Growth Functions for Bases of Commutator Subgroups

by

© Michael Watson

A thesis submitted to the School of Gradate Studies in partial fulfillment of the requirements for the degree of Master of Science.

Department of Mathematics and Statistics

Memorial University

September 2016

St. John's, Newfoundland and Labrador, Canada

Abstract

In this paper, we compare growth functions of different bases of commutator subgroups of free groups. Of the bases that we consider, the geodesic Schreier basis appears to be the fastest. In connection with groups, we consider a basis for the free Lie commutator subalgebra. In contrast with groups, the growth of a basis for a free Lie algebra does not depend on the choice of basis.

Acknowledgements

The completion of this thesis would not have been possible without the assistance of Dr. Bahturin. He has helped me to discover the beauty and significance of algebra, and encouraged me to strive for my best. He has my deepest appreciation for the patience and support he has provided for the past two years. I would also like to thank Dr. Baird and Dr. Kotchetov for their efforts in helping me revise this thesis.

Statement of contribution

This thesis has been written in collaboration with Dr. Bahturin. In particular, he has provided me with propositions 8 and 9, and their proofs, in Section 2.2. He also provided the method for constructing an upper bound on the growth function in Section 2.5.

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List of symbols

$\langle X \rangle$	group generated by the elements of X
F	a free group
F'	the commutator subgroup of F
$[x, y] = x^{-1}y^{-1}xy$	the commutator of elements x and y.
$[x_1, x_2, \ldots, x_n] = [[\ldots[x_1, x_2], x_3], \ldots x_n]$	the left-normed commutator

Chapter 1

Free Groups and Free Lie Algebras

1.1 Free Groups

We start our discussion of free groups with the statement of their universal property, and provide a few basic results. In the sections that follow, we provide a construction to prove that such free groups exist, and discuss the methods of Nielsen which are used to show that every finitely generated subgroup of a free group is free. This result was later proven for any subgroup (finitely generated or not), and is known as the Nielsen-Schreier theorem.

Definition 1. *A group F is free on a set X if given any group G and a map $f : X \to G$, there exists a unique homomorphism $\phi : F \to G$ such that $\phi \circ \iota = f$, where ι is the inclusion map. The set X is called the free basis of F.*

In other words, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & F \arrow[d, dotted, "\phi"] \\ & G \end{tikzcd}$$

Proposition 1 ([6], Section 1.1). *Let F_1 and F_2 be free groups with bases X_1 and X_2, respectively. Then F_1 and F_2 are isomorphic if and only if $|X_1| = |X_2|$.* $\square$

Corollary 1 ([6], Section 1.1). *All bases for a given free group F have the same cardinality, called the rank of F.* $\square$

Proposition 2 ([6], Section 1.1). *Let X be a subset of a group G such that $X \cap X^{-1} = \emptyset$. Then X is a basis for a free subgroup of G if and only if no product $x_1 \dots x_n$ is trivial, where $n \geq 1, x_i \in X^{\pm 1}$, and all $x_i x_{i+1} \neq 1$.* $\square$

We have covered some of the basic properties of free groups that follow from the universal property. Yet so far, we have only given an abstract definition of a free group. In the following section, we provide a construction of free groups to prove existence.

1.2 Construction of Free Groups

Now that free groups have been defined in terms of their universal property, we give a construction. Let X be a (not-necessarily finite) set. We let $\Omega(X)$ be the monoid consisting of words, that is all finite products, of letters in the "alphabet" $X^{\pm 1} = X \bigcup \{x^{-1} : x \in X\}$. The associative binary operation is the juxtaposition of words and we denote by 1 the empty word. Given a word $w = a_1 a_2 \dots a_n$, where $a_i \in X^{\pm 1}$, the length of w is given by $|w| = n$. We may then define the word $w^{-1} = a_n^{-1} \dots a_1^{-1}$. Next, define an equivalence relation $\sim$ on $\Omega(X)$ as follows:

Two words $u, v \in \Omega(X)$ are equivalent if there exists a sequence of words $u = w_0, w_1, \dots, w_k = v$ where w_i is obtained from w_{i-1} by insertion or deletion of a subword of the form xx^{-1}, where $x \in X^{\pm 1}$. From this definition, it follows that if $u_1 \sim u_2$ and $v_1 \sim v_2$, then $u_1 v_1 \sim u_2 v_2$, and $u_1^{-1} \sim u_2^{-1}$.

It is easy to verify that this relation is an equivalence relation. We then form the quotient monoid $\Omega(X)/\sim$, consisting of equivalence classes of words. Every class has an inverse: given a word u, it follows that $uu^{-1} \sim 1$. Therefore the quotient monoid is a group. We will show that it is in fact a free group.

A word $w \in \Omega(X)$ is said to be reduced if it contains no subword $xx^{-1}, x \in X^{\pm 1}$. We must show that each equivalence class contains a unique reduced word. It is clear that each equivalence class contains at least one reduced word, since deletions of the form xx^{-1} will eventually lead to a reduced word. We must now show that distinct reduced words u, v are not equivalent. Assume on the contrary that we have some sequence of words $u = w_0, w_1, \ldots, w_n = v$, where w_i is obtained from w_{i-1} by insertion or deletion of subwords of the form xx^{-1}, where $N = \sum |w_i|$ is a minimum. Since u and v are both reduced, and $u \neq v$, then $n > 0$ and $|w_1| > |w_0|, |w_{n-1}| > |w_n|$. It follows that for some $i, 0 < i < n$, $|w_i| > |w_{i-1}|, |w_{i+1}|$. Now w_{i-1} is obtained from w_i by deletion of a subword of the form aa^{-1}, and w_{i+1} is obtained by deletion of a subword bb^{-1}, where $a, b \in X^{\pm 1}$. Now if these subwords coincide, then $w_{i-1} = w_{i+1}$, contrary to the minimality of N. If the subwords overlap without coinciding, then w_i contains a subword of the form $aa^{-1}a$. So w_{i-1} and w_{i+1} are both obtained by replacing $aa^{-1}a$ by a, and again we have $w_{i-1} = w_{i+1}$. The last possibility is that these subwords aa^{-1} and bb^{-1} do not overlap at all, in which case w_i can be replaced by the result of removing both aa^{-1} and bb^{-1}, resulting in $N' = N - 4$, again contrary to minimality of N.

Proposition 3 ([6], Section 1.1)**.** *The group $F = \Omega(X)/\sim$ is a free group with basis the set $[X]$ of equivalence classes of elements from X, and $|[X]| = |X|$.*

Proof. Let G be any group, and let f map the set $[X]$ of equivalence classes of elements $x \in X$ into G. We first show that $|[X]| = |X|$. Let $x_1, x_2 \in X$ and $x_1 \neq x_2$. Then $[x_1] \neq [x_2]$ since x_1 and x_2 are distinct reduced one-letter words. Then f determines a map $g : X \to G$ by $g(x) = f([x])$. Define an extension ϕ of g from $\Omega(X)$ into G by setting $\phi(w) = \phi(x_1^{\beta_1} \dots x_n^{\beta_n}) = (g(x_1))^{\beta_1} \dots (g(x_n))^{\beta_n}$, where $x_i \in X, \beta_i = \pm 1$. If w_1 and w_2 are equivalent, then $\phi(w_1) = \phi(w_2)$, i.e. ϕ maps equivalent words onto the same element in G, thereby inducing a map $\phi : F \to G$ that is clearly a homomorphic extension of f.

$\square$

1.3 Nielsen's Method

The methods of Nielsen focus largely on the cancellations of elements in a free group. Given a finite generating set U for a subgroup of a free group, we can use Nielsen transformations to transform U into a free basis. The methods in this chapter are used to prove the Nielsen Subgroup Theorem, obtained in 1921, which states that finitely generated subgroups of free groups are free. This result was later generalized by Schreier who proved that any subgroup (finitely generated or not) of a free group is free; this is the well-known Nielsen-Schreier Theorem. We begin this section with a discussion on Nielsen transformations.

Consider a well-ordered subset $U = \{u_1, u_2, \dots\}$, either finite or infinite, of a group G. We define three transformations on this set:

(T1) replace some u_i by u_i^{-1};

(T2) replace some u_i by $u_i u_j$, where $j \neq i$;

(T3) delete some u_i where $u_i = 1$.

These transformations leave fixed each u_h where $h \neq i$. These three types of transformations are known as <u>elementary Nielsen transformations</u>. A composition of such transformations is a <u>Nielsen transformation</u>, and is called regular if there is no factor of type (T3). Clearly transformations of type (T1) and (T2) have inverses, so the regular Nielsen transformations form a group. This group contains every permutation fixing all but finitely many of the u_i, and it also contains every transformation carrying u_i into one of $u_i u_j, u_i u_j^{-1}, u_j u_i, u_j^{-1} u_i$, where $j \neq i$, and fixing every u_h where $h \neq i$.

Proposition 4 ([6], Section 1.2)**.** *If U is carried into V by a Nielsen transformation, then $\langle U \rangle = \langle V \rangle$.* □

We now introduce the notion of Nielsen-reduced sets, or N-reduced for short. We will see in a bit that every well-ordered finite generating set can be carried to a Nielsen-reduced set via Nielsen transformations, and it is these Nielsen-reduced sets that form free bases for subgroups of free groups. Let F be a free group with basis X and let $U = \{u_1, u_2, \ldots\}$ be a well-ordered set, where each u_i is in F. As before, the length of a word, $|w|$, is the length of the reduced word over X representing w. A set U is called N-reduced if the following conditions hold for all triples of elements v_1, v_2, v_3, where each v_j is of the form $u_i^{\pm 1}$:

(N0) $v_1 \neq 1$;

(N1) $v_1 v_2 \neq 1 \Rightarrow |v_1 v_2| \geq |v_1|, |v_2|$;

(N2) $v_1 v_2 \neq 1$ and $v_2 v_3 \neq 1 \Rightarrow |v_1 v_2 v_3| > |v_1| - |v_2| + |v_3|$.

