Using Spacetime Surgeries to Model Bouncing Null Fluids and Null Fluid Radiation

by

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Abstract

In 1991 Ori proposed a construction to remedy energy condition violations in the ingoing Vaidya-Reissner-Nordström (VRN) spacetime. The model replaces the violation regions of VRN with an outgoing VRN spacetime. The two spacetimes are attached along a spacelike hypersurface and the hybrid spacetime models the ingoing matter bouncing at this surface. Such a spacetime surgery was claimed to produce no thin shell at the junction surface. In 2015 Booth showed that in the special case of extremal Reissner-Nordström a thin shell will exist under the same construction.

In this work the Ori construction is generalized to Husain null fluids, a one parameter generalization of the VRN solution. Along the way, the tension between the extremal case and the VRN case will be resolved: there are several ways to match coordinates along the junction surface and the two cases chose different matchings. Further, the case of a timelike junction surface will be analyzed as a classical model of null radiation from a black hole.

Acknowledgements

I would like to thank my fiancé Emily for enduring many lengthy physics chats and being supportive through this entire experience. Unfortunately for her such chats will not cease with the completion of this thesis.

I would like to thank Dr. Ivan Booth for supervising this thesis and taking the time to discuss things, related to relativity or otherwise, each and every week. My experience with the entire gravity group at Memorial has been very positive.

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Statement of contributions

The idea to generalize Ori's Vaidya-Reissner-Nordström construction to Husain Null fluids, and to investigate the seemingly conflicting results of [30] and [2], originated from the author's supervisor Dr. Ivan Booth (IB). The calculations involving thin shells and the matching conditions were a collaborative effort between IB and the author, with IB doing thin shell stress-energy calculations and the linear matter example. The author then provided a check by independently doing the same calculations under the direction of IB. Further, the author wrote the first draft of [4] to appear in Physical Review D. From [4], Amos Ori pointed out a calculational error involving the quantity U_{vv}, equation (4.35) in this text, which was subsequently corrected.

The idea to use a timelike junction surface to model black hole radiation originated when IB visited with Dr. Matt Visser at the Victoria University of Wellington. There the thin shell stress energy was calculated for two matching Vaidya solutions. Later on, the author, supervised by IB, generalized this calculation to Husain null fluids and calculated the linear matter example along with the constant r example.

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$(-,+,+,+)$ -	Spacetime Signature
$\alpha, \beta, \gamma, ...$ -	Spacetime Indices 0,1,2,3,4
$a, b, c, ...$ -	Hypersurface Indices 1,2,3
$g_{\alpha\beta}$ -	Spacetime metric tensor
$R_{\alpha\beta}$ -	Ricci Tensor
$G_{\alpha\beta}$ -	Einstein Tensor
$T_{\alpha\beta}$ -	Stress-Energy Tensor
h_{ab} -	Induced metric
$F_{\alpha\beta}$ -	Faraday Tensor
$E_{\alpha\beta}$ -	Electromagnetic Field Tensor

Chapter 1

Introduction

The year 2016 marked the centenary of the publication of Einstein's theory of general relativity. In the one hundred years following 1916, spectacular progress has been made in developing the physics and mathematics of general relativity. One of the most striking developments has been the theory of black holes. In 1916, Karl Schwarzschild gave the first non-trivial exact solution to the vacuum Einstein equations [34]. The Schwarzschild solution describes the spacetime outside of a spherically symmetric static mass and, after many years of relativity research, it was found to also describes a black hole. Generalizations to a charged mass were given by Reissner and Nordström independently [28, 32], and the inclusion of angular momentum was given by Kerr [21]. Finally, charge and rotation were combined in the Kerr-Newman black hole [27].

Another avenue of generalization for the Schwarzschild spacetime is through the description of dynamical black holes. That is, having a black hole solution which is time-dependent. Such a generalization was accomplished by Vaidya and describes null dust radiating from a white hole or irradiating a black hole [36]. The Vaidya solution has been generalized to charged null dust [1] and also to a null fluid with tangential pressure [17]. The case of a pressurized null fluid, which we will refer to as a Husain null fluid, will be particularly important later on.

The energy conditions of general relativity provide criteria for describing physically realistic solutions to the Einstein equations. Although the exact form of the energy conditions will be provided later, it will be noted here that in the case of Vaidya-Reissner-Nordström (VRN) or Husain null fluids there exist regions within the spacetime wherein these conditions are violated [17, 25, 30]. Physically this violation will correspond to regions where an observer would measure a negative energy density. For collapsing charged matter, the region will be where the charged matter continues to collapse inward despite electromagnetic forces becoming stronger than the gravitational attraction [30].

In [30], Ori analyzed the VRN spacetime and proposed a physically motivated construction to excise the violation regions. Looking at the Lorentz force on the charged dust and including a Lorentz force term in the equations of motion, he was able to show that on the boundary between the violation region and the non-violating region of spacetime the wave-vector of the dust vanishes. The vanishing of the wave-vector prompted a reinterpretation of VRN: at the surface of vanishing wave-vector, the matter bounces and instantaneously transitions from ingoing to outgoing charged dust. Mathematically, this new interpretation corresponds to a new hybrid spacetime composed of violation free regions of ingoing and outgoing charged Vaidya spacetimes. See Figure 1.1 for an illustration of the construction.

In general, when constructing a hybrid spacetime one may obtain a thin shell discontinuity in the stress-energy tensor along the "seam" of the surgery. Such a thin-shell is described by jumps in the extrinsic curvature tensor across the boundary. Conditions for when the new hybrid spacetime contains such thin shells are given by the Israel-Darmois junction conditions [18], which will be described in Chapter 2.

Generalizations to Ori's procedure have been applied to f(R) theories of gravity [3]

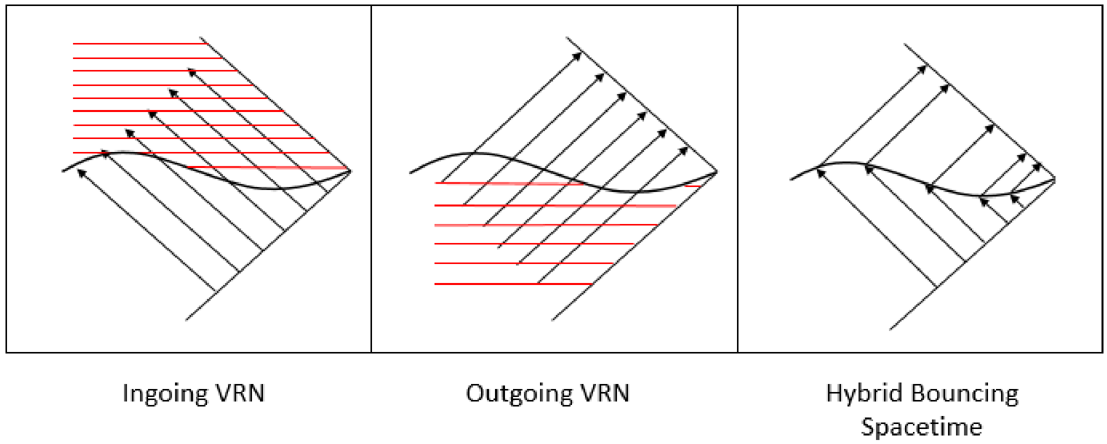

Figure 1.1: Schematic Spacetime Diagram of Ori's spacetime surgery construction The horizontal lines indicate regions of energy condition violations. The ingoing and outgoing arrows represent ingoing and outgoing charged dust. The curved horizontal line indicates the junction surface.

and also to extremal Reissner-Nordström [2]. However, in [2] a possible inconsistency with [30] was noted. Along the hybrid junction surface, it was found that a thin shell discontinuity exists whereas in [30] no such thin shell was present. Since extremal Reissner-Nordström is a very special case of charged Vaidya, such a discrepancy is concerning and will be investigated.

This work has three overall goals: to investigate and resolve the tension between the two results of [30] and [2], to generalize the Ori construction to Husain null fluids, and to consider the case of a timelike junction surface. It will be shown that the tension arises due to the freedom one has in how to match the two component spacetime coordinates along the junction surface; two different choices are available for VRN which satisfy the Israel-Darmois junction conditions, whereas only one is available for the extremal case. Ori chooses the matching condition which is unavailable in the extremal case and therefore one is able to conclude that [30] and [2] do not contradict each other. Further, it will be shown that in the generalization to Husain null fluids there is no thin shell in the Ori matching scenario. An example will also be given to illustrate the construction.

In the Ori construction, and the Husain null fluid generalization provided here, the junction surface is also assumed to be spacelike. This restriction gives the instantaneous bouncing condition, but mathematically is not a necessity. In Chapter 4 we also present the timelike junction surface case. The computations of the junction conditions will be very similar to the spacelike case, but the physical interpretation is quite different. By having the junction surface a timelike surface, we look to model null radiation from a black hole. That is, outside the black hole horizon we have a radiating surface which radiates a positive energy density null fluid to null infinity and a negative energy density null fluid into the black hole. It will be shown that the results regarding thin shells for the spacelike case carry over to the timelike case; no thin shell will be present in the Ori matching scenario.

The idea of using the Vaidya solution, and even spacetime surgery, to model black hole radiation is not new. Thermal radiation emitted from a black hole, Hawking radiation, was first discussed by Hawking in [12] and the Vaidya solution has been extensively used to model both radiating spheres [5, 14, 23, 24, 35, 37] and in semi-classical approximations of evaporating black holes [8, 9, 13, 15, 16, 19, 20, 22]. In [15] formation and evaporation of a black hole was modelled via spacetime surgery connecting an ingoing modified Vaidya to an outgoing Vaidya along a timelike hypersurface. The timelike hypersurface models the surface of pair production with negative energy density null dust falling into the black hole and positive energy density null dust escaping to null infinity. The model was further studied in [8] in the context of the information loss paradox (See references within [8] for details of this paradox). In this work the Vaidya evaporation model (as described in [8, 15]) will be generalized to Husain null fluids. Mathematically this is equivalent to the Ori construction with a timelike hypersurface which is not necessarily at the violation region boundary.

The rest of the work is organized as follows. In Chapter 2 all necessary background material is presented, including the energy conditions, hypersurface geometry, the junction conditions, Vaidya spacetimes and its generalization to Vaidya-Reissner-Nordström. In Chapter 3 the Husain null fluid spacetime is reviewed and the regions of energy condition violations are found. In Chapter 4 the main results of the spacelike null fluid bounce scenario and the timelike radiating model are presented and analyzed with a few examples. In Chapter 5 a summary is provided.

The work done on the null bounce model in this thesis has been expanded upon in [4] and will subsequently appear in Physical Review D. Further, a current collaboration exists between the author, supervisor, and Dr. Matt Visser and Jessica Santiago from the University of Victoria in Wellington. From this collaboration, a publication regarding the timelike junction model in the Vaidya spacetime should appear in the near future. It should also be emphasized here that all computations using a computer algebra system were done using the Physics package of Maple.

