STRONGLY CLEAN RINGS AND g(x)-CLEAN RINGS

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Strongly Clean Rings and $g(x)$-Clean Rings

by

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Abstract

Let R be an associative ring with identity $1 \neq 0$. An element $a \in R$ is called clean if there exists an idempotent e and a unit u in R such that $a = e + u$, and a is called strongly clean if, in addition, $eu = ue$. The ring R is called clean (resp., strongly clean) if every element of R is clean (resp., strongly clean). The notion of a clean ring was given by Nicholson in 1977 in a study of exchange rings and that of a strongly clean ring was introduced also by Nicholson in 1999 as a natural generalization of strongly π-regular rings. Besides strongly π-regular rings, local rings give another family of strongly clean rings.

The main part of this thesis deals with the question of when a matrix ring is strongly clean. This is motivated by a counter-example discovered by Sánchez Campos and Wang-Chen respectively to a question of Nicholson whether a matrix ring over a strongly clean ring is again strongly clean. They both proved that the 2×2 matrix ring $\mathbb{M}_2(\mathbb{Z}_{(2)})$ is not strongly clean, where $\mathbb{Z}_{(2)}$ is the localization of $\mathbb{Z}$ at the prime ideal (2). The following results are obtained regarding this question:

- Various examples of non-strongly clean matrix rings over strongly clean rings.
- Completely determining the local rings R (commutative or noncommutative) for which $\mathbb{M}_2(R)$ is strongly clean.
- A necessary condition for $\mathbb{M}_2(R)$ over an arbitrary ring R to be strongly clean.
- A criterion for a single matrix in $\mathbb{M}_n(R)$ to be strongly clean when R has IBN and

every finitely generated projective R-module is free.

- A sufficient condition for the matrix ring $\mathbb{M}_n(R)$ over a commutative ring R to be strongly clean.
- Necessary and sufficient conditions for $\mathbb{M}_n(R)$ over a commutative local ring R to be strongly clean.
- A family of strongly clean triangular matrix rings.
- New families of strongly π-regular (of course strongly clean) matrix rings over non-commutative local rings or strongly π-regular rings.

Another part of this thesis is about the so-called $g(x)$-clean rings. Let $C(R)$ be the center of R and let $g(x)$ be a polynomial in $C(R)[x]$. An element $a \in R$ is called $g(x)$-clean if $a = e + u$ where $g(e) = 0$ and u is a unit of R. The ring R is $g(x)$-clean if every element of R is $g(x)$-clean. The $(x^2 - x)$-clean rings are precisely the clean rings. The notation of a $g(x)$-clean ring was introduced by Camillo and Simón in 2002. The relationship between clean rings and $g(x)$-clean rings is discussed here.

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Contents

List of symbols

R	associative ring with identity $1 \neq 0$
M_R, ${}_RM$	unitary right (left) R-module
$\mathrm{End}(M_R)$	endomorphism ring of a right R-module M
$C(R)$	center of a ring R
$r(a)$	right annihilator of $a \in R$
$l(a)$	left annihilator of $a \in R$
$U(R)$	group of multiplicative units in a ring R
$J(R)$	Jacobson radical of a ring R
$\mathbb{N}$	natural numbers excluding 0
$\mathbb{Z}$	ring of integers
$\mathbb{Q}$	field of rational numbers
$\mathbb{C}$	field of complex numbers
PID	principal ideal domain
UFD	unique factorization domain
ED	Euclidean domain
IBN	invariant basis number
$\mathbb{Z}_{(p)}$	localization of $\mathbb{Z}$ at the prime ideal (p)
$\widehat{\mathbb{Z}}_p$	completion of $\mathbb{Z}$ at the prime ideal (p), or ring of p-adic integers

$R[x]$	polynomial ring over R in indeterminate x
$R[[x]]$	formal power series ring over R in indeterminate x,
	or completion of $R[x]$ at ideal (x) in $R[x]$
$\mathfrak{m}$	maximal ideal of a ring R
$\mathfrak{p}$	prime ideal of a ring R
$\mathbb{M}_n(R)$	$n \times n$ matrix ring over a ring R
$\mathbb{T}_n(R)$	$n \times n$ upper triangular matrix ring over a ring R
$\oplus_{i\in I} A_i$	direct sum of modules or other algebraic systems
$\Pi_{i\in I} A_i$	direct product of modules or other algebraic systems
$\det A, \|A\|$	determinant of a matrix A over a commutative ring
$\mathrm{tr}(A), \mathrm{tr}\, A$	trace of a matrix A
$GL_n(R)$	general linear group over R,
	or multiplicative group of units in $\mathbb{M}_n(R)$
$\gcd(f(x), g(x))$	greatest common divisor of $f(x)$ and $g(x)$ in a UFD $R[x]$
	which is monic if R is a field
$\deg(f(x))$	degree of the polynomial$f(x)$
$\Re(f,g)$	resultant of polynomials$f(t), g(t) \in R[t]$
$\mathrm{Max}(R)$	maximal spectrum$\{\mathfrak{m} : \mathfrak{m}$ is a maximal ideal in $R\}$
	of a commutative ring R
$S^{-1}R$	fractionization of a commutative ring R
	by a multiplicatively closed set S
$rad(M_R)$	Jacobson radical of the right R-module M
$\chi_A(t)$	characteristic polynomial of the matrix $A \in \mathbb{M}_n(R)$
	over a commutative ring R
$Q_c(R)$	quotient field of an integral domain R
$\mathbb{Z}[\omega]$	domain $\{n + m\omega : n, m \in \mathbb{Z}\}$ where $\omega \in \mathbb{C}\backslash\mathbb{Q}, \omega^2 \in \mathbb{Z}$

Introduction

Let R be an associative ring with identity $1 \neq 0$, $C(R)$ be the center of R, and $g(x)$ be a polynomial in the polynomial ring $C(R)[x]$. By Nicholson [51, 52], an element a in a ring R is called **clean** if there exist an idempotent e and a unit u in R such that $a = e + u$ and a is called **strongly clean** if, in addition, $eu = ue$. The ring R is called **clean** (resp., **strongly clean**) if every element of R is clean (resp., strongly clean). Following Camillo-Simón [18] and Nicholson-Zhou [54], an element $a \in R$ is called **$g(x)$-clean** if $a = e + u$ where $g(e) = 0$ and u is a unit of R and R is $g(x)$-clean if every element of R is $g(x)$-clean. Thus, the $(x^2 - x)$-clean rings are precisely the clean rings. An element a in a ring R is called **strongly π-regular** if both chains $aR \supseteq a^2R \supseteq \cdots$ and $Ra \supseteq Ra^2 \supseteq \cdots$ terminate and the ring R is called **strongly π-regular** if every element of R is strongly π-regular [10], or equivalently, the chain $aR \supseteq a^2R \supseteq \cdots$ terminates for all $a \in R$ [26].

This thesis deals with some aspects of clean rings, strongly clean rings, strongly π-regular rings and $g(x)$-clean rings. The subject falls under the area of study of exchange rings and largely overlaps with the study of von Neumann regular rings.

In 1964, Crawley and Jónsson introduced the well-known exchange property [25] when they worked on direct sum refinements for algebraic systems. Let τ be a cardinal number. A module M is said to have the **τ-exchange property** if for every module X and each direct decomposition $X = M' \oplus Y = \oplus_{i \in I} N_i$ with $M' \cong M$ and $card(I) \leq \tau$, there are submodules $N_i' \leq N_i$, $i \in I$, such that $X = M' \oplus (\oplus_{i \in I} N_i')$, and M is said to have the **exchange property** (or to be an **exchange module**) if M has the τ-exchange property for every cardinal number τ. A module M is said to have the **finite exchange property** if M has the n-exchange property for every positive integer n. Modules with the exchange property often have isomorphic refinements for direct sum decompositions [31, pp. 39-41, Theorem 2.9 and Theorem 2.10]. In 1972, Warfield introduced exchange rings [63]. A ring R is called an **exchange ring** if the regular module R_R has the exchange property (equivalently, R is an exchange ring if for any $a \in R$, there exists an idempotent $e \in R$ such that $e \in aR$ and $1 - e \in (1 - a)R$ by [51, p.167] or [37, Theorem 2.4]). It is well known that the definition of an exchange ring is left-right symmetric and a module M_R has the finite exchange property iff the endomorphism ring $End(M_R)$ is an exchange ring [63]. A ring is **semiregular** if it is von Neumann regular modulo the Jacobson radical and idempotents lift modulo Jacobson radical. For example, semiperfect rings are semiregular. Exchange rings include semiregular rings, π-regular rings (including von Neumann regular rings), unital C^*-algebras of real rank zero [3, Theorem 7.2], and many other classes of rings.

In [51], Nicholson proved that clean rings are exchange rings and an exchange ring whose idempotents are central is clean. In 1994, Camillo and Yu observed that a ring constructed by Bergman [39] is exchange but not clean. Thus, for the first time, people know that clean rings form a proper class of exchange rings. Since the publication of Camillo-Yu's paper, clean rings have attracted more and more authors and they are now a quite active subject and much progress has been made. Recall that a ring R is **unit regular** if every element $a \in R$ can be written as $a = ava$ with some $v \in U(R)$, or equivalently, $a = eu$ for some idempotent $e \in R$ and some unit $u \in U(R)$. Thus, clean rings are the additive analogs of unit regular rings. Surprisingly, every unit regular ring is clean by Camillo, Khurana and Yu in [15] and [17] where it is also proved that every semiperfect ring is clean. In 2006, Camillo, Khurana, Lam, Nicholson and Zhou proved that the endomorphism ring of a continuous module is clean [16]. These results show that the class of clean rings is quite large.

In 1999, Nicholson discovered the nice connection between the well known Fitting's Lemma and a certain class of clean rings which he called strongly clean rings. Local rings are strongly clean [52]. One notices that strongly clean rings are the additive analogs of **strongly regular rings** where each element a can be written as $a = eu = ue$ with e an idempotent and u a unit. In 1947, Arens and Kaplansky [5] first investigated rings that are now called strongly π-regular elements and rings. Azumaya [10] defined left and right π-regular elements and strongly π-regular rings. He proved that if $a \in R$ is strongly π-regular, then there exist $b \in R$ and $n > 0$ such that $a^n = a^{n+1}b$ and $ab = ba$ [10, Theorem 3]. Strongly π-regular rings include one-sided perfect rings, strongly regular rings and algebraic algebras over a field. In 1988, Burgess and Menal [14] proved that strongly π-regular rings are strongly clean (so strongly regular rings and one-sided perfect rings are strongly clean). For an element α in the endomorphism ring $End(M_R)$ of the right R-module M_R, Armendariz, Fisher and Snider [6] proved that α is strongly π-regular iff it satisfies Fitting's Lemma, that is, there exists an $n \in \mathbb{N}$ such that $M = Im\alpha^n \oplus Ker\alpha^n$. Nicholson [52] observed that α is strongly π-regular iff there exists a direct sum decomposition $M_R = P_R \oplus Q_R$ such that the restriction $\alpha|_P : P \to P$ is an isomorphism and $\alpha|_Q : Q \to Q$ is nilpotent; and that α is strongly clean iff it satisfies the general Fitting's Lemma, that is, there exists a direct sum decomposition $M_R = P_R \oplus Q_R$ such that the restriction $\alpha|_P : P \to P$ and $(1-\alpha)|_Q : Q \to Q$ are isomorphisms. Thus, he not only proved that every strongly π-regular element is strongly clean but also showed that strongly clean rings are a natural generalization of strongly π-regular rings. Thus, various questions can be asked whether certain properties of a strongly π-regular ring can be extended to a strongly clean ring. In considering the Morita invariant property of strongly clean rings, Nicholson [52] raised two questions: Let R be strongly clean with $e^2 = e \in R$. Is eRe strongly clean? Is $\mathbb{M}_n(R)$ strongly clean? In her 2002 unpublished manuscript [60], Sánchez Campos answered the first question affirmatively and gave a counter-example to the second question. In 2004, Wang and Chen [62], independently, published a counter-example to the second question. Thus, an interesting question follows naturally: (*) When is the matrix ring $\mathbb{M}_n(R)$ strongly clean?

In 2000, Camillo and Simón proved that if V is a countable dimensional vector space over a division ring D and if $g(x) \in C(D)[x]$ has two distinct roots in $C(D)$, then $End(V_D)$ is $g(x)$-clean [18]. In 2004, Nicholson and Zhou generalized Camillo and Simón's result by proving that $End(_RM)$ is $g(x)$-clean where $_RM$ is a semisimple module over an arbitrary ring R and $g(x) \in (x-a)(x-b)C(R)[x]$ with $a, b \in C(R)$ and b, $b-a \in U(R)$ [54]. So one may ask: What is the relation between clean rings and $g(x)$-clean rings?

In this thesis, partial answers to question $(*)$ are obtained when the underlying ring is local or strongly π-regular. Thus, new families of strongly clean rings are obtained. Some of these strongly clean rings are neither local nor strongly π-regular. We also discuss the strongly clean property for triangular matrix rings over local rings and the strongly π-regular property of matrix rings over strongly π-regular rings or local rings. At last, $g(x)$-clean rings are touched. Related to question $(*)$, a recent result of Borooah, Diesl and Dorsey [12] shows that the matrix ring $\mathbb{M}_n(R)$ over a commutative local ring R is strongly clean iff R is an n-SRC ring (see Definition 3.1.8).

The thesis is organized as follows:

In chapter 1, two important classes of local rings through localization and completion are introduced for later use.

In chapter 2, various non-strongly clean matrix rings over strongly clean rings are presented; a criterion for a single matrix in $\mathbb{M}_n(R)$ to be strongly clean is given when R has IBN and every finitely generated projective R-module is free; a criterion for the matrix ring $\mathbb{M}_2(R)$ to be strongly clean is given when R is commutative local; a complete characterization of the local ring R is obtained for $\mathbb{M}_2(R)$ to be strongly clean and many more examples of strongly clean rings are obtained; and at last, a necessary condition for $\mathbb{M}_2(R)$ over an arbitrary ring R to be strongly clean is obtained.

In chapter 3, the SRC factorization is generalized from a commutative local ring to a commutative ring; a sufficient condition for the matrix ring $\mathbb{M}_n(R)$ over a commutative ring R to be strongly clean is proved; and necessary and sufficient conditions for the matrix ring $\mathbb{M}_n(R)$ over a commutative local ring R to be strongly clean are given.

In chapter 4, a family of strongly clean triangular matrix rings over some local rings are obtained.

Chapter 5 is about when a matrix ring is strongly π-regular.

In chapter 6, $g(x)$-clean rings are discussed.

Chapter 1

Preliminaries

Local rings are one of the classes of rings considered in this thesis. Later we will see that there are two kinds of local rings that behave totally differently with respect to the strongly clean property of matrix rings over them. In this chapter, we briefly mention several properties of local rings and give a number of examples of them, including localization and the ring of p-adic integers for later use. A special class of local rings, Henselian rings, is also introduced. More detailed information on local rings and Henselian rings can be found in [8, 29, 59].

1.1 Local rings

A proper ideal $\mathfrak{m}$ is called **maximal** if there is no proper ideal of R strictly containing $\mathfrak{m}$. Recall that a ring R is **local** if the non-invertible elements of R form an ideal. The results in the next theorem are well known.

Theorem 1.1.1 *[2, Theorem 15.15] For a ring R, the following statements are equivalent:*

1. *R is a local ring.*

2. *R has a unique maximal left ideal.*

3. *$J(R)$ is a maximal left ideal.*

4. *The set of elements of R without a left inverse is closed under addition.*

5. $J(R) = \{x \in R | Rx \neq R\}$.

6. *$R/J(R)$ is a division ring.*

7. $J(R) = \{x \in R : x$ *is not invertible*$\}$.

8. *If $x \in R$, then either x or $1 - x$ is invertible.*

In the rest of this section, all rings are commutative.

Definition 1.1.2 *A subset S of a ring R is* ***multiplicatively closed*** *if $1 \in S, 0 \notin S$, and $s_1 s_2 \in S$ for all $s_1, s_2 \in S$.*

Theorem 1.1.3 *Let R be a commutative ring and S be a multiplicatively closed subset in R.*

1. *Define a relation $\sim$ on $R \times S$: $(r_1, s_1) \sim (r_2, s_2)$ iff there exists some $s \in S$ such that $(r_1 s_2 - r_2 s_1)s = 0$. Then $\sim$ is an equivalence relation.*

2. *Denote the equivalence class of (a, s) as $\frac{a}{s}$ or a/s. Define addition $a/s + b/t = (at + bs)/st$ and multiplication $(a/s)(b/t) = ab/st$. Then these operations are well-defined.*

3. *The set of all these equivalence classes with the addition and multiplication in (2) forms a ring, denoted as $S^{-1}R$.*

4. *$\theta : R \to S^{-1}R$, $\theta(r) = r/1$ is a ring homomorphism with $\theta(s)$ invertible in $S^{-1}R$ for all $s \in S$.*

5. *Let $R^{'}$ be a commutative ring and $\psi : R \longrightarrow R^{'}$ a ring homomorphism with $\psi(s)$ invertible in $R^{'}$ for all $s \in S$. Then there exists a unique ring homomorphism $\varphi : S^{-1}R \longrightarrow R^{'}$ such that the following diagram commutes:*

$$\begin{tikzcd} R \arrow[r, "\theta"] \arrow[d, "\psi"'] & S^{-1}R \arrow[dl, dotted, "\varphi"] \\ R^{'} & \end{tikzcd}$$

Proof We only prove (*5*).

For existence, define $\varphi : S^{-1}R \longrightarrow R^{'}$ by $\varphi(r/s) = \psi(r)\psi(s)^{-1}$. Suppose that $r/s = r^{'}/s^{'}$. Then there exists some $t \in S$ such that $(rs^{'} - r^{'}s)t = 0$. Therefore, $\left(\psi(r)\psi(s^{'}) - \psi(r^{'})\psi(s)\right)\psi(t) = 0$. Notice that $\psi(t)$ is invertible. So $\psi(r)\psi(s^{'}) - \psi(r^{'})\psi(s) = 0$ and thus, $\psi(r)\psi(s)^{-1} = \psi(r')\psi(s^{'})^{-1}$. That is, φ is well-defined. Clearly, φ is a homomorphism and the diagram commutes.

For uniqueness, suppose $h : S^{-1}R \longrightarrow R^{'}$ is another homomorphism that makes the diagram commute:

$$\begin{tikzcd} R \arrow[r, "\theta"] \arrow[d, "\psi"'] & S^{-1}R \arrow[dl, dotted, "h"] \\ R^{'} & \end{tikzcd}$$

Then $h(r/1) = h\theta(r) = \psi(r)$ for all $r \in R$. So $h(1/s) = h((s/1)^{-1}) = [h(s/1)]^{-1} = \psi(s)^{-1}$ for all $s \in S$. Hence, $h(r/s) = h(r/1)h(1/s) = \psi(r)\psi(s)^{-1}$ for all $r/s \in S^{-1}R$. Notice that $\varphi(r/s) = \psi(r)\psi(s)^{-1}$ for all $r/s \in S^{-1}R$. Therefore, $h = \varphi$. $\square$

Definition 1.1.4 *Let S be a multiplicatively closed subset of a commutative ring R. Then a* ***fraction ring*** *of R with respect to S is a commutative ring, denoted by $S^{-1}R$ too, and a ring homomorphism $\theta : R \longrightarrow S^{-1}R$ such that $\theta(s)$ is invertible for every $s \in S$*

and $S^{-1}R$ is universal with the property: If R' is a commutative ring and $\psi : R \longrightarrow R'$ is a ring homomorphism with $\psi(s)$ invertible for all $s \in S$, then there exists a unique ring homomorphism $\varphi : S^{-1}R \longrightarrow R'$ with $\varphi\theta = \psi$, i.e., the following diagram is commutative:

$$\begin{tikzcd} R \arrow[r, "\theta"] \arrow[d, "\psi"'] & S^{-1}R \arrow[dl, dotted, "\varphi"] \\ R' & \end{tikzcd}$$

So the ring $S^{-1}R$ constructed in Theorem 1.1.3 is a fraction ring of R with respect to S.

Corollary 1.1.5 *If S contains no zero divisors, then $\theta : R \longrightarrow S^{-1}R$ is monic; if R is an integral domain and $S = R\backslash\{0\}$, then we call $S^{-1}R$ the quotient field and denoted by $Q_c(R)$; if S is any multiplicatively closed subset of R, then $S^{-1}R$ is a subring of $Q_c(R)$.*

Proof Suppose $\theta(r) = r/1 = 0$. Then $r/1 = 0/s$ for some $s \in S$. So $(rs - 0)t = 0$ for some $t \in S$. Hence $rst = 0$. Since S contains no zero divisor, we get $r = 0$. That is, θ is monic. The rest is easy to prove. $\square$

A proper ideal $\mathfrak{p}$ in a commutative ring R is **prime** if $xy \in \mathfrak{p}$ implies $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. It is well known that $\mathfrak{p}$ is a prime ideal of R iff $R/\mathfrak{p}$ is a domain and $\mathfrak{m}$ is a maximal ideal of R iff $R/\mathfrak{m}$ is a field.

Theorem 1.1.6 *Let $\mathfrak{p}$ be a prime ideal of a commutative ring R. Then $S = R\backslash\mathfrak{p}$ is a multiplicatively closed set and $S^{-1}R$ is a local ring, denoted by $R_\mathfrak{p}$ ($R_\mathfrak{p}$ is called the* ***localization*** *of R at the prime ideal $\mathfrak{p}$).*

Corollary 1.1.7 *Let $\mathbb{Z}$ be the ring of integers. Then for any prime number $p \in \mathbb{Z}$, $\mathbb{Z}_{(p)} = \{m/n \in \mathbb{Q} : m, n \in \mathbb{Z}, n \neq 0, p$ and n coprime $\}$ is a local ring and $\mathbb{Z}$ is a subring of $\mathbb{Z}_{(p)}$.*

Some other local rings appear as power series rings $R[[x]]$ where R is local and the ring of p-adic integers $\widehat{\mathbb{Z}}_p$.

Example 1.1.8 *Let R be a local ring. Then the power series ring $R[[x]]$ is a local ring with $\mathfrak{m} = (x) = R[[x]]x + J(R)$. In particular, if F is a field, then $F[[x]]$ is a local ring with $\mathfrak{m} = (x) = F[[x]]x$.*

Example 1.1.9 *Let $p \in \mathbb{Z}$ be a prime number. The ring of p-adic integers $\widehat{\mathbb{Z}}_p$ is a local ring.*

The ring of p-adic integers is $\widehat{\mathbb{Z}}_p = \{\sum_{i=0}^{\infty} a_i p^i : a_i \in \{0, 1, \cdots, p-1\}\}$. Let $x = \sum_{i=0}^{\infty} a_i p^i$ and $y = \sum_{i=0}^{\infty} b_i p^i$ in $\widehat{\mathbb{Z}}_p$. Define $x + y = \sum_{i=0}^{\infty} c_i p^i$ where the coefficients c_i are defined inductively: by the division algorithm in $\mathbb{Z}$, there exist unique integers $0 \le c_0 < p$ and $h_0 \le 1$ such that $a_0 + b_0 = c_0 + ph_0$; and unique integers $0 \le c_1 < p$ and $h_1 \in \mathbb{Z}$ such that $a_1 + b_1 + h_0 = c_1 + ph_1$; and inductively, unique integers $0 \le c_k < p$ and $h_k \in \mathbb{Z}$ such that $a_k + b_k + h_{k-1} = c_k + ph_k$, $k \in \mathbb{N}$. Similarly, define $xy = \sum_{i=0}^{\infty} d_i p^i$ where $a_0 b_0 = d_0 + ph_0$, $a_k b_0 + a_{k-1} b_1 + \cdots + a_0 b_k + h_{k-1} = d_k + ph_k$, $d_0, d_k \in \{0, 1, \cdots, p-1\}$, and $h_0, h_k \in \mathbb{Z}$ $(k = 1, 2, \cdots)$. Then $\widehat{\mathbb{Z}}_p$ forms a ring. It is a local ring with $\mathfrak{m} = p\widehat{\mathbb{Z}}_p$.

In commutative algebra, an important class of local rings is constructed by the completion of a ring with respect to certain ideals. In fact, for a local ring R, $R[[x]]$ is the completion of $R[x]$ with respect to the ideal $(x) = xR[x]$ and $\widehat{\mathbb{Z}}_p$ is the completion of $\mathbb{Z}$ with respect to the ideal $p\mathbb{Z}$ (see [29]).

1.2 Hensel's Lemma and Henselian rings

In this section, we introduce Hensel's Lemma and Henselian rings. Later we will see that matrix rings over them are strongly clean.

Let I be an ideal of a ring R. For $f(t) = a_0 + a_1 t + \cdots + a_n t^n \in R[t]$, we write $\overline{f}(t) = \overline{a_0} + \overline{a_1} t + \cdots + \overline{a_n} t^n \in \frac{R}{I}[t]$.

Definition 1.2.1 *(**Hensel's Lemma**) Let R be a commutative ring with a maximal* *ideal* $\mathfrak{m}$. *We say that* R *satisfies Hensel's Lemma if* R *satisfies the following property: For any monic polynomial* $f(t) \in R[t]$, *if* $\overline{f}(t) = \alpha(t)\,\beta(t)$ *such that* $\alpha(t)$ *is monic,* $\alpha(t)$ *and* $\beta(t)$ *are coprime in* $\frac{R}{\mathfrak{m}}[t]$, *then there exist unique polynomials* $g(t), h(t) \in R[t]$ *with* $g(t)$ *monic such that* $f(t) = g(t)h(t)$, $\overline{g}(t) = \alpha(t)$ *and* $\overline{h}(t) = \beta(t)$.

Definition 1.2.2 *A commutative local ring* R *is called a* ***Henselian ring*** *if* R *satisfies Hensel's Lemma [11, 49].*

The following is a generalization of a Henselian ring which will be used later.

Definition 1.2.3 *[7] A local ring* R *(may not be commutative) with* $\overline{R} = R/J(R)$ *being a field is called a* ***general Henselian ring*** *if* R *satisfies the following condition : For any monic polynomial* $f(t) \in R[t]$, *if* $\overline{f}(t) = \alpha(t)\beta(t)$ *with* $\alpha(t)$, $\beta(t) \in \overline{R}[t]$ *coprime and* $\alpha(t)$ *monic, then there exist unique polynomials* $g(t), h(t) \in R[t]$ *with* $g(t)$ *monic such that* $f(t) = g(t)h(t)$, $\overline{g}(t) = \alpha(t)$, *and* $\overline{h}(t) = \beta(t)$.

It is well known in commutative algebra [29, Theorem 7.18] that the ring of p-adic integers $\widehat{\mathbb{Z}}_p$ and the formal power series ring $F[[x]]$ are Henselian rings where p is a prime number in $\mathbb{Z}$ and F is a field.

Chapter 2

Strongly Clean Matrix Ring $\mathbb{M}_2(R)$

As we mentioned in the introduction, the matrix ring $\mathbb{M}_2(\mathbb{Z}_{(2)})$ over the local domain $\mathbb{Z}_{(2)}$ is not strongly clean [60, 62]. In section 2.1, more negative examples are given. In section 2.2, we give a criterion for a single matrix in $\mathbb{M}_n(R)$ to be strongly clean when R has IBN (see Definition 2.2.3) and every finitely generated projective R-module is free and then we easily get a criterion for the matrix ring $\mathbb{M}_2(R)$ to be strongly clean when R is commutative local. In section 2.3, we determine when $\mathbb{M}_2(R)$ is strongly clean where R is a local ring. In section 2.4, many examples of strongly clean 2×2 matrix rings over local rings are given. At last, in section 2.5, we give a necessary condition for $\mathbb{M}_2(R)$ over an arbitrary ring R to be strongly clean. Section 2.1 and some part of section 2.2 come from [22, 23].

2.1 Non-strongly clean matrix rings over commutative local rings

If R is a commutative local domain, when is $\mathbb{M}_n(R)$ strongly clean? In this section, we prove that $\mathbb{M}_n(R)$ is not strongly clean if R is any of the following types:

- $\mathbb{Z}_{(p)}$, the localization of $\mathbb{Z}$ at a prime ideal (p).
- $S[x]_{\mathfrak{p}}$, the localization of the polynomial ring $S[x]$ at a prime ideal $\mathfrak{p}$, where S is a commutative domain.
- $\mathbb{Z}[\omega]_{\mathfrak{p}}$, the localization of $\mathbb{Z}[\omega]$ at a prime ideal $\mathfrak{p}$, where $\omega \in \mathbb{C}\backslash\mathbb{Q}$ with $\omega^2 \in \mathbb{Z}$ such that $\mathbb{Z}[\omega]$ is a UFD (Unique Factorization Domain).

We first notice that, for any ring R and for integers $n \geq m \geq 1$, if $\mathbb{M}_n(R)$ is strongly clean, then so is $\mathbb{M}_m(R)$. This observation follows from the next result of Sánchez Campos [60]. For an element $a \in R$, $r(a)$ and $l(a)$ denote the **right** and **left annihilators** of a in R respectively. If $a = e + u$ with $e^2 = e, u \in U(R)$ and $eu = ue$, then we say $a = e + u$ is a **strongly clean expression** of a.

Theorem 2.1.1 *[60, Theorem 2.3] Let R be a strongly clean ring. Then, for any $e^2 = e \in R$, eRe is strongly clean.*

Proof Let $a \in eRe$ with $a = g + u$ where $g^2 = g \in R, u \in U(R)$, and $gu = ug$. For any $x \in r(a)$, $ax = 0$ implies $gx = -ux$. So $x = -u^{-1}gx = -gu^{-1}x$. Hence, $gx = x$. So $x \in r(1-g)$, i.e., $r(a) \subseteq r(1-g)$. Similarly, we have $l(a) \subseteq l(1-g)$. So $(1-g)(1-e) = (1-e)(1-g) = 0$ because $(1-e) \in r(a) \cap l(a)$. Hence $eg = ge = ege$ is an idempotent in eRe. So $eu = ue = eue \in U(eRe)$ because e, g, a, and u commute. Therefore, $a = ege + eue$ is a strongly clean expression of a in eRe. So eRe is strongly clean. □

The next theorem gives a necessary condition for the 2×2 matrix ring over a commutative ring to be strongly clean.

Theorem 2.1.2 *Let R be a commutative ring. If $\mathbb{M}_2(R)$ is strongly clean, then, for any $w \in J(R)$, $x^2 - x = w$ is solvable in R.*

Proof For $w \in J(R)$, let $A = \begin{pmatrix} 1 & -k \\ 1 & 0 \end{pmatrix}$ where $k = w(1+4w)^{-1}$. By hypothesis, let $A = E + U$ be a strongly clean expression of A in $\mathbb{M}_2(R)$ where $E = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $U = \begin{pmatrix} 1-a & -k-b \\ 1-c & -d \end{pmatrix}$. The invertibility of U gives

$$\det U = d(a-1) + (k+b)(1-c) \in U(R). \tag{2.1.1}$$

By $EU = UE$ we get

$$b = -kc, \;\; c = a - d. \tag{2.1.2}$$

Since $k \in J(R)$, $b \in J(R)$ by (2.1.2). So $(k+b)(1-c) \in J(R)$. Thus, (2.1.1) gives

$$d \in U(R) \text{ and } a - 1 \in U(R). \tag{2.1.3}$$

$E^2 = E$ implies

$$a - a^2 = bc, \;\; d - d^2 = bc. \tag{2.1.4}$$

Since $b \in J(R)$, it follows by (2.1.3) and (2.1.4) that $a, 1-d \in J(R)$. So $1+a-d \in J(R)$. Hence, $a - d \in U(R)$. But by (2.1.4), $a - a^2 = d - d^2$, and so $(a+d-1)(a-d) = 0$. Thus, $a + d = 1$. Hence, $c = a - d = a - (1-a) = 2a - 1$ by (2.1.2). So we have

$$\begin{aligned} a - a^2 = bc &= -kc^2 = -k(2a-1)^2 \\ &= -k(4a^2 - 4a + 1) = 4k(a - a^2) - k, \end{aligned}$$

where the first equality follows from (2.1.4) and the second by (2.1.2). So $(1-4k)(a^2-a) = k$. Hence, $a^2 - a = (1-4k)^{-1}k = w$ by $k = w(1+4w)^{-1}$. Thus, a is a solution of $x^2 - x = w$. $\square$

Later we will see that this condition is also sufficient for commutative local rings (Corollary 2.2.12) and in addition we will generalize this to arbitrary rings (Theorem 2.5.1).