Proposition 5 ([6], Section 1.2)**.** *If $U = \{u_1, u_2, \ldots u_n\}$ is finite, then U can be carried by a Nielsen transformation into some V such that V is N-reduced.* □

Proposition 6 ([6], Section 1.2)**.** *If U is N-reduced, then $\langle U \rangle$ is free with U as a basis.* □

Theorem 1 (The Nielsen Subgroup Theorem [6], Section 1.2)**.** *Every finitely generated subgroup of a free group is free.*

Proof. Let F be a free group, and G a subgroup generated by some finite subset U of F. By the previous proposition, U can be carried to V by a Nielsen transformation such that V is N-reduced. Then $\langle U \rangle = \langle V \rangle$, and G is free with V as a free basis.

$\square$

Theorem 2 (The Nielsen-Schreier Subgroup Theorem [6], Section 1.2)**.** *Every subgroup of a free group is free.* $\square$

There exists a topological approach to the Nielsen-Schreier Theorem using covering spaces; we direct the reader to [6]. We conclude this section by noting that there may be several distinct free bases for a subgroup of a free group. In the next section, we discuss the Schreier method for constructing free bases for subgroups of free groups.

1.4 The Schreier Method

The methods of Schreier provide a way to form a basis for any subgroup H (finitely generated or not) of a free group F of rank r. Given a right coset Hu, we denote the length of this coset to be the minimal length of elements in the coset Hu. A subset $T \subset F$ is a Schreier transversal for H if the following conditions are satisfied:

(1) $\forall u, v \in T, Hu = Hv \Rightarrow u = v$;

(2) $F = \bigcup_{u \in T} Hu$;

(3) any prefix of a word in T is itself in T.

If $u \in T, w \in F$, and $Hu = Hw$, then we write $u = \overline{w}$. A Schreier transversal is called geodesic if each element $u \in T$ is of minimal length among all representatives of Hu. Given a Schreier transversal T for H, we may construct a free basis in the following way. For $u \in T, x \in X^{\pm 1}$, we consider $ux(\overline{ux})^{-1}$. Then the set of all of these elements different from the identity forms a free basis for H.

It will be convenient to use the equivalent language of graphs when finding a Schreier basis. We consider a directed graph Γ, which we will call the coset graph, with vertex set $V = F/H$. The edge set appears as follows. Given a coset Hu, we form an edge $e = (Hu, Hux)$, for each $x \in X^{\pm 1}$, where the label of e is denoted by $\mathrm{lab}(e) = x$. The inverse edge is given by $e^{-1} = (Hux, Huxx^{-1}) = (Hux, Hu)$, with label x^{-1}. We define the initial point of an edge $e = (Hu, Hux)$ to be $\alpha(e) = Hu$, and the terminal point to be $\omega(e) = Hux$. If we have a sequence $e_1, e_2, \ldots, e_n$ such that $\omega(e_i) = \alpha(e_{i+1})$ for each $i = 1, \ldots, n-1$, then we may define the path $\pi = e_1 e_2 \ldots e_n$, where $\alpha(\pi) = \alpha(e_1)$, and $\omega(\pi) = \omega(e_n)$. The label of π is given by $lab(\pi) = \mathrm{lab}(e_1 \ldots e_n) = \mathrm{lab}(e_1) \ldots \mathrm{lab}(e_n)$. If a path π has no parts of the form ee^{-1}, then $lab(\pi)$ is a reduced word in F_r. The length of Hu is then the shortest path on the graph from the origin H to Hu. We may then define the sphere V_n consisting of those cosets of length n.

Let $V(n) = \bigcup_{k=0}^{n} V_k$. Let $\Gamma(n)$ be the induced subgraph of Γ with vertex set $V(n)$. The notion of a (geodesic) Schreier transversal is analogous to a (geodesic) maximal subtree. A maximal tree $\mathcal{T}$ will be constructed as the union $\mathcal{T}(n)$ of maximal subtrees in $\Gamma(n)$, where $\mathcal{T}(0) = \Gamma(0)$. Once $\mathcal{T}(n-1)$ has been constructed, we define the vertex set by $V(\mathcal{T}(n)) = V(\mathcal{T}(n-1)) \bigcup V_n$. For the edge set, we start with $E(\mathcal{T}(n-1))$, and add one (double) edge from $\Gamma(n)$ connecting each vertex in V_n with a vertex in V_{n-1}. We see that the length of Hu corresponds to the length of the unique path π in $\mathcal{T}$ from H to Hu. We now describe how to find the Schreier generators. The edge set of the graph Γ can be defined as the disjoint union of the tree edges and non-tree edges. Take any non-tree edge e with label x, and let π_1 be the path in $\mathcal{T}$ from H to $\alpha(e)$, and π_2 the path in $\mathcal{T}$ from H to $\omega(e)$. Set $u = lab(\pi_1), v = lab(\pi_2)$. Then the

Schreier generators are exactly the elements uxv^{-1}.

Note that our construction of a maximal tree $\mathcal{T}$ leads to a geodesic tree. That is, the length of each tree path to a vertex is minimal. However it is not a requirement when forming a Schreier basis that the transversal, or equivalently maximal tree, be geodesic.

In the proof of the Nielsen theorem, it is shown that every finitely generated subgroup of a free group is freely generated by some Nielsen-reduced set. The Schreier bases that arise in this section are also connected with Nielsen-reduced sets.

Theorem 3 ([7], Section 3.2)**.** *Any geodesic Schreier basis for a subgroup H of a free group F is Nielsen-reduced. Conversely, every Nielsen reduced set of generators for H is (up to inverses) a geodesic Schreier basis for H.* □

1.5 Lie Algebras

Lie algebras arise naturally as vector spaces of linear transformations, together with a new multiplication that is neither commutative nor associative. We begin our discussion of Lie algebras with their definition.

Definition 2. *A Lie algebra is a vector space L over a field F with multiplication defined via the Lie bracket $[\cdot,\cdot] : L \times L \to L$ which has the following properties:*

(L1) The bracket operation is bilinear;

(L2) $[x, x] = 0$ *for all* $x \in L$*;*

(L3) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, *for all* $x, y, z \in L$.

Condition $(L3)$ is called the **Jacobi identity**. From $(L1)$ and $(L2)$, it follows that the Lie bracket is anti-commutative:

$$0 = [x + y, x + y] = [x, y] + [y, x]$$

In a similar fashion, a Lie ring is an abelian group endowed with a bracket satisfying properties $(L1)$ through $(L3)$.

Lie algebras can naturally be obtained from associative algebras in the following way. Given an associative algebra A, and $x, y \in A$, we define the Lie bracket by $[x, y] = xy - yx$, and it is easy to verify that this definition satisfies $(L1)$ through $(L3)$. If a Lie algebra is obtained in such a way, we denote it as A_L. It was discussed in [5] that every Lie algebra is isomorphic to a subalgebra of some A_L, which is equivalent to showing that every Lie algebra is isomorphic to a Lie algebra of linear transformations.

1.6 Free Lie Algebras

Similar to the notion of a free group is a free Lie algebra. We give the definition of a free Lie algebra in terms of its universal property, and provide a construction. In particular, we show that free Lie algebras may be constructed from free groups.

Definition 3. *Let L be a Lie algebra generated by a set $X \subset L$. We say L is free on X if, given any map $f : X \to M$ into a Lie algebra M, there exists a unique homomorphism $\phi : L \to M$ such that $f = \phi \circ \iota$, where ι is the inclusion map.*

In a similar fashion, we define the free Lie ring by replacing, in the previous definition, each occurrence of the word "algebra" with "ring". Analogous to the Nielsen-Schreier theorem for free groups, we have the following:

Theorem 4 (Shirshov-Witt)**.** *Every Lie subalgebra of a free Lie algebra is itself free.*

□

In particular, bases of free Lie algebras arise as so-called basic commutators (Section 1.7). Next we demonstrate the relationship between Lie algebras and groups, as discussed in [1].

Let G be any group, and consider the lower central series:

$$G = G_1 \supseteq G_2 \supseteq G_3 \supseteq \dots$$

where $G_{i+1} = [G_i, G]$. Set $L_i = G_i/G_{i+1}$. Each L_i is abelian. Then the set $L(G) = \bigoplus_{i=0}^{\infty} L_i$ is an abelian group. Addition of homogeneous elements is supplied by the group operation:

$$xG_{i+1} + yG_{i+1} = (xy)G_{i+1},$$

where $x, y \in G_i$. We make $L(G)$ into a Lie ring by defining the Lie bracket, which we denote by $[.,.]_L$ to distinguish it from the group commutator:

$$[uG_{i+1}, vG_{j+1}]_L = [u, v]G_{i+j+1}$$

for $u \in G_i, v \in G_j$, where $[u, v]$ is the commutator of the group elements u and v. This definition makes sense since for terms G_i, G_j in the lower central series, $[G_i, G_j] \subseteq G_{i+j}$. The axioms of a Lie ring follow from the Witt-Hall identities, which will be discussed in Section 1.8. To turn a Lie ring into a Lie algebra, we need to allow for coefficients in some field K. We do this by forming the tensor product by K over $\mathbb{Z}$. Thus $L(X) = \bigoplus_{i=0}^{\infty} L_i \bigotimes K$ is a Lie algebra. To form the free Lie algebra, we take for G a free group F, and consider its lower central series. We discuss in section 1.8 that the quotients F_i/F_{i+1} are free abelian groups. Thus, if we know a free basis for each term F_i/F_{i+1}, then this determines a linear basis for a free Lie algebra. In the next section, we define a collection process which produces a free basis for each quotient F_i/F_{i+1}. These basis elements are known as basic commutators.

1.7 Commutator Collecting Process

The commutator collecting process was described by M. Hall [3]. It provides a way to write any group element as a product of generators and their higher commutators, arranged in a specific order.

Let $X = \{x_1, \ldots, x_r\}$ be a free basis for a group F. We define formal commutators c_j and weights $w(c_j)$ by the following rules:

- $c_i = x_i, i = 1, \ldots, r$, are the commutators of weight 1; that is, $w(x_i) = 1$;
- If c_i and c_j are commutators, and $i \neq j$, then $c_k = [c_i, c_j]$ is a commutator of weight $w(c_k) = w(c_i) + w(c_j)$.

The commutators $c_i = x_i$ of weight 1 are ordered by their subscripts $i = 1, \ldots, r$. We then assign an arbitrary ordering to commutators of higher weight, with the rule that $c_{i_1} < c_{i_2}$ if $w(c_{i_1}) < w(c_{i_2})$.