Chapter 2

Background and Review

In this chapter the necessary background is given in order to construct and analyze the bounce and radiation models later on. This material is also presented in detail in [11, 31, 38].

2.1 Einstein's Equations and the Energy Conditions

The Einstein equations are a set of ten, nonlinear, coupled, partial differential equations of the metric tensor $g_{\alpha\beta}$. In their most general form they are given by

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}, \tag{2.1}$$

where $R_{\alpha\beta}$ is the Ricci tensor, R the Ricci scalar, and Λ the cosmological constant. In this work Λ is always taken to be zero. On the right hand side, $T_{\alpha\beta}$ is the stress-energy tensor constructed from the physical matter present within the spacetime. Defining the Einstein tensor $G_{\alpha\beta}$ by

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}, \tag{2.2}$$

the equations may be written in the more compact form $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$. The Einstein equations relate geometric quantities concerning spacetime curvature, $G_{\alpha\beta}$, to the physical stress-energy encapsulated by $T_{\alpha\beta}$. By prescribing matter within the spacetime, one hopes to solve for the geometry of spacetime given by $g_{\alpha\beta}$. If there is no stress-energy, then $T_{\alpha\beta} = 0$ and one has the vacuum Einstein equations $R_{\alpha\beta} = 0$.

A natural way to interpret solving Einstein's equations for a specific spacetime is through the prescription of a stress-energy tensor and then solving for the curvature caused by such matter. However, nothing prohibits working in the opposite direction: prescribe the spacetime by some properly defined metric tensor $g_{\alpha\beta}$, compute the Einstein tensor $G_{\alpha\beta}$, and then demand that the Einstein equations be satisfied by setting $8\pi T_{\alpha\beta} = G_{\alpha\beta}$. This method would lead to a mathematically valid solution, but the physical validity of the metric tensor $g_{\alpha\beta}$ could still be questioned.

The energy conditions are criteria on the stress-energy tensor $T_{\alpha\beta}$ that encapsulate physical ideas that one has about properties of matter. By demanding that the energy conditions are satisfied, the majority of the Einstein tensors $G_{\alpha\beta}$ that one could dream up are ruled out on physical grounds. Therefore, since the Einstein equations don't make any distinction between exotic, unrealistic types of matter (i.e. matter with negative energy density for example) and the matter which has properties that we expect, the energy conditions are an extra condition which we must impose to limit ourselves to more physically meaningful solutions in classical general relativity.

The energy conditions do provide classical criteria for physically meaningful solutions but are not the last word on the physicality of a proposed stress-energy tensor. Quantum fields can have non-positive local energy density [6] and the Casimir effect is a quantum effect which violates all energy conditions [7]. Further, the classical theory of traversable wormholes can lead to violations of all energy conditions [26]. In the timelike junction surface model of black hole radiation there will necessarily be energy condition violations on the ingoing side in order to satisfy local conservation of energy. In the spacelike bounce model, the energy conditions are demanded; nothing too "exotic" is happening in the model (it is a collapsing null fluid) and so we expect the energy conditions to be satisfied.

The energy conditions which one might demand in a given situation are the weak condition, the null condition, the strong condition, and the dominant condition. It should be noted that, despite the naming, they are mostly distinct conditions; the strong condition does not necessarily imply the weak condition.

2.1.1 Hawking-Ellis Stress-Energy Classification

In [11], stress-energy tensors $T_{\alpha\beta}$ are classified by their eigenvectors. In four dimensions there are mathematically four distinct possibilities, though here we only consider Type I and Type II. The Husain null fluid stress-energy tensors we will describe later will be Type II, but we also include a description of Type I here as most types of matter are of that form.

A Type I stress-energy tensor will have four distinct eigenvalues $\{\mu, p_1, p_2, p_3\}$ and one timelike eigenvector e_0^α. The eigenvalue μ corresponds to the energy density as measured an observer moving in the direction of ξ^α, and p_1, p_2, and p_3 are the principal pressures. With respect to a unit tetrad $\{e_0^\alpha, e_1^\alpha, e_2^\alpha, e_3^\alpha\}$, where e_0^α is the timelike unit eigenvector, the Type I stress energy tensor can be written as

$$T^{\alpha\beta} = \mu e_0^\alpha e_0^\beta + p_1 e_1^\alpha e_1^\beta + p_2 e_2^\alpha e_2^\beta + p_3 e_3^\alpha e_3^\beta. \tag{2.3}$$

A Type II stress-energy tensor has a double null eigenvector. Letting N^α and ℓ^α be cross-normalized null vectors, so that $N^\alpha \ell_\alpha = -1$, and $\{e_2^\alpha, e_3^\alpha\}$ be spacelike orthonormal vectors which are also orthogonal to N^α and ℓ^α, the stress energy tensor can be written as

$$T^{\alpha\beta} = \mu N^\alpha N^\beta + \rho(N^\alpha \ell^\beta + N^\beta \ell^\alpha) + P(g^{\alpha\beta} + N^\alpha \ell^\beta + N^\beta \ell^\alpha). \tag{2.4}$$

Here N^α is the null eigenvector and the flux of energy is in the N^α direction. For an observer with timelike four velocity v^a, the measured energy-momentum current J_a is given by $J^a = T^a{}_b v^b$.

2.1.2 The Weak and Null Conditions

The weak condition states that $T_{\alpha\beta} v^\alpha v^\beta \geq 0$ holds for any timelike vector v^α [31]. The quantity $T_{\alpha\beta} v^\alpha v^\beta$ represents the energy density of matter as measured by an observer with 4-velocity v^α, and so the weak energy condition is the statement that energy density of matter must always be measured to be non-negative. Since the weak condition specifies any timelike vector v^α, all observers must agree that the energy density is non-negative.

Focusing on the Type II stress energy tensor (2.4), the weak condition can be restated as

$$\mu \geq 0, \quad \rho \geq 0, \quad 0 \leq P \leq \rho. \tag{2.5}$$

The null energy condition can be obtained through continuity from the weak energy condition. As the timelike vector v^α approaches a null vector k^α the weak energy condition should still hold (since it holds for all timelike vectors) and a condition similar to the weak condition should hold for all null vectors k^α. The null condition may be stated as

$$T_{\alpha\beta} k^\alpha k^\beta \geq 0 \tag{2.6}$$

for all possible null vectors k^α.

2.1.3 The Strong Condition

The strong condition states that $T_{\alpha\beta}v^\alpha v^\beta \geq -\frac{1}{2}T$, where T is the trace of the stress-energy tensor and v^α is any timelike vector [38]. For a Type II stress tensor, the strong condition is equivalent to

$$\mu \geq 0, \quad P \geq 0, \quad \rho + P \geq 0. \tag{2.7}$$

The strong condition may also be recast into a geometrical condition on the spacetime via Einstein's equations. By contracting Einstein's equations, the Ricci scalar R can be written in terms of the trace of the stress-energy tensor:

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

$$\Rightarrow g^{\alpha\beta}R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}g^{\alpha\beta} = 8\pi T_{\alpha\beta}g^{\alpha\beta} \tag{2.8}$$

$$\Rightarrow R = -8\pi T.$$

Letting v^α be any normalized timelike vector, by Einstein's equations we have the following:

$$R_{\alpha\beta}v^\alpha v^\beta - \frac{1}{2}Rg_{\alpha\beta}v^\alpha v^\beta = 8\pi T_{\alpha\beta}v^\alpha v^\beta$$

$$\Leftrightarrow R_{\alpha\beta}v^\alpha v^\beta = 8\pi\left(T_{\alpha\beta}v^\alpha v^\beta + \frac{1}{2}T\right). \tag{2.9}$$

Therefore the strong energy condition is equivalent to the geometric condition

$$R_{\alpha\beta}v^{\alpha}v^{\beta} \geq 0, \tag{2.10}$$

where v^{α} is a unit timelike vector. The strong condition is often written in terms of the Ricci tensor.

2.1.4 The Dominant Condition

The dominant condition can be physically understood to be a restriction on the flow of energy to be less than the speed of light [38]. The mathematical statement is that for any future directed timelike vector v^{α}, $-T^{\alpha}{}_{\beta}v^{\beta}$ is a timelike or null future directed vector. Note that the dominant condition does imply the weak condition. For a Type II stress tensor the dominant energy condition is equivalent to

$$\mu \geq 0, \quad \rho \geq 0, \quad |P| \leq |\rho|. \tag{2.11}$$

2.1.5 Summary of Energy Conditions for Type-II

Here we collect the results of this section and produce a summary of energy conditions for Type-II stress tensors of the form (2.4) in Table 2.1.

2.2 Hypersurfaces

To properly describe a spacetime surgery of two solutions of the Einstein equations, there must be a "seam", or junction surface, at which the two spacetimes are attached. This seam will be a three dimensional surface within the hybrid spacetime (See Figure 2.1). In order to describe whether or not two spacetimes can be matched continuously,

Energy Condition	Conditions
Weak	$\mu \geq 0 \quad \rho \geq 0 \quad \rho + P \geq 0$
Null	$\mu \geq 0 \quad \rho + P \geq 0$
Strong	$\mu \geq 0 \quad P \geq 0 \quad \rho + P \geq 0$
Dominant	$\mu \geq 0 \quad \rho \geq 0 \quad \lvert P \rvert \leq \lvert \rho \rvert$
All Simultaneously	$\mu \geq 0 \quad \rho \geq 0 \quad 0 \leq P \leq \rho$

Table 2.1: Summary of Energy Conditions for Type-II Stress-Energy Tensors
It should be noted that the energy conditions here differ from those in [11] where it was found that the energy conditions for type-II stress-energy tensors are incorrect individually. See [4] for more details.

one must describe this 3 dimensional surface from the vantage of each component spacetime and compare. We must first review the basic mathematics of hypersurfaces here in order to understand and apply the Israel-Darmois junction conditions in the next section. For more details on hypersurfaces see [18, 31, 38].

For a manifold $\mathcal{M}$ with coordinates $\{x^\alpha\}$ of dimension n, a *hypersurface* Σ is a submanifold of dimension $(n-1)$. Since Σ is a manifold itself, one may install intrinsic coordinates $\{y^a\}$ on its surface. Using the hypersurface coordinates $\{y^a\}$, Σ can be described parametrically as

$$x^\alpha = x^\alpha(y^a), \tag{2.12}$$

or by a constraint equation of the form

$$f(x^\alpha) = 0. \tag{2.13}$$

Although a precise definition of hypersurfaces as embedded submanifolds exists, as in [38], we simply stick the simpler definition as a surface with a single constraint on the coordinates $\{x^\alpha\}$.

A basis tetrad involving three linearly independent tangent vectors and the normal to the surface can be formed on the hypersurface using the intrinsic coordinates $\{y^a\}$.

