GRACEFULLY LABELLED TREES FROM SKOLEM AND RELATED SEQUENCES

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GRACEFULLY LABELLED TREES FROM SKOLEM AND RELATED SEQUENCES

by

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Abstract

In this thesis we use Skolem sequences, hooked Skolem sequences, and periodic odd sequences to find graceful labellings of trees.

Using a particular Skolem sequence of order n we will produce a graceful labelling of a certain tree on $2n$ vertices. Additionally, the following two theorems will be established.

- A Skolem sequence of order $n \equiv 0, 1 \pmod 4$ implies the existence of a graceful tree on $2n$ vertices which has a perfect matching or a matching on $2n - 2$ vertices
- A hooked Skolem sequence of order $n \equiv 2, 3 \pmod 4$ implies the existence of a graceful tree on $2n + 1$ vertices which has a matching on either $2n$ or $2n - 2$ vertices.

The periodic odd sequence will be used to show a particular class of trees to be graceful. Given a tree T, consider one of its longest paths P_T, which is not necessarily unique. We define T to be m-distant if no vertices of T are a distance greater than m away from P_T. We will show that all 3-distant graphs with the following properties are graceful.

(1) They have perfect matchings.
(2) They can be constructed by the attachment of paths of length two to the vertices of a 1-distant tree (caterpillar), by identifying an end vertex of each path with a vertex of the 1-distant tree.

Consequently, all 2-distant trees (lobsters) having perfect matchings are graceful.

Dedication

This thesis is dedicated to my Mother and to the memory of my Father. Although they never really understood any of the mathematics I studied, they always encouraged me to do my best.

Acknowledgements

I would like to begin by thanking my partner Angela French whose confidence in me has been my largest source of motivation and inspiration. Her interest and pride in my work has been a great honour.

I would like to thank my supervisor, Professor Nabil Shalaby, for his commitment and assistance over the past two years. He has taught me many things about mathematical research, the most important of which has been that I still have a lot to learn.

I would also like to thank Professor Rolf Rees who helped me refine raw ideas into solid mathematics. His assistance with the content of Chapter one has been greatly appreciated.

Finally, I would like to thank Professor Alex Rosa and Professor David Pike for reviewing my thesis and providing helpful comments and criticisms. Throughout the course of this thesis they have helped to add precision and clarity to my writing, for which I am grateful.

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Introduction

The Ringel and Kotzig conjectures

Two of the most important conjectures in the study of graph labellings are the conjectures of Ringel [**14**] and Kotzig [**9**].

CONJECTURE 0.1 (Ringel). *The complete graph on $2n+1$ vertices, K_{2n+1}, can be decomposed into $2n+1$ copies of a given tree with n edges.*

CONJECTURE 0.2 (Kotzig/Ringel-Kotzig). *The complete graph on $2n+1$ vertices, K_{2n+1}, can be cyclically decomposed into $2n+1$ copies of a given tree with n edges.*

In 1967, Rosa [**15**] defined several graph labellings to address Kotzig's Conjecture. One of these was the ρ-labelling.

DEFINITION 0.1. *A labelling of a graph G is a mapping $f : V_G \longrightarrow \mathbb{Z}$.*

DEFINITION 0.2. *A ρ-labelling of a graph G is an injective mapping $f : V_G \longrightarrow \{0, 1, \dots, 2|E_G|\}$ such that the associated mapping $g : E_G \longrightarrow \{1, \dots, 2|E_G|\}$ defined by $g(\{u,v\}) \in \{|f(u)-f(v)|, (2|E_G|+1) - |f(u)-f(v)|\}$ is injective and the set of edge values is $\{x_1, \dots, x_{|E_G|}\}$, where $x_i = i$ or $x_i = 2|E_G|+1-i$.*

A ρ-labelling of P_3 is given in Figure 0.1.

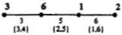


FIGURE 0.1. A ρ-labelling.

In the same work, Rosa proved the following theorem which shows that Kotzig's Conjecture holds if and only if all trees have ρ-labellings

THEOREM 0.3 (Rosa). *The complete graph on $2n+1$ vertices, K_{2n+1}, can be cyclically decomposed into $2n+1$ copies of a given graph with n edges if and only if there exists a ρ-labelling of that graph.*

Another graph labelling defined by Rosa was the β-labelling.

DEFINITION 0.4. *A β-labelling of a graph G is an injective mapping $f : V_G \longrightarrow \{0, 1, \dots, |E_G|\}$ such that the associated mapping $g : E_G \longrightarrow \{1, \dots, |E_G|\}$ defined by $g(\{u,v\}) = |f(u)-f(v)|$ is bijective.*

The β-labelling has since been termed a graceful labelling by Golomb [**7**], where a graph which exhibits a graceful labelling is said to be graceful. A graceful labelling of K_4 is given in Figure 0.2.

From their definitions, one can see that a graceful labelling of a graph is also a ρ-labelling. Using this fact, Rosa then addressed the Kotzig Conjecture from

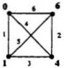


FIGURE 0.2. A graceful labelling of K_4.

the viewpoint of graceful labellings, stating that if a given tree with n edges has a graceful labelling, then K_{2n+1} can be cyclically decomposed into $2n+1$ copies of that tree. This statement prompts the question "Are all trees graceful?".

In the same work, Rosa then provided techniques for gracefully labelling paths and caterpillars, where a caterpillar is a tree containing a path from which all the vertices of the graph are a distance at most one (we will later refer to a path as a 0-distant tree, and a caterpillar as a 1-distant tree). Figures 0.3 and 0.4 give graceful labellings of a path and a caterpillar, respectively.

FIGURE 0.3. A graceful labelling of a path of length five.

FIGURE 0.4. A graceful labelling of a caterpillar.

The gracefulness of lobsters

Since the statement of Kotzig's Conjecture over 40 papers have been written on the gracefulness of trees. In 1979, Bermond [2] conjectured that all lobsters are graceful, where a lobster is a tree containing a path from which all the vertices of the graph are a distance at most two (we will later refer to a lobster as a 2-distant tree). An example of a lobster is given in Figure 0.5.

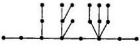


FIGURE 0.5. A lobster.

Most advancements towards verifying Bermond's conjecture consider only very special cases. Ng [12] gives that all lobsters of the forms shown in Figure 0.6 are graceful. In [5], Chen, Lu and Yeh define two families of lobsters, firecrackers and banana trees, as given below (before we define firecrackers and banana trees, we will need the definition of a q-star). In the same work, they show each of these types of lobsters to be graceful, as well as those shown in Figure 0.7. Examples of firecrackers and banana trees can be found in Figures 0.9 and 0.10.

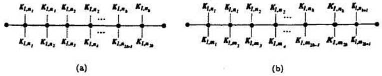


FIGURE 0.6. The graceful lobsters found by Ng [12].

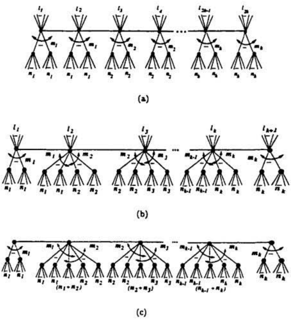


FIGURE 0.7. The graceful lobsters found by Chen, Lu, and Yeh [5].

DEFINITION 0.5. *A q-star on $lq+1$ vertices is the graph formed by adjoining l paths of length $q-1$, each by a leaf, to a single vertex. This single vertex is known as the central vertex of the q-star.*

An example of a 1-star and a 2-star are given in Figure 0.8.

FIGURE 0.8. A 1-star and a 2-star.

DEFINITION 0.6. *A firecracker is a tree consisting of a series of 1-stars adjoined by their central vertices to the vertices of a path, such that each vertex of the path is adjoined to exactly one 1-star.*

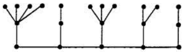


FIGURE 0.9. A firecracker.

DEFINITION 0.7. *A banana tree is a tree consisting of a series of 1-stars, each of which is adjoined by a leaf to a single vertex, such that the i^{th} 1-star has $i-1$ edges.*

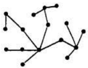


FIGURE 0.10. A banana tree.

Before the publication of [5], Bhat-Nayak and Deshmukh [3] introduced a slightly different definition of banana trees which did not include the condition that the i^{th} star have $i-1$ edges. They show the following three types of these banana trees to be graceful. Note that each banana tree can be described by the adjoined stars.

- $K_{1,1}, K_{1,2}, \dots, K_{1,t-1}, (\alpha+1)K_{1,t}, K_{1,t+1}, \dots, K_{1,n}$, where $\alpha \geq 0$,
- $2K_{1,1}, 2K_{1,2}, \dots, 2K_{1,t-1}, (\alpha+2)K_{1,t}, 2K_{1,t+1}, \dots, 2K_{1,n}$, where $0 \leq \alpha < t$, and
- $3K_{1,1}, 3K_{1,2}, \dots, 3K_{1,n}$.

As the gracefulness of banana trees has been resolved with respect to the definition by Chen, Lu, and Yeh, future research on banana trees will use the definition of Bhat-Nayak and Deshmukh. A survey on graph labellings by Gallian [6] indicates that new results on the gracefulness of banana trees will soon appear. These are attributable to Murugan, Arumugam and Vilfred.

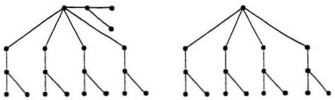
"Graphting"

A 1973 result by Stanton and Zarnke [18] uses graceful trees to create larger graceful trees using a technique called "graphting". Before we consider this process we require the definition of a balanced tree.

DEFINITION 0.8. *A tree T is balanced if there exists a subtree S for which the subgraph on V_T with edge set $E_T \setminus E_S$ is either*

(1) *$|V_S|$ copies of some tree (Type 1), or*
(2) *$|V_S| - 1$ copies of some tree unioned with K_1 (Type 2).*

The definition of a balanced tree can also be viewed as follows. Consider two trees T_1 and T_2, where T_1 and T_2 may be isomorphic. A Type 1 balanced tree is obtained by attaching copies of T_2, by identifying a fixed vertex of T_2, with every vertex of T_1. A Type 2 balanced tree is obtained by attaching copies of T_2, by identifying a fixed vertex of T_2, with all but one vertex of T_1. Examples of balanced trees are given in Figure 0.11.



(a) Type 1 (b) Type 2

FIGURE 0.11. Balanced trees.

The process of "graphting" gives that if T_1 and T_2 are graceful, then any balanced tree created from them is also graceful. Assuming T_1 and T_2 to be graceful, let the number of vertices in T_1 and T_2 be n_{T_1} and n_{T_2}, respectively. As well, let the number of copies of T_2 to be "graphted" to T_1 be λ, where $\lambda = n_{T_1}$ or $\lambda = n_{T_1} - 1$. The "grapht" of T_2 to T_1 can be made graceful by doing the following.

(1) Relabel the vertices of T_1 by multiplying their labels by n_{T_2}. If $\lambda = n_{T_1}$, then a constant c, $0 \leq c < n_{T_2}$, can be added to these new labels. If $\lambda = n_{T_1} - 1$ then no constant can be added. Our example will use $\lambda = n_{T_1} - 1$, and we will call the newly labelled graph T_1'. This is illustrated in Figures 0.12 and 0.13.

