Testing for Parallel Carryover Effects in Redundant Systems

by

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Abstract

Redundancy is an important approach to increase the reliability and availability of systems. There has been a recent interest in analyzing failure data from redundant systems to detect the effects of adverse events. A carryover effect is defined as an effect which may cause a temporary increase in the event intensity after the occurrence of a condition or an event. We consider a parallel type of carryover effect in which the event intensity of a process is temporarily increased after event occurrences in other processes. The main goal of this thesis is to develop formal tests for the assessment of parallel carryover effects in redundant systems with repairable components connected in parallel. We, therefore, develop partial score tests for the presence of parallel carryover effects, and discuss their asymptotic properties analytically as well as through simulations. A data set based on the information obtained from a power company is analyzed to illustrate the methods developed.

To Yuna

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Statement of contribution

Dr. Candemir Cigsar proposed the research question that was investigated throughout this thesis. Dr. Candemir Cigsar and Yongho Lim jointly designed the study. Yongho Lim implemented the algorithms, conducted the simulation study, and drafted the manuscript. Dr. Candemir Cigsar supervised the study and contributed to the final manuscript.

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Chapter 1

Introduction

In this chapter, our main goal is to introduce some concepts related to the reliability of power systems and research topics. In Section 1.1, we first introduce fundamental concepts in reliability analysis of redundant systems, and next discuss the type of data sets. In Section 1.2, we give an outline of the thesis.

1.1 Reliability of Power Systems and Redundancy

The demand of consumers of manufacturers, systems or service providers on quality, productivity and availability of a product, system or service is getting higher than ever. Reliability attached to those characteristics is of critical importance. In basic terms, reliability is defined as the probability that a product, system or service operates under the operating conditions for some specified period of time. Improving reliability is, therefore, a very important issue for manufacturers and service providers to be competitive.

In this thesis, we consider the reliability of systems with multiple components. Our focus is on diesel operated power systems, which are often used to generate electricity in isolated, hard-to-reach communities to meet the power demand, but methods and models developed can be applied in other settings as well. The function of a power system is to supply electrical energy to its consumers on demand as reliably and safely as possible and in an economically justifiable manner (Billinton and Allan, 1984). Power generation companies may sell their power to utility, industrial, residential and commercial customers. The unavailability of a power system when it is needed can create serious problems for its consumers as well as resulting in severe financial loss to a power company. The expectation of customers from a power company is, therefore, a continuous supply of least-cost electricity whenever there is a demand for power. This is not always possible in reality. There are many events causing unwanted system stoppages, which are beyond the control of system engineers. Such events, to which we refer to as *failures*, are often in a recurrent nature. Modelling and analysis of recurrent failures can be useful for identifying opportunities for reliability improvements in power systems especially in the operation phase.

There are different ways of improving the reliability of a system. As Billinton and Allan (1992, Section 1.3) denoted, *redundancy* is an important approach to affect the reliability and availability of a system. In this approach, there is one or more back-ups of the components in a system so that the function of a failed component is absorbed by other unfailed components. There are two types of redundancy. The first type is called *standby redundancy*, in which the redundant component waits in a standby position until the failure of active components, and starts operating when one or more active components fail. In the second type of redundancy, called *active redundancy*, components operate and share a function together. In case of a failure of a component, remaining components absorb the load of the failed component (Billinton and Allan, 1992). In this thesis, we focus on the active redundancy as it fits a natural model for the type of data sets considered. A system having either standby or active redundancy is called a *redundant system*. An example of a redundant system with active redundancy is a (fully) *parallel system* with K components, in which all components are connected in parallel. In such systems, a system failure occurs if and only if all of the components fail. This type of systems are sometimes called *fully redundant*. A system is called a *series system*, if all of its componenets are connected in series. A series system is sometimes called *zero redundant*. There are also *partially redundant systems*, in which some components are connected in series and some components are connected in parallel. In this thesis, we focus on fully redundant systems consisting of K parallely connected components.

Systems or their components can be classified as repairable or nonrepairable (Rigdon and Basu, 2000). A *nonrepairable system* is a system that is discarded when a failure occurs. Cost of replacement is usually low in nonrepairable systems. A *repairable system* is a system that can be restored to an operating condition after some repair other than replacement of the entire system when a failure occurs. As discussed by Rigdon and Basu (2000), repairable systems are generally more complex than nonrepairable systems. Our focus in this thesis is on systems with repairable components. Statistical analysis of repairable systems usually needs assumptions on the nature of repairs, which plays an important role on the development of a model for failure data. Ascher (1968) introduces the concept of a *minimal repair*, in which a repair is assumed to bring a failed system or component to the operating condition just like before the failure. In this case, a repair is sometimes called *as-bad-as-old* (Ascher, 1968). A *perfect repair* is a repair that brings the condition of a repairable system to that of a completely new system after a failure. This type of repair is sometimes called *as-good-as-new* (Rigdon, 2007; Misra, 2012). As Lindqvist (2006) denoted, the assumption of a minimal repair is usually applied when only a minor part of the system is repaired or replaced after a failure. This type of repairs are better suited in our study, so we assume that a minimal repair occurs after each failure of components of a system. In Section 2.1, we introduce models corresponding to the assumption of minimal and perfect repairs. There are also *imperfect repair* models (Brown and Proschan, 1983; Baker, 2001; Lindqvist, 2006). In this thesis, we do not discuss such repairs, but state them as a future work in the final chapter.

In the next two subsections, we first introduce the motivating example of this research and then present the main target and outline of this thesis. It is well known that there are major problems in reliability when it comes to data collection (Lawless, 1983). For example, failure data are often incomplete or biased in the sense that not all types of failures are reported. We do not consider such difficulties in this thesis.

1.1.1 Example: Diesel Plants of a Power Company

The framework of the data sets used in this thesis are obtained from a power company. Unexpected power outages in the remote, isolated communities in different regions is a major concern for this power company and its customers. There are 25 diesel plants operating in those isolated locations to provide electricity. Environmental conditions in these regions are usually extremely severe, especially in the winter times. Also, some of the power plants are 35-40 years old. Mainly because of these reasons, power plants operating in these communities fail frequently, and require increasing attention for maintenance, refurbishment and replacement. Most communities that receive their electricity from diesel plants have an operator living in the community who can respond to unexpected power outages that may occur on site and complete regular maintenance. Therefore, a failed power plant can be sometimes repaired in a short time. However, there are many occasions when a failed power plant requires long repair times, which may substantially increase the total cost of operation and create unpleasant situations for customers.

To improve the reliability of operations, the power company implements the redundancy approach. Most of the power plants operating in isolated communities have multiple diesel operated engines working together in order to supply electricity demand of a community. These engines are parallely connected to share the electricity demand. If one of the engines fails, the remaining engines in operating conditions share the load of the failed one and operate in an increased capacity. The power plant has an unwanted system stoppage if and only if all of the engines do not operate. This framework is, therefore, suitable for the analysis within the fully redundant systems. In the remainder of this thesis, we consider a power plant as a system and its engines as parallely connected components operating under redundancy.

1.2 Main Goal and Outline of the Thesis

Modelling and detection of the effects of adverse events on repairable systems has been a major research area in the analysis of reliability data. Redundancy is an important approach to increase the reliability. In redundant systems with active redundancy, the failure of a component may temporarily increase the probability of failures of other components. We refer to this phenomenon as a *parallel carryover effect*. In other words, a parallel carryover effect is a temporary adverse effect resulting in an increased risk of failures in redundant components of an active redundant system during the downtime of the failed components. The presence of parallel carryover effects causes temporary clustering of events (failures) together in the redundant components.

The main goal of this thesis is to provide a thorough discussion of parallel carryover effects, develop formal tests for the absence of them in various settings, and investigate their properties. The outcome of this thesis can be beneficial for power generation companies to improve their reliability programs and to determine their maintenance programs, which minimize the total cost of operation and maximize the availability of repairable systems.

The remainder of this thesis is arranged as follows. In Chapter 2, we introduce the notation used in the thesis, fundamental models for recurrent event processes as well as mathematical concepts and simulation procedures applied in the following chapters.

In Chapter 3, we discuss testing for the presence of parallel carryover effects in redundant systems with two components. We first consider testing for parallel carryover effects in redundant systems where repair times of the redundant component are negligible. Then, we consider testing for parallel carryover effects in redundant systems where repair times of the redundant component are not negligible. Asymptotic properties of the test statistics are discussed analytically as well as through simulations in two different setups; (*i*) when the observation period increases for a single system, and (*ii*) when the number of systems increases for multiple systems.

In Chapter 4, we discuss testing for parallel carryover effects in redundant systems with three components as an extension of Chapter 3. We first consider testing for parallel carryover effects in redundant systems with three components when repair times of the redundant component are negligible. Next, we discuss the same issue when repair times of the redundant component are not negligible. We discuss the asymptotic properties of test statistics within the same setups of Chapter 3 via simulations. Finally, we illustrate the methods developed in Chapter 4 by analyzing a simulated data set. The simulation of the data set is based on the information received from a power company on their diesel power plants.

In Chapter 5, we discuss testing for the presence of parallel carryover effects in redundant systems, which are subject to monotonically increasing time trends in the rate of event occurrences due to stochastic aging. This type of trends is often seen in repairable systems. In this chapter, we also extend our methodology to deal with external covariates in the models. We first consider systems with two components and then extended the methods to three components case. We also discuss the asymptotic properties of test statistics through simulations. Finally, we analyze a simulated failure data set in power systems to illustrate the methods.

In the final chapter, we give a summary of the results of the previous chapters and present our conclusions. Some important future research topics are also discussed.

Chapter 2

Basic Concepts and Models

In this chapter, we introduce basic concepts and models that are useful in this thesis. In Section 2.1, we introduce basic terminology and concepts in recurrent event processes. We focus mostly on the Poisson processes. In Section 2.2, we give the methodology based on the likelihood functions to test a hypothesis with score tests. We explain simulation procedures in Section 2.3.

2.1 Basic Concepts and Models

In this section, we introduce the basic notation and terminology frequently used in the remaining part of this thesis. We introduce the notation for a single process observed over a fixed time interval. Extensions to multiple processes and other observation schemes are given later in the thesis whenever it is needed.

As defined in the previous section, reliability is the probability that a product, system or service operates under the operating conditions for some specified period of time (Meeker and Escobar, 1998). In this thesis, we focus on systems with multiple components, in which components are parallely connected. Figure 2.1 shows the diagram of such a system with two parallelly connected components. We consider only binary components with operational (*up*) and nonoperational (*down*) states, in which either up or down state is possible for any component at any given time instant. We define any event that results in an unplanned stoppage in a component as a failure in the corresponding component. Furthermore, a system is in down state if and only if all of its components are down. We assume components are repairable and provide possibly recurrent failure (event) data.

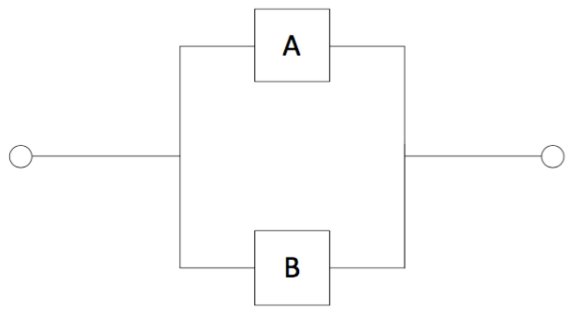

Figure 2.1: A system with two components (Component A and Component B) connected in parallel.

The statistical analysis of repairable systems is usually constructed within the point process framework. Let random variables T_1, T_2, $\ldots$, with the property that $0 < T_1 < T_2 < \cdots$ denote the occurrence times of a well-defined event along a time axis. The T_i are called event times or arrival times. We then define the jth gap time W_j as the interarrival time between the jth and $j+1$st event times; that is, $W_j = T_j - T_{j-1}$, $j = 1, 2, \ldots$, where by mathematical convention $T_0 = 0$. We also let $t_1, t_2, \ldots$ and $w_1, w_2, \ldots$ denote realizations of $T_1, T_2, \ldots$ and $W_1, W_2, \ldots$, respectively.

Let the random variable $N(t)$ denote the number of event occurrences over a time interval $(0, t]$, where $t > 0$. We also let $N(s, t)$ denote the number of event occurrences over $(s, t]$ so that $N(s, t) = N(t) - N(s)$ for all $0 \leq s < t < \infty$. We assume that $N(0) = 0$ and $E\{N(t)\} < \infty$ for each t, where E denotes expectation. The stochastic process $\{N(t), t > 0\}$ is then called a *counting process*. Many properties of counting processes and their related functions are given, for example, in Daley and Vere-Jones (2003).

We next define the intensity function of a counting process, but to do this we first define the history of a stochastic process. We let $\mathcal{H}(t) = \{N(u), 0 \leq u < t\}$ denote *the history of the process* $\{N(t), t > 0\}$ *at time t*. The history $\mathcal{H}(t)$ includes all information about the counting process $\{N(t), t > 0\}$ from time 0 to just prior to time t. In our simple setup, this information includes event occurrence times t_1, t_2, ..., and the number of events at each time point in $[0, t)$. More information on histories of stochastic processes can be found in Daley and Vere-Jones (2003, pp. 423–427). We let dt denote an infinitesimal positive valued real number and $dN(t)$ be an infinitesimal increment in $N(t)$; that is, $dN(t) = N((t + dt)^{-}) - N(t^{-})$, which gives the number of events in $[t, t + dt)$. We are now in a position to define the intensity function. Let $\lambda(t|\mathcal{H}(t))$ denote the intensity function of a counting process $\{N(t), t > 0\}$ (with respect to its history $\mathcal{H}(t)$), which is mathematically defined as

$$\lambda(t|\mathcal{H}(t)) = \lim_{\Delta t \to 0} \frac{\Pr\{N((t + \Delta t)^{-}) - N(t^{-}) = 1 \,|\, \mathcal{H}(t)\}}{\Delta t}, \tag{2.1}$$

where $\Delta t > 0$. The intensity function gives the instantaneous probability of an event occurring in $[t, t + dt)$, given the process history $\mathcal{H}(t)$. Assuming that two or more events cannot occur together at the same instant, the intensity function completely specifies a recurrent event process (Cook and Lawless, 2007). In this case, since $dN(t)$ is a 0-1 valued (binary) random variable, it can be shown that $\lambda(t \,|\, \mathcal{H}(t))\, dt = E\{dN(t) \,|\, \mathcal{H}(t)\}$.

Other important concepts include mean, rate and hazard functions. The *mean function*, denoted by $\mu(t)$, is a nondecreasing, right continuous function which gives the expected number of events up to time t; that is, $\mu(t) = E\{N(t)\}$ for $t > 0$. We also use the notation $\mu(s, t)$ to denote the expected number of events in any finite interval $(s, t]$, where $0 \leq s < t < \infty$. Thus, $\mu(s, t) = E\{N(s, t)\}$. We let $\rho(t)$ denote the instantaneous rate of change of the expected number of events with respect to time. We refer to $\rho(t)$ as the *rate of occurrence of failures function* (ROCOF) or, simply, the *rate function*. Since rate and mean functions are not conditioned on the history, they are called marginal properties of a point process. Assuming the derivative of $\mu(t)$ exists for $t > 0$, by definition, $\rho(t) = (d/dt)\mu(t)$. The *hazard function* of a positive random variable W is defined by

$$h(w) = \lim_{\Delta t \to 0} \frac{\Pr\{W < w + \Delta t \,|\, W > w\}}{\Delta t}, \qquad w > 0. \tag{2.2}$$

If $f(w)$, $w > 0$, denotes the probability density function (p.d.f.) of W and $F(w) = \Pr\{W \leq w\} = \int_0^w f(u)\, du$ is the cumulative density function (c.d.f.) of W, it is well known that $h(w) = f(w)/[1 - F(w)]$ for $w > 0$.

The data generating mechanism of a counting process $\{N(t), t > 0\}$ in the continuous case is governed by its intensity function. The specification of the intensity function (2.1), therefore, defines a statistical model for recurrent event processes. We now introduce some fundamental models for recurrent event processes through the specification of their intensity functions. We start with Poisson processes and then introduce renewal processes and some ramifications of them.

Poisson processes are usually useful if there is an interest in modelling the number of event occurrences. The process $\{N(t), t > 0\}$ is called a Poisson process if its associated intensity function is given by

$$\lambda(t \,|\, \mathcal{H}(t)) = \rho(t), \qquad t > 0, \tag{2.3}$$

where $\rho(t)$ is the rate function of the process (Cook and Lawless, 2007). It is clear from (2.3) that the intensity function of a Poisson process depends on t, but it is independent from the previous event occurrences over $[0, t)$. Therefore, Poisson processes have the Markov property (see, e.g., Thompson, 1988). A Poisson process $\{N(t), t > 0\}$ is called a *homogeneous Poisson process* (HPP) if the intensity function (2.3) is constant for any $t > 0$. Otherwise, it is called a *nonhomogeneous Poisson process* (NHPP).

In a reliability context, Poisson processes are canonical models for repairable systems if minimal repairs are applied after after each failure. The distinction between a HPP and a NHPP is an important modelling issue. As denoted by Thompson (1988, p. 22), homogeneous Poisson processes (HPPs) are often used for modelling recurrent event data mainly because of their simple properties, but their adequacy should be always investigated especially in reliability studies of repairable systems. For example, since HPPs have a constant rate function, they are not appropriate models when there is a time trend due to stochastic aging (Lai and Xie, 2006). This is a major limitation of their use in the reliability studies of repairable systems because many repairable systems are more prone to fail as they age. A constant rate function cannot be adequate in such cases. Nonetheless, HPPs have some applications in reliability studies, in particular, when the observation periods of systems are short. A NHPP is a canonical model for repairable system if there is stochastic aging in the system due to a wear-out phenomenon or due to reliability growth (Thompson, 1988, p. 53; Lai and Xie, 2006, p. 7).

Poisson processes and their generalizations are well discussed in the literature. Their properties can be found in many stochastic processes or point processes texts (see, e.g., Thompson, 1988; Kingman, 1993; Daley and Vere-Jones, 2003; Nakagawa, 2011). Here, we state only some of the useful properties, and refer to literature for their proofs. For example, let $\{N(t), t > 0\}$ be a Poisson process with the intensity function (or equivalently, the rate function) $\rho(t)$, where $t > 0$. Then, the number of events in any finite interval $(s, t]$, where $0 \leq s < t < \infty$, has a Poisson distribution with the mean $\mu(s, t)$ (Daley and Vere-Jones, 2003, p. 34). We have, therefore,

$$\Pr\{N(s,t) = n\} = \frac{[\mu(s,t)]^n}{n!} e^{-\mu(s,t)}, \qquad n = 0, 1, 2, \ldots, \tag{2.4}$$

where $\mu(s,t) = \int_s^t \rho(u)\, du$. Let $V(t)$ denote the variance function. Since the mean and the variance of any Poisson distributed random variable are equal, the variance function of the number of events in a Poisson process $\{N(t), t > 0\}$ is given by $V(t) = Var\{N(t)\}$, where Var stands for the variance and $Var\{N(t)\} = \mu(t)$ for $t > 0$.

The following property is useful to simulate realizations of a HPP. The counting process $\{N(t), t > 0\}$ is a HPP with a constant rate function ρ, where $\rho > 0$, if and only if the gap times W_j, $j = 1, 2, \ldots$, are independent and identically distributed (i.i.d.) exponential variables with mean ρ^{-1}. A proof of this statement can be found, for example, in Rigdon and Basu (2000, pp. 45–49). Another important result which can be used to simulate realizations of a NHPP is given as follows. Let $\{N(t), t > 0\}$ be a NHPP with mean function $\mu(t)$ and $\{N^*(s), s > 0\}$ be a HPP with mean function $\mu^* = 1$. By letting $s = \mu(t)$, we can show that $N^*(s) = N(\mu^{-1}(s))$ for $s > 0$ (Daley and Vere-Jones, 2003, p. 258). We used these two results in simulation and data analysis sections of the next chapters. We explain the simulation procedure in Section 1.4.

Another important class of models for recurrent event processes can be based on *renewal processes*. A renewal process $\{N(t), t > 0\}$ is a point process in which the gap times W_1, W_2, $\ldots$, are i.i.d. In this case, the intensity function (2.1) of $\{N(t), t > 0\}$ takes the form of

$$\lambda(t \,|\, \mathcal{H}(t)) = h(t - t_{N(t^-)}), \qquad t > 0, \tag{2.5}$$

where h is the hazard function defined in (2.2). In a reliability context, a renewal process implies that there is a perfect (i.e., as-good-as-new) repair after each failure of a repairable system, which brings the system to a brand new condition. In some cases, this can be a reasonable assumption; for example, if a complete overhaul is performed after each failure of a repairable system. However, the assumption of i.i.d. gap times is a very strong one, and its validity needs to be carefully checked in each application. Many basic properties of the renewal processes can be found in stochastic processes texts. For example, see Nakagawa (2011, Chapter 3). As discussed by Thompson (1988, Section 5.2), a renewal process cannot model systems that is wearing out. Therefore, we do not provide a detailed background on renewal processes. However, one important relation is that, if the gap times W_j, $j = 1, 2, \ldots$, of renewal process $\{N(t), t > 0\}$ are i.i.d. exponential random variables with mean $E\{W_j\} = \rho^{-1}$, where $0 < \rho < \infty$, then the process $\{N(t), t > 0\}$ is a HPP with rate function ρ.

2.1.1 Covariates in Recurrent Event Processes

In many studies, there are covariates of interest. In such cases, models can be extened to include covariates. An excellent discussion of this issue is given by Kalbfleisch and Prentice (2002); also, see Andersen et al. (1996), Daley and Vere-Jones (2003) and Cook and Lawless (2007). The basic idea is to consider the covariates as a vector of stochastic processes and then extend the history by including their path information. Following the notation of Daley and Vere-Jones (2003, pp. 237–238) and Cook and Lawless (2007, Section 2.2.2), this can be done as follows.

Suppose that we observe a p-dimensional vector of stochastic processes denoted by $\{X(t), t > 0\}$, where $\{X(t), t > 0\} = \{X_1(t), \ldots, X_p(t); 0 < t < \infty\}$. Let $\mathcal{H}^X(t)$ denote the history of the process $X(t)$ over the time interval $[0, t]$. Thus, $\mathcal{H}^X(t)$ includes paths of covariate processes $X_j(t)$, $j = 1, \ldots, p$, in [0,t]. Now suppose that $\{N(t), 0 < t\}$ is a counting process with the intensity function $\lambda_0(t \,|\, \mathcal{H}^N(t))$, where $\mathcal{H}^N(t) = \{N(u), 0 \le u < t\}$ is the history of the counting process. Then, a model of multiplicative form is given by the following intensity function.

$$\lambda(t|\mathcal{H}(t)) = \lambda_0(t|\mathcal{H}^N(t))\, \psi(X_1(t), \ldots, X_p(t)), \qquad t > 0, \tag{2.6}$$

where λ_0 is called the baseline intensity function, ψ is a nonnegative valued function and $\mathcal{H}(t)$ is the extended history including information on both $\mathcal{H}^N(t)$ and $\mathcal{H}^X(t)$. If we specify $\lambda_0(t \,|\, \mathcal{H}^N(t)) = \rho(t)$ and $\log \psi(X_1, \ldots, X_p) = \sum_{j=1}^{p} \beta_j X_j$ in (2.6), we obtain the intensity function of the multiplicative form

$$\rho(t)\, \exp[\beta_1 X_1(t) + \cdots + \beta_p X_p(t)], \qquad t > 0, \tag{2.7}$$

where $\rho(t)$ is the rate function of $\{N(t), t > 0\}$ and the β_j are regression parameters. The model (2.7) is called the *modulated Poisson process* (Cook and Lawless, 2007).

Instead of $\rho(t)$, if we specify the baseline rate function with $h(t - t_{N(t^-)})$, we obtain the modulated renewal process with the intensity function of the form

$$h(t - t_{N(t^-)}) \exp[\beta_1 X_1(t) + \cdots + \beta_p X_p(t)], \qquad t > 0, \tag{2.8}$$

where h is the hazard function defined in (2.2). The model (2.8) is discussed by Cox (1972).

It should be noted that multiplicative models of the from (2.6) specify multiplicative effects of covariates on the intensity function. The validity of this assumption should be checked. Some methods for checking this assumption are discussed by Cook and Lawless (2007, Section 3.7.2). As an alternative to multiplicative models, additive models can also be used (Aalen et al., 2008). In this case, the general intensity function can be written as

$$\lambda(t|\mathcal{H}(t)) = \lambda_0(t|\mathcal{H}^N(t)) + \psi(X_1(t), \ldots, X_p(t)), \qquad t > 0. \tag{2.9}$$

There is an important remark regarding to the inference with models involving covariates in recurrent event processes. The full likelihood based inference requires that the evolution of the covariate processes $\{X_j(t), t > 0\}$, $j = 1, \ldots, p$, should be independent from the counting process $\{N(t), t > 0\}$. Kalbfleisch and Prentice (2003, p. 196–198) refer to such covariates as *external*, which means, in their words, that the covariate process $\{X(u), 0 \leq u \leq t\}$ may influence the probabilistic characteristics of event occurrences over time, but its future path up to any time t, where $t > u$, is not affected by the occurrence of an event at time u. In this case, as explained in the next section, a full likelihood approach can be based on the models of the multiplicative form (2.6). If a covariate is not external, the likelihood function should be considered for both $\{X(t), t > 0\}$ and $\{N(t), t > 0\}$ together. In general, such a likelihood function is too complicated and the treatment of covariates requires care. Unless otherwise stated, we restrict the discussion in this thesis only to external covariates, in which their values are known at time t and probability laws do not include the parameters in the event generating model under study. All the models and probabilities are conditional on the values of the covariates. For notational purposes, we use the notation $\mathcal{H}(t) = \{N(u), X(s); 0 \leq u < t, 0 \leq s \leq t\}$ to denote the history of the processes $\{N(t), t > 0\}$ and $\{X(t), t > 0\}$.

2.2 Likelihood Methods and Score Tests

In this section, we give the likelihood function for a recurrent event process and develop partial score test procedures for testing composite hypothesis. Suppose that $\{N(t), t > 0\}$ is a counting process with the associated intensity function $\lambda(t|\mathcal{H}(t))$. The likelihood function for the outcome that n events occur at times $0 < t_1 < t_2 < \cdots < t_n < \tau$ in the time interval $(0, \tau]$, given the history of the process $\mathcal{H}(t)$, is of the form (Cook and Lawless, 2007, p. 30)

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} \lambda(t_i|\mathcal{H}(t_i)) \exp\left\{-\int_0^{\tau} \lambda(s|\mathcal{H}(s))\, ds\right\}, \tag{2.10}$$

where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)'$ is a $p \times 1$ vector of parameters. Let $\hat{\boldsymbol{\theta}}$ be the *maximum likelihood estimator* of $\boldsymbol{\theta}$ which maximizes $L(\boldsymbol{\theta})$ and let $\ell(\boldsymbol{\theta})$ be the log likelihood function; that is, $\ell(\boldsymbol{\theta}) = \log L(\boldsymbol{\theta})$. Let $\boldsymbol{U}(\boldsymbol{\theta}) = (U_1(\boldsymbol{\theta}), \ldots, U_p(\boldsymbol{\theta}))'$ be the $p \times 1$ score vector with entries

$$U_j(\boldsymbol{\theta}) = \frac{\partial \ell(\boldsymbol{\theta})}{\partial \theta_j}, \qquad j = 1, \ldots, p, \tag{2.11}$$

Usually the maximum likelihood estimator $\hat{\boldsymbol{\theta}}$ can be obtained by solving $\boldsymbol{U}(\boldsymbol{\theta}) = \mathbf{0}$ where $\mathbf{0}$ is a $p \times 1$ vector of zeros. Let $\boldsymbol{I}(\boldsymbol{\theta})$ be the $p \times p$ *information matrix* where the entries of $\boldsymbol{I}(\boldsymbol{\theta})$ are defined by

$$I_{ij}(\boldsymbol{\theta}) = -\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j}, \qquad i, j = 1, \ldots, p, \tag{2.12}$$

Also, let $\boldsymbol{J}(\boldsymbol{\theta})$ be the $p \times p$ *expected information matrix* with entries

$$J_{ij}(\boldsymbol{\theta}) = E\left(-\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j}\right), \qquad i, j = 1, \ldots, p. \tag{2.13}$$

Under mild regularity conditions, $E(\boldsymbol{U}(\boldsymbol{\theta})) = \mathbf{0}$ and variance-covariance matrix of $\boldsymbol{U}(\boldsymbol{\theta})$ is $J(\boldsymbol{\theta})$.

Assuming that the model is a regular model and inverse of $\boldsymbol{J}(\boldsymbol{\theta})$ exists, a test statistic for testing $H_0 : \boldsymbol{\theta} = \boldsymbol{\theta_0}$ is of the form

$$\boldsymbol{U}(\boldsymbol{\theta_0})^{'}\,\boldsymbol{J}^{-1}(\boldsymbol{\theta_0})\,\boldsymbol{U}(\boldsymbol{\theta_0}) \tag{2.14}$$

which is called *a score statistic* and the test based on (2.14) is called *a score test.* Under regularity conditions, the test statistic (2.14) is asymptotically chi-squared distributed with p degrees of freedom under H_0, (Boos, 1992).

If we are interested in only a part of the parameters in $\boldsymbol{\theta}$, we can make a partition of it as $\boldsymbol{\theta} = (\boldsymbol{\alpha}^{'}, \boldsymbol{\beta}^{'})^{'}$ where $\boldsymbol{\alpha}$ is $k \times 1$ vector of nuisance parameters and $\boldsymbol{\beta}$ is $q \times 1$ vector of parameters of interest. Then $\boldsymbol{U}(\boldsymbol{\theta})$ can be partitioned accordingly into two parts denoted by $\boldsymbol{U}_\alpha(\boldsymbol{\theta})$ and $\boldsymbol{U}_\beta(\boldsymbol{\theta})$ where $\boldsymbol{U}_\alpha(\boldsymbol{\theta})$ is a $k \times 1$ vector of score functions with entries $\boldsymbol{U}_{\alpha_j}(\boldsymbol{\theta}) = \partial\ell(\boldsymbol{\theta})/\partial\alpha_j$, $j = 1, \ldots, k$, and $\boldsymbol{U}_\beta(\boldsymbol{\theta})$ is a $q \times 1$ vector of score functions with entries $\boldsymbol{U}_{\beta_j}(\boldsymbol{\theta}) = \partial\ell(\boldsymbol{\theta})/\partial\beta_j$, $j = 1, \ldots, q$. Similarly, $\boldsymbol{J}(\boldsymbol{\theta})$ can be partitioned as follows.

$$\boldsymbol{J}(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{J}_{\alpha\alpha}(\boldsymbol{\theta}) & \boldsymbol{J}_{\alpha\beta}(\boldsymbol{\theta}) \\ \boldsymbol{J}_{\beta\alpha}(\boldsymbol{\theta}) & \boldsymbol{J}_{\beta\beta}(\boldsymbol{\theta}) \end{pmatrix}, \tag{2.15}$$

where $\boldsymbol{J}_{\alpha\alpha}(\boldsymbol{\theta})$ is a $k \times k$ matrix, $\boldsymbol{J}_{\alpha\beta}(\boldsymbol{\theta}) = \boldsymbol{J}_{\beta\alpha}(\boldsymbol{\theta})^{'}$ is a $k \times q$ matrix and $\boldsymbol{J}_{\beta\beta}(\boldsymbol{\theta})$ is a $q \times q$ matrix. Observed information matrix $\boldsymbol{I}(\boldsymbol{\theta})$ can be partitioned in the same way. Assuming they exist, the inverse matrix of (2.15) can be written as

$$\boldsymbol{J}(\boldsymbol{\theta})^{-1} = \begin{pmatrix} \boldsymbol{J}^{\alpha\alpha}(\boldsymbol{\theta}) & \boldsymbol{J}^{\alpha\beta}(\boldsymbol{\theta}) \\ \boldsymbol{J}^{\beta\alpha}(\boldsymbol{\theta}) & \boldsymbol{J}^{\beta\beta}(\boldsymbol{\theta}) \end{pmatrix}. \tag{2.16}$$

Let $\boldsymbol{\beta} = \boldsymbol{\beta_0}$, where $\boldsymbol{\beta_0}$ is specified value of $\boldsymbol{\beta}$. An estimator of $\boldsymbol{\alpha}$ under this specified case can be found by maximizing $L(\boldsymbol{\alpha}, \boldsymbol{\beta_0})$ or $\ell(\boldsymbol{\alpha}, \boldsymbol{\beta_0})$. Such an estimator is called *a restricted maximum likelihood estimator* of $\boldsymbol{\alpha}$ and denoted by $\boldsymbol{\alpha}(\boldsymbol{\beta_0})$ and if $\boldsymbol{\beta_0} = 0$, then shortly $\tilde{\boldsymbol{\alpha}}$. In this case, the function $L(\boldsymbol{\alpha}, \boldsymbol{\beta_0})$ is called *profile likelihood function* for $\boldsymbol{\beta}$ and $\ell(\boldsymbol{\alpha}, \boldsymbol{\beta_0})$ is called *profile log likelihood function* for $\boldsymbol{\beta}$. If we denote that $\tilde{\boldsymbol{\theta}}_0 = (\tilde{\boldsymbol{\alpha}}(\boldsymbol{\beta_0}), \boldsymbol{\beta_0})$ then under $H_0 : \boldsymbol{\beta} = \boldsymbol{\beta_0}$,

$$\boldsymbol{U}_\beta(\tilde{\boldsymbol{\theta}}_0)^{'}\,\boldsymbol{J}^{\beta\beta}(\tilde{\boldsymbol{\theta}}_0)\,\boldsymbol{U}_\beta(\tilde{\boldsymbol{\theta}}_0) \tag{2.17}$$

is asymptotically a chi-squared distributed with q degrees of freedom. A test of H_0 : $\boldsymbol{\beta} = \boldsymbol{\beta_0}$ based on (2.17) is called *a partial score test.* We can replace $\boldsymbol{J}^{\beta\beta}(\tilde{\boldsymbol{\theta}}_0)$ with a consistent estimator of $\boldsymbol{J}^{\beta\beta}(\boldsymbol{\theta_0})$, because this will also give the same asymptotic result

(Boos, 1992).

Suppose that $\{N(t), t > 0\}$ is a counting process and the associated intensity function is given by

$$\lambda(t|\mathcal{H}(t);\boldsymbol{\theta}) = \rho(t;\boldsymbol{\alpha})\exp\{\boldsymbol{X}(t)'\boldsymbol{\beta}\}, \quad t > 0, \tag{2.18}$$

where $\boldsymbol{X}(t)' = (X_1(t), \ldots, X_q(t))'$ is a $q \times 1$ vector, ρ is a baseline intensity function, and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)'$ is a vector of unknown nuisance parameters, and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_q)'$ is a vector of unknown regression parameters that is of interest. Then the likelihood function is given as follows.

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} \rho(t_i;\boldsymbol{\alpha})\exp\{\boldsymbol{X}(t_i)'\boldsymbol{\beta}\}\exp\left\{-\int_0^\tau \rho(s;\boldsymbol{\alpha})\exp\{\boldsymbol{X}(s)'\boldsymbol{\beta}\}ds\right\}. \tag{2.19}$$

The likelihood (2.19) is a partial likelihood and it is discussed by Cook and Lawless (2007, pp. 47–49). The log likelihood function $\ell(\boldsymbol{\theta}) = \log L(\boldsymbol{\theta})$ is given by

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{n} \log \rho(t_i;\boldsymbol{\alpha}) + \sum_{i=1}^{n} \boldsymbol{X}(t_i)'\boldsymbol{\beta} - \int_0^\tau \rho(s;\boldsymbol{\alpha})\exp\{\boldsymbol{X}(s)'\boldsymbol{\beta}\}ds. \tag{2.20}$$

Then the score vector is $\boldsymbol{U}(\boldsymbol{\theta}) = (\boldsymbol{U}_\alpha(\boldsymbol{\theta})', \boldsymbol{U}_\beta(\boldsymbol{\theta})')'$, where $\boldsymbol{U}_\alpha(\boldsymbol{\theta})$ is a $k \times 1$ vector of score functions with entries $\boldsymbol{U}_{\alpha_l}(\boldsymbol{\theta}) = \partial\ell(\boldsymbol{\theta})/\partial\alpha_l$, $l = 1, \ldots, k$, and $\boldsymbol{U}_\beta(\boldsymbol{\theta})$ is a $q \times 1$ vector of score functions with entries $\boldsymbol{U}_{\beta_j}(\boldsymbol{\theta}) = \partial\ell(\boldsymbol{\theta})/\partial\beta_j$, $j = 1, \ldots, q$. Therefore, the components $\boldsymbol{U}_{\alpha_l}(\boldsymbol{\theta})$ and $\boldsymbol{U}_{\beta_j}(\boldsymbol{\theta})$ are given as follows.

$$\boldsymbol{U}_{\alpha_l}(\boldsymbol{\theta}) = \sum_{i=1}^{n} \left(\frac{\partial}{\partial\alpha_l}\log\rho(t_i;\boldsymbol{\alpha})\right) - \int_0^\tau \left(\frac{\partial}{\partial\alpha_l}\rho(s;\boldsymbol{\alpha})\right)\exp\{\boldsymbol{X}(s)'\boldsymbol{\beta}\}ds, \tag{2.21}$$

and

$$\boldsymbol{U}_{\beta_j}(\boldsymbol{\theta}) = \sum_{i=1}^{n} X_j(t_i) - \int_0^\tau \rho(s;\boldsymbol{\alpha})X_j(s)\exp\{\boldsymbol{X}(s)'\boldsymbol{\beta}\}ds. \tag{2.22}$$

The observed information matrix is partitioned as follows.

$$\boldsymbol{I}(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{I}_{\alpha\alpha}(\boldsymbol{\theta}) & \boldsymbol{I}_{\alpha\beta}(\boldsymbol{\theta}) \\ \boldsymbol{I}_{\beta\alpha}(\boldsymbol{\theta}) & \boldsymbol{I}_{\beta\beta}(\boldsymbol{\theta}) \end{pmatrix}, \tag{2.23}$$

where $\boldsymbol{I}_{\alpha\alpha}(\boldsymbol{\theta})$ is a $k \times k$ matrix with entries $I_{\alpha_u\alpha_v} = -(\partial^2/\partial\alpha_u\partial\alpha_v)\ell(\boldsymbol{\theta})$, $u, v = 1, \ldots, k$, so that

$$I_{\alpha_u\alpha_v}(\boldsymbol{\theta}) = -\sum_{i=1}^{n}\left(\frac{\partial^2}{\partial\alpha_u\partial\alpha_v}\log\rho(t_i;\boldsymbol{\alpha})\right) - \int_0^\tau\left(\frac{\partial^2}{\partial\alpha_u\partial\alpha_v}\rho(s;\boldsymbol{\alpha})\right)\exp\{\boldsymbol{X}(s)^{'}\boldsymbol{\beta}\}ds, \tag{2.24}$$

$\boldsymbol{I}_{\alpha\beta}(\boldsymbol{\theta}) = \boldsymbol{I}_{\beta\alpha}(\boldsymbol{\theta})^{'}$ is a $k \times q$ matrix with entries $I_{\alpha_u\beta_v} = -(\partial^2/\partial\alpha_u\partial\beta_v)\ell(\boldsymbol{\theta})$, $u = 1\ldots,k$, $v = 1,\ldots,q$, so that

$$I_{\alpha_u\beta_v}(\boldsymbol{\theta}) = \int_0^\tau X_v(s)\left(\frac{\partial}{\partial\alpha_u}\rho(s;\boldsymbol{\alpha})\right)\exp\{\boldsymbol{X}(s)^{'}\boldsymbol{\beta}\}ds, \tag{2.25}$$

$\boldsymbol{I}_{\beta\beta}(\boldsymbol{\theta})$ is a $q \times q$ matrix with entries $I_{\beta_u\beta_v} = -(\partial^2/\partial\beta_u\partial\beta_v)\ell(\boldsymbol{\theta})$, $u, v = 1,\ldots,q$, is given by

$$I_{\beta_u\beta_v}(\boldsymbol{\theta}) = \int_0^\tau \rho(s;\boldsymbol{\alpha})X_u(s)\,X_v(s)\exp\{\boldsymbol{X}(s)^{'}\boldsymbol{\beta}\}ds. \tag{2.26}$$

For testing the null hypothesis $H_0 : \boldsymbol{\beta} = \boldsymbol{\beta_0}$ against the alternative hypothesis H_a : $\boldsymbol{\beta} \neq \boldsymbol{\beta_0}$, a partial score test can be used to test H_0. If we let $\boldsymbol{\theta_0} = (\boldsymbol{\alpha}_0^{'}, \boldsymbol{\beta}_0^{'})^{'}$ be a $p \times 1$ vector where $\boldsymbol{\alpha}_0$ is the true value of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}_0$ is the value of $\boldsymbol{\beta}$ under the null hypothesis. If we let $\boldsymbol{\beta_0} = 0$ so that we want to test $H_0 : \boldsymbol{\beta} = \boldsymbol{0}$, then $\boldsymbol{\theta_0} = (\boldsymbol{\alpha}_0^{'}, \boldsymbol{0}^{'})^{'}$. Let $\tilde{\boldsymbol{\alpha}}$ be a restricted maximum likelihood function of $\boldsymbol{\alpha}$ under $H_0 : \boldsymbol{\beta} = \boldsymbol{0}$, then the score statistic for testing $H_0 : \boldsymbol{\beta} = \boldsymbol{0}$ is of the form

$$\boldsymbol{U}_\beta(\tilde{\boldsymbol{\theta}}_0)^{'}\,\boldsymbol{J}^{\beta\beta}(\tilde{\boldsymbol{\theta}}_0)\,\boldsymbol{U}_\beta(\tilde{\boldsymbol{\theta}}_0), \tag{2.27}$$

where $\boldsymbol{U}_\beta(\tilde{\boldsymbol{\theta}}_0)$ is a $q \times 1$ vector of score functions with entries $\boldsymbol{U}_{\beta_j}(\tilde{\boldsymbol{\theta}}_0)$, $j = 1,\ldots,q$; that is,

$$\boldsymbol{U}_{\beta_j}(\tilde{\boldsymbol{\theta}}_0) = \sum_{i=1}^{n} X_j^{'}(t_i) - \int_0^\tau \rho(s;\tilde{\boldsymbol{\alpha}})X_j(s)ds, \tag{2.28}$$

and $q \times q$ matrix $\boldsymbol{J}^{\beta\beta}(\tilde{\boldsymbol{\theta}}_0)$ is given by

$$\left\{\boldsymbol{J}_{\beta\beta}(\tilde{\boldsymbol{\theta}}_0) - \boldsymbol{J}_{\beta\alpha}(\tilde{\boldsymbol{\theta}}_0)\boldsymbol{J}_{\alpha\alpha}(\tilde{\boldsymbol{\theta}}_0)^{-1}\boldsymbol{J}_{\alpha\beta}(\tilde{\boldsymbol{\theta}}_0)\right\}^{-1}. \tag{2.29}$$

Replacing $\boldsymbol{J}^{\beta\beta}(\tilde{\boldsymbol{\theta}}_0)$ with $\boldsymbol{I}^{\beta\beta}(\tilde{\boldsymbol{\theta}}_0)$ in (2.27) gives the same asymptotic results.

2.3 Simulation Procedures

In this section, we explain the simulation procedures used in the subsequent chapters of this thesis. This simulation procedures can be used (i) to study the null distribution of the test statistic, (ii) to study the distribution of the test statistic under the alternative hypothesis and also obtain the power, (iii) to obtain the p-value.

A key result for generating a realization of a counting process is given in the following. Its proof can be found, e.g., in Cook and Lawless (2007, p. 30). For a counting process $\{N(t), t > 0\}$, with the associated intensity fuction $\lambda(t|\mathcal{H}(t))$,

$$\Pr\{N(s,t) = 0|\mathcal{H}(s^+)\} = \exp\left(-\int_s^t \lambda(u|\mathcal{H}(u))du\right), \tag{2.30}$$

where in the exponential term $\mathcal{H}(u)) = \{\mathcal{H}(s^+), N(s,u) = 0\}$. Therefore, we can show that $\Pr\{N(t_{j-1}, t_{j-1} + w) = 0|\mathcal{H}(t_{j-1}^+)\} = \exp\{-\int_{t_{j-1}}^{t_{j-1}+w} \lambda(u|\mathcal{H}(u))du\}$.