The normal to Σ, denoted now as $\tilde{n}_\alpha$, is given by $f_{,\alpha}$. Since f is zero along the surface, the direction of the gradient of f is orthogonal to the surface and so is proportional to the normal. One may scale the normal to unit length:

$$n_\alpha = \frac{\epsilon}{f_{,\beta}f^{,\beta}}\tilde{n}_\alpha, \tag{2.14}$$

where ϵ is 1 for a spacelike normal and -1 for a timelike normal. The null case is not of interest here and will not be discussed. The three linearly independent tangent vectors, which we denote as $\{e^\alpha{}_a\}$, are obtained by differentiating the parametric equations (2.12):

$$e^\alpha{}_a = \frac{\partial x^\alpha}{\partial y^a}. \tag{2.15}$$

Hypersurfaces can be classified as either *timelike*, *spacelike*, or *null* by the orientation of the normal. That is, a hypersurface is timelike if its normal is spacelike, spacelike if its normal is timelike, and null if its normal is null.

Since Σ is a $(n-1)$-dimensional manifold in its own right, it is natural to seek information about its intrinsic geometry. A metric h_{ab} can be induced on Σ so as to allow for the computation of lengths. We call h_{ab} the induced metric or first fundamental form. Denoting the line element of the hypersurface by $d\Sigma^2$, it can be written as

$$d\Sigma^2 = h_{ab}dy^a dy^b. \tag{2.16}$$

By pulling back the spacetime metric $g_{\alpha\beta}$ onto the surface via the tangent vectors $\{e^\alpha{}_a\}$ the induced metric is

$$h_{ab} = g_{\alpha\beta}e^\alpha{}_a e^\beta{}_b. \tag{2.17}$$

The induced metric h_{ab} is an intrinsic geometric entity; once obtained, it discards information about the spacetime $\mathcal{M}$ in which Σ is embedded. However, when Σ is viewed from the larger spacetime $\mathcal{M}$ there could be obvious differences between two hypersurfaces even if the induced metrics are the same. As an example, consider a plane and a parabolic cylinder in Euclidean three space. Both are intrinsically flat and are described by the same induced metric. However, when viewed as an imbedding in 3-space there is an obvious difference: the parabolic cylinder "bends" and the plane does not. Therefore another quantity is needed to fully describe hypersurfaces embedded in a parent spacetime. Such a quantity is called the extrinsic curvature.

The extrinsic curvature tensor K_{ab} of the hypersurface Σ is defined, using the parametric description, as

$$K_{ab} \equiv n_{\alpha;\beta} e^{\alpha}{}_{a} e^{\beta}{}_{b}, \tag{2.18}$$

and it describes how a hypersurface is embedded in the surrounding manifold $\mathcal{M}$ by looking at how the normal to the surface changes whilst moving around Σ.

By combining the induced metric h_{ab} and the extrinsic curvature K_{ab}, one gets a full description of the intrinsic and extrinsic properties of Σ embedded within $\mathcal{M}$. The language of hypersurfaces will play an important role in the Israel-Darmois junction conditions and subsequent spacetime surgeries in the next sections.

2.3 Junction Conditions

Spacetime surgery is an effective way of generating models for general relativistic phenomena using pre-existing exact solutions. The aim is to take two or more spacetimes and sew them together to form a hybrid spacetime describing the physical phenomenon one wishes to describe. A classic example of such a surgery is the Oppenheimer-Snyder collapse, which models the collapse of a star to a black hole [29].

The boundary, or junction surface, between the separate solutions is a hypersurface which we shall denote by Σ. Although Σ could be a null hypersurface, here we will

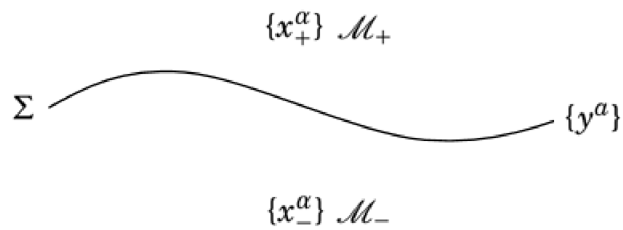

Figure 2.1: Coordinate and junction condition set up.

only be concerned with spacelike and timelike junction surfaces. The Israel-Darmois junction conditions [18] provide criteria for when two solutions can be properly formed into a hybrid solution without issues of continuity in the metric or Riemann tensor across Σ. We shall presently review a derivation for the junction conditions similar to that given in [31].

Let $(\mathcal{M}_+, g^+_{\alpha\beta})$ and $(\mathcal{M}_-, g^-_{\alpha\beta})$ be the two spacetimes we wish to join along a timelike or spacelike boundary Σ. Further assume coordinates $\{x^\alpha_+\}$ and $\{x^\alpha_-\}$ are coordinates on $\mathcal{M}_+$ and $\mathcal{M}_-$ respectively, and $\{y^a\}$ are intrinsic coordinates installed on Σ. See Figure 2.1 for a representation of the construction.

We shall also introduce coordinates $\{x^\alpha\}$ which exist on both sides of Σ. In a general situation such coordinates will not exist, but we introduce them in order to facilitate calculations. In the end the junction conditions will be statements independent of $\{x^\alpha\}$.

A useful idea to introduce here is that of a distribution. Letting Σ be pierced by a congruence of geodesics parametrized by proper distance (or proper time) ℓ and scaled such that $\ell = 0$ on Σ, $\ell > 0$ in $\mathcal{M}_+$ and $\ell < 0$ in $\mathcal{M}_-$, we introduce the Heaviside distribution

$$\Theta(\ell) = \begin{cases} 1 & : \ell > 0 \\ 0 & : \ell < 0, \end{cases} \tag{2.19}$$

which is undefined at $\ell = 0$. The coordinate distance along the congruence from Σ is

$$dx^\alpha = n^\alpha d\ell, \tag{2.20}$$

which, after a contraction with n_α, leads to the expression

$$n_\alpha = \epsilon \partial_\alpha \ell. \tag{2.21}$$

The parameter ϵ determines whether Σ is spacelike or timelike just as before.

2.3.1 First Junction Condition

Using the Heaviside distribution, the metric $g_{\alpha\beta}$ for the hybrid spacetime can be written as

$$g_{\alpha\beta} = \Theta(\ell) g^+_{\alpha\beta} + \Theta(-\ell) g^-_{\alpha\beta}, \tag{2.22}$$

and we require that this distributional metric satisfy the Einstein field equations in a distributional sense. That is, the Reimann tensor $R^\alpha{}_{\beta\gamma\delta}$ should be well-defined as a distribution. Since the Reimann tensor is constructed from the Christoffel symbols, which in turn are constructed from derivatives of the metric, the first course of action is to calculate the derivative of (2.22):

$$\begin{aligned} \partial_\gamma g_{\alpha\beta} &= \partial_\gamma \left(\Theta(\ell) g^+_{\alpha\beta} + \Theta(-\ell) g^-_{\alpha\beta} \right) \\ &= \Theta(\ell) \partial_\gamma g^+_{\alpha\beta} + \Theta(-\ell) \partial_\gamma g^-_{\alpha\beta} + (g^+_{\alpha\beta} - g^-_{\alpha\beta}) \delta(\ell) \partial_\gamma \ell \\ &= \Theta(\ell) \partial_\gamma g^+_{\alpha\beta} + \Theta(-\ell) \partial_\gamma g^-_{\alpha\beta} + \epsilon \delta(\ell) n_\gamma [g_{\alpha\beta}]. \end{aligned} \tag{2.23}$$

Here, and in the subsequent work, we use the the notation

$$[A] = A^{+} - A^{-} \tag{2.24}$$

where A is tensor and the $\pm$ notation indicates to which side of the hypersurface A is measured. Although the derivative of the metric is well-defined, there will be issues with the δ term in (2.23) when one attempts to construct the Christoffel symbols

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(\partial_{\beta} g_{\delta\gamma} + \partial_{\gamma} g_{\beta\delta} - \partial_{\delta} g_{\beta\gamma} \right). \tag{2.25}$$

Looking at only the first term in (2.25) and utilizing (2.23),

$$\begin{aligned} g^{\alpha\delta}\partial_{\beta} g_{\delta\gamma} &= \left(\Theta(\ell) g_{+}^{\alpha\beta} + \Theta(-\ell) g_{-}^{\alpha\beta} \right) \left(\Theta(\ell)\partial_{\gamma} g^{+}_{\alpha\beta} + \Theta(-\ell)\partial_{\gamma} g^{-}_{\alpha\beta} + \epsilon\delta(\ell) n_{\gamma}[g_{\alpha\beta}] \right) \\ &= g_{+}^{\alpha\beta}\Theta(\ell)\partial_{\gamma} g^{+}_{\alpha\beta} + g_{-}^{\alpha\beta}\Theta(-\ell)\partial_{\gamma} g^{-}_{\alpha\beta} + \left(g_{+}^{\alpha\beta}\Theta(\ell) + g_{-}^{\alpha\beta}\Theta(-\ell) \right) \epsilon\delta(\ell) n_{\gamma}[g_{\alpha\beta}]. \end{aligned} \tag{2.26}$$

The last term is proportional to $\Theta(\ell)\delta(\ell)$ which is not well-defined as a distribution. If we are to construct a hybrid solution to the Einstein field equations, the δ term must vanish in (2.23); this is only satisfied if

$$[g_{\alpha\beta}] = 0. \tag{2.27}$$

Although (2.27) makes the definition of the Christoffel symbols make sense distributionally, it is plagued by the fact that it depends on the coordinates $\{x^{\alpha}\}$ we defined on both sides of Σ. In general, we will not have such coordinates. To obtain an expression which is independent of the coordinates $\{x^{\alpha}\}$, we use the tangent vectors

${e_a}^\alpha$ to the surface Σ defined by

$$ {e_a}^\alpha = \frac{\partial x^\alpha}{\partial y^a}. \tag{2.28} $$

Note that

$$ \left[\frac{\partial x^\alpha}{\partial y^a} \right] = 0 \tag{2.29} $$

since the coordinates $\{x^\alpha\}$ are defined on both sides of Σ. Then, using (2.29) in conjunction with (2.27), we have

$$ \left[g_{\alpha\beta} \frac{\partial x^\alpha}{\partial y^a} \frac{\partial x^\beta}{\partial y^b} \right] = 0, \tag{2.30} $$

which is equivalent to saying

$$ [h_{ab}] = 0. \tag{2.31} $$

Here (2.31) is written in terms of the intrinsic coordinates $\{y^a\}$ and it serves as the first junction condition. The first junction condition, intuitively, is quite reasonable; both $\mathcal{M}_+$ and $\mathcal{M}_-$ must induce the same metric on Σ.

2.3.2 Second Junction Condition

The second junction condition is more involved than the first and arises from calculating the distributional Riemann tensor from (2.22). It will be shown that one, in general, obtains another δ term which can be interpreted as a thin shell of matter. From this point, we demand that the first junction condition is satisfied (or else the distributional solution $g_{\alpha\beta}$ is ill-defined from the start).