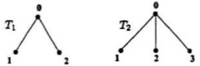


FIGURE 0.12. Two graceful graphs. T_2 is to be "graphted" to T_1.

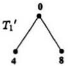


FIGURE 0.13. The relabelling of T_1 where $c = 0$.

(2) Now relabel the copies of T_2 in the following manner. Fix a vertex v of T_2. We label the i^{th} copy of T_2, denoted T_{2_i}, $0 \leq i \leq \lambda - 1$, such that vertices of even distance from v will have value $a + i(n_{T_2})$, and vertices of odd distance will have value $a + (\lambda - 1 - i)(n_{T_2})$, where a was the original value of the vertex in the graceful labelling of T_2. This is illustrated in Figure 0.14.

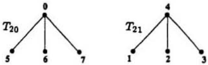


FIGURE 0.14. The relabelling of T_2 where $\lambda = n_{T_1} - 1$.

(3) Then attach the copies of T_2 to T_1' by identifying the vertex labelled α in T_1' with the vertex labelled α in the copy of T_2 in which it exists. This is illustrated in Figure 0.15.

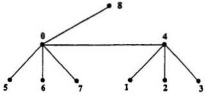


FIGURE 0.15. The graceful "grapht" of T_2 to T_1 using Stanton and Zarnke's technique, where $\lambda = n_{T_1} - 1$.

Skolem sequences

Given that the value of an edge in a graceful labelling is determined by the absolute value of the difference of the labels of its end vertices, it becomes evident that we can use pre-existing mathematical structures based on differences to create graceful labellings. One such mathematical structure is a Skolem sequence, which was defined by Skolem [17] in 1957 to generate Steiner triple systems.

DEFINITION 0.9. *A Skolem sequence of order n is a sequence* $(s_0, s_1, \ldots, s_{2n-1})$, *which has the following properties.*

(1) *Its entries are taken from the set* $\{1, \ldots, n\}$.
(2) $\forall k \in \{1, 2, \ldots, n\}$, *there are exactly two subscripts* $i(k)$ *and* $j(k)$, *for which* $s_{i(k)} = s_{j(k)} = k$.
(3) $\forall k \in \{1, 2, \ldots, n\}$, $|i(k) - j(k)| = k$.

As an example, consider $(6, 1, 1, 2, 8, 2, 6, 7, 3, 4, 5, 3, 8, 4, 7, 5)$ which is a Skolem sequence of order 8. It is common to view a Skolem sequence of order n as a partition of the set $\{0, 1, \ldots, 2n-1\}$ into pairs of the form $\{i(k), j(k)\}, 1 \le k \le n$ such that $j(k) - i(k) = k$. For example, the above Skolem sequence gives the pairs $\{0,6\}, \{1,2\}, \{3,5\}, \{4,12\}, \{7,14\}, \{8,11\}, \{9,13\}$, and $\{10,15\}$.

In the same paper in which he defined Skolem sequences, Skolem established the necessary and sufficient conditions for their existence. For more details on Skolem sequences and their properties see Shalaby [16].

THEOREM 0.10. *There exists a Skolem sequence of order n if and only if* $n \equiv 0, 1 \pmod 4$.

PROOF. (Necessity) Assume there exists a Skolem sequence of order n. Let the pairs determined by this sequence be $\{i(k), j(k)\}, 1 \le k \le n$ as described above and consider the sums

$$A = \sum_{k=1}^{n} [j(k) - i(k)] = \sum_{p=1}^{n} p = \frac{(n)(n+1)}{2}$$

and

$$B = \sum_{k=1}^{n} [j(k) + i(k)] = \sum_{p=0}^{2n-1} p = \frac{(2n-1)(2n)}{2} = (2n-1)(n).$$

Now

$$\begin{aligned} A + B = 2\sum_{k=1}^{n} j(k) &= \frac{n(5n-1)}{2} \\ \Longrightarrow \sum_{k=1}^{n} j(k) &= \frac{n(5n-1)}{4} \\ \Longrightarrow n &\equiv 0, 1 \pmod 4, \end{aligned}$$

since $\sum_{k=1}^{n} j(k)$ is an integer.

(Sufficiency) The sufficiency of $n \equiv 0, 1 \pmod 4$ is proven by constructions of Skolem's. Consider the following pairs for a Skolem sequence of order $4k$, $k \geq 2$.

$$\begin{array}{ll} \{4k+r-1, 8k-r-1\}, & 0 \leq r \leq 2k-1 \\ \{r, 4k-r-3\}, & 0 \leq r \leq k-2 \\ \{k+r+1, 3k-r-2\}, & 0 \leq r \leq k-3 \\ \{2k-1, 4k-2\} & \\ \{2k, 6k-1\} & \\ \{k-1, k\} & \end{array}$$

When $k = 1$ consider the sequence $(1, 1, 4, 2, 3, 2, 4, 3)$.

As well, consider the following pairs for a Skolem sequence of order $4k+1$, $k \geq 2$.

$$\begin{array}{ll} \{4k+r+1, 8k-r+1\}, & 0 \leq r \leq 2k-1 \\ \{r, 4k-r-1\}, & 0 \leq r \leq k-1 \\ \{k+r+2, 3k-r-1\}, & 0 \leq r \leq k-3 \\ \{2k, 6k+1\} & \\ \{2k+1, 4k\} & \\ \{k, k+1\} & \end{array}$$

When $k = 1$ consider the sequence $(1, 1, 5, 2, 4, 2, 3, 5, 4, 3)$ □

Previous work by Abrham [1] has shown relations between certain 2-regular graphs and certain Skolem sequences. Consider a graceful 2-regular graph on $n = 4t$ vertices, which contains only cycles of even length. Abrham's construction gives a Skolem sequence $(s_1, s_2, \ldots, s_{2n+2})$ of order $n+1$ which has the property that if $1 \leq k \leq n$ and if $s_i = s_{i+k} = k$ then either $i + k \leq n+1$ or $i \geq n+2$. Conversely, Skolem sequences of order $4t+1$ exhibiting this property can be used to construct graceful labellings of 2-regular graphs on $n = 4t$ vertices which contain only cycles of even length.

As well, consider a graceful 2-regular graph on $n = 4t-1$ vertices containing exactly one cycle of odd length, where this odd cycle contains the edge of value $\frac{(n+1)}{2}$. Abrham's construction gives a Skolem sequence $(s_1, s_2, \ldots, s_{2n+2})$ of order $n+1$ which has the following properties.

(1) If $1 \leq l \leq n, l \neq \frac{(n+1)}{2}$ and if $s_i = s_{i+l} = l$ then either $i + l \leq n+1$ or $i \geq n+2$.

(2) If $s_i = s_{i+\frac{(n+1)}{2}} = \frac{(n+1)}{2}$ then $i \leq n+1$ and $i + \frac{(n+1)}{2} \geq n+2$.

Conversely, Skolem sequences of order $4t$ exhibiting these properties can be used to construct graceful labellings of 2-regular graphs on $n = 4t-1$ vertices containing exactly one cycle of odd length, where this cycle includes the edge having value $\frac{(n+1)}{2}$.

Abrham's application of certain Skolem sequences to the graceful labelling of 2-regular graphs raises the question of how an arbitrary Skolem sequence can be used to create a graceful graph. In this thesis we will generate a graceful tree using an arbitrary Skolem sequence. As well, we will use a particular Skolem sequence construction to create two families of graceful trees.

Contents of chapters

Chapter one describes an algorithm for creating graceful trees from Skolem and hooked Skolem sequences. To do this one must first consider a bijection between the positions of the sequence and the vertices of a graph, where the position i maps to the vertex labelled i. Given this, a Skolem sequence of order n will then provide n disjoint edges which comprise the edge values $1, 2, \ldots, n$. To create a graceful tree, one need only add additional edges, with appropriate values, while being careful not to create cycles. This is the purpose of the algorithm we will develop. Additionally in Chapter one, Skolem sequences are used to gracefully label two infinite families of trees, where in particular the techniques are open to small variations which can produce additional infinite families of graceful trees.

Chapter two presents a method for gracefully labelling all lobsters with perfect matchings. As in Chapter one, we consider a bijection between the positions in a sequence and the vertices of a graph. Given this, a periodic odd sequence of order n will then provide n disjoint edges which comprise the odd edge values from 1 to $2n-1$. As with Skolem sequences, one can use these edges to create a graceful tree, however the task is made a little simpler as the edge values to be placed are smaller and are all even. Using this we prove that all lobsters with perfect matchings to be graceful as a corollary to slightly larger theorem involving big lobsters, a term to be defined in Chapter two.

Chapter three summarizes the major results of the thesis and suggests directions for future research.

CHAPTER 1

Graceful Trees From Skolem and Hooked Skolem Sequences

1. Graceful trees from Skolem sequences

Given a Skolem sequence of order n, $S = (s_0, s_1, \dots, s_{2n-1})$, we will consider a correlation between sequences and labelled trees, where the i^{th} position corresponds to the vertex labelled i. It should be noted that a Skolem sequence of order n was first viewed by Skolem as a partition P of $\mathbb{Z}_{2n}$ into n subsets of size 2, such that for $\{a_i, b_i\} \in P$,

$$\bigcup_{i=1}^{n} \{|a_i - b_i|\} = \{1, \dots, n\}.$$

As one considers the Skolem pairs $\{a_i, b_i\}$, $1 \leq i \leq n$, it is evident that these represent edges having values $1, 2, \dots, n$, where the edge of value i is incident with the vertices labelled a_i and b_i. These edges constitute a perfect matching of a graph on the vertices $0, 1, \dots, 2n-1$. To create graceful trees we will use these edges and add edges of value $n+1, n+2, \dots, 2n-1$ without creating cycles. Before we show how to do this we require some new definitions and notations which will be useful for our technique of edge addition.

DEFINITION 1.1. *Consider a Skolem sequence of order n, $S = (s_0, s_1, \dots, s_{2n-1})$.*

- *The core of the sequence C_S is the central 2 positions. Namely $n-1$ and n.*
- *Given a vertex labelled i, its complement $\bar{i}$ is the vertex labelled $2n-1-i$.*
- *Given a vertex labelled $i \notin C_S$, the maximal connection of i will be the addition of the edge $\{0, i\}$ if $i > n$, or $\{i, 2n-1\}$ otherwise. Note that for each $i \in \{0, 1, \dots, 2n-1\}$ the maximal connection of i achieves the same edge value as the maximal connection of $\bar{i}$.*
- *Given a position i, its primary follower $f_p(i) = j : s_{\bar{j}} = s_i, i \neq \bar{j}$. In general, the primary followers of i are elements of the form $f_p{}^{(k)}(i)$ where $f_p{}^{(k)}(i) = f_p(f_p{}^{(k-1)}(i))$.*
- *Given a position i, its secondary follower $f_s(i) = j : s_j = s_{\bar{i}}, j \neq \bar{i}$. The secondary followers of i are of the same form as its primary followers where f_p is replaced by f_s.*

We begin by proving the following lemmata.

