.19), we have

$$
\begin{aligned}
V_{\log,m}(E,K) &= \sup_{y\in\mathbb{R}^n} I_{\log,m}(E,K;y) \\
&= \sup_{y\in\mathbb{R}^n} \int_E \left(\log \frac{1}{\|x-y\|_K}\right)^m dx \\
&\le \sup_{y\in\mathbb{R}^n} \int_{B_r^K(y)} \left(\log \frac{1}{\|x-y\|_K}\right)^m dx \\
&\le \frac{V(E)}{n^m} \sum_{i=0}^m \frac{m!}{(m-i)!} \left(\log \frac{V(K)}{V(E)}\right)^{m-i}.
\end{aligned}
$$

Next is to check the equality situation of (3.17).

On the one hand, if E is almost a $K-$ball, that is, $\exists y_0 \in \mathbb{R}^n$ and $r_0 = \left(\frac{V(E)}{V(K)}\right)^{1/n}$ such that

$$
V\left(E^c \cap B_{r_0}^K(y_0)\right) = V\Big(\left(B_{r_0}^K(y_0)\right)^c \cap E\Big) = 0,
$$

then the equalities in (3.18) hold:

$$
\int_{E^c \cap B_{r_0}^K(y_0)} \left(\log \frac{1}{\|x-y_0\|_K}\right)^m dx = \int_{\left(B_{r_0}^K(y_0)\right)^c \cap E} \left(\log \frac{1}{\|x-y_0\|_K}\right)^m dx = 0,
$$

which implies (3.19) holds, i.e.,

$$\int_E \left(\log \frac{1}{\|x-y_0\|_K}\right)^m dx = \int_{B^K_{r_0}(y_0)} \left(\log \frac{1}{\|x-y_0\|_K}\right)^m dx.$$

By (3.21), (3.22) and $r_0 = \left(\frac{V(E)}{V(K)}\right)^{1/n}$, it follows that

$$\begin{aligned} &I_{\log,m}(E,K;y_0) \\ &= \int_{B^K_{r_0}(y_0)} \left(\log \frac{1}{\|x-y_0\|_K}\right)^m dx \\ &= \frac{V(E)}{n^m} \sum_{i=0}^{m} \frac{m!}{(m-i)!} \left(\log \frac{V(K)}{V(E)}\right)^{m-i}. \end{aligned}$$

Consequently, the equality in (3.17) holds.

On the other hand, by Lemma 3.1.5, there exists $y_1 \in \mathbb{R}^n$ such that

$$V_{\log,m}(E,K) = \sup_{y\in\mathbb{R}^n} I_{\log,m}(E,K;y) = I_{\log,m}(E,K;y_1),$$

and if E is not almost a K-ball, then for $r_1 = \left(\frac{V(E)}{V(K)}\right)^{1/n} > 0$, it follows that

$$V\left(E^c \cap B^K_{r_1}(y_1)\right) \neq 0 \;\;\&\;\; V\left(B^K_{1}(y_1)^c \cap E\right) \neq 0.$$

Consequently, inequalities in (3.18) are strict, then

$$\int_{E^c\cap B^K_r(y_1)} \left(\log \frac{1}{\|x-y_1\|_K}\right)^m dx > \int_{\left(B^K_r(y_1)\right)^c\cap E} \left(\log \frac{1}{\|x-y_1\|_K}\right)^m dx.$$

Thus, inequality in (3.19) is also strict. Consequently,

$$V_{\log,m}(E,K)$$

$$
\begin{aligned}
&= I_{\log,m}(E,K;y_1)\\
&= \int_E \left(\log \frac{1}{\|x-y_1\|_K}\right)^m dx\\
&< \int_{B_r^K(y_1)} \left(\log \frac{1}{\|x-y_1\|_K}\right)^m dx\\
&= \frac{V(E)}{n^m} \sum_{i=0}^{m} \frac{m!}{(m-i)!} \left(\log \frac{V(K)}{V(E)}\right)^{m-i}.
\end{aligned}
$$

So inequality (3.17) is strict. Hence, in order to have equality in (3.17), E must be almost a K-ball. □

Remark 3.2.2. *Two comments are in order:*

(i) In particular, if E is star-shaped with respect to the origin, then inequality (3.17) holds with equality if and only if K and E are dilates.

(ii) Note that the kernels of $V_\alpha(E,K)$ and $V_{\log,1}(E,K)$ have the relation in a derivative way (see (3.2)*). From Theorem 2.2.1 (i) and Theorem 3.2.1 it is interestingly seen that their optimal upper bounds have exact the same relation:*

$$
\begin{aligned}
&\frac{\partial}{\partial\alpha}\left(\frac{n}{n-\alpha}V(E)^{\frac{n-\alpha}{n}}V(K)^{\frac{\alpha}{n}}\right)\Bigg|_{\alpha=0}\\
&= V(E)\left[\left(\frac{V(K)}{V(E)}\right)^{\frac{\alpha}{n}}\frac{\partial}{\partial\alpha}\left(\frac{n}{n-\alpha}\right)+\frac{n}{n-\alpha}\frac{\partial}{\partial\alpha}\left(\frac{V(K)}{V(E)}\right)^{\frac{\alpha}{n}}\right]\Bigg|_{\alpha=0}\\
&= \frac{V(E)}{n}\log\frac{eV(K)}{V(E)}.
\end{aligned}
$$

3.3 Dual polynomial log-Minkowski inequality

For the application in convex geometry analysis, the dual polynomial log-Minkowski inequality can be implied from Theorem 3.2.1.

Theorem 3.3.1. *Let K, L be two star bodies in $\mathbb{R}^n$ and m be an odd number. Then*

$$\sum_{i=0}^{m} \frac{n^{m-i} m!}{(m-i)!} \int_{S^{n-1}} \left(\log \frac{\rho_K(u)}{\rho_L(u)}\right)^{m-i} dV_L(u) \le \sum_{i=0}^{m} \frac{m!}{(m-i)!} \left(\log \frac{V(K)}{V(L)}\right)^{m-i}, \tag{3.23}$$

where $dV_L(u)$ is the normalized cone-volume measure:

$$dV_L(u) = \left(\frac{\rho_L^n(u)}{nV(L)}\right) dS(u). \tag{3.24}$$

Here, $dS(\cdot)$ denotes the standard surface area measure on the unit sphere S^{n-1}. The equality in (3.23) *holds if and only if K and L are dilates.*

Proof. Suppose K and L are two star bodies. By Theorem 3.2.1, we have

$$\begin{aligned} I_{\log,m}(L, K; 0) &= \int_L \left(\log \frac{1}{\|x\|_K}\right)^m dx \\ &\le \sup_{y\in\mathbb{R}^n} \int_L \log \frac{1}{\|x-y\|_K} dx \\ &\le \frac{V(L)}{n^m} \sum_{i=0}^{m} \frac{m!}{(m-i)!} \left(\log \frac{V(K)}{V(L)}\right)^{m-i}. \end{aligned} \tag{3.25}$$

By using the polar coordinates and integration by parts for m times, it follows that

$$\begin{aligned} & I_{\log,m}(L, K; 0) \\ &= \int_L (\log \rho_K(x))^m \, dx \\ &= \int_{S^{n-1}} \int_0^{\rho_L(u)} (\log \rho_K(ru))^m \, r^{n-1} \, dr du \\ &= n^{-1} \int_{S^{n-1}} \int_0^{\rho_L(u)} \left(\log \frac{\rho_K(u)}{r}\right)^m dr^n du \\ &= n^{-1} \int_{S^{n-1}} \rho_L(u)^n \left(\log \frac{\rho_K(u)}{\rho_L(u)}\right)^m du \\ &\quad + \frac{m}{n} \int_{S^{n-1}} \int_0^{\rho_L(u)} r^{n-1} \left(\log \frac{\rho_K(u)}{r}\right)^{m-1} dr du \end{aligned}$$

$$\vdots$$

$$== V(L) \int_{S^{n-1}} \sum_{i=0}^{m} \frac{m!}{n^i (m-i)!} \log \left(\frac{\rho_K(u)}{\rho_L(u)} \right)^{m-i} \, dV_L(u),$$

where dV_L are defined in (3.24). This, together with (3.25), implies (3.23). The equality condition of (3.23) also follows from the equality condition in Theorem 3.2.1.

□

Let $m = 1$, Theorem 3.3.1 reduces to the dual log-Minkowski inequality developed by Gardner-Hug-Weil-Ye in [30] and by Wang-Liu in [66].

Corollary 3.3.2. *Let K and L be two star bodies in $\mathbb{R}^n$. Then*

$$\int_{S^{n-1}} \log \left(\frac{\rho_K(u)}{\rho_L(u)} \right) \, dV_L(u) \leq n^{-1} \log \frac{V(K)}{V(L)},$$

with equality holds if and only if K and L are dilates.

3.4 Star bodies by anisotropic potentials

In this section, we characterize the star body with respect to the origin in terms of anisotropic Riesz-potentials in Chapter 2 and logarithmic potentials in this chapter for all $\mathbb{R}^{n \geq 2}$, as a convex geometric extension of [59, Theorem] (characterizing a Euclidean $\mathbb{R}^3$-sphere by means of single-layer potentials) from the physical space $\mathbb{R}^3$ to the Euclidean $2 \leq n$-dimensional space $\mathbb{R}^n$.

Theorem 3.4.1. *Suppose*

(i) *$K, \Omega \subset \mathbb{R}^n$ are star bodies with respect to the origin.*

(ii) *f is a continuous function on $\mathbb{R}^n$ enjoying the following two properties:*

(ii-a) $\max_{x\in\Omega} f(x)$ *and* $\min_{x\in\Omega} f(x)$ *are only attainable on* $\partial\Omega$*, the boundary of* Ω*. Moreover, let*

$$\begin{cases} \frac{\partial f(x)}{\partial_K \vec{x}^-} = \lim_{t\to 0^+} \frac{f(x)-f(x-tx)}{\|tx\|_K}; \\ \frac{\partial f(x)}{\partial_K \vec{x}^+} = \lim_{t\to 0^+} \frac{f(x+tx)-f(x)}{\|tx\|_K}. \end{cases}$$

$\exists x_1, x_2 \in \partial\Omega$*, such that*

$$\begin{cases} \max_{x\in\Omega} f(x) = f(x_1); \\ \min_{x\in\Omega} f(x) = f(x_2); \\ \frac{\partial f(x_1)}{\partial_K \vec{x_1}^-} - \frac{\partial f(x_1)}{\partial_K \vec{x_1}^+} \le \frac{\partial f(x_2)}{\partial_K \vec{x_2}^-} - \frac{\partial f(x_2)}{\partial_K \vec{x_2}^+}. \end{cases}$$

(ii-b) $\kappa_n > 0$ *is a dimensional constant with*

$$f(x) = \begin{cases} \kappa_n \|x\|_K^{2-n} & \text{as } x \in \Omega^c = \mathbb{R}^{n\ge 3} \setminus \Omega; \\ -\kappa_2 \log \|x\|_K & \text{as } x \in \Omega^c = \mathbb{R}^2 \setminus \Omega. \end{cases}$$

Then Ω *is a dilation of* K*, namely,* $\Omega = \lambda K$ *for a constant* $\lambda > 0$*.*

Remark 3.4.2. *If* f *is a non constant harmonic function in* Ω°*, the interior of* Ω*, then the maximum principle of harmonic function implies* $\max_{x\in\Omega} f(x)$ *and* $\min_{x\in\Omega} f(x)$ *are only attainable on* $\partial\Omega$*. For example, if* K *is an ellipsoid (ellipse in two dimension) and* f *is the following anisotropic potential*

$$f_K(x) = \begin{cases} \int_{\partial\Omega} \|x-y\|_K^{2-n}\, d\sigma(y) & \text{as } n \ge 3; \\ \int_{\partial\Omega} \log \|x-y\|_K^{-1}\, d\sigma(y) & \text{as } n = 2, \end{cases}$$

where σ *denotes the surface area measure, then* f_K *is harmonic in* Ω°*, i.e. it satisfies* *the Laplace equation $\Delta f_K = 0$. This can be seen from the following computation:*

$$f_K(x) = f_{T(B(o,1))}(x)$$
$$= \begin{cases} \int_{\partial\Omega} \|x-y\|^{2-n}_{T(B(o,1))}\, d\sigma(y) = \int_{T^{-1}(\partial\Omega)} \frac{\det(T)}{|\tilde{x}-\tilde{y}|^{n-2}}\, d\sigma(\tilde{y}) & as\ n \geq 3; \\ \int_{\partial\Omega} \log \|x-y\|^{-1}_{T(B(o,1))}\, d\sigma(y) = \int_{T^{-1}(\partial\Omega)} \frac{\det(T)}{(-\log|\tilde{x}-\tilde{y}|)^{-1}}\, d\sigma(\tilde{y}) & as\ n = 2, \end{cases}$$

where T is a matrix with positive determinant. However, for a general K we are led to adopt [58, Theorem 1.7.2] to compute $\Delta f = 0$, thereby finding a geometric condition on K such that f_K is harmonic - in other words - f_K is not always harmonic.

Moreover, if $\Omega = K = B(o,1)$ then one has not only, $\forall x \in \partial B(o,1)$

$$\frac{\partial f_{B(o,1)}(x)}{\partial_{B(o,1)}\vec{x}^-} - \frac{\partial f_{B(o,1)}(x)}{\partial_{B(o,1)}\vec{x}^+} = \begin{cases} (n-2)\omega_{n-1} & as\ n \geq 3; \\ 2\pi & as\ n = 2, \end{cases}$$

by [65, Theorem 1.11], but also for $x \in (B(o,1))^c$,

$$f_K(x) = \begin{cases} \int_{\partial B(o,1)} |x-y|^{2-n}\, d\sigma(y) = \omega_{n-1}|x|^{2-n} & as\ n \geq 3; \\ \int_{\partial B(o,1)} \log|x-y|^{-1}\, d\sigma(y) = 2\pi \log|x|^{-1} & as\ n = 2, \end{cases}$$

by the mean value property of the harmonic function, where ω_{n-1} is the surface area of $B(o,1)$. This is the initial reason to consider an extension of [59, Theorem].

Proof of Theorem 3.4.1. Let

$$\begin{cases} \lambda_1 = \sup\{\lambda : \lambda K \subseteq \Omega\}; \\ x_1 \in \lambda_1 K \cap \partial\Omega; \\ \lambda_2 = \inf\{\lambda : \Omega \subseteq \lambda K\}; \\ x_2 \in \lambda_2 K \cap \partial\Omega. \end{cases}$$

Obviously,

$$\|x_2\|_K = \lambda_2 \geq \lambda_1 = \|x_1\|_K.$$

If we can verify

$$\|x_2\|_K \leq \|x_1\|_K,$$

then we will have

$$\lambda_2 = \lambda_1 = \lambda \quad \& \quad \Omega = \lambda K,$$

as desired. So, it remains to validate the last inequality. This validation consists of the three steps as seen below.

Step 1. We claim

$$\begin{cases} \max_{x\in\Omega} f(x) = f(x_1); \\ \min_{x\in\Omega} f(x) = f(x_2), \end{cases} \tag{3.26}$$

and then we can choose x_1, x_2 as the points satisfying (ii-a). Evidently, it is enough to check the maximum case and the minimum case follows in a similar way.

Since Ω is a star-shaped body at the origin, f is continuous and (ii-b), we have

$$f(x) = \begin{cases} \kappa_n \|x\|_K^{2-n} & \text{as } x \in \Omega^c \cup \partial\Omega; \\ \kappa_2 \log \|x\|_K^{-1} & \text{as } x \in \Omega^c \cup \partial\Omega. \end{cases}$$

If the first equation in (3.26) is invalid, then $\max_{x\in\Omega} f(x)$ would be attained by another point $x_1' \in \partial\Omega$, and hence:

$$\begin{cases} \kappa_n \|x_1\|_K^{2-n} = f(x_1) < \max_{x\in\Omega} f(x) = f(x_1') = \kappa_n \|x_1'\|_K^{2-n} \quad \text{as} \quad n \geq 3; \\ \kappa_2 \log \|x_1\|_K^{-1} = f(x_1) < \max_{x\in\Omega} f(x) = f(x_1') = \kappa_2 \log \|x_1'\|_K^{-1} \quad \text{as} \quad n = 2. \end{cases}$$

This would in turn imply

$$\|x_1\|_K > \|x_1'\|_K,$$

whence violating

$$\lambda_1 = \sup\{\lambda : \lambda K \subseteq \Omega\} \ \ \& \ \ x_1 \in \lambda_1 K \cap \partial\Omega \Rightarrow \|x_1\|_K \le \|x_1'\|_K.$$

Consequently, we get

$$\begin{cases} f(x_1) = \max_{x\in\Omega} f(x) \ge f(x_1 - tx_1) \quad \forall \ \ t \in [0,1] \quad \& \quad x_1 - tx_1 \in \lambda_1 K \subseteq \Omega; \\ f(x_2) = \min_{x\in\Omega} f(x) \le f(x_2 - tx_2) \quad \forall \ \ t \in [0,1] \quad \& \quad x_2 - tx_2 \in \Omega, \end{cases}$$

where $x_2 - tx_2 \in \Omega$ since Ω is a star-shaped body at the origin, thereby arriving at

$$\begin{cases} \frac{\partial f(x_1)}{\partial_K \vec{x_1}^-} = \lim_{t\to 0^+} \frac{f(x_1)-f(x_1-tx_1)}{\|tx_1\|_K} \ge 0; \\ \frac{\partial f(x_2)}{\partial_K \vec{x_2}^-} = \lim_{t\to 0^+} \frac{f(x_2)-f(x_2-tx_2)}{\|tx_2\|_K} \le 0. \end{cases}$$

Step 2. Using the fact that $\|\cdot\|_K$ is homogeneous one, we have

$$\begin{aligned} \frac{\partial f(x)}{\partial_K \vec{x}^+} &= \lim_{t\to 0^+} \frac{f(x+tx)-f(x)}{\|tx\|_K} \\ &= \lim_{t\to 0^+} \frac{f(x+tx)-f(x)}{t\|x\|_K} \\ &= \lim_{t\to 0^+} \frac{f(x+tx)-f(x)}{(1+t)\|x\|_K - \|x\|_K} \\ &= \lim_{t\to 0^+} \frac{f(x+tx)-f(x)}{\|(1+t)x\|_K - \|x\|_K} \\ &= \frac{\partial f(x)^+}{\partial \|x\|_K^+}, \end{aligned}$$

where $\partial f(x)^+$ and $\partial\|x\|_K^+$ denote respectively the directional derivative of these functions in the direction $\vec{x}$ at the point x. Similarly, we also have

$$\frac{\partial f(x)}{\partial_K \vec{x}^-} = \frac{\partial f(x)^-}{\partial\|x\|_K^-},$$

where $\partial f(x)^-$ and $\partial\|x\|_K^-$ denote respectively the directional derivative of these functions in the direction $-\vec{x}$ at the point x.

Step 3. With the help of the previous computations, we obtain

$$\begin{cases} \begin{cases} \frac{\partial f(x_1)}{\partial_K \vec{x_1}^-} - \frac{\partial f(x_1)}{\partial_K \vec{x_1}^+} = \frac{\partial f(x_1)}{\partial_K \vec{x_1}^-} + \kappa_n(n-2)\|x_1\|_K^{1-n} \geq \kappa_n(n-2)\|x_1\|_K^{1-n}; \\ \frac{\partial f(x_2)}{\partial_K \vec{x_2}^-} - \frac{\partial f(x_2)}{\partial_K \vec{x_2}^+} = \frac{\partial f(x_2)}{\partial_K \vec{x_2}^-} + \kappa_n(n-2)\|x_2\|_K^{1-n} \leq \kappa_n(n-2)\|x_2\|_K^{1-n}. \end{cases} & \text{as } n \geq 3; \\ \begin{cases} \frac{\partial f(x_1)}{\partial_K \vec{x_1}^-} - \frac{\partial f(x_1)}{\partial_K \vec{x_1}^+} = \frac{\partial f(x_1)}{\partial_K \vec{x_1}^-} + \kappa_2\|x_1\|_K^{-1} \geq \kappa_2\|x_1\|_K^{-1}; \\ \frac{\partial f(x_2)}{\partial_K \vec{x_2}^-} - \frac{\partial f(x_2)}{\partial_K \vec{x_2}^+} = \frac{\partial f(x_2)}{\partial_K \vec{x_2}^-} + \kappa_2\|x_2\|_K^{-1} \leq \kappa_2\|x_2\|_K^{-1}. \end{cases} & \text{as } n = 2. \end{cases}$$

As a consequence, we achieve

$$\begin{cases} \kappa_n(n-2)\|x_1\|_K^{1-n} \leq \kappa_n(n-2)\|x_2\|_K^{1-n} \quad \text{as} \quad n \geq 3; \\ \kappa_2\|x_1\|_K^{-1} \leq \kappa_2\|x_2\|_K^{-1} \quad \text{as} \quad n = 2, \end{cases}$$

by (ii-a), whence getting the required inequality $\|x_2\|_K \leq \|x_1\|_K$. $\square$

Chapter 4

A functional mixed volume induced by the Orlicz addition for measures and its optimization problem

4.1 Orlicz addition for functions

Throughout this chapter, Ω is a nonempty set equipped with measure μ and distance d, and $m \geq 1$ is an integer. Denote $\mathcal{F}$ the set of nonnegative real-valued measurable functions defined on Ω. We use $\mathcal{F}^+$ to denote the set of all functions in $\mathcal{F}$ which are positive, and $\mathcal{F}^{+c}$ for the set of all functions in $\mathcal{F}^+$ which are also continuous.

