HEREDITARY SEMIGROUP RINGS AND MAXIMAL ORDERS

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Hereditary Semigroup Rings and Maximal Orders

by

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Abstract

In this thesis, we study several problems concerning semigroup algebras $K[S]$ of a semigroup S over a field K.

In Chapter 1 and Chapter 2 we give some background on semigroups and semigroup rings. In Chapter 3, we discuss the global dimension of semigroup rings $R[S]$ where R is a ring and S is a monoid with a sequence of ideals $S = I_1 \supset I_2 \supset \cdots \supset I_n \supset I_{n+1}$ such that each I_i/I_{i+1} is a non-null Rees matrix semigroup.

In Chapter 4, we investigate when a semigroup algebra has right global dimension at most 1, that is, when is it right hereditary. As an application of the results in Chapter 3, we describe when $K[S^1]$ is hereditary for a non-null Rees semigroup S. For arbitrary semigroups that are nilpotent in the sense of Malcev, we describe when its semigroup algebra is hereditary Noetherian prime. And for cancellative semigroups we obtain a description of when its semigroup algebra is hereditary Noetherian.

In Chapter 5, we generalize the concept of unique factorization monoid and investigate Noetherian unique factorization semigroup algebras of submonoids of torsion-free polycyclic-by-finite groups.

In Chapter 6, we investigate when a semigroup algebra $K[S]$ is a polynomial identity domain which is also a unique factorization ring. In order to prove this result we describe first the height one prime ideals of such algebras.

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Introduction

Maximal orders in simple Artinian rings of quotients have attracted considerable interest. In particular, it has been shown that various algebraic ring constructions yield examples of Noetherian maximal orders or of maximal orders satisfying a polynomial identity. For a field K and a commutative monoid S Chouinard proved that the monoid algebra $K[S]$ is a Krull domain if and only if S is a Krull order in its group of quotients. Moreover, the class group of $K[S]$ equals the class group of S. This shows, in particular, that the height one primes of $K[S]$ determined by the minimal primes of S are crucial. Brown described when a group algebra $K[G]$ of a polycyclic-by-finite group G is a prime Noetherian maximal order. It is always the case if G is torsion-free. If G is a finitely generated torsion-free abelian-by-finite group (equivalently, $K[G]$ is a Noetherian PI domain) then all height one primes are principally generated by a normal element. So, in the terminology of Chatters and Jordan, $K[G]$ is a unique factorization ring.

It remains an unsolved problem to characterize when an arbitrary semigroup algebra $K[S]$ over a field K is a prime maximal order that is Noetherian or satisfies a polynomial identity.

Apart from the two cases mentioned above, an answer to the question has been obtained only for some special classes of semigroups, such as Malcev nilpotent semigroups, or for some special classes of maximal orders, such as principal ideal rings.

In this thesis we continue these investigations. We investigate when a semigroup algebra is hereditary Noetherian prime or a unique factorization ring in the sense of Chatters and Jordan. The former part is basically a question of Okninski, Problem 37 in **[52]**. For a ring to be (right) hereditary one needs the (right) global dimension to be at most one. Hence, our first contribution to the the subject is to control the global dimension of certain types of matrix semigroups.

We now briefly outline the content of each chapter. Chapters 1 and 2 cover some notation and background on semigroups and semigroup rings.

In **[45]**, Kuzmanovich and Teply determined a lower and upper bound for the homological dimension of $K[S]$ for the class of finite monoids S that have a sequence of ideals $S = I_1 \supset I_2 \supset \cdots \supset I_n \supset I_{n+1}$ such that all the Rees factors I_i/I_{i+1} are non-null Rees matrix semigroups. In Chapter 3 we sharpen their upper bound. We also include some examples of semigroups which have a null Rees factor. These examples indicate that in this case the solution is yet rather unclear. Hence the solution to arbitrary finite semigroups is still open.

As an application of the results in Chapter 3 we first determine in Chapter 4 when the (contracted) semigroup algebra $K_0[S]$ of a finite non-null Rees matrix semigroup S is hereditary. Next we characterize when $K_0[S]$ is a hereditary Noetherian prime ring when S is an arbitrary nilpotent semigroup (in the sense of Malcev). It turns out that such a ring is a prime principal ideal ring. In the last part of this chapter we fully describe when a semigroup algebra of a cancellative monoid is a Noetherian hereditary ring. Our results rely on the solution of the problem for group algebras. These were obtained by Goursaud and Valette for nilpotent groups and Dicks for arbitrary groups.

In Chapter 5 we investigate when a monoid algebra $K[S]$ of a cancellative monoid is a Noetherian unique factorization ring. Such monoids S have a group of fractions, say G. Because of Quinn's result on graded rings, $K[S]$ is (right) Noetherian if and only if S satisfies the ascending chain condition on right ideals. Since $K[G]$ also is a Noetherian unique factorization ring and because these have only been described for groups G that are polycyclic-by-finite, we restrict to this situation. In case G is also torsion-free, we show that the problem is closely related to group algebras $K[G]$ and the monoid S, and actually the monoid $N(S)$ consisting of the normalizing elements of S. Hence in the first part of the chapter we investigate unique factorization monoids, and more generally Krull monoids. As in the ring case it turns out that S is a unique factorization monoid if and only if S is a Krull order with trivial normalizing class group.

In the final Chapter, we investigate when a monoid algebra $K[S]$ of a cancellative monoid S is a domain satisfying a polynomial identity and which is a unique factorization ring (the Noetherian condition is not assumed). In this case S has a group of fractions that is torsion-free abelian-by-finite group G and the group algebra $K[G]$ is a unique factorization ring. First we show that for such a monoid S, if P is a prime ideal of $K[S]$ with $P \cap S \neq \emptyset$ then $K[S \cap P]$ is also a prime ideal. It follows that, if $K[S]$ is a Krull order, then the height one prime ideals intersecting S are precisely the ideals of the form $K[Q]$ with Q a minimal prime ideal of S. The proof of this result relies on the structure theory of skew linear semigroups, as developed by Okniński. This result on prime ideals is the crucial step for us to investigate when $K[S]$ is a unique factorization ring.

CHAPTER 1

Semigroups

In this chapter, we give some definitions and structural descriptions of certain important classes of semigroups. For more information, the reader is referred to [**13**], [**28**] and [**52**].

1.1. Some basic Definitions

A *semigroup* S is a multiplicatively closed set such that the operation is associative. A *subsemigroup* T of S is a non-empty subset which is closed under multiplication. A *subgroup* G of S is a subsemigroup which is a group.

1.1.1. An element e of S is called a *left identity* of S if $ea = a$ for all $a \in S$. Similarly one defines *right identity* and an *identity* of S if it is an element that is both a left and a right identity. A semigroup S may have multiple right or left identities, but if it has a right identity and a left identity, they must necessarily coincide and in this case S has a unique identity.

A semigroup S is called a *monoid* if S contains an identity element 1. Then u is a *right unit* of S if there is a $v \in S$ such that $uv = 1$. Similarly, one defines *left unit* and u is a *unit* if it is both a left and right unit. We write $U(S)$ for the set of units of S.

1.1.2. An element z of S is called *left zero* if $za = z$ for every $a \in S$. Similarly one defines *right zero* and z is called a *zero* element if it is both a left and right zero element. As for identity elements, a semigroup S may have multiple right or left zeros, but if has a right zero and a left zero, they must necessarily coincide and in this case S has a unique zero. If S has a zero element, it will usually be denoted θ. A semigroup S with zero element θ will be called a *zero* or *null semigroup* if $ab = \theta$ for all $a, b \in S$.

1.1.3. Let S be any semigroup, and let 1 be a symbol not representing any element of S. Extend the given binary operation in S to one in $S \cup \{1\}$ by defining $11 = 1$ and $1a = a1 = a$ for every $a \in S$. Obviously $S \cup \{1\}$ is a monoid. Let

$$S^1 = \begin{cases} S & \text{if S has an identity element,} \\ S \cup \{1\} & \text{otherwise;} \end{cases}$$

Similarly one can adjoin a zero element θ to S, denoted by $S^0 = S \cup \{\theta\}$.

1.1.4. An element $e \in S$ which satisfies $e = e^2$ is called an *idempotent*. We write $E(S)$ for the set of idempotent elements of a semigroup S. The set $E(S)$ can be partially ordered by $e \leq f$ if and only if $ef = fe = e$. If S contains a zero element θ, then $\theta \leq e$ for every idempotent $e \in E$. A *band* is a semigroup S every element of which is idempotent.

1.1.5. A *homomorphism* of a semigroup S into a semigroup T is a mapping $\phi : S \to T$ which preserves products:

$$\phi(xy) = \phi(x)\phi(y) \text{ for all } x, y \in S.$$

If $\phi : S \to T$ and $\psi : T \to U$ are homomorphisms, then so is the composite mapping $\psi \circ \phi : S \to U$. An *isomorphism* of semigroups is a bijective homomorphism.

1.1.6. By a *left* ideal of a semigroup we mean a non-empty subset I of S such that $S^1 I \subseteq I$. Similarly one defines a *right* ideal and I is a *two-sided ideal*, or simply *ideal*, if I is both a left and right ideal of S. If S has a zero element θ, then $\{\theta\}$ is always an ideal of S.

If $\{I_\alpha \mid \alpha \in A\}$ is a family of ideals of a semigroup S then $\cup I_\alpha$ and $\cap I_\alpha$ are also ideals of S, the latter provided that it is non-empty. The same is true for the family of left or right ideals.

If $a \in S$ then the right ideal generated by a is denoted by aS^1; clearly $aS^1 = aS \cup \{a\}$. Similarly, the left ideal generated by a is denoted by S^1a. The ideal generated by a is $S^1aS^1 = SaS \cup Sa \cup aS \cup \{a\}$.

1.1.7. An equivalence relation ρ is called a *right congruence* on a semigroup S if $a\rho b$ implies that $ac\rho bc$ for every $a, b, c \in S$. A *left congruence* is defined similarly. A *congruence* is an equivalence relation ρ on S which is both a left and a right congruence.

Let I be an ideal of a semigroup S and $a, b \in S$. Define $a\rho b$ if either $a = b$ or else both a and b belong to I. We call ρ the *Rees congruence modulo I*. The equivalence classes of S mod ρ are I itself and every one element set $\{a\}$ with

$a \in S \setminus I$. We shall write S/I instead of S/ρ, and we call S/I the *Rees factor semigroup of S modulo I.*

1.1.8. For the Rees factors of semigroups, we have analogues of two of the isomorphism theorems for groups.

THEOREM 1.1. *(Theorem 2.36 in* **[13]***) Let J be an ideal and T a subsemigroup of a semigroup S and $J \cap T \neq \emptyset$. Then $J \cap T$ is an ideal of T, $J \cup T$ is a subsemigroup of S, and*

$$(J \cup T)/J \cong T/(J \cap T).$$

THEOREM 1.2. *(Theorem 2.37 in* **[13]***) Let J be an ideal of a semigroup S, and let δ be the natural homomorphism of S upon the Rees factor S/J. Then δ induces a one-to-one, inclusion-preserving mapping $A \to \delta(A) = A/J$ of the set of all ideals A of S containing J upon the set of all ideals of S/J, and*

$$(S/J)/(A/J) \cong S/A.$$

1.2. Green relations

1.2.1. The *Green relations* on a semigroup S are the equivalence relations, which are denoted respectively by $\mathcal{L}, \mathcal{R}, \mathcal{H}$, and $\mathcal{J}$. These were introduced by

Green in 1951 and defined as follows, for $a, b \in S$,

$$
\begin{array}{lll}
a\mathcal{L}b & \text{if and only if} & S^1a = S^1b, \\
a\mathcal{R}b & \text{if and only if} & aS^1 = bS^1, \\
a\mathcal{H}b & \text{if and only if} & aS^1 = bS^1 \text{ and } S^1a = S^1b, \\
a\mathcal{J}b & \text{if and only if} & S^1aS^1 = S^1bS^1.
\end{array}
$$

Clearly $\mathcal{L}$ is an equivalence relation such that $a\mathcal{L}b$ implies that $ac\mathcal{L}bc$ for all $c \in S$, that is, $\mathcal{L}$ is a right congruence. If $a\mathcal{L}b$, we say that a and b are $\mathcal{L}$-equivalent. By L_a we mean the set of all elements of S which are $\mathcal{L}$-equivalent to a, that is, the equivalence class of S mod $\mathcal{L}$; we call L_a the *$\mathcal{L}$-class* containing a.

Similarly R_a, H_a, and J_a denote respectively the $\mathcal{R}, \mathcal{H}$, and *$\mathcal{J}$-class* containing a.

LEMMA 1.3 (Theorem 2.16 in [**13**]). *For any $\mathcal{H}$-class H of a semigroup S the following are equivalent:*

1. *$ab \in H$ for some $a, b \in H$;*
2. *H contains an idempotent;*
3. *H is a subsemigroup of S;*
4. *H is a subgroup of S.*

COROLLARY 1.4. *The maximal subgroups of a semigroup S coincide with the $\mathcal{H}$-classes of S which contain idempotents. They are pairwise disjoint. Each subgroup of S is contained in exactly one maximal subgroup of S.*

1.3. Regular semigroups and Inverse semigroups

1.3.1. An *inverse* of an element a in a semigroup S is an element b of S such that

$$aba = a \text{ and } bab = b;$$

the elements a and b are also called *mutually inverse*. As shown in the next example, an element can have many inverses.

EXAMPLE 1.5. *Let X and Y be two sets, and define a binary operation on $S = X \times Y$ as follows:*

$$(x_1, y_1)(x_2, y_2) = (x_1, y_2), \quad x_1, x_2 \in X, y_1, y_2 \in Y.$$

This semigroup is called the rectangular band on $X \times Y$. In such a rectangular band S, every two elements are mutually inverse.

1.3.2. An element a of a semigroup S is called *regular* if $a \in aSa$, that is, if $axa = a$ for some $x \in S$. In this case, ax is an idempotent. Note we have the following equivalent conditions.

LEMMA 1.6 (Lemma II.2.2 in [**28**]). *For an element a of a semigroup S the following are equivalent:*

1. *a is regular;*
2. *a has an inverse;*
3. *R_a contains an idempotent;*

4. *L_a contains an idempotent.*

In other words, a is regular if and only if $aS^1 = eS^1$ ($S^1a = S^1e$) for some idempotent element e, i.e. the principal right (left) ideal of S generated by a has an idempotent generator e.

A semigroup is called *regular* if all its elements are regular. From the equivalent definitions of regular elements, we know that S is a regular semigroup if and only if every $\mathcal{R}$-class of S contains an idempotent, if and only if every $\mathcal{L}$-class of S contains an idempotent, if and only if every principal right (left) ideal of S is generated by an idempotent.

1.3.3. An *inverse* semigroup is a semigroup such that every element has a unique inverse.

THEOREM 1.7 (Proposition II.2.6 in [**28**]). *The following conditions on a semigroup S are equivalent:*

1. *S is an inverse semigroup;*
2. *every $\mathcal{R}$-class of S contains exactly one idempotent and every $\mathcal{L}$-class of S contains exactly one idempotent.*
3. *S is regular and the idempotents of S commute with each other.*

1.4. 0-Minimal Ideals and 0-Simple Semigroups.

1.4.1. A semigroup S is *left simple* if it does not properly contain any left ideal. Similarly we can define a *right simple* semigroup and a *simple* semigroup S if it does not properly contain a two-sided ideal.

A two-sided (left, right) ideal M of a semigroup S is called *minimal* if it does not properly contain any two-sided (left, right) ideal of S. If A is any other ideal of S of the same type as M, either $M \subseteq A$ or $M \cap A = \emptyset$. In particular, two distinct minimal ideals of the same type are disjoint.

Since two two-sided ideals A and B of a semigroup S always contain the set product AB, it follows that there can be at most one minimal two-sided ideal of S. If S has a minimal two-sided ideal K, then K is called the *kernel* of S. Since K is contained in any two-sided ideal of S, it may be characterized as the intersection of all the two-sided ideals of S. If the intersection is empty, then S does not have a kernel. It has been proved by Suschkewsch that any finite semigroup has a kernel.

1.4.2. According with the theory of minimal ideals in rings, we introduce the notion of 0-minimality. A two-sided (left, right) ideal M of S with zero θ is called *0-minimal* if $M \neq \theta$ and θ is the only two-sided (left, right) ideal of S properly contained in M.

If M is a 0-minimal two-sided ideal (left, right) ideal of a semigroup S with zero θ, then M^2 is an ideal of the same type as M contained in M, so we must have either $M^2 = \{\theta\}$ or $M^2 = M$.

1.4.3. A semigroup S is *0-simple* if $S^2 \neq \{\theta\}$ and $\{\theta\}$ is the only proper two-sided ideal of S. Let S be a semigroup with zero θ such that $\{\theta\}$ is the only proper two-sided ideal of S. Then either S is 0-simple or S is the null semigroup of order 2. Furthermore, S is 0-simple if and only if $SaS = S$ for every element $a \neq \theta$ of S.

Moreover, Clifford proved the following.

THEOREM 1.8 (Theorem 2.29 in [**13**]). *Let M be a 0-minimal ideal of a semigroup S with zero θ. Then either $M^2 = \theta$ or M is a 0-simple subsemigroup of S.*

Furthermore, by using Theorem 1.2 and Theorem 1.8, we have the following Corollary.

COROLLARY 1.9.

1. *An ideal J of a semigroup is maximal (proper) ideal of S if and only if S/J has no proper non-zero ideal, hence if and only if S/J is either 0-simple or the null semigroup of order two.*
2. *If J and J' are ideals of S with $J \subset J'$, then J is maximal in J' if and only if J'/J is a 0-minimal ideal of S/J. If this is the case, then J'/J is either a 0-simple semigroup or a null semigroup.*

1.4.4. Let S be a semigroup without zero, and let $S^0 = S \cup \{\theta\}$. Then $A \to A \cup \{\theta\}$ is a one-to-one mapping of the set of all two-sided (left, right) ideals A of S upon the set of all non-zero two-sided (left, right) ideals of S^0.

This mapping preserves inclusion, and, in particular, A is minimal if and only if $A \cup \{\theta\}$ is 0-minimal. Consequently, any theorem concerning 0-minimal ideals implies an evident corollary concerning minimal ideals in a semigroup without zero. Similarly, any theorem concerning 0-simple semigroups implies an evident corollary concerning simple semigroups. For example, Theorem 1.8 implies that

COROLLARY 1.10. *If a semigroup S contains a kernel K, then K is a simple subsemigroup of S.*

1.4.5. Let $s \in S$. The principal ideal S^1sS^1 of S generated by s is denoted by J_s, while the subset of J_s consisting of non-generators of J_s (as an ideal of S) is denoted by I_s. Thus $I_s = \emptyset$ if and only if J_s is a minimal ideal of S, and if it is not the case, then I_s is an ideal of S. Each Rees factor semigroup J_s/I_s, with $s \in S$, is called a *principal factor* of S. Obviously I_s is maximal in J_s, then we have

COROLLARY 1.11. *Each principal factor of any semigroup S is 0-simple, simple, or null of order two. Only if S has a kernel is there a simple principal factor, and in this case the kernel is the only simple principal factor.*

1.4.6. A semigroup S is *semisimple* if every principal factor of the semigroup is 0-simple or simple. This amounts to excluding null factors. Note any regular semigroup is semisimple since $S^1axaS^1 = S^1aS^1$ for some $x \in S$ implies that $(S^1aS^1)^2$ contains the element axa which is still a generator of S^1aS^1.

1.4.7. A *principal series* of a semigroup S is a chain

$$S = S_1 \supset S_2 \supset \cdots \supset S_m \supset S_{m+1}$$

of ideals $S_i(i = 1, \cdots, m)$ of S, beginning with S and ending with S_{m+1}, which is the empty set if S does not contain a zero, otherwise, $S_{m+1} = \{\theta\}$, and there is no ideal of S strictly between S_i and $S_{i+1}(i = 1, \cdots, m)$. By the *factors* of the principal series we mean the Rees factor semigroups $S_i/S_{i+1}(i = 1, \cdots, m)$. By Theorem 1.8, S_i/S_{i+1} is either 0-simple, simple, or null.

THEOREM 1.12 (Proposition II.4.9 in [**28**]). *Let S be a semigroup admitting a principal series,*

$$S = S_1 \supset S_2 \supset \cdots \supset S_m \supset S_{m+1}.$$

Then the factors of this series are isomorphic in some order to the principal factors of S. In particular, any two principal series of S have isomorphic factors.

1.5. Completely 0-Simple Semigroups

1.5.1. Let E be the set of idempotents of a semigroup S. Recall that $e \leq f$ if and only if $e = ef = fe$ for e, f idempotents. An idempotent f is called *primitive* if $f \neq \theta$ and if $e \leq f$ implies that $e = \theta$ or $e = f$.

By a *completely 0-simple semigroup* we mean a 0-simple semigroup that has a primitive idempotent.

For example, any finite 0-simple semigroup is completely 0-simple. It is been shown by E. H. Moore that some power of every element of a finite semigroup is idempotent, hence any finite 0-simple semigroup must contain an idempotent, that is, $E \neq \emptyset$. Furthermore, $E \neq \{\theta\}$, since $E = \{\theta\}$ implies that every element of S, and hence S itself, is nilpotent, contradicting $S^2 = S$. It is then clear that the finite partially ordered set $E \setminus \{\theta\}$ must contain a minimal element, that is, a primitive idempotent.

1.5.2. We have the following descriptions of completely 0-simple semigroup which is due to Clifford.

THEOREM 1.13 (Theorem 2.48 in [**13**]). *Let S be a 0-simple semigroup. Then S is completely 0-simple if and only if it contains at least one 0-minimal left ideal and at least one 0-minimal right ideal of S. In fact, a completely 0-simple semigroup is the union of its 0-minimal left (right) ideals.*

1.6. Rees Theorem

The Rees Theorem gives a complete construction of all completely 0-simple semigroups using groups and sets. To show this result we recall the definition of a Rees matrix semigroup over a group G.

1.6.1. Let G^0 be a group with zero adjoined, and let I, Λ be two sets. By a *Rees $I \times \Lambda$ matrix over G^0* we mean a $I \times \Lambda$ matrix over G^0 with at most one nonzero entry. If $g \in G, i \in I$, and $\lambda \in \Lambda$, then $(g)_{i\lambda}$ denotes the Rees matrix over

G^0 having g in the ith row and λth column, its remaining entries being 0. For any $i \in I$ and $\lambda \in \Lambda$, the expression $(0)_{i\lambda}$ will mean the $I \times \Lambda$ zero matrix, which simply will be denoted by θ.

Further, let $P = (p_{\lambda i})_{\lambda \in \Lambda, i \in I}$ be a generalized $\Lambda \times I$ matrix over G^0, that is, every $p_{\lambda i}$ lies in G^0. We use P to define a binary operation on the set of Rees $I \times \Lambda$ matrices over G^0 as follows:

$$AB = A \circ P \circ B,$$

where $\circ$ means the usual matrix multiplication. If A and B are Rees $I \times \Lambda$ matrices over G^0, then so is AB. In fact, if $A = (a)_{i\lambda}$ and $B = (b)_{j\mu}$ then we easily find that

$$(a)_{i\lambda}(b)_{j\mu} = (ap_{\lambda j}b)_{i\mu} \ (a, b \in G; i, j \in I, \lambda, \mu \in \Lambda).$$

The set of all Rees $I \times \Lambda$ matrices over G^0 is a semigroup with respect to the above defined operation. We call it the *Rees $I \times \Lambda$ matrix semigroup over the group with zero G^0 with sandwich matrix P*, and denote it by $\mathcal{M}^0(G; I, \Lambda; P)$. The group G is called the *structure group* of $\mathcal{M}^0(G; I, \Lambda; P)$ and P is called the *sandwich matrix*. In fact, G is a maximal subgroup.

Actually, any nonzero element of $\mathcal{M}^0(G; I, \Lambda; P)$ is uniquely determined by its nonzero entry, and so it may be denoted by (g, i, m), where $g \in G, i \in I, m \in \Lambda$. Therefore, $\mathcal{M}^0(G; I, \Lambda; P)$ may be treated as the set of all triples (g, i, m), $g \in G^0, i \in I, m \in \Lambda$, with the multiplication given by

$$(g, i, m)(h, j, n) = (gp_{mj}h, i, n) \text{ for } g, h \in G^0, i, j \in I, m, n \in \Lambda.$$

All triples (θ_G, i, m) are identified with the zero element θ of $\mathcal{M}^0(G; I, \Lambda; P)$.

1.6.2. The sandwich matrix P is said to be *regular* in case for each $i \in I$ there exist $\lambda \in \Lambda$ such that $p_{\lambda i} \neq \theta$, and for each $\lambda \in \Lambda$ there exists $i \in I$ such that $p_{\lambda i} \neq \theta$. The importance of Rees matrix semigroups comes from the following fundamental result which is known as the **Rees Theorem.**

THEOREM 1.14 (Lemma 3.1 in [**13**]). *Let S be a semigroup. Then S is completely 0-simple if and only if S is isomorphic to $\mathcal{M}^0(G; I, \Lambda; P)$, a Rees matrix semigroup for some group G, nonempty sets I and Λ, and regular sandwich matrix P.*

1.6.3. For a completely 0-simple semigroup S, Theorem 1.7 tells us that to be an inverse semigroup, each row and column of the regular sandwich matrix P does not contain more than one non-zero element. This remark will be used in Chapter 6. Moreover, we have a nice representation for this type of semigroups.

A *Brandt semigroup* is a Rees matrix semigroup $B(G; I) = \mathcal{M}^0(G; I, \Lambda; P)$ in which G is a group, $\Lambda = I \neq \emptyset$ and P is the identity matrix ($p_{ii} = 1 \in G$, $p_{ij} = 0$ if $i \neq j$).

COROLLARY 1.15 (Theorem 3.9 in [**13**]). *A completely 0-simple semigroup is an inverse semigroup if and only if it is isomorphic to a Brandt semigroup.*

1.7. Cancellative semigroups

An element a of a semigroup S is said to be *left (right) cancellable* if, for any $x, y \in S$, $ax = ay$ ($xa = ya$) implies $x = y$. We say that S is *cancellative* if every element of S is left and right cancellable. A cancellative semigroup S has a group G of right fractions if and only if S satisfies the right Ore condition, that is, for every $s, t \in S$,

$$sS \cap tS \neq \emptyset.$$

Then G is unique, up to isomorphism, and may be identified with SS^{-1}. If S also satisfies the left Ore condition (defined symmetrically) then $G = SS^{-1} = S^{-1}S$ is called *the group of fractions* of S. We give two natural classes for which a semigroup has a group of right fractions.

THEOREM 1.16 (Lemma 7.1, Proposition 7.12 in [**52**]). *Let S be a cancellative semigroup such that either of the following conditions hold:*

1. *S has no non-commutative free subsemigroups.*
2. *S has the ascending chain condition on right ideals.*

Then S has a group of right fractions.

1.8. Nilpotent semigroups

Let x, y be elements of a semigroup S and let $w_1, w_2, \cdots$ be elements of the monoid S^1. Consider the sequence of elements defined inductively as follows:

$$x_0 = x, \; y_0 = y,$$

and

$$x_{n+1} = x_n w_{n+1} y_n, \; y_{n+1} = y_n w_{n+1} x_n, \text{ for } n \geq 0.$$

We say that the identity $X_n = Y_n$ is satisfied in S if $x_n = y_n$ for all $x, y \in S$, $w_1, w_2, \cdots \in S^1$. A semigroup S is called *(generalized) nilpotent* of class n if S satisfies the identity $X_n = Y_n$ and n is the least positive integer with this property.