The Riemann tensor is a (1,3) tensor which, in index notation, is given by

$$ R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha{}_{\beta\delta} - \partial_\delta \Gamma^\alpha{}_{\beta\gamma} + \Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta} - \Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\beta\gamma}. \tag{2.32} $$

The Christoffel symbols ${\Gamma^\alpha}_{\beta\gamma}$ were given in (2.25). Computing the derivative of the Christoffel symbols yields

$$\partial_\delta {\Gamma^\alpha}_{\beta\gamma} = \Theta(\ell)\partial_\delta {\Gamma^{+\alpha}}_{\beta\gamma} + \Theta(-\ell)\partial_\delta {\Gamma^{-\alpha}}_{\beta\gamma} + \epsilon\delta(\ell)[{\Gamma^\alpha}_{\beta\gamma}]n_\delta. \tag{2.33}$$

Plugging (2.33) into (2.32) and simplifying gives

$$R^\alpha{}_{\beta\gamma\delta} = \Theta(\ell)R^{+\alpha}{}_{\beta\gamma\delta} + \Theta(-\ell)R^{-\alpha}{}_{\beta\gamma\delta} + \delta(\ell)A^\alpha{}_{\beta\gamma\delta}, \tag{2.34}$$

where

$$A^\alpha{}_{\beta\gamma\delta} = \epsilon\left([{\Gamma^\alpha}_{\beta\delta}]n_\gamma - [{\Gamma^\alpha}_{\beta\gamma}]n_\delta\right). \tag{2.35}$$

The δ term in (2.34) indicates that the Riemann tensor has a discontinuity at the junction surface Σ. However, such a discontinuity is quite mild and can be physically interpreted via the Einstein equations as a thin shell of stress-energy. It can be shown, in [31] for example, that one may write the stress energy tensor of the hybrid spacetime as

$$T^{\alpha\beta} = \Theta(\ell)T^{\alpha\beta}_+ + \Theta(-\ell)T^{\alpha\beta}_- + \delta(\ell)S^{ab}e^\alpha{}_a e^\beta{}_b, \tag{2.36}$$

where

$$S^{ab} = -\frac{\epsilon}{8\pi}\left([K_{ab}] - [K]h_{ab}\right). \tag{2.37}$$

Therefore in order for there to be no discontinuity in the Riemann tensor along the junction, we must have

$$[K_{ab}] = 0. \tag{2.38}$$

Condition (2.38) is the second junction condition. As can be seen from (2.36), if $[K_{ab}] \neq 0$ then we still get a valid solution just with an instantaneous stress-energy contribution. It is in this sense that we call S^{ab} a thin shell or surface layer.

2.4 Vaidya Spacetime

In this section we review the Vaidya spacetime as a dynamical, spherically symmetric solution of Einstein's equations describing ingoing or outgoing dust [36]. The Vaidya solution provides the jumping off point for the remainder of this analysis and several generalizations will be reviewed subsequently. More details on the Vaidya solution than are presented here can be found in [10, 31].

The Vaidya metric is a generalization of the Schwarzschild spacetime, allowing the mass M to vary with retarded time v. The line element in ingoing Eddington-Finkelstein coordinates $\{v, r, \theta, \phi\}$, known as the ingoing Vaidya metric, is given by

$$ds^2 = -\left(1 - \frac{2M(v)}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2, \qquad (2.39)$$

where $d\Omega^2$ is the two sphere line element

$$d\Omega^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (2.40)$$

Likewise one may write the outgoing Vaidya metric in terms of outgoing Eddington-Finkelstein coordinates $\{u, r, \theta, \phi\}$:

$$ds^2 = -\left(1 - \frac{2M(u)}{r}\right) du^2 - 2dudr + r^2 d\Omega^2. \qquad (2.41)$$

A convenient way of looking at both the ingoing and outgoing cases simultaneously is to use a coordinate w and parameter ϵ such that whenever $\epsilon = 1$, $w = v$ and whenever $\epsilon = -1$, $w = u$. Then both ingoing and outgoing line elements can then be written as

$$ds^2 = -\left(1 - \frac{2M(w)}{r}\right) du^2 + 2\epsilon dwdr + r^2 d\Omega^2. \qquad (2.42)$$

Unlike Schwarzschild, the Vaidya solution is not a vacuum solution but rather a dynamical solution in which the mass function $M(w)$ can be interpreted as a spherically symmetric body absorbing or emitting null dust. To see this, one may calculate the stress-energy tensor $T_{\alpha\beta}$ from the metric (2.42). The only non-zero component is

$$T_{ww} = \frac{\epsilon}{4\pi r}\frac{dM(w)}{dw}. \tag{2.43}$$

Relative to the radially outward and radially inward null vectors

$$\ell = \frac{\partial}{\partial v} + \frac{\epsilon}{2}\left(1 - \frac{2M(v)}{r}\right)\frac{\partial}{\partial r}, \tag{2.44}$$

and

$$N = -\epsilon\frac{\partial}{\partial r} \tag{2.45}$$

the stress energy tensor may be written in the form

$$T_{\alpha\beta} = \mu N_\alpha N_\beta, \tag{2.46}$$

where

$$\mu = \frac{\epsilon}{4\pi r}\frac{dM(w)}{dw}. \tag{2.47}$$

A stress-energy tensor with the form given by (2.46) is a perfect fluid with no pressure, usually called relativistic or null dust [33]. The energy flux in the direction N is given by μ. The energy conditions are satisfied if $\mu > 0$. If $\epsilon = 1$, then we require $M_v > 0$ and the mass is absorbing radiation, and if $\epsilon = -1$ then we require $M_u < 0$ with the mass emitting radiation. These two cases are black and white hole solutions respectively.

2.5 Vaidya-Reissner-Nordström and Energy Condition Violations

In this section we review the Vaidya-Reissner-Nordström spacetime (VRN hereafter), which is sometimes called charged Vaidya [30], as a generalization of both the Vaidya spacetime and the Reissner-Nordström spacetime. It will be shown that there exist regions in which all energy conditions are violated. The ingoing VRN metric is given by

$$ds^2 = -\left(1 - \frac{2M(v)}{r} + \frac{Q(v)^2}{r^2}\right) dv^2 + 2dvdr + r^2 d\Omega^2, \tag{2.48}$$

and the outgoing VRN solution is given by

$$ds^2 = -\left(1 - \frac{2M(u)}{r} + \frac{Q(u)^2}{r^2}\right) dv^2 - 2dudr + r^2 d\Omega^2. \tag{2.49}$$

Using the same notation as in the previous section, both ingoing and outgoing metrics can be written simultaneously as

$$ds^2 = -\left(1 - \frac{2M(w)}{r} + \frac{Q(w)^2}{r^2}\right) dw^2 + 2\epsilon dwdr + r^2 d\Omega^2. \tag{2.50}$$

The function $Q(v)$ is interpreted as charge, and $M(v)$ retains the role as the mass function from the previous section. It is assumed here that $M(v)$ and $Q(v)$ are positive definite functions. The Faraday tensor, $F_{\alpha\beta}$, will have non-vanishing components

$$F_{rv} = -F_{vr} = \frac{Q(v)}{r^2}, \tag{2.51}$$

corresponding to a radial electric field and no magnetic field. The electromagnetic field tensor is given generically by [31]

$$E^{\alpha}{}_{\beta} = \frac{1}{4\pi}\left(F^{\alpha\mu}F_{\beta\mu} - \frac{1}{4}\delta^{\alpha}{}_{\beta}F^{\mu\nu}F_{\mu\nu}\right), \tag{2.52}$$

and so Einstein's equations may be written as

$$G_{\alpha\beta} = \frac{1}{8\pi}\left(M_{\alpha\beta} + E_{\alpha\beta}\right), \tag{2.53}$$

where $M_{\alpha\beta}$ represents the non-electromagnetic matter components of the stress-energy. By separating out the electromagnetic portion of the stress-energy, as in (2.53), we can show that $M_{\alpha\beta}$ may be written in the form of null dust. Since the matter component $M_{\alpha\beta}$ is null dust, VRN represents a generalization of Vaidya. Solving (2.53) for $M_{\alpha\beta}$ using (2.52) yields the non-zero component

$$T_{ww} = \frac{M_w r - QQ_w}{4\pi r^3}. \tag{2.54}$$

Therefore, relative to the null vectors ℓ and N given in (2.44) and (2.45), we again have a null dust stress-energy tensor

$$T_{\alpha\beta} = \mu N_\alpha N_\beta, \tag{2.55}$$

with energy flux given by

$$\mu = \frac{M_w r - QQ_w}{4\pi r^3}. \tag{2.56}$$

One difference between the VRN case and the pure Vaidya case is that the energy conditions are not trivially satisfied; four different scenarios can occur. As outlined in Section 2.1, all of the energy conditions are satisfied if $\mu > 0$. That is,

$$M_w r - QQ_w > 0. \tag{2.57}$$

Defining a hypersurface R_0 by

$$R_0 = Q\left|\frac{Q_w}{M_w}\right|, \tag{2.58}$$

this particular hypersurface defines a boundary beyond which energy condition violations occur. Depending on the respective parities of M_w and Q_w, there will exist regions of the spacetime where energy condition violations occur. Table 2.2 enumerates the regions of violation. Physically the most natural case to consider is when $M_v > 0$ and $Q_v > 0$, which corresponds to an accreting charged null fluid. In [30], a resolution to these regions of violation is proposed wherein the violation region is excised and replaced by an outgoing VRN solution in a spacetime surgery. It was further claimed that such a surgery could be done without having a thin shell discontinuity arising from the junction conditions. We will return to this resolution in the more general context of Husain null fluid spacetimes in Chapter 4.

M_w Parity	Q_w Parity	Violation Regions
$M_w > 0$	$Q_w > 0$	$r < R_0$
$M_w > 0$	$Q_w < 0$	No violations
$M_w < 0$	$Q_w > 0$	All values of r
$M_w < 0$	$Q_w < 0$	$r > R_0$

Table 2.2: Regions of energy condition violations in VRN spacetime

Chapter 3

Husain Null Fluid Spacetimes and Energy Condition Violations

In this chapter we review the Husain null fluid spacetime presented in [17] as a one parameter generalization of the VRN spacetime. After reviewing the construction, we look at how energy condition violations can arise in this more general spacetime.

3.1 Husain Null Fluid Spacetimes

The basic idea behind the derivation of the Husain metric is to assume an arbitrary spherically symmetric spacetime and write the stress-energy tensor as a type-II null fluid. By the imposition of an equation of state relating tangential pressure P to the fluid energy density ρ, one can restrict the spacetime to those obeying the dominant energy condition. However, it will be shown that there will still exist regions of violation if all energy conditions are assumed to be satisfied.