It was proved in [60] and in [62] that $\mathbb{M}_2(\mathbb{Z}_{(2)})$ is not strongly clean. This is a special case of the following result.

Corollary 2.1.3 *For any prime $p \in \mathbb{Z}$, $\mathbb{M}_n(\mathbb{Z}_{(p)})$ is not strongly clean for every $n \geq 2$.*

Proof Notice that $p \in J(\mathbb{Z}_{(p)})$ and $x^2 - x + p = 0$ has no solution in $\mathbb{Q}$ because the discriminant $1 - 4p < 0$. So by Theorem 2.1.2, $\mathbb{M}_2(\mathbb{Z}_{(p)})$ is not strongly clean. Hence, $\mathbb{M}_n(\mathbb{Z}_{(p)})$ is not strongly clean by Theorem 2.1.1. □

Corollary 2.1.4 *Let S be a commutative domain, $\mathfrak{p}$ a prime ideal of $S[x]$, and $S[x]_{\mathfrak{p}}$ the localization of $S[x]$ at $\mathfrak{p}$. Then $\mathbb{M}_n(S[x]_{\mathfrak{p}})$ is not strongly clean for every $n \geq 2$.*

Proof Take $h(x) \in J(S[x]_{\mathfrak{p}})$ with $h(x) \in S[x]$ such that the degree, $\deg h$, of $h(x)$ is an odd number. We claim that $y^2 - y = h(x)$ has no solution in $S[x]_{\mathfrak{p}}$; so $\mathbb{M}_2(S[x]_{\mathfrak{p}})$ is not strongly clean by Theorem 2.1.2. Otherwise, there exists $\frac{f(x)}{g(x)} \in S[x]_{\mathfrak{p}}$ such that

$$(\frac{f(x)}{g(x)})^2 - \frac{f(x)}{g(x)} = h(x).$$

That is

$$f(x)[f(x) - g(x)] = h(x)g(x)^2.$$

Either $\deg f > \deg g$ or $\deg f < \deg g$ or $\deg f = \deg g$ clearly leads to a contradiction. Hence, $\mathbb{M}_n(S[x]_{\mathfrak{p}})$ is not strongly clean for every $n \geq 2$ by Theorem 2.1.1. □

We can give more negative examples after the following lemmas.

Lemma 2.1.5 *Let R be a commutative domain and $A \in \mathbb{M}_2(R)$. Then A is an idempotent iff $A = 0$ or $A = I$ or $A = \left(\begin{smallmatrix} a & b \\ c & 1-a \end{smallmatrix}\right)$ where $bc = a - a^2$ in R.*

Proof The verification is straightforward. □

An element a in a ring R is called a **square** if $a = b^2$ for some $b \in R$. The trace and the determinant of a square matrix A over a commutative ring are denoted by $\mathrm{tr}A$ and $\det A$ respectively.

Lemma 2.1.6 *Let R be a commutative domain and $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{M}_2(R)$ with $s = a_{11} - a_{22}$ and $t = (\mathrm{trA})^2 - 4\det \mathrm{A}$. If A and $I - A$ are non-invertible and if A is a strongly clean element in $\mathbb{M}_2(R)$, then s^2t is a square in R.*

Proof Since A and $A - I$ are non-invertible in $\mathbb{M}_2(R)$ and A is strongly clean in $\mathbb{M}_2(R)$, by Lemma 2.1.5, there exist $a, b, c \in R$ with $bc = a - a^2$ such that

$$A = E + (A - E), \text{ where } E = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix},$$

is a strongly clean expression of A in $\mathbb{M}_2(R)$. It follows from $E(A - E) = (A - E)E$ that

$$sb = a_{12}(2a - 1), \ \ sc = a_{21}(2a - 1).$$

Since $bc = a - a^2$ and $t = s^2 + 4a_{12}a_{21}$, we have $a_{12}a_{21}(2a-1)^2 = s^2bc = s^2(a - a^2)$, which gives $(s^2 + 4a_{12}a_{21})a^2 - (s^2 + 4a_{12}a_{21})a + a_{12}a_{21} = 0$. That is, $ta^2 - ta + a_{12}a_{21} = 0$. It follows that $[t(2a - 1)]^2 = t(4ta^2 - 4ta + t) = t(-4a_{12}a_{21} + t) = t(s^2 - t + t) = s^2t$. □

Corollary 2.1.7 *Let R be a commutative domain and $p \in R$ be a nonunit and $q \in R$. If $A = \begin{pmatrix} p+1 & p \\ q & p \end{pmatrix}$ is a strongly clean element in $\mathbb{M}_2(R)$, then $4qp + 1$ is a square in R.*

Proof Since p is a nonunit of R, A and $A - I$ are non-invertible in $\mathbb{M}_2(R)$. In this case, $s = (p + 1) - p = 1$ and $t = (\mathrm{tr}A)^2 - 4\det A = (2p + 1)^2 - 4(p^2 + p - pq) = 1 + 4pq$. So by Lemma 2.1.6, $4pq + 1 = s^2t$ is a square in R. □

Throughout the following discussion, let ω denote a complex number such that $\omega^2 \in \mathbb{Z}$ and $\omega \notin \mathbb{Q}$ and let $\mathbb{Z}[\omega] = \{n + m\omega : n, m \in \mathbb{Z}\}$. Then $\mathbb{Z}[\omega]$ is a domain and the representation $n + m\omega$ of elements of $\mathbb{Z}[\omega]$ is unique. The study of such domains $\mathbb{Z}[\omega]$ has evolved into a subject in algebraic number theory. Every nonzero nonunit in $\mathbb{Z}[\omega]$ is a product of irreducibles, but it is difficult to determine which choices of ω make $\mathbb{Z}[\omega]$ a UFD, a PID (Principal Ideal Domain), or an ED (Euclidean Domain) [61]. One of the main results of this section is the following theorem. In this theorem, any subring of $\mathbb{Z}[\omega]$ is assumed to contain the natural number 1.

Theorem 2.1.8 *Suppose that $\mathbb{Z}[\omega]$ is a UFD. Let R be a subring of $Q_c(\mathbb{Z}[\omega])$ such that $S \subseteq R \subseteq Q_c(S)$ for some subring S of $\mathbb{Z}[\omega]$. Then the following are equivalent:*

1. $R = Q_c(R)(= Q_c(S))$.
2. $\mathbb{M}_n(R)$ *is strongly clean for all* $n \geq 1$.
3. $\mathbb{M}_n(R)$ *is strongly clean for some* $n > 1$.
4. $\mathbb{M}_2(R)$ *is strongly clean.*

Proof "(*1*) $\Rightarrow$ (*2*)" because being artinian is a property of Morita invariant and artinian rings are one-sided perfect and one-sided perfect rings are strongly clean. "(*2*) $\Rightarrow$ (*3*)" is clear.

"(*3*) $\Rightarrow$ (*4*)". This is by Theorem 2.1.1.

"(*4*) $\Rightarrow$ (*1*)". Suppose that $R \neq Q_c(R)$.

Case 1. There exists a nonzero nonunit $p \in S$ such that $p \notin \mathbb{Z}$. Since $p \in \mathbb{Z}[\omega]$, write $p = u + v\omega$ with $u, v \in \mathbb{Z}$. Then $v \neq 0$. Choose $q \in \mathbb{Z}$ to be a prime number such that

$$q > max\{(2v)^2|\omega^2| + 1, 4|u|\}.$$

By (*4*), $A = \begin{pmatrix} p+1 & p \\ q & p \end{pmatrix}$ is a strongly clean element of $\mathbb{M}_2(R)$. Therefore, by Corollary 2.1.7, $4qp + 1$ is a square in R. Because $4qp + 1 \in \mathbb{Z}[\omega]$ and $\mathbb{Z}[\omega]$ is a UFD, $4qp + 1 = x^2$ is solvable in $Q_c(\mathbb{Z}[\omega])$ if and only if the equation is solvable in $\mathbb{Z}(\omega)$. Therefore, there exists $\xi \in \mathbb{Z}[\omega]$ such that

$$\xi^2 = 4qp + 1. \tag{2.1.5}$$

Write $\xi = n + m\omega$ with $n, m \in \mathbb{Z}$. Then it follows from (2.1.5) that

$$4qu + 1 = n^2 + m^2\omega^2, \tag{2.1.6}$$

$$2qv = nm. \tag{2.1.7}$$

Since q is a prime number, either $q|n$ or $q|m$. If $q|n$, write $n = qn_1$. Then $2v = n_1m$ by (2.1.7). Since $v \neq 0$, $n_1 \neq 0$, so $m = \frac{2v}{n_1}$. Thus, (2.1.6) yields

$$4qu + 1 = q^2n_1^2 + (\frac{2v}{n_1})^2\omega^2, \tag{2.1.8}$$

showing that

$$q \;\Big|\; [(\frac{2v}{n_1})^2\omega^2 - 1]\,. \tag{2.1.9}$$

Note that

$$|(\frac{2v}{n_1})^2\omega^2 - 1| \leq (\frac{2v}{n_1})^2|\omega^2| + 1 \leq (2v)^2|\omega^2| + 1 < q,$$

so it follows from (2.1.9) that $(\frac{2v}{n_1})^2\omega^2 - 1 = 0$. Thus, (2.1.8) yields $4qu = q^2n_1^2$, so $q = \frac{4u}{n_1^2}$, contrary to the fact that $q > 4|u|$.

So it must be that $q|m$. Write $m = qm_1$. Then $2v = nm_1$ by (2.1.7). Since $v \neq 0$, $m_1 \neq 0$, so $n = \frac{2v}{m_1}$. By (2.1.6), we have

$$4qu + 1 = (\frac{2v}{m_1})^2 + q^2m_1^2\omega^2, \quad \text{that is,} \quad q(4u - qm_1^2\omega^2) = (\frac{2v}{m_1})^2 - 1.$$

Since $(\frac{2v}{m_1})^2 - 1 \geq 0$ and $q > (\frac{2v}{m_1})^2 - 1$ (because $q > (2v)^2|\omega^2| + 1$), it must be that $(\frac{2v}{m_1})^2 - 1 = 0$. This shows that $q = \frac{4u}{m_1^2\omega^2} = \frac{4|u|}{m_1^2|\omega^2|} \leq 4|u|$, a contradiction.

Case 2. Every nonzero nonunit $z \in S$ is an element of $\mathbb{Z}$. We claim that $S = \mathbb{Z}$. If not, then there exists $n + m\omega \in S$ with $n, m \in \mathbb{Z}$ and $m \neq 0$. Because $R \neq Q_c(R)$, R has a nonzero nonunit z such that $z \in S$. By hypothesis, $z \in \mathbb{Z}$. So $z(n + m\omega)$ is a nonzero nonunit of S. But $z(n + m\omega) \notin \mathbb{Z}$ since $zm \neq 0$. This contradiction shows that $S = \mathbb{Z}$. Thus, by hypothesis, $\mathbb{Z} \subseteq R \subset \mathbb{Q}$. Take a prime number $p \in R$ but $\frac{1}{p} \notin R$. Choose a prime number q with $q > p + 2$. Then by (*4*), $A = \begin{pmatrix} p+1 & p \\ q & p \end{pmatrix}$ is a strongly clean element of $\mathbb{M}_2(R)$. Therefore, by Corollary 2.1.7, $4qp + 1$ is a square in R. Since $R \subseteq \mathbb{Q}$ and $4qp + 1 \in \mathbb{Z}$, there exists $m \in \mathbb{Z}$ such that $m^2 = 4qp + 1$ and $m > 1$. Therefore,

$$qp = \frac{m+1}{2} \cdot \frac{m-1}{2}.$$

It can be verified that

$$\begin{aligned}
\frac{m+1}{2} &= 1 \Longrightarrow m = 1, \\
\frac{m+1}{2} &= p \Longrightarrow q = p - 1, \\
\frac{m+1}{2} &= q \Longrightarrow q = p + 1, \text{ and} \\
\frac{m+1}{2} &= qp \Longrightarrow qp = 2.
\end{aligned}$$

But this is impossible by the choice of q. The proof is complete. □

Corollary 2.1.9 *Let $R = \mathbb{Z}[\omega]$ be a UFD (for example, $\omega = \sqrt{-1}, \sqrt{-2}, \sqrt{2}, \sqrt{3}$, etc.). Then for any prime ideal $\mathfrak{p}$ of R, $\mathbb{M}_n(R_{\mathfrak{p}})$ is not strongly clean for every $n \geq 2$.*

Proof By $R_{\mathfrak{p}} \subseteq R_{\mathfrak{p}} \subseteq Q_c(R_{\mathfrak{p}})$ and Theorem 2.1.8, we get the result. □

2.2 Strongly clean matrices via similarity

In this section, we give a necessary and sufficient condition for a matrix of $\mathbb{M}_n(R)$ to be strongly clean where R has IBN (see Definition 2.2.3) and every finitely generated projective R-module is free. As an easy consequence of this result, a criterion for a 2×2 matrix over a commutative local ring to be strongly clean and a criterion for $\mathbb{M}_2(R)$ over a commutative local ring R to be strongly clean are obtained. At the end of this section we present a family of non-trivial strongly clean matrix rings.

A matrix $A \in \mathbb{M}_n(R)$ is called **singular** if A is non-invertible and **nonsingular** if A is invertible. Here we give a more detailed definition related to singularity of a matrix.

Definition 2.2.1 *A singular matrix $A \in \mathbb{M}_n(R)$ is called* ***purely singular*** *if $I - A$ is singular and* ***semi-purely singular*** *if $I - A$ is nonsingular. A nonsingular matrix $A \in \mathbb{M}_n(R)$ is called* ***purely nonsingular*** *if $I - A$ is nonsingular and* ***semi-purely nonsingular*** *if $I - A$ is singular.*

Every matrix belongs to exactly one of the above four types. All types of matrices are strongly clean except purely singular ones. So we have the following lemma.

Lemma 2.2.2 *The matrix ring $\mathbb{M}_n(R)$ is strongly clean if and only if its purely singular matrices are strongly clean.*

Definition 2.2.3 *[45, Definition 1.3] A ring R is said to have right* IBN *(Invariant Basis Number) if, for any natural numbers n, m, $(R^n)_R \cong (R^m)_R$ implies that $n = m$.*

Notice that this definition means that any two bases of a finitely generated free module F_R have the same finite number of elements. This common number is defined to be the ***rank*** of F_R. Similarly, we can define left IBN. It is known that a ring has right IBN iff it has left IBN. So we can speak of the IBN property of a ring without distinction of "left" or "right".

The following lemma will be useful later.

Lemma 2.2.4 *[52, Theorem 3] Let M_R be a module. Then the following are equivalent for $\varphi \in \mathrm{End}(M_R)$:*

1. *φ is strongly clean in* $\mathrm{End}(M_R)$.
2. *There is a decomposition $M = P \oplus Q$ where P and Q are φ-invariant, and $\varphi|_P$ and $(1-\varphi)|_Q$ are isomorphisms.*

Pictorially, φ is strongly clean iff M_R has a PQPQ-decomposition:

$$\begin{array}{cccccc} M_R & = & P & \oplus & Q \\ & & {\scriptstyle\varphi}\big\downarrow{\scriptstyle\cong} & & {\scriptstyle 1-\varphi}\big\downarrow{\scriptstyle\cong} \\ M_R & = & P & \oplus & Q. \end{array}$$

Similar characterizations for φ to be clean, strongly π-regular, or strongly regular were given in [16].

It is well known that $\mathbb{M}_n(R) \cong \mathrm{End}((R^n)_R)$ and $(R^n)_R$ is a left $\mathbb{M}_n(R)$-module. Fix a basis of $(R^n)_R$. Then every element $v \in (R^n)_R$ can be considered as an $n \times 1$ matrix.

Furthermore, on one hand, every right R-module endomorphism $\varphi \in \mathrm{End}((R^n)_R)$ corresponds to exactly one matrix $T \in \mathbb{M}_n(R)$ and $\varphi(v) = Tv$ where Tv is the multiplication of matrices T and v; and on the other hand, every matrix $T \in \mathbb{M}_n(R)$ corresponds to exactly one right R-module endomorphism $\varphi \in \mathrm{End}((R^n)_R)$ and $Tv = \varphi(v)$. So for every matrix $T \in \mathbb{M}_n(R)$, we always use $\varphi_T \in \mathrm{End}((R^n)_R)$ to correspond to T and $\varphi_T(v) = Tv$. For convenience, in section 2.5 and section 3.1, we will directly use $T \in \mathbb{M}_n(R)$ to denote the endomorphism $\varphi_T \in \mathrm{End}((R^n)_R)$ and we say "the kernel of T" instead of " the kernel of φ_T" and "the image of T" instead of "the image of φ_T".

Now we can prove the following theorem:

Theorem 2.2.5 *Let R be a ring having* IBN *and every finitely generated projective R-module be free. Then a purely singular matrix $T \in \mathbb{M}_n(R)$ is strongly clean iff T is similar to $C = \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}$ where T_0 is semi-purely nonsingular and T_1 is semi-purely singular.*

Proof "$\Rightarrow$". Suppose T is purely singular and strongly clean. Let $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_n\}$ be a basis of $(R^n)_R$ and, under this basis, T is the matrix corresponding to φ_T. Then

$$\varphi_T(\epsilon_1, \epsilon_2, \cdots, \epsilon_n) = (\varphi_T(\epsilon_1), \varphi_T(\epsilon_2), \cdots, \varphi_T(\epsilon_n)) = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n)T.$$

By Lemma 2.2.4, there exist $R_1 \neq 0$ and $R_2 \neq 0$ such that

$$\varphi_T : (R^n)_R = R_1 \oplus R_2 \to (R^n)_R = R_1 \oplus R_2$$

with $\varphi_T|_{R_1}$ and $(1 - \varphi_T)|_{R_2}$ being right R-module isomorphisms. The direct summands R_1 and R_2 are projective right R modules and so they are both free. In addition, they satisfy

$$n = rank((R^n)_R) = rank(R_1) + rank(R_2) \tag{2.2.1}$$

since R has IBN. Suppose $rank(R_1) = k$. Then by equality (2.2.1), we can assume that $\{\eta_1, \eta_2, \cdots, \eta_n\}$ is a basis of $(R_R)^n$ where $\{\eta_1, \eta_2, \cdots, \eta_k\}$ is a basis of R_1 and $\{\eta_{k+1}, \eta_{k+2}, \cdots, \eta_n\}$ is a basis of R_2. Since $\varphi_T|_{R_1} : R_1 \to R_1$ and $(1 - \varphi_T)|_{R_2} : R_2 \to R_2$ are both isomorphisms, we have

$$\varphi_T|_{R_1}(\eta_1, \eta_2, \cdots, \eta_k) = (\eta_1, \eta_2, \cdots, \eta_k)T_0 \tag{2.2.2}$$

with some T_0 being nonsingular and

$$(1-\varphi_T)|_{R_2}(\eta_{k+1},\eta_{k+2},\cdots,\eta_n)=(\eta_{k+1},\eta_{k+2},\cdots,\eta_n)T_1^{'} \tag{2.2.3}$$

with some $T_1^{'}$ being nonsingular. By equality (2.2.3), we get

$$\begin{aligned}&(\varphi_T|_{R_2})(\eta_{k+1},\eta_{k+2},\cdots,\eta_n)\\&=(\eta_{k+1},\eta_{k+2},\cdots,\eta_n)I_{n-k}-(\eta_{k+1},\eta_{k+2},\cdots,\eta_n)T_1^{'}\\&=(\eta_{k+1},\eta_{k+2},\cdots,\eta_n)(I_{n-k}-T_1^{'}).\end{aligned}$$

Claim. $T_1=I_{n-k}-T_1^{'}$ is singular.

Let $C=\begin{pmatrix}T_0&0\\0&T_1\end{pmatrix}$. If T_1 is nonsingular, then

$$\begin{aligned}&\varphi_T(\eta_1,\eta_2,\cdots,\eta_n)\\&=(\varphi_T(\eta_1),\varphi_T(\eta_1),\cdots,\varphi_T(\eta_n))\\&=(\eta_1,\eta_2,\cdots,\eta_n)\begin{pmatrix}T_0&0\\0&T_1\end{pmatrix}\\&=(\eta_1,\eta_2,\cdots,\eta_n)C.\end{aligned}$$

Now $\begin{pmatrix}T_0&0\\0&T_1\end{pmatrix}\begin{pmatrix}T_0^{-1}&0\\0&T_1^{-1}\end{pmatrix}=\begin{pmatrix}T_0^{-1}&0\\0&T_1^{-1}\end{pmatrix}\begin{pmatrix}T_0&0\\0&T_1\end{pmatrix}=I_n$. So C is nonsingular and φ_T is an isomorphism under the basis $\{\eta_1,\eta_2,\cdots,\eta_n\}$. But φ_T is not an isomorphism because T is purely singular. Hence, $T_1=I_{n-k}-T_1^{'}$ is singular. We proved the claim.

Since $\{\epsilon_1,\epsilon_2,\cdots,\epsilon_n\}$ and $\{\eta_1,\eta_2,\cdots,\eta_n\}$ are both bases of $(R^n)_R$, we have

$$(\epsilon_1,\epsilon_2,\cdots,\epsilon_n)=(\eta_1,\eta_2,\cdots,\eta_n)P_1,$$

$$(\eta_1,\eta_2,\cdots,\eta_n)=(\epsilon_1,\epsilon_2,\cdots,\epsilon_n)P_2.$$

So $(\epsilon_1,\epsilon_2,\cdots,\epsilon_n)=(\epsilon_1,\epsilon_2,\cdots,\epsilon_n)P_2P_1=(\epsilon_1,\epsilon_2,\cdots,\epsilon_n)I_n$.

By the uniqueness of the expression of every element of a free module, we get $P_2P_1=I_n$. Similarly, we get $P_1P_2=I_n$. Hence, $P_2P_1=P_1P_2=I_n$. Let $P=P_1=P_2^{-1}$. Now

$$\varphi_T(\epsilon_1,\epsilon_2,\cdots,\epsilon_n)=(\epsilon_1,\epsilon_2,\cdots,\epsilon_n)T,$$

$$\varphi_T(\epsilon_1,\epsilon_2,\cdots,\epsilon_n)=\varphi_T((\eta_1,\eta_2,\cdots,\eta_n)P_1)$$

$$
\begin{aligned}
&= \varphi_T((\eta_1, \eta_2, \cdots, \eta_n))P_1 \\
&= (\eta_1, \eta_2, \cdots, \eta_n) \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix} P_1 \\
&= (\epsilon_1, \epsilon_2, \cdots, \epsilon_n) P_2 \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix} P_1 \\
&= (\epsilon_1, \epsilon_2, \cdots, \epsilon_n) P^{-1} \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix} P.
\end{aligned}
$$

So $PTP^{-1} = \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}$. Since $I_{n-k} - T_1 = T_1'$ is nonsingular, we get T_1 is semi-purely singular.

Let $I_k - T_0 = T_0'$. Then

$$
\begin{aligned}
&P^{-1}(I_n - T)P = I_n - P^{-1}TP \\
&= I_n - \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix} = \begin{pmatrix} I_k & 0 \\ 0 & I_{n-k} \end{pmatrix} - \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix} \\
&= \begin{pmatrix} I_k - T_0 & 0 \\ 0 & I_{n-k} - T_1 \end{pmatrix} = \begin{pmatrix} T_0' & 0 \\ 0 & T_1' \end{pmatrix}.
\end{aligned}
$$

If T_0' is nonsingular, then $I_n - T$ is nonsingular which contradicts the fact that T is purely singular. So T_0 is semi-purely non-singular.

"$\Leftarrow$". Suppose there exists $P \in GL_n(R)$ such that $P^{-1}TP = C = \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}$ where T_0 is semi-purely nonsingular of order k and T_1 is semi-purely singular. Then

$$
C = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-k} \end{pmatrix} + \begin{pmatrix} T_0 & 0 \\ 0 & T_1 - I_{n-k} \end{pmatrix}
$$

where $\begin{pmatrix} 0 & 0 \\ 0 & I_{n-k} \end{pmatrix}$ is an idempotent and $\begin{pmatrix} T_0 & 0 \\ 0 & T_1 - I_{n-k} \end{pmatrix}$ is a nonsingular matrix. Since C is strongly clean and the matrix which is similar to a strongly clean matrix is also strongly clean, T is strongly clean. $\square$

Every PID has IBN and every submodule of a finitely generated free module over a PID is a free module [55, Theorem VI.1]. Every local ring has IBN [45, Example 1.6] and every projective module over a local ring is free (see [2, Corollary 26.7] or [44, Theorem 19.29]). The well-known Quillen-Suslin Theorem [59, p.149] says that if R is a PID, then every finitely generated projective $R[t_1, \cdots, t_k]$-module is free. So the class of rings having IBN and every finite generated projective module over them free is large. Here we get a criterion for a single matrix over such a ring to be strongly clean.

Corollary 2.2.6 *Let R be a ring having* IBN, *assume every finitely generated projective R-module be free, and let $T \in \mathbb{M}_2(R)$ be purely singular. Then T is strongly clean iff T is similar to* $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ *with $v-1, u \in U(R)$ and $v, u-1 \notin U(R)$.*

Proof By Theorem 2.2.5 and Lemma 2.2.2. □

The claims of the next two examples follow by Corollary 2.2.6.

Example 2.2.7 *Suppose that both $A \in \mathbb{M}_2(\mathbb{Z})$ and $I - A$ are non-invertible, then A is strongly clean iff A is similar to one of the elements in* $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \right\}$.

Example 2.2.8 *Let $\mathbb{Z}[\mathbf{i}] = \{a + b\mathbf{i} : a, b \in \mathbb{Z}\}$ be the ring of the Gaussian integers. If $A \in \mathbb{M}_2(\mathbb{Z}[\mathbf{i}])$ and $I - A$ are non-invertible, then A is strongly clean iff A is similar to one of the elements in* $\left\{ \begin{pmatrix} t_0 & 0 \\ 0 & t_1 \end{pmatrix} : t_0 \in \{1, -1, \mathbf{i}, -\mathbf{i}\}, t_1 \in \{0, 2, 1-\mathbf{i}, 1+\mathbf{i}\} \right\}$.

For a local ring, we have

Corollary 2.2.9 *Let R be a local ring and let $T \in \mathbb{M}_2(R)$ be purely singular. Then T is strongly clean iff it is similar to* $\begin{pmatrix} 1+j_0 & 0 \\ 0 & j_1 \end{pmatrix}$ *with $j_0, j_1 \in J(R)$.*

Proof By Corollary 2.2.6. □

Corollary 2.2.10 *Let R be a local ring. Then the following are equivalent:*

1. *$\mathbb{M}_2(R)$ is strongly clean.*
2. *Every purely singular matrix in $\mathbb{M}_2(R)$ is similar to $\begin{pmatrix} 1+j_0 & 0 \\ 0 & j_1 \end{pmatrix}$ with $j_0, j_1 \in J(R)$.*
3. *Every purely singular matrix in $\mathbb{M}_2(R)$ is similar to a diagonal matrix.*

Proof "(*1*) $\Rightarrow$ (*2*)". By Lemma 2.2.2 and Corollary 2.2.9.

"(*2*) $\Rightarrow$ (*3*)". This is trivially true.

"(*3*) $\Rightarrow$ (*1*)". By Lemma 2.2.2 and the fact that every diagonal matrix over a local ring is strongly clean. $\square$

Using the techniques of [22] and [23], the author of [46, Theorem 2.6] proved the equivalences (*1*) $\Leftrightarrow$ (*3*) $\Leftrightarrow$ (*4*) of the next result. Here we give a much simpler proof.

Corollary 2.2.11 *Let R be a commutative local ring. Then the following are equivalent for the matrix $A \in \mathbb{M}_2(R)$:*

1. *A is purely singular and strongly clean.*
2. *A is similar to $\begin{pmatrix} t_0 & 0 \\ 0 & t_1 \end{pmatrix}$, where $1 - t_0 \in J(R)$ and $t_1 \in J(R)$.*
3. *$|A| \in J(R)$ and $1 - \mathrm{tr}(A) \in J(R)$ and A is similar to a diagonal matrix.*
4. *$|A| \in J(R)$ and $1 - \mathrm{tr}(A) \in J(R)$ and $x^2 - \mathrm{tr}(A)x + |A| = 0$ is solvable in R.*

Proof "(*1*) $\Rightarrow$ (*2*)". It follows by Corollary 2.2.9.

"(*2*) $\Rightarrow$ (*3*)". It is clear.

"(*3*) $\Rightarrow$ (*4*)". Suppose that (*3*) holds and assume A is similar to $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Since similarity preserves the determinants and the traces of matrices, one obtains that $|A| = ab$ and $\mathrm{tr}(A) = a+b$. So both a and b are roots in R of $x^2 - \mathrm{tr}(A)x + |A|$. Hence, (*4*) holds.

"*(4)* $\Rightarrow$ *(1)*". Suppose that *(4)* holds. Let $a \in R$ be a root of $x^2 - \mathrm{tr}(A)x + |A|$. Then $b = \mathrm{tr}(A) - a$ is also a root of $x^2 - \mathrm{tr}(A)x + |A|$. Thus, $a + b = \mathrm{tr}(A)$ and $ab = |A|$. Since $\mathrm{tr}(A) \in U(R)$ and $|A| \in J(R)$, one of a, b must be a unit and the other must be in $J(R)$. Without loss of generality, we assume that $a \in U(R)$ and $b \in J(R)$. Write $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. From $a_{11} + a_{22} = \mathrm{tr}(A) \in U(R)$, either a_{11} or a_{22} is a unit. Without loss of generality, we may assume that $a_{22} \in U(R)$. Let $P = \begin{pmatrix} a_{21} & a - a_{11} \\ b - a_{22} & a_{12} \end{pmatrix}$, and thus $P \in \mathrm{GL}_2(R)$ since $|P| = aa_{22} + b(a_{11} - a) - |A| \in U(R)$. Then

$$\begin{aligned} PAP^{-1} &= \frac{1}{|P|}\begin{pmatrix} a_{21} & a - a_{11} \\ b - a_{22} & a_{12} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} a_{12} & a_{11} - a \\ a_{22} - b & a_{21} \end{pmatrix} \\ &= \frac{1}{|P|}\begin{pmatrix} * & a_{21}(-a^2 + \mathrm{tr}(A)a - |A|) \\ a_{12}(-b^2 + \mathrm{tr}(A)b - |A|) & * \end{pmatrix} \\ &= \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \end{aligned}$$

is strongly clean in $\mathbb{M}_2(R)$, and so is A. By direct calculation, the hypothesis shows that $A \notin \mathrm{GL}_2(R)$ and $I - A \notin \mathrm{GL}_2(R)$, that is, A is purely singular. $\square$

The following is essentially [23, Theorem 8] and is also contained in [12, Proposition 24].