Obviously every power nilpotent semigroup, that is, a semigroup S with zero such that $S^m = \theta$ for some integer $m \geq 1$, satisfies the identity $X_m = Y_m$, and so it is nilpotent.

1.8.1. Actually for a group G, this definition coincides with the classical notion of nilpotency.

THEOREM 1.17 (Theorem 7.2 in [**52**]). *Let $n \geq 1$. Then the following conditions are equivalent for a group G.*

1. *G is a nilpotent group of class n in the classical sense.*
2. *n is the least positive integer for which the identity $X_n = Y_n$ is satisfied in G.*

Note that a subsemigroup of a nilpotent group is a nilpotent semigroup.

1.8.2. Note that the condition $X_n = Y_n$ is a bit stronger than the one required by Malcev, who required elements w_i in S only (see [**52**]). However the definitions agree on the class of cancellative semigroups. Indeed, to prove the next result one only needs to use $w_i \in S$.

THEOREM 1.18 (Theorem 7.3 in [**52**]). *Let S be a cancellative Malcev nilpotent semigroup of class n. Then S has a group of fractions that is nilpotent of class n.*

1.8.3. An inverse semigroup $S = \mathcal{M}^0(G; M, M; I)$ (with I an $M \times M$ identity matrix) of matrix type over a nilpotent group G, i.e. an inverse completely 0-simple semigroup, satisfies the identity $X_{n+2} = Y_{n+2}$, where n is the nilpotency class of G. Moreover,

PROPOSITION 1.19 (Lemma 2.1 in [**31**]). *Let S be a completely 0-simple semigroup over a maximal group G. Then S is nilpotent if and only if G is nilpotent and S is an inverse semigroup.*

CHAPTER 2

Semigroup Rings

In this chapter, we give some background on semigroup rings.

2.1. Basic definitions

Semigroup rings have been extensively studied. See, for example, Gilmer's book **[23]** for commutative semigroup rings, and Okniński's book **[52]** for the non-commutative case.

2.1.1. Let R be a ring and S a semigroup. The *semigroup ring* $R[S]$ is the ring whose elements are all formal sums

$$\sum_{s\in S} r_s s$$

with each coefficient $r_s \in R$ and all but finitely many of the coefficients equal to zero. Addition is defined component-wise so that

$$\Sigma_{s\in S} r_s s + \Sigma_{s\in S} q_s s = \Sigma_{s\in S}(r_s + q_s)s.$$

Multiplication is given by the rule

$$(r_s s)(q_t t) = (r_s q_t)(st)$$

which is extended distributively so that

$$(\Sigma_{s\in S}r_s s)(\Sigma_{s\in S}q_s s) = \Sigma_{s\in S}(\Sigma_{uv=s}r_u q_v)s.$$

This is the natural generalization of group ring. For $a = \Sigma r_s s \in R[S]$, the set $\{s \in S \mid r_s \neq 0\}$ is called the support of a and is denoted by $supp(a)$. If $R = K$ is a field, then $K[S]$ is called a *semigroup algebra.*

2.1.2. Let T be another semigroup and $\phi : S \to T$ is a semigroup homomorphism. By $\bar{\phi}$ we mean the extension of ϕ to the ring homomorphism of $R[S]$ into $R[T]$ given by the formula $\bar{\phi}(\Sigma\alpha_s s) = \Sigma\alpha_s\phi(s)$.

If S has a zero element θ, we write $R_0[S]$ for the quotient $R[S]/R\theta$; $R_0[S]$ is called a *contracted semigroup ring.* Thus $R_0[S]$ may be identified with the set of finite sums $\Sigma r_s s$ with $r_s \in R, s \in S \setminus \{\theta\}$, subject to the component-wise addition and multiplication given by the rule

$$st = \begin{cases} st & \text{if } st \neq \theta \\ 0 & \text{if } st = \theta \end{cases}$$

defined on the basis $S \setminus \{\theta\}$. If S has no zero element, then by definition $R_0[S] = R[S]$. For any $a = \Sigma r_s s \in R[S]$, by $supp_0(a)$ we mean the set $\{s \in S \setminus \{\theta\} \mid r_s \neq 0\}$. Thus, $supp_0(a) = supp(a) \setminus \{\theta\}$. The following lemma shows that in the study of semigroup rings one needs to consider contracted semigroup rings.

LEMMA 2.1 (Lemma 4.7, Corollary 4.9 in [**52**]). *Let I be an ideal of a semigroup S. Then $R_0[S/I] \cong R[S]/R[I]$. In particular, if S has a zero element θ, then $R[S] \cong R \oplus R_0[S]$.*

2.1.3. Let K be a field and S be a semigroup. This section explores the nice relationship between the the set of right congruences on S and the set of right ideals of $K[S]$.

Let ρ be a right congruence on S, that is, ρ is an equivalence relation such that, for any $s, t, x \in S$, we have $(sx, tx) \in \rho$ whenever $(s, t) \in \rho$. If $\phi_\rho : S \to S/\rho$ is the natural mapping onto the set S/ρ of ρ-classes in S, then we denote by $I(\rho)$ the right ideal of $K[S]$ generated by the set $\{s - t \mid s, t \in S, (s, t) \in \rho\}$. Since ρ is a right congruence on S, then $I(\rho)$ coincides with the right K-subspace generated by the set $\{s - t \mid s, t \in S, (s, t) \in \rho\}$. Moreover, the K-vector space $K[S/\rho]$ is a right $K[S]$-module under the natural action defined by $\phi_\rho(s) \circ t = \phi_\rho(st)$ for $s, t \in S$. With this notation, we have the following result.

LEMMA 2.2 (Lemma 4.1, [**52**]). *For any right congruence ρ on S, $\bar{\phi}_\rho : K[S] \to K[S/\rho]$ is a homomorphism of right $K[S]$-modules such that*

$$Ker(\bar{\phi}_\rho) = I(\rho) = \Sigma_{s \in S} w_s(\rho)$$

where $w_s(\rho) = \{\Sigma_{i=1}^m \alpha_i s_i \in K[S] \mid m \geq 1, \Sigma_{i=1}^m \alpha_i = 0, (s, s_i) \in \rho \text{ for all } i = 1, 2, \cdots, m\}$, and $K[S/\rho] \cong K[S]/I(\rho)$ as right $K[S]$-modules. Moreover, the correspondence $\rho \to I(\rho)$ establishes a one-to-one order preserving mapping of the set of right congruences on S into the set of right ideals of $K[S]$.

Combining Lemma 2.2 with its left-right symmetric analog, we derive the following consequence.

COROLLARY 2.3. *For any congruence* ρ *on* S, $\bar{\phi}_\rho : K[S] \to K[S/\rho]$ *is a homomorphism of algebras such that* $ker(\bar{\phi}_\rho) = I(\rho)$ *and* $K[S/\rho] \cong K[S]/I(\rho)$ *as* K*-algebras. Consequently,* $\rho \to I(\rho)$ *is an order-preserving mapping of the set of congruences of* S *into the set of two-sided ideals of* $K[S]$.

It is clear that the trivial congruence of S determines the zero ideal of $K[S]$. The universal congruence $S \times S$ leads to the ideal $I = \{s - t \mid s, t \in S\}K = \{\Sigma\alpha_s s \in K[S] \mid \Sigma\alpha_s = 0\}$. This ideal is usually denoted by $\omega(K[S])$ and is called the augmentation ideal of $K[S]$, where the corresponding homomorphism $K[S] \to K$ is called the augmentation map.

From Lemma 2.2 we know that the right ideal $I(\rho)$ determines a right congruence $\rho = \{(s,t) \in S \times S \mid s - t \in I(\rho)\}$. More general, any right ideal of $K[S]$ determines a right congruence on S. Let J be a right ideal of $K[S]$, define a relation ρ_J on S by $\rho_J = \{(s,t) \in S \times S \mid s - t \in J\}$.

LEMMA 2.4 (Lemma 4.5, [**52**]). *Let* J *be a right ideal of* $K[S]$. *Then,*

1. ρ_J *is a right congruence on* S *such that* $I(\rho_J) \subseteq J$.
2. *There exist natural right* $K[S]$*-module homomorphisms,* $K[S] \to K[S/\rho_J] \to K[S]/J$.

3. *If J is a two-sided ideal of $K[S]$, then ρ_J is a congruence on S, the mappings in (2) are homomorphisms of K-algebras, and the semigroup S/ρ_J embeds into the multiplicative semigroup of the algebra $K[S]/J$.*

2.2. Munn algebras

In this section, we describe an important class of semigroup algebras arising from completely 0-simple semigroups.

2.2.1. Let K be a field and let R be any algebra over K. Let I and Λ be indexing sets, and P be a $\Lambda \times I$ matrix with entries in R. By $\hat{R} = \mathcal{M}(R; I, \Lambda; P)$ one defines the following associative K-algebra. The elements of $\hat{R}$ are all $I \times \Lambda$ matrices over R with finitely many non-zero entries. Addition is the usual addition of matrices, and the scalar multiplication by elements of K is component-wise. Matrices multiply by insertion of the sandwich matrix P, that is, if A and B are two elements of $\hat{R}$, then the product $A \circ B$ in $\hat{R}$ is defined by

$$A \circ B = APB.$$

The K-algebra $\hat{R} = \mathcal{M}(R; I, \Lambda; P)$ is called the *Munn $I \times \Lambda$ matrix algebra over R with sandwich matrix P*. If $| I |= m$ and $| \Lambda |= n$, then denote this algebra by $\hat{R} = \mathcal{M}(R; m, n; P)$. The crucial example and motivation comes from the following observation.

LEMMA 2.5 (Lemma 5.17, [**13**]). *The contracted algebra $K_0[S]$ of Rees matrix semigroup $S = \mathcal{M}^0(G; I, \Lambda; P)$ over a field K is isomorphic with the Munn algebra $\mathcal{M}(K[G]; I, \Lambda; P)$.*

2.2.2. It is well known when Munn algebras are semisimple, see for example, Theorem 5.19 in [**13**].

THEOREM 2.6. *Let K be a field. A Munn algebra $\hat{R} = \mathcal{M}(R; m, n; P)$ over a finite dimensional K-algebra R is semisimple if and only if*

1. *R is semisimple,*
2. *$m = n$ and P is non-singular (that is, P is invertible in the matrix ring $M_n(R)$).*

If this is the case, then $\hat{R} \cong M_n(R)$.

Recall that a semisimple algebra contains an identity. The following property states that condition two in the theorem is equivalent with $\hat{R}$ having an identity element.

THEOREM 2.7 (Lemma 5.18 [**13**]). *Let K be a field. The Munn algebra $\hat{R} = \mathcal{M}(R; m, n; P)$ over a finite dimensional K-algebra R contains an identity if and only if*

1. *R contains an identity,*
2. *the sandwich matrix P is non-singular (in particular $m = n$).*

If this is the case, then the mapping $A \to AP$ is an isomorphism of $\hat{R}$ onto the full matrix algebra $M_n(R)$.

Maschke's Theorem describes when the group algebra $K[G]$ of a finite group G is semisimple Artinian: $K[G]$ is semisimple if and only if the characteristic of K does not divide the order of G. For the contracted semigroup algebras $K_0[S]$ of a finite completely 0-simple semigroup $S = \mathcal{M}^0(G; m, n; P)$, we have the following corollary.

COROLLARY 2.8. *Let $S = \mathcal{M}^0(G; m, n; P)$ be a finite completely 0-simple semigroup and K a field, Then $K[S]$ is semisimple if and only if (i) the characteristic of K does not divide the order of G. (ii) P is non-singular (in particular m=n) regarded as a matrix over $K[G]$.*

Zelmanov showed that $K[S]$ is right Artinian implies that S is a finite semigroup and the converse holds if S is a monoid (see also Theorem 14.23 in [**52**]). More generally, Munn and Poinzovskii obtained independently the following generalization of Maschke's Theorem.

THEOREM 2.9 (Theorem 14. 24, [**52**]). *Let S be a semigroup and K a field. The semigroup algebra $K[S]$ is semisimple Artinian if and only if S has a chain of ideals $S = S_n \supseteq S_{n-1} \supseteq \cdots \supseteq S_1$ such that every S_i/S_{i-1} and S_1 is a completely 0-simple semigroup with a Rees representation of the type $\mathcal{M}^0(G; m, m; P)$ for some*

$m \geq 1$, and an invertible matrix P in $M_m(K[G])$, where G is a finite group of order not divisible by the characteristic of K.

2.3. Semigroup Algebras and Group Algebras

In this section, S will be a cancellative semigroup and K a field. If S has a group of (right) fractions $G = SS^{-1}$ then the group algebra $K[G]$ is a localization of the semigroup algebra $K[S]$. Since group algebras have been well studied, this fact can be exploited in the study of semigroup algebras. Therefore we recall some properties of localization and group algebras.

2.3.1. We start with the following basic result (see [**54**], Lemma 10.2.13 or Lemma 7.13 in [**52**]).

LEMMA 2.10. *Let T be a right Ore subset of a ring R. Then,*

1. *for every $a_1, \cdots, a_n \in RT^{-1}$, there exists $t \in T$ such that $a_i t \in R$, for all $i = 1, \cdots, n$.*
2. *for every right ideal I of RT^{-1}, we have $(I \cap R)RT^{-1} = I$.*

Moreover, if Z is a right Ore subset of cancellative semigroup S, then Z is a right Ore subset of $K[S]$ and $K[S]Z^{-1} = K[SZ^{-1}]$.

Hence one has the following observation on the behavior of primeness and semiprimeness under localizations.

LEMMA 2.11. *Let B be a ring that is the localization of its subring A with respect to a right Ore subset Z. Then,*

1. *B is prime (semiprime) whenever A is prime (semiprime, respectively).*
2. *If Z is contained in the centre $Z(B)$ of B or B is right Noetherian, then the converse holds.*

2.3.2. We now state a result that will be crucial for studying the relationship between the properties of $K[S]$ and $K[SS^{-1}]$ (see [**52**] Lemma 7.15).

LEMMA 2.12. *Let G be a group of right fractions of its subsemigroup S. Then,*

1. *For any right ideals $R_1 \subseteq R_2$ of $K[G]$, we have $R_1 \cap K[S] \subseteq R_2 \cap K[S]$.*
2. *If P is a prime ideal of $K[G]$ and $K[G]/P$ is a Goldie ring (or $K[G]$ is a right Noetherian ring), then $P \cap K[S]$ is a prime ideal of $K[S]$.*

Let S be a semigroup with a group G of right fraction. We now solve when $K[S]$ is prime or semiprime. That these conditions are equivalent with $K[G]$ being prime or semiprime was shown by Okniński (Theorem 7.19 in [**52**]). The equivalence of the other conditions is well known (see for example [**54**], Section II.2).

THEOREM 2.13. *Assume S has a group G of right fractions. Then the following conditions are equivalent.*

1. *$K[S]$ is prime (semiprime).*
2. *$K[G]$ is prime (semiprime).*

3. *G has no non-trivial finite normal subgroups ($ch(K) = 0$, or $ch(K) = p > 0$ and G has no finite normal subgroups of order divisible by p).*
4. *The FC-center $\Delta(G)$ (defined in Chapter 5) is torsion-free abelian ($ch(K) = 0$, or $ch(K) = p > 0$ and $\Delta(G)$ has no p-torsion).*
5. *$Z(K[G])$ is prime (semiprime).*

CHAPTER 3

Global Dimensions of Semigroup Rings

In [**50**] and [**51**], Nico discussed the upper bound for homological dimensions of semigroup rings $R[S]$ of a finite regular semigroup S over a commutative ring R. Recently, in [**45**], Kuzmanovich and Teply discovered bounds for homological dimensions of semigroup rings $R[S]$ of semigroups S which are monoids with a chain of ideals such that each factor semigroup is a finite non-null Rees matrix semigroup: the bounds are in terms of the dimension of the coefficient ring R and the structure of the semigroup S. In this Chapter we continue these investigations. Amongst other results we show that the upper bound obtained by Kuzmanovich and Teply can be sharpened. The results proved in this chapter will appear in [**37**].

We now outline the contents of this chapter. We will discuss the global dimension of semigroup rings $R[S]$ where R is a ring and S is a monoid with a sequence of ideals $S = I_1 \supset I_2 \supset \cdots \supset I_n \supset I_{n+1}$ such that I_i/I_{i+1} is a non-null Rees matrix semigroup $\mathcal{M}^0(G_i; m_i, n_i; P_i)$ (for all $1 \le i \le n$) and $I_{n+1} = \{\theta\}$ or $\emptyset$.

In Section 3.1 we recall the definition of global dimension of a ring. In Section 3.2 we recall some results on the global dimension of group algebras. In Section 3.3, we show that the global dimension of $R[S]$ is bounded by the global dimension of

$R[G_i]$ and a parameter $\mu_j(S)$ which somehow depends on the sandwich matrices P_i of the Munn algebras $\mathcal{M}(R[G_i]; m_i, n_i; P_i)$. In Section 3.4, we apply these results to finite semigroups. We obtain the exact global dimension of $K[S]$ where K is a field and S is a monoid extension of a finite non-null Rees matrix semigroup (the latter is not necessary a completely 0-simple semigroup). Specific examples are given in Section 3.5.

The above mentioned results are a step toward handling the global dimension of a semigroup algebra $K[S]$ of an arbitrary finite semigroup S. The remaining step is to deal with semigroups which have a principal factor that is a null semigroup. The examples given in Section 3.6 indicate that the solution here is rather unclear. Indeed we give two examples (with a null factor), but one has finite global dimension and the other does not.

3.1. Global Dimensions

3.1.1. *Projective dimension* of a right module M_R, written *pd* M_R, is the shortest length n of a projective resolution

$$0 \to P_n \to \cdots \to P_0 \to M \to 0$$

or ∞ if no finite length projective resolution exists.

In fact, the following numbers are all equal:

1. $sup\{pd\ M \mid M$ any right R-module$\}$;
2. $sup\{pd\ M \mid M$ any cyclic right R-module$\}$;

The common number is called the **right global dimension** of R, written *rgld* R.

3.1.2. $rgld\ R = 0$ means precisely that R is a semisimple Artinian ring. $rgld\ R \leq 1$ means that R is a right hereditary ring. Note a right hereditary ring means every right ideal is projective or equivalently every submodule of projective module R_R is projective. (For a right hereditary ring, every submodule of a free module is isomorphic to a direct sum of right ideals).

3.1.3. Similarly one can define left global dimension $lgld\ R$. In general $lgld\ R \neq rgld\ R$. However, if R is left and right Noetherian, then $lgld\ R = rgld\ R$ (see [**48**] 7.1.11).

3.1.4. We list several properties on the global dimension. For more details, we refer the reader to [**48**] and [**54**].

1. Consider a short exact sequence of right R-modules

$$0 \to A \to B \to C \to 0$$

   If two of the modules A, B, C have finite projective dimension then so does the third. Moreover, we have the following three possibilities:

   (a) if $pd\ B < pd\ A$, then $pd\ C = pd\ A + 1$.

   (b) if $pd\ B = pd\ A$, then $pd\ C \leq pd\ B + 1$.

   (c) if $pd\ B > pd\ A$, then $pd\ C = pd\ B$.

2. $rgld\ R = sup\{pd\ I \mid I \lhd_r R\} + 1$ unless R is semisimple.

3. If ϕ is a right denominator set in a ring R then $rgld\ R\phi^{-1} \leq rgld\ R$ where $R\phi^{-1}$ is a localization ring of R.

4. If σ is an automorphism of R, then $rgld\ R[x, \sigma] = rgld\ R + 1$.

3.2. Global dimensions of group rings

Most of the following results come from [**48**] and [**54**].

3.2.1. In this subsection, we consider the global dimension of a group ring $R[G]$.

LEMMA 3.1 (Theorem 7.5.6 in [**48**]). *Let R be a ring, G be finite group with $|G|$ a unit in R, and then rgld $R[G] =$ rgld R.*

We say that G is a *polycyclic-by-finite group* if G has a finite subnormal series

$$\langle 1 \rangle = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

with each quotient G_{i+1}/G_i infinite cyclic or finite. Particularly, if each quotient is infinite cyclic, then we call G a *poly-infinite cyclic* group. The *Hirsch number* of a polycyclic-by-finite group G, written $h(G)$, is defined to be the number of infinite cyclic quotients which occur in the above series. It is well known that $R[G]$ is right Noetherian if R is right Noetherian and G is a polycyclic-by-finite group. But the converse is still an open question.

LEMMA 3.2 (Corollary 7.5.6, [**48**]). *Let R be a ring and G be a group. Then*

1. *rgld $R \leq$ rgld $R[G]$;*
2. *If G is poly-infinite cyclic with Hirsch number h then rgld $R[G] \leq$ rgld $R+h$;*
3. *If R is a $\mathbf{Q}$-algebra and G is polycyclic-by-finite with Hirsch number h then*

$$rgld\ R[G] \leq rgld\ R + h.$$

3.2.2. It is well known when a group algebra $K[G]$ has zero global dimension, that is, when it is semisimple Artinian. This is known by Maschke's Theorem.

THEOREM 3.3. *Let G be a finite group. Then $K[G]$ is semisimple if and only if $\operatorname{char} K = 0$ or $\operatorname{char} K = p$ and G is a p'-group.*

In the next chapter we will investigate the description of when $rgld\ K[G] = 1$. But now we list some properties on when the global dimension is finite (see Chapter 10 in [**54**]).

THEOREM 3.4. *Let $K[G]$ be a group algebra. The following properties hold.*

1. *If V is the principal right $K[G]$ module, then $rgld\ K[G] = pd\ V$.*
2. *If H is a subgroup of G, then $rgld\ K[H] \leq rgld\ K[G]$.*
3. *Let H be a normal subgroup of G. If $K[H]$ and $K[G/H]$ have finite global dimensions, then so does $K[G]$, and we have $rgld\ K[G] \leq rgld\ K[H] + rgld\ K[G/H]$.*
4. *If $rgld\ K[G] < \infty$ and if $\operatorname{char} K = p$, then G is a p'-group. In particular, if G is finite group, $rgld\ K[G] < \infty$ implies that $K[G]$ is semisimple.*
5. *If $G = \langle x_i \mid i \in I \rangle$ is a nontrivial free group, then the augmentation ideal $\omega(K[G])$ is a free right $K[G]$-module with free generators $\{x_i - 1 \mid i \in I\}$. Furthermore, $rgld\ K[G] = 1$.*
6. *Let H be a subgroup of G of finite index. If $rgld\ K[H] < \infty$ and if G has no element of order p in case $\operatorname{char} K = p$, then $rgld\ K[G] < \infty$.*

7. *Let G be a polycyclic-by-finite group. Then $rgld\ K[G] < \infty$ if and only if G has no elements of order p in case K has characteristic p. Furthermore, $rgld\ K[G] = h(G)$, the Hirsch number of G.*

3.3. Monoid extensions of Rees matrix semigroups

In this section, we will investigate the global dimension of semigroup rings $R[S]$, where R is a ring with an identity and S is a monoid with a chain of ideals $S = S_1 \supset S_2 \supset \cdots \supset S_\iota$ such that each factor semigroup S_i/S_{i+1} is a non-null Rees matrix semigroup $\mathcal{M}^0(G_i; m_i, n_i; P_i)$. In particular, any finite semisimple semigroup S satisfies the above assumption. In [**45**], Kuzmanovich and Teply showed

THEOREM 3.5 (Theorem 3.7 in [**45**]). *Let R be a ring with identity and S be a monoid with a chain of ideals $S = S_1 \supset S_2 \supset \cdots \supset S_\iota \supset S_{\iota+1}$ such that each factor semigroup S_i/S_{i+1} is a finite, non-null Rees matrix semigroup $\mathcal{M}^0(G_i; m_i, n_i; P_i)$ and $S_{\iota+1} = \emptyset$ or $\{\theta\}$. Then the global dimension of $R[S]$ is finite if and only if each $R[G_i]$ has finite global dimension. In this case, $rgldR[G_i] = rgldR$ and then*

$$rgldR \leq rgldR[S] \leq rgldR + 2\iota - 2.$$

3.3.1. We will sharpen the above upper bound. First we note that for a Rees matrix semigroup $\mathcal{M}^0(G; m, n; P)$ with non-null multiplication, there is no loss of generality in assuming that $P_{11} = 1$ (Remark 3.5 in [**45**]). To see this, suppose that $P_{ij} = g$ for an element $g \in G$. Let Q be the $m \times m$ permutation matrix corresponding to the transposition $(1, i)$. Let Q_1 be the $n \times n$ permutation matrix corresponding to the transposition $(1, j)$. By definition, $Q \circ Q = I_m$ and $Q_1 \circ Q_1 = I_n$. Define $\phi : \mathcal{M}^0(G; m, n; P) \to \mathcal{M}^0(G; m, n; Q_1 \circ P \circ Q)$ by $\phi(A) = Q \circ A \circ Q_1$. Note that $\phi(A)\phi(B) = (Q \circ A \circ Q_1) \circ (Q_1 \circ P \circ Q) \circ (Q \circ B \circ Q_1) = Q \circ A \circ P \circ B \circ Q_1 = \phi(AB)$. It follows that ϕ is an isomorphism. Clearly the $(1,1)$-entry of the sandwich matrix $Q_1 \circ P \circ Q$ is g. Hence we may assume that $P_{11} = g$. Now let W be the $n \times n$ diagonal matrix given by $W = diag(g^{-1}, 1, \cdots, 1)$, and define $\psi : \mathcal{M}^0(G; m, n; P) \to \mathcal{M}^0(G; m, n; W \circ P)$ by $\psi(A) = A \circ W^{-1}$. It follows that ψ is an isomorphism and the sandwich matrix $W \circ P$ has 1 on its $(1,1)$-entry. So indeed we may assume that $P_{11} = 1$.

THEOREM 3.6 (Lemma 3.6 in [**45**]). *Let S be a monoid with an ideal U which is isomorphic to a non-zero Rees matrix semigroup $\mathcal{M}^0(G; m, n; P)$. Then the ideal $I = R_0[U]$ of $\Lambda = R_0[S]$ satisfies the following properties:*

1. *There exist subsets A, B of U and an idempotent $e \in U$ such that $I = \bigoplus_{a \in A} aI = \bigoplus_{b \in B} Ib$. Moreover, $I = IeI = \Lambda e \Lambda$, $e\Lambda = eI$ and $\Lambda e = Ie$.*
2. *For any $a \in A$ and $b \in B$, $ae = a$, $eb = b$ and thus $ba = ebae \in G \bigcup \{\theta\}$, where θ denotes the zero element.*
3. *As a right Λ-module, $I = \bigoplus_{a \in A} aI$ is projective. Similarly, $I = \bigoplus_{b \in B} Ib$ is a left projective Λ-module.*
4. *Ie is a left projective Λ-module. Considered as a right $R[G]$-module, $Ie \cong \bigoplus_{a \in A} aR[G]$ is free.*

5. *Any nonzero element of I can be expressed as a sum of $a\alpha b$ where $a \in A, b \in B$, and $\alpha \in R[G]$.*

Proof. Without loss of generality, we can assume $P_{11} = 1$. Abusing notation, we identify $G\bigcup\{\theta\}$ with $\{(g, 1, 1) \mid g \in G\bigcup\{\theta\}\}$.