The most general spherically symmetric metric in ingoing Eddington-Finkelstein coordinates (r, v, θ, ϕ) is given by

$$ds^2 = -e^{2\psi(r,v)}F(r,v)dv^2 + 2e^{\psi(r,v)}dvdr + r^2 d\Omega^2, \tag{3.1}$$

where $\psi(r,v)$ and $F(r,v)$ are arbitrary functions and $r \in [0,\infty)$ and $v \in (-\infty,\infty)$. It is assumed for simplicity that $\psi(r,v) = 0$, and so the metric is reduced to

$$ds^2 = -\left(1 - \frac{2m(r,v)}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2, \tag{3.2}$$

where $F(r,v)$ has been rewritten in terms of an arbitrary mass function $m(r,v)$. The Einstein tensor $G_{\alpha\beta}$ has non-zero components

$$G^v{}_v = -\frac{2m_r}{r^2}, \tag{3.3}$$

$$G^r{}_v = \frac{2m_v}{r^2}, \tag{3.4}$$

and

$$G^\theta{}_\theta = G^\phi{}_\phi = -\frac{m_{rr}}{r}, \tag{3.5}$$

By Einstein's equations, these quantities can be related to the stress-energy tensor $T_{\alpha\beta}$ by

$$m_r = -4\pi r^2 T^v{}_v, \tag{3.6}$$

and

$$m_v = 4\pi r^2 T^v{}_r. \tag{3.7}$$

Relative to the two null vectors ℓ and N given by

$$\ell = \frac{\partial}{\partial v} + \frac{1}{2}\left(1 - \frac{2m(v,r)}{r}\right)\frac{\partial}{\partial r} \tag{3.8}$$

and

$$N = -\frac{\partial}{\partial r}, \tag{3.9}$$

the stress-energy tensor for the null fluid spacetime can be expressed as

$$T_{\alpha\beta} = \mu N_\alpha N_\beta + \rho(r,v)(\ell_\alpha N_\beta + N_\alpha \ell_\beta) + P(r,v)(g_{\alpha\beta} + \ell_\alpha N_\beta + \ell_\beta N_\alpha), \tag{3.10}$$

where

$$\rho = \frac{m_r}{4\pi r^2}, \tag{3.11}$$

$$P = -\frac{m_{rr}}{8\pi r}, \tag{3.12}$$

and

$$\mu = \frac{m_v}{4\pi r^2}. \tag{3.13}$$

The stress-energy tensor can be interpreted as an inward falling, self-interacting null fluid with energy flux μ in the direction of N, self-interaction energy density ρ, and tangential pressure P. In order for all of the energy conditions to hold we must have

$$\mu \geq 0, \quad \rho \geq 0, \quad \text{and} \quad 0 \leq P \leq \rho. \tag{3.14}$$

In terms of the mass function $m(v,r)$, these conditions translate to the conditions

$$m_v \geq 0, \quad m_r \geq 0, \quad \text{and} \quad -2m_r \leq r m_{rr} \leq 0. \tag{3.15}$$

In order to satisfy the latter two conditions one may impose the general fluid equation of state

$$P = k\rho^a, \tag{3.16}$$

where $a \leq 1$. Only the case $a = 1$ and $k > 1/2$ are considered here because $k < 1/2$

leads to the loss of asymptotic flatness [17]. The equation of state, (3.11), and (3.12), provide an integrable differential equation

$$-\frac{m_{rr}}{8\pi r} = \frac{km_r}{4\pi r^2}, \tag{3.17}$$

having the general solution

$$m(r,v) = f(v) - \frac{g(v)}{r^{2k-1}}, \tag{3.18}$$

where the functions $f(v)$ and $g(v)$ are arbitrary. The line element (3.2) then becomes

$$ds^2 = -\left(1 - \frac{2f(v)}{r} + \frac{2g(v)}{r^{2k}}\right) dv^2 + 2dvdr + r^2 d\Omega^2. \tag{3.19}$$

The energy conditions guarantee that $g(v) \geq 0$ and so we may define a new function $\Xi(v)$ by

$$\Xi(v)^{2k} = 2g(v). \tag{3.20}$$

Rewriting the metric as

$$ds^2 = -\left(1 - \frac{2f(v)}{r} + \left(\frac{\Xi(v)}{r}\right)^{2k}\right) dv^2 + 2dvdr + r^2 d\Omega^2, \tag{3.21}$$

it now has the form of the Husain metric to be used in the remainder of this work. Choosing $k = 1$ reduces to VRN and choosing $\Xi(v) = 0$ reduces to Vaidya. Using the expression for the mass function $m(r,v)$ in (3.18) we can obtain explicit expressions for both the energy density and pressure:

$$\rho = \frac{(2k-1)}{16\pi}\left(\frac{\Xi(v)}{r}\right)^{2k}, \text{ and } \quad P = \frac{k(2k-1)}{16\pi}\left(\frac{\Xi(v)}{r}\right)^{2k}. \tag{3.22}$$

3.2 Energy Condition Violations

The energy conditions which must be satisfied are, from (3.14),

$$\mu \geq 0, \quad \rho \geq 0, \quad \text{and} \quad 0 \leq P \leq \rho. \tag{3.23}$$

Requiring $\rho \geq 0$ gives the condition of $g(v) \geq 0$, which we already used for the definition of $\Xi(v)$. Futher requiring $0 \leq P \leq \rho$ places an upper bound $k \leq 1$ on the parameter k. Therefore we must have $1/2 < k \leq 1$.

The inequality which creates regions of violations is the same as in the VRN case. Requiring $\mu \geq 0$ yields

$$M_v - k\Xi_v \left(\frac{\Xi}{r}\right)^{2k-1} \geq 0. \tag{3.24}$$

This inequality cannot be satisfied by a limitation of parameter definitions like in the previous cases. Just as in the VRN situation, violations arise depending on the parity of M_v and Ξ_v. Defining

$$R_0 = \Xi \left(k \left|\frac{\Xi_v}{M_v}\right|\right)^{1/(2k-1)}, \tag{3.25}$$

the four possible cases are presented in Table 3.1. The surface defined by $r = R_0(v)$, in the appropriate scenarios of Table 3.1, gives the boundary beyond which energy condition violations occur. This $\mu = 0$ surface is the junction hypersurface used in [30] for the VRN spacetime.

M_v Parity	Q_v Parity	Violation Regions
$M_v \leq 0$	$Q_v > 0$	All values of r
$M_v > 0$	$Q_v > 0$	$r < R_0$
$M_v < 0$	$Q_v < 0$	$r > R_0$
$M_v \geq 0$	$Q_v < 0$	No Violations

Table 3.1: Regions of energy condition violations in ingoing Husain spacetime

The outgoing Husain spacetime is defined analogously to the outgoing VRN spactime. The line element,

$$ds^2 = -\left(1 - \frac{2M(u)}{r} + \left(\frac{\Xi(u)}{r}\right)^{2k}\right) du^2 - 2dudr + r^2 d\Omega^2, \tag{3.26}$$

gives regions of violation, as indicated in Table 3.2, with the surface beyond which violations occur given by

$$\tilde{R}_0 = \Xi \left(k \left|\frac{\Xi_u}{M_u}\right|\right)^{1/(2k-1)}. \tag{3.27}$$

M_u Parity	Q_u Parity	Violation Regions
$M_u \leq 0$	$Q_u > 0$	All values of r
$M_u > 0$	$Q_u > 0$	$r < \tilde{R}_0$
$M_u < 0$	$Q_u < 0$	$r > \tilde{R}_0$
$M_u \geq 0$	$Q_u < 0$	No Violations

Table 3.2: Regions of energy condition violations in outgoing Husain spacetime

Therefore the Husain spacetime, as a 1-parameter generalization of VRN, suffers the same violations as VRN. That is, the VRN spacetime is not a special case in which violations occur based on some special dependence on k. The solution to removing violating regions can be taken from Ori's construction in [30]: remove the regions of violation and replace them with the outgoing Husain solution. Of course, there could be thin shells that are introduced in this more general setting and so the junction conditions must be checked explicitly.

Chapter 4

Bouncing and Radiation Models

The purpose of this chapter is two-fold: first we generalize Ori's construction to Husain null fluids, and second we construct a model of black hole radiation using Husain null fluid spacetimes.

In the first section we discuss the intrinsic and extrinsic geometry of an arbitrary spherically symmetric hypersurface. Limiting to the case of spacelike hypersurfaces in the second section, Ori's construction is generalized using more general matching conditions. Following this construction, a simple model example is given. Turning then to the case of timelike hypersurfaces, we examine a model of black hole radiation.

4.1 Geometry of Hypersurfaces

In order to check the Israel-Darmois junction conditions later on, we first study the intrinsic and extrinsic geometry of spherically symmetric hypersurfaces. Consider the metric

$$ds^2 = -f(w, r)dw^2 + 2\epsilon dwdr + r^2 d\Omega^2, \tag{4.1}$$

where $\epsilon = \pm 1$ corresponds to ingoing ($w = v$) for $\epsilon = 1$ and outgoing ($w = u$) for $\epsilon = -1$ just as before. For a Husain null fluid spacetime

$$f(w,r) = 1 - \frac{2M(w)}{r} + \left(\frac{\Xi(w)}{r}\right)^{2k}, \tag{4.2}$$

but for the sake of clarity and tidiness we will keep the more general function $f(w,r)$ in the subsequent construction.

Let Σ be a spherically symmetric hypersurface parametrized by $r = R(\lambda)$ and $w = W(\lambda)$ in a spacetime $(\mathcal{M}, g)$. The induced metric on Σ can be computed by pulling the spacetime metric back to the hypersurface via a map $\Phi : \mathcal{M} \to \Sigma$ given by

$$\Phi^{-1}(\lambda, \theta, \phi) = (W(\lambda), R(\lambda), \theta, \phi), \tag{4.3}$$

and its line element is

$$d\Sigma^2 = h_{ab}dx^a dx^b = (-f\dot{W}^2 + 2\epsilon \dot{W}\dot{R})d\lambda^2 + R^2 d\Omega^2, \tag{4.4}$$

with dots indicating a derivative with respect to the parameter λ.

4.2 Spacelike Case: Bouncing Null Fluids

In order to discuss the extrinsic geometry of Σ it is useful to construct a unit tetrad. The timelike unit vector $\hat{u}$ pointing in the positive w direction is

$$\hat{u}^\alpha \partial_\alpha = \frac{\epsilon}{\sqrt{2\epsilon R_w - f}} \left(\frac{\partial}{\partial w} + (\epsilon f - R_w)\frac{\partial}{\partial r}\right). \tag{4.5}$$

The spacelike unit vector $\hat{n}$ pointing towards positive r is

$$\hat{n}^{\alpha}\partial_{\alpha} = \frac{\epsilon}{\sqrt{2\epsilon R_w - f}}\left(\frac{\partial}{\partial w} + R_w \frac{\partial}{\partial r}\right) \tag{4.6}$$

and the unit angular tangent vectors are given by

$$\hat{e}^{\alpha}_{\theta}\partial_{\alpha} = \frac{1}{r}\frac{\partial}{\partial \theta} \tag{4.7}$$

and

$$\hat{e}^{\alpha}_{\phi}\partial_{\alpha} = \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}. \tag{4.8}$$

Here $R_w = \frac{\dot{R}}{\dot{W}} = \frac{dR}{dw}$ where Σ is reparametrized as $r = R(w)$.