Let Φ_m denote the set of all continuous functions $\varphi : [0, \infty)^m \to [0, \infty)$ that are strictly increasing in each component with $\varphi(o) = 0$ and $\lim_{t\to\infty} \varphi(tz) = \infty$ for each nonzero $z \in [0, \infty)^m$. Hereafter $o = (0, \cdots, 0)$ stands for the origin of $\mathbb{R}^m$. Let Ψ_m denote the set of all continuous functions $\varphi : (0, \infty)^m \to (0, \infty)$ that are strictly decreasing in each component with $\lim_{t\to 0} \varphi(tz) = \infty$ and $\lim_{t\to\infty} \varphi(tz) = 0$ for each $z \in (0, \infty)^m$. Note that $\varphi(x) = x_1^p + \cdots + x_m^p$ belongs to Φ_m if $p > 0$ and belongs to

Ψ_m if $p < 0$.

The Orlicz addition of functions is defined as follows.

Definition 4.1.1. *For $\varphi \in \Phi_m$, $\widetilde{+}_\varphi(p_1, \cdots, p_m)$, the Orlicz addition of functions $p_1, \cdots, p_m \in \mathcal{F}$, is (uniquely and implicitly) defined by*

$$\varphi\left(\frac{p_1(x)}{\widetilde{+}_\varphi(p_1, \cdots, p_m)(x)}, \cdots, \frac{p_m(x)}{\widetilde{+}_\varphi(p_1, \cdots, p_m)(x)}\right) = 1, \tag{4.1}$$

if $p_1(x) + \cdots + p_m(x) > 0$, and otherwise by

$$\widetilde{+}_\varphi(p_1, \cdots, p_m)(x) = 0.$$

If $\varphi \in \Psi_m$ and in addition $p_1, \cdots, p_m \in \mathcal{F}^+$, the Orlicz addition $\widetilde{+}_\varphi(p_1, \cdots, p_m)$ is also defined by equation (4.1).

Clearly if $\varphi \in \Phi_m$, then $\widetilde{+}_\varphi(p_1, \cdots, p_m)(x) = 0$ implies that $p_1(x) = \cdots = p_m(x) = 0$. Moreover, $\widetilde{+}_\varphi(p_1, \cdots, p_m) \in \mathcal{F}$, if $p_1, \cdots, p_m \in \mathcal{F}$. For simplicity, in later context, the functions $p_1, \cdots, p_m$ in $\widetilde{+}_\varphi(p_1, \cdots, p_m)$ for $\varphi \in \Psi_m$ are always assumed to be in $\mathcal{F}^+$.

It is worth to mention that if $\varphi \in \Phi_m$, $\widetilde{+}_\varphi(p_1, \cdots, p_m)(x)$ for $x \in \Omega$ given in Definition 4.1.1 is equal to the infimum of $\Lambda(x) \subset \mathbb{R}$, where

$$\Lambda(x) = \left\{\lambda > 0 : \varphi\left(\frac{p_1(x)}{\lambda}, \cdots, \frac{p_m(x)}{\lambda}\right) \leq 1\right\}.$$

If $\varphi \in \Psi_m$ and $p_i \in \mathcal{F}^+$ for all $i = 1, \cdots, m$, then $\widetilde{+}_\varphi(p_1, \cdots, p_m)(x)$ for $x \in \Omega$ is equal to the supremum of $\Lambda(x) \subset \mathbb{R}$. To this end, if $\varphi \in \Phi_m$ and $p_1(x) = \cdots = p_m(x) = 0$, then $\Lambda(x) = \{\lambda : \lambda > 0\}$ and hence $\inf \Lambda(x) = 0$ as desired. Now assume that

$\sum_{j=1}^m p_j(x) > 0$ which yields

$$(p_1(x), \cdots, p_m(x)) \neq o.$$

It is easy to see that

$$\widetilde{+}_\varphi(p_1, \cdots, p_m)(x) \in \Lambda(x)$$

by formula (4.1). On the other hand, the fact that $\lim_{t\to\infty} \varphi(tz) = \infty$ for each nonzero $z \in [0,\infty)^m$ implies $\widetilde{+}_\varphi(p_1, \cdots, p_m)(x) > 0$. Formula (4.1) together with the fact that φ is strictly increasing in each component implies that for all $0 < \lambda < \widetilde{+}_\varphi(p_1, \cdots, p_m)(x)$,

$$\varphi\left(\frac{p_1(x)}{\lambda}, \cdots, \frac{p_m(x)}{\lambda}\right) > 1.$$

Thus, $\Lambda(x) = \big[\widetilde{+}_\varphi(p_1, \cdots, p_m)(x), \infty\big)$. This yields the desired equality:

$$\widetilde{+}_\varphi(p_1, \cdots, p_m)(x) = \inf \Lambda(x).$$

Along the same lines, one can get the desired argument for the case $\varphi \in \Psi_m$.

Now we prove the basic properties of $\widetilde{+}_\varphi(p_1, \cdots, p_m)(x)$ where $p_1, \cdots, p_m \in \mathcal{F}$.

Theorem 4.1.2. *Let $m \geq 2$ and $\varphi \in \Phi_m$, $p_1, \cdots, p_m \in \mathcal{F}$.*

(i) *For $r \geq 0$, one has,*

$$\widetilde{+}_\varphi(rp_1, \cdots, rp_m) = r \cdot \widetilde{+}_\varphi(p_1, \cdots, p_m).$$

(ii) *Assume that $\varphi \in \Phi_m$ satisfies $\varphi(e_j) = 1$ for all $j = 1, \cdots, m$, where $\{e_1, \cdots, e_m\}$ is the standard orthonormal basis of $\mathbb{R}^m$. Then, for $j = 1, \cdots, m$, one has*

$$\widetilde{+}_\varphi(0, \cdots, 0, p_j, 0, \cdots, 0) = p_j.$$

(iii) *If* $q_1, \cdots, q_m \in \mathcal{F}$ *such that* $p_j \le q_j$ *for all* $j = 1, \cdots, m$, *then*

$$\widetilde{+}_\varphi(p_1, \cdots, p_m) \le \widetilde{+}_\varphi(q_1, \cdots, q_m).$$

In particular,

$$\widetilde{+}_\varphi(p_1, \cdots, p_m)(x) \le {\tau_0}^{-1} \cdot \sum_{j=1}^{m} p_j(x), \tag{4.2}$$

where $\tau_0 > 0$ *satisfies* $\varphi(\tau_0, \cdots, \tau_0) = 1$.

(iv) *Assume that* $p_{ij} \in \mathcal{F}$ *for* $j = 1, \cdots, m$ *and* $i = 1, 2, \cdots$, *such that, for all* $x \in \Omega$ *and for all* $j = 1, \cdots, m$,

$$\lim_{i \to \infty} p_{ij}(x) = p_j(x).$$

Then, for all $x \in \Omega$,

$$\lim_{i \to \infty} \widetilde{+}_\varphi(p_{i1}, \cdots, p_{im})(x) = \widetilde{+}_\varphi(p_1, \cdots, p_m)(x).$$

(v) *Let* p_{ij} *be as in (iv) and let* $S \subset \Omega$ *be a compact set. Assume that all functions* p_{ij} *are positive and continuous on* S, *and the sequence* p_{ij} *is uniformly convergent to a positive function* p_j *on* S *as* $i \to \infty$. *Then* $\widetilde{+}_\varphi(p_{i1}, \cdots, p_{im})$ *is convergent to* $\widetilde{+}_\varphi(p_1, \cdots, p_m)$ *uniformly on* S *as* $i \to \infty$.

The above statements except statement (ii) still hold true when $\varphi \in \Psi_m$ *and all functions involved are positive, except that* $r \ge 0$ *should be replaced by* $r > 0$ *in (i).*

Proof. The proof of this theorem is similar to those in [30]. For completeness, we include a brief proof with modifications emphasized.

(i) $\forall x \in \Omega$, the equality holds trivially if $r = 0$ or $(p_1(x), \cdots, p_m(x)) = o$. The desired equality for $r > 0$ and $(p_1(x), \cdots, p_m(x)) \neq o$ follows from the fact that, the equation

$$\varphi\left(\frac{p_1(x)}{\lambda}, \cdots, \frac{p_m(x)}{\lambda}\right) = 1$$

has a unique solution and the fact that

$$\begin{aligned} 1 &= \varphi\left(\frac{rp_1(x)}{\widetilde{+}_\varphi(rp_1, \cdots, rp_m)(x)}, \cdots, \frac{rp_m(x)}{\widetilde{+}_\varphi(rp_1, \cdots, rp_m)(x)}\right) \\ &= \varphi\left(\frac{p_1(x)}{\left(\frac{\widetilde{+}_\varphi(rp_1, \cdots, rp_m)(x)}{r}\right)}, \cdots, \frac{p_m(x)}{\left(\frac{\widetilde{+}_\varphi(rp_1, \cdots, rp_m)(x)}{r}\right)}\right). \end{aligned}$$

(ii) $\forall j \in \{1, \cdots, m\}$, if $p_j(x) = 0$, then

$$\widetilde{+}_\varphi(0, \cdots, 0, p_j, 0, \cdots, 0)(x) = 0.$$

Assume that $p_j(x) \neq 0$. Formula (4.1) implies that

$$\varphi\left(0, \cdots, 0, \frac{p_j(x)}{\widetilde{+}_\varphi(0, \cdots, 0, p_j, 0, \cdots, 0)(x)}, 0, \cdots, 0\right) = 1.$$

Together with the facts that $\varphi(e_j) = 1$ and φ is strictly increasing in each component, one gets

$$\widetilde{+}_\varphi(0, \cdots, 0, p_j, 0, \cdots, 0)(x) = p_j(x).$$

(iii) Assume that $p_j \leq q_j$ for all $j = 1, \cdots, m$. Note that φ is strictly increasing in each component. Then,

$$\begin{aligned} 1 &= \varphi\left(\frac{q_1(x)}{\widetilde{+}_\varphi(q_1, \cdots, q_m)(x)}, \cdots, \frac{q_m(x)}{\widetilde{+}_\varphi(q_1, \cdots, q_m)(x)}\right) \\ &= \varphi\left(\frac{p_1(x)}{\widetilde{+}_\varphi(p_1, \cdots, p_m)(x)}, \cdots, \frac{p_m(x)}{\widetilde{+}_\varphi(p_1, \cdots, p_m)(x)}\right) \end{aligned}$$

$$\leq \varphi\left(\frac{q_1(x)}{\widetilde{+}_\varphi(p_1,\cdots,p_m)(x)},\cdots,\frac{q_m(x)}{\widetilde{+}_\varphi(p_1,\cdots,p_m)(x)}\right).$$

Again by the fact that φ is strictly increasing in each component, one gets

$$\widetilde{+}_\varphi(p_1,\cdots,p_m) \leq \widetilde{+}_\varphi(q_1,\cdots,q_m).$$

In particular, let $q_1 = \cdots = q_m = \sum_{j=1}^m p_j$, then

$$\widetilde{+}_\varphi(p_1,\cdots,p_m) \leq \widetilde{+}_\varphi\left(\sum_{j=1}^m p_j,\cdots,\sum_{j=1}^m p_j\right).$$

The right hand side is equal to ${\tau_0}^{-1} \cdot \sum_{j=1}^m p_j(x)$ which follows directly from

$$\varphi(\tau_0,\cdots,\tau_0) = 1.$$

(iv) $\forall x \in \Omega$, assume that $\sum_{j=1}^m p_j(x) = 0$. As

$$\lim_{i\to\infty} p_{ij}(x) = p_j(x),$$

then for all $\epsilon > 0$, there is $i(\epsilon) \in \mathbb{N}$, such that for $i > i(\epsilon)$,

$$\sum_{j=1}^m p_{ij}(x) < \epsilon.$$

By formula (4.2), one has, for all $i > i(\epsilon)$,

$$\begin{aligned} 0 &\leq \widetilde{+}_\varphi(p_{i1},\cdots,p_{im})(x) \\ &\leq {\tau_0}^{-1} \cdot \sum_{j=1}^m p_{ij}(x) < {\tau_0}^{-1}\epsilon. \end{aligned}$$

Consequently, one has,

$$\lim_{i\to\infty} \widetilde{+}_\varphi(p_{i1},\cdots,p_{im})(x) = 0 = \widetilde{+}_\varphi(p_1,\cdots,p_m)(x)$$

because $\sum_{j=1}^m p_j(x) = 0$.

Now assume that $\sum_{j=1}^m p_j(x) > 0$ and $\lim_{i\to\infty} p_{ij}(x) = p_j(x)$. Then,

$$\lim_{i\to\infty} \sum_{j=1}^m p_{ij}(x) = \sum_{j=1}^m p_j(x) > 0.$$

Then there is $i_0 \in \mathbb{N}$, such that, $\sum_{j=1}^m p_{ij}(x) > 0$ for all $i > i_0$ and hence

$$1 = \varphi\left(\frac{p_{i1}(x)}{\widetilde{+}_\varphi(p_{i1},\cdots,p_{im})(x)},\cdots,\frac{p_{im}(x)}{\widetilde{+}_\varphi(p_{i1},\cdots,p_{im})(x)}\right).$$

Taking the limit as $i\to\infty$, the desired conclusion follows from the continuity of φ and the uniqueness of the solution of (4.1). That is, for all $x\in\Omega$,

$$\lim_{i\to\infty} \widetilde{+}_\varphi(p_{i1},\cdots,p_{im})(x) = \widetilde{+}_\varphi(p_1,\cdots,p_m)(x).$$

(v) Assume that all functions p_{ij} are positive and continuous on S, and the sequence p_{ij} is uniformly convergent to a positive function p_j on S. Then, there exist $c_1, c_2 > 0$ such that for all $x \in S$, $c_1 \le p_{ij}(x) \le c_2$ for all i and j. Part (iv) implies that $\widetilde{+}_\varphi(p_{i1},\cdots,p_{im})$ converges to $\widetilde{+}_\varphi(p_1,\cdots,p_m)$ pointwise on S.

If the convergence is not uniform on S, then there exist $\varepsilon_0 > 0$ and $n_i > i$, such that, $x_{n_i} \in S$ with $x_{n_i} \to x_0$ (due to the compactness of S), and

$$\left|\widetilde{+}_\varphi(p_{n_i1},\cdots,p_{n_im})(x_{n_i}) - \widetilde{+}_\varphi(p_1,\cdots,p_m)(x_{n_i})\right| \ge \varepsilon_0. \tag{4.3}$$

Part (iii) and the fact $\varphi(\tau_0, \cdots, \tau_0) = 1$ imply

$$c_1/\tau_0 \le \widetilde{+}_\varphi(p_{n_i1}, \cdots, p_{n_im})(x) \le c_2/\tau_0,$$

That is, $\{\widetilde{+}_\varphi(p_{n_i1}, \cdots, p_{n_im})(x_{n_i})\}_{i\in\mathbb{N}}$ is a bounded sequence and hence has a convergent subsequence. Without loss of generality, assume that

$$\lim_{i\to\infty} \widetilde{+}_\varphi(p_{n_i1}, \cdots, p_{n_im})(x_{n_i}) = c_0 > 0,$$

where $c_0 > 0$ is a constant. This, together with $x_{n_i} \to x_0$ and p_{n_ij} converges to p_j uniformly on S, further implies that

$$\begin{aligned} 1 &= \varphi\left(\frac{p_{n_i1}(x_{n_i})}{\widetilde{+}_\varphi(p_{n_i1}, \cdots, p_{n_im})(x_{n_i})}, \cdots, \frac{p_{n_im}(x_{n_i})}{\widetilde{+}_\varphi(p_{n_i1}, \cdots, p_{n_im})(x_{n_i})}\right) \\ &\to \varphi\left(\frac{p_1(x_0)}{c_0}, \cdots, \frac{p_m(x_0)}{c_0}\right), \quad \text{as } i \to \infty. \end{aligned}$$

It follows that $c_0 = \widetilde{+}_\varphi(p_1, \cdots, p_m)(x_0)$, which leads to a contradiction with

$$|c_0 - \widetilde{+}_\varphi(p_1, \cdots, p_m)(x_0)| \ge \varepsilon_0,$$

after taking $i \to \infty$ from both sides of (4.3). Hence, the desired uniform convergence follows. □

Our definition of the Orlicz addition for measures is motivated by the recently introduced Orlicz addition for convex bodies and star bodies, which are the foundation of the newly initiated Orlicz-Brunn-Minkowski theory for convex bodies and its dual theory [29, 30, 67, 74].

Next we briefly discuss these Orlicz additions in geometry and show how it can be linked with our Orlicz addition for measures (or functions). Notations and concepts for geometry below are standard, and more details can be found in [58].

The radial Orlicz sum of star bodies $K_1, \cdots, K_m$, denoted by $\widetilde{+}_\varphi(K_1, \ldots, K_m)$, is (uniquely and implicitly) defined by its radial function $\rho_{\widetilde{+}_\varphi(K_1,\ldots,K_m)}$, the unique solution of the following equation [30]: for $u \in S^{n-1}$,

$$\varphi\left(\frac{\rho_{K_1}(u)}{\rho_{\widetilde{+}_\varphi(K_1,\ldots,K_m)}(u)}, \ldots, \frac{\rho_{K_m}(u)}{\rho_{\widetilde{+}_\varphi(K_1,\ldots,K_m)}(u)}\right) = 1. \tag{4.4}$$

Formula (4.4) can be used to define the radial Orlicz sum of star bodies $K_1, \cdots, K_m$ with respect to $\varphi \in \Psi_m$. In fact, the radial Orlicz sum of star bodies $K_1, \cdots, K_m$ defined by formula (4.4) is a special case of the Orlicz addition of functions defined by formula (4.1); it can be obtained by letting $\Omega = S^{n-1}$ and letting $p_j = \rho_{K_j}$. Alternatively, it can be also obtained by letting $\Omega = \mathbb{R}^n \setminus \{o\}$ and $p_j(x) = r^{-1} \rho_{K_j}(u)$ for all $x = ru \neq o$.

An arguably better way to characterize convex body K is by its support function h_K. Let $\Omega = S^{n-1}$, $p_j = h_{K_j}$, and let $\varphi \in \Phi_m$ be convex, then formula (4.1) becomes: for $u \in S^{n-1}$,

$$\varphi\left(\frac{h_{K_1}(u)}{h_{+_\varphi(K_1,\ldots,K_m)}(u)}, \ldots, \frac{h_{K_m}(u)}{h_{+_\varphi(K_1,\ldots,K_m)}(u)}\right) = 1. \tag{4.5}$$

This is exactly the Orlicz addition of convex bodies, $+_\varphi(K_1, \ldots, K_m)$, given in [29].

4.2 Orlicz addition for measures and a functional Orlicz-Brunn-Minkowski inequality

Let μ be a measure on Ω such that $\mu(\Omega) > 0$. Denote $\mathcal{M}$ the set of finite measures on Ω that are absolutely continuous with respect to μ and whose density functions with respect to μ are in $\mathcal{F}$. That is, $P \in \mathcal{M}$ has the density function p with respect to μ such that $p \in \mathcal{F}$, $P(\Omega) < \infty$, and

$$P(A) = \int_A p(x)\, d\mu(x), \quad \text{for all measurable } A \subseteq \Omega.$$

In this chapter, we always assume that $\mathcal{M} \neq \emptyset$. Let $\mathcal{M}^+$ and $\mathcal{M}^{+c}$ denote the sets of all measures in $\mathcal{M}$ whose density functions are in $\mathcal{F}^+$ and in $\mathcal{F}^{+c}$, respectively. Note that $\widetilde{+}_\varphi(p_1, \cdots, p_m) \in \mathcal{F}$ and

$$\begin{aligned}\int_\Omega \widetilde{+}_\varphi(p_1, \cdots, p_m)(x)\, d\mu(x) &\leq \int_\Omega {\tau_0}^{-1} \cdot \sum_{j=1}^m p_j(x)\, d\mu(x) \\ &= {\tau_0}^{-1} \cdot \sum_{j=1}^m \int_\Omega p_j(x)\, d\mu(x) \\ &< \infty,\end{aligned}$$

where the first inequality follows from inequality (4.2). That is, $\widetilde{+}_\varphi(p_1, \cdots, p_m)$ can be the density function of a measure in $\mathcal{M}$. This observation leads to our definition for the Orlicz addition of measures.

