GRADINGS BY FINITE GROUPS ON LIE ALGEBRAS OF TYPE D_4

JASON MELVIN McGRAW

Gradings by finite groups on Lie algebras of type D_4

by

©Jason Melvin McGraw

A Thesis Submitted to the School of Graduate Studies in partial fulfillment of the requirement for the degree of Master of Science

Department of Mathematics and Statistics
Memorial University of Newfoundland

October, 2006

St. John's, Newfoundland, Canada

Your file Votre référence
ISBN: 978-0-494-30488-4
Our file Notre référence
ISBN: 978-0-494-30488-4

Abstract

In this thesis we explore the gradings by finite groups on Lie algebras of type D_4 over the field of complex numbers. For gradings on simple Lie algebras several approaches have been studied. In [9], Onishchik and Vinberg give an exposition of the results of V. Kac who had classified all automorphisms of finite order in all simple Lie algebras, hence classified the gradings of such algebras by finite cyclic groups.

J. Patera and co-authors [5], [6], [7] have focused on "fine" gradings and approach this with the help of maximal Abelian subgroups (MAD-subgroups) of diagonizable automorphisms in $\mathrm{Aut}(gl(n, \mathbb{C}))$. More recently Y. Bahturin, I. Shestakov, M. Zaicev [1] have approached gradings on simple Lie algebras by finite groups by looking at the dual group action which will be the main approach used in this paper.

The gradings on simple Lie algebras of type D_l, $l > 4$, have been described by Y. Bahturin and M. Zaicev in [4]. This was done by looking at gradings on the full matrix algebras and noting that for a realisation of a Lie algebra of type D_l, $l > 4$, as $K(M_{2l}, *)$, the skew-symmetric matrices with respect to a transpose involution $*$, the automorphisms of $K(M_{2l}, *)$ can be lifted to

automorphisms of the full matrix algebra. The gradings on Lie algebras of type D_4 were not described in [1] or [6] because some of the automorphisms of these Lie algebras cannot be lifted to the full matrix algebra.

In this thesis we apply the same approach as [1], [2], [3], [4] to describe all gradings that can be lifted to the full matrix algebra and all gradings that are isomorphic to these gradings. We also develop an approach inspired by [9] which may be fruitful for describing the remaining gradings. We give examples of some of these gradings.

Acknowledgements

I would like to acknowledge Dr. Bahturin for his instrumental guidance and financial support throughout my Masters of Science degree. Our various meetings and reading courses have given me invaluable mathematical insights. Many thanks to my family for their continued support which fuels my pursuit of higher education. I thank the Department of Mathematics and Statistics of Memorial University of Newfoundland for providing a friendly atmosphere and facilities. I also thank the School of Graduate Studies and the Department of Mathematics and Statistics for financial support. I also want to thank H. McIntosh and N. Kirby for their friendship and moral support throughout my years at Memorial University of Newfoundland.

List of Figures

Contents

Introduction

The gradings on Lie algebras of type D_l, $l > 4$, have been described in [4] and [6] and we apply similar techniques as in [4] to describe gradings for the D_4 case. The motivation behind this thesis is that for the D_4 case, not all gradings were described in [4] or [6]. The D_4 case is different because not all gradings are matrix gradings.

We approach the D_4 case by looking at the actions of the dual group $\widehat{G}$ associated to a grading by a finite abelian group G and use a group homorphism f from the dual group actions to automorphisms of Lie algebras. We prove that $f(\widehat{G}) \simeq G$ when G is generated by its support.

It is well known that the support of a grading of a simple Lie algebra generates an abelian group and we use this to impose restrictions on $f(\widehat{G})$. We show that we can always express $f(\widehat{G})$ as the direct product of a cyclic group and a subgroup of inner automorphisms. Since when G is generated by its support $f(\widehat{G}) \simeq G$, we can use the restriction on $f(\widehat{G})$ on G to express G as the direct product of a cyclic group $\langle z \rangle$ and a subgroup A of G. Since all the gradings by cyclic groups were described by V. Kac we know all the natural grading by $G/A \simeq \langle z \rangle$, which will allow us to view the grading by G

as a refinement of the natural grading.

We also classify gradings on a Lie algebra of type D_4 into two classes: matrix gradings and non-matrix gradings.

We describe all matrix gradings as well as a technique which looks at gradings on subalgebras that are point-wise fixed by an automorphism of L, which allows the gradings to extend to the whole Lie algebra.

The technique could be fruitful for finding certain non-matrix gradings if we knew the gradings on Lie algebras of type G_2. We also give an example of a non-matrix grading by a group G, for which there is a natural matrix grading by G/K for some normal subgroup K of G.

Chapter 1

Definitions of gradings on a Lie algebra and actions by automorphisms

1.1 Definitions and various types of gradings.

Through this work the base field of coefficients is always the field of complex numbers $\mathbb{C}$. Let us start with some definitions.

Definition 1.1.1 *A vector space L over a field $\mathbb{C}$, with an operation $L \times L \longrightarrow L$, denoted $(x, y) \mapsto [x, y]$ and called the bracket or commutator of x and y, is called a* Lie algebra over $\mathbb{C}$ *if the following axioms are satisfied:*

1. *The bracket operation is bilinear.*
2. $[x, x] = 0$, $\forall x \in L$.

3. $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0 \quad \forall x,y,z \in L.$

We denote the endomorphism of L which sends $y \mapsto [x,y]$ *by* $\operatorname{ad} x$.

Definition 1.1.2 *An element x of a Lie algebra L is called* ad-nilpotent *if there exists a positive n such that* $(\operatorname{ad} x)^n = 0$.

Definition 1.1.3 *An* isomorphism of Lie algebras φ *is a linear bijective mapping sending a Lie algebra L into a Lie algebra L',* $\varphi: L \longrightarrow L'$ *and* $\varphi[x,y] = [\varphi(x), \varphi(y)]$, $\forall x, y \in L$.