The events "$N(t_{j-1}, t_{j-1} + w) = 0|\mathcal{H}(t_{j-1}^+)$" and "$W_j > w|T_{j-1} = t_{j-1}, \mathcal{H}(t_{j-1})$" are equivalent almost surely. Therefore,

$$\Pr\{W_j > w|T_{j-1} = t_{j-1}, \mathcal{H}(t_{j-1})\} = \exp\left(-\int_{t_{j-1}}^{t_{j-1}+w} \lambda(u|\mathcal{H}(u))du\right). \tag{2.31}$$

Now, if we let $E_j = \int_{t_{j-1}}^{t_{j-1}+W_j} \lambda(u|\mathcal{H}(u))du$ where $j = 1, 2, \ldots$, from (2.31) the random variable E_j has an exponential distribution with mean 1, given t_{j-1}, $t_0 = 0$ and $\mathcal{H}(t)$ (Cook and Lawless, 2007, p. 44). Therefore, $U_j = \exp\left(-\int_{t_{j-1}}^{t_{j-1}+W_j} \lambda(u|\mathcal{H}(u))du\right) = \exp\{-E_j\}$ has a uniform distribution on $(0, 1)$. Then we can obtain each event T_j by solving an equation $E_j = \int_{t_{j-1}}^{t_{j-1}+W_j} \lambda(u|\mathcal{H}(u))du$ for W_j. The simulation algorithm given for this procedure is given below:

1. Set $j = 1$, $t_0 = 0$.

2. Generate U_j from the standard uniform distribution.

3. Obtain E_j by transforming $\exp\{-E_j\} = U_j$, so that $E_j = -\log(U_j)$.

4. Obtain w_j by solving an equation $E_j = \int_{t_{j-1}}^{t_{j-1}+W_j} \lambda(u|\mathcal{H}(u))du$ for W_j.

5. If $T_j \leq \tau$, then increase j by 1 and go to the step 2, otherwise stop the loop.

In generating a HPP with the rate function ρ, steps 2-4 give $W_J = -\log(U_j)/\rho$, where $j = 1, 2, \ldots$. In generation of a NHPP with the rate function $\rho(t)$, mean function $\mu(t)$ can be generated by using $N^*(s) = N(\mu^{-1}(s))$, where $\{N^*(s), s > 0\}$ is a HPP with mean function $\mu^* = 1$ (see Section 1.2). Therefore, in generation of NHPP, steps 2-4 give HPP with rate 1 as $\mu(T_j) = \mu(t_{j-1}) + E_j$, where $j = 1, 2, \ldots$ and $T_j = t_{j-1} + W_j$ is the jth event time. Then $T_j = \mu^{-1}(\mu(t_{j-1}) + E_j)$ gives the jth event time for NHPP, where μ^{-1} is the inverse transformation of μ.

The above algorithm has been used and recorded widely in the literature (see, e.g., Lewis and Shedler, 1976; Daley and Vare-Jones, 1988; Cook and Lawless, 2007). It should be noted that the integral in step 4 may not have a closed form. In such cases, numerical methods can be applied to obtain $W_j = w_j$.

2.3.1 The Use of the Simulations

We are able to discuss the distributions of the statistics and obtain p-values by generating data under null and alternative hypotheses. For example, suppose that $\{N(t), t > 0\}$ is a counting process with the intensity function $\lambda(t|\mathcal{H}(t)) = \rho(t; \boldsymbol{\alpha}) \exp\{\boldsymbol{X}(t)'\boldsymbol{\beta}\}$, and testing following composite hypotheses is of interests.

$$H_0 : \boldsymbol{\beta} = \mathbf{0}, \boldsymbol{\alpha} \in \mathbb{R}^k \qquad \text{vs.} \qquad H_a : \boldsymbol{\beta} \neq \mathbf{0}, \boldsymbol{\alpha} \in \mathbb{R}^k \tag{2.32}$$

We generate B realizations of recurrent event processes under the null hypothesis to assess the asymptotic distribution of a statistic. For each realization, we obtain $\tilde{\boldsymbol{\theta}}_0 = (\tilde{\boldsymbol{\alpha}}', \tilde{\mathbf{0}}')'$, the partial score vector $\boldsymbol{U}_\beta(\tilde{\boldsymbol{\theta}}_0)'$ and the matrix $\boldsymbol{J}^{\beta\beta}(\tilde{\boldsymbol{\theta}}_0)$, where $\tilde{\boldsymbol{\theta}}_0$ is an estimate of $\boldsymbol{\theta}$ under the null hypothesis in (2.32). Then we can use these to study the distribution of a score test statistic, under the null hypothesis.

Similarly, the power of the test can be obtained by simulation. We let the power function of the test of hypothesis (2.32) as $P(\boldsymbol{\beta_a}) = \Pr\{\text{reject } H_0 | \boldsymbol{\beta} = \boldsymbol{\beta_a}\}$. We can obtain the power by data generation under the alternative hypothesis in (2.32) and by obtaining $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\alpha}}', \hat{\boldsymbol{\beta}}')'$, the partial score vector $\boldsymbol{U}_\beta(\hat{\boldsymbol{\theta}})'$ and the matrix $\boldsymbol{J}^{\beta\beta}(\hat{\boldsymbol{\theta}})$, where $\hat{\boldsymbol{\theta}}$ is an estimate of $\boldsymbol{\theta}$ under the alternative hypothesis in (2.32).

We can obtain p-value by simulation. We first generate B realizations of recurrent event processes under the null hypothesis. Then we calculate the partial score statistic Z_i, $i = 1, \dots, B$. Then p-value can be estimated by obtaining

$$\frac{\sum_{i=1}^{B} I(Z_i > Z_{test})}{B}, \tag{2.33}$$

where Z_{test} is the test statistic based on the given data set. Under the null hypothesis in (2.32), $\boldsymbol{U}_\beta(\tilde{\boldsymbol{\theta}}_0)' \, \boldsymbol{J}^{\beta\beta}(\tilde{\boldsymbol{\theta}}_0) \, \boldsymbol{U}_\beta(\tilde{\boldsymbol{\theta}}_0)$ is asymptotically chi-squared distributed with q degrees freedom. Then p-value can be also obtained by $Pr\{\chi_q^2 \geq \boldsymbol{U}_\beta(\tilde{\boldsymbol{\theta}}_0)' \, \boldsymbol{J}^{\beta\beta}(\tilde{\boldsymbol{\theta}}_0) \, \boldsymbol{U}_\beta(\tilde{\boldsymbol{\theta}}_0)\}$, where $\tilde{\boldsymbol{\theta}}_0$ is an estimate of $\boldsymbol{\theta}$ under the null hypothesis in (2.32).

Chapter 3

Redundant Systems with Two Components

In this chapter, we consider a system which includes two components; a primary component and a redundant component working in parallel. Our goal is to develop a formal method to assess whether or not there is an adverse effect of repairs of a failed component on the redundant component. We therefore develop partial score tests and discuss their asymptotic properties analytically and through simulations.

3.1 Introduction

As discussed previously, systems consisting of parallely connected components can be seen in many industrial settings. Some examples include light bulbs in an automobile, batteries in a laptop computer, computer server nodes and so forth. In repairable systems settings, components of a system are subject to repairs and repair times may not be negligible. For example, in fully redundant systems, the remaining components share the duty of a failed component during its downtimes. This type of downtimes due to unwanted failures in components may have some temporary residual effects resulting in an increased risk of failures in the redundant components of a system during the downtime of the failed components. We refer to such adverse effects as parallel carryover effects. If a parallel carryover effect is significant, the system may not perform well, and the cost of operation can be considerably high. Therefore, there is a room for improving the reliability of a system by detecting parallel carryover effects.

In this section, we develop tests for the presence of parallel carryover effects, and discuss their asymptotic properties. To this end, we consider partial score tests. Partial score tests provide a convenient way of testing for the presence of a parallel carryover effect because they do not require to obtain the maximum likelihood estimates of the parameters under the alternative hypothesis.

3.2 Models and Tests for Parallel Carryover Effects

In this section, we first extend the notation introduced in the previous chapter to the "two components" case. Then, we discuss models for a single system and multiple systems under two different settings according to the duration of repairs in the redundant component.

Let $\{N_1(t), N_2(t), \ldots, N_K(t); t \geq 0\}$ be a K-variate process. If each $\{N_i(t); t \geq 0\}$, $i = 1, \ldots, K$, is a counting process and no two or more of the processes jump at the same time, the K-variate process $\{N_1(t), N_2(t), \ldots, N_K(t); t \geq 0\}$ is called a *multivariate counting process* (Fleming and Harrington, 1991). In this case, we let $\mathcal{H}(t)$ denote the history of the multivariate counting process at time t, where we assume that $\mathcal{H}(t)$ includes all information on the event times and the number of events of each counting process $\{N_i(t), t \geq 0\}$ $(i = 1, \ldots, K)$ in $[0, t)$.

Now, suppose that there is a bivariate counting process $\{N_A(t), N_B(t); t \geq 0\}$, where $\{N_A(t), t \geq 0\}$ is a counting process for Component A and $\{N_B(t), t \geq 0\}$ is a counting process for Component B in a system of two parallely connected components. We let $t_{A1}, t_{A2}, \ldots,$ where $0 < t_{A1} < t_{A2} < \cdots$, and $t_{B1}, t_{B2}, \ldots,$ where $0 < t_{B1} < t_{B2} < \cdots$, denote the failure times of Components A and B, respectively. The components are subject to repairs and repair times cannot be ignored. Let Δ_A and Δ_B denote the repair times of Components A and B, respectively. In other words, if Component A fails, for example, we assume that the repair takes Δ_A time units. Similarly, if Component B fails, it takes Δ_B time units to repair it. We also need to define at-risk indicators. For $K = A, B$, the function $Y_K(t)$ is called *at-risk indicator* of the process $\{N_K(t), t > 0\}$, which takes the value of 1 when Component K is up and the process $\{N_K(t), t > 0\}$ is under observation; otherwise, it is equal to 0. For example, if at time t Component A is up and under observation and Component B is down, then $Y_A(t) = 1$ and $Y_B(t) = 0$.

A model including parallel carryover effects in Component A is given by

$$\lambda_A(t|\mathcal{H}(t)) = Y_A(t)\,\alpha_A\,\exp\{\beta_A\,X_A(t)\}, \qquad t > 0, \tag{3.1}$$

where $Y_A(t)$ is the at-risk function of Component A, $\alpha_A > 0$ is a baseline rate function, and

$$X_A(t) = I\{N_B(t^-) > 0\}I\{t - t_{BN_B(t^-)} \leq \Delta_B\}, \tag{3.2}$$

and β_A is a regression parameter. The intensity function (3.1) increases from α_A to $\alpha_A\,\exp\{\beta_A\}$ at each failure time of Component B for Δ_B time units. After Δ_B time units, it reduces to α_A. This behaviour is what we refer to as a parallel carryover effect. In the remaining part of this section, we develop formal tests for the presence of such effects. It should be noted that Δ_B defines the duration of a carryover effect period in Component A. We therefore call Δ_B the carryover effect period in Component A. In this study, we assume that carryover effect periods are constant. We discuss issues related to the choice of the carryover effect periods in the final chapter.

A model for parallel carryover effects can also be similarly defined for Component B. In this case, the intensity function of $\{N_B(t),\, t \geq 0\}$ is given by

$$\lambda_B(t|\mathcal{H}(t)) = Y_B(t)\,\alpha_B\,\exp\{\beta_B\,X_B(t)\}, \qquad t > 0, \tag{3.3}$$

where $Y_B(t)$ is the at-risk function of Component B, $\alpha_B > 0$ is a baseline rate function, and

$$X_B(t) = I\{N_A(t^-) > 0\}I\{t - t_{AN_A(t^-)} \leq \Delta_A\}. \tag{3.4}$$

Similarly, parallel carryover effects in Component B can be investigated through model (3.3) with (3.4).

It should be noted that the models (3.1) and (3.3) can be equivalently written in a simple form as follows. For $K, J = A, B$ and $K \neq J$,

$$\lambda_K(t|\mathcal{H}(t)) = Y_K(t)\exp\{\beta_K(1 - Y_J(t))\}, \qquad t > 0. \tag{3.5}$$

From (3.5), it is easy to see that the system is down if and only if both components are down; that is, $Y_A(t) = 0$ and $Y_B(t) = 0$ at time t. In the following development in this chapter, we mostly focus on the models of the former types because they provide explicit relation to the duration of repairs.

Let m denote the number of systems in a study, each with two components. In the next subsections, we develop partial score tests for the presence of parallel carryover effects in four different cases; (i) a single system is under observation and repair times of Component A are negligible ($m = 1$, $\Delta_A = 0$), (ii) multiple systems are under observation and repair times of Component A are negligible ($m > 1$, $\Delta_A = 0$), (iii) a single system is under observation and repair times of Component A are not negligible ($m = 1$, $\Delta_A > 0$), and (iv) multiple systems are under observation and repair times of Component A are not negligible ($m > 1$, $\Delta_A > 0$). In all cases, we assume the baseline rate functions are constants. We consider the settings in which baseline rate functions depend on time in Chapter 5.

3.2.1 Case 1: $m = 1$, $\Delta_A = 0$

We first consider a single system ($m = 1$) with two components; Component A and Component B. In this setting, we assume that repair times of one of the components (say, Component A) is negligible; that is, $\Delta_A = 0$ so that failures of Component A does not affect the probabilistic characteristics of the failure occurrences in Component B. Furthermore, we assume that failure occurrences are governed by HPPs. Under these assumptions and following the notation given previously, the model for Component A is of the form

$$\lambda_A(t|\mathcal{H}(t)) = Y_A(t)\,\alpha_A\,\exp\{\beta_A\,X_A(t)\}, \qquad t > 0, \tag{3.6}$$

where $X_A(t)$ is defined in (3.2) and $\mathcal{H}(t) = \{N_A(u), N_B(u); 0 \leq u < t\}$. Since $\Delta_A = 0$ and $\Delta_B > 0$, the intensity function of Component B is given by

$$\lambda_B(t|\mathcal{H}(t)) = Y_B(t)\,\alpha_B, \qquad t > 0. \tag{3.7}$$

A test for a parallel carryover effect in Component A can be developed by considering the following composite hypothesis.

$$H_0 : \beta_A = 0,\ \alpha_A > 0, \qquad \text{vs.} \qquad H_1 : \beta_A \neq 0,\ \alpha_A > 0, \tag{3.8}$$

where α_A is a nuisance parameter.

Suppose that such a system with its components is under observation over the followup period $[0, \tau]$, where τ is a fixed end-of-followup time. Notice that, since Component A is continuously under observation over $[0, \tau]$ and its repair times are negligible, we can safely drop $Y_A(t)$ from the model (3.6). Let n_A, where $n_A \geq 0$, denote the number of failures of Component A in $[0, \tau]$ and t_{A1}, t_{A2}, ..., t_{An_A} be the failure times of Component A. The likelihood function with the outcome "$N_A(\tau) = n_A$ failures of Component A at times $t_{A1} \leq t_{A2} \leq \cdots \leq t_{An_A}$ in $[0, \tau]$" is then given by

$$L(\boldsymbol{\theta}) = \prod_{j=1}^{n_A} \alpha_A \exp\{\beta_A X_A(t_{Aj})\} \exp\{-\int_0^\tau \alpha_A \exp\{\beta_A X_A(s)\} \, ds\}, \tag{3.9}$$

where $\boldsymbol{\theta} = (\alpha_A, \beta_A)$. The log likelihood function $\ell(\boldsymbol{\theta}) = \log L(\boldsymbol{\theta})$ is given by

$$\ell(\boldsymbol{\theta}) = n_A \log \alpha_A + \sum_{j=1}^{n_A} \beta_A X_A(t_{Aj}) - \int_0^\tau \alpha_A \, e^{\beta_A X_A(s)} \, ds. \tag{3.10}$$

The components of the score vector $\boldsymbol{U}(\boldsymbol{\theta}) = (U_{\alpha_A}(\boldsymbol{\theta}), U_{\beta_A}(\boldsymbol{\theta}))'$, where $U_{\alpha_A}(\boldsymbol{\theta}) = (\partial/\partial\alpha_A)\ell(\boldsymbol{\theta})$ and $U_{\beta_A}(\boldsymbol{\theta}) = (\partial/\partial\beta_A)\ell(\boldsymbol{\theta})$ can be written as follows.

$$U_{\alpha_A}(\boldsymbol{\theta}) = \frac{n_A}{\alpha_A} - \int_0^\tau \exp\{\beta_A X_A(s)\} \, ds, \tag{3.11}$$

and

$$U_{\beta_A}(\boldsymbol{\theta}) = \sum_{j=1}^{n_A} X_A(t_{Aj}) - \alpha_A \int_0^\tau X_A(s) \exp\{\beta_A X_A(s)\} \, ds. \tag{3.12}$$

The observed information matrix $\boldsymbol{I}(\boldsymbol{\theta})$ is given by

$$\boldsymbol{I}(\boldsymbol{\theta}) = \begin{pmatrix} I_{\alpha_A\alpha_A}(\boldsymbol{\theta}) & I_{\alpha_A\beta_A}(\boldsymbol{\theta}) \\ I_{\beta_A\alpha_A}(\boldsymbol{\theta}) & I_{\beta_A\beta_A}(\boldsymbol{\theta}) \end{pmatrix}, \tag{3.13}$$

with the components

$$I_{\alpha_A\alpha_A}(\boldsymbol{\theta}) = \frac{n_A}{{\alpha_A}^2}, \tag{3.14}$$

$$I_{\alpha_A\beta_A}(\boldsymbol{\theta}) = I_{\beta_A\alpha_A}(\boldsymbol{\theta}) = \int_0^\tau X_A(s) \exp\{\beta_A X_A(s)\} \, ds, \tag{3.15}$$

$$I_{\beta_A\beta_A}(\boldsymbol{\theta}) = \alpha_A \int_0^\tau X_A(s) \exp\{\beta_A X_A(s)\} \, ds. \tag{3.16}$$

Let $\tilde{\alpha}_A$ denote the restricted maximum likelihood estimator of α_A under the null hypothesis $H_0 : \beta_A = 0$. Letting $\beta_A = 0$ in (3.11), and then solving $U_{\alpha_A}(\alpha_A, 0) = 0$ for α_A, we find that $\tilde{\alpha}_A = n_A/\tau$, where $\tilde{\alpha}_A$ is the restricted maximum likelihood estimator of α_A. For notational convenience, we define the integral

$$\mathcal{I}(\tau, \beta_A, \Delta_B) = \int_0^\tau X_A(s) \exp\{\beta_A X_A(s)\} \, ds. \tag{3.17}$$

The integral $\mathcal{I}(\tau, \beta_A, \Delta_B)$ is a function of the followup period $[0, \tau]$, the regression parameter β_A, and the duration of the repairs of Component B; that is, Δ_B. With this notation, the partial score statistic $U_{\beta_A}(\tilde{\alpha}_A, 0)$, where $U_{\beta_A}(\alpha_A, \beta_A)$ is given in (3.12), can be written as

$$U_{\beta_A}(\tilde{\alpha}_A, 0) = \sum_{j=1}^{n_A} X_A(t_{Aj}) - \tilde{\alpha}_A \mathcal{I}(\tau, 0, \Delta_B), \tag{3.18}$$

and the estimated variance of $U_{\beta_A}(\tilde{\alpha}_A, 0)$ is given by

$$\widehat{Var}(U_{\beta_A}(\tilde{\alpha}_A, 0)) = I_{\beta_A \beta_A}(\tilde{\alpha}, 0) - I_{\beta_A \alpha_A}(\tilde{\alpha}, 0) \, I^{-1}_{\alpha_A \alpha_A}(\tilde{\alpha}, 0) \, I_{\alpha_A \beta_A}(\tilde{\alpha}, 0), \tag{3.19}$$

$$= \left(\frac{n_A}{\tau^2}\right) \mathcal{I}(\tau, 0, \Delta_B) \, [\tau - \mathcal{I}(\tau, 0, \Delta_B)]. \tag{3.20}$$

The standardized partial score statistic for testing H_0 in (3.8) is then

$$Z = \frac{U_\beta(\tilde{\alpha}_A, 0)}{\widehat{Var}(U_\beta(\tilde{\alpha}_A, 0))^{\frac{1}{2}}}, \tag{3.21}$$

where $U_\beta(\tilde{\alpha}_A, 0)$ and $\widehat{Var}(U_{\beta_A}(\tilde{\alpha}_A, 0))$ are given in (3.18) and (2.20), respectively. As we show in Section 3.3, the distribution of (3.21) is asymptotically standard normal as $\tau \to \infty$. The p–values for H_0 in (3.8) can be obtained from this approximation. Alternatively, when τ is small, the p–values can be computed via simulation. In Section 3.4, we discuss the asymptotic properties and power of the test statistic (3.21) through simulations under various scenarios.

3.2.2 Case 2: $m > 1$, $\Delta_A = 0$

We now consider the case in which the number of systems m is greater than 1 and repair times of Component A are negligible; that is, $m > 1$ and $\Delta_A = 0$. We consider a similar setup given in Section 3.2.1, so each system has two components connected in parallel and failures are governed by homogeneous Poisson processes (HPPs). We assume that failures of Component B affect the probabilistic characteristics of failure occurrences of Component A, while failures of Component A do not affect the probabilistic characteristics of failure occurrences of Component B.

Since we consider $m > 1$ in this case, we need to include an index to denote the system under observation. Therefore, we adapt the notation of the previous subsection to this situation. Let's suppose that we have m independent systems under observation, each with two components; Component A and Component B. Furthermore, suppose that there are m bivariate counting processes $\{N_{Ai}(t), N_{Bi}(t); t \geq 0\}$, $i = 1, \ldots, m$, where $\{N_{Ai}(t); t \geq 0\}$ is a counting process for the failures of Component A in the ith system and $\{N_{Bi}(t); t \geq 0\}$ is a counting process for the failures of Component B in the ith systems. For $i = 1, \ldots, m$, we let $t_{Ai1}, t_{Ai2}, \ldots$, where $0 < t_{Ai1} < t_{Ai2} < \cdots$, denote the failure times of Component A in the ith system. Similarly, for $i = 1, \ldots, m$, we let $t_{Bi1}, t_{Bi2}, \ldots$, where $0 < t_{Bi1} < t_{Bi2} < \cdots$, denote the failure times of Component B in the ith system. Thus, t_{Kij} is the jth failure time of Component K in the ith system, where $K = A, B$; $i = 1, \ldots, m$; and $j = 1, 2, \ldots$. We use the notation $Y_{Ai}(t)$ and $Y_{Bi}(t)$ to denote the at-risk indicators of Component A and Component B in the ith system, respectively, so that Y_{Ki}, $K = A, B$, takes the value of 0 when Component K in the ith system is down; otherwise, it is equal to 1. Finally, we let $\mathcal{H}_i(t) = \{N_{Ai}(u), N_{Bi}(u); 0 \leq u < t\}$ denote the history of the bivariate process $\{N_{Ai}(t), N_{Bi}(t); t \geq 0\}$, where $i = 1, \ldots, m$.

The intensity function of Component A in the ith system, $i = 1, \ldots, m$, is given by

$$\lambda_{Ai}(t|\mathcal{H}_i(t)) = Y_{Ai}(t)\, \alpha_A \exp\{\beta_A X_{Ai}(t)\}, \qquad t > 0, \tag{3.22}$$

where $X_{Ai}(t) = I\{N_{Bi}(t^-) > 0\} I\{t - t_{BiN_{Bi}(t^-)} \leq \Delta_B\}$ and Δ_B is the duration of the repairs of Component B. Since $\Delta_A = 0$ and $\Delta_B > 0$, the intensity function of Component B in the ith system , where $i = 1, \ldots, m$, is given by

$$\lambda_{Bi}(t|\mathcal{H}_i(t)) = Y_{Bi}(t)\, \alpha_B, \qquad t > 0. \tag{3.23}$$

Once again, we consider the following hypothesis for a test of parallel carryover effects.

$$H_0 : \beta_A = 0,\ \alpha_A > 0, \qquad \text{vs.} \qquad H_1 : \beta_A \neq 0,\ \alpha_A > 0, \tag{3.24}$$

where α_A is a nuisance parameter.

Suppose that m such independent systems are under observation over the interval

$[0, \tau_i]$, where τ_i is the end-of-followup time of the ith system and $i = 1, \ldots, m$. Also, let $N_{Ai}(\tau_i) = n_{Ai}$ and $t_{Ai1}, \ldots, t_{Ain_{Ai}}$ be the failure times of Component A in the ith system for $i = 1, \ldots, m$. The likelihood function for m independent systems is given by

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{m} L_i(\boldsymbol{\theta}), \tag{3.25}$$

where $\boldsymbol{\theta} = (\alpha_A, \beta_A)$ and

$$L_i(\boldsymbol{\theta}) = \prod_{j=1}^{n_{Ai}} \alpha_A \, \exp\{\beta_A X_{Ai}(t_{Aij})\} \, \exp\{-\int_0^{\tau_i} \alpha_A \, \exp\{\beta_A X_{Ai}(s)\} \, ds\}, \tag{3.26}$$

which is the likelihood contribution of the ith system for the outcome "$N_{Ai}(\tau_i) = n_{Ai}$ failures of Component A in the ith system at times $t_{Ai1} < t_{Ai2} < \cdots < t_{Ain_{Ai}}$". Once again, we would like to note that the at-risk indicators Y_{Ai}, $i = 1, \ldots, m$, are not needed in the likelihood function (3.25). The corresponding log likelihood function $\ell(\boldsymbol{\theta}) = \log L(\boldsymbol{\theta})$ is given by

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{m} \ell_i(\boldsymbol{\theta}), \tag{3.27}$$

where

$$\ell_i(\boldsymbol{\theta}) = n_{Ai} \log \alpha_A + \beta_A \sum_{j=1}^{n_{Ai}} X_{Ai}(t_{Aij}) - \alpha_A \int_0^{\tau_i} \exp\{\beta_A X_{Ai}(s)\} \, ds. \tag{3.28}$$

Once again for notational convenience, we define the functions

$$\mathcal{I}_1(m, \beta_A, \Delta_B) = \sum_{i=1}^{m} \int_0^{\tau_i} \exp\{\beta_A X_{Ai}(s)\} \, ds, \tag{3.29}$$

and

$$\mathcal{I}_2(m, \beta_A, \Delta_B) = \sum_{i=1}^{m} \int_0^{\tau_i} X_{Ai}(s) \, \exp\{\beta_A X_{Ai}(s)\} \, ds. \tag{3.30}$$

The functions (3.29) and (3.30) depend on the number of systems m, the duration of the downtimes of Component B; that is, Δ_B and the parameter β_A. They also depend on the duration of followups $[0, \tau_i]$, $i = 1, \ldots, m$, but since we are not interested in the asympotic distribution of the test statistic developed later in this section when τ_i

increases, we do not emphasize them in (3.29) and (3.30). With this notation, components of the score vector $\boldsymbol{U}(\boldsymbol{\theta}) = (U_{\alpha_A}(\boldsymbol{\theta}), U_{\beta_A}(\boldsymbol{\theta}))'$, where $U_{\alpha_A}(\boldsymbol{\theta}) = (\partial/\partial\alpha_A)\ell(\boldsymbol{\theta})$ and $U_{\beta_A}(\boldsymbol{\theta}) = (\partial/\partial\beta_A)\ell(\boldsymbol{\theta})$ are given by

$$U_{\alpha_A}(\boldsymbol{\theta}) = \sum_{i=1}^{m} \frac{n_{Ai}}{\alpha_A} - \mathcal{I}_1(m, \beta_A, \Delta_B), \tag{3.31}$$

and

$$U_{\beta_A}(\boldsymbol{\theta}) = \sum_{i=1}^{m} \sum_{j=1}^{n_{Ai}} X_{Ai}(t_{Aij}) - \alpha_A \mathcal{I}_2(m, \beta_A, \Delta_B). \tag{3.32}$$

The components of the 2×2 observed information matrix $\boldsymbol{I}(\boldsymbol{\theta})$, where

$$\boldsymbol{I}(\boldsymbol{\theta}) = \begin{pmatrix} I_{\alpha_A\alpha_A}(\boldsymbol{\theta}) & I_{\alpha_A\beta_A}(\boldsymbol{\theta}) \\ I_{\beta_A\alpha_A}(\boldsymbol{\theta}) & I_{\beta_A\beta_A}(\boldsymbol{\theta}) \end{pmatrix}, \tag{3.33}$$

are given by

$$I_{\alpha_A\alpha_A}(\boldsymbol{\theta}) = \sum_{i=1}^{m} \frac{n_{Ai}}{\alpha_A^2}, \tag{3.34}$$

$$I_{\alpha_A\beta_A}(\boldsymbol{\theta}) = I_{\beta_A\alpha_A}(\boldsymbol{\theta}) = \mathcal{I}_2(m, \beta_A, \Delta_B), \tag{3.35}$$

$$I_{\beta_A\beta_A}(\boldsymbol{\theta}) = \alpha_A \mathcal{I}_2(m, \beta_A, \Delta_B). \tag{3.36}$$

Let $\tilde{\alpha}_A$ be the restricted maximum likelihood estimator of α_A under the null hypothesis $H_0 : \beta_A = 0$. By solving $U_{\alpha_A}(\alpha_A, 0) = 0$ in (3.32) for $\alpha_A = \tilde{\alpha}_A$, we can obtain

$$\tilde{\alpha}_A = \frac{\sum_{i=1}^{m} n_{Ai}}{\sum_{i=1}^{m} \tau_i}. \tag{3.37}$$

Following the score procedures explained in Chapter 1, we obtain the standardized partial score statistic

$$Z = \frac{U_\beta(\tilde{\alpha}_A, 0)}{\widehat{Var}(U_\beta(\tilde{\alpha}_A, 0))^{\frac{1}{2}}}, \tag{3.38}$$

for testing the null hypothesis $H_0 : \beta_A = 0$, where

$$U_\beta(\tilde{\alpha}_A, 0) = \sum_{i=1}^{m} \sum_{j=1}^{n_{Ai}} X_{Ai}(t_{Aij}) - \tilde{\alpha}_A \mathcal{I}_2(m, 0, \Delta_B), \tag{3.39}$$

and

$$\widehat{Var}(U_\beta(\tilde{\alpha}_A, 0)) = \left(\frac{\sum_{i=1}^m n_{Ai}}{(\sum_{i=1}^m \tau_i)^2}\right) \mathcal{I}_2(m, 0, \Delta_B) \left(\sum_{i=1}^m \tau_i - \mathcal{I}_2(m, 0, \Delta_B)\right). \tag{3.40}$$

We discuss the asymptotic distribution of the test statistic (3.38) analytically as well as through simulations in Sections 3.3 and 3.4, respectively. In Section 3.3, we showed that the standardized partial score statistic Z in (3.38) converges to a standard normal distribution under the null hypothesis stated in (3.24) as $m \to \infty$ for fixed observation periods. Therefore, this result can be used to calculate p–values for testing the presence of parallel carryover effects in the redundant component (Component A) when m is large. For small m values, p–values can be obtained via simulations.

3.2.3 Case 3: $m = 1$, $\Delta_A > 0$

In some systems, one of the components operates in a constant or full operating capacity in the up state and does not change its load following the failures of other components, while other components are still redundant. We now focus on such a system with two components; Component A and Component B. The setup of this section is similar to that of Section 3.2.1 except that both Components A and B are subject to non-negligible repair times.

The model for Component A is given by

$$\lambda_A(t|\mathcal{H}(t)) = Y_A(t)\, \alpha_A \, \exp\{\beta_A X_A(t)\}, \qquad t > 0, \tag{3.41}$$

where $X_A(t)$ is defined in (3.2) and $\mathcal{H}(t) = \{N_A(u), N_B(u); 0 \leq u < t\}$. It should be noted that this is the same model for Component A given in Section 3.2.1, but in this case, since $\Delta_A > 0$, we cannot drop $Y_A(t)$ from the model. The model for the Component B is once again defined by

$$\lambda_B(t|\mathcal{H}(t)) = Y_B(t)\, \alpha_B, \qquad t > 0. \tag{3.42}$$

Similarly, we want to test the null hypothesis $H_0 : \beta_A = 0$ against the alternative hypothesis $H_1 : \beta_A \neq 0$.

Following the setup in Section 3.2.1, the likelihood function with the outcome that "$N_A(\tau) = n_A$ failures of Component A at times $t_{A1} < t_{A2} < \cdots < t_{An_A}$ in a fixed interval $[0,\tau]$" can be written as follows.

$$L(\boldsymbol{\theta}) = \prod_{j=1}^{n_A} \alpha_A \exp\{\beta_A X_A(t_{Aj})\} \exp\{-\int_0^\tau Y_A(t)\,\alpha_A \exp\{\beta_A X_A(s)\}\,ds\}. \tag{3.43}$$

where $\boldsymbol{\theta} = (\alpha_A, \beta_A)$. The log likelihood function $\ell(\boldsymbol{\theta})$ is given by

$$l(\boldsymbol{\theta}) = n_A \log \alpha_A + \sum_{j=1}^{n_A} \beta_A X_A(t_{Aj}) - \int_0^\tau Y_A(t)\,\alpha_A \exp\{\beta_A X_A(s)\}\,ds \tag{3.44}$$

The components of the score vector $U(\boldsymbol{\theta})$ are given by

$$U_{\alpha_A}(\boldsymbol{\theta}) = \frac{n_A}{\alpha_A} - \int_0^\tau Y_A(t) \exp\{\beta_A X_A(s)\}\,ds, \tag{3.45}$$

and

$$U_{\beta_A}(\boldsymbol{\theta}) = \sum_{j=1}^{n_A} X_A(t_{Aj}) - \alpha_A \int_0^\tau Y_A(t)\,X_A(s) \exp\{\beta_A X_A(s)\}\,ds. \tag{3.46}$$

Also, components of the 2×2 observation matrix $\boldsymbol{I}(\boldsymbol{\theta})$ are given by

$$I_{\alpha_A\alpha_A}(\boldsymbol{\theta}) = \frac{n_A}{{\alpha_A}^2}, \tag{3.47}$$

$$I_{\alpha_A\beta_A}(\boldsymbol{\theta}) = I_{\beta_A\alpha_A}(\boldsymbol{\theta}) = \int_0^\tau Y_A(s)\,X_A(s) \exp\{\beta_A X_A(s)\}\,ds, \tag{3.48}$$

$$I_{\beta_A\beta_A}(\boldsymbol{\theta}) = \alpha_A \int_0^\tau Y_A(s)\,X_A(s) \exp\{\beta_A X_A(s)\}\,ds. \tag{3.49}$$

Let $\tilde{\alpha}_A$ be the restricted maximum likelihood estimator of α_A under the null hypothesis $H_0 : \beta_A = 0$. Then, we obtain

$$\tilde{\alpha}_A = \frac{n_A}{\int_0^\tau Y_A(s)ds}. \tag{3.50}$$

Notice that the restricted maximum likelihood estimator $\tilde{\alpha}_A$ is the ratio of the observed number of failures in Component A over $[0,\tau]$ to the total time that Component A stays in the up state in $[0,\tau]$. For convenience, we define

$$\mathcal{I}(\tau, \beta_A, \Delta_B) = \int_0^\tau Y_A(s)\,X_A(s) \exp\{\beta_A X_A(s)\}ds, \tag{3.51}$$

so that, under the null hypothesis $H_0 : \beta_A = 0$, $\mathcal{I}(\tau, 0, \Delta_B)$ depends on the end of the follow up time τ and the choice of Δ_B. Then ,

$$U_\beta(\tilde{\alpha}_A, 0) = \sum_{j=1}^{n_A} X_A(t_j) - \tilde{\alpha}_A \mathcal{I}(\tau, 0, \Delta_B), \tag{3.52}$$

and

$$\widehat{Var}(U_\beta(\tilde{\alpha}_A, 0)) = \frac{n_A \mathcal{I}(\tau, 0, \Delta_B)}{\{\int_0^\tau Y_A(s)ds\}^2} \left[\int_0^\tau Y_A(s)ds - \mathcal{I}(\tau, 0, \Delta_B) \right]. \tag{3.53}$$

Therefore, we obtain the partial score statistic for testing $H_0 : \beta_A = 0$ as follows

$$Z = \frac{U_\beta(\tilde{\alpha}_A, 0)}{\widehat{Var}(U_\beta(\tilde{\alpha}_A, 0))^{\frac{1}{2}}}. \tag{3.54}$$

Under mild regularity conditions on $Y_A(t)$, it can be shown with a very similar method given in Section 3.3 that the distribution of the test statistic (3.53) asymptotically converges to a standard normal distribution as τ increases. We discuss this via simulations in Section 3.4.

3.2.4 Case 4: $m > 1$, $\Delta_A > 0$

We now consider the case where multiple systems are under observation and repair times of failures in Component A are not negligible; that is, $m > 1$ and $\Delta_A > 0$. This case is, therefore, an extension of the case given in Section 3.2.2.

Following the notation introduced in Section 3.2.2 using the model (3.22) for Component A and the model (3.23) for Component B, the likelihood function for m independent systems can be written as follows.

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{m} L_i(\boldsymbol{\theta}), \tag{3.55}$$

where $\boldsymbol{\theta} = (\alpha_A, \beta_A)$ and the likelihood contribution of Component A in the ith system is given by

$$L_i(\boldsymbol{\theta}) = \prod_{j=1}^{n_{Ai}} \alpha_A \exp\{\beta_A X_{Ai}(t_{Aij})\} \exp\{-\int_0^{\tau_i} Y_{Ai}(s)\alpha_A \exp\{\beta_A X_{Ai}(s)\}\, ds\}. \tag{3.56}$$

The log likelihood function is then given by $\ell(\boldsymbol{\theta}) = \log L(\boldsymbol{\theta}) = \sum_{i=1}^{m} \log L_i(\boldsymbol{\theta})$ where

$\log L_i(\boldsymbol{\theta})$ is

$$n_{Ai} \log \alpha_A + \beta_A \sum_{j=1}^{n_{Ai}} X_{Ai}(t_{Aij}) - \alpha_A \int_0^{\tau_i} Y_{Ai}(s)\ \exp\{\beta_A X_{Ai}(s)\}\, ds. \tag{3.57}$$

Taking the derivatives of $\ell(\boldsymbol{\theta})$ with respect to α_A and β_A, we obtain the score functions as follows.

$$U_{\alpha_A}(\boldsymbol{\theta}) = \sum_{i=1}^{m} \frac{n_{Ai}}{\alpha_A} - \sum_{i=1}^{m} \int_0^{\tau_i} Y_{Ai}(s) \exp\{\beta_A X_{Ai}(s)\}\, ds, \tag{3.58}$$

and

$$U_{\beta_A}(\boldsymbol{\theta}) = \sum_{i=1}^{m} \sum_{j=1}^{n_{Ai}} X_{Ai}(t_{Aij}) - \alpha_A \sum_{i=1}^{m} \int_0^{\tau_i} Y_{Ai}(s)\, X_{Ai}(s)\ \exp\{\beta_A X_{Ai}(s)\}\, ds. \tag{3.59}$$

Now consider the null hypothesis $H_0 : \beta_A = 0$. Under this null hypothesis, solving $U_{\alpha_A}(\alpha_A, 0) = 0$ for $\alpha_A = \tilde{\alpha}_A$ gives

$$\tilde{\alpha}_A = \frac{\sum_{i=1}^{m} n_{Ai}}{\sum_{i=1}^{m} \int_0^{\tau_i} Y_{Ai}(s)\, ds}, \tag{3.60}$$

which is the restricted maximum likelihood estimator of α_A. For convenience, we define

$$\mathcal{I}(m, \beta_A, \Delta_B) = \sum_{i=1}^{m} \int_0^{\tau_i} Y_{Ai}(s)\, X_{Ai}(s) \exp\{\beta_A\, X_A\}\, ds, \tag{3.61}$$

Then, the partial score statistic for testing $H_0 : \beta_A = 0$ is given by

$$Z = \frac{U_\beta(\tilde{\alpha}_A, 0)}{\widehat{Var}(U_\beta(\tilde{\alpha}_A, 0))^{\frac{1}{2}}}, \tag{3.62}$$

where

$$U_\beta(\tilde{\alpha}_A, 0) = \sum_{i=1}^{m} \sum_{j=1}^{n_{Ai}} X_{Ai}(t_{Aij}) - \tilde{\alpha}_A\, \mathcal{I}(m, 0, \Delta_B), \tag{3.63}$$

and

$$\widehat{Var}(U_\beta(\tilde{\alpha}_A, 0)) = \frac{\mathcal{I}(m, 0, \Delta_B) \sum_i^m n_{Ai}}{(\sum_{i=1}^{m} \int_0^{\tau} Y_{Ai}(s)\, ds)^2} \left[\sum_{i=1}^{m} \int_0^{\tau_i} Y_{Ai}(s)\, ds - \mathcal{I}(m; \Delta_B) \right]. \tag{3.64}$$

By following the method given in Section 3.3, it can be shown that, the asymptotic distribution of the test statistics (3.62) is a standard normal distribution as $m \to \infty$. We discuss this convergence result through simulations in Section 3.4.

3.3 Asymptotic Properties of Test Statistics

In this section, we discuss the asymptotic properties of the test statistics (3.21) and (3.38) analytically. We consider the asymptotic distribution of the former statistic when the observation period τ increases. We utilize simple results from the martingale theory for this purpose. As for the latter statistic, we consider the asymtotic distribution when the number of systems m increases for a fixed observation period. In this case, we show that our model belongs to the family of point process models considered in Andersen et al. (1993, Chapter VI.1.2) and Peña (1998), and satisfies the conditions stated by them to obtain the large sample properties of the test statistic. The development in this section is primarily based on Cigsar (2010, Section 2.3).