LEMMA 1.2. *Both f_p and f_s are permutations of $\{0, 1, \dots, 2n-1\}$.*

PROOF. We will show that f_p is such a permutation, where the proof for f_s is analogous and will be left to the reader. Assume that it is not such a permutation and thereby there exists $i, i' \in \{0, 1, \ldots, 2n-1\}$ for which $f_p(i) = f_p(i')$. Hence

$$\begin{aligned} j = j' &: s_{\bar{j}} = s_i, i \neq \bar{j}, s_{\bar{j'}} = s_{i'}, i' \neq \bar{j'} \\ \Longrightarrow s_{\bar{j}} = s_{\bar{j'}} = s_i = s'_i &: i \neq \bar{j} = \bar{j'} \neq i' \\ \Longrightarrow i = i' & \end{aligned}$$

which contradicts our assumption. □

LEMMA 1.3. $f_p = f_s^{-1}$.

PROOF. Let $i \in \{0, 1, \ldots, 2n-1\}$. Now

$$\begin{aligned} f_p(f_s(i)) &= f_p(j) : s_j = s_{\bar{i}}, j \neq \bar{i} \\ &= l : s_{\bar{l}} = s_j, \bar{l} \neq j \\ &= l : s_{\bar{l}} = s_j = s_{\bar{i}}, \bar{i} \neq j \neq \bar{l} \\ &= l : \bar{l} = \bar{i} \\ &= i. \end{aligned}$$

As well,

$$\begin{aligned} f_s(f_p(i)) &= f_s(j) : s_{\bar{j}} = s_i, \bar{j} \neq i \\ &= l : s_l = s_{\bar{j}}, l \neq \bar{j} \\ &= l : s_l = s_{\bar{j}} = s_i, l \neq \bar{j} \neq i \\ &= l : l = i \\ &= i \end{aligned}$$

giving the desired result. □

Now consider the following definition.

DEFINITION 1.4. *Let $O_s(i)$ be the secondary orbit of i defined by f_s. That is*

$$O_s(i) = \left(i \; f_s(i) \; f_s^{(2)}(i) \; \ldots \; f_p(i) = f_s^{(-1)}(i)\right).$$

As well, let

$$\overline{O_s(i)} = \left(\bar{i} \; \overline{f_s(i)} \; \overline{f_s^{(2)}(i)} \; \ldots \; \overline{f_p(i)} = \overline{f_s^{(-1)}(i)}\right).$$

We "trace" $O_s(i)$ by sequentially maximally connecting $f_s(i)$, $f_s^{(2)}(i)$, $\ldots$, $f_p(i)$, and finally i. Similarly, let $O_p(i)$ be the primary orbit of i defined by f_p.

Given this definition, we will establish a series of lemmata mostly involving f_s and O_s, where analogous observations can be made regarding f_p and O_p.

LEMMA 1.5. *For each $i \in \{0, 1, \ldots, 2n-1\}$, $f_p^{(m)}(i) = \overline{f_s^{(m)}(\bar{i})}$.*

PROOF. We will prove this using induction on m, where the result is trivial for $m = 0$. Additionally, $f_p(i) = j : s_{\bar{j}} = s_i, i \neq \bar{j}$, giving that

$$\begin{aligned}\overline{f_s(\bar{i})} &= j' : s_{\bar{j'}} = s_i, \bar{j'} \neq i \\ &= j' : s_{\bar{j'}} = s_i = s_{\bar{j}}, \bar{j'} \neq i \neq \bar{j} \\ &= j' : \bar{j'} = \bar{j} \\ &= j' : j' = j \\ &= f_p(i),\end{aligned}$$

proving the result for $m = 1$.

Assume that for $l < m$, $m \geq 1$, the result holds. Now

$$\begin{aligned}f_p{}^{(m)}(i) &= f_p(f_p{}^{(m-1)}(i)) \\ &= f_p(\overline{f_s{}^{(m-1)}(\bar{i})}) \text{ [by the induction hypothesis]} \\ &= \overline{f_s(f_s{}^{(m-1)}(\bar{i}))} \text{ [by the argument for } m{=}1] \\ &= \overline{f_s{}^{(m)}(\bar{i})}\end{aligned}$$

as desired. □

Lemma 1.5 gives that for any $i \in \{0, 1, \ldots, 2n-1\}$, the maximal connection of the elements of $O_p(i)$ achieves the same edge values as the maximal connection of the elements of $O_s(\bar{i})$.

LEMMA 1.6. *For each* $i \in \{0, 1, \ldots, 2n-1\}, \bar{i} \notin O_s(i)$.

PROOF. We wish to prove that $f_s{}^{(m)}(i) \neq \bar{i}$ for all $m \geq 0$, which we will show by induction on m. When $m = 0$, $i = \bar{i} \Longrightarrow i = 2n - 1 - i \Longrightarrow i \notin \mathbf{Z}$, which is a contradiction. As well, when $m = 1$, $f_s(i) = \bar{i} \Longrightarrow j = \bar{i} : s_j = s_{\bar{i}}, j \neq \bar{i}$, which is also a contradiction. Now, assume that the result holds for $p < m$, for any $m \geq 2$, but that it does not hold for m. That is there exists an i for which $f_s{}^{(m)}(i) = \bar{i}$. Thereby

$$\begin{aligned}f_s{}^{(m-2)}(f_s(i)) &= f_s{}^{(m-1)}(i) \\ &= f_p(f_s{}^{(m)}(i)) \text{ [by Lemma 1.3]} \\ &= f_p(\bar{i}) \\ &= \overline{f_s(i)} \text{ [by Lemma 1.5]}\end{aligned}$$

contradicting the induction hypothesis, so the desired result follows. □

LEMMA 1.7. *For any* $i, j \in \{0, 1, \ldots, 2n-1\}, j = f_s{}^{(m)}(\bar{i}) \Longleftrightarrow i = f_s{}^{(m)}(\bar{j})$.

PROOF. We will prove this by induction on m, where the case $m = 0$ is trivial. Assume that for $l < m$, $m \geq 1$, the result holds. Now

$$\begin{aligned}j = f_s{}^{(m)}(\bar{i}) &\Longleftrightarrow f_p(j) = f_s{}^{(m-1)}(\bar{i}) \text{ [by Lemma 1.3]} \\ &\Longleftrightarrow i = f_s{}^{(m-1)}(\overline{f_p(j)}) \text{ [by the induction hypothesis]} \\ &\Longleftrightarrow i = f_s{}^{(m-1)}(f_s(\bar{j})) \text{ [by Lemma 1.5]} \\ &\Longleftrightarrow i = f_s{}^{(m)}(\bar{j})\end{aligned}$$

as desired. □

LEMMA 1.8. *For each* $i \in \{0,1,\ldots,2n-1\}, O_s(\bar{i}) = \overline{O_s(i)}$.

PROOF. Consider the following.

$$\begin{aligned} j \in O_s(\bar{i}) &\iff j = f_s^{(m)}(\bar{i}) \text{ [for some } m \geq 0] \\ &\iff i = f_s^{(m)}(\bar{j}) \text{ [by Lemma 1.7]} \\ &\iff \bar{j} \in O_s(i) \\ &\iff j \in \overline{O_s(i)} \end{aligned}$$

giving the desired result. □

The consequence of Lemma 1.6 and Lemma 1.8 is that f_s partitions $\{0,1,\ldots,2n-1\}$ into disjoint complementary orbits.

THEOREM 1.9. *The existence of a Skolem sequence of order* n *implies the existence of a graceful tree on* $2n$ *vertices which exhibits a perfect matching or a matching on* $2n-2$ *vertices.*

PROOF. (Algorithmic) Consider a Skolem sequence, $S = (s_0, s_1, \ldots, s_{2n-1})$, and let the i^{th} position corresponds to the vertex labelled i. For all pairs $\{a_i, b_i\}$, where $b_i - a_i = i$, add the edge $\{a_i, b_i\}$. This gives a perfect matching on the vertices labelled $0, \ldots, 2n-1$. This is pictured in Figure 1.1.

We now add edges of value $n+1, \ldots, 2n-1$, without creating any cycles, by performing the following procedures.

(1) Trace $O_s(n)$ until one of the following occurs.
   (a) Position n is reached, in which case do not connect it as it is in the core.
   (b) Position 0 or $2n-1$ is reached. Assume without loss of generality that it is 0. Maximally connect 0, then trace $O_p(\bar{0}) = O_p(2n-1)$ until position $n-1$ is reached. Do not maximally connect it as it is in the core. Note that by Lemma 1.5 these maximal connections achieve the same edge values as the maximal connections of the elements of $O_s(n)$ and that stopping at $n-1$ in $O_p(\bar{0}) = O_p(2n-1)$ is equivalent to stopping at n in $O_s(n)$. As well, note that the vertices adjacent to 0 and $2n-1$ in the perfect matching are not maximally connected.

   This step is shown in Figure 1.2.

   Given this step we now term $O_s(n)$ and $O_s(\bar{n})$ to be "traced". If there are additional untraced secondary orbits proceed as follows.
(2) Choose a position i, for which $O_s(i)$ is untraced. Note that i is not in the core. Trace $O_s(\bar{i})$.

   If $0 \in O_s(\bar{i})$, then Step 1(b) did not occur and vice-versa. Given this, we remove the edge of value l adjacent to $2n-1 = \bar{0}$ which occurred in the perfect matching. The deletion of this edge excludes two vertices from the perfect matching. Note that if $l = n-1$ or $l = n$, then i is in the core, which is a contradiction, so $l < n-1$. This edge length must now be added back in as $\{n, n+l\}$, which can easily be seen not to be a multiple edge. As well, note that the vertex adjacent to 0 in the perfect matching is not maximally connected.

If $0 \notin O_s(\bar{i})$ but rather $2n-1 \in O_s(\bar{i})$, do the same as above, switching 0 with $2n-1$.

(3) If there are any remaining untraced secondary orbits repeat Step 2.

Steps 2 and 3 are shown in Figure 1.3.

To show that the constructed graph is a graceful tree it remains to verify three things.

(1) 0 and $2n-1$ are maximally connected.
This follows from the fact that whenever $O_s(0)$ is traced the algorithm maximally connects either 0 or $2n-1$.

(2) No cycles have been created.
Consider an edge in the perfect matching. The endpoints exist in complementary orbits, so the tracing of secondary orbits guarantees that the only possible violation would be with the edges containing 0 and $2n-1$. We must consider two cases.

(a) **$O_s(0)$ contains a core element.** Without loss of generality let it be n. Step 1(b) will be used, so neither of the vertices adjacent to 0 and $2n-1$ in the perfect matching are maximally connected and no cycles have been created.

(b) **$O_s(0)$ does not contain a core element.** Given this, Step 2 of the algorithm will occur. Consequently the vertex adjacent to 0 in the perfect matching is not maximally connected and the edge incident with $2n-1$ is deleted (or vice-versa). Additionally, there will be no interference from the edge that is added back into the graph, as it is adjacent to the edge containing n in the perfect matching which remained isolated. Hence no cycles are created.