An isomorphism sending L into itself is called an automorphism. *Denote the group of all automorphism of L as* Aut L.

Definition 1.1.4 *An* ideal I *of a Lie algebra L is a subalgebra of L such that $[x,z] \in I$ for all $x \in L$ and $z \in I$. A Lie algebra L is called* simple *if* $[L,L] \neq 0$ *and L has no ideals other than L and $\{0\}$.*

Definition 1.1.5 *For a Lie algebra L we define the* derived series *of L by setting $L^{(0)} = L$ and $L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$ for $i \geq 1$. A Lie algebra L is called* solvable *if $L^{(n)} = \{0\}$ for some $n \geq 0$. L is called* semisimple *if it has no solvable ideals other than $\{0\}$.*

Definition 1.1.6 *A* grading by a group G, *also called a* G-grading, $R = \bigoplus_{g \in G} R_g$, *on an algebra R is the decomposition of R as the direct sum of subspaces R_g such that $R_{g'} R_{g''} \subset R_{g'g''}$ for all g', $g'' \in G$. Any element $x \in R_g$ is called* homogeneous of degree g *and a subspace V of R is called* a graded subspace *if* $V = \bigoplus_{g \in G} (V \cap R_g)$.

Definition 1.1.7 *The set* Supp $R = \{g \in G \mid R_g \neq \{0\}\}$ *is called the* support *of the grading* $R = \bigoplus_{g\in G} R_g$. *By* $S(G)$ *we denote the subgroup of* G *generated by* Supp R.

Definition 1.1.8 *An isomorphism* ϕ, $\phi : L \to L'$ *for graded Lie algebras* $L = \bigoplus_{g\in G} L_g$ *and* $L' = \bigoplus_{g\in G} L'_g$ *is called an* isomorphism of gradings *if* $\phi(L_g) = L'_g$ *for all* $g \in G$. *Gradings on Lie algebras* L, L' *are called* isomorphic *if there exists an isomorphism of gradings* ϕ, $\phi : L \to L'$.

It is well known [3], for a grading by a group G on a finite-dimensional simple Lie algebra, that $S(G)$ is an abelian group. We now state some useful observations.

Observation 1.1.9 *Any grading by a group* G *on Lie algebra* L, $L = \bigoplus_{g\in G} L_g$, *can be viewed as a grading by* $S(G)$.

Observation 1.1.10 *Let* L *and* L' *be Lie algebras and* φ *an isomorphism of Lie algebras* $\varphi : L \to L'$. *For any grading by a group* G *on* L, $L = \bigoplus_{g\in G} L_g$, *there is a grading by* G *on* L', $L' = \bigoplus_{g\in G} L'_g$ *where* $L'_g = \varphi(L_g)$.

Observation 1.1.11 *Let* $L = \bigoplus_{g\in G} L_g$ *be a grading by a group* G *on Lie algebra* L *and* K *a normal subgroup of* G. *The* natural G/K grading of L *can be defined if one sets* $L = \bigoplus_{\overline{g}\in G/K} L_{\overline{g}}$, *where* $\overline{g} = gK$, *with* $L_{\overline{g}} = \bigoplus_{k\in K} L_{gk}$.

Observation 1.1.12 *For any grading by a group* G *on Lie algebra* L, $L = \bigoplus_{g\in G} L_g$, L_e *is a subalgebra of* L, *where* e *is the identity of* G.

Gradings on simple Lie algebras are closely connected to the gradings on (associative) matrix algebras. Let us define two types of gradings on the matrix algebras, *fine gradings* in the sense of [2] and *elementary gradings.* The notion of fine gradings in the sense of [5], [6], [7] differs from that in [2]. These types of gradings on matrix algebras have been used in [4] to describe all gradings on Lie algebras of type D_l for $l > 4$ and are also useful to describe the gradings of Lie algebras of type D_4.

Definition 1.1.13 *A grading by a group G on $R = M_n$, the full $n \times n$ matrix algebra,*

$$R = \bigoplus_{g \in G} R_g$$

is called an elementary grading if there exist an n-tuple $\tau = (g_1, \ldots, g_n) \in G^n$ such that any matrix unit e_{ij}, $1 \leq i, j \leq n$, is homogeneous and $e_{ij} \in R_g \Leftrightarrow g = g_i^{-1} g_j$. We can always set one $g_i = e$, $1 \leq i \leq n$ since the tuple $(g_i^{-1} g_1, \ldots, g_i^{-1} g_{i-1}, e, g_i^{-1} g_{i+1}, \ldots, g_i^{-1} g_n)$ defines the same grading.

Definition 1.1.14 *A grading by a group G on $R = M_n$, the full $n \times n$ matrix algebra,*

$$R = \bigoplus_{g \in G} R_g$$

is called a fine grading if $\dim R_g = 1$ *for all $g \in$* Supp G. *A particular case of a fine grading by G is a so-called ε-*grading, *where ε is an n^{th} primitive root of 1. Let $G = \langle a \rangle_n \times \langle b \rangle_n$ and*

$$X_a = \begin{pmatrix} \varepsilon^{n-1} & 0 & \cdots & 0 \\ 0 & \varepsilon^{n-2} & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad X_b = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Set $X_g = X_a^i X_b^j$ for $g = a^i b^j$. If we define $R_g = \mathrm{Span}\{X_g\}$ for $g = a^i b^j$, then we obtain a fine grading on R by G, called an ε-grading.