Corollary 2.2.12 *Let R be a commutative local ring. Then $\mathbb{M}_2(R)$ is strongly clean iff for every $w \in J(R)$, $t^2 - t = w$ is solvable in R.*

Proof "$\Leftarrow$". Let $A \in \mathbb{M}_2(R)$ and assume that A is a purely singular matrix. Then $|A| \in J(R)$ and $1 - \mathrm{tr}(A) \in J(R)$. Thus, $\mathrm{tr}(A) \in U(R)$. By hypothesis, there exists $a \in R$ such that $a^2 - a + \frac{|A|}{\mathrm{tr}(A)^2} = 0$. Thus, $(\mathrm{tr}(A)a)^2 - \mathrm{tr}(A)(\mathrm{tr}(A)a) + |A| = 0$. So $t^2 - \mathrm{tr}(A)t + |A| = 0$ is solvable in R. Hence A is strongly clean in $\mathbb{M}_2(R)$ by Corollary 2.2.11.

"$\Rightarrow$". Let $w \in J(R)$ and $A = \begin{pmatrix} 1 & w \\ 1 & 0 \end{pmatrix}$. Then $|A| = |I - A| = -w \in J(R)$. So A is a purely singular strongly clean matrix. Thus, by Corollary 2.2.11, $t^2 - \mathrm{tr}(A)t + |A| = 0$ is solvable in R. That is, $t^2 - t = w$ is solvable in R (This direction can be proved by

Theorem 2.1.2). $\square$

Let $C_2 = \{1, g\}$ be the abelian group of order 2. The proof of (*2*) of the next lemma for the group ring RC_2 is contained in the proof of [38, Proposition 3] and (*1*) follows from [50]. Here we give a simple proof.

Lemma 2.2.13 *Let R be a commutative local ring.*

1. *If $2 \in J(R)$, then $J(RC_2) = \{r_0+r_1g : r_0+r_1 \in J(R)\}$ and $RC_2/J(RC_2) \cong R/J(R)$. In particular, RC_2 is local.*

2. *If $2 \in U(R)$, then $RC_2 \cong R \oplus R$.*

Proof (*1*). Write $RC_2 = \{a + bg : a, b \in R\}$. Note that if $a^2 - b^2 \in U(R)$, then $(a+bg)^{-1} = (a^2-b^2)^{-1}(a-bg)$. Let $\Delta = \{a+bg : a+b \in J(R)\}$. Then Δ is an ideal of RC_2. For any $a+bg \in \Delta$, $1+(a+bg) = (1+a)+bg \in U(RC_2)$ because $(1+a)^2 - b^2 = 1 + [2a + (a+b)(a-b)] \in U(R)$. So $\Delta \subseteq J(RC_2)$. But it is clear that $J(RC_2) \subseteq \Delta$, so $\Delta = J(RC_2)$. Thus, $R/J(R) \to RC_2/J(RC_2)$ given by $r + J(R) \mapsto r + J(RC_2)$ is a ring isomorphism.

(*2*). This is because $\theta : RC_2 \to R \oplus R$, $a + bg \mapsto (a+b, a-b)$, is an isomorphism. $\square$

Lemma 2.2.14 *Let R be a local ring, $w \in J(R)$, and $u \in U(R)$ be central. The following are equivalent:*

1. *$x^2 - ux = w$ is solvable in R.*

2. *$x^2 - ux = w$ is solvable in $U(R)$.*

3. *$x^2 - ux = w$ is solvable in $J(R)$.*

Proof If x_0 satisfies the equation, then so does $u - x_0$, and in this case $x_0(u - x_0) = -w \in J(R)$. Hence one of x_0 and $u - x_0$ is in $J(R)$ and the other belongs to $U(R)$. $\square$

Corollary 2.2.15 *Let R be a commutative local ring. The following are equivalent:*

1. *$\mathbb{M}_2(R)$ is strongly clean.*

2. *$\mathbb{M}_2(RC_2)$ is strongly clean.*

Proof "*(2)* $\Rightarrow$ *(1)*". This is because $\mathbb{M}_2(R)$ is an image of $\mathbb{M}_2(RC_2)$.

"*(1)* $\Rightarrow$ *(2)*". Let $S = RC_2$.

Case 1. $2 \in U(R)$. By Lemma 2.2.13, $RC_2 \cong R \oplus R$. So $\mathbb{M}_2(RC_2) \cong \mathbb{M}_2(R) \oplus \mathbb{M}_2(R)$ is strongly clean.

Case 2. $2 \in J(R)$. By Lemma 2.2.13, RC_2 is a commutative local ring. For $w \in J(S)$, we show that there exist $x_0, x_1 \in R$ such that $x^2 - x = w$ where $x = x_0 + x_1 g$. By Lemma 2.2.13, $w = r_0 + r_1 g$ where $r_0 + r_1 \in J(R)$. By Corollary 2.2.12, there exists $a_0 \in R$ such that $a_0^2 - a_0 = r_0 + r_1$. Let $x_0 = a_0 - x_1$. Then

$$x^2 - x = w \Longleftrightarrow 2x_1^2 + (1 - 2a_0)x_1 = -r_1.$$

So it suffices to show that $2y^2 + (1 - 2a_0)y = -r_1$ is solvable in R. Because $2a_0 - 1 \in U(R)$, the substitution $y = (2a_0 - 1)z$ shows that

$$2y^2 + (1 - 2a_0)y = -r_1 \Longleftrightarrow 2z^2 - z = b$$

where $b = -r_1(2a_0 - 1)^{-2}$. So it suffices to show that $2z^2 - z = b$ is solvable in R. Because $2b \in J(R)$, Corollary 2.2.12 ensures that there exists $z_0 \in R$ such that $z_0^2 - z_0 = 2b$. And by Lemma 2.2.14 we can assume that $z_0 \in J(R)$; so $1 - z_0 \in U(R)$. Then $z = b(z_0 - 1)^{-1}$ satisfies $2z^2 - z = b$. $\square$

The first known strongly clean matrix ring over a local ring which is not a division ring is $\mathbb{M}_2(\widehat{\mathbb{Z}}_p)$ ([22, Theorem 2.4]). Here it follows easily from the following.

Corollary 2.2.16 *$\mathbb{M}_2(R)$ is strongly clean if R is a Henselian ring. In particular, $\mathbb{M}_2(\widehat{\mathbb{Z}}_p)$ and $\mathbb{M}_2(F[[x]])$ are strongly clean where F is a field.*

Proof Let $\omega \in J(R)$. We prove $t^2 - t = \omega$ is solvable in R. Let $\theta : R \to R/J(R)$, $r \mapsto \overline{r}$, be the natural ring homomorphism. Then $\theta' : R[t] \to \frac{R}{J(R)}[t]$, $f(t) = a_0 + \cdots + a_n t^n \mapsto \overline{f}(t) = \overline{a_0} + \cdots + \overline{a_n} t^n$, is a map. Let $g(t) = t^2 - t - \omega \in R[t]$. Then $\overline{g}(t) = t(t-1)$ with $\gcd(t, t-1) = 1$. So, by Hensel's Lemma (Definition 1.2.1), $g(t) = (t - \xi_1)(t - \xi_2)$ for some $\xi_1 \in R$ and $\xi_2 \in R$. That is, $t^2 - t = \omega$ is solvable. So $\mathbb{M}_2(R)$ is strongly clean. $\square$

The authors of [12] proved that, for any $n \geq 1$, $\mathbb{M}_n(R)$ is strongly clean when R is Henselian. We will discuss this in Chapter 3.

2.3 When is $\mathbb{M}_2(R)$ over a local ring R strongly clean?

In section 2.2 we got the criteria for $\mathbb{M}_2(R)$ over a commutative local ring R to be strongly clean. In this section, we completely determine when $\mathbb{M}_2(R)$ over a local ring (probably not commutative) is strongly clean.

Lemma 2.3.1 *Let R be a local ring and let $A \in \mathbb{M}_2(R)$. Then either A is invertible or $I - A$ is invertible or A is similar to $\begin{pmatrix} 1+w_0 & 1 \\ w_1 & 0 \end{pmatrix}$ where $w_0, w_1 \in J(R)$.*

Proof Write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and assume neither A nor $I - A$ is invertible. We proceed with three cases.

Case 1. $b \in U(R)$. Let $P = \begin{pmatrix} 1 & 0 \\ -bdb^{-1} & b \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 0 \\ (1-d)b^{-1} & 1 \end{pmatrix}$. Then $P^{-1} = \begin{pmatrix} 1 & 0 \\ db^{-1} & b^{-1} \end{pmatrix}$ and $Q^{-1} = \begin{pmatrix} 1 & 0 \\ -(1-d)b^{-1} & 1 \end{pmatrix}$. Moreover, $B := PAP^{-1} = \begin{pmatrix} a + bdb^{-1} & 1 \\ bc - bdb^{-1}a & 0 \end{pmatrix}$ and $Q(I-A)Q^{-1} = \begin{pmatrix} 1 - a + b(1-d)b^{-1} & -b \\ (1-d)b^{-1}(1-a) - c & 0 \end{pmatrix}$. Since neither A nor $I - A$ is invertible, it follows that $c - db^{-1}a = b^{-1}(bc - bdb^{-1}a) \in J(R)$ and $(1-d)b^{-1}(1-a) - c \in J(R)$. So

$$b^{-1} - db^{-1} - b^{-1}a \in J(R) \ \text{ or } \ 1 - bdb^{-1} - a \in J(R).$$

Let $w_0 = -1 + bdb^{-1} + a$ and $w_1 = b(c - db^{-1}a)$. Then $w_0, w_1 \in J(R)$ and A is similar to $B = \begin{pmatrix} 1+w_0 & 1 \\ w_1 & 0 \end{pmatrix}$.

Case 2. $c \in U(R)$. Let $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $PAP^{-1} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$. The lemma holds by Case 1.

Case 3. $b \notin U(R)$ and $c \notin U(R)$. Then $b, c \in J(R)$. It follows that $a \in U(R)$ or $d \in U(R)$. Because $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$, we may assume that $a \in U(R)$. Because A is not invertible, d is not a unit of R, so $d \in J(R)$. Let $P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then $P^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ and $PAP^{-1} = \begin{pmatrix} a-b & b \\ a+c-b-d & b+d \end{pmatrix}$. Since $a+c-b-d \in U(R)$, the lemma holds by Case 2. $\square$

For a ring R and a polynomial $f(t) = a_0 + a_1 t + a_2 t^2 \cdots + a_n t^n \in R[t]$, an element $r \in R$ is called a **left** (respectively, **right**) **root** of $f(t)$ if $a_0 + ra_1 + r^2 a_2 + \cdots + r^n a_n = 0$ (respectively, $a_0 + a_1 r + a_2 r^2 + \cdots + a_n r^n = 0$). It should be noted that a left root of $f(t)$ need not be a right root although $f(t)$ can be rewritten as $f(t) = a_0 + ta_1 + t^2 a_2 + \cdots + t^n a_n$.

Lemma 2.3.2 *Let R be a local ring and let $u \in U(R)$ and $w_0, w_1 \in J(R)$. Then $\begin{pmatrix} 1+w_0 & u \\ w_1 & 0 \end{pmatrix}$ is strongly clean iff $t^2 - (1+w_0)t - uw_1$ has two left roots, one in $1 + J(R)$ and the other in $J(R)$.*

Proof "$\Rightarrow$". Suppose that $A = \begin{pmatrix} 1+w_0 & u \\ w_1 & 0 \end{pmatrix}$ is strongly clean. Clearly, neither A nor $I - A$ is invertible. So, by Corollary 2.2.9, there exists an invertible matrix $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $PAP^{-1} = \begin{pmatrix} t_0 & 0 \\ 0 & t_1 \end{pmatrix}$ where $1 - t_0, t_1 \in J(R)$.

If $a, d \in U(R)$, let $Q = \begin{pmatrix} a^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}$. Then $QP = \begin{pmatrix} 1 & a^{-1}b \\ d^{-1}c & 1 \end{pmatrix}$ and $(QP)A(QP)^{-1} = Q(PAP^{-1})Q^{-1} = \begin{pmatrix} a^{-1}t_0 a & 0 \\ 0 & d^{-1}t_1 d \end{pmatrix}$ with $1 - a^{-1}t_0 a, d^{-1}t_1 d \in J(R)$.

If $a \notin U(R)$ or $d \notin U(R)$, then $a \in J(R)$ or $d \in J(R)$. Since P is invertible, it follows that both b and c are in $U(R)$. Let $Q = \begin{pmatrix} 0 & c^{-1} \\ b^{-1} & 0 \end{pmatrix}$. Then $QP = \begin{pmatrix} 1 & c^{-1}d \\ b^{-1}a & 1 \end{pmatrix}$ and $(QP)A(QP)^{-1} = Q(PAP^{-1})Q^{-1} = \begin{pmatrix} c^{-1}t_1 c & 0 \\ 0 & b^{-1}t_0 b \end{pmatrix}$ with $1 - b^{-1}t_0 b, c^{-1}t_1 c \in J(R)$.

Therefore, replacing P by QP, we may assume that $P = \begin{pmatrix} 1 & b \\ c & 1 \end{pmatrix}$ such that $PAP^{-1} = \begin{pmatrix} t_0 & 0 \\ 0 & t_1 \end{pmatrix}$ where $1 - t_0, t_1 \in J(R)$ or $1 - t_1, t_0 \in J(R)$. Notice that

$$PAP^{-1} = \begin{pmatrix} t_0 & 0 \\ 0 & t_1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 + w_0 + bw_1 & u \\ c(1+w_0) + w_1 & cu \end{pmatrix} = \begin{pmatrix} t_0 & t_0 b \\ t_1 c & t_1 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} 1 + w_0 + bw_1 = t_0 & (1) \\ u = t_0 b & (2) \\ c(1+w_0) + w_1 = t_1 c & (3) \\ cu = t_1 & (4). \end{cases}$$

By (1), $t_0 \in U(R)$, so the case that $1 - t_1, t_0 \in J(R)$ cannot happen. Thus, it must be that $1 - t_0 \in J(R)$ and $t_1 \in J(R)$. Thus, by (2) and (4), $b \in U(R)$ and $c \in J(R)$. Clearly, (3) and (4) give

$$c(1+w_0) + w_1 = cuc \text{ or } uc(1+w_0) + uw_1 = ucuc.$$

Hence $\lambda_1 = uc \in J(R)$ is a left root of $t^2 - (1+w_0)t - uw_1$.

On the other hand, (1) and (2) give

$$(1+w_0)b + bw_1 b = u \text{ or } b^{-1}(1+w_0) + w_1 = b^{-1}ub^{-1}.$$

Let $\lambda_2 = ub^{-1} \in U(R)$. Then $(\lambda_2)^2 - \lambda_2(1+w_0) - uw_1 = 0$. Thus, $\lambda_2 - 1 - w_0 = (\lambda_2)^{-1}uw_1 \in J(R)$. So λ_2 is also a left root of $t^2 - (1+w_0)t - uw_1$ which is in $1 + J(R)$

"$\Leftarrow$". Suppose that $\lambda_1 \in J(R)$ and $\lambda_2 \in 1 + J(R)$ are two left roots of $t^2 - (1+w_0)t - uw_1$. Let $t_0 = \lambda_2$ and $t_1 = u^{-1}\lambda_1 u$ and let $P = \begin{pmatrix} 1 & \lambda_2^{-1}u \\ u^{-1}\lambda_1 & 1 \end{pmatrix}$. It is easy to see that P is invertible. Moreover,

$$PA = \begin{pmatrix} 1 & \lambda_2^{-1}u \\ u^{-1}\lambda_1 & 1 \end{pmatrix} \begin{pmatrix} 1+w_0 & u \\ w_1 & 0 \end{pmatrix} = \begin{pmatrix} 1 + w_0 + \lambda_2^{-1}uw_1 & u \\ u^{-1}\lambda_1(1+w_0) + w_1 & u^{-1}\lambda_1 u \end{pmatrix}$$

$$\begin{pmatrix} t_0 & 0 \\ 0 & t_1 \end{pmatrix} P = \begin{pmatrix} \lambda_2 & 0 \\ 0 & u^{-1}\lambda_1 u \end{pmatrix} \begin{pmatrix} 1 & \lambda_2^{-1}u \\ u^{-1}\lambda_1 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_2 & u \\ u^{-1}\lambda_1^2 & u^{-1}\lambda_1 u \end{pmatrix}.$$

So $PA = \begin{pmatrix} t_0 & 0 \\ 0 & t_1 \end{pmatrix} P$ because λ_1, λ_2 are left roots of $t^2 - (1 + w_0)t - uw_1$. Hence A is similar to $\begin{pmatrix} t_0 & 0 \\ 0 & t_1 \end{pmatrix}$. So A is strongly clean. $\square$

Lemma 2.3.3 *Let R be a ring with $w_0, w_1, t_0 \in R$. Consider two polynomials $f(t) = t^2 - (1+w_0)t - w_1$ and $g(t) = t^2 - (1-w_0)t - (w_0+w_1)$ over R. Then the following hold for $t_0 \in R$:*

1. *t_0 is a left root of $f(t)$ iff $1 + w_0 - t_0$ is a right root of $f(t)$.*

2. *t_0 is a left root of $f(t)$ iff $1 - t_0$ is a left root of $g(t)$.*

Proof (*1*). This is because that $(1+w_0-t_0)^2 - (1+w_0)(1+w_0-t_0) - w_1 = t_0^2 - t_0(1+w_0) - w_1$.

(*2*). This follows by the fact that $(1-t_0)^2 - (1-t_0)(1-w_0) - (w_0+w_1) = t_0^2 - t_0(1+w_0) - w_1$.

$\square$

Theorem 2.3.4 *The following are equivalent for a local ring R:*

1. *$\mathbb{M}_2(R)$ is strongly clean.*

2. *For any $A \in \mathbb{M}_2(R)$, either A is invertible or $I - A$ is invertible or A is similar to a diagonal matrix.*

3. *For any $w_0, w_1 \in J(R)$, $\begin{pmatrix} 1+w_0 & 1 \\ w_1 & 0 \end{pmatrix}$ is strongly clean.*

4. *For any $w_0, w_1 \in J(R)$, $\begin{pmatrix} 0 & w_1 \\ 1 & 1+w_0 \end{pmatrix}$ is strongly clean.*

5. *For any $w_0, w_1 \in J(R)$, $t^2 - (1+w_0)t - w_1$ has two left roots, one in $1 + J(R)$ and the other in $J(R)$.*

6. *For any $w_0, w_1 \in J(R)$, $t^2 - (1+w_0)t - w_1$ has a left root in $J(R)$.*

7. *For any $w_0, w_1 \in J(R)$, $t^2 - (1+w_0)t - w_1$ has a left root in $1 + J(R)$.*

8. *The right version of (5) or (6) or (7) holds.*

Proof "(*1*) $\Leftrightarrow$ (*2*)". "(*1*) $\Rightarrow$ (*2*)" is by Corollary 2.2.10, and "(*2*) $\Rightarrow$ (*1*)" is clear.

"(*1*) $\Rightarrow$ (*3*)". It is obvious.

"(*3*) $\Leftrightarrow$ (*4*)". This holds because $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1+w_0 & 1 \\ w_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & w_1 \\ 1 & 1+w_0 \end{pmatrix}$.

"(*3*) $\Leftrightarrow$ (*5*)". This is by Lemma 2.3.2.

"(*5*) $\Rightarrow$ (*1*)". Let $A \in \mathbb{M}_2(R)$ and assume that neither A nor $I - A$ is invertible. By Lemma 2.3.1, A is similar to $\begin{pmatrix} 1+w_0 & 1 \\ w_1 & 0 \end{pmatrix}$ where $w_0, w_1 \in J(R)$. By (*5*), $t^2 - (1+w_0)t - w_1$ has two left roots, one in $1+J(R)$ and the other in $J(R)$. So, by Lemma 2.3.2, $\begin{pmatrix} 1+w_0 & 1 \\ w_1 & 0 \end{pmatrix}$ is strongly clean. Hence A is strongly clean.

"(*5*) $\Rightarrow$ (*6*)". This is clear.

"(*6*) $\Rightarrow$ (*5*)". Given $w_0, w_1 \in J(R)$, $t^2 - (1+w_0)t - w_1$ has a left root in $J(R)$ by (*6*). Again by (*6*), $t^2 - (1-w_0)t - (w_0 + w_1)$ has a left root $\lambda \in J(R)$. By Lemma 2.3.3(*2*), $1 - \lambda \in 1 + J(R)$ is a left root of $t^2 - (1+w_0)t - w_1$. So (*5*) holds.

"(*5*) $\Leftrightarrow$ (*7*)". The proof is the same as that of "(*5*) $\Leftrightarrow$ (*6*)".

Furthermore, Lemma 2.3.3 (*1*) proves the equivalence of (*5*) and its right version; the equivalence of (*6*) and the right version of (*7*) and the equivalence of (*7*) and the right version of (*6*) both follow because of Lemma 2.3.3 (*2*). □

Corollary 2.3.5 *[Corollary 2.2.12] Let R be a commutative local ring. Then $\mathbb{M}_2(R)$ is strongly clean iff for every $w \in J(R)$, $t^2 - t = w$ is solvable in R.*

Proof We prove that the solvability of $t^2 - t = w$ in R for all $w \in J(R)$ implies that of $t^2 - (1+w_0)t = w_1$ with every $w_0, w_1 \in J(R)$. Let $t = (1+w_0)x$. Then we have

$$x^2 - x = \frac{w_1}{(1+w_0)^2} \tag{2.3.1}$$

with $\frac{w_1}{(1+w_0)^2} \in J(R)$. If $t^2 - t = w$ is solvable in R for all $w \in J(R)$, then (2.3.1) is solvable. Hence, $t^2 - (1 + w_0)t = w_1$ with every $w_0, w_1 \in J(R)$ is solvable. $\square$

2.4 Applications and examples

Conditions (*5*)-(*8*) of Theorem 2.3.4 are "easy-to-verify" criteria for a 2×2 matrix ring over a local ring to be strongly clean. In this section we use them to give new families of strongly clean rings.

The authors of [12] proved that matrix rings over Henselian rings are all strongly clean. For a general Henselian ring (see Definition 1.2.3), we have the following theorem.

Theorem 2.4.1 *Let R be a general Henselian ring. Then $\mathbb{M}_2(R)$ is strongly clean.*

Proof Let $w_0, w_1 \in J(R)$ and let $f(t) = t^2 - (1 + w_0)t - w_1$. Then $\bar{f}(t) = t^2 - t = t(t-1) \in \overline{R}[t]$. By hypothesis, there exist monic polynomials $t - a, t - b \in R[t]$ such that $f(t) = (t - a)(t - b)$ and $t - \bar{a} = t$ and $t - \bar{b} = t - 1$. It follows that $a \in J(R)$ is a left root of $f(t)$. Hence $\mathbb{M}_2(R)$ is strongly clean by Theorem 2.3.4. $\square$

The next example, which appeared in [7, Example 16], gives a general Henselian ring that is not commutative.

Example 2.4.2 *Let R be a (not necessarily commutative) ring and $d : R \to R$ a derivation, that is , $d(ab) = d(a)b + ad(b)$ for $a, b \in R$. Consider the set of the formal expressions*

$$R((\partial^{-1})) = \Big\{ \sum_{i>-\infty}^{n} a_i\partial^i : a_i \in R \Big\}.$$

For $\alpha = \sum_{i>-\infty}^{n} a_i\partial^i, \beta = \sum_{j>-\infty}^{m} b_j\partial^j \in R((\partial^{-1}))$, define addition $\alpha + \beta$ componentwise and define multiplication according to the Leibnitz rule:

$$\alpha\beta = (\sum_{i>-\infty}^{n} a_i\partial^i)(\sum_{j>-\infty}^{m} b_j\partial^j) := \sum_{i,j;k\geq 0} \binom{i}{k} a_i d^k(b_j)\partial^{i+j-k},$$

where for any $i \in \mathbb{Z}$,

$$\binom{i}{k} = \begin{cases} \frac{i(i-1)\cdots(i-k+1)}{k(k-1)\cdots 1} & , \text{ if } k > 0, \\ 1 & , \text{ if } k = 0. \end{cases}$$

Thus, for $a \in R$,

$$\begin{aligned} [\partial, a] &= \partial a - a\partial = d(a) \text{ and} \\ [\partial^{-1}, a] &= \partial^{-1}a - a\partial^{-1} \\ &= -d(a)\partial^{-2} + d^2(a)\partial^{-3} + \cdots + (-1)^k d^k(a)\partial^{-(k+1)} + \cdots. \end{aligned}$$

Hence $\partial^n a = \sum_{i=0}^{\infty}\binom{n}{i}d^i(a)\partial^{n-i}$for any negative integer n. By [58], $R((\partial^{-1}))$ is a ring, called the ring of formal pseudo-differential operators (Volterra operators) with coefficients from R. These rings are extensively used in applied mathematics and analysis. Our interest here is in the subring $R[[\partial^{-1}]] = \left\{\sum_{i>-\infty}^{0} a_i\partial^i : a_i \in R\right\}$ is a subring of $\subseteq R((\partial^{-1}))$.

When $R = F$ is a field, $F[[\partial^{-1}]]$ is a local ring that is clearly not commutative with Jacobson radical $F[[\partial^{-1}]]\partial^{-1}$ by [58, Proposition 1(i)], and $F[[\partial^{-1}]]$ is a general Henselian ring by [7, Example 16]. So $\mathbb{M}_2(F[[\partial^{-1}]])$ is strongly clean by Theorem 2.4.1.

In order to give another family of strongly clean matrix rings, we need a new notion.

Following [13], a local ring R is called **bleached** if, for all $j \in J(R)$ and $u \in U(R)$, the additive abelian group endomorphisms $l_u - r_j : R \to R$ $(x \mapsto ux - xj)$ and $l_j - r_u : R \to R$ $(x \mapsto jx - xu)$ are surjective. By [13, Example 13], some of the bleached local rings include: commutative local rings, division rings, local rings R with $J(R)$ nil, local rings R for which some power of each element of $J(R)$ is central in R, local rings R for which some power of each element of $U(R)$ is central in R, power series rings over bleached local rings, and skew power series rings $R[[x;\sigma]]$ of a bleached local ring R with σ an automorphism of R.

Definition 2.4.3 *A local ring R is called **weakly bleached** if, for all $j_1, j_2 \in J(R)$, the additive abelian group endomorphisms $l_{1+j_1} - r_{j_2}$ and $l_{j_2} - r_{1+j_1}$ are surjective.*

By Nicholson [52, Example 2] (also see [13, Theorem 18]), a local ring R is weakly bleached iff the 2×2 upper triangular matrix ring $\mathbb{T}_2(R)$ is strongly clean. Because there exists a local ring R such that $\mathbb{T}_2(R)$ is not strongly clean by [13], local rings need not be weakly bleached. On the other hand, bleached rings are clearly weakly bleached. We now cite an example of [13] to show that weakly bleached rings need not be bleached.

Example 2.4.4 *Let k be a field, and let $R = k[t_1, t_2, \cdots]_{(t_1)}$ be a ring of polynomials in countably many indeterminates, localized at the prime ideal (t_1). Let σ be the endomorphism of $k[t_1, t_2, \cdots]$ that is the identity on k and satisfies $\sigma(t_i) = t_{i+1}$ for all i. Then σ extends to the localization R. By [13, Example 38], the local ring $R[[x;\sigma]]$ is not bleached. However, $\mathbb{T}_n(R[[x;\sigma]])$ is strongly clean for all $n \geq 1$ by [13, Theorem 40]. Hence $R[[x;\sigma]]$ is weakly bleached.*

Theorem 2.4.5 *Let R be a weakly bleached local ring and let $\sigma : R \to R$ be an endomorphism with $\sigma(J(R)) \subseteq J(R)$. Then the following are equivalent for $n \geq 1$:*

1. *$\mathbb{M}_2(R)$ is strongly clean.*
2. *$\mathbb{M}_2(R[[x;\sigma]])$ is strongly clean.*
3. *$\mathbb{M}_2(R[x;\sigma]/(x^n))$ is strongly clean.*

Proof "*(2) ⇒ (3) ⇒ (1)*". This follows because any image of a strongly clean ring is again strongly clean.

"*(1) ⇒ (2)*". Let $S = R[[x;\sigma]]$. Note that $J(S) = J(R) + Sx$. By Theorem 2.3.4, it suffices to show that, for any $w_0, w_1 \in J(S)$, $t^2 - (1 + w_0)t - w_1$ has a left root in $J(S)$. Write

$$w_0 = b_0 + b_1 x + \cdots,$$

$$w_1 = c_0 + c_1 x + \cdots,$$

$$t = t_0 + t_1 x + \cdots,$$

where $b_0, c_0 \in J(R)$. Then $t^2 - t(1+w_0) - w_1 = 0 \Leftrightarrow$

$$\begin{cases} t_0^2 - t_0(1+b_0) - c_0 & = 0 \qquad\qquad\qquad\qquad\qquad\qquad (P_0) \\ t_k[1 - \sigma^k(t_0) + \sigma^k(b_0)] - t_0 t_k & = [t_1\sigma(t_{k-1}) + \cdots + t_{k-1}\sigma^{k-1}(t_1)] \\ & \quad -[t_0 b_k + \cdots + t_{k-1}\sigma^{k-1}(b_1)] - c_k \qquad (P_k) \end{cases}$$

for $k = 1, 2, \cdots$. By Theorem 2.3.4, $t^2 - (1+b_0)t - c_0$ has a left root $t_0 \in J(R)$. Thus, $1 - \sigma^k(t_0) + \sigma^k(b_0) \in 1 + J(R)$, so (P_k) is solvable for t_k (because R is weakly bleached) for $k = 1, 2, \cdots$. Thus, $\Sigma_i t_i x^i \in J(S)$ is a left root of $t^2 - (1+w_0)t - w_1$. The proof is complete. □

The next result is [23, Theorem 9] when $\sigma = 1_R$.

Corollary 2.4.6 *Let R be a commutative local ring and let $\sigma : R \longrightarrow R$ be an endomorphism with $\sigma(J(R)) \subseteq J(R)$. Then the following are equivalent for $n \geq 1$:*

1. $\mathbb{M}_2(R)$ *is strongly clean.*

2. $\mathbb{M}_2(R[[x;\sigma]])$ *is strongly clean.*

3. $\mathbb{M}_2(R[x;\sigma]/(x^n))$ *is strongly clean.*

It is unknown whether Henselian rings are exactly those commutative local rings over which the matrix rings are strongly clean (see [12]). But the next example gives a local ring R that is not general Henselian such that $\mathbb{M}_2(R)$ is strongly clean.

Example 2.4.7 *Let D be a division ring and σ an endomorphism of D. Then $\mathbb{M}_2(D[[x;\sigma]])$ is strongly clean by Theorem 2.4.5. If, in particular, $D = \mathbb{C}$ and σ is the complex conjugation, then $D[[x;\sigma]]$ is not general Henselian by [7, Example 17].*

The next corollary follows by Theorems 2.4.1 and 2.4.5.

Corollary 2.4.8 *If R is a weakly bleached general Henselian ring and σ is an endomorphism of R with $\sigma(J(R)) \subseteq J(R)$, then $\mathbb{M}_2(R[[x;\sigma]])$ is strongly clean.*

2.5 A necessary condition for $\mathbb{M}_2(R)$ over an arbitrary ring R to be strongly clean

In this section, we give a necessary condition for the matrix ring $\mathbb{M}_2(R)$ over an arbitrary ring R to be strongly clean. This is a generalization of Theorem 2.1.2 and [27, Theorem 3.7.2]. It is also related to 2.3.4. The method comes from [27].