Consider the setup given in Section 3.2.1, where a single system ($m = 1$) with two components (Components A and B) is under observation over a time interval $[0, \tau]$, where τ is a prespecified end-of-followup time. Also, Δ_A and Δ_B denote the repair times of Components A and B, respectively, where $\Delta_A = 0$ and $\Delta_B > 0$. Let $\{N_A(t), t > 0\}$ be the counting process for failure occurrences in Component A with the associated intensity function

$$\lambda_A(t \,|\, \mathcal{H}(t)) = \alpha_A \exp\{\beta_A X_A(t)\}, \tag{3.65}$$

where $X_A(t) = I\{N_B(t^-) > 0\} I\{t - t_{BN_B(t^-)} \leq \Delta_B\}$, and $\{N_B(t), t > 0\}$ is the counting process for failure occurrences in Component B with the intensity function given in (3.7), and $\mathcal{H}(t) = \{N_A(u), N_B(u); 0 \leq u < t\}$. In Section 3.2.1, we consider testing the hypothesis $H_0 : \beta_A = 0$ against $H_1 : \beta_A \neq 0$, and develop the partial score test statistic

$$Z = \frac{U_\beta(\tilde{\alpha}_A, 0)}{\widehat{Var}(U_\beta(\tilde{\alpha}_A, 0))^{\frac{1}{2}}}, \tag{3.66}$$

where $\tilde{\alpha}_A = n_A/\tau = (1/\tau) \int_0^\tau dN_A(t)$, and

$$U_{\beta_A}(\tilde{\alpha}_A, 0) = \sum_{j=1}^{n_A} X_A(t_j) - \tilde{\alpha}_A \int_0^\tau X_A(t)\, dt \tag{3.67}$$

and

$$\widehat{Var}(U_{\beta_A}(\tilde{\alpha}_A, 0)) = \left(\frac{n_A}{\tau^2}\right) \int_0^\tau X_A(t)\, dt \left[\tau - \int_0^\tau X_A(t)\, dt\right]. \tag{3.68}$$

We want to show that, under the null hypothsis $H_0 : \beta_A = 0$, the test statistic Z in (3.66) converges in distribution to a standard normal random variable as τ increases; that is, under H_0, $Z \xrightarrow{d} N(0, 1)$ as $\tau \to \infty$. Let α_{A0} be the true value of the parameter α_A assuming that the null hypothesis is true. In the following development, we use τ in the superscripts of score functions to show their dependence on the observation period. From the score function given in (3.18), we can write that

$$\frac{1}{\sqrt{\tau}} U^\tau_{\beta_A}(\tilde{\alpha}_A, 0) = \frac{1}{\sqrt{\tau}} U^\tau_{\beta_A}(\alpha_0) - \sqrt{\tau}(\tilde{\alpha}_A - \alpha_0)\frac{1}{\tau} \int_0^\tau X_A(t)\, dt, \tag{3.69}$$

where

$$\frac{1}{\sqrt{\tau}} U^\tau_{\beta_A}(\alpha_0) = \frac{1}{\sqrt{\tau}} \int_0^\tau X_A(t)\, dM_A(t), \tag{3.70}$$

and $M_A(t) = \int_0^t [dN_A(s) - \alpha_0\, ds]$ is a *bona fide* martingale with respect to the history $\mathcal{H}(t)$ (Daley and Vere-Jones, 2003, p. 428). We assume that $X_A(t)$ is measurable with respect to $\mathcal{H}(t)$ at time t^- in $[0, \tau]$ so it is predictable with respect to $\mathcal{H}(t)$ (Daley and Vere-Jones, 2003, p. 425). Also, notice that, for any t in $[0, \Delta_B]$, $X_A(t) = 1$ if there is at least one failure of Component B in $[0, \Delta_B]$; otherwise, $X_A(t) = 0$. Similarly, for any t in $[\Delta_B, \tau]$, $X_A(t) = 1$ if there is at least one failure of Component B in $[t - \Delta_B, t]$; otherwise, $X_A(t) = 0$. Therefore,

$$E\left\{\int_0^\tau X_A(t)\, dt\right\} = \int_0^{\Delta_B} \left(1 - e^{-\alpha_B t}\right)\, dt + \int_{\Delta_B}^\tau \left(1 - e^{-\alpha_B \Delta_B}\right)\, dt. \tag{3.71}$$

From (3.71), we can show that

$$\lim_{\tau\to\infty} (1/\tau)\, E\left\{\int_0^\tau X_A(t)\, dt\right\} = 1 - e^{-\alpha_B \Delta_B}. \tag{3.72}$$

Let $\vartheta = 1 - e^{-\alpha_B \Delta_B}$, where $0 < \vartheta < 1$. Therefore, by a weak law of large numbers, as $\tau \to \infty$, we obtain

$$\frac{1}{\tau} \int_0^\tau X_A(t)\, \alpha_0\, dt \xrightarrow{p} \alpha_0 \vartheta. \tag{3.73}$$

Also, notice that, for every $\varepsilon > 0$ and sufficiently large τ, almost surely $I(|X_A(t)| >$

$\varepsilon\sqrt{\tau}) = 0$ for all $t \in [0, \tau]$. Therefore, for every $\varepsilon > 0$, we have

$$\lim_{\tau\to\infty} (1/\tau)\, E\left[\int_0^\tau X_A^2(t)\, I(|X_A(t)| > \varepsilon\sqrt{\tau})\, \alpha_0\, dt\right] = 0. \tag{3.74}$$

From the results (3.72) and (3.74) and a central limit theorem for point process martingales (see, Karr, 1991, pp. 421–422, for the theorem and its proof), we obtain the convergence

$$\frac{1}{\sqrt{\tau}} U_{\beta_A}^\tau(\alpha_0) \xrightarrow{d} N(0, \alpha_0\vartheta), \tag{3.75}$$

as $\tau \to \infty$. A convergence result for $\sqrt{\tau}(\tilde{\alpha}_A - \alpha_0)$ can be shown similarly. First notice that

$$\sqrt{\tau}(\tilde{\alpha}_A - \alpha_0) = \frac{1}{\sqrt{\tau}} \int_0^\tau dM_A(t). \tag{3.76}$$

Therefore, by taking $X_A(t) = 1$ in (3.72) and (3.74), we have

$$\sqrt{\tau}(\tilde{\alpha}_A - \alpha_0) \xrightarrow{d} N(0, \alpha_0), \tag{3.77}$$

as $\tau \to \infty$. Since $E\{dM_A(t)\} = E\{dN_A(t) - \alpha_0\, dt\} = 0$ under the null hypothesis, we have

$$\begin{aligned} Cov\left\{\frac{1}{\sqrt{\tau}} U_{\beta_A}^\tau(\alpha_0), \sqrt{\tau}(\tilde{\alpha}_A - \alpha_0)\right\} &= \frac{1}{\tau} Cov\left\{\int_0^\tau X_A(t)\, dM_A(t), \int_0^\tau dM_A(t)\right\}. \\ &= \frac{1}{\tau} E\left\{\int_0^\tau X_A(t)\, \alpha_0\, dt\right\}. \end{aligned} \tag{3.78}$$

Therefore, from (3.72), we obtain

$$Cov\left\{\frac{1}{\sqrt{\tau}} U_{\beta_A}^\tau(\alpha_0), \sqrt{\tau}(\tilde{\alpha}_A - \alpha_0)\right\} \to \alpha_0\vartheta, \tag{3.79}$$

as $\tau \to \infty$. From the convergence results in (3.75), (3.77) and (3.79), and by applying Slutsky's Theorem, $(1/\tau) U_{\beta_A}^\tau(\tilde{\alpha}_A, 0)$ in (3.69) converges in distribution to a zero mean normal distribution with the asymptotic variance $\alpha_0\vartheta(1 - \vartheta)$ as $\tau \to \infty$ under the null hypothesis. Therefore, we obtain the following result. Under the null hypothesis $H_0 : \beta_A = 0$, as $\tau \to \infty$,

$$\frac{\frac{1}{\sqrt{\tau}} U_{\beta_A}^\tau(\tilde{\alpha}_A, 0)}{\sqrt{\alpha_0\vartheta(1 - \vartheta)}} \xrightarrow{d} N(0, 1). \tag{3.80}$$

The terms α_0 and ϑ in (3.80) can be replaced by their consistent estimators. In this case, we obtain the variance estimator given in (3.68).

In the remaining part of this section, we discuss the asymptotic properties of the test statistic given in (3.38) as $m \to \infty$ for a fixed τ value. Andersen et al. (1993, pp. 420–421) consider a general intensity model which depends on a vector of parameters $\boldsymbol{\theta}$ for a multivariate counting process. They state five conditions (Conditions A–E) to derive the large sample properties of the maximum likelihood estimators $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$. These conditions have to be checked for each specific model under study. We do not restate these conditions here or give the proofs, but our goal is to show that the model (3.22) satisfies these conditions, and therefore we can apply the results of the theorems. In particular, under these conditions, the score vector $\boldsymbol{U}(\boldsymbol{\theta}) = \mathbf{0}$ has a solution $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\theta}}$ converges in probability to $\boldsymbol{\theta}_0$ as m increases, where $\boldsymbol{\theta}_0$ is the real value of $\boldsymbol{\theta}$. In notation, that is $\hat{\boldsymbol{\theta}} \xrightarrow{p} \boldsymbol{\theta}_0$ as $m \to \infty$ (Andersen et al., 1993, Theorem VI.1.1., p. 422). Furthermore, if $\hat{\boldsymbol{\theta}}$ is a consistent solution of $\boldsymbol{U}(\boldsymbol{\theta}) = \mathbf{0}$, then $(1/\sqrt{m})\,\boldsymbol{U}(\hat{\boldsymbol{\theta}}) \xrightarrow{d} N(0, \boldsymbol{\Sigma})$ as $m \to \infty$, where $\boldsymbol{\Sigma}$ is defined in Condition D (Andersen et al., pp. 424–426).

Following the setup given in Section 3.2.2, in our case the intensity function of Component A in the ith system, $i = 1, \ldots, m$, is given by

$$\lambda_{Ai}(t|\mathcal{H}_i(t)) = Y_{Ai}(t)\,\alpha_A\,\exp\{\beta_A X_{Ai}(t)\}, \qquad t > 0, \tag{3.81}$$

where $X_{Ai}(t) = I\{N_{Bi}(t^-) > 0\}I\{t - t_{BiN_{Bi}(t^-)} \le \Delta_B\}$ and $\mathcal{H}_i(t) = \{N_{Ai}(u), N_{Bi}(u); 0 \le u < t\}$.

Let $\boldsymbol{\theta} = (\alpha_A, \beta_A)'$. We now check Conditions A–E of Andersen et al. (1993, pp. 420–421) for the above model under the null hypthesis $H_0 : \beta_A = 0$. Suppose that, under the null hypothesis, the true value of α_A is α_0, where $\alpha_0 > 0$, and $\boldsymbol{\theta}_0 = (\alpha_0, 0)'$. Condition A is a Cramèr-type regularity condition on $\lambda_{Ai}(t|\mathcal{H}_i(t))$, $\log \lambda_{Ai}(t|\mathcal{H}_i(t))$ and the log likelihood function given in (3.28). It is easy to see that Condition A holds in our case. Now notice that, for a fixed $\tau > 0$, we have

$$\frac{1}{m}\sum_{i=1}^{m}\int_0^\tau \frac{Y_{Ai}(s)}{\alpha_0}\,ds \xrightarrow{p} \sigma_{\alpha_A\alpha_A}, \qquad \text{as } m \to \infty. \tag{3.82}$$

Since τ is a fixed value and $\Delta_A = 0$, $Y_{Ai}(t) = 1$ in $[0, \tau]$. Therefore, $\sigma_{\alpha_A\alpha_A} = \tau/\alpha_0$. Next, we need to show the following convergence result.

$$\frac{1}{m}\sum_{i=1}^{m}\int_0^\tau Y_{Ai}(s)X_{Ai}(s)\,ds \xrightarrow{p} \sigma_{\alpha_A\beta_A}, \qquad \text{as } m \to \infty, \tag{3.83}$$

where $\sigma_{\alpha_A\beta_A} > 0$. Since τ is fixed and $Y_{Ai}(t) = 1$ in $[0, \tau]$, the result in (3.83) follows by a weak law of large numbers, where $\sigma_{\alpha_A\beta_A}$ is the right hand side of (3.71). It should be noted that if $Y_{Ai}(t)$ is not always 1 (e.g., when $\Delta_A > 0$), some conditions needs to be applied. Similarly, for $\tau > 0$, we have

$$\frac{1}{m}\sum_{i=1}^{m}\int_0^{\tau} Y_{Ai}(s)\,\alpha_0\,X_{Ai}(s)\,ds \xrightarrow{p} \sigma_{\beta_A\beta_A}, \qquad \text{as } m \to \infty, \tag{3.84}$$

where $\sigma_{\beta_A\beta_A} = \alpha_0\,\sigma_{\alpha_A\beta_A}$. Therefore, under the null hypothesis $H_0 : \beta_A = 0$, Condition B is satisfied with $\sigma_{\alpha_A\alpha_A}$, $\sigma_{\alpha_A\beta_A}$ and $\sigma_{\beta_A\beta_A}$ given above.

As for Condition C, we need to show that, for all $\varepsilon > 0$,

$$\frac{1}{m}\sum_{i=1}^{m}\int_0^{\tau} Y_{Ai}(s)\frac{1}{\alpha_0}\,I\left\{\frac{1}{\alpha_0\sqrt{m}} > \varepsilon\right\}ds \xrightarrow{p} 0, \qquad \text{as } m \to \infty, \tag{3.85}$$

and

$$\frac{1}{m}\sum_{i=1}^{m}\int_0^{\tau} Y_{Ai}(s)X_{Ai}^2(s)\,I\left\{\frac{X_{Ai}(s)}{\sqrt{m}} > \varepsilon\right\}ds \xrightarrow{p} 0, \qquad \text{as } m \to \infty. \tag{3.86}$$

Notice that, for a fixed τ, $Y_{Ai}(s) = 1$, where $s \in [0, \tau]$, and $\alpha_0 > 0$, the left hand side of (3.85) is $(\tau/\alpha_0)I\left\{\frac{1}{\sqrt{m}} > \alpha_0\,\varepsilon\right\}$, which converges to 0 as $m \to \infty$. Therefore, the convergence in (3.85) holds. The latter convergence result can be shown with similar arguments which lead to the convergence result in (3.74). In this case, $X_A(s)$ in (3.74) needs to be replaced by $X_{Ai}(s)$, and limit should be taken on the average of the expectation of the terms when $m \to \infty$, instead of $\tau \to \infty$. The result follows from the weak law of large numbers. Therefore, Condition C holds for the model in (3.81).

Let $\mathbf{\Sigma}$ be a 2×2 matrix defined as follows.

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_{\alpha_A\alpha_A} & \sigma_{\alpha_A\beta_A} \\ \sigma_{\beta_A\alpha_A} & \sigma_{\beta_A\beta_A} \end{pmatrix}, \tag{3.87}$$

where $\sigma_{\alpha_A\alpha_A}$, $\sigma_{\alpha_A\beta_A}$ and $\sigma_{\beta_A\beta_A}$ are given above, and $\sigma_{\beta_A\alpha_A} = \sigma_{\alpha_A\beta_A}$. Condition D requires that the matrix $\mathbf{\Sigma}$ should be positive definite. Notice that $\sigma_{\alpha_A\alpha_A} = \tau/\alpha_0$ is positive because $\alpha_0 > 0$, $\tau > 0$ and τ is fixed. We also need to show that

$\sigma_{\alpha_A\beta_A}(\tau - \sigma_{\alpha_A\alpha_B}) > 0$. Recall that

$$\begin{aligned} 0 < \sigma_{\alpha_A\beta_A} &= \int_0^{\Delta_B} \left(1 - e^{-\alpha_B t}\right)\, dt + \int_{\Delta_B}^{\tau} \left(1 - e^{-\alpha_B \Delta_B}\right)\, dt \\ &< \Delta_B(1 - e^{-\alpha_B\Delta_B}) + (\tau - \Delta_B)(1 - e^{-\alpha_B\Delta_B}) = \tau(1 - e^{-\alpha_B\Delta_B}) < \tau. \end{aligned} \tag{3.88}$$

Therefore, the matrix $\mathbf{\Sigma}$ is positive definite, and Condition D holds. Condition E is required for the boundedness of the third derivaties of $\lambda_{Ai}(t|\mathcal{H}_i(t))$ and $\log \lambda_{Ai}(t|\mathcal{H}_i(t))$ given in (3.81) with respect to $\boldsymbol{\theta}$ under the null hypothesis, as well as for the regularity of the remainder term of a Taylor series expansion. It is easy to see that the requirements on the boundedness is satified for the model (3.81). Also, requirements on the remainder term hold for the model (3.81). It is worth noting that these conditions can be shown for the model (2.23) by following a similar method. In this case, the above approach should include mild regularity conditions on the at-risk indicator $Y_{Bi}(t)$, $i = 1, \ldots, m$, such as the integrals $\int_0^\tau Y_{Bi}(s)\, ds$ should not be equal to 0.

Since ConditionA–E given in Andersen et al. (1993, pp. 420–421) are satisfied for the model (3.81). Therefore, from Andersen et al. (1993, p. 424–426), we conclude that, under the null hypothesis $H_0 : \beta_A = 0$, $\alpha_A > 0$,

$$\frac{\frac{1}{\sqrt{m}} U_{\beta_A}(\tilde{\alpha}_A, 0)}{\sigma(\alpha_0)} \xrightarrow{d} N(0, 1), \qquad \text{as } m \to \infty, \tag{3.89}$$

where $\tilde{\alpha}_A = \sum_{i=1}^m n_{Ai} / \sum_{i=1}^m \tau_i$ is the restricted maximum likelihood estimator of α_A and $\sigma^2(\alpha_0) = \alpha_{\beta_A\beta_A} - (\sigma^2_{\alpha_A\beta_A}/\sigma_{\alpha_A\alpha_A})$. We obtain the final convergence result by replacing $\sigma^2(\alpha_0)$ with a consistent estimator of it. Therefore, we obtain that, by Slutsky's theorem, under the null hypothesis $H_0 : \beta_A = 0$, $\alpha_A > 0$,

$$Z = \frac{U_\beta(\tilde{\alpha}_A, 0)}{\widehat{Var}(U_\beta(\tilde{\alpha}_A, 0))^{\frac{1}{2}}} \xrightarrow{d} N(0, 1), \qquad \text{as } m \to \infty, \tag{3.90}$$

where Z is given in (3.38).

3.4 Simulation Studies

In this section, we present the results of simulation studies carried out to assess when asymptotic normal approximation for testing parallel carryover effects of redundant system is satisfactory. We consider two settings; (i) where $m = 1$ and $\tau \to \infty$, and (ii) $m \to \infty$ and τ is fixed.

We consider testing for parallel carryover effects in a single system (m=1) where repair times of Component A are negligible; that is, $m = 1$, $\Delta_A = 0$, $\Delta_B > 0$. The composite hypothesis of no parallel carryover effects in Component A in this case is $H_0 : \beta_A = 0$, $\alpha_A > 0$ and the standardized partial score statistic associated with the null hypothesis, as given in Section 3.2.1, is

$$Z = \frac{U_{\beta_A}(\tilde{\alpha}_A, 0)}{\widehat{Var}(U_{\beta_A}(\tilde{\alpha}_A, 0))^{\frac{1}{2}}}, \tag{3.91}$$

where the score statistic $U_{\beta_A}(\tilde{\alpha}_A, 0)$ is given by (3.18) and its estimated variance $\widehat{Var}(U_{\beta_A}(\tilde{\alpha}_A, 0))^{\frac{1}{2}}$ is given by (3.19). Our goal is to investigate the null distribution of Z and to assess the adequacy of the standard normal approximation when τ increases. We generated 10,000 realizations of HPPs based on (3.6) and (3.7). In the simulation, we took $\alpha_A = \alpha_B = 0.1$ and $\beta_A = 0$. We consider $\Delta_B = 1, 3, 7$ and 14.

Normal quantile-quantile (Q-Q) plots of 10,000 values of Z are presented in Figure 3.1 when $\Delta_B = 1$ for $\tau = 100, 200, 500$ and 1000. The standard normal approximations are not quite accurate in those four cases. However, it is noted that as τ increases, there is an apparent improvement in the standard normal approximation. Figure 3.2 shows the results when $\Delta_B = 3$. In this scenario, the standard normal approximations are suitable for $\tau = 500$ and 1000. In Figure 3.3, we can see that the standard normal approximations are quite accurate for $\tau = 500$ and $\tau = 1000$ when $\Delta_B = 7$. Figure 3.4 shows also that, when $\Delta_B = 14$, the standard normal approximations are adequate for all $\tau = 100, 200, 500$ and 1000.

Table 3.1 presents estimates of Q_p and $Pr(Z > Q_p)$ where $\hat{Q}_p$ is the empirical pth quantile of Z computed from 10,000 samples, Q_p is the pth quantile of the standard normal distribution, and $\hat{Pr}(Z > Q_p) = 1 - p$ where $p = 0.950, 0.975$ and 0.990. In all the scenarios with $\Delta_B = 1, 3, 7$ and 14, the standard normal approximations are quite accurate.

The power of the statistic (3.91) against the alternative hypothesis $H_A : \beta_A \neq 0$, $\alpha_A > 0$ is investigated. We use Monte Carlo simulation methods to obtain the power of the test. We use 0.95 quantile of the standard normal distribution and the empirical 0.95 quantile of the test statistic in the 10,000 simulations runs that are under the null hypothesis with different τ and Δ_B values as discussed above. We generated

1,000 processes under the model (3.6) and (3.7). We took $\alpha_A = \alpha_B = 0.1$. We set β_A at 0.693 and 0.916, where $\exp\{0.693\} = 2$ and $\exp\{0.916\} = 2.5$. Here, $\exp\{\beta_A\}$ means that while Component B is under repair, the rate of occurrences of failures in Component A is $\exp\{\beta_A\}$ times higher comparing with the rate while Component B is in the up state. The power results are provided in Table 3.2, where entries are the proportions of the values of Z in 1,000 samples which are larger than the quantile values. This shows that the power of the test is high overall except that when $\exp\{\beta_A\} = 2$ and τ is small.

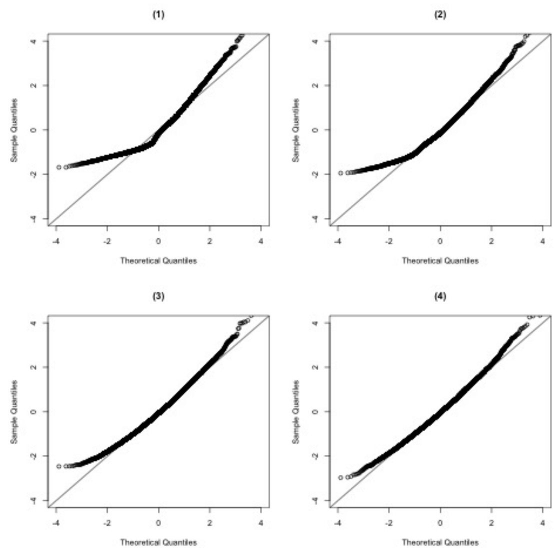

Figure 3.1: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_B = 1$, and (1) $\tau = 100$, (2) $\tau = 200$, (3) $\tau = 500$, (4) $\tau = 1000$

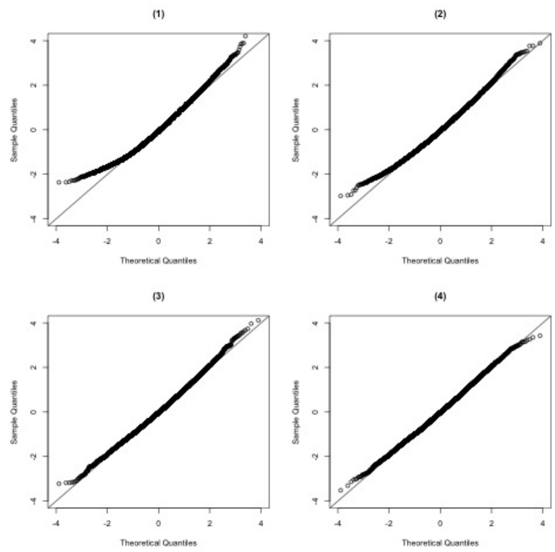

Figure 3.2: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_B = 3$, and (1) $\tau = 100$, (2) $\tau = 200$, (3) $\tau = 500$, (4) $\tau = 1000$

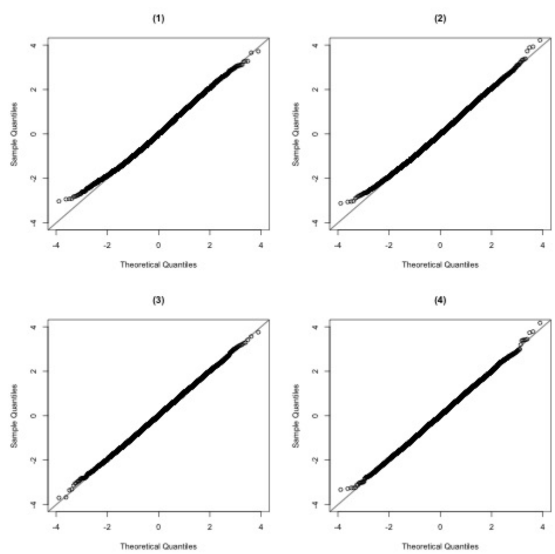

Figure 3.3: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_B = 7$, and (1) $\tau = 100$, (2) $\tau = 200$, (3) $\tau = 500$, (4) $\tau = 1000$

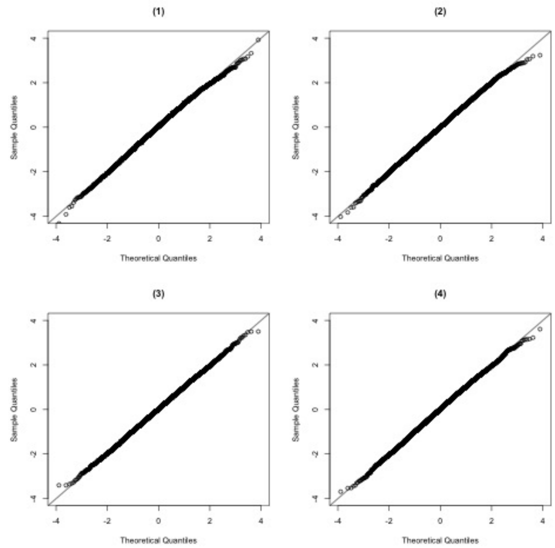

Figure 3.4: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_B = 14$, and (1) $\tau = 100$, (2) $\tau = 200$, (3) $\tau = 500$, (4) $\tau = 1000$

Δ_B	τ	$\hat{Q}_{0.950}$	$\hat{Q}_{0.975}$	$\hat{Q}_{0.990}$	$\hat{Pr}(Z > 1.645)$	$\hat{Pr}(Z > 1.960)$	$\hat{Pr}(Z > 2.326)$
1	100	1.854	2.356	2.940	0.0657	0.0445	0.0267
	200	1.840	2.265	2.712	0.0676	0.0416	0.0228
	500	1.766	2.118	2.675	0.0608	0.0349	0.0168
	1000	1.753	2.169	2.586	0.0605	0.0355	0.0181
3	100	1.775	2.204	2.663	0.0636	0.0387	0.0198
	200	1.699	2.036	2.440	0.0563	0.0296	0.0125
	500	1.709	2.097	2.457	0.0557	0.0314	0.0144
	1000	1.694	2.005	2.449	0.0547	0.0278	0.0119
7	100	1.707	2.029	2.434	0.0576	0.0318	0.0127
	200	1.690	2.001	2.401	0.0553	0.0281	0.0116
	500	1.636	1.939	2.312	0.0496	0.0238	0.0099
	1000	1.637	1.975	2.334	0.0494	0.0259	0.0102
14	100	1.643	1.909	2.224	0.0499	0.0232	0.0075
	200	1.590	1.876	2.239	0.0452	0.0209	0.0080
	500	1.605	1.869	2.254	0.0456	0.0211	0.0078
	1000	1.622	1.944	2.289	0.0482	0.0243	0.0087

Table 3.1: $\hat{Q}_p$ is the empirical pth quantile of Z computed from 10,000 samples when $m = 1$. $\hat{Pr}(Z > Q_p)$ is the proportion of the values of Z in 10,000 samples which are larger than the pth quantile of a standard normal distribution

Δ_B	τ	$\hat{Pr}\{Z > 1.645 \mid H_a\}$ $e^{\beta_A} = 2$	$\hat{Pr}\{Z > \hat{Q}_{0.950} \mid H_a\}$ $e^{\beta_A} = 2$	$\hat{Pr}\{Z > 1.645 \mid H_a\}$ $e^{\beta_A} = 6$	$\hat{Pr}\{Z > \hat{Q}_{0.950} \mid H_a\}$ $e^{\beta_A} = 6$
1	100	0.254	0.214	0.861	0.836
	200	0.357	0.312	0.974	0.964
	500	0.565	0.528	1.000	1.000
	1000	0.805	0.782	1.000	1.000
3	100	0.327	0.292	0.974	0.971
	200	0.523	0.493	0.999	0.998
	500	0.818	0.800	1.000	1.000
	1000	0.975	0.971	1.000	1.000
7	100	0.382	0.345	0.995	0.994
	200	0.548	0.536	1.000	1.000
	500	0.885	0.886	1.000	1.000
	1000	0.991	0.991	1.000	1.000
14	100	0.308	0.309	0.995	0.995
	200	0.566	0.588	1.000	1.000
	500	0.877	0.881	1.000	1.000
	1000	0.992	0.992	1.000	1.000

Table 3.2: Power of Z : $m = 1, \Delta_A = 0$

We now consider testing for the presence of parallel carryover effects in multiple systems where repair times of Component A are negligible, which is Case 2 of Section 3.2.2, where $m > 1$, $\Delta_A = 0$ and $\Delta_B > 0$. The hypothesis of no parallel carryover effects in Component A is conducted by using the statistic Z in (3.38). We generated 10,000 realizations of m HPPs under the null hypothesis with fixed values of the parameters, where $\alpha_A = \alpha_B = 0.1$, and $\beta_A = 0$. We considered $\Delta_B = 1, 3, 7$ and 14 and fixed τ at 100 this time. Normal quantile-quantile (Q-Q) plots of 10,000 values of Z are given in Figures 3.5, 3.6, 3.7 and 3.8 with various combinations of m and Δ_B. When $\Delta_B = 1$ in Figure 3.5, the standard normal approximation is appropriate except when $m = 10$. The standard normal approximations are quite suitable when $\Delta_B = 3, 7$ and 14.

Table 3.3 shows estimates of Q_p and $Pr(Z > Q_p)$ when $p = 0.950, 0.975$ and 0.990. Table 3.3 also indicates that the standard normal approximations are overall accurate for all m and Δ_B values except that when $m = 10$ and 20 and $\Delta_B = 1$ and 3 in the tails.

The power of the statistic (3.38) against the alternative hypothesis $H_A : \beta_A \neq 0$, $\alpha_A > 0$ is investigated by using Monte Carlo simulation methods. We use the 0.95 quantile of the standard normal distribution and the empirical 0.95 quantile of the test statistic in the 10,000 simulations runs that are under the null hypothesis with different m and Δ_B. We generated 1,000 processes under the alternative model where we took $\alpha_A = \alpha_B = 0.1$ when $\exp\{\beta_A\} = 2$ or 2.5. The power results are presented in Table 3.4 where entries are the proportions of the values of Z in 1,000 samples which are larger than the quantile values. This indicates that the power of the test is quite high overall. It can also easily seen that the power of the test increases as m increases.

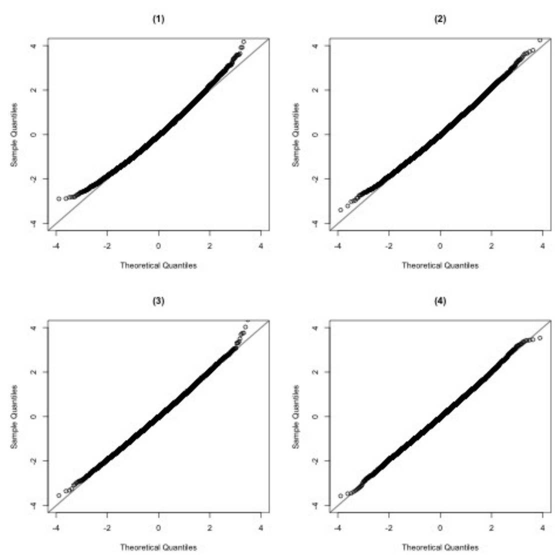

Figure 3.5: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\tau = 100$, $\Delta_B = 1$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$, (4) $m = 100$

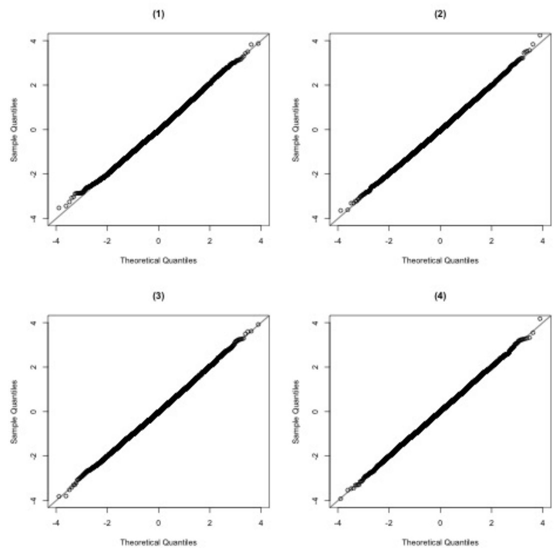

Figure 3.6: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\tau = 100$, $\Delta_B = 3$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$, (4) $m = 100$

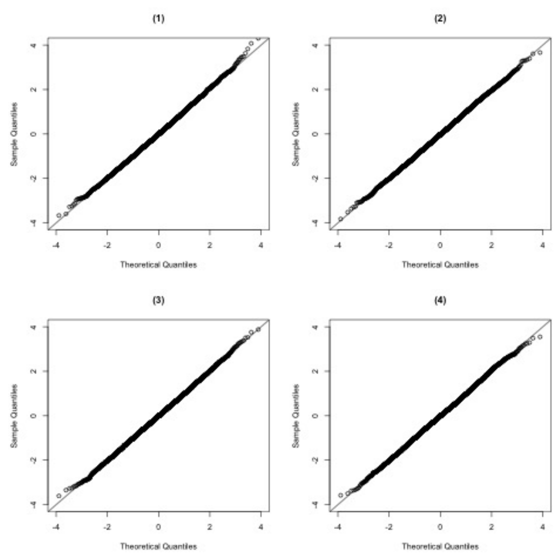

Figure 3.7: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\tau = 100$, $\Delta_B = 7$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$, (4) $m = 100$

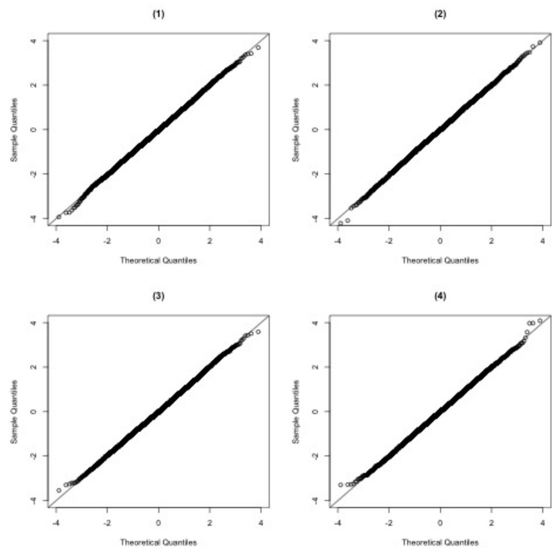

Figure 3.8: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\tau = 100$, $\Delta_B = 14$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$, (4) $m = 100$

Δ_B	m	$\hat{Q}_{0.950}$	$\hat{Q}_{0.975}$	$\hat{Q}_{0.990}$	$\hat{Pr}(Z > 1.645)$	$\hat{Pr}(Z > 1.960)$	$\hat{Pr}(Z > 2.326)$
1	10	1.728	2.130	2.516	0.059	0.0340	0.0158
	20	1.721	2.079	2.508	0.056	0.0329	0.0159
	50	1.665	1.982	2.369	0.0516	0.0267	0.0110
	100	1.678	2.010	2.417	0.0544	0.0272	0.0123
3	10	1.705	2.036	2.388	0.0551	0.0301	0.0115
	20	1.735	2.083	2.476	0.0583	0.0319	0.0140
	50	1.646	1.971	2.394	0.0502	0.0258	0.0119
	100	1.624	1.947	2.290	0.0477	0.0245	0.0092
7	10	1.688	2.013	2.368	0.0548	0.0295	0.0116
	20	1.657	2.007	2.438	0.0513	0.0272	0.0125
	50	1.653	1.983	2.334	0.0508	0.026	0.0105
	100	1.691	1.982	2.413	0.0541	0.0266	0.0126
14	10	1.636	1.950	2.287	0.0490	0.0243	0.0095
	20	1.631	1.958	2.312	0.0490	0.0249	0.0249
	50	1.638	1.964	2.302	0.0493	0.0253	0.0097
	100	1.620	1.918	2.359	0.0475	0.0221	0.0109

Table 3.3: $\hat{Q}_p$ is the empirical pth quantile of Z computed from 10,000 samples when $m > 1$ and $\tau = 100$. $\hat{Pr}(Z > Q_p)$ is the proportion of the values of Z in 10,000 samples which are larger than the pth quantile of a standard normal distribution

Δ	m	$\hat{P}r\{Z > 1.645 \mid H_a\}$ $e^{\beta_A} = 2$	$\hat{P}r\{Z > \hat{Q}_{0.950} \mid H_a\}$ $e^{\beta_A} = 2$	$\hat{P}r\{Z > 1.645 \mid H_a\}$ $e^{\beta_A} = 2.5$	$\hat{P}r\{Z > \hat{Q}_{0.950} \mid H_a\}$ $e^{\beta_A} = 2.5$
1	10	0.790	0.770	0.950	0.950
	20	0.970	0.960	1.000	1.000
	50	1.000	1.000	1.000	1.000
	100	1.000	1.000	1.000	1.000
3	10	0.970	0.970	1.000	1.000
	20	1.000	1.000	1.000	1.000
	50	1.000	1.000	1.000	1.000
	100	1.000	1.000	1.000	1.000
7	10	0.990	0.990	1.000	1.000
	20	1.000	1.000	1.000	1.000
	50	1.000	1.000	1.000	1.000
	100	1.000	1.000	1.000	1.000
14	10	0.980	0.990	1.000	1.000
	20	1.000	1.000	1.000	1.000
	50	1.000	1.000	1.000	1.000
	100	1.000	1.000	1.000	1.000

Table 3.4: Power of Z : $m > 1, \Delta_A = 0$

We also considered the score statistic (3.54) where $m = 1$, $\Delta_A > 0$, $\Delta_B > 0$. We generated 10,000 realizations of HPPs based on (3.41) and (3.42). We took $\alpha_A = \alpha_B = 0.1$ and $\beta_A = 0$. We considered $\Delta_B = 1, 3, 7$ and 14. For $\Delta_B = 1$ in Figure 3.9, the normal approximation is not quite adequate, however with large τ values, the approximation gets better. For $\Delta_B = 3$ (Figure 3.10), the normal approximation is adequate when $\tau = 1000$. For $\Delta_B = 7$ and 14 (Figures 3.11 and 3.12, respectively), the approximations are overall good. Table 3.4 shows estimated Q_p and $\hat{P}r(Z > Q_p)$ values with $p = 0.950, 0.975$ and 0.990, and this indicates also that the standard normal approximation is accurate for large τ.

The power of the statistic (3.54) against the alternative hypothesis $H_A : \beta_A \neq 0$, $\alpha_A > 0$ is investigated by using Monte Carlo simulation methods as well. We used 0.95 quantile of the standard normal distribution and the empirical 0.95 quantile of the test statistic obtained from 10,000 simulations runs with different τ and Δ_B

combinations. We generated 1,000 processes under the model (3.41) and (3.42) taking $\alpha_A = \alpha_B = 0.1$ when $\beta_A = 0.693$ or 1.098, where $\exp\{0.693\} = 2$ and $\exp\{1.098\} = 3$. Table 3.6 provides the power results. Entries in Table 3.6 are the proportions of the values of Z in 1,000 samples which are larger than the quantile values. Table 3.6 shows that the power of the test is high overall, and power increases as τ increases.

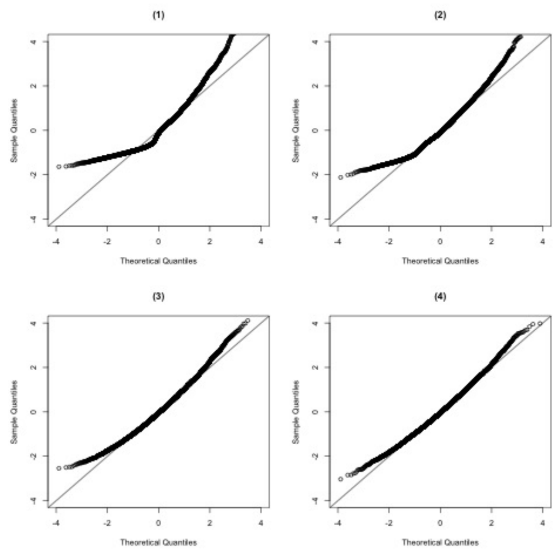

Figure 3.9: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_B = 1$, $\Delta_A > 0$, and (1) $\tau = 100$, (2) $\tau = 200$, (3) $\tau = 500$, (4) $\tau = 1000$

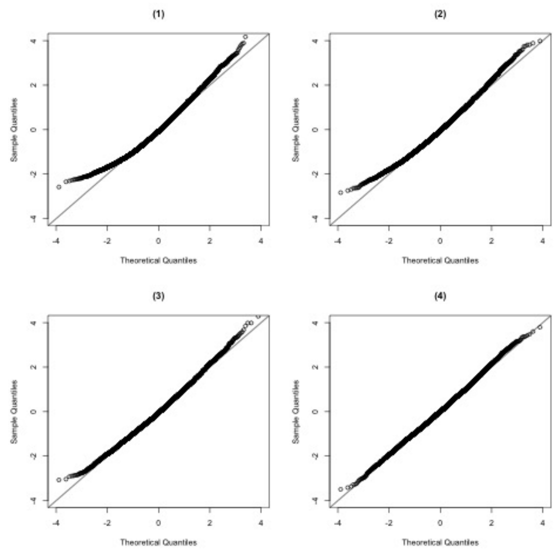

Figure 3.10: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_B = 3$, $\Delta_A > 0$, and (1) $\tau = 100$, (2) $\tau = 200$, (3) $\tau = 500$, (4) $\tau = 1000$

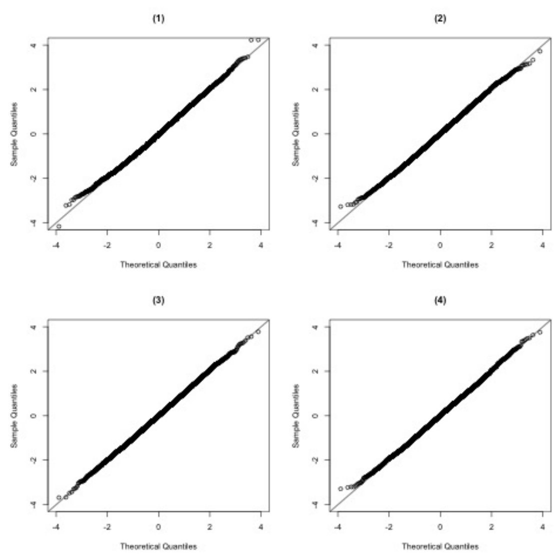

Figure 3.11: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_B = 7$, $\Delta_A > 0$, and (1) $\tau = 100$, (2) $\tau = 200$, (3) $\tau = 500$, (4) $\tau = 1000$

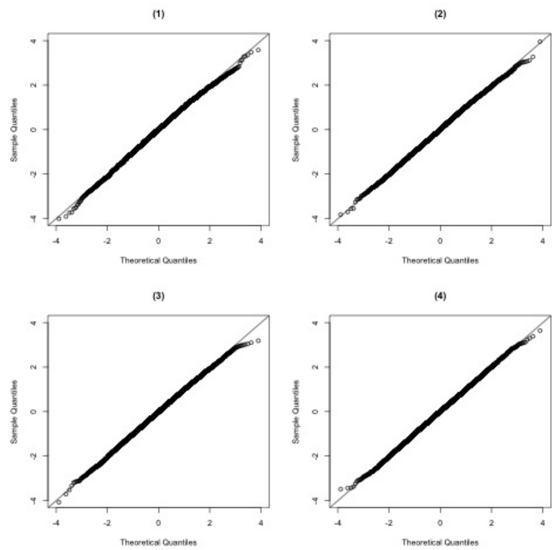

Figure 3.12: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_B = 14$, $\Delta_A > 0$, and (1) $\tau = 100$, (2) $\tau = 200$, (3) $\tau = 500$, (4) $\tau = 1000$

Δ_B	τ	$\hat{Q}_{0.950}$	$\hat{Q}_{0.975}$	$\hat{Q}_{0.990}$	$\hat{Pr}(Z > 1.645)$	$\hat{Pr}(Z > 1.960)$	$\hat{Pr}(Z > 2.326)$
1	100	1.985	2.571	3.101	0.079	0.052	0.034
	200	1.868	2.317	2.874	0.071	0.044	0.025
	500	1.787	2.194	2.690	0.061	0.037	0.021
	1000	1.715	2.064	2.600	0.059	0.032	0.016
3	100	1.830	2.231	2.694	0.065	0.038	0.020
	200	1.769	2.168	2.620	0.062	0.036	0.018
	500	1.715	2.093	2.493	0.056	0.032	0.014
	1000	1.730	2.110	2.493	0.058	0.033	0.015
7	100	1.677	1.994	2.344	0.055	0.026	0.011
	200	1.679	2.033	2.345	0.053	0.028	0.011
	500	1.683	2.009	2.366	0.055	0.028	0.012
	1000	1.638	1.976	2.399	0.049	0.026	0.012
14	100	1.633	1.918	2.188	0.049	0.021	0.006
	200	1.620	1.917	2.244	0.047	0.022	0.008
	500	1.603	1.891	2.228	0.045	0.021	0.007
	1000	1.640	1.977	2.342	0.050	0.026	0.011

Table 3.5: $\hat{Q}_p$ is the empirical pth quantile of Z computed from 10,000 samples when $m = 1$ and $\Delta_A > 0$. $\hat{Pr}(Z > Q_p)$ is the proportion of the values of Z in 10,000 samples which are larger than the pth quantile of a standard normal distribution

Δ	τ	$\hat{P}r\{Z > 1.645 \vert H_a\}$ $e^{\beta_A} = 2$	$\hat{P}r\{Z > \hat{Q}_{0.950} \vert H_a\}$ $e^{\beta_A} = 2$	$\hat{P}r\{Z > 1.645 \vert H_a\}$ $e^{\beta_A} = 3$	$\hat{P}r\{Z > \hat{Q}_{0.950} \vert H_a\}$ $e^{\beta_A} = 3$
1	100	0.950	0.910	1.000	1.000
	200	0.950	0.940	1.000	1.000
	500	0.940	0.940	1.000	1.000
	1000	0.940	0.950	1.000	1.000
3	100	0.960	0.950	1.000	1.000
	200	0.940	0.940	1.000	1.000
	500	0.940	0.940	1.000	1.000
	1000	0.960	0.960	1.000	1.000
7	100	0.950	0.950	1.000	1.000
	200	0.950	0.940	1.000	1.000
	500	0.960	0.960	0.99	1.000
	1000	0.950	0.950	1.000	1.000
14	100	0.950	0.950	1.000	1.000
	200	0.930	0.930	1.000	1.000
	500	0.950	0.950	1.000	1.000
	1000	0.950	0.960	0.99	1.000

Table 3.6: Power of Z : $m = 1, \Delta_A > 0$

Finally, we considered the score statistic (3.62) when $m > 1$, $\Delta_A > 0$, $\Delta_B > 0$. We fixed τ at 100, and generated 10,000 realizations of m HPPs taking $\alpha_A = \alpha_B = 0.1$, $\beta_A = 0$ for various m values with $\Delta_B = 1, 3, 7$ and 14. We fixed Δ_A at 1. Normal probability plots (Figures 3.13, 3.14, 3.15 and 3.16) show that normal approximations are quite suitable for overall scenarios with these various combinations with Δ_B and m except for some cases in which $\Delta_B = 1$. Table 3.7 presents the value of estimated Q_p and $\hat{P}r(Z > Q_p)$ where $p = 0.950, 0.975$ and 0.990. This shows that for $\Delta_B = 3, 7$ and 14, the standard normal approximations are suitable.