Definition 1.1.15 *A map $*$ is called an involution on an associative algebra A if*

1. $(a^*)^* = a, \ \forall a \in A,$
2. $(ab)^* = b^* a^*, \ \forall a, b \in A,$
3. $(a+b)^* = a^* + b^*, \ \forall a, b \in A.$

For an involution $*$ on an algebra A we define

$$K(A, *) = \{x \in A \,|\, x^* = -x\}.$$

The elements of $K(A,*)$ are called skew-symmetric *with respect to $*$. We set $K(M_n) = K(M_n, *)$ where $*$ is the matrix transpose.*

Observation 1.1.16 *For an involution $*$ on M_n, $L = K(M_n, *)$ is a Lie algebra under the commutator $[x,y] = xy - yx$ for all $x, y \in L$ since $(xy - yx)^* = y^*x^* - x^*y^* = -(xy - yx)$ and the axioms of Definition 1.1.1 hold.*

Definition 1.1.17 *A grading by a group G on a Lie algebra L, $L = \bigoplus_{g \in G} L_g$, is called a* matrix grading *if L is graded isomorphic to $K(M_n, *) = L' = \bigoplus_{g \in G} L'_g$, n a positive integer and $*$ an involution, such that there exists a G-grading on M_n, $M_n = R = \bigoplus_{g \in G} R_g$, such that L' is a graded subspace.*

1.2 Gradings and actions by automorphisms.

In this section we introduce the action by the *dual group* $\widehat{G}$ on a Lie algebra L graded by an abelian group G so that the study of gradings by finite abelian groups is equivalent to the study of actions of finite abelian groups by automorphims. Unless otherwise specified, a grading by a group G on a Lie algebra L will mean a grading by the finite abelian group G on a finite dimensional Lie algebra L where $G = S(G)$. The restriction of $G = S(G)$ may reduce the number of subspaces $L_g = \{0\}$, $g \in G$.

Definition 1.2.1 *The dual group $\widehat{G}$ of an abelian group G is the group of homomorphisms taking G into $\mathbb{C}^*$, i.e., $\widehat{G} = \{\chi \mid \chi : G \to \mathbb{C}^*\,,\ \chi$ is a homorphism$\}$.*

The following result is well known.

Theorem 1.2.2 *Let G be a finite abelian group. We can express G and $\widehat{G}$ as*

$$G = \langle g_1 \rangle_{k_1} \times \cdots \times \langle g_n \rangle_{k_n}, \tag{1.1}$$

$$\widehat{G} = \langle \chi_1 \rangle_{k_1} \times \cdots \times \langle \chi_n \rangle_{k_n}, \tag{1.2}$$

with $\chi_i : G \to \mathbb{C}^*$, $\chi_i(g_i) = e^{2\pi i/k_i} = \varepsilon_i$ and $\chi_j(g_i) = 1$ for $1 \le i, j \le n$, $i \ne j$.

This gives us that for a finite abelian group G, $G \simeq \widehat{G}$. Now let L be a simple Lie algebra and $L = \bigoplus_{g \in G} L_g$ a grading by a group G. Since L is simple G, is abelian. Any element $x \in L$ can be uniquely decomposed as the sum of homogeneous components, $x = \sum_{g \in G} x_g$, $x_g \in L_g$. Given $\chi \in \widehat{G}$ we can define the action of χ as

$$\chi * x = \sum_{g \in G} \chi(g) x_g. \tag{1.3}$$

Theorem 1.2.3 *Let $L = \bigoplus_{g \in G} L_g$ be a grading by a group G. A subspace $V \subset L$ is a graded subspace if and only if V is invariant under the action of $\widehat{G}$.*

Proof

Let V be a graded subspace of L, i.e. $V = \bigoplus_{g\in G} V \cap L_g$. Let $x = \sum_{g\in G} x_g$, $x_g \in V \cap L_g$ and $\chi \in \widehat{G}$. Then $\chi * x = \sum_{g\in G} \chi(g)x_g$ which implies $\chi * x \in V$ since $\chi(g)x_g \in V \cap L_g$. Hence $\chi * V = V$ for $\chi \in \widehat{G}$.

Now we assume for contradiction that $\widehat{G} * V = V$ and $V \neq \bigoplus_{g\in G}(L_g \cap V)$ for the other claim. We can choose a non-zero $x \in V$ such that $x = \sum_{g\in K} x_g$, $K = \{g_1, \ldots, g_n\}$ is a subset of G, $x_g \in L_g$ and $x_g \notin V$ for all $g \in K$. Using Theorem 1.2.2 we can choose a $\chi \in \widehat{G}$ such that $\chi(g_1) \neq \chi(g_n)$. We set

$$\begin{aligned} x' = \ & \chi * x - \chi(g_n)x = (\textstyle\sum_{g\in K} \chi(g)x_g) - \chi(g_n)x \\ = \ & \textstyle\sum_{g\in K}(\chi(g) - \chi(g_n))x_g \\ = \ & \textstyle\sum_{g\in K} x'_g \end{aligned}$$

where $x'_g = (\chi(g) - \chi(g_n))x_g \in L_g$ for all $g \in K$. We now express x' as $x' = \sum_{g\in K'} x'_g$ where $K' \subset K$ such that $x'_{g'} \neq 0$ for all $g' \in K'$. K' is a non-zero proper subset of K since $x'_{g_1} = (\chi(g_1) - \chi(g_n))x_g \neq 0$ and $x'_{g_n} = (\chi(g_n) - \chi(g_n))x_g = 0$. Since $\widehat{G} * V = V$ and $x \in V$, $x' \in V$. It is clear that if we keep on repeating this process, we end up with a multiple of x_{g_1} as an element of V which is a contradiction. ∎

It is easy to observe that Equation (1.3) defines a $\widehat{G}$-action on L by automorphisms of L. We define f to be the group homomorphism $f : \widehat{G} \to \text{Aut } L$ by setting for each $\chi \in \widehat{G}$

$$f(\chi)(x) = \chi * x.$$

We can express the subspaces L_g as

$$L_g = \{x \in L \,|\, \chi * x = \chi(g)x, \ \forall \chi \in \widehat{G}\} = \{x \in L \,|\, f(\chi)(x) = \chi(g)x, \ \forall \chi \in \widehat{G}\}.$$

Theorem 1.2.4 *Let $L = \bigoplus_{g\in G} L_g$ be a grading by a group G. If $G = S(G)$ then $f(\widehat{G}) \simeq \widehat{G}$.*