Theorem 2.5.1 *Let R be a ring for which $\mathbb{M}_2(R)$ is strongly clean. Then*

1. *For any $w_0, w_1 \in J(R)$, the polynomial $t^2 - (1+w_0)t - w_1$ has a right root in $J(R)$ and a right root in $1 + J(R)$.*

2. *For any $w_0, w_1 \in J(R)$, the polynomial $t^2 - (1+w_0)t - w_1$ has a left root in $J(R)$ and a left root in $1 + J(R)$.*

Proof (*1*). Let $A = \begin{pmatrix} 1+w_0 & 1 \\ w_1 & 0 \end{pmatrix}$ and $\left\{ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be the standard basis for R^2. As we discussed before Theorem 2.2.5, under this basis, A corresponds to φ_A. For computation simplicity, we identify the matrix A with the corresponding endomorphism $\varphi_A \in \mathrm{End}((R^2)_R)$. It is clear that A and $I - A$ are non-invertible. So $(R^2)_R$ has a non-trivial $R_1R_2R_1R_2$-decomposition

$$\begin{array}{ccccc} (R^2)_R & = & R_1 & \oplus & R_2 \\ & & A \big\downarrow \cong & & I-A \big\downarrow \cong \\ (R^2)_R & = & R_1 & \oplus & R_2. \end{array}$$

with $0 \neq R_1 < (R^2)_R$ and $0 \neq R_2 < (R^2)_R$. For notation convenience, let bar denote the natural epimorphisms. For example, the natural homomorphism $R \longrightarrow \overline{R} = R/J(R)$ is denoted by $r \longmapsto \overline{r} = r + J(R)$. Let $rad(R^2)$ be the Jacobson radical of the module $(R^2)_R$. Since $A : R_1 \longrightarrow R_1$ is an isomorphism, we get an isomorphism $\overline{A} : (R_1 + rad(R^2))/rad(R^2) \longrightarrow (R_1 + rad(R^2))/rad(R^2)$ with $\overline{A}(\overline{r}) = \overline{A(r)}$. Similarly, $\overline{I} - \overline{A} : (R_2 + rad(R^2))/rad(R^2) \longrightarrow (R_2 + rad(R^2))/rad(R^2)$ is also an isomorphism. For $\overline{A}$ and $\overline{I} - \overline{A}$, we have

$$\overline{R_1} = \overline{A}(\overline{R_1}) \subseteq \overline{A}(\overline{R}^2) = Im \begin{pmatrix} \overline{1} & \overline{1} \\ \overline{0} & \overline{0} \end{pmatrix} = \overline{e}_1 \overline{R}, \tag{2.5.1}$$

and

$$\overline{R_2} = (\overline{I} - \overline{A})(\overline{R_2}) \subseteq (\overline{I} - \overline{A})(\overline{R}^2) = Im \begin{pmatrix} \overline{0} & -\overline{1} \\ \overline{0} & \overline{1} \end{pmatrix} = (\overline{e}_2 - \overline{e}_1)\overline{R}. \tag{2.5.2}$$

Since $R_1 \oplus R_2 = R^2$, we have $\overline{R_1} \oplus \overline{R_2} = \overline{R}^2$. By $\overline{e}_1\overline{R} \oplus (\overline{e}_2 - \overline{e}_1)\overline{R} = \overline{R}^2$, (2.5.1), and (2.5.2), we get $\overline{R_1} = \overline{e}_1\overline{R}$ and $\overline{R_2} = (\overline{e}_2 - \overline{e}_1)\overline{R}$. Let $E : R^2 = R_1 \oplus R_2 \to R_1$ be the projection onto R_1 with kernel R_2. Then $I - E : R^2 = R_1 \oplus R_2 \to R_2$ is the projection onto R_2 with kernel R_1. Let $\eta_1 = Ee_2, \eta_2 = (I - E)e_2$. Then $\eta_1 \in R_1$ and $\eta_2 \in R_2$. So $Ee_2 + (I - E)e_2 = e_2 = e_1 + (e_2 - e_1)$. Hence, $\overline{E}\overline{e}_2 + (\overline{I} - \overline{E})\overline{e}_2 = \overline{e}_2 = \overline{e}_1 + (\overline{e}_2 - \overline{e}_1)$. Since $\overline{E}\overline{e}_2$ and $\overline{e}_1$ are in $\overline{R_1}$, $(\overline{I} - \overline{E})\overline{e}_2$ and $(\overline{e}_2 - \overline{e}_1)$ are in $\overline{R_2}$, and $\overline{R_1} \oplus \overline{R_2} = \overline{R}^2$, we get $\overline{\eta}_1 = \overline{E}\overline{e}_2 = \overline{e}_1$ and $\overline{\eta}_2 = (\overline{I} - \overline{E})\overline{e}_2 = (\overline{e}_2 - \overline{e}_1)$. So $\{\overline{\eta}_1, \overline{\eta}_2\} = \{\overline{E}\overline{e}_2, (\overline{I} - \overline{E})\overline{e}_2\} = \{\overline{e}_1, (\overline{e}_2 - \overline{e}_1)\}$ is a basis for $\overline{R}^2$. Let $\theta : R^2 \to \overline{R}^2$ with $\theta(\eta_1) = \overline{\eta}_1 = \overline{E}\overline{e}_2 = \overline{e}_1$ and $\theta(\eta_2) = \overline{\eta}_2 = (\overline{I} - \overline{E})\overline{e}_2 = (\overline{e}_2 - \overline{e}_1)$. Then $\eta_1 R + \eta_2 R + rad(R^2) = R^2$. However, $rad(R^2)$ is superfluous in R^2 by Nakayama's Lemma, so we get $\eta_1 R + \eta_2 R = R^2$. That is, $\{\eta_1, \eta_2\}$ generate R^2 as a right R-module. Let $\eta_1 r_1 + \eta_2 r_2 = 0$. Then $\eta_1 r_1 = 0$ and $\eta_2 r_2 = 0$ because $\eta_1 \in R_1$, and $\eta_2 \in R_2$. Let $\eta_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\eta_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$. Then by $\overline{\eta}_1 = \overline{E}\overline{e}_2 = \overline{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\overline{\eta}_2 = (\overline{I} - \overline{E})\overline{e}_2 = (\overline{e}_2 - \overline{e}_1) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, we get $x_1, y_2 \in 1 + J(R)$, $y_1 \in J(R)$, and $x_2 \in -1 + J(R)$. So $r_1 = 0$ by $\eta_1 r_1 = \begin{pmatrix} x_1 r_1 \\ y_1 r_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $r_2 = 0$ by $\eta_2 r_2 = \begin{pmatrix} x_2 r_2 \\ y_2 r_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. So η_1 and η_2 are R-linearly independent. Hence, $\{\eta_1, \eta_2\}$ is a basis for R^2. If $r_1 \in R_1$ such that $r_1 = \eta_1 l_1 + \eta_2 l_2$ with $l_1, l_2 \in R$, then $(r_1 - \eta_1 l_1) - \eta_2 l_2 = 0$. Hence, $r_1 - \eta_1 l_1 = 0$ and $\eta_2 l_2 = 0$. So $r_1 = \eta_1 l_1$ and l_1 is uniquely determined because $r_1 = \begin{pmatrix} x_1 l_l \\ y_1 l_1 \end{pmatrix}$ with $x_1 \in U(R)$. So η_1 is a basis for R_1. Similarly, $R_2 = \eta_2 R$ is free with basis η_2. Let $\eta_1' = \eta_1 x_1^{-1} = \begin{pmatrix} 1 \\ x \end{pmatrix}$ with $x = y_1 x_1^{-1} \in J(R)$. Then η_1' is also a basis for R_1. Now $A : R_1 \to R_1$ is an isomorphism. Let $A\eta_1' = \eta_1' r$ with $r \in R$. That is, $\begin{pmatrix} 1 + w_0 & 1 \\ w_1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} = \begin{pmatrix} 1 \\ x \end{pmatrix} r$. So $\begin{pmatrix} 1 + w_0 + x \\ w_1 \end{pmatrix} = \begin{pmatrix} r \\ xr \end{pmatrix}$. Hence, $r = 1 + w_0 + x$.

Notice that $\begin{pmatrix} 1+w_0 & 1 \\ w_1 & 0 \end{pmatrix}^2 - \begin{pmatrix} 1+w_0 & 1 \\ w_1 & 0 \end{pmatrix}(1+w_0) - Iw_1 = 0$. So

$$\begin{aligned}&\left(\begin{pmatrix} 1+w_0 & 1 \\ w_1 & 0 \end{pmatrix}^2 - \begin{pmatrix} 1+w_0 & 1 \\ w_1 & 0 \end{pmatrix}(1+w_0) - Iw_1\right)(\eta_1') \\ &= A^2\eta_1' - A(1+w_0)\eta_1' - w_1\eta_1' = 0.\end{aligned} \tag{2.5.3}$$

By direct computation, we get

$$A^2\eta_1' = \eta_1'(1+w_0+x)^2 = \begin{pmatrix} 1 \\ x \end{pmatrix}(1+w_0+x)^2 = \begin{pmatrix} (1+w_0+x)^2 \\ x(1+w_0+x)^2 \end{pmatrix}, \tag{2.5.4}$$

$$A(1+w_0)\eta_1' = \begin{pmatrix} (1+w_0)^2 & 1+w_0 \\ w_1(1+w_0) & 0 \end{pmatrix}\begin{pmatrix} 1 \\ x \end{pmatrix} = \begin{pmatrix} (1+w_0)^2+(1+w_0)x \\ w_1(1+w_0) \end{pmatrix}, \tag{2.5.5}$$

and

$$w_1\eta_1' = \begin{pmatrix} w_1 \\ w_1 x \end{pmatrix}. \tag{2.5.6}$$

Comparing (2.5.3), (2.5.4), (2.5.5), and (2.5.6), we get $(1+w_0+x)^2 - (1+w_0)(1+w_0+x) - w_1 = 0$. That is, $1+w_0+x \in 1+J(R)$ is a right root of $t^2-(1+w_0)t-w_1=0$. By Lemma 2.3.3, we know $t^2-(1+w_0)t-w_1=0$ also has a right root in $J(R)$.

(*2*). The proof is similar to the above. □

Chapter 3

Strongly Clean Matrix Ring $\mathbb{M}_n(R)$

In this chapter, we discuss the strongly clean property of $\mathbb{M}_n(R)$ where R is commutative or R is commutative local. In [13], the authors determined the commutative local rings R for which $\mathbb{M}_n(R)$ is strongly clean when n is an arbitrary and fixed positive integer by considering the so-called SRC factorization of polynomials of $R[t]$. In section 3.1, we generalize the SRC factorization from a commutative local ring to a commutative ring. We obtain a sufficient condition for $\mathbb{M}_n(R)$ over a commutative ring R to be strongly clean. We also obtain a necessary and sufficient condition for $\mathbb{M}_n(R)$ to be strongly clean where R has IBN and every finitely generated projective R-module is free. In section 3.2, we prove that the strongly clean property of $\mathbb{M}_n(R)$ implies that of $\mathbb{M}_n(R[[x]])$ and of $\mathbb{M}_n\left(\frac{R[x]}{(x^k)}\right)$ if R is commutative local and n, k are positive integers. We also discuss the strongly clean property of $\mathbb{M}_n(RC_2)$ where R is commutative local and C_2 is the abelian group of order two. Section 3.2 comes from [66].

3.1 SRC factorization and strongly clean matrices

In this section, we discuss the SRC factorization of polynomials of $R[t]$ and the strongly clean property of $\mathbb{M}_n(R)$ where R is a commutative ring or R has IBN and every finitely generated projective R-module is free.

Let R be a commutative ring and $A \in \mathbb{M}_n(R)$. For $f(t) = a_0 + a_1 t + \cdots + a_n t^n \in R[t]$, write $f(A) = a_0 I_n + a_1 A + \cdots + a_n A^n \in \mathbb{M}_n(R)$ and let $R[A] = \{f(A) : f(t) \in R[t]\}$. Then $R[A]$ is a subring of $\mathbb{M}_n(R)$. Because $R[t] \to R[A]$, $f(t) \mapsto f(A)$, is a ring homomorphism and $R[A] \to \mathbb{M}_n(R)$ is the inclusion homomorphism, we obtain that $(R^n)_R$ is a left $R[t]$-module with $tv = Av$ for all $v \in (R^n)_R$. As discussed before Theorem 2.2.5, fixing a basis of $(R^n)_R$, we identify a matrix $A \in \mathbb{M}_n(R)$ with the corresponding endomorphism $\varphi_A \in \mathrm{End}((R^n)_R)$, and we say "the kernel of A" instead of "the kernel of φ_A" and "the image of A" instead of "the image of φ_A", and so on.

Theorem 3.1.1 *Let R be a commutative ring and $A \in \mathbb{M}_n(R)$ with characteristic polynomial $\chi_A(t) = \det(tI - A)$. If there exist monic polynomials $f_i(t) \in R[t]$, polynomials $a(t), b(t), c(t) \in R[t]$, and $e_i^2 = e_i \in R$ $(i = 0, 1)$ such that $\chi_A(t) = f_0(t) f_1(t)$ with $f_i(e_i) \in U(R)$(i=0,1) and $f_0(t)a(t) + f_1(t)b(t) = c(t)$ with $c(A) \in GL_n(R)$, then A is strongly clean.*

Proof Let $A \in \mathbb{M}_n(R)$ and let $f_0(t), f_1(t), a(t), b(t), c(t), e_0$, and e_1 be given as in the theorem. Then $f_0(t)a(t) + f_1(t)b(t) = c(t)$ implies $f_0(A)a(A) + f_1(A)b(A) = c(A)$. Notice that $f_0(A), f_1(A), a(A), b(A)$ and $c(A)$ commute with each other. So we get

$$f_0(A)a(A)c(A)^{-1} + f_1(A)b(A)c(A)^{-1} = I_n. \tag{3.1.1}$$

Claim 1. $R^n = \mathrm{Ker}(f_0(A)) \oplus \mathrm{Ker}(f_1(A))$.

Let $x \in R^n$. Then by (3.1.1)

$$x = \left[f_0(A)a(A)c(A)^{-1}\right](x) + \left[f_1(A)b(A)c(A)^{-1}\right](x).$$

By the Cayley-Hamilton Theorem for characteristic polynomials, we have

$$\left[f_0(A)a(A)c(A)^{-1}\right](x) \in \mathrm{Ker}(f_1(A)) \text{ and } \left[f_1(A)b(A)c(A)^{-1}\right](x) \in \mathrm{Ker}(f_0(A)).$$

If $x \in \mathrm{Ker}(f_0(A)) \bigcap \mathrm{Ker}(f_1(A))$, then

$$x = \left[f_0(A)a(A)c(A)^{-1}\right](x) + \left[f_1(A)b(A)c(A)^{-1}\right](x) = 0.$$

So $R^n = \mathrm{Ker}(f_0(A)) \oplus \mathrm{Ker}(f_1(A))$.

Claim 2. $(A - e_0 I)|_{\mathrm{Ker}(f_0(A))}$ and $(e_1 I - A)|_{\mathrm{Ker}(f_1(A))}$ are isomorphisms.

For simplicity, we write $(A - e_0 I)_{\mathrm{Ker}(f_0(A))}$ in stead of $(A - e_0 I)|_{\mathrm{Ker}(f_0(A))}$ and similarly for others. Let $f_0(t) = t^k + \sum_{i=0}^{k-1} a_i t^i$ and $f_1(t) = t^{n-k} + \sum_{i=0}^{n-k-1} b_i t^i$ with $f_0(e_0) \in U(R)$ and $f_1(e_1) \in U(R)$. Then

$$\begin{aligned}
& f_0(A)|_{\mathrm{Ker}(f_0(A))} = 0 \\
& \Rightarrow a_0 I + a_1 A + \cdots + a_{k-1} A^{k-1} + A^k = 0 \text{ on } \mathrm{Ker}(f_0(A)) \\
& \Rightarrow a_0 I + a_1 (A - e_0 I + e_0 I) + \cdots + a_{k-1} (A - e_0 I + e_0 I)^{k-1} \\
& \qquad + (A - e_0 I + e_0 I)^k = 0 \text{ on } \mathrm{Ker}(f_0(A)) \\
& \Rightarrow f_0(e_0) I + g(A)(A - e_0 I) = 0 \text{ on } \mathrm{Ker}(f_0(A)) \text{ for some } g(t) \in R[t] \\
& \Rightarrow (e_0 I - A) g(A) \left(f_0(e_0)\right)^{-1} = I \text{ on } \mathrm{Ker}(f_0(A)).
\end{aligned}$$

By direct computation, $\mathrm{Ker}(f_0(A))$ is both $e_0 I$- and A-invariant. So $\mathrm{Ker}(f_0(A))$ is both $(e_0 I - A)$- and $(g(A) f_0(e_0)^{-1})$-invariant. Let $e_0 I - A = V$. Then, $e_0 I - A = V$ is an isomorphism on $\mathrm{Ker}(f_0(A))$, i.e., $A_{\mathrm{Ker}(f_0(A))} = (e_0 I)_{\mathrm{Ker}(f_0(A))} + V_{\mathrm{Ker}(f_0(A))}$. $(e_0 I)_{\mathrm{Ker}(f_0(A))} V_{\mathrm{Ker}(f_0(A))} = V_{\mathrm{Ker}(f_0(A))} (e_0 I)_{\mathrm{Ker}(f_0(A))}$ because e_0 is a central idempotent.

$$\begin{aligned}
& f_1(A)|_{\mathrm{textKer}(f_1(A))} = 0 \\
& \Rightarrow f_1(A) = b_0 I + b_1 A + \cdots + b_{n-k-1} A^{n-k-1} + A^{n-k} = 0 \text{ on textKer}(f_1(A)) \\
& \Rightarrow f_1(A) = b_0 I + b_1 (A - e_1 I + e_1 I) + \cdots + (A - e_1 I + e_1 I)^{n-k} = 0 \text{ on textKer}(f_1(A)) \\
& \Rightarrow f_1(e_1) I + h(A)(A - e_1 I) = 0 \text{ on } \mathrm{Ker}(f_1(A)) \text{ for some } h(t) \in R[t] \\
& \Rightarrow (e_1 I - A) h(A) \left(f_1(e_1)\right)^{-1} = I \text{ on } \mathrm{Ker}(f_1(A)).
\end{aligned}$$

Similarly, $\mathrm{Ker}(f_1(A))$ is both $(e_1 I)$- and A-invariant. Let $A - e_1 I = W$. Then $A - e_1 I = W$ is an isomorphism on $\mathrm{Ker}(f_1(A))$, i.e., $A_{\mathrm{Ker}(f_1(A))} = (e_1 I)_{\mathrm{Ker}(f_1(A))} + W_{\mathrm{Ker}(f_1(A))}$. Clearly, $(e_1 I)_{\mathrm{Ker}(f_1(A))} W_{\mathrm{Ker}(f_1(A))} = W_{\mathrm{Ker}(f_1(A))} (e_1 I)_{\mathrm{Ker}(f_1(A))}$.

Let $E = \left[(e_0 I)_{\mathrm{Ker}(f_0(A))} \oplus (e_1 I)_{\mathrm{Ker}(f_1(A))}\right]$ and $U = \left[V_{\mathrm{Ker}(f_0(A))} \oplus W_{\mathrm{Ker}(f_1(A))}\right]$. Then by the above argument, E is an idempotent and U is a unit and $EU = UE$. So $A = \left[(e_0 I)_{\mathrm{Ker}(f_0(A))} \oplus (e_1 I)_{\mathrm{Ker}(f_1(A))}\right] + \left[V_{\mathrm{Ker}(f_0(A))} \oplus W_{\mathrm{Ker}(f_1(A))}\right] = E + U$ is the strongly clean expression. $\square$

Recall that, for a commutative ring R, a pair of polynomials $(f_0(t), f_1(t))$ in $R[t]$ is **unimodular** if $f_0(t)R[t] + f_1(t)R[t] = R[t]$ or equivalently, $f_0(t)a(t) + f_1(t)b(t) = 1$ with $a(t)$ and $b(t)$ in $R[t]$.

Corollary 3.1.2 *Let R be a commutative ring and let $(f_0(t), f_1(t))$ be a unimodular pair of monic polynomials in $R[t]$ with $f_i(e_i) \in U(R)$ for some $e_i^2 = e_i \in R$ $(i = 0, 1)$. Then any matrix A with characteristic polynomial $\chi_A(t) = f_0(t)f_1(t)$ is strongly clean.*

Proof This is because $f_0(t)a(t)+f_1(t)b(t) = 1$ where $a(t), b(t), c(t) \in R[t]$ with $c(t) = 1$. $\square$

Example 3.1.3 *In $\left(\mathbb{Z}\left[\sqrt{-5}\right]\right)[t]$ (Notice this ring is not a UFD), $f(t) = t^4 - 4t^3 + 5t^2 - 2t$ has a factorization $f(t) = f_0(t)f_1(t)$ where $f_0(t) = t^2 - 2t + 1$, $f_1(t) = t^2 - 2t$, $e_0 = 0$, and $e_1 = 1$ satisfy the assumption of Corollary 3.1.2. So every matrix in $\mathbb{M}_4\left(\mathbb{Z}\left[\sqrt{-5}\right]\right)$ with characteristic polynomial $f(t)$ is strongly clean. In particular,*

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -5 & -4 & 3 \\ 0 & -8 & -8 & 6 \end{pmatrix} \in \mathbb{M}_4\left(\mathbb{Z}\left[\sqrt{-5}\right]\right) \textit{ is strongly clean. In fact,}$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -6 & -5 & 3 \\ 0 & -8 & -8 & 5 \end{pmatrix} \textit{ is a strongly clean expression with}$$

$$E^2 = E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \textit{ and } U = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -6 & -5 & 3 \\ 0 & -8 & -8 & 5 \end{pmatrix} \in GL_4\left(\mathbb{Z}\left[\sqrt{-5}\right]\right).$$

For a commutative ring R, we use *Max(R)* to denote the maximal spectrum of R, that is, $\text{Max}(R) = \{\mathfrak{m} : \mathfrak{m} \text{ is a maximal ideal in } R\}$. For each $\mathfrak{m} \in \text{Max}(R)$, the natural ring homomorphism $R \to \frac{R}{\mathfrak{m}}$ with $r \mapsto \overline{r} = r + \mathfrak{m}$ induces a map $R[t] \to \frac{R}{\mathfrak{m}}[t]$ with $f(t) = a_0 + a_1 t + \cdots + a_n t^n \mapsto \overline{f}(t) = \overline{a_0} + \overline{a_1} t + \cdots + \overline{a_n} t^n$.

Definition 3.1.4 *A commutative ring R is said to have the* **weakly unimodular** *property if, for any pair $(f_0(t), f_1(t))$ of monic polynomials in $R[t]$, the unimodularity of $(\overline{f_0}(t), \overline{f_1}(t))$ in $\frac{R}{\mathfrak{m}}[t]$ for all $\mathfrak{m} \in Max(R)$ implies the unimodularity of $(f_0(t), f_1(t))$ in $R[t]$.*

A ring R is **semilocal** if $R/J(R)$ is semisimple. A commutative ring is semilocal iff it has finitely many maximal ideals.

Proposition 3.1.5 *Commutative semilocal rings have the weakly unimodular property.*

Proof Let R be a commutative semilocal ring. Then R has finitely many maximal ideals, say $\mathfrak{m}_1, \cdots, \mathfrak{m}_n$. Let $f_0(t), f_1(t) \in R[t]$ be monic polynomials and $(\overline{f_0}(t), \overline{f_1}(t))$ be unimodular in $\frac{R}{\mathfrak{m}_k}[t]$ for $k = 1, 2, \cdots, n$. Since $\overline{f_0}(t)\frac{R}{\mathfrak{m}_k}[t] + \overline{f_1}(t)\frac{R}{\mathfrak{m}_k}[t] = \frac{R}{\mathfrak{m}_k}[t]$, we get $f_0(t)R[t] + f_1(t)R[t] + \mathfrak{m}_k[t] = R[t]$. Hence, $f_0(t)a_k(t) + f_1(t)b_k(t) + c_k(t) = 1$ for some $a_k(t), b_k(t) \in R[t]$ and $c_k(t) \in \mathfrak{m}_k[t]$. Therefore,

$$1 = \Pi_{k=1}^{n} \left(f_0(t)a_k(t) + f_1(t)b_k(t) + c_k(t)\right) = f_0(t)a'(t) + f_1(t)b'(t) + c'(t)$$

for some $a'(t), b'(t) \in R[t]$ and $c'(t) \in J(R)[t]$. Thus, $R[t] = f_0(t)R[t] + f_1(t)R[t] + c'(t)R[t] = f_0(t)R[t] + f_1(t)R[t] + J(R)R[t]$. Notice that $\frac{R[t]}{f_0(t)R[t]+f_1(t)R[t]}$ is a finitely generated R-module and $J(R)\frac{R[t]}{f_0(t)R[t]+f_1(t)R[t]} = \frac{J(R)R[t]+f_0(t)R[t]+f_1(t)R[t]}{f_0(t)R[t]+f_1(t)R[t]} = \frac{R[t]}{f_0(t)R[t]+f_1(t)R[t]}$. So, $f_0(t)R[t] + f_1(t)R[t] = R[t]$ by Nakayama's Lemma. Therefore, $(f_0(t), f_1(t))$ is unimodular in $R[t]$. $\square$

Corollary 3.1.6 *Commutative local rings have the weakly unimodular property.*

Proof Local rings are semilocal rings. □

When $R[t]$ is a UFD, we let $\gcd(h(t), g(t))$ be the **greatest common divisor** of the polynomials $h(t), g(t) \in R[t]$. If R is a field, we require $\gcd(h(t), g(t))$ to be the monic greatest common divisor of the polynomials $h(t), g(t) \in R[t]$.

Proposition 3.1.7 *Every* UFD *has the weakly unimodular property.*

Proof Let $f_0(t), f_1(t) \in R[t]$ be monic polynomials and $\left(\overline{f_0}(t), \overline{f_1}(t)\right)$ be unimodular in $\frac{R}{\mathfrak{m}}[t]$ for every $\mathfrak{m} \in \mathrm{Max}(R)$. Then $\gcd\left(\overline{f_0}(t), \overline{f_1}(t)\right) = 1$ in $\frac{R}{\mathfrak{m}}[t]$. We want to prove that $\gcd(f_0(t), f_1(t))$ is a unit in $R[t]$. Suppose $\gcd(f_0(t), f_1(t))$ is not a unit.

Case 1. $\gcd(f_0(t), f_1(t)) = m \in R$ but $m \notin U(R)$.

Then there exists $\mathfrak{m}_o \in \mathrm{Max}(R)$ such that $m \in \mathfrak{m}_o$. So $\gcd\left(\overline{f_0}(t), \overline{f_1}(t)\right) = \overline{m} = 0$ in $\frac{R}{\mathfrak{m}_o}[t]$. This is a contradiction.

Case 2. $\gcd(f_0(t), f_1(t)) = g(t) \in R[t]$ with $\deg(g(t)) \geq 1$ in $R[t]$.

Then for any $\mathfrak{m} \in \mathrm{Max}(R)$, $\gcd\left(\overline{f_0}(t), \overline{f_1}(t)\right) \neq 1$ in $\frac{R}{\mathfrak{m}}[t]$ because the coefficient of the leading term of $g(t)$ is a unit.

Hence, $(f_0(t), f_1(t))$ is unimodular in $R[t]$. □

In [12], the authors defined SRC factorization as the following.

Definition 3.1.8 *Let R be a commutative local ring. A factorization $h(t) = h_0(t)h_1(t)$ in $R[t]$ of a monic polynomial $h(t)$ is said to be an* SR ***factorization*** *if $h_0(t)$ and $h_1(t)$ are monic and $h_0(0)$, $h_1(1) \in U(R)$. The ring R is an n-*SR ***ring*** *if every monic polynomial of degree n in $R[t]$ has an* SR *factorization. A factorization $h(t) = h_0(t)h_1(t)$ in $R[t]$ of a monic polynomial $h(t)$ is said to be an* SRC ***factorization*** *if it is an* SR *factorization and $\overline{h_0}(t), \overline{h_1}(t)$ are co-prime in the PID $\bar{R}[t]$ $(= \frac{R}{J(R)}[t])$. The ring R is an n-*SRC ***ring***

if every monic polynomial of degree n in $R[t]$ has an SRC *factorization.*

A local ring has only two idempotents 0 and 1. Commutative local rings are weakly unimodular. By Corollary 3.1.6, we know that, for a commutative local ring R and for monic polynomials $f_0(t)$ and $f_1(t)$ in $R[t]$, $\gcd\left(\overline{f_0}(t), \overline{f_1}(t)\right) = 1$ iff $\left(\overline{f_0}(t), \overline{f_1}(t)\right)$ is unimodular in $\frac{R}{J(R)}[t]$ iff $(f_0(t), f_1(t))$ is unimodular in $R[t]$. So we generalize Definition 3.1.8 as follows:

Definition 3.1.9 *Let R be a commutative ring and let $f(t) \in R[t]$ be a monic polynomial of degree n. A factorization $f(t) = f_0(t)f_1(t)$ in $R[t]$ is called an n-*SR ***factorization*** *if $f_i(t)$ is monic in $R[t]$ and $f_i(e_i) \in U(R)$ for some $e_i^2 = e_i \in R$ $(i = 0, 1)$. The factorization $f(t) = f_0(t)f_1(t)$ is an n-*SRC ***factorization*** *if, in addition, $(f_0(t), f_1(t))$ is unimodular in $R[t]$. The ring R is an n-*SR ***ring*** *if every monic polynomial of degree n has an* SR *factorization and R is an n-*SRC ***ring*** *if every monic polynomial of degree n has an* SRC *factorization. We call R an* SR *(*SRC*)* ***ring*** *if every monic polynomial in $R[t]$ has an* SR *(*SRC*) factorization.*

From now on, the notions "SR" and "SRC" are in the sense of Definition 3.1.9.

Proposition 3.1.10 *Let R be a commutative ring. Then R is strongly clean iff R is* 1-SR *iff R is* 1-SRC.

Proof Suppose that R is strongly clean. Let $f(t) = t+a \in R[t]$. Write $-a = e+u$ where $e^2 = e \in R$, $u \in U(R)$ and $eu = ue$. So $f(e) = -u \in U(R)$. Hence, $f(t) = f_0(t)f_1(t)$ with $f_0(t) = t + a$ and $f_1(t) = 1$ is an SR factorization. Obviously, this is also an SRC factorization.