Definition 4.2.1. *Let $P_1, \cdots, P_m \in \mathcal{M}$ with density functions $p_1, \cdots, p_m \in \mathcal{F}$. For $\varphi \in \Phi_m$, the Orlicz addition of measures $P_1, \cdots, P_m$, denoted by $\widetilde{+}_\varphi(P_1, \cdots, P_m)$, is the measure in $\mathcal{M}$ whose density function is $\widetilde{+}_\varphi(p_1, \cdots, p_m)$. Similarly, the Orlicz addition of $P_1, \cdots, P_m \in \mathcal{M}^+$ for $\varphi \in \Psi_m$ is a measure in $\mathcal{M}^+$ whose density function is $\widetilde{+}_\varphi(p_1, \cdots, p_m)$.*

Remark 4.2.2. *Clearly, if $P_1, \cdots, P_m \in \mathcal{M}^+$ (or $\mathcal{M}^{+c}$, respectively), then*

$$\widetilde{+}_\varphi(P_1, \cdots, P_m) \in \mathcal{M}^+$$

(or $\mathcal{M}^{+c}$, respectively). In later context, for $\varphi \in \Psi_m$, the measures $P_1, \cdots, P_m$ in $\widetilde{+}_\varphi(P_1, \cdots, P_m)$ are always assumed to be in $\mathcal{M}^+$.

The following theorem provides a functional dual Orlicz-Brunn-Minkowski inequality for the Orlicz addition of measures.

Theorem 4.2.3. *Let $m \geq 2$ and let $P_j \in \mathcal{M}$ with density functions p_j for $j = 1, \cdots, m$. Assume that $A \subset \Omega$ is measurable with $\mu(A) > 0$ such that $\sum_{j=1}^m p_j(x) > 0$ for $x \in A$ almost everywhere with respect to μ. If $\varphi \in \Phi_m \cup \Psi_m$ is concave, then*

$$\varphi\left(\frac{P_1(A)}{\widetilde{+}_\varphi(P_1, \cdots, P_m)(A)}, \cdots, \frac{P_m(A)}{\widetilde{+}_\varphi(P_1, \cdots, P_m)(A)}\right) \geq 1. \tag{4.6}$$

If φ is convex, the inequality holds with $\geq$ replaced by $\leq$.

If φ is strictly concave or convex, and $P_1, \cdots, P_m \in \mathcal{M}^{+c}$, equality holds if and only if there are constants $a_j > 0$ such that $p_j = a_j p_1$ for $2 \leq j \leq m$.

Proof. Let $\varphi \in \Phi_m$ and $\sum_{j=1}^m p_j(x) > 0$ for $x \in A$ almost everywhere with respect to μ. By inequality (4.2), for $x \in A$ almost everywhere with respect to μ,

$$0 < \widetilde{+}_\varphi(p_1, \cdots, p_m)(x) \leq \tau_0^{-1} \sum_{j=1}^m p_j(x).$$

Together with $\mu(A) > 0$, one has

$$0 < \widetilde{+}_\varphi(P_1, \cdots, P_m)(A) < \infty.$$

Hence, we can define a probability measure $d\nu$ on A by

$$d\nu = \frac{\widetilde{+}_\varphi(p_1, \cdots, p_m)}{\widetilde{+}_\varphi(P_1, \cdots, P_m)(A)}\, d\mu.$$

Assume that $\varphi \in \Phi_m$ is concave. By (4.1) and Jensen's inequality (see e.g. Proposition

2.2 in [30]), one has,

$$
\begin{aligned}
1 &= \int_A \varphi\left(\frac{p_1(x)}{\widetilde{+}_\varphi(p_1,\cdots,p_m)(x)},\cdots,\frac{p_m(x)}{\widetilde{+}_\varphi(p_1,\cdots,p_m)(x)}\right) d\nu(x)\\
&\le \varphi\left(\int_A \frac{p_1(x)}{\widetilde{+}_\varphi(p_1,\cdots,p_m)(x)}\, d\nu(x),\cdots,\int_A \frac{p_m(x)}{\widetilde{+}_\varphi(p_1,\cdots,p_m)(x)}\, d\nu(x)\right)\\
&= \varphi\left(\frac{P_1(A)}{\widetilde{+}_\varphi(P_1,\cdots,P_m)(A)},\cdots,\frac{P_m(A)}{\widetilde{+}_\varphi(P_1,\cdots,P_m)(A)}\right).
\end{aligned}
$$

If $\varphi \in \Phi_m$ is a convex function, the above inequality holds with $\le$ replaced by $\ge$.

Assume that φ is strictly concave or strictly convex. Note that $P_j \in \mathcal{M}^{+c}$ has continuous and positive density functions p_j for all $j = 1, \cdots, m$. This yields that

$$
\frac{p_j}{\widetilde{+}_\varphi(p_1,\cdots,p_m)} \quad \text{for all } j = 1,\cdots,m
$$

are positive and continuous on A. Hence, equality holds in (4.6) if and only if there are constants $b_j > 0$, such that, for all $x \in A$ and for all $j = 1, \cdots, m$,

$$
\frac{p_j(x)}{\widetilde{+}_\varphi(p_1,\cdots,p_m)(x)} = b_j.
$$

Equivalently, there are constants $a_j > 0$ such that $p_j = a_j p_1$ for $2 \le j \le m$.

The proof for the case $\varphi \in \Psi_m$ follows along the same lines, and hence is omitted.

□

Corollary 4.2.4. *Let $m \ge 2$ and let $P_j \in \mathcal{M}$ with density functions p_j for $j = 1, \cdots, m$. Assume that $A \subset \Omega$ is measurable such that $P_{j_0}(A) > 0$ for some $j_0 \le m$. If $\mathcal{M}^+ \neq \emptyset$ or $\mu(A) < \infty$, then for any concave function φ in Φ_m such that $\varphi(e_j) = 1$ for $j = 1, \cdots, m$, one has,*

$$
\varphi\left(\frac{P_1(A)}{\widetilde{+}_\varphi(P_1,\cdots,P_m)(A)},\cdots,\frac{P_m(A)}{\widetilde{+}_\varphi(P_1,\cdots,P_m)(A)}\right) \ge 1, \tag{4.7}
$$

while the inequality holds with $\geq$ replaced by $\leq$ if $\varphi \in \Phi_m$ is convex.

Proof. Assume that $\mu(A) < \infty$ and $\varphi \in \Phi_m$ such that $\varphi(e_j) = 1$ for $j = 1, \cdots, m$. Let $p_j \in \mathcal{F}$ be density functions of $P_j \in \mathcal{M}$ for $j = 1, \cdots, m$. Let $\varepsilon > 0$ and p_j^ε be functions defined on A by

$$p_j^\varepsilon(x) = p_j(x) + \varepsilon, \qquad \text{for} \;\; x \in A. \tag{4.8}$$

It is clear that $p_j^\varepsilon \downarrow p_j$ pointwise on A as $\varepsilon \downarrow 0$. By the arguments of (iii) and (iv) in Theorem 4.1.2, we get

$$\widetilde{+}_\varphi(p_1^\varepsilon, \cdots, p_m^\varepsilon) \downarrow \widetilde{+}_\varphi(p_1, \cdots, p_m)$$

pointwise on A as $\varepsilon \downarrow 0$. The Lebesgue dominated convergence theorem (as $\mu(A) < \infty$) implies that, as $\varepsilon \downarrow 0$,

$$\int_A p_j^\varepsilon(x)\, d\mu(x) \downarrow \int_A p_j(x)\, d\mu(x)$$

for $j = 1, \cdots, m$ and

$$\int_A \widetilde{+}_\varphi(p_1^\varepsilon, \cdots, p_m^\varepsilon)(x)\, d\mu(x) \downarrow \int_A \widetilde{+}_\varphi(p_1, \cdots, p_m)(x)\, d\mu(x),$$

where $\int_A \widetilde{+}_\varphi(p_1, \cdots, p_m)(x)\, d\mu(x)$ is bounded by (4.2) and $\int_A \widetilde{+}_\varphi(p_1^\varepsilon, \cdots, p_m^\varepsilon)(x)\, d\mu(x)$ is also bounded when ε is small enough.

The statements (ii) and (iii) in Theorem 4.1.2, together with the assumption that $P_{j_0}(A) > 0$ for some $j_0 \leq m$, imply

$$\begin{aligned} \widetilde{+}_\varphi(P_1, \cdots, P_m)(A) &= \int_A \widetilde{+}_\varphi(p_1, \cdots, p_m)(x)\, d\mu(x) \\ &\geq \int_A \widetilde{+}_\varphi(0, \cdots, 0, p_{j_0}, 0, \cdots, 0)(x)\, d\mu(x) \end{aligned}$$

$$= P_{j_0}(A).$$

Then, for all $\varepsilon > 0$,

$$\int_A \widetilde{+}_\varphi(p_1^\varepsilon, \cdots, p_m^\varepsilon)(x)\, d\mu(x) \geq P_{j_0}(A) > 0.$$

Assume that $\varphi \in \Phi_m$ is concave. By inequality (4.6), one has,

$$1 \leq \varphi\left(\frac{\int_A p_1^\varepsilon(x)\, d\mu(x)}{\int_A \widetilde{+}_\varphi(p_1^\varepsilon, \cdots, p_m^\varepsilon)(x)\, d\mu(x)}, \cdots, \frac{\int_A p_m^\varepsilon(x)\, d\mu(x)}{\int_A \widetilde{+}_\varphi(p_1^\varepsilon, \cdots, p_m^\varepsilon)(x)\, d\mu(x)} \right).$$

Letting $\varepsilon \downarrow 0$ and by the continuity of φ, one gets the desired inequality (4.7).

The proof for the other case $\mathcal{M}^+ \neq \emptyset$ follows along the same lines with p_j^ε in (4.8) replaced by

$$p_j^\varepsilon = p_j + \varepsilon p$$

where p is the density function of any given measure $P \in \mathcal{M}^+$. □

Next we discuss some special cases and applications. The above functional Orlicz-Brunn-Minkowski inequalities for the Orlicz addition of measures are important and have many interesting consequences. We will list some of them in both geometry and analysis.

The first one is the following fundamental dual Orlicz-Brunn-Minkowski inequality for star bodies [30]. See [74] for a spacial case. For $\varphi \in \Phi_m \cup \Psi_m$, let

$$\varphi_n(z) = \varphi(z_1^{1/n}, \ldots, z_m^{1/n})$$

for $z = (z_1, \cdots, z_m) \in [0, \infty)^m$ if $\varphi \in \Phi_m$ and for $z \in (0, \infty)^m$ if $\varphi \in \Psi_m$. Let $V_n(K)$

stand for the n-dimensional volume of K. When K is a star body,

$$V_n(K) = \frac{1}{n}\int_{S^{n-1}} \rho_K^n(u)\, d\sigma(u),$$

with σ the spherical measure on S^{n-1}.

Theorem 4.2.5. *Let $m, n \geq 2$. If $\varphi \in \Phi_m \cup \Psi_m$ such that φ_n is concave, then for all star bodies $K_1, \cdots, K_m$,*

$$\varphi_n\left(\frac{V_n(K_1)}{V_n(\widetilde{+}_\varphi(K_1,\ldots,K_m))},\ldots,\frac{V_n(K_m)}{V_n(\widetilde{+}_\varphi(K_1,\ldots,K_m))}\right) \geq 1,$$

while if φ_n is convex, the inequality holds with $\geq$ replaced by $\leq$.

If φ_n is strictly concave (or convex, as appropriate), equality holds if and only if there exist constants $a_j > 0$ such that $\rho_{K_j} = a_j \rho_{K_1}$ for $j = 2, \ldots, m$.

In fact, Theorem 4.2.5 follows from Theorem 4.2.3 directly by letting $\Omega = S^{n-1}$, $\mu = \sigma$, $p_j = \rho^n_{K_j}$ for $j = 1, \ldots, m$, and by using the fact that

$$\left[\rho_{\widetilde{+}_\varphi(K_1,\ldots,K_m)}\right]^n = \widetilde{+}_{\varphi_n}(\rho^n_{K_1}, \cdots, \rho^n_{K_m}).$$

Let $A \subset \Omega$ be a measurable subset with $\mu(A) > 0$. Define

$$\|p\|^s_{s,A} = \int_A p(x)^s\, d\mu(x),$$

for $p \in \mathcal{F}$ if $s > 0$ and for $p \in \mathcal{F}^+$ if $s < 0$. Denote by $\mathcal{L}_{s,A}$ the set of functions with finite $\|\cdot\|^s_{s,A}$, that is, if $p \in \mathcal{L}_{s,A}$, then $\|p\|^s_{s,A} < \infty$. Let $\varphi_s(z) = \varphi(z_1^{1/s}, \cdots, z_m^{1/s})$. We have the following theorem regarding $\|\cdot\|_{s,A}$.

Theorem 4.2.6. *Let $m \geq 2$ and let $p_j \in \mathcal{F}^+$ with $0 < \|p_j\|_{s,A} < \infty$. Let $\varphi \in \Phi_m \cup \Psi_m$*

such that φ_s is concave. Then

$$\varphi\left(\frac{\|p_1\|_{s,A}}{\|\widetilde{+}_\varphi(p_1,\cdots,p_m)\|_{s,A}},\cdots,\frac{\|p_m\|_{s,A}}{\|\widetilde{+}_\varphi(p_1,\cdots,p_m)\|_{s,A}}\right)\geq 1.$$

If φ_s is convex, the inequality holds with $\geq$ replaced by $\leq$.

If φ_s is strictly concave or convex, and $p_1, \cdots, p_m \in \mathcal{F}^{+c}$, equality holds if and only if there are constants $a_j > 0$ such that $p_j(x) = a_j p_1(x)$ for all $x \in A$ and for $2 \leq j \leq m$.

Proof. The desired result follows from Theorem 4.2.3 and the following equality:

$$\begin{aligned}1 &= \varphi\left(\frac{p_1(x)}{\widetilde{+}_\varphi(p_1,\cdots,p_m)(x)},\cdots,\frac{p_m(x)}{\widetilde{+}_\varphi(p_1,\cdots,p_m)(x)}\right)\\ &= \varphi_s\left(\frac{[p_1(x)]^s}{[\widetilde{+}_\varphi(p_1,\cdots,p_m)(x)]^s},\cdots,\frac{[p_m(x)]^s}{[\widetilde{+}_\varphi(p_1,\cdots,p_m)(x)]^s}\right).\end{aligned}$$

□

A special case of Theorem 4.2.6 is the standard Minkowski inequality for the L_s norm of functions with $s \geq 1$. Here, the L_s norm of g is

$$\|g\|_{L^s(A)} = \||g|\|_{s,A}.$$

In fact, let $m = 2$ and $\varphi(x_1, x_2) = x_1 + x_2$, then

$$\begin{aligned}\|g_1+g_2\|_{L^s(A)} &\leq \||g_1|+|g_2|\|_{s,A}\\ &\leq \||g_1|\|_{s,A} + \||g_2|\|_{s,A}\\ &= \|g_1\|_{L^s(A)} + \|g_2\|_{L^s(A)},\end{aligned}$$

where the second inequality follows from Theorem 4.2.6.

A fundamental object in convex geometry is the L_s mixed volume. Define $V_s(K, L)$, the L_s mixed volume of convex bodies K, L with the origin in their interior, by

$$V_s(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L^s(u) h_K^{1-s}(u) \, dS_K(u), \tag{4.9}$$

where S_K on S^{n-1} is the surface area measure of K. Note that $V_s(K, K) = V_n(K)$, the volume of K. Let $\Omega = S^{n-1}$ and $n \cdot d\mu = h_K^{1-s} \, dS_K$, then

$$V_s(K, L) = \|h_L(u)\|_{s, S^{n-1}}^s.$$

Together with Theorem 4.2.6 and formula (4.5), one gets the following Orlicz-Brunn-Minkowski type inequality for the L_s mixed volumes, which is new to the literature of geometry.

Theorem 4.2.7. *Let $m \geq 2$ and let $K, K_1, \cdots, K_m$ be convex bodies with the origin in their interiors. Let $\varphi \in \Phi_m \cup \Psi_m$ be such that φ_s is concave. Then*

$$\varphi_s \left(\frac{V_s(K, K_1)}{V_s(K, +_\varphi(K_1, \cdots, K_m))}, \cdots, \frac{V_s(K, K_1)}{V_s(K, +_\varphi(K_1, \cdots, K_m))} \right) \geq 1.$$

If φ_s is convex, the inequality holds with $\geq$ replaced by $\leq$.

If φ_s is strictly concave or convex, equality holds if and only if there are constants $a_j > 0$ such that $h_{K_j} = a_j h_{K_1}$ for all $2 \leq j \leq m$.

4.3 An interpretation of the f-divergence

A special case of the Orlicz addition of functions $p_1, \cdots, p_m$ in Definition 4.1.1 is the linear Orlicz addition, where φ in formula (4.1) is replaced by

$$\varphi(x_1, \cdots, x_m) = \sum_{j=1}^{m} \alpha_j \varphi_j(x_j),$$

with all $\alpha_j > 0$, and with φ_j either all in Φ_1 or all in Ψ_1. To obtain an interpretation for the f-divergence, we consider $m = 2$, $\alpha_1 = 1$, and $\alpha_2 = \varepsilon > 0$. That is, $\varphi(x_1, x_2) = \varphi_1(x_1) + \epsilon\varphi_2(x_2)$ for $\varphi_1, \varphi_2 \in \Phi_1$ and $p_1 \widetilde{+}_{\varphi,\varepsilon} p_2$ is defined by

$$\varphi_1\left(\frac{p_1(x)}{p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x)}\right) + \varepsilon\varphi_2\left(\frac{p_2(x)}{p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x)}\right) = 1, \tag{4.10}$$

if $p_1(x) + p_2(x) > 0$, and otherwise by 0. We use the same formula for $p_1 \widetilde{+}_{\varphi,\varepsilon} p_2$ if $\varphi_1, \varphi_2 \in \Psi_1$ and $p_1, p_2 \in \mathcal{F}^+$.

The following theorem is needed for our interpretation of the f-divergence. Denote by $(\varphi_1)'_l(1)$ and $(\varphi_1)'_r(1)$ the left and, respectively, the right derivatives of φ_1 at $t = 1$ if they exist. Let $\Phi_1^{(1)}$ and $\Psi_1^{(1)}$ stand for the set of functions $\varphi \in \Phi_1$ and, respectively, $\varphi \in \Psi_1$, such that $\varphi(1) = 1$.