Proof

We prove this by showing that if $G = S(G)$ then the *kernel* of f, $\ker f = \{\chi \in \widehat{G} \mid f(\chi) = id_{\mathrm{Aut}\,L}\}$, only contains the identity of $\widehat{G}$. Let $G = S(G)$ and we express G and $\widehat{G}$ as in (1.1), (1.2) respectively. Let $\chi \in \ker f$, $g \in \mathrm{Supp}\, G$ and $x_g \in L_g$ be non-zero. We have $x_g = f(\chi)(x_g) = id_{\mathrm{Aut}\,L} x_g = x_g$. Hence $\chi(g) = 1$ for all $g \in \mathrm{Supp}\, G$. Since $\mathrm{Supp}\, G$ generates G, it follows that $\chi(G) = 1$, which implies that χ is the identity of $\widehat{G}$. ■

By the above, $\widehat{G}$ defines a finite abelian subgroup $f(\widehat{G})$ of Aut L. In most cases we can view $\widehat{G}$ itself as a subgroup of Aut L, that is, we can identify $\widehat{G}$ with $f(\widehat{G})$ by means of f. It is also useful to work the other way, in other words to start with an abelian subgroup of Aut L and get a corresponding grading.

Theorem 1.2.5 *Let $K = \langle\kappa_1\rangle_{k_1} \times \cdots \times \langle\kappa_n\rangle_{k_n}$, be an abelian subgroup of* Aut L. *There is a unique grading by a group $G = \langle g_1\rangle_{k_1} \times \cdots \times \langle g_n\rangle_{k_n}$, $L = \bigoplus_{g\in G} L_g$ such that*

$$L_{g_1^{m_1}\cdots g_n^{m_n}} = \{x \in L \mid \kappa_1^{p_1}\cdots\kappa_n^{p_n}(x) = \varepsilon_1^{m_1p_1}\cdots\varepsilon_n^{m_np_n}x,\ \varepsilon_j = e^{2\pi i/k_j},\ 1 \le j \le n\}.$$

Proof

Let $K = \langle\kappa_1\rangle_{k_1} \times \cdots \times \langle\kappa_n\rangle_{k_n}$ be an abelian group contained in Aut L. Let G and $\widehat{G}$, as defined previously (1.1), (1.2), be isomorphic to $f(\widehat{G})$. Define $f : \widehat{G} \to \mathrm{Aut}\, L$ by $f(\chi_i) = \kappa_i$. $L = \bigoplus_{g\in G} L_g$ is a grading by G if we set

$L_{g_1{}^{m_1}\ldots g_n^{m_n}} = \{x \in L \,|\, \kappa_j(x) = e^{2m_j\pi i/k_j}x,\ 1 \leq j \leq n\}.$ ■

Chapter 2

Lie algebras of type D_4

2.1 Lie algebras of type D_4 and their Cartan decompositions.

As was mentioned in Observation 1.1.16, given an associative algebra A we can define the commutator $[\ ,\ ]$ by $[a, b] = ab - ba$ for all a, $b \in A$ which makes A a Lie algebra. Any finite dimensional Lie algebra can be realised as a subalgebra of M_n by Ado's Theorem. The Lie subalgebra L of M_8 consisting skew symmetric matrices with respect to the matrix transpose is a simple Lie algebra of type D_4. This realisation can be instrumental for finding gradings of L by a finite abelian group, via the correspondence between gradings and actions by automorphisms.

In order to describe the group Aut L of automorphisms of a simple Lie algebra L we will define *maximal toral subalgebras*, *root systems*, *bases* and *Dynkin diagrams.*

The following definitions and basic facts are classical and can be found e.g. in [8].

Definition 2.1.1 *A subset Φ of the* Euclidean space *E, i.e., a finite dimensional vector space over $\mathbb{R}$ endowed with a positive definite form $(\ ,\ )$, is called a* root system in *E if the following axioms are satisfied:*

1. *Φ is finite, spans E, and does not contain 0.*
2. *If $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$.*
3. *If $\alpha \in \Phi$, then the* reflection *σ_α leaves Φ invariant where* $\sigma_\alpha(\beta) = \beta - \langle\beta, \alpha\rangle\alpha,\ \forall\beta \in \Phi.$
4. *If α, $\beta \in \Phi$, then $\langle\beta, \alpha\rangle := 2(\beta, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$.*

A root system Φ is called irreducible *if it cannot be partitioned into the union of two proper subsets such that each root in one set is orthogonal to each root in the other.*

The *rank* of a root system Φ is the dimension of its corresponding Euclidean space E. The elements of a root system are called *roots.* The *length* of a root α is $\sqrt{(\alpha, \alpha)}$. It is shown in [8] that in each irreducible root system there are at most two root lengths. When two root lengths occur, the roots of smaller length are called *short roots* and all others *long roots.* Useful subsets of Φ are bases.

Definition 2.1.2 *A subset Δ of a root system Φ with corresponding Euclidean space E is called a* base *and its elements are called* simple roots *if the following* *axioms are satisfied:*

1. Δ *is a basis of* E.
2. *each root* β *can be written as*

$$\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha \tag{2.1}$$

with integral coefficients k_α *all non-negative or all non-positive.*

The roots with non-negative coefficients in the decomposition (2.1) are called *positive roots* and all others are called *negative roots.* It is known that if α, $\beta \in \Phi$ and $\langle \alpha, \beta \rangle < 0$, then $\alpha + \beta$ is a root. It is also known (see [8]) that any root $\beta \in \Phi$ can be written as

$$\beta = \pm \sum_{j=1}^{k} \alpha_{i_j}$$

where every partial sum is a root, $\alpha_{i_j} \in \Delta$, and k is some integer depending on β.

We can recover the structure of a root system if we know all $\langle \alpha_i, \alpha_j \rangle$. The idea is to start with a simple root and to keep adding as many simple roots as we can until we exhaust all the possibilities. Repeat the process for all other simple roots to obtain all the positive roots.