Suppose that R is 1-SR. Let $a \in R$. Then $f(t) = t - a$ has an SR factorization in $R[t]$. It must be that $f(t) = f_0(t)$ or $f(t) = f_1(t)$. So there exists $e^2 = e \in R$ such that $f(e) = e - a \in U(R)$. Thus, a is strongly clean. $\square$

Proposition 3.1.11 *Let R be a commutative ring and let $f(t) \in R[t]$ be a monic polynomial of degree $\deg(f(t)) = n \geq 1$. If $f(t)$ has an n-SRC factorization, then all matrices with characteristic polynomial $f(t)$ are strongly clean.*

Proof This is essentially the case in Corollary 3.1.2. $\square$

Theorem 3.1.12 *Let R be a commutative ring. If R is an n-SRC ring, then the matrix ring $\mathbb{M}_n(R)$ is strongly clean.*

Proof For any matrix $A \in \mathbb{M}_n(R)$, the characteristic polynomial, $\chi_A(t)$, of A has an n-SRC factorization. So A is strongly clean by Proposition 3.1.11. That is, $\mathbb{M}_n(R)$ is strongly clean. $\square$

Corollary 3.1.13 *[12] Every Henselian ring R is n-SRC for each positive integer n. That is, R is SRC. So matrix rings over a Henselian ring are strongly clean.*

Proof Let $f(t)$ be any monic polynomial in $R[t]$. Then $\overline{f}(t)$ in $\frac{R}{J(R)}[t]$ can be factorized as $\overline{f}(t) = \overline{f_0}(t)\overline{f_1}(t)$ with $\overline{f_i}(t) \in \frac{R}{J(R)}[t]$ monic, $\gcd\left(\overline{f_0}(t), \overline{f_1}(t)\right) = 1$, and $\overline{f_i}(i) \in U(R)$ $(i = 0, 1)$. So by Hensel's Lemma (see Definition 1.2.1), there exist monic polynomials $f_i(t) \in R[t]$ such that $f_i(i) \in U(R)$ $(i = 0, 1)$. By Corollary 3.1.6, $(f_0(t), f_1(t))$ is unimodular. So R is n-SRC for each positive integer n. The rest follows from Corollary 3.1.12. $\square$

If R is commutative local, then R is n-SRC iff $\mathbb{M}_n(R)$ is strongly clean by [12]. But for a commutative ring R, R being n-SRC ring is not a necessary condition for $\mathbb{M}_n(R)$ to be strongly clean.

Example 3.1.14 *Let R be a Boolean ring with more than 2 elements. Then R is not a 2-SRC ring. But $\mathbb{M}_n(R)$ is strongly clean for any positive integer n.*

Proof Since R is a Boolean ring with more than 2 elements, there exists a polynomial

$f(t) = t^2 + et \in R[t]$ with $e \neq 0, 1$. Suppose that $f(t) = f_0(t)f_1(t)$ is an SRC factorization in $R[t]$.

Case 1. If one of $f_i(t)$ $(i = 0, 1)$ is $f(t)$, say $f_0(t) = f(t) = t^2 + et$, then there exists an element $a \in R$ such that $f_0(a) = a + ea = 1$. So $1 + e = 1$. Hence, $e = 0$. This is a contradiction.

Case 2. If $f_0(t) = t - a$ and $f_1(t) = t - b$, then there exist $e_0, e_1 \in R$ such that $f_0(e_0) = e_0 - a = 1$ and $f_1(e_1) = e_1 - b = 1$. Notice that $f(t) = t^2 - (a+b)t + ab = t^2 + et$. So we have

$$\begin{aligned} a + b &= e_0 + e_1 = e \text{ and} \\ ab &= (1 + e_0)(1 + e_1) = 0. \end{aligned} \tag{3.1.2}$$

By equality (3.1.2), we get $(1 + e)(1 + e_1) = 0$. So $1 + e_1 \in eR$. Hence, there exists some $r \in R$ such that $e_1 = 1 + er = 1 + ere$. Therefore, by the first equality in (3.1.2), $e_0 = 1 + e + ere$. So $f_0(t) = t + e + ere$ and $f_1(t) = t + ere$. If there exist $m(t), n(t) \in R[t]$ such that $f_0(t)m(t) + f_1(t)n(t) = 1$, then $(f_1(t) + e)m(t) + f_1(t)n(t) = 1$. So $f_1(t)(m(t) + n(t)) + em(t) = 1$. Let $t = 0$. Then we get $ere(m(0) + n(0)) + em(0) = 1$, i.e., $e[ere(m(0) + n(0)) + m(0)] = 1$. So $e = 1$. This is also a contradiction.

So R is not a 2-SRC ring.

By Remark 5.1.4, if R is Boolean, then $\mathbb{M}_n(R)$ is strongly clean for any positive integer $n \geq 1$. □

Now we give some necessary conditions for a matrix to be strongly clean.

In the following, we always consider $e_0 = 0$ and $e_1 = 1$ for the SR factorization because of Theorem 2.2.5.

Proposition 3.1.15 *Let R be a commutative ring such that every finitely generated projective R-module is free. If $T \in \mathbb{M}_n(R)$ is strongly clean, then $\chi_T(t)$ has an n-SR factorization.*

Proof If T is semi-purely nonsingular, then $\chi_T(t) = \det(tI - T) = f_0(t)f_1(t) = \chi_T(t) \cdot 1$ with $f_0(t) = \chi_T(t)$ and $f_1(t) = 1$. If T is semi-purely singular, then $\chi_T(t) = \det(tI-T) = f_0(t)f_1(t) = 1 \cdot \chi_T(t)$ with $f_0(t) = 1$ and $f_1(t) = \chi_T(t)$. If T is purely singular, then, by Theorem 2.2.5, T is similar to $C = \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}$ where T_0 is semi-purely nonsingular and T_1 is semi-purely singular. So $\chi_T(t) = \chi_{T_0}(t) \cdot \chi_{T_1}(t)$ with $f_0(t) = \chi_{T_0}(t)$ and $f_1(t) = \chi_{T_1}(t)$ is an n-SR factorization. □

Example 3.1.16 *In $\mathbb{M}_2(\mathbb{Z})$, $A = \begin{pmatrix} 2 & 3 \\ 6 & 4 \end{pmatrix}$ is not strongly clean. In fact, $f(t) = \chi_A(t) = t^2 - 6t - 10$ does not have an* SR *factorization because $f(0) = -10, f(1) = -15$ and $f(t) = (t - \frac{6-\sqrt{76}}{2})(t - \frac{6+\sqrt{76}}{2})$.*

Proposition 3.1.15 shows that if T is not purely singular, then $\chi_T(t)$ has a **trivial** SRC **factorization**, that is, one of the factors is 1 and the other is $\chi_T(t)$ itself.

Given a monic polynomial $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0 \in R[t]$, the matrix

$$C_f = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$$

is called the **companion matrix** of $f(t)$.

Lemma 3.1.17 *[43, Theorem VII.4.3] Let F be a field and $f(t)$ be a monic polynomial in $F[t]$. Then $f(t)$ is the characteristic and minimal polynomial of the companion matrix C_f.*

Proposition 3.1.18 *Let R be a commutative ring such that every finitely generated projective R-module is free and $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0 \in R[t]$. If the companion matrix C_f is strongly clean, then $\chi_{C_f}(t) = f(t)$ has an n-*SRC *factorization.*

Proof If C_f is not purely singular, then $f(t)$ has a trivial SRC factorization. So we can assume C_f is purely singular. Then by Theorem 2.2.5, there exists $P \in \mathbb{M}_n(R)$ such that $P^{-1}C_fP = \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}$ with T_0 being $k \times k$ semi-purely nonsingular matrix and T_1

being $(n-k) \times (n-k)$ semi-purely singular matrix where $0 < k < n$. Then for every maximal ideal $\mathfrak{m}$ in R, $\overline{C_f} \in \mathbb{M}_n(\frac{R}{\mathfrak{m}})$ and $\overline{C_f} = C_{\overline{f}}$ has $\overline{f}(t) \in \frac{R}{\mathfrak{m}}[t]$ as the characteristic and minimal polynomial by Lemma 3.1.17. So $\overline{f}(t) = \chi_{\overline{C_f}}(t) = \chi_{\overline{T}_0}(t) \cdot \chi_{\overline{T}_1}(t) = \det(tI_k - \overline{T}_0) \cdot \det(tI_{n-k} - \overline{T}_1)$. If $\gcd(\det(tI_k - \overline{T}_0), \det(tI_{n-k} - \overline{T}_0)) = g(t)$ with degree $\deg(g(t)) \geq 1$, then the minimal polynomial of $\overline{C_f}$ is $\frac{\det(tI_k - \overline{T}_0)\det(tI_{n-k} - \overline{T}_1)}{g(t)}$ which has degree less than $\deg(\chi_{\overline{C_f}}) = deg(f)$. This is a contradiction. So $f_0(t) = \det(tI - T_0)$ and $f_1(t) = \det(tI - T_1)$ give an n-SRC factorization for $\chi_{C_f}(t) = f(t)$. □

Theorem 3.1.19 *Let R be a commutative ring such that every finitely generated projective R-module is free and let $f(t) \in R[t]$ be a monic polynomial of degree $\deg(f(t)) = n$. Then the following are equivalent:*

1. *For all $A \in \mathbb{M}_n(R)$ with $\chi_A(t) = f(t)$, A is strongly clean.*
2. *The companion matrix C_f is strongly clean.*
3. *$f(t)$ has an n-SRC factorization.*

Proof "(1) ⇒ (2)". This is clear.

"(2) ⇒ (3)". By Proposition 3.1.18.

"(3) ⇒ (1)". By Proposition 3.1.11. □

Corollary 3.1.20 *Let R be a commutative ring such that every finitely generated projective R-module is free. Then a purely singular matrix $A \in \mathbb{M}_n(R)$ is strongly clean iff $\chi_A(t)$ has an n-SR factorization $\chi_A(t) = f_0(t)f_1(t)$ and A is similar to $\begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}$ where $\chi_{T_0}(t) = f_0(t)$ and $\chi_{T_1}(t) = f_1(t)$.*

Proof "⇒". By Theorem 2.2.5, A is similar to $\begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}$ where T_0 is semi-purely nonsingular and T_1 is semi-purely singular. By the proof of Proposition 3.1.15, $\chi_A(t)$ has an n-SR factorization $\chi_A(t) = f_0(t)f_1(t)$ where $\chi_{T_0}(t) = f_0(t)$ and $\chi_{T_1}(t) = f_1(t)$.

"$\Leftarrow$". By Theorem 3.1.19, T_0 and T_1 are strongly clean because $\chi_{T_0}(t) = f_0(t)$ and $\chi_{T_1}(t) = f_1(t)$ have trivial SRC factorizations. So A is strongly clean because the strongly clean property is invariant under similarity. □

Example 3.1.21 *Let $R = \mathbb{Z}_{(p)}$ with $p \equiv 3 \pmod 4$. Then the monic polynomial $f(t) = [(t-1)(t^2+1)+p][t(t^2+1)+p]$ with $f_0(t) = (t-1)(t^2+1)+p$ and $f_1(t) = t(t^2+1)+p$ is the only* SR *factorization (so it does not have* SRC *factorization) [27, Example 3.17].*

Let $A = \begin{pmatrix} 0 & 0 & 1-p & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -p \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$. *By direct computation, we obtain $\chi_A(t) = f(t)$ and A is similar to* $\begin{pmatrix} C_{f_0} & 0 \\ 0 & C_{f_1} \end{pmatrix}$. *So by Corollary 3.1.20, A is strongly clean.*

The class of rings R having IBN such that every finitely generated projective R-module is free is bigger than the class of local rings. By Theorem 3.1.22, matrix rings over rings in the first class do not produce more strongly clean rings than the matrix rings over local rings. But Theorem 2.2.5, Theorem 3.1.19 and Corollary 3.1.20 can be used to obtain all strongly clean matrices over these rings. This is one of the reasons that we introduce Definition 3.1.9.

A ring R is called an ***I*-finite ring** if R does not have an infinite set of non-zero orthogonal idempotents. Camillo-Yu [17] proved that R is semiperfect iff R is I-finite and clean. Here, for rings R having IBN such that every finitely generated projective R-module is free, we have the following result.

Theorem 3.1.22 *Let R be a ring having IBN such that every finitely generated projective R-module is free. The following are equivalent:*

1. *R is a strongly clean ring.*

2. *R is a clean ring.*

3. R is a local ring.

4. R is an exchange ring.

5. R is a semiperfect ring.

If, in addition, R is commutative, then the above are equivalent to each of the following:

6. R is 1-SR.

7. R is 1-SRC.

Proof "(*3*) $\Rightarrow$ (*1*) $\Rightarrow$ (*2*)". This is clear.

"(*2*) $\Rightarrow$ (*4*)". This is by [51].

"(*4*) $\Rightarrow$ (*3*)". We prove R has only 0 and 1 as its idempotents. Suppose $e^2 = e \in R$. Then $R = Re \oplus R(1-e)$. Since R has IBN and every finitely generated projective R-module is free, we get $Re = 0$ or $R(1-e) = 0$. So $e = 0$ or $e = 1$. Now let $r \notin U(R)$. Then because R is an exchange ring, there exists $e^2 = e$ such that $e \in Rr$ and $1-e \in R(1-r)$. That is, $1 \in Rr$ or $1 \in R(1-r)$. But $r \notin U(R)$, so $1 \in R(1-r)$. Similarly, $1 \in (1-r)R$. So $1-r \in U(R)$. Therefore, R is local.

"(*3*) $\Rightarrow$ (*5*)". This is clear.

"(*5*) $\Rightarrow$ (*2*)". This is by [17].

For the rest of the proof, let R be commutative.

"(*3*) $\Rightarrow$ (*7*)". Suppose that $f(t) = t + a$. Let $f_0(t) = t + a$ and $f_1(t) = 1$ if $a \in U(R)$; and $f_0(t) = 1$ and $f_1(t) = t + a$ if $a \in J(R)$. Then $f(t) = f_0(t)f_1(t)$ is an SRC factorization.

"(*7*) $\Rightarrow$ (*6*)". This is clear.

"(*6*) $\Rightarrow$ (*3*)". Let $a \notin U(R)$. Then $f(t) = t + ra$ has an SR factorization by (*6*). But $f(t)$ only has the trivial factorization. So $f_1(t) = f(t)$. That is, $1 + ra \in U(R)$. Therefore, R is local. $\square$

By Theorem 3.1.19 and Theorem 3.1.22, we get the following:

Corollary 3.1.23 *Let R be a commutative ring such that every finitely generated projective R-module is free. Then $\mathbb{M}_n(R)$ is strongly clean iff R is a local n-*SRC *ring.*

Corollary 3.1.24 *[12] Let R be a commutative local ring. Then $\mathbb{M}_n(R)$ is strongly clean iff R is an n-*SRC *ring.*

We defined SR and SRC factorization over commutative rings. In fact, we can define them over any noncommutative ring. Here we define them over local rings.

Definition 3.1.25 *Let R be a local ring and $\overline{R}[t] = \frac{R}{J(R)}[t]$. A monic polynomial $f(t) \in R[t]$ is said to have an* SR *factorization if $f(t) = g_0(t)g_1(t) = h_1(t)h_0(t)$, where $g_0(t), g_1(t)$, $h_0(t), h_1(t) \in R[t]$ are monic polynomials such that $g_0(0), g_1(1)$, $h_0(0), h_1(1) \in U(R)$. If, in addition, $\overline{R}[t]\bar{g_0}(t) + \overline{R}[t]\bar{g_1}(t) = \overline{R}[t]$ and $\bar{h_0}(t)\overline{R}[t] + \bar{h_1}(t)\overline{R}[t] = \overline{R}[t]$ hold, then $f(t)$ is said to have an* SRC *factorization.*

It is interesting to compare the next result with [12, Corollary 15, Proposition 17], which states that, for a commutative local ring R, $\mathbb{M}_2(R)$ is strongly clean, iff every companion matrix in $\mathbb{M}_2(R)$ is strongly clean, iff every monic quadratic polynomial over R has an SR factorization, iff every monic quadratic polynomial over R has an SRC factorization.

Theorem 3.1.26 *The following are equivalent for a local ring R:*

1. *$\mathbb{M}_2(R)$ is strongly clean.*
2. *Every companion matrix in $\mathbb{M}_2(R)$ is strongly clean.*
3. *Every monic quadratic polynomial over R has an* SR *factorization.*
4. *Every monic quadratic polynomial over R has an* SRC *factorization.*

Proof "*(1)* $\Leftrightarrow$ *(2)*". This holds by "*(1)* $\Leftrightarrow$ *(4)*" of Theorem 2.3.4.

"*(1)* $\Rightarrow$ *(4)*". Suppose $f(t) = t^2 + at + b \in R[t]$.

Case 1. $f(0) \in U(R)$.

Let $f_0(t) = t^2 + at + b$ and $f_1(t) = 1$. Then $f(t) = f_0(t)f_1(t)$ is an SRC factorization.

Case 2. $f(1) \in U(R)$.

Let $f_0(t) = 1$ and $f_1(t) = t^2 + at + b$. Then $f(t) = f_0(t)f_1(t)$ is an SRC factorization.

Case 3. $f(0), f(1) \in J(R)$.

Then $b \in J(R)$ and $-a = 1 + (b - f(1)) \in 1 + J(R)$. By Theorem 2.3.4, $f(t)$ has a left root $t_0 \in J(R)$ and a left root $t_1 \in 1 + J(R)$. Thus, $f(t) = (t - t_1)(t + a + t_1) = (t - t_0)(t + a + t_0)$ is clearly an SRC factorization.

"*(4)* $\Rightarrow$ *(3)*". It is obvious.

"*(3)* $\Rightarrow$ *(1)*". For $w_0, w_1 \in J(R)$, $f(t) = t^2 - (1 + w_0)t - w_1$ has an SR factorization. This clearly shows that $f(t)$ has a left root in $J(R)$ and a left root in $1 + J(R)$ by *(3)*. Hence, *(1)* holds by Theorem 2.3.4. $\square$

3.2 Strongly clean matrix rings over commutative local rings

In this section, all rings are assumed to be commutative local. If $n \geq 2$ and if R is a commutative local ring such that $\mathbb{M}_n(R)$ is strongly clean, we prove that both $\mathbb{M}_n(R[[x]])$ and $\mathbb{M}_n\left(\frac{R[x]}{(x^k)}\right)$ $(k \geq 1)$ are strongly clean and that $\mathbb{M}_n(RC_2)$ is strongly clean when $2 \in U(R)$ or $2 = 0$ in R. We do not know whether $\mathbb{M}_n(RC_2)$ is still strongly clean when $0 \neq 2 \in J(R)$. These generalize results in [32]. This section comes from [66].

For a ring homomorphism $\theta : R \longrightarrow S$, define $\theta' : R[x] \longrightarrow S[x]$ by $\theta'(\sum r_i x^i) = \sum \theta(r_i)x^i$. In particular, for a maximal ideal $\mathfrak{m}$ in R, we use η to represent the natural ring epimorphism $\eta : R \longrightarrow \frac{R}{\mathfrak{m}}$ with $\eta(r) = r + \mathfrak{m} = \overline{r}$. Then η induces a map $\eta' : R[t] \longrightarrow \frac{R}{\mathfrak{m}}[t] = \overline{R}[t]$ with $\eta'(\sum_{i=0}^{n} r_i t^i) = \sum_{i=0}^{n} \eta(r_i)t^i = \sum_{i=0}^{n} \overline{r}_i t^i$.

Lemma 3.2.1 *Let $\theta : R \to S$ be an onto ring homomorphism. If R is n-SRC, then S is n-SRC.*

Proof Notice that R and S are commutative local. The following diagram is commutative where $\overline{\theta} : R/J(R) \to S/J(S), r + J(R) \mapsto \theta(r) + J(S)$, is an isomorphism:

$$\begin{array}{ccc} R & \xrightarrow{\theta} & S \\ {\scriptstyle \eta_R}\downarrow & & \downarrow{\scriptstyle \eta_S} \\ \frac{R}{J(R)} & \xrightarrow{\overline{\theta}} & \frac{S}{J(S)}. \end{array}$$

It induces the commutative diagram with $\overline{\theta}'$ an isomorphism:

$$\begin{array}{ccc} R[t] & \xrightarrow{\theta'} & S[t] \\ {\scriptstyle \eta'_R}\downarrow & & \downarrow{\scriptstyle \eta'_S} \\ \frac{R}{J(R)}[t] & \xrightarrow{\overline{\theta}'} & \frac{S}{J(S)}[t]. \end{array}$$

Let $h'(t) \in S[t]$ be a monic polynomial of degree n. Then there exists a monic polynomial $h(t) \in R[t]$ of degree n such that $\theta'(h(t)) = h'(t)$. Since R is an n-SRC ring, there exists an SRC factorization $h(t) = h_0(t)h_1(t)$ in $R[t]$. Let $\theta'(h_i(t)) = h_i'(t)$, $i = 0, 1$. Then $h'(t) = h_0'(t)h_1'(t)$ with $h_i'(i) = \theta'(h_i(i)) \in U(S)$. By the commutativity of the latter diagram, $\overline{\theta}'\eta_R'(h_i(t)) = \eta_S'\theta'(h_i(t)) = \eta_S'(h_i'(t))$ for $i = 0, 1$. Because $\overline{\theta}'$ is an isomorphism and $\gcd(\eta_R'(h_0(t)), \eta_R'(h_1(t))) = 1$, we get $\gcd(\eta_S'(h_0'(t)), \eta_S'(h_1'(t))) = 1$. So $h'(t) = h_0'(t)h_1'(t)$ is an SRC factorization in $S[t]$. Hence S is an n-SRC ring. $\square$

For a ring epimorphism $\theta : R \to S$, S being n-SRC does not imply that R is n-SRC. For example, let $\theta : \mathbb{Z}_{(p)} \to \mathbb{Z}_p$ be the natural ring epimorphism. Then $\mathbb{M}_n(\mathbb{Z}_{(p)})$ is not strongly clean for any $n > 1$ by Corollary 2.1.3. So $\mathbb{Z}_{(p)}$ is not n-SRC by Corollary 3.1.24, but $\mathbb{Z}_p$ is certainly n-SRC.

Let R be a commutative ring. For $f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x +$

$\cdots + b_m x^m$ in $R[x]$, the $(n+m)\times(n+m)$ determinant

$$\Re(f,g) = \left|\begin{array}{ccccccccc} a_n & a_{n-1} & \cdots & \cdots & a_0 & & & & \\ & a_n & a_{n-1} & \cdots & \cdots & a_0 & & & \\ & & \ddots & & & & \ddots & & \\ & & & a_n & a_{n-1} & \cdots & \cdots & a_0 & \\ b_m & b_{m-1} & \cdots & \cdots & b_0 & & & & \\ & b_m & b_{m-1} & \cdots & \cdots & b_0 & & & \\ & & \ddots & & & & \ddots & & \\ & & & b_m & b_{m-1} & \cdots & \cdots & b_0 & \end{array}\right| \begin{array}{l} \left.\vphantom{\begin{array}{c} a\\a\\a\\a \end{array}}\right\} m \\ \left.\vphantom{\begin{array}{c} a\\a\\a\\a \end{array}}\right\} n \end{array}$$

is called the **resultant** of $f(x)$ and $g(x)$ [19, 47].

The following lemma is known. We give the proof here.

Lemma 3.2.2 *[19, Lemma 2, p.321] Let E be an algebraically closed field. Let $f(x) = a_0+a_1x+\cdots+a_nx^n$ $(a_n \neq 0)$, and $g(x) = b_0+b_1x+\cdots+b_mx^m$ $(b_m \neq 0)$ be two polynomials in $E[x]$ such that $f(\alpha_i) = 0$ and $g(\beta_j) = 0$ where α_i and $\beta_j \in E$ for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$. Then $\Re(f, g) = a_n^m g(\alpha_1)g(\alpha_2)\cdots g(\alpha_n) = b_m^n f(\beta_1)f(\beta_2)\cdots f(\beta_m)$.*

Proof The proof is by induction. Suppose $n = 1$. Then $f(x) = a_0+a_1x$ and $\alpha = -a_0/a_1$ is the root of $f(x)$ and

$$\Re(f,g) = \left|\begin{array}{ccccc} a_1 & a_0 & & & \\ & a_1 & a_0 & & \\ & & \ddots & \ddots & \\ & & & a_1 & a_0 \\ b_m & b_{m-1} & \cdots & b_1 & b_0 \end{array}\right| .$$

Adding α times column j to column $j+1$, $j = 1, \cdots, m$, one obtains

$$\Re(f,g) = \begin{vmatrix} a_1 & a_1\alpha + a_0 & & & \\ & a_1 & a_1\alpha + a_0 & & \\ & & \ddots & \ddots & \\ & & & a_1 & a_1\alpha + a_0 \\ b_m & b_{m-1} + b_m\alpha & \cdots & b_1 + \cdots + b_m\alpha^{m-1} & g(\alpha) \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & 0 & & & \\ & a_1 & 0 & & \\ & & \ddots & \ddots & \\ & & & a_1 & 0 \\ b_m & b_{m-1} & \cdots & b_1 & g(\alpha) \end{vmatrix} = a_1^m g(\alpha).$$

Suppose the lemma holds for $n = k$. We want to prove it when $n = k+1$. Let $\alpha, \alpha_1, \cdots, \alpha_k$ be the roots of $f(x)$. Then $f(x) = (x-\alpha)f_1(x)$ where $f_1(x) = c_0 + c_1x + c_2x^2 + \cdots + c_{k-1}x^{k-1} + a_{k+1}x^k$. Hence, the coefficients of $f(x)$ and $f_1(x)$ have the following relations:

$$\begin{cases} a_0 + c_0\alpha = 0 \\ a_1 + c_1\alpha = c_0 \\ a_2 + c_2\alpha = c_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{k-1} + c_{k-1}\alpha = c_{k-2} \\ a_k + a_{k+1}\alpha = c_{k-1} \end{cases}$$

and

$$\Re(f,g) = \begin{vmatrix}
a_{k+1} & a_k & \cdots & \cdots & a_1 & a_0 & & & \\
 & a_{k+1} & a_k & \cdots & \cdots & a_1 & a_0 & & \\
 & & \ddots & & & & \ddots & & \\
 & & & & a_{k+1} & a_k & \cdots & \cdots & a_1 \quad a_0 \\
b_m & b_{m-1} & \cdots & b_1 & b_0 & & & & \\
 & b_m & b_{m-1} & \cdots & b_1 & b_0 & & & \\
 & & & & & & \ddots & & \\
 & & & & b_m & b_{m-1} & \cdots & \cdots & b_1 \quad b_0
\end{vmatrix}.$$

Adding α times column j to column $j+1$, $j = 1, 2, \cdots, m+k$, we get $\Re(f,g) =$

$$\begin{vmatrix}
a_{k+1} & c_{k-1} & \cdots & \cdots & & c_0 & 0 & & \\
 & a_{k+1} & c_{k-1} & & \cdots & & c_0 & 0 & \\
 & & \ddots & & & & \ddots & & \\
 & & & & a_{k+1} & c_{k-1} & \cdots & c_0 & 0 \\
b_m & b_m\alpha + b_{m-1} & \cdots & g(\alpha) & \alpha g(\alpha) & \cdots & & \alpha^{k-1}g(\alpha) & \alpha^k g(\alpha) \\
 & b_m & b_m\alpha + b_{m-1} & \cdots & g(\alpha) & \cdots & & \alpha^{k-2}g(\alpha) & \alpha^{k-1}g(\alpha) \\
 & & \ddots & & & & \ddots & & \\
 & & & & b_m & b_m\alpha + b_{m-1} & \cdots & \alpha g(\alpha) & g(\alpha)
\end{vmatrix}.$$

Adding $-\alpha$ times row $j+1$ to row j, $j = m+1, \cdots, m+k$, we get

$$\Re(f,g) = \begin{vmatrix} a_{k+1} & c_{k-1} & \cdots & \cdots & c_0 & 0 & & & & \\ & a_{k+1} & c_{k-1} & \cdots & \cdots & c_0 & 0 & & & \\ & & \ddots & & & & & \ddots & & \\ & & & a_{k+1} & c_{k-1} & & \cdots & \cdots & c_0 & 0 \\ b_m & b_{m-1} & \cdots & b_1 & b_0 & & & & & \\ & b_m & b_{m-1} & \cdots & b_1 & b_0 & & & & \\ & & \ddots & & & & & \ddots & & \\ & & & b_m & b_{m-1} & \cdots & \cdots & b_1 & b_0 & \\ & & & & b_m & b_{m-1} & \cdots & \cdots & b_1 & g(\alpha) \end{vmatrix}.$$

$$= g(\alpha) \begin{vmatrix} a_{k+1} & c_{k-1} & \cdots & \cdots & c_0 & & & & \\ & a_{k+1} & c_{k-1} & \cdots & \cdots & c_0 & & & \\ & & \ddots & & & & & \ddots & \\ & & & & a_{k+1} & c_{k-1} & \cdots & \cdots & c_0 \\ b_m & b_{m-1} & \cdots & b_1 & b_0 & & & & \\ & b_m & b_{m-1} & \cdots & b_1 & b_0 & & & \\ & \ddots & & & & & \ddots & & \\ & & & b_m & b_{m-1} & \cdots & \cdots & b_1 & b_0 \\ & & & & b_m & b_{m-1} & \cdots & \cdots & b_1 & b_0 \end{vmatrix}$$

$$= g(\alpha)\Re(f_1, g).$$

By induction hypothesis, $\Re(f_1, g) = a_{k+1}^m g(\alpha_1)g(\alpha_2)\cdots g(\alpha_k)$. So $\Re(f,g) = a_{k+1}^m g(\alpha)g(\alpha_1)g(\alpha_2)\cdots g(\alpha_k)$. Similarly, we can prove the other equality. □

The next lemma is an exercise in [47, I.D.8].

Lemma 3.2.3 *Let R be a commutative local ring, $\eta_R : R \longrightarrow \frac{R}{J(R)}$ be the natural ring homomorphism, and $A = (r_{ij}) \in \mathbb{M}_n(R)$, $\overline{A} = (\overline{r_{ij}}) \in \mathbb{M}_n(\frac{R}{J(R)})$. Then* $\det A \in U(R)$ *iff* $\det \overline{A} \neq 0$.

Proof Define $\tau(j_1j_2\cdots j_n) = \begin{cases} 1, & \text{if } j_1j_2\cdots j_n \text{ is an even permutation,} \\ -1\ , & \text{if } j_1j_2\cdots j_n \text{ is an odd permutation.} \end{cases}$ By definition of determinant,

$$\begin{aligned}
\det \overline{A} &= \sum_{j_1j_2\cdots j_n} (-1)^{\tau(j_1j_2\cdots j_n)} \overline{r_{1j_1}} \cdot \overline{r_{2j_2}} \cdot \cdots \cdot \overline{r_{nj_n}} \\
&= \sum_{j_1j_2\cdots j_n} (-1)^{\tau(j_1j_2\cdots j_n)} \eta_R(r_{1j_1})\eta_R(r_{2j_2})\cdots\eta_R(r_{nj_n}) \\
&= \sum_{j_1j_2\cdots j_n} (-1)^{\tau(j_1j_2\cdots j_n)} \eta_R(r_{1j_1} \cdot r_{2j_2} \cdot \cdots \cdot r_{nj_n}) \\
&= \sum_{j_1j_2\cdots j_n} \eta_R[(-1)^{\tau(j_1j_2\cdots j_n)} r_{1j_1} \cdot r_{2j_2} \cdot \cdots \cdot r_{nj_n}] \\
&= \eta_R(\det(A)).
\end{aligned}$$

So $detA \in U(R)$ iff $det\overline{A} \in U(S)$ since θ is an epimorphism. $\square$

Theorem 3.2.4 *Let R be a commutative local ring and let $n \geq 1$. Then R is an n-*SRC *ring iff so is $R[[x]]$.*

Proof "$\Rightarrow$". Clearly $R[[x]]$ is a commutative local ring with $J(R[[x]]) = J(R) + xR[[x]]$. Define $\theta : R[[x]] \to R$ by $\theta(\sum_{i\geq 0} r_i x^i) = r_0$, and $\overline{\theta} : \frac{R[[x]]}{J(R[[x]])} \to \frac{R}{J(R)}$ by $\overline{\theta}(r + J(R[[x]])) = \theta(r) + J(R) = r + J(R)$, $r \in R$. Then θ is onto, $\overline{\theta}$ is an isomorphism, and the following diagram is commutative:

$$\begin{array}{ccc}
R[[x]] & \xrightarrow{\theta} & R \\
\Big\downarrow{\scriptstyle \eta_{R[[x]]}} & & \Big\downarrow{\scriptstyle \eta_R} \\
\frac{R[[x]]}{J(R[[x]])} & \xrightarrow{\overline{\theta}} & \frac{R}{J(R)}.
\end{array}$$

Also it induces the commutative diagram

$$\begin{array}{ccc}
R[[x]][t] & \xrightarrow{\theta'} & R[t] \\
\Big\downarrow{\scriptstyle \eta'_{R[[x]]}} & & \Big\downarrow{\scriptstyle \eta'_R} \\
\frac{R[[x]]}{J(R[[x]])}[t] & \xrightarrow{\overline{\theta}'} & \frac{R}{J(R)}[t]
\end{array}$$

with $\overline{\theta}'$ an isomorphism. Let $h(t) = t^n + \sum_{i=0}^{n-1} f_i t^i \in R[[x]][t]$ with $f_i = \sum_{j\geq 0} r_{ij}x^j \in R[[x]]$.