A string $c_{i_1} \ldots c_{i_m}$ of commutators is said to be in collected form if $i_1 \leq i_2 \leq \ldots \leq i_m$, so that the commutators are read in order from left to right. An arbitrary string of commutators $c_{i_1} \ldots c_{i_m} c_{i_{m+1}} \ldots c_{i_n}$ has a collected part $c_{i_1} \ldots c_{i_m}$ if $i_1 \leq i_2 \leq \ldots \leq i_m$, and $i_m \leq i_j$ for $j = m+1, \ldots, n$. The string will then have an uncollected part $c_{i_{m+1}} \ldots c_{i_n}$, where i_{m+1} is not the least of $i_j, j = m+1, \ldots, n$.

We now define a collecting process for a string of commutators. Suppose that c_u is in the uncollected part, and is of minimal weight in the uncollected part. Let $c_{i_j} = c_u$

be the left-most uncollected c_u. Then we replace

$$c_{i_1} \dots c_{i_m} \dots c_{i_{j-1}} c_{i_j} \dots c_{i_n}$$

by

$$c_{i_1} \dots c_{i_m} \dots c_{i_j} c_{i_{j-1}} [c_{i_{j-1}}, c_{i_j}] \dots c_{i_n}.$$

We have in effect moved c_{i_j} to the left, and introduced a new commutator $[c_{i_{j-1}}, c_{i_j}]$. This commutator is clearly of higher weight than that of c_{i_j}. Therefore, c_{i_j} still has minimal weight among the uncollected part. After some finite number of steps, c_{i_j} will be moved to the $(m+1)$st position, and will then join the collected part. We note that since each step introduces a new commutator, it is not guaranteed that this process will terminate.

During the collection process, only certain commutators arise. For example, $[x_2, x_1]$ may arise, but not $[x_1, x_2]$, since x_1 is collected before x_2. The commutators that might actually arise are known as <u>basic commutators</u>. They are defined inductively as follows:

- $c_i = x_i, i = 1, \dots, r$ are the basic commutators of weight 1;
- Suppose that basic commutators of weight $< n$ have been defined. Then the basic commutators of weight n are of the form $c_k = [c_i, c_j]$, where

  (a) c_i and c_j are basic and $w(c_i) + w(c_j) = n$

  (b) $c_i > c_j$, and if $c_i = [c_s, c_t]$, then $c_j \geq c_t$;
- The commutators of weight n follow those of weight less than n, and are ordered arbitrarily with respect to each other.

We discuss in Section 1.8 that the basic commutators of weight k form a free basis of the free abelian group F_k/F_{k+1}, where F_i denotes the i-th term of the lower central series.

1.8 The Basis Theorem

The structure of the quotients F_i/F_{i+1} of terms of the lower-central series of a free group F can be studied from the theory of Lie rings and formal power series. We start be discussing Hall's identities, which show a relationship between the commutation of group elements and the Lie bracket. We will then show how free groups may be mapped into free associative algebras. In doing so, we will need to allow for inverses of the form

$$(1-x)^{-1} = 1 + x - x^2 + x^3 - x^4 + \ldots + (-1)^{n-1}x^n + \ldots$$

The commutation and multiplication of group elements is closely related to the multiplication and addition of elements in a Lie algebra. P. Hall originally discussed the group theory of lower central series in 1933, and it was later expanded upon by Magnus.

Definition 4. *Let a, b be elements of a group G. Then we write*

$$a^b = b^{-1}ab, a^{-b} = b^{-1}a^{-1}b.$$

Theorem 5 ([7], Section 5.2)**.** *For any three elements a, b, c of a group G, the following equations hold:*

1. $[a, b][b, a] = 1$;

2. $[a, bc] = [a, c][a, b][[a, b], c]$;

3. $[ab, c] = [a, c][[a, c], b][b, c]$;

4. $[[a, b], c^a][[c, a]b^c][[b, c]a^b] = 1$;

5. $[[a, b], c][[b, c], a][[c, a], b] = [b, a][c, a][c, b]^a[a, b][a, c]^b[b, c]^a[a, c][c, a]^b$.

□

Theorem 6 ([7], Section 5.3)**.** *Let a, b, c be elements of a group G. Let k, m, n be positive integers such that $a \in G_k, b \in G_m, c \in G_n$, where G_i is the i-th term of the lower central series. Then*

1. $ab \equiv ba \mod G_{k+m}$;

2. $[a, bc] \equiv [a, b][a, c] \mod G_{k+m+n}$;

3. $[ab, c] \equiv [a, c][b, c] \mod G_{k+m+n}$;

4. $[a, b, c][b, c, a][c, a, b] \equiv 1 \mod G_{k+m+n+1}$.

□

This theorem shows a close connection between the properties of commutation of group elements and the axioms of a Lie ring. In fact, if the equations of Theorem 6 were replaced by equalities, rather than congruences, we would obtain a Lie ring, where the ring addition is supplied by group multiplication, and ring multiplication is the commutation of group elements.

Definition 5. *A free associative $\mathbb{Z}$-algebra on a set $X = \{x_1, \ldots, x_r\}$ is an algebra $A(r)$ with a map $\iota : X \to A(r)$ such that for any associative $\mathbb{Z}$-algebra R and a map $f : X \to R$, there exists a unique homomorphism $\phi : A(r) \to R$ such that $f = \phi \circ \iota$.*

The free associative $\mathbb{Z}$-algebra on a set $X = \{x_1, \ldots, x_r\}$ may be described as the $\mathbb{Z}$-algebra for which the monomials $x_{n_1}^{e_1} x_{n_2}^{e_2} \ldots x_{n_k}^{e_k}$, (where $n_j \neq n_{j+1}$, and $n_1, n_2, \ldots, n_k \in \{1, \ldots, r\}$, and $k, e_1, e_2, \ldots, e_k$ are positive integers) form a $\mathbb{Z}$-linear basis.

From the free associative algebra $A = A(X)$, we may form the algebra $\hat{A}$ by admitting infinite sums. The elements of $\hat{A}$ appear as formal power series in the non-commuting variables $x_1, \ldots, x_r$:

$$v = \sum_{n=0}^{\infty} u_n$$

where u_n is a homogeneous element of degree n belonging to $A(r)$. Addition and multiplication in $\hat{A}$ are defined in the natural way; multiplication is well-defined since in the product of two infinite sums, only finitely many terms in each sum will contribute to a component of a given degree.

Let $\Lambda(r)$ be the free Lie algebra of rank r, generated by $\varepsilon_1, \ldots, \varepsilon_r$.

Proposition 7 ([7], Section 5.6)**.** *There exists a mapping μ of the free Lie algebra $\Lambda(r)$ on free generators ε_p into the free associative algebra $A(r)$ on free generators x_p with the following properties:*

1. *Every element $\phi \in \Lambda(r)$ has exactly one image $\mu(\phi)$ in $A(r)$;*

2. *$\mu(\varepsilon_p) = x_p, (p = 1, 2, \ldots, r)$;*

3. *If ϕ, ψ are in $\Lambda(r)$ and if $\alpha \in \mathbb{Z}$, then*

$$\mu(\alpha\phi) = \alpha\mu(\phi)$$

$$\mu(\phi + \psi) = \mu(\phi) + \mu(\psi)$$

$$\mu(\phi \circ \psi) = [\mu(\phi), \mu(\psi)]$$

□

Definition 6. *An element of $A(r)$ that is the image of some element $\Lambda(r)$ under the mapping μ is called a Lie element of $A(r)$.*

Let $M \subseteq \hat{A}$ be the ideal consisting of all power series with zero constant term. We now observe how group structures arise from these formal power series.

Proposition 8 ([7], Section 5.5)**.** *Let $G = 1 + M \subseteq \hat{A}$ consist of all formal power series with constant term 1. Then G is a group under multiplication. Moreover, if $g = 1 + h$, then*

$$g^{-1} = 1 - h + h^2 - h^3 + \ldots + (-1)^n h^n + \ldots$$

□

Theorem 7 ([7], Section 5.5)**.** *If A is freely generated by $x_1, \ldots, x_r$, then the elements*

$$a_p = 1 + x_p, \quad p = 1, \ldots, r$$

of $\hat{A}$ are free generators of a free group F of rank r. Moreover,

$$a_p^{-1} = 1 - x_p + x_p^2 - x_p^3 + \ldots + (-1)^n x_p^n + \ldots$$

□

Theorem 8 (Basis Theorem [7], Section 5.6)**.** *In A, the free associative algebra of rank at least 2, there exists a sequence $z_1, z_2, \ldots$ of homogeneous Lie elements with non-decreasing degrees in the free generators of A such that*

(i) The elements z_v form a linear basis (over $\mathbb{Z}$) for the Lie elements of A.

(ii) The products

$$z_{v_1}^{e_1} z_{v_2}^{e_2} \dots z_{v_k}^{e_k}, \quad 1 \leq v_1 < v_2 < \dots < v_k, \quad k \geq 1$$

with integral exponents $e_1, \dots, e_k$*, together with the identity* 1 *form a linear basis for all elements of* A

(iii) All Lie elements z_v *of degree* ≥ 2 *in the free generators of* A *can be written in the form*

$$z_v = [z_\lambda, z_\mu], \quad 1 \leq \mu < \lambda$$

(iv) Morever, every element z_v *of degree* ≥ 2 *can be written in the form above in such a way that for each* τ *with* $\mu \leq \tau < \lambda$*, the element*

$$[[z_\lambda, z_\mu], z_\tau] = [z_v, z_\tau]$$

occurs in the sequence $z_1, z_2, z_3, \dots$ *We call the elements* $z_1, z_2, z_3, \dots$ *basic Lie elements*

□

Theorem 9 ([7], Section 5.7)**.** *The quotient groups* $F_n/F_{n+1}, n = 1, 2, 3, \dots,$ *of the lower central series of a free group* F *freely generated by* $a_1, \dots, a_r$ *are isomorphic, as abelian groups, to the submodules* L_n *of homogeneous elements of degree* n *in the Lie algebra* L *freely generated by* $\xi_1, \dots, \xi_r$ □

Chapter 2

Growth of Bases for Commutator Subgroups

2.1 Schreier-Type Formulas

Given a free group F of rank r, and a subgroup H of finite index $[F : H]$, the Schreier formula allows us to find the rank of H as follows:

$$\text{rank}H = (\text{rank}F - 1)[F : H] + 1.$$

However, given a free group F of rank at least 2, there always exists a non-trivial subgroup of infinite index. Let x and y be distinct elements in a free basis for F. Let $U = \{x^k y x^{-k} : k \in \mathbb{Z}\}$. This set is clearly Nielsen-reduced, and is a free basis of infinite rank for the group it generates. Therefore, we consider an infinite basis B, along with the subsets $B_n = \{b \in B : \ell(b) = n\}$, with respect to some length function ℓ. If each B_n is finite, then we may form the Hilbert series, which is a formal power series of the form:

$$H(B,t) = \sum_{n=1}^{\infty} |B_n| t^n$$

If rank$H$ is finite, then it follows that rank$H = \mathcal{H}(B,1)$.