Definition 2.1.3 *For a root system* Φ *of rank* l, *the* $l \times l$ *matrix* $C = (\langle \alpha_i, \alpha_j \rangle)_{ij}$ *is called the* Cartan matrix *of* Φ *and its entries are called the* Cartan integers.

This information can be displayed graphically by so called *Dynkin diagrams.*

Definition 2.1.4 *Fix a base $\Delta = \{\alpha_1, \dots, \alpha_l\}$ of Φ. Define the Dynkin diagram of Φ to be a graph having l vertices and the i^{th} vertex is joined to the j^{th} vertex $(i \neq j)$ by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges. We denote the k^{th} vertex by v_{α_k} where α_k is a simple root of the corresponding base Δ of Φ. When a double or triple edge occurs, we add an arrow in between the vertices pointing to the vertex that corresponds to the shorter of the two roots.*

We can recover the Cartan matrix of Φ from the Dynkin diagram (see [8]). First we take any two distinct simple roots α_i and α_j with $(\alpha_i, \alpha_i) \leq (\alpha_j, \alpha_j)$, i.e., α_i is the shorter of the two roots if two root lengths occur. Then $\langle \alpha_j, \alpha_i \rangle = -e(i,j)$ where $e(i,j)$ is the number of edges between the i^{th} and j^{th} vertices. Also $\langle \alpha_i, \alpha_j \rangle = -1$ if $\langle \alpha_j, \alpha_i \rangle \neq 0$ and $\langle \alpha_i, \alpha_j \rangle = 0$ if $\langle \alpha_j, \alpha_i \rangle = 0$.

Now we relate root systems to semisimple Lie algebras. These are the algebras with non-singular *Killing form* $\kappa(x,y) = Tr(\operatorname{ad} x, \operatorname{ad} y)$. For any semisimple Lie algebra L, L possesses Lie subalgebras consisting of ad-diagonalizable elements. These subalgebras are called *toral subalgebras.* The toral subalgebras are abelian, i.e. the commutator of any two elements is 0. If we fix a maximal toral subalgebra H for a semisimple Lie algebra L then $\operatorname{ad} H$ is a set of endomorphisms of L that can be simultaneously diagonalised. The restriction of κ to H and its natural dual to H^* is non-singular.

The *Cartan decomposition of* L with respect to H is defined by

$$L = H \bigoplus_{\alpha \in \Phi} L_\alpha$$

where $L_\alpha = \{x \in L \mid [h,x] = \alpha(h)x \ \forall h \in H\}$ and $\Phi \subset H^*$. It turns out that each L_α are one-dimensional, the real linear span E of Φ in H^* is a Euclidean space, with respect to κ and Φ is a root system. For any two roots α, α' with $\alpha+\alpha' \in \Phi$, $[L_\alpha, L_{\alpha'}] \subset L_{\alpha+\alpha'}$ and any two roots β, β' with $\beta+\beta' \notin \Phi$, $[L_\beta, L_{\beta'}] = \{0\}$.

Having H, Φ and Δ fixed for a semisimple Lie algebra L, one can choose a basis

$$\{h_{\alpha_i}, x_\alpha, y_\alpha \mid \alpha_i \in \Delta,\ \alpha \text{ is a positive root of } \Phi\}$$

of L such that

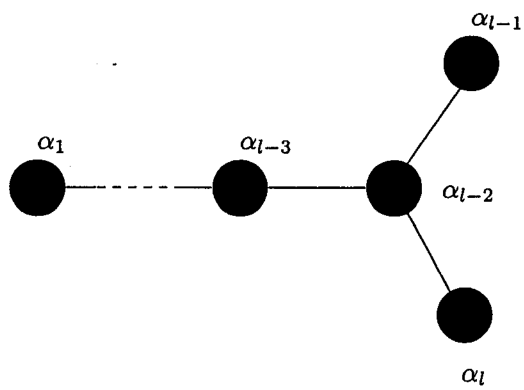
$$\begin{aligned} & x_\alpha \in L_\alpha, && \text{for all } \alpha, \in \Phi, \\ & y_\alpha \in L_{-\alpha}, && \text{for all } \alpha, \in \Phi, \\ & h_{\alpha_i} \in H, && \\ & [x_\alpha, y_\alpha] = h_\alpha, && \text{for all } \alpha \in \Phi \\ & [h_\alpha, z_{\alpha'}] = \langle \alpha, \alpha' \rangle z_{\alpha'}, && \text{for all } \alpha, \alpha' \in \Phi,\ z_{\alpha'} \in L_{\alpha'}. \end{aligned} \tag{2.2}$$

We call the above basis a *canonical basis of* L.

At this point we should mention that bases are not to their corresponding root system and maximal toral subalgebras are not unique to their corresponding semisimple Lie algebras. We do know however, that the maximal toral subalgebras of a simple Lie algebra L are all conjugate, i.e., for any two maximal toral subalgebras H and H' of L there exists a $\lambda \in \text{Int } L \subset \text{Aut } L$ (see Section 2.2) such that $\lambda(H) = H'$. Also for any two maximal toral subalgebra H, H' of a semisimple Lie algebra L with root systems Φ, Φ' respectively, the corresponding Euclidean spaces E, E' are isomorphic and there is an isomorphism that sends Φ to Φ'.

Definition 2.1.5 *Lie algebras isomorphic to the Lie algebra* $L = K(M_{2l})$ *with commutator* [,] *defined by* $[a,b] = ab - ba$ *for all* a, $b \in L$ *are called* Lie algebras of type D_l, $l \geq 4$.