1. Let $e = (1, 1, 1)$, that is, e has a 1 in $(1, 1)$ entry and zero elsewhere. Then $e^2 = e \circ P \circ e = e$ is an idempotent and thus $e\Lambda = eI$ and $\Lambda e = Ie$ (again $\circ$ means the ordinary product of matrices). Clearly, $IeI \subseteq I$. Now we need to show that $I \subseteq IeI$. It is sufficient to show that, for an arbitrary element $a \in R[G]$, IeI contains a matrix that has a as its (i, j) entry and zero in its other entries. Indeed, let A_i be the matrix with 1 in the $(i, 1)$ entry and all other entries 0, and let C_j be the matrix with a in the $(1, j)$ entry and all other entries 0. Then A_ieC_j has a as its (i, j) entry and zero for all its other entries. Hence $IeI = I$. Let $A = \{(1, i, 1) | 1 \le i \le n\}$. Choose A_i as before, clearly A_iI is the i-th row of I, so $I = \bigoplus_{A_i \in A} A_iI$. Similarly, $I = \bigoplus_{B_j \in B} IB_j$ where $B = \{(1, 1, j) | 1 \le j \le m\}$ and B_j is the matrix with 1 in $(1, j)$ entry and all other entries 0.
2. This follows from the proof of 1. For example, $A_ie = A_i \circ P \circ e = A_i$.
3. Since $A_ie = A_i$, a direct computation shows that left multiplication by A_i yields a (right) Λ-isomorphism from eI to A_iI. So $I = \oplus_{A_i \in A} A_iI \cong \oplus_{A_i \in A} eI$ is projective as a right Λ-module.

4. As $Ie = \Lambda e$, it is clear that Ie is a left projective Λ-module. Hence, from the above,

$$\begin{aligned} Ie &= \bigoplus_{A_i \in A} A_i I e \\ &= \bigoplus_{A_i \in A} A_i e I e \\ &\cong \bigoplus_{A_i \in A} A_i R[G] \end{aligned}$$

Since $A_i R[G] \cong R[G]$ as $R[G]$ modules, we obtain that Ie is a free right $R[G]$-module.

5. Similar as in the proof of the first part (replace e by $e\alpha \in R[G]$ and C_j by B_j). □

3.3.2. With notations as in 3.3.1, for any left Λ-module M, we define two modules M^{*} and M^{**} via the natural exact sequences as in [**50**]:

$$0 \longrightarrow \Lambda e M \longrightarrow M \longrightarrow M^{*} \longrightarrow 0,$$

$$0 \longrightarrow M^{**} \longrightarrow \Lambda e \otimes_{R[G]} eM \xrightarrow{\delta} \Lambda e M \longrightarrow 0.$$

Here $IeM = \Lambda eM$ is a submodule of M, eM is also a left $R[G]$-module; the map δ in the second sequence is given by $\beta \otimes m \longmapsto \beta m$.

Then we have the following lemma generalizing that in the completely-0-simple case discussed by Nico in [**50**].

LEMMA 3.7. *With M^{*} and M^{**} defined as above, $xM^{*} = xM^{**} = 0$ for all $x \in I$. Moreover, if the subalgebra I has a left identity, then $M^{**} = 0$ for every left $R_0[S]$-module M.*

Proof. Because of Lemma 3.6, $\Lambda eM = \Lambda e\Lambda M = IM$ and thus $M^{*} = M/IM$. Hence $xM^{*} = 0$ is obvious. By Lemma 3.6.(4), any element $\alpha \in M^{**}$ can be written as $\alpha = \sum_{a\in A} a \otimes m_a$, where $m_a \in eM$ and $\sum_{a\in A} am_a = 0$ in ΛeM. Now, let $x \in I$. By Lemma 3.6.(5), write $x = \sum a'\alpha' b'$ with $a' \in A, b' \in B$, and $\alpha' \in R[G]$. For each term $a \otimes m_a$ of α, if $b'a \in R[G]$, then $a'\alpha' b'a \otimes m_a = a' \otimes \alpha' b' am_a$, and if $b'a = 0$, then $a'\alpha' b'a \otimes m_a = 0$. Hence, always, $a'\alpha' b'\alpha = \sum_{a\in A} a' \otimes \alpha' b' am_a = a' \otimes \alpha' b'(\sum_{a\in A} am_a) = 0$. Therefore $x\alpha = 0$, as desired. The last part of the statement of the lemma is obvious by using x equal the left identity of I. □

It follows from the Lemma 3.7 that for any left $R_0[S]$-module M both modules M^{*} and M^{**} are left $R_0[S/I]$-modules. We also mention the following well known lemma on change of rings (see Proposition 7.2.2 in [**48**]).

LEMMA 3.8. *Let R, S be rings with identity. If $R \longrightarrow S$ is a ring homomorphism, then for any left S-module M,*

$$pd_R(M) \leq pd_S(M) + pd_R(S).$$

3.3.3. In order to show the main theorem of this section, we also need the following lemma.

LEMMA 3.9. *Assume S is a monoid with an ideal U that is isomorphic to a non-null Rees matrix semigroup $\mathcal{M}^0(G;m,n;P)$ and $S \neq U$. Let $T = S/U$. Consider $R_0[T]$ as a left $R_0[S]$-module, then $pd_{R_0[S]}(R_0[T]) \leq 1$. Furthermore, $R_0[T]$ is projective if and only if $R_0[U]$ has a right identity.*

Proof. Obviously, we have a short exact sequence

$$0 \longrightarrow R_0[U] \longrightarrow R_0[S] \longrightarrow R_0[T] \longrightarrow 0.$$

By Lemma 3.6, $R_0[U]$ is a projective $R_0[S]$-module and thus $pd_{R_0[S]}(R_0[T]) \leq 1$. Furthermore, $R_0[T]$ is projective if and only if the sequence splits, or equivalently if $R_0[U]$ has a right identity. □

THEOREM 3.10. *Let S be a monoid, U an ideal which is isomorphic to a non-null Rees matrix semigroup $\mathcal{M}^0(G;m,n;P)$ and let $T = S/U$. Then, for any ring R with identity,*

$$lgld\ (R_0[S]) \leq max\{lgld\ (R[G]), lgld\ (R_0[T]) + \sigma(U)\},$$

where

$$\sigma(U) = \begin{cases} 0, & \textit{if } R_0[U] \textit{ has an identity.} \\ 1, & \textit{if } R_0[U] \textit{ has a left or right identity, but not an identity.} \\ 2, & \textit{if } R_0[U] \textit{ has neither a left nor a right identity.} \end{cases}$$

Proof. As before, put $\Lambda = R_0[S]$. Consider the following exact sequences:

$$0 \longrightarrow \Lambda eM \longrightarrow M \longrightarrow M^{*} \longrightarrow 0,$$

$$0 \longrightarrow M^{\cdot\cdot} \longrightarrow \Lambda e \otimes_{R[G]} eM \xrightarrow{\delta} \Lambda eM \longrightarrow 0.$$

for any left Λ-module M. By 3.1.4,

$$pd_\Lambda(M) \leq \max\{pd_\Lambda(\Lambda eM), pd_\Lambda(M^{\cdot})\},$$

and

$$pd_\Lambda(\Lambda eM) \leq \max\{pd_\Lambda(\Lambda e \otimes_{R[G]} eM), pd_\Lambda(M^{\cdot\cdot}) + 1\}.$$

By Lemma 3.6, $\Lambda e = Ie$ is a free right $R[G]$-module, hence $pd_\Lambda(\Lambda e \otimes_{R[G]} eM) \leq pd_{R[G]}(eM)$. Thus the second inequality becomes

$$pd_\Lambda(\Lambda eM) \leq \max\{pd_{R[G]}(eM), pd_\Lambda(M^{\cdot\cdot}) + 1\}.$$

And by Lemma 3.8 and Lemma 3.9, we have,

$$\begin{aligned} pd_\Lambda(M^{\cdot}) &\leq pd_{R_0[T]}(M^{\cdot}) + pd_\Lambda(R_0[T]) \\ &\leq \begin{cases} lgld(R_0[T]), & \text{if } R_0[U] \text{ has a right identity.} \\ lgld(R_0[T]) + 1, & \text{otherwise.} \end{cases} \end{aligned}$$

If $R_0[U]$ does not have a left identity, then again by Lemma 3.8 and Lemma 3.9,

$$\begin{aligned} pd_\Lambda(M^{\cdot\cdot}) &\leq pd_{R_0[T]}(M^{\cdot\cdot}) + pd_\Lambda(R_0[T]) \\ &\leq \begin{cases} lgld(R_0[T]), & \text{if } R_0[U] \text{ has a right identity.} \\ lgld(R_0[T]) + 1, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} pd_\Lambda(M) &\leq \max\{pd_\Lambda(\Lambda e M), pd_\Lambda(M^{*})\} \\ &\leq \max\{pd_{R[G]}(eM), pd_\Lambda(M^{**}) + 1, pd_\Lambda(M^{*})\} \\ &\leq max\{lgld(R[G]), lgld(R_0[T]) + \sigma(U)\}. \end{aligned}$$

So the result follows in this case. On the other hand, if $R_0[U]$ has a left identity, then, by Lemma 3.7, $M^{**} = 0$ for any left Λ-module M. From the second exact sequence, we then get,

$$\begin{aligned} pd_\Lambda(\Lambda e M) &= pd_\Lambda(\Lambda e \otimes_{R[G]} eM) \\ &\leq pd_{R[G]}(eM) \\ &\leq lgld(R[G]) \end{aligned}$$

Hence

$$\begin{aligned} pd_\Lambda(M) &\leq max\{pd_\Lambda(\Lambda e M), pd_\Lambda(M^{*})\} \\ &\leq max\{lgld(R[G]), lgld(R_0[T]) + \sigma(U)\}. \end{aligned}$$

So the result follows. □

COROLLARY 3.11. *Let $S = \mathcal{M}^0(G; m, n; P)$ with nonzero sandwich matrix P. Then S^1 is a monoid with an ideal S which is isomorphic to a non-null Rees matrix semigroup. Thus*

$$lgld\ R_0[S^1] \leq max\{lgld\ (R[G]), lgld\ (R) + \sigma(S)\}.$$

where

$$\sigma(S) = \begin{cases} 0, & \textit{if } R_0[S] \textit{ has an identity.} \\ 1, & \textit{if } R_0[U] \textit{ has a left or right identity, but not an identity.} \\ 2, & \textit{if } R_0[S] \textit{ has neither a left nor a right identity.} \end{cases}$$

3.3.4. Theorem 3.10 also allows us to find the following upper bound for the left global dimension of contracted semigroup algebras $R_0[S]$ of more general semigroups S.

THEOREM 3.12. *Let S be a monoid with a sequence of ideals $S = I_1 \supset I_2 \supset \cdots \supset I_n \supset I_{n+1}$, where $I_{n+1} = \{\theta\}$ or $\emptyset$. Let R be a ring with an identity. Assume that, for all $1 \le i \le n$, I_i/I_{i+1} is a non-null Rees matrix semigroup $\mathcal{M}^0(G_i; m_i, n_i; P_i)$. Let σ be defined as in Theorem 3.10 and let $\mu_j(S) = \sigma(I_j/I_{j+1}) + \cdots + \sigma(I_n/I_{n+1})$, for $1 \le j \le n$. Let $\mu_{n+1}(S) = 0$. Then*

$$lgld\ R_0[S] \le max\{lgld(R[G_j]) + \mu_{j+1}(S) : j = 1, \cdots, n\}.$$

Proof. If $n = 1$, that S being a monoid implies $m_1 = n_1 = 1$. Hence $R_0[S] = R[G]$, a group ring. In this situation the assertion is obvious. We now prove the result by induction on n. For $n \ge 2$ consider the factor semigroup S/I_n. By Theorem 3.10, we have

$$lgld\ R_0[S] \le max\{lgld\ (R[G_n]), lgld\ (R_0[S/I_n]) + \sigma(I_n)\}.$$

By the induction hypothesis,

$$lgld\ R_0[S/I_n] \le max\{lgld\ (R[G_j]) + \mu_{j+1}(S/I_n) : j = 1, \cdots, n-1\}.$$

As $\mu_{j+1}(S/I_n) = \sigma(I_{j+1}/I_{j+2}) + \cdots + \sigma(I_{n-1}/I_n)$, we therefore get

$$\begin{aligned} lgld\ R_0[S] \leq\ & max\{lgld\ (R[G_n]), lgld\ (R[G_j]) + \mu_{j+1}(S/I_n) + \\ & \sigma(I_n) : j = 1, \cdots, n-1\} \\ \leq\ & max\{lgld\ (R[G_j]) + \mu_{j+1}(S) : j = 1, \cdots, n\} \end{aligned}$$

□

Obviously Theorem 3.12 is applicable to finite regular semigroups. This case was also investigated by Nico in [**50**], [**51**].

3.4. Applications

3.4.1. First recall the following result on $m \times n$ matrix A over a division ring D (Corollary 11.2.3 in [**14**]).

LEMMA 3.13. *For a $m \times n$ matrix A over a division ring D the following conditions are equivalent:*

1. *A is left regular, that is, $XA = 0$ implies $X = 0$.*
2. *A has a right inverse, that is, $AB = I$ for some $n \times m$ matrix B.*

Moreover, when (1) and (2) hold, then $m \leq n$, with equality if and only if (1) and (2) are equivalent to their left-right analogue.

3.4.2. As an application of Theorem 3.10, we obtain the exact value of the global dimension of $K_0[S^1]$ for non-null Rees matrix semigroups S and K a field.

THEOREM 3.14. *Let S be a non-null Rees matrix semigroup $\mathcal{M}^0(G;n,m;P)$ with G a finite group. If K is a field of characteristic not dividing the order of G, then lgld $(K_0[S^1]) = \sigma(S) =$ lgld $(K[S^1])$.*

Proof. The assumption implies that $lgld(K[G]) = 0$. From Theorem 3.10, we have $lgld(K_0[S^1]) \leq 1$ provided that $\sigma(S) \leq 1$. It is obvious that $lgld(K_0[S^1]) = 0$ if and only if $\mu_1(S^1) = \sigma(S) = 0$. Hence the theorem holds for $\sigma(S) \leq 1$.

Next assume $\sigma(S) = 2$. We may assume $lgld(K_0[S^1]) \geq 1$. Hence by 3.1.4,

$$lgld(K_0[S^1]) = 1 + \sup\{pd_{K_0[S^1]}(I) : I \subseteq K_0[S^1] \text{ is a left ideal}\}$$

So to prove the theorem, it is sufficient to find a left ideal of projective dimension 1. Since $K[G]$ is semisimple, say $K[G] = M_{k_1}(D_1) \bigoplus \cdots \bigoplus M_{k_r}(D_r)$ for some division rings D_i, we can decompose $K_0[S]$ naturally as the sum of $\mathcal{M}(M_{k_i}(D_i); n_i, m_i; P_i)$ for $1 \leq i \leq r$. Here $P = P_1 \bigoplus \cdots \bigoplus P_r$ and entries of P_i belong to $M_{k_i}(D_i)$ for all $1 \leq i \leq r$. Since $K_0[S]$ does not have a left identity, there exists i_0 such that $\mathcal{M}(M_{k_{i_0}}(D_{i_0}); n_{i_0}, m_{i_0}; P_{i_0}) \cong \mathcal{M}(D_{i_0}; k_{i_0}n_{i_0}, k_{i_0}m_{i_0}; \bar{P_{i_0}})$ does not have a left identity. Here $\overline{P_{i_0}}$ denotes the $k_{i_0}m_{i_0} \times k_{i_0}n_{i_0}$ matrix obtained from P_{i_0} by erasing the matrix brackets of all the entries. Hence $\overline{P_{i_0}}$ does not have a left inverse. From Lemma 3.13, we have $ann_r(\mathcal{M}(M_{n_{i_0}}(D_{i_0}); n_{i_0}, m_{i_0}; P_{i_0})) \neq 0$ and thus $ann_r(K_0[S]) \neq 0$. (By $ann_r(\ )$ we denote the right annihilator.) Let $0 \neq \delta \in ann_r(K_0[S])$ and let $I = K_0[S^1]\delta$. Clearly $K_0[S^1]\delta \cong K$ as left $K_0[S^1]$ modules. By Lemma 3.9 and the fact that $K_0[S]$ does not have a right identity, $pd_{K_0[S^1]}(K) = 1$. The result follows. That $lgld\ K_0[S^1] = lgld\ K[S^1]$ is obvious. □

3.5. Examples

The following examples show that in Theorem 3.14 all possible values for the global dimension can be reached. It follows that in Theorem 3.10 the upper bound obtained can not be sharpened.

EXAMPLE 3.15. *Let* $G = \{1\}$ *be the trivial group and let* $S = \mathcal{M}^0(G; 2, 2; P)$ *with* $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. *Thus, for any field* K, *by Theorem 2.9,* $K_0[S^1]$ *is semisimple and then* $rgld\ K_0[S^1] = 0$.

EXAMPLE 3.16. *Let* G *be the trivial group and let* $S = \mathcal{M}^0(G; 1, 2; P)$ *with* $P = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. *So* S *is a completely 0-simple semigroup. Let* K *be a field. Because of Theorem 2.7,* $K_0[S]$ *does not have an identity element. However, any nonzero element of* S *is a left identity of* $K_0[S]$. *Hence, from Theorem 3.14,* $rgld\ K_0[S^1] = 1$.

EXAMPLE 3.17. *Let* G *be the trivial group and let* $S = \mathcal{M}^0(G; 1, 2; P)$ *with* $P = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. *So* S *is not completely 0-simple semigroup, as* P *is not a regular matrix. Let* K *be a field. Again* $K_0[S]$ *does not have an identity element, but it has* $(1, 1, 2)$ *as a left identity. So, again by Theorem 3.14,* $rgld\ K_0[S^1] = 1$.

EXAMPLE 3.18 (Example 4.1 [**45**]). *Let* $G = \{1\}$ *be the trivial group and let* $S = \mathcal{M}^0(G; 2, 2; P)$ *with* $P = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. *Let* K *be a field. Again by Theorem 2.7,* $K_0[S]$ *does not have an identity and it is easily verified that it neither contains a left nor a right identity. So, again by Theorem 3.14,* $\text{rgld}\ K_0[S^1] = 2$.

3.6. Null factors

If S contains A null factor, then it is still unknown when $K[S]$ has finite global dimension or not. We give two examples, one of finite global dimension (given by Kuzmanovich and Teply, Example 4.1 in [**45**]) and one of infinite global dimension.

3.6.1. By a *graph*, say Γ, we understand a system consisting of a nonempty set, $V(\Gamma)$, whose elements are called *vertices* of Γ, and a set $E(\Gamma)$, whose elements are called *edges* of Γ, and an incidence map $(i, t) : E(\Gamma) \to V(\Gamma) \times V(\Gamma)$. For any edge e of Γ, $i(e), t(e)$ are called *initial* and *terminal* vertices of e, respectively. By a path β in Γ we mean a sequence of edges $\alpha_1, \cdots, \alpha_m$, written $\beta = \alpha_1 \cdots \alpha_m$, with $o(\alpha_{i+1}) = t(\alpha_i)$ for $1 \leq i \leq m - 1$. Define $o(\beta) = o(\alpha_1)$ and $t(\beta) = t(\alpha_m)$. Let B be the set of all paths in Γ; we regard each vertex as a path of length 0 with $t(v_i) = o(v_i) = v_i$, so $V(\Gamma) \subseteq B$. Then the path algebra $K\Gamma$ of Γ over the field K is the K-vector space with basis the set B, and multiplication defined via $\beta \cdot \gamma = \beta\gamma$ if $t(\beta) = o(\gamma)$ and $\beta \cdot \gamma = 0$ otherwise, for $\beta, \gamma \in B$ (note $v_i a_i = a_i$ and $a_i v_{i+1} = a_i$). A familiar example is that of the path algebra $K\Gamma$ where Γ is the directed graph $v_1 \xrightarrow{a_1} v_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} v_n$. The algebra $K\Gamma$ is isomorphic with the ring $T_n(K)$ of upper triangular $n \times n$ matrices over a field K. (see 3.6 in **[18]**)

If K is an algebraically closed field then any finite dimensional K-algebra A is isomorphic with a quotient $K\Gamma/\langle\rho\rangle$ of a path algebra $K\Gamma$, where $\langle\rho\rangle$ is the two-sided ideal of $K\Gamma$ generated by a set ρ in Γ.

Again take Γ as $v_1 \xrightarrow{a_1} v_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} v_n$ and $\rho = \{\text{paths of length } 2\} = \{a_i a_{i+1}\}$, then $K\Gamma/\langle\rho\rangle \cong T_n(K)/(rad\ T_n(K))^2$, a ring known to have global dimension $n-1$ (see **[19]**). In fact $\Lambda = K\Gamma/\langle\rho\rangle$ is a semigroup algebra. To see this, let T be the set of all paths in Γ with a special element θ (zero) adjoined. Then T becomes a semigroup where the product of paths α and β is defined to the conjunction of α and β (as before). Let U be all the paths with length greater than and equal to 2 and θ, then U is an ideal of T. Let S be the Rees factor semigroup T/U. Obviously, $\Lambda \cong K_0[S]$.

For the semigroup $S = \{\theta, v_1, \cdots, v_n, a_1, \cdots, a_{n-1}\}$, by induction, we can construct the ideal chain as follows:

$$S = \bigcup_{i=1}^{n} Sv_iS \supseteq \bigcup_{i=1}^{n-1} Sv_iS \supseteq \bigcup_{i=2}^{n-1} Sv_iS \supseteq \cdots \supseteq Sv_{n-1}S \supseteq Sa_{n-1}S.$$

Denote the above chain by

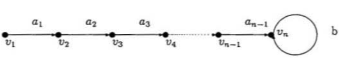
$$S \supseteq S_1 \supseteq \cdots \supseteq S_n,$$

where $S_n = Sa_{n-1}S$ and $S_i = \bigcup_{j=i}^{n-1} Sv_jS$ for all $1 \leq i \leq n-1$. Note that $Sv_{n-1}S = \{v_{n-1}, a_{n-2}, a_{n-1}, \theta\}$ and $Sa_{n-1}S = \{\theta, a_{n-1}\}$. Obviously, $S_i/S_{i+1} = \{v_i, a_{i-1}, \theta\} \cong \mathcal{M}^0(1; 2, 1; P)$ with $P = (1, 1, 1)$ for $2 \leq i \leq n$. And S_n is a null semigroup. Note also $S/S_1 = \{v_n, \theta\}$, and $S_1/S_2 = \{v_1, \theta\}$ are trivial groups with zero adjoined.

This example shows that the global dimension of $K_0[S]$ is finite and S has a principal ideal chain with a null factor semigroup.

3.6.2. The following example based on the above shows that there exists a semigroup S with a null factor semigroup but the global dimension of the contracted semigroup algebra $K_0[S]$ is infinite.

Let Γ be a directed graph as follows:



with the relations $\rho = \{a_1a_2, \cdots, a_{n-2}a_{n-1}, a_{n-1}b, b^2\}$. Then $A = K\Gamma/\langle\rho\rangle$ has infinite global dimension (see Example 1.1 and 1.2 in **[25]**). We can also think of A as a semigroup algebra. If $S = \{\theta, v_1, \cdots, v_n, a_1, \cdots, a_{n-1}, b\}$, then we have principal ideal chain as follows:

$$S = \bigcup_{i=1}^{n} Sv_iS \supseteq \bigcup_{i=2}^{n} Sv_iS \supseteq \cdots \supseteq Sv_{n-1}S \bigcup Sv_nS \supseteq Sv_nS \supseteq SbS$$

Denote the above chain by

$$S = S_1 \supseteq \cdots \supseteq S_n \supseteq S_{n+1}.$$

where $S_{n+1} = SbS = \{b, \theta\}$ and $S_i = \bigcup_{j=i}^{n} Sv_jS$ for all $1 \leq i \leq n$. Note also that $S_i/S_{i+1} = \{v_i, a_{i-1}\} \cong \mathcal{M}^0(1; 2, 1; P)$ with $P = (1, 1, 1)$ for $2 \leq i \leq n$ are non-null semigroups. Obviously S_{n+1} is a null semigroup. And $S_1/S_2 = \{v_1, \theta\}$ is a trivial group with zero adjoined.

CHAPTER 4

Hereditary Group Rings and Semigroup Rings

In this chapter we investigate when a semigroup algebra has right global dimension at most 1, that is, when is it right hereditary. Hence we consider a question posed by Okniński in **[52]**: characterize hereditary semigroup algebras. Recall that if a ring is left and right Noetherian, then the left and right global dimension are equal. It follows that such a ring is left hereditary if and only if it is right hereditary. Such rings we simply call hereditary rings.

Hereditary and semihereditary rings have been the subject of considerable study. Many interesting examples of these rings arise as group rings or semigroup rings. Dicks has characterized the hereditary group rings [**17**], earlier, Goursaud and Valette classified hereditary group rings of nilpotent groups [**26**]. Cheng and Wong [**16**] characterized the hereditary monoid rings that are also domains.

In Section 1 we recall the characterizations of group algebras that are right hereditary. As said above, for semigroup algebras a solution is known only in case $K[S]$ is a domain. In Section 2, as an application of the results in Chapter 3, we describe when $K_0[S^1]$ is hereditary for a finite non-null Rees Matrix semigroup S. Under the extra assumption that $K[S]$ is Noetherian we are able in Section 6 to solve the problem in case S is a cancellative monoid. In section 5 we are able to describe, for arbitrary nilpotent semigroups S (thus not necessarily cancellative), when a semigroup algebra is hereditary Noetherian prime.

Recall that a Noetherian commutative domain is hereditary if and only if it is a Dedekind domain, in particular it is completely integrally closed, that is, a maximal order. Also in the non commutative setting there is a close relationship with maximal order. For example a hereditary Noetherian prime ring which is P.I. is equivalent to a Dedekind prime ring. Hence, in some sense, it is not surprising that our answer in Section 5 relates our description to a special class of maximal orders, in particular to principal ideal rings. We therefore recall some background on maximal orders in Section 3 and in Section 4 we recall some results on group and semigroup rings that are principal ideal rings. The results proved in this chapter will appear in [**37**].

4.1. Hereditary Group Rings

4.1.1. In the case where G is a nilpotent group, Goursaud and Valette [**26**] (see for example in [**52**]) showed the following result.

LEMMA 4.1. *Let G be a nilpotent group. Then $K[G]$ is right hereditary if and only if either of the following holds:*

1. *G is finite-by-(infinite cyclic), and the order of the torsion subgroup of G is invertible in K.*
2. *G is locally finite and countable, and the order of every element of G is invertible in K.*

4.1.2. A very deep result on right hereditary group rings for any group G was described by Dicks. To state this result, we need some preparation. Let us fix a connected graph X. We may view X as a small category with object set $ob(X) = E(X) \cup V(X)$ and non-identity morphism $i_e : e \to i(e), t_e : e \to t(e)$ for $e \in E(X)$. A functor $\mathcal{G} : X \to Groups$, into the category of groups, is called a *connected graph of groups*. For vertices v of X, the $\mathcal{G}(v)$ will be called *vertex groups* of $\mathcal{G}$, and for edges e of X, the $\mathcal{G}(e)$ will be called the *edge groups* of $\mathcal{G}$. The image of $x \in \mathcal{G}(e)$ under homomorphism $\mathcal{G}(i_e) : \mathcal{G}(e) \to \mathcal{G}(i(e))$ will be denoted by x^{i_e}, and a similar notation is used for t_e. Let T be a *spanning tree* for X, that is, a subgraph with the same vertex set and with a minimal edge set so that the subgraph is still connected. The *fundamental group* of $\mathcal{G}$ with respect to T is defined as the group $\pi(\mathcal{G}, T)$ universal with respect to the following properties:

1. for each vertex v of X, there is a group homomorphism $\mathcal{G}(v) \to \pi(\mathcal{G}, T)$.
2. for each edge e of X, there is an element $q(e)$ of $\pi(\mathcal{G}, T)$ such that $q(e)^{-1}x^{i_e}q(e) = x^{t_e}$ for all $x \in \mathcal{G}(e)$, and if e is an edge of T, then $q(e) = 1$.