Let Γ be the directed coset graph discussed in section 1.4, with vertex set $V = F/H$. We recall that the set V_n consists of those cosets of minimal distance n from the origin. So we may form a Hilbert series for the quotient F/H as follows:

$$\mathcal{H}(F/H,t) = \sum_{n=1}^{\infty} |V_n| t^n.$$

We say that a subgroup is <u>even</u> if it is generated by elements of even length. In the event that H is an even subgroup, $r = \text{rank}F$, and B is a geodesic Schreier basis, it was shown in [2] that $\mathcal{H}(B,t)$ and $\mathcal{H}(F/H,t)$ are related by the following formula:

$$\mathcal{H}(B,t) = 2\left(\frac{2rt^2}{t^2+1} - 1\right)\mathcal{H}(F/H,t^2) + 2. \tag{2.1}$$

Thus the Hilbert series for a geodesic Schreier basis B can be obtained if we know the Hilbert series for the quotient F/H. This formula holds independent of choice of geodesic Schreier transversal.

An analogue of the Schreier formula exists in the case of Lie subalgebras of free Lie algebras [9]. A <u>finitely graded set</u> is a countable set X, equipped with a weight function $wt : X \to \mathbb{N}$ such that the subsets $X_i = \{x \in X : wt(x) = i\}$ are finite for all $i \in \mathbb{N}$. For any monomial $y = x_{i_1} \ldots x_{i_n}$, $x_j \in X$, we set $wt(y) = wt(x_{i_1}) + \ldots + wt(x_{i_n})$. Let Y be the set of all monomials generated by X, which is also finitely graded. Let A be an algebra generated by a finitely graded set X. Then A has a filtration $\bigcup_{i=1}^{\infty} A^i$, where A^i is the space spanned by monomials of length at most i. If A is freely generated by X, then define

$$\mathcal{H}_X(A,t) = \mathcal{H}(Y,t) = \sum_{t=1}^{\infty} |Y_i| t^i.$$

If B is a vector subspace of A, then the quotient space A/B acquires a filtration:

$$(A/B)^n = (A^n + B)/B \cong A^n/(B \cap A^n).$$

In [9], Petrogradsky defines on operator ε on $\mathbb{Z}[[t]]$, the ring of formal power series in the indeterminate t over $\mathbb{Z}$:

$$\varepsilon : \sum_{i=0}^{\infty} a_i t^i \mapsto \prod_{i=0}^{\infty} \frac{1}{(1-t^i)^{a_i}}.$$

Petrogradsky then showed that for any Lie algebra L generated by a finitely graded set X, and any Lie subalgebra K, there exists a free generating set Z such that:

$$\mathcal{H}(Z) = (\mathcal{H}(X) - 1)\, \varepsilon\, (\mathcal{H}(L/K)) + 1.$$

In particular, Petrogradsky showed that the Hilbert series $\mathcal{H}(Z)$ does not depend on the choice of homogeneous Z. However, in the case of groups, we consider specific bases which give rise to non-equivalent growth functions, and thus distinct Hilbert series.

2.2 Growth of Bases

Let F be a free group with basis X, and let B be a free basis of a free subgroup. Set $B_n = \{b \in B : \ell_X(b) = n\}$, where $\ell_X(b)$ denotes the length of the reduced word over X. We define the cumulative growth function of a basis B by $\gamma(n) = \sum_{i=1}^{n} |B_i|$. The strict growth function of the basis B is given by $\lambda(n) = \gamma(n) - \gamma(n-1) = |B_n|$. Given two functions, f_1 and f_2, we may define $f_1 \preceq f_2$ if there exist positive constants c_1, d_1 such that $f_1(n) \preceq c_1 f_2(d_1 n)$ for all $n > 0$. Functions f_1 and f_2 are equivalent, written as $f_1 \sim f_2$, if $f_1 \preceq f_2$ and $f_2 \preceq f_1$.

Thanks to Dr. Bahturin and Dr. Olshanskii, we have the following two propositions:

Proposition 9. *Let T_1 and T_2 be distinct geodesic Schreier transversals for a subgroup H of a free group F of rank r. Let $\lambda_{T_1}(n)$ and $\lambda_{T_2}(n)$ be the growth functions of the Schreier bases corresponding to T_1 and T_2. Then $\lambda_{T_1}(n) = \lambda_{T_2}(n)$, for all $n \in \mathbb{N}$.*

Proof. We will show that the growth function is independent of choice of geodesic Schreier transversal. Let $\Gamma = \Gamma(F/H)$ be the coset graph, as described in Section 1.4. Let T be any geodesic maximal sub-tree. Let $S_n = \{Hv \in F/H : \ell(Hv) = n\}$ be the sphere of radius n, where $\ell(Hv)$ is the minimal distance from H to Hv. The cardinality $|S_n|$ is independent of choice of maximal tree. The Schreier generators that appear are in one-to-one correspondence with the non-tree edges. For each edge $e = (Hv, Hvx), x \in X$, there exists some $n \in \mathbb{N}$ such that the initial point $\alpha(e) = Hv \in S_n$, and the terminal point $\omega(e) \in S_n \bigcup S_{n+1}$. Note that we do not include the edges with terminal point $\omega(e) \in S_{n-1}$, since this would give rise to an inverse of a Schreier generator already described. Furthermore, $\omega(e)$ cannot be in S_m, $m > n+1$, since this would result in a a vertex $Hw \in S_m, m > n+1$ being reached by a path of length $n+1$, contrary to the definition of S_m. The Schreier generator is formed as the label of the tree path from H to Hv, times the label of the non-tree edge, times the label of the tree path from Hvx to H. If $\omega(e) \in S_{n+1}$, then the length of the Schreier generator is $2n+2$. The number of tree edges from S_n to S_{n+1} is $|S_{n+1}|$. Therefore, the Schreier generators of length $2n+2$ are independent of choice of geodesic tree. An edge with both the initial and terminal vertices in S_n clearly is not an edge of a geodesic tree. So the number of resulting Schreier generators does not depend on the choice of maximal tree. Since the number of Schreier generators of a given length is independent of choice of geodesic tree, then it is independent of choice of geodesic Schreier transversal. Therefore, it follows that the strict growth functions corresponding to any two geodesic Schreier transversals are the same.

□

Proposition 10. *Let T_0 be a geodesic Schreier transversal, and T a non-geodesic Schreier transveral for a subgroup H of a free group F. Let $\gamma_{T_0}(n)$ and $\gamma_T(n)$ be the growth functions of the Schreier bases corresponding to T_0 and T. Then for some $N' \in \mathbb{N}, \gamma_{T_0}(n) > \gamma_T(n)$, for all $n \geq N'$.*

Proof. If T is geodesic, then from the previous proposition we have that $\lambda_{T_0} = \lambda_T$. If T is non-geodesic, then there exists a vertex $Hv \in S_N$ such that the length of the tree-path from H to Hv is $m > N$. We consider a non-tree edge $e = (Hv, Hvx)$ with label $x \in X$, where $Hvx \in S_N \bigcup S_{N+1}$; again, the edges with terminal point $Hvx \in S_{N-1}$ give rise to the inverse of a Schreier generator. The Schreier generator that results is the label of the tree path from H to Hv, times the label x of the non-tree edge, times the label of the tree path from Hvx back to H. The length is at least $m+N+2 > 2N+2$, or at least $m+N+1 > 2N+1$. So the number of Schreier generators of length $2N+2$ or $2N+1$ will be less than in the case of maximal geodesic tree T_0. Since in the case of a geodesic tree, the number of generators of length $2N+2$ and $2N+1$ is maximal, it follows that $\gamma_{T_0}(n) > \gamma_T(n)$ for all $n \geq 2N+1 = N'$.

□

Let H be a subgroup of F and W a Schreier basis for H. Let $H(n)$ be the subgroup of H generated by all elements of length at most n with respect to X, and let $r_H(n)$ denote the cardinality of the minimum generating set of $H(n)$. Let

$W(n) = \{w \in W : \ell_X(w) \leq n\}$.

Proposition 11 ([8]). *Let H be a subgroup of a free group F. Then $r_H(n) = |W(n)|$.* □

Furthermore, $r_H(n)$ is closely related to the geodesic Schreier transversal, where H is a normal subgroup. Let $\gamma_{F/H}(n)$ be the growth function of F/H, defined by $\gamma_{F/H}(n) = |\{gH \in FH : \ell(gH) \leq n\}|$. The length $\ell(gH)$ of a coset is given as the minimal path length from the origin H to the vertex gH in the coset graph $\Gamma(F/H)$. Then we have the following:

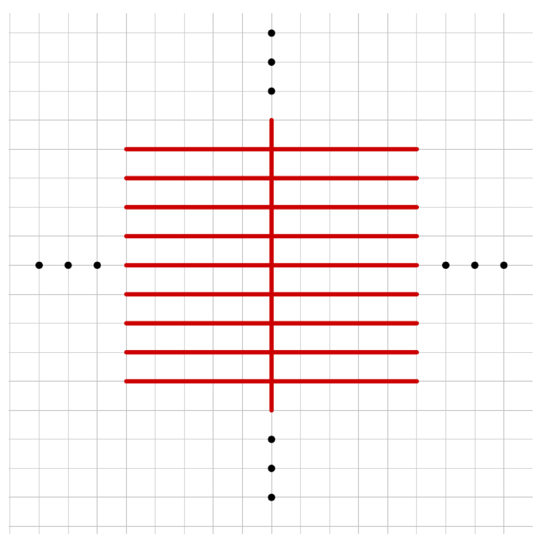
Theorem 10 ([8]). *Let H be a normal subgroup of a free group F. Then $r_H \sim \gamma_{F/H}$* □

Our goal in the sections that follow is to discuss the growth functions for different bases of the commutator subgroup. Of the different bases that we consider for the commutator subgroup, the geodesic Schreier bases appear to be "the fastest".