Lie algebras of type D_l have the following Dynkin diagram:



(Figure 1)

For a Lie algebra L of type D_4, we can define a maximal toral subalgebra $H = \text{Span}\{h_{\alpha_1}, h_{\alpha_2}, h_{\alpha_3}, h_{\alpha_4}\}$, a root system Φ corresponding to H and a base $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ of Φ, where the following are the positive roots:

$$\begin{array}{llll}
\alpha_2, & \alpha_1, & \alpha_3, & \alpha_4, \\
\alpha_2 + \alpha_1, & \alpha_2 + \alpha_3, & \alpha_2 + \alpha_4, & \\
\alpha_2 + \alpha_1 + \alpha_3, & \alpha_2 + \alpha_1 + \alpha_4, & \alpha_2 + \alpha_3 + \alpha_4, & \\
\alpha_2 + \alpha_1 + \alpha_3 + \alpha_4, & \alpha_2 + \alpha_1 + \alpha_3 + \alpha_4 + \alpha_2. & &
\end{array}$$

The Cartan integers are:

$$\begin{array}{ll}
\langle \alpha_i, \alpha_i \rangle = 2, & 1 \leq i \leq 4, \\
\langle \alpha_2, \alpha_j \rangle = -1, & j \in \{1,3,4\}, \\
\langle \alpha_j, \alpha_2 \rangle = -1, & j \in \{1,3,4\}, \\
\langle \alpha_j, \alpha_k \rangle = 0, & \text{for distinct } j,k \in \{1,3,4\}.
\end{array}$$

Here are the definitions for the other simple Lie algebras.

Definition 2.1.6 *Let $Tr(x)$ be the trace of x. Lie algebras isomorphic to*

$$L = \{x \in M_{l+1} \,|\, Tr(x) = 0\}$$

with commutator $[\ ,\ ]$ defined by $[a, b] = ab - ba$ for all a, $b \in L$ are called Lie algebras of type A_l, $l \geq 1$.

Definition 2.1.7 *Lie algebras isomorphic to the Lie algebra*

$$L = K(M_{2l+1})$$

with commutator $[\ ,\ ]$ defined by $[a, b] = ab - ba$ for all a, $b \in L$ are called Lie algebras of type B_l, $l \geq 2$.

Definition 2.1.8 *Let ψ be an involution on M_{2l}, $l \geq 2$, defined by $\psi(x) = Sx^*S^{-1}$ where $*$ is the matrix transpose and*

$$S = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}.$$

Lie algebras isomorphic to the Lie algebra

$$L = K(M_{2l}, \psi),$$

with commutator $[\ ,\ ]$ defined by $[a, b] = ab - ba$ for all a, $b \in L$ are called Lie algebras of type C_l, $l \geq 3$.

Definition 2.1.9 *A simple Lie algebra, whose Cartan matrix is*

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix},$$

is callded a Lie algebra of type G_2.

2.2 Realisation of a Lie algebra of type D_4 as $K(M_8, *)$.

Let L denote the following realisation of the Lie algebra of type D_4 over $\mathbb{C}$,

$$L = K(M_8) = \{x \in M_8(\mathbb{C}) \mid x = -x^*\},$$

where x^* is the transpose of x. A basis for L is the set $\{E_{ij} = e_{ij} - e_{ji} \mid 1 \leq i < j \leq 8,\}$, where e_{ij} is the 8×8 matrix with 1 in the i^{th}, j^{th} position and zero everywhere else.

The commutator of E_{ij} and E_{kl} for $i \neq j$ and $k \neq l$ is

$$\begin{aligned}
[E_{ij}, E_{kl}] &= [e_{ij} - e_{ji}, e_{kl} - e_{lk}] \\
&= e_{ij}e_{kl} - e_{ij}e_{lk} - e_{ji}e_{kl} + e_{ji}e_{lk} - e_{kl}e_{ij} + e_{kl}e_{ji} + e_{lk}e_{ij} - e_{lk}e_{ji} \\
&= \delta_{jk}E_{il} - \delta_{jl}E_{ik} - \delta_{ik}E_{jl} + \delta_{il}E_{jk}.
\end{aligned}$$

Let $H = \text{Span}\{E_{25}, E_{36}, E_{47}, E_{18}\}$. Then H is a 4-dimensional toral subalgebra, hence a maximal toral subalgebra of L. Let $L = H \bigoplus_{\alpha\in\Phi} L_\alpha$ be the Cartan decomposition with respect to H, root system Φ of H and base $\Delta = \{\alpha_i \mid 1 \leq i \leq 4\}$ of Φ. A canonical basis of L corresponding to this decomposition takes the following form.

$$\begin{aligned}
h_{\alpha_1} &= i(E_{25} - E_{36}), & x_{\alpha_1} &= \{E_{23} + iE_{26} + iE_{35} + E_{56}\}/2, \\
h_{\alpha_2} &= i(E_{36} - E_{47}), & x_{\alpha_2} &= \{E_{34} + iE_{37} + iE_{46} + E_{67}\}/2, \\
h_{\alpha_3} &= i(E_{47} - E_{18}), & x_{\alpha_3} &= \{-E_{14} + iE_{17} + iE_{48} + E_{78}\}/2, \\
h_{\alpha_4} &= i(E_{47} + E_{18}), & x_{\alpha_4} &= \{-E_{14} + iE_{17} - iE_{48} - E_{78}\}/2,
\end{aligned}$$

$$x_{\alpha_2+\alpha_1} = \{-E_{24} - iE_{27} - iE_{45} - E_{57}\}/2,$$
$$x_{\alpha_2+\alpha_1+\alpha_3} = \{E_{12} - iE_{15} - iE_{28} - E_{58}\}/2,$$
$$x_{\alpha_2+\alpha_1+\alpha_3+\alpha_4} = \{E_{24} - iE_{27} + iE_{45} - E_{57}\}/2,$$
$$x_{\alpha_2+\alpha_1+\alpha_3+\alpha_4+\alpha_2} = \{-E_{23} + iE_{26} - iE_{35} + E_{56}\}/2,$$
$$x_{\alpha_2+\alpha_1+\alpha_4} = \{E_{12} - iE_{15} + iE_{28} + E_{58}\}/2,$$
$$x_{\alpha_2+\alpha_3} = \{-E_{13} + iE_{16} + iE_{38} + E_{68}\}/2,$$
$$x_{\alpha_2+\alpha_3+\alpha_4} = \{-E_{34} + iE_{37} - iE_{46} + E_{67}\}/2,$$
$$x_{\alpha_2+\alpha_4} = \{-E_{13} + iE_{16} - iE_{38} - E_{68}\}/2,$$
$$y_\alpha = -\overline{x_\alpha^*}, \text{ where } \overline{x} \text{ is the conjugate of } x.$$