Case 1. If $h(0) \in U(R[[x]])$, then let $h_0(t) = h(t)$, $h_1(t) = 1$; and if $h(1) \in U(R[[x]])$, then let $h_0(t) = 1$, $h_1(t) = h(t)$. In either case, $h(t)$ has a trivial SRC factorization in $R[[x]][t]$.

Case 2. If $h(0) = f_0 \in J(R[[x]])$ and $h(1) = 1 + \sum_{i=0}^{n-1} f_i \in J(R[[x]])$, then $r_{00} \in J(R)$ and $1 + \sum_{i=0}^{n-1} r_{i0} \in J(R)$. Let $h'(t) = \theta'(h(t))$. Then $h'(t) = t^n + \sum_{i=0}^{n-1} r_{i0}t^i$, $h'(0) = r_{00} \in J(R)$, and $h'(1) = 1 + \sum_{i=0}^{n-1} r_{i0} \in J(R)$. Since R is n-SRC, there exist $h_0'(t) = t^k + \sum_{i=0}^{k-1} a_{i0}t^i$ and $h_1'(t) = t^{n-k} + \sum_{i=0}^{n-k-1} b_{i0}t^i$ in $R[t]$ such that $h_0'(0) \in U(R)$, $h_1'(1) \in U(R)$, $\gcd(\eta_R'(h_0'(t)), \eta_R'(h_1'(t))) = 1$, and $h'(t) = h_0'(t)h_1'(t)$. Let $h_0(t) = t^k + \sum_{i=0}^{k-1} A_i t^i \in R[[x]][t]$ with $A_i = \sum_{j\geq 0} a_{ij}x^j$, and $h_1(t) = t^{n-k} + \sum_{i=0}^{n-k-1} B_i t^i \in R[[x]][t]$ with $B_i = \sum_{j\geq 0} b_{ij}x^j$. Next we prove that there exist $A_i, B_j \in R[[x]]$ $(i = 0, \cdots, k-1$ and $j = 0, \cdots, n-k-1)$ such that $h(t) = h_0(t)h_1(t)$. Notice that

$$\begin{aligned}
& h(t) = h_0(t)h_1(t) \\
& \Leftrightarrow t^n + \sum_{i=0}^{n-1} f_i t^i = \left(t^k + \sum_{i=0}^{k-1} A_i t^i\right)\left(t^{n-k} + \sum_{i=0}^{n-k-1} B_i t^i\right) \\
& \Leftrightarrow t^n + \sum_{i=0}^{n-1}\left(\sum_{j=0}^{\infty} r_{ij}x^j\right) t^i = \left[t^k + \sum_{i=0}^{k-1}\left(\sum_{j=0}^{\infty} a_{ij}x^j\right) t^i\right]\left[t^{n-k} + \sum_{i=0}^{n-k-1}\left(\sum_{j=0}^{\infty} b_{ij}x^j\right) t^i\right] \\
& \Leftrightarrow t^n + \sum_{i=0}^{n-1} r_{i0}t^i + \sum_{j=1}^{\infty}\left(\sum_{i=0}^{n-1} r_{ij}t^i\right) x^j \\
& \quad = \left[t^k + \sum_{i=0}^{k-1} a_{i0}t^i + \sum_{j=1}^{\infty}\left(\sum_{i=0}^{k-1} a_{ij}t^i\right) x^j\right]\left[t^{n-k} + \sum_{i=0}^{n-k-1} b_{i0}t^i + \sum_{j=1}^{\infty}\left(\sum_{i=0}^{n-k-1} b_{ij}t^i\right) x^j\right] \\
& \Leftrightarrow \text{the conditions } (P_0) \text{ and } (P_m) \text{ hold for all } m \in \mathbb{N},
\end{aligned}$$

where

$$(P_0): \left(t^k + \sum_{i=0}^{k-1} a_{i0}t^i\right)\left(t^{n-k} + \sum_{i=0}^{n-k-1} b_{i0}t^i\right) = t^n + \sum_{i=0}^{n-1} r_{i0}t^i,$$

$$(P_m): \left(t^k + \sum_{i=0}^{k-1} a_{i0}t^i\right)\left(\sum_{i=0}^{n-k-1} b_{im}t^i\right) + \left(\sum_{i=0}^{k-1} a_{im}t^i\right)\left(t^{n-k} + \sum_{i=0}^{n-k-1} b_{i0}t^i\right)$$
$$+ \sum_{j=1}^{m-1}\left[\left(\sum_{i=0}^{k-1} a_{ij}t^i\right)\left(\sum_{i=0}^{n-k-1} b_{i,m-j}t^i\right)\right] = \sum_{i=0}^{n-1} r_{im}t^i.$$

Notice that by the choice of $h_0^{'}(t)$ and $h_1^{'}(t)$, (P_0) holds for suitable $a_{i0}(0 \le i \le k-1)$ and $b_{i0}(0 \le i \le n-k-1)$. Assume that for $s \ge 1$, there exist $a_{ij}(0 \le i \le k-1, 0 \le j \le s-1)$ and $b_{ij}(0 \le i \le n-k-1, 0 \le j \le s-1)$ in R such that (P_m) holds for all $m = 0, 1, \cdots s-1$. We next show that there exist $a_{is}(0 \le i \le k-1)$ and $b_{js}(0 \le j \le n-k-1)$ in R such that (P_s) holds. Notice that (P_s) is equivalent to

$$(*): \left(t^k + \sum_{i=0}^{k-1} a_{i0}t^i\right)\left(\sum_{i=0}^{n-k-1} b_{is}t^i\right) + \left(\sum_{i=0}^{k-1} a_{is}t^i\right)\left(t^{n-k} + \sum_{i=0}^{n-k-1} b_{i0}t^i\right)$$
$$= \sum_{i=0}^{n-1} r_{is}t^i - \sum_{j=1}^{s-1}\left[\left(\sum_{i=0}^{k-1} a_{ij}t^i\right)\left(\sum_{i=0}^{n-k-1} b_{i,s-j}t^i\right)\right]$$
$$= r_{0s}^{'} + r_{1s}^{'}t + \cdots + r_{n-1,s}^{'}t^{n-1}$$

where all $r_{is}^{'}$ are known elements of R. Thus, $(*)$ is equivalent to:

$$(**)\begin{cases} b_{n-k-1,s} + a_{k-1,s} = r_{n-1,s}^{'} \\ a_{k-1,0}b_{n-k-1,s} + b_{n-k-2,s} + b_{n-k-1,0}a_{k-1,s} + a_{k-2,s} = r_{n-2,s}^{'} \\ \vdots \\ a_{00}b_{1s} + a_{10}b_{0s} + b_{10}a_{0s} + b_{00}a_{1s} = r_{1s}^{'} \\ a_{00}b_{0s} + b_{00}a_{0s} = r_{0s}^{'}. \end{cases}$$

As a linear system in variables $a_{is}(0 \le i \le k-1)$ and $b_{js}(0 \le j \le n-k-1)$, the matrix form of (**) is $AX = B$ where

$$A^T = \begin{pmatrix} 1 & b_{n-k-1,0} & \cdots & \cdots b_{00} & & & & \\ & 1 & b_{n-k-1,0} & \cdots & \cdots & b_{00} & & \\ & & \ddots & & & & \ddots & \\ & & & 1 & b_{n-k-1,0} & \cdots & \cdots & b_{00} \\ 1 & a_{k-1,0} & \cdots & \cdots & a_{00} & & & \\ & 1 & a_{k-1,0} & \cdots & \cdots & a_{00} & & \\ & & \ddots & & & & \ddots & \\ & & & 1 & a_{k-1,0} & \cdots & \cdots & a_{00} \end{pmatrix},$$

$$X^T = \begin{pmatrix} a_{k-1,s} & a_{k-2,s} & \cdots & a_{1,s} & a_{0,s} & b_{n-k-1,s} & b_{n-k-2,s} & \cdots & b_{1,s} & b_{0,s} \end{pmatrix},$$

$$B^T = \begin{pmatrix} r'_{n-1,s} & r'_{n-2,s} & \cdots & r'_{1s} & r'_{0s} \end{pmatrix}.$$

Denote $\eta'_R h'_i(t) = \overline{h'_i}(t)$ $(i = 0, 1)$. Since $\gcd(\overline{h'_0}(t), \overline{h'_1}(t))= \gcd(\eta'_R h'_0(t), \eta'_R h'_1(t)) = 1$, there exist $\overline{g'_i}(t)$ $(i = 0, 1)$ such that

$$\overline{h'_0}(t) \cdot \overline{g'_0}(t) + \overline{h'_1}(t) \cdot \overline{g'_1}(t) = 1. \tag{3.2.1}$$

Let E be an algebraically closed extension field of $R/J(R)$ and suppose $\overline{h'_0}(\alpha_i) = 0$ where $\alpha_i \in E$ for $i = 1, 2, \cdots, k$. Then, by Lemma 3.2.2, $det\overline{A} = \Re(\overline{h'_1}(t), \overline{h'_0}(t)) = \overline{h'_1}(\alpha_1)\overline{h'_1}(\alpha_2)\cdots\overline{h'_1}(\alpha_k) \neq 0$ (by (3.2.1)). So $\overline{A}$ is invertible. By Lemma 3.2.3, A is invertible, so $AX = B$ is solvable. This proves the existence of $a_{is}(0 \le i \le k-1)$ and $b_{js}(0 \le j \le n-k-1)$ such that (P_s) holds. Hence there exist $h_0(t)$ and $h_1(t)$ in $R[[x]][t]$ as claimed before such that $h(t) = h_0(t)h_1(t)$.

Because $\overline{\theta}'$ is an isomorphism and because $\gcd(\eta'_R\theta'(h_0(t)), \eta'_R\theta'(h_1(t))) = 1$, we have $\gcd(\eta'_{R[[x]]}(h_0(t)), \eta'_{R[[x]]}(h_1(t))) = 1$. So $h(t)$ has an SRC factorization. Hence $R[[x]]$ is an n-SRC ring.

"$\Leftarrow$" holds by Lemma 3.2.1. $\square$

Theorem 3.2.5 *Let R be a commutative local ring and let $n, k \in \mathbb{N}$. Then the following are equivalent:*

1. *$\mathbb{M}_n(R)$ is strongly clean.*

2. *$\mathbb{M}_n(R[[x]])$ is strongly clean.*

3. *$\mathbb{M}_n\left(\dfrac{R[x]}{(x^k)}\right)$ is strongly clean.*

4. *$\mathbb{M}_n(R[[x_1, x_2, \cdots, x_k]])$ is strongly clean.*

5. *$\mathbb{M}_n\left(\dfrac{R[x_1, x_2, \cdots, x_k]}{(x_1^{n_1}, x_2^{n_2}, \cdots, x_k^{n_k})}\right)$ is strongly clean.*

Proof Note that all underlying rings are commutative local.

"*(1)* $\Leftrightarrow$ *(2)*". This follows by Corollary 3.1.24 and Theorem 3.2.4.

"*(2)* $\Rightarrow$ *(3)* $\Rightarrow$ *(1)*". Since R is an image of $\dfrac{R[x]}{(x^k)}$ and $\dfrac{R[x]}{(x^k)}$ is an image of $R[[x]]$, the implications follow by Corollary 3.1.24 and Lemma 3.2.1.

"*(1)* $\Leftrightarrow$ *(4)*". By the equivalence *(1)* $\Leftrightarrow$ *(2)* and induction.

"*(1)* $\Leftrightarrow$ *(5)*". By the equivalence of *(1)* $\Leftrightarrow$ *(3)* and induction. □

Example 3.2.6 *If R is a Henselian ring and $m, s, n_1, \cdots, n_s \in \mathbb{N}$, then, by Corollary 3.1.24 and Theorem 3.2.5, $\mathbb{M}_n(R[[x_1, x_2, \cdots, x_s]])$ and $\mathbb{M}_n\left(\dfrac{R[x_1, x_2, \cdots, x_s]}{(x_1^{n_1}, x_2^{n_2}, \cdots, x_s^{n_s})}\right)$ are strongly clean.*

Corollary 2.2.15 proved that for a commutative local ring R, $\mathbb{M}_2(R)$ is strongly clean iff so is $\mathbb{M}_2(RC_2)$. Next, we extend this result from 2 to an arbitrary positive integer n.

Theorem 3.2.7 *Let R be a commutative local ring with $2 \in U(R)$ or* char$R = 2$. *Then $\mathbb{M}_n(R)$ is strongly clean iff so is $\mathbb{M}_n(RC_2)$.*

Proof "$\Leftarrow$". This holds because $\mathbb{M}_n(R)$ is an image of $\mathbb{M}_n(RC_2)$.

"$\Rightarrow$". If $2 \in U(R)$, then $RC_2 \cong R \times R$ by Lemma 2.2.13. So $\mathbb{M}_n(RC_2) \cong \mathbb{M}_n(R) \oplus \mathbb{M}_n(R)$ is strongly clean.

Now assume that char$R = 2$. Then RC_2 is commutative local by [50]. We can assume $n \geq 2$. Write $C_2 = \{1, g\}$ and let $f(x) = x^n + \sum_{i=0}^{n-1}(r_i + r_i^{'}g)x^i \in (RC_2)[x]$ such that

$f(0) = r_0 + r_0^{'}g \in J(RC_2)$ and $f(1) = 1 + \sum_{i=0}^{n-1}(r_i + r_i^{'}g) \in J(RC_2)$. Let $\omega : RC_2 \to R$, $a + bg \mapsto a + b$, be the augmentation map. As in the proof of Theorem 3.2.4, we have two commutative diagrams with $\overline{\omega}$ and $\overline{\omega}^{'}$ isomorphisms:

$$\begin{array}{ccc} RC_2 & \xrightarrow{\omega} & R \\ \eta_{RC_2}\Big\downarrow & & \Big\downarrow\eta_R \\ \frac{RC_2}{J(RC_2)} & \xrightarrow{\overline{\omega}} & \frac{R}{J(R)}, \end{array}$$

$$\begin{array}{ccc} RC_2[x] & \xrightarrow{\omega^{'}} & R[x] \\ \eta^{'}_{RC_2}\Big\downarrow & & \Big\downarrow\eta^{'}_R \\ \frac{RC_2}{J(RC_2)}[x] & \xrightarrow{\overline{\omega}^{'}} & \frac{R}{J(R)}[x]. \end{array}$$

Since $\mathbb{M}_n(R)$ is strongly clean, $f^{'}(x) := \omega^{'}(f(x)) = x^n + \sum_{i=0}^{n-1}(r_i + r_i^{'})x^i$ has a non-trivial SRC factorization $f^{'}(x) = f_0^{'}(x)f_1^{'}(x)$ in $R[x]$. Write $f_0^{'}(x) = a_0 + a_1x + \cdots + a_{m-1}x^{m-1} + x^m$ and $f_1^{'}(x) = b_0 + b_1x + \cdots + b_{n-m-1}x^{n-m-1} + x^{n-m}$ where $1 \le m < n$. Next we show that there exist $y_i, z_j \in R$ $(i = 0, \cdots, m-1, j = 0, \cdots, n-m-1)$ such that

$$\begin{aligned} f_0(x) &= x^m + \sum_{i=0}^{m-1}\left[y_i + (a_i - y_i)\,g\right]x^i, \\ f_1(x) &= x^{n-m} + \sum_{i=0}^{n-m-1}\left[z_i + (b_i - z_i)\,g\right]x^i, \\ f(x) &= f_0(x)f_1(x). \end{aligned} \tag{3.2.2}$$

The equality $f(x) = f_0(x)f_1(x)$ is equivalent to

$$\begin{aligned} x^n + \sum_{i=0}^{n-1} r_ix^i &= \left(x^m + \sum_{i=0}^{m-1} y_ix^i\right)\left(x^{n-m} + \sum_{i=0}^{n-m-1} z_ix^i\right) \\ &\quad + \left[\sum_{i=0}^{m-1}(a_i - y_i)\,x^i\right]\left[\sum_{i=0}^{n-m-1}(b_i - z_i)\,x^i\right], \\ \sum_{i=0}^{n-1} r_i^{'}x^i &= (x^m + \sum_{i=0}^{m-1} y_ix^i)[\sum_{i=0}^{n-m-1}(b_i - z_i)x^i] \\ &\quad + \left(x^{n-m} + \sum_{i=0}^{n-m-1} z_ix^i\right)\left[\sum_{i=0}^{m-1}(a_i - y_i)\,x^i\right]. \end{aligned} \tag{3.2.3}$$

Clearly, the second equality of (3.2.3) follows from $f'(x) = f_0'(x)f_1'(x)$ and from the first equality of (3.2.3). So it suffices to show that there exist $y_i, z_j \in R$ $(i = 0, \cdots, m-1, j = 0, \cdots, n-m-1)$ that make the first equality of (3.2.3) hold. The first equality of (3.2.3) is equivalent to

$$\begin{cases} y_0z_0 + (a_0 - y_0)(b_0 - z_0) = r_0 \\ y_0z_1 + y_1z_0 + (a_0 - y_0)(b_1 - z_1) + (a_1 - y_1)(b_0 - z_0) = r_1 \\ \qquad \vdots \\ y_{m-2} + y_{m-1}z_{n-m-1} + z_{n-m-2} + (a_{m-1} - y_{m-1})(b_{n-m-1} - z_{n-m-1}) = r_{n-2} \\ y_{m-1} + z_{n-m-1} = r_{n-1}, \end{cases}$$

which, since $\mathrm{char}(R) = 2$, is equivalent to

$$\begin{cases} c_0 := r_0 + a_0b_0 = b_0y_0 + a_0z_0 \\ c_1 := r_1 + a_0b_1 + a_1b_0 = b_1y_0 + b_0y_1 + a_0z_1 + a_1z_0 \\ \qquad \vdots \\ c_{n-2} := r_{n-2} + a_{m-1}b_{n-m-1} \\ \qquad = y_{m-2} + b_{n-m-1}y_{m-1} + z_{n-m-2} + a_{m-1}z_{n-m-1} \\ c_{n-1} := r_{n-1} = y_{m-1} + z_{n-m-1}. \end{cases} \tag{3.2.4}$$

As a linear system in variables $y_i (i = 0, \cdots, m-1)$ and $z_i (i = 0, \cdots, n-m-1)$, the matrix form of (3.2.4) is $AX = C$ where

$$A^T = \begin{pmatrix} 1 & b_{n-m-1} & \cdots & \cdots & b_1 & b_0 & & & \\ & 1 & b_{n-m-1} & \cdots & \cdots & b_1 & b_0 & & \\ & & \ddots & & & & \ddots & & \\ & & & 1 & b_{n-m-1} & \cdots & \cdots & b_1 & b_0 \\ 1 & a_{m-1} & \cdots & \cdots & a_1 & a_0 & & & \\ & 1 & a_{m-1} & \cdots & \cdots & a_1 & a_0 & & \\ & & \ddots & & & & \ddots & & \\ & & & 1 & a_{m-1} & \cdots & \cdots & a_1 & a_0 \end{pmatrix},$$

$$X^T = (y_{m-1}, y_{m-2} \cdots, y_0, z_{n-m-1}, \cdots, z_0), \text{ and}$$

$$C^T = (c_{n-1}, c_{n-2}, \cdots, c_0).$$

An argument similar to the proof of Theorem 3.2.4 shows that A is invertible. So $AX = C$ is solvable. This shows the existence of the y_i and z_j such that $f(x) = f_0(x)f_1(x)$. Hence $\mathbb{M}_n(RC_2)$ is strongly clean. □

Proposition 3.2.8 *Let R be a commutative local ring with $0 \neq 2 \in J(R)$ and let $\mathbb{M}_3(R)$ be strongly clean. If for any $m, n \in R$ and $u \in U(R)$, $4x^3 - 2mx^2 + ux + n = 0$ is solvable in R, then $\mathbb{M}_3(RC_2)$ is strongly clean.*

Proof The two diagrams in the proof of Theorem 3.2.7 are still valid. Let

$$f(x) = (r_0 + r_0'g) + (r_1 + r_1'g)x + (r_2 + r_2'g)x^2 + x^3 \in RC_2[x]$$

$$\text{with } f(0) = r_0 + r_0'g \in J(RC_2) \text{ and}$$

$$f(1) = (r_0 + r_0'g) + (r_1 + r_1'g) + (r_2 + r_2'g) + 1 \in J(RC_2).$$

Then

$$f'(x) = \omega'(f(x)) = (r_0 + r_0') + (r_1 + r_1')x + (r_2 + r_2')x^2 + x^3 \in R[x]$$

$$\text{with } f'(0) = r_0 + r_0' \in J(R) \text{ and}$$

$$f'(1) = (r_0 + r_0') + (r_1 + r_1') + (r_2 + r_2') + 1 \in J(R).$$

Since $\mathbb{M}_3(R)$ is strongly clean, $f'(x)$ has a non-trivial SRC factorization $f'(x) = f_0'(x)f_1'(x)$ in $R[x]$. We can assume that $\{f_0'(x), f_1'(x)\} = \{a_0 + x, b_0 + b_1x + x^2\}$. Then

$$\begin{cases} r_0 + r_0' = a_0b_0 \\ r_1 + r_1' = a_0b_1 + b_0 \\ r_2 + r_2' = a_0 + b_1. \end{cases} \tag{3.2.5}$$

Next we show that there exist $y_0, z_0, z_1 \in R$ such that $f(x) = f_0(x)f_1(x)$ and $f_i'(x) = \omega'(f_i(x))$ ($i = 0, 1$) where $\{f_0(x), f_1(x)\} = \{[y_0 + (a_0 - y_0)g] + x, [z_0 + (b_0 - z_0)g] + [z_1 + (b_1 - z_1)g]x + x^2\}$. The condition $f(x) = f_0(x)f_1(x)$ is equivalent to

$$\begin{cases} r_0 = y_0z_0 + (a_0 - y_0)(b_0 - z_0) \\ r_1 = y_0z_1 + (a_0 - y_0)(b_1 - z_1) + z_0 \\ r_2 = z_1 + y_0 \\ r_0' = z_0(a_0 - y_0) + y_0(b_0 - z_0) \\ r_1' = z_1(a_0 - y_0) + y_0(b_1 - z_1) + b_0 - z_0 \\ r_2' = b_1 - z_1 + a_0 - y_0. \end{cases} \tag{3.2.6}$$

Since the first three equalities of (3.2.6) and (3.2.5) clearly imply the last three equalities of (3.2.6), it suffices to show that there exist $y_0, z_0, z_1 \in R$ such that the first three equalities of (3.2.6) hold. Rewrite the first three equations of (3.2.6) as

$$\begin{cases} 2y_0z_0 - b_0y_0 - a_0z_0 = r_0 - a_0b_0 \\ 2y_0z_1 - b_1y_0 + z_0 - a_0z_1 = r_1 - a_0b_1 \\ z_1 = r_2 - y_0. \end{cases} \tag{3.2.7}$$

Clearly (3.2.7) is equivalent to

$$\begin{cases} 4y_0^3 - 2my_0^2 + uy_0 + n = 0 \\ z_0 = 2y_0^2 - (2r_2 - b_1 + a_0)y_0 + a_0(r_2 - b_1) + r_1 \\ z_1 = r_2 - y_0. \end{cases} \tag{3.2.8}$$

where $m = (2r_2 + 2a_0 - b_1), u = (4a_0r_2 - 2a_0b_1 + 2r_1 - b_0 - a_0b_1 + a_0^2)$, and $n = -a_0^2r_2 + a_0^2b_1 - a_0r_1 + a_0b_0 - r_0$. As in the last part of the proof of Theorem 3.2.4, $b_0 - a_0b_1 + a_0^2 = \Re(f_0'(x), f_1'(x)) \in U(R)$. So $u \in U(R)$. By hypothesis, the first equation of (3.2.8) is solvable for y_0 in R. Hence, (3.2.8) is solvable for y_0, z_0 and z_1 in R. So $\mathbb{M}_3(RC_2)$ is strongly clean. $\square$

Corollary 3.2.9 *If R is a Henselian ring, then $\mathbb{M}_3(RC_2)$, $\mathbb{M}_3((RC_2)[[x]])$, and $\mathbb{M}_3\left(\frac{(RC_2)[x]}{(x^k)}\right)$ are strongly clean for any $k \in \mathbb{N}$.*

Proof We show that $\mathbb{M}_3(RC_2)$ is strongly clean. By Corollary 3.1.24 and Proposition 3.2.8, it suffices to show that when $2 \in J(R)$ and for any $m, n \in R$ and $u \in U(R)$, $h(x) = 4x^3 - 2mx^2 + ux + n$ has a root in R. Let $h_1'(x) = x + \frac{n}{u}$ and $h_0'(x) = u$. Then $\eta_R'(h(x)) = \eta_R'(h_0'(x))\eta_R'(h_1'(x))$. By Hensel's Lemma, there exist $h_1(x) = x + s_3$ and $h_0(x)$ in $R[x]$ such that $\eta_R'(h_1(x)) = \eta_R'(h_1'(x))$, $\eta_R'(h_0(x)) = \eta_R'(h_0'(x))$, and $h(x) = h_1(x)h_0(x)$. So $h(x)$ has a solution $x = -s_3 \in R$. Hence $\mathbb{M}_3(RC_2)$ is strongly clean. By Theorem 3.2.4, $\mathbb{M}_3((RC_2)[[x]])$, and $\mathbb{M}_3\left(\frac{(RC_2)[x]}{(x^k)}\right)$ are strongly clean. $\square$

Chapter 4

Strongly Clean Triangular Matrix Rings

Our main result states that the triangular matrix rings over commutative local rings are strongly clean. This chapter comes from [22].

4.1 Strongly clean triangular matrix rings

For each $n \geq 1$, a ring R is clean iff $\mathbb{T}_n(R)$ is clean [38]. When is $\mathbb{T}_n(R)$ strongly clean? Several efforts have been made towards this question. By [52, Example 2], $\mathbb{T}_2(R)$ is strongly clean if R is a commutative local ring. It was proved in [62] that if R is a commutative local ring for which every element is uniquely the sum of an idempotent and a unit (or equivalently, $R/J(R) \cong \mathbb{Z}_2$ by [1, Corollary 22]), then $\mathbb{T}_n(R)$ is a strongly clean ring for every $n \geq 1$. The main result is the following Theorem 4.1.1. For a ring R, we write R^n (resp., R_n) for the set of all $1 \times n$ (resp., $n \times 1$) matrices over R. For $\beta \in R^n$, $\beta^T \in R_n$ denotes the transpose of β.

Theorem 4.1.1 *If R is a commutative local ring, then $\mathbb{T}_n(R)$ is a strongly clean ring for every $n \geq 1$.*

Proof We prove the claim by induction on n. For $n = 1$, the result holds since local rings are strongly clean.

Assume that $n > 1$ and every $(a_{ij}) \in \mathbb{T}_{n-1}(R)$ has a strongly clean expression $(a_{ij}) = (e_{ij}) + (u_{ij})$ such that, for any $1 \le i \le n-1$, $e_{ii} = 0$ if $a_{ii} \in U(R)$. Now let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in \mathbb{T}_n(R).$$

Claim. There exist $(e_{ij})^2 = (e_{ij}) \in \mathbb{T}_n(R)$ and $(u_{ij}) \in U(\mathbb{T}_n(R))$ such that

$$(a_{ij}) = (e_{ij}) + (u_{ij}), (e_{ij})(u_{ij}) = (u_{ij})(e_{ij})$$

and that, for any $1 \le i \le n$,

$$e_{ii} = 0 \text{ if } a_{ii} \in U(R).$$

Write

$$A = \begin{pmatrix} A_1 & \alpha \\ 0 & a_{nn} \end{pmatrix} \text{ where } A_1 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} \\ 0 & a_{22} & \cdots & a_{2,n-1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1} \end{pmatrix} \text{ and } \alpha = (a_{1n}, \cdots, a_{n-1,n})^T.$$

By the induction hypothesis, A_1 has a strongly clean expression

$$A_1 = E + U, \text{ where } E = (e_{ij}), U = (u_{ij}) \tag{4.1.1}$$

such that

$$\text{for any } 1 \le i \le n-1, \ e_{ii} = 0 \text{ if } a_{ii} \in U(R). \tag{4.1.2}$$

Case 1. $a_{nn} \in J(R)$. Take $e_{nn} = 1$ and $u_{nn} = a_{nn} - 1$. Then $U - (u_{nn}+1)I$ is a unit in $\mathbb{T}_{n-1}(R)$. Let $\delta_1 = [U - (u_{nn}+1)I]^{-1}(E - I)\alpha$ and let

$$F = \begin{pmatrix} E & \delta_1 \\ 0 & e_{nn} \end{pmatrix}, \text{ and } V = \begin{pmatrix} U & \alpha - \delta_1 \\ 0 & u_{nn} \end{pmatrix} \in \mathbb{T}_n(R).$$

Because E and $U - (u_{nn}+1)I$ commute, E and $[U - (u_{nn}+1)I]^{-1}$ commute, so $E\delta_1 = 0$. Thus, it follows that $F^2 = F$. By the definition of δ_1, we have

$$(E - I)\alpha = [U - (u_{nn}+1)I]\delta_1 = U\delta_1 - u_{nn}\delta_1 - \delta_1, \text{ so}$$

$$U\delta_1 + (\alpha - \delta_1) = E\alpha + u_{nn}\delta_1 = E(\alpha - \delta_1) + \delta_1 u_{nn}.$$

It follows that $FV = VF$. Moreover, it is easily seen that $A = F + V$ and $V \in U(\mathbb{T}_n(R))$. Therefore, the claim is proved in the case where $a_{nn} \notin U(R)$.

Case 2. $a_{nn} \in U(R)$. Take $e_{nn} = 0$ and $u_{nn} = a_{nn}$. To prove the claim in this case, it suffices to show that there exists $\gamma_1 \in R_{n-1}$ such that

$$F^2 = F = \begin{pmatrix} E & \gamma_1 \\ 0 & e_{nn} \end{pmatrix}, \quad FV = VF, \text{ and } V = \begin{pmatrix} U & \alpha - \gamma_1 \\ 0 & u_{nn} \end{pmatrix}.$$

Note that

$$\begin{aligned} F^2 = F &\Longleftrightarrow E\gamma_1 = \gamma_1, \text{ and} \\ FV = VF &\Longleftrightarrow (U + E - a_{nn}I)\gamma_1 = E\alpha \\ &\Longleftrightarrow (A_1 - a_{nn}I)\gamma_1 = E\alpha. \end{aligned}$$

Thus, it suffices to show that the system

$$\begin{cases} EX &= X \qquad\qquad (\Lambda) \\ (A_1 - a_{nn}I)X &= E\alpha \qquad\quad (\Gamma) \end{cases}$$

has a solution $X = (x_1, \cdots, x_{n-1})^T$ in R_{n-1}.