Theorem 4.3.1. *Let $\varphi_1, \varphi_2 \in \Phi_1^{(1)}$ be such that $(\varphi_1)'_l(1)$ exists and is positive. Let $A \subset \Omega$ be measurable with $\mu(A) > 0$, and $p_1 \in \mathcal{F}^+ \cap \mathcal{L}_{s,A}$ and $p_2 \in \mathcal{F} \cap \mathcal{L}_{s,A}$ such that,*

$$\sup_{x \in A} \left(\frac{p_2(x)}{p_1(x)}\right) < a_1$$

for some constant $a_1 < \infty$. Then, for $0 \neq s \in \mathbb{R}$, one has,

$$(\varphi_1)'_l(1) \lim_{\varepsilon \to 0^+} \frac{\|p_1 \widetilde{+}_{\varphi,\varepsilon} p_2\|_{s,A}^s - \|p_1\|_{s,A}^s}{s \cdot \varepsilon} = \int_A \varphi_2\left(\frac{p_2(x)}{p_1(x)}\right) \left[p_1(x)\right]^s d\mu(x). \tag{4.11}$$

If $\varphi_1, \varphi_2 \in \Psi_1^{(1)}$ *satisfy that* $(\varphi_1)'_r(1)$ *exists and is nonzero, and if* $p_1, p_2 \in \mathcal{F}^+ \cap \mathcal{L}_{s,A}$ *such that*

$$\inf_{x\in A}\left(\frac{p_2(x)}{p_1(x)}\right) > a_2$$

for some constant $a_2 > 0$*, then* (4.11) *holds with* $(\varphi_1)'_l(1)$ *replaced by* $(\varphi_1)'_r(1)$*.*

Proof. Let $\varphi_1, \varphi_2 \in \Phi_1^{(1)}$ and $\varepsilon \in (0,1]$. As φ_1 and φ_2 are strictly increasing, one can easily check, by an argument similar to the proof of Theorem 4.1.2 (iii), that for all $\varepsilon \in (0,1]$,

$$p_1(x) \le p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x) \le p_1 \widetilde{+}_{\varphi,1} p_2(x), \quad \text{for all } x \in A.$$

This together formula (4.10) yields for all $x \in A$,

$$\begin{aligned}
1 &= \varphi_1\left(\frac{p_1(x)}{p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x)}\right) + \varepsilon\varphi_2\left(\frac{p_2(x)}{p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x)}\right) \\
&\le \varphi_1\left(\frac{p_1(x)}{p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x)}\right) + \varepsilon\varphi_2\left(\frac{p_2(x)}{p_1(x)}\right) \\
&\le \varphi_1\left(\frac{p_1(x)}{p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x)}\right) + \varepsilon\varphi_2(a_1),
\end{aligned}$$

where we have used the assumption

$$\sup_{x\in A}\left(\frac{p_2(x)}{p_1(x)}\right) < a_1 < \infty.$$

The above assumption also implies that there is a constant $b_1 < \infty$, such that, for all $\varepsilon \in (0,1]$,

$$1 \le \frac{p_1 \widetilde{+}_{\varphi,\varepsilon} p_2}{p_1} \le \frac{p_1 \widetilde{+}_{\varphi,1} p_2}{p_1} < b_1 \quad \text{on } A.$$

Let ε be small enough so that $1 - \varepsilon\varphi_2(a_1) > 0$. Then,

$$\varphi_1^{-1}\big(1 - \varepsilon\varphi_2(a_1)\big) \le \frac{p_1(x)}{p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x)},$$

and hence, for all $x \in A$,

$$0 \le \frac{p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x) - p_1(x)}{p_1(x)} \tag{4.12}$$
$$= \left(\frac{p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x)}{p_1(x)} \right) \cdot \left(1 - \frac{p_1(x)}{p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x)} \right)$$
$$\le b_1 \cdot \left(1 - \varphi_1^{-1} \left(1 - \varepsilon \varphi_2(a_1) \right) \right).$$

Taking $\varepsilon \to 0^+$, (4.12) yields

$$\frac{p_1 \widetilde{+}_{\varphi,\varepsilon} p_2}{p_1} \to 1, \text{ uniformly on } A \text{ as } \varepsilon \to 0^+. \tag{4.13}$$

For convenience, let

$$w(\varepsilon, x) = \frac{p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x) - p_1(x)}{p_1(x)}.$$

Then $w(\varepsilon, x) \to 0^+$ as $\varepsilon \to 0^+$ by (4.13). For $x \in A$, by (4.10) and (4.13),

$$\lim_{\varepsilon \to 0^+} \frac{w(\varepsilon, x)}{\varepsilon} = \lim_{\varepsilon \to 0^+} \left(\frac{p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x)}{p_1(x)} \right) \cdot \lim_{\varepsilon \to 0^+} \frac{\left(1 - \frac{p_1(x)}{p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x)} \right)}{\varepsilon} \tag{4.14}$$
$$= \lim_{\varepsilon \to 0^+} \left(\frac{1 - z(\varepsilon)}{1 - \varphi_1(z(\varepsilon))} \right) \cdot \lim_{\varepsilon \to 0^+} \varphi_2 \left(\frac{p_2(x)}{p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x)} \right)$$
$$= \frac{1}{(\varphi_1)'_l(1)} \cdot \varphi_2 \left(\frac{p_2(x)}{p_1(x)} \right)$$

where we have used

$$z(\varepsilon) = \varphi_1^{-1} \left(1 - \varepsilon \varphi_2 \left(\frac{p_2(x)}{p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x)} \right) \right) \to 1^-$$

as $\varepsilon \to 0^+$ (note that $\varphi_1 \in \Phi_1^{(1)}$ is increasing). This further implies that for all $x \in A$,

$$
\begin{aligned}
(4.15) \qquad 0 &\le \lim_{\varepsilon\to 0^+} \frac{\left[p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x)\right]^s - \left[p_1(x)\right]^s}{s \cdot \varepsilon} \\
&\lim_{\varepsilon\to 0^+} \frac{\left[1 + w(\varepsilon, x)\right]^s - 1}{s \cdot \varepsilon} \cdot \left[p_1(x)\right]^s \\
&= \lim_{\varepsilon\to 0^+} \frac{\left[1 + w(\varepsilon, x)\right]^s - 1}{s \cdot w(\varepsilon, x)} \cdot \lim_{\varepsilon\to 0^+} \frac{w(\varepsilon, x)}{\varepsilon} \cdot \left[p_1(x)\right]^s \\
&= \frac{1}{(\varphi_1)'_l(1)} \cdot \varphi_2\left(\frac{p_2(x)}{p_1(x)}\right) \cdot \left[p_1(x)\right]^s.
\end{aligned}
$$

Moreover, by inequality (4.12) and a calculation similar to (4.15), we get, for $\varepsilon < 1/\varphi_2(a_1)$, for $0 \neq s \in \mathbb{R}$ and for all $x \in A$,

$$
\begin{aligned}
0 &\le \lim_{\varepsilon\to 0^+} \frac{\left[p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x)\right]^s - \left[p_1(x)\right]^s}{s \cdot \varepsilon} \\
&\le \lim_{\varepsilon\to 0^+} \frac{\left[1 + u(\varepsilon)\right]^s - 1}{s \cdot \varepsilon} \cdot \left[p_1(x)\right]^s \\
&= \frac{b_1 \cdot \varphi_2\left(a_1\right)}{(\varphi_1)'_l(1)} \cdot \left[p_1(x)\right]^s,
\end{aligned}
$$

where $u(\varepsilon) = b_1 \cdot \left(1 - \varphi_1^{-1}\left(1 - \varepsilon\varphi_2(a_1)\right)\right)$ and

$$
\lim_{\varepsilon\to 0^+} \frac{u(\varepsilon)}{\varepsilon} = \frac{b_1 \cdot \varphi_2\left(a_1\right)}{(\varphi_1)'_l(1)}
$$

follows from a calculation similar to (4.14). Hence, for $0 \neq s \in \mathbb{R}$, one can find $\varepsilon_0 < 1/\varphi_2(a_1)$, such that, for all $0 < \varepsilon < \varepsilon_0$ and for all $x \in A$,

$$
\frac{\left[p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x)\right]^s - \left[p_1(x)\right]^s}{s \cdot \varepsilon} \le \frac{2b_1 \cdot \varphi_2\left(a_1\right)}{(\varphi_1)'_l(1)} \cdot \left[p_1(x)\right]^s.
$$

Note that $p_1 \in \mathcal{F}^+ \cap \mathcal{L}_{s,A}$, hence

$$
0 \le \int_A \frac{2b_1 \cdot \varphi_2\left(a_1\right)}{(\varphi_1)'_l(1)} \cdot \left[p_1(x)\right]^s d\mu(x) < \infty.
$$

The desired formula (4.11) then follows by the Lebesgue dominant convergent theorem. That is,

$$\begin{aligned}
&(\varphi_1)'_l(1) \lim_{\varepsilon\to 0^+} \frac{\|p_1 \widetilde{+}_{\varphi,\varepsilon} p_2\|^s_{s,A} - \|p_1\|^s_{s,A}}{s\cdot\varepsilon} \\
&= (\varphi_1)'_l(1) \int_A \lim_{\varepsilon\to 0^+} \frac{\big[1+w(\varepsilon,x)\big]^s - 1}{s\cdot\varepsilon} \cdot \big[p_1(x)\big]^s \, d\mu(x) \\
&= \int_A \varphi_2\left(\frac{p_2(x)}{p_1(x)}\right) \big[p_1(x)\big]^s \, d\mu(x).
\end{aligned}$$

The case for $\varphi_1, \varphi_2 \in \Psi_1^{(1)}$ can be proved along the same lines. For completeness, we include a brief proof with modifications emphasized. Assume that

$$\inf_{x\in A} \left(\frac{p_2(x)}{p_1(x)}\right) > a_2 > 0.$$

If $\varphi_1, \varphi_2 \in \Psi_1^{(1)}$ and $\varepsilon \in (0,1]$, then for $x \in A$,

$$\frac{p_1 \widetilde{+}_{\varphi,1} p_2(x)}{p_1(x)} \le \frac{p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x)}{p_1(x)} \le 1,$$

where $p_1, p_2 \in \mathcal{F}^+$. Note that φ_1 and φ_2 are decreasing. Hence, formula (4.10) yields, for all $x \in A$,

$$1 \le \varphi_1\left(\frac{p_1(x)}{p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x)}\right) + \varepsilon\varphi_2(a_2).$$

Let $\varepsilon < 1/\varphi_2(a_2)$. Similar to inequality (4.12), one has,

$$\begin{aligned}
0 &\le \frac{p_1(x) - p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x)}{p_1(x)} \\
&= \left(\frac{p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x)}{p_1(x)}\right) \cdot \left(\frac{p_1(x)}{p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x)} - 1\right) \\
&\le \varphi_1^{-1}\left(1 - \varepsilon\varphi_2(a_2)\right) - 1.
\end{aligned}$$

This yields (4.13) if we let $\varepsilon \to 0^+$. Moreover,

$$\lim_{\varepsilon\to 0^+} \frac{\varphi_1^{-1}(1-\varepsilon\varphi_2(a_2))-1}{\varepsilon} = -\lim_{\varepsilon\to 0^+} \frac{\bar{z}(\varepsilon)-1}{\varphi_1(\bar{z}(\varepsilon))-1} \cdot \varphi_2(a_2) = -\frac{\varphi_2(a_2)}{(\varphi_1)'_r(1)}, \tag{4.16}$$

because $\bar{z}(\varepsilon) = \varphi_1^{-1}(1-\varepsilon\varphi_2(a_2)) \to 1^+$ as $\varepsilon \to 0^+$ (note that φ_1 is decreasing). Hence, one can find $\varepsilon_0 < 1/\varphi_2(a_2)$, such that for all $0 < \varepsilon < \varepsilon_0$ and for all $x \in A$,

$$\frac{\big[p_1(x)\big]^s - \big[p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x)\big]^s}{p_1(x)\cdot\varepsilon} \le -\frac{2\varphi_2(a_1)}{(\varphi_1)'_r(1)} \cdot \big[p_s(x)\big]^s.$$

Follows the calculations for (4.15) and (4.16), one can get, for all $x \in A$,

$$0 \le \lim_{\varepsilon\to 0^+} \frac{\big[p_1(x)\big]^s - \big[p_1 \widetilde{+}_{\varphi,\varepsilon} p_2(x)\big]^s}{s\cdot\varepsilon} = -\frac{1}{(\varphi_1)'_r(1)} \cdot \varphi_2\left(\frac{p_2(x)}{p_1(x)}\right) \cdot \big[p_1(x)\big]^s.$$

The desired formula (4.11) then follows by the Lebesgue dominant convergent theorem. □

Let p_1 and p_2 be density functions of measures $P_1 \in \mathcal{M}^+$ and $P_2 \in \mathcal{M}$ respectively. Consider $A = \Omega$ and $s = 1$. Under the assumptions stated in Theorem 4.3.1, formula (4.11) becomes, if one notices the definition of the f-divergence given in (1.2),

$$\lim_{\varepsilon\to 0^+} \frac{P_1 \widetilde{+}_{\varphi,\varepsilon} P_2(\Omega) - P_1(\Omega)}{\varepsilon} = \frac{1}{(\varphi_1)'_l(1)} \cdot D_{\varphi_2}(P_2, P_1), \tag{4.17}$$

where the measure $P_1 \widetilde{+}_{\varphi,\varepsilon} P_2$ refers to the measure with the density function $p_1 \widetilde{+}_{\varphi,\varepsilon} p_2$. In other words, we provide an interpretation for the f-divergence by the linear Orlicz addition of measures.

Again, with suitable selections of Ω, μ, P_1, P_2 etc, one could lead to many interesting and important results. For a continuous function $\phi : [0, \infty) \to \mathbb{R}$, define $\widetilde{V}_\phi(K, L)$, the dual Orlicz mixed volume of star bodies K and L, by

$$\widetilde{V}_\phi(K, L) = \frac{1}{n} \int_{S^{n-1}} \phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right) [\rho_K(u)]^n \, d\sigma(u). \tag{4.18}$$

The dual Orlicz mixed volume is a central concept in the dual Orlicz-Brunn-Minkowski theory. It can be obtained by formula (4.11), if we let $\Omega = S^{n-1}$, $\mu = \sigma$, $s = n$, $p_1 = \rho_K$, $p_2 = \rho_L$ and the star body $K \widetilde{+}_{\varphi,\varepsilon} L$ given by, for $u \in S^{n-1}$,

$$\varphi_1\left(\frac{\rho_K(u)}{\rho_{K\widetilde{+}_{\varphi,\varepsilon}L}(u)}\right) + \varepsilon\varphi_2\left(\frac{\rho_L(u)}{\rho_{K\widetilde{+}_{\varphi,\varepsilon}L}(u)}\right) = 1.$$

Please see Theorem 5.4 in [30] for more precise statements.

Now we prove the following theorem regarding the L_s mixed volume given by (4.9). Let K, L be convex bodies with the origin in their interiors. Let $\Omega = S^{n-1}$ and $n \cdot d\mu = h_K^{1-s} \, dS_K$. Define the convex body $K +_{\varphi,\varepsilon} L$ by its support function $h_{K+_{\varphi,\varepsilon}L}$, the unique solution of

$$\varphi_1\left(\frac{h_K(u)}{h_{K+_{\varphi,\varepsilon}L}(u)}\right) + \varepsilon\varphi_2\left(\frac{h_L(u)}{h_{K+_{\varphi,\varepsilon}L}(u)}\right) = 1,$$

for $u \in S^{n-1}$ and for convex functions $\varphi_1, \varphi_2 \in \Phi_1^{(1)}$.

Corollary 4.3.2. *Let K, L be convex bodies with the origin in their interiors. Assume that convex functions $\varphi_1, \varphi_2 \in \Phi_1^{(1)}$ satisfy the conditions in Theorem 4.3.1. Then, for $0 \neq s \in \mathbb{R}$,*

$$(\varphi_1)'_l(1) \lim_{\varepsilon \to 0^+} \frac{V_s(K, K +_{\varphi,\varepsilon} L) - V_n(K)}{s \cdot \varepsilon} = V_{\varphi_2}(K, L),$$

where $V_\phi(K,L)$ is the Orlicz ϕ-mixed volume ([29, 67, 71]) defined by

$$V_{\varphi_2}(K,L) = \frac{1}{n}\int_{S^{n-1}} \varphi_2\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u)\, dS_K(u).$$

Proof. Let $\Omega = S^{n-1}$ and $n \cdot d\mu = h_K^{1-s}\, dS_K$. Let $p_1 = h_K$ and $p_2 = h_L$. Note that if K, L are convex bodies, then p_1 and p_2 satisfy the assumptions in Theorem 4.3.1 automatically. The corollary follows immediately from Theorem 4.3.1 and the fact $V_s(K,L) = \|h_L(u)\|_{s,S^{n-1}}^s$. □

In other words, we provide a new interpretation for the Orlicz ϕ-mixed volume, which is different from the one given by [29, 67]:

$$(\varphi_1)'_l(1) \lim_{\varepsilon\to 0^+} \frac{V_n(K +_{\varphi,\varepsilon} L) - V_n(K)}{n\cdot\varepsilon} = V_{\varphi_2}(K,L).$$

It is worth to mention that the Orlicz ϕ-mixed volume is a fundamental object in the Orlicz-Brunn-Minkowski theory for convex bodies; and it plays important roles in, e.g., the Orlicz-Minkowski inequality [29, 67], and the Orlicz affine and geominimal surface areas [71].

4.4 An inequality equivalent to Jensen's inequality

With the linear Orlicz addition of functions, we can prove that the classical Jensen's inequality has an equivalent form. For $\alpha_1, \alpha_2 > 0$, let

$$\varphi(x_1, x_2) = \alpha_1\varphi_1(x_1) + \alpha_2\varphi_2(x_2), \tag{4.19}$$

with φ_1, φ_2 are either both in Φ_1 or both in Ψ_1. For this special φ, the functional Orlicz-Brunn-Minkowski inequality in Theorem 4.2.3 can be rewritten as:

$$\alpha_1\varphi_1\left(\frac{P_1(\Omega)}{\widetilde{+}_\varphi(P_1,P_2)(\Omega)}\right)+\alpha_2\varphi_2\left(\frac{P_2(\Omega)}{\widetilde{+}_\varphi(P_1,P_2)(\Omega)}\right)\geq 1 \tag{4.20}$$

if φ_1, φ_2 are concave; and the direction of the inequality is reversed if φ_1, φ_2 are convex. On the other hand, by Jensen's inequality, one can obtain the following inequality:

$$\begin{aligned} D_\phi(P_2,P_1) &= \int_\Omega \phi\left(\frac{p_2(x)}{p_1(x)}\right)p_1(x)\,d\mu(x) \\ &\leq P_1(\Omega)\cdot\phi\left(\frac{P_2(\Omega)}{P_1(\Omega)}\right), \end{aligned} \tag{4.21}$$

if ϕ is concave; the direction of the inequality is reversed if ϕ is convex. If ϕ is strictly concave or convex and $p_1, p_2 \in \mathcal{F}^{+c}$, equality holds if and only if p_2/p_1 is a constant on Ω.

Note that Hölder's and Jensen's inequalities are special cases of inequality (4.21).

Theorem 4.4.1. *Let $p_1, p_2, \varphi_1, \varphi_2$ satisfy the conditions in Theorem 4.3.1. The functional Orlicz-Brunn-Minkowski inequality (4.20) is equivalent to inequality (4.21) in the following sense: if one of them holds, the other one also holds.*

Moreover, if the convexity or concavity of functions involved is strict and $p_1, p_2 \in \mathcal{F}^{+c}$, these two inequalities have the same characterization for equality.

Proof. We only prove the case when $\varphi_1, \varphi_2 \in \Phi_1^{(1)}$ are concave. The proofs for other cases can be proved along the same lines.

Let φ be as in (4.19) for some constants $\alpha_1, \alpha_2 > 0$. First, recall that $p_1 \in \mathcal{F}^+ \cap \mathcal{L}_{s,\Omega}$. Statements (ii)-(iii) of Theorem 4.1.2 yield

$$0 < P_1 \widetilde{+}_\varphi P_2(\Omega) < \infty,$$

where $P_1\widetilde{+}_\varphi P_2$ is the measure with density function $p_1\widetilde{+}_\varphi p_2$ given by, for $x \in \Omega$,

$$\alpha_1\varphi_1\left(\frac{p_1(x)}{p_1\widetilde{+}_\varphi p_2(x)}\right) + \alpha_2\varphi_2\left(\frac{p_2(x)}{p_1\widetilde{+}_\varphi p_2(x)}\right) = 1. \tag{4.22}$$

Suppose that inequality (4.21) holds true. For the concave functions φ_1, φ_2,

$$\begin{cases} \frac{D_{\varphi_2}(P_2, P_1\widetilde{+}_\varphi P_2)}{P_1\widetilde{+}_\varphi P_2(\Omega)} \le \varphi_2\left(\frac{P_2(\Omega)}{P_1\widetilde{+}_\varphi P_2(\Omega)}\right), \\ \frac{D_{\varphi_1}(P_1, P_1\widetilde{+}_\varphi P_2)}{P_1\widetilde{+}_\varphi P_2(\Omega)} \le \varphi_1\left(\frac{P_1(\Omega)}{P_1\widetilde{+}_\varphi P_2(\Omega)}\right). \end{cases}$$

It can be checked by (4.22) that

$$\begin{aligned} 1 &= \alpha_1\frac{D_{\varphi_1}(P_1, P_1\widetilde{+}_\varphi P_2)}{P_1\widetilde{+}_\varphi P_2(\Omega)} + \alpha_2\frac{D_{\varphi_2}(P_2, P_1\widetilde{+}_\varphi P_2)}{P_1\widetilde{+}_\varphi P_2(\Omega)} \\ &\le \alpha_1\varphi_1\left(\frac{P_1(\Omega)}{\widetilde{+}_\varphi(P_1, P_2)(\Omega)}\right) + \alpha_2\varphi_2\left(\frac{P_2(\Omega)}{\widetilde{+}_\varphi(P_1, P_2)(\Omega)}\right). \end{aligned}$$

That is the desired inequality (4.20) holds.