(3) For each $i \in \{1, 2, \dots, n-2\}$ exactly one of i and $\bar{i}$ is maximally connected.
When $O_s(i)$ is traced we also consider $O_s(\bar{i})$ to be traced. This, in combination with Lemma 1.8, Lemma 1.5, and the exclusive tracing of secondary orbits (or the equivalent, in the case of Step 1(b)) ensure the desired property.

Recall that the edges $\{a_i, b_i\}$ determine a perfect matching on the vertices $1, \dots, 2n$. The algorithmic addition of edges removes at most one edge from this perfect matching giving either a perfect matching or a matching on $2n-2$ vertices. □

1.1. Examples of graceful trees constructed from Skolem sequences. Consider the Skolem sequence, $(1,1,8,5,2,6,2,7,5,4,8,6,3,4,7,3)$, which we will use to create a graceful tree. The edges determined by this Skolem sequence are $\{0,1\}$, $\{2,10\}$, $\{3,8\}$, $\{4,6\}$, $\{5,11\}$, $\{7,14\}$, $\{9,13\}$, and $\{12,15\}$, as shown in Figure 1.1.

In this Skolem sequence

$f_p(0) = 14$	$f_p(4) = 9$	$f_p(8) = 12$	$f_p(12) = 0$
$f_p(1) = 15$	$f_p(5) = 4$	$f_p(9) = 2$	$f_p(13) = 6$
$f_p(2) = 5$	$f_p(6) = 11$	$f_p(10) = 13$	$f_p(14) = 8$
$f_p(3) = 7$	$f_p(7) = 1$	$f_p(11) = 10$	$f_p(15) = 3$,

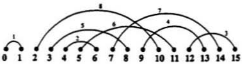


FIGURE 1.1. The perfect matching obtained from the Skolem sequence.

giving the primary orbits

$$O_p(0) = O_p(14) = O_p(8) = O_p(12) = (0\ 14\ 8\ 12)$$
$$O_p(1) = O_p(15) = O_p(3) = O_p(7) = (1\ 15\ 3\ 7)$$
$$O_p(2) = O_p(5) = O_p(4) = O_p(9) = (2\ 5\ 4\ 9)$$
$$O_p(6) = O_p(11) = O_p(10) = O_p(13) = (6\ 11\ 10\ 13).$$

As well,

$f_s(0) = 12$	$f_s(4) = 5$	$f_s(8) = 14$	$f_s(12) = 8$
$f_s(1) = 7$	$f_s(5) = 2$	$f_s(9) = 4$	$f_s(13) = 10$
$f_s(2) = 9$	$f_s(6) = 13$	$f_s(10) = 11$	$f_s(14) = 0$
$f_s(3) = 15$	$f_s(7) = 3$	$f_s(11) = 6$	$f_s(15) = 1,$

giving the primary orbits

$$O_s(0) = O_s(12) = O_s(8) = O_s(14) = (0\ 12\ 8\ 14)$$
$$O_s(1) = O_s(7) = O_s(3) = O_s(15) = (1\ 7\ 3\ 15)$$
$$O_s(2) = O_s(9) = O_s(4) = O_s(5) = (2\ 9\ 4\ 5)$$
$$O_s(6) = O_s(13) = O_s(10) = O_s(11) = (6\ 13\ 10\ 11).$$

In Step 1, tracing $O_s(8) = (0\ 14\ 8\ 12)$ from 8 will add the edge $\{14, 0\}$, then invoke Step 1(b). This adds the edge $\{0, 15\}$, then traces $O_p(\overline{0}) = O_p(\overline{15}) = (1\ 15\ 3\ 7)$. This adds the edge $\{3, 15\}$, then stops as $7 = \overline{8}$ is reached. This is shown in Figure 1.2.

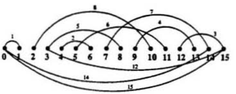


FIGURE 1.2. Step 1 of the algorithm. Note that Step 1(b) has been invoked.

Given this we now consider $O_s(8) = (0\ 12\ 8\ 14)$, and $O_s(\overline{8}) = O_s(7) = (1\ 7\ 3\ 15)$ to be traced. Step 2 can now be invoked as $O_s(2) = (2\ 9\ 4\ 5)$ has not been traced. Tracing $O_s(\overline{2}) = O_s(13) = (6\ 13\ 10\ 11)$ from 13 adds the edges $\{10, 0\}$, $\{11, 0\}$, $\{6, 15\}$, and $\{13, 0\}$, as shown in Figure 1.3. We now consider $O_s(2) = (2\ 9\ 4\ 5)$, and $O_s(\overline{2}) = O_s(13) = (6\ 13\ 10\ 11)$ to be traced. As there are no untraced secondary orbits remaining, we are done. The resulting graceful tree is shown in Figure 1.3.

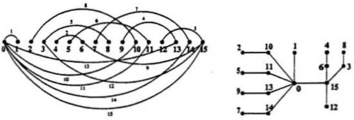


FIGURE 1.3. Steps 2 and 3 of the algorithm. Note that Step 1(b) has been invoked and $O_s(0)$ contains a core element.

In the case where there is a perfect matching, the tree obtained is always a 2-star, having central vertex 0, with the following graphs attached.

- A P_1, by identifying one of its vertices with 0.
- A 2-star, by identifying its central vertex with $2n-1$, which is adjacent to 0.

The number of vertices in each 2-star is dependent on the Skolem sequence chosen. However if the number of vertices in the 2-stars having central vertices 0 and $2n-1$ are $2p+1$ and $2q+1$ respectively, then $p+q=n-1$. As well if $d(i)$ is the degree of vertex i, then $d(0) = p+1$ and $d(2n-1) = q+2$, giving $d(0) + d(2n-1) = p+q+3 = n+2$.

Consider the Skolem sequence, $(3, 6, 7, 3, 1, 1, 8, 6, 5, 7, 2, 4, 2, 5, 8, 4)$, which we will use to create a graceful tree. The edges determined by this Skolem sequence are $\{0, 3\}$, $\{1, 7\}$, $\{2, 9\}$, $\{4, 5\}$, $\{6, 14\}$, $\{8, 13\}$, $\{10, 12\}$, and $\{11, 15\}$, as shown in Figure 1.4.

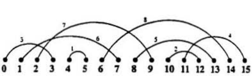


FIGURE 1.4. The perfect matching obtained from the Skolem sequence.

In this Skolem sequence

$$\begin{array}{llll} f_p(0) = 12 & f_p(4) = 10 & f_p(8) = 2 & f_p(12) = 5 \\ f_p(1) = 8 & f_p(5) = 11 & f_p(9) = 13 & f_p(13) = 7 \\ f_p(2) = 6 & f_p(6) = 1 & f_p(10) = 3 & f_p(14) = 9 \\ f_p(3) = 15 & f_p(7) = 14 & f_p(11) = 0 & f_p(15) = 4, \end{array}$$

giving the primary orbits

$$O_p(0) = O_p(12) = O_p(5) = O_p(11) = (0\ 12\ 5\ 11)$$
$$O_p(1) = O_p(8) = O_p(2) = O_p(6) = (1\ 8\ 2\ 6)$$
$$O_p(3) = O_p(15) = O_p(4) = O_p(10) = (3\ 15\ 4\ 10)$$
$$O_p(7) = O_p(14) = O_p(9) = O_p(13) = (7\ 14\ 9\ 13).$$

As well,

$$\begin{array}{llll} f_s(0) = 11 & f_s(4) = 15 & f_s(8) = 1 & f_s(12) = 0 \\ f_s(1) = 6 & f_s(5) = 12 & f_s(9) = 14 & f_s(13) = 9 \\ f_s(2) = 8 & f_s(6) = 2 & f_s(10) = 4 & f_s(14) = 7 \\ f_s(3) = 10 & f_s(7) = 13 & f_s(11) = 5 & f_s(15) = 3, \end{array}$$

giving the primary orbits

$$O_s(0) = O_s(11) = O_s(5) = O_s(12) = (0\ 11\ 5\ 12)$$
$$O_s(1) = O_s(6) = O_s(2) = O_s(8) = (1\ 6\ 2\ 8)$$
$$O_s(3) = O_s(10) = O_s(4) = O_s(15) = (3\ 10\ 4\ 15)$$
$$O_s(7) = O_s(13) = O_s(9) = O_s(14) = (7\ 13\ 9\ 14).$$

In Step 1, tracing $O_s(8) = (1\ 6\ 2\ 8)$ from 8 will add the edges $\{1, 15\}$, $\{6, 15\}$, and $\{2, 15\}$, then invoke Step 1(a). This is shown in Figure 1.5.

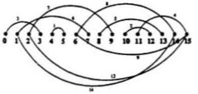


FIGURE 1.5. Step 1 of the algorithm. Note that Step 1(a) has been invoked.

Given this we now consider $O_s(8) = (1\ 6\ 2\ 8)$, and $O_s(\bar{8}) = O_s(7) = (7\ 13\ 9\ 14)$ to be traced. Step 2 can now be invoked as $O_s(3) = (3\ 10\ 4\ 15)$ has not been traced. Tracing $O_s(\bar{3}) = O_s(12) = (0\ 11\ 5\ 12)$ from 12 adds the edges $\{0, 15\}$, $\{11, 0\}$, $\{5, 15\}$, and $\{12, 0\}$, however since $0 \in O_s(12)$, we must remove the edge $\{11, 15\}$ from the original perfect matching and add the edge $\{8, 12\}$. We now consider $O_s(3) = (3\ 10\ 4\ 15)$, and $O_s(\bar{3}) = O_s(12) = (0\ 11\ 5\ 2)$ to be traced.

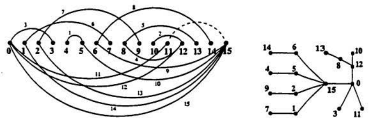


FIGURE 1.6. Obtaining a graceful tree with a perfect matching on $2n-2$ vertices.

As there are no untraced secondary orbits remaining, we are done. The resulting graceful tree is shown in Figure 1.6.

In the case where there is a matching on only $2n-2$ vertices consider the vertex $v \in \{0, 2n-1\}$ from which the edge in the perfect matching is deleted. Again, let this edge have value $l < n-1$. Additionally let $\overline{v}$ be the other vertex. The tree obtained is always a 2-star, having central vertex v, with the following attached.

- A 2-star, by identifying its central vertex with $\overline{v}$, which is adjacent to v.
- A P_1, by identifying one of its vertices with $\overline{v}$.
- A P_2, by identifying one of its vertices with $n + l \neq 2n - 1$.

The number of vertices in each 2-star is dependent on the Skolem sequence chosen. However if the number of vertices in the 2-stars having central vertices v and $\overline{v}$ are $2p+1$ and $2q+1$, then $p+q=n-2$. As well $d(v)=p$ and $d(v')=q+3$, giving $d(0)+d(2n-1)=d(v)+d(v')=p+(q+3)=n+1$.