For this purpose, fix some notation and let

$$A_i = \begin{pmatrix} a_{ii} & a_{i,i+1} & \cdots & a_{i,n-1} \\ 0 & a_{i+1,i+1} & \cdots & a_{i+1,n-1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1} \end{pmatrix}, \quad E_i = \begin{pmatrix} e_{ii} & e_{i,i+1} & \cdots & e_{i,n-1} \\ 0 & e_{i+1,i+1} & \cdots & e_{i+1,n-1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e_{n-1,n-1} \end{pmatrix},$$

$$U_i = \begin{pmatrix} u_{ii} & u_{i,i+1} & \cdots & u_{i,n-1} \\ 0 & u_{i+1,i+1} & \cdots & u_{i+1,n-1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & u_{n-1,n-1} \end{pmatrix},$$

and write

$$A_i = \begin{pmatrix} a_{ii} & \beta_i \\ 0 & A_{i+1} \end{pmatrix}, \quad E_i = \begin{pmatrix} e_{ii} & e_i \\ 0 & E_{i+1} \end{pmatrix} \tag{4.1.3}$$

where

$$\beta_i = (a_{i,i+1}, \cdots, a_{i,n-1}), e_i = (e_{i,i+1}, \cdots, e_{i,n-1});$$

and write

$$X_i = (x_i, \cdots, x_{n-1})^T, \text{ and } \alpha_i = (a_{in}, \cdots, a_{n-1,n})^T.$$

Thus, equation $i, i+1, \cdots, n-1$ in (Λ) form

$$E_i X_i = X_i \qquad (\Lambda(i)).$$

That is

$$\begin{cases} e_{ii}x_i + e_i X_{i+1} &= x_i \qquad (\Lambda_i) \\ E_{i+1}X_{i+1} &= X_{i+1} \qquad (\Lambda(i+1)). \end{cases}$$

And equation $i, i+1, \cdots, n-1$ in (Γ) become a system

$$(A_i - a_{nn}I)X_i = E_i\alpha_i \qquad (\Gamma(i))$$

which is

$$\begin{cases} (a_{ii} - a_{nn})x_i + \beta_i X_{i+1} &= e_{ii}a_{in} + e_i\alpha_{i+1} \qquad (\Gamma_i) \\ (A_{i+1} - a_{nn}I)X_{i+1} &= E_{i+1}\alpha_{i+1} \qquad (\Gamma(i+1)). \end{cases}$$

First we consider the following two equations:

$$\begin{cases} e_{n-1,n-1}x_{n-1} &= x_{n-1} \qquad (\Lambda_{n-1}) \\ (a_{n-1,n-1} - a_{nn})x_{n-1} &= e_{n-1,n-1}a_{n-1,n} \quad (\Gamma_{n-1}). \end{cases}$$

If $a_{n-1,n-1} \in U(R)$, then $e_{n-1,n-1} = 0$ by the induction hypothesis; so $x_{n-1} = 0$ satisfies both (Λ_{n-1}) and (Γ_{n-1}). If $a_{n-1,n-1} \notin U(R)$, then it must be that $e_{n-1,n-1} = 1$ because $a_{n-1,n-1} = e_{n-1,n-1} + u_{n-1,n-1}$. Since $a_{nn} \in U(R)$ by our assumption, $a_{n-1,n-1} - a_{nn} \in U(R)$; so $x_{n-1} = (a_{n-1,n-1} - a_{nn})^{-1}a_{n-1,n}$ is a solution of both (Λ_{n-1}) and (Γ_{n-1}). Therefore, $x_{n-1} \in R$ exists to satisfy both (Λ_{n-1}) and (Γ_{n-1}).

Now assume that $i \leq n-2$ and there exists $X_{i+1} \in R_{n-i-1}$ satisfying $(\Lambda(i+1))$ and $(\Gamma(i+1))$. We next show that there exists $x_i \in R$ such that X_i satisfies both $(\Lambda(i))$ and $(\Gamma(i))$, or, equivalently, x_i satisfies both (Λ_i) and (Γ_i). We proceed with two cases.

Subcase 1. $a_{ii} \in U(R)$. Then $e_{ii} = 0$ by the induction hypothesis. Choose $x_i = e_i X_{i+1}$. Thus, x_i satisfies (Λ_i). Next we show that x_i satisfies (Γ_i) as well. It follows from (4.1.1) that $A_i = E_i + U_i$ is a strongly clean expression of A_i. Because $e_{ii} = 0$, $E_i^2 = E_i$ implies that

$$e_i = e_i E_{i+1}. \tag{4.1.4}$$

Note that $e_{ii} = 0$ implies that $u_{ii} = a_{ii}$, so it follows from $E_i U_i = U_i E_i$ (using (4.1.3)) that

$$e_{ii}(\beta_i - e_i) + e_i U_{i+1} = u_{ii} e_i + (\beta_i - e_i) E_{i+1},$$

which gives

$$e_i(A_{i+1} - E_{i+1}) = e_i U_{i+1} = a_{ii} e_i + \beta_i E_{i+1} - e_i E_{i+1},$$

showing

$$e_i A_{i+1} = a_{ii} e_i + \beta_i E_{i+1}. \tag{4.1.5}$$

The left hand side of (Γ_i) is

$$\begin{aligned}
(a_{ii} - a_{nn}) x_i + \beta_i X_{i+1} &= (a_{ii} - a_{nn}) e_i X_{i+1} + \beta_i (E_{i+1} X_{i+1}) \text{ (by } (\Lambda(i+1))) \\
&= [(a_{ii} e_i + \beta_i E_{i+1}) - a_{nn} e_i] X_{i+1} \\
&= (e_i A_{i+1} - a_{nn} e_i) X_{i+1} \text{ (by (4.1.5))} \\
&= e_i (A_{i+1} - a_{nn} I) X_{i+1} \\
&= e_i E_{i+1} \alpha_{i+1} \text{ by } (\Gamma(i+1)) \\
&= e_i \alpha_{i+1} \text{ (by (4.1.4))} \\
&= e_{ii} a_{in} + e_i \alpha_{i+1} \text{ (because } e_{ii} = 0).
\end{aligned}$$

Hence $x_i = e_i X_{i+1}$ satisfies both (Λ_i) and (Γ_i).

Subcase 2. $a_{ii} \notin U(R)$. It must be that $e_{ii} = 1$ because $a_{ii} = e_{ii} + u_{ii}$ is a strongly clean expression of a_{ii}. Thus, $a_{ii} - a_{nn} \in U(R)$. Choose

$$x_i = (a_{ii} - a_{nn})^{-1} (e_{ii} a_{in} + e_i \alpha_{i+1} - \beta_i X_{i+1}).$$

Thus, x_i satisfies (Γ_i). Next we show that x_i satisfies (Λ_i), that is , $e_i X_{i+1} = 0$.

Because $e_{ii} = 1$, $E_i^2 = E_i$ implies that $e_i E_{i+1} = 0$. So, by $(\Lambda(i+1))$, we have $e_i X_{i+1} = e_i(E_{i+1}X_{i+1}) = 0$. Thus, x_i satisfies (Λ_i). Therefore, by the induction principle, there exists $X \in R_{n-1}$ satisfying (Λ) and (Γ). So the claim is proved in this case. The proof of Theorem 4.1.1 is now complete. □

Remark 4.1.2 *If $a = e_1 + u_1$ and $b = e_2 + u_2$ are strongly clean expressions of a and b in R respectively and $v \in R$, there do not always exist $\alpha_1, \alpha_2 \in R$ such that*

$$\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} = \begin{pmatrix} e_1 & \alpha_1 \\ 0 & e_2 \end{pmatrix} + \begin{pmatrix} u_1 & \alpha_2 \\ 0 & u_2 \end{pmatrix}$$

is a strongly clean expression in $\mathbb{T}_2(R)$. For example, $a = 1 + 4, b = 0 + 2$ are strongly clean expressions in $\mathbb{Z}_{(3)}$, but $\begin{pmatrix} 5 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & \alpha_1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 4 & \alpha_2 \\ 0 & 2 \end{pmatrix}$ cannot be a strongly clean expression for any $\alpha_1, \alpha_2 \in \mathbb{Z}_{(3)}$.

Corollary 4.1.3 *If $R = \prod R_i$ is a direct product of commutative local rings R_i, then $\mathbb{T}_n(R)$ is strongly clean for every $n \geq 1$.*

Proof $\mathbb{T}_n(R) \cong \prod \mathbb{T}_n(R_i)$ is strongly clean because each $\mathbb{T}_n(R_i)$ is strongly clean by Theorem 4.1.1. □

Corollary 4.1.4 *Let R be a commutative semilocal ring. The following are equivalent:*

1. *R is semiperfect.*
2. *$\mathbb{T}_n(R)$ is strongly clean for every $n \geq 1$.*
3. *$\mathbb{T}_n(R)$ is strongly clean for some $n \geq 1$.*

Proof "*(1)* ⇒ *(2)*". Since R is semiperfect, there exist orthogonal local idempotents $e_i, i = 1, \cdots, m$, such that $1 = e_1 + \cdots + e_m$. So $R = e_1Re_1 \times \cdots \times e_mRe_m$ is a direct product of commutative local rings, so the implication follows by Corollary 4.1.3.

"*(2)* $\Rightarrow$ *(3)*". This is clear.

"*(3)* $\Rightarrow$ *(1)*". Let $e \in \mathbb{T}_n(R)$ whose $(1,1)$-entry is 1 and all other entries are 0. Then $R \cong e\mathbb{T}_n(R)e$ is clean by Theorem 2.1.1. So idempotents lift modulo $J(R)$. Hence R is semiperfect. $\square$

We mention a related result. It was proved by Chen [21] that, for a bimodule ${}_RM_S$ over two rings R and S, $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is strongly π-regular iff both R and S are strongly π-regular. It follows that if R is a strongly π-regular ring then $\mathbb{T}_n(R)$ is strongly π-regular and hence is strongly clean.

We point out that Theorem 4.1.1 can be generalized to any "skew" triangular matrix ring over a commutative local ring defined as follows: For a ring R and an endomorphism σ of R, let $\mathbb{T}_n(R,\sigma) = \{(a_{ij})_{n\times n} : a_{ij} \in R \text{ and } a_{ij} = 0 \text{ if } i > j\}$. For $(a_{ij}), (b_{ij}) \in \mathbb{T}_n(R,\sigma)$, define

$$(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}) \text{ and } (a_{ij}) * (b_{ij}) = (c_{ij}),$$

where $c_{ij} = 0$ for $i > j$, and $c_{ij} = \Sigma_{k=i}^{j} a_{ik}\sigma^{k-i}(b_{kj})$ for $i \leq j$. It can be easily verified that $\mathbb{T}_n(R,\sigma)$ is a ring, called the skew triangular matrix ring over R. Clearly, $\mathbb{T}_n(R, 1_R) = \mathbb{T}_n(R)$, and $\mathbb{T}_2(R,\sigma)$ coincides with the formal triangular matrix ring $\begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$ where ${}_RM = {}_RR$ with $xr = x\sigma(r)$ for $x \in M, r \in R$. If σ is an automorphism, then $\mathbb{T}_n(R) \cong \mathbb{T}_n(R,\sigma)$ via $(a_{ij}) \mapsto (b_{ij})$ where $b_{ij} = \sigma^{1-i}(a_{ij})$. The proof of Theorem 4.1.1 can be slightly modified to prove the following

Theorem 4.1.5 *If R is a commutative local ring and σ is an endomorphism of R with $\sigma(J(R)) \subseteq J(R)$, then $\mathbb{T}_n(R,\sigma)$ is a strongly clean ring for every $n \geq 1$.*

We conclude by giving an example showing that there exist endomorphisms σ of a commutative local domain R which are not automorphisms such that $\sigma(J(R)) \subseteq J(R)$.

Example 4.1.6 *Let $R = \mathbb{Z}[x]_{(x)} = \{\frac{f(x)}{g(x)} : f(x), g(x) \in \mathbb{Z}[x], g(0) \neq 0\}$ be the localization* *of $\mathbb{Z}[x]$ at (x), and let $\sigma : R \to R$ be given by $\frac{f(x)}{g(x)} \mapsto \frac{f(0)}{g(0)}$. Then $\sigma(J(R)) = \sigma(xR) = 0 \subseteq J(R)$, but σ is neither monic nor epic.*

We would like to point out that Theorem 4.1.1 has been extended recently in [13] from a commutative local ring to a bleached local ring (see the definition before Definition 2.4.3).

Chapter 5

Strongly π-Regular Rings

Strongly π-regular rings are strongly clean. To enlarge the class of strongly π-regular rings is a task itself. Furthermore, this work also enlarges the class of strongly clean rings. In section 5.1, we get a new class of strongly π-regular rings using a result of Hirano. In particular, matrix rings over Boolean or strongly regular rings are strongly π-regular (and hence strongly clean). This section comes from [66]. In section 5.2, we present a new family of strongly π-regular rings which are matrix rings over local rings.

5.1 Finite extensions of strongly π-regular rings

Let S be a ring and R be a subring of S such that they share the same identity. The ring S is called a **finite extension** of R if S, as a right R-module, is generated by a finite set X of generators.

Theorem 5.1.1 *[42] Let R be a ring whose prime factor rings are artinian. Then every finite extension of R is strongly π-regular.*

Note that, by [20], there exists a strongly π-regular ring R such that $\mathbb{M}_2(R)$ is not strongly π-regular. A ring R is called **right duo** if every right ideal is an ideal.

Corollary 5.1.2 *Let R be a right duo strongly π-regular ring, let G be a locally finite group, and let $n \geq 1$. Then $\mathbb{M}_n(RG)$ is strongly π-regular.*

Proof To show the claim, without loss of generality, we may assume that G is a finite group. Then $\mathbb{M}_n(RG)$ is a finite extension of RG and RG is a finite extension of R. So $\mathbb{M}_n(RG)$ is a finite extension of R. Since R is right duo strongly π-regular, every prime factor ring $\overline{R} = R/I$ is again right duo strongly π-regular. So $\overline{R}$ must be a strongly π-regular domain. Hence, $\overline{R}$ is a division ring (of course artinian). The claim now follows by Theorem 5.1.1. □

Corollary 5.1.3 *Let R be a right duo strongly π-regular ring, let G be a locally finite group, and let $n, k \geq 1$. Then $\mathbb{M}_n((RG)[[x]])$ and $\mathbb{M}_n\left(\frac{(RG)[x]}{(x^k)}\right)$ are strongly clean.*

Proof By Corollary 5.1.2, the matrix ring $\mathbb{M}_n(RG)$ is strongly π-regular. Then $\mathbb{M}_n((RG)[[x]]) \cong \mathbb{M}_n(RG)[[x]]$ and $\mathbb{M}_n\left(\frac{(RG)[x]}{(x^k)}\right)$ are strongly clean by [24, Corollary 2.2]. □

Remark 5.1.4 *Notice that Boolean rings are strongly regular rings and strongly regular rings are right duo and strongly π-regular.*

5.2 A criterion for $\mathbb{M}_2(R)$ over a local ring R to be strongly π-regular

It is pointed out in [12] that, for a commutative local ring R, $\mathbb{M}_n(R)$ is strongly π-regular iff R is strongly π-regular iff $J(R)$ is nil. In this section, we characterize the local rings R for which $\mathbb{M}_2(R)$ is strongly π-regular.

Lemma 5.2.1 *[52] Let M_R be a module. The following are equivalent for $\varphi \in \operatorname{End}(M_R)$:*

1. *φ is strongly π-regular in* $\operatorname{End}(M_R)$.

2. *There is a decomposition $M = P \oplus Q$ where P and Q are φ-invariant, and $\varphi|_P$ is an isomorphism and $\varphi|_Q$ is nilpotent.*

Units and nilpotent elements of a ring are clearly strongly π-regular elements. These are called the **trivial strongly π-regular elements**. A strongly π-regular element is called **non-trivial** if it is not trivial. Fixing a basis of $(R^n)_R$, we know there is a one-to-one correspondence between the matrix in $\mathbb{M}_n(R)$ and the endomorphism in $\mathrm{End}((R^n)_R)$. So, in the following, for $A \in \mathbb{M}_n(R)$, we denote $\varphi_A \in \mathrm{End}((R^n)_R)$. Using Lemma 5.2.1, we can prove the following theorem.

Theorem 5.2.2 *Let R be a ring having* IBN *such that every finitely generated projective R-module is free. Then $A \in \mathbb{M}_n(R)$ is a non-trivial strongly π-regular matrix iff A is similar to* $\begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}$, *where T_0 is an invertible matrix and T_1 is a nilpotent matrix.*

Proof "$\Rightarrow$". Suppose T is a non-trivial strongly π-regular matrix. Then by Lemma 5.2.1, there exist $R_1 \neq 0$ and $R_2 \neq 0$ such that

$$\varphi_T : (R_R)^n = R_1 \oplus R_2 \longrightarrow (R_R)^n = R_1 \oplus R_2$$

with $\varphi_T|_{R_1}$ being a right R-module isomorphism and $\varphi_T|_{R_2}$ being a nilpotent right R-module endomorphism. The direct summands R_1 and R_2 are projective right R-modules and so they are both free right R-modules. They satisfy

$$n = rank((R_R)^n) = rank(R_1) + rank(R_2) \tag{5.2.1}$$

since R has IBN. Suppose $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_n\}$ is a basis of $(R_R)^n$ and under this basis, φ_T is the endomorphism corresponding to the matrix T. Then

$$\varphi_T(\epsilon_1, \epsilon_2, \cdots, \epsilon_n) = (\varphi_T(\epsilon_1), \varphi_T(\epsilon_2), \cdots, \varphi_T(\epsilon_n)) = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n)T.$$

Suppose $rank(R_1) = k$. Then by equality (5.2.1), we can assume that $\{\eta_1, \eta_2, \cdots, \eta_n\}$ is a basis of $(R_R)^n$ where $\{\eta_1, \eta_2, \cdots, \eta_k\}$ is a basis of R_1 and $\{\eta_{k+1}, \eta_{k+2}, \cdots, \eta_n\}$ is a basis of R_2. Since $\varphi_T|_{R_1} : R_1 \to R_1$ is an isomorphism and $\varphi_T|_{R_2} : R_2 \to R_2$ is nilpotent, we have

$$\varphi_T|_{R_1}(\eta_1, \eta_2, \cdots, \eta_k) = (\eta_1, \eta_2, \cdots, \eta_k)T_0 \tag{5.2.2}$$

with some T_0 being invertible and

$$\varphi_T|_{R_2}(\eta_{k+1}, \eta_{k+2}, \cdots, \eta_n) = (\eta_{k+1}, \eta_{k+2}, \cdots, \eta_n)T_1 \tag{5.2.3}$$

with some T_1 being nilpotent. Let $C = \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}$. Then

$$\varphi_T(\eta_1, \eta_2, \cdots, \eta_n) = (\varphi_T(\eta_1), \varphi_T(\eta_1), \cdots, \varphi_T(\eta_n))$$
$$= (\eta_1, \eta_2, \cdots, \eta_n)\begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}$$
$$= (\eta_1, \eta_2, \cdots, \eta_n)C.$$

Since $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_n\}$ and $\{\eta_1, \eta_2, \cdots, \eta_n\}$ are both bases of $(R_R)^n$, we have

$$(\epsilon_1, \epsilon_2, \cdots, \epsilon_n) = (\eta_1, \eta_2, \cdots, \eta_n)P_1,$$
$$(\eta_1, \eta_2, \cdots, \eta_n) = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n)P_2.$$

So $(\epsilon_1, \epsilon_2, \cdots, \epsilon_n) = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n)P_2P_1 = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n)I_n$

and $P_2P_1 = I_n$. Similarly, we get $P_1P_2 = I_n$.

Hence, $P_2P_1 = P_1P_2 = I_n$. That is, $P := P_1 = P_2^{-1}$.

Now

$$\varphi_T(\epsilon_1, \epsilon_2, \cdots, \epsilon_n) = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n)T,$$
$$\varphi_T(\epsilon_1, \epsilon_2, \cdots, \epsilon_n) = \varphi_T((\eta_1, \eta_2, \cdots, \eta_n)P_1)$$
$$= \varphi_T((\eta_1, \eta_2, \cdots, \eta_n))P_1$$
$$= (\eta_1, \eta_2, \cdots, \eta_n)\begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}P_1$$
$$= (\epsilon_1, \epsilon_2, \cdots, \epsilon_n)P_2\begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}P_1$$
$$= (\epsilon_1, \epsilon_2, \cdots, \epsilon_n)P^{-1}\begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}P.$$

So $PTP^{-1} = \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}$ where T_0 is invertible and T_1 is nilpotent.

"$\Leftarrow$". Suppose there exists $P \in GL(n,R)$ such that $P^{-1}TP = C = \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}$ where $T_0 \in \mathbb{M}_k(R)$ is invertible and $T_1 \in \mathbb{M}_{n-k}(R)$ is nilpotent. Let the nilpotency index of T_1 be m. Then $C^m = \begin{pmatrix} T_0^m & 0 \\ 0 & 0 \end{pmatrix}$. So $(\mathbb{M}_n(R))C^m = (\mathbb{M}_n(R))C^{m+1}$ and $C^m(\mathbb{M}_n(R)) = C^{m+1}(\mathbb{M}_n(R))$. Hence, T is strongly π-regular. $\square$

Corollary 5.2.3 *Let R be a local ring. Then $A \in \mathbb{M}_2(R)$ is a non-trivial strongly π-regular matrix iff A is similar to $\begin{pmatrix} t_0 & 0 \\ 0 & t_1 \end{pmatrix}$, where $t_0 \in U(R)$ and $t_1 \in R$ is nilpotent.*

Corollary 5.2.4 *Let R be a commutative local ring. Then the following are equivalent for $A \in \mathbb{M}_2(R)$:*

1. *A is a non-trivial strongly π-regular matrix.*
2. *A is similar to $\begin{pmatrix} t_0 & 0 \\ 0 & t_1 \end{pmatrix}$, where $t_0 \in U(R)$ and $t_1 \in R$ is nilpotent.*
3. *$|A| \in R$ is nilpotent and $\operatorname{tr}(A) \in U(R)$ and A is similar to a diagonal matrix.*
4. *$|A| \in R$ is nilpotent, $\operatorname{tr}(A) \in U(R)$, and $x^2 - \operatorname{tr}(A)x + |A| = 0$ is solvable in R.*

Proof "*(1)* $\Rightarrow$ *(2)*". It follows by Corollary 5.2.3.

"*(2)* $\Rightarrow$ *(3)*". It is clear.

"*(3)* $\Rightarrow$ *(4)*". Same as the proof of "*(3)* $\Rightarrow$ *(4)*" of Corollary 2.2.11.

"*(4)* $\Rightarrow$ *(1)*". Suppose that *(4)* holds. Let $a \in R$ be a root of $x^2 - \operatorname{tr}(A)x + |A|$. Then $b := \operatorname{tr}(A) - a$ is also a root of $x^2 - \operatorname{tr}(A)x + |A|$. Thus, $a + b = \operatorname{tr}(A)$ and $ab = |A|$. Hence one of a, b must be a unit and the other must be nilpotent. Without loss of generality, we assume that $a \in U(R)$ and b is nilpotent. Write $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. From $a_{11} + a_{22} = \operatorname{tr}(A) \in U(R)$, either a_{11} or a_{22} is a unit. Without loss of generality, we may assume that $a_{22} \in U(R)$. Let $P = \begin{pmatrix} a_{21} & a - a_{11} \\ b - a_{22} & a_{12} \end{pmatrix}$. Then $P \in \mathrm{GL}_2(R)$ since

$|P| = aa_{22} + b(a_{11} - a) - |A| \in U(R)$. Thus

$$\begin{aligned} PAP^{-1} &= \frac{1}{|P|}\begin{pmatrix} a_{21} & a - a_{11} \\ b - a_{22} & a_{12} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} a_{12} & a_{11} - a \\ a_{22} - b & a_{21} \end{pmatrix} \\ &= \frac{1}{|P|}\begin{pmatrix} * & a_{21}(-a^2 + \mathrm{tr}(A)a - |A|) \\ a_{12}(-b^2 + \mathrm{tr}(A)b - |A|) & * \end{pmatrix} \\ &= \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \end{aligned}$$

for some $c, d \in R$. Since $|A| = cd$ and $\mathrm{tr}(A) = c + d$, one of c and d must be a unit and the other must be nilpotent. Thus, by direct calculation we know $\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$ is a non-trivial strongly π-regular matrix. Hence, A is strongly π-regular. □

As pointed out in [12], it follows from the results in the literature that for any commutative ring R, $\mathbb{M}_n(R)$ is strongly π-regular iff so is R and that, for a commutative local ring R, $\mathbb{M}_n(R)$ is strongly π-regular iff so is R, iff $J(R)$ is nil. Let R be the commutative local ring of p-adic integers. Then $\mathbb{M}_2(R)$ is a strongly clean ring but it is not strongly π-regular. Below, we characterize the local rings R for which $\mathbb{M}_2(R)$ is strongly π-regular.

Lemma 5.2.5 *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}_2(R)$ *where R is a local ring. If* $A \notin \mathbb{M}_2(J(R)) \cup \mathrm{GL}_2(R)$*, then A is similar to* $\begin{pmatrix} r & 1 \\ w & 0 \end{pmatrix}$ *where $r \in R$ and $w \in J(R)$.*

Proof Case 1. $b \in U(R)$. Let $P = \begin{pmatrix} 1 & 0 \\ -bdb^{-1} & b \end{pmatrix}$. Then $PAP^{-1} = \begin{pmatrix} a + bdb^{-1} & 1 \\ b(c - db^{-1}a) & 0 \end{pmatrix}$. Since $PAP^{-1} \notin \mathrm{GL}_2(R)$, we have $b(c - db^{-1}a) \in J(R)$. So the claim holds.

Case 2. $c \in U(R)$. Since $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$, the claim holds by Case 1.

Case 3. $b, c \in J(R)$. By hypothesis, either $a \in U(R)$ and $d \in J(R)$ or $a \in J(R)$ and $d \in U(R)$. We may assume that $a \in U(R)$ and $d \in J(R)$. Then $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a - c & a + b - c - d \\ c & c + d \end{pmatrix}$. Since $a + b - c - d \in U(R)$, the claim holds by Case 1. □

Lemma 5.2.6 *Let R be a local ring and $A = \begin{pmatrix} u & 1 \\ w & 0 \end{pmatrix}$ where $u \in U(R)$ and $w \in J(R)$. Then A is strongly π-regular iff $t^2 - ut - w$ has two left roots, one in $U(R)$ and the other nilpotent.*

Proof "$\Rightarrow$". It is clear that $A \notin \mathrm{GL}_2(R)$ and A is not nilpotent. Since A is strongly π-regular, by Corollary 5.2.3, there exists $P \in \mathrm{GL}_2(R)$ such that $PAP^{-1} = \begin{pmatrix} v & 0 \\ 0 & j \end{pmatrix}$ where either $v \in U(R)$ and j is nilpotent or v is nilpotent and $j \in U(R)$. As shown in the proof of Lemma 2.3.2, there exists $P = \begin{pmatrix} 1 & b \\ c & 1 \end{pmatrix} \in \mathrm{GL}_2(R)$ such that $PAP^{-1} = \begin{pmatrix} v & 0 \\ 0 & j \end{pmatrix}$, where either $v \in U(R)$ and j is nilpotent or v is nilpotent and $j \in U(R)$. From $PA = \begin{pmatrix} v & 0 \\ 0 & j \end{pmatrix} P$, one obtains

$$\begin{cases} u + bw = v \\ 1 = vb \\ cu + w = jc \\ c = j. \end{cases}$$

Thus, v can not be nilpotent, so it must be that $v \in U(R)$ and j is nilpotent. It follows that $c^2 - cu - w = 0$ and $(b^{-1})^2 - b^{-1}u - w = 0$. So the implication holds.

"$\Leftarrow$". Assume that $t^2 - ut - w$ has two left roots b, c with $b \in U(R)$ and c being nilpotent. Let $P = \begin{pmatrix} 1 & b^{-1} \\ c & 1 \end{pmatrix}$ and $D = \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}$. Then P is invertible and $PA = DP$. So A is strongly π-regular by Corollary 5.2.3. □

Theorem 5.2.7 *The following are equivalent for a local ring R:*

1. *$\mathbb{M}_2(R)$ is strongly π-regular.*

2. *$\mathbb{M}_2(J(R))$ is nil and, for any $u \in U(R)$ and $w \in J(R)$, $t^2 - ut - w$ has two left roots, one in $U(R)$ and the other in $J(R)$.*

3. *$\mathbb{M}_2(J(R))$ is nil and, for any $u \in U(R)$ and $w \in J(R)$, $t^2 - ut - w$ has two right roots, one in $U(R)$ and the other in $J(R)$.*

Proof "(*1*) $\Rightarrow$ (*2*)". (*1*) clearly implies that $\mathbb{M}_2(J(R))$ is nil. For $u \in U(R)$ and $w \in J(R)$, let $A = \begin{pmatrix} u & 1 \\ w & 0 \end{pmatrix}$. By (*1*), A is strongly π-regular. Hence, by Lemma 5.2.6, $t^2 - ut - w = 0$ has two left roots, one in $U(R)$ and the other is nilpotent. So (*2*) holds.

"(*2*) $\Rightarrow$ (*1*)". Let $A \in \mathbb{M}_2(R)$. We want to show that A is strongly π-regular. Because of (*2*), we may assume that $A \notin \mathbb{M}_2(J(R))$ and $A \notin \mathrm{GL}_2(R)$. Thus, by Lemma 5.2.5, we may assume that $A = \begin{pmatrix} u & 1 \\ w & 0 \end{pmatrix}$ where $u \in R$ and $w \in J(R)$. If $u \in J(R)$, then $A^2 \in \mathbb{M}_2(J(R))$. So A is nilpotent and hence is strongly π-regular. Therefore, we may further assume that $u \in U(R)$. By (*2*), $t^2 - ut - w = 0$ has two left roots, one in $U(R)$ and the other in $J(R)$ which is nilpotent. Thus, by Lemma 5.2.6, A is strongly π-regular.

"(*1*) $\Leftrightarrow$ (*3*)". Similar to the proof of "(*1*) $\Leftrightarrow$ (*2*)". □

As mentioned before, for a commutative local ring R, $\mathbb{M}_2(R)$ is strongly π-regular iff $J(R)$ is nil. As a contrast of this, there exists a local ring R with $J(R)$ locally nilpotent (thus, $\mathbb{M}_2(J(R))$ is nil), but $\mathbb{M}_2(R)$ is not strongly π-regular by [20]. Notice that for a left perfect ring R, $\mathbb{M}_n(R)$ is again left perfect, so it is strongly π-regular. Our concluding example gives a noncommutative local ring that is not one-sided perfect such that $\mathbb{M}_2(R)$ is strongly π-regular.

Example 5.2.8 *Let $G = \{\alpha_r : 0 \le r \in \mathbb{R}\}$ be a semigroup with multiplication defined by $\alpha_r\alpha_s = \alpha_{r+s}$. Then G has identity α_0. Let DG be the semigroup ring of G over a division ring D that is not a field and let (α_1) be the ideal of DG generated by α_1. Let $R = (DG)/(\alpha_1)$ be the quotient ring. Thus, $R = \bigoplus\{D\alpha_r : 0 \le r < 1\}$ is a left vector space over D with a basis $\{\alpha_r : 0 \le r < 1\}$ and the multiplication of R is given by*

$$\alpha_r\alpha_s = \begin{cases} \alpha_{r+s}, & \text{if } r+s<1, \\ 0, & \text{if } r+s \ge 1. \end{cases}$$

The ring R (with unity α_0) is noncommutative and the following hold:

1. *R is a local ring with $J(R)$ locally nilpotent.*

2. *R is not one-sided perfect.*

3. *$\mathbb{M}_2(R)$ is strongly π-regular.*

Proof *(1)*. It is clear that $J(R) = \bigoplus\{D\alpha_r : 0 < r < 1\}$ is locally nilpotent. Since $R/J(R) \cong D$, R is local.