By assuming Σ is spacelike, the extra requirement

$$\dot{W}\left(-f\dot{W} + 2\epsilon\dot{R}\right) > 0 \tag{4.9}$$

must be satisfied. The hypersurface adapted tetrad, Equations (4.5 - 4.8), has unit normal $\hat{u}$. Then the extrinsic curvature three tensor K_{ij} is given by

$$K_{ij} = \hat{e}^{\alpha}_{i}\hat{e}^{\beta}_{j}\nabla_{\alpha}\hat{u}_{\beta}, \tag{4.10}$$

where $i, j = 1, 2, 3$. The extrinsic curvature can be expressed as

$$K = \left(\frac{-\epsilon(2R_{ww} + ff_r) + (f_w + 3f_r R_w)}{2(2\epsilon R_w - f)^{3/2}}\right)(\hat{n}\otimes\hat{n}) + \left(\frac{\epsilon f - R_w}{R\sqrt{2\epsilon R_w - f}}\right)(\hat{e}_{\theta}\otimes\hat{e}_{\theta} + \hat{e}_{\phi}\otimes\hat{e}_{\phi}). \tag{4.11}$$

Thus far no matching has taken place and the analysis has been completely general with the function $f(w, r)$ and spherically symmetric spacelike hypersurface Σ.

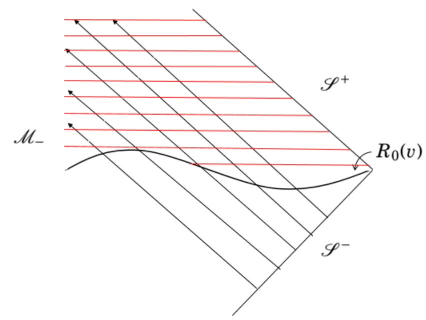

Figure 4.1: Ingoing Husain Spacetime diagram illustrating violations.
The horizontal lines indicate the region of energy condition violations which we wish to exise.

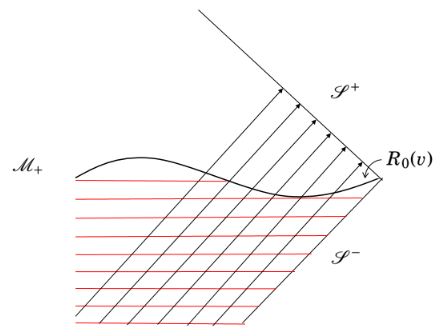

Figure 4.2: Outgoing Husain Spacetime diagram illustrating violations.
The horizontal lines indicate the region of energy condition violations.

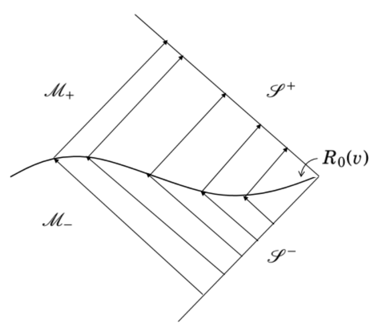

Figure 4.3: Hybrid Spacetime diagram with spacelike junction surface.
The two solutions are attached via the $\mu = 0$ surface R_0 and no energy conditions are violated. The ingoing and outgoing lines are ingoing and outgoing geodesics characterized by lines of constant v and constant u respectively.

4.2.1 Matching Conditions

In this subsection we look at matching the infalling ($\mathcal{M}_-$) and radiating Husain solutions ($\mathcal{M}_+$) along a spacelike hypersurface Σ (See Figures 4.1 - 4.3 for the construction diagram). The metrics are

$$ds^2 = -f(v,r)dv^2 + 2dvdr + r^2 d\Omega^2, \tag{4.12}$$

and

$$ds^2 = -\tilde{f}(u,\tilde{r})du^2 - 2dud\tilde{r} + \tilde{r}^2 d\Omega^2 \tag{4.13}$$

respectively. Parametrize Σ by

$$(v,r) = (V(\lambda), R(\lambda)) \tag{4.14}$$

and

$$(u,\tilde{r}) = (U(\lambda), \tilde{R}(\lambda)) \tag{4.15}$$

for each of the respective spacetimes. The first junction condition states that the induced metric h_{ab} must be the same as viewed from either $\mathcal{M}_-$ or $\mathcal{M}_+$. The induced metrics are

$$d\Sigma_-^2 = (-f\dot{V}^2 + 2\dot{V}\dot{R})d\lambda^2 + R^2 d\Omega^2, \tag{4.16}$$

and

$$d\Sigma_+^2 = (-\tilde{f}\dot{U}^2 - 2\dot{U}\dot{\tilde{R}})d\lambda^2 + \tilde{R}^2 d\Omega^2, \tag{4.17}$$

respectively from (4.4). The first junction condition immediately gives

$$R(\lambda) = \tilde{R}(\lambda), \tag{4.18}$$

and therefore we discard the tilde and simply use $R(\lambda)$.

The more interesting matching cases arise from matching the (λ, λ) components:

$$f\dot{V}^2 - 2\dot{V}\dot{R} = \tilde{f}\dot{U}^2 + 2\dot{U}\dot{R}. \tag{4.19}$$

We assume that the matter on the hypersurface is the same as described by both sides. That is,

$$M^-(V(\lambda)) = M^+(U(\lambda)) \quad \text{and} \quad \Xi^-(V(\lambda)) = \Xi^+(U(\lambda)), \tag{4.20}$$

which reduces (4.19) to

$$f\dot{V}^2 - 2\dot{V}\dot{R} = f\dot{U}^2 + 2\dot{U}\dot{R}. \tag{4.21}$$

The assumption of (4.20) is a physical assumption, not a mathematical one. In the special $k = 1$ case of VRN, M and Ξ represent the physical quantities of mass and charge and it is reasonable to require a condition such as (4.19) because of this. Factoring (4.21) yields

$$(\dot{V} + \dot{U})\left(f(\dot{V} - \dot{U}) - 2\dot{R}\right) = 0, \tag{4.22}$$

which indicates two possible matchings:

$$\textbf{Reflective: } \dot{U} = -\dot{V} \quad \Rightarrow \quad U_v = -1, \tag{4.23}$$

and

$$\textbf{Ori: } \dot{U} = \dot{V} - \frac{2\dot{R}}{f} \quad \Rightarrow \quad U_v = 1 - \frac{2R_v}{f}. \tag{4.24}$$

The right hand side corresponds to choosing the parameter λ to be the ingoing coordinate v.

It should be noted that in both cases $U_v < 0$. Such an observation is obvious for the reflective case, and the Ori case can be written as

$$U_v = 1 - \frac{2R_v}{f} = \frac{1}{f}(f - 2R_v) < 0 \tag{4.25}$$

because $f > 0$ by assumption and $f - 2R_v < 0$ by the spacelike nature of Σ. As a consequence of U_v always being negative and the matching condition, one may show that

$$\frac{1}{\sqrt{-f - 2R_u}} = -\frac{U_v}{\sqrt{-f + 2R_v}}. \tag{4.26}$$

To see this identity, we start with the more general matching condition (4.21) yet again:

$$f - 2R_v = fU_v^2 + 2U_v R_v. \tag{4.27}$$

This may be rewritten as

$$\begin{aligned} -f + 2R_v &= U_v^2(-f - 2R_u) \\ \Rightarrow \frac{1}{-f - 2R_u} &= \frac{U_v^2}{-f + 2R_v} \\ \Rightarrow \frac{1}{\sqrt{-f - 2R_u}} &= -\frac{U_v}{\sqrt{-f + 2R_v}} \end{aligned}$$

after using $R_u = \frac{R_v}{U_v}$ and taking the negative square root due to the fact that $U_v < 0$. Thus the identity is established and one may trade denominators with R_u terms for denominators with R_v terms. Such an identity proves to be useful in calculating the second junction condition.

The time reflective case corresponds to the matching condition used in [2] for the extremal Reissner-Nordström case, whereas the Ori condition is (as the name implies) the one used in [30]. For the static case, pure Schwarzschild or Reissner-Nordström, the Ori transformation is the one which reparametrizes the surface from ingoing to outgoing coordinates.

In both matching choices the second junction condition can be checked to calculate the stress-energy induced by any discontinuities introduced in the construction. If the extrinsic curvatures of Σ in $\mathcal{M}_-$ and $\mathcal{M}_+$ do not match, then there exists a thin shell of matter located at Σ with a stress-energy tensor given by (2.37):

$$S_{ij} = -\frac{1}{8\pi}\left([K_{ij}] - [K]h_{ij}\right). \tag{4.28}$$

The radial and tangential pressure densities are given by

$$S_{nn} = \frac{1}{4\pi}[K_{\theta\theta}], \tag{4.29}$$

and

$$S_{\theta\theta} = S_{\phi\phi} = \frac{1}{8\pi}\left([K_{\theta\theta}] + [K_{nn}]\right). \tag{4.30}$$

Using the extrinsic curvature given by (4.11), the components are

$$[K_{\theta\theta}] = \frac{2R_v - f(1-U_v)}{R\sqrt{2R_v - f}} \tag{4.31}$$

and

$$[K_{nn}] = \frac{2R_{vv}(1-U_v) + 2R_vU_{vv} + ff_r(1-U_v^3) - (f_v + 3f_rR_v)(1+U_v^2)}{2(2R_v - f)^{3/2}}. \tag{4.32}$$

In both of the above calculations we have eliminated R_u using $R_u = \frac{R_v}{U_v}$ and traded denominators using (4.26).

Specializing now to the Reflective matching, the thin shell stress-energy components are calculated to be

$$S_{nn}^{\text{ref}} = \frac{R_v - f}{2\pi R\sqrt{2R_v - f}} = 2[K_{nn}^{+}], \tag{4.33}$$

and

$$S_{\theta\theta}^{\text{ref}} = \frac{R_v - f}{4\pi R\sqrt{2R_v - f}} + \frac{2R_v v + f_r(f - 3f_r R_v) - f_v}{8\pi(2R_v - f)^{3/2}} = 2[K_{22}^{+}]. \tag{4.34}$$

Using

$$U_{vv} = -\frac{2R_{vv}}{f} + \frac{2f_r R_v^2}{f^2} + \frac{2f_v R_v}{f^2}, \tag{4.35}$$

the Ori components are

$$S_{nn}^{\text{Ori}} = 0 \tag{4.36}$$

and

$$S_{\theta\theta}^{\text{Ori}} = \frac{f_v}{4\pi R\sqrt{2R_v - f}}. \tag{4.37}$$

In both cases there will be generically a thin shell present. In the reflective matching case this corresponds to a doubling effect caused by $u = -v$ on the surface; that is, the extrinsic curvatures are negatives of each other and so the jump is a doubling of the extrinsic curvature of one of the sides.

In the Ori case the thin shell can be eliminated only if $f_v = 0$. In the original Ori construction for VRN, the junction surface was chosen to be the $\mu = 0$ hypersurface beyond which energy conditions are violated. Since μ is defined by m_v (equivalently f_v here) in (3.15), along this choice of hypersurface the thin shell will vanish and be consistent with the statements of [30].