2. Graceful trees from hooked Skolem sequences

DEFINITION 1.10. *A hooked Skolem sequence of order* n *is a sequence*

$$(s_0, s_1, \dots, s_{2n-1}, s_{2n}),$$

which has the following properties.

(1) *Its entries are taken from the set* $\{0, 1, \dots, n\}$.
(2) $\forall k \in \{1, 2, \dots, n\}$, *there are exactly two subscripts* $i(k)$ *and* $j(k)$ *for which* $s_{i(k)} = s_{j(k)} = k$.
(3) $\forall k \in \{1, 2, \dots, n\}$, $|i(k) - j(k)| = k$.
(4) $s_{2n-1} = 0$. *This zero is referred to as a "hook".*

As an example of a hooked Skolem sequence consider $(6,1,1,4,5,3,6,4,3,5,2,0,2)$ which is of order 6. The notion of a hooked Skolem sequence was developed by O'Keefe [13] who found the necessary and sufficient conditions for the existence of a hooked Skolem sequence of order n to be $n \equiv 2, 3 \pmod 4$. For more detail on hooked Skolem sequences and their properties see Shalaby [16].

As with Skolem sequences, the partition determined by a hooked Skolem sequence gives a matching on $2n$ of the $2n+1$ vertices, leaving the vertex labelled $2n-1$ isolated. To create graceful trees from these hooked Skolem sequences we will use these edges then add edges of values $n+1, n+2, \ldots, 2n-1, 2n$ while not creating cycles.

The creation of graceful trees from hooked Skolem sequences is analogous to the creation using Skolem sequences, however some of the lemmata regarding orbits must be modified. The majority of definitions and lemmata will see only minor changes.

DEFINITION 1.11. *Consider a hooked Skolem sequence $S = (s_0, s_1, \ldots, s_{2n-1}, s_{2n})$, which is of order n.*

- *The core of the sequence C_S is the central position, namely n.*
- *Given a vertex labelled i its complement $\bar{i}$ is the vertex labelled $2n - i$.*
- *Given a vertex labelled $i \notin C_S$, the maximal connection of i will be the addition of the edge $\{0, i\}$ if $i > n$, or $\{i, 2n\}$ otherwise. Note that for each $i \in \{0, 1, \ldots, 2n\}$, the maximal connection of i achieves the same edge value as the maximal connection of $\bar{i}$.*
- *Given a position i for which $s_i \neq 0$, its primary follower $f_p(i) = j : s_{\bar{j}} = s_i, i \neq \bar{j}$. If $s_i = 0$, then $f_p(i) = \bar{i}$. In general, the primary followers of i are elements of the form $f_p{}^{(k)}(i)$ where $f_p{}^{(k)}(i) = f_p(f_p{}^{(k-1)}(i))$.*
- *Given a position i for which $s_{\bar{i}} \neq 0$, its secondary follower $f_s(i) = j : s_j = s_{\bar{i}}, j \neq \bar{i}$. If $s_{\bar{i}} = 0$, then $f_s(i) = \bar{i}$. The secondary followers of i are of the same form as its primary followers where f_p is replaced by f_s.*

Similar observations regarding f_p and f_s as those made for Skolem sequences will be made for hooked Skolem sequences. We will leave most of them unproven as the only additional verification required involves the case when either $s_i = 0$ or $s_{\bar{i}} = 0$.

LEMMA 1.12. *The functions f_p and f_s are permutations of $\{0, 1, \ldots, 2n\}$.*

LEMMA 1.13. $f_p = f_s{}^{-1}$.

LEMMA 1.14. *For each $i \in \{0, 1, \ldots, 2n\}, f_p{}^{(m)}(i) = \overline{f_s{}^{(m)}(\bar{i})}$.*

Lemma 1.14 gives that for any $i \in \{0, 1, \ldots, 2n\}$, the maximal connection of the elements of $O_p(i)$ achieves the same edge values as the maximal connection of the elements of $O_s(\bar{i})$.

Now consider the following definition.

DEFINITION 1.15. *The secondary orbit and primary orbit which contain the hook position are called the hooked secondary orbit and hooked primary orbit respectively. We will denote them by H_s and H_p.*

LEMMA 1.16. *For any $i, j \in \{0, 1, \ldots, 2n\}, j = f_s{}^{(m)}(\bar{i}) \iff i = f_s{}^{(m)}(\bar{j})$.*

LEMMA 1.17. *For each $i \in \{0, 1, \ldots, 2n\}, O_s(\bar{i}) = \overline{O_s(i)}$.*

LEMMA 1.18. *For each $i \in \{0, 1, \ldots, 2n\}, \bar{i} \in O_s(i) \iff i \in O_s(n)$.*

PROOF. ($\Longleftarrow$)

$$\begin{aligned} i \in O_s(n) &\Longleftrightarrow \exists k : i = f_s^{(k)}(n) \\ &\Longleftrightarrow f_p^{(k)}(i) = n \text{ [by Lemma 1.13]} \\ &\Longleftrightarrow \overline{f_s^{(k)}(\bar{i})} = n \text{ [by Lemma 1.14]} \\ &\Longleftrightarrow f_s^{(k)}(\bar{i}) = n \\ &\Longleftrightarrow \bar{i} \in O_s(n) \\ &\Longrightarrow \bar{i} \in O_s(i). \end{aligned}$$

($\Longrightarrow$) Consider any secondary orbit for which $\exists i : \bar{i} \in O_s(i)$. If $j \in O_s(i)$, then by Lemma 1.17, $\bar{j} \in O_s(i)$. That is to say there is only one such orbit of odd length, namely the orbit for which $\exists k : \bar{k} = k$, which is $O_s(n)$.

Now consider a secondary orbit $O_s(i)$ for which $\bar{i} \in O_s(i)$ but $O_s(i) \neq O_s(n)$.

$$\begin{aligned} \bar{i} = f_s^{(2l)}(i) &\Longleftrightarrow f_p^{(l)}(\bar{i}) = f_s^{(l)}(i) \text{ [by Lemma 1.13]} \\ &\Longleftrightarrow \overline{f_s^{(l)}(i)} = f_s^{(l)}(i) \text{ [by Lemma 1.14]} \\ &\Longleftrightarrow f_s^{(l)}(i) = n, \end{aligned}$$

which is a contradiction. Thereby

$$\begin{aligned} \bar{i} = f_s^{(2l+1)}(i) &\Longleftrightarrow f_p^{(l)}(\bar{i}) = f_s^{(l+1)}(i) \text{ [by Lemma 1.13]} \\ &\Longleftrightarrow \overline{f_s^{(l)}(i)} = f_s[f_s^{(l)}(i)] \text{ [by Lemma 1.14]} \\ &\Longleftrightarrow f_s^{(l)}(i) = 1. \end{aligned}$$

That is $1 \in O_s(i)$. Yet $O_s(n)$ is an orbit of odd length, giving $n = f_s^{(2k+1)}(n) \Longrightarrow 1 \in O_s(n)$. This gives $n \in O_s(i)$, which is a contradiction. Hence $\bar{i} \in O_s(i) \Longrightarrow i \in O_s(n)$. □

An immediate consequence of Lemma 1.18 is that $O_s(n) = H_s$ as $2n - 1 = \bar{1} \in O_s(1)$. As well, Lemma 1.18 and Lemma 1.17 give that f_s partitions $\{0, 1, \dots, 2n\}$ into disjoint complementary orbits, with the noticeable exception of $O_s(n) = O_s(\bar{n}) = \overline{O_s(n)} = H_s$.

THEOREM 1.19. *The existence of a hooked Skolem sequence of order n implies the existence of a graceful tree on $2n+1$ vertices which exhibits a matching on either $2n$ or $2n-2$ vertices.*

PROOF. (Algorithmic) Consider a hooked Skolem sequence $S = (s_0, s_1, \dots, s_{2n})$, and let the i^{th} position correspond to the vertex labelled i. For all pairs $\{a_i, b_i\}$, where $b_i - a_i = i$, add the edge $\{a_i, b_i\}$. This gives a perfect matching on the vertices labelled $\{0, \dots, 2n-2, 2n\}$. This is pictured in Figure 1.7.

We now add edges of value $n+1, \dots, 2n$ without creating any cycles by performing the following procedures.

(1) Trace $O_s(n)$ until one of the following occurs.
   (a) Position 1 is reached, in which case connect it. Note that position $2n-1$ remains isolated.
   (b) Position 0 or $2n-1$ is reached. Assume without loss of generality that it is 0. Maximally connect 0, then trace $O_p(\bar{0}) = O_p(2n)$ until

position $2n-1$ is reached. Maximally connect it, as position 1 has not yet been maximally connected. Note that by Lemma 1.5 these maximal connections achieve the same edge values as the maximal connections of the elements of $O_s(n)$ and that stopping at $2n-1$ in $O_p(\bar{0}) = O_p(2n)$ is equivalent to stopping at 1 in $O_s(n)$. As well, note that the vertices adjacent to 0 and $2n$ in the perfect matching are not maximally connected.

This step is shown in Figure 1.8.

Given this step we now term $O_s(n) = O_s(\bar{n})$ to be "traced". If there are additional untraced secondary orbits proceed as follows.

(2) Choose a position i, for which $O_s(i)$ is untraced. Note that i is not in the core. Trace $O_s(\bar{i})$.

If $0 \in O_s(\bar{i})$, then Step 1(b) did not occur and vice-versa. Given this we remove the edge of value l adjacent to $2n = \bar{0}$ which occurred in the perfect matching. The deletion of this edge excludes two vertices from the perfect matching. Note that if $l = n$, then i is in the core, which is a contradiction, so $l < n$. This edge value must now be added back in as $\{2n-1, 2n-1-l\}$, which cannot be a multiple edge as $2n-1$ is isolated. As well, note that the vertex adjacent to 0 in the perfect matching is not maximally connected.

If $0 \notin O_s(\bar{i})$ but rather $2n \in O_s(\bar{i})$, do the same as above, switching 0 with $2n$. In this context we consider both $O_s(i)$ and $O_s(\bar{i})$ to have been traced.

(3) If there are any remaining untraced secondary orbits repeat Step 2.

Steps 2 and 3 are shown in Figure 1.9.

To show that the constructed graph is a graceful tree it remains to verify three things. The first two follow from analogous reasoning to that used for Skolem sequences.