(2). If let $x_i = \alpha_{2^{-i}} \in J(R)$ for $i = 1, 2, \cdots$, then for any $n > 0$, $x_n \cdots x_2 x_1 = x_1 x_2 \cdots x_n = \alpha_r \neq 0$ where $r = 1 - \frac{1}{2^n}$. So $J(R)$ is neither left nor right T-nilpotent and thus R is not one-sided perfect.

(3). Because $J(R)$ is locally nilpotent, $\mathbb{M}_2(J(R))$ is nil. So, by Theorem 5.2.7, it suffices to show that, for any $u \in U(R)$ and any $w \in J(R)$, $t^2 - ut - w$ has two left roots, one in $U(R)$ and the other in $J(R)$. Write

$$u = u_0\alpha_{r_0} + u_1\alpha_{r_1} + \cdots + u_n\alpha_{r_n} \text{ with } 0 \neq u_0 \in D,$$

$$w = w_0\alpha_{r_0} + w_1\alpha_{r_1} + \cdots + w_n\alpha_{r_n} \text{ with } w_0 = 0,$$

where $0 = r_0 < r_1 < \cdots < r_n < 1$. Rewrite $u = \Sigma u_r\alpha_r$ and $w = \Sigma w_r\alpha_r$ where $u_r = w_r = 0$ for $r \notin \{r_0, \cdots, r_n\}$. Write $t = \Sigma t_r\alpha_r$. Then

$$t^2 - tu - w = 0 \Leftrightarrow t(t-u) = w$$

$$\Leftrightarrow \begin{cases} t_0(t_0 - u_0) = 0 \\ \sum_{r+s=k} t_r(t_s - u_s) = w_k \text{ for } 0 < k < 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} t_0(t_0 - u_0) = 0 & (P_0) \\ t_k(t_0 - u_0) + t_0t_k = w_k + t_0u_k - \sum_{r+s<k} t_r(t_s - u_s) \text{ for } 0 < k < 1 & (P_k) \end{cases}$$

From (P_0), $t_0 = 0$ or $t_0 = u_0$. For $t_0 = 0$ or $t_0 = u_0$, we next show that one can find $t_k \in D$ for each $0 < k < 1$ such that almost all t_k are zero and (P_k) holds for all $0 \leq k < 1$. Thus, $\Sigma_{r\geq 0} t_r\alpha_r$ gives a left root of $t^2 - ut - w$ in $J(R)$ when $t_0 = 0$ and a left root in $U(R)$ when $t_0 = u_0$.

Let $m > 0$ be an integer such that $mr_1 \geq 1$ and let

$$C = \Big\{r_{i_0} + r_{i_1} + \cdots + r_{i_j} : j \in \{0, 1, \cdots, m\}, r_{i_0}, r_{i_1}, \cdots, r_{i_j} \in \{r_0, \cdots, r_n\}\Big\}.$$

Then C is a finite set, so we can write $C = \{s_0, s_1, s_2, \cdots\}$ with $0 = s_0 < s_1 < \cdots$. Take $t_0 = 0$ or $t_0 = u_0$ and choose $t_r = 0$ if $r \notin C$. Suppose that, for $k = s_l$ where $l > 0$ is an integer, we have chosen t_r for all $0 \leq r < k$. Since (P_k) is solvable for t_k, we then choose t_k to be the (unique) solution to (P_k). A simple induction shows that all the required t_r's exist. The proof is complete. □

Chapter 6

$g(x)$-Clean Rings

In section 6.1, we discuss some general properties of $g(x)$-clean rings which are similar to those of clean rings. In section 6.2, we focus on $(x^2 + cx + d)$-clean rings, in particular, on $(x^2 - 2x)$-clean and $(x^2 - nx)$-clean rings. In section 6.3, we consider $(x^n - x)$-clean rings. Theorem 6.2.2 and Theorem 6.2.5 are the main results. This chapter comes from [33].

6.1 $g(x)$-clean rings

In this section, we discuss some general properties similar to those of clean rings.

Definition 6.1.1 *Let $g(x)$ be a fixed polynomial in $C(R)[x]$. An element $r \in R$ is called $g(x)$-* ***clean*** *if $r = e + u$ where $g(e) = 0$ and $u \in U(R)$. Following Camillo and Simón [18], we say that R is $g(x)$-**clean** if every element of R is $g(x)$-clean.*

The $(x^2 - x)$-clean rings are precisely the clean rings. The following two examples explain the relations between $g(x)$-clean rings and clean rings.

Example 6.1.2 *There exists an $(x^4 - x)$-clean ring which is not clean. Recall that*

$\mathbb{Z}_{(7)} = \{\frac{m}{n} \in \mathbb{Q} : \gcd(7,n) = 1\}$. *The proof of [67, Theorem 3.1] shows that* $\mathbb{Z}_{(7)}C_3$ *is an* $(x^4 - x)$*-clean ring. But* $\mathbb{Z}_{(7)}C_3$ *is not clean by [38, Example 1].*

Example 6.1.3 *Let* R *be a Boolean ring containing more than two elements and let* $c \in R$ *with* $0 \neq c \neq 1$. *Let* $g(x) = x^2 + (1+c)x + c = (x+1)(x+c)$. *Then* R *is not* $g(x)$*-clean:*

If $c = e + u$ *where* u *is a unit and* $g(e) = 0$, *then it must be that* $u = 1$ *and so* $e = c - 1 = c + 1$. *But, clearly,* $g(c+1) \neq 0$. *However,* R *is certainly clean.*

Let R and S be rings and $\theta : C(R) \to C(S)$ be a ring homomorphism with $\theta(1) = 1$. For $g(x) = \Sigma a_i x^i \in C(R)[x]$, let $\theta'(g(x)) = \Sigma\theta(a_i)x^i \in C(S)[x]$. Then θ induces a map θ' from $C(R)[x]$ to $C(S)[x]$. Clearly, if $g(x)$ is a polynomial with coefficients in $\mathbb{Z}$, then $\theta'(g(x)) = g(x)$.

Proposition 6.1.4 *Let* $\theta : R \to S$ *be a ring epimorphism. If* R *is* $g(x)$*-clean, then* S *is* $\theta'(g(x))$*-clean.*

Proof Let $g(x) = a_0 + a_1x + \cdots + a_nx^n \in C(R)[x]$. Then $\theta'(g(x)) = \theta(a_0) + \theta(a_1)x + \cdots + \theta(a_n)x^n \in C(S)[x]$. For any $s \in S$, there exists $r \in R$ such that $\theta(r) = s$. Since R is $g(x)$-clean, there exist $e \in R$ and $u \in U(R)$ such that $r = e + u$ and $g(e) = 0$. Then $s = \theta(r) = \theta(e) + \theta(u)$ with $\theta(u) \in U(S)$ and $\theta'(g(\theta(e)) = 0$, that is, S is $\theta'(g(x))$-clean. $\square$

Let $R \to R/I, r \mapsto \overline{r} = r + I$, $g(x) \in C(R)[x]$, and $\overline{g}(x) \in \frac{R}{I}[x]$.

Corollary 6.1.5 *Let* R *be* $g(x)$*-clean. Then, for any ideal* I *of* R, R/I *is* $\overline{g}(x)$*-clean where* $\overline{g}(x) \in C(R/I)[x]$.

We say R is **lifting g-roots modulo I** if $\overline{g}(\overline{a}) = 0$, $a \in R$, implies $g(b) = 0$ for $b \in R$ and $b - a \in I$. This is the generalization of lifting idempotents modulo I where $g(x) = x^2 - x$.

Proposition 6.1.6 *Let $I \subsetneq J(R)$ be an ideal of R, $\eta : R \to R/I$ with $\eta(r) = r + I = \overline{r}$, and $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R)[x]$ with $\overline{g}(x) = \sum_{i=0}^{n} \overline{a}_i x^i \in C(R/I)[x]$. If R/I is $\overline{g}(x)$-clean and R is lifting g-roots modulo I, then R is $g(x)$-clean .*

Proof For any $r \in R$, let $r + I = \overline{r} = \overline{e} + \overline{u}$ where $\overline{g}(\overline{e}) = 0$ and $\overline{u} \in U(R/I)$. Because roots of $\overline{g}(x)$ lift modulo I, we can assume $e \in R$ such that $g(e) = 0$. So $r - e - u = i$ for some $i \in I$. Hence $r = e + (u + i)$ with $u + i \in U(R)$. Thus, r is $g(x)$-clean, that is, R is $g(x)$-clean. $\square$

We omit the argument of the following proposition since the proof is standard.

Proposition 6.1.7 *Let $g(x) \in \mathbb{Z}[x]$ and let $\{R_i\}_{i\in I}$ be a family of rings. Then the direct product $\prod_{i\in I} R_i$ is $g(x)$-clean iff every R_i, $i \in I$, is $g(x)$-clean.*

Canonically, we can identify a ring R with $\{aI_n : a \in R\}$, a subring of $M_n(R)$, where I_n is the identity matrix of $M_n(R)$. Thus, we can identity $g(x) = \sum a_i x^i \in C(R)[x]$ with $\sum a_i I_n x^i \in C(M_n(R))[x]$.

Proposition 6.1.8 *Let R be a ring, $g(x) \in C(R)[x]$, and $n \in \mathbb{N}$. Then R is $g(x)$-clean iff the upper triangular matrix ring $\mathbb{T}_n(R)$ is $g(x)$-clean.*

Proof "$\Rightarrow$". Let $A = (a_{ij}) \in \mathbb{T}_n(R)$ with $a_{ij} = 0$ for $1 \le j < i \le n$. Since R is $g(x)$-clean, for any $1 \le i \le n$, there exist $e_{ii} \in R$ and $u_{ii} \in U(R)$ such that $a_{ii} = e_{ii} + u_{ii}$ with $g(e_{ii}) = 0$. Suppose $g(x) = \sum_{i=0}^{m} a_i x^i \in C(R)[x]$. Let $A = E + U$ with

$$E = \begin{pmatrix} e_{11} & 0 & \cdots & 0 \\ 0 & e_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_{nn} \end{pmatrix} \text{ and } U = \begin{pmatrix} u_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1n} \\ 0 & u_{22} & \cdots & a_{2,n-1} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & u_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & \cdots & 0 & u_{nn} \end{pmatrix}.$$

Then $U \in GL_n(R)$ and $g(E) = a_0 I_n + a_1 E + \cdots + a_m E^m =$

$$= \begin{pmatrix} a_0 & 0 & \cdots & 0 \\ 0 & a_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 \end{pmatrix} + \begin{pmatrix} a_1 e_{11} & 0 & \cdots & 0 \\ 0 & a_1 e_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_1 e_{nn} \end{pmatrix} + \cdots +$$

$$+ \begin{pmatrix} a_m e_{11}^m & 0 & \cdots & 0 \\ 0 & a_m e_{22}^m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_m e_{nn}^m \end{pmatrix} = \begin{pmatrix} g(e_{11}) & 0 & \cdots & 0 \\ 0 & g(e_{22}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g(e_{nn}) \end{pmatrix} = 0.$$

So $\mathbb{T}_n(R)$ is $g(x)$-clean.

"$\Leftarrow$". Define $\theta : \mathbb{T}_n(R) \to R$ by $\theta(A) = a_{11}$. Then θ is a ring epimorphism. By a proof similar to that of Proposition 6.1.4, we have R is $g(x)$-clean. $\square$

In [38], the authors proved that if R is clean, then so is $M_n(R)$ for all $n \geq 1$. Here we have a similar result for $g(x)$-clean rings.

Proposition 6.1.9 *Let R be a ring and $g(x) \in C(R)[x]$. If R is $g(x)$-clean, then $M_n(R)$ is $g(x)$-clean for all $n \geq 1$.*

Proof We prove the claim by induction on n. The case $n = 1$ is clear. Assume the claim holds for $M_{n-1}(R)$ where $n > 1$. If $\alpha \in M_n(R)$, write $\alpha = \begin{pmatrix} A & X \\ Y & b \end{pmatrix}$ in block form where $A \in M_{n-1}(R)$ and $b \in R$. By hypothesis, $A = E+U$ where $E \in M_{n-1}(R)$ is a root of $g(x)$ and U is a unit of $M_{n-1}(R)$. Then $b - YU^{-1}X \in R$. So, since R is $g(x)$-clean, we have $b - YU^{-1}X = e+u$ where $e \in R$ is a root of $g(x)$ and $u \in U(R)$. Then $\alpha - \begin{pmatrix} E & 0 \\ 0 & e \end{pmatrix} = \beta$, where $\beta = \begin{pmatrix} U & X \\ Y & u+YU^{-1}X \end{pmatrix}$. We obtain

$$\begin{pmatrix} I_{n-1} & 0 \\ -YU^{-1} & 1 \end{pmatrix} \begin{pmatrix} U & X \\ Y & u+YU^{-1}X \end{pmatrix} \begin{pmatrix} I_{n-1} & -U^{-1}X \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & u \end{pmatrix}.$$

So β is a unit of $M_n(R)$. Since $\begin{pmatrix} E & 0 \\ 0 & e \end{pmatrix}$ is a root of $g(x)$, $\alpha \in M_n(R)$ is $g(x)$-clean. $\square$

Proposition 6.1.10 *Let R be a ring and $g(x) \in C(R)[x]$. Then the formal power series ring $R[[t]]$ is $g(x)$-clean iff R is $g(x)$-clean.*

Proof " $\Leftarrow$". Let $f = \sum_{i\geq 0} a_i t^i \in R[[t]]$. Since R is $g(x)$-clean, $a_0 = e + u$ where $e \in R$, $u \in U(R)$ and $g(e) = 0$. Then $f = e + (u + \sum_{i\geq 1} a_i t^i)$ with $u + \sum_{i\geq 1} a_i t^i \in U(R[[t]])$. So f is $g(x)$-clean. Hence, $R[[t]]$ is $g(x)$-clean.

"$\Rightarrow$". Since $\theta : R[[t]] \to R$, $\sum_{i\geq 0} a_i t^i \mapsto a_0$, is a ring epimorphism. By Proposition 6.1.4, R is $g(x)$-clean. $\square$

Remark 6.1.11 *Generally the polynomial ring $R[t]$ is not $g(x)$-clean for a non-zero polynomial $g(x) \in C(R)[x]$. For example, the polynomial ring $R[t]$ with R commutative is not $(x^2 - x)$-clean in [38] and is not $(x^n - x)$-clean by [67].*

6.2 $(x^2 + cx + d)$-clean rings

In this section, we consider some types of $(x^2 + cx + d)$-clean rings.

If V is a countably infinite dimensional vector space over a division ring D, then $\mathrm{End}(V_D)$ is clean by Nicholson and Varadarajan [53]. Further, Camillo and Simón [12] proved that $\mathrm{End}(V_D)$ is $g(x)$-clean provided that $g(x) \in C(D)[x]$ has two distinct roots in $C(D)$. Recently, this result has been extended as the following.

Example 6.2.1 *[54] Let R be a ring and M_R be a semisimple module over R. If $g(x) \in (x-a)(x-b)C(R)[x]$ where $a, b \in C(R)$ are such that b and $b - a$ are both units in R, then $\mathrm{End}(M_R)$ is $g(x)$-clean.*

Example 6.2.1 implies that the endomorphism ring of a semisimple module is clean (let $a = 0$ and $b = 1$). But it is surprising that Example 6.2.1 does not say more than this.

Theorem 6.2.2 *Let R be a ring and $g(x) \in (x-a)(x-b)C(R)[x]$ where $a, b \in C(R)$ are such that $b - a \in U(R)$. Then the following hold:*

1. *R is clean iff R is $(x-a)(x-b)$-clean.*

2. *If R is clean, then R is $g(x)$-clean.*

Proof (*1*). "$\Rightarrow$". Let $r \in R$. Since R is clean, $\frac{r-a}{b-a} = e + u$ where $e^2 = e \in R$ and $u \in U(R)$. Thus, $r = [e(b-a)+a] + u(b-a)$, where $u(b-a) \in U(R)$ and $e(b-a)+a$ is a root of $(x-a)(x-b)$. Hence R is $(x-a)(x-b)$-clean.

"$\Leftarrow$". Let $r \in R$. Since R is $(x-a)(x-b)$-clean, $r(b-a)+a = e+u$ where e is a root of $(x-a)(x-b)$ and $u \in U(R)$. Thus, $r = \frac{e-a}{b-a} + \frac{u}{b-a}$, where $\frac{u}{b-a}$ is a unit of R and $(\frac{e-a}{b-a})^2 = \frac{(e-a)(e-b+b-a)}{(b-a)^2} = \frac{(e-a)(b-a)}{(b-a)^2} = \frac{e-a}{b-a}$. So R is clean.

(*2*). This follows from (*1*). □

Note that the converse of (*2*) need not hold by Example 6.1.2 and Example 6.1.3.

Corollary 6.2.3 *Let R be a ring. Then R is clean iff R is (x^2+x)-clean.*

Proof This is the case of Theorem 6.2.2 (*1*) when $a = 0$ and $b = -1$. □

Remark 6.2.4 *Though the clean rings are just the (x^2+x)-clean rings, a clean element need not be an (x^2+x)-clean element. For example, $1+1 = 2 \in \mathbb{Z}$ is clean but it is not (x^2+x)-clean in $\mathbb{Z}$.*

For any $n \in \mathbb{N}$, let $U_n(R)$ denote the set of elements of R that can be expressed as a sum of k units of R with $1 \leq k \leq n$ [40]. Rings generated by units are discussed in many papers (see, for example, [40, 41, 57]).

It is an open question whether or not the clean property of the matrix ring $\mathbb{M}_n(R)$ $(n > 1)$ implies that of R [38]. But the (x^2-2x)-clean property of R and of the matrix ring $\mathbb{M}_n(R)$ $(n > 1)$ are equivalent and the (x^2-2x)-clean rings are precisely those rings whose elements can be expressed as the sum of a unit and a square root of 1. The equivalence "*(7)* $\Leftrightarrow$ *(8)*" in the next theorem belongs to [35] and "*(6)* $\Rightarrow$ *(7)*" has been proved by Camillo and Yu [17].

Theorem 6.2.5 *Let R be a ring and $m, n, k \in \mathbb{N}$. Then the following are equivalent:*

1. *R is $(x^2 - 2^n x)$-clean.*

2. *R is $(x^2 + 2^n x)$-clean.*

3. *R is $(x^2 - 2x)$-clean.*

4. *R is $(x^2 + 2x)$-clean.*

5. *R is $(x^2 - 1)$-clean.*

6. *R is clean and $2 \in U(R)$.*

7. *For any $a \in R$, a can be expressed as $a = u + v$ where $u \in U(R)$ and $v^2 = 1$.*

8. *$\mathbb{M}_k(R)$ is $(x^2 - 2x)$-clean.*

9. *$\mathbb{M}_k(R[[t]])$ is $(x^2 - 2x)$-clean.*

10. *$\mathbb{M}_k\left(\dfrac{R[t]}{(t^m)}\right)$ is $(x^2 - 2x)$-clean.*

Proof "*(1)* $\Rightarrow$ *(6)*". We prove $2 \in U(R)$. Suppose $2 \notin U(R)$. Then $\overline{R} = R/(2^n R) \neq 0$. Let $2^n = e + u$ with $e^2 - 2^n e = 0$ and $u \in U(R)$. Since $\overline{0} = \overline{2^n} = \overline{e} + \overline{u}$, we have $\overline{e} = -\overline{u} \in U(\overline{R})$. But $\overline{e}^2 = \overline{e^2} = \overline{2^n e} = \overline{0}$. This is a contradiction. So $2 \in U(R)$. Then R is clean by *(1)* of Theorem 6.2.2 with $a = 0$ and $b = 2^n$.

"*(6)* $\Rightarrow$ *(1)*". By *(1)* of Theorem 6.2.2, R is $(x^2 - 2^n x)$-clean.

Similarly, we can prove "*(2)* $\Leftrightarrow$ *(6)*", "*(3)* $\Leftrightarrow$ *(6)*" and "*(4)* $\Leftrightarrow$ *(6)*".

"*(6)* $\Rightarrow$ *(7)*". Let $a \in R$. By "*(3)* $\Leftrightarrow$ *(6)*", $1 - a = e + u$ where $e^2 = 2e$ and $u \in U(R)$. Then $a = (-u) + (1 - e)$ with $-u \in U(R)$ and $(1 - e)^2 = 1$ ([17, Proposition 10]).

"(*7*) $\Rightarrow$ (*6*)". Let $a \in R$. By (*7*), $1 - a = u + v$ where $u \in U(R)$ and $v^2 = 1$. Thus, $a = (-u) + (1 - v)$ with $-u \in U(R)$ and $(1 - v)^2 = 2(1 - v)$. By "(*3*) $\Leftrightarrow$ (*6*)", we proved that (*7*) implies (*6*).

"(*5*) $\Rightarrow$ (*7*)". If R is $(x^2 - 1)$-clean, then for any $r \in R$, there exist $v, u \in U(R)$ such that $r = v + u$ and $v^2 = 1$.

"(*7*) $\Rightarrow$ (*5*)". Let $a \in R$. Then a can be expressed as $a = u + v$ with $u, v \in U(R)$ and $v^2 = 1$. So v is the root of $x^2 - 1$. Hence R is $(x^2 - 1)$-clean.

"(*8*) $\Leftrightarrow$ (*7*)". By [35, Theorem 1.5].

"(*9*) $\Leftrightarrow$ (*3*)". Since R is $(x^2 - 2x)$-clean iff $R[[t]]$ is $(x^2 - 2x)$-clean by Proposition 6.1.10, we get the equivalence of (*9*) and (*3*) by "(*8*) $\Leftrightarrow$ (*3*)".

"(*10*) $\Leftrightarrow$ (*3*)". By Proposition 6.1.4, "(*3*) $\Rightarrow$ (*9*) $\Rightarrow$ (*10*) $\Rightarrow$ (*3*)". $\square$

Remark 6.2.6 *Let $m, k \in \mathbb{N}$. Similar to Theorem 6.2.5, it can be proved that, for a ring R and a fixed integer $n > 0$, the following are equivalent:*

1. *R is $(x^2 - n^m x)$-clean.*

2. *R is $(x^2 + n^k x)$-clean.*

3. *R is $(x^2 - nx)$-clean.*

4. *R is $(x^2 + nx)$-clean.*

5. *R is a clean ring with $n \in U(R)$.*

But the other corresponding items in Theorem 6.2.5 are unknown if $2 \notin U(R)$.

Example 6.2.7 *Let R be a ring with $n \in U(R)$. Then, for any continuous or discrete R-module M (see definition in [48]), the endomorphism ring $End_R(M)$ is an $(x^2 - nx)$-clean ring.*

Proof By a result in [16], every endomorphism ring of continuous or discrete module is clean. So by Theorem 6.2.2, $End_R(M)$ is an $(x^2 - nx)$-clean ring. For example, when R is a division ring, $End_R(M)$ is an $(x^2 - nx)$-clean ring for any $n \in \mathbb{N}$. □

Let $C(X)$ denote the ring of all continuous real valued functions from a topological space X to the real number field $\mathbb{R}$ and $C^*(X)$ denote the subring of $C(X)$ consisting of all bounded functions in $C(X)$ [34, pp. 10-11]. A topological space X is called **strongly zero-dimensional** if X is a non-empty completely regular Hausdorff space and every finite functionally open cover $\{U_i\}_{i=1}^k$ of the space X has a finite open refinement $\{V_i\}_{i=1}^m$ such that $V_i \cap V_j = \emptyset$ for any $i \neq j$ [30].

Example 6.2.8 *Let X be a strongly zero-dimensional topological space. Then both $\mathbb{M}_k(C(X))$ and $\mathbb{M}_k(C^*(X))$ are $(x^2 - nx)$-clean rings for any $n, k \in \mathbb{N}$.*

Proof By [9, Theorem 2.5], $C(X)$ and $C^*(X)$ are clean. So they are $(x^2 - nx)$-clean by Theorem 6.2.2 and n is invertible in $C(X)$ and $C^*(X)$. Then, by Proposition 6.1.9, Theorem 6.2.5 and Remark 6.2.6, $\mathbb{M}_k(C(X))$ and $\mathbb{M}_k(C^*(X))$ are $(x^2 - nx)$-clean rings for any $n, k \in \mathbb{N}$. □

Example 6.2.9 *Let F be a field with characteristic char $F = c$, let V be an infinite dimensional vector space over F, and let R be the subring of* $\mathrm{End}({}_FV)$ *generated by the identity and the finite rank transformations. Then $\mathbb{M}_k(R)$ is an $(x^2 - nx)$-clean ring where $n, k \in \mathbb{N}$ and c does not divide n.*

Proof By [36, Example 5.15], R is a unit-regular ring. So by [38], R is clean. Then R is an $(x^2 - nx)$-clean ring since $n \in R$ is a unit. Hence, by Proposition 6.1.8, $\mathbb{M}_k(R)$ is an $(x^2 - nx)$-clean ring for any $n, k \in \mathbb{N}$. □

Example 6.2.10 *Ehrlich [28] defined the unit-regular rings. She proved that if R is a* *unit-regular ring with $2 \in U(R)$, then every element $rur = r \in R$ with certain $u \in U(R)$ can be expressed as $r = \frac{2ru-1}{2}u^{-1} + \frac{1}{2}u^{-1}$, that is, $R = U_2(R)$. In fact, for every unit-regular ring with $2 \in U(R)$, the matrix ring $\mathbb{M}_k(R)$, for any $k \in \mathbb{N}$, is an $(x^2 - 2x)$-clean ring by [38] and Theorem 6.2.5.*

Proposition 6.2.11 *Let R be a ring with $d \in U(R)$. If R is $(x^2 + cx + d)$-clean, then $R = U_2(R)$. In particular, if R is $(x^2 + x + 1)$-clean, then $R = U_2(R)$ is $(x^4 - x)$-clean.*

Proof Let $r \in R$. Then $r = e + u$ with $u \in U(R)$ and $e^2 + ce + d = 0$. So $e(e + c) = (e + c)e = -d \in U(R)$. Hence $e \in U(R)$. That is, $r \in U_2(R)$. Therefore, $R = U_2(R)$. Now the other conclusion is easy. □

6.3 $(x^n - x)$-clean rings

A ring R is called **potent** if idempotents lift modulo $J(R)$ and every left (or right) ideal not contained in $J(R)$ contains a nontrivial idempotent (an idempotent that is not 0 or 1 is called a nontrivial idempotent). Every exchange ring is potent, so is every clean ring [51]. Notice that any potent ring containing no infinite family of orthogonal nonzero idempotents is a semiperfect ring [17]. Since $\mathbb{Z}_{(7)}C_3$ is not a semiperfect ring [64] but is a Noetherian ring, it is not potent (hence not exchange). Thus, by Example 6.1.2, an $(x^4 - x)$-clean ring need not be potent. By Ye [67, p. 5624], the directly infinite regular ring with 2 invertible constructed by Bergman [39, Example 1] is not $(x^n - x)$-clean for every $n \geq 2$.

Proposition 6.3.1 *Let R be a ring with $n \in \mathbb{N}$. Then R is $(ax^{2n} - bx)$-clean iff R is $(ax^{2n} + bx)$-clean.*

Proof " $\Rightarrow$ ". Suppose R is $(ax^{2n} - bx)$-clean. Then for any $r \in R$, $-r = e+u$, $ae^{2n} - be = 0$ and $u \in U(R)$. So $r = (-e) + (-u)$ where $(-u) \in U(R)$ and $a(-e)^{2n} + b(-e) = 0$. Hence, r is $(ax^{2n} + bx)$-clean. Therefore, R is $(ax^{2n} + bx)$-clean.

" $\Leftarrow$ ". Suppose R is $(ax^{2n}+bx)$-clean. Let $r \in R$. Then there exist e and u such that $-r = e+u$, $ae^{2n}+be = 0$ and $u \in U(R)$. So $r = (-e)+(-u)$ satisfies $a(-e)^{2n}-b(-e) = 0$. Hence, R is $(ax^{2n} - bx)$-clean. $\square$

By Proposition 6.3.1, we get that $\mathbb{Z}_{(7)}C_3$ is also $(x^4 + x)$-clean.

Example 6.3.2 *Let $2 \leq n \in \mathbb{N}$. If for every $a \in R$, $a = u + v$ where $u \in U(R)$ and $v^{n-1} = 1$, then R is $(x^n - x)$-clean.*

The following lemma is well-known.

Lemma 6.3.3 *Let $a \in R$. The following are equivalent for $n \geq 1$:*

1. *$a = a(ua)^n$ for some $u \in U(R)$.*
2. *$a = ve$ for some $e^{n+1} = e$ and some $v \in U(R)$.*
3. *$a = fw$ for some $f^{n+1} = f$ and some $w \in U(R)$.*

Proof "(*1*) $\Rightarrow$ (*2*)". Suppose that (*1*) holds and let $e = ua$. Then $a = u^{-1}e$ with $e^{n+1} = e$.

"(*2*) $\Rightarrow$ (*3*)". Suppose that (*2*) holds and let $f = vev^{-1}$. Then $a = fv$ with $f^{n+1} = f$.

"(*3*) $\Rightarrow$ (*1*)". Suppose that (*3*) holds. Then $(aw^{-1})^{n+1} = f^{n+1} = f = aw^{-1}$. It follows that $a = fw = (aw^{-1})^{n+1}w = a(w^{-1}a)^n$. $\square$

Proposition 6.3.4 *Let R be an $(x^n - x)$-clean ring where $n \geq 2$ and $a \in R$. Then either (i) $a = u + v$ where $u \in U(R)$ and $v^{n-1} = 1$; or (ii) both aR and Ra contain nontrivial idempotents.*

Proof Write $a = u + e$ where u is a unit and $e^n = e$. Then $ae^{n-1} = ue^{n-1} + e$. So $a(1 - e^{n-1}) = u(1 - e^{n-1})$. Since $1 - e^{n-1}$ is an idempotent, by Lemma 6.3.3, $u(1 - e^{n-1}) = fw$ where $w \in U(R)$ and $f^2 = f \in R$. So $f = a(1 - e^{n-1})w^{-1} \in aR$. Suppose (i) does not hold. Then $1 - e^{n-1} \neq 0$. Hence, $f \neq 0$. Thus, aR contains a non trivial idempotent. Similarly, Ra contains a non trivial idempotent. □

An element $r \in R$ is called n-**clean** if $r = e + u_1 + \cdots + u_n$ with $e^2 = e \in R$ and $u_i \in U(R)$ for $1 \leq i \leq n$. And R is called n-**clean** if every element of R is n-clean [65].

Proposition 6.3.5 *Let $n \in \mathbb{N}$. If the ring R is $(x^n - x)$-clean, then R is 2-clean.*

Proof Let $r \in R$. Then $r = t + v$ for some $t^n = t$ and $v \in U(R)$. Since $t(= t^n)$ is a strongly π-regular element and strongly π-regular element is strongly clean(it is of course clean) [52], $t = e + u$ for some $e^2 = e \in R$ and $u \in U(R)$. So $r = e + u + v$ is 2-clean. Hence, R is a 2-clean ring. □

In fact, all $(x^2 - x)$-clean rings and $(x^2 + cx + d)$-clean rings with $d \in U(R)$ discussed above are 2-clean rings.

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