On the other hand, assume that inequality (4.20) holds for all $\alpha_1, \alpha_2 > 0$, in particular for $\alpha_1 = 1$ and $\alpha_2 = \varepsilon$. Then,

$$\varphi_1\left(\frac{P_1(\Omega)}{P_1\widetilde{+}_{\varphi,\varepsilon} P_2(\Omega)}\right) + \varepsilon\varphi_2\left(\frac{P_2(\Omega)}{P_1\widetilde{+}_{\varphi,\varepsilon} P_2(\Omega)}\right) \ge 1$$

which is equivalent to, for ε small enough,

$$\frac{P_1(\Omega)}{P_1\widetilde{+}_{\varphi,\varepsilon} P_2(\Omega)} \ge \varphi_1^{-1}\left(1 - \varepsilon\varphi_2\left(\frac{P_2(\Omega)}{P_1\widetilde{+}_{\varphi,\varepsilon} P_2(\Omega)}\right)\right).$$

Together with (4.17), one gets,

$$\frac{D_{\varphi_2}(P_2, P_1)}{P_1(\Omega)} = (\varphi_1)'_l(1) \lim_{\varepsilon\to 0^+} \frac{P_1\widetilde{+}_{\varphi,\varepsilon} P_2(\Omega) - P_1(\Omega)}{\varepsilon \cdot P_1(\Omega)}$$

$$
\begin{aligned}
&= (\varphi_1)'_l(1) \lim_{\varepsilon\to 0^+} \frac{1-\left(\frac{P_1(\Omega)}{P_1 \widetilde{+}_{\varphi,\varepsilon} P_2(\Omega)}\right)}{\varepsilon} \\
&\le (\varphi_1)'_l(1) \lim_{\varepsilon\to 0^+} \frac{1-\varphi_1^{-1}\left(1-\varepsilon\varphi_2\left(\frac{P_2(\Omega)}{P_1 \widetilde{+}_{\varphi,\varepsilon} P_2(\Omega)}\right)\right)}{\varepsilon} \\
&= \varphi_2\left(\frac{P_2(\Omega)}{P_1(\Omega)}\right)
\end{aligned}
$$

where the limit in the last equality can be obtained by a calculation similar to (4.14). Hence, inequality (4.21) holds.

Note that if the functions involved are strictly concave and $p_1, p_2 \in \mathcal{F}^{+c}$, these two inequalities have the same characterization for equality; that is, there is a constant $\alpha > 0$ such that $p_1 = \alpha p_2$ on Ω. □

4.5 An optimization problem for the f-divergence and related affine isoperimetric inequalities

A general optimization problem for the Csiszár's f-divergence can be described as follows: for a fixed measure $P_1 \in \mathcal{M}$ and a set of measures $\mathcal{E} \subset \mathcal{M}$, find

$$
\inf_{P_2 \in \mathcal{E}} D_f(P_2, P_1) \quad \text{or} \quad \sup_{P_2 \in \mathcal{E}} D_f(P_2, P_1), \tag{4.23}
$$

where the infimum and supremum depend on the convexity and concavity of f. The optimization problem (4.23) contains many important objects in the information theory as special cases, such as the famous I-divergence geometry of probability distributions (see e.g., the highly cited paper by Csiszár [22]).

With appropriate selections of geometric measures on convex (or star) bodies, the optimization problem (4.23) leads to fundamental geometric notions, for instance, the dual Orlicz affine and geominimal surface areas [72]. Let $\varphi : (0,\infty) \to (0,\infty)$ such that $\varphi_n(t) = \varphi(t^{1/n})$ for all $t \in (0, \infty)$ is decreasing and strictly convex. The dual Orlicz geominimal surface area of a star body K is defined by

$$\widetilde{G}_{\varphi}^{Orlicz}(K) = \inf_{L \in \mathcal{K}} \left\{ n\widetilde{V}_{\varphi}(K, L) \right\}$$

where $\mathcal{K}$ is the set of convex bodies with the following properties: if $L \in \mathcal{K}$, then L is a convex body with its centroid at o and with $V_n(L^\circ) = V_n(B(o, 1))$. Here, $B(o, 1)$ is the unit Euclidean ball of $\mathbb{R}^n$ and L° is the polar body of L defined by

$$L^\circ = \{y \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \quad \text{for all } x \in L\}.$$

Translating to the language of the f-divergence, one can let $\Omega = S^{n-1}$, $\mu = \sigma$ the spherical measure on S^{n-1}, $dP = \rho_K^n \, d\mu$ and $dQ = \rho_L^n \, d\mu$. Then for $\varphi \in \Phi$,

$$\widetilde{G}_{\varphi}^{Orlicz}(K) = \inf_{Q \in \mathcal{E}} D_{\varphi_n}(Q, P),$$

where $\mathcal{E}$ contains all measures $dQ = \rho_L^n \, d\mu$ with $L \in \mathcal{K}$.

An arguably more important concept is the Orlicz geominimal surface area for convex bodies, which can be defined by, if $\varphi(t^{-1/n})$ is convex on $t \in (0, \infty)$,

$$G_{\varphi}^{Orlicz}(K) = \inf_{L \in \mathcal{K}} \left(\int_{S^{n-1}} \varphi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u) \, dS_K(u) \right).$$

Translating to the language of the f-divergence, one can let $\Omega = S^{n-1}$, $\mu = S_K$ the surface area measure of K on S^{n-1}, $dP = h_K \, d\mu$, $dQ = h_L \, d\mu$, and $\mathcal{E}$ be the set containing all measures $dQ = h_L \, d\mu$ with $L \in \mathcal{K}$. Then,

$$G_{\varphi}^{Orlicz}(K) = \inf_{Q \in \mathcal{E}} D_{\varphi}(Q, P).$$

Motivated by the connection between the optimization problem (4.23) and Orlicz affine and geominimal surface areas, we propose the dual functional affine and geominimal surface areas for functions and/or measures. To simplify our arguments, we make the following assumptions (and more general arguments could be made by slight modifications). Let $\Omega = \mathbb{R}^n$, μ be the Lebesgue measure on $\mathbb{R}^n$, and γ_n be the Gaussian function. That is, $\gamma_n(x) = e^{-\frac{\|x\|_2^2}{2}}$ for $x \in \mathbb{R}^n$ where $\|\cdot\|_2$ denotes the usual Euclidean norm on $\mathbb{R}^n$.

For $p \in \mathcal{F}^+$, define $p^\circ_{x_0} : \mathbb{R}^n \to [0,\infty]$, the polar dual function of p with respect to $x_0 \in \mathbb{R}^n$, by

$$p^\circ_{x_0}(y) = \inf_{x\in\mathbb{R}^n} \left(\frac{e^{-\langle x,y\rangle}}{p(x-x_0)} \right).$$

In particular, the polar dual function of $p \in \mathcal{F}^+$ (with respect to o) is

$$p^\circ(y) = \inf_{x\in\mathbb{R}^n} \left(\frac{e^{-\langle x,y\rangle}}{p(x)} \right).$$

Note that $\gamma_n^\circ = \gamma_n$ and hence γ_n can be viewed as the "unit Euclidean ball" of functions (in terms of the polar dual for functions). Consequently, the Gaussian function γ_n serves as the optimizers of many optimization problems in topics, such as, probability theory and information theory.

Let $\mathcal{D} \subset \mathcal{F}^+$ be the set given by

$$\mathcal{D} = \left\{ p \in \mathcal{F}^+ : \mu(p)\mu(p^\circ) \le \big[\mu(\gamma_n)\big]^2 \right\},$$

where for simplicity,

$$\mu(p) = \int_{\mathbb{R}^n} p(x)\, dx.$$

Clearly, $\mathcal{D} \neq \emptyset$ as $\gamma_n \in \mathcal{D}$. Note that the choice of the set $\mathcal{D}$ is not ad-hoc; it comes from the geometry of log-concave functions. In fact, the functional Blaschke-Santaló inequality for log-concave functions (see e.g., [10, 25, 39]) states that for a log-concave function p (where p can be written as $p = e^{-\psi}$ with ψ a convex function), there exists $z_0 \in \mathbb{R}^n$ (indeed z_0 can be assumed to be the center of mass of p) such that

$$\mu(p)\mu(p_{z_0}^{\circ}) \le \big[\mu(\gamma_n)\big]^2 = (2\pi)^n. \tag{4.24}$$

Denote by $\mathcal{L}_c$ the set of all log-concave functions; and clearly all log-concave functions with barycenters at o are in $\mathcal{D}$.

Let $\varphi : (0,\infty) \to (0,\infty)$ be either in Φ or in Ψ with

$$\begin{cases} \Phi = \{\varphi : \varphi \text{ is decreasing and strictly convex on } (0,\infty)\}; \\ \Psi = \{\varphi : \varphi \ \text{ is increasing and strictly concave on } (0,\infty)\}. \end{cases}$$

When we say a measure $Q \in \mathcal{D}$, we mean that Q is a measure whose density q is in $\mathcal{D}$.

Now, we define the dual functional Orlicz affine and geominimal surface areas for functions and/or measures. Write by q the density function for $Q \in \mathcal{M}$.

Definition 4.5.1. *For fixed measure $P \in \mathcal{M}^+$, the dual functional Orlicz affine surface area for P is defined by*

$$\widetilde{\Omega}_{\varphi}^{orlicz}(P) = \inf_{Q\in\mathcal{D}} D_{\varphi}\left(\frac{\mu(q^{\circ})}{\mu(\gamma_n)}Q, P\right) \tag{4.25}$$

for $\varphi \in \Phi$; while for $\varphi \in \Psi$, $\widetilde{\Omega}_{\varphi}^{orlicz}(P)$ is defined similarly but with "inf" replaced by "sup".

In a similar way, with $\mathcal{D}$ replaced by $\mathcal{D} \cap \mathcal{L}_c$, we can define $\widetilde{G}_{\varphi}^{orlicz}(P)$, the dual functional Orlicz geominimal surface area for P.

It can be easily checked that if φ is a constant $\alpha > 0$, then $\widetilde{\Omega}_{\varphi}^{orlicz}(P) = \widetilde{G}_{\varphi}^{orlicz}(P) =$

$\alpha P(\mathbb{R}^n)$ for any measure $P \in \mathcal{M}^+$. It is also clear that

$$\widetilde{\Omega}_\varphi^{orlicz}(P) \leq \widetilde{G}_\varphi^{orlicz}(P)$$

if $\varphi \in \Phi$; while if $\varphi \in \Psi$, $\widetilde{\Omega}_\varphi^{orlicz}(P) \geq \widetilde{G}_\varphi^{orlicz}(P)$.

In general, it is not easy to calculate $\widetilde{\Omega}_\varphi^{orlicz}(P)$ and $\widetilde{G}_\varphi^{orlicz}(P)$, except when P is a Gaussian measure. To this end, for $c > 0$ a constant, let $(\gamma_n \circ c)(x) = \gamma_n(cx)$ for all $x \in \mathbb{R}^n$. Note that $(\gamma_n \circ c)^\circ = \gamma_n \circ c^{-1}$. By letting $q = \gamma_n \circ c$, then

$$\begin{aligned}\widetilde{\Omega}_\varphi^{orlicz}(\gamma_n \circ c) &= \inf_{Q \in \mathcal{D}} D_\varphi\left(\frac{\mu(q^\circ)}{\mu(\gamma_n)} Q, \gamma_n \circ c\right) \\ &\leq \varphi(c^n) \cdot \int_{\mathbb{R}^n} e^{-\frac{\|cx\|_2^2}{2}}\, dx.\end{aligned}$$

On the other hand, as $\varphi \in \Phi$ is convex, Jensen's inequality implies that

$$\begin{aligned}\widetilde{\Omega}_\varphi^{orlicz}(\gamma_n \circ c) &\geq \inf_{Q \in \mathcal{D}} \mu(\gamma_n \circ c) \cdot \varphi\left(\frac{\mu(q)\mu(q^\circ)}{\mu(\gamma_n)\mu(\gamma_n \circ c)}\right) \\ &\geq \inf_{Q \in \mathcal{D}} \varphi(c^n) \cdot \int_{\mathbb{R}^n} e^{-\frac{\|cx\|_2^2}{2}}\, dx \\ &= \varphi(c^n) \cdot \int_{\mathbb{R}^n} e^{-\frac{\|cx\|_2^2}{2}}\, dx,\end{aligned}$$

where the second inequality follows from the definition of $\mathcal{D}$ and the fact that $\varphi \in \Phi$ is decreasing. That is, if $\varphi \in \Phi$, then

$$\begin{aligned}\widetilde{\Omega}_\varphi^{orlicz}(\gamma_n \circ c) &= \varphi(c^n) \cdot \int_{\mathbb{R}^n} e^{-\frac{\|cx\|_2^2}{2}}\, dx \\ &= \left(\frac{\sqrt{2\pi}}{c}\right)^n \cdot \varphi(c^n).\end{aligned} \tag{4.26}$$

This result also holds for $\varphi \in \Psi$. Moreover, if $\varphi \in \Phi \cup \Psi$,

$$\widetilde{G}_{\varphi}^{orlicz}(\gamma_n \circ c) = \left(\frac{\sqrt{2\pi}}{c}\right)^n \cdot \varphi(c^n).$$

Let T be a linear transform on $\mathbb{R}^n$ with determinant ± 1. First of all, for all $p \in \mathcal{F}^+$,

$$\begin{aligned}(p \circ T)^\circ(y) &= \inf_{x \in \mathbb{R}^n} \left(\frac{e^{-\langle x,y\rangle}}{(p \circ T)(x)}\right) \\ &= \inf_{z \in \mathbb{R}^n} \left(\frac{e^{-\langle T^{-1}z,y\rangle}}{p(z)}\right) \\ &= p^\circ(T^{-t}y),\end{aligned}$$

where T^{-1} denotes the inverse of T and T^{-t} the transpose of T^{-1}. An easy argument by the substitution $z = Tx$ yields

$$\mu(p \circ T) = \int_{\mathbb{R}^n} (p \circ T)(x)\, dx = \int_{\mathbb{R}^n} p(z)\, dz.$$

Similarly, $\mu\big((p \circ T)^\circ\big) = \mu(p^\circ)$ and hence $p \circ T \in \mathcal{D}$ if $p \in \mathcal{D}$.

On the other hand, we can check that

$$\begin{aligned}D_\varphi(Q \circ T, P \circ T) &= \int_{\mathbb{R}^n} \varphi\left(\frac{(q \circ T)(x)}{(p \circ T)(x)}\right)(p \circ T)(x)\, dx \\ &= \int_{\mathbb{R}^n} \varphi\left(\frac{q(z)}{p(z)}\right) p(z)\, dz \\ &= D_\varphi(Q, P).\end{aligned}$$

Taking the infimum if $\varphi \in \Phi$ (or supremum if $\varphi \in \Psi$) over $\mathcal{D}$, one gets

$$\widetilde{\Omega}_{\varphi}^{orlicz}(P \circ T) = \widetilde{\Omega}_{\varphi}^{orlicz}(P).$$

In fact, we have proved the following result, which asserts that both $\widetilde{\Omega}_{\varphi}^{orlicz}(\cdot)$ and $\widetilde{G}_{\varphi}^{orlicz}(\cdot)$ are invariant under the volume preserving (invertible) linear transforms.

Theorem 4.5.2. *Let T be a linear transform on $\mathbb{R}^n$ with determinant ± 1. For any $P \in \mathcal{M}^+$, one has,*

$$\widetilde{\Omega}_{\varphi}^{orlicz}(P \circ T) = \widetilde{\Omega}_{\varphi}^{orlicz}(P)$$

where $P \circ T \in \mathcal{M}^+$ is the measure with density function $p \circ T(x) = p(Tx)$ for all $x \in \mathbb{R}^n$; and

$$\widetilde{G}_{\varphi}^{orlicz}(P \circ T) = \widetilde{G}_{\varphi}^{orlicz}(P).$$

The functional affine isoperimetric inequality aims to provide upper and/or lower bounds for an affine invariant functional defined on functions. Here, an affine invariant functional $\mathcal{G} : \mathcal{F}^+ \to \mathbb{R}$ is a functional such that

$$\mathcal{G}(p) = \mathcal{G}(p \circ T)$$

for all invertible linear transform T on $\mathbb{R}^n$ with determinant ± 1. For example, $\mu(p)\mu(p^\circ)$ is an affine invariant functional, and the celebrated functional Blaschke-Santaló inequality (4.24) is a typical example of the functional affine isoperimetric inequality.

Another example of such affine invariant functionals is

$$\mathcal{G}(p) = \widetilde{\Omega}_{\varphi}^{orlicz}(P).$$

The following functional affine isoperimetric inequality provides upper and/or lower bounds for $\widetilde{\Omega}_{\varphi}^{orlicz}(P)$.

Theorem 4.5.3. *For $\varphi \in \Phi$, one has,*

$$\widetilde{G}_{\varphi}^{orlicz}(P) \geq \widetilde{\Omega}_{\varphi}^{orlicz}(P) \geq \widetilde{\Omega}_{\varphi}^{orlicz}(\gamma_n \circ c) = \widetilde{G}_{\varphi}^{orlicz}(\gamma_n \circ c),$$

where $c > 0$ is the constant determined by

$$c = \left(\frac{\mu(\gamma_n)}{\mu(p)}\right)^{1/n}.$$

The inequalities hold for $\varphi \in \Psi$ with "$\geq$" replaced by "$\leq$".

Proof. Note that the function $\varphi \in \Phi$ is decreasing and strictly convex. Jensen's inequality implies that

$$\begin{aligned}
\widetilde{\Omega}_{\varphi}^{orlicz}(P) &= \inf_{Q \in \mathcal{D}} D_{\varphi}\left(\frac{\mu(q^{\circ})}{\mu(\gamma_n)}Q, P\right) \\
&\geq \mu(p) \inf_{Q \in \mathcal{D}} \varphi\left(\frac{\mu(q)\mu(q^{\circ})}{\mu(\gamma_n)\mu(p)}\right) \\
&\geq \mu(p)\varphi\left(\frac{\mu(\gamma_n)}{\mu(p)}\right) \\
&= \varphi(c^n)c^{-n}\mu(\gamma_n) \\
&= \widetilde{\Omega}_{\varphi}^{orlicz}(\gamma_n \circ c)
\end{aligned}$$

where the second equality follows from the fact that φ is decreasing and $\mu(q)\mu(q^{\circ}) \leq \mu(\gamma_n)^2$, and the last equality follows from formula (4.26).

For $\varphi \in \Psi$, which is increasing and strictly concave, Jensen's inequality implies that

$$\begin{aligned}
\widetilde{\Omega}_{\varphi}^{orlicz}(P) &= \sup_{Q \in \mathcal{D}} D_{\varphi}\left(\frac{\mu(q^{\circ})}{\mu(\gamma_n)}Q, P\right) \\
&\leq \mu(p) \sup_{Q \in \mathcal{D}} \varphi\left(\frac{\mu(q)\mu(q^{\circ})}{\mu(\gamma_n)\mu(p)}\right)
\end{aligned}$$

$$\leq \mu(p)\varphi\left(\frac{\mu(\gamma_n)}{\mu(p)}\right)$$
$$= \widetilde{\Omega}_{\varphi}^{orlicz}(\gamma_n \circ c)$$

where the second equality follows from the fact that φ is increasing and $\mu(q)\mu(q^\circ) \leq \mu(\gamma_n)^2$, and the last equality follows from formula (4.26). □

Theorem 4.5.3 states that, among all measures $P \in \mathcal{M}^+$, the dual functional Orlicz affine and geominimal surface areas for $\varphi \in \Phi$ attain their minimums at the Gaussian measures; while if $\varphi \in \Psi$, their maximums are attained at the Gaussian measures.

The following functional affine isoperimetric inequality provides an upper bound for $\widetilde{\Omega}_{\varphi}^{orlicz}(P)$. It states that, among all measures $P \in \mathcal{D}$, the dual functional Orlicz affine surface area for $\varphi \in \Phi$ attain its maximum at the Gaussian measures.