J. P. Serre showed that the isomorphism class of $\pi(\mathcal{G}, T)$ is independent of choice of T. For this reason one usually speaks of the fundamental group of $\mathcal{G}$, without reference to a spanning tree. For more details, proofs, and applications, the reader is referred to [**17**].

LEMMA 4.2 (Dicks, Theorem 1 in [**17**] or Theorem 17.4 in [**52**]). *Let G be a group. Then $K[G]$ is right hereditary if and only if*

**) G is the fundamental group of a connected graph of finite groups with invertible orders in K.*

Moreover, if G is finitely generated, then the above is equivalent to any of the following conditions:

1. *G has a free subgroup of finite index, and the orders of finite subgroups of G are invertible in K.*
2. *G is the fundamental group of a finite connected graph of finite groups of orders invertible in K.*

4.1.3. It is well known when a fundamental group G of a connected graph of finite groups has no free subgroup of rank 2. By $G_1 * G_2$ we denote the free product of the groups G_1 and G_2. The cyclic group of order two is denoted by C_2.

LEMMA 4.3 (Dicks, Theorem 2, [**17**]). *A fundamental group G of a connected graph of finite groups has no free subgroup of rank 2 if and only if either of the following holds:*

1. *G is countable locally finite.*
2. *G is finite-by-(infinite cyclic).*
3. *G is finite-by-($C_2 * C_2$).*

The *infinite dihedral group* D_∞ is the group with presentation $\{s, t \mid t^2 = 1, tst^{-1} = s^{-1}\}$. It is well known that $C_2 * C_2 \cong D_\infty$.

4.1.4. Monoid algebras that are hereditary domains are characterized in the following result.

LEMMA 4.4 (Chen and Wong, [**16**]). *The following conditions are equivalent for a monoid S.*

1. $K[S]$ *is a hereditary domain.*
2. $K[S]$ *is hereditary, and* S *is torsion free and weakly cancellative.*
3. S *is a free product of a free group and a free monoid.*
4. $K[S]$ *is a free ideal ring (fir).*

Here, torsion free means that, for any $a \in S$, we have $a = 1$ whenever $a^n = 1$ for some $n \geq 1$, and weakly cancellative means that either of the equalities $ab = a, ba = a$ implies that $b = 1$ and that $aub = ab$ for some $u \in U(S)$ implies that $u = 1$.

4.2. Finite Non-null Rees Matrix Semigroups

Let S be a non-null Rees matrix semigroup $\mathcal{M}^0(G; n, m; P)$ with G a finite group. In Chapter 3, we showed that if K is a field of characteristic not dividing the order of G, then $lgld(K_0[S^1]) = \sigma(S)$. We now describe when $K_0[S^1]$ is hereditary.

THEOREM 4.5. *Let* S *be a non-null Rees matrix semigroup* $\mathcal{M}^0(G; n, m; P)$ *with* G *a finite group. Let* K *be a field. Then the following are equivalent:*

1. $K_0[S^1]$ *is hereditary.*
2. $K[G]$ *is semisimple and* $K_0[S]$ *has a left or right identity,*

3. *$K[G]$ is semisimple and there exists $a \in K_0[S]$ that is not a right or not a left divisor of zero in $K_0[S]$.*
4. *$K[G]$ is semisimple and $ann_r(K_0[S]) = 0$ or $ann_l(K_0[S]) = 0$.*

Proof. First we show that (1) implies (2). Since $K[S^1]$ is hereditary, $K[G] \cong eK_0[S^1]e$ is also hereditary by Proposition 7.8.9 in [**48**]. Since G is finite, Theorem 3.4 implies that $K[G]$ is semisimple. So (2) follows and that (2) implies (1) follows from Theorem 3.14. That (2) implies (3) and (3) implies (4) are clear. (4) implies (2) is shown in the last part of the proof of Theorem 3.14 in Chapter 3. □

Remark: If G is trivial, then the above conditions are equivalent to $rank(P) = \min\{m, n\}$. In general, the above conditions are equivalent to $rank(P_i) = k_i \cdot \min\{m_i, n_i\}$ for all $1 \leq i \leq r$ when $K[G] = M_{k_1}(D_1) \bigoplus \cdots \bigoplus M_{k_r}(D_r)$ and $K[S] = \bigoplus_{i=1}^r \mathcal{M}(D_i; k_i n_i, k_i m_i; P_i)$. Here $rank(P_i)$ is defined as the dimension of the column space of P_i (see [**52**]).

4.3. Maximal Orders

Recall that a ring Q is a quotient ring if every regular element of Q is a unit. Given a quotient ring Q, a subring R is called a *right order* in Q if each $q \in Q$ has the form rs^{-1} for some $r, s \in R$ (s regular in R). A *left order* is defined analogously; and a left and right order is called an *order*. For a right order R, we have the following results.

LEMMA 4.6 (Lemma 3.1.6 in [48]). *Let R be a right order in a quotient ring Q and let S be subring of Q. Then*

1. *If there are units $a, b \in Q$ such that $aRb \subseteq S$, then S is a also a right order in Q.*
2. *If $R \subseteq S \subseteq Q$ then S is a right order in Q.*
3. *If R is a prime right Goldie ring, A is a nonzero ideal of R, and S is a subring of R with $A \subseteq S \subseteq R$ then S is a prime right Goldie ring, and has the same right quotient ring as R.*

Let R_1, R_2 be two right orders in a fixed quotient ring Q. If there are units $a_1, a_2, b_1, b_2 \in Q$ such that $a_1R_1b_1 \subseteq R_2$ and $a_2R_2b_2 \subseteq R_1$, then R_1, R_2 are called *equivalent* right orders. Further, Let R be a right or left order in a quotient Q, then R is *maximal* right or left order if it is maximal within its equivalence class as above.

Recall that a commutative domain R is completely integrally closed in its quotient field Q if, for a and q in Q with $a \neq 0$, $aq^n \in R$ for all n implies $q \in R$. A commutative integral domain is a maximal order in its quotient field if and only if it is completely integrally closed.

4.3.1. Let R be a right or left order in a quotient ring Q. Then a *fractional right* R-ideal is a submodule I of Q_R such that $aI \subseteq R$ and $bR \subseteq I$ for some units $a, b \in Q$. In a similar fashion *fractional left R-ideal* and *fractional (two-sided) R-ideal* are defined. Further, if $I \subseteq R$, then I is called an (integral) R-ideal.

The *right order* and *left order* of a fractional right (or left) R-ideal I are defined respectively to be

$$O_r(I) = \{q \in Q \mid Iq \subseteq I\},$$

$$O_l(I) = \{q \in Q \mid qI \subseteq I\}.$$

THEOREM 4.7 (Proposition 5.1.4 in [**48**]). *If R is a right order in Q then the following conditions are equivalent:*

1. *R is a maximal right order;*
2. *$O_r(I) = O_l(I) = R$ for all fractional R -ideals I;*
3. *$O_r(I) = O_l(I) = R$ for all R -ideals I.*

4.3.2. We will use the following notations. Let R be an order in a quotient ring Q. For subsets A, B of Q we denote $(A :_l B) = \{q \in Q \mid qB \subseteq A\}$ and $(A :_r B) = \{q \in Q \mid Bq \subseteq A\}$. In particular for a fractional R-ideal I, $(I :_l I) = O_l(I)$. A fractional R-ideal I is *invertible* if there exist a fractional R-ideal B such that $IB = BI = R$. In this case B is usually denoted by I^{-1}. Note that R is a maximal order if and only if $(I :_l I) = (I :_r I) = R$ for every fractional R-ideal I. Hence, if R is a maximal order, then $(R :_l I) = (R :_r I)$ for any fractional R-ideal I. Indeed, let $q \in (R :_l I)$, then $qI \subseteq R$. Hence $IqI \subseteq IR \subseteq I$ and thus $Iq \subseteq R$. Therefore $(R :_l I) = (R :_r I)$ by the symmetry. We simply denote this fractional R-ideal by $(R : I)$ or by I^{-1}. Recall that I is *divisorial* if $I = I^*$, where $I^* = (R : (R : I))$. The divisorial product $I * J$ of two divisorial ideals I and J is defined as $(IJ)^*$.

A prime Goldie ring R such that every nonzero ideal is invertible is called an *Asano prime ring* or an *Asano order*. It is equivalent with R being a maximal order so that every ideal of R is divisorial (Proposition 5.2.6 in [**48**]).

For an hereditary ring, every ideal is projective. Further, a projective ideal is divisorial (5.1.7 in [**48**]), and thus in an hereditary ring every ideal is divisorial. Rings satisfying the following conditions are called *Dedekind prime rings*.

LEMMA 4.8. *The following conditions on a ring R are equivalent:*

1. *R is a hereditary Noetherian prime ring and is a maximal order.*
2. *R is a hereditary Noetherian Asano order.*

4.4. Principal Ideal Rings

In this section we recall some results on semigroup rings that are principal ideal rings. We first state a result of Passman [**55**] on the group ring case $K[G]$ (as mentioned in [**42**] one can allow the coefficient to be a matrix algebra). Fisher and Sehgal had dealt with the case that G is a nilpotent group [**21**].

LEMMA 4.9 (Theorem 1.1, [**42**]). *Let G be a group and $R = M_n(K)$ a matrix ring over a field K. The following conditions are equivalent:*

1. *$R[G]$ is a left principal ideal ring;*
2. *$R[G]$ is left Noetherian and the augmentation ideal $\omega(R[G])$ is a left principal ideal;*

3. *if* $\operatorname{char} K = 0$, *then* G *is finite or finite-by-infinite cyclic; if* $\operatorname{char} K = p > 0$, *then* G *is (finite* p'*)-by-cyclic* p *or* G *is (finite* p'*)-by-infinite cyclic.*

In **[42]** Jespers and Wauters obtained the following extension to semigroup algebras $K[S]$ of cancellative monoids S.

LEMMA 4.10 (Theorem 2.1 in **[42]**). *Let* S *be a cancellative monoid and* K *a field. The following conditions are equivalent:*

1. $K[S]$ *is a left principal ideal ring;*
2. *either* S *is a group satisfying the conditions of Lemma 4.9 or* S *contains a finite subgroup* H *and a non-periodic element* x *such that* $xH = Hx$ *and* $S = \cup_{i \in N} Hx^i$, *if* $\operatorname{char} K = p > 0$, *then* H *is a* p'*- group; moreover the central idempotents of* $K[H]$ *remain central in* $K[S]$.

In particular $K[S]$ *is a left principal ideal ring if and only if* $K[S]$ *is a right principal ideal ring.*

For contracted semigroup algebras of arbitrary nilpotent semigroups Jespers and Wauters proved the following.

LEMMA 4.11 (Theorem 1.5 in **[42]**). *Let* S *be a nilpotent semigroup and* K *be a field such that* $K_0[S]$ *is a prime ring. Then the following conditions are equivalent:*

1. $K_0[S]$ *is an Asano-order;*

2. $K_0[S]$ *is a left principal ideal ring;*

3. $S \cong \mathcal{M}^0(\{e\}; n, n; \Delta)$ *or* $S \cong \mathcal{M}^0(\{x^i | i \in N\}; n, n; \Delta)$ *or* $S \cong \mathcal{M}^0(\{x^i \mid i \in Z\}; n, n; \Delta)$ *(Δ denotes the identity matrix) and thus* $K_0[S] \cong M_n(K)$ *or* $K_0[S] \cong M_n(K[X])$ *or* $K_0[S] \cong M_n(K[X, X^{-1}])$.

In particular $K[S]$ is a left principal ideal ring if and only if $K[S]$ is a right principal ideal ring.

Note that Jespers and Okniński in [**32**] describe arbitrary semigroup algebras that are principal. In particular it is shown that such algebras are P.I.. Since we do not need this result we will not go into the details.

4.5. Nilpotent semigroups

It is well known that a Noetherian commutative domain is hereditary if and only if it is a Dedekind domain. For the non-commutative case, this conclusion is false in general. But when R is a hereditary Noetherian prime (HNP) ring satisfying a polynomial identity(PI), then R is obtained from a Dedekind prime ring by a finite iteration process of forming idealizers of generative isomaximal right ideals, i.e. R is equivalent to a Dedekind prime ring (Theorem 13.7.15, 5.6.12 and 5.6.8, [**48**]). In this section, we prove that if S is a nilpotent semigroup and $K[S]$ is a HNP, even without the PI condition, then $K[S]$ is a Dedekind prime ring and thus $K[S]$ is a maximal order.

4.5.1. Jespers and Okniński have given a structural description of semigroup algebras of nilpotent semigroups (this ultimately is based on the structure theorem of Okniński on linear semigroups, [**53**]). We will exploit one of their structural theorems. Recall that a semigroup S is called *uniform* if it embeds into a completely 0-simple semigroup U such that S intersects non-trivially all $\mathcal{H}$-classes of U (every maximal subgroup G of the least completely 0-simple subsemigroup $\hat{S}$ of U containing S is then generated by $S \cap G$). Recall (from Lemma 2.4) that for a prime ideal P of $K[S]$, S/ρ_P is a subsemigroup of $K[S]/P$.

LEMMA 4.12 (Theorem 3.5 in [**31**]). *Let S be a nilpotent semigroup, K a field and P a prime ideal of $K[S]$ such that $K[S]/P$ is left Goldie with classical ring of quotients $M_n(D)$ and D a division ring. Then the semigroup S/ρ_P has an ideal chain*

$$S/\rho_P = I_r \supseteq I_{r-1} \supseteq \cdots \supseteq I_1 = I \supseteq I_0,$$

where $I_0 = \{\theta\}$ if S has a zero element and $I_0 = \emptyset$ otherwise, and for all $j > 0$, I_j consists of matrices of rank less than or equal to $j + i - 1$ (i is the rank of elements in I) of $S/\rho_P \subseteq M_n(D)$; in particular I is the ideal of elements of S/ρ_P of minimal nonzero rank in $M_n(D)$:

1. *I is uniform in a completely 0-simple inverse subsemigroup $\hat{I}$ of $M_n(D)$ with finitely many idempotents and $\hat{S} = (S/\rho_P) \bigcup \hat{I}$ is a nilpotent subsemigroup of $M_n(D)$.*

2. $K\{I\} \subseteq K[S]/P \subseteq K\{\hat{I}\} = K\{\hat{S}\}$, *where* $K\{\hat{I}\}$ *denotes the subalgebra of* $M_n(D)$ *generated by* $\hat{I}$; *moreover* $M_n(D)$ *is the common classical ring of quotients of these three algebras and* $K\{\hat{I}\}$ *is a left and right localization of* $K\{I\}$ *with respect to an Ore set.*
3. *Denote by* G *a maximal subgroup of* $\hat{I}$ *, there exists a prime ideal* Q *of* $K[G]$ *such that* $K[G]/Q$ *is a Goldie ring and* $K\{\hat{I}\} \cong M_q(K[G]/Q)$, *where* q *is the number of nonzero idempotents of* $\hat{I}$; *moreover* G *is the group of quotients of* $I \bigcap G$.

From this Theorem, we have another lemma considering prime Goldie rings.

LEMMA 4.13 (Theorem 1.6 in [**42**]). *Let* S *be a nilpotent semigroup ,* K *a field ,* $P = K \cdot \theta$ *if* S *has a zero element, otherwise* $P = \{0\}$. *If* $K_0[S] = K[S]/P$ *is a prime left Goldie ring satisfying the ascending chain condition on two sided ideals, then, with notations as in Lemma 4.12, we have* $Q = 0$, $K\{\hat{I}\} = K_0[\hat{I}]$, G *is poly-infinite cyclic and* $q = n$.

4.5.2. From this lemma, we can show that an HNP semigroup algebra of a nilpotent semigroup is a maximal order.

PROPOSITION 4.14. *Let* S *be a nilpotent semigroup. If* $K_0[S]$ *is hereditary prime left Goldie ring satisfying ascending chain condition on two-sided ideals, then* G *is infinite cyclic or trivial. In the latter case,* $S \cong \mathcal{M}^0(\{e\}; n, n; \Delta)$. *In particular,* $K_0[S]$ *is a maximal order.*

Proof. We use the same notation as in Lemma 4.13. Note here that $K_0[\hat{I}] \cong M_q(K[G])$ is a localization of $K_0[I]$ with respect to an Ore set C. First we show that $K_0[\hat{I}] \cong M_q(K[G])$ is also a localization of $K_0[S]$ with respect to the Ore set C. Since elements of C are regular, it is suffices to show that C satisfies the Ore condition in $K_0[S]$. Let $s \in K_0[S]$ and $c \in C$. Then $sc^{-1} \in K_0[\hat{I}] = C^{-1}K_0[I]$, so $sc^{-1} = d^{-1}r$ for some $d \in C$ and $r \in K_0[I]$. Hence $ds = rc$ and thus $Cs \bigcap K_0[S]c \neq \emptyset$.

Now since $K_0[S]$ is hereditary, so is $K_0[\hat{I}] \cong M_q(K[G])$. By Lemma 4.1, G is either finite-by-(infinite cyclic), and the order of the torsion subgroup of G is invertible in K, or G is locally finite and countable, and the order of every element of G is invertible in K. From the proof of Lemma 4.13, we know that G is torsion-free, hence in this case, G is infinite cyclic or trivial. Obviously in the latter case, $S \cong \mathcal{M}^0(\{e\}; n, n; \Delta)$.

Now, we show that $K_0[S]$ is a maximal order when G is infinite cyclic. For this it is sufficient to show that $K_0[S]$ is a Dedekind prime ring. Because of Proposition 5.6.3 in [**48**](A hereditary Noetherian prime ring R is Dedekind if and only if it has no idempotent ideals other than 0 and R), we only have to show that any idempotent ideal of $K_0[S]$ is trivial. So suppose I is a nontrivial idempotent ideal of $K_0[S]$. Then since $K_0[\hat{I}]$ is Noetherian, $IK_0[\hat{I}]$ is an idempotent ideal of $K_0[\hat{I}]$.(Theorem 1.31 in [**11**]). This is a contradiction since $K_0[\hat{I}]$ is a left principal ideal ring by the result of Lemma 4.9 and the fact that a prime principal ideal ring does not contains a nontrivial idempotent ideal. Hence $K_0[S]$ has no nontrivial idempotent ideal and thus we have done. □

4.5.3. Once we know $K_0[S]$ is a maximal order, we have the following structure theorem:

THEOREM 4.15 (Theorem 3.4 in [**37**]). *Let S be a nilpotent semigroup. The following conditions are equivalent:*

1. *$K_0[S]$ is HNP.*
2. *$K_0[S]$ is a prime Asano-order.*
3. *$K_0[S]$ is a prime left principal ideal ring.*
4. *$S \cong \mathcal{M}^0(\{e\}; n, n; \Delta)$ or $S \cong \mathcal{M}^0(\{x^i | i \in N\}; n, n; \Delta)$ or $S \cong \mathcal{M}^0(\{x^i \mid i \in Z\}; n, n; \Delta))$ (Δ denotes the identity matrix) and thus $K_0[S] \cong M_n(K)$ or $K_0[S] \cong M_n(K[X])$ or $K_0[S] \cong M_n(K[X, X^{-1}])$.*

Proof. Note that because of Theorem 4.11, the last three conditions are equivalent. $3 \Longrightarrow 1$ is obvious and it remains to show that $1 \Longrightarrow 2$. If $K_0[S]$ is a HNP, then from Proposition 4.14 it follows that $K_0[S]$ is a Dedekind prime ring and thus a prime Asano order. □.

Since $K_0[S]$ is a HNP, then from proposition 4.14 it follows that G is trivial or infinite cyclic. A structural proof of $1 \Longrightarrow 2$ can be done similarly to that in [**42**].

4.6. Cancellative semigroups

4.6.1. We first recall a simple but useful lemma on cancellative semigroups (Lemma 1.3 in [**36**]). For a completeness' sake, we include the proof.

LEMMA 4.16. *Let S be a cancellative semigroup. If S satisfies the ascending chain condition on right ideals, then for any $a, b \in S$, there exists a positive integer r such that $a^r b \in bS^1$.*

Proof. Let $a, b \in S$. Consider the following ascending chain of right ideals of S:

$$abS^1 \subseteq abS^1 \cup a^2bS^1 \subseteq abS^1 \cup a^2bS^1 \cup a^3bS^1 \subseteq \cdots$$

Since S satisfies the ascending chain condition on right ideals, there exists positive integer $n > i$ such that

$$a^n b \in a^i bS^1.$$

Because, by assumption, S is cancellative, it follows that $a^{n-i}b \in bS^1$. □

4.6.2. Now we prove the following result.

THEOREM 4.17 (Theorem 4.3 in [**37**]). *Let S be a cancellative monoid and K a field of characteristic p (not necessarily nonzero). Then the following are equivalent:*

1. *The semigroup algebra $K[S]$ is a Noetherian hereditary ring.*
2. *The semigroup S satisfies one of the following conditions:*

(a) *S is a finite p'-group.*

(b) *S is a finite p'-by-infinite cyclic group.*

(c) *S contains a finite p'-subgroup H and a non-periodic element x such that $S = \bigcup_{i \in N} Hx^i$, $xH = Hx$, and every central idempotent of $K[H]$ remains central in $K[S]$.*

(d) *S is a (finite p')-by-$C_2 * C_2$ group ($char P \neq 2$).*

Proof. If S satisfies one of conditions (a) or (b), then the result is obvious. If S satisfies case (c), then $K[S]$ is a skew polynomial ring $K[H][g, \sigma]$ with $rgld\ K[S] = rgld\ K[H] + 1$ (See Theorem 7.5.3 in [**48**] or Remark 3.1.4 in Chapter 3). Thus K[S] is hereditary. If S satisfies case (d), then (1) follows from the result of W. Dicks (Lemma 4.2 and Lemma 4.3).

Conversely, assume $K[S]$ is hereditary and Noetherian. Then S has a group G of fractions by Theorem 1.16. So $K[G]$ is a localization of $K[S]$ and $K[G]$ is also hereditary and Noetherian.

Since $K[G]$ is Noetherian, the group G satisfies the ascending chain condition on subgroups. So from Lemma 4.2 and Lemma 4.3 we obtain that G is either finite or finite-by-(infinite cyclic) or finite-by-$C_2 * C_2$ ($char P \neq 2$), and moreover, the orders of finite subgroups of G are invertible in K. In the first case, we get that $K[G]$ and thus $K[S] = K[G]$ is semisimple Artinian.

Now we discuss the second case, that is, G is finite-by-(infinite cyclic). We will prove $K[S]$ is a principal left ideal ring. First note that by Lemma 4.9, $K[G]$ is a principal (left and right) ideal ring.

Because all finite subgroups of G have invertible order in K, Theorem 2.13 implies that the semigroup algebra $K[S]$ is semiprime (so is $K[G]$).

Now we claim $K[S]$ is a maximal order. Since $K[S]$ is a semiprime Noetherian hereditary ring, the semigroup algebra $K[S]$ can be decomposed into a finite direct sum of hereditary Noetherian prime rings (see for example Theorem 5.4.6 in [**48**]):

$$K[S] = \bigoplus_{i=1}^{n} e_i K[S], n \geq 1,$$

where each e_i is a primitive central idempotent. Hence to prove the claim it is sufficient to show that each $e_i K[S]$ is a Dedekind prime ring, and thus we only need to show that each $e_i K[S]$ has no nontrivial idempotent ideal (Proposition 5.6.3 in [**48**]). So suppose I is an idempotent ideal of $e_i K[S]$. Since $K[G]$ is a Noetherian ring and a localization of $K[S]$, it follows that $e_i K[G]$ is also a Noetherian ring and a localization of $e_i K[S]$. Hence $Ie_i K[G]$ is a two-sided idempotent ideal of $e_i K[G]$. But the latter is prime principal ideal ring and thus $Ie_i K[G] = 0$ or $Ie_i K[G] = e_i K[G]$, as required.

Finally, we prove $K[S]$ is a left principal ideal ring. Let H be a finite normal subgroup of G and $g \in G$ so that $G/H = \langle gH \rangle$ is an infinite cyclic group. Then $K[G] = K[H] * (G/H) = K[H][g, g^{-1}; \sigma]$, a skew Laurent polynomial ring over $K[H]$. Obviously, $K[S] \subseteq K[H][g, g^{-1}, \sigma]$ and $S \subseteq G = \langle g, H \rangle$. Let $A = \{i \in Z : S \cap Hg^i \neq \emptyset\}$. Clearly A is a nontrivial subsemigroup of Z. If A is a group, then $A = mZ$ for some $m \geq 1$. Hence $S \subseteq \bigcup_{i \in Z} Hg^{im} \subseteq G$. But $G = \langle S, S^{-1} \rangle$ implies $\bigcup_{i \in Z} Hg^{im} = G$ and thus $m = 1$.

Assume now A is not a group. Then, without loss of generality, we may assume $A \subseteq \mathbf{N}\bigcup\{0\}$. Since submonoids of $\mathbf{N} \cup \{0\}$ are well known (see for example Theorem 2.4 in **[23]**), there exists K_0 such that $k \in A$ for all $k \geq K_0$. Denote $H_k = \{h \in H | hg^k \in S\}$. Hence $H_k \neq \emptyset$ for all $k \geq K_0$. Because H is finite, the automorphism σ has finite order, say α. Let $j = \alpha \cdot |H| \cdot K_0$, then $g^j \in S$ and thus $1 \in H_j$. Obviously, $H_j \subseteq H_{2j} \subseteq \cdots \subseteq H_{nj} \subseteq \cdots$. Since H is a finite group, there exists a multiple j_0 of j such that $H_{mj_0} = H_{j_0}$ for any $m \geq 1$. So H_{j_0} is a subgroup since it is multiplicatively closed. Clearly also

$$H_{j_0+1} \subseteq H_{2j_0+1} \subseteq \cdots \subseteq H_{nj_0+1} \subseteq \cdots$$

$$\cdots$$

$$H_{2j_0-1} \subseteq H_{3j_0-1} \subseteq \cdots \subseteq H_{(n+1)j_0-1} \subseteq \cdots$$

Hence as H is finite, there exists a multiple v of j_0 such that

$$H_v = H_{2v} = \cdots = H_{nv} = \cdots$$

$$H_{v+1} = H_{2v+1} = \cdots = H_{nv+1} = \cdots$$

$$\cdots$$

$$H_{2v-1} = H_{3v-1} = \cdots = H_{(n+1)v-1} = \cdots$$

We claim that $H_v = H$. Let $h \in H \subseteq G = SS^{-1}$. Then $h = p^{-1}t$ for some $p, t \in S$. So $h = p^{-v}(p^{v-1}t)$. Replacing p by p^v we may assume $p = h_{vk}g^{vk}$ for some $h_{vk} \in H_{vk}$ and some positive number k. Hence $t = h'_{vk}g^{vk}$ for some $h'_{vk} \in H_{vk}$.

As g^{vk} acts trivially on H, we get $h = (h_{vk}g^{vk})^{-1}(h'_{vk}g^{vk}) = g^{-vk}h_{vk}^{-1}h'_{vk}g^{vk} = h_{vk}^{-1}h'_{vk} \in H_{vk}$ Hence there exists a positive integer k such that $H = H_{vk} = H_v$.