The power of the statistic (3.62) against the alternative hypothesis H_A : $\beta_A \neq 0$, $\alpha_A > 0$ is investigated by using Monte Carlo simulation study. To obtain the power, 0.95 quantile of the standard normal distribution and the empirical 0.95 quantile of the test statistic in the 10,000 simulations runs with different m and Δ_B used. The power results are provided in Table 3.8. This shows that the power of the test is overall quite high.

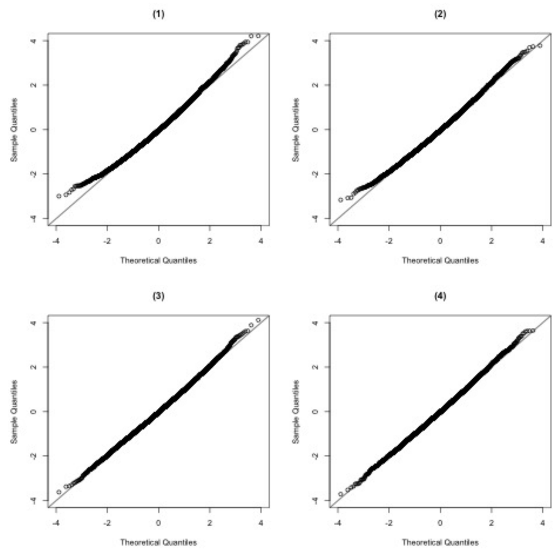

Figure 3.13: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\tau = 100$, $\Delta_A = 1$, $\Delta_B = 1$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$, (4) $m = 100$

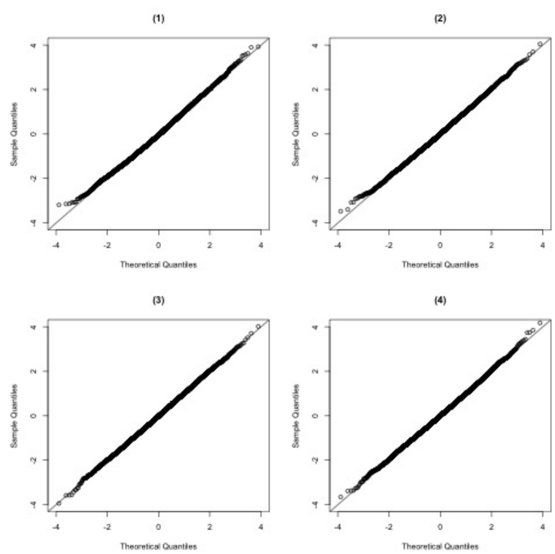

Figure 3.14: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\tau = 100$, $\Delta_A = 1$, $\Delta_B = 3$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$, (4) $m = 100$

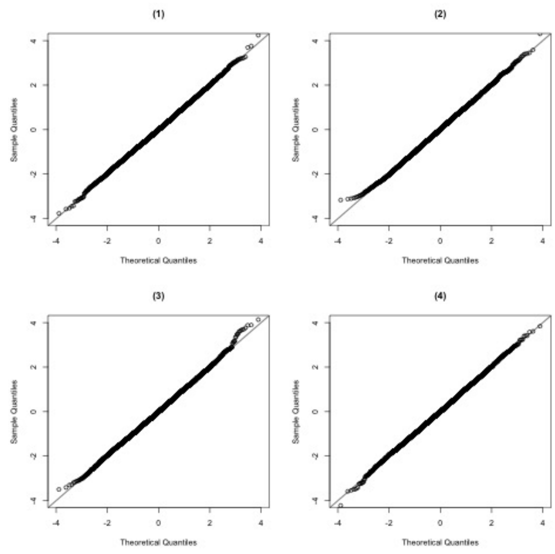

Figure 3.15: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\tau = 100$, $\Delta_A = 1$, $\Delta_B = 7$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$, (4) $m = 100$

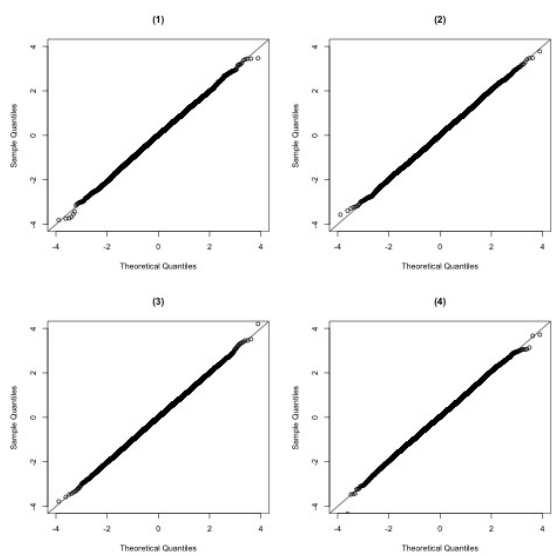

Figure 3.16: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\tau = 100$, $\Delta_A = 1$, $\Delta_B = 14$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$, (4) $m = 100$

Δ_B	m	$\hat{Q}_{0.950}$	$\hat{Q}_{0.975}$	$\hat{Q}_{0.990}$	$\hat{Pr}(Z > 1.645)$	$\hat{Pr}(Z > 1.960)$	$\hat{Pr}(Z > 2.326)$
1	10	1.781	2.111	2.540	0.061	0.034	0.015
	20	1.719	2.074	2.490	0.056	0.032	0.015
	50	1.675	1.996	2.426	0.052	0.028	0.012
	100	1.729	2.083	2.496	0.058	0.032	0.015
3	10	1.685	1.990	2.384	0.054	0.027	0.011
	20	1.668	1.990	2.396	0.052	0.027	0.012
	50	1.673	1.989	2.344	0.053	0.027	0.010
	100	1.647	1.996	2.419	0.050	0.026	0.013
7	10	1.650	1.953	2.350	0.050	0.024	0.011
	20	1.669	1.983	2.406	0.053	0.026	0.012
	50	1.653	1.950	2.350	0.051	0.024	0.010
	100	1.650	1.985	2.385	0.051	0.026	0.012
14	10	1.633	1.950	2.340	0.049	0.024	0.011
	20	1.681	2.005	2.369	0.054	0.028	0.011
	50	1.639	1.963	2.327	0.050	0.025	0.010
	100	1.685	2.032	2.369	0.055	0.029	0.011

Table 3.7: $\hat{Q}_p$ is the empirical pth quantile of Z computed from 10,000 samples when $m > 1$, $\Delta_A = 1$ and $\tau = 100$. $\hat{Pr}(Z > Q_p)$ is the proportion of the values of Z in 10,000 samples which are larger than the pth quantile of a standard normal distribution

Δ_B	m	$\hat{Pr}\{Z > 1.645 \mid H_a\}$ $e^{\beta_A} = 2$	$\hat{Pr}\{Z > \hat{Q}_{0.950} \mid H_a\}$ $e^{\beta_A} = 2$	$\hat{Pr}\{Z > 1.645 \mid H_a\}$ $e^{\beta_A} = 3$	$\hat{Pr}\{Z > \hat{Q}_{0.950} \mid H_a\}$ $e^{\beta_A} = 3$
1	10	0.830	0.850	0.990	0.990
	20	0.950	0.960	1.000	1.000
	50	1.000	1.000	1.000	1.000
	100	1.000	1.000	1.000	1.000
3	10	0.940	0.940	1.000	1.000
	20	1.000	1.000	1.000	1.000
	50	1.000	1.000	1.000	1.000
	100	1.000	1.000	1.000	1.000
7	10	0.980	0.980	1.000	1.000
	20	1.000	1.000	1.000	1.000
	50	1.000	1.000	1.000	1.000
	100	1.000	1.000	1.000	1.000
14	10	0.970	0.970	1.000	1.000
	20	1.000	1.000	1.000	1.000
	50	1.000	1.000	1.000	1.000
	100	1.000	1.000	1.000	1.000

Table 3.8: Power of Z : $m > 1, \Delta_A = 1$

Chapter 4

Redundant Systems with Three Components

In this chapter, as an extension of Chapter 3, we consider repairable systems with three components working in parallel; a redundant component and two primary components. In Section 4.2, we develop a test of parallel carryover effects in redundant systems with three components. We present simulation results in Section 4.3. In Section 4.4, we illustrate methods by analyzing a failure data set from diesel operated power plants.

4.1 Introduction

Redundant systems may include more than two components working in parallel. In this section, we discuss testing for the presence of parallel carryover effects in redundant systems with three components. We develop partial score tests of parallel carryover effects in such systems, and discuss asymptotic properties of them through simulations.

4.2 Models and Tests for Parallel Carryover Effects

In this section, we consider three components in a system, instead of two. We discuss models for a single system and multiple systems under two different settings based on the duration of repairs in the redundant component.

Suppose that there is a multivariate counting processes $\{N_A(t), N_B(t), N_C(t); t \geq 0\}$, where $\{N_A(t); t \geq 0\}$ is a counting process for Component A, $\{N_B(t); t \geq 0\}$ is a counting process for Component B and $\{N_C(t); t \geq 0\}$ is a counting process for Component C. The three components are working in parallel in a system. We let $t_{A1}, t_{A2}, \dots,$ where $0 < t_{A1} < t_{A2} < \dots$, $t_{B1}, t_{B2}, \dots,$ where $0 < t_{B1} < t_{B2} < \dots$, and $t_{C1}, t_{C2}, \dots,$ where $0 < t_{C1} < t_{C2} < \dots$, denote the failure times of Components A, B and C, respectively. All components are subject to repairs and repair times in the primary components cannot be ignored. Let Δ_A, Δ_B and Δ_C denote the repair times of Components A, B and C, respectively. For $K = A, B$ and C, $Y_K(t)$ is the *at-risk indicator* of process $\{N_K(t); t > 0\}$.

A model including parallel carryover effects for Component A in a redundant system having three components is given by

$$\lambda_A(t|\mathcal{H}(t)) = Y_A(t)\,\alpha_A\,\exp\{\beta_A X_{AB}(t) + \beta_A X_{AC}(t)\}, \quad t > 0, \tag{4.1}$$

where $Y_A(t)$ is the at-risk indicator function of Component A, $\alpha_A > 0$ is a baseline rate function, and

$$X_{AB}(t) = I\{N_B(t^-) > 0\} I\{t - t_{B\,N_B(t^-)} \leq \Delta_B\}, \tag{4.2}$$

and

$$X_{AC}(t) = I\{N_C(t^-) > 0\} I\{t - t_{C\,N_C(t^-)} \leq \Delta_C\}, \tag{4.3}$$

and β_A is a regression parameter. Notice that, for $K = B, C$, $X_{AK}(t)$ takes the value of 1 while Component K is under a repair; otherwise, it equals 0. Also, we assume that one parameter β_A represents the effect of repairs of Component B or C on Component A. This assumption is applicable in many settings, but models can be extended by including separate parameters for the effects of the repairs of Components B and C. For simplicity, we do not pursue this case in this thesis. However, if there is a need for more detailed modelling, methods in this section can be applied after obvious modifications.

While Component A is in the up state at time t (i.e., $Y_A(t) = 1$), there are four possibilities: (i) Both Components B and C are in the up state, (ii) Component B is in the down state and Component C is in the up state, (iii) Component B is in the up state and Component C is in the down state, and (iv) both Components B and C are in the down state. In case (i), failures of Component A are governed by a HPP with rate function α_A. In case (ii), the intensity function of Component A jumps from α_A to $Y_A(t)\,\alpha_A\,\exp\{\beta_A\}$ after each failure in Component B, and stays there for Δ_B time units as long as there is no failures in Component A (i.e., $Y_A(t)=1$) and Component C (i.e., $Y_C(t)=1$). Similarly, in case (iii), the intensity function of Component A jumps from α_A to $Y_A(t)\,\alpha_A\,\exp\{\beta_A\}$ after each failure in Component C, and stays there for Δ_C time units as long as there is no failures in Component A (i.e., $Y_A(t)=1$) and Component B (i.e., $Y_B(t)=1$). In the last case, when both Components B and C are in the down state, the intensity function (4.1) becomes $\alpha_A\exp\{2\beta_A\}$, and stays there as long as there is no failures in Component A (i.e., $Y_A(t)=1$) and both Components B and C are under repair ($Y_B(t)=0$ and $Y_C(t)=0$).

A model for parallel carryover effects can be also similarly defined for Component B. In this case, the intensity function of $\{N_B(t); t\geq 0\}$ is given by

$$\lambda_B(t|\mathcal{H}(t)) = Y_B(t)\,\alpha_B\,\exp\{\beta_B X_{BA}(t) + \beta_B X_{BC}(t)\}, \quad t>0, \tag{4.4}$$

where $\alpha_B>0$ is a baseline rate function, and

$$X_{BA}(t) = I\{N_A(t^-)>0\}I\{t - t_{A\,N_A(t^-)} \leq \Delta_A\}, \tag{4.5}$$

and

$$X_{BC}(t) = I\{N_C(t^-)>0\}I\{t - t_{C\,N_C(t^-)} \leq \Delta_A\}. \tag{4.6}$$

Similarly, the intensity function of $\{N_C(t); t\geq 0\}$ is given by

$$\lambda_C(t|\mathcal{H}(t)) = Y_C(t)\,\alpha_C\,\exp\{\beta_C X_{CA}(t) + \beta_C X_{CA}(t)\}, \quad t>0, \tag{4.7}$$

where $\alpha_C>0$ is a baseline rate function, and

$$X_{CA}(t) = I\{N_A(t^-)>0\}I\{t - t_{A\,N_A(t^-)} \leq \Delta_A\}, \tag{4.8}$$

and

$$X_{CB}(t) = I\{N_B(t^-)>0\}I\{t - t_{B\,N_B(t^-)} \leq \Delta_B\}. \tag{4.9}$$

Let m denote the number of systems, each with three components. In the remaining part of this section, we develop partial score tests for the presence of parallel carryover effects in four different cases; (i) a single system is under observation and repair times of Components A and B are negligible ($m=1$, $\Delta_A=0$, $\Delta_B=0$,

$\Delta_C > 0$), (ii) a single system is under observation and repair times of Component A are negligible ($m = 1$, $\Delta_A = 0$, $\Delta_B > 0$, $\Delta_C > 0$), (iii) a single system is under observation and repair times of Component A are not negligible ($m = 1$, $\Delta_A > 0$, $\Delta_B > 0$, $\Delta_C > 0$), and (iv) multiple systems are under observation and repair times of Component A are not negligible ($m > 1$, $\Delta_A > 0$, $\Delta_B > 0$, $\Delta_C > 0$). In all cases, we assume the baseline rate functions are constants.

4.2.1 Case 1: $m = 1$, $\Delta_A = 0$, $\Delta_B = 0$, $\Delta_C > 0$

We first consider a simple case in a system with 3 components connected in parallel; Components A, B and C. In this setting, we assume that repair times of Components A and B are negligible; that is $\Delta_A = 0$ and $\Delta_B = 0$ so that failures of Component A and failures of Component B do not affect the probabilistic characteristics of failure occurrences in Component C. We assume also that failure occurrences are governed by HPPs. In this case, the intensity functions for Components A, B and C are given by

$$\text{Component A:} \qquad \lambda_A(t|\mathcal{H}(t)) = Y_A(t)\,\alpha_A\,\exp\{\beta_A X_A(t)\}, \quad t > 0, \tag{4.10}$$

where $Y_A(t)$ is at-risk indicator function $\alpha_A > 0$ is a baseline rate function and $X_A(t) = I\{N_C(t^-) > 0\}I\{t - T_{N_C(t^-)} \leq \Delta_C\}$, and

$$\text{Component B:} \qquad \lambda_B(t|\mathcal{H}(t)) = Y_B(t)\,\alpha_B\,\exp\{\beta_B X_B(t)\}, \quad t > 0, \tag{4.11}$$

where $Y_B(t)$ is at-risk indicator function $\alpha_B > 0$ is a baseline rate function and $X_B(t) = I\{N_C(t^-) > 0\}I\{t - T_{N_C(t^-)} \leq \Delta_C\}$, and since $\Delta_A = 0$ and $\Delta_B = 0$

$$\text{Component C:} \qquad \lambda_C(t|\mathcal{H}(t)) = Y_C(t)\,\alpha_C, \quad t > 0, \tag{4.12}$$

where $\alpha_C > 0$ is a baseline rate function and $\mathcal{H}(t) = \{N_A(u), N_B(u), N_A(u), Y_C(s); 0 \leq u < t, 0 \leq s \leq t\}$.

A test for the presence of parallel carryover effects in Component A can be developed by considering the following composite hypothesis.

$$H_0 \,:\, \beta_A = 0, \alpha_A > 0, \quad \text{vs.} \quad H_a \,:\, \beta_A \neq 0, \alpha_A > 0, \tag{4.13}$$

where β_A is the parameter of interest and α_A is a nuisance parameter. We suppose that the system is under observation over the followup period $[0, \tau]$, where τ is the end-of-followup time. Note that, since $\Delta_A = 0$ and $\Delta_B = 0$, we can safely drop at-risk indicator functions $Y_A(t)$ and $Y_B(t)$ from the model (4.10) and (4.11), respectively. Let n_A, where $n_A \geq 0$, denote the number of failures of Component A over $[0, \tau]$, and $t_{A1}, t_{A2}, \ldots, t_{An_A}$ be the failure times of Component A in $[0, \tau]$. Then the likelihood function of the outcome "$N_A(\tau) = n_A$ failures of Component A at times $t_{A1} \leq t_{A2} \leq \cdots \leq t_{An_A}$ in $[0, \tau]$" is given by

$$L(\boldsymbol{\theta}) = \prod_{j=1}^{n_A} \alpha_A \exp\{\beta_A X_A(t_{Aj})\} \exp\{-\int_0^\tau \alpha_A \exp\{\beta_A X_A(s)\}\, ds\}, \tag{4.14}$$

where $\boldsymbol{\theta} = (\alpha_A, \beta_A)$. The likelihood function (4.14) is of the same form with the likelihood function (3.9) given in Section 3.2.1, and thus, the partial score test statistic based on (4.14) is the same with the one obtained in Section 3.2.1. We only state the form of the test statistic here, but not discuss its derivation. Let

$$\mathcal{I}(\tau, \beta_A, \Delta_B) = \int_0^\tau X_A(s) \exp\{\beta_A X_A(s)\}\, ds. \tag{4.15}$$

Then, the partial score test for testing the null hypothesis $H_0 : \beta_A = 0$ is given by

$$Z = \frac{U_\beta(\tilde{\alpha}_A, 0)}{\widehat{Var}(U_\beta(\tilde{\alpha}_A, 0))^{\frac{1}{2}}}, \tag{4.16}$$

where

$$U_{\beta_A}(\tilde{\alpha}_A, 0) = \sum_{j=1}^{n_A} X_A(t_j) - \tilde{\alpha}_A \mathcal{I}(\tau, 0, \Delta_B), \tag{4.17}$$

and

$$\widehat{Var}(U_{\beta_A}(\tilde{\alpha}_A, 0)) = \mathcal{I}(\tau, 0, \Delta_B)\, [\tau - \mathcal{I}(\tau, 0, \Delta_B)]. \tag{4.18}$$

The asymptotic properties of the test statistic (4.16) are discussed in Section 3.3 analytically and in Section 3.4 through simulations.

4.2.2 Case 2: $m = 1$, $\Delta_A = 0$, $\Delta_B > 0$, $\Delta_C > 0$

In this section, we consider a case in which there is a single system under observation and the repair times are negligible for Component A, but not negligible for

Components B and C. This means that $m = 1$ and $\Delta_A = 0$, $\Delta_B > 0$ and $\Delta_C > 0$.

The intensity functions of Components A, B and C are respectively given as follows.

$$\text{Component A:} \qquad \lambda_A(t|\mathcal{H}(t)) = Y_A(t)\,\alpha_A\,\exp\{\beta_A X_{AB}(t) + \beta_A X_{AC}(t)\}, \quad t > 0, \tag{4.19}$$

where $Y_A(t)$ is the at-risk indicator of Component A, $X_{AB}(t)$ and $X_{AC}(t)$ are respectively defined in (4.2) and (4.3), β_A is a regression parameter, $\Delta_A = 0$, and $\alpha_A > 0$ is a baseline rate function.

$$\text{Component B:} \qquad \lambda_B(t|\mathcal{H}(t)) = Y_B(t)\,\alpha_B\,\exp\{\beta_B X_{BC}(t)\}, \quad t > 0, \tag{4.20}$$

where $Y_B(t)$ is the at-risk indicator function of Component B, $\Delta_B > 0$, $X_{BC}(t)$ is defined in (4.6), and $\alpha_B > 0$ is a baseline rate function.

$$\text{Component C:} \qquad \lambda_C(t|\mathcal{H}(t)) = Y_C(t)\,\alpha_C\,\exp\{\beta_C X_{CB}(t)\}, \quad t > 0, \tag{4.21}$$

where $Y_C(t)$ is the at-risk indicator function of Component C, $X_{CB}(t)$ is defined in (4.9), $\Delta_C > 0$, and $\alpha_C > 0$ is a baseline rate function and $\mathcal{H}(t) = \{N_A(u), N_B(u), N_C(u), Y_B(s), Y_C(s); 0 \le u < t, 0 \le s \le t\}$.

Then the likelihood function of the outcome that "$N_A(\tau) = n_A$ failures of Component A occur at times $t_{A1} \le t_{A2} \le \cdots \le t_{An_A}$ in $[0, \tau]$" is given by

$$L(\boldsymbol{\theta}) = \prod_{j=1}^{n_A} \alpha_A\, e^{\beta_A X_{AB}(t_{Aj}) + \beta_A X_{AC}(t_{Aj})}\ \exp\{-\int_0^\tau \alpha_A\, e^{\beta_A X_{AB}(s) + \beta_A X_{AC}(s)}\, ds\}. \tag{4.22}$$

Then the log likelihood function is given by

$$\ell(\boldsymbol{\theta}) = n_A \log \alpha_A + \sum_{j=1}^{n_A} \{\beta_A X_{AB}(t_{Aj}) + \beta_A X_{AC}(t_{Aj})\} - \alpha_A \int_0^\tau e^{\beta_A X_{AB}(s) + \beta_A X_{AC}(s)}\, ds. \tag{4.23}$$

The score vector is then defined by $\boldsymbol{U}(\boldsymbol{\theta}) = (U_{\alpha_A}(\boldsymbol{\theta}), U_{\beta_A}(\boldsymbol{\theta}))'$ with components

$$U_{\alpha_A}(\boldsymbol{\theta}) = \frac{n_A}{\alpha_A} - \int_0^\tau e^{\beta_A (X_{AB}(s) + X_{AC}(s))}\, ds, \tag{4.24}$$

and

$$U_{\beta_A}(\boldsymbol{\theta}) = \sum_{j=1}^{n_A} X_{AB}(t_{Aj}) + \sum_{j=1}^{n_A} X_{AC}(t_{Aj}) \\ - \alpha_A \int_0^{\tau} (X_{AB}(s) + X_{AC}(s))\, e^{\beta_A(X_{AB}(s)+X_{AC}(s))}\, ds. \tag{4.25}$$

Also, the observed information matrix $\boldsymbol{I}(\boldsymbol{\theta})$ is given by

$$\boldsymbol{I}(\boldsymbol{\theta}) = \begin{pmatrix} I_{\alpha_A\alpha_A}(\boldsymbol{\theta}) & I_{\alpha_A\beta_A}(\boldsymbol{\theta}) \\ I_{\beta_A\alpha_A}(\boldsymbol{\theta}) & I_{\beta_A\beta_A}(\boldsymbol{\theta}) \end{pmatrix}, \tag{4.26}$$

where the components are given by

$$I_{\alpha_A\alpha_A}(\boldsymbol{\theta}) = \frac{n_A}{\alpha_A^2}, \tag{4.27}$$

$$I_{\alpha_A\beta_A}(\boldsymbol{\theta}) = I_{\beta_A\alpha_A}(\boldsymbol{\theta}) = \int_0^{\tau} (X_{AB}(s) + X_{AC}(s))\, e^{\beta_A X_{AB}(s) + \beta_A X_{AC}(s)}\, ds, \tag{4.28}$$

$$I_{\beta_A\beta_A}(\boldsymbol{\theta}) = \alpha_A \int_0^{\tau} (X_{AB}(s) + X_{AC}(s))^2\, e^{\beta_A X_{AB}(s) + \beta_A X_{AC}(s)}\, ds. \tag{4.29}$$

Let $\tilde{\alpha}_A$ be the restricted maximum likelihood estimator of α_A under $H_0 : \beta_A = 0$. By solving $U_{\alpha_A}(\alpha_A, 0) = 0$ in (4.24) with $\beta_A = 0$ for $\alpha_A = \tilde{\alpha}_A$, we obtain

$$\tilde{\alpha}_A = \frac{n_A}{\tau} \tag{4.30}$$

Therefore, the standardized partial score statistic for testing the hypotheses (4.13) is given by

$$Z = \frac{U_{\beta_A}(\tilde{\alpha}_A, 0)}{\widehat{Var}(U_{\beta_A}(\tilde{\alpha}_A, 0))^{\frac{1}{2}}}, \tag{4.31}$$

where

$$U_{\beta}(\tilde{\alpha}_A, 0) = \sum_{j=1}^{n_A} X_{AB}(t_{Aj}) + \sum_{j=1}^{n_A} X_{AC}(t_{Aj}) - \tilde{\alpha}_A \int_0^{\tau} (X_{AB}(s) + X_{AC}(s)) ds, \tag{4.32}$$

and

$$\widehat{Var}(U_\beta(\tilde{\alpha}_A, 0)) = I_{\beta_A\beta_A}(\tilde{\alpha}_A, 0) - I_{\beta_A\alpha_A}(\tilde{\alpha}_A, 0) I^{-1}_{\alpha_A\alpha_A}(\tilde{\alpha}_A, 0) I_{\alpha_A\beta_A}(\tilde{\alpha}_A, 0) \tag{4.33}$$

$$= \frac{n_A}{\tau^2}\left[\tau \int_0^\tau (X_{AB}(s) + X_{AC}(s))^2 ds - \left[\int_0^\tau (X_{AB}(s) + X_{AC}(s))\, ds\right]^2\right] \tag{4.34}$$

We discuss the asymptotic properties of this test statistic in Section 4.3 through simulations. However, a proof based on the method in Section 3.3 can be useful to show its asymptotic distribution analytically.

4.2.3 Case 3: $m = 1$, $\Delta_A > 0$, $\Delta_B > 0$, $\Delta_C > 0$

This section presents the derivation of a test statistics developed for testing the presence of parallel carryover effects in redundant systems. In this case, there is a single system under observation and the repair times are not negligible for Components A, B and C; that is, $m = 1$ and $\Delta_A > 0$, $\Delta_B > 0$ and $\Delta_C > 0$. Component A is the redundant component in the system.

In this case, the intensity functions of Components A, B and C are given below:

$$\text{Component A:} \qquad \lambda_A(t|\mathcal{H}(t)) = Y_A(t)\, \alpha_A\, \exp\{\beta_A X_{AB}(t) + \beta_A X_{AC}(t)\}, \quad t > 0, \tag{4.35}$$

where $Y_A(t)$ is the at-risk indicator function of Component A, $X_{AB}(t)$ and $X_{AC}(t)$ are respectively defined in (4.2) and (4.3), $\alpha_A > 0$ is a baseline rate function, β_A is a regression parameter.

$$\text{Component B:} \qquad \lambda_B(t|\mathcal{H}(t)) = Y_B(t)\, \alpha_B\, \exp\{\beta_B X_{BC}(t)\}, \quad t > 0, \tag{4.36}$$

where $Y_B(t)$ is the at-risk indicator function of Component B, $X_{BC}(t)$ is defined in (3.6), $\alpha_B > 0$ is a baseline rate function, and β_B is a regression parameter.

$$\text{Component C:} \qquad \lambda_C(t|\mathcal{H}(t)) = Y_C(t)\, \alpha_C\, \exp\{\beta_C X_{CB}(t)\}, \quad t > 0. \tag{4.37}$$

where $Y_C(t)$ is the at-risk indicator function of Component C, $X_{CB}(t)$ is defined in (4.9), $\alpha_C > 0$ is a baseline rate function, and β_C is a regression parameter. The history of the processes is given by $\mathcal{H}(t) = \{N_A(u), N_B(u), N_C(u), Y_A(s), Y_B(s), Y_C(s); 0 \leq$

$u < t, 0 \leq s \leq t\}$.

As explained previously in Section 4.2, in this case the intensity function (4.35) jumps from α_A to $\alpha_A \exp\{\beta_A\}$ when either one of Component B or Component C fails. Then, it increases from $\alpha_A \exp\{\beta_A\}$ to $\alpha_A \exp\{2\beta_A\}$ when both of Components B and C fail.

In this case, the likelihood function of the outcome that "$N_A(\tau) = n_A$ failures of Component A occur at times $t_{A1} \leq t_{A2} \leq \cdots \leq t_{An_A}$ in $[0, \tau]$, where τ is fixed" is given by

$$L(\boldsymbol{\theta}) = \prod_{j=1}^{n_A} \alpha_A \, e^{\beta_A X_{AB}(t_{Aj}) + \beta_A X_{AC}(t_{Aj})} \, \exp\{-\int_0^\tau Y_A(s) \, \alpha_A \, e^{\beta_A X_{AB}(s) + \beta_A X_{AC}(s)} \, ds\}, \tag{4.38}$$

where $\boldsymbol{\theta} = (\alpha_A, \beta_A)$. Then, the log likelihood function $\ell(\boldsymbol{\theta}) = \log L(\boldsymbol{\theta})$ is given by

$$n_A \log \alpha_A + \sum_{j=1}^{n_A} \{\beta_A X_{AB}(t_{Aj}) + \beta_A X_{AC}(t_{Aj})\} - \alpha_A \int_0^\tau Y_A(s) \, e^{\beta_A X_{AB}(s) + \beta_A X_{AC}(s)} \, ds. \tag{4.39}$$

The components of the score vector $\boldsymbol{U}(\boldsymbol{\theta})$ are given by

$$U_{\alpha_A}(\boldsymbol{\theta}) = \frac{n_A}{\alpha_A} - \int_0^\tau Y_A(s) \, e^{\beta_A (X_{AB}(s) + X_{AC}(s))} \, ds, \tag{4.40}$$

and

$$\begin{aligned} U_{\beta_A}(\boldsymbol{\theta}) = \sum_{j=1}^{n_A} X_{AB}(t_{Aj}) + & \sum_{j=1}^{n_A} X_{AC}(t_{Aj}) \\ & - \alpha_A \int_0^\tau Y_A(s) \, (X_{AB}(s) + X_{AC}(s)) \, e^{\beta_A (X_{AB}(s) + X_{AC}(s))} \, ds. \end{aligned} \tag{4.41}$$

By solving $U_{\alpha_A}(\alpha_A, 0) = 0$ in (4.40) with $\beta_A = 0$, we can easily show that the restricted maximum likelihood estimator $\tilde{\alpha}_A$ of α_A is given as $\tilde{\alpha}_A = n_A / (\int_0^\tau Y_A(s) \, ds)$.

For testing the presence of parallel carryover effects in Component A, we consider the hypothesis (4.13). The partial score function under the null hypothesis $H_0 : \beta_A = 0$ can be written as follows.

$$U_\beta(\tilde{\alpha}_A, 0) = \sum_{j=1}^{n_A} (X_{AB}(t_{Aj}) + X_{AC}(t_{Aj})) - \tilde{\alpha}_A \int_0^\tau Y_A(s) \, (X_{AB}(s) + X_{AC}(s)) ds. \tag{4.42}$$

The components of the observed information matrix $\boldsymbol{I}(\alpha_A, \beta_A)$ evaluated at $(\alpha_A, \beta_A) = (\tilde{\alpha}_A, 0)$ are given by

$$I_{\alpha_A\alpha_A}(\tilde{\alpha}_A, 0) = \frac{n_A}{\tilde{\alpha}_A^2}, \tag{4.43}$$

$$I_{\alpha_A\beta_A}(\tilde{\alpha}_A, 0) = I_{\beta_A\alpha_A}(\tilde{\alpha}_A, 0) = \int_0^\tau Y_A(s)\,(X_{AB}(s) + X_{AC}(s))ds, \tag{4.44}$$

$$I_{\beta_A\beta_A}(\tilde{\alpha}_A, 0) = \tilde{\alpha}_A \int_0^\tau Y_A(s)\,(X_{AB}(s) + X_{AC}(s))^2\,ds. \tag{4.45}$$

From the score procedures explained in Chapter 1, we obtain the following standardized partial score statistic for testing the presence of parallel carryover effects in Component A.

$$Z = \frac{U_{\beta_A}(\tilde{\alpha}_A, 0)}{\widehat{Var}(U_{\beta_A}(\tilde{\alpha}_A, 0))^{\frac{1}{2}}}, \tag{4.46}$$

where $U_\beta(\tilde{\alpha}_A, 0)$ is given in (3.42) and $\widehat{Var}(U_\beta(\tilde{\alpha}_A, 0))$ is given by

$$\frac{n_A}{(\int_0^\tau Y_A(s)\,ds)^2}\left[\left(\int_0^\tau Y_A(s)\,ds\right)\mathcal{I}_1(\tau, \beta_A, \Delta_B, \Delta_C) - \mathcal{I}_2(\tau, \beta_A, \Delta_B, \Delta_C)\right], \tag{4.47}$$

with

$$\mathcal{I}_1(\tau, \beta_A, \Delta_B, \Delta_C) = \int_0^\tau Y_A(s)\,(X_{AB}(s) + X_{AC}(s))^2 ds, \tag{4.48}$$

and

$$\mathcal{I}_2(\tau, \beta_A, \Delta_B, \Delta_C) = \left[\int_0^\tau Y_A(s)\,(X_{AB}(s) + X_{AC}(s))\,ds\right]^2. \tag{4.49}$$

Asymptotic properties of the test statistic (4.46) are discussed in Section 4.3 through simulations. Under some mild regularity conditions on the at-risk indicator $Y_A(t)$ (see, Karr, 1991, p. 421), the asymptotic distribution of the test statistic (4.46) can be analytically shown with a similar method given in Section 3.3.

4.2.4 Case 4: $m > 1, \Delta_A > 0$, $\Delta_B > 0$, $\Delta_C > 0$

In this section, we consider a similar setting of Section 4.2.3, but here we assume that m independent systems are under observation, each with three components: Component A, B and C. The repair times of the components are not negligible; that is, $\Delta_A > 0$, $\Delta_B > 0$, $\Delta_C > 0$. Furthermore, we assume that failures of Components B and C affect the probabilistic characteristics of failure occurrences of Component

A, while failures of Component A do not affect the probabilistic characteristics of failure occurrences of Components B and C. Therefore, Component A operates as a redundant component in the system.

Now, suppose that there are m independent multivariate counting processes $\{N_{Ai}(t), N_{Bi}(t), N_{Ci}(t);\, t \geq 0\}$, $i = 1, \ldots, m$, where $\{N_{Ki}(t); t \geq 0\}$, for $K = A$, B or C, is a counting process of failure occurrences in Component A in the ith system, $i = 1, \ldots, m$. For $K = A$, B or C, we let $t_{Ki1}, t_{Ki2}, \ldots,$ where $0 < t_{Ki1} < t_{Ki2} < \cdots$, denote the failure times of the Kth component in the ith system, $i = 1, \ldots, m$. Let Δ_{Ai}, Δ_{Bi} and Δ_{Ci} denote the repair times of Components A, B and C in the ith system, respectively. For $K = A$, B or C, the at-risk indicator $Y_{Ki}(t)$ of the process $\{N_{Ki}(t);\, t > 0\}$ takes value of 1 when Component K is up and the process $\{N_{Ki}(t);\, t > 0\}$ is under observation; otherwise, it equals to 0. The history of the multivariate processes $\{N_{Ai}(t), N_{Bi}(t), N_{Ci}(t);\, t \geq 0\}$, $i = 1, \ldots, m$, is denoted by $\mathcal{H}_i(t) = \{N_{Ai}(u), N_{Bi}(u), N_{Ci}(u);\, 0 \leq u < t\}$.

In this case, the intensity functions of Components A, B and C in the ith system are given as follows.

$$\text{Component A:} \qquad \lambda_{Ai}(t|\mathcal{H}_i(t)) = Y_{Ai}(t)\, \alpha_A\, \exp\{\beta_A X_{ABi}(t) + \beta_A X_{ACi}(t)\}, \quad t > 0, \tag{4.50}$$

where $Y_{Ai}(t)$ is the at-risk indicator function of Component A in ith system, $\alpha_A > 0$ is a baseline rate function, β_A is a regression parameter, and

$$\begin{aligned} X_{ABi}(t) &= I\{N_{Bi}(t^-) > 0\} I\{t - t_{BiN_{Bi}(t^-)} \leq \Delta_{Bi}\}, \\ X_{ACi}(t) &= I\{N_{Ci}(t^-) > 0\} I\{t - t_{CiN_{Ci}(t^-)} \leq \Delta_{Ci}\}. \end{aligned} \tag{4.51}$$

$$\text{Component B:} \qquad \lambda_{Bi}(t|\mathcal{H}_i(t)) = Y_{Bi}(t)\, \alpha_B\, \exp\{\beta_B X_{BCi}(t)\}, \quad t > 0, \tag{4.52}$$

where $Y_{Bi}(t)$ is the at-risk indicator function of Component B in the ith system, $\alpha_B > 0$ is a baseline rate function, and

$$X_{BCi}(t) = I\{N_{Ci}(t^-) > 0\} I\{t - t_{CiN_{Ci}(t^-)} \leq \Delta_{Ci}\}. \tag{4.53}$$

$$\text{Component C:} \qquad \lambda_{Ci}(t|\mathcal{H}_i(t)) = Y_{Ci}(t)\, \alpha_C\, \exp\{\beta_C X_{CBi}(t)\}, \quad t > 0, \tag{4.54}$$

where $Y_{Ci}(t)$ is the at-risk indicator function of Component C in the ith system,

$\alpha_C > 0$ is a baseline rate function, and

$$X_{CBi}(t) = I\{N_{Bi}(t^-) > 0\} I\{t - T_{BiN_{Bi}(t^-)} \le \Delta_{Bi}\}. \tag{4.55}$$

By considering the following composite hypothesis, a test of the presence of parallel carryover effects in Component A can be developed.

$$H_0 \,:\, \beta_A = 0, \alpha_A > 0, \quad \text{vs.} \quad H_a \,:\, \beta_A \neq 0, \alpha_A > 0, \tag{4.56}$$

where α_A is a nuisance parameter and β_A is the parameter of interest.