Theorem 4.5.4. *For measures $P \in \mathcal{D}$ and for $\varphi \in \Phi$, one has,*

$$\widetilde{\Omega}_{\varphi}^{orlicz}(P) \leq \widetilde{\Omega}_{\varphi}^{orlicz}(\gamma_n \circ c_1),$$

where $c_1 > 0$ is the constant determined by

$$c_1 = \left(\frac{\mu(p^\circ)}{\mu(\gamma_n)}\right)^{1/n}.$$

Proof. By (4.25) and $P \in \mathcal{D}$, one has, for $\varphi \in \Phi$,

$$\widetilde{\Omega}_{\varphi}^{orlicz}(P) = \inf_{Q \in \mathcal{D}} D_{\varphi}\left(\frac{\mu(q^\circ)}{\mu(\gamma_n)}Q, P\right)$$
$$\leq D_{\varphi}\left(\frac{\mu(p^\circ)}{\mu(\gamma_n)}P, P\right)$$
$$= \mu(p)\varphi\left(\frac{\mu(p^\circ)}{\mu(\gamma_n)}\right)$$
$$\leq \varphi(c_1^n)c_1^{-n}\mu(\gamma_n)$$

$$= \widetilde{\Omega}_{\varphi}^{orlicz}(\gamma_n \circ c_1)$$

where the first inequality follows by letting $Q = P$, the second inequality follows from $\mu(q)\mu(q^\circ) \leq \mu(\gamma_n)^2$, and the last equality follows from formula (4.26). □

Along the same lines, we can prove the following functional affine isoperimetric inequality for $\widetilde{G}_{\varphi}^{orlicz}(P)$. It states that, among all log-concave measures $P \in \mathcal{D}$, the dual functional Orlicz geominimal surface area for $\varphi \in \Phi$ attain its maximum at the Gaussian measures.

Theorem 4.5.5. *Let $P \in \mathcal{D}$ be a log-concave measure whose density function $p \in \mathcal{F}^+$ is a log-concave function. Then, for $\varphi \in \Phi$, one has,*

$$\widetilde{\Omega}_{\varphi}^{orlicz}(P) \leq \widetilde{G}_{\varphi}^{orlicz}(P) \leq \widetilde{G}_{\varphi}^{orlicz}(\gamma_n \circ c_1),$$

where $c_1 > 0$ is the constant given in Theorem 4.5.4.

When $\varphi : (0, \infty) \to (0, \infty)$ is an increasing and strictly convex function, one can also define the dual functional Orlicz affine surface area for $P \in \mathcal{M}$ by

$$\widetilde{\Omega}_{\varphi}^{orlicz}(P) = \inf_{Q \in \mathcal{D}} D_{\varphi}\left(\frac{\mu(q^\circ)}{\mu(\gamma_n)} Q, P\right),$$

and the dual functional Orlicz geominimal surface area for P with $\mathcal{D}$ replaced by $\mathcal{D} \cap \mathcal{L}_c$. These functionals are again affine invariant, but we are not able to calculate $\widetilde{\Omega}_{\varphi}^{orlicz}(\gamma_n \circ c)$ and $\widetilde{G}_{\varphi}^{orlicz}(\gamma_n \circ c)$ precisely. However, we are still able to prove the following functional affine isoperimetric inequalities.

Theorem 4.5.6. *Let $\varphi : (0,\infty) \to (0,\infty)$ be an increasing and strictly convex function. For measures $P \in \mathcal{D}$, one has,*

$$\begin{aligned} \widetilde{\Omega}_{\varphi}^{orlicz}(P) &\leq \mu(p)\varphi\left(\frac{\mu(p^{\circ})}{\mu(\gamma_n)}\right) \\ &\leq \varphi(c_1^n)c_1^{-n}\mu(\gamma_n), \end{aligned}$$

where $c_1 > 0$ is the constant given in Theorem 4.5.4.

These inequalities also hold for the dual functional Orlicz geominimal surface area if in addition $P \in \mathcal{D}$ is a log-concave measure.

Chapter 5

Cordes-Nirenberg's embedding and restriction with application to an elliptic equation

5.1 Definitions of the function spaces

The function spaces are defined as follows:

Definition 5.1.1. *Let* $(p, \lambda) \in [1, \infty) \times [0, n]$. *The Morrey space* $L^{p,\lambda}$ *consists of all* $f \in L^p_{\mathrm{loc}}$ *such that*

$$\|f\|_{L^{p,\lambda}} = \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} \left(r^{\lambda-n} \int_{B(x,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}} < \infty,$$

where L^p_{loc} *denotes the class of all functions* f *with* $|f|^p$ *being locally integrable with respect to the Lebesgue measure* dy.

Note that if $\lambda = n$, $L^{p,n}$ is just the L^p space.

Definition 5.1.2. *Let* $(s, \kappa) \in [1, \infty) \times [0, n]$.

(i) The space $H^{s,\kappa}$ *consists of all* $f \in L^s_{\mathrm{loc}}$ *with*

$$\|f\|_{H^{s,\kappa}} = \inf_{\omega \in B_1^{n-\kappa}} \left(\int_{\mathbb{R}^n} |f(y)|^s \big(\omega(y)\big)^{1-s} \, dy \right)^{\frac{1}{s}} < \infty,$$

where $B_1^{n-\kappa}$ *comprises all nonnegative functions* w *on* $\mathbb{R}^n$ *obeying*

$$\int_{\mathbb{R}^n} \omega \, d\Lambda^{(\infty)}_{n-\kappa} = \int_0^\infty \Lambda^{(\infty)}_{n-\kappa}(\{y \in \mathbb{R}^n : \omega(y) > t\}) \, dt \le 1$$

where

$$\Lambda^{(\infty)}_{n-\kappa}(E) = \inf \sum_j r_j^{n-\kappa}$$

is the $(n-\kappa)$*-dimensional Hausdorff capacity of* $E \subset \mathbb{R}^n$*, where the infimum is taken over all countable coverings of* E *by open balls of radius* r_j*.*

(ii) The space $H^{\infty,\kappa}$ *consists of all* $f \in L^\infty_c$ *satisfying*

$$\|f\|_{H^{\infty,\kappa}} = \inf_{\omega \in B_1^{n-\kappa}} \|f\omega^{-1}\|_{L^\infty} < \infty,$$

where L^∞_c *is the class of all* L^∞*-functions with compact support.*

Note that if $s = 1$ then $H^{1,k}$ is just the space L^1. For $s \in (0, \infty]$, the space $H^{s,\kappa}$ is an associate Morrey space (cf. [5, 57, 69, 12]):

$$(L^{s',\kappa})^* = H^{s,\kappa} \quad \& \quad (H^{s,\kappa})^* = L^{s',\kappa},$$

where $s' = \frac{s}{s-1}$ is the conjugate number of s and

$$\begin{cases} (L^{s',\kappa})^* = \left\{ f \in L^s_{\mathrm{loc}} : \|f\|_{(H^{s,\kappa})^*} = \sup_{\|g\|_{L^{s',\kappa}} \le 1} \int_{\mathbb{R}^n} f(y)g(y)\, dy \right\}; \\ (H^{s,\kappa})^* = \left\{ f \in L^{s'}_{\mathrm{loc}} : \|f\|_{(H^{s,\kappa})^*} = \sup_{\|g\|_{H^{s,\kappa}} \le 1} \int_{\mathbb{R}^n} f(y)g(y)\, dy \right\}. \end{cases}$$

Let μ be a Radon measure on $\mathbb{R}^n$ with

$$|||\mu|||_\beta = \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} r^{-\beta}\mu(B(x,r)),$$

where $B(x,r)$ denotes the Euclidean ball centered at x with radius r. We say μ is admissible if $\exists k > 0$ such that $\mu(B_1) \approx \mu(B_2)$, where $B_1, B_2 \subsetneqq \mathbb{R}^n$ are two balls with the same radius $r > 0$ and Euclidean distance $\mathrm{dist}(B_1, B_2) = kr$.

Definition 5.1.3. *Let $(q,\eta) \in [1,\infty)\times[0,\infty)$ and μ be a Radon measure on $\mathbb{R}^n$. The μ-based Campanato space $\mathcal{L}_\mu^{q,\eta}$ consists of all $f \in L^q_{\mathrm{loc},\mu}$ satisfying*

$$\|f\|_{\mathcal{L}_\mu^{q,\eta}} = \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} \left(r^{\eta-n}\int_{B(x,r)} |f(y) - f_{B(x,r),\mu}|^q \, d\mu(y)\right)^{\frac{1}{q}} < \infty,$$

where

$$f_{B(x,r),\mu} = \big(\mu(B(x,r))\big)^{-1}\int_{B(x,r)} f \, d\mu$$

and $L^q_{\mathrm{loc},\mu}$ denotes the class of all functions f with $|f|^p$ being locally μ-integrable. When μ is the Lebesgue measure on $\mathbb{R}^n$, $\mathcal{L}_\mu^{q,\eta}$ and its special cases BMO (the John-Nirenberg space of all functions with bounded mean oscillation) and C^γ (the space of all Hölder continuous functions with order $\gamma \in (0,1]$) can be found in [32, Chapter III].

For the Dirichlet problem (5.17) based on a bounded open set $\Omega \subset \mathbb{R}^n$ with C^2 boundary $\partial\Omega$, we need the function spaces $L^{p,\lambda}(\Omega)$, $\mathcal{L}_\mu^{q,\eta}(\Omega)$ and $CN^{p,\tilde{\alpha}}(\Omega)$ which are obtained via substituting Ω and $B(x,r)\cap\Omega$ for $\mathbb{R}^n$ and $B(x,r)$ in Definitions 5.1.1, 5.1.3 and 2.4.1, respectively.

Definition 5.1.4. *Let $\Omega \subsetneqq \mathbb{R}^n$ be an open bounded set with C^2 boundary, $k \in \mathbb{N}$ and $p \in [1,\infty)$. The Sobolev space $W^{k,p}(\Omega)$ (respectively $W_0^\infty(\Omega)$) is the closure of $C^\infty(\Omega)$, the set of all the smooth functions on Ω (respectively $C_0^\infty(\Omega)$ the set of all the smooth* *functions on* Ω *with compact support), with respect to the norm*

$$\|f\|_{W^{p,k}(\Omega)} = \|f\|_{L^p(\Omega)} + \left\| \left(\sum_{|\vec{\alpha}|=k} |D^{\vec{\alpha}} f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)},$$

where $f \in C^\infty(\Omega)$ *(respectively* $C_0^\infty(\Omega)$*),* $\vec{\alpha} = (\alpha_1, \cdots, \alpha_n)$ *is a multi-index with* $|\vec{\alpha}| = \sum_{j=1}^n \alpha_j$ *and*

$$D^{\vec{\alpha}} f = \frac{\partial^{|\vec{\alpha}|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Moreover, for $\tilde{\alpha} \in [0, n)$ *let* $W^{k,p,(\tilde{\alpha})}(\Omega)$ *consist of all functions* $f \in W^{k,p}(\Omega)$ *with* $D^\alpha f \in CN^{p,\tilde{\alpha}}(\Omega)$ *for* $|\vec{\alpha}| = k$ *and*

$$\|f\|_{W^{k,p,(\tilde{\alpha})}(\Omega)} = \|f\|_{L^p(\Omega)} + \left\| \left(\sum_{|\vec{\alpha}|=k} |D^{\vec{\alpha}} f|^2 \right)^{\frac{1}{2}} \right\|_{CN^{p,\tilde{\alpha}}(\Omega)} < \infty.$$

Next, for $\alpha \in (0, n)$ let I_α be the Riesz operator:

$$I_\alpha f(y) = \int_{\mathbb{R}^n} \frac{f(x)}{|x-y|^{n-\alpha}}\, dx,$$

where f is a Lebesgue measurable function. Note that if $\Gamma(\cdot)$ is the standard gamma function then

$$u = \left(\frac{\Gamma((n-\alpha)/2)}{\pi^{n/2} 2^\alpha \Gamma(\alpha/2)} \right) I_\alpha f$$

solves the $\frac{\alpha}{2}$-th order Laplace equation

$$(-\Delta)^{\frac{\alpha}{2}} u = f \tag{5.1}$$

under the Fourier transform (see [4]). Moreover, we refer to [61, 38, 60] for a vast amount of theory developed for the fractional equation (5.1).

5.2 Cordes-Nirenberg's embedding

The aim of this section is to establish the Cordes-Nirenberg's embedding.

Theorem 5.2.1. *If* $\exists m > 0$ *such that*

$$(5.2) \qquad \begin{cases} 1 \le s \le \infty; \\ 1 \le q \le p < \infty; \\ 0 \le \tilde{\alpha} < n; \\ 0 < \lambda \le n; \\ 0 < \kappa < n; \\ pn \le (1+m)^{-1}(1+ms)(n-\tilde{\alpha}), \end{cases}$$

then

(i) $CN^{p,\tilde{\alpha}} \subseteq L^{q,\lambda} \Longleftrightarrow \frac{p}{q} = \frac{n-\tilde{\alpha}}{\lambda}$.

(ii) $H^{s,\kappa} \subseteq CN^{p,\tilde{\alpha}} \Longleftrightarrow \frac{p\kappa}{s'} = n(p-1) + \tilde{\alpha}$.

In order to do this, we first investigate the embeddings between Cordes-Nirenberg spaces and Morrey spaces.

Lemma 5.2.2. *If*

$$(5.3) \qquad \begin{cases} 0 \le \widetilde{\alpha} < n; \\ 0 < \lambda \le n; \\ 1 \le q \le p < \infty; \\ \frac{p}{q} = \frac{n-\widetilde{\alpha}}{\lambda}, \end{cases}$$

then $CN^{p,\widetilde{\alpha}} \subseteq L^{q,\lambda}$. *Additionally,* $CN^{p,\widetilde{\alpha}} \subsetneq L^{q,\lambda}$ *if and only if* $\lambda \in (0,n)$.

Proof. Assume (5.3) holds. If

$$f \in CN^{p,\widetilde{\alpha}} \;\; \& \;\; (x,y,z,r) \in \mathbb{R}^n \times B(x,r) \times B(x,r) \times (0,\infty),$$

then $|y-z| \le 2r$ and hence an application of the Hölder inequality with $1 \le p/q$ gives

$$\begin{aligned}
\left(r^{\lambda-n}\int_{B(x,r)} |f|^q \, dv\right)^{\frac{1}{q}} &\lesssim r^{\frac{\lambda-n}{q}} \left(\int_{B(x,r)} |f|^p \, dv\right)^{\frac{1}{p}} \left(\int_{B(x,r)} dv\right)^{\frac{1}{q}-\frac{1}{p}} \\
&\approx \left(r^{-\widetilde{\alpha}}\int_{B(x,r)} |f|^p \, dv\right)^{\frac{1}{p}} \\
&\lesssim \left(\int_{B(x,r)} \frac{|f(z)|^p}{|z-y|^{\widetilde{\alpha}}} \, dz\right)^{\frac{1}{p}} \\
&\lesssim \left(\int_{\mathbb{R}^n} \frac{|f(z)|^p}{|z-y|^{\widetilde{\alpha}}} \, dz\right)^{\frac{1}{p}}.
\end{aligned} \tag{5.4}$$

Taking the supremum over x, r, y to both sides of (5.4), we have $\|f\|_{L^{q,\lambda}} \lesssim \|f\|_{CN^{p,\widetilde{\alpha}}}$ and so $CN^{p,\widetilde{\alpha}} \subseteq L^{q,\lambda}$.

Additionally, if $\lambda \in (0,n)$, let $g(z) = |z|^{-\frac{\lambda}{q}}$, then, by Theorem 2.2.1, we have

$$\begin{aligned}
\|g\|_{L^{q,\lambda}_{\mu}} &= \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} \left(r^{\lambda-n}\int_{B(x,r)} |z|^{-\lambda} \, dz\right)^{\frac{1}{q}} \\
&= \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} \left(r^{\lambda-n} I_\lambda(B(x,r), B(o,1); o)\right)^{\frac{1}{q}} \\
&\le \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} \left(r^{\lambda-n} V_\lambda(B(x,r), B(o,1))\right)^{\frac{1}{q}} \\
&\approx \sup_{r\in(0,\infty)} \left(r^{\lambda-n} r^{n-\lambda}\right)^{\frac{1}{q}} \\
&\approx 1,
\end{aligned}$$

and

$$\sup_{y\in\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|g(z)|^p}{|z-y|^{\widetilde{\alpha}}}\,dz\right)^{\frac{1}{p}}\geq\left(\int_{\mathbb{R}^n}\frac{|g(z)|^p}{|z|^{\widetilde{\alpha}}}\,dz\right)^{\frac{1}{p}}=\left(\int_{\mathbb{R}^n}|z|^{-\frac{\lambda p}{q}-\widetilde{\alpha}}\,dz\right)^{\frac{1}{p}}=\infty,$$

which implies $g\notin CN^{p,\widetilde{\alpha}}$ and hence $CN^{p,\widetilde{\alpha}}\subsetneqq L^{q,\lambda}$.

On the other hand, if $\lambda=n$, by (5.3), we have

$$1\leq\frac{p}{q}=\frac{n-\widetilde{\alpha}}{n}\leq 1,$$

which implies $\widetilde{\alpha}=0$ and $p=q$, then

$$CN^{p,\widetilde{\alpha}}=L^p=L^q=L^{q,\lambda}$$

by the Definitions 5.1.1 & 2.4.1. Hence, if $CN^{p,\widetilde{\alpha}}\subsetneqq L^{q,\lambda}$, then $\lambda\in(0,n)$. □

For the necessaries of the embeddings between Cordes-Nirenberg spaces and Morrey spaces, we have the following lemma.

Lemma 5.2.3. *Let*

$$\begin{cases}1\leq p,q<\infty;\\ 0\leq\widetilde{\alpha}<n;\\ 0\leq\lambda\leq n.\end{cases}$$

If $CN^{p,\widetilde{\alpha}}\subseteq L^{q,\lambda}$ or $L^{q,\lambda}\subseteq CN^{p,\widetilde{\alpha}}$, then $p\lambda=(n-\widetilde{\alpha})q$.

Proof. **Case $CN^{p,\widetilde{\alpha}}\subseteq L^{q,\lambda}$:** $\forall\widetilde{r}>0$, let $f=\widetilde{r}^{\frac{\widetilde{\alpha}-n}{p}}\chi_{B(o,\widetilde{r})}$, then, by Theorem 2.2.1, it follows that

$$\|f\|^p_{CN^{p,\widetilde{\alpha}}}=\sup_{y\in\mathbb{R}^n}\widetilde{r}^{\widetilde{\alpha}-n}\int_{B(o,\widetilde{r})}\frac{1}{|x-y|^{\widetilde{\alpha}}}\,dx=\widetilde{r}^{\widetilde{\alpha}-n}V_{\widetilde{\alpha}}(B(o,\widetilde{r}),B(o,1))\lesssim\widetilde{r}^{\widetilde{\alpha}-n}\widetilde{r}^{n-\widetilde{\alpha}}\approx 1,$$

which implies $f \in CN^{p,\widetilde{\alpha}}$ and hence $f \in L^{q,\lambda}$. Note that

$$\begin{aligned}
(5.5)\qquad \|f\|^q_{L^{q,\lambda}} &= \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} r^{\lambda-n}\int_{B(x,r)} |\widetilde{r}^{\frac{\widetilde{\alpha}-n}{p}}\chi_{B(o,\widetilde{r})}|^q\,dv \\
&= \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} r^{\lambda-n}\widetilde{r}^{\frac{q(\widetilde{\alpha}-n)}{p}} V\left(B(x,r)\cap B(o,\widetilde{r})\right) \\
&= \max\Bigg\{ \sup_{(x,r)\in\mathbb{R}^n\times(0,\widetilde{r}]} \frac{r^{\lambda-n}}{\widetilde{r}^{\frac{q(n-\widetilde{\alpha})}{p}}} V\left(B(x,r)\cap B(o,\widetilde{r})\right), \\
&\qquad \sup_{(x,r)\in\mathbb{R}^n\times(\widetilde{r},\infty)} \frac{r^{\lambda-n}}{\widetilde{r}^{\frac{q(n-\widetilde{\alpha})}{p}}} V\left(B(x,r)\cap B(o,\widetilde{r})\right)\Bigg\} \\
&\approx \max\left\{ \sup_{r\in(0,\widetilde{r}]} r^{\lambda}\widetilde{r}^{\frac{q(\widetilde{\alpha}-n)}{p}},\ \sup_{r\in(\widetilde{r},\infty)} r^{\lambda-n}\widetilde{r}^{\frac{q(\widetilde{\alpha}-n)}{p}+n}\right\} \\
&\approx \widetilde{r}^{\lambda+\frac{q(\widetilde{\alpha}-n)}{p}},
\end{aligned}$$

which, together with $f \in L^{q,\lambda}$, implies $\lambda + \frac{q(\widetilde{\alpha}-n)}{p} = 0$ and hence $p\lambda = (n-\widetilde{\alpha})q$, since $\widetilde{r} > 0$ is arbitrary.