2.3 Geodesic Schreier Basis for F_2'

Let F be a free group of rank r. It is well-known that $F/F' \cong \mathbb{Z}^r$, where F' is the commutator subgroup of F. In the case where we have rank 2, the quotient F/F' can be viewed as the set of integral points in the real plane. We start by considering a directed coset graph Γ with vertex set $V = F/F'$, and labels of edges given by $X = \{x, y\}$. We have two methods for constructing the Hilbert series for the Schreier basis: either find the Hilbert series directly, or use the formula for even subgroups given above. Both of these methods have been discussed by Shaqaqha [10]. Following his method, we compute the Hilbert series for the symmetric basis (consisting of a free basis and their inverses); in later chapters, we compute Hilbert series only for free bases, not symmetric bases. We divide the result by 2 to obtain a Hilbert series for the free basis. As we will see, the method of finding the Hilbert series using the formula for even subgroups is easier than finding it directly.

We start with the Hilbert series for the quotient F/F'. We associate with every coset $Hy^l x^k$ the ordered pair (k,l). Given a vertex (k,l), we may take an edge e with label y, respectively y^{-1}, with initial point $\alpha(e) = (k,l)$, and terminal point $\omega(e) = (k, l+1)$, respectively $\omega(e) = (k, l-1)$. In a similar way, we may take edges with label x or x^{-1}. Since F_2/F_2' is the free abelian group of rank 2, then one possible minimal path π from the origin $(0,0)$ to a vertex (k,l) has label $lab(\pi) = y^l x^k$. So we may take as geodesic Schreier transversal $T = \{y^l x^k : k, l \in \mathbb{Z}\}$. The sphere of radius n, denoted by S_n consists of all points (k,l) such that $|k| + |l| = n$. For $n \geq 1$, S_n is the rhombus of size $|S_n| = 4n$, and $|S_0| = 1$.



We may now form the Hilbert series:

$$\mathcal{H}(F_2/F_2', t) = 1 + \sum_{n=1}^{\infty} 4nt^n = \frac{(1+t)^2}{(1-t)^2}.$$

Plugging this into equation 2.1 for even subgroups, we obtain:

$$\mathcal{H}(B,t) = 2\left(\frac{4t^2}{t^2+1} - 1\right)\frac{(1+t^2)^2}{(1-t^2)^2} + 2 = \frac{8t^4}{(1-t^2)^2}.$$

We now confirm this formula by calculating $\mathcal{H}(B,t)$ directly. Given our coset graph Γ, and geodesic Schreier transversal $T = \{y^l x^k : k, l \in \mathbb{Z}\}$, we consider a maximal subtree $\mathcal{T}$ with "trunk" the vertical line $k = 0$, and we have as branches the horizontal lines $l = c$. For every point (k,l) with $k \neq 0$, there are two vertical non-tree edges with labels y and y^{-1}. These edges go from (k,l) to $(k,l+1)$ and $(k,l-1)$. We may then construct two loops λ_1 and λ_{-1}. The loop λ_1 starts at the origin $(0,0)$, follows the tree path (with label $y^l x^k$) to the point (k,l), takes the non-tree edge (with label y) to the point $(k,l+1)$, and follows the tree path (with label $x^{-k}y^{-(l+1)}$) back to the origin. Then we have that $lab(\lambda_1) = y^l x^k y x^{-k} y^{-(l+1)}$. Similarly, if λ_{-1} takes as non-tree edge labelled by y^{-1}, then it has label $lab(\lambda_2) = y^l x^k y^{-1} x^{-k} y^{-(l-1)}$.

Thus for every vertex (k,l), where $k, l \neq 0$, we get one Schreier generator of length $2|k| + 2|l| + 2$, and one of length $2|k| + 2|l|$. If $k \neq 0, l = 0$, then both generators have length $2|k| + 2|l| + 2 = 2|l| + 2$. It follows that all, except four, points on the rhombus $|k| + |l| = n$ ($4n - 4$ in total) produce $4n - 4$ generators of length $2n + 2$, and $4n - 4$ generators of length $2n$. The vertices $(n,0)$ and $(-n,0)$ together produce 4 generators of length $2n + 2$. The vertices $(0,n)$ and $(0,-n)$ do not produce Schreier generators, since there are no non-tree edges to follow. Therefore, the number of free generators of length $2n$ is $4(n-1) + (4n-4) = 8n - 8$. Thus we may form the Hilbert series

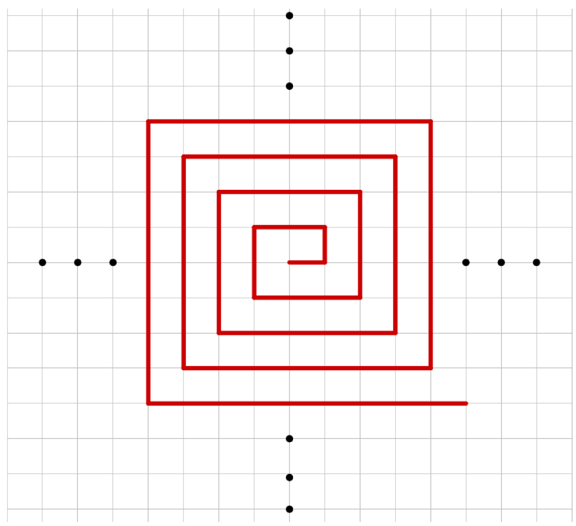
$$\mathcal{H}(B,t) = \sum_{n=1}^{\infty}(8n-8)t^{2n} = \frac{8t^4}{(1-t^2)^2},$$

which agrees with the Hilbert series obtained above. The strict growth function of this basis is given by $\lambda(2n) = 8(n-1), \lambda(2n+1) = 0$. Thus $\lambda(2n) \sim n$, and this geodesic

Schreier basis has linear growth. However this is the growth for the symmetric Schreier basis. For the Schreier free basis, we divide by two to obtain $\lambda(2n) = 4(n-1)$. Using the strict growth function and the fact that F' is an even subgroup, the cumulative growth function is $\gamma(2n) = \gamma(2n+1) = 2n^2 - 2n$.

2.4 Non-Geodesic Schreier Basis

There are many choices for the Schreier transversal; in fact, it need not even be geodesic as in the previous section. For a maximal tree, we may take the following spiral:



Unfortunately for us, we may not use the formula for Hilbert series of even subgroups discussed in [2] since the transversal is not geodesic. We must find the Hilbert series directly.

Geometrically, it makes sense to partition the vertices into the sets: upper left corners (UL), upper right corners (UR), lower left corners (LL), lower right corners (LR), interior left (IL), interior top (IT), interior right (IR), and interior bottom (IB). We give formal definitions:

1. $UL = \{Hx^k y^l : l > 0, k = -l\}$

2. $UR = \{Hx^k y^l : k = l > 0\}$

3. $LL = \{Hx^k y^l : k = l < 0\}$

4. $LR = \{Hx^k y^l : k > 0, k = 1 - l\}$

5. $IL = \{Hx^k y^l : k < 0, 0 < |l| < k\}$

6. $IT = \{Hx^k y^l : l > 0, 0 < |k| < l\}$

7. $IR = \{Hx^k y^l : k > 0, -(k-1) < l < k\}$

8. $IB = \{Hx^k y^l : l < 0, l < k < 1 - l\}$

A vertex $Hx^k y^l$ is on level n if:

1. $Hx^k y^l$ is not an interior-bottom vertex and $max\{|k|, |l|\} = n$, or if

2. $Hx^k y^l$ is an interior-bottom vertex and $-l = n - 1$

With some simple inductive arguments, given a corner vertex $Hx^k y^l$ on level n, we may find the length of the path, following the spiral, from the origin to $Hx^k y^l$. For example, the distance to the lower left corner vertex on level n is $2n(2n+1)$. Clearly this holds for level 1. Suppose it holds for level n (we say the origin is on level 0). The length of the bottom and the right sides of level n is given by $2n - 1$, while the length of the top and left sides is given by $2n$. So the distance from the origin to the lower left vertex on level $n+1$ is the distance to the lower left vertex on level n, plus the sum of the lengths of the sides on level $n+1$. The result is

$$2n(2n+1) + 2(2(n+1) - 1) + 2(2(n+1)) = 2(n+1)((2(n+1) + 1)$$

Corner	Length
UL	$(2n)^2$
LL	$2n(2n+1)$
UR	$2n(2n-1)$
LR	$(2n-1)^2$

Now given an interior vertex, it is easy to count the length of its transversal counterpart by counting backwards / forwards from the corners. It is easy to see that every non-trivial vertex of our graph allows two non-tree edges. Given a vertex Hx^ky^l on level n and non-tree edge e, we say that the Schreier generator is outer if $\omega(e)$ is on a level $> n$, and inner otherwise. Clearly, every outer Schreier generator is the inverse of some inner Schreier generator. So we need only determine lengths of the inner Schreier generators. Also, the corner vertices give rise only to outer Schreier generators. So we need only determine lengths of inner Schreier generators formed from the interior vertices.

First, we need to determine the length, following the spiral, to any vertex Hx^ky^l. We describe this process for the interior-bottom vertices, since the other vertices follow similarly. Let Hx^ky^l be an interior-bottom vertex on level n. We start by counting the distance to the lower left corner vertex on level $n-1$, which from the table above is $2(n-1)(2(n-1)+1) = 4n^2 - 6n + 2$. The x-coordinate of the lower-left corner is $-(n-1)$. So the distance from the lower-left corner to Hx^ky^l is $k+(n-1)$. Therefore, the length following the spiral from the origin to Hx^ky^l is given by:

$$4n^2 - 6n + 2 + k + (n-1) = 4n^2 - 5n + 1 + k$$

Furthermore, the range for the x-coordinate is $-(n-1) < k < n$. In a similar fashion, we determine lengths for the other 3 types of interior vertices. The results are summarized in the table below.