We suppose that m systems are independent and under observation period over $[0, \tau_i]$, where τ_i is the fixed end-of-followup time of the ith system, $i = 1, 2, \ldots, m$. We let $t_{Ai1}, t_{Ai1}, \ldots, t_{Ain_{Ai}}$ be the failure times of Component A in the ith system, and let $N_{Ai}(\tau_i) = n_{Ai}$ be the observed number of failures of Component A in the ith system over the time interval $[0, \tau_i]$. The likelihood function of the outcome that "$N_{Ai}(\tau_i) = n_{Ai}$ failures of Component A observed at times $t_{Ai1}, t_{Ai1}, \ldots, t_{Ain_{Ai}}$ in $[0, \tau_i]$" is given by

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{m} L_i(\boldsymbol{\theta}), \tag{4.57}$$

where

$$L_i(\boldsymbol{\theta}) = \prod_{j=1}^{n_{Ai}} \alpha_A \, e^{\beta_A X_{ABi}(t_{Aij}) + \beta_A X_{ACi}(t_{Aij})} \, \exp\{-\int_0^{\tau_i} Y_{Ai}(s)\, \alpha_A \, e^{\beta_A X_{ABi}(s) + \beta_A X_{ACi}(s)} \, ds\}. \tag{4.58}$$

Then, the log likelihood function $\ell(\boldsymbol{\theta}) = \log L(\boldsymbol{\theta})$ is given by

$$\begin{aligned} &\sum_{i=1}^{m} n_{A_i} \log \alpha_A + \beta_A \sum_{i=1}^{m} \sum_{j=1}^{n_{A_i}} (X_{ABi}(t_{Aij}) + X_{ACi}(t_{Aij})) \\ &\qquad\qquad - \alpha_A \sum_{i=1}^{m} \int_0^{\tau_i} Y_{Ai}(s) \, e^{\beta_A X_{ABi}(s) + \beta_A X_{ACi}(s)} \, ds, \end{aligned} \tag{4.59}$$

The components of the score vector $U(\boldsymbol{\theta})$ are followed by

$$U_{\alpha_A}(\boldsymbol{\theta}) = \frac{\sum_{i=1}^{m} n_{A_i}}{\alpha_A} - \sum_{i=1}^{m} \int_0^{\tau_i} Y_{Ai}(s) \, e^{\beta_A X_{ABi}(s) + \beta_A X_{ACi}(s)} \, ds, \tag{4.60}$$

and

$$
\begin{aligned}
U_{\beta_A}(\boldsymbol{\theta}) = &\sum_{i=1}^{m}\sum_{j=1}^{n_{A_i}}(X_{ABi}(t_{Aij}) + X_{ACi}(t_{Aij})) \\
&- \alpha_A \sum_{i=1}^{m}\int_0^{\tau_i} Y_{Ai}(s)\,(X_{ABi}(s) + \beta_A X_{ACi})\, e^{\beta_A X_{ABi}(s)+\beta_A X_{ACi}(s)}\, ds.
\end{aligned} \tag{4.61}
$$

Also, the components of the observed information matrix $\boldsymbol{I}(\boldsymbol{\theta})$ are given by

$$
I_{\alpha_A\alpha_A}(\boldsymbol{\theta}) = \frac{\sum_{i=1}^{m} n_{Ai}}{{\alpha_A}^2}, \tag{4.62}
$$

$$
I_{\alpha_A\beta_A}(\boldsymbol{\theta}) = I_{\beta_A\alpha_A}(\boldsymbol{\theta}) = \sum_{i=1}^{m}\int_0^{\tau_i} Y_{Ai}(s)\,(X_{ABi}(s) + X_{ACi}(s))\, e^{\beta_A X_{ABi}(s)+\beta_A X_{ACi}(s)}\, ds, \tag{4.63}
$$

$$
I_{\beta_A\beta_A}(\boldsymbol{\theta}) = \sum_{i=1}^{m}\int_0^{\tau_i} Y_{Ai}(s)\,(X_{ABi}(s) + X_{ACi}(s))^2\, e^{\beta_A X_{ABi}(s)+\beta_A X_{ACi}(s)}\, ds. \tag{4.64}
$$

Under the null hypothesis $H_0 : \beta_A = 0$, we obtain the restricted maximum likelihood estimator $\tilde{\alpha}_A$ of α_A as $\tilde{\alpha}_A = (\sum_{i=1}^{m} n_{A_i})/\sum_{i=1}^{m}\int_0^{\tau_i} Y_{Ai}(s)\,ds$. Then, from the score function $U_{\beta_A}(\boldsymbol{\theta})$ given in (4.61), we can write the partial score function evaluated at $(\alpha_A, \beta_A) = (\tilde{\alpha}_A, 0)$ as follows.

$$
U_{\beta_A}(\tilde{\alpha}_A, 0) = \sum_{i=1}^{m}\sum_{j=1}^{n_{A_i}}(X_{ABi}(t_{Aij})+X_{ACi}(t_{Aij}))-\tilde{\alpha}_A \sum_{i=1}^{m}\int_0^{\tau_i} Y_{Ai}(s)\,(X_{ABi}(s)+\beta_A X_{ACi})\, ds. \tag{4.65}
$$

We define the integrals $\mathcal{I}_1(m, \beta_A, \Delta_B, \Delta_C)$ and $\mathcal{I}_2(m, \beta_A, \Delta_B, \Delta_C)$, for notational convenience, as

$$
\mathcal{I}_1(m, \beta_A, \Delta_B, \Delta_C) = \sum_{i=1}^{m}\int_0^{\tau_i} Y_{Ai}(s)\,(X_{ABi}(s) + X_{ACi}(s))^2\, ds, \tag{4.66}
$$

and

$$
\mathcal{I}_2(m, \beta_A, \Delta_B, \Delta_C) = \left(\sum_{i=1}^{m}\int_0^{\tau_i} Y_{Ai}(s)\,(X_{ABi}(s) + X_{ACi}(s))\, ds\right)^2. \tag{4.67}
$$

Then, the estimated variance of $U_{\beta_A}(\tilde{\alpha}_A, 0)$, which is denoted by $\widehat{Var}(U_{\beta_A}(\tilde{\alpha}_A, 0))$, can be written as

$$\frac{\sum_{i=1}^{m} n_{A_i}}{(\sum_{i=1}^{m}\int_0^{\tau_i} Y_{A_i}(s)\,ds)^2}\left[\left(\sum_{i=1}^{m}\int_0^{\tau_i} Y_{A_i}(s)\,ds\right)\mathcal{I}_1(m,\beta_A,\Delta_B,\Delta_C)-\mathcal{I}_2(m,\beta_A,\Delta_B,\Delta_C)\right]. \tag{4.68}$$

With the partial score function (4.65) and its estimated variance (4.68), we obtain the standardized partial score statistic for testing $H_0 : \beta_A = 0$ as

$$Z = \frac{U_\beta(\tilde{\alpha}_A, 0)}{\widehat{Var}(U_\beta(\tilde{\alpha}_A, 0))^{\frac{1}{2}}}\,. \tag{4.69}$$

Under some mild convergence conditions on the at-risk indicators Y_{Ai}, $i = 1, \ldots, m$, (see, Andersen et al., 1993, pp. 426–427 for an example), we can obtain the asymptotic distribution of the test statistic (4.69) as $m \to \infty$. We discuss some of the large sample properties of the test statistic (4.69) in Section 4.3 through simulations.

4.3 Simulation Studies

In this section, we present the results of simulation studies conducted to assess when asymptotic normal approximation for the test statistics developed in Section 4.2.2, Section 4.2.3 and Section 4.2.4 are satisfactory. We consider two settings (i) $m = 1$ and $\tau \to \infty$ and (ii) $m \to \infty$ and τ fixed.

We first consider the setting given in Section 4.2.2; that is, Case 2: $m = 1$, $\Delta_A = 0$, $\Delta_B > 0$, $\Delta_C > 0$. The hypothesis of no parallel carryover effects in Component A is $H_0 :\ \beta_A = 0$, $\alpha_A > 0$ and the test statistic for testing the null hypothesis is

$$Z = \frac{U_{\beta_A}(\tilde{\alpha}_A, 0)}{\widehat{Var}(U_{\beta_A}(\tilde{\alpha}_A, 0))^{\frac{1}{2}}}, \tag{4.70}$$

where the score statistic $U_{\beta_A}(\tilde{\alpha}_A, 0)$ is given by (4.32) and its estimated variance $\widehat{Var}(U_{\beta_A}(\tilde{\alpha}_A, 0))^{\frac{1}{2}}$ is given by (4.34). We look into the null distribution of Z and assess the standard normal approximation when τ increases.

We generated 10,000 realizations of HPPs based on (4.19), (4.20), and (4.21) where $\alpha_A = \alpha_B = \alpha_C = 0.1$, and $\beta_A = 0$, $\beta_B = \beta_C = 0.693$ so that $\exp\{0.693\} = 2$. We consider Δ_B, $\Delta_C = 1, 3, 7$ and 14 time units. Normal quantile-quantile (Q-Q) plots of 10,000 values of Z are presented with Δ_B, $\Delta_C = 1$ for four different end-of-followup times τ in Figure 4.1. The normal approximation is not adequate for small τ values. The distribution of Z in (4.70) converges to a standard normal distribution as τ increases, but the convergence rate is slow. Figure 4.2 shows the results when $\Delta_B = \Delta_C = 3$. In this case, the distribution converges faster to the standard normal approximation. When $\Delta_B = \Delta_C = 7$ in Figure 4.3, the standard normal approximation is quite accurate even for small τ values, except for the tails when $\tau = 100$. Figure 4.4 shows similar results to those of Figure 4.3.

We let Q_p be the pth quantile of the standard normal distribution and $\hat{Q}_p$ be the empirical pth quantile of Z computed from 10,000 samples. We obtain the estimates of $Pr(Z > Q_p) = 1 - p$, where $p = 0.950, 0.975$ and 0.990 in Table 4.1. The results are in line with those obtained from Figures 4.1–4.4. Therefore, except the case in which repair times of primary components are too small, the standard normal approximations are adequate when approximately 20 or more failures are observed in the redundant component over the followup period.

The power of the statistic (4.70) was investigated by Monte Carlo simulation methods. We used the 0.95 quantile of the standard normal distribution as well as the empirical 0.95 quantile of the test statistic obtained from 10,000 simulations runs under the null hypothesis. We generated 1,000 processes under the models

$$\begin{aligned}
\lambda_A(t|\mathcal{H}(t)) &= Y_A(t)\,\alpha_A\,\exp\{\beta_A X_{AB}(t) + \beta_A X_{AC}(t)\}, \\
\lambda_B(t|\mathcal{H}(t)) &= Y_B(t)\,\alpha_B\,\exp\{\beta_B X_{BC}(t)\}, \\
\lambda_C(t|\mathcal{H}(t)) &= Y_C(t)\,\alpha_C\,\exp\{\beta_C X_{CB}(t)\},
\end{aligned} \tag{4.71}$$

where we took $\alpha_A = \alpha_B = \alpha_C = 0.1$. We took (i) $\beta_A = \beta_B = \beta_C = 0.693$ and (ii) $\beta_A = \beta_B = \beta_C = 1.386$, so that $e^{0.693} = 2$ and $e^{1.386} = 4$. The power results are provided in Table 4.2, where entries are the proportions of the values of Z in 1,000 samples which are larger than the quantile values. Table 4.2 shows poor results when Δ_B and Δ_C are small. This is not surprising because the expected number of followup failures for Δ_B and Δ_C times are small, especially when $\exp\{\beta_A\} = 2$. However, power of the test is increasing as τ and/or Δ_B and Δ_C increase. Also, power increases as e^{β_A} increases.

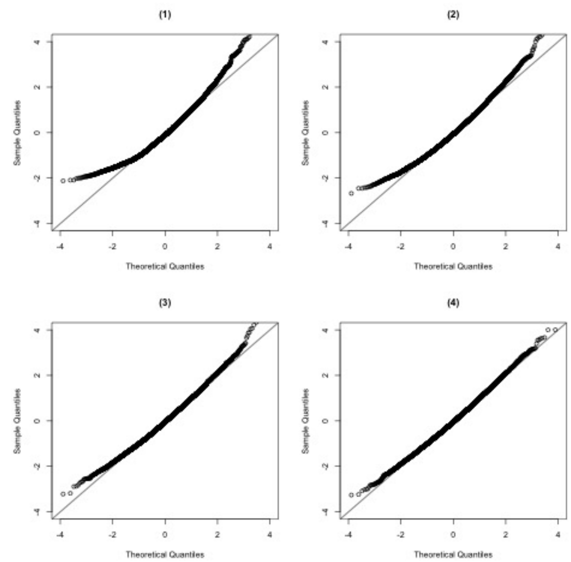

Figure 4.1: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_A = 0$, $\Delta_B = 1$, $\Delta_C = 1$, and (1) $\tau = 100$, (2) $\tau = 200$, (3) $\tau = 500$, (4) $\tau = 1000$

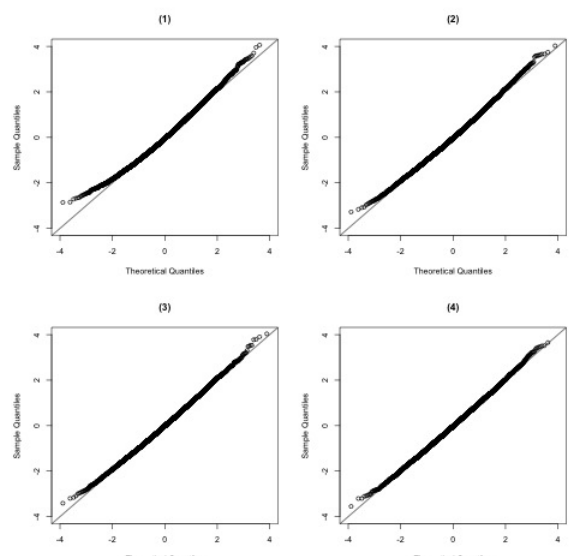

Figure 4.2: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when$\Delta_A = 0$, $\Delta_B = 3$, $\Delta_C = 3$, and (1) $\tau = 100$, (2) $\tau = 200$, (3) $\tau = 500$, (4) $\tau = 1000$

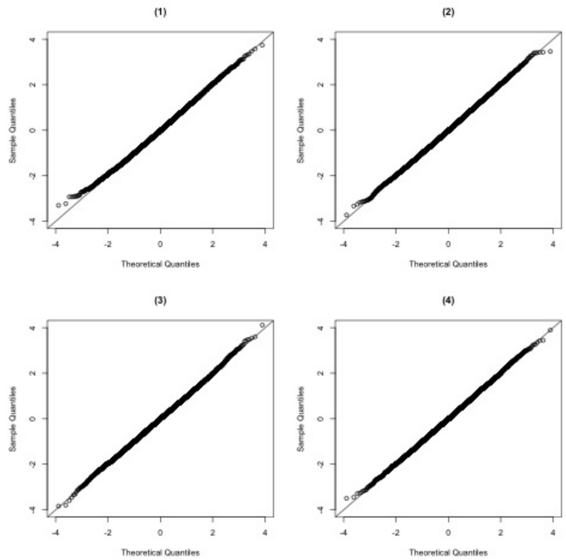

Figure 4.3: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_A = 0$, $\Delta_B = 7$, $\Delta_C = 7$, and (1) $\tau = 100$, (2) $\tau = 200$, (3) $\tau = 500$, (4) $\tau = 1000$

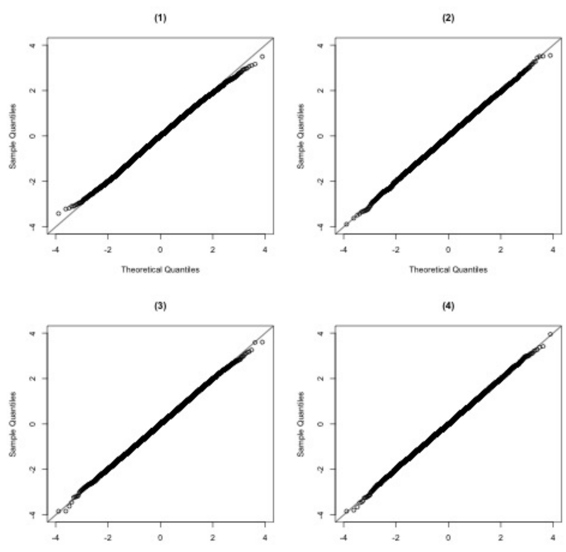

Figure 4.4: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_A = 0$, $\Delta_B = 14$, $\Delta_C = 14$, and (1) $\tau = 100$, (2) $\tau = 200$, (3) $\tau = 500$, (4) $\tau = 1000$

Δ_B and Δ_C	τ	$\hat{Q}_{0.950}$	$\hat{Q}_{0.975}$	$\hat{Q}_{0.990}$	$\hat{Pr}(Z > 1.645)$	$\hat{Pr}(Z > 1.960)$	$\hat{Pr}(Z > 2.326)$
1	100	1.814	2.248	2.854	0.0630	0.0406	0.0229
	200	1.794	2.193	2.67	0.0640	0.0381	0.0197
	500	1.722	2.070	2.517	0.0560	0.0315	0.0143
	1000	1.686	2.046	2.433	0.0539	0.0295	0.0136
3	100	1.731	2.102	2.55	0.0585	0.0348	0.0146
	200	1.718	2.095	2.488	0.0587	0.0318	0.0150
	500	1.693	2.048	2.385	0.0550	0.0301	0.0116
	1000	1.643	1.986	2.398	0.0499	0.0263	0.0119
7	100	1.662	1.997	2.371	0.0518	0.0274	0.0114
	200	1.659	1.993	2.361	0.0515	0.0273	0.0106
	500	1.615	1.936	2.336	0.0472	0.0237	0.0103
	1000	1.652	1.968	2.386	0.0508	0.0252	0.0110
14	100	1.607	1.859	2.178	0.0454	0.0195	0.0070
	200	1.638	1.906	2.261	0.0490	0.0224	0.0085
	500	1.627	1.947	2.287	0.0484	0.0244	0.0092
	1000	1.657	1.944	2.325	0.0515	0.0235	0.0100

Table 4.1: $\hat{Q}_p$ is the empirical pth quantile of Z computed from 10,000 samples when $m = 1$, $\Delta_A = 0$, $\Delta_B > 0$, $\Delta_C > 0$. $\hat{Pr}(Z > Q_p)$ is the proportion of the values of Z in 10,000 samples which are larger than the pth quantile of a standard normal distribution

Δ_B and Δ_C	τ	$\hat{Pr}\{Z > 1.645\}$ $e^{\beta_A} = 2$	$\hat{Pr}\{Z > \hat{Q}_{0.950}\}$ $e^{\beta_A} = 2$	$\hat{Pr}\{Z > 1.645\}$ $e^{\beta_A} = 4$	$\hat{Pr}\{Z > \hat{Q}_{0.950}\}$ $e^{\beta_A} = 4$
3	100	0.185	0.174	0.286	0.263
	200	0.227	0.216	0.396	0.383
	500	0.324	0.313	0.575	0.575
	1000	0.433	0.433	0.726	0.733
7	100	0.510	0.506	0.664	0.653
	200	0.678	0.675	0.857	0.852
	500	0.921	0.924	0.993	0.993
	1000	0.995	0.995	1.000	1.000
14	100	0.586	0.592	0.656	0.655
	200	0.778	0.779	0.870	0.873
	500	0.982	0.982	0.997	0.997
	1000	1.000	1.000	1.000	1.000

Table 4.2: Power of Z : $m = 1$, $\Delta_A = 0$, $\Delta_B > 0$, $\Delta_C > 0$

We also considered the test statistic (4.46) developed in Section 4.2.3, where $m = 1$, $\Delta_A > 0$, $\Delta_B > 0$, $\Delta_C > 0$. We generated 10,000 realizations of HPPs based on (4.35), (4.36), and (4.37) with $\alpha_A = \alpha_B = \alpha_C = 0.1$, and $\beta_A = 0$, $\beta_B = \beta_C = 0.693$. We considered Δ_B, $\Delta_C = 1, 3, 7$ and 14. For Δ_B, $\Delta_C = 3$, the results are presented in Figure 4.6, which shows that the normal approximations are quite accurate when $\tau = 500$ and 1000. For Δ_B, $\Delta_C = 5$ in Figure 4.7, the normal approximations are adequate when $\tau = 500$ and 1000. For Δ_B, $\Delta_C = 14$ (See Figure 4.8), the approximations are good when $\tau = 200, 500$ and 1000. Table 3.3 shows estimated Q_p and $\hat{Pr}(Z > Q_p)$ values where $p = 0.950, 0.975$ and 0.990. The results in this table indicate also that the standard normal approximation is adequate for large τ.

The power of the statistic (4.46) against the alternative hypothesis $H_A : \beta_A \neq 0$, $\alpha_A > 0$ is investigated by using Monte Carlo simulation methods. The 0.95 quantile of the standard normal distribution and the empirical 0.95 quantile of the test statistic obtained from 10,000 simulations runs were used to obtain the power. We generated 1,000 processes under the models

$$\begin{aligned}
\lambda_A(t|\mathcal{H}(t)) &= Y_A(t)\,\alpha_A\,\exp\{\beta_A X_{AB}(t) + \beta_A X_{AC}(t)\},\\
\lambda_B(t|\mathcal{H}(t)) &= Y_B(t)\,\alpha_B\,\exp\{\beta_B X_{BC}(t)\},\\
\lambda_C(t|\mathcal{H}(t)) &= Y_C(t)\,\alpha_C\,\exp\{\beta_C X_{CB}(t)\},
\end{aligned} \tag{4.72}$$

where we took $\alpha_A = \alpha_B = \alpha_C = 0.1$. We took (i) $\beta_A = \beta_B = \beta_C = 0.916$ and (ii) $\beta_A = \beta_B = \beta_C = 1.098$, so that $e^{0.916} = 2.5$ and $e^{1.098} = 3$. The power results are provided in Table 4.4. Each entry in Table 4.4 is the proportions of the values of Z in 1,000 samples which are larger than the quantile values and this shows that the power of the test is high overall, and power increases as τ increases.

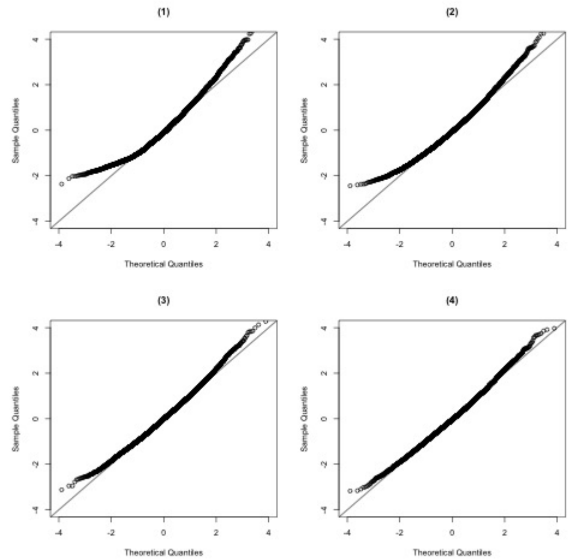

Figure 4.5: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_A = 1$, $\Delta_B = 1$, $\Delta_C = 1$, and (1) $\tau = 100$, (2) $\tau = 200$, (3) $\tau = 500$, (4) $\tau = 1000$

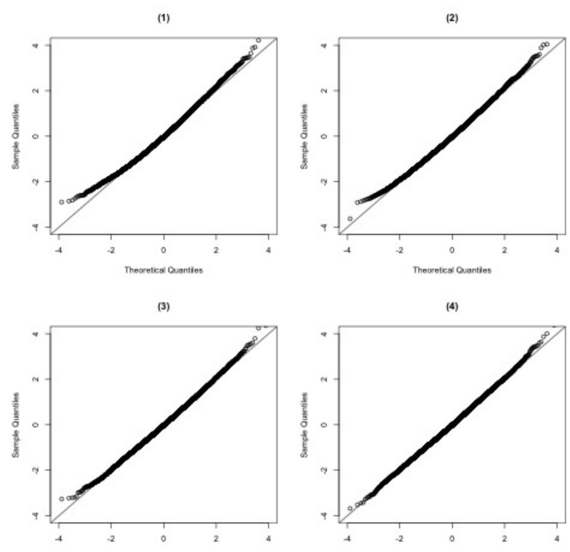

Figure 4.6: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_A = 1$, $\Delta_B = 3$, $\Delta_C = 3$, and (1) $\tau = 100$, (2) $\tau = 200$, (3) $\tau = 500$, (4) $\tau = 1000$

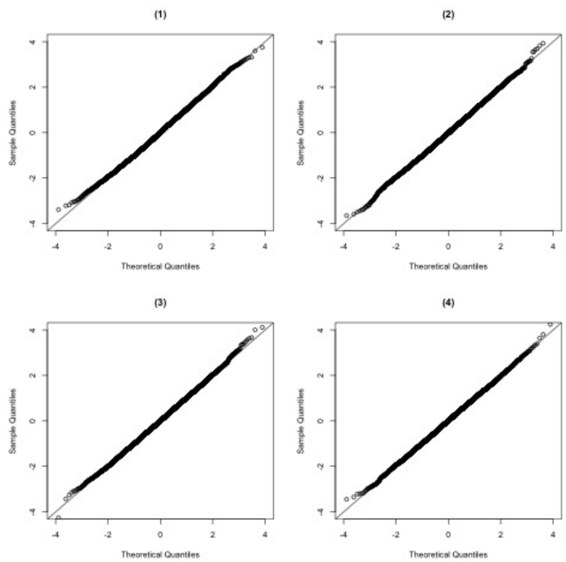

Figure 4.7: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_A = 1$, $\Delta_B = 7$, $\Delta_C = 7$, and (1) $\tau = 100$, (2) $\tau = 200$, (3) $\tau = 500$, (4) $\tau = 1000$

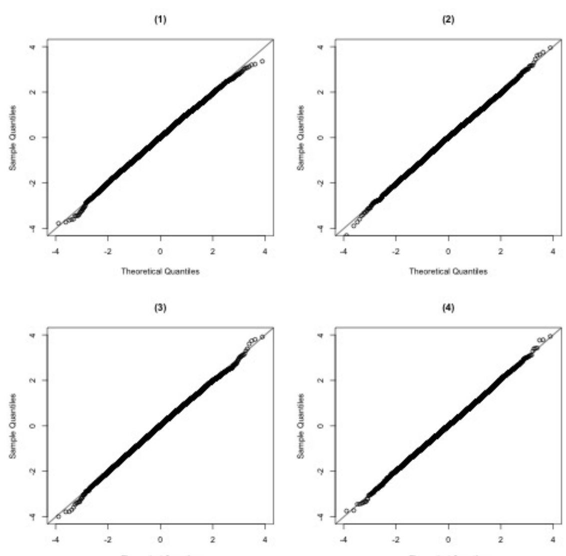

Figure 4.8: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_A = 1$, $\Delta_B = 14$, $\Delta_C = 14$, and (1) $\tau = 100$, (2) $\tau = 200$, (3) $\tau = 500$, (4) $\tau = 1000$

Δ_B and Δ_C	τ	$\hat{Q}_{0.950}$	$\hat{Q}_{0.975}$	$\hat{Q}_{0.990}$	$\hat{Pr}(Z > 1.645)$	$\hat{Pr}(Z > 1.960)$	$\hat{Pr}(Z > 2.326)$
1	100	1.879	2.280	2.839	0.0695	0.0426	0.0227
	200	1.830	2.246	2.725	0.0652	0.0391	0.0215
	500	1.755	2.128	2.609	0.0611	0.0327	0.0171
	1000	1.716	2.122	2.516	0.0565	0.0343	0.0163
3	100	1.767	2.133	2.553	0.0625	0.0355	0.0171
	200	1.714	2.080	2.511	0.0578	0.032	0.0162
	500	1.677	2.037	2.422	0.0537	0.0287	0.0122
	1000	1.694	1.995	2.387	0.0557	0.0272	0.0116
7	100	1.667	2.004	2.405	0.0537	0.028	0.013
	200	1.658	1.974	2.359	0.0511	0.0264	0.0111
	500	1.646	1.990	2.341	0.0501	0.0267	0.0104
	1000	1.605	1.924	2.294	0.0451	0.0231	0.0092
14	100	1.587	1.906	2.259	0.0449	0.0222	0.0084
	200	1.614	1.906	2.294	0.0474	0.0219	0.0092
	500	1.637	1.951	2.258	0.0492	0.0242	0.0081
	1000	1.649	1.990	2.353	0.0506	0.0271	0.0115

Table 4.3: $\hat{Q}_p$ is the empirical pth quantile of Z computed from 10,000 samples when $m = 1$ and $\Delta_A = 1$, $\Delta_B > 0$, $\Delta_C > 0$. $\hat{Pr}(Z > Q_p)$ is the proportion of the values of Z in 10,000 samples which are larger than the pth quantile of a standard normal distribution

Δ_B and Δ_C	τ	$\hat{Pr}\{Z > 1.645\}$ $e^{\beta_A} = 2.5$	$\hat{Pr}\{Z > \hat{Q}_{0.950}\}$ $e^{\beta_A} = 2.5$	$\hat{Pr}\{Z > 1.645\}$ $e^{\beta_A} = 3$	$\hat{Pr}\{Z > \hat{Q}_{0.950}\}$ $e^{\beta_A} = 3$
3	100	0.116	0.099	0.108	0.092
	200	0.104	0.098	0.112	0.104
	500	0.104	0.100	0.120	0.116
	1000	0.100	0.094	0.113	0.108
7	100	0.243	0.239	0.278	0.277
	200	0.203	0.202	0.315	0.312
	500	0.249	0.249	0.389	0.389
	1000	0.315	0.326	0.453	0.464
14	100	0.400	0.409	0.509	0.512
	200	0.478	0.484	0.575	0.581
	500	0.579	0.579	0.788	0.788
	1000	0.735	0.735	0.929	0.929

Table 4.4: Power of Z : $m = 1$, $\Delta_A = 1$, $\Delta_B > 0$, $\Delta_C > 0$

We now considered the score statistic (4.69) given in Section 4.2.4, where $m > 1$, $\Delta_A > 0$, $\Delta_B > 0$, $\Delta_C > 0$. We fixed τ at 100, and generated 10,000 realizations of m HPPs based on (4.50), (4.52), and (4.54) with $\alpha_A = \alpha_B = \alpha_C = 0.1$, and $\beta_A = 0$, $\beta_B = \beta_C = 0.693$. We considered Δ_B, $\Delta_C = 1, 3, 7$ and 14, and $m = 10, 20, 50$ and 100. Normal Q-Q plots (Figures 4.9, 4.10, 4.11 and 4.12) show that normal approximations are quite suitable with these various combinations with Δ_B, Δ_C and m. Table 4.5 presents the value of estimated Q_p and $\hat{Pr}(Z > Q_p)$ where $p = 0.950, 0.975$ and 0.990. This shows that when Δ_B, $\Delta_C = 1, 3, 7$ and 14, the normal approximation is suitable.

The power of the statistic (4.69) under the alternative hypothesis $H_A : \beta_A \neq 0$, $\alpha_A > 0$ is investigated by using Monte Carlo simulation study. To obtain the power, 0.95 quantile of the standard normal distribution and the empirical 0.95 quantile of the test statistic obtained from 10,000 simulations runs with different m, Δ_B and Δ_C used. The power results are provided in Table 4.6, which shows that the power of the test is high for long τ and/or m values. Also, power increases as e^{β_A} increases.

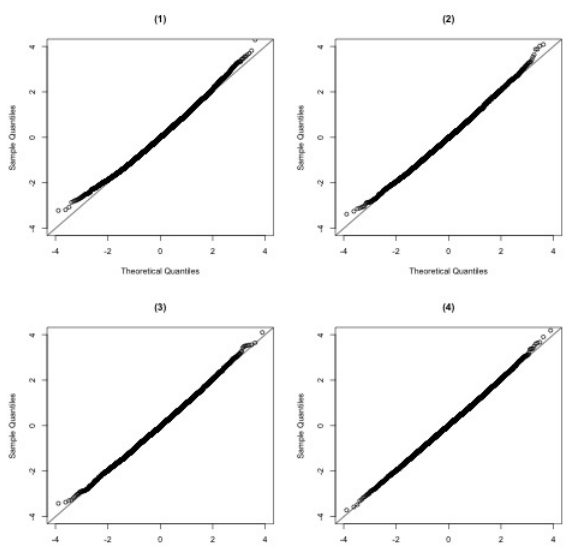

Figure 4.9: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_A = 1$, $\Delta_B = 1$, $\Delta_C = 1$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$, (4) $m = 100$

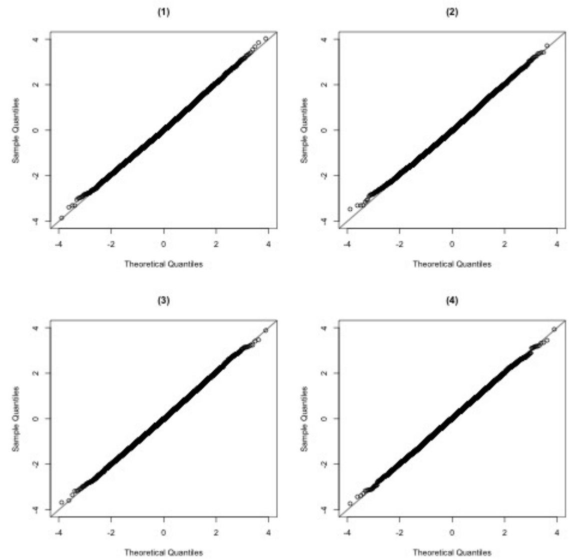

Figure 4.10: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_A = 1$, $\Delta_B = 3$, $\Delta_C = 3$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$, (4) $m = 100$

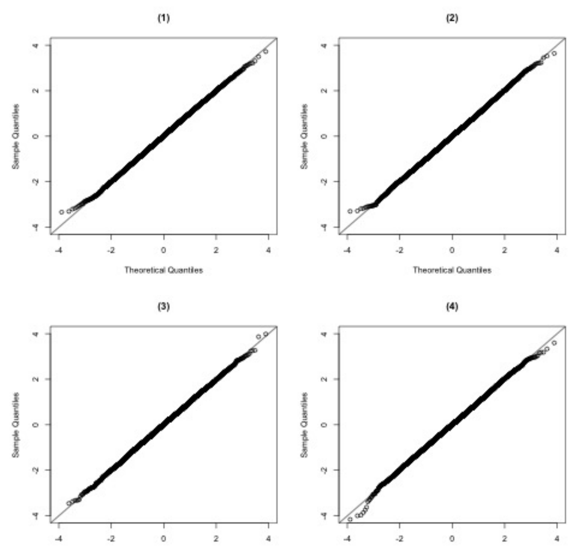

Figure 4.11: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_A = 1$, $\Delta_B = 7$, $\Delta_C = 7$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$, (4) $m = 100$

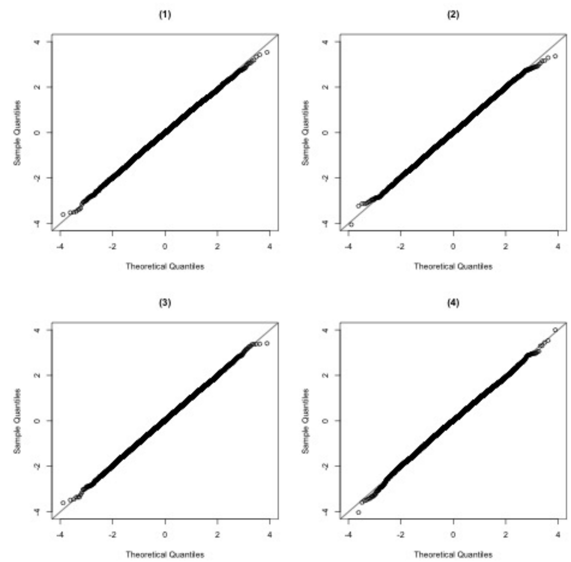

Figure 4.12: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_A = 1$, $\Delta_B = 14$, $\Delta_C = 14$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$, (4) $m = 100$

Δ_B and Δ_C	m	$\hat{Q}_{0.950}$	$\hat{Q}_{0.975}$	$\hat{Q}_{0.990}$	$\hat{Pr}(Z > 1.645)$	$\hat{Pr}(Z > 1.960)$	$\hat{Pr}(Z > 2.326)$
1	10	1.717	2.071	2.540	0.0579	0.0307	0.0151
	20	1.714	2.041	2.458	0.0573	0.0306	0.0132
	50	1.662	1.987	2.392	0.0514	0.0264	0.0122
	100	1.674	1.999	2.381	0.0526	0.0271	0.0112
3	10	1.702	2.032	2.406	0.0570	0.0288	0.0121
	20	1.703	1.979	2.366	0.0551	0.0268	0.0112
	50	1.689	2.011	2.398	0.0548	0.0274	0.0114
	100	1.674	1.984	2.350	0.0531	0.0265	0.0109
7	10	1.636	1.952	2.304	0.0493	0.0246	0.0096
	20	1.654	1.983	2.376	0.0508	0.0263	0.0112
	50	1.637	1.928	2.308	0.0485	0.0237	0.0097
	100	1.664	1.984	2.329	0.0517	0.0263	0.0101
14	10	1.570	1.845	2.220	0.0408	0.0195	0.0076
	20	1.638	1.950	2.318	0.0495	0.0245	0.0097
	50	1.649	1.938	2.349	0.0509	0.0240	0.0103
	100	1.596	1.880	2.261	0.0456	0.0209	0.0083

Table 4.5: $\hat{Q}_p$ is the empirical pth quantile of Z computed from 10,000 samples when $m > 1$. $\hat{Pr}(Z > Q_p)$ is the proportion of the values of Z in 10,000 samples which are larger than the pth quantile of a standard normal distribution

Δ_B and Δ_C	m	$\hat{Pr}\{Z > 1.645\}$ $e^{\beta_A} = 2.5$	$\hat{Pr}\{Z > \hat{Q}_{0.950}\}$ $e^{\beta_A} = 2.5$	$\hat{Pr}\{Z > 1.645\}$ $e^{\beta_A} = 3$	$\hat{Pr}\{Z > \hat{Q}_{0.950}\}$ $e^{\beta_A} = 3$
3	10	0.120	0.113	0.149	0.137
	20	0.110	0.103	0.162	0.149
	50	0.154	0.143	0.181	0.176
	100	0.207	0.204	0.277	0.270
7	10	0.324	0.324	0.444	0.445
	20	0.483	0.480	0.614	0.611
	50	0.725	0.726	0.824	0.826
	100	0.901	0.896	0.960	0.958
14	10	0.646	0.662	0.772	0.781
	20	0.823	0.823	0.907	0.908
	50	0.984	0.984	0.996	0.996
	100	1.000	1.000	1.000	1.000

Table 4.6: Power of Z : $m > 1$, $\Delta_A = 1$, $\Delta_B > 0$, $\Delta_C > 0$

4.4 Application

We give an illustration of the methods considered in Section 4.2.4 where we discuss testing for parallel carryover effects when $m > 1$, $\Delta_A > 0$, $\Delta_B > 0$ and $\Delta_C > 0$ case. In Section 1.1.1, we introduced an overview of a data set of failures of diesel operated power plants.

The data set received from the power company has information on locations of communities, the number of generators in each community, maximum capacity and connected capacity of the generators in each community, engine hours effective of each generator on December 31, 2009, engine hours effective of each generator on December 31, 2011. Other information includes model, purchase year and operating speed of each generator. We also received an availability data set, which includes failure data only for a few communities for about two years. This availability data set includes times when generators are out of service as well as times when generators return back to service.

Since there is very limited information on the failure times, we generated a failure data set based on the available information received from the company, which includes limited failure data, capacities of generators and purchase year of generators. In the information sheet received from the company, there are 22 communities where each community has 3, 4 or 5 generators working together, but most of the communities have 3 generators to provide the electricity to the community. Therefore, we first assume that all communities have 3 generators working in parallel. We assume also that the redundant component (Component A) is identical for all communities. However, components in a system maybe different from system to system.

We analyze the data for 16 communities. These are the communities with 3 power generators. The purchase dates of each generator are provided in the original data. Therefore, we used the purchase dates as the start of the observation times of each generator. The observation period of each generator ends on September 1, 2015. In each community, the redundant component is the oldest generator. Thus, the start time of the observation (i.e. $t = 0$) is the purchase date of the redundant component in a system.

Repair times are identical within communities; that is, $\Delta_{Ai} = \Delta_{Bi} = \Delta_{Ci} = d_i$, where $d_i > 0$ is a constant and $i = 1, 2, \ldots, 16$. However, the repair times between communities can be different, because repair times also include travelling time from the headquarters of the company and the distances from the headquarters are diverse. Therefore, we let repair times be $15, 30$ or 45 days according to their distances from the headquarters. The average, median and variance of the number of failures of the redundant generators in the communities during the observation periods is 30.6, 29 and 105.8, respectively, and ranging from 13 to 56. A part of the data generated is given in Appendix A.1. It is worthy noting that we did not consider time trends in the data generation.

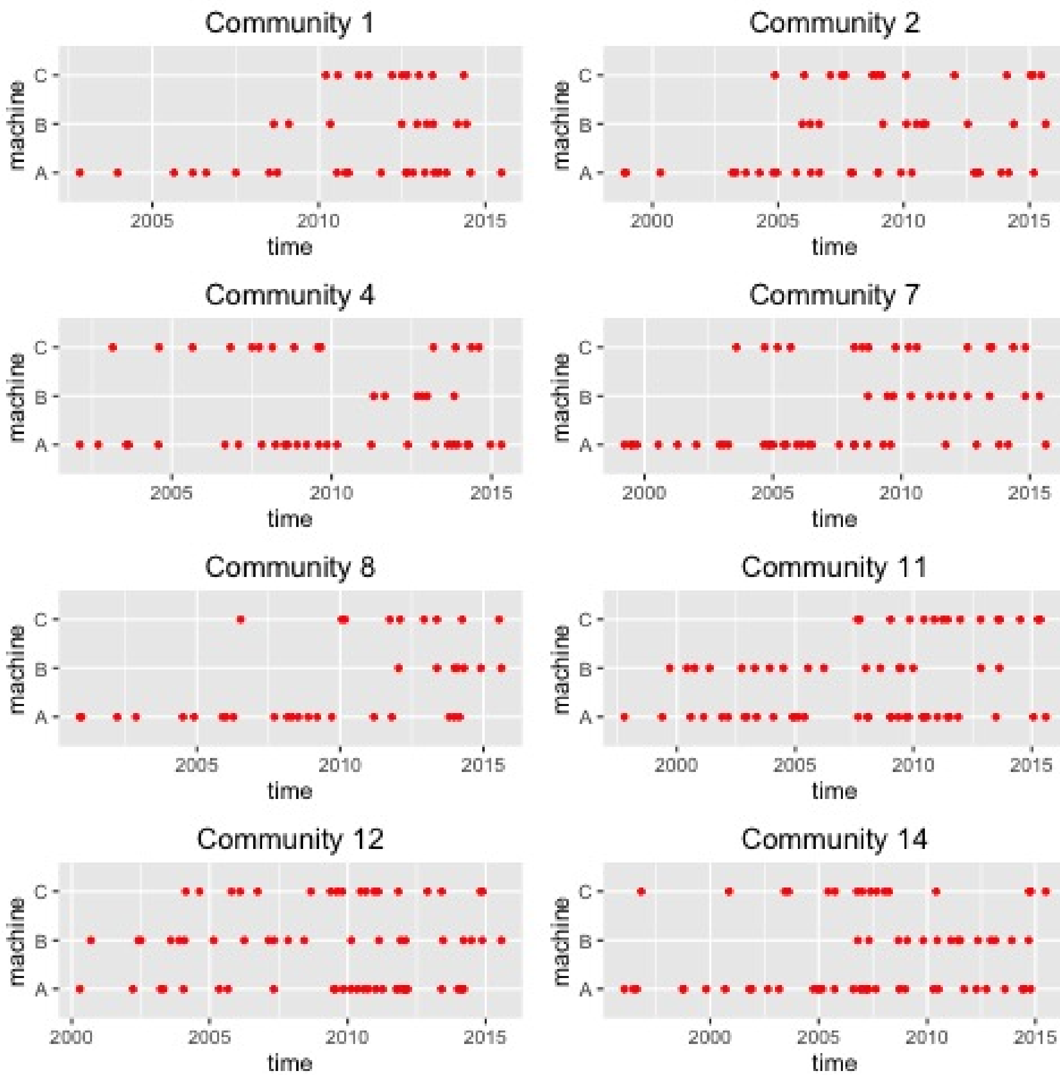

Figure 4.13: Dot plots of failures of generators in communities

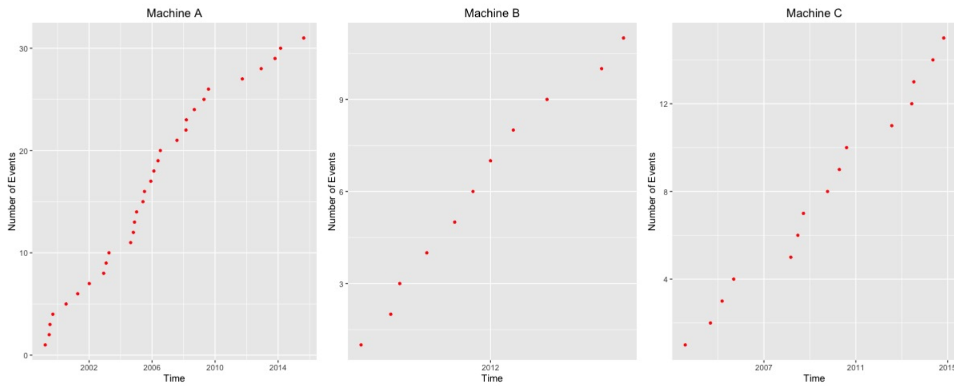

Figure 4.14: Cumulative failures of community 7

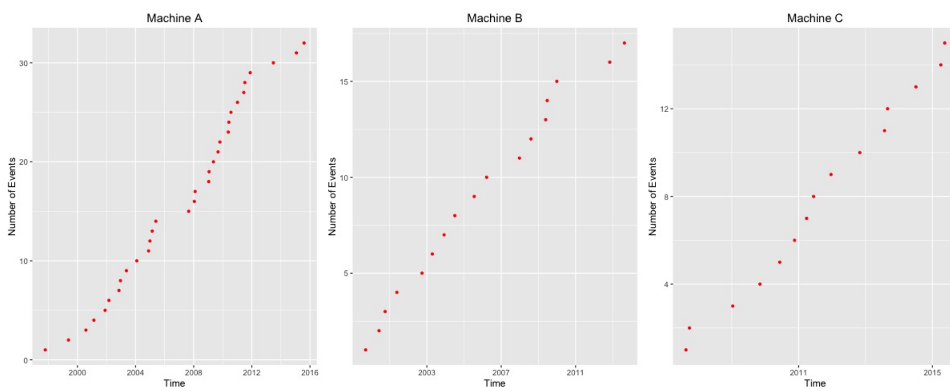

Figure 4.15: Cumulative failures of community 11

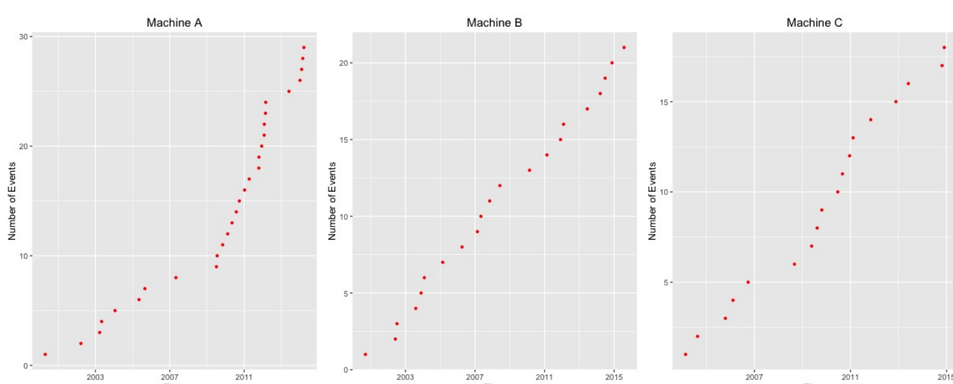

Figure 4.16: Cumulative failures of community 12

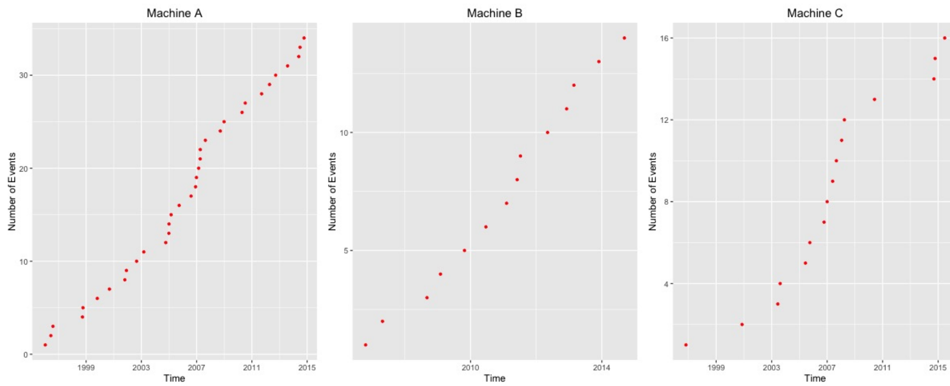

Figure 4.17: Cumulative failures of community 14

Figure 4.13 shows the dot plots of the generated failure times of power generators for 8 communities. There are some clusters of failure events togeher in the redundant generators (called Machine A in the plots) soon after failures of other generators. Figures 4.14, 4.15, 4.16 and 4.17 present the plots of cumulative failures of generators (Machine A, Machine B and Machine C) with respect to the calendar time in the communities 7, 11, 12 and 14, respectively. There are some clusters of failures in the redundant generator (Machine A) noted in the plots.