Therefore the spacelike construction and spacetime surgery of Husain null fluids accomplishes two things. First, it resolves the conflict between [2] and [30]: the reason one calculated a thin shell stress energy and the other didn't was due to a difference choice of matching conditions. In the Extremal Reissner-Nordström case, the reflective matching was chosen. In fact, since the extremal Reissner-Nordström case is defined by $f(v,r) = 0$, the Ori matching is not available as a coordinate matching. Second, the construction generalizes the Ori procedure to the wider class of Husain spacetimes which do not necessarily need to include charge but are still plagued by the same energy condition violations.

4.2.2 Linear Matter

In this section we test the validity of the assumptions of the construction given in the previous section. That is, does there exist a f such that the $\mu = 0$ surface is spacelike and not within a trapped surface?

The first thing we must establish is the location of trapped and untrapped regions within our spacetime. A trapped surface is one in which both the expansions of the ingoing and outgoing null congruences, θ, are non-positive. In the ℓ and N directions the expansions are

$$\theta_{(\ell)} = \frac{\epsilon f}{r} \tag{4.38}$$

and

$$\theta_{(N)} = -\frac{2\epsilon}{r}. \tag{4.39}$$

If $\epsilon = 1$, corresponding to the infalling null fluid, $\theta_{(\ell)} < 0$ only if $f < 0$ and $\theta_{(N)}$ is always negative. Therefore the surface is trapped when $f < 0$. An apparent horizon ($\theta_{(\ell)} = 0$) occurs at $f = 0$. If $\epsilon = -1$ then the opposite situation occurs an the totally untrapped region ($\theta_\ell, \theta_N > 0$) must have $f < 0$. The apparent horizon is still present at $f = 0$. Therefore the regular part of spacetime is when $f > 0$.

Therefore we may conclude that the surface $\mu = 0$ is outside of the black and white hole regions if we can choose $M(v)$ and $\Xi(v)$ such that

$$2\epsilon R_w - f(w, R(w)) > 0 \tag{4.40}$$

and

$$f(w, R(w)) > 0. \tag{4.41}$$

Consider the case of linear matter $\Xi(w) = \xi M(w)$, where $\xi > 0$ is a constant. In the $k = 1$ case of VRN, ξ corresponds to the charge to mass ratio of the fluid. From (3.25) the $\mu = 0$ surface is

$$R(w) = \chi M(w), \tag{4.42}$$

where $\chi = \xi(k\xi)^{1/(2k-1)}$. It follows that

$$f(w, R(w)) = 1 - \frac{2}{\chi} + \left(\frac{\xi}{\chi}\right)^{2k} = 1 - \frac{1}{\chi}\left(\frac{2k-1}{k}\right), \tag{4.43}$$

and therefore $f(w, R(w))$ is constant and is positive if

$$\chi > \frac{2k-1}{k}. \tag{4.44}$$

In terms of the constant ξ the inequality becomes

$$\xi > \frac{(2k-1)^{(2k-1)/2k}}{k}. \tag{4.45}$$

Returning to the VRN case ($k = 1$), the condition requires that the charge to mass ratio must be greater than 1.

From (4.40), the junction surface $R(w)$ is spacelike in the ingoing spacetime if

$2R_v \geq f$. That is, we need

$$M_v > \frac{f}{2\chi} = \frac{1}{2k\chi^2}(k\chi + 1 - 2k). \tag{4.46}$$

Thus for any choice of (ξ, χ) we have a lower bound on M_v and so the surface $R(w)$ can be chosen to be spacelike. The lower bound represents a minimum rate of expansion of the black hole.

The thin shell stress energy can be calculated for this special case using (4.33) - (4.37). In the Ori condition, since we are matching along the $\mu = 0$ surface, no thin shell will be present from the general construction results. However, in the time reflection case in general there will be a thin shell given by the components

$$S_{nn}^{\text{ref}} = \frac{M_v - f/\chi}{2\pi M\sqrt{2M_v\chi - f}} \tag{4.47}$$

and

$$S_{\theta\theta}^{\text{ref}} = \frac{1}{4\pi\sqrt{\chi}}\left(\frac{M_v - f/\chi}{M\sqrt{2M_v - f/\chi}} + \frac{M_{vv}}{(2M_v - f/\chi)^{3/2}}\right), \tag{4.48}$$

where f is given by (4.43). The thin shell components will vanish in the restrictive case of linear accretion, $M_v = (f/\chi)v$, but in general still will not vanish.

4.3 Timelike Case: Null Fluid Hawking Radiation

By requiring the junction surface to be timelike rather than spacelike, a different physical model can be considered. Although the timelike construction is very similar to that of the null fluid bounce model, the physical interpretation is quite different; this of course illustrates the usefulness of spacetime surgeries in easily creating physical models.

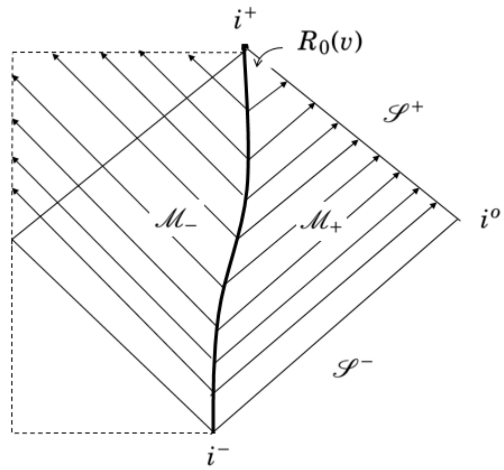

Figure 4.4: Schematic Hybrid Spacetime diagram with time junction surface. The two solutions are attached via the timelike surface R_0. The ingoing and outgoing lines are ingoing and outgoing geodesics characterized by lines of constant v and constant u respectively.

Assume now that Σ is a timelike hypersurface necessarily satisfying

$$\dot{W}(-f\dot{W} + 2\epsilon\dot{R}) < 0. \tag{4.49}$$

Further, the spacelike normal to the surface is

$$\hat{n}^\alpha \partial_\alpha = \frac{\epsilon}{\sqrt{f - 2\epsilon R_w}} \left(\frac{\partial}{\partial w} + (\epsilon f - R_w) \frac{\partial}{\partial r} \right). \tag{4.50}$$

and the timelike tangent vector is

$$\hat{n}^\alpha \partial_\alpha = \frac{\epsilon}{\sqrt{f - 2\epsilon R_w}} \left(\frac{\partial}{\partial w} + R_w \frac{\partial}{\partial r} \right). \tag{4.51}$$

The angular tangent vectors are the same as in Equations (4.7) and (4.8).

4.3.1 Matching Conditions

Since the metrics which are being matched along Σ are the same, the first junction condition will produce the same matching conditions. These are repeated here as

$$\textbf{Reflective:}\ \dot{U} = -\dot{V} \quad \Rightarrow \quad U_v = -1,$$

$$\textbf{Ori:}\ \dot{U} = \dot{V} - \frac{2\dot{R}}{f} \quad \Rightarrow \quad U_v = 1 - \frac{2R_v}{f}. \tag{4.52}$$

The extrinsic curvature, given by

$$K_{ij} = e_i^{\alpha} e_j^{\beta} \nabla_{\alpha} \hat{n}_{\beta}, \tag{4.53}$$

has the explicit form

$$K = \left(\frac{3\epsilon f_r R_w - 2R_{ww} - f f_r + \epsilon f_w}{2(f - 2\epsilon R_w)^{3/2}} \right) (\hat{u} \otimes \hat{u}) + \left(\frac{f - \epsilon R_w}{R\sqrt{f - 2\epsilon R_w}} \right) (e_\theta \otimes e_\theta + e_\phi \otimes e_\phi). \tag{4.54}$$

The jumps in extrinsic curvature components for the time reflective case can be calculated and using (2.37) the stress-energy components are given as

$$S_{\theta\theta}^{\text{ref}} = 0 \tag{4.55}$$

and

$$S_{uu}^{\text{ref}} = 0. \tag{4.56}$$

Note that this result is different than the doubling result in the bounce model. The difference occurs due to the normal changing from a timelike vector in the spacelike bouncing model to a spacelike vector in the timelike radiation model. Therefore there will be no thin shell for any choice of f in the reflective timelike case. In the Ori case we compute

$$S_{\theta\theta}^{\mathrm{Ori}} = -\frac{f_v}{8\pi f\sqrt{f-2R_v}} \tag{4.57}$$

and

$$S_{uu}^{\mathrm{Ori}} = 0. \tag{4.58}$$

The timelike case yields a similar result for the Ori matching as in the bounce model. The only way for the thin shell to vanish is if $f_v = 0$; if we choose Σ to be the $\mu = 0$ surface again, then there will be no thin shell. One difference between the null radiation model and the null bounce model is the choice of surface Σ. In the null bounce model, the $\mu = 0$ surface is the only physically meaningful choice of junction surface since it represents the boundary of the violation region. However, in the null radiation model we are not concerned with the removal of energy condition violations; in fact, the radiation falling into the black hole must violate the weak energy condition in order for local energy conservation to be satisfied at Σ. This negative energy density ingoing fluid would model the evaporation of the black hole. Although the $\mu = 0$ still represents a "good choice" of Σ, we have freedom in the choice of radiating hypersurface. This choice will allow us to examine a second example in addition to the linear matter example for the timelike case.

4.3.2 Linear Matter

The linear matter model is almost identical to the spacelike case mathematically with the exception that we must demand the timelike condition (4.49) rather than the spacelike condition. We choose Σ to be the $\mu = 0$ surface again. Since the $\mu = 0$ surface represents the surface beyond which energy conditions are violated (at smaller r values), it is natural, though not necessary, to choose this as the radiating surface. Although energy condition violations are not too concerning in this case, as elaborated on above, we still want all energy conditions to be satisfied in the outgoing fluid portion of the spacetime; an observer outside of the radiating surface should not measure violations.

Recall the linear matter condition

$$\Xi(w) = \xi M(w) \tag{4.59}$$

and the timelike condition

$$2\epsilon R_w - f(w, R(w)) < 0. \tag{4.60}$$

Just as in (4.43) the junction hypersurface is $R(w) = \chi M(w)$ and

$$f(w, R(w)) = 1 - \frac{1}{\chi}\left(\frac{2k-1}{k}\right). \tag{4.61}$$

We then require

$$\xi > \frac{(2k-1)^{(2k-1)/2k}}{k} \tag{4.62}$$

just as in the spacelike case (4.45). The difference in models occurs when demanding that the timelike condition (4.60) holds. From $R(w) = \chi M(w)$ it follows that

$$M_v < \frac{1}{2k\chi^2}(k\chi + 1 - 2k). \tag{4.63}$$

Unlike the spacelike case, there is now an upper bound for each choice of (k, ξ) on the rate at which mass is accreting onto the black hole. Further, no thin shell will be present in either matching since the $\mu = 0$ surface was chosen.

4.3.3 Constant Radius Radiating Surface

In the next example we consider the case where $R(w) = R =$ constant. Since $R_w = 0$, the timelike condition and the positivity of f are the same condition:

$$f(w, R) > 0. \tag{4.64}$$

Further, we require that the radiating side of R satisfies the energy conditions. Since the $\mu = 0$ surface consitutes the surface beyond which violations occur, we choose R to be greater than the maximum r value attained by the $\mu = 0$ surface. Such a maximum will exist since negative energy density matter in accreting onto the black hole and so $m_v < 0$ in the sense of (3.13) and $R(w)$ is decreasing.