Position	Length	Range
IB	$4n^2 - 5n + 1 + k$	$-(n-1) < k < n$
IR	$4n^2 - 3n + l$	$-(n-1) < l < n$
IT	$4n^2 - n - k$	$-n < k < n$
IL	$4n^2 + n - l$	$-n < l < n$

Notice that these formulas for interior lengths agree on the corners of the spiral. Furthermore, given an interior bottom vertex Hx^ky^l on level n, and an inner Schreier generator formed by an edge e, the terminal point $\omega(e) = Hx^ky^{l+1}$ will be on level $n-1$ and either an interior-bottom vertex, or lower left corner or lower right corner vertex. Therefore, we may determine the length of the resulting Schreier generator from our formulas. It is the sum of the length of the tree path to the vertex Hx^ky^l, plus 1 for the non-tree edge, plus the length from the vertex Hx^ky^{l+1} to the origin. The result is

$$
\begin{aligned}
&(4n^2 - 5n + 1 + k) + 1 + (4(n-1)^2 - 5(n-1) + 1 + k) \\
=&8n^2 - 18n + 12 + 2k
\end{aligned}
$$

The lengths of Schreier generators formed from the other interior edges are determined in a similar fashion. The lengths of their respective inner Schreier generators are summarized in the following table:

Position	Length	Range
IB	$8n^2 - 18n + 12 + 2k$	$-(n-1) < k < n$
IR	$8n^2 - 14n + 8 + 2l$	$-(n-1) < l < n$
IT	$8n^2 - 10n + 6 - 2k$	$-n < k < n$
IL	$8n^2 - 6n + 4 - 2l$	$-n < l < n$

Looking at any row of the above table, we see that the lengths of Schreier generators increase by 2 as k or l increases by 1. In other words, as we traverse the spiral, the lengths of inner Schreier generators increases by 2. This pattern still holds as we transition from interior bottom points to interior right points, from bottom vertices to right vertices, from right vertices to top vertices, and from top vertices to left vertices. For example, the longest interior bottom Schreier generator on round n is of length $8n^2 - 18n + 12 + 2(n-1) = 8n^2 - 16n + 10$. The shortest inner right generator is of length $8n^2 - 14n + 8 + 2(-(n-2)) = 8n^2 - 16n + 12$. So clearly as we transition from inner bottom to inner right generators, the pattern of increasing by 2 still holds.

It follows that for each $n \geq 2$, there is exactly one Schreier generator of length $2n$. Therefore, the strict growth function is given by $\lambda(2n) = 1, n \geq 2$. The cumulative growth function is then $\gamma(2n) = \gamma(2n+1) = n$.

2.5 Ward's Basis

In [4], the authors expand on the basic commutators that arise from the collection process to form a basis for each term of the lower central series. In particular, if we consider a free group F of rank r, then its commutator subgroup F' is generated by all elements of the form $[b_0, b_1^{\beta_1}, \ldots, b_q^{\beta_q}]$ where each b_i is in a free generating set X for F, $\beta_i = \pm 1$, $b_0 > b_1 \leq b_2 \leq \ldots \leq b_q$, and $b_i = b_j \Rightarrow \beta_i = \beta_j$.

Suppose that $X = \{x, y\}$ is a basis for F, and ordered such that $x < y$. Then the generating set for F' splits into six general cases:

(1) $w_k = [y, \underbrace{x, \ldots, x}_{k}]$

(2) $u_k = [y, \underbrace{x^{-1}, \ldots, x^{-1}}_{k}]$

(3) $v_{k,l} = [w_k, \underbrace{y, \ldots, y}_{l}]$

(4) $t_{k,l} = [w_k, \underbrace{y^{-1}, \ldots, y^{-1}}_{l}]$

(5) $z_{k,l} = [u_k, \underbrace{y, \ldots, y}_{l}]$

(6) $p_{k,l} = [u_k, \underbrace{y^{-1}, \ldots, y^{-1}}_{l}]$

There are two notions of "length" that we may consider in this case. First, we may consider the lengths of the reduced words representing each basic commutator. As we will see, construction of the growth function in this way is quite messy. Alternatively, one may consider the notion of a weight. The set of <u>invertators</u> over a generating set X is the closure of X under the operation $x, y \to [x^{\pm 1}, y^{\pm 1}]$. A <u>weight</u> wt_X is defined on the set of inverators of X such that $wt_X(x) = 1$ for any $x \in X$, and $wt_X([x^{\pm 1}, y^{\pm 1}]) = wt_X(x) + wt_X(y)$. There is some ambiguity in this definition. For example, $wt_x([x, x]) = 2$, while $wt_X([x, x, x]) = 3$, even though $[x, x] = [x, x, x]$. So we say that the weight is a function of the representation of an invertator, and not necessarily of the group element itself.

2.5.1 Characterization by Weight

We wish to know for each $n \in \mathbb{N}$, how many basic commutators have weight n. We have that $wt([y, \underbrace{x^{\beta_1}, \ldots, x^{\beta_1}}_{k}, \underbrace{y^{\beta_2}, \ldots, y^{\beta_2}}_{l}]) = k + l + 1$.

First we see that $k \in \{1, \ldots, n-1\}$, and $\beta_i \in \{1, -1\}$. If $k \in \{1, \ldots, n-2\}$, then $l = n-1-k$. Four possibilities then arise for our exponents β_i, leading to $4(n-2)$ options. If $k = n-1$, then $l = 0$, and $\beta_1 \in \{1, -1\}$, so we have only two possibilities in this case. In total, there are $4(n-2)+2 = 4n-6$ ways to form a basic commutator of degree $n \geq 3$. Thus $\lambda(n) = 4n-6, n \geq 3$. The cumulative growth function is given as $\gamma(n) = 2 + \sum_{i=3}^{n}(4i-6) = 2(n-1)^2$, for $n \geq 2$.

2.5.2 Characterization by Length

We now must determine the lengths over $\{x, y\}$ of the reduced words representing each of these basic commutators. We provide formulas for w_k and $v_{k,l}$, and from there it will be obvious that $\ell(w_k) = \ell(u_k)$, and $\ell(v_{k,l}) = \ell(t_{k,l}) = \ell(z_{k,l}) = \ell(p_{k,l})$. First we determine $\ell(w_k)$.

Observe that:

$$
\begin{aligned}
w_1 &= [y, x] = y^{-1}x^{-1}yx \\
w_2 &= [w_1, x] = [y^{-1}x^{-1}yx, x] = x^{-1}\underbrace{y^{-1}xyx^{-1}y^{-1}x^{-1}y}_{\overline{w_2}}x^2 \\
&= x^{-1}\overline{w_2}x^2 \\
w_3 &= [w_2, x] = [x^{-1}\overline{w_2}x^2, x] = x^{-2}\underbrace{\overline{w_2}^{-1}(xx^{-1})x^{-1}\overline{w_2}}_{\overline{w_3}}x^3 \\
&= x^{-2}\overline{w_3}x^3 \\
w_4 &= [w_3, x] = [x^{-2}\overline{w_3}x^3, x] = x^{-3}\underbrace{\overline{w_3}^{-1}(x^2x^{-2})x^{-1}\overline{w_3}}_{\overline{w_4}}x^4 \\
&= x^{-3}\overline{w_4}x^4
\end{aligned}
$$

where each $\overline{w_i}$ is conjugated by some power of y. So then we have that:

$$\ell(w_1) = 4$$

$$\ell(w_2) = 2\ell(w_1) + 2 = 5 \cdot 2$$

$$\ell(w_3) = 2\ell(w_2) = 5 \cdot 2^2$$

$$\ell(w_4) = 2\ell(w_3) - 2 = 5 \cdot 2^3 - 2$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$\ell(w_k) = 2\ell(w_{k-1}) - 2(k-3) = 5 \cdot 2^{k-1} - \sum_{i=1}^{k-3} i2^{k-2-i}$$

Solving this recurrence relation yields

$$\ell(w_k) = 2^{k+1} + 2k - 2, k \geq 3$$

Finding the length $\ell(v_{k,l})$ is more tedious. We start by finding a formula for $\ell(v_{1,l})$, and then generalize to $\ell(v_{k,l})$. First, observe that:

$$
\begin{aligned}
v_{1,1} &= [w_1, y] = [y^{-1}x^{-1}yx, y] = \underbrace{x^{-1}y^{-1}x(yy^{-1})y^{-1}x^{-1}yx}_{\overline{v_{1,1}}}\,y \\
&= \overline{v_{1,1}}y \\
v_{1,2} &= [v_{1,1}, y] = [\overline{v_{1,1}}y, y] = y^{-1}\,\underbrace{\overline{v_{1,1}}^{-1}y^{-1}\overline{v_{1,1}}}_{\overline{v_{1,2}}}\,y^2 \\
&= y^{-1}\overline{v_{1,2}}y^2 \\
v_{1,3} &= [v_{1,2}, y] = [y^{-1}\overline{v_{1,2}}y^2, y] = y^{-2}\,\underbrace{\overline{v_{1,2}}^{-1}(yy^{-1})y^{-1}\overline{v_{1,2}}}_{\overline{v_{1,3}}}\,y^3 \\
&= y^{-2}\overline{v_{1,3}}y^3 \\
v_{1,4} &= [v_{1,3}, y] = [y^{-2}\overline{v_{1,3}}y^3, y] = y^{-3}\,\underbrace{\overline{v_{1,3}}^{-1}(y^2y^{-2})y^{-1}\overline{v_{1,3}}}_{\overline{v_{1,4}}}\,y^4 \\
&= y^{-3}\overline{v_{1,4}}y^4 \\
&\cdot \\
&\cdot \\
&\cdot \\
v_{1,l} &= y^{-(l-1)}\overline{v_{1,l}}y^l
\end{aligned}
$$