Clustering of failures in Figures 4.13–4.17 suggests the presence of parallel carryover effects in redundant components. Therefore, we consider the following model for the redundant generators: For, $i = 1, \ldots, 16$,

$$\lambda_{Ai}(t|\mathcal{H}_i(t)) = Y_{Ai}(t)\,\alpha_A\,\exp\{\beta_A X_{ABi}(t) + \beta_A X_{ACi}(t)\}, \quad t > 0, \tag{4.73}$$

where $Y_{Ai}(t)$ is the at-risk indicator function of Component A (the redundant generator) in the ith system, $\alpha_A > 0$ is a baseline rate function, β_A is a regression parameter, and

$$X_{ABi}(t) = I\{N_{Bi}(t^-) > 0\}I\{t - t_{BiN_{Bi}(t^-)} \leq \Delta_{Bi}\},$$
$$X_{ACi}(t) = I\{N_{Ci}(t^-) > 0\}I\{t - t_{CiN_{Ci}(t^-)} \leq \Delta_{Ci}\}.$$

We test the presence of parallel carryover effects in the redundant components by testing $H_0 : \beta_A = 0, \alpha_A > 0$. Table 4.7 presents the maximum likelihood estimates $\hat{\alpha}_A$ and $\hat{\beta}_A$ of α_A and β_A, respectively, and the restricted maximum likelihood estimate $\tilde{\alpha}_A$ of α_A, when $\beta_A = 0$, as well as their standard errors within parentheses. The value of the test statistic Z is given in Table 4.8 which is 1.915; the standard normal approximation gives a two-sided p-value of 0.0554 and 1000 simulation run gives a two-sided p-value of 0.071. These results suggest that there is some evidence for the presence of parallel carryover effects in the redundant components.

$\hat{\alpha}_A$	$\hat{\beta}_A$	$\tilde{\alpha}_A$
0.004716 (0.000217)	0.645239 (0.138141)	0.005008 (0.000228)

Table 4.7: Estimates of α_A, β_A and α_A when $\beta_A = 0$. The numbers in parentheses are the standard errors

$U_{\beta_A}(\tilde{\boldsymbol{\theta}})$	$\widehat{Var}(U_{\beta_A}(\tilde{\boldsymbol{\theta}}))$	Z	$\ell(\hat{\boldsymbol{\theta}})$	$\ell(\tilde{\boldsymbol{\theta}})$
13.15811	47.20691	1.915096	3019.458	3028.644

Table 4.8: The test statistic Z, $\ell(\hat{\boldsymbol{\theta}})$ and $\ell(\tilde{\boldsymbol{\theta}})$

It should be noted that, even though we did not generate the data from a trend model, the convex shapes of plots of cumulative failures of generators against calendar time given in Figures 4.14–4.17 suggest the presence of monotonic trends. For example, see the plot of Machine A in Figure 3.16. We discuss modeling trends with parallel carryover effects in the next chapter. However, to investigate this issue with the current data, we now consider the model (4.73) with a trend term as follows.

$$\lambda_{Ai}(t|\mathcal{H}_i(t)) = Y_{Ai}(t)\,\alpha_A\, \exp\{\beta_A X_{ABi}(t) + \beta_A X_{ACi}(t) + \gamma\, t\}, \tag{4.74}$$

where $i = 1, 2, \ldots, 16$. We test the absence of trend by testing $H_0 : \gamma = 0$ with the pooled data. Table 4.9 shows the maximum likelihood estimates of α_A, β_A and γ in model (4.74), and their standard errors. A Wald-type statistic $W = \hat{\gamma}^2/se^2(\hat{\gamma})$ gives a value of 3.536449 in Table 4.9; The p-value is 0.060033. This result suggests that there is a mild increasing trend in the data.

$\hat{\alpha}_A$	$\hat{\beta}_A$	$\hat{\gamma}$
0.015488 (0.001331)	0.693139 (0.101230)	0.000035 (0.0000101)
$W = \hat{\gamma}^2/se^2(\hat{\gamma})$	p-value	
3.536449	0.060033	

Table 4.9: Estimates of α_A, β_A, γ and Wald type statistic W and p-value. The numbers in parentheses are the standard errors

Chapter 5

Redundant Systems with Trends and Covariates

In this chapter, we consider the tests for parallel carryover effects in redundant systems with trends. This chapter is organized as follows. In Section 5.1, we briefly introduce the concept of trend in recurrent event processes. In Section 5.2, we provide tests for the presence of parallel carryover effects when monotonic trends due to stochastic aging are present. We present the results of simulation studies in Section 4.2. We develop a score test for the presence of parallel carryover effects in models with external covariates and trends in Section 5.3. In Section 5.4, we illustrate the methods by analyzing a generated data set.

5.1 Introduction

Stochastic aging is an important concept in the analysis of repairable systems. If there is no effect of the age of a system on probabilistic characteristics of event occurrences, we say that there is no stochastic aging. On the contrary, if probabilistic characteristics of event occurrences of a system depend on the age of a system, we say that there is stochastic aging (Lai and Xie, 2006). Many repairable systems are subject to stochastic aging. In this chapter, we discuss the assessment of parallel carryover effects in repairable systems subject to stochastic aging.

In recurrent event processes, stochastic aging is usually studied within the context of time trends (Cox and Lewis, 1966; Cook and Lawless, 2007). As discussed by Lawless et al. (2012), the definition of a trend in recurrent event processes is elusive.

Trends can appear in processes because of various reasons including stochastic aging due to wear-out phenomenon, previous number of events in a process or some external factors. There are monotonically increasing or decreasing trends as well as non-monotonic trends such as seasonal or bathtub type. Trends due to stochastic aging are usually in a monotonically increasing nature because repairable systems are more prone to fail as they age (Thompson, 1988). In this case, a trend can be defined as a systematic change in the rate function of a recurrent event process (Lawless et al., 2012). Non-homogenous Poisson processes (NHPPs) are useful to model increasing time trends. We discussed NHPPs in Section 2.1.

In the remainder of this chapter, we consider models for parallel carryover effects and monotonic trends together, and develop tests for the presence of parallel carryover effects in various settings. We study the asymptotic properties of the tests developed. We also extend our models to include external covariates, and develop tests for parallel carryover effects when external covariates are present.

5.2 Models and Tests for Parallel Carryover Effects with Trends

In this section, we discuss models and tests for parallel carryover effects with monotonic trends in a single system and multiple systems. We first introduce the notation.

Suppose that there is a bivariate counting processes $\{N_A(t), N_B(t); t \geq 0\}$, where $\{N_A(t); t \geq 0\}$ is a counting process for Component A and $\{N_B(t); t \geq 0\}$ is a counting process for Component B in a system with two components working in parallel. We let $t_{A1}, t_{A2}, \dots$, where $0 < t_{A1} < t_{A2} < \dots$, and $t_{B1}, t_{B2}, \dots$, where $0 < t_{B1} < t_{B2} < \dots$ denote the failure times of Component A and Component B, respectively. The components A and B are subject to repairs, and repair times cannot be ignored. Let Δ_A and Δ_B denote the repair times of Component A and Component B, respectively. For $K = A, B$, $Y_K(t)$ is the at-risk indicator of process $\{N_k(t); t > 0\}$, which takes value of 1 when Component K is up and the process $\{N_k(t); t > 0\}$ is under observation; otherwise, it equals to 0.

A model including parallel carryover effects for Component A with monotonic trend due to stochastic aging is given by

$$\lambda_A(t|\mathcal{H}(t)) = Y_A(t)\, \rho_A(t)\, \exp\{\beta_A X_A(t)\}, \quad t > 0, \tag{5.1}$$

where $Y_A(t)$ is the at-risk indicator of Component A, β_A is a parameter and $\rho_A(t) > 0$ is a time dependent baseline rate function. The rate function $\rho_A(t)$ is given by $\rho_A(t) = \alpha_A \exp(\gamma_A t)$, where $\gamma_A \in \mathbb{R}$ and $t > 0$. Then the intensity function (5.1) can be written as

$$\lambda_A(t|\mathcal{H}(t)) = Y_A(t)\,\alpha_A\, \exp\{\beta_A X_A(t) + \gamma_A t\}, \quad t > 0, \tag{5.2}$$

and

$$X_A(t) = I\{N_B(t^-) > 0\} I\{t - t_{B\,N_B(t^-)} \leq \Delta_B\}, \tag{5.3}$$

where function $X_A(t)$ takes the value of 1, if Component B is in the down state at time t; otherwise it is 0. The intensity function (5.2) jumps from $\alpha_A \exp\{\gamma_A t\}$ to $\alpha_A \exp\{\beta_A + \gamma_A t\}$ when Component B fails, and stays for Δ_B time units; otherwise, it remains $\alpha_A \exp\{\gamma_A t\}$ if Component B is in the up states.

Similarly, the intensity function of the counting process $\{N_B(t); t > 0\}$ is given by

$$\lambda_A(t|\mathcal{H}(t)) = Y_B(t)\,\alpha_B\, \exp\{\beta_B X_B(t) + \gamma_B t\}, \quad t > 0, \tag{5.4}$$

and

$$X_B(t) = I\{N_A(t^-) > 0\} I\{t - t_{A\,N_A(t^-)} \leq \Delta_A\}, \tag{5.5}$$

We let m denote the number of systems. In the remaining part of this section, we develop partial score tests for the presence of parallel carryover effects in five different cases; (i) a single system with monotonic trend is under observation and repair times of Components A and B are negligible ($m = 1$, $\Delta_A = 0$, $\Delta_B > 0$), (ii) multiple systems with monotonic trend are under observation and repair times of Component A are negligible ($m > 1$, $\Delta_A = 0$, $\Delta_B > 0$), (iii) a single system with monotonic trend is under observation and repair times of Component A are not negligible ($m = 1$, $\Delta_A > 0$, $\Delta_B > 0$), (iv) multiple systems with monotonic trend are under observation and repair times of Component A are not negligible ($m > 1$, $\Delta_A > 0$, $\Delta_B > 0$), (v) a single 3–component system with monotonic trend is under observation and repair times of Component A are not negligible ($m = 1$, $\Delta_A > 0$, $\Delta_B > 0$, $\Delta_C > 0$). The asymptotic distribution of the test statistic developed in this chapter are discussed through simulations in Section 5.3.

5.2.1 Case 1: $m = 1$, $\Delta_A = 0$, $\Delta_B > 0$

In this section, we consider a redundant system with two components; Components A and B. We first discuss a model with a single system and repair times of Component A are negligible. Which means that $m = 1$ and $\Delta_A = 0$. Since we consider monotonic trend in the model, we use NHPPs for this purpose. Under these assumptions, the intensity function of Component A is given by

$$\lambda_A(t|\mathcal{H}(t)) = Y_A(t)\,\alpha_A\,\exp(\beta_A\,X_A(t) + \gamma_A t), \quad t > 0, \tag{5.6}$$

where $Y_A(t)$ is at-risk indicator of Component A, $\alpha_A > 0$ is a baseline rate function and β_A and γ_A are parameters and $X_A(t) = I\{N_B(t^-) > 0\}I\{t - t_{B\,N_B(t^-)} \leq \Delta_B\}$. Since $\Delta_A = 0$ and $\Delta_B > 0$, the intensity function of Component B is given by

$$\lambda_B(t|\mathcal{H}(t)) = Y_B(t)\,\alpha_B\,\exp(\gamma_A t) \quad t > 0, \tag{5.7}$$

where $Y_B(t)$ is at-risk indicator of Component B, $\alpha_B > 0$ is a baseline rate function, γ_B is a parameter and $\mathcal{H}(t) = \{N_A(u), N_B(u), Y_B(s); 0 \leq u < t, 0 \leq s \leq t\}$. A test for the presence of a parallel carryover effect in Component A can be developed by considering the following hypothesis.

$$H_0 \;:\; \beta_A = 0, \alpha_A > 0, \gamma_A \in \mathbb{R}, \quad \text{vs.} \quad H_a \;:\; \beta_A \neq 0, \alpha_A > 0, \gamma_A \in \mathbb{R}, \tag{5.8}$$

where α_A, γ_A are nuisance parameters and β_A is the parameter of interest.

We suppose that a system is under observation over the followup period $[0, \tau]$, where τ is a fixed end-of-followup time. Note that, we can safely drop the at-risk indicator $Y_A(t)$ from the model (5.6), because Component A is continuously under observation in $[0, \tau]$ and its repair times are negligible. Let n_A denote the number of failures of Component A in $[0, \tau]$. The likelihood function of the outcome "$N_A(\tau) = n_A$ failures of Component A at times $t_{A1} \leq t_{A2} \leq \cdots \leq t_{An_A}$ in $[0, \tau]$" is given as follows.

$$L(\boldsymbol{\theta}) = \prod_{j=1}^{n_A} \alpha_A\, e^{\beta_A\, X_A(t_{Aj}) + \gamma_A t_{Aj}}\ \exp\{-\int_0^{\tau} \alpha_A\, e^{\beta_A\, X_A(s) + \gamma_A s}\, ds\}, \tag{5.9}$$

where $\boldsymbol{\theta} = (\alpha_A, \gamma_A, \beta_A)$. The log likelihood function $\ell(\boldsymbol{\theta}) = \log L(\boldsymbol{\theta})$ is given as follows.

$$\ell(\boldsymbol{\theta}) = n_A \log \alpha_A + \sum_{j=1}^{n_A} \beta_A X_A(t_{Aj}) + \sum_{j=1}^{n_A} \gamma_A t_{Aj} - \int_0^{\tau} \alpha_A \, e^{\beta_A X_A(s) + \gamma_A s} \, ds. \tag{5.10}$$

Let $\tilde{\alpha}_A$, $\tilde{\gamma}_A$ denote the restricted maximum likelihood estimator of α_A, γ_A, respectively under the null hypothesis in (5.8). They can be obtained by maximizing $l(\boldsymbol{\theta}_0)$ where $\boldsymbol{\theta}_0 = (\alpha_A, \gamma_A, 0)$. This can be done with an optimizing software such as the `nlm` package in `R` (R core team, 2013). The score vector is then defined by $\boldsymbol{U}(\boldsymbol{\theta}) = (U_{\alpha_A}(\boldsymbol{\theta}), U_{\gamma_A}(\boldsymbol{\theta}), U_{\beta_A}(\boldsymbol{\theta}))'$ with components

$$U_{\alpha_A}(\boldsymbol{\theta}) = \frac{n_A}{\alpha_A} - \int_0^{\tau} e^{\beta_A X_A(s) + \gamma_A s} \, ds, \tag{5.11}$$

$$U_{\gamma_A}(\boldsymbol{\theta}) = \sum_{j=1}^{n_A} t_{Aj} - \alpha_A \int_0^{\tau} s \, e^{\beta_A X_A(s) + \gamma_A s} \, ds, \tag{5.12}$$

and

$$U_{\beta_A}(\boldsymbol{\theta}) = \sum_{j=1}^{n_A} X_A(t_{Aj}) - \alpha_A \int_0^{\tau} X_A(s) \, e^{\beta_A X_A(s) + \gamma_A s} \, ds. \tag{5.13}$$

Then the observed information matrix $\boldsymbol{I}(\boldsymbol{\theta})$ is given by

$$\boldsymbol{I}(\boldsymbol{\theta}) = \begin{pmatrix} I_{\alpha_A \alpha_A}(\boldsymbol{\theta}) & I_{\alpha_A \gamma_A}(\boldsymbol{\theta}) & I_{\alpha_A \beta_A}(\boldsymbol{\theta}) \\ I_{\gamma_A \alpha_A}(\boldsymbol{\theta}) & I_{\gamma_A \gamma_A}(\boldsymbol{\theta}) & I_{\gamma_A \beta_A}(\boldsymbol{\theta}) \\ I_{\beta_A \alpha_A}(\boldsymbol{\theta}) & I_{\beta_A \gamma_A}(\boldsymbol{\theta}) & I_{\beta_A \beta_A}(\boldsymbol{\theta}) \end{pmatrix}, \tag{5.14}$$

where its components are given as follows.

$$
\begin{aligned}
I_{\alpha_A\alpha_A}(\boldsymbol{\theta}) &= \frac{n_A}{\alpha_A^2},\\
I_{\alpha_A\gamma_A}(\boldsymbol{\theta}) = I_{\gamma_A\alpha_A}(\boldsymbol{\theta}) &= \int_0^\tau s\, e^{\beta_A X_A(s)+\gamma_A s}\, ds,\\
I_{\alpha_A\beta_A}(\boldsymbol{\theta}) = I_{\beta_A\alpha_A}(\boldsymbol{\theta}) &= \int_0^\tau X_A(s)\, e^{\beta_A X_A(s)+\gamma_A s}\, ds,\\
I_{\gamma_A\gamma_A}(\boldsymbol{\theta}) &= \alpha_A \int_0^\tau s^2\, e^{\beta_A X_A(s)+\gamma_A s}\, ds,\\
I_{\gamma_A\beta_A}(\boldsymbol{\theta}) = I_{\beta_A\gamma_A}(\boldsymbol{\theta}) &= \alpha_A \int_0^\tau s\, X_A(s)\, e^{\beta_A X_A(s)+\gamma_A s}\, ds,\\
I_{\beta_A\beta_A}(\boldsymbol{\theta}) &= \alpha_A \int_0^\tau \{X_A(s)\}^2\, e^{\beta_A X_A(s)+\gamma_A s}\, ds.
\end{aligned}
$$

Let $\tilde{\boldsymbol{\theta}}_0 = (\tilde{\alpha}_A, \tilde{\gamma}_A, 0)$. Then, the standardized partial score statistic for testing the presence of parallel carryover effects is given by

$$
Z = \frac{U_{\beta_A}(\tilde{\boldsymbol{\theta}}_0)}{\widehat{Var}(U_\beta(\tilde{\boldsymbol{\theta}}_0)^{\frac{1}{2}}}, \tag{5.15}
$$

where $U_{\beta_A}(\tilde{\boldsymbol{\theta}}_0)$ is given by

$$
U_{\beta_A}(\tilde{\alpha}_A, \tilde{\gamma}_A, 0) = \sum_{j=1}^{n_A} X_A(t_{Aj}) - \tilde{\alpha}_A \int_0^\tau X_A(s)\, e^{\tilde{\gamma}_A s}\, ds, \tag{5.16}
$$

and variance estimate $\widehat{Var}(U_\beta(\tilde{\boldsymbol{\theta}}_0))$ is

$$
I_{\beta_A\beta_A}(\tilde{\boldsymbol{\theta}}_0) - \begin{pmatrix} I_{\beta_A\alpha_A}(\tilde{\boldsymbol{\theta}}_0) & I_{\beta_A\gamma_A}(\tilde{\boldsymbol{\theta}}_0) \end{pmatrix} \begin{pmatrix} I_{\alpha_A\alpha_A}(\tilde{\boldsymbol{\theta}}_0) & I_{\alpha_A\gamma_A}(\tilde{\boldsymbol{\theta}}_0) \\ I_{\gamma_A\alpha_A}(\tilde{\boldsymbol{\theta}}_0) & I_{\gamma_A\gamma_A}(\tilde{\boldsymbol{\theta}}_0) \end{pmatrix}^{-1} \begin{pmatrix} I_{\alpha_A\beta_A}(\tilde{\boldsymbol{\theta}}_0) \\ I_{\gamma_A\beta_A}(\tilde{\boldsymbol{\theta}}_0) \end{pmatrix}. \tag{5.17}
$$

5.2.2 Case 2: $m > 1$, $\Delta_A = 0$, $\Delta_B > 0$

We now consider multiple systems with monotonic trend. We assume that repair times of Component A are negligible while the repair times of Component B are not negligible; That is, $m = 1$ and, $\Delta_A = 0$ and $\Delta_B > 0$. We assume that there are m independent systems under observation each with two components; Components

A and B. We let $\{N_{Ai}(t), N_{Bi}(t); t \geq 0\}$, $i = 1, 2, \ldots, m$, where, for $K = A, B$, $\{N_{Ki}(t); t \geq 0\}$ is a counting process for Component K in the ith system. We let $t_{Ki1}, t_{Ki2}, \ldots$, where $0 < t_{Ki1} < t_{Ki2} < \ldots$, denote the failure times of Component K ($K = A, B$) in the ith system. The components are subject to repairs, and repair times cannot be ignored. Let Δ_{Ai} and Δ_{Bi} denote the repair times of Components A and B in the ith system, respectively. For $K = A, B$, $Y_{Ki}(t)$ is the at-risk indicator of the process $\{N_{Ki}(t); t > 0\}$. History of the ith counting process $\{N_{Ai}(t), N_{Bi}(t); t \geq 0\}$, $i = 1, 2, \ldots, m$, is denoted by $\mathcal{H}_i(t) = \{N_{Ai}(u), N_{Bi}(u); 0 \leq u < t\}$.

A model including parallel carryover effects for Component A in the ith system with monotonic trends is given by

$$\lambda_{Ai}(t|\mathcal{H}_i(t)) = Y_{Ai}(t)\,\rho_A(t)\,\exp\{\beta_A X_{Ai}(t)\}, \quad t > 0, \tag{5.18}$$

where β_A is a regression parameter and $\rho_A(t) > 0$ is a time dependent baseline rate function. The rate function $\rho_A(t)$ is given by $\rho_A(t) = \alpha_A \exp(\gamma_A t)$. The intensity function (5.18) can be rewritten as

$$\lambda_{Ai}(t|\mathcal{H}_i(t)) = Y_{Ai}(t)\,\alpha_A\,\exp\{\beta_A X_{Ai}(t) + \gamma_A t\}, \quad t > 0, \tag{5.19}$$

where γ_A is a real valued parameter and

$$X_{Ai}(t) = I\{N_{Bi}(t^-) > 0\} I\{t - t_{Bi\,N_{Bi}(t^-)} \leq \Delta_B\}. \tag{5.20}$$

Since $\Delta_{Ai} = 0$ and $\Delta_{Bi} > 0$, the probabilistic characteristics of failure occurrences of Component B are not affected by failures of Component A. Therefore, the intensity function of Component B in ith system is then given by

$$\lambda_{Bi}(t|\mathcal{H}_i(t)) = Y_{Bi}(t)\,\alpha_B\,\exp\{\gamma_B t\}, \quad t > 0, \tag{5.21}$$

where γ_B is a real valued parameter, and

$$X_{Bi}(t) = I\{N_{Ai}(t^-) > 0\} I\{t - t_{Ai\,N_{Ai}(t^-)} \leq \Delta_A\}. \tag{5.22}$$

A test of parallel carryover effects in Component A in this case can be developed by considering the hypothesis given in (5.8).

We suppose that m independent systems are under observation over $[0, \tau_i]$, where τ_i

is a fixed end-of-followup time of the ith system, $i = 1, 2, \ldots, m$. We let $t_{Ai1}, t_{Ai2}, \ldots, t_{Ain_{Ai}}$ be the failure times of Component A in the ith system, and let $N_{Ai}(\tau_i) = n_{Ai}$. Then the likelihood function of the outcome "$N_{Ai}(\tau_i) = n_{Ai}$ failures of Component A at times $t_{Ai1} \le t_{Ai2} \le \cdots \le t_{Ain_{Ai}}$ in $[0, \tau_i]$" is given by

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{m} L_i(\boldsymbol{\theta}), \tag{5.23}$$

where

$$L_i(\boldsymbol{\theta}) = \prod_{j=1}^{n_{Ai}} \alpha_A \, e^{\beta_A X_{Ai}(t_{Aij}) + \gamma_A t_{Aij}} \ \exp\{-\int_0^{\tau_i} \alpha_A \, e^{\beta_A X_{Ai}(s) + \gamma_A s} \, ds\}, \tag{5.24}$$

where $\boldsymbol{\theta} = (\alpha_A, \beta_A, \gamma)$. The log likelihood function $\ell(\boldsymbol{\theta}) = \log L(\boldsymbol{\theta})$ is given by

$$\log \alpha_A \sum_{i=1}^{m} n_{Ai} + \beta_A \sum_{i=1}^{m} \sum_{j=1}^{n_{Ai}} X_{Ai}(t_{Aij}) + \gamma_A \sum_{i=1}^{m} \sum_{j=1}^{n_{Ai}} t_{Aij} \ - \alpha_A \sum_{i=1}^{m} \int_0^{\tau_i} e^{\beta_A X_{Ai}(s) + \gamma_A s} \, ds. \tag{5.25}$$

The components of score vector $\boldsymbol{U}(\boldsymbol{\theta}) = (U_{\alpha_A}(\boldsymbol{\theta}), U_{\gamma_A}(\boldsymbol{\theta}), U_{\beta_A}(\boldsymbol{\theta}))'$ are followed by

$$U_{\alpha_A}(\boldsymbol{\theta}) = \frac{\sum_{i=1}^{m} n_{Ai}}{\alpha_A} - \sum_{i=1}^{m} \int_0^{\tau_i} e^{\beta_A X_{Ai}(s) + \gamma_A s} \, ds, \tag{5.26}$$

$$U_{\gamma_A}(\boldsymbol{\theta}) = \sum_{i=1}^{m} \sum_{j=1}^{n_{Ai}} t_{Aij} \ - \alpha_A \sum_{i=1}^{m} \int_0^{\tau_i} s \, e^{\beta_A X_{Ai}(s) + \gamma_A s} \, ds, \tag{5.27}$$

and

$$U_{\beta_A}(\boldsymbol{\theta}) = \sum_{i=1}^{m} \sum_{j=1}^{n_{Ai}} X_{Ai}(t_{Aij}) - \alpha_A \sum_{i=1}^{m} \int_0^{\tau_i} X_{Ai}(s) \, e^{\beta_A X_{Ai}(s) + \gamma_A s} \, ds. \tag{5.28}$$

Then the elements of the observed information matrix $\boldsymbol{I}(\boldsymbol{\theta})$ are given by

$$
\begin{aligned}
I_{\alpha_A\alpha_A}(\boldsymbol{\theta}) &= \frac{\sum_{i=1}^{m} n_{Ai}}{\alpha_A^2},\\
I_{\alpha_A\gamma_A}(\boldsymbol{\theta}) = I_{\gamma_A\alpha_A}(\boldsymbol{\theta}) &= \sum_{i=1}^{m}\int_0^{\tau_i} s\, e^{\beta_A X_{Ai}(s)+\gamma_A s}\, ds,\\
I_{\alpha_A\beta_A}(\boldsymbol{\theta}) = I_{\beta_A\alpha_A}(\boldsymbol{\theta}) &= \sum_{i=1}^{m}\int_0^{\tau_i} X_{Ai}(s)\, e^{\beta_A X_{Ai}(s)+\gamma_A s}\, ds,\\
I_{\gamma_A\gamma_A}(\boldsymbol{\theta}) &= \alpha_A \sum_{i=1}^{m}\int_0^{\tau_i} s^2\, e^{\beta_A X_{Ai}(s)+\gamma_A s}\, ds,\\
I_{\gamma_A\beta_A}(\boldsymbol{\theta}) = I_{\beta_A\gamma_A}(\boldsymbol{\theta}) &= \alpha_A \sum_{i=1}^{m}\int_0^{\tau_i} s\, X_{Ai}(s)\, e^{\beta_A X_{Ai}(s)+\gamma_A s}\, ds,\\
I_{\beta_A\beta_A}(\boldsymbol{\theta}) &= \alpha_A \sum_{i=1}^{m}\int_0^{\tau_i} X_{Ai}(s)\, e^{\beta_A X_{Ai}(s)+\gamma_A s}\, ds.
\end{aligned}
$$

We let $\tilde{\boldsymbol{\theta}}_{\mathbf{0}} = (\tilde{\alpha}_A, \tilde{\gamma}_A, 0)$. When $\beta_A = 0$, the restricted maximum likelihood estimators $\tilde{\alpha}_A$ and $\tilde{\gamma}_A$ can be obtained by optimizing software packages. The standardized partial score statistic for testing the presence of parallel carryover effects is then given by

$$
Z = \frac{U_{\beta_A}(\tilde{\boldsymbol{\theta}}_0)}{\widehat{Var}(U_\beta(\tilde{\boldsymbol{\theta}}_0))^{\frac{1}{2}}}, \tag{5.29}
$$

where the partial score function is given by

$$
U_{\beta_A}(\tilde{\boldsymbol{\theta}}_{\mathbf{0}}) = \sum_{i=1}^{m}\sum_{j=1}^{n_{Ai}} X_{Ai}(t_{Aij}) - \tilde{\alpha}_A \sum_{i=1}^{m}\int_0^{\tau_i} X_{Ai}(s)\, e^{\tilde{\gamma}_A s}\, ds, \tag{5.30}
$$

and variance estimate $\widehat{Var}(U_\beta(\tilde{\boldsymbol{\theta}}_0))$ is given by

$$
I_{\beta_A\beta_A}(\tilde{\boldsymbol{\theta}}_0) - \begin{pmatrix} I_{\beta_A\alpha_A}(\tilde{\boldsymbol{\theta}}_0) & I_{\beta_A\gamma_A}(\tilde{\boldsymbol{\theta}}_0) \end{pmatrix} \begin{pmatrix} I_{\alpha_A\alpha_A}(\tilde{\boldsymbol{\theta}}_0) & I_{\alpha_A\gamma_A}(\tilde{\boldsymbol{\theta}}_0) \\ I_{\gamma_A\alpha_A}(\tilde{\boldsymbol{\theta}}_0) & I_{\gamma_A\gamma_A}(\tilde{\boldsymbol{\theta}}_0) \end{pmatrix}^{-1} \begin{pmatrix} I_{\alpha_A\beta_A}(\tilde{\boldsymbol{\theta}}_0) \\ I_{\gamma_A\beta_A}(\tilde{\boldsymbol{\theta}}_0) \end{pmatrix}. \tag{5.31}
$$

5.2.3 Case 3: $m = 1$, $\Delta_A > 0$, $\Delta_B > 0$

In this section, we consider a single system with two components, in which repair times of Component A are not ignorable, but failures of Component A do not affect the probabilistic characteristics of the failure occurrences of Component B. In this case, the intensity functions of Components A, and B are given by

$$\text{Component A:} \qquad \lambda_A(t|\mathcal{H}(t)) = Y_A(t)\,\alpha_A\,\exp(\beta_A\,X_A(t) + \gamma_A t), \quad t > 0, \tag{5.32}$$

where $Y_A(t)$ is the at-risk indicator of Component A, $\alpha_A > 0$ is a baseline rate function, β_A and γ_A are parameters and

$$\text{Component B:} \qquad \lambda_B(t|\mathcal{H}(t)) = Y_B(t)\,\alpha_B\,\exp(\gamma_A t), \quad t > 0, \tag{5.33}$$

where $Y_B(t)$ is the at-risk indicator, $\alpha_B > 0$ is a baseline rate function and γ_A is a parameter and $\mathcal{H}(t) = \{N_A(u), N_B(u), Y_B(s); 0 \le u < t, 0 \le s \le t\}$. Then the likelihood function of the outcome "$N_A(\tau) = n_A$ failures of Component A at times $t_{A1} \le t_{A2} \le \cdots \le t_{An_A}$ in $[0,\tau]$" is given as follows.

$$L(\boldsymbol{\theta}) = \prod_{j=1}^{n_A} \alpha_A\, e^{\beta_A\,X_A(t_{Aj}) + \gamma_A t_{Aj}}\ \exp\{-\int_0^\tau Y_A(s)\,\alpha_A\,e^{\beta_A\,X_A(s)+\gamma_A s}\,ds\}, \tag{5.34}$$

where $\boldsymbol{\theta} = (\alpha_A, \beta_A, \gamma)$. The log likelihood function $\ell(\boldsymbol{\theta}) = \log L(\boldsymbol{\theta})$ is given as follows.

$$\ell(\boldsymbol{\theta}) = n_A \log\alpha_A + \sum_{j=1}^{n_A} \beta_A X_A(t_{Aj}) + \sum_{j=1}^{n_A} \gamma_A t_{Aj} - \int_0^\tau Y_A(s)\,\alpha_A\,e^{\beta_A X_A(s)+\gamma_A s}\,ds. \tag{5.35}$$

Let $\tilde{\alpha}_A$ and $\tilde{\gamma}_A$ denote the restricted maximum likelihood estimator of α_A and γ_A when $\beta_A = 0$. $\tilde{\alpha}_A$ and $\tilde{\gamma}_A$ can be obtained by maximizing $l(\boldsymbol{\theta}_0)$ where $\boldsymbol{\theta}_0 = (\alpha_A, \gamma_A, 0)$. The score vector is then defined by $\boldsymbol{U}(\boldsymbol{\theta}) = (U_{\alpha_A}(\boldsymbol{\theta}), U_{\gamma_A}(\boldsymbol{\theta}), U_{\beta_A}(\boldsymbol{\theta}))'$ with components

$$U_{\alpha_A}(\boldsymbol{\theta}) = \frac{n_A}{\alpha_A} - \int_0^\tau Y_A(s)\,e^{\beta_A X_A(s)+\gamma_A s}\,ds, \tag{5.36}$$

$$U_{\gamma_A}(\boldsymbol{\theta}) = \sum_{j=1}^{n_A} t_{Aj} - \alpha_A \int_0^\tau Y_A(s)\,s\,e^{\beta_A X_A(s)+\gamma_A s}\,ds, \tag{5.37}$$

and

$$U_{\beta_A}(\boldsymbol{\theta}) = \sum_{j=1}^{n_A} X_A(t_{Aj}) - \alpha_A \int_0^\tau Y_A(s)\,X_A(s)\,e^{\beta_A X_A(s)+\gamma_A s}\,ds. \tag{5.38}$$

The observed information matrix $\boldsymbol{I}(\boldsymbol{\theta})$ is given by

$$\boldsymbol{I}(\boldsymbol{\theta}) = \begin{pmatrix} I_{\alpha_A\alpha_A}(\boldsymbol{\theta}) & I_{\alpha_A\gamma_A}(\boldsymbol{\theta}) & I_{\alpha_A\beta_A}(\boldsymbol{\theta}) \\ I_{\gamma_A\alpha_A}(\boldsymbol{\theta}) & I_{\gamma_A\gamma_A}(\boldsymbol{\theta}) & I_{\gamma_A\beta_A}(\boldsymbol{\theta}) \\ I_{\beta_A\alpha_A}(\boldsymbol{\theta}) & I_{\beta_A\gamma_A}(\boldsymbol{\theta}) & I_{\beta_A\beta_A}(\boldsymbol{\theta}) \end{pmatrix}, \tag{5.39}$$

where

$$\begin{aligned}
I_{\alpha_A\alpha_A}(\boldsymbol{\theta}) &= \frac{n_A}{\alpha_A^2}, \\
I_{\alpha_A\gamma_A}(\boldsymbol{\theta}) &= I_{\gamma_A\alpha_A}(\boldsymbol{\theta}) = \int_0^\tau Y_A(s)\, s\, e^{\beta_A X_A(s)+\gamma_A s}\, ds, \\
I_{\alpha_A\beta_A}(\boldsymbol{\theta}) &= I_{\beta_A\alpha_A}(\boldsymbol{\theta}) = \int_0^\tau Y_A(s)\, X_A(s)\, e^{\beta_A X_A(s)+\gamma_A s}\, ds, \\
I_{\gamma_A\gamma_A}(\boldsymbol{\theta}) &= \alpha_A \int_0^\tau Y_A(s)\, s^2\, e^{\beta_A X_A(s)+\gamma_A s}\, ds, \\
I_{\gamma_A\beta_A}(\boldsymbol{\theta}) &= I_{\beta_A\gamma_A}(\boldsymbol{\theta}) = \alpha_A \int_0^\tau Y_A(s)\, s\, X_A(s)\, e^{\beta_A X_A(s)+\gamma_A s}\, ds, \\
I_{\beta_A\beta_A}(\boldsymbol{\theta}) &= \alpha_A \int_0^\tau Y_A(s)\, \{X_A(s)\}^2\, e^{\beta_A X_A(s)+\gamma_A s}\, ds.
\end{aligned}$$

Let $\tilde{\boldsymbol{\theta}}_0 = (\tilde{\alpha}_A, \tilde{\gamma}_A, 0)$. The standardized partial score statistic for testing the presence of parallel carryover effects is given by

$$Z = \frac{U_{\beta_A}(\tilde{\boldsymbol{\theta}}_0)}{\widehat{Var}(U_\beta(\tilde{\boldsymbol{\theta}}_0)^{\frac{1}{2}}}, \tag{5.40}$$

where $U_{\beta_A}(\tilde{\boldsymbol{\theta}}_0)$ is given by

$$U_{\beta_A}(\tilde{\alpha}_A, \tilde{\gamma}_A, 0) = \sum_{j=1}^{n_A} X_A(t_{Aj}) - \tilde{\alpha}_A \int_0^\tau Y_A(s)\, X_A(s)\, e^{\tilde{\gamma}_A s}\, ds, \tag{5.41}$$

and variance estimate $\widehat{Var}(U_\beta(\tilde{\boldsymbol{\theta}}_0))$ is

$$I_{\beta_A\beta_A}(\tilde{\boldsymbol{\theta}}_0) - \begin{pmatrix} I_{\beta_A\alpha_A}(\tilde{\boldsymbol{\theta}}_0) & I_{\beta_A\gamma_A}(\tilde{\boldsymbol{\theta}}_0) \end{pmatrix} \begin{pmatrix} I_{\alpha_A\alpha_A}(\tilde{\boldsymbol{\theta}}_0) & I_{\alpha_A\gamma_A}(\tilde{\boldsymbol{\theta}}_0) \\ I_{\gamma_A\alpha_A}(\tilde{\boldsymbol{\theta}}_0) & I_{\gamma_A\gamma_A}(\tilde{\boldsymbol{\theta}}_0) \end{pmatrix}^{-1} \begin{pmatrix} I_{\alpha_A\beta_A}(\tilde{\boldsymbol{\theta}}_0) \\ I_{\gamma_A\beta_A}(\tilde{\boldsymbol{\theta}}_0) \end{pmatrix}. \tag{5.42}$$

5.2.4 Case 4: $m > 1$, $\Delta_A > 0$, $\Delta_B > 0$

We now consider multiple systems with two components; Components A and B. Repair times of Component A are not negligible. We assume that the failures of redundant component (Component A) in the ith system, where $i = 1, 2, \ldots, m$. Failures of Component A do not affect the probabilistic characteristics of failure occurrences of Component B while failures of Component B affect the probabilistic characteristics of failure occurrences of Component A.

In this case, the intensity functions of Components A and B in the ith system can be defined by

$$\text{Component A:} \qquad \lambda_{Ai}(t|\mathcal{H}_i(t)) = Y_{Ai}(t)\, \alpha_A\, \exp\{\beta_A\, X_{Ai}(t)\gamma_A t\}, \quad t > 0, \tag{5.43}$$

and

$$\text{Component B:} \qquad \lambda_{Bi}(t|\mathcal{H}_i(t)) = Y_{Bi}(t)\, \alpha_B\, \exp\{\gamma_A t\}, \quad t > 0, \tag{5.44}$$

where $X_{Ai}(t) = I\{N_{Bi}(t^-) > 0\} I\{t - t_{Bi\, N_{Bi}(t^-)} \leq \Delta_B\}$, $Y_{Ai}(t)$ is the at-risk indicator of Component A in the ith system, β_A and γ_A are parameters, and $\mathcal{H}_i(t) = \{N_{Ai}(u), N_{Bi}(u), Y_{Ai}(s), Y_{Bi}(s); 0 \leq u < t, 0 \leq s \leq t\}$.