For a more concrete situation, assume a linear matter case with $\Xi(v) = \xi M(v)$. Further assume that the evaporation starts at v_0. The $\mu = 0$ surface is located at $R_0 = \xi^2 M(v_0)$ and so choose $R > R_0$ as the timelike radiating junction surface.

The time reversal thin shell will still vanish from the previous general results (4.55, 4.56). However, for the Ori matching,

$$[K_{uu}] = \frac{\frac{2M_v}{R}\left(1 - \frac{\xi^2 M(v)}{R}\right)}{\left(1 - \frac{2M(v)}{R} + \frac{\xi^2 M(v)^2}{R^2}\right)^{3/2}} \tag{4.65}$$

and

$$[K_{\theta\theta}] = 0. \tag{4.66}$$

Since $m_v < 0$ and never constant, there is no choice of constant R which will cause the thin shell to vanish for all time v.

The choice of a constant R surface is plausible in the timelike model unlike in the bounce model, but physically the $\mu = 0$ surface is still the ideal choice. In the constant

R situation, even when the black hole evaporates to a small size the radiating surface will remain at the same r value. Such a situation, although perhaps mathematically easier to analyze, is not very physical.

Chapter 5

Conclusion

In classical general relativity, the energy conditions provide physically motivated restrictions on the stress-energy tensor in order to obtain physically meaningful solutions. Although these conditions have been shown to be violated even in classical scenarios (i.e. traversable wormholes), they still provide excellent criteria for physically reasonable models and should not be disregarded. When violations of these energy conditions occur in a spacetime, it is important to look at whether there exists physics unaccounted for by the solution. This is precisely the agenda of [30] for the VRN spacetime. By carefully including a Lorentz force term in the equations of motion, it was shown that the ingoing null dust bounces at the boundary of the violation region. Physically, the bounce occurs when the Lorentz force overpowers gravitational attraction. Mathematically, this model is constructed via a spacetime surgery, replacing the violation region with an outgoing VRN spacetime. It was found that no thin shell was introduced in such a construction.

Violations also occur in the extremal Reissner-Nordström solution, which is a special case of VRN with $M = Q$. In [2], under the same construction as in [30], it was found that there is a thin shell present. Since Extremal Reissner-Nordström is a special case of VRN, there exists tension between the results of [2] and [30] which is worthy of investigation.

In [17], a null fluid model is presented: a one parameter generalization of the VRN spacetime. The model assumes an equation of state $P = k\rho$ and represents a null fluid with tangential pressure P and energy density ρ. When $k = 1$, and the pressure is interpreted as arising from charged matter, VRN is recovered. The Husain null fluid family of spacetimes not only cover the charged case of VRN but also the physically different scenario of pressurized null fluids.

With the different physics represented by Husain null fluid solutions and the conflicting results regarding thin shells in [30] and [2], it is worthwhile to examine the construction of Ori in the more general case of Husain null fluid spacetimes. Through this generalization, questions of coordinate matching along the junction surface arise and the freedom of choice resolves the tension satisfactorily; in extremal Reissner-Nordström one must choose the time reversal matching, whereas in the VRN case one has the extra option of the Ori matching. Further, one can assess the second junction condition for the Husain null fluid construction for each matching condition: with the Ori matching, along the $\mu = 0$ surface, no thin shell will be present and in the time reversal case the jump in extrsinic curvature is double the extrinsic curvature as measured by either side.

The Ori bouncing construction relies on the spacelike nature of the junction hypersurface, but there is nothing which forbids taking a timelike hypersurface instead. The major difference in taking the junction surface to be timelike is in the physics the model then explains. Although the construction is nearly identical, the timelike surface no longer corresponds to the bouncing surface since one does not have a solution which described both ingoing and outgoing null dust simultaneously. In the timelike case then, one can model null fluid radiation from a black hole by constructing a radiating junction surface with an outgoing solution on one side and an ingoing solution on the other. The outgoing solution radiates a positive energy density null fluid to null infinity and the ingoing solution radiates negative a energy density null fluid into the black hole so as to satisfy local conservation of energy. The timelike hybrid spacetime can be interpreted as a classical model of Hawking radiation similar to Vaidya models like [15].

The junction conditions for the timelike case were calculated and yielded different results than the spacelike bouncing model for the time reflection case. However, they were identical for the Ori case. In the time reversal case it was found that no thin shell is present at all, a difference to the doubling result of the spacelike case. In the Ori case there will be no thin shell present so long as the $\mu = 0$ surface is chosen again. However, unlike the spacelike null bounce model where the $\mu = 0$ surface is the only choice in which energy conditions are fully satisfied throughout the hybrid spacetime, there will necessarily be regions in which the weak energy condition violation occurs on the ingoing side (negative energy is needed to satisfy local energy conversation at the radiating surface). So long as energy conditions are satisfied on the outgoing side, one has freedom in where to place the radiating surface. Both a $\mu = 0$ surface example and a constant R surface example were explored using linear matter $\Xi = \xi M$.

The junction conditions provide an excellent resource when building general relativistic models. Hybrid spacetimes can capture many phenomena that singular solutions cannot. However, care must be taken in accounting for any thin shells or choices in matchings that arise from these spacetime surgeries. In this work we have shown that much can be obtained from simple hybrid solutions and, further, that simple assumptions regarding matching coordinates, or the nature of the junction surface, can lead to vastly different mathematical results and physical models.

Bibliography

[1] W. B. Bonnor and P. C. Vaidya. Spherically symmetric radiation of charge in Einstein-Maxwell theory. *Gen. Rel. Grav.*, 1:127–130, 1970.

[2] I. Booth. Evolutions from extremality. *Phys. Rev. D*, 93, 2016.

[3] S. Chatterjee, S. Ganguli, and A. Virmani. Charged Vaidya Solution Satisfies Weak Energy Condition. *Gen. Rel. Grav.*, 48(7):91, 2016.

[4] B. Creelman and I. Booth. Collapse and bounce of null fluids. 2016.

[5] M. Demianski and J. Lasota. Contracting and radiating bodies. *Astrophysical Letters*, 1:205, 1968.

[6] H. Epstein, V. Glaser, and A. Jaffe. Nonpositivity of the energy density in quantized field theories. *Il Nuovo Cimento (1955-1965)*, 36(3):1016–1022, 1965.

[7] L. H. Ford and T. A. Roman. Averaged energy conditions and quantum inequalities. *Physical Review D*, 51(8):4277, 1995.

[8] V. P. Frolov. Information loss problem and a 'black hole' model with a closed apparent horizon. *Journal of High Energy Physics*, 2014(5):49, 2014.

[9] F. Girotto and A. Saa. Semianalytical approach for the vaidya metric in double-null coordinates. *Physical Review D*, 70(8):084014, 2004.

[10] J. B. Griffiths and J. Podolskỳ. *Exact space-times in Einstein's general relativity.* Cambridge University Press, 2009.

[11] S. Hawking and G. Ellis. The large scale structure of the universe. *CUP, Cambridge*, 1973.

[12] S. W. Hawking. Particle creation by black holes. *Communications in mathematical physics*, 43(3):199–220, 1975.

[13] S. A. Hayward. Formation and evaporation of regular black holes. *Phys. Rev. Lett.*, 96:031103, 2006.

[14] L. Herrera, A. Di Prisco, and J. Ospino. Some analytical models of radiating collapsing spheres. *Physical Review D*, 74(4):044001, 2006.

[15] W. A. Hiscock. Models of evaporating black holes. i. *Phys. Rev. D*, 23:2813–2822, Jun 1981.

[16] W. A. Hiscock. Models of evaporating black holes. ii. effects of the outgoing created radiation. *Physical Review D*, 23(12):2823, 1981.

[17] V. Husain. Exact solutions for null fluid collapse. *Phys. Rev. D*, 53(4), 1996.

[18] W. Israel. Singular hypersurfaces and thin shells in general relativity. *Il Nuovo Cimento*, (1), 1966.

[19] Y. Kaminaga. A dynamical model of an evaporating charged black hole and quantum instability of cauchy horizons. *Classical and Quantum Gravity*, 7(7):1135, 1990.

[20] H. Kawai, Y. Matsuo, and Y. Yokokura. A Self-consistent Model of the Black Hole Evaporation. *Int. J. Mod. Phys.*, A28:1350050, 2013.

[21] R. Kerr. Gravitational field of a spinning mass as an example of algebraically special metrics. *Physical Review Letters*, 5(11), 1963.

[22] Y. Kuroda. Vaidya spacetime as an evaporating black hole. *Progress of theoretical physics*, 71(6):1422–1425, 1984.

[23] K. Lake. Collapse of radiating imperfect fluid spheres. *Physical Review D*, 26(2):518, 1982.

[24] K. Lake and C. Hellaby. Collapse of radiating fluid spheres. *Phys. Rev. D*, 24:3019–3022, Dec 1981.

[25] K. Lake and T. Zannias. Structure of singularities in the spherical gravitational collapse of a charged null fluid. *Phys. Rev.*, D43(6):1798, 1991.

[26] M. S. Morris and K. S. Thorne. Wormholes in spacetime and their use for interstellar travel: A tool for teaching general relativity. *American Journal of Physics*, 56(5):395–412, 1988.

[27] E. T. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence. Metric of a rotating, charged mass. *Journal of mathematical physics*, 6(6):918–919, 1965.

[28] G. Nordström. On the energy of the gravitational field in einstein's theory. *Verhandl. Koninkl. Ned. Akad. Wetenschap., Afdel. Natuurk., Amsterdam.*, (26), 1918.

[29] J. R. Oppenheimer and H. Snyder. On continued gravitational contraction. *Phys. Rev.*, 56:455–459, Sep 1939.

[30] A. Ori. Charged null fluid and the weak energy condition. *Class. Quantum Grav.*, (8):1559–1575, 1991.

[31] E. Poisson. *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics*. Cambridge University Press, Cambridge, UK, 2004.

[32] H. Reissner. ber die eigengravitation des elektrischen feldes nach der einsteinschen theorie. *Annalen der Physik*, (50), 1916.

[33] B. Schutz. *A First Course in General Relativity*. Cambridge University Press, Cambridge, UK, 2009.

[34] K. Schwarzschild. On the gravitational field of a mass point according to einstein's theory. *arXiv preprint physics/9905030*, 1999.

[35] W. G. Unruh. Collapse of radiating fluid spheres and cosmic censorship. *Phys. Rev. D*, 31:2693–2694, May 1985.

[36] P. Vaidya. Nonstatic solutions of einstein's field equations for spheres of fluids radiating energy. *Phys. Rev.*, 83(10), 1951.

[37] P. C. Vaidya. Nonstatic solutions of einstein's field equations for spheres of fluids radiating energy. *Phys. Rev.*, 83:10–17, Jul 1951.

[38] R. M. Wald. *General Relativity*. University of Chicago Press, Chicago, USA, 1984.