Case $L^{q,\lambda} \subseteq CN^{p,\widetilde{\alpha}}$: $\forall \widetilde{r} > 0$, let $g = \widetilde{r}^{-\frac{\lambda}{q}}\chi_{B(o,\widetilde{r})}$, then by a similar way as in (5.5), we have

$$\|f\|^q_{L^{q,\lambda}} \approx \max\left\{ \sup_{r\in(0,\widetilde{r}]} \left(\frac{r}{\widetilde{r}}\right)^{\frac{\lambda}{q}},\ \sup_{r\in(\widetilde{r},\infty)} \left(\frac{r}{\widetilde{r}}\right)^{\frac{n-\lambda}{q}}\right\} \approx 1,$$

which implies $g \in L^{q,\lambda}$ and hence $g \in CN^{p,\widetilde{\alpha}}$. By the equality case in Theorem 2.2.1, it follows that

$$\|g\|^p_{CN^{p,\widetilde{\alpha}}} = \sup_{y\in\mathbb{R}^n} \widetilde{r}^{-\frac{\lambda p}{q}} \int_{B(o,\widetilde{r})} \frac{1}{|x-y|^{\widetilde{\alpha}}}\,dx = \widetilde{r}^{-\frac{\lambda p}{q}} V_{\widetilde{\alpha}}(B(o,\widetilde{r}), B(o,1)) \approx \widetilde{r}^{-\frac{\lambda p}{q}+n-\widetilde{\alpha}},$$

which implies $-\frac{\lambda p}{q} + n - \widetilde{\alpha} = 0$ and hence $p\lambda = (n-\widetilde{\alpha})q$, since $\widetilde{r} > 0$ is arbitrary. □

Proof of Theorem 5.2.1: $CN^{p,\widetilde{\alpha}} \subseteq L^{q,\lambda}$. This follows immediately from Lemmas 5.2.2 & 5.2.3. □

Next, in order to completely establish associate Morrey spaces embedding $H^{s,k} \subseteq$

$CN^{p,\widetilde{\alpha}}$ for all $s \in [1, \infty]$, we first need a lemma (see [69, Lemma 4.1]) for $H^{\infty,k}$, $k \in (0, n)$.

Lemma 5.2.4. *For $k \in (0, n)$, let $\mathrm{L}^{1,\beta}$ be the Morrey space of all signed Radon (locally finite regular signed Borel) measure μ whose total variation measures $|\mu| \equiv \widetilde{\mu}$ obey $\||\mu|\|_k < \infty$, and set $L^1_{\Lambda_k^\infty}$ be the class of all Λ_k^∞-quasi continuous functions f on $\mathbb{R}^n$ for which*

$$\|f\|_{L^1_{\Lambda_k^\infty}} = \int_{\mathbb{R}^n} |f| \, d\Lambda_k^\infty < \infty.$$

If

$$\left[L^1_{\Lambda_k^\infty}\right]^* = \left\{ \text{signed Radon measure } \mu : \|\mu\|_{[L^1_{\Lambda_k^\infty}]^*} = \sup_{\|g\|_{L^1_{\Lambda_k^\infty}} \le 1} \left| \int_{\mathbb{R}^n} g \, d\mu \right| < \infty \right\}$$

and

$$\left[H^{\infty,k}\right]^* = \left\{ f \in L^1_{\mathrm{loc}} : \|f\|_{[H^{\infty,k}]^*} = \sup_{\|g\|_{H^{\infty,k}} \le 1} \left| \int_{\mathbb{R}^n} f g \, dv \right| < \infty \right\},$$

then

$$\left[L^1_{\Lambda_k^\infty}\right]^* = \mathrm{L}^{1,\beta} \quad \& \quad \left[H^{\infty,k}\right]^* = L^{1,\beta}.$$

Consequently, $L^1_{\Lambda_{n-k}^\infty}$ exists as a subspace of $H^{\infty,k}$.

There are atom decompositions for $H^{s,k}$ spaces, $s \in (1, \infty)$ and $L^1_{\Lambda_k^\infty}$ spaces (see [5, Theorem 3.3] and [5, Remark 3.4] respectively).

Lemma 5.2.5. *Let $s \times k \in (1, \infty) \times (0, n)$.*

(i) If $f \in H^{s,k}$, then

$$\|f\|_{H^{s,k}} \approx \inf \left\{ \sum_j |c_j| : f = \sum_j c_j a_j \right\},$$

where the infimum is taken for all such decompositions and a_j is a $(s,k)-$atom, in other words, a_j is supported in a ball $B_j \subseteq \mathbb{R}^n$ with radius r_j and satisfies

$$\|a_j\|_{L^p} \leq (V(B_j))^{\frac{k-n}{ns'}} \approx r_j^{\frac{k-n}{s'}}.$$

(ii) If $f \in L^1_{\Lambda_k^\infty}$, then

$$\|f\|_{L^1_{\Lambda_k^\infty}} \approx \inf\left\{\sum_j |d_j| : f = \sum_j d_j b_j\right\},$$

where the infimum is taken for all such decompositions and b_j is a $(\infty,k)-$atom, in other words, b_j is supported in a ball $\widetilde{B}_j \subseteq \mathbb{R}^n$ with radius $\widetilde{r}_j$ and satisfies

$$\|b_j\|_{L^\infty} \leq (V(\widetilde{B}_j))^{\frac{k-n}{n}} \approx \widetilde{r}_j^{k-n}.$$

Proof of Theorem 5.2.1: $H^{s,k} \subseteq CN^{p,\widetilde{\alpha}}$. Under the given conditions (5.2), we consider three cases as seen below.

Case $1 < s < \infty$: Note that

$$\exists m > 0,\ pn \leq \frac{(1+ms)(n-\widetilde{\alpha})}{1+m} \Leftrightarrow pn < s(n-\widetilde{\alpha}),$$

hence

$$\begin{cases} p < s; \\ \frac{s\widetilde{\alpha}}{s-p} < \frac{s\left(n-\frac{pn}{s}\right)}{s-p} < n. \end{cases}$$

Then, if $pk = s'(n(p-1)+\widetilde{\alpha})$, $\forall f \in H^{s,k}$, by duality, Lemma 5.2.5 (i), Hölder's inequality and Theorem 2.2.1, we have

$$\begin{aligned}
\|f\|_{CN^{p,\widetilde{\alpha}}} &= \sup_{y\in\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|f(x)|^p}{|x-y|^{\widetilde{\alpha}}}\,dx\right)^{\frac{1}{p}}\\
&= \sup_{y\in\mathbb{R}^n}\sup_{\|g\|_{L^{p'}}\le 1}\int_{\mathbb{R}^n}\frac{g(x)|f(x)|}{|x-y|^{\frac{\widetilde{\alpha}}{p}}}\,dx\\
&\le \sum_j |c_j| \sup_{y\in\mathbb{R}^n}\sup_{\|g\|_{L^{p'}}\le 1}\int_{B_j}\frac{|g(x)a_j(x)|}{|x-y|^{\frac{\widetilde{\alpha}}{p}}}\,dx\\
&\le \sum_j |c_j|\|a_j\|_{L^s}\sup_{y\in\mathbb{R}^n}\sup_{\|g\|_{L^{p'}}\le 1}\left(\int_{B_j}\frac{|g(x)|^{s'}}{|x-y|^{\frac{\widetilde{\alpha}s'}{p}}}\,dx\right)^{\frac{1}{s'}}\\
&\le \sum_j |c_j| r_j^{\frac{k-n}{s'}}\sup_{y\in\mathbb{R}^n}\sup_{\|g\|_{L^{p'}}\le 1}\|g\|_{L^{p'}}^{s'}\left(\int_{B_j}\frac{1}{|x-y|^{\frac{\widetilde{\alpha}s'}{p}\left(\frac{p'}{s'}\right)'}}\,dx\right)^{\left(s'\left(\frac{p'}{s'}\right)'\right)^{-1}}\\
&= \sum_j |c_j| r_j^{\frac{k-n}{s'}}\sup_{y\in\mathbb{R}^n}\left(\int_{B_j}\frac{1}{|x-y|^{\frac{s\widetilde{\alpha}}{s-p}}}\,dx\right)^{\frac{s-p}{sp}}\\
&= \sum_j |c_j| r_j^{\frac{k-n}{s'}}\left(V_{\frac{s\widetilde{\alpha}}{s-p}}(B_j, B(o,1))\right)^{\frac{s-p}{sp}}\\
&\lesssim \sum_j |c_j| r_j^{\frac{k-n}{s'}+\left(n-\frac{s\widetilde{\alpha}}{s-p}\right)\frac{s-p}{sp}}\\
&\approx \sum_j |c_j|,
\end{aligned}$$

which implies $\|f\|_{CN^{p,\widetilde{\alpha}}} \lesssim \|f\|_{H^{s,k}}$ by taking the infimum to both sides. Hence $H^{s,k} \subseteq CN^{p,\widetilde{\alpha}}$.

On the other hand, let $r > 0$ and

$$f_0 = r^{\frac{k-n}{s'}-\frac{n}{s}}\chi_{B(o,r)}$$

then f_0 is supported in $B(o,r)$ and

$$\|f_0\|_{L^s} = \left(\int_{B(o,r)} r^{\frac{(k-n)s}{s'}-n}\,dx\right)^{\frac{1}{s}} \approx r^{\frac{k-n}{s'}},$$

which imply f_0 is an $(s,k)-$atom and hence $f_0 \in H^{s,k}$.

If $H^{s,k} \subseteq CN^{p,\widetilde{\alpha}}$, then $f_0 \in CN^{p,\widetilde{\alpha}}$. By the equality case in Theorem 2.2.1, we have

$$\begin{aligned}\|f_0\|^p_{CN^{p,\widetilde{\alpha}}} &= \sup_{y\in\mathbb{R}^n} r^{\frac{(k-n)p}{s'}-\frac{np}{s}} \int_{B(o,r)} \frac{1}{|x-y|^{\widetilde{\alpha}}}\,dx \\ &= r^{\frac{(k-n)p}{s'}-\frac{np}{s}} V_{\widetilde{\alpha}}(B(o,r),B(o,1)) \\ &\approx r^{\frac{(k-n)p}{s'}-\frac{np}{s}+n-\widetilde{\alpha}},\end{aligned}$$

which implies

$$\frac{(k-n)p}{s'} - \frac{np}{s} + n - \widetilde{\alpha} = 0$$

whence

$$pk = s'(n(p-1)+\widetilde{\alpha}),$$

since $r > 0$ is arbitrary.

Case $s=\infty$**:** If

$$pk = s'(n(p-1)+\widetilde{\alpha}) = n(p-1)+\widetilde{\alpha},$$

then $\forall f \in H^{\infty,k}$, $\forall w \in B_1^{n-k} \subseteq L^1_{\Lambda^{\infty}_{n-k}}$, by duality, Lemma 5.2.5 (ii), Hölder's inequality and Theorem 2.2.1, we have

$$
\begin{aligned}
(5.6)\qquad \|f\|_{CN^{p,\widetilde{\alpha}}} &= \sup_{y\in\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|f(x)|^p}{|x-y|^{\widetilde{\alpha}}}\,dx\right)^{\frac{1}{p}}\\
&= \sup_{y\in\mathbb{R}^n}\sup_{\|g\|_{L^{p'}}\le 1}\int_{\mathbb{R}^n}\frac{g(x)|f(x)|}{|x-y|^{\frac{\widetilde{\alpha}}{p}}}\,dx\\
&\le \|fw^{-1}\|_{L^\infty}\sup_{y\in\mathbb{R}^n}\sup_{\|g\|_{L^{p'}}\le 1}\int_{\mathbb{R}^n}\frac{|g(x)w(x)|}{|x-y|^{\frac{\widetilde{\alpha}}{p}}}\,dx\\
&= \|fw^{-1}\|_{L^\infty}\sum_j |d_j|\sup_{y\in\mathbb{R}^n}\sup_{\|g\|_{L^{p'}}\le 1}\int_{\widetilde{B}_j}\frac{|g(x)b_j(x)|}{|x-y|^{\frac{\widetilde{\alpha}}{p}}}\,dx\\
&\le \|fw^{-1}\|_{L^\infty}\sum_j |d_j|\widetilde{r}_j^{\,k-n}\sup_{y\in\mathbb{R}^n}\sup_{\|g\|_{L^{p'}}\le 1}\int_{\widetilde{B}_j}\frac{|g(x)|}{|x-y|^{\frac{\widetilde{\alpha}}{p}}}\,dx\\
&\le \|fw^{-1}\|_{L^\infty}\sum_j |d_j|\widetilde{r}_j^{\,k-n}\sup_{y\in\mathbb{R}^n}\sup_{\|g\|_{L^{p'}}\le 1}\|g\|_{L^{p'}}\left(\int_{\widetilde{B}_j}\frac{1}{|x-y|^{\widetilde{\alpha}}}\,dx\right)^{\frac{1}{p}}\\
&= \|fw^{-1}\|_{L^\infty}\sum_j |d_j|\widetilde{r}_j^{\,k-n}\sup_{y\in\mathbb{R}^n}\left(V_{\widetilde{\alpha}}(\widetilde{B}_j,B(o,1))\right)^{\frac{1}{p}}\\
&\lesssim \|fw^{-1}\|_{L^\infty}\sum_j |d_j|\widetilde{r}_j^{\,k-n+\frac{n-\widetilde{\alpha}}{p}}\\
&\approx \|fw^{-1}\|_{L^\infty}\sum_j |d_j|\\
&\approx \|fw^{-1}\|_{L^\infty},
\end{aligned}
$$

where $\sum_j |d_j| \lesssim 1$ since $\|w\|_{L^1_{\Lambda^\infty_{n-k}}} \le 1$. This implies $\|f\|_{CN^{p,\widetilde{\alpha}}} \lesssim \|f\|_{H^{s,k}}$ by taking the infimum for w to both sides of (5.6), then $H^{\infty,k} \subseteq CN^{p,\widetilde{\alpha}}$.

On the other hand, let $\widetilde{r} > 0$ and

$$
g_0 = \widetilde{r}^{k-n}\chi_{B(o,\widetilde{r})}
$$

then g_0 is supported in $B(o,\widetilde{r})$ and $\|g_0\|_{L^\infty} = \widetilde{r}^{k-n}$, which imply g_0 is a $(\infty, n-k)-$atom and hence $g_0 \in L^1_{\Lambda^\infty_{n-k}}$. By Lemma 5.2.4, $L^1_{\Lambda^\infty_{n-k}}$ exists as a subspace of $H^{\infty,k}$, then $g_0 \in H^{\infty,k}$.

If $H^{\infty,k} \subseteq CN^{p,\widetilde{\alpha}}$, then $g_0 \in CN^{p,\widetilde{\alpha}}$. By the equality case in Theorem 2.2.1, we have

$$\begin{aligned}\|g_0\|^p_{CN^{p,\widetilde{\alpha}}} &= \sup_{y\in\mathbb{R}^n} \widetilde{r}^{p(k-n)} \int_{B(o,\widetilde{r})} \frac{1}{|x-y|^{\widetilde{\alpha}}}\,dx \\ &= \widetilde{r}^{p(k-n)} V_{\widetilde{\alpha}}(B(o,\widetilde{r}), B(o,1)) \\ &\approx \widetilde{r}^{p(k-n)+n-\widetilde{\alpha}},\end{aligned}$$

which implies $p(k-n)+n-\widetilde{\alpha} = 0$ and hence $pk = n(p-1)+\widetilde{\alpha}$, since $\widetilde{r} > 0$ is arbitrary.

Case $s = 1$**:** If $r > 0$ and $f_0 = r^{-n}\chi_{B(o,r)}$, then

$$\|f_0\|_{H^{1,k}} = \|f_0\|_{L^1} = \int_{B(o,r)} r^{-n}\,dx \approx 1,$$

and hence $f_0 \in H^{1,k}$.

If $H^{1,k} \subseteq CN^{p,\widetilde{\alpha}}$, then $f_0 \in CN^{p,\widetilde{\alpha}}$, and hence the equality case in Theorem 2.2.1 implies

$$\|f_0\|^p_{CN^{p,\widetilde{\alpha}}} = \sup_{y\in\mathbb{R}^n} r^{-np} \int_{B(o,r)} \frac{1}{|x-y|^{\widetilde{\alpha}}}\,dx = r^{-np} V_{\widetilde{\alpha}}(B(o,r), B(o,1)) \approx r^{-np+n-\widetilde{\alpha}},$$

which implies

$$-np+n-\widetilde{\alpha} = 0 \ \ \& \ \ n(p-1)+\widetilde{\alpha} = 0 = \frac{pk}{s'}$$

since $r > 0$ is arbitrary.

On the other hand, note that

$$1 \le p < \infty \ \ \& \ \ 0 \le \widetilde{\alpha} < n.$$

So

$$\exists m > 0,\ pn \le \frac{(1+ms)(n-\widetilde{\alpha})}{1+m} = n - \widetilde{\alpha} \Rightarrow p = 1 \ \&\ \widetilde{\alpha} = 0.$$

Hence

$$CN^{p,\widetilde{\alpha}} = CN^{1,0} = L^1 = H^{1,k} = H^{s,k}$$

from Definitions 5.1.2 & 2.4.1.

□

5.3 Cordes-Nirenberg's restricting

In this section, we establish the restriction of Riesz-Cordes-Nirenberg potentials.

Theorem 5.3.1. *Let*

$$(5.7) \qquad \begin{cases} 1 \le p, q < \infty; \\ 0 \le \alpha < n; \\ 0 \le \widetilde{\alpha} = n - \lambda < n; \\ (n-\alpha)q < \beta \le n; \\ \alpha p < p + \lambda; \\ \eta = q\left(\frac{\lambda}{p} - \alpha\right) + n - \beta \le n. \end{cases}$$

(i) If $|||\mu|||_\beta < \infty$, then $I_\alpha : CN^{p,\widetilde{\alpha}} \subseteq L^{p,\lambda} \to \mathcal{L}^{q,\eta}_\mu$ is bounded.

(ii) If $I_\alpha : CN^{p,\widetilde{\alpha}} \subseteq L^{p,\lambda} \to \mathcal{L}^{q,\eta}_\mu$ is bounded and μ is admissible, then $|||\mu|||_\beta < \infty$.

Proof. Assume (5.7) holds. Then $CN^{p,\widetilde{\alpha}} \subseteq L^{p,\lambda}$ follows from Lemma 5.2.2.