Notice that each $v_{1,i}$ begins with x^{-1} and ends with x. Since the $v_{1,i}$ are conjugated by powers of y, then the only cancellations we need to consider are those inside of the $v_{1,l}$. As we see again, the pattern does not arise until $l \geq 3$. Two cancellations occur in $v_{1,1}$, none in $v_{1,2}$, two in $v_{1,3}$, four in $v_{1,4}$, and $2 \cdot (l-2)$ cancellations in $v_{1,l}, l \geq 4$. Again, we list the lengths of each of these commutators:

$$
\begin{aligned}
\ell(v_{1,1}) &= 8 \\
\ell(v_{1,2}) &= 2\ell(v_{1,1}) + 2 = 9 \cdot 2 \\
\ell(v_{1,3}) &= 2\ell(v_{1,2}) = 9 \cdot 2^2 \\
\ell(v_{1,4}) &= 2\ell(v_{1,3}) - 2(1) = 9 \cdot 2^3 - 2^1(1) \\
\ell(v_{1,5}) &= 2\ell(v_{1,4}) - 2(2) = 9 \cdot 2^4 - 2^2(1) - 2^1(2) \\
&. \\
&. \\
&. \\
\ell(v_{1,l}) &= 2\ell(v_{1,l-1}) - 2(l-2) = 9 \cdot 2^{l-1} - \sum_{i=1}^{l-3} i \cdot 2^{l-2-i}
\end{aligned}
$$

Solving this recurrence relation yields

$$\ell(v_{1,l}) = 2^{l+2} + 2l - 2, l \geq 3$$

Next we find the degree function for $v_{k,l}$ where $k \geq 2$. Again, we see that our pattern starts at $l = 3$.

$$
\begin{aligned}
v_{k,1} &= [w_k, y] = \underbrace{w_k^{-1}y^{-1}w_k}_{\overline{v_{k,1}}}y \\
&= \overline{v_{k,1}}y \\
v_{k,2} &= [v_{k,1}, y] = [\overline{v_{k,1}}, y] = y^{-1}\underbrace{\overline{v_{k,1}}^{-1}}_{\overline{v_{k,2}}}y^{-1}\overline{v_{k,1}}y^2 \\
&= y^{-1}\overline{v_{k,2}}y^2 \\
v_{k,3} &= [v_{k,2}, y] = [y^{-1}\overline{v_{k,2}}y^2, y] = y^{-2}\overline{v_{k,2}}^{-1}(yy^{-1})y^{-1}\overline{v_{k,2}}y^3 \\
&= y^{-2}\overline{v_{k,3}}y^3 \\
v_{k,4} &= [y^{-2}\overline{v_{k,3}}y^3, y] = y^{-3}\underbrace{\overline{v_{k,3}}^{-1}(y^2y^{-2})y^{-1}\overline{v_{k,3}}}_{\overline{v_{k,4}}}y^4 \\
&= y^{-3}\overline{v_{k,4}}y^4 \\
&. \\
&. \\
&. \\
v_{k,l} &= y^{-(l-1)}\overline{v_{k,l}}y^l
\end{aligned}
$$

As before, each $v_{k,l}$ begins with x^{-1} and ends with x. The pattern for counting cancellations arises for $l \geq 4$. We have the degrees:

$$
\begin{aligned}
\ell(v_{k,1}) &= 2\ell(w_k) + 2 \\
\ell(v_{k,2}) &= 2\ell(v_{k,1}) + 2 = 4 \cdot \ell(w_k) + 6 \\
\ell(v_{k,3}) &= 2\ell(v_{k,2}) = 8 \cdot \ell(w_k) + 12 \\
\ell(v_{k,4}) &= 2\ell(v_{k,3}) - 2 \cdot 1 = 2[8\ell(w_k) + 12] - 2 \cdot 1 \\
\ell(v_{k,5}) &= 2\ell(v_{k,4}) - 2 \cdot 2 = 2^2[8 \cdot \ell(w_k) + 12] - 2^2 \cdot 1 - 2 \cdot 2 \\
&. \\
&. \\
&. \\
\ell(v_{k,l}) &= 2 \cdot \ell(v_{k,l-1}) - 2 \cdot (l-3) \\
&= 2^{l-3}[8 \cdot \ell(w_k) + 12] - \sum_{m=1}^{l-3} m \cdot 2^{l-2-m}.
\end{aligned}
$$

We have the following finite lengths, and then can form a general pattern:

$$\ell(v_{2,1}) = 22, \ell(v_{2,2}) = 46, \ell(v_{2,3}) = 92$$

$$\ell(v_{3,1}) = 42, \ell(v_{3,2}) = 86, \ell(v_{3,3}) = 172$$

$$\ell(v_{k,l}) = 2^{k+l+1} + k \cdot 2^{l+1} - 2^l + 2l - 2,$$

for $k, l \geq 3$.

Our goal now is to determine the cumulative growth function for this basis. It can be described by the inverse of the length function. We will be concerned with the cumulative growth, and not the strict growth, since the strict growth is a bit more complicated. First consider a basis element of the form w_k, where $k \geq 3$, and whose length is given by $n = \ell(w_k) = 2^{k+1} + 2k - 2$.

$$
\begin{aligned}
n &\geq 2^{k+1} + 2k - 2 \\
&\geq 2^k \\
\Rightarrow k &\leq \log_2(n)
\end{aligned}
$$

Since the elements of the form u_k have the same length, we multiply by two to obtain $2\log_2(n)$. This gives us an upper bound for the number of basis elements of the form w_k and u_k of length at most n.

Next, we consider an element of the form $v_{1,l}$, where $l \geq 3$, with length given by $\ell(v_{1,l}) = 2^{l+2} + 2l - 2$. As before, we take a lower bound of this function and invert:

$$
\begin{aligned}
n = &\geq 2^{l+2} + 2l - 2 \\
&\geq 2^l \\
\Rightarrow l &\leq \log_2(n)
\end{aligned}
$$

Since $\ell(v_{1,l}) = \ell(t_{1,l}) = \ell(z_{1,l}) = \ell(p_{1,l})$, we multiply by 4 to obtain $4\log_2(n)$.

Last, we consider the elements of the form $v_{k,l}, k, l \geq 3$.

$$
\begin{aligned}
n &\geq 2^{k+l+1} + k \cdot 2^{l+1} - 2^l + 2l - 2 \\
&\geq 2^{l+k} \\
\Rightarrow k + l &\leq \log_2(n)
\end{aligned}
$$

To determine how many elements of the form $v_{k,l}$ give rise to generators of length at most n, we consider the rhombus $\{(k,l) : k + l \leq \log_2(n)\}$. Since $k, l > 0$, we restrict to the first quadrant of the Euclidean plane. For each fixed value $\log_2(n)$, the points in this rhombus lie on or below the line that connects the points $(0, \log_2(n))$ and $(\log_2(n), 0)$. The number of such integral points is half the area of the given square. Therefore, the size of this rhombus is estimated as

$$|\{(k,l) : k + l \leq \log_2(n)\}| \leq \frac{(\log_2(n))^2}{2}$$

Since $\ell(v_{k,l}) = \ell(t_{k,l}) = \ell(z_{k,l}) = \ell(p_{k,l})$, we multiply this value by 4 to obtain $2(\log_2(n))^2$. Summing the following values, we obtain an upper bound for the cumulative growth function:

$$\begin{aligned}\gamma(n) &\leq 2\log_2(n) + 4\log_2(n) + 2(\log_2(n))^2 \\ &\leq 8(\log_2(n))^2\end{aligned}$$

2.6 $\{[x^k, y^l] : k, l \neq 0\}$

We consider the set $B = \{[x^k, y^l] : k, l \neq 0\}$, which is clearly contained in the commutator subgroup F'. In fact, it is a free basis for F'. First, we show that it is a free basis for the subgroup it generates by showing that it is Nielsen-reduced.

The length of $[x^k, y^l]$ is given by $\ell([x^k, y^l]) = 2(|k| + |l|)$. Clearly no element is the identity, so $(N0)$ is satisfied. Furthermore, we have that

$$[x^{k_1}, y^{l_1}][x^{k_2}, y^{l_2}] = x^{-k_1}y^{-l_1}x^{k_1}y^{l_1}x^{-k_2}y^{-l_2}x^{k_2}y^{l_2},$$

and

$$[x^{k_1}, y^{l_1}][x^{k_2}, y^{l_2}][x^{k_3}, y^{l_3}] = x^{-k_1}y^{-l_1}x^{k_1}y^{l_1}x^{-k_2}y^{-l_2}x^{k_2}y^{l_2}x^{-k_3}y^{-l_3}x^{k_3}y^{l_3},$$

whence it follows that:

$$\ell\left([x^{k_1}, y^{l_1}][x^{k_2}, y^{l_2}]\right) = 2 \cdot (|k_1| + |l_1| + |k_2| + |l_2|)$$
$$\ell\left([x^{k_1}, y^{l_1}][x^{k_2}, y^{l_2}][x^{k_3}, y^{l_3}]\right) = 2 \cdot (|k_1| + |l_1| + |k_2| + |l_2| + |k_3| + |l_3|)$$

Therefore $(N1)$ and $(N2)$ are satisfied and this set is Nielsen-reduced. Consequently, it forms a free basis for the subgroup it generates.

Now we must show that it generates the commutator subgroup F'. Any element in F' can be written in the form

$$x^{k_1}y^{l_1}x^{k_2}y^{l_2}\dots x^{k_s}y^{l_s}$$

where $\sum_{i=1}^{s} k_i = \sum_{j=1}^{s} l_j = 0$.