(1) 0 and $2n-1$ are maximally connected.
(2) No cycles have been created.
(3) For each $i = 1, 2, \ldots, n-1$, exactly one of i and $\bar{i}$ is maximally connected.
We must consider two cases.
 (a) $\mathbf{i \notin O_s(n)}$. When $O_s(i)$ is traced we also consider $O_s(\bar{i})$ to be traced. This fact, in combination with Lemma 1.17, Lemma 1.14, and the exclusive tracing of secondary orbits (or the equivalent, in the case of Step 1(b)), ensure the desired property.
 (b) $\mathbf{i \in O_s(n)}$. Let l be the smallest natural number for which $\bar{i} = f_s^{(l)}(i)$ and let k be the smallest natural number for which $i = f_s^{(k)}(\bar{i})$.
If $l \equiv 0 \pmod 2$, then by the argument in Lemma 1.14 n is between i and $\bar{i}$ in $O_s(n)$. Also, since $O_s(n)$ is of odd length we have that $k \equiv 1 \pmod 2$ and, by the argument in Lemma 1.14, 1 is between $\bar{i}$ and i in $O_s(n)$. That is $O_s(n)$ is of the form

$$(\ldots i \ldots n \ldots \bar{i} \ldots 1 \ldots).$$

But $O_s(n)$ is traced from n until 1 is reached. This fact, in combination with Lemma 1.17, Lemma 1.14, and the exclusive tracing of secondary orbits (or the equivalent, in the case of step 1(b)), ensure the desired property.

If $l \equiv 1 \pmod 2$, then $O_s(n)$ is of the form

$$(\ldots \bar{i} \ldots n \ldots i \ldots 1 \ldots),$$

and the same argument ensures that the desired property holds.

Recall that the edges $\{a_i, b_i\}$ determine a matching on the vertices $\{0, 1, \ldots, 2n-2, 2n-1\}$. The algorithmic addition of edges removes at most one edge from this matching giving a matching on either $2n$ or $2n-2$ vertices. ◻

2.1. Examples of graceful trees from hooked Skolem sequences. Consider the hooked Skolem sequence $(6, 1, 1, 4, 5, 3, 6, 4, 3, 5, 2, 0, 2)$, which we will use to create a graceful tree. The edges determined by this hooked Skolem sequence are $\{0, 6\}$, $\{1, 2\}$, $\{3, 7\}$, $\{4, 9\}$, $\{5, 8\}$, and $\{10, 12\}$, as shown in Figure 1.7.

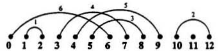


FIGURE 1.7. The matching obtained from the hooked Skolem sequence.

For this hooked Skolem sequence

$f_p(0) = 6$	$f_p(3) = 5$	$f_p(6) = 12$	$f_p(9) = 8$
$f_p(1) = 10$	$f_p(4) = 3$	$f_p(7) = 9$	$f_p(10) = 0$
$f_p(2) = 11$	$f_p(5) = 4$	$f_p(8) = 7$	$f_p(11) = 1$
			$f_p(12) = 2,$

giving the primary orbits (0 6 12 2 11 1 10), (3 5 4), and (7 9 8). The hooked primary orbit H_p, is (0 6 12 2 11 1 10).

As well,

$f_s(0) = 10$	$f_s(3) = 4$	$f_s(6) = 0$	$f_s(9) = 7$
$f_s(1) = 11$	$f_s(4) = 5$	$f_s(7) = 8$	$f_s(10) = 1$
$f_s(2) = 12$	$f_s(5) = 3$	$f_s(8) = 9$	$f_s(11) = 2$
			$f_s(12) = 6,$

giving the secondary orbits (0 10 1 11 2 12 6), (3 4 5), and (7 8 9). The hooked secondary orbit H_s, is (0 10 1 11 2 12 6).

In Step 1, tracing $H_s = O_s(6) = (0\ 10\ 1\ 11\ 2\ 12\ 6)$ from 6 will invoke Step 1(b). We then add the edge $\{0, 12\}$, then trace $O_p(\bar{0}) = O_p(12) = (0\ 6\ 12\ 2\ 11\ 1\ 10)$ from 12. This adds the edges $\{2, 12\}$, and $\{11, 0\}$, then stops as 11 has been reached. This is shown in Figure 1.8.

Given this we now consider $H_s = O_s(6) = O_s(\bar{6}) = (0\ 10\ 1\ 11\ 2\ 12\ 6)$ to be traced. Step 2 can now be invoked as $O_s(3) = (3\ 4\ 5)$ has not been traced. Tracing $O_s(\bar{3}) = O_s(9) = (7\ 8\ 9)$ from 9 adds the edges $\{7, 0\}$, $\{8, 0\}$, and $\{9, 0\}$, as shown in Figure 1.9. We now consider $O_s(3) = (3\ 4\ 5)$ and $O_s(\bar{3}) = O_s(9) = (7\ 8\ 9)$ to be traced. As there are no remaining untraced secondary orbits, we are done. The resulting graceful tree is shown in Figure 1.9.

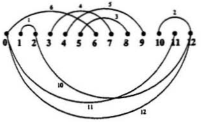


FIGURE 1.8. Step 1 of the algorithm. Note that Step 1(b) has been invoked.

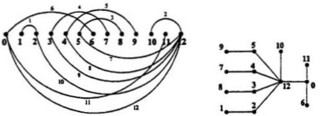


FIGURE 1.9. Steps 2 and 3 of the algorithm. Note that Step 1(b) has been invoked and $O_s(0)$ contains a core element.

In the case where there is a matching on $2n$ vertices the tree obtained is always a 2-star, having central vertex 0, with the following graphs attached.

- Two paths of length one, by identifying one vertex of each with 0.
- A 2-star, by identifying its central vertex with $2n$, which is adjacent to 0.

The number of vertices in each 2-star is dependent on the hooked Skolem sequence chosen. However if the number of vertices in the 2-stars having central vertices 0 and $2n$ are $2p+1$ and $2q+1$ respectively, then $p+q = n-1$. As well, if $d(i)$ is the degree of vertex i, then $d(0) = p+2$ and $d(2n) = q+2$, giving $d(0)+d(2n) = p+q+4 = n+3$.

Consider the hooked Skolem sequence $(2,5,2,6,1,1,5,3,4,6,3,0,4)$, which we will use to create a graceful tree. The edges determined by this hooked Skolem sequence are $\{0,2\}$, $\{1,6\}$, $\{3,9\}$, $\{4,5\}$, $\{7,10\}$, and $\{8,12\}$, as shown in Figure 1.10.

FIGURE 1.10. The matching obtained from the hooked Skolem sequence.

For this hooked Skolem sequence

$f_p(0) = 10$	$f_p(3) = 3$	$f_p(6) = 11$	$f_p(9) = 9$
$f_p(1) = 6$	$f_p(4) = 7$	$f_p(7) = 2$	$f_p(10) = 5$
$f_p(2) = 12$	$f_p(5) = 8$	$f_p(8) = 0$	$f_p(11) = 1$
			$f_p(12) = 4,$

giving the primary orbits (0 10 5 8), (1 6 11), (2 12 4 7), (3), and (9). The hooked primary orbit H_p, is (1 6 11).

As well,

$f_s(0) = 8$	$f_s(3) = 3$	$f_s(6) = 1$	$f_s(9) = 9$
$f_s(1) = 11$	$f_s(4) = 12$	$f_s(7) = 4$	$f_s(10) = 0$
$f_s(2) = 7$	$f_s(5) = 10$	$f_s(8) = 5$	$f_s(11) = 6$
			$f_s(12) = 2,$

giving the secondary orbits (0 8 5 10), (1 11 6), (2 7 4 12), (3), and (9). The hooked secondary orbit H_s, is (1 11 6).

In Step 1, tracing $H_s = O_s(6) = (1\ 11\ 6)$ from 6 will reach 1 and then invoke Step 1(a). This adds the edge $\{1, 12\}$, which is shown in Figure 1.11.

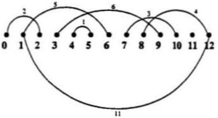


FIGURE 1.11. Step 1 of the algorithm. Note that Step 1(a) has been invoked.

Given this we now consider $H_s = O_s(6) = O_s(\bar{6}) = (1\ 11\ 6)$ to be traced. Step 2 can now be invoked as $O_s(2) = (2\ 7\ 4\ 12)$ has not been traced. Tracing $O_s(\bar{2}) = O_s(10) = (0\ 8\ 5\ 10)$ from 10 adds the edges $\{0, 12\}$, $\{8, 0\}$, and $\{5, 12\}$, however since $0 \in O_s(\bar{2})$ we must delete the edge $\{8, 12\}$ and then add the edge $\{11, 7\}$ as shown in Figure 1.12.

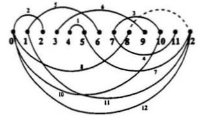


FIGURE 1.12. Step 2 of the algorithm.

We now consider $O_s(2) = (2\ 7\ 4\ 12)$ and $O_s(\overline{2}) = O_s(10) = (0\ 8\ 5\ 10)$ to be traced. As $O_s(3) = (3)$ has not yet been traced so Step 3 can be invoked. Tracing $O_s(\overline{3}) = O_s(9) = (9)$ will add the edge $\{9, 0\}$, at which point we consider $O_s(3) = (3)$ and $O_s(\overline{3}) = O_s(9) = (9)$ traced. As there are no remaining untraced secondary orbits, we are done. The resulting graceful tree is shown in Figure 1.13.

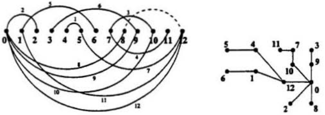


FIGURE 1.13. Obtaining a graceful tree with a perfect matching on $2n-2$ vertices

In the case where there is a matching on only $2n-2$ vertices consider the vertex $v \in \{0, 2n\}$ from which the edge in the perfect matching is deleted. Again, let this edge have value $l < n$. The tree obtained is always a 2-star, having central vertex v, with the following attached.

- A 2-star, by identifying its central vertex with $\overline{v}$, which is adjacent to v.
- A P_1, by identifying one of its vertices with $\overline{v}$.
- A P_1, by identifying one of its vertices with $2n-1-l$.

The number of vertices in each 2-star is dependent on the Skolem sequence chosen. However if the number of vertices in the 2-stars having central vertices v and $\overline{v}$ are $2p+1$ and $2q+1$, then $p+q = n-1$. As well, $d(v) = p$ and $d(v') = q+3$, giving $d(0) + d(2n) = d(v) + d(v') = p + (q+3) = n+2$.

It should be noted that a graceful tree on $2n$ vertices, $n \equiv 0, 1 \pmod 4$, does not necessarily give yield to a Skolem sequence of order n as the perfect matching may not contain the edges of value $\{1, \ldots, n\}$. Moreover, there may be no perfect matching at all. The same can be said for graceful trees on $2n+1$ vertices with regards to hooked Skolem sequences.

3. Specific constructions of graceful trees from Skolem sequences

We will now construct graceful trees from Skolem sequences. Recall that the necessary and sufficient condition for the existence of a Skolem sequence of order n is that $n \equiv 0, 1 \pmod 4$.

3.1. Skolem sequences of order $4k, k \geq 3$. Consider the following construction of Skolem pairs as edges of a tree on $8k$ vertices.