A convenient way to use this information is to notice that if the indices which equal each other are not both in the first or second position ($E_{\text{first,second}}$) then $[E_{ij}, E_{kl}] = E_{ab}$ where a, b are the other two indices in the order that they appear. If the indices that equal each other are both in the first or second position then $[E_{ij}, E_{kl}] = -E_{ab}$ where a, b are the other two indices in the order that they appear. If all indices are distinct or $\{i,j\} = \{k,l\}$ then $[E_{ij}, E_{kl}] = 0$.

Examples

$[E_{21}, E_{13}] = E_{23}$, $[E_{12}, E_{32}] = -E_{13}$, $[E_{25}, E_{18}] = 0$, $[E_{25}, E_{52}] = 0$.

2.3 Automorphisms of Lie algebras of type D_4.

As mentioned, the automorphisms of a Lie algebra L are important in finding gradings by a group G on L. Some automorphisms are defined by the automorphisms of root systems. Let us fix a semisimple Lie algebra L, a Cartan subalgebra H of L, a root system Φ of H, a base Δ of Φ and a canonical basis as described in Section 2.1. Let P be the set of permutations on the simple roots of the base $\Delta = \{\alpha_1, \dots, \alpha_l\}$ such that $\langle \overline{\eta}(\alpha_i), \overline{\eta}(\alpha_j)\rangle = \langle \alpha_i, \alpha_j\rangle$, $1 \le i, j \le l$, for all $\overline{\eta} \in P$. The group P is in one-to-one correspondence with the group of graph automorphisms of the Dynkin diagram since for $\overline{\eta}(\alpha_i) = \alpha_p$ and $\overline{\eta}(\alpha_j) = \alpha_q$,

$$\begin{aligned} e(i,j) = &\ \langle \alpha_i, \alpha_j\rangle\langle \alpha_j, \alpha_i\rangle = \langle \overline{\eta}(\alpha_i), \overline{\eta}(\alpha_j)\rangle\langle \overline{\eta}(\alpha_j), \overline{\eta}(\alpha_i)\rangle \\ = &\ \langle \alpha_p, \alpha_q\rangle\langle \alpha_q, \alpha_p\rangle = e(p,q) \end{aligned}$$

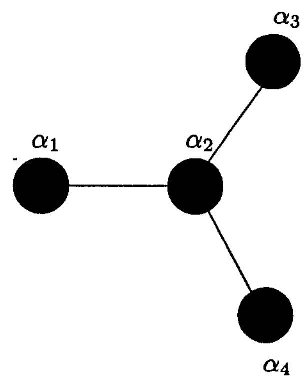
and conversely if $e(i,j) = e(p,q)$ then $\langle \alpha_i, \alpha_j\rangle = \langle \overline{\eta}(\alpha_i), \overline{\eta}(\alpha_j)\rangle$. We define $\operatorname{Aut}\Delta$ to be the set of automorphism of L satisfying the following property: For all $\eta \in \operatorname{Aut}\Delta$ there exists an $\overline{\eta} \in P$ such that

$$\begin{aligned} \eta(x_{\sum \alpha_i}) &= x_{\sum \overline{\eta}(\alpha_i)}, \\ \eta(y_{\sum \alpha_i}) &= y_{\sum \overline{\eta}(\alpha_i)}, \\ \eta(h_{\sum \alpha_i}) &= h_{\sum \overline{\eta}(\alpha_i)}, \end{aligned}$$

It follows that $\operatorname{Aut}\Delta$ is a subgroup of $\operatorname{Aut} L$.

In [10] it is shown that $\operatorname{Aut} L$ is the semidirect product of $\operatorname{Int} L$ by $\operatorname{Aut}\Delta$ where $\operatorname{Int} L = \{\exp(\operatorname{ad} x) \mid x \text{ is ad-nilpotent}, x \in L\}$ is a normal subgroup of $\operatorname{Aut} L$ called the *inner automorphism of L.*

For a Lie algebra of type D_4, with base $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, the Dynkin diagram is the following.



(Figure 2)

From the diagram we see that Aut $\Delta \simeq S_3 = \{\text{id}, (13), (14), (34), (134), (143)\}$. We define the group operation in S_3 by applying the permutation to the right, i.e., (134)(34)=(13). Aut Δ is generated by σ_{ij} and ρ_{ijk}, for distinct i, j, $k \in \{1, 3, 4\}$, where $\overline{\sigma_{ij}}$ permutes the simple roots α_i and α_j while $\overline{\rho_{ijk}}$ permutes the simple roots α_i, α_j and α_k in that order.

Since Aut L is the semidirect product of Aut Δ and Int L, we can express any automorphism $\psi \in$ Aut L uniquely as $\psi = \lambda\pi$ where $\lambda \in$ Int L and $\pi \in$ Aut Δ. The group operation is defined as

$$(\lambda\pi)(\lambda'\pi') = (\lambda(\pi\lambda'\pi^{-1}))(\pi\pi').$$

In this paper the semidirect property of Aut L will be used extensively and we use the fact that for $\pi, \pi' \in$ Aut Δ and $\lambda, \lambda' \in$ Int L there exist a $\lambda'' \in$ Int L such that $(\lambda\pi)(\lambda'\pi') = \lambda''(\pi\pi')$.