Since $H_{v+i} \neq \emptyset$, there exists $h_0 \in H_{v+i}$ such that $h_0 g^{v+i} \in S$ for $1 \leq i \leq v$. Hence $g^{2v+i} = h_0^{-1} g^v \cdot h_0 g^{v+i} \in S$ (again we use that v is a multiple of $\alpha \cdot |H|$). Thus $g^l \in S$ for all $l \geq 2v$ and $H_t = H$ for all $t \geq 3v$. Consider the ideal $I = \bigoplus_{t \geq 3v} K[H]g^t$ of $K[S]$. It follows that $HI \subseteq I$ and $gI \subseteq I$. Since $K[S]$ is a maximal order we obtain that $g \in S$ and $H \subseteq S$. Therefore, $S = \bigcup_{i \in N} Hg^i$ and $K[S] = \bigoplus_{i \in N} K[H]g^i \cong K[H][g, \sigma]$.

We now show that the central idempotents in $K[H]$ remain central in $K[S]$. Write $K[H] = A_1 \bigoplus \cdots \bigoplus A_n$, where each A_i is simple Artinian with unit element, say e_i. It is sufficient to prove that each e_i is central in $K[S]$. We do this for $i = 1$. Since conjugation by σ permutes the idempotents $e_1, \cdots, e_n$, we get $g^{-1}A_1 = A_m g^{-1}$, for some $1 \leq m \leq n$. We need to show that $m = 1$. Suppose the contrary. Then consider the left ideal $L = A_1 + K[S]g$ of $K[S]$. Calculating in $K[G]$ we get $(e_1 g^{-1}) \cdot L = e_1 g^{-1}(A_1 + K[S]g) = e_1 g^{-1} A_1 + e_1 g^{-1} K[S] g = e_1 A_m g^{-1} + e_1 K[S] \subseteq L$ because $e_1 A_m = 0$ and $e_1 K[H] \subseteq A_1$. Since $K[S]$ is a maximal order, it follows $e_1 g^{-1} \in K[S]$, a contradiction. Hence condition (c) is satisfied.

Now we discuss the third case, that is, G contains a finite p'-subgroup H and $G/H \cong < a, b \mid bab = a^{-1}, b^2 = 1 >$; where $p(\neq 2)$ is the characteristic of the field K. We can express any element of G as $hx^i y$ or $\bar{h}x^j$ where $h, \bar{h} \in H$, $i, j \in Z$, and x, y are pre-images in G of a and b respectively. Because G is the group of quotients of S, there must exist an element in S with form $hx^i y$ with $h \in H$, $i \in Z$. Consider the abelian subgroup $N = < x >$ of G. Because N has finite index in G and $G = SS^{-1}$, we get $N = (S \bigcap N)(S \bigcap N)^{-1}$ by Lemma 7.5 in [**52**].

We now claim that, if $x^t \in S$ for some positive integer t, then $x^{-kt} \in S$ for some $k \geq 1$. Indeed, since $K[S]$ is Noetherian, by Lemma 4.16, for any $c, d \in S$, there exists a positive integer r such that $c^r d \in dS^1$. We apply this to $c = x^t$ and $d = hx^i y$. Then $c^r d = x^{rt} hx^i y = hx^i yg$ for some $g \in S$. Hence it is easily seen that there exists $h' \in H$ such that $g = yx^{rt} yh' = x^{-rt} h'$ and thus $x^{-kt} \in S$ for some positive integer k. This proves the claim. It follows that $S \bigcap N$ is a subgroup of N. Hence $N = (S \bigcap N)(S \bigcap N)^{-1} = S \bigcap N$. So $N \subseteq S$ and thus $S = G$ is a (finite p')-by-$C_2 * C_2$ group. □

Note also, by Lemma 4.10 and Theorem 2.13 in [**54**], the semigroup algebra $K[S]$ is a semiprime principal left ideal ring if and only if one of conditions $(a), (b), (c)$ holds. However, (d) does not give a principal left ideal ring. Indeed, it is well known (see for example in [**2**]) that the group algebra of the infinite dihedral group $C_2 * C_2$ is not a maximal order.

CHAPTER 5

Noetherian Unique Factorization Semigroup Rings

In [**12**], Chatters and Jordan defined a unital ring R to be a Noetherian unique factorization ring (or simply, a Noetherian UFR) if R is a prime left and right Noetherian ring such that every non-zero prime ideal of R contains a non-zero principal prime ideal. It is shown in [**12**] that if R is a Noetherian unique factorization ring then R is a maximal order (with trivial normalizing class group). In [**10**] , Chatters, Gilchrist, and Wilson studied arbitrary unique factorization rings (that is, without the Noetherian restriction) and unique factorization domains (or simply UFD), that is, R is an integral domain such that every non-zero prime ideal contains a completely prime element.

In [**2**], [**8**] and [**9**], several authors studied the problem of when a group ring is a unique factorization ring. For G an abelian group and R a ring which satisfies a polynomial identity, Chatters and Clark [**9**] showed that the group ring $R[G]$ is a UFR if and only if R is a UFR and G is a torsion free group satisfying the ascending chain condition on cyclic subgroups. For G a polycyclic-by-finite group and R a Noetherian commutative UFD, Brown [**2**] showed that $R[G]$ is a Noetherian UFR if and only if $\Delta^{+}(G) = \{1\}$, G is dihedral free, and every plinth of G is centric. Chatters and Clark [**9**] proved that this result still holds for any commutative coefficient ring R which is a UFD. In [**8**] Chatters proved that a prime group ring $R[G]$ which satisfies a polynomial identity is a UFR if and only if R is a UFR and G is a dihedral-free group satisfying the ascending chain condition on cyclic subgroups.

For an abelian torsion free cancellative monoid S and an integral domain D, Gilmer [**23**] showed that the semigroup ring $D[S]$ is UFD if and only if D is a UFD and S is a unique factorization monoid which satisfies the ascending chain condition on cyclic submonoids. For semigroup algebras $K[S]$ of arbitrary semigroups over a field K several related arithmetical structures have been investigated. Jespers and Okniński in [**32**] obtained a complete description of left and right principal ideal semigroup algebras $K[S]$. For a submonoid S of a finitely generated torsion free nilpotent group Jespers and Okniński in [**34**] showed that $K[S]$ is a Noetherian maximal order precisely when S modulo its unit group is a finitely generated abelian monoid which is a maximal order in its group of quotients. In particular, S is a normalizing cancellative monoid which is a Krull order in the sense of Wauters in [**59**].

Recently, Jespers and Okniński in [**36**] investigated submonoids S of polycyclic-by-finite groups. It is described when $K[S]$ is left and right Noetherian, and in this case the prime ideals of $K[S]$ are studied. Using these results, we investigate in this Chapter when Noetherian semigroup algebras of submonoids of torsion free polycyclic-by-finite groups are unique factorization rings. The results proved will appear in [**38**].

In Section 1, we recall the definition and give some background on unique factorization rings. In Section 2 we recall the description of group rings that are unique factorization rings. In Section 3, we generalize the concept of an abelian unique factorization monoid to the non-commutative setting. In Section 4 we recall results for monoids that are Krull orders. In Section 5 we first recall some important properties of semigroup algebras of submonoids of a polycyclic-by-finite group, such as the description of when these algebras are Noetherian and the description of the height one primes. Next we describe when such monoids S are unique factorization monoids. In Section 6 we investigate when the semigroup algebra of a submonoid of a torsion-free polycyclic-by-finite group is a Noetherian unique factorization ring. Finally in Section 7 we give some examples.

5.1. Unique factorization rings

We first recall the definition of a unique factorization ring.

5.1.1. Let R be a prime ring. An element x of R is normal if $xR = Rx$. A *prime element* of R is a non-zero normal element p such that pR is a proper prime ideal. Such a prime p is said to be *completely prime* if R/pR is an integral domain. We say R is a *unique factorization ring* (UFR) if every non-zero prime ideal of R contains a prime element, and that R is a unique factorization domain (UFD) if R is an integral domain and every non-zero prime ideal contains a completely prime element.

A ring R is said to be a prime *Krull order* if R is a prime maximal order and R satisfies ascending chain condition on divisorial ideals contained in R.

If R is Noetherian, R is Krull order if and only if R is maximal order. Chatters and Jordan in [**12**] showed that a Noetherian UFR is a maximal order (Krull order).

5.1.2. Furthermore, for a UFR, the following results (cf. [**10**]) hold:

LEMMA 5.1. *Let R be a UFR. Then*

1. *Every non-zero ideal of R contains a product of prime elements.*
2. *Let x be a non-zero element of R. Then there are only finitely-many non-associated prime elements p of R such that $x \in pR$.*
3. *The prime ideal pR with p a prime element of R has height 1.*
4. $\cap_{n=1}^{\infty} p^n R = 0$ *where p is a prime element of R.*
5. $C(pR) \subseteq C(p^n R)$ *for every positive integer n, where $C(I)$ denotes the set of elements of R which are regular modulo I.*
6. *The elements of $C(pR)$ are regular as elements of R.*
7. *Let x be normal element of R with $x \not\in pR$. Then $xR \cap p^n R = xp^n R$ for every positive integer n.*
8. *Every non unit nonzero normal element x is a product of prime elements of R.*
9. *For every non unit normal element x, there are non-associated prime elements $p_1, \cdots, p_n$ of R and non-negative integers $a(1), \cdots, a(n)$ such that*

$$xR = p_1^{a(1)} \cdots p_n^{a(n)} R = p_1^{a(1)} R \cap \cdots \cap p_n^{a(n)} R.$$

10. *Let P be a prime ideal of R which is minimal over a normal element x, then height$(P) = 1$ and $P = pR$ for some prime element p of R.*
11. *The set of principal ideals of R is closed under finite intersection and satisfies the ascending chain condition.*

Hence a ring R is a UFR if and only if every nonzero ideal of R contains nonzero normal element and every non unit nonzero normal element of R is a product of primes. Thus the notion of UFR is an extension to the non-commutative situation of that of commutative unique factorization domains.

5.1.3. A ring R is *conformal* if every non-zero ideal of R contains a non-zero normal element of R. Denote by $N(R)$ the set of all normal elements in R. Jordan showed the following result for Noetherian prime rings.

LEMMA 5.2 (Proposition 2.2, [**44**]). *Let R be a prime Noetherian ring. Then R is UFR if and only if R is conformal and every irreducible element of $N(R)$ is prime in R.*

5.2. Unique factorization group rings

5.2.1. Several authors studied the problem of when a group ring $R[G]$ is a unique factorization ring. First we recall the commutative situation given by Gilmer and Parker (see for example [**23**]). For G an torsion-free abelian group and R a commutative integral domain, $R[G]$ is a UFD if and only if R is a UFD and G is cyclically Noetherian (i.e., G satisfies ascending chain condition on cyclic subgroups). Actually the latter condition is equivalent to every rank 1 subgroup of G is cyclic (Lemma 4.2.13 in [**43**]). Chatters and Clark in [**9**] then extended the result as follows. For G an abelian group and R a ring which satisfies a polynomial identity, the group ring $R[G]$ is a UFR if and only if R is a UFR and G is a cyclically Noetherian torsion free group.

5.2.2. The best known results are due to Brown [**2, 3**]. To state his results we recall some notions. The *F.C. subgroup* of G, denoted by $\Delta(G)$, is the set of elements of G which have only a finite number of distinct conjugates. Clearly $\Delta(G)$ contains all proper finite normal subgroups of G. The torsion elements of $\Delta(G)$ form a subgroup, denoted by $\Delta^+(G)$, and $\Delta(G)/\Delta^+(G)$ is torsion-free abelian (see for example Lemma 4.1.6 in [**54**]). A subgroup H of G is *orbital* in G if H has only a finite number of distinct conjugates by elements of G, or equivalently, $N_G(H)$ has finite index in G (where $N_G(H)$ is the normalizer of H in G). We say G is *dihedral free* if G has no orbital infinite dihedral subgroups.

A *plinth* of G is a torsion free abelian orbital subgroup A of G such that $A\otimes_{\mathbf{Z}}\mathbf{Q}$ is an irreducible $\mathbf{Q}T$-module for every subgroup T of finite index in $N_G(A)$. The plinth A is *centric* if its centralizer $C_G(A)$ has a finite index in G, or equivalently, A has rank one. Otherwise, A is *eccentric*.

LEMMA 5.3 (Theorem D in [**2**]). *Let R be a Noetherian commutative UFD and G a polycyclic-by-finite group. Then RG is a UFR if and only if $\Delta^+(G) = \{1\}$, G is dihedral free, and every plinth of G is centric.*

Chatters and Clark [**9**] proved that this result still holds for any commutative coefficient ring R which is a UFD. Brown in [**2**] also showed that

LEMMA 5.4. *Let G be a polycyclic-by-finite group. If $\Delta^+(G) = 1$, then the following statements are equivalent:*

1. *every non-zero ideal of $R[G]$ contains an invertible ideal;*
2. *every non-zero ideal of $R[G]$ contains a non-zero normal element;*
3. *every non-zero ideal of $R[G]$ contains a non-zero central element;*
4. *every plinth of G is centric.*

5.2.3. Also in [**2, 3**], Brown described when group rings of polycyclic-by-finite groups are UFD.

LEMMA 5.5 (Theorem E', [**3**]). *Let R be a commutative Noetherian UFD, and let G be a polycyclic-by-finite group. Then $R[G]$ is a UFD if and only if*

1. *G is torsion free,*
2. *all plinths of G are central,*
3. *$G/\Delta(G)$ is torsion free.*

In fact, by using group theoretic techniques, MacKenzie [**47**] found a nice relationship between UFR and UFD when describing group rings of polycyclic-by-finite groups. Note that a subgroup H of G is called *characteristic* in G if $\phi(H) \subseteq H$ for every automorphism ϕ of G.

LEMMA 5.6 (Theorem 2.1, [**47**]). *Let R be a commutative Noetherian UFD and G a polycyclic-by-finite group. If $R[G]$ is a UFR then G has a normal (in fact, characteristic) subgroup H of finite index that $R[H]$ is a UFD.*

If G is a torsion-free finitely generated nilpotent group, the proof of the previous Lemma yields the following result.

COROLLARY 5.7. *Let R be a commutative Noetherian UFD and G a torsion-free finitely generated nilpotent group. Then $R[G]$ is a UFD.*

5.3. Unique Factorization Monoids

As will be proven in the next section, also in the non-commutative situation the notion of the arithmetical structure on S will play a crucial role. We therefore will generalize the definitions of unique factorization monoid from the commutative setting to non commutative situation. This will be done similar to the ring setting.

First let us fix some definitions. As in ring theory, an element c of a monoid S is said to be *normal* (*invariant* in [**15**]) if $cS = Sc$. The submonoid of normal elements is denoted $N(S)$. If $N(S) = S$ then S is called a *normalizing semigroup* (or an *invariant monoid* in [**15**]). A non-invertible element $p \in S$ is called *irreducible* (or an *atom*) if it cannot be written as the product of two non-invertible elements in

S. A normal element $p \in S$ is said to be *prime* if Sp is a prime ideal in S, that is, for any $a, b \in S$, $aSb \subseteq Sp$ implies $a \in Sp$ or $b \in Sp$. Given two normal elements $a, b \in S$, if $a = bu$ for some unit u of $U(S)$, then we say a and b are *associated.*

Recall that an ideal P of a semigroup S is prime if $aSb \subseteq P$ implies $a \in P$ or $b \in P$. Furthermore, if $S \setminus P$ is a subsemigroup of S then P is called completely prime. Denote by $Spec(S)$ the set of all prime ideals of S and by $X^1(S)$ the set of all minimal prime ideals of S. For any ideal I of S we denote by $C(I)$ the set of elements of S which are regular (i.e., not zero-divisors) modulo I. Set $C(S) = \bigcap C(P)$, where the intersection is taken over all $P \in X^1(S)$.

5.3.1. For completeness' sake we recall the following definition. An abelian cancellative monoid S is a *unique factorization monoid (factorial)* if each principal ideal of S can be written as a finite product of prime ideals of S. As mentioned above Gilmer showed that they can be described as follows.

LEMMA 5.8 (Theorem 6.8, [**23**]). *Let G be a group, let $\{Z_\alpha\}_{\alpha \in A}$ be a family of monoids, each isomorphic to the additive monoid of nonnegative integers. Then monoid $G \bigoplus \sum_{\alpha \in A} Z_\alpha$ is factorial. Conversely, each factorial monoid S is isomorphic to such a monoid $G \bigoplus \sum_{\alpha \in A} Z_\alpha$.*

5.3.2. In [**15**] Cohn defines a UF-monoid (unique factorization monoid) as normalizing cancellative monoid S for which the quotient monoid $S/U(S)$ is free abelian. This clearly generalizes the previous definition.

A monoid S is said to satisfy *right* ACC_1 if S satisfies the ascending chain condition on principal right ideals. Similarly, one defines *left* ACC_1. Of course, these two notions coincide in a normalizing monoid. Note that a normalizing monoid S satisfying ACC_1 is *atomic*, which means every element of S is either a unit or a product of irreducible elements (atoms) (see for example [**15**]).

We have the following descriptions of UF-monoid, cf. Theorem 3.1.1. in [**15**]

LEMMA 5.9. *Let S be a normalizing cancellative monoid. The following conditions are equivalent:*

1. *S is a UF-monoid,*
2. *S satisfies ascending chain condition on principal ideals (ACC_1) and any two elements have a greatest common divisor,*
3. *S satisfies ACC_1 and any two elements have a least common multiple,*
4. *S is atomic and every atom of S is prime.*

5.3.3. More generally, we define a monoid to be a *UF-monoid* (*unique factorization monoid*) if every prime ideal of S contains a principal prime ideal P, i.e., $P = Sn$ for some normal element n of S. Note that (as for the ring case [**12**]), if S is cancellative, then $P = Sn$ is equivalent with $P = Sa = bS$ for some $a, b \in S$. Indeed, if $Sa = bS$, then $a = bs$ and $b = ta$. Since Sa is an two sided ideal we get $as = s'a$ for some $s' \in S$. Hence $a = tas = ts'a$, and thus $ts' = 1$. So $t \in U(S)$ and therefore $bS = Sa = Sta = Sb$.

LEMMA 5.10 (Lemma 2.1 in [**39**]). *Let S be a submonoid of a group. Then,*

(1) *$C(pS) \subseteq C(p^n S)$ for every positive integer n and prime element p of S.*

(2) *If p is a prime element of S and $x \in N(S)$ with $x \notin pS$, then $xS \cap p^n S = xp^n S$, for any positive integer n. In particular, for any non-associate prime elements $p_1, \cdots, p_n$ of S and non-negative integers $a(1), \cdots, a(n)$*

$$p_1^{a(1)} \cdots p_n^{a(n)} S = p_1^{a(1)} S \cap \cdots \cap p_n^{a(n)} S.$$

If, furthermore, S is a UF-monoid, then the following conditions hold.

(3) *Every proper ideal of S contains a product of prime elements.*

(4) *For any prime element $p \in S$, $\cap_{n=1}^{\infty} p^n S = \emptyset$.*

(5) *$C(S) \cap N(S) = U(S)$.*

(6) *Let x be a non-invertible element of S, then there are only finitely many non-associated prime elements p of S such that $x \in pS$.*

(7) *Every non-invertible normal element of S can be written as a product of prime elements.*

Proof. (1) Let $c \in C(pS)$ and suppose that $c \in C(p^k S)$ for some positive integer k. We must show that $c \in C(p^{k+1} S)$. Let $s \in S$ and $cs \in p^{k+1} S$. Then $cs \in p^k S$ and thus $s \in p^k S$. So $s = yp^k$ for some $y \in S$. Also $cs = zp^{k+1}$ for some $z \in S$. Hence $cyp^k = zp^{k+1}$, and so $cy = zp$. But $c \in C(pS)$ implies $y \in pS$ and hence $s \in p^{k+1} S$, as required.

(2) Let n be a positive integer. Obviously $xS \cap p^n S \supseteq xp^n S$. Conversely, suppose $p^n a = xb \in p^n S \cap xS$. Because of the assumption $x \in C(pS)$ we get from (1) that $x \in C(p^n S)$. Therefore $b \in p^n S$ and thus $xb \in xp^n S$, as desired.

(3) For otherwise let I be an ideal of S maximal for the condition that it does not contain a product of prime elements. Clearly I is a prime ideal. Since by assumption S is a UF-monoid, I contains a prime element, a contradiction.

(4) Set $I = \bigcap_{n=1}^{\infty} p^n S$ and suppose $I \neq \emptyset$. Obviously $I = pI$. From (3), I contains a product $p_1 \cdots p_n \in I$ for some prime elements $p_1, \cdots, p_n$ of S. Since $p_1 \cdots p_n \in pS$, we get that $p_i \in pS$ for some i. Write $p_i = pt$ for some $t \in S$. Hence $p_i S = pS$ (indeed, for if $p_i S \subsetneq pS$, then $pt \in p_i S$ implies that $t = sp_i$ for some $s \in S$. Therefore $ps = 1$, a contradiction). Thus $I = pI = p_i I$ for some i. So $p_1 \cdots p_n \in I = p_i I$. Because S is cancellative and p_i are normal we obtain that I contains a product of $n-1$ prime elements. Repeating this argument several times, we get that I contains an invertible element, a contradiction.

(5) Indeed, if $n \in C(S) \bigcap N(S)$ then, by (3), there exists $x \in S$ and prime elements $p_i \in S$ such that $xn = p_1 \cdots p_n$. Since each p_i is normal and $n \in C(S)$, it follows that $x \in Sp_i$, for every i. Since S is cancellative this implies $n \in U(S)$.

(6) Let x be a non-invertible element of S, then by (3) there are prime elements $p_1, \cdots, p_n$ of S such that $p_1 p_2 \cdots p_n \in SxS$. Let p be a prime element of S such that $x \in pS$. Then $p_1 p_2 \cdots p_n \in pS$ and therefore $p_i \in pS$ for some i. Thus (as in the proof of (4)) $p_i S = pS$. So there are only finitely-many possibilities for pS.

(7) Let x be a non-invertible normal element of S. Because of (4), for each prime element p of S there is a positive integer n such that $x \notin p^n S$. By (6) there only finitely-many non-associated prime elements p of S such that $x \in pS$. From these two statements it follows that there are prime elements $p_1, \cdots, p_n$ of S such

that $x = p_1 \cdots p_n y$, where y is an element of S such that there is no prime element p of S with $y \in pS$. Since x, p_i are normal, we have $y \in N(S)$. Moreover, $y \in C(S)$ and thus y is a unit by (5). Therefore x is a product of prime elements $p_1, \cdots, p_n y$. $\square$

PROPOSITION 5.11 (Proposition 2.2 in [**39**]). *Let S be a submonoid of a group. Then the following conditions are equivalent.*

1. *S is a UF-monoid.*
2. *Every ideal of S contains a normal element and every non-invertible normal element of S can be written as a product of prime elements.*
3. *Every ideal of S contains a normal element and every irreducible element in $N(S)$ is prime in S and S satisfies the ascending chain condition on principal ideals generated by a normal element.*

Proof. (1) implies (2). This follows from (3) and (7) in Lemma 5.10.

(2) implies (3). It is obvious that every irreducible element of $N(S)$ is prime in S. To show that S satisfies the ascending chain condition on principal ideals generated by a normal element, it is sufficient to show that there are only finitely many principal ideals of S which contain x for any non-invertible normal element $x \in S$.

Let y be a normal element of S such that $xS \subseteq yS$. Because of (2) and (7) in Lemma 5.10, there are non-associated prime elements $p_1, \cdots, p_n, q_1, \cdots, q_m$ of S and non-negative integers $a(1), \cdots, a(n), b(1), \cdots, b(m)$ such that $xS = p_1^{a(1)}S \cap$

$\cdots \cap p_n^{a(n)}S = p_1^{a(1)} \cdots p_n^{a(n)}S$ and $yS = q_1^{b(1)}S \cap \cdots \cap q_m^{b(m)}S = q_1^{b(1)} \cdots q_m^{b(m)}S$. It is routine to show that for each $1 \leq j \leq m$ there exists an i, $1 \leq i \leq n$, such that $q_jS = p_iS$ and that $b(j) \leq a(i)$. Hence (3) follows.

(3) implies (1). Let P be a prime ideal of S. Then P contains a non-invertible normal element of S, say n. First we show that n is a product of irreducible elements of $N(S)$. Note that, by assumption, S satisfies the ascending chain condition on principal ideals generated by a normal element. Hence $N(S)$ satisfies ACC_1 and thus each element of $N(S)$ is either a unit or a product of irreducible elements. Therefore n is a product of irreducible elements of $N(S)$, and thus by the assumption, n is a product of prime elements of S. Consequently, P contains a prime element. □

Note that if S is a normalizing cancellative monoid, then the description of UF-monoid obtained in Proposition 5.11 corresponds with the one obtained by Cohn in Lemma 5.9.

5.4. Monoid Krull orders

In Chapter 4 we recalled several notions concerning rings that are maximal orders. Wauters in [59] introduced non-commutative monoids that are maximal orders in a group of quotients. Although the definitions are similar to in the ring case we state them here for completeness' sake.

A cancellative monoid S which has a left and right group of quotients G is called an *order*. A fractional left S-ideal I is a subset of G such that $SI \subseteq I$ and $S\alpha \subseteq I \subseteq S\beta$ for some $\alpha, \beta \in G$. Similarly one defines fractional right S-ideal. A (two-sided) fractional S-ideal is a subset of G that is both a fractional left and right S-ideal. If A and B are subsets of G, we put $(A :_l B) = \{x \in G \mid xB \subseteq A\}$ and $(A :_r B) = \{x \in G \mid Bx \subseteq A\}$. An order S is a maximal order if $(I :_l I) = S = (I :_r I)$ for each fractional S-ideal. It follows in this case that, for any fractional S-ideal I, $(S :_l I) = (S :_r I)$. This fractional ideal we simply denote as $(S : I)$. Note that being a maximal order is equivalent with the condition that there does not exist a submonoid S' of G properly containing S and such that $aS'b \subseteq S$ for some $a, b \in S$. A fractional S-ideal I is said to be divisorial if $I = (S : (S : I))$. One says that S is a *Krull order* if S a maximal order satisfying the ascending chain condition on divisorial ideals contained in S. The following result can be found in [**59**].

LEMMA 5.12. *Let S be a maximal order, then the set of divisorial ideals $D(S)$ is a commutative group, where $I * J = (S : (S : IJ))$ and $I, J \in D(S)$. Furthermore, if S is a Krull order, then $D(S)$ is a free abelian group.*

Let S be a Krull order, then $D(S) \cong Z^{\Lambda}$ for a certain index set Λ, and this isomorphism is order-preserving. Of course, the order relation on Z^{Λ} is defined by $(a_\lambda)_{\lambda\in\Lambda} \leq (b_\lambda)_{\lambda\in\Lambda}$ if and only if $a_\lambda \leq b_\lambda$ for all $\lambda \in \Lambda$. Let $\phi : D(S) \to Z^{\Lambda}$ be an order preserving isomorphism. Put $e_i = (\delta_{i\lambda})_{\lambda\in\Lambda}$, and let $P_i = \phi^{-1}(e_i)$. Here

$\delta_{i\lambda} = 1$ when $i = \lambda$ and 0 otherwise. Thus any element A of $D(S)$ can be written as $A = P_1^{n_1} * \cdots * P_k^{n_k}(\ n_i \in Z)$. It is obvious that P_i is a prime ideal of S. Indeed, let $x, y \in S$ such that $xSy \in P_i$. Then $\overline{SxS} * \overline{SyS} = \overline{SxSyS} \subseteq \overline{P_i} = P_i$ where $\overline{B} = (S : (S : B))$ denotes the divisorial closure of an ideal B. Furthermore $\phi(\overline{SxS}) = \Sigma n_j e_j$ and $\phi(\overline{SyS}) = \Sigma m_j e_j$ with $n_j, m_j \geq 0$. In particular, $\phi(\overline{SxS}) + \phi(\overline{SyS}) = \Sigma(n_j + m_j)e_j \geq \phi(P_i) = e_i$. Therefore $n_i \geq 1$ or $m_i \geq 1$ yield that either $x \in \overline{SxS} \subseteq P_i$ or $y \in \overline{SyS} \subseteq P_i$.