Edges		Edge Values	
$\{8k-2, 8k-1\}$		1	
$\{6k-r-2, 6k+r-1\}$	$1 \leq r \leq k-1$	$2r+1$	$1 \leq r \leq k-1$
$\{k-r-1, 3k+r-2\}$	$1 \leq r \leq k-3$	$2k+2r-1$	$1 \leq r \leq k-3$
$\{1, 4k-4\}$		$4k-5$	
$\{0, 4k-3\}$		$4k-3$	
$\{k-1, 5k-2\}$		$4k-1$	
$\{2k-r-1, 2k+r-1\}$	$1 \leq r \leq k-1$	$2r$	$1 \leq r \leq k-1$
$\{4k-2, 6k-2\}$		$2k$	
$\{5k-r-2, 7k+r-2\}$	$1 \leq r \leq k-1$	$2k+2r$	$1 \leq r \leq k-1$
$\{2k-1, 6k-1\}$		$4k$	

These $4k$ edges are disjoint having values $1, 2, \ldots, 4k$. To create a graceful tree we must now add in the edges of values $4k+1, \ldots, 8k-1$ while being careful not to create cycles. The additional edges given below will create an infinite class of graceful trees.

Edges		Edge Values	
$\{0, 8k-1\}$		$8k-1$	
$\{1, 8k-1\}$		$8k-2$	
$\{0, 6k+r-1\}$	$1 \leq r \leq k-1$	$6k+r-1$	$1 \leq r \leq k-1$
$\{0, 7k+r-2\}$	$1 \leq r \leq k-1$	$7k+r-2$	$1 \leq r \leq k-1$
$\{2k+r-1, 8k-1\}$	$1 \leq r \leq k-1$	$6k-r$	$1 \leq r \leq k-1$
$\{3k+r-2, 8k-1\}$	$1 \leq r \leq k-3$	$5k-r+1$	$1 \leq r \leq k-3$
$\{k-3, 5k-2\}$		$4k+1$	
$\{2k-4, 6k-2\}$		$4k+2$	
$\{2k-4, 6k-1\}$		$4k+3$	

For $k \geq 5$ these additional edges generate a graceful labelling of the 2-star on $4k-3$ vertices onto which are attached four additional graphs. This 2-star has central vertex $8k-1$ and edges

$$\{8k-1, 2k+r\}, \{2k+r-1, 2k-r-1\}, 1 \leq r \leq k-1$$
$$\{8k-1, 3k+r-2\}, \{3k+r-2, k-r-1\}, 1 \leq r \leq k-3$$
$$\{8k-1, 0\}, \{0, 4k-3\}$$
$$\{8k-1, 1\}, \{1, 4k-4\}.$$

The attached graphs are as follows.

- A P_1, by identifying one of its vertices with the central vertex $8k-1$. This edge is $\{8k-2, 8k-1\}$.

- A 2-star on $4k - 3$ vertices, by identifying its central vertex with 0. Its edges are

$$\{0, 6k + r - 1\}, \{6k + r - 1, 6k - r - 2\}, 1 \leq r \leq k - 1$$
$$\{0, 7k + r - 2\}, \{7k + r - 2, 5k - r - 2\}, 1 \leq r \leq k - 1.$$

- A P_4, by identifying its central vertex with $2k - 4$. Its edges are

$$\{2k - 4, 6k - 2\}, \{6k - 2, 4k - 2\}$$
$$\{2k - 4, 6k - 1\}, \{6k - 1, 2k - 1\}.$$

- A P_2, by identifying an end vertex with $k - 3$. Its edges are

$$\{k - 3, 5k - 2\}, \{5k - 2, k - 1\}.$$

An example of this construction is provided in Figure 1.14. For $3 \leq k < 5$ this construction also generates graceful trees, however the P_2 and the P_4 do not attach in the above described positions.

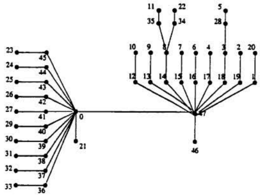


FIGURE 1.14. The graceful tree obtained from the described construction for order $4k$, where $k = 6$.

3.2. Skolem sequences of order $4k+1, k \geq 4$. Consider the following construction of Skolem pairs as edges of a tree on $8k+2$ vertices.

Edges		Edge Values	
$\{8k, 8k+1\}$		1	
$\{6k-r-2, 6k+r-1\}$	$1 \leq r \leq k-3$	$2r+1$	$1 \leq r \leq k-3$
$\{5k, 7k-3\}$		$2k-3$	
$\{5k-1, 7k-2\}$		$2k-1$	
$\{k-r-1, 3k+r-2\}$	$1 \leq r \leq k-3$	$2k+2r-1$	$1 \leq r \leq k-3$
$\{1, 4k-4\}$		$4k-5$	
$\{0, 4k-3\}$		$4k-3$	
$\{k-1, 5k-2\}$		$4k-1$	
$\{4k-2, 8k-1\}$		$4k+1$	
$\{2k-r-1, 2k+r-1\}$	$1 \leq r \leq k-1$	$2r$	$1 \leq r \leq k-1$
$\{5k-r-2, 7k+r-2\}$	$1 \leq r \leq k-1$	$2k+2r$	$1 \leq r \leq k-1$
$\{2k-1, 6k-1\}$		$4k$	
$\{6k-2, 8k-2\}$		$2k$	

These $4k+1$ edges are disjoint having values $1, 2, \ldots, 4k+1$. To construct a graceful tree we must now add in the edges of values $4k+2, \ldots, 8k+1$ while being careful not to create cycles. The additional edges given below will create an infinite class of graceful trees.

Edges		Edge Values	
$\{0, 8k+1\}$		$8k+1$	
$\{1, 8k+1\}$		$8k$	
$\{0, 8k-1\}$		$8k-1$	
$\{0, 8k-2\}$		$8k-2$	
$\{0, 7k+r-2\}$	$1 \leq r \leq k-1$	$7k+r-2$	$1 \leq r \leq k-1$
$\{0, 6k+r+1\}$	$1 \leq r \leq k-3$	$6k+r+1$	$1 \leq r \leq k-3$
$\{2k+r-1, 8k+1\}$	$1 \leq r \leq k-1$	$6k-r+2$	$1 \leq r \leq k-1$
$\{3k+r-2, 8k+1\}$	$1 \leq r \leq k-3$	$5k-r+3$	$1 \leq r \leq k-3$
$\{2k-4, 6k-1\}$		$4k+3$	
$\{2k-4, 6k\}$		$4k+4$	
$\{2k-4, 6k+1\}$		$4k+5$	
$\{k-4, 5k-2\}$		$4k+2$	

For $k \geq 6$ these additional edges generate a graceful labelling of the 2-star on $4k-3$ vertices onto which are attached four additional graphs. This 2-star has central vertex $8k+1$ and edges given by

$$\begin{aligned}
&\{8k+1, 2k+r-1\}, \{2k+r-1, 2k-r-1\}, 1 \leq r \leq k-1\\
&\{8k+1, 3k+r-2\}, \{3k+r-2, k-r-1\}, 1 \leq r \leq k-3\\
&\{8k+1, 1\}, \{1, 4k-4\}\\
&\{8k+1, 0\}, \{0, 4k-3\}.
\end{aligned}$$

The attached graphs are as follows.

- A P_1, by identifying one of its vertices with the central vertex $8k+1$. This edge will be $\{8k, 8k+1\}$.

- A 2-star on $4k-3$ vertices, by identifying its central vertex with 0. Its edges are

$$\begin{array}{l}\{0,6k+r+1\},\{6k+r+1,6k-r-2\},1\leq r\leq k-3\\ \{0,7k+r-2\},\{7k+r-2,5k-r-2\},1\leq r\leq k-1\\ \{0,8k-1\},\{8k-1,4k-2\}\\ \{0,8k-2\},\{8k-2,6k-2\}.\end{array}$$

- A 2-star on 7 vertices, by identifying its central vertex with $2k-4$. Its edges are

$$\begin{array}{l}\{2k-4,6k-1\},\{6k-1,2k-1\}\\ \{2k-4,6k\},\{6k,6k-3\}\\ \{2k-4,6k+1\},\{6k+1,6k-4\}.\end{array}$$

- A P_2, by identifying an end vertex with $k-4$. Its edges are

$$\{k-4,5k-2\},\{5k-2,k-1\}.$$

An example of this construction is provided in Figure 1.15. For $4\leq k<6$ this construction also generates graceful trees, however the P_2 and the 2-star on 7 vertices do not attach in the above described positions.

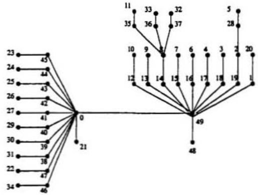


FIGURE 1.15. The graceful tree obtained from Skolem's construction of order $4k+1$ where $k=6$.

Using particular constructions of Skolem sequences we have generated two new classes of graceful trees. It should be noted that we can now apply the result of Stanton and Zarnke to make graceful balanced trees from them.

Each of these trees has a perfect matching which is evident from the partition of $\mathbf{Z}_{2n}$ into 2-subsets obtained from the Skolem sequence. Additionally, the structure of each tree is similar to that of a lobster, differing only by the added 2-stars and paths. From these sequences additional graceful trees should be obtainable, however the main objective of this work is not to find each of these but merely to illustrate the use of Skolem sequences in obtaining graceful trees.

CHAPTER 2

Graceful Trees From Periodic Odd Sequences

1. Using periodic odd sequences to create graceful trees

We have previously seen the utility of Skolem sequences in finding graceful labellings of trees, where a Skolem sequence of order n provides edges of value $1, 2, \ldots, n$. However, the addition of edges of value $n+1, \ldots, 2n-1$ without creating cycles presents less options than if it were required to add edges of lesser value. For instance, an edge of value $2n+1$ can only be added between 0 and $2n+1$, yet an edge of value 2 can be added between 0 and 2, 1 and 3, ... , or $2n-3$ and $2n-1$. In particular, there are at most $2n-i$ possible choices for the addition of a length i. Consequently, difference sequences which contain larger differences than those provided by Skolem sequences may prove to be more useful in creating graceful trees.

DEFINITION 2.1. *The periodic odd sequence of order* n, $(p_0, p_1, \ldots, p_{2n-1})$, *is the sequence of length* $2n$ *for which* $p_i = p_{2n-1-i} = 2n-2i-1$ *for all* $i : 0 \leq i \leq n-1$.

Using the bijection between sequence positions and vertices that was described in Chapter one, the periodic odd sequence of order n thereby provides edges of odd value between 1 and $2n-1$. To create a graceful tree now requires the addition of edges of even value so as to not create cycles. As compared with Skolem sequence, this not only provides more choices for the addition of edges, but a sense of uniformity as the edges to be added are all of even value. To add an even edge requires connecting two vertices of the same parity, both of which are adjacent to vertices of the opposite parity in the perfect matching resulting from the periodic odd sequence. Before we begin to explore the utility of periodic odd sequences in creating graceful trees, we will need the following definitions of a big lobster and an m-distant tree.

DEFINITION 2.2. *A big lobster is a tree which contains a path from which all vertices are a distance at most three. We will later refer to a big lobster as a* 3*-distant tree.*

This definition can be found in a paper by Ling [10]. From this definition we can see that all caterpillars and lobsters are big lobsters, however neither converse is true. An example of a big lobster which is neither a caterpillar nor a lobster is given in Figure 2.1.