The likelihood function of the outcome "$N_{Ai}(\tau_i) = n_{Ai}$ failures of Component A at times $t_{Ai1} \leq t_{Ai2} \leq \cdots \leq t_{Ain_{Ai}}$ in $[0, \tau_i]$" can be written as

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{m} L_i(\boldsymbol{\theta}), \tag{5.45}$$

where

$$L_i(\boldsymbol{\theta}) = \prod_{j=1}^{n_{Ai}} \alpha_A\, e^{\beta_A\, X_{Ai}(t_{Aij}) + \gamma_A t_{Aij}}\ \exp\{-\int_0^{\tau_i} Y_{Ai}(s)\, \alpha_A\, e^{\beta_A\, X_{Ai}(s) + \gamma_A s}\, ds\}, \tag{5.46}$$

where $\boldsymbol{\theta} = (\alpha_A, \beta_A, \gamma)$. The log likelihood function $\ell(\boldsymbol{\theta}) = \log L(\boldsymbol{\theta})$ is given by

$$\log \alpha_A \sum_{i=1}^{m} n_{Ai} + \beta_A \sum_{i=1}^{m} \sum_{j=1}^{n_{Ai}} X_{Ai}(t_{Aij}) \\ + \gamma_A \sum_{i=1}^{m} \sum_{j=1}^{n_{Ai}} t_{Aij} - \alpha_A \sum_{i=1}^{m} \int_0^{\tau_i} Y_{Ai}(s)\, e^{\beta_A\, X_{Ai}(s) + \gamma_A s}\, ds. \tag{5.47}$$

Then, the components of score vector $\boldsymbol{U}(\boldsymbol{\theta}) = (U_{\alpha_A}(\boldsymbol{\theta}), U_{\gamma_A}(\boldsymbol{\theta}), U_{\beta_A}(\boldsymbol{\theta}))'$ are followed by

$$U_{\alpha_A}(\boldsymbol{\theta}) = \frac{\sum_{i=1}^{m} n_{Ai}}{\alpha_A} - \sum_{i=1}^{m} \int_0^{\tau_i} Y_{Ai}(s)\, e^{\beta_A X_{Ai}(s) + \gamma_A s}\, ds, \tag{5.48}$$

$$U_{\gamma_A}(\boldsymbol{\theta}) = \sum_{i=1}^{m} \sum_{j=1}^{n_{Ai}} t_{Aij} - \alpha_A \sum_{i=1}^{m} \int_0^{\tau_i} Y_{Ai}(s) s\, e^{\beta_A X_{Ai}(s) + \gamma_A s}\, ds, \tag{5.49}$$

and

$$U_{\beta_A}(\boldsymbol{\theta}) = \sum_{i=1}^{m} \sum_{j=1}^{n_{Ai}} X_{Ai}(t_{Aij}) - \alpha_A \sum_{i=1}^{m} \int_0^{\tau_i} Y_{Ai}(s)\, X_{Ai}(s)\, e^{\beta_A X_{Ai}(s) + \gamma_A s}\, ds. \tag{5.50}$$

The elements of observed information matrix $\boldsymbol{I}(\boldsymbol{\theta})$ are given by

$$\begin{aligned}
I_{\alpha_A\alpha_A}(\boldsymbol{\theta}) &= \frac{\sum_{i=1}^{m} n_{Ai}}{\alpha_A^2}, \\
I_{\alpha_A\gamma_A}(\boldsymbol{\theta}) = I_{\gamma_A\alpha_A}(\boldsymbol{\theta}) &= \sum_{i=1}^{m} \int_0^{\tau_i} Y_{Ai}(s)\, s\, e^{\beta_A X_{Ai}(s) + \gamma_A s}\, ds, \\
I_{\alpha_A\beta_A}(\boldsymbol{\theta}) = I_{\beta_A\alpha_A}(\boldsymbol{\theta}) &= \sum_{i=1}^{m} \int_0^{\tau_i} X_{Ai}(s)\, Y_{Ai}(s)\, e^{\beta_A X_{Ai}(s) + \gamma_A s}\, ds, \\
I_{\gamma_A\gamma_A}(\boldsymbol{\theta}) &= \alpha_A \sum_{i=1}^{m} \int_0^{\tau_i} Y_{Ai}(s)\, s^2\, e^{\beta_A X_{Ai}(s) + \gamma_A s}\, ds, \\
I_{\gamma_A\beta_A}(\boldsymbol{\theta}) = I_{\beta_A\gamma_A}(\boldsymbol{\theta}) &= \alpha_A \sum_{i=1}^{m} \int_0^{\tau_i} Y_{Ai}(s)\, s\, X_{Ai}(s)\, e^{\beta_A X_{Ai}(s) + \gamma_A s}\, ds, \\
I_{\beta_A\beta_A}(\boldsymbol{\theta}) &= \alpha_A \sum_{i=1}^{m} \int_0^{\tau_i} Y_{Ai}(s)\, X_{Ai}(s)\, e^{\beta_A X_{Ai}(s) + \gamma_A s}\, ds.
\end{aligned}$$

Let $\tilde{\boldsymbol{\theta}}_0 = (\tilde{\alpha}_A, \tilde{\gamma}_A, 0)$. $\tilde{\alpha}_A$ are $\tilde{\gamma}_A$ restricted maximum likelihood estimators of α_A and γ_A, respectively, when $\beta_A = 0$. Good optimizing software packages such as `nlm` in `R`, give those restricted maximum likelihood estimators without analytical derivations.

Then the standardized partial score statistic for testing parallel carryover effects is given by

$$Z = \frac{U_{\beta_A}(\tilde{\boldsymbol{\theta}}_0)}{\widehat{Var}(U_\beta(\tilde{\boldsymbol{\theta}}_0))^{\frac{1}{2}}}, \tag{5.51}$$

where the partial score function is given by

$$U_{\beta_A}(\tilde{\boldsymbol{\theta}}_0) = \sum_{i=1}^{m} \sum_{j=1}^{n_{Ai}} X_{Ai}(t_{Aij}) - \tilde{\alpha}_A \sum_{i=1}^{m} \int_0^{\tau_i} Y_{Ai}(s)\, X_{Ai}(s)\, e^{\tilde{\gamma}_A s}\, ds, \tag{5.52}$$

and variance estimate $\widehat{Var}(U_\beta(\tilde{\boldsymbol{\theta}}_0))$ is

$$I_{\beta_A\beta_A}(\tilde{\boldsymbol{\theta}}_0) - \begin{pmatrix} I_{\beta_A\alpha_A}(\tilde{\boldsymbol{\theta}}_0) & I_{\beta_A\gamma_A}(\tilde{\boldsymbol{\theta}}_0) \end{pmatrix} \begin{pmatrix} I_{\alpha_A\alpha_A}(\tilde{\boldsymbol{\theta}}_0) & I_{\alpha_A\gamma_A}(\tilde{\boldsymbol{\theta}}_0) \\ I_{\gamma_A\alpha_A}(\tilde{\boldsymbol{\theta}}_0) & I_{\gamma_A\gamma_A}(\tilde{\boldsymbol{\theta}}_0) \end{pmatrix}^{-1} \begin{pmatrix} I_{\alpha_A\beta_A}(\tilde{\boldsymbol{\theta}}_0) \\ I_{\gamma_A\beta_A}(\tilde{\boldsymbol{\theta}}_0) \end{pmatrix}. \tag{5.53}$$

5.2.5 Case 5: $m = 1$, $\Delta_A > 0$, $\Delta_B > 0$, $\Delta_C > 0$

In this section, we consider a single system with 3 components; Components A, B and C. We consider the case where repair times of Components A, B and C are not negligible. That is, $m = 1$, $\Delta_A > 0$, $\Delta_B > 0$ and $\Delta_C > 0$. However, one of the components in the system, that is Component A, is a redundant component. Therefore, failures of Component A do not affect the probabilistic characteristics of failure occurrences of Components B and C. Failures of Component B affect the probabilistic characteristics of failure occurrences of Component C. Similarly, failures of Component C affect the probabilistic characteristics of failure occurrences of Component B.

We now give extensions of notations given in previous sections. Suppose that there is a multivariate counting processes $\{N_A(t), N_B(t), N_C(t); t \geq 0\}$, where $\{N_A(t); t \geq 0\}$ is a counting process for Component A, $\{N_B(t); t \geq 0\}$ is a counting process for Component B and $\{N_C(t); t \geq 0\}$ is a counting process for Component C in a system with three components working in parallel. We let $t_{A1}, t_{A2}, \ldots,$ where $0 < t_{A1} < t_{A2} < \ldots,$ $t_{B1}, t_{B2}, \ldots,$ where $0 < t_{B1} < t_{B2} < \ldots,$ and $t_{C1}, t_{C2}, \ldots,$ where $0 < t_{C1} < t_{C2} < \ldots,$ denote the failure times of Components A, B and C, respectively. The components are subject to repairs and repair times cannot be ignored. Let Δ_A, Δ_B and Δ_C denote the repair times of Components A, B and C, respectively. For $K = A, B$ and C, $Y_K(t)$ is the at-risk indicator of process $\{N_K(t); t > 0\}$.

A model including parallel carryover effects for Component A is given by

$$\lambda_A(t|\mathcal{H}(t)) = Y_A(t)\, \alpha_A \, \exp\{\beta_A X_{AB}(t) + \beta_A X_{AC}(t) + \gamma_A t\}, \quad t > 0, \tag{5.54}$$

where $\alpha_A > 0$ is a baseline rate function, β_A and γ_A are parameters, $X_{AB}(t) =$

$I\{N_B(t^-) > 0\}I\{t - t_{B\,N_B(t^-)} \le \Delta_B\}$, and $X_{AC}(t) = I\{N_C(t^-) > 0\}I\{t - t_{C\,N_C(t^-)} \le \Delta_C\}$.

When Component A is in the up state at time t, (i) the intensity function (5.54) becomes $\alpha_A \exp\{2\beta_A + \gamma_A t\}$ when both Component B and Component C fails, (ii) $\alpha_A \exp\{\beta_A + \gamma_A t\}$ when one of Components B or C fails, (iii) $\alpha_A \exp\{\gamma_A t\}$ if none of Components B and C fails. This is a similar situation that we explained in Section 3.2.

A model for parallel carryover effects in this case can be also defined for Component B. In this case, the intensity function of Component B is given by

$$\lambda_B(t|\mathcal{H}(t)) = Y_B(t)\,\alpha_B\,\exp\{\beta_B\,X_{BC}(t) + \gamma_B t\}, \quad t > 0, \tag{5.55}$$

where $\alpha_B > 0$ is a baseline rate function, γ_B is a parameter, and $X_{BC}(t) = I\{N_C(t^-) > 0\}I\{t - t_{C\,N_C(t^-)} \le \Delta_C\}$.

Similarly, the intensity function of Component C is given by

$$\lambda_C(t|\mathcal{H}(t)) = Y_C(t)\,\alpha_C\,\exp(\beta_C\,X_{CB}(t) + \gamma_C t), \quad t > 0, \tag{5.56}$$

where $\alpha_C > 0$ is a baseline rate function, γ_C is a parameter, $X_{CB}(t) = I\{N_B(t^-) > 0\}I\{t - t_{B\,N_B(t^-)} \le \Delta_B\}$.

A test for the presence of a parallel carryover effect in Component A can be developed by considering the following composite hypothesis given in (5.8).

We suppose that a single system is under observation over the followup period $[0, \tau]$, where τ is fixed end-of-followup time. Let n_A, where $n_A \ge 0$, denote the number of failures of Component A over $[0, \tau]$ and $t_{A1}, t_{A1}, \ldots, t_{An_A}$ be the failure times of Component A.

Let $\boldsymbol{\theta} = (\alpha_A, \gamma_A, \beta_A)$. The likelihood function of the outcome "$N_A(\tau) = n_A$ failures of Component A at times $t_{A1}, t_{A2}, \ldots, t_{An_A}$ in $[0, \tau]$" is given by

$$L(\boldsymbol{\theta}) = \prod_{j=1}^{n_A} \alpha_A\, e^{\beta_{AB}\,(X_{AB}(t_{Aj}) + X_{AC}(t_{Aj})) + \gamma_A t_{Aj}}\ \exp\{-\int_0^{\tau} Y_A(s)\,\alpha_A\, e^{\beta_A\,(X_{AB}(s) + X_{AC}(s)) + \gamma_A s}\, ds\}, \tag{5.57}$$

The log likelihood function $\ell(\boldsymbol{\theta}) = \log L(\boldsymbol{\theta})$ is given by

$$\ell(\boldsymbol{\theta}) = n_A \log \alpha_A + \beta_A \sum_{j=1}^{n_A} (X_{AB}(t_{Aj}) + X_{AC}(t_{Aj})) + \sum_{j=1}^{n_A} \gamma_A t_{Aj} \tag{5.58}$$

$$- \alpha_A \int_0^{\tau} Y_A(s)\, e^{\beta_A(X_{AB}(s)+X_{AC}(s))+\gamma_A s}\, ds. \tag{5.59}$$

Let $\tilde{\boldsymbol{\theta}}_0 = (\tilde{\alpha}_A, \tilde{\gamma}_A, 0)$ where $\tilde{\alpha}_A$ and $\tilde{\gamma}_A$ are restricted maximum likelihood estimators of α_A and γ_A, respectively, when $\beta_A = 0$.

Then, we obtain the following standardized partial score test statistic for testing parallel carryover effects

$$Z = \frac{U_{\beta_A}(\tilde{\boldsymbol{\theta}}_0)}{\widehat{Var}(U_{\beta}(\tilde{\boldsymbol{\theta}}_0))^{\frac{1}{2}}}, \tag{5.60}$$

where the partial score function $U_{\beta_A}(\tilde{\boldsymbol{\theta}}_0)$ is given by

$$U_{\beta_A}(\tilde{\boldsymbol{\theta}}_0) = \sum_{j=1}^{n_A} (X_{AB}(t_{Aj}) + X_{AC}(t_{Aj})) - \tilde{\alpha}_A \int_0^{\tau} Y_A(s)\,(X_{AB}(s) + X_{AC}(s))\, e^{\tilde{\gamma}_A s}\, ds. \tag{5.61}$$

Variance estimate $\widehat{Var}(U_{\beta}(\tilde{\boldsymbol{\theta}}_0))$ is followed by

$$I_{\beta_A\beta_A}(\tilde{\boldsymbol{\theta}}_0) - \begin{pmatrix} I_{\beta_A\alpha_A}(\tilde{\boldsymbol{\theta}}_0) & I_{\beta_A\gamma_A}(\tilde{\boldsymbol{\theta}}_0) \end{pmatrix} \begin{pmatrix} I_{\alpha_A\alpha_A}(\tilde{\boldsymbol{\theta}}_0) & I_{\alpha_A\gamma_A}(\tilde{\boldsymbol{\theta}}_0) \\ I_{\gamma_A\alpha_A}(\tilde{\boldsymbol{\theta}}_0) & I_{\gamma_A\gamma_A}(\tilde{\boldsymbol{\theta}}_0) \end{pmatrix}^{-1} \begin{pmatrix} I_{\alpha_A\beta_A}(\tilde{\boldsymbol{\theta}}_0) \\ I_{\gamma_A\beta_A}(\tilde{\boldsymbol{\theta}}_0) \end{pmatrix}, \tag{5.62}$$

where

$$\begin{aligned}
I_{\alpha_A\alpha_A}(\tilde{\boldsymbol{\theta}}_0) &= \frac{n_A}{\tilde{\alpha}_A^2}, \\
I_{\alpha_A\gamma_A}(\tilde{\boldsymbol{\theta}}_0) &= I_{\gamma_A\alpha_A}(\tilde{\boldsymbol{\theta}}_0) = \int_0^\tau Y_A(s)\, s\, e^{\tilde{\gamma}_A s}\, ds, \\
I_{\alpha_A\beta_A}(\tilde{\boldsymbol{\theta}}_0) &= I_{\beta_A\alpha_A}(\tilde{\boldsymbol{\theta}}_0) = \int_0^\tau Y_A(s)\, (X_{AB}(s) + X_{AC}(s))\, e^{\tilde{\gamma}_A s}\, ds, \\
I_{\gamma_A\gamma_A}(\tilde{\boldsymbol{\theta}}_0) &= \tilde{\alpha}_A \int_0^\tau Y_A(s)\, s^2\, e^{\tilde{\gamma}_A s}\, ds, \\
I_{\gamma_A\beta_A}(\tilde{\boldsymbol{\theta}}_0) &= I_{\beta_A\gamma_A}(\tilde{\boldsymbol{\theta}}_0) = \tilde{\alpha}_A \int_0^\tau Y_A(s)\, s\, (X_{AB}(s) + X_{AC}(s))\, e^{\tilde{\gamma}_A s}\, ds, \\
I_{\beta_A\beta_A}(\tilde{\boldsymbol{\theta}}_0) &= \tilde{\alpha}_A \int_0^\tau Y_A(s)\, (X_{AB}(s) + X_{AC}(s))^2\, e^{\tilde{\gamma}_A s}\, ds.
\end{aligned}$$

5.3 Simulation Studies

In this section, we present the results of simulation studies conducted to assess when asymptotic normal approximation for test statistics developed in Section 5.2.3, Section 5.2.4 and Section 5.2.5 are satisfactory where

(*i*) $m = 1$, $\Delta_A > 0$, $\Delta_B > 0$
(*ii*) $m > 1$, $\Delta_A > 0$, $\Delta_B > 0$,
(*iii*) $m = 1$, $\Delta_A > 0$, $\Delta_B > 0$, $\Delta_C > 0$.

We consider two settings where $m = 1$ and $\tau \to \infty$, and $m \to \infty$ and τ is fixed in each of the 3 cases given above.

We first consider testing for presence of parallel carryover effects in a single system with monotonic trend. In this case, $m = 1$, $\Delta_A > 0$ and $\Delta_B > 0$ so we consider the models (5.32) and (5.33). The hypothesis of no parallel carryover effects in Component A is $H_0 : \beta_A = 0, \alpha_A > 0, \gamma_A \in \mathbb{R}$ and this is conducted by using the statistic Z in (5.40). We generated 10,000 realizations of NHPPs under the null hypothesis with fixed values of the parameters for all scenarios where $\alpha_A = \alpha_B = 0.1$, $\beta_A = 0$, $\beta_B = 0.693$, and $\gamma_A = \gamma_B = 0.001$. We considered $\Delta_A = 1$ and $\Delta_B = 1, 3, 7,$ and 14. We fixed Δ_A at 1 . Normal quantile-quantile (Q-Q) plots of 10,000 values of Z are given in Figures 5.1 when $\Delta_A = 1$ for $\tau = 100, 200, 500$ and 1000. The standard normal approximations are not accurate in those cases when $\tau = 100, 200$

and 500. However, it is noted that as τ increases, the standard normal approximation improves. Figure 5.2 shows the results when $\Delta = 3$. In this scenario, the standard normal approximations are accurate when $\tau = 500$ and 1000. Figures 5.3 and 5.4 present the results when $\Delta = 7$ and $\Delta = 14$, respectively. In those scenarios, the standard normal approximations are accurate, as τ increases.

Table 5.1 presents estimates of Q_p and $Pr(Z > Q_p)$ where $p = 0.950, 0.975$ and 0.990. This also indicates that the standard normal approximation is not adequate for small and moderate τ values and small Δ_B values, but the approximation becomes accurate as τ increses.

The power of the statistic (5.40) against the alternative hypothesis $H_A : \beta_A \neq 0$ is investigated by Monte Carlo simulation methods. We use the 0.95 quantile of the standard normal distribution and the empirical 0.95 quantile of the test statistic obtained from 10,000 simulations runs conducted under the null hypothesis with different τ, Δ_B and Δ_C values as discussed above. We generated 1,000 processes under the alternative model where we took $\alpha_A = \alpha_B = 0.1$ and $\beta_A = \beta_B = 0.693$. The power results are presented in Table 4.2 where entries are the proportions of the values of Z in 1,000 samples which are larger than the quantile values. Table 5.2 shows that the power of the test is high overall, and power increases as τ increases.

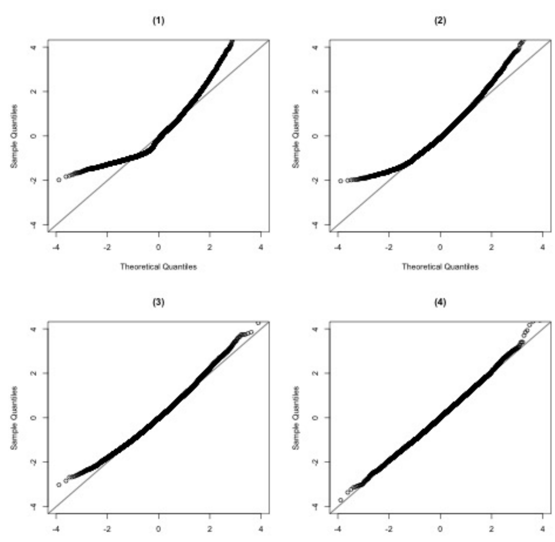

Figure 5.1: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_B = 1$, $\Delta_A = 1$, and (1) $\tau = 100$, (2) $\tau = 200$, (3) $\tau = 500$, (4) $\tau = 1000$

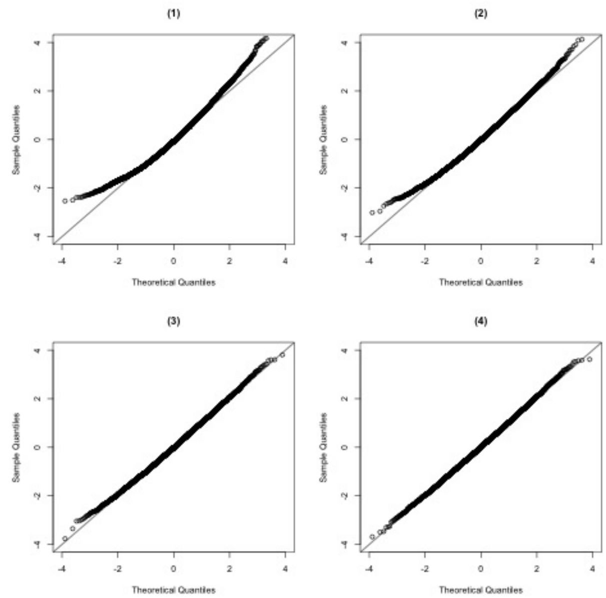

Figure 5.2: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_B = 3$, $\Delta_A = 1$, and (1) $\tau = 100$, (2) $\tau = 200$, (3) $\tau = 500$, (4) $\tau = 1000$

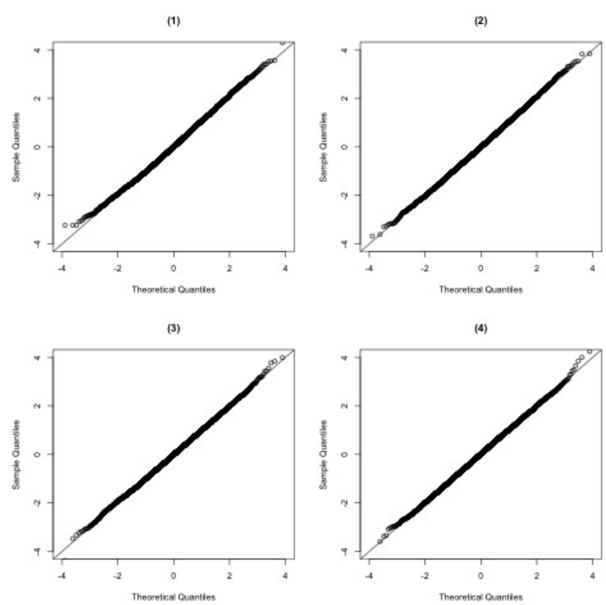

Figure 5.3: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_B = 7$, $\Delta_A = 1$, and (1) $\tau = 100$, (2) $\tau = 200$, (3) $\tau = 500$, (4) $\tau = 1000$

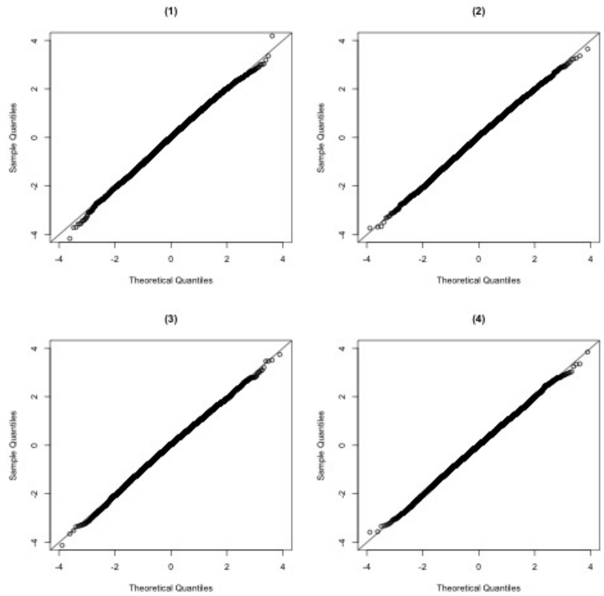

Figure 5.4: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_B = 14$, $\Delta_A = 1$, and (1) $\tau = 100$, (2) $\tau = 200$, (3) $\tau = 500$, (4) $\tau = 1000$

Δ_B	τ	$\hat{Q}_{0.950}$	$\hat{Q}_{0.975}$	$\hat{Q}_{0.990}$	$\hat{Pr}(Z > 1.645)$	$\hat{Pr}(Z > 1.960)$	$\hat{Pr}(Z > 2.326)$
1	100	1.901	2.517	3.191	0.0781	0.0527	0.0324
	200	1.874	2.314	2.845	0.0699	0.0447	0.0244
	500	1.780	2.172	2.614	0.0629	0.0365	0.019
	1000	1.693	2.011	2.459	0.0546	0.0278	0.014
3	100	1.870	2.267	2.729	0.0710	0.0436	0.0226
	200	1.736	2.091	2.502	0.0590	0.0342	0.0158
	500	1.685	2.030	2.375	0.0537	0.0275	0.0117
	1000	1.696	2.046	2.448	0.0557	0.0299	0.0133
7	100	1.759	2.058	2.476	0.0622	0.0321	0.0152
	200	1.692	2.029	2.434	0.0538	0.0299	0.0127
	500	1.639	1.947	2.32	0.0488	0.0244	0.0100
	1000	1.656	1.974	2.321	0.0513	0.0257	0.0099
14	100	1.687	1.988	2.318	0.0551	0.0268	0.0097
	200	1.669	1.944	2.278	0.0528	0.0239	0.0089
	500	1.619	1.890	2.281	0.0483	0.0205	0.0085
	1000	1.593	1.942	2.348	0.0460	0.0241	0.0103

Table 5.1: $\hat{Q}_p$ is the empirical pth quantile of Z computed from 10,000 samples when m=1. $\hat{Pr}(Z > Q_p)$ is the proportion of the values of Z in 10,000 samples which are larger than the pth quantile of a standard normal distribution

Δ_B	τ	$\hat{P}r\{Z > 1.645\}$ $e^{\beta_A} = 2$	$\hat{P}r\{Z > \hat{Q}_{0.950}\}$ $e^{\beta_A} = 2$	$\hat{P}r\{Z > 1.645\}$ $e^{\beta_A} = 2.5$	$\hat{P}r\{Z > \hat{Q}_{0.950}\}$ $e^{\beta_A} = 2.5$
1	100	0.284	0.219	0.390	0.309
	200	0.398	0.330	0.574	0.512
	500	0.724	0.691	0.927	0.910
	1000	0.989	0.988	1.000	1.000
3	100	0.319	0.258	0.506	0.432
	200	0.509	0.483	0.723	0.694
	500	0.876	0.871	0.988	0.986
	1000	0.998	0.998	1.000	1.000
7	100	0.369	0.338	0.516	0.476
	200	0.580	0.568	0.787	0.771
	500	0.918	0.918	0.996	0.996
	1000	0.999	0.999	1.000	1.000
14	100	0.310	0.294	0.461	0.443
	200	0.528	0.515	0.765	0.750
	500	0.874	0.880	0.991	0.993
	1000	0.994	0.994	1.000	1.000

Table 5.2: Power of Z : $m = 1$, $\Delta_A > 0$, $\Delta_B > 0$

We now consider testing for the presence of parallel carryover effects in multiple systems with trend. This is the case in which $m > 1$, $\Delta_A > 0$, $\Delta_B > 0$, and the models are given in (5.43) and (5.44). The hypothesis of no parallel carryover effects in Component A is conducted by employing the statistic Z in (5.51). We generated 10,000 realizations of m NHPPs under the null hypothesis with the parameter values $\alpha_A = \alpha_B = 0.1$, $\beta_A = 0$, $\beta_B = 0.693$, and $\gamma_A = \gamma_B = 0.001$. We considered $\Delta_B = 1, 3, 7$ and 14 and we fixed Δ_A at 1 and τ at 100. Normal quantile-quantile (Q-Q) plots of 10,000 values of Z are given in Figures 5.5, 5.6, 5.7, and 5.8 with various combinations of m and Δ_B. The standard normal approximation is very accurate in each setting with all m values. Table 5.3 presents estimates of Q_p and $Pr(Z > Q_p)$ when $p = 0.950, 0.975$ and 0.990. This also indicates that the standard normal approximation is adequate as τ increases.

The power of the statistic (5.51) against the alternative hypothesis $H_A : \beta_A \neq 0$ is investigated by Monte Carlo simulation methods. We use the 0.95 quantile of the standard normal distribution and the empirical 0.95 quantile of the test statistic obtained from 10,000 simulations runs calculated under the null hypothesis with different m and Δ_B values. We generated 1,000 processes under the alternative model where we took $\alpha_A = \alpha_B = 0.1$ and $\gamma_A = \gamma_B = 0.001$ when $e^{\beta_A} = e^{\beta_B} = 1.5$ or 2. The power results are presented in Table 5.4 where entries are the proportions of the values of Z in 1,000 samples which are larger than the quantile values. And Table 5.4 indicates that the power of the test is high as m increases.

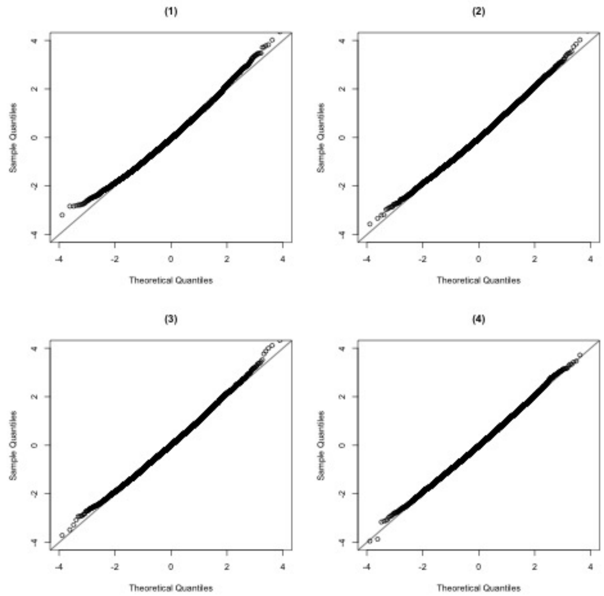

Figure 5.5: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\tau = 100$, $\Delta_B = 1$, $\Delta_A = 1$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$, (4) $m = 100$

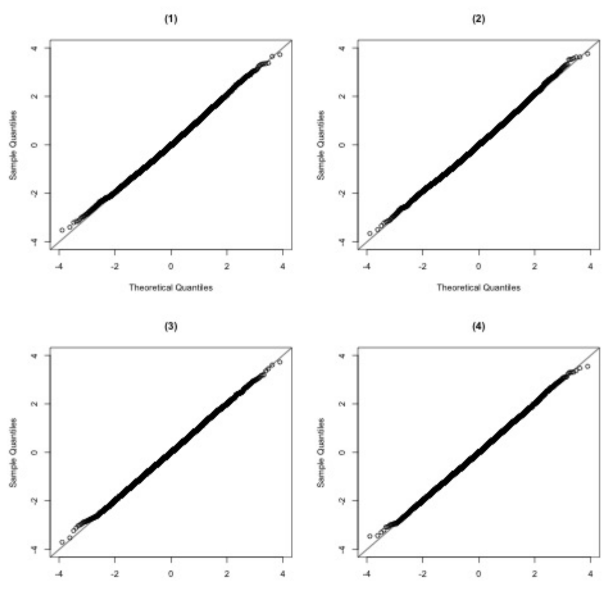

Figure 5.6: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\tau = 100$, $\Delta_B = 3$, $\Delta_A = 1$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$, (4) $m = 100$

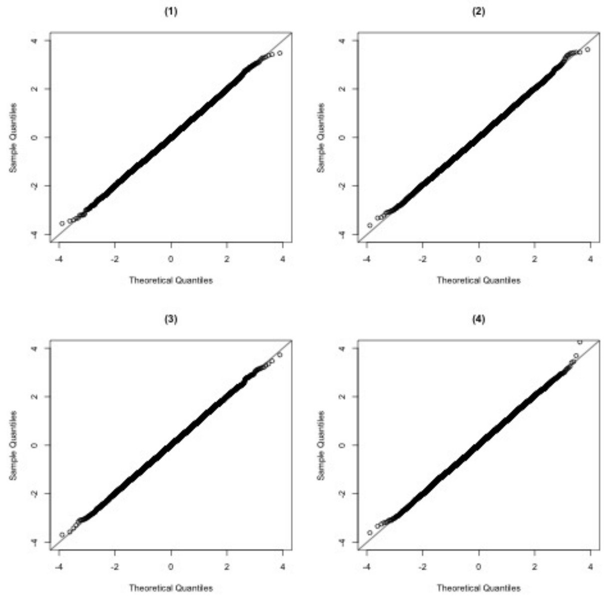

Figure 5.7: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\tau = 100$, $\Delta_B = 7$, $\Delta_A = 1$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$, (4) $m = 100$

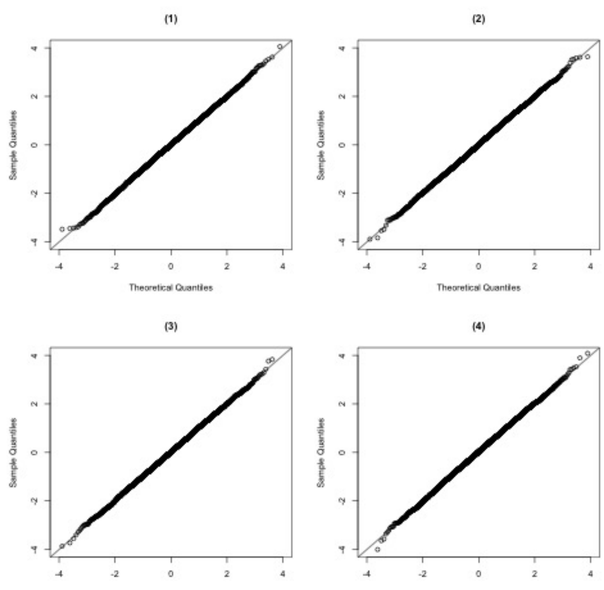

Figure 5.8: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\tau = 100$, $\Delta_B = 14$, $\Delta_A = 1$, and (1) $m = 10$, (2) $m = 20$, (3) $m = 50$, (4) $m = 100$

Δ_B	m	$\hat{Q}_{0.950}$	$\hat{Q}_{0.975}$	$\hat{Q}_{0.990}$	$\hat{Pr}(Z > 1.645)$	$\hat{Pr}(Z > 1.960)$	$\hat{Pr}(Z > 2.326)$
1	10	1.694	2.110	2.552	0.0543	0.0316	0.0161
	20	1.714	2.061	2.471	0.0562	0.0301	0.0136
	50	1.729	2.093	2.471	0.0581	0.0328	0.0134
	100	1.670	2.004	2.426	0.0531	0.0276	0.0121
3	10	1.707	2.026	2.457	0.0552	0.0286	0.0132
	20	1.699	2.056	2.443	0.0549	0.0305	0.0129
	50	1.684	1.962	2.368	0.0539	0.0256	0.0105
	100	1.655	1.977	2.381	0.0510	0.0263	0.0116
7	10	1.622	1.935	2.326	0.0477	0.0241	0.0101
	20	1.667	1.975	2.303	0.0522	0.0263	0.0095
	50	1.668	1.977	2.327	0.0532	0.0258	0.0101
	100	1.690	1.995	2.321	0.0546	0.0272	0.0099
14	10	1.664	1.943	2.316	0.0519	0.024	0.0098
	20	1.648	1.960	2.319	0.0504	0.0251	0.0099
	50	1.650	1.974	2.342	0.0513	0.0254	0.0108
	100	1.675	1.989	2.321	0.0535	0.0275	0.0099

Table 5.3: $\hat{Q}_p$ is the empirical pth quantile of Z computed from 10,000 samples when $m > 1$, $\Delta_A = 1$, and $\tau = 100$. $\hat{Pr}(Z > Q_p)$ is the proportion of the values of Z in 10,000 samples which are larger than the pth quantile of a standard normal distribution

Δ_B	m	$\hat{Pr}\{Z > 1.645\}$ $e^{\beta_A} = 1.5$	$\hat{Pr}\{Z > \hat{Q}_{0.950}\}$ $e^{\beta_A} = 1.5$	$\hat{Pr}\{Z > 1.645\}$ $e^{\beta_A} = 2$	$\hat{Pr}\{Z > \hat{Q}_{0.950}\}$ $e^{\beta_A} = 2$
1	10	0.463	0.448	0.834	0.822
	20	0.686	0.660	0.978	0.978
	50	0.947	0.934	1.000	1.000
	100	0.998	0.998	1.000	1.000
3	10	0.643	0.624	0.975	0.970
	20	0.868	0.854	0.999	0.999
	50	0.996	0.996	1.000	1.000
	100	1.000	1.000	1.000	1.000
7	10	0.682	0.693	0.987	0.987
	20	0.918	0.913	1.000	1.000
	50	1.000	1.000	1.000	1.000
	100	1.000	1.000	1.000	1.000
14	10	0.678	0.673	0.987	0.985
	20	0.927	0.927	1.000	1.000
	50	0.999	0.999	1.000	1.000
	100	1.000	1.000	0.999	0.999

Table 5.4: Power of Z : $m > 1$, $\Delta_A = 1$, $\Delta_B > 0$

We consider the score statistic (5.60) given in Section 5.2.5 where $m = 1$, $\Delta_A > 0$, $\Delta_B > 0$, $\Delta_C > 0$. We use the models (5.54), (5.55), and (5.56) for Components A, B and C, respectively. We generated 10,000 realizations of NHPPs with $\alpha_A = \alpha_B = \alpha_C = 0.1$, $\beta_A = 0$, $\beta_B = \beta_C = 0.693$, $\gamma_A = \gamma_B = \gamma_C = 0.001$. We considered Δ_B, $\Delta_C = 1, 3, 7$ and 14 when $\tau = 100, 200, 500$ and 1000.

For Δ_B, $\Delta_C = 1$, from the Q-Q plots in Figure 5.9, the normal approximations are quite accurate when $\tau = 500$ and 1000. For Δ_B, $\Delta_C = 1$ from the Q-Q plots in Figure 5.9, the normal approximation is adequate when $\tau = 1000$. For Δ_B, $\Delta_C = 3$ from the Q-Q plots in Figure 5.10, the normal approximations are adequate when $\tau =$ 500 and 1000. For Δ_B, $\Delta_C = 7$ from the Q-Q plots in Figure 5.11, the approximations are accurate when $\tau = 200, 500$ and 1000. For Δ_B, $\Delta_C = 14$ from the Q-Q plots in Figure 5.12, the approximations are quite accurate at $\tau = 500$ and 1000.

Table 5.5 shows estimated Q_p and $\hat{Pr}(Z > Q_p)$ values where $p = 0.950, 0.975$ and 0.990. Table 5.5 indicates also that the standard normal approximations are adequate for large τ when Δ_B $\Delta_C = 1, 3$ and 7. However, the normal approximation is less accurate when Δ_B and Δ_C are 14 time units.

We next consider the power of the tests with size 0.05. So, we used 0.95 quantile of the standard normal distribution and 0.95 empirical quantile of the test statistic obtained from 10,000 simulations runs with various combinations of τ, Δ_B and Δ_C. We generated 1,000 processes where we took $\alpha_A = \alpha_B = \alpha_C = 0.1$, $\beta_A = \beta_B = \beta_C = 0.693$, and $\gamma_A = \gamma_B = \gamma_C = 0.001$. The results of the power of the test are presented in Table 5.6 where entries are the proportions of the values of Z in 1,000 samples which are larger than the quantile values. It shows that the power of the test is high overall, and power increases as τ increases.