THEOREM 5.13. *Let S be a Krull order. If every ideal of S contains a normal element, then $D(S)$ is generated by the minimal prime ideals of S.*

Proof. First we claim that every prime ideal P of S contains P_i for some $i \in \Lambda$. Indeed, let n be a non-invertible normal element contained in P. Then $Sn \subseteq P$. Since $Sn \in D(S)$, we may write $P \supseteq Sn = P_1^{n_1} * \cdots * P_k^{n_k} \supseteq P_1^{n_1} \cdots P_k^{n_k}$ where all $n_i \geq 0$. Therefore $P \supseteq P_i$ for some i.

Suppose P is minimal prime ideal. By the above claim, $P = P_i$ for some i. Conversely, let P be a prime generator of $D(S)$. If P is not a minimal prime, then $Q \subset P$ for some prime ideal Q of S. Again, by the claim, $P_i \subseteq Q \subset P$ for some $i \in \Lambda$. Therefore $\phi(P) < \phi(P_i) = e_i$, a contradiction because $\phi(P) > 0$. □

5.5. Submonoids of polycyclic-by-finite groups

Let S be a submonoid of a polycyclic-by-finite group G. In **[36]** (Corollary 3.3) it has been described when $K[S]$ is left and right Noetherian. Because of its importance for our investigations we state this result. The equivalence of the first two conditions is an immediate consequence of Quinn's result (see for example [**56**]). That S is finitely generated in this case follows from Corollary 3.5 in [**36**].

PROPOSITION 5.14. *Let S be a submonoid of a polycyclic-by-finite group. The following conditions are equivalent:*

1. *S satisfies the ascending chain condition on right and left ideals,*
2. *$K[S]$ is left and right Noetherian,*
3. *S has a group of quotients $G = SS^{-1}$ which contains a normal subgroup H of finite index and a normal subgroup $F \subseteq H$ such that $S \cap H$ is finitely generated, $F \subseteq U(S)$ and H/F is abelian.*

Moreover, in this case, S and $N(S)$ are finitely generated monoids, and every ideal of S intersects $N(S)$; in particular $G = SN(S)^{-1}$.

5.5.1. We will characterize UF-submonoids S of polycyclic-by-finite groups that satisfy the ascending chain condition on left and right ideals. For this we first need the following fundamental property of Jespers and Okniński [**36**]. For a field K, by $X^1_k(K[S])$ we denote the set of height one prime ideals of $K[S]$ intersecting S. The set of all prime ideals is denoted by $Spec(K[S])$. The last part of the statement is the real hard part. This is proved in [**36**] making use of the structure theorem of Okniński on linear semigroups. We only include a proof of (2) and (3) which are not stated in [**36**].

PROPOSITION 5.15 ([**36**], see also Proposition 2.2 in [**38**]). *Let K be a field and S a submonoid of polycyclic-by-finite group. If S satisfies the ascending chain condition on left and right ideals, then the following properties hold.*

1. *The set $X_h^1(K[S])$ is finite.*
2. *For any $P \in Spec(S)$, there exists $I_P \in X_h^1(K[S])$ such that $\emptyset \neq I_P \bigcap S \subseteq P$.*
3. *Every $P \in Spec(S)$ contains a minimal prime in S.*

Moreover, if G is torsion free, then $X_h^1(K[S]) = \{K[Q] \mid Q \in X^1(S)\}$ and $K[P \cap S] \in Spec(K[S])$ provided $P \in Spec(K[S])$ and $P \cap S \neq \emptyset$.

Proof. (2) Let $P \in \operatorname{Spec}(S)$. Because of Proposition 5.14, let $n \in N(S) \cap P$. Note $K[P]$ is an ideal of $K[S]$ such that $K[P] \bigcap S = P$. Let I be an ideal of $K[S]$ maximal for the condition that $I \bigcap S = P$. Since P is prime, $I \in Spec(K[S])$. By the Principal Ideal Theorem there exists $I_P \in Spec(K[S])$ minimal over $K[Sn]$ and $I_P \subseteq I$. Moreover, I_P has height 0 or 1. Since $I_P \bigcap S \neq \emptyset$ the height has to be one (see [**36**, comments after Proposition 4.2]). So $I_P \in X_h^1(K[S])$ and $I_P \bigcap S \subseteq I \bigcap S = P$.

(3) Let $P \in Spec(S)$, by (2), there exists $I_P \in X_h^1(K[S])$ such that $P \supseteq I_P \bigcap S$. If $I_P \bigcap S \notin X^1(S)$, then $I_P \bigcap S \supsetneq P_1$ with $P_1 \in Spec(S)$. So as before, we have $P_1 \supseteq I_{P_1} \bigcap S \neq \emptyset$ with $I_{P_1} \in X_h^1(K[S])$. Note that $I_P \neq I_{P_1}$ because, otherwise $P_1 \supseteq I_{P_1} \bigcap S = I_P \bigcap S \supsetneq P_1$, a contradiction. Repeating this argument we get a descending chain

$$P \supseteq I_P \bigcap S \supsetneq P_1 \supseteq I_{P_1} \bigcap S \supsetneq \cdots$$

Since $X_h^1(K[S])$ is finite, this chain must stop after finitely many steps, i.e. there exists i such that $I_{P_i} \cap S \in X^1(S)$. $\square$

5.5.2. We now give a description of when a submonoid of a polycyclic-by-finite group is a UF-monoid.

THEOREM 5.16 (Theorem 2.3 in [**38**]). *Let S be a submonoid of a polycyclic-by-finite group G. If S satisfies the ascending chain condition on right and left ideals, then the following conditions are equivalent:*

1. *S is a UF-monoid.*
2. *Every non-invertible normal element of S can be written as a product of prime elements of S.*
3. *Every irreducible element in $N(S)$ is prime in S.*
4. *Every minimal prime ideal of S is generated by a prime element.*

Proof. Proposition 5.14 and Proposition 5.11 yield that (1) implies (2) and (2) implies (3).

To prove (3) implies (4), let P be a minimal prime ideal of S. By Proposition 5.14, P contains a normal element n. Since S satisfies the ascending chain condition on left and right ideals, so does $N(S)$. Hence every element of $N(S)$ is a product of irreducible normal elements. Thus, because of condition (3), n is a product of prime elements of S. Since n belongs to the prime ideal P we get therefore that P contains a prime element p. Hence (4) follows.

By Proposition 5.15 every prime ideal of S contains a minimal prime ideal. Hence, (4) implies (1) is clear. □

5.5.3. As in the ring case, we find that a UF-monoid S is a maximal order provided S satisfies the ascending chain condition on left and right ideals.

PROPOSITION 5.17 (Proposition 2.4 in [**38**]). *Let S be a submonoid of a polycyclic - by - finite group. Assume S satisfies the ascending chain conditions on left and right ideals. If S is a UF-monoid, then S is a maximal order.*

Proof. Let G be the group of quotients of S. Suppose $qI \subseteq I$ for some $q \in G \setminus S$ and I an ideal of S. Because S satisfies the ascending chain condition on left and right ideals, we can choose I maximal with respect to the property that such q exists. Write $q = z^{-1}c$ for $z \in N(S)$ and $c \in S$. Because of Theorem 5.16, there exist prime elements $p_1, \cdots, p_n$ of S such that $z = p_1 \cdots p_n$. Of course we may assume that z and c have no common factor that is a normal element. In other words, we may assume $c \notin p_i S$ for every i.

Now $cI \subseteq zI \subseteq p_i S$. Because $p_i S$ is prime and $c \notin p_i S$ we get that $I \subseteq p_i S$. Thus $p_i^{-1} I \subseteq S$ and hence $p_i^{-1} I$ is an ideal of S and it contains I. Since $p_i \in N(S)$ there exist $c' \in S$ and $z' \in N(S)$ so that $c' p_i^{-1} = p_i^{-1} c$ and $z' p_i^{-1} = p_i^{-1} z$. Then

$$q = z^{-1}c = (z' p_i^{-1})^{-1} c' p_i^{-1} = p_i z'^{-1} c' p_i^{-1}$$

and thus $z'^{-1} c' \notin S$. However, as $p_i^{-1} cI \subseteq p_i^{-1} zI$, we get

$$z'^{-1} c' (p_i^{-1} I) \subseteq (p_i^{-1} I).$$

The maximality of I therefore implies that $I = p_i^{-1}I$. Thus

$$\bigcap_{n=1}^{\infty} p_i^n I = \bigcap_{n=1}^{\infty} I = I.$$

However this contradicts with (2) of Lemma 5.10. Therefore S is a maximal order. □

For a Krull order S, the normalizing class group $Cl(S)$ is defined by $Cl(S) = D(S)/P(S)$, where $P(S)$ is the set of principal ideals generated by normal elements of S.

COROLLARY 5.18. *Let S be a submonoid of a polycyclic-by-finite group. Assume S satisfies the ascending chain conditions on left and right ideals. Then S is a UF-monoid if and only if S is a Krull order and the normalizing class group is trivial.*

Proof. Assume S is a UF-monoid. Because S satisfies ascending chain condition on left and right ideals, Proposition 5.17 yields that S is also Krull order. Moreover, by Proposition 5.14 every ideal of S contains a normal element. Hence Theorem 5.13 implies that $D(S)$ is generated by the minimal prime ideals and thus the normalizing class group is trivial. Conversely, assume that S is a Krull order with trivial normalizing class group. Proposition 5.14 implies that every ideal of S contains a normal element, and thus Theorem 5.13 implies that $D(S)$ is generated by minimal prime ideals. Since $Cl(S)$ is trivial, all minimal prime ideals are principal. Hence Theorem 5.16 yields that S is a UF-monoid. □

5.5.4. We now show that our UF-monoids are (as in the ring case) the intersection of local UF-monoids. These proofs are standard, but again for completeness' sake they are included.

PROPOSITION 5.19. *Let S be a UF-submonoid of a polycyclic-by-finite group G and assume S satisfies ascending chain condition on left and right ideals. Let $P = pS$ be a minimal prime ideal of S and $W = \{$product of such q's $\mid q$ are prime elements of S but not associated with $p\}$. Denote by S_W the localized monoid SW^{-1}. Then*

1. *S_W is a UF-monoid satisfying ascending chain condition on left and right ideals and $X^1(S_W) = \{S_W p\}$.*
2. *$C(P)$ is an Ore set in S, and the localized monoid $S_{C(P)}$ equals $S(N(S) \setminus P)^{-1} = S_W$.*
3. *$S = \cap_{P \in X^1(S)} S_{C(P)}$.*

Proof. (1). As a localization of S, it is clear that $T = S_W$ also satisfies the ascending chain condition on left and right ideals.

Now we show that p is also a prime element of T. First we show that pT is an ideal. Let $w \in W$. Then by Lemma 5.10 $pwS = wS \cap pS = wpS$. Note $pT = pwT = pwST = wpST = wpT$, and thus $w^{-1}pT = pT$. So $TpT = pT$. For symmetry reasons, $Tp = TpT = pT$ is an ideal of T. Next we show that $pT \cap S = pS$. Take $ptw^{-1} = s \in pT \cap S$, then $pt = sw \in pS$. Since $w \in C(pS)$, we have $s \in pS$. Thus we have proved $pT \cap S \subseteq pS$. The converse is obvious. Now we can show that pT is also a prime ideal. Since ideals of T are generated by their intersection with S, the equality $pT \cap S = pS$ clearly implies that indeed pT is a prime ideal.

To show that T is a UF-monoid, let P be a prime ideal of T. Thus $S \cap P$ is an ideal of S. Since S is a UF-monoid, there exists an element $n \in N(S)$ such that $n \in P \cap S$ and $n = q_1 \cdots q_k$ for prime elements q_i of S with $1 \leq i \leq k$. If none of the q_i are associated with p, then $n \in W$ and thus n is a unit in T and therefore $P = T$, a contradiction. Then there exists a q_{i_0} such that q_{i_0} is associated with p. Therefore $p \in P$ and thus T is a UF-monoid. Actually the previous shows also that every prime element of T is associated with p. Thus $X^1(S_W) = \{S_W p\}$.

(2) The equality $pT \cap S = pS$ clearly implies that $C(pS) \subseteq C(pT)$. Next we show that any element c of $C(pT)$ is a unit in T. Because $T \subseteq G = SN(S)^{-1}$, there exist $y \in S$ and $n \in N(S)$ such that $cyn^{-1} = 1$ and thus $cy = n$. Since S is a UF-monoid, then n is a product of prime elements q_i of S where $1 \leq i \leq k$. If none of the q_i is associated with p, then $n \in W$ and thus is a unit in T. Otherwise, $n = up^i$ for some $u \in W$ and some i. Since $c \in C(pT)$, we obtain that $y \in pT$. Therefore $cy'p = up^i$ for some $y' \in T$ and thus $cy' = up^{i-1}$. Continuing this procedure, we get c is a unit in T.

From the above we know every element of $C(pS)$ is a unit in T. Thus $C(pS)$ is an Ore set of S. Indeed, for any $c \in C(pS)$ and $s \in S$, we have $cys = s$ for some $y \in T$. Write $y = y'w^{-1}$ and then $cy's'w^{-1} = s$ for some $y', s' \in S$ and $w \in W$.

Therefore $cy's' = sw$ where $y's' \in S$ and $w \in C(pS)$. Moreover, $S_W \subseteq S_{C(pS)}$ implies that $S_W = S_{C(pS)}$.

(3) Obviously $S \subseteq \cap_{P \in X^1(S)} S_{C(P)}$. We will show that $\cap_{P \in X^1(S)} S_{C(P)} \subseteq S$. Note that every $P \in X^1(S)$ is generated by a prime element of S since S is a UF-monoid. Let $q \in \cap_{P \in X^1(S)} S_{C(P)}$. Since $\cap_{P \in X^1(S)} S_{C(P)} \subseteq G = SN(S)^{-1}$, we get $q = r(p_1 \cdots p_n)^{-1}$ where p_i are prime elements of S. Furthermore, we write also $q = r(p_1 \cdots p_n)^{-1} = s_i c_i^{-1}$ where $s_i \in S$ and $c_i \in C(p_i S)$. Because $c_n \in C(p_n S)$ and because of (2), we know $c_n s = p_1 \cdots p_n t$ for some $s \in S$ and $t \in C(p_n S)$. However $c_n \in C(p_n S)$ implies that $s = vp_n$ for some $v \in S$. Hence $rt = qp_1 \cdots p_n t = qc_n s = qc_n v p_n$. Note that $t \in C(p_n S)$, we obtain $r \in p_n S$. Therefore $q \in S(p_1 \cdots p_{n-1})^{-1}$. Repeating this process, we get $q \in S$. Thus $\cap_{P \in X^1(S)} S_{C(P)} = S$. $\square$

5.6. Noetherian Unique Factorization Semigroup rings

Now let us turn back to semigroup algebras $K[S]$. Recall that $X^1(K[S])$ denotes the set of height one prime ideals of $K[S]$ and $X_h^1(K[S])$ denotes the set of height one prime ideals of $K[S]$ intersecting S. The set of all prime ideals is denoted by $Spec(K[S])$. For any ideal I of $K[S]$ we denote by $C(I)$ the set of elements of $K[S]$ which are regular modulo I. Set $C(K[S]) = \bigcap C(P)$, where P ranges over all the height one prime ideals of $K[S]$.

In this section we investigate when a semigroup algebra of a submonoid of a torsion-free polycyclic-by-finite group is a Noetherian UFR.

5.6.1. First we need two propositions. For an element $f = \sum_{s\in S} k_s s \in K[S]$, we write $\mathrm{supp}(f) = \{s \in S \mid 0 \neq k_s \in K\}$, the support of f.

LEMMA 5.20 (Lemma 3.1, [**38**]). *Let K be a field, S be a submonoid of a torsion free polycyclic-by-finite group and suppose S satisfies the ascending chain condition on left and right ideals. Then the following conditions hold.*

(1) *$C(pK[S]) \subseteq C(p^k K[S])$ for any prime element $p \in S$ and any positive integer k.*

(2) *If p is a prime element of S and x is a normal element of S with $x \notin pK[S]$, then $xK[S] \bigcap p^m K[S] = xp^m K[S]$ for any positive integer m. So, if $p_1, \cdots, p_n$ are non-associate prime elements of S, then $p_1^{t_1} \cdots p_n^{t_n} K[S] = p_1^{t_1} k[S] \bigcap \cdots \bigcap p_n^{t_n} K[S]$.*

(3) *Suppose S is a UF-monoid, then each nonzero element f of $K[S]$ can be written in the form $f = hn$ for some $h \in K[S]$ and $n \in N(S)$, where $h \notin pK[S]$ for any prime element p of S. Furthermore, n is unique up to inverses.*

Proof. (1). This is a special case of the elementary result that if x is a regular normal element of an arbitrary ring R then $C(xR) \subseteq C(x^k R)$ for every positive integer k.

Since S is a submonoid of torsion free polycyclic-by-finite group, then every prime element of S is also a prime element of $K[S]$. Thus (1) is obvious.

(2) Suppose $xa = p^m b$ for some $a, b \in K[S]$. Because x is a normal element and $x \notin pK[S]$, $x \in C(pK[S])$. Hence by (1), $x \in C(p^m K[S])$. It follows that $a \in p^m K[S]$. Hence $xa \in xp^m K[S]$. Hence (2) follows.

(3) Let $0 \neq f \in K[S]$. Because $K[S]$ is right Noetherian there exists $h \in K[S]$ so that $hK[S]$ is maximal with respect to the condition $f = hn$, for some $n \in N(S)$. By the maximality condition on $hK[S]$, the element h does not have any prime element of S as a factor. So the first part of (3) follows.

For the last part, assume $hn = h'n'$, with $n, n' \in N(S)$ and $h, h' \in K[S]$, and h, h' do not belong to any $pK[S]$, with p a prime element in S (that is, $\mathrm{supp}(h) \not\subseteq pS$ and $\mathrm{supp}(h') \not\subseteq pS$). Since S is cancellative, $\mathrm{supp}(h)n = \mathrm{supp}(h')n'$, and thus $\mathrm{supp}(h)Sn = \mathrm{supp}(h')Sn'$. Hence, for any prime $p \in S$, $n \in pS$ if and only if $n' \in pS$. Since n and n' are products of prime elements in S and because S is cancellative it follows that $n = n'u$ for some u in the unit group of S. □

As for commutative semigroup algebras (with notations as in (3)) the element $n \in N(S)$ is called a *homogeneous content* of $f \in K[S]$. If n is a unit, then we say that f is *homogeneous primitive* (h-primitive). It is easily verified that this definition is left-right symmetric. Note that, if $S = N(S)$, then n is a greatest common divisor of the elements in $\mathrm{supp}(f)$.

LEMMA 5.21 (Lemma 3.2 in [**38**]). *Let K be a field and S be a torsion free submonoid of a polycyclic-by-finite group. Suppose S satisfies the ascending chain condition on left and right ideals. If S is a UF-monoid and h is a homogeneously* *primitive element in $N(K[S])$, then $h \in C(wK[S])$ for any non-invertible element $w \in N(S)$.*

Proof. Let $w \in (N(S) \setminus U(S))$. Write $w = q_1^{t_1} \cdots q_n^{t_n}$, with each q_i a prime element in S and each t_i a positive integer. By Lemma 5.20, $wK[S] = q_1^{t_1}K[S] \cap \cdots \cap q_n^{t_n}K[S]$. Since $h \notin q_iK[S]$ and because h is normal, $h \in C(q_iK[S])$. Then, again by Lemma 5.20, $h \in C(q_i^{t_i}K[S])$ and thus $h \in C(wK[S])$. □

5.6.2. We also need the following Lemma due to Menal [**49**, Lemma 2].

LEMMA 5.22. *Let R be a ring and M a nontrivial monoid so that $R[M]$ is a domain. Let $a \in M$ such that $a = \alpha\beta$ for some $\alpha, \beta \in R[M]$. If supp(α) contains a unit then α is a unit.*

Proof. Denote by $a \sim b$ if $a = ub$ for some $u \in U(S)$. Then $\sim$ is an equivalence relation. Let

$$a = \alpha\beta = (U + X)(Ya + Z_1 + \cdots + Z_n). \ (*)$$

where U is such that $supp(U) \subseteq U(M)$ and X is such that $supp(X) \cap U(M) = \emptyset$, Ya denotes the sum of terms in β that have a as a right factor, and $Z_1, \cdots, Z_n$ denote the sum of equivalent terms in β that do no t have a as a right factor.

Note that by assumption $U \neq 0$. We want to show that $n = 0$. Suppose $n > 0$. Since $R[M]$ is a domain, for any k we have $UZ_k \neq 0$ and its support can not contain left multiples of a. So, there exists $u \in supp(U)$ such that $uz_k = x'z_j$ for some z_k and z_j in Z_k and Z_j respectively where x' is a non-unit. From this we see that for each $z_k \in Z_k$ there is an index j with $z_k = x'' z_j$ for some $z_j \in Z_j$ and some non-unit x''.

Re-indexing, we obtain:

$$\begin{aligned} z_1 &= x_1 z_2, \\ z_2 &= x_2 z_3, \\ \cdots & \quad \cdots\cdots \\ z_t &= x_t z_1, \end{aligned}$$

where $t \geq 1$ and the x_i's are non-units. But then

$$z_1 = x_1 x_2 \cdots x_t z_1$$

implies x_1 is a unit, a contradiction. Hence $n = 0$ and thus $\beta = Ya$. Therefore α is a unit. □

5.6.3. Let S be a submonoid of a torsion free polycyclic-by-finite group G. We now determine when $K[S]$ is a left and right Noetherian UFR with trivial central class group (or equivalently, all height one prime ideals are generated by a central element). Actually, we show a more general result. For this we recall that Brown in **[2, 4]** showed that $K[G]$ is a Noetherian maximal order with class group isomorphic with the first cohomology group $H^1(G/C_G(\Delta(G)), K^{\bullet} \times \Delta(G))$ (here $C_G(\Delta(G))$ denotes the centraliser of $\Delta(G)$ in G). In particular, if $K[G]$ is a UFR, then all height one primes are generated by a central element if and only if $H^1(G/C_G(\Delta(G)), K^{\bullet} \times \Delta(G)) = \{1\}$. More generally, Wauters in **[60]** showed that if $K[G]$ is a UFR (and K has characterisitc zero) with all

$H^1(G/C_G(\Delta(G)), \Delta(G)) = \{1\}$ then every height one prime ideal is generated by a semi-invariant. Recall that $0 \neq r \in K[G]$ is called a semi-invariant if there exists $\lambda \in Hom(G, K^*)$ so that $grg^{-1} = \lambda(g)r$ for each $g \in G$ (λ is called the weight of r). Note that a semi-invariant element is normal in $K[G]$. In [**60**], Theorem 5.3, it is described when $H^1(G/C_G(\Delta(G)), \Delta(G))$ is trivial.

Also recall that it still is an open problem when a group algebra $K[G]$ of a torsion free polycyclic-by-finite group only has trivial units, that is, all units in $K[G]$ are of the form kg with $0 \neq k \in K$ and $g \in G$. It is conjectured that this is always the case. In case G is a right ordered group (for example a poly-infinite cyclic group) then it is well known and easy to show that the conjecture holds.

THEOREM 5.23 (Theorem 3.3 in [**38**]). *Let S be a monoid with a torsion free polycyclic-by-finite group of quotients G. Assume S satisfies the ascending chain condition on left and right ideals. Suppose that $K[S]$ is a (Noetherian) UFR, then $K[G]$ is a UFR, and if furthermore all units in $K[G]$ are trivial, then S is a UF-monoid. Conversely, suppose that $K[G]$ is a UFR such that every height one prime ideal of $K[G]$ is generated by a semi-invariant and S is a UF-monoid, then $K[S]$ is a (Noetherian) UFR.*

Proof. Note that since G is torsion free, the group algebra $K[G]$, and thus also $K[S]$, is a domain (see for example [**56**, Theorem 37.5]).

Assume that $K[S]$ is a Noetherian UFR. Since $K[G]$ is a Noetherian localization of $K[S]$ it is easily shown that $K[G]$ is a UFR.

Because of Theorem 5.16, to prove that S is a UF–monoid, it is sufficient to show that if $s \in N(S)$ is an irreducible element in $N(S)$, then s is prime in S. Actually we will show that s is prime in $K[S]$. Because of Lemma 5.2, we only need to show that s is irreducible in $N(K[S])$. So, assume $s = \alpha\beta$ with $\alpha, \beta \in N(K[S])$. Then $1 = (s^{-1})\alpha\beta \in K[G]$. So α and β are units in $K[G]$. Because of the assumption that units are trivial in $K[G]$, we get that $\mathrm{supp}(\alpha) = \{x\}$ and hence $x \in S \cap N(K[S]) = N(S)$. Similarly $\mathrm{supp}(\beta) = \{y\}$ and $y \in N(S)$; and also $s = xy$. The irreducibility of s in $N(S)$ therefore gives that x or y, and thus α or β is a unit in $K[S]$ (Lemma 5.22), as required.

For the converse we assume S is a UF-monoid, $K[G]$ is a UFR, and every height one prime ideal of $K[G]$ is generated by a semi-invariant. We prove $K[S]$ is a UFR. Take any prime ideal P of $K[S]$. We need to prove that P contains a principal prime ideal of $K[S]$. If $P \bigcap S \neq \emptyset$ then, by Proposition 5.15, $K[P \cap S]$ is a prime ideal of $K[S]$. Hence $P \cap S$ is a prime ideal in S. So by assumption (2), $P \cap S$ contains a prime ideal $Sp = pS \in X^1(S)$, $p \in S$. Because of Proposition 5.15, $K[Sp] \in X^1(K[S])$, as required.