DEFINITION 2.3. *Given a tree* T, *let* P_T *be one of its longest paths, not necessarily unique. If all of its vertices are a distance at most* m *from* P_T, *then* T *is* m*-distant.*

An initial observation is that all m-distant trees are also m'-distant trees for $m' > m$. As well, one can see that paths are 0-distant, caterpillars are 1-distant,

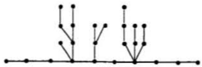


FIGURE 2.1. A big lobster.

lobsters are 2-distant, big lobsters are 3-distant, and vice-versa. The introduction of the definition of m-distant should eliminate the existing clutter of terminology. An example of a 4-distant tree is given in Figure 2.2.

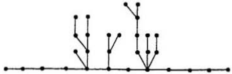


FIGURE 2.2. A 4-distant tree.

2. Gracefulness of certain 3-distant trees

Using a periodic odd sequence of order n, P_{2n-1}, the path on $2n$ vertices, can be created by the addition of the edges $\{i, 2n-2-i\}, 0 \leq i \leq n-2$, as shown in in Figure 2.3. Additionally, it should be noted that the addition of the edge $\{j,k\}$ of value $|k-j|$, which connects the edges $\{j, 2n-1-j\}$ and $\{k, 2n-1-k\}$ in the perfect matching, achieves the same edge value and connects the same edges as the addition of $\{2n-1-j, 2n-1-k\}$. The consequence of this is that the edges in the perfect matching of P_{2n-1} can be "pushed up" to form a 1-distant tree with a perfect matching. This labelling of a 1-distant tree is illustrated in Figure 2.3. Given that 0-distant and 1-distant trees can be gracefully labelled using periodic odd sequences, one wonders whether or not 2-distant trees can be gracefully labelled using periodic odd sequences.

THEOREM 2.4. *All 3-distant graphs with the following properties are graceful.*

1. *They have perfect matchings.*
2. *They can be constructed by the attachment of paths of length two to the vertices of a 1-distant tree (caterpillar), by identifying an end vertex of each path with a vertex of the 1-distant tree.*

It should be noted that we can now apply the result of Stanton and Zarnke to make graceful balanced trees from these graceful 3-distant trees. An example of a 3-distant tree which has a perfect matching but does not satisfy the second property is given in Figure 2.4.

PROOF. Consider a 3-distant tree T on $n = 2m$ vertices with the properties required by Theorem 2.4. Let C_T be a maximal subcaterpillar to which the P_2's

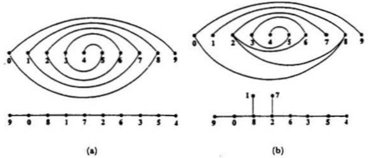


FIGURE 2.3. A labelling of 0-distant and 1-distant trees using the periodic odd sequence.

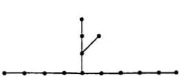


FIGURE 2.4. An example of a 3-distant tree which is not of the form required by Theorem 2.4. Note that this is the minimal such tree with a perfect matching.

are attached. Given that T had a perfect matching, so does C_T as an induced subgraph. We will denote the edge sets that constitute these perfect matchings by M_T and M_{C_T} respectively.

Let P_{C_T} be a maximal path in C_T and let $e_{0,1}$ be an edge containing a pendant vertex of P_{C_T}. Note that in M_T $e_{0,1}$ is also an edge containing a pendant vertex. For $e_{i,1}$, let $e_{i+1,1}$ be the unique edge in M_{C_T} which is at distance 1 from $e_{i,1}$ and furthest away from $e_{0,1}$. Now let the other edges in M_T which are at distance 1 from $e_{i,1}$ be $e_{i,2}, \dots, e_{i,f(i)}$. Note that T consists exclusively of the $\frac{n}{2}$ edges of the form $e_{i,j}$ and the $\frac{n}{2} - 1$ edges that connect them. As an example see Figure 2.5.

Let $S(e_{i,j}) = j - 1 + \sum_{k=0}^{i-1} f(k)$ and label the vertices of $e_{i,j}$ as $S(e_{i,j})$ and $n - 1 - S(e_{i,j})$. We first note that

$$S(e_{0,1}) = 0. \tag{1}$$

As well,

$$S(e_{i,j+1}) - S(e_{i,j}) = (j+1) - 1 + \sum_{k=0}^{i-1} f(k) - \left[j - 1 + \sum_{k=0}^{i-1} f(k) \right] = 1 \tag{2}$$

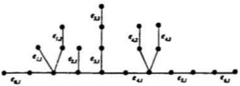


FIGURE 2.5. The edges of the perfect matching in an admissible 3-distant tree as described by Theorem 2.4.

and

$$S(e_{i+1,1}) - S(e_{i,f(i)}) = 1 - 1 + \sum_{k=0}^{i} f(k) - \left[f(i) - 1 + \sum_{k=0}^{i-1} f(k) \right] = 1. \tag{3}$$

Given that there are $\frac{n}{2}$ such $S(e_{i,j})$, we have $S(e_{i,j}) \leq \frac{n}{2} - 1$ and $n - 1 - S(e_{i,j}) \geq \frac{n}{2}$. This, in combination with (1), (2), and (3) gives that the vertex labels are unique and use all the values $0, \ldots, n-1$. Additionally, the edge $e_{i,j}$ has value $[n - 1 - S(e_{i,j})] - S(e_{i,j}) = n - 1 - 2S(e_{i,j})$ so the edges $e_{i,j}$ of M_T are distinct and use all the odd values ranging from 1 to $n - 1$.

It only remains to orient the labels of the edges of M_T such that the $\frac{n}{2} - 1$ edges in $E_T \setminus M_T$ are uniquely assigned even edge values. We must consider two cases.

- Edges between $e_{i,j}$ and $e_{i,j+1}$. Place $S(e_{i,j}) \in e_{i,j}$ adjacent to $n - 1 - S(e_{i,j+1}) \in e_{i,j+1}$ or $n - 1 - S(e_{i,j}) \in e_{i,j}$ adjacent to $S(e_{i,j+1}) \in e_{i,j+1}$. In either case the edge value generated is

$$n - 2 - 2S(e_{i,j}). \tag{4}$$

- Edges between $e_{i,f(i)}$ and $e_{i+1,1}$. Place $S(e_{i,f(i)}) \in e_{i,f(i)}$ adjacent to $n - 1 - S(e_{i+1,1}) \in e_{i+1,1}$ or $n - 1 - S(e_{i,f(i)}) \in e_{i,f(i)}$ adjacent to $S(e_{i+1,1}) \in e_{i+1,1}$. In either case the edge value generated is

$$n - 2 - 2S(e_{i,f(i)}). \tag{5}$$

Consequently, we need only orient the labels of $e_{i,j}$ (or $e_{i+1,1}$) according to the label of $e_{i,j-1}$ (or $e_{i,f(i)}$) from which it is at distance 1. From (4) and (5), the edges between $e_{i,j}$ and $e_{i,j+1}$, in combination with the edges between $e_{i,f(i)}$ and $e_{i+1,1}$, uniquely obtain the even edge values from 0 to $n-2$. Thereby T is graceful. □

An immediate consequence of Theorem 2.4 is the following corollary which makes significant progress toward resolving Bermond's conjecture that all lobsters are graceful.

COROLLARY 2.5. *All 2-distant trees (lobsters) which have perfect matchings are graceful.*

It should be noted that particular cases of Corollary 2.5 can be obtained by using the Stanton and Zarnke "graphting" technique in [18]. More specifically, a 2-distant tree with a perfect matching which is formed from a 1-distant tree, by identifying an end vertex of a P_1 with each vertex of the 1-distant tree, is graceful. This gives the following corollary.

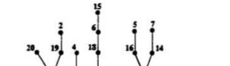
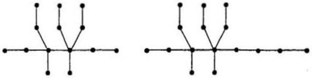


FIGURE 2.6. The graceful labelling of an admissible 3-distant tree as described by Theorem 2.4.

COROLLARY 2.6 (Stanton, Zarnke). *The existence of a graceful 1-distant tree (caterpillar), implies the existence of a graceful labelling of a 2-distant tree (lobster).*

An example of a 2-distant tree which can be obtained from Corollary 2.6 is pictured in Figure 2.7. Koh, Rogers, and Tan extend the results of Stanton and Zarnke in [8], allowing more flexibility in the choice of graphs which can be "graphted" to the 1-distant tree. Consequently, additional cases of Corollary 2.5 can be resolved, however a thorough examination of [8], as well as other literature on this topic, indicate that Corollary 2.5 has not been obtained in its entirety.



(a) A graceful 2-distant tree with a perfect matching which can be obtained from Corollary 2.6.

(b) A graceful 2-distant tree with a perfect matching which cannot be obtained by from Corollary 2.6.

FIGURE 2.7.

CHAPTER 3

Conclusions and Future Research

This thesis makes progress towards resolving the Kotzig Conjecture that K_{2n+1} can be cyclically decomposed into $2n+1$ copies of any given tree with n edges.

In Chapter one algorithms are established which use Skolem sequences and hooked Skolem sequences to create graceful trees. The results established in these contexts are given below.

THEOREM 3.1. *The existence of a Skolem sequence of order n implies the existence of a graceful tree on 2n vertices which has a perfect matching or a matching on 2n − 2 vertices.*

THEOREM 3.2. *The existence of a hooked Skolem sequence of order n implies the existence of a graceful tree on 2n + 1 vertices which has a matching on either 2n or 2n − 2 vertices.*

Additionally, in Chapter one graceful labellings for two classes of trees are derived from Skolem sequences. If the order of the sequence is $n = 4k$, where $k \geq 5$, the graph, which is on $2n$ vertices, will be composed of a 2-star to which are attached another 2-star, as well as a P_1, a P_2, and a P_4. If $n = 4k + 1$, where $k \geq 6$, it is composed of a 2-star to which are attached two additional 2-stars, as well as a P_1, and a P_2.

In Chapter two the periodic odd sequence is used to show the following theorem and corollary. The corollary makes significant progress toward verifying Bermond's conjecture that all 2-distant trees (lobsters) are graceful.

THEOREM 3.3. *All 3-distant graphs with the following properties are graceful.*

(1) *They have perfect matchings.*
(2) *They can be constructed by the attachment of paths of length two to the vertices of a 1-distant tree (caterpillar), by identifying an end vertex of each path with a vertex of the 1-distant tree.*

COROLLARY 3.4. *All 2-distant trees (lobsters) which have perfect matchings are graceful.*

Future research on topics presented in this thesis could include the following.

- Exploring the utility of the algorithms found in the proofs of Theorems 3.1 and 3.2 towards the general case of extended Langford sequences with multiple defects. Initial observation indicates that the algorithm should be changed to accommodate a core whose size is dependent on the starting defect, additional defects, and the number of hooks. With additional hooks one would also need to consider more hooked orbits each of which must contain exactly two hooks.
- Verifying Bermond's conjecture that all lobsters are graceful.

- Using periodic odd sequences to establish graceful labellings of m-distant trees for $m > 3$.

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