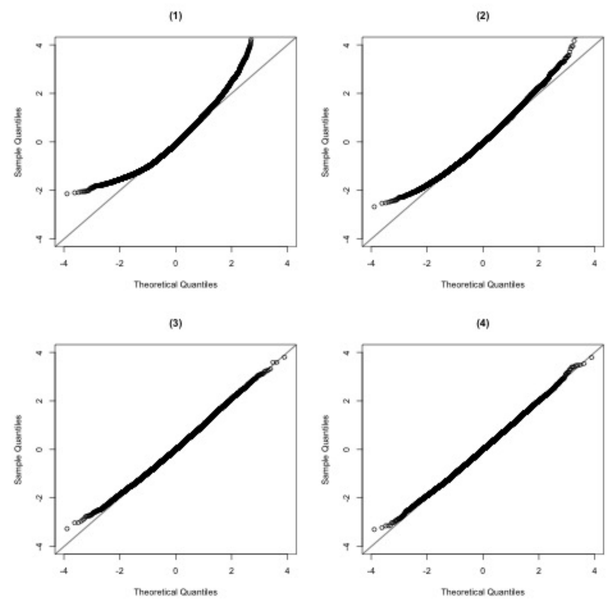

Figure 5.9: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_B = \Delta_C = 1$, $\Delta_A = 1$, and (1) $\tau = 100$, (2) $\tau = 200$, (3) $\tau = 500$, (4) $\tau = 1000$

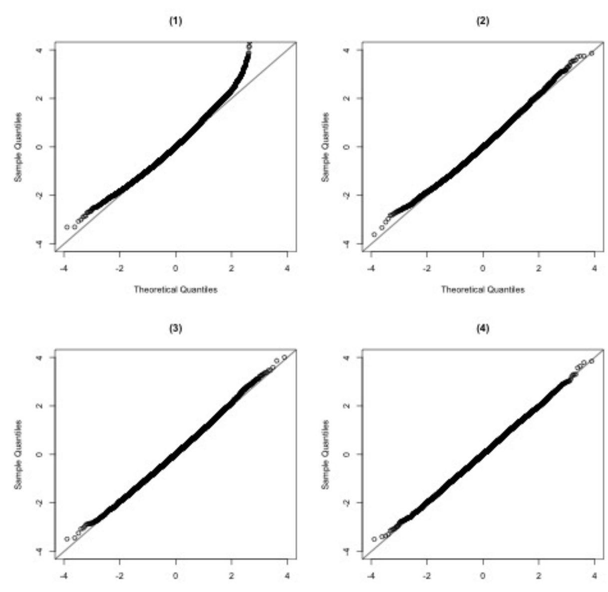

Figure 5.10: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_B = \Delta_C = 3$, $\Delta_A = 1$, and (1) $\tau = 100$, (2) $\tau = 200$, (3) $\tau = 500$, (4) $\tau = 1000$

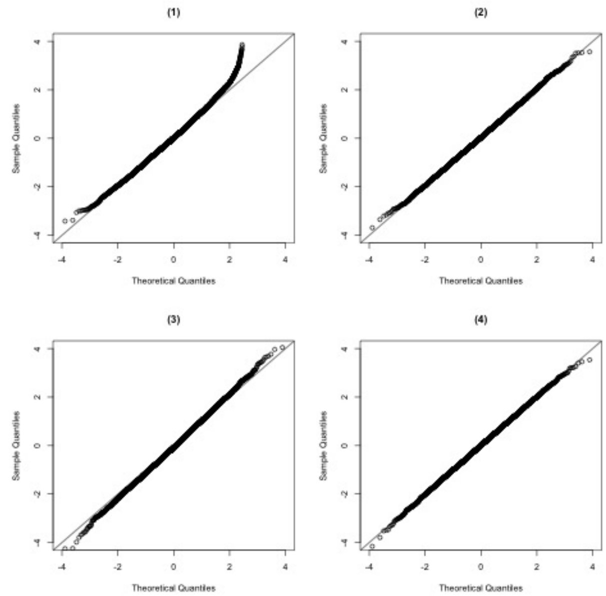

Figure 5.11: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_B = \Delta_C = 7$, $\Delta_A = 1$, and (1) $\tau = 100$, (2) $\tau = 200$, (3) $\tau = 500$, (4) $\tau = 1000$

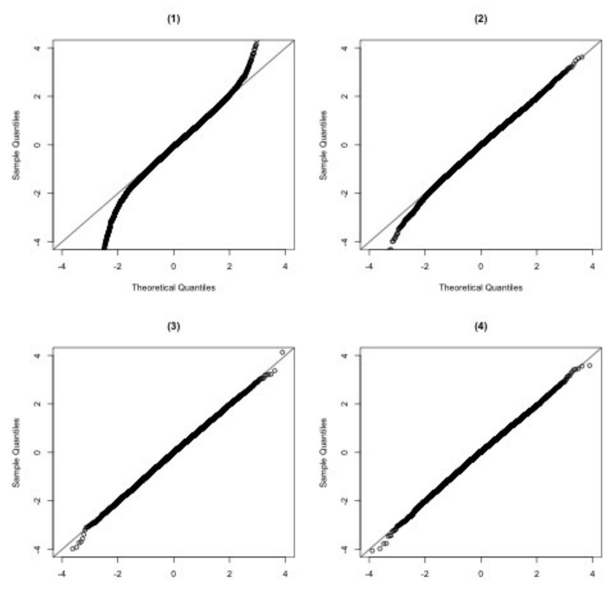

Figure 5.12: Normal Q-Q plots of 10,000 simulated values of the test statistic Z when $\Delta_B = \Delta_C = 14$, $\Delta_A = 1$, and (1) $\tau = 100$, (2) $\tau = 200$, (3) $\tau = 500$, (4) $\tau = 1000$

Δ_B Δ_C	τ	$\hat{Q}_{0.950}$	$\hat{Q}_{0.975}$	$\hat{Q}_{0.990}$	$\hat{Pr}(Z > 1.645)$	$\hat{Pr}(Z > 1.960)$	$\hat{Pr}(Z > 2.326)$
1	100	1.910	2.430	3.084	0.0719	0.0468	0.0283
	200	1.823	2.197	2.603	0.0671	0.0392	0.0182
	500	1.711	2.040	2.419	0.0584	0.0296	0.0121
	1000	1.646	1.963	2.297	0.0501	0.0255	0.0093
3	100	1.910	2.270	2.919	0.0768	0.0459	0.0233
	200	1.755	2.118	2.524	0.0604	0.0355	0.0146
	500	1.737	2.067	2.518	0.0601	0.0317	0.0151
	1000	1.655	1.935	2.323	0.0512	0.0237	0.0100
7	100	1.842	2.236	3.002	0.0695	0.0413	0.0217
	200	1.663	2.017	2.408	0.0521	0.0283	0.0115
	500	1.734	2.057	2.513	0.0605	0.0323	0.0145
	1000	1.659	1.968	2.339	0.051	0.0256	0.0103
14	100	1.653	2.008	2.509	0.0507	0.0274	0.0138
	200	1.588	1.897	2.288	0.0442	0.0223	0.0094
	500	1.597	1.928	2.270	0.0446	0.0234	0.0087
	1000	1.600	1.897	2.265	0.0451	0.0212	0.0088

Table 5.5: $\hat{Q}_p$ is the empirical pth quantile of Z computed from 10,000 samples when m=1. $\hat{Pr}(Z > Q_p)$ is the proportion of the values of Z in 10,000 samples which are larger than the pth quantile of a standard normal distribution

Δ_B and Δ_C	τ	$\hat{Pr}\{Z > 1.645\}$ $e^{\beta_A} = 2.5$	$\hat{Pr}\{Z > \hat{Q}_{0.950}\}$ $e^{\beta_A} = 2.5$	$\hat{Pr}\{Z > 1.645\}$ $e^{\beta_A} = 3$	$\hat{Pr}\{Z > \hat{Q}_{0.950}\}$ $e^{\beta_A} = 3$
3	100	0.051	0.030	0.087	0.061
	200	0.059	0.050	0.083	0.069
	500	0.078	0.069	0.091	0.079
	1000	0.076	0.074	0.174	0.173
7	100	0.127	0.096	0.156	0.120
	200	0.160	0.156	0.234	0.230
	500	0.220	0.199	0.425	0.402
	1000	0.381	0.376	0.691	0.687
14	100	0.168	0.326	0.243	0.593
	200	0.313	0.565	0.443	0.460
	500	0.525	0.549	0.725	0.734
	1000	0.725	0.741	0.912	0.917

Table 5.6: Power of Z : $m = 1, \Delta_A = 1$ with Stochastic Aging

5.4 Redundant Systems with Trends and Covariates

In this section, we consider a redundant system with trend due to stochastic aging and external covariates. We consider multiple systems with 3 components; Components A, B and C. We consider the case where repair times of Components A, B and C are not negligible. That is, $m > 1$, $\Delta_A > 0$, $\Delta_B > 0$ and $\Delta_C > 0$. However, one of the components in the system, that is Component A, is a redundant component. For $K = A, B$ and C, $\boldsymbol{z}_{Ki}(t) = (z_{1,Ki}(t), z_{2,Ki}(t), \ldots, z_{p,Ki}(t))'$ be a $p \times 1$ matrix to include possibly time-varying external covariates and fixed covariates. The intensity functions of Components A, B and C in the ith system, $i = 1, \ldots, m$, are given by

$$\lambda_{Ai}(t|\mathcal{H}(t)) = Y_{Ai}(t)\,\alpha_A\,\exp\{\beta_A\,(X_{ABi}(t)+X_{ACi}(t))+\gamma_A t+\boldsymbol{\zeta}'\boldsymbol{z}_{Ai}(t)\}, \quad t > 0, \tag{5.63}$$

$$\lambda_{Bi}(t|\mathcal{H}(t)) = Y_{Bi}(t)\,\alpha_B\,\exp\{\beta_B\,X_{BCi}(t) + \gamma_B t + \boldsymbol{\zeta}'\boldsymbol{z}_{Bi}(t)\}, \quad t > 0, \tag{5.64}$$

and

$$\lambda_{Ci}(t|\mathcal{H}(t)) = Y_{Ci}(t)\,\alpha_C\,\exp\{\beta_C\,X_{CBi}(t) + \gamma_C t + \boldsymbol{\zeta}'\boldsymbol{z}_{Ci}(t)\}, \quad t > 0, \tag{5.65}$$

respectively. Let $\boldsymbol{\theta} = (\alpha_A, \beta_A, \gamma_A, \boldsymbol{\zeta})$. The likelihood function $L(\boldsymbol{\theta})$ of the outcome "$N_{Ai}(\tau_i) = n_{Ai}$ failures of Component A at times $t_{Ai1} \leq t_{Ai2} \leq \cdots \leq t_{Ain_{Ai}}$ in $[0, \tau_i]$" can be written as follows.

$$\begin{aligned}\prod_{i=1}^{m}\prod_{j=1}^{n_A} \alpha_A\,\exp(\beta_A(X_{ABi}(t_{Aij}) + X_{ACi}(t_{Aij})) + \gamma_A t_{Aij} + \boldsymbol{\zeta}'\boldsymbol{z}_{Ai}(t_{Aij}))& \\ \exp\{-\int_0^{\tau_i} Y_{Ai}(s)\,\alpha_A\,e^{\beta_A\,(X_{ABi}(s)+X_{ACi}(s))+\gamma_A s+\boldsymbol{\zeta}'\boldsymbol{z}_{Ai}(s)}ds\}.&\end{aligned} \tag{5.66}$$

Then, log likelihood function $\ell(\boldsymbol{\theta}) = \log L(\boldsymbol{\theta})$ is given by

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{m} n_{Ai} \log \alpha_A + \beta_A \sum_{i=1}^{m} \sum_{j=1}^{n_{Ai}} (X_{ABi}(t_{Aij}) + X_{ACi}(t_{Aij})) + \gamma_A \sum_{i=1}^{m} \sum_{j=1}^{n_{Ai}} t_{Aij}$$
$$+ \sum_{i=1}^{m} \sum_{j=1}^{n_{Ai}} \boldsymbol{\zeta}' \boldsymbol{z}_{Ai}(t_{Aij}) - \alpha_A \sum_{i=1}^{m} \int_0^{\tau_i} Y_{Ai}(s)\, \alpha_A\, e^{\beta_A (X_{ABi}(s)+X_{ACi}(s))+\gamma_A s+\boldsymbol{\zeta}' \boldsymbol{z}_{Ai}(s)} ds.$$

The components of the score vector $\boldsymbol{U}(\boldsymbol{\theta})$ are given by

$$U_{\alpha_A}(\boldsymbol{\theta}) = \frac{\sum_{i=1}^{m} n_{Ai}}{\alpha_A} + \sum_{i=1}^{m} \int_0^{\tau_i} Y_{Ai}(s)\, e^{\beta_A (X_{ABi}(s)+X_{ACi}(s))+\gamma_A s+\boldsymbol{\zeta}' \boldsymbol{z}_{Ai}(s)} ds, \tag{5.67}$$

$$U_{\gamma_A}(\boldsymbol{\theta}) = \sum_{i=1}^{m} \sum_{j=1}^{n_{Ai}} t_{Aij} - \alpha_A \sum_{i=1}^{m} \int_0^{\tau_i} Y_{Ai}(s)\, s\, e^{\beta_A (X_{ABi}(s)+X_{ACi}(s))+\gamma_A s+\boldsymbol{\zeta}' \boldsymbol{z}_{Ai}(s)} ds, \tag{5.68}$$

$$\boldsymbol{U}_{\boldsymbol{\zeta}}(\boldsymbol{\theta}) = \sum_{i=1}^{m} \sum_{j=1}^{n_{Ai}} \boldsymbol{z}_{Ai}(t_{Aij}) - \alpha_A \sum_{i=1}^{m} \int_0^{\tau_i} Y_{Ai}(s)\, \boldsymbol{z}_{Ai}(s)\, e^{\beta_A (X_{ABi}(s)+X_{ACi}(s))+\gamma_A s+\boldsymbol{\zeta}' \boldsymbol{z}_{Ai}(s)} ds, \tag{5.69}$$

and

$$U_{\beta_A}(\boldsymbol{\theta}) = \sum_{i=1}^{m} \sum_{j=1}^{n_{Ai}} X_{ABi}(t_{Aij}) + X_{ACi}(t_{Aij})$$
$$- \alpha_A \sum_{i=1}^{m} \int_0^{\tau_i} Y_{Ai}(s)\, (X_{ABi}(s) + X_{ACi}(s))\, e^{\beta_A (X_{ABi}(s)+X_{ACi}(s))+\gamma_A s+\boldsymbol{\zeta}' \boldsymbol{z}_{Ai}(s)} ds. \tag{5.70}$$

The observed information matrix $\boldsymbol{I}(\boldsymbol{\theta}) = \partial \ell(\boldsymbol{\theta}) / \partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'$ is given by

$$\boldsymbol{I}(\boldsymbol{\theta}) = \begin{pmatrix} I_{\alpha_A \alpha_A}(\boldsymbol{\theta}) & I_{\alpha_A \gamma_A}(\boldsymbol{\theta}) & I_{\alpha_A \boldsymbol{\zeta}_A}(\boldsymbol{\theta}) & I_{\alpha_A \beta_A}(\boldsymbol{\theta}) \\ I_{\gamma_A \alpha_A}(\boldsymbol{\theta}) & I_{\gamma_A \gamma_A}(\boldsymbol{\theta}) & I_{\gamma_A \boldsymbol{\zeta}_A}(\boldsymbol{\theta}) & I_{\gamma_A \beta_A}(\boldsymbol{\theta}) \\ I_{\boldsymbol{\zeta}_A \alpha_A}(\boldsymbol{\theta}) & I_{\boldsymbol{\zeta}_A \gamma_A}(\boldsymbol{\theta}) & I_{\boldsymbol{\zeta}_A \boldsymbol{\zeta}_A}(\boldsymbol{\theta}) & I_{\boldsymbol{\zeta}_A \beta_A}(\boldsymbol{\theta}) \\ I_{\beta_A \alpha_A}(\boldsymbol{\theta}) & I_{\beta_A \gamma_A}(\boldsymbol{\theta}) & I_{\beta_A \boldsymbol{\zeta}_A}(\boldsymbol{\theta}) & I_{\beta_A \beta_A}(\boldsymbol{\theta}) \end{pmatrix}, \tag{5.71}$$

where

$$
\begin{aligned}
I_{\alpha_A\alpha_A}(\boldsymbol{\theta}) &= \frac{\sum_{i=1}^{m} n_{Ai}}{\alpha_A^2}, \\
I_{\alpha_A\gamma_A}(\boldsymbol{\theta}) &= \sum_{i=1}^{m}\int_0^{\tau_i} Y_{Ai}(s)\, s\, e^{\beta_A\,(X_{ABi}(s)+X_{ACi}(s))+\gamma_A s+\boldsymbol{\zeta}'\boldsymbol{z}_{Ai}(s)} ds, \\
I_{\alpha_A\boldsymbol{\zeta}_A}(\boldsymbol{\theta}) &= \frac{\sum_{i=1}^{m} n_{Ai}}{\alpha_A} + \sum_{i=1}^{m}\int_0^{\tau_i} Y_{Ai}(s)\, \boldsymbol{z}_{Ai}(s)\, e^{\beta_A\,(X_{ABi}(s)+X_{ACi}(s))+\gamma_A s+\boldsymbol{\zeta}'\boldsymbol{z}_{Ai}(s)} ds, \\
I_{\alpha_A\beta_A}(\boldsymbol{\theta}) &= \sum_{i=1}^{m}\int_0^{\tau_i} Y_{Ai}(s)\, (X_{ABi}(s)+X_{ACi}(s))\, e^{\beta_A\,(X_{ABi}(s)+X_{ACi}(s))+\gamma_A s+\boldsymbol{\zeta}'\boldsymbol{z}_{Ai}(s)} ds, \\
I_{\gamma_A\gamma_A}(\boldsymbol{\theta}) &= \alpha_A \sum_{i=1}^{m}\int_0^{\tau_i} Y_{Ai}(s)\, s^2\, e^{\beta_A\,(X_{ABi}(s)+X_{ACi}(s))+\gamma_A s+\boldsymbol{\zeta}'\boldsymbol{z}_{Ai}(s)} ds, \\
I_{\gamma_A\boldsymbol{\zeta}_A}(\boldsymbol{\theta}) &= \alpha_A \sum_{i=1}^{m}\int_0^{\tau_i} Y_{Ai}(s)\, s\, \boldsymbol{z}_{Ai}(s)\, e^{\beta_A\,(X_{ABi}(s)+X_{ACi}(s))+\gamma_A s+\boldsymbol{\zeta}'\boldsymbol{z}_{Ai}(s)} ds, \\
I_{\gamma_A\beta_A}(\boldsymbol{\theta}) &= \alpha_A \sum_{i=1}^{m}\int_0^{\tau_i} Y_{Ai}(s)\, s\, (X_{ABi}(s)+X_{ACi}(s))\, e^{\beta_A\,(X_{ABi}(s)+X_{ACi}(s))+\gamma_A s+\boldsymbol{\zeta}'\boldsymbol{z}_{Ai}(s)} ds, \\
I_{\boldsymbol{\zeta}_A\boldsymbol{\zeta}_A}(\boldsymbol{\theta}) &= \alpha_A \sum_{i=1}^{m}\int_0^{\tau_i} Y_{Ai}(s)\, \boldsymbol{z}_{Ai}(s)\, s\, e^{\beta_A\,(X_{ABi}(s)+X_{ACi}(s))+\gamma_A s+\boldsymbol{\zeta}'\boldsymbol{z}_{Ai}(s)} ds, \\
I_{\boldsymbol{\zeta}_A\beta_A}(\boldsymbol{\theta}) &= \alpha_A \sum_{i=1}^{m}\int_0^{\tau_i} Y_{Ai}(s)\, \boldsymbol{z}_{Ai}(s)\, (X_{ABi}(s)+X_{ACi}(s))\, e^{\beta_A\,(X_{ABi}(s)+X_{ACi}(s))+\gamma_A s+\boldsymbol{\zeta}'\boldsymbol{z}_{Ai}(s)} ds, \\
I_{\beta_A\beta_A}(\boldsymbol{\theta}) &= \alpha_A \sum_{i=1}^{m}\int_0^{\tau_i} Y_{Ai}(s)\, \{X_{ABi}(s)+X_{ACi}(s)\}^2\, e^{\beta_A\,(X_{ABi}(s)+X_{ACi}(s))+\gamma_A s+\boldsymbol{\zeta}'\boldsymbol{z}_{Ai}(s)} ds.
\end{aligned}
$$

For testing $H_0 : \beta_A = 0$, the restricted maximum likelihood estimators of $\boldsymbol{\theta}_0 = (\alpha_A, \gamma_A, \boldsymbol{\zeta}, 0)$ can be obtained maximizing $\ell(\boldsymbol{\theta}_0)$. This can be done by an optimizing software package such as `nlm` in `R`. Then, let $\tilde{\boldsymbol{\theta}}_0 = (\tilde{\alpha}_A, \tilde{\gamma}_A, \tilde{\boldsymbol{\zeta}}, 0)$ denote the restricted maximum likelihood estimator of $\boldsymbol{\theta}_0$.

Then the standardized partial score statistic for testing parallel carryover effects is given by

$$
Z = \frac{U_\beta(\tilde{\boldsymbol{\theta}}_0)}{\widehat{Var}(U_\beta(\tilde{\boldsymbol{\theta}}_0))^{\frac{1}{2}}}, \tag{5.72}
$$

where the partial score function is given by

$$\begin{aligned} U_{\beta_A}(\boldsymbol{\theta}) = &\sum_{i=1}^{m}\sum_{j=1}^{n_{Ai}} X_{ABi}(t_{Aij}) + X_{ACi}(t_{Aij}) \\ &- \alpha_A \sum_{i=1}^{m} \int_0^{\tau_i} Y_{Ai}(s)\left(X_{ABi}(s) + X_{ACi}(s)\right) e^{\gamma_A s + \boldsymbol{\zeta}' \boldsymbol{z}_{Ai}(s)} ds, \end{aligned} \tag{5.73}$$

and $\widehat{Var}(U_\beta(\tilde{\boldsymbol{\theta}}_0))$ is given by

$$I_{\beta_A\beta_A}(\tilde{\boldsymbol{\theta}}_0) - \boldsymbol{I_1}(\tilde{\boldsymbol{\theta}}_0)\, \boldsymbol{I_2}(\tilde{\boldsymbol{\theta}}_0)^{-1}\, \boldsymbol{I_1}(\tilde{\boldsymbol{\theta}}_0)', \tag{5.74}$$

where

$$\boldsymbol{I_1}(\tilde{\boldsymbol{\theta}}_0) = \begin{pmatrix} I_{\beta_A\alpha_A}(\tilde{\boldsymbol{\theta}}_0) & I_{\beta_A\gamma_A}(\tilde{\boldsymbol{\theta}}_0) & I_{\beta_A\boldsymbol{\zeta}_A}(\tilde{\boldsymbol{\theta}}_0) \end{pmatrix}, \tag{5.75}$$

and

$$\boldsymbol{I_2}(\tilde{\boldsymbol{\theta}}_0) = \begin{pmatrix} I_{\alpha_A\alpha_A}(\tilde{\boldsymbol{\theta}}_0) & I_{\alpha_A\gamma_A}(\tilde{\boldsymbol{\theta}}_0) & I_{\alpha_A\boldsymbol{\zeta}_A}(\tilde{\boldsymbol{\theta}}_0) \\ I_{\gamma_A\alpha_A}(\tilde{\boldsymbol{\theta}}_0) & I_{\gamma_A\gamma_A}(\tilde{\boldsymbol{\theta}}_0) & I_{\gamma_A\boldsymbol{\zeta}_A}(\tilde{\boldsymbol{\theta}}_0) \\ I_{\boldsymbol{\zeta}_A\alpha_A}(\tilde{\boldsymbol{\theta}}_0) & I_{\boldsymbol{\zeta}_A\gamma_A}(\tilde{\boldsymbol{\theta}}_0) & I_{\boldsymbol{\zeta}_A\boldsymbol{\zeta}_A}(\tilde{\boldsymbol{\theta}}_0) \end{pmatrix}. \tag{5.76}$$

5.5 Application

The purpose of this section is to illustrate the methods developed in this chapter. Therefore, we use a generated data set for testing the presence of parallel carryover effects in multiple systems with three components: Components A, B and C. We consider the case given in Section 5.2.5, where the number of systems $m > 0$, and the repair times of Components A, B and C are not negligible; i.e., $\Delta_A > 0$, $\Delta_B > 0$ and $\Delta_C > 0$. In this setup, Component A is the redundant component and Components B and C are primary components.

As explained in Sections 1.1.1 and 4.4, information and limited availability data on diesel power engines (i.e., power generators) operating in 22 remote communities were provided by a power company. Among those communities, 4 of them have relatively old generators comparing with the other generators operating in the same community; the community number 3, the community number 5, the community number 9 and the community number 15. Since ages of some generators are more than 30 years, it is useful to consider increasing time trends in the failure occurrences of these power generators. Since the availability data, including failure times and return to service times, are very limited, we generated a failure data set for the power generators operating in those 4 communities with old power generators. The values of parameters used in the data generation process were based on the limited availability data set, capacities of the generators and purchase year of the generators. In the following discussion, we use the term "system" to denote the power system operating in a community, and "component" to denote the power generators, and use them interchangeably. Each of those 4 systems includes three components.

We let the oldest generator in each of these systems be the redundant component (Component A). Other two generators (Components B and C) are then the primary generators. Starting time of the followups (i.e., $t = 0$) of the systems are the purchase date of the redundant component. The end-of-followup time τ is September 1, 2015, which is the same for all 4 systems. Since the redundant components are the oldest ones, we consider monotonic time trends only in them. Repair times are based on the distances of the communities from the headquarters of the company. This means that repair times are identical for the components operating in the same system. However, repair times can vary between communities. Therefore, we chose the repair times of the generators as follows: $\Delta_3 = 30$ days, $\Delta_5 = 30$ days, $\Delta_9 = 15$ days, and $\Delta_{15} = 45$ days for the commnity numbers 3, 5, 9, and 15, respectively.

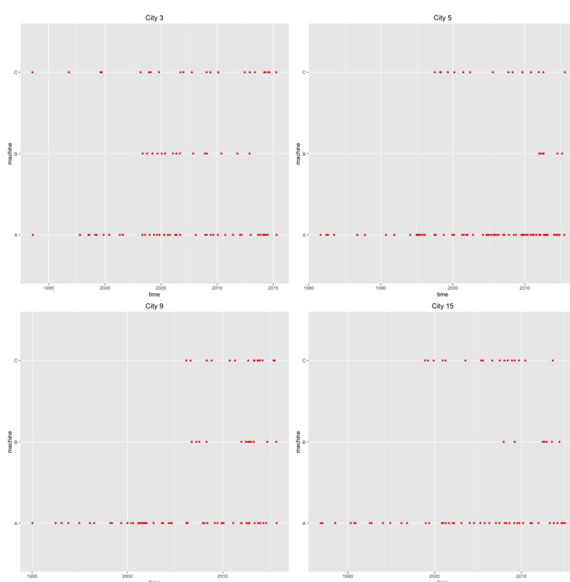

Figure 5.13: Dot plots of failures of the redundant generators operating in the community number 3, 5, 9 and 15

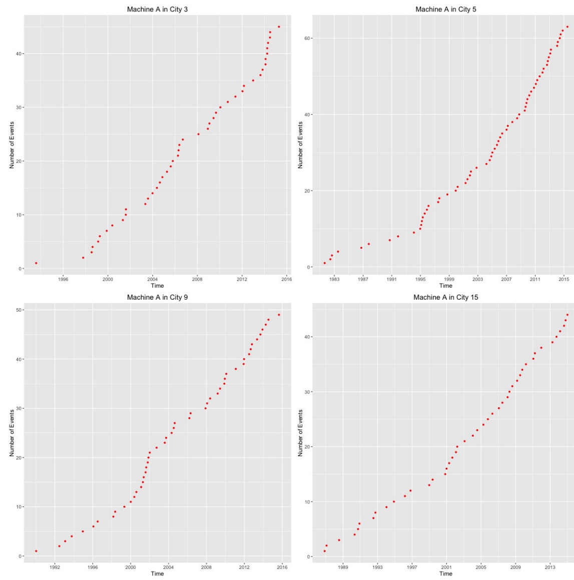

Figure 5.14: Plots of cumulative failures of the redundant generators in the community number 3, 5, 9 and 15 versus operating time

Observation periods of the redundant generators in the community number 3, 5, 9 and 15 are 9131, 12783, 9496 and 10592 days, respectively. Also, total number of failures of the redundant generators operating in the community number 3, 5, 9 and 15 over their corresponding observation period are 45, 63, 49 and 44, respectively. The observation periods of the generators B and C in the community number 3, 5, 9 and 15 are different from each others. Dot plots of the failures of the redundant generators are given in Figure 5.13. The cumulative number of failures of redundant components against the operating times are presented in Figure 5.14. The plots in these figures show that there are clustering of failures over time. Furthermore, mild convex shape observed in the plots presented in Figure 5.13 indicates an increasing trend in the failure occurrences in redundnat generators.

The absence of parallel carryover effects in a system can be tested by considering the following model. For $i = 3$, 5, 9, and 15, the model for Component A in the ith system is given by

$$\lambda_{Ai}(t|\mathcal{H}(t)) = Y_{Ai}(t)\,\alpha_{Ai}\,\exp\{\beta_{Ai}X_{ABi}(t) + \beta_{Ai}X_{ACi}(t) + \gamma_i\,t\}, \qquad t > 0, \qquad (5.77)$$

where $Y_{Ai}(t)$ is the at-risk indicator of Component A in the ith system, α_{Ai}, β_{Ai} and γ_i are model parameters, and $X_{ABi}(t)$ and $X_{ACi}(t)$ are defined in Section 5.2.5.

We present the restricted maximum likelihood estimates $\tilde{\alpha}_{Ai}$ and $\tilde{\gamma}_{Ai}$ of α_{Ai} and γ_{Ai}, respectively, and their standard errors in Table 5.7 when $\beta_{Ai} = 0$. We used the `nlm` package in `R` to obtain these estimates and their standard errors. We test the null hypothesis $H_0 : \beta_{Ai} = 0$ against the alternative hypothesis $H_1 : \beta_{Ai} \neq 0$. The observed values of the test statistic Z are given in Table 5.8 along with the p–values based on the standard normal distribution as well as based on 1000 simulation runs (denoted by p^*–value). The results in Table 5.8 suggest that there is some evidence against the null hypothesis $H_0 : \beta_{A_3} = 0$ and $H_0 : \beta_{A_9} = 0$. Therefore, we conclude that there is parallel carryover effects on the redundant components operating in community 3 and community 9.

$\tilde{\alpha}_{A_3}$	0.002501 (0.001232)	$\tilde{\alpha}_{A_5}$	0.002499 (0.000914)
$\tilde{\gamma}_3$	0.000182 (0.000050)	$\tilde{\gamma}_5$	0.000131 (0.000022)
$\tilde{\alpha}_{A_9}$	0.002504 (0.000823)	$\tilde{\alpha}_{A_{15}}$	0.002501(0.000840)
$\tilde{\gamma}_9$	0.000144 (0.000033)	$\tilde{\gamma}_{15}$	0.000119 (0.000030)

Table 5.7: Estimates of α_{Ai}, β_{Ai}, γ_i and α_{Ai}, γ_i when $\beta_{Ai} = 0$ where i = 3, 5, 9, and 15. The numbers in parentheses are the standard errors of the estimates.

i	Δ_i	$U_{\beta_{Ai}}(\tilde{\boldsymbol{\theta}})$	$Var(U_{\beta_{Ai}}(\tilde{\boldsymbol{\theta}}))$	Z	p-value	p^*-value
3	30	4.729459	6.231403	1.894605	0.058	0.069
5	30	1.874224	4.122099	0.923129	0.355	0.355
9	15	3.305245	2.682509	2.018056	0.049	0.058
15	45	1.537437	4.950848	0.690967	0.489	0.507

Table 5.8: Statistic Z and p-values

Figure 5.14 suggests that there are some trends in the number of failures of the redundant generators as we generated data from a trend model. It should be noted that we took $\gamma = 0.0001$ in the model (5.77) while generating the data. Therefore, the rate function increases very slowly as time increases. We now test of absence of monotonic trend of the systems. For this purpose, we use the Laplace test (Cox and Lewis, 1966), which is given by

$$LA = \frac{\{\sum_{j=1}^{n_i} T_{ij} - n_i(\tau_i)/2\}}{\{n_i\,(\tau_i)^2/12\}^{1/2}} \tag{5.78}$$

The observed values of the Laplace statistic (5.78) are presented in Table 5.9. The standard normal approximation gives a two-sided p-value of 0.0003258, 0.0000268, 0.0495566 and 0.1258483. These results show that there is a strong evidence of trend in the redundant components in community 3, 5 and 9.

i	LA	p-value
3	3.593810	0.0003258
5	4.198454	0.0000268
9	1.963771	0.0495566
15	1.530681	0.1258483

Table 5.9: Statistic LA and p-values

Chapter 6

Summary and Future Research

In this chapter, we present a summary of the results obtained in this thesis. We also briefly discuss some of the limitations of our approach and future research topics.

6.1 Summary and Conclusions

The statistical analysis of failure data from repairable systems has been a major research area in statistics and reliability engineering. In this thesis, we considered a reliability improvement technique called redundancy, which is often applied to power systems. Redundancy can significantly increase the availability of systems. However, implementation of it incurs cost. If the cost of repairs of components in a redundant system is expensive, the reliability program of a company may not be cost-efficient. Therefore, it is important to detect the reasons of failures of components in redundant systems. In some cases, failures of components may result in a temporary increase in the risk of failures in the redundant components while the failed ones are under repair. In such cases, failures may cluster together over time. We referred to this phenomenon as a parallel carryover effect. In this thesis, we developed simple tests for parallel carryover effects and discuss their asymptotic properties in various settings. The tests developed are easy to implement and have good overall power.

It is well known that the data acquisition is notoriously difficult in reliability studies (see, Lawless, 1983; Blischke and Murthy, 2003). In this thesis, we analyzed two randomly generated data sets as explained below (also, see Sections 4.4 and 5.5). The data sets were generated according to the limited information received from a power company. Our purpose with these analyses was to illustrate the methodology developed in the thesis. We did not consider issues related to the data sets. However, the results of these analyses showed that the methods developed can be applied to real life data sets as well.

In Chapter 3, we investigated testing for the presence of parallel carryover effects in redundant systems with two components. We considered the cases in which repair times of the redundant component are negligible and non-negligible. Asymptotic normal approximations of the test statistics given in Sections 3.2.1 and 3.2.2 were discussed analytically as well as by simulations under two different settings; (i) when the observation period increases in a single system, and (ii) when the number of systems approaches infinity for a fixed observation period. Simulation studies showed that the standard normal approximation is adequate in both cases. The results of a simulation study conducted to investigate the power of the tests were presented under various scenarios. We found that the overall power of the test is high in overall.

In Chapter 4, we discussed testing for parallel carryover effects in redundant systems with three components. We considered cases in which repair times of the redundant component are negligible and non-negligible. We presented two settings; (i) when a single system is under observation, and (ii) when multiple systems are under observation. We investigated asymptotic properties of the test statistics through simulations. The results of our simulation studies indicated that the standard normal approximation is accurate in settings with large m and/or τ values. An application of the methods was given by analyzing a simulated data set in the context of power systems with multiple generators.

In Chapter 5, we discussed testing for parallel carryover effects in redundant systems with stochastic aging and covariates. The models developed in Chapters 2 and 3 were extended accordingly. We developed partial score tests for the presence of parallel carryover effects in redundant systems with two or three components. In this chapter, we assumed that the components are subject to monotonic trends due to stochastic aging. Once again, we considered two settings; (i) when a single system is under observation, and (ii) when multiple systems are under observation. The adequacy of the standard normal approximations for the test statistics was investigated through simulations under various settings. Finally, we analyzed a generated data set similar to that of the previous chapter, but included a trend in the rate functions of failure occurrences of the redundant generator.

An issue related to our methodology is the choice of repair times Δ of a failed component. The tests developed in this thesis require a fixed value for the repair times of a failed component. This information can be obtained from experts or history data. However, care is needed because too long or too short time specification for the repair times may result in difficulties in the estimation of parallel carryover effects. In an extreme case, when $\Delta \to \infty$ or $\Delta \to 0$, parallel carryover effects are not estimable. This is not an important issue in the context of repairable systems because repair times of failed components are usually not that short or long.

Another issue is the misspecification of Δ. We studied this issue through a simulation study. Table B.1 and Table B.2 in Appendix B show that the power of the test statistic Z in (3.21) developed in Section 3.2.1 when Δ_{B_0} is misspecified. In Table B.1, Δ_{B_0} denotes the true value of the repair times and Δ_B is the value of the repair times used in the test. For example, we generated data with $\Delta_{B_0} = \frac{2}{3}\Delta_B$, where $\Delta_B = 7$ time units was used in the test. The factors of the simulation study were $\Delta_B = 7$, 14, $\Delta_{B_0} = \frac{2}{3}\Delta_B, \frac{4}{3}\Delta_B$, $e^{\beta_A} = 2$, 4, and $\tau = 100$, 200, 500, and 1000. The results of Tables B.1 and B.2 indicates that there is a small loss in the power if repair times specified little smaller or larger than the true value of repair times. However, power is increasing as τ increases. We also investigated the power of the test statistic Z in (3.38) developed in Section 3.2.2 when $m = 10$, 20, 50, 100 and $\tau = 100$. In this case, we present the results with the same factors of Tables B.1 and B.2, but $e^{\beta_A} = 2$, 2.5. The results are presented in Tables B.3 and B.4, and the conclusions are similar to those obtained from Tables B.1 and B.2.

In this thesis, we considered the settings where parallel carryover effects can be classified as time varying external covariates. This is because of the fact that failures of primary components temporarily change the probabilistic characteristics of failure occurrences in a redundant component, but failures of a redundant component do not affect the probabilistic characteristics of failure occurrences of primary components. As discussed in Chapter 1, in this case, carryover effects can be classified as an external covariate (Kalbfleisch and Prentice, 2002). However, if failures of a redundant component also affects the probabilistic characteristics of failure occurrences on the primary components, then joint modelling of the at-risk indicators and counting processes is needed. This is because the at risk indicator does not evolve independently of the counting process. In this case, the full likelihood based inference may become unmanageable.

6.2 Alternating Two-State Processes

In this thesis, we assumed for simplicity that parallel carryover effect periods are constant. In applications, the duration of repair times may vary. Alternating two-state processes are useful when the duration of repair times varies (See Cook and Lawless, 2007; Nakagawa, 2008). An alternating two-state process involves the "up" state and the "down" state, and a process can be either in the up state or the down state at time t. We can specify a counting process model, which is similar to the models developed in this thesis, for "up to down" transitions. Then, this model can be extended to include varying repair times which are generated from a model. Such a method is considered by Hong et al. (2013), where they use a truncated lognormal distribution as a model for event durations. We can develop such a method to investigate the presence of carryover effects with varying repair times. This topic will be investigated in the future.

6.3 Redundant Systems with Imperfect Repairs

Imperfect repairs are common in applications. There are many imperfect repair models proposed for the repairable systems. A model that can incorporate imperfect repairs as well as effects of maintenance activities of repairable systems is proposed by Cigsar (2010). In this model, an internal carryover effect is specified to reflect imperfect repairs or maintenance activities so that it is assumed that after each repair or maintenance activities the risk of a failure temporarily changes for the same process. Our models in this thesis can be extended to include such internal carryover effects as well. For example, with the settings of Section 3.2.1, this can be done by specifying the intensity function as $\lambda_A(t|\mathcal{H}(t)) = Y_A(t)\,\alpha_A \exp\{\beta_{A_1} X_A(t) + \beta_{A_2} Z_A(t)\}$, where $X_A(t) = I\{N_B(t^-) > 0\} I\{t - t_{BN_B(t^-)} \le \Delta_B\}$ and $Z_A(t) = I\{N_A(t^-) > 0\} I\{t - t_{AN_A(t^-)} \le \Delta\}$. So function $X_A(t)$ detects parallel carryover effects and function $Z_A(t)$ detects transient carryover effects. Here, Δ_B is the repair times of Component B and Δ is specified time of transient carryover of Component A. We will explore such models as a future research.

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Appendix A

Data Sets

A.1 Data 1

Part of data set used in Section 3.4

Comp : 1=Component A, 2=Component B, 3=Component C

Region	City	Comp	Start	End
1	1	1	60	90
1	1	1	474	504
1	1	1	1092	1122
1	1	1	1296	1326
1	1	1	1445	1475
1	1	1	1770	1800
1	1	1	2137	2167
1	1	1	2224	2254
1	1	1	2879	2909
1	1	1	2968	2998
1	1	1	3006	3036
1	1	1	3362	3392
1	1	1	3638	3668
1	1	1	3641	3671
1	1	1	3711	3741

Region	City	Comp	Start	End
1	1	1	3995	4025
1	1	1	4076	4106
1	1	1	4342	4372
1	1	1	4680	4710
1	1	2	2187	2217
1	1	2	2350	2380
1	1	2	2807	2837
1	1	2	3589	3619
1	1	2	3757	3787
1	1	2	3863	3893
1	1	2	3936	3966
1	1	2	4200	4230
1	1	2	4297	4327
1	1	3	2756	2786
1	1	3	2890	2920

A.2 Data 2

City 3 $\Delta = 30$ $\tau = 9131$			City 5 $\Delta = 30$ $\tau = 12783$			City 9 $\Delta = 15$ $\tau = 9496$			City 15 $\Delta = 45$ $\tau = 10592$		
A	B	C	A	B	C	A	B	C	A	B	C
1065	4640	1052	358	11444	6156	132	6239	6030	75	7778	4469
2601	4782	2240	649	11524	6417	1026	6410	6195	155	8242	4588
2877	4958	3267	729	11638	6453	1251	6528	6813	685	9418	4831
2908	5122	3310	1038	11674	6813	1502	6804	7001	1344	9495	5210
3091	5259	4569	2223	12364	7138	1930	8139	7693	1490	9588	5331
3147	5367	4845	2604	12610	7597	2338	8312	7897	1547	9806	6170
3371	5625	4901	3672		7943	2502	8395	8400	2139	10122	6831
3553	5736	5171	4095		9097	3107	8449	8617	2223		6908
3898	5848	5871	4903		9887	3179	8513	8638	2692		7291
3994	6282	5965	5223		10095	3531	8612	8757	3007		7611
4632	6663	6240	5274		10596	3778	9135	8825	3473		7806
4720	6729	6717	5309		11027	3914	9472	8844	3710		7944
4867	7201	6841	5353		11423	3996		8948	4502		8125
5010	7720	7097	5452		11661	4179		9372	4647		8235
5107	8114	7959	5565		12747	4245		9413	5179		8425
5190		8122	5643			4277			5236		8688
5339		8293	6137			4350			5344		9844
5462		8594	6197			4373			5479		
5529		8643	6605			4432			5632		
5698		8732	7032			4461			5686		
5718		8763	7130			4510			5998		
5743		8986	7531			4774			6347		
5856			7628			5089			6534		
6368			7754			5140			6793		
6674			7804			5353			6985		
6726			8090			5435			7173		
6860			8595			5472			7449		
6940			8765			6036			7598		
7084			8844			6084			7819		
7326			8897			6659			7897		
7576			9017			6722			8018		
7806			9132			6836			8221		
7853			9204			7128			8351		
8154			9292			7222			8439		
8394			9395			7384			8596		
8463			9621			7404			8908		
8558			9682			7460			8974		
8613			9912			7826			9245		
8640			10158			8129			9716		
8704			10264			8145			9887		
8997			10543			8339			10041		
			10587			8401			10208		
			10633			8448			10274		
			10679			8646			10351		
			10777			8777					
			10871			8855					
			11014			8981					
			11111			9085					
			11180			9489					
			11303								
			11449								
			11506								
			11678								
			11718								
			11769								
			11844								
			11878								
			12196								
			12227								
			12329								
			12373								
			12476								
			12718								

Table A.1: Data set used in Section 4.5, times (in days) are failure times in each component, repair times, Δ, are the same among the components within a city

Appendix B

Misspecification of Δ_B

τ	Δ_{B_0}	$\hat{P}r\{Z > 1.645\}$ $e^{\beta_A} = 2$	$\hat{P}r\{Z > \hat{Q}_{0.950}\}$ $e^{\beta_A} = 2$	$\hat{P}r\{Z > 1.645\}$ $e^{\beta_A} = 4$	$\hat{P}r\{Z > \hat{Q}_{0.950}\}$ $e^{\beta_A} = 4$
100	$\frac{2}{3}\Delta_B$	0.232	0.230	0.753	0.748
	Δ_B	0.369	0.369	0.996	0.995
	$\frac{4}{3}\Delta_B$	0.237	0.237	0.704	0.700
200	$\frac{2}{3}\Delta_B$	0.401	0.379	0.954	0.958
	Δ_B	0.537	0.518	1.000	1.000
	$\frac{4}{3}\Delta_B$	0.381	0.360	0.925	0.916
500	$\frac{2}{3}\Delta_B$	0.679	0.687	1.000	1.000
	Δ_B	0.885	0.890	1.000	1.000
	$\frac{4}{3}\Delta_B$	0.717	0.736	1.000	1.000
1000	$\frac{2}{3}\Delta_B$	0.901	0.898	1.000	1.000
	Δ_B	0.991	0.991	1.000	1.000
	$\frac{4}{3}\Delta_B$	0.924	0.920	1.000	1.000

Table B.1: Results of the power study when Δ_{B_0} is misspecified, where $\Delta_B = 7$ is assumed

τ	Δ_{B_0}	$\hat{P}r\{Z > 1.645\}$ $e^{\beta_A} = 2$	$\hat{P}r\{Z > \hat{Q}_{0.950}\}$ $e^{\beta_A} = 2$	$\hat{P}r\{Z > 1.645\}$ $e^{\beta_A} = 4$	$\hat{P}r\{Z > \hat{Q}_{0.950}\}$ $e^{\beta_A} = 4$
100	$\frac{2}{3}\Delta_B$	0.195	0.199	0.657	0.660
	Δ_B	0.290	0.293	0.995	0.995
	$\frac{4}{3}\Delta_B$	0.198	0.205	0.499	0.503
200	$\frac{2}{3}\Delta_B$	0.333	0.345	0.926	0.928
	Δ_B	0.519	0.535	1.000	1.000
	$\frac{4}{3}\Delta_B$	0.276	0.299	0.790	0.802
500	$\frac{2}{3}\Delta_B$	0.610	0.610	1.000	1.000
	Δ_B	0.882	0.882	1.000	1.000
	$\frac{4}{3}\Delta_B$	0.576	0.577	0.988	0.988
1000	$\frac{2}{3}\Delta_B$	0.884	0.885	1.000	1.000
	Δ_B	0.994	0.994	1.000	1.000
	$\frac{4}{3}\Delta_B$	0.832	0.832	1.000	1.000

Table B.2: Results of the power study when Δ_{B_0} is misspecified, where $\Delta_B = 14$ is assumed

m	Δ_{B_0}	$\hat{P}r\{Z > 1.645\}$ $e^{\beta_A} = 2$	$\hat{P}r\{Z > \hat{Q}_{0.950}\}$ $e^{\beta_A} = 2$	$\hat{P}r\{Z > 1.645\}$ $e^{\beta_A} = 2.5$	$\hat{P}r\{Z > \hat{Q}_{0.950}\}$ $e^{\beta_A} = 2.5$
10	$\frac{2}{3}\Delta_B$	0.887	0.884	0.992	0.992
	Δ_B	0.988	0.988	1.000	1.000
	$\frac{4}{3}\Delta_B$	0.912	0.912	0.994	0.994
20	$\frac{2}{3}\Delta_B$	0.993	0.992	1.000	1.000
	Δ_B	1.000	1.000	1.000	1.000
	$\frac{4}{3}\Delta_B$	0.996	0.996	1.000	1.000
50	$\frac{2}{3}\Delta_B$	1.000	1.000	1.000	1.000
	Δ_B	1.000	1.000	1.000	1.000
	$\frac{4}{3}\Delta_B$	1.000	1.000	1.000	1.000
100	$\frac{2}{3}\Delta_B$	1.000	1.000	1.000	1.000
	Δ_B	1.000	1.000	1.000	1.000
	$\frac{4}{3}\Delta_B$	1.000	1.000	1.000	1.000

Table B.3: Results of the power study when Δ_{B_0} is misspecified, where $\Delta_B = 7$ is assumed

m	Δ_{B_0}	$\hat{Pr}\{Z > 1.645\}$ $e^{\beta_A} = 2$	$\hat{Pr}\{Z > \hat{Q}_{0.950}\}$ $e^{\beta_A} = 2$	$\hat{Pr}\{Z > 1.645\}$ $e^{\beta_A} = 2.5$	$\hat{Pr}\{Z > \hat{Q}_{0.950}\}$ $e^{\beta_A} = 2.5$
10	$\frac{2}{3}\Delta_B$	0.884	0.884	0.996	0.996
	Δ_B	0.990	0.990	1.000	1.000
	$\frac{4}{3}\Delta_B$	0.792	0.792	0.955	0.955
20	$\frac{2}{3}\Delta_B$	0.995	0.995	1.000	1.000
	Δ_B	1.000	1.000	1.000	1.000
	$\frac{4}{3}\Delta_B$	0.972	0.972	0.997	0.997
50	$\frac{2}{3}\Delta_B$	1.000	1.000	1.000	1.000
	Δ_B	1.000	1.000	1.000	1.000
	$\frac{4}{3}\Delta_B$	1.000	1.000	1.000	1.000
100	$\frac{2}{3}\Delta_B$	1.000	1.000	1.000	1.000
	Δ_B	1.000	1.000	1.000	1.000
	$\frac{4}{3}\Delta_B$	1.000	1.000	1.000	1.000

Table B.4: Results of the power study when Δ_{B_0} is misspecified, where $\Delta_B = 14$ is assumed