Proposition 12. *Any element of the form $x^{k_1}y^{l_1}\dots x^{k_s}y^{l_s}$ can be written as a product $cx^{\overline{k_s}}y^{\overline{j_s}}$, where $c \in \langle B\rangle$, $\overline{k_s} = \sum_{i=1}^{s} k_i$, and $\overline{l_s} = \sum_{j=1}^{s} l_j$.*

Proof. We proceed on induction on s. If $s = 1$, then the result holds with c being equal to the identity. Now assume that the result holds for $s - 1$. By the inductive step, we have that

$$(x^{k_1}y^{l_1}\dots x^{k_{s-1}}y^{l_{s-1}})x^{k_s}y^{l_s} = (cx^{\overline{k_{s-1}}}y^{\overline{l_{s-1}}})x^{k_s}y^{l_s}$$

where $c \in \langle B\rangle$. Observe:

$$\begin{aligned} cx^{\overline{k_{s-1}}}y^{\overline{l_{s-1}}}x^{k_s}y^{l_s} &= cx^{\overline{k_{s-1}}}y^{\overline{l_{s-1}}}(x^{-\overline{k_{s-1}}}y^{-\overline{l_{s-1}}}y^{\overline{l_{s-1}}}x^{\overline{k_{s-1}}})x^{k_s}y^{l_s} \\ &= c[x^{-\overline{k_{s-1}}}, y^{-\overline{l_{s-1}}}]y^{\overline{l_{s-1}}}x^{\overline{k_s}}y^{l_s} \\ &= c[x^{-\overline{k_{s-1}}}, y^{-\overline{l_{s-1}}}]y^{\overline{l_{s-1}}}x^{\overline{k_s}}(y^{-\overline{l_{s-1}}}x^{-\overline{k_s}}x^{\overline{k_s}}y^{\overline{l_{s-1}}})y^{l_s} \\ &= c[x^{-\overline{k_{s-1}}}, y^{-\overline{l_{s-1}}}][y^{-\overline{l_{s-1}}}, x^{-\overline{k_s}}]x^{\overline{k_s}}y^{\overline{l_s}}. \end{aligned}$$

Clearly $c[x^{-\overline{k_{s-1}}}, y^{-\overline{l_{s-1}}}][y^{-\overline{l_{s-1}}}, x^{-\overline{k_s}}] \in \langle B \rangle$. The result follows.

□

In the case where $x^{k_1}y^{l_1}\dots x^{k_s}y^{l_s}$ is in the commutator subgroup, then $\overline{k_s} = \overline{l_s} = 0$, whence every element of F' can be written as a product of elements in $B^{\pm 1}$. Since B is Nielsen-reduced, it follows that B is a free basis for the commutator subgroup. We may now find the growth functions and determine the Hilbert series for this basis.

We define the sets

$$\begin{aligned} B_n &= \{[x^k, y^l] : k, l \neq 0, 2(|k| + |l|) = n\} \\ &= \left\{[x^k, y^l] : k, l \neq 0, |k| + |l| = \frac{n}{2}\right\} \end{aligned}$$

Clearly $|B_n| = 0$ for odd n, or for $n = 2$. Now suppose that $n > 2$ is even. Then

$$k, l \in \left\{-\left(\frac{n}{2} - 1\right), \dots, -1, 1, \dots, \left(\frac{n}{2} - 1\right)\right\}.$$

Consequently, $|B_n| = \left(2\left(\frac{n}{2} - 1\right)\right)2 = 2n - 4 \Rightarrow |B_{2n}| = 4(n-1)$. Thus we have the

Hilbert series

$$\mathcal{H}(B,t) = \sum_{n=1}^{\infty} |B_{2n}| t^{2n} = \sum_{n=2}^{\infty} 4(n-1)t^{2n} = \frac{4t^4}{(1-t^2)^2}$$

For the strict growth function, we have $\lambda(2n) = 4(n-1)$, and this basis has linear growth. The cumulative growth function is $\gamma(2n) = 2n^2 - 2n$. Notice that these growth functions are identical to the growth functions in the case of geodesic Schreier basis. However, as we will now show, this basis is not Schreier.

If B were non-geodesic Schreier, then as we showed in Section 2.2, the growth functions would not be equal to those in the case of geodesic Schreier. Therefore, we must show that the basis B does not arise from any maximal geodesic tree in the coset graph $\Gamma = \Gamma(F/F')$. We now consider the graph $\mathbb{Z}\times\mathbb{Z} \cong F/F'$, where edges are labelled by x and y. We need only consider the three elements of B: $[x^{-1}, y^{-1}], [x^{-2}, y^{-1}], [x^{-1}, y^{-2}]$. In assuming that B is geodesic Schreier, we must have that the y- and x- axes be contained in the maximal tree, by requirement that the tree is geodesic. Therefore, exactly one of the edges in each of the three generators listed above must be non-tree edges. Consider the generator $[x^{-1}, y^{-1}]$. Now the edge $e_1 = ((1,0),(1,1))$ cannot be a non-tree edge. If it were, then $[x^{-1}, y^{-1}]$ and $[x^{-1}, y^{-2}]$ would either be formed from the same non-tree edge (which is nonsense), or else $[x^{-1}, y^{-2}]$ would have two non-tree edges. So then it must be that $e_2 = ((1,1),(0,1))$ is a non-tree edge which corresponds to the generator $[x^{-1}, y^{-1}]$. Now consider the generator $[x^{-2}, y^{-1}]$. Neither the edges $e_3 = ((2,0),(2,1))$ nor $e_4 = ((2,1),(1,2))$ can be non-tree edges, lest $[x^{-2}, y^{-1}]$ be formed from two non-tree edges.

Thus in considering these three generators, we have shown that the basis B cannot be formed from a maximal geodesic tree, and is thus not a geodesic Schreier system. Furthermore, since the growth functions are the same as in the geodesic Schreier case,

B cannot be non-geodesic Schreier either. Consequently, we have given an example of a basis that is not Schreier, but grows as fast as the geodesic Schreier systems.

2.7 Basis for Free Lie Commutator Subalgebra

In the case of free Lie algebras generated by a set X, we consider the filtration associated to X, where $S_n(X)$ is the subspace spanned by all monomials of length at most n. For example, $S_1(X) = \langle X \rangle$, the vector space spanned by elements of X, $S_2(X) = \langle X, [X, X] \rangle, S_3(X) = \langle X, [X, X], [X, X, X] \rangle$, and so on.

The growth function of the Lie algebra L is given by

$$\gamma_S(n) = \dim S_n(X).$$

In contrast with groups, the growth functions of bases of Lie algebras do not depend on the choice of homogeneous basis [9].

Let $X = \{x, y\}$ be a basis for a free Lie algebra L, and suppose that it is well-ordered such that $y > x$. It was shown in [1] that the commutator subalgebra L' is generated by the set $\{[y, \underbrace{x, \ldots, x}_{l}, \underbrace{y, \ldots, y}_{k}] : k \in \mathbb{N} \bigcup 0, l \in \mathbb{N}\} = \{u_{k,l} : l \neq 0\}$.

The weight is given by $wt(u_{k,l}) = k + l + 1$. Therefore,

$$B_n = \{u_{k,l} : k + l + 1 = n\} = \{u_{k,l} : k + l = n - 1\}$$

Since $l \in \{1, 2, \ldots, n-1\}$ and then $k = n - l - 1$, we have that $|B_n| = (n - 1)$. Then we form the Hilbert series:

$$\mathcal{H}(B, t) = \sum_{n=2}^{\infty} |B_n| t^n = \sum_{n=2}^{\infty} (n - 1) t^n = \frac{t^2}{(1 - t)^2}.$$

In this case, the strict growth function is given by $\lambda(n) = n - 1$, for $n \geq 2$. The cumulative growth function is given by $\gamma(n) = \frac{n(n-1)}{2}$.

Alternatively, we may determine the Hilbert series via Petrogradsky's formula:

$$\mathcal{H}(B) = (\mathcal{H}(X) - 1)\,\varepsilon\,(\mathcal{H}(L/K)) + 1$$

where B is the homogeneous basis for the Lie commutator subalgebra, and $X = \{x, y\}$.

Clearly $\mathcal{H}(X) = 2t$. We must determine the Hilbert series for L/L'. Recall that given a filtration in L, we define a filtration in L/L' by $(L/L')^n = (L^n + L')/L'$. Since L has basis $\{x, y\}$, then the basis elements of degree 1 in L/L' are $x + L', y + L'$. Then for $n \geq 2$, we have that $\dim(L/L')^n = \dim((L^n + L')/L') = 0$, since $L^n \subseteq L'$, for $n \geq 2$. Thus, the Hilbert series is given by $\mathcal{H}(L/L') = 2t$, and thus $\varepsilon\,(\mathcal{H}(L/L')) = \frac{1}{(1-t)^2}$. Plugging into his formula, we obtain

$$\begin{aligned}\mathcal{H}(B) &= (\mathcal{H}(X) - 1)\,\varepsilon\,(\mathcal{H}(L/K)) + 1\\ &= (2t - 1)\left(\frac{1}{(1-t)^2}\right) + 1\\ &= \frac{t^2}{(1-t)^2}\end{aligned}$$

2.8 Summary

As we have shown, growth functions of bases of a group are not guaranteed to be the same, or even equivalent. Of the different bases that we have considered, the geodesic Schreier bases appear to be the fastest. The geodesic Schreier bases share the same growth functions with the basis of the form $\{x^k, y^l : k, l \neq 0\}$. Growing slightly slower are the non-geodesic Schreier generators appearing from the spiral. Ward's basis appeared the slowest. In contrast with groups, the growth of bases for Lie algebras does not depend on the choice of homogeneous generating set. As we have seen, the commutator of group elements is closely connected with the Lie bracket. The growth of the Lie basis we considered is a scalar multiple of the growth of geodesic Schreier basis for the commutator subgroup. The results are summarized in the table below.

Basis	Degree	Strict Growth $\lambda(n)$	Cumulative Growth $\gamma(n)$
Lie Algebras	Weight	$\lambda(n) = n - 1$	$\gamma(n) = \frac{n(n-1)}{2}$
Geodesic Schreier	Length	$\lambda(2n) = 4(n-1)$, $n \geq 2$; $\lambda(2n+1) = 0$	$\gamma(2n) = \gamma(2n+1) = 2n^2 - 2n$
$\{[x^k, y^l] : k, l \neq 0\}$	Length	$\lambda(2n) = 4(n-1)$, $n \geq 2$; $\lambda(2n+1) = 0$	$\gamma(2n) = \gamma(2n+1) = 2n^2 - 2n$
Non-Geodesic Schreier	Length	$\lambda(2n) = 1$, $n \geq 2$; $\lambda(2n+1) = 0$	$\gamma(2n) = \gamma(2n+1) = n$
Ward's Basis	Length		$\gamma(n) \leq 8(\log_2(n))^2$
Ward's Basis	Weight	$\lambda(n) = 4n - 6, n \geq 3$	$\gamma(n) = 2(n-1)^2$

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