(i) Let $\|\|\mu\|\|_\beta < \infty$. It suffices to prove that $\forall B(x,r)$, $\exists c > 0$ such that

$$J = \left(r^{\eta-n} \int_{B(x,r)} |I_\alpha f(y) - c|^q \, d\mu(y) \right)^{\frac{1}{q}} \lesssim \|\|\mu\|\|_\beta^{\frac{1}{q}} \|f\|_{CN^{p,\tilde{\alpha}}}. \tag{5.8}$$

Actually, by the Minkowski's inequality and Hölder's inequality, we have

$$\begin{aligned}
&\left(\int_{B(x,r)} |I_\alpha f(y) - (I_\alpha f)_{B(x,r),\mu}|^q \, d\mu(y) \right)^{\frac{1}{q}} \\
&\leq \left(\int_{B(x,r)} |I_\alpha f(y) - c|^q \, d\mu(y) \right)^{\frac{1}{q}} + (\mu(B(x,r)))^{\frac{1}{q}} |c - (I_\alpha f)_{B(x,r),\mu}| \\
&= \left(\int_{B(x,r)} |I_\alpha f(y) - c|^q \, d\mu(y) \right)^{\frac{1}{q}} \\
&\quad + (\mu(B(x,r)))^{\frac{1}{q}} \left| \frac{1}{\mu(B(x,r))} \int_{B(x,r)} (I_\alpha f(y) - c) \, d\mu(y) \right| \\
&\leq 2 \left(\int_{B(x,r)} |I_\alpha f(y) - c|^q \, d\mu(y) \right)^{\frac{1}{q}},
\end{aligned}$$

which, together with (5.8), implies

$$\begin{aligned}
\|I_\alpha f\|_{\mathcal{L}_\mu^{q,\eta}} &= \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} \left(r^{\eta-n} \int_{B(x,r)} |I_\alpha f(y) - (I_\alpha f)_{B(x,r),\mu}|^q \, d\mu(y) \right)^{\frac{1}{q}} \\
&\leq 2 \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} \inf_{c\in\mathbb{R}} \left(r^{\eta-n} \int_{B(x,r)} |I_\alpha f(y) - c|^q \, d\mu(y) \right)^{\frac{1}{q}} \\
&\lesssim \|\|\mu\|\|_\beta^{\frac{1}{q}} \|f\|_{CN^{p,\tilde{\alpha}}}.
\end{aligned}$$

Since

$$\eta = q\left(\frac{\lambda}{p} - \alpha\right) + n - \beta,$$

if $f_1 = f\chi_{B(x,4r)}$ and $f_2 = f\chi_{B(x,4r)^c}$ then

$$J \lesssim \left(r^{q\left(\frac{\lambda}{p}-\alpha\right)-\beta} \int_{B(x,r)} |I_\alpha f_1|^q \, d\mu \right)^{\frac{1}{q}} + \left(r^{q\left(\frac{\lambda}{p}-\alpha\right)-\beta} \int_{B(x,r)} |I_\alpha f_2(y) - c|^q \, d\mu(y) \right)^{\frac{1}{q}}$$

$=: J_1 + J_2$

For J_1**:** By Minkowski's inequality, Fubini's theorem, (5.7) and Hölder's inequality, we have

$$
\begin{aligned}
& r^{\frac{\beta}{q}+\alpha-\frac{\lambda}{p}} J_1 \\
& \le \left[\int_{B(x,r)} \left(\int_{B(x,4r)} \frac{|f(z)|}{|z-y|^{n-\alpha}}\,dz\right)^q d\mu(y)\right]^{\frac{1}{q}} \\
& \le \int_{B(x,4r)} \left(\int_{B(x,r)} \frac{|f(z)|^q}{|z-y|^{q(n-\alpha)}}\,d\mu(y)\right)^{\frac{1}{q}} dz \\
& \approx \int_{B(x,4r)} |f(z)| \left(\int_{\{x:|x-y|\le r\}} \int_{|x-y|}^{\infty} t^{q(\alpha-n)-1}\,dt\,d\mu(y)\right)^{\frac{1}{q}} dz \\
& \approx \int_{B(x,4r)} |f(z)| \left[\int_0^r t^{q(\alpha-n)-1} \left(\int_{\{x:|x-y|\le t\}} d\mu(y)\right) dt \right. \\
& \quad \left. + \int_r^{\infty} t^{q(\alpha-n)-1} \left(\int_{\{x:|x-y|\le r\}} d\mu(y)\right) dt\right]^{\frac{1}{q}} dz \\
& \approx |||\mu|||_{\beta}^{\frac{1}{q}} \int_{B(x,4r)} |f(z)| \left(\int_0^r t^{\beta+q(\alpha-n)-1}\,dt + r^{\beta}\int_r^{\infty} t^{q(\alpha-n)-1}\,dt\right)^{\frac{1}{q}} dz \\
& \approx r^{\frac{\beta}{q}+\alpha-n} |||\mu|||_{\beta}^{\frac{1}{q}} \int_{B(x,4r)} |f(z)|\,dz \\
& \approx r^{\frac{\beta}{q}+\alpha-\frac{\lambda}{p}} |||\mu|||_{\beta}^{\frac{1}{q}} \left(r^{\lambda-n}\int_{B(x,4r)} |f(z)|^p\,dz\right)^{\frac{1}{p}},
\end{aligned}
$$

which implies

$$
J_1 \lesssim |||\mu|||_{\beta}^{\frac{1}{q}} \|f\|_{L^{p,\lambda}}. \tag{5.9}
$$

For J_2**:** Let

$$
c = \frac{1}{\mu(B(x,r))}\int_{B(x,r)} I_\alpha f_2\,d\mu.
$$

Then, $\forall y \in B(x,r)$, by mean value theorem, (5.7) and Hölder's inequality, it follows that

$$
\begin{aligned}
&|I_\alpha f_2(y) - c| \\
&\quad \le \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |I_\alpha f_2(y) - I_\alpha f_2(z)| \, d\mu(z) \\
&\quad \le \frac{1}{\mu(B(x,r))} \int_{B(x,r)} \int_{B(x,4r)^c} |f(u)| \left| \frac{1}{|u-y|^{n-\alpha}} - \frac{1}{|u-z|^{n-\alpha}} \right| du \, d\mu(z) \\
&\quad \lesssim \frac{1}{\mu(B(x,r))} \int_{B(x,r)} \int_{B(x,4r)^c} |f(u)| \sup_{\xi=\theta y+(1-\theta)z} \frac{|y-z|}{|u-\xi|^{n-\alpha+1}} \, du \, d\mu(z) \\
&\quad \approx \frac{1}{\mu(B(x,r))} \int_{B(x,r)} \int_{B(x,4r)^c} |f(u)| \frac{|y-z|}{|u-x|^{n-\alpha+1}} \, du \, d\mu(z) \\
&\quad \lesssim r \int_{B(x,4r)^c} \frac{|f(u)|}{|u-x|^{n-\alpha+1}} \, du \\
&\quad \approx r \sum_{k=2}^{\infty} \int_{2^k r < |u-x| \le 2^{k+1} r} \frac{|f(u)|}{|u-x|^{n-\alpha+1}} \, du \\
&\quad \lesssim r \sum_{k=2}^{\infty} (2^k r)^{\alpha-n-1} \int_{2^k r < |u-x| \le 2^{k+1} r} |f(u)| \, du \\
&\quad \lesssim r \sum_{k=2}^{\infty} (2^k r)^{\frac{\alpha p - \lambda - p}{p}} \left((2^k r)^{\lambda - n} \int_{2^k r < |u-x| \le 2^{k+1} r} |f(u)|^p \, du \right)^{\frac{1}{p}} \\
&\quad \lesssim r^{\frac{\alpha p - \lambda}{p}} \|f\|_{L^{p,\lambda}},
\end{aligned}
$$

which implies

$$
\begin{aligned}
J_2 &= \left(r^{q(\frac{\lambda}{p} - \alpha) - \beta} \int_{B(x,r)} |I_\alpha f_2(y) - c|^q \, d\mu(y) \right)^{\frac{1}{q}} \\
&\lesssim (r^{-\beta} \mu(B(x,r)))^{\frac{1}{q}} \|f\|_{L^{p,\lambda}} \\
&\lesssim \||\mu\||_{\beta}^{\frac{1}{q}} \|f\|_{L^{p,\lambda}}.
\end{aligned} \tag{5.10}
$$

Hence (5.8) holds from combining (5.9) & (5.10).

(ii) Let $I_\alpha : CN^{p,\widetilde{\alpha}} \to \mathcal{L}_\mu^{q,\eta}$ be bounded and μ be admissible. Then

$$\|I_\alpha f\|_{\mathcal{L}_\mu^{q,\eta}} \lesssim \|f\|_{CN^{p,\widetilde{\alpha}}} \quad \forall f \in CN^{p,\widetilde{\alpha}}. \tag{5.11}$$

Let $r > 0$, $x_0, x_1, x_2 \in \mathbb{R}^n$ and x_1 be in the line segment connecting x_0, x_2 with

$$\begin{cases} |x_0 - x_1| = 3r; \\ |x_1 - x_2| = (jk+2)r, \end{cases} \tag{5.12}$$

where $j \in \mathbb{N}$ such that $jk > 2$. Let $B_i = B(x_i, r)$, $i = 0, 1, 2$ and $B = B(x_1, (jk+3)r)$, then $B_2 \subseteq B$, since (5.12).

Let $f_0 = r^{\frac{\widetilde{\alpha}-n}{p}} \chi_{B_0}$, then, by Theorem 2.2.1, it follows that

$$\|f_0\|_{CN^{p,\widetilde{\alpha}}}^p = \sup_{y \in \mathbb{R}^n} r^{\widetilde{\alpha}-n} \int_{B_0} \frac{1}{|x-y|^{\widetilde{\alpha}}}\, dx = r^{\widetilde{\alpha}-n} V_{\widetilde{\alpha}}(B_0, B(o,1)) \lesssim r^{\widetilde{\alpha}-n} r^{n-\widetilde{\alpha}} \approx 1,$$

which implies $f_0 \in CN^{p,\widetilde{\alpha}}$ and hence

$$\|I_\alpha f_0\|_{\mathcal{L}_\mu^{q,\eta}} \lesssim 1,$$

due to (5.11).

By Minkowski's inequality, it follows that

$$\begin{aligned} &\left(r^{\eta-n} \int_{B_1} |I_\alpha f_0(y) - (I_\alpha f_0)_{B_2,\mu}|^q \, d\mu(y) \right)^{\frac{1}{q}} \\ &\leq \left(r^{\eta-n} \int_{B_1} |I_\alpha f_0(y) - (I_\alpha f_0)_{B_1,\mu}|^q \, d\mu(y) \right)^{\frac{1}{q}} \\ &\quad + \left(r^{\eta-n} \mu(B_1) \right)^{\frac{1}{q}} |(I_\alpha f_0)_{B_1,\mu} - (I_\alpha f_0)_{B_2,\mu}| \\ &=: I_1 + I_2. \end{aligned} \tag{5.13}$$

It is clear to see

$$I_1 \leq \|I_\alpha f_0\|_{\mathcal{L}_\mu^{q,\eta}} \lesssim 1. \tag{5.14}$$

For I_2, note that $B_2 \subseteq B$ and $\mu(B_1) \approx \mu(B_2)$ as well as μ is admissible. So, by Hölder's inequality we have

$$\begin{aligned}
&\left(r^{\eta-n}\mu(B_1)\right)^{-\frac{1}{q}} I_2 \\
&\leq |(I_\alpha f_0)_{B_1,\mu} - (I_\alpha f_0)_{B,\mu}| + |(I_\alpha f_0)_{B,\mu} - (I_\alpha f_0)_{B_2,\mu}| \\
&\leq \frac{1}{\mu(B_1)} \int_{B_1} |I_\alpha f_0(y) - (I_\alpha f_0)_{B,\mu}| \, d\mu(y) \\
&\quad + \frac{1}{\mu(B_2)} \int_{B_2} |I_\alpha f_0(y) - (I_\alpha f_0)_{B,\mu}| \, d\mu(y) \\
&\leq \left(\frac{1}{\mu(B_1)} \int_{B_1} |I_\alpha f_0(y) - (I_\alpha f_0)_{B,\mu}|^q \, d\mu(y)\right)^{\frac{1}{q}} \\
&\quad + \left(\frac{1}{\mu(B_2)} \int_{B_2} |I_\alpha f_0(y) - (I_\alpha f_0)_{B,\mu}|^q \, d\mu(y)\right)^{\frac{1}{q}} \\
&\leq \left(\frac{1}{\mu(B_1)} \int_{B} |I_\alpha f_0(y) - (I_\alpha f_0)_{B,\mu}|^q \, d\mu(y)\right)^{\frac{1}{q}} \\
&\quad + \left(\frac{1}{\mu(B_2)} \int_{B} |I_\alpha f_0(y) - (I_\alpha f_0)_{B,\mu}|^q \, d\mu(y)\right)^{\frac{1}{q}} \\
&\approx \left(\frac{1}{\mu(B_1)} \int_{B} |I_\alpha f_0(y) - (I_\alpha f_0)_{B,\mu}|^q \, d\mu(y)\right)^{\frac{1}{q}},
\end{aligned}$$

which implies

$$I_2 \lesssim \left(r^{\eta-n} \int_{B} |I_\alpha f_0(y) - (I_\alpha f_0)_{B,\mu}|^q \, d\mu(y)\right)^{\frac{1}{q}} \lesssim \|I_\alpha f_0\|_{\mathcal{L}_\mu^{q,\eta}} \lesssim 1. \tag{5.15}$$

Applying (5.14) & (5.15) to (5.13), we have

$$\left(r^{\eta-n} \int_{B_1} |I_\alpha f_0(y) - (I_\alpha f_0)_{B_2,\mu}|^q \, d\mu(y)\right)^{\frac{1}{q}} \lesssim 1. \tag{5.16}$$

On the other hand, note that for $y \in B_1$, $z \in B_2$ and $w \in B_0$ one has

$$\begin{cases} |y-w| \le |y-x_1| + |x_1-x_0| + |x_0-w| \le 5r; \\ |z-w| \ge |x_2-x_0| - |z-x_2| - |x_0-w| \ge (jk+3)r. \end{cases}$$

Thus

$$\begin{aligned} &|I_\alpha f_0(y) - (I_\alpha f_0)_{B_2,\mu}| \\ &\quad = \left| \frac{1}{\mu(B_2)} \int_{B_2} (I_\alpha f_0(y) - I_\alpha f_0(z))\, d\mu(z) \right| \\ &\quad = r^{\frac{\tilde{\alpha}-n}{p}} \left| \frac{1}{\mu(B_2)} \int_{B_2} \int_{B_0} \left(\frac{1}{|y-w|^{n-\alpha}} - \frac{1}{|z-w|^{n-\alpha}} \right) dw d\mu(z) \right| \\ &\quad \ge r^{\frac{\tilde{\alpha}-n}{p}} \left| \frac{1}{\mu(B_2)} \int_{B_2} \int_{B_0} \left((5r)^{\alpha-n} - ((jk+3)r)^{\alpha-n} \right) dw d\mu(z) \right| \\ &\quad = (5^{\alpha-n} - (jk+3)^{\alpha-n}) r^{\alpha + \frac{\tilde{\alpha}-n}{p}}, \end{aligned}$$

which, together with (5.16) & (5.7), implies

$$\begin{aligned} 1 &\gtrsim \left(r^{\eta-n} \int_{B_1} |I_\alpha f_0(y) - (I_\alpha f_0)_{B_2,\mu}|^q \, d\mu(y) \right)^{\frac{1}{q}} \\ &\gtrsim \left(r^{\eta-n+q\left(\frac{\tilde{\alpha}-n}{p}+\alpha\right)} \mu(B_1) \right)^{\frac{1}{q}} \approx \left(r^{-\beta} \mu(B_1) \right)^{\frac{1}{q}}. \end{aligned}$$

Hence $|||\mu|||_\beta < \infty$, which completes the proof. □

To find an application to partial differential equations, we restrict some related function spaces on a bounded set $\Omega \subsetneq \mathbb{R}^n$, where $\mathbb{R}^n$ and $B(x,r)$ in the definitions of the function spaces are replaced, respectively, by Ω and $B(x,r) \cap \Omega$, thereby obtaining

Corollary 5.3.2. *Let*

$$\begin{cases} 1 < p < \infty; \\ 1 \le q < \infty; \\ 0 \le \widetilde{\alpha} < n, \end{cases}$$

Ω be with C^2 boundary and $u \in W^{2,p,(\widetilde{\alpha})}(\Omega) \cap W_0^{1,p}(\Omega)$ solve the Dirichlet problem

$$\begin{cases} \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f(x) \in CN^{p,\widetilde{\alpha}}(\Omega) & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.17}$$

with the following three constraints:

(a) $a_{ij} \in L^{\infty}(\Omega)$ *&* $a_{ij}(x) = a_{ij}(x)$, $i, j = 1, 2, \cdots, n$;

(b) (Strong ellipticity condition) $\exists \nu > 0$ *such that*

$$\sum_{i,j=1}^{n} a_{i,j}(x) \xi_i \xi_j \ge \nu |\xi|^2, \text{ a.e. } x \in \Omega, \ \forall \xi \in \mathbb{R}^n;$$

(c) (Cordes-type condition) $\exists K \in [0, 1)$ *such that*

$$\frac{(\sum_{i,j=1}^{n} a_{i,j}(x))^2}{\sum_{i,j=1}^{n} (a_{i,j}(x))^2} \ge n - \frac{K^2}{C_{MT}^2}, \text{ a.e. } x \in \Omega,$$

where C_{MT} is a constant in [41, Theorem 3.2].

(i) If

$$\begin{cases} (n-2)q < n; \\ p < n - \tilde{\alpha}; \\ \eta = q \left(\frac{n - \tilde{\alpha}}{p} - 2 \right) \le n, \end{cases} \tag{5.18}$$

then

$$u \in \mathcal{L}^{q,\eta}(\Omega) = \begin{cases} L^{q,\eta}(\Omega), & as\ \eta \in (0,n]; \\ BMO(\Omega), & as\ \eta = 0; \\ C^{-\frac{\eta}{q}}(\overline{\Omega}), & as\ \eta \in [-q,0). \end{cases} \tag{5.19}$$

(ii) If

$$\begin{cases} (n-1)q < n; \\ \eta = q\left(\frac{n-\tilde{\alpha}}{p} - 1\right) \le n, \end{cases} \tag{5.20}$$

then

$$\nabla u \in \left(\mathcal{L}^{q,\eta}(\Omega)\right)^n = \begin{cases} \left(L^{q,\eta}(\Omega)\right)^n, & as\ \eta \in (0,n]; \\ \left(BMO(\Omega)\right)^n, & as\ \eta = 0; \\ \left(C^{-\frac{\eta}{q}}(\overline{\Omega})\right)^n, & as\ \eta \in [-q,0). \end{cases} \tag{5.21}$$

.

Proof. For $u \in W^{2,p,(\tilde{\alpha})}(\Omega) \cap W_0^{1,p}(\Omega)$, we use [41, (32)] to get the following inequality for a constant $K_0 \in (0,1)$:

$$\|\Delta u\|_{CN^{p,\tilde{\alpha}}(\Omega)} \le \frac{\|f\|_{CN^{p,\tilde{\alpha}}(\Omega)}}{\nu(1-K_0)}.$$

Moreover, we refer to [2, 49] for the following representation of $f \in C_c^\infty$ for a constant

$c_{n,m}$:

$$
\begin{cases} f = c_{n,m} I_m(\nabla^m f) & \text{as m is even;} \\ |f| \lesssim I_m(|\nabla^m f|) & \text{as m is odd,} \end{cases} \tag{5.22}
$$

where C_c^∞ denotes the set of all smooth functions with compact support and

$$
\nabla^m f = \begin{cases} (-\Delta)^{\frac{m}{2}} f & \text{as m is even;} \\ \nabla(-\Delta)^{\frac{m-1}{2}} f & \text{as m is odd.} \end{cases}
$$

Observe that (5.22) holds for $f \in C^\infty(\Omega)$ (which is dense in $W^{2,p,(\tilde{\alpha})}(\Omega)$). Thus, following the argument as in [49, Section 1.1.10, Theorem 1], we can check that when $m = 1$ or 2, (5.22) also holds for $f \in W^{2,p,(\tilde{\alpha})}(\Omega) \cap W_0^{1,p}(\Omega)$.

(i) If (5.18) holds, then Theorem 5.3.1(i) (with $d\mu$ being the n-dimensional Lebesgue measure) and (5.22) are used to derive

$$
\|u\|_{\mathcal{L}^{q,\eta}(\Omega)} \approx \|I_2(-\Delta u)\|_{\mathcal{L}^{q,\eta}(\Omega)} \lesssim \|\Delta u\|_{CN^{p,\tilde{\alpha}}(\Omega)} \lesssim \|f\|_{CN^{p,\tilde{\alpha}}(\Omega)},
$$

which yields $u \in \mathcal{L}^{q,\eta}(\Omega)$. For the classification of $\mathcal{L}^{q,\eta}(\Omega)$ in (5.19), we refer to [32, Chapter III] and [55].

(ii) With the help of (5.22), we achieve

$$
\begin{cases} \nabla u(y) = c_{n,2}(n-2)\big(\mathcal{R}_1 \Delta u(y), ..., \mathcal{R}_n \Delta u(y)\big); \\ \mathcal{R}_j \Delta u(y) := \int_{\mathbb{R}^n} (x_j - y_j)|x-y|^{-n} \Delta u(x)\, dx \quad \forall \quad j \in \{1, ..., n\}. \end{cases} \tag{5.23}
$$

If (5.20) holds, then a similar way of proving Theorem 5.3.1(i) (under $d\mu$ being the

n-dimensional Lebesgue measure), plus a minor modification, yields

$$\|\mathcal{R}_i \Delta u\|_{\mathcal{L}^{q,\eta}(\Omega)} \lesssim \|\Delta u\|_{CN^{p,\tilde{\alpha}}(\Omega)} \lesssim \|f\|_{CN^{p,\tilde{\alpha}}(\Omega)}, \quad \forall \quad i \in \{1, ..., n\}$$

which, along with (5.23), implies $\nabla u \in \left(\mathcal{L}^{q,\eta}(\Omega)\right)^n$. □

Remark 5.3.3. *The Dirichlet problem* (5.17) *has been broadly studied (see [41] and its references for more details). Note that* (5.19) *is new and* (5.21) *covers and complements [41, Corollary 4.1 (i)] - if* $q \in (1, n]$ *and* $\eta \in [-q, 0)$*, then*

$$\nabla u \in \left(\mathcal{L}^{q,\eta}(\Omega)\right)^n = \left(C^{-\frac{\eta}{q}}(\overline{\Omega})\right)^n.$$

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