Next assume $P \bigcap S = \emptyset$, or equivalently, $P \cap N(S) = \emptyset$. It follows that $PK[G] \cap K[S] = P$ and $PK[G] = K[G]P$ is a prime ideal of $K[G]$. Since $K[G]$ is a UFR, the prime ideal $PK[G]$ contains a prime element of $K[G]$. By assumption, $K[G]z \subseteq K[[G]P$ for some semi-invariant z (with weight λ) of $K[G]$. Then $gzg^{-1} = \lambda(g)z$ for any $g \in G$ and thus $K[G]z = zK[G]$ and $K[S]z = zK[S]$. Write $z = hst^{-1}$ for a homogeneous primitive element $h \in K[S]$ and $s, t \in N(S)$. Then we have $K[S]h = K[S]zts^{-1} = zK[S]ts^{-1} = zts^{-1}K[S] = hK[S]$ and thus $h \in N(K[S])$. Note also that, if $x \in hK[G] \bigcap K[S]$, then $x = hyf^{-1}$ with $y \in K[S]$ and $f \in N(S)$. So $xf = hy$. Since $h \in N(K[S])$, Lemma 5.21 implies that $y = y'f$ for some $y' \in K[S]$ and thus $x = hy' \in hK[S]$. So indeed $hK[G] \bigcap K[S] = hK[S]$. Since $hK[G]$ is a prime ideal and because $K[G]$ is Noetherian localization of $K[S]$, we get that (see for example Lemma 7.15 in [**52**]) $hK[S] = K[S]h$ is a prime ideal of $K[S]$ and thus h is prime element of $K[S]$. Obviously $hK[S] \subseteq P$. This proves that $K[S]$ is a UFR. □

COROLLARY 5.24 (Corollary 3.4, [**38**]). *Let S be a monoid satisfying the ascending chain condition on left and right ideals. Assume S has a torsion free polycyclic-by-finite group of quotients. Then, $K[S]$ is a UFR with all height one prime ideals generated by a central element if and only if S is a UF-monoid with all minimal prime ideals generated by a central element and $K[G]$ is a UFR with all height one prime ideals generated by a central element.*

Proof. Because of Theorem 5.23 (and its proof) we only need to show that if $K[S]$ is a UFR with all height one prime ideals generated by a central element then S is a UF-monoid with all minimal primes generated by a central element. Again, as in the proof of Theorem 5.23, it is sufficient to show that every irreducible element $s \in N(S)$ is irreducible in $N(K[S])$ and central in $K[S]$. Since s is a normal element and $K[S]$ is a UFR with all height one prime ideals generated by central element, write $s = p_1^{n_1} \cdots p_k^{n_k}$ for some central prime elements p_i of $K[S]$. Clearly

$K[G] = K[G]s$ and thus each p_i is a central unit in $K[G]$. Obviously, $p_i \in K[\Delta(G)]$. Since ΔG is a torsion free abelian group (and thus ordered), each p_i is a trivial unit. Hence $p_i = k_i x_i$ for some $k_i \in K$ and $x_i \in Z(S)$. Thus $s = x_1^{n_1} \cdots x_k^{n_k}$ and each $x_i \in Z(S)$. The irreducibility of s in $N(S)$ implies that $s = x_1$. So indeed, s is central in $K[S]$ and is irreducible in $N(K[S])$. □.

5.6.4. Let S be a UF-monoid such that every minimal prime is completely prime, it is easy to show that $S = N(S)$ (a similar argument can be found later in Theorem 5.27). For the normalizing semigroups, we have a stronger result. First we need the following lemma (this is Gauss Lemma in case S is abelian).

LEMMA 5.25 (Lemma 3.5 in [**38**]). *Assume S is a normalizing UF-monoid and K a field and $K[S]$ a domain. Assume that f, g are h-primitive. Then fg is h-primitive.*

Proof. The proof is similar to that of the commutative situation (see for example Theorem 14.4 in [**23**]). However, for completness' sake we include a proof. Since S is a normalizing UF-monoid, $S/\mathcal{U}(S) = T$ is a free abelian semigroup. Let $\leq$ be a linear order on T and let $\overline{s}$ denote the natural image of $s \in S$ in T. So we can consider $K[S]$ as a ring graded by the ordered monoid T.

Write $f = \Sigma_{i=1}^{n} a_i s_i$ and $g = \Sigma_{j=1}^{m} b_j t_j$, where $a_i, b_j \in K[U]$ and $s_i, t_j \in S$ so that $\overline{s_1} < \overline{s_2} < \cdots < \overline{s_n}$ and $\overline{t_1} < \overline{t_2} < \cdots < \overline{t_m}$. Since S is a normalizing UF-monoid, to prove that fg is h-primitive, it suffices to show that each prime element $w \in S$ does not divide all elements in the support of fg.

Now, $\gcd(\mathrm{supp}(a_is_i)) = s_i$ and $\gcd(\mathrm{supp}(b_jt_j)) = t_j$. Because, by assumption, f is h-primitive, there exists an index i so that $s_i \notin Sw$ and $s_k \in Sw$ for all $1 \leq k < i$. Similarly, there exists an index j so that $t_j \notin Sw$ and $t_l \in Sw$ for all $1 \leq k < j$. Now, for the T-gradation on $K[S]$, the $\overline{s_it_j}$-component of fg has the form

$$a_is_ib_jt_j + (\text{a sum of terms } a_xs_xb_yt_y, \text{ where either } x < i \text{ or } y < j\ ).$$

Clearly w then divides each such $a_xs_xb_yt_y$. On the other hand $\mathrm{supp}(a_is_ib_jt_j) \subseteq \mathcal{U}(S)s_it_j$ and $s_i, t_j \notin Sw$. Thus $\mathrm{supp}(a_is_ib_js_j) \not\subseteq Sw$. Since $\mathrm{supp}(a_is_ib_jt_j) \subseteq \mathrm{supp}(fg)$, the result follows. □

THEOREM 5.26 (Theorem 3.6 in [**38**]). *Let S be a normalizing monoid with a torsion free polycyclic-by-finite group of quotients G. Assume S satisfies the ascending chain condition on left and right ideals. Then $K[S]$ is a (Noetherian) UFR if and only if $K[G]$ is a UFR and S is a UF-monoid.*

Proof. The proof follows the line of that of Theorem 5.23. Hence we only prove those claims in the proof that require a different argument. First assume that $K[S]$ is a U.F.R.. To show that S is a UF-monoid it is sufficient to show that each irreducible element $p \in S$ is prime in S. Again by Proposition 5.16 it is sufficient to show that p is irreducible in $K[S]$. For this, suppose that $p = \alpha\beta$ with $\alpha, \beta \in K[S]$. Hence $p = ab$ for some $a \in \mathrm{supp}(\alpha)$ and $b \in \mathrm{supp}(\beta)$. Because of the irreducibility of p in S we get that $a \in \mathcal{U}(S)$ or $b \in \mathcal{U}(S)$. It then follows from Lemma 2 in [**49**] that α or β is a unit in $K[S]$, as desired.

Conversely, assume $K[G]$ is a U.F.R. and S is a UF-monoid. With the same notation as in the proof of Theorem 5.23, let P be a prime ideal of $K[S]$ so that $P \cap S = \emptyset$. Let $z = hst^{-1}$ be a prime element in $PK[G]$, with h a homogeneous primitive element and $s, t \in N(S) = S$. Then $K[G]h = K[G]z = hK[G]$. Suppose now that $x \in hK[G] \cap K[S]$. Then $x = hyf^{-1}$ with $y \in K[S]$ and $f \in N(S)$. So $xf = hy$. Write $y = y'c$, with $c \in N(S)$ and y' an h-primitive element in $K[S]$. Hence $xf = hy'c$. By Lemma 5.25, we know hy' is h-primitive. It thus follows that $c \in Sf$ and so there exists $c' \in S$ so that $x = hy'c' \in hK[S]$. Hence we have shown that $hK[G] \bigcap K[S] = hK[S]$. Similarly $K[S]h = K[G]h \bigcap K[S]$. Consequently, $K[S]h = hK[S]$ is a prime ideal contained in P. The result therefore follows. □

5.6.5. Moreover, we can determine when $K[S]$ is a UFD.

THEOREM 5.27 (Theorem 3.7 in [**38**]). *Let S be a submonoid of a torsion free polycyclic-by-finite group of quotients G. Then, $K[S]$ is a Noetherian UFD if and only if the following conditions are satisfied:*

1. *$K[G]$ is a UFD,*
2. *S satisfies the ascending chain condition on right and left ideals,*
3. *$S = N(S)$ is a UF-monoid.*

Proof. Assume the three conditions are satisfied. By Theorem 5.26, $K[S]$ is a Noetherian UFR. Now we need to prove that every height-1 prime ideal P of $K[S]$

is completely prime. If $P \cap S = \emptyset$, then $K[G]P \in X^1(K[G])$ and $K[S]/P \subseteq K[G]/K[G]P$. So $K[S]/P$ is a domain. On the other hand, if $P \cap S \neq \emptyset$, then $P = K[p]$ with $p = S \cap P$ a minimal prime in S. It follows that $K[S]/P \cong K_0[S/p]$, a contracted semigroup algebra. Since, by assumption, $S \setminus p$ is a subsemigroup of S, we get that $K[S]/P \cong K[S \setminus p]$. As a subring of $K[G]$, the latter is therefore a domain as well.

Conversely, assume $K[S]$ is a UFD. As a Noetherian localization of a UFD, the group algebra $K[G]$ is a UFD. Clearly condition (2) is satisfied as $K[S]$ is Noetherian.

To obtain condition (3), by Theorem 5.26, we only need to show $S = N(S)$. So let $s \in S \setminus U(S)$. Because $SN(S)^{-1}$ is a group, there exist $t \in S$ and $n \in N(S)$ so that $stn^{-1} = 1$ and thus $st = n$. As n is a normal element in the UFD $K[S]$, $n = p_1 \cdots p_k$, a product of prime elements p_i in $K[S]$. Since each ideal $p_i K[S]$ is completely prime and because $K[S]$ is a domain it follows that $s = \alpha c$ and $t = \beta c'$ for some $c, c' \in N(K[S])$, $\alpha, \beta \in K[S]$, and $cc' \in Sn$. Hence $n = \alpha\beta' cc'$, for some $\beta' \in K[S]$. It follows that $\alpha, \beta \in U(K[S]) \subseteq N(K[S])$. Hence $s, t \in N(K[S]) \cap S = N(S)$, as required. $\Box$

5.6.6. Let S be a submonoid of a nilpotent group. In **[35]** it is shown that $K[S]$ is left Noetherian if and only if $K[S]$ is right Noetherian, in other words, the ascending chain condition on left ideals of S is equivalent with the ascending chain condition on right ideals of S.

COROLLARY 5.28 (Corollary 3.8 in **[38]**). *Let S be a submonoid of a torsion free finitely generated nilpotent group of quotients G. Assume S satisfies the ascending chain condition on left ideals. Then the following conditions are equivalent.*

1. *$K[S]$ is a UFR.*
2. *$K[S]$ is a UFD.*
3. *$S = N(S)$ is a UF-monoid*

Proof. Note that G is a poly-infinite cyclic group and thus all units in $K[G]$ are trivial. Since G is a torsion free finitely generated nilpotent group, Corollary 5.7 yields that $K[G]$ is a UFD. So, by Theorem 5.27, conditions (2) and (3) are equivalent.

It remains to show that (1) implies (3). Now, if $K[S]$ is a (Noetherian) UFR, and thus a maximal order, then we know from **[34]** that $S = N(S)$. From Theorem 5.23 we get that S is a UF-monoid. □

5.7. Examples

5.7.1. Consider submonoids of $G = \langle x, y \mid y^{-1}xy = x^{-1}\rangle$. Note G is not a nilpotent group. In the following we always use the equivalent group condition $xyx = y$. Obviously $\Delta(G) = \langle x, y^2 \mid xy^2 = y^2x\rangle$ is a torsion free abelian group. First $K[G]$ is not a UFD by Lemma 5.5. However, because $K[G]$ is a prime PI-algebra, it follows from Lemma 5.3 and Lemma 5.4 that $K[G]$ is a UFR.

Now let us look at the submonoids of G.

1. Let S_1 be a submonoid generated by $1, x, y^2$. Because $xy^2 = y^2x$, this submonoid is abelian. Obviously $S_1 \cong Z_1 \bigoplus Z_2$ is factorial, where Z_i is isomorphic to the additive monoid of nonnegative integers. Hence $K[S_1]$ is a UFD.
2. Let S_2 be the normalizing submonoid generated by x, x^{-1}, y. Since $U(S_2) = \langle x, x^{-1} \rangle$ and $S_2/U(S_2)$ is free abelian, S_2 is a UF-monoid. By Theorem 5.23, $K[S_2]$ is a Noetherian UFR.
3. Let S_3 be the submonoid generated by $1, x, y$. Then the set of normal elements is generated by $1, y^2, yxy, yx^2y, \cdots, yx^iy, \cdots$. Note in this monoid, $y^2S_3 \subset yxyS_3 \subset yx^2yS_3 \subset \cdots$ $(yx^{i+1}y \notin yx^iyS_3$). Hence we have a strictly ascending chain on principal ideals of S. Hence $K[S_3]$ is not a Noetherian UFR. In fact $K[S]$ is not even a UFR.

5.7.2. Consider submonoids of $G = \langle x_1, x_2, a \mid x_2x_1 = x_1x_2a, a \text{ is central}\rangle$. Obviously G is a torsion free nilpotent group. So by Corollary 5.7, $K[G]$ is a UFD. Let us look at the submonoids of G.

1. Let S_1 be a submonoid generated by $1, a, x_1, x_2$. Then S_1 does not satisfy the ascending chain condition on right ideals as $x_1x_2S_1 \subset x_1x_2S_1 \cup x_1^2x_2S_1 \subset \cdots \subset x_1x_2S_1 \cup \cdots \cup x_1^nx_2S_1 \subset \cdots$. Thus $K[S_1]$ is not a Noetherian UFR.
2. Let S_2 be the submonoid generated by x_1, x_2^{-1}, x_2, a. Then we have a strictly ascending ideal chain $x_2^{-1}x_1S_2 \subset x_2^{-2}x_1S_2 \subset x_2^{-3}x_1S_2 \subset \cdots$. Thus $K[S_2]$ is not a Noetherian UFR.

3. Let S_3 be the submonoid generated by a, a^{-1}, x_1, x_2. Obviously S_3 is a normalizing submonoid. It follows that $K[S_3]$ is a Noetherian UFR (UFD) since S_3 is a UF-monoid and satisfies the ascending chain condition on left and right ideals.

CHAPTER 6

Unique Factorization Semigroup Rings with a Polynomial Identity

In this chapter we investigate when a semigroup algebra $K[S]$ of a cancellative monoid S is a PI domain which is a unique factorization ring. We do not require that $K[S]$ is Noetherian. In other words, our monoids are submonoids of torsion free abelian-by-finite groups. In order to tackle this problem we have to investigate prime ideals of $K[S]$. More specifically, we have to prove an analogue of Proposition 5.15 in the setting of PI semigroup rings. Since a prime PI ring is embedded in a matrix algebra we are again in a position to apply the theory of linear semigroups, and hence we are able to prove such an analogue. This is done in Section 2 while certain properties on unique factorization rings with a PI are recalled in Section 1. Then we investigate the unique factorization semigroup rings with a PI in Section 3. Finally examples are given in Section 4. The results proved in this chapter will appear in [**39**].

6.1. PI algebras and unique factorization rings

6.1.1. Let A be a commutative ring. For any integer $m \geq 1$, we denote by $A\langle x_1, \cdots, x_m\rangle$ free A-algebra in m free generators $x_1, \cdots, x_m$. A A-algebra R is said to satisfy a polynomial identity (shortly, R is a PI-algebra) if there exists an integer n and a nonzero element $f = f(x_1, \cdots, x_n) \in A\langle x_1, \cdots, x_n\rangle$ such that $f(a_1, \cdots, a_n) = 0$ for every $a_1, \cdots, a_n \in R$ and one of the monomials of f of highest (total) degree has coefficient 1. Also R is called a PI-ring if R satisfies a polynomial identity with $A = \mathbf{Z}$. Commutative rings, nilpotent rings, and matrix algebras are basic examples of PI-rings.

We are interested in prime PI rings R. In this case, the well-known Posner Theorem yields that R is an order in its classical ring of quotient $Q_{cl}(R)$ and $Q_{cl}(R) = R(Z(R) \setminus \{0\})^{-1}$.

6.1.2. Group algebras satisfying a polynomial identity have been completely characterized. The following result is due to Isaacs and Passman if $ch(K) = 0$, and to Passman if $ch(K) > 0$. Let G be a group. Then $K[G]$ is a PI-algebra if and only if G is abelian-by-finite or G is a p-abelian-by-finite group if $ch(K) = p > 0$. A p-abelian group A is a group such that the commutator subgroup A' is a finite p-group. If $K[G]$ is a prime PI algebra, then G is a (torsion-free abelian)-by -finite group. In case of domain, G is torsion free.

Suppose now that S is a cancellative semigroup. Because of Theorem 1.16 one obtains the following result.

LEMMA 6.1 (Theorem 20.1 in **[52]**). *Let S be a cancellative semigroup. If $K[S]$ is a polynomial identity ring, then S has a group of fractions G and $K[G]$ is a PI-algebra satisfying the same multilinear identities as $K[S]$.*

6.1.3. In [**10**, Corollary 4.8], Chatters, Gilchrist, and Wilson showed that a UFR with a PI is also a maximal order. Let I be a right ideal of a prime Goldie ring R which has a classical quotient ring Q. We call I *closed* if $I = \{x \in R : xK \subseteq I$ for some right ideal K of R with $(R :_l K) = R\}$. Theorem 4.19 in [**10**] also yields that if R is a UFR with PI then R satisfies the ascending chain condition for closed right ideals. Note that if R is a maximal order then divisorial ideals are closed. To justify this claim, it is sufficient to show that if $I = (R : (R : I))$ is an integral divisorial ideal of R then $I = \{x \in R : x(R : I)I \subseteq I\}$ and $(R :_l (R : I)I) = R$. Indeed, for any $x \in I$, $x(R : I) \subseteq R$ and then $x(R : I)I \subseteq I$. Conversely, let $x(R : I)I \subseteq I$ and $x \in R$. Since R is a maximal order, $x(R : I) \subseteq R$ and thus $x \in I$. Since R is a maximal order we also have $(R : (R : I)I) = ((R : I) : (R : I)) = R$ as required. Therefore, R is a maximal order that satisfies ascending chain condition on integral divisorial ideals, that is, R is a Krull order.

Abbasi, Kobayashi, Marubayashi, and Ueda in [**1**] give a different definition of UFR: an order R is called a UFR if R is a Krull order so that all its divisorial ideals are principal, i.e., a Krull order with the trivial normalizing class group. Also in [**1**] (page 195, Remark (2)) they give an example of group algebra $K[G]$ which is a UFR in their sense but not a UFR in our sense (i.e., as defined by Chatters and Jordan). The construction of G is given as follows: let $H = \langle x \rangle$ be an infinite cyclic group, where $x = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in SL_2(\mathbf{Z})$, the 2×2 special linear group over $\mathbf{Z}$. Assume that the eigenvalues of x are of the form $\alpha = a + b\sqrt{c}$, $\beta = a - b\sqrt{c}$, where $a, b \in \mathbf{Q}, 2a \in \mathbf{Z}, c \in \mathbf{Z}$ and c is not square (for example, $e = 2$ and $f = g = h = 1$). Let $A = \langle y \rangle \times \langle z \rangle$ be a direct product of infinite cyclic groups $\langle y \rangle$ and $\langle z \rangle$. Define an action of H on A as follows; $y^x = y^e z^g$ and $z^x = y^f z^h$. With this action, we can construct a semi-direct product $G = A \rtimes H$ of A by H. Since no proper pure subgroup of A is G-orbital, A is a plinth of G. Since A has rank two, we get that A is eccentric. Brown's result (Lemma 5.3) implies that G is not a UFR in our sense. But by Theorem 4.3 in [**1**], $K[G]$ is a UFR in their sense, which means every divisorial prime ideal is principal. Also this example tells us that there exists a height one prime ideal of $K[G]$ that is not divisorial.

6.1.4. However, if R is a PI Krull order then it is well known (see for example [**5**], [**6**], or [**29**]) that the divisorial prime ideals are precisely height one prime ideals. Hence one gets the following result.

THEOREM 6.2. *Let R be a prime PI ring. Then R is a UFR if and only if R is a Krull order with the trivial normalizing class group.*

In [**8**, Theorem 2.1] Chatters describes when a prime PI group ring is a UFR.

LEMMA 6.3. *Suppose $R[G]$ is a prime PI ring. Then $R[G]$ is a UFR if and only if R is a UFR and G is a dihedral-free group satisfying the ascending chain condition on cyclic subgroups.*

6.2. Prime ideals

Let S be a cancellative monoid. In this section we investigate prime ideals in prime semigroup algebras $K[S]$ that satisfy a polynomial identity. By Lemma 6.1, such a monoid S has a group of quotients $G = S^{-1}S = SS^{-1}$ and $K[G]$ is also a prime ring satisfying a polynomial identity. Hence G is a (torsion-free abelian)-by-finite group.

LEMMA 6.4 (Lemma 1.1 in [**39**]). *Let S be a submonoid of a torsion-free abelian-by-finite group. Let G be the group of quotients of S. Then $G = SZ(S)^{-1}$.*

Proof. Because of the assumption G is torsion-free abelian-by-finite. Hence for a field K, the group algebra $K[G]$ and thus also the semigroup algebra $K[S]$ is a prime PI ring. Hence $K[S]$ and $K[G]$ have the same classical ring of quotients $Q = Q_{cl}(K[S])$, and Q is obtained from $K[S]$ by inverting the nonzero elements in the centre $Z(K[S])$ of $K[S]$. So, for any element $g \in G$, there exists a central element $\alpha \in Z(K[S])$ such that $g\alpha \in K[S]$. Hence we have $g(supp(\alpha)) \subseteq S$. Now, for any $h \in G$, $h\alpha h^{-1} = \alpha$. Hence h supp(α) $h^{-1} =$ supp$(\alpha) \subseteq S$. Therefore $ghx^t h^{-1} \in S$ for any $h \in G$, $x \in supp(\alpha)$ and positive integer t. Since G is abelian-by-finite, there exists a power n so that $x^n \in S \cap \Delta(G)$. Since x^n has only finitely many conjugates in G, say $g_1x^ng_1^{-1}, \cdots, g_mx^ng_m^{-1}$, then $z = (g_1xg_1^{-1})^n \cdots (g_mxg_m^{-1})^n \in Z(S)$. Since $gz = g(g_1xg_1^{-1})^n \cdots (g_mxg_m^{-1})^n \in S$, we obtain that $g \in SZ(S)^{-1}$. □

Let now S be a submonoid of a torsion-free abelian-by-finite group and let K be a field. Of course the semigroup algebra $K[S]$ has a natural S (and $SZ(S)^{-1}$) gradation. We now prove that the homogeneous part of a prime ideal in $K[S]$ is again a prime ideal. This result is well known for rings graded by torsion-free abelian groups and, more generally, for rings graded by unique product groups [**30**]. As stated in Proposition 5.14 it is also valid for the semigroup algebra of a submonoid of a torsion-free polycyclic-by-finite group which satisfies the ascending chain condition on right and left ideals.

Since the classical ring of quotients of a prime PI algebra $K[S]$ is a matrix ring $M_n(D)$ over a skew field D, we consider S as a skew linear semigroup. The latter have been extensively studied by Okniński. For definitions and needed results on this topic we refer the reader to [**53**].

THEOREM 6.5 (Theorem 1.2 in [**39**]). *Let S be a submonoid of a torsion free abelian-by-finite group and let K be a field. The following properties hold.*

1. *If P is a prime ideal of $K[S]$ with $P \cap S \neq \emptyset$, then $K[S \cap P]$ is a prime ideal in $K[S]$.*
2. *If Q be a prime of S, then $K[Q]$ is also a prime ideal in $K[S]$.*
3. *The height one prime ideals of $K[S]$ intersecting S are of the form $K[Q]$ where Q is a minimal prime ideal of S.*

Proof. (1) Let P be a prime ideal of $K[S]$ with $P \cap S \neq \emptyset$. We will show $K[S]/K[S \cap P]$ is a prime ring, i.e. the contracted semigroup algebra $K_0[S/(S \cap$

$P)] \cong K[S]/K[S \cap P]$ of the Rees factor monoid $S' = S/(S \cap P)$ is a prime ring. Since $K[S]/P$ is a prime PI ring, $Q_{cl}(K[S]/P) = M_n(D)$, with D a division algebra.

Let $\phi : S' \longrightarrow M_n(D)$ be the natural monoid homomorphism and $S'' = \phi(S')$. It follows from the structure theorem of skew linear semigroups ([**53**]) that the monoid S'' has an ideal U contained in a completely 0-simple subsemigroup $\hat{U}$ of $M_n(D)$ such that U is uniform in $\hat{U}$, the completely 0-simple closure of U (see 4.5.1). Furthermore, the nonzero elements of U are the elements of minimal nonzero rank of S''. Let A be an abelian subgroup of G of finite index in the group of quotients of S. Let A'' be the natural image of $S \cap A$ in S''.

Now we claim that $\hat{U}$ is an inverse semigroup. Since $\hat{U}$ is a completely 0-simple, we can write $\hat{U} = \mathcal{M}(\bar{G}; I, \Lambda; \bar{P})$, where $\bar{G}$ is a maximal subgroup of $\hat{U}$, $\bar{P}$ is the sandwich matrix with $\mid \Lambda \mid$ rows and $\mid I \mid$ columns. We only need to show that each row and column of $\bar{P}$ does not contain more than one non-zero element. Suppose $p_{ij} \neq 0$ and $p_{ik} \neq 0$. Since U is uniform, the 0-cancellative parts $U_{ji} = \hat{U}_{ji} \cap U$ and $U_{ki} = \hat{U}_{ki} \cap U$ of U are nonzero. Let $0 \neq u_{ji} \in U_{ji}$ and $0 \neq u_{ki} \in U_{ki}$, then $u_{ji}^n u_{ki}^n \neq 0$ and $u_{ji}^n, u_{ki}^n \in A''$. By the commutativity of A'', we have $u_{ji}^n u_{ki}^n = u_{ki}^n u_{ji}^n \neq 0$. Therefore $j = k$. This shows that each row of $\bar{P}$ contains exactly one nonzero entry. Similarly, one shows that each column contains exactly one nonzero entry. So $\hat{U}$ is an inverse semigroup. Since the K-algebra generated by $\hat{U}$ is contained in $M_n(D)$ and $M_n(D)$ does not have an infinite set of orthogonal idempotents, it follows that $\mid I \mid=\mid \Lambda \mid= t < \infty$. If necessary, rearranging the entries of $\bar{P}$, we get $\hat{U} = \mathcal{M}(\bar{G}; t, t; E)$, where E is the identity matrix.

Let $I' \subseteq S'$ be the inverse image of U under ϕ. Then

$$K_0[I'] = \Sigma_H K_0[\phi^{-1}(H) \cup \{\theta\}],$$

where H runs through the set of intersections of U with the different $\mathcal{H}$-classes of $\hat{U}$. Note that each $\phi^{-1}(H) \cup \{\theta\}$ is a semigroup with zero element θ and thus $K_0[\phi^{-1}(H) \cup \{\theta\}]$ is a contracted semigroup algebra contained in $K_0[S']$. The sum is direct as K-vector spaces and we thus get a Munn algebra pattern. Since $\hat{U}$ is an inverse semigroup we thus get

$$R = K_0[I'] = \Sigma_{1 \leq i,j \leq t} R_{ij},$$

with each $R_{ij} = K_0[\phi^{-1}(H) \cup \{\theta\}]$ for some H. Furthermore,

$$\phi(R_{ij} R_{kl}) = \{0\} \quad (\text{ for } j \neq k)$$

and thus, as $R_{ij} R_{kl}$ is S-homogeneous,

$$R_{ij} R_{kl} = \{0\} \text{ if } j \neq k$$

and, in general,

$$R_{ij} R_{kl} \subseteq R_{il}.$$

We now show that $K_0[I']$ is an essential ideal of $K_0[S']$ and also that $K_0[I']$ is a prime ring. It then follows that $K_0[S']$ is prime, i.e., $K[S \cap P]$ is a prime ideal of $K[S]$.

To show that $K_0[I']$ is an essential ideal of $K_0[S']$, it is sufficient to show that the right (respectively left) annihilator of $K[I']$ in $K[S']$ is zero. Suppose $K_0[I']x' = 0$ for some $x' \in K_0[S']$. Let I be the inverse image of I' in S, and x an inverse image of x' in $K[S]$. So $Ix \in K[S \cap P]$ and thus $I\ supp(x) \subseteq (S \cap P)$. Since I is an ideal of S and $I \not\subseteq S \cap P$, we have $\text{supp}(x) \subseteq (S \cap P)$. Therefore $x' = 0$ as required.