2.4 Finite abelian subgroups of the group of automorphisms of a Lie algebra of type D_4.

We showed in the previous chapter that a grading by a finite abelian group G on a Lie algebra L with $G = S(G)$ has a corresponding abelian subgroup $f(\widehat{G}) \subset \text{Aut } L$. We are going to use this information to impose some restrictions on $f(\widehat{G})$ for gradings by a finite abelian group G on a Lie algebra L of type D_4 by looking at properties of finite abelian groups of Aut L. This section uses an approach similar to that in [3].

Theorem 2.4.1 *For any finite abelian subgroup K of* Aut L, *we can express K as*

$$K = \langle \varphi \rangle \times A$$

where $A \subset \text{Int } L$ and φ is of order p^n, p is 2 or 3, for some non-negative integer n.

Proof

Since K is a finite abelian group we can express K as

$$K = \Gamma_2 \times \Gamma_3 \times \Gamma',$$

where Γ_2, Γ_3 are the 2-Sylow subgroup and 3-Sylow subgroup of K, respectively, and Γ' is the direct product of all other p-Sylow subgroups of K. Also Γ_2 can be expressed as

$$\Gamma_2 = \langle \varphi_{i_1} \rangle \times \cdots \times \langle \varphi_{i_n} \rangle$$

where the order of φ_{i_k} is 2^{i_k} and $i_l \leq i_{l+1}$, $1 \leq l \leq n-1$. Similarly we express Γ_3 as

$$\Gamma_3 = \langle \varphi_{j_1} \rangle \times \cdots \times \langle \varphi_{j_m} \rangle$$

where the order of φ_{j_k} is 3^{j_k} and $j_l \leq j_{l+1}$, $1 \leq l \leq m-1$. Hence we can express K as a direct product of cyclic subgroups and Γ'

$$K = \langle \varphi_{i_1} \rangle \times \cdots \times \langle \varphi_{i_n} \rangle \times \langle \varphi_{j_1} \rangle \times \cdots \times \langle \varphi_{j_m} \rangle \times \Gamma'$$

and set $\varphi_r = \lambda_r \pi_r$ for some unique pair $\lambda_r \in \text{Int } L$, $\pi_r \in \text{Aut } \Delta$. For any $\phi = \lambda\pi$, $\lambda \in \text{Int } L$ and $\pi \in \text{Aut } \Delta$, of order $o(\phi) = t$, it follows that $\text{id}_{\text{Aut } L} = \phi^t = \lambda' \pi^t$ for some $\lambda' \in \text{Int } L$ which implies $\lambda' = \pi^t = \text{id}_{\text{Aut } L}$. Hence $\Gamma' \subset \text{Int } L$. Also

$$\langle \varphi_r \rangle \not\subseteq \text{Int } L \Leftrightarrow \varphi_r \notin \text{Int } L.$$

For $K \subset \text{Int } L$, the theorem follows. If $K \not\subseteq \text{Int } L$, then either there exists only one index a with $\pi_a \neq \text{id}_{\text{Aut } L}$, and then our claim follows, or there are indices a and b with π_a, $\pi_b \neq \text{id}_{\text{Aut } L}$ and $a \leq b$. Since K is abelian,

$$\varphi_a \varphi_b = (\lambda_a \pi_a)(\lambda_b \pi_b) = \lambda'(\pi_a \pi_b) = \varphi_b \varphi_a = (\lambda_b \pi_b)(\lambda_a \pi_b) = \lambda''(\pi_b \pi_a)$$

for some λ', $\lambda'' \in \text{Int } L$ which implies $\pi_a \pi_b = \pi_b \pi_a$. The only abelian subgroups of $\text{Aut } \Delta \simeq S_3$ are

$$\langle \text{id}_{\text{Aut } L} \rangle, \quad \langle \sigma_{ij} \rangle \quad \text{and} \quad \langle \rho_{134} \rangle \quad \text{for distinct } i,\, j \in \{1, 3, 4\}.$$

It follows that all π_r are in one of the 4 latter subgroups above.

If $\pi_a = \pi_b = \sigma_{ij}$, then φ_a, $\varphi_b \in \Gamma_2$ since the order of φ_a, φ_b is divisible by 2 and $\Gamma_3 \subset \text{Int } L$ by the abelian subgroups of $\text{Aut } \Delta$ argument. Also

$$\begin{aligned} &\varphi_a^{-1} = \sigma_{ij} \lambda_a^{-1}, \\ &\varphi_b \varphi_a^{-1} = (\lambda_b \sigma_{ij})(\sigma_{ij} \lambda_a^{-1}) = \lambda_b \lambda_a^{-1} \in \text{Int } L. \end{aligned}$$

We can now write K as

$$K = \langle \phi_{i_1} \rangle \times \cdots \times \langle \phi_{i_n} \rangle \times \Gamma_3 \times \Gamma'$$

where $\phi_{i_l} = \varphi_{i_l}$ for $1 \leq l \leq n$, $i_l \neq b$ and $\phi_b = \varphi_b \varphi_a^{-1}$ since for any p-Sylow subgroup $P = \langle g \rangle \times \langle h \rangle$ where the order of g is less or equal than order of h,

$$P = \langle g \rangle \times \langle h \rangle = \langle g \rangle \times \langle h\, g^{-1} \rangle.$$

Repeating this process for other $\varphi_r \notin \text{Int } L$ we can express K as

$$K = \langle \varphi_a \rangle \times A$$

where $A \subset \text{Int } L$.

Similarly, for $\pi_a = \rho_{134}^c$, $\pi_b = \rho_{134}^d$, $1 \leq c, d \leq 2$, we have that φ_a, $\varphi_b \in \Gamma_3$ since the order of φ_a, φ_b is divisible by 3, and $\Gamma_2 \subset \text{Int } L$ by the abelian subgroups of Aut Δ argument. We have either $c + d = 3$ or $c = d$.

For the first case of $c + d = 3$, we have

$$\varphi_b \varphi_a = (\lambda_b \rho_{134}^d)(\lambda_a \rho_{134}^{-d}) = \lambda_b(\rho_{134}^d \lambda_a \rho_{134}^{-d}) \in \text{Int } L.$$

For the