Finally we show that $K_0[I']$ is prime. Now, note that for each $1 \leq i \leq t$ there exists an $\mathcal{H}$-class H of $\hat{U}$ so that

$$R_{ii} = K_0[\phi^{-1}(H) \cup \{\theta\}] = K[\phi^{-1}(H)]$$

and $\phi^{-1}(H)$ is a subsemigroup of S. The torsion free assumption on the group of quotients of S implies that $K[S]$ is a domain (see for example [**56**, Theorem 37.5]). Hence each diagonal component R_{ii} is domain. Therefore, to prove that $K_0[I']$ is prime, it is sufficient to show that $R_{11}JR_{11} \neq \{0\}$ for every nonzero ideal J of $K_0[I']$. First we show that $JR_{11} \neq \{0\}$. Suppose the contrary, i.e., $JR_{11} = \{0\}$. Then for any $(a_{ij}) \in J$, we get $(a_{ij})RR_{11} = \{0\}$. Hence

$$(a_{ij})I'(I')_{11} = \{0\}.$$

Note that $\{0\} \neq I'(I')_{11} \subseteq S'$. Now take a, I, I_{11} as inverse images of $(a_{ij}) \in K_0[I']$ in $K[S]$ and I', I'_{11} in S, then we get

$$aII_{11} \in K[S \cap P].$$

So

$$sII_{11} \subseteq S \cap P \quad \text{for any } s \in supp(a).$$

Since $II_{11} \not\subseteq S \cap P$, we get $s \in S \cap P$ for any $s \in supp(a)$. Therefore $a \in K[S \cap P]$ and thus $(a_{ij}) = 0$. This proves that indeed $JR_{11} \neq \{0\}$. Similarly, if $R_{11}R(a_{ij}) = 0$ for $(a_{ij}) \in JR_{11}$, then $(a_{ij}) = 0$. Thus $R_{11}JR_{11} \neq \{0\}$, as required.

(2) Let Q be a prime ideal of S. Then there exists an ideal P of $K[S]$ maximal with respect to the condition $P \cap S = Q$. Clearly P is a prime ideal of $K[S]$. By (1), $K[Q] = K[P \cap S]$ is a prime ideal of $K[S]$.

(3) Let P be a height one prime ideal of $K[S]$ with $P \cap S \neq \emptyset$. Then, by (1), $K[P \cap S]$ is also a prime ideal. Since P has height one, we get $P = K[P \cap S]$. If $P \cap S$ is not minimal, then there exists a prime ideal $P_1 \subset P \cap S$. By (2) $K[P_1] \subset K[P \cap S]$ is a prime ideal, in contradiction with $P \in X^1(K[S])$. Hence $P \cap S$ is a minimal prime ideal of S. □

COROLLARY 6.6 (Corollary 1.3 in **[39]**). *Let S be a submonoid of torsion free abelian-by-finite group and let K be a field. If $K[S]$ is Noetherian or if $K[S]$ is a Krull order then every prime ideal of $K[S]$ which intersects S non trivially contains a height one prime ideal. In particular, the height one prime ideals of $K[S]$ that intersect S non trivially are precisely the ideals of the form $K[P]$ with P a minimal prime ideal in S.*

Proof. Because of the assumptions and Lemma 6.4 every ideal of S contains a central element of S, and thus also a central element of $K[S]$. If $K[S]$ is Noetherian, then Theorem 6.5 and the Principal Ideal Theorem imply that for every prime ideal Q of S, the ideal $K[Q]$ of $K[S]$ contains a height one prime ideal P with $P \cap S \neq \emptyset$. Again by Theorem 6.5, it then follows that $P = K[P \cap S]$ with $P \cap S$ a minimal prime in S. This proves the result in the Noetherian situation. On the other hand, if $K[S]$ is a PI Krull order then we know that every prime ideal contains a divisorial prime ideal and hence contains a height one prime ideal. So again the result follows from Theorem 6.5. □

6.3. Unique factorization semigroup rings with PI

In this section, we discuss when semigroup rings of submonoids of torsion-free abelian-by-finite groups are unique factorization semigroup rings with PI.

6.3.1. For a submonoid S of a torsion-free abelian-by-finite group, the descriptions of UF-monoid can be obtained from Lemma 6.4 and Proposition 5.11.

COROLLARY 6.7 (Corollary 2.3 in [**39**]). *Let S be a submonoid of a torsion free abelian-by-finite group. Then the following conditions are equivalent.*

1. *S is a UF-monoid.*
2. *Every non-invertible normal element of S can be written as a product of prime elements.*
3. *Every irreducible element in $N(S)$ is prime in S and S satisfies the ascending chain condition on principal ideals generated by a normal element.*

LEMMA 6.8 (Lemma 2.4 in [**39**]). *Let S be a submonoid of a torsion-free abelian-by-finite group and let K be a field. If S is a UF-monoid then for any $f \in K[S]$ there exist $n \in N(S)$ and $f_1 \in K[S]$ so that $f = f_1 n$ and $f_1 \notin pK[S]$ for any prime element $p \in N(S)$.*

Proof. Let $f \in K[S]$. If $\text{supp}(f) \subseteq Sn_1$, for $n_1 \in N(S)$, then $f = f_1 n_1$ for some $f_1 \in K[S]$. The same argument applied to f_1 yields $f_1 = f_2 n_2$ with $n_2 \in N(S)$ so that $\text{supp}(f_1) \subseteq Sn_2$. Repeating this argument we get normal elements $n_i \in N(S)$ and $f_i \in K[S]$ so that

$$f_i = f_{i+1} n_{i+1}.$$

If for some i the ideal generated by $\text{supp}(f_i)$ is not contained in any Sn with $n \in N(S) \setminus U(S)$, then

$$f = f_i n_i \cdots n_1$$

and $f_i \notin K[S]p$ for any prime element $p \in N(S)$, as desired. So assume that the previous does not hold for any i, i.e., for any i the normal element n_i is not a unit. Now the ideal of S generated by $\text{supp}(f)$ contains a central element z. So

$$z \in Sn_k n_{k-1} \cdots n_1$$

for any positive integer k. Because of Corollary 6.7 the element z can be written uniquely (modulo inverses) as a product of prime elements. But since also each n_i is a product of prime elements this yields a contradiction. □

As in the previous chapter, an element $f_1 \in K[S]$ that does not belong to any $K[S]p$, with p a prime element of S, is called an homogeneously primitive element (or simply, an h-primitive element).

6.3.2. For a submonoid S of a torsion-free abelian-by-finite group we know from Theorem 6.5 that any prime element $p \in S$ is also a prime element in $K[S]$. So we obtain from Lemma 5.10 the following facts.

1. $C(pK[S]) \subseteq C(p^k K[S])$ for any prime element $p \in S$ and any positive integer k.
2. If p is a prime of S and x is a normal element of S with $x \notin pK[S]$, then $xK[S] \cap p^m K[S] = xp^m K[S]$ for any positive integer m. So, if $p_1, \cdots, p_n$ are non-associated prime elements of S, then $p_1^{t_1} \cdots p_n^{t_n} K[S] = p_1^{t_1} K[S] \cap \cdots \cap p_n^{t_n} K[S]$.
3. If, furthermore, S is UF-monoid and h is a h-primitive element in $N(K[S])$, then $h \in C(wK[S])$ for any non-invertible $w \in N(S)$, where $C(wK[S])$ denotes the set of regular elements of $K[S]$ modulo $wK[S]$.

Let S be a submonoid of a torsion-free abelian-by-finite group G. We now determine when $K[S]$ is a (PI) UFR with trivial central class group, that is a UFR with all height one prime ideals generated by a central element. We actually prove a more general theorem.

THEOREM 6.9 (Theorem 2.5 in [**39**]). *Let S be a submonoid of torsion-free abelian-by-finite group. Let G be the group of quotients of S. Assume that S is a UF-monoid. Then $K[S]$ is a UFR implies that $K[G]$ is a UFR. Conversely, if $K[G]$ is a UFR such that every height one prime ideal is generated by a semi-invariant, then $K[S]$ is a UFR.*

Proof. Because of Lemma 6.4, the group algebra $K[G]$ is the localization of $K[S]$ with respect to the central Ore set $Z(S)$. Hence, for any prime ideal Q of $K[G]$, the intersection $Q \cap K[S]$ is a prime ideal in $K[S]$. It then easily follows that $K[G]$ is a UFR if $K[S]$ is a UFR.

Conversely, assume $K[G]$ is a UFR. We show that $K[S]$ is a UFR. Let P be a prime ideal of $K[S]$. We have to prove that P contains a principal prime ideal. In case $P \cap S \neq \emptyset$ then, by Theorem 6.5, $K[P \cap S]$ is also a prime ideal of $K[S]$. By assumption, S is a UF-monoid. Hence the prime ideal $P \cap S$ of S contains a prime element $p \in S$. Again by Theorem 6.5, $K[Sp]$ is a prime ideal of $K[S]$ contained in P.

So we now consider the case that $P \cap S = \emptyset$, and thus $PK[G] = K[G]P$ is a prime ideal of $K[S]$ so that $PK[G] \cap K[S] = P$. Since $K[G]$ is a UFR, the prime ideal $PK[G]$ contains a prime element q of $K[G]$. By assumption, choose q as a semi-invariant. then $gqg^{-1} = \lambda(g)q$ for any $g \in G$ and some $\lambda \in Hom(G, K^{*})$. Moreover, $qK[S] = K[S]q$. Because S is a UF-monoid, we obtain from Lemma 6.8 that $q = hst^{-1}$ for an homogeneous primitive element $h \in K[S]$ and $s, t \in N(S)$.

Then we have $K[G]h = K[G]qts^{-1} = qK[G]ts^{-1} = qK[G] = hK[G]$ and $K[S]h = K[S]qts^{-1} = qts^{-1}K[S] = hK[S]$. Now we show that $hK[G] \bigcap K[S] = hK[S]$. Hence it follows that $hK[S] = K[S]h$ is a prime ideal contained in P, as required. So let $x \in hK[G] \bigcap K[S]$, then $x = hyn^{-1}$ with $y \in K[S]$ and $n \in N(S)$. So $xn = hy$. Since By the earlier remarks, if $n \notin U(S)$, then $h \in C(nK[S])$. Hence $y = y'n$ for some $y' \in K[S]$. So $x = hy' \in hK[S]$. So indeed $hK[G] \bigcap K[S] = hK[S]$. □

If in the above theorem G also is finitely generated, then by Brown's result $K[G]$ is always a UFR. If, furthermore, the first cohomology group $H^1(G/C_G(\Delta(G)), \Delta(G))$ is trivial, then Wauters showed in **[60]** that every height one prime of $K[G]$ is generated by a semi-invariant. So, in this case it follows that $K[S]$ is a UFR provided that S is a UF-monoid. The converse holds if all units in $K[G]$ are trivial.

COROLLARY 6.10 (Corollary 2.6 in **[39]**). *Let S be a submonoid of a torsion-free abelian-by-finite group and K a field. Let G be the group of quotients of S and assume that all units of $K[G]$ are trivial. If $K[S]$ is a UFR, then the following conditions are satisfied:*

1. *S is UF-monoid.*
2. *G satisfies the ascending chain condition on cyclic subgroups.*

Proof. From Chatters' result (Lemma 6.3) we know that $K[G]$ is a UFR if and only if G satisfies the ascending chain condition on cyclic subgroups. Hence, because of Theorem 6.9 it is sufficient to prove that if $K[S]$ is a UFR, then S is a UF-monoid.

Now from Theorem 6.2 we know that $K[S]$ satisfies the ascending chain condition on principal ideals generated by normal elements. Hence S satisfies the ascending chain condition on principal ideals generated by normal elements. So, because of Proposition 5.11 it remains to show that if n is an irreducible element in $N(S)$, then n is prime in $K[S]$. Since $K[S]$ is a UFR and because of Lemma 5.1, it is actually sufficient to show that n is irreducible in $N(K[S])$. This is proved as in the last part of the proof of Theorem 5.23. If $n = \alpha\beta$ with $\alpha, \beta \in N(K[S])$ then α, β are units in $K[G]$. Therefore $|\operatorname{supp}(\alpha)| = |\operatorname{supp}(\beta)| = 1$ and thus $\alpha, \beta \in S \cap N(K[S]) = N(S)$. The irreducibility of n in $N(S)$ implies that α or β is a unit in S and thus in $K[S]$. $\square$.

In general it remains an open problem whether the the two conditions listed in Corollary 6.10 are necessary and sufficient for $K[S]$ to be a UFR. However, we now state a solution to the problem under the extra assumption that the central class group is trivial.

COROLLARY 6.11 (Corollary 2.7 in [**39**]). *Let S be a submonoid of a torsion free abelian-by-finite group. Let G be the group quotients of S. Then, $K[S]$ is a UFR with all height one prime ideals generated by a central element if and only if S is a UF-monoid with all minimal prime ideals generated by a central element and $K[G]$ is a UFR with all height one prime ideals generated by a central element.*

Proof. Because of Theorem 6.9 (and its proof) we only need to show that if $K[S]$ is a UFR with all height one prime ideals generated by a central element then S is a UF-monoid with all minimal primes generated by a central element. Again, as in the proof of Theorem 6.9, it is sufficient to show that every irreducible element $s \in N(S)$ is irreducible in $N(K[S])$ and central in $K[S]$. Since s is a normal element and $K[S]$ is a UFR with all height one prime ideals generated by central element, write $s = p_1^{n_1} \cdots p_k^{n_k}$ for some central prime elements p_i of $K[S]$. Clearly $K[G] = K[G]s$ and thus each p_i is a central unit in $K[G]$. Obviously, $p_i \in K[\Delta(G)]$. Since ΔG is a torsion free abelian group (and thus ordered), each p_i is a trivial unit. Hence $p_i = k_i x_i$ for some $k_i \in K$ and $x_i \in Z(S)$. Thus $s = x_1^{n_1} \cdots x_k^{n_k}$ and each $x_i \in Z(S)$. The irreducibility of s in $N(S)$ implies that $s = x_1$. So indeed, s is central in $K[S]$ and is irreducible in $N(K[S])$. □.

6.3.3. In case S is a normalizing monoid, that is $S = N(S)$, then we have a complete solution to the UFR problem.

PROPOSITION 6.12 (Proposition 2.8 in [**39**]). *Let S be a normalizing monoid (i.e. $S = N(S)$) with a torsion free abelian-by-finite group of quotients G. Then the following conditions are equivalent:*

(1) *$K[S]$ is a UFR,*

(2) *$K[G]$ is a UFR and S is a UF-monoid,*

(3) *$\mathcal{U}(S)$ satisfies the ascending chain condition on cyclic and $S/\mathcal{U}(S)$ is an abelian UF-monoid.*

Proof. That (1) and (2) are equivalent is proved similarly as in the proof of Theorem 5.26.

We now prove that (2) and (3) are equivalent. Since $S = N(S)$ we know from Section 5.3.2 and Propos ition 5.11 that S is a UF-monoid if and only if $S/\mathcal{U}(S)$ is abelian UF-monoid, that is, $S/\mathcal{U}(S)$ is a free abelian monoid. Since a free abelian group satisfies the ascending chain condition on cyclic subgroups, G satisfies the ascending chain condition on cyclic subgroups if and only if $\mathcal{U}(S)$ satisfies the ascending chain condition on cyclic subgroups. Hence the result follows. □

The above results relate the unique factorization property of a prime ring R to its cancellative submonoid $N(R)^{\times}$ of nonzero normalizing elements. Also in **[44]** Jordan investigated this relationship. It is shown that this relationship is not as strong as one might hope for. For example, an example is given of a Noetherian prime ring R so that all nonzero ideals contain a nonzero normal element (that is, R is conformal) and $N(R)^{\times}$ is a UF-monoid, however, R is not a UFR.

6.4. Examples

6.4.1. In **[24]** Gateva-Ivanova and Van den Bergh introduced the class of type I monoids. A special subclass is that of the binomial monoids. These are studied in **[33]** and are defined as monoids generated by a finite set $X = \{x_1, \cdots, x_n\}$ subject to precisely $n(n-1)/2$ quadratic relations (one for each $n \geq j > i \geq 1$)

$$x_j x_i = x_{i'} x_{j'},$$

satisfying the following conditions:

B1. $i' < j'$ and $i' < j$;

B2. as we vary (i, j), every pair (i', j') occurs exactly once;

B3. the overlaps $x_k x_j x_i = (x_k x_j) x_i = x_k (x_j x_i)$ do not give rise to new relations in S.

In [**24, 33**] it is shown that the semigroup algebra $K[S]$ shares several properties with commutative polynomial algebras. In particular they are Noetherian PI domains that are a maximal order, and S is UF-monoid. Also G is a finitely generated torsion-free abelian-by-finite group and thus Theorem 6.9 implies at once the following result.

COROLLARY 6.13 (Corollary 3.1 in [**39**]). *Let S be a binomial semigroup and K a field. Let G be the group of quotients of S. Then $K[S]$ is a unique factorization ring provided that $H^1(G/C_G(\Delta(G)), \Delta(G)) = 1$.*

If $G/C_G(\Delta(G))$ is a cyclic group of order n with generator g, then it is well known that (see for example [**20**]) $H^1(G/C_G(\Delta(G)), \Delta(G)) = \Delta(G)_T / I\Delta(G)$ where $\Delta(G)_T$ consists of all elements a of $\Delta(G)$ such that $a \cdot a^g \cdot a^{g^2} \cdots a^{g^{n-1}} = 1$ and $I\Delta(G)$ consists of all elements of form $a^{-1} \cdot a^g$ for any $a \in \Delta(G)$. With this description it is then easily verified that the following three examples of binomial semigroups satisfy the triviality of the mentioned first cohomology group and hence yield examples of PI unique factorisation algebras.

EXAMPLE 6.14. *The monoid algebra of each of the following binomial monoids is a PI Noetherian UFR:*

1. $S_1 =< x_1, x_2, x_3 >$ *subject to the relations* $x_1x_2 = x_2x_1$, $x_3x_1 = x_2x_3$, *and* $x_3x_2 = x_1x_3$; $\Delta(G) =< x_1, x_2, x_3^2 >$.
2. $S_2 =< x_1, x_2, x_3, x_4 >$ *subject to the relations* $x_1x_2 = x_2x_1, x_3x_1 = x_2x_3, x_4x_1 = x_2x_4, x_3x_2 = x_1x_3, x_4x_2 = x_1x_4, x_4x_3 = x_3x_4$; $\Delta(G) =< x_1, x_2, x_3^2, x_4^2, x_3x_4 >$.
3. $S_3 =< x_1, x_2, x_3, x_4 >$ *subject to the relations* $x_2x_1 = x_1x_2, x_3x_1 = x_2x_4, x_4x_1 = x_2x_3, x_3x_2 = x_1x_4, x_4x_2 = x_1x_3, x_4x_3 = x_3x_4$; $\Delta(G) =< x_1^2, x_2^2, x_1x_2, x_3^2, x_4^2, x_3x_4 >$.

6.4.2. Finally we show that via semi-direct products one easily can construct non-Noetherian examples of unique factorization semigroup algebras that are PI. Indeed, let H be a torsion-free abelian group such that $K[H]$ is a UFR but non-Noetherian. Let φ be an automorphism of H of finite order and define the monoid

$$S = H \rtimes_{\varphi} \{z^n \mid n \geq 0\},$$

that is, as a set S is the direct product of the group H and an infinite cyclic monoid, and the product is defined as follows:

$$(h_1z^n)(h_2z^m) = h_1(\varphi(h_2))^n z^{n+m}.$$

It follows that S is a normalizing monoid with a group of quotients G that is torsion-free abelian-by-finite. As S/H is infinite cyclic, it follows that S is a UF-monoid. Since $K[G]$ is a UFR we again get from Theorem 6.9 that K[S] is UFR.

Bibliography

[1] G.Q. Abbasi, S. Kobayashi, H. Marubayashi, and A. Ueda, Non commutative unique factorization ring, Comm. Algebra, 19(1), 167-198(1991).

[2] K. A. Brown, Height one primes of polycyclic group rings, J. London Math. Soc., (2) 32 (1985) 426-438.

[3] K. A. Brown, Corrigendum and Appendum to height one primes of polycyclic group rings, J. London Math. Soc., (2) 38 (1985) 421-422.

[4] K. A. Brown, Class groups and automorphism groups of group rings, Glasgow Math. J. 28 (1986), 79–86.

[5] M. Chamarie, Anneaus de Krull non commutatifs, J. Algebra 72 (1981), 210-222.

[6] M. Chamarie, Anneaus de Krull non commutatifs, Thése, Université Clausle Bernard, Lyon I, 1981.

[7] A. W. Chatters, Non-commutative unique factorization domains, Math. Proc. Camb. Phil. Soc. (1984), 95, 49-54.

[8] A. W. Chatters, Unique factorization in P.I. group-rings, J. Austral.Math. Soc.(Series A)59 (1995), 232-243.

[9] A. W. Chatters and J. Clark, Group rings which are unique factorization rings, Comm. Algebra, 19(2), 585-598, 1991.

[10] A. W. Chatters, M. P. Gilchrist and D. Wilson, Unique factorization rings, Proc. Edinburgh Math. Soc. (2) 35 (1992), 255-269.

[11] A. W. Chatters and C. R. Hajarnavis, Rings with chain conditions, Pitman Publishing Limited, 1980.

[12] A. W. Chatters and D. A. Jordan, Non-commutative unique factorization rings, J. London Math. Soc. (2) 33 (1986), 22-32.

[13] A. H. Clifford and G. B. Preston, "The algebraic theory of semigroups", Vol. I, Amer. Math. Soc., Providence, 1961

[14] P. M. Cohn, "Algebra", vol 3, John Wiley & Sons Ltd, England, 1982.

[15] P. M. Cohn, "Free rings and their relations", Academic Press Inc. Ltd., London, 1985.

[16] C. C. Chen and R. W. Wong, Hereditary monoid rings, Amer J. Math., 104 (1982), pp935-942.

[17] W. Dicks, Hereditary group rings, J. London Math. Soc. 20 (1979), 27-38.

[18] Yu. A. Drozd and V. V. Kirichenko, "Finite dimensional algebras", Springer-Verlag Berlin Heidelberg, 1994.

[19] S. Elenberg, H. Nagao, and T. Nakayama, On the dimension of modules and algebras, IV. Dimension of residue rings of hereditary rings, Nagoya Math. J. 10 (1956), 87-95.

[20] L. Evens, The cohomology of groups, Oxford University Press, New York, 1991.

[21] J. L. Fisher and S. K. Seghal, Principal ideal group rings, Comm. Algebra 4 (1976), 319-325.

[22] M. P. Gilchrist and M. K. Smith, Non-commutative UFD's are often PID's, Math. Proc. Camb. Phil. Soc. (1984), 95, 417-419.

[23] R. Gilmer, "Commutative semigroup algebras", The University of Chicago Press, Chicago and London, 1984.

[24] Gateva-Ivanova T. and Van den Bergh M., Semigroups of I-type, J. Algebra, 206 (1) (1998), 97-112.

[25] E. L. Green, E. Kirkman, and J. Kuzmanovich, Finitistic Dimensions of Finite Dimensional Monomial Algebras, J. Algebra, 136 (1991), 37-50.

[26] J. Goursaud and J. Valette, Anneaux de groupe hereditaries et semihereditaires, J. Algebra 34 (1975), 205-212.

[27] K. R. Gooodearl and R. B. Warfield. "An introduction to non-commutative Noetherian rings", Cambridge University Press 1989.

[28] P. Q. Grillet, "Semigroups", Marcel Dekker, New York, 1995.

[29] E Jespers, de Le Bruyn, and P. Wauters, A characterization of Central Ω-Krull Rings, J. Algebra 81, 165-179 (1983).

[30] E. Jespers, J. Krempa, and E. R. Puczylowski, On radicals of graded rings, Comm. Algebra, 10(17), 1849-1854, 1982.

[31] E. Jespers and J. Okniński, Nilpotent semigroups and semigroup algebras, Journal of Algebra, 169(3)(1994), 984-1011.

[32] E. Jespers and J. Okniński, Semigroup algebras that are principal ideal rings, Journal of Algebra, 183, 837-863(1996).

[33] E. Jespers and J. Okniński, Binomial semigroups, Journal of Algebra, 202(1)(1998), 250-275.

[34] E. Jespers and J. Okniński, Semigroup algebras and maximal orders, Canad. Math. Bull. 42 (1999), no. 3, 298-306.

[35] E. Jespers and J. Okniński, Noetherian semigroup algebras, J. Algebra 218 (1999), no. 2, 543-562.

[36] E. Jespers and J. Okniński, Submonoids of Polycyclic-by-Finite Groups and Their algebras, Algebras and Representation Theory, to appear.

[37] E. Jespers and Q. Wang, Hereditary semigroup algebras, J. Algebra (to appear).

[38] E. Jespers and Q. Wang, Noetherian Unique Factorization Semigroup Algebras, submitted.

[39] E. Jespers and Q. Wang, Height one prime ideals in semigroup algebras satisfying a polynomial identity, submitted.

[40] E. Jespers and P. Wauters, On central Ω-Krull rings and their class groups, Can. J. Math. Vol. XXXVI, No.2. 1984, pp. 206-239.

[41] E. Jespers and P. Wauters, A general notion of non-commutative Krull rings, J. Algebra 112, 388-415 (1988).

[42] E. Jespers and P. Wauters, Principal ideal semigroup algebras, Comm. Algebra 23(1995), 5057-5076.

[43] G. Karpilovsky, "Commutative group algebras", Marcel Dekker, Inc, New York, 1983.

[44] D. A. Jordan, Unique factorization of normal elements in non-commutative rings, Glasgow Math. J. 31 (1989), 103–113.

[45] J. Kuzmanovich and M. L. Teply, Homological dimensions of Semigroup rings, Comm. Algebra 25(9)(1997), 2817-2837.

[46] L. Le Bruyn, M. Van den Bergh and F. Van Oystaeyen, "Graded order", Birkhauser Boston, 1988.

[47] K. W. MacKenzie, Polycyclic group rings and unique factorization rings, Glasgow Math. J. 36(1994), 135-144.

[48] J. C. McConnell and J. C. Robson, "Noncommutative Noetherian rings", Wiley-interscience, New York, 1987.

[49] P. Menal, The monoid rings that are Bezout domains, Arch.Math., Vol. 37,43-47(1981).

[50] W. R. Nico, Homological dimension in semigroup algebras, J. Algebra 18(1971), 404-413.

[51] W. R. Nico, An improvement upper bound for global dimension of semigroup algebras, Proc. Amer. Soc. 35(1972), 34-36.

[52] J. Okniński, "Semigroup algebras", Dekker, New York, 1991.

[53] J. Okniński, "Semigroups of Matrices", World Scientific, Singapore, 1998.

[54] D. S. Passman, "The algebraic structure of group rings", Wiley-interscience, New York, 1977.

[55] D. S. Passman, Observations on group rings, Comm. Algebra, 5 (11), 1119-1162 (1977).

[56] D. S. Passman, "Infinite Crossed Products", Academic Press, New York, 1989.

[57] M. Petrich, "Introduction to semigroups", Bell & Howell Company, Columbus, 1973

[58] L. H. Rowen, "Ring Theory II", Academic Press, London, 1988.

[59] P. Wauters, On some subsemigroups of noncommutative krull rings, Comm. Algebra, 12(14), 1751-1765(1984)

[60] P. Wauters, The semicentre of a group algebra, Proc. Edin. Math. Soc. (1999) 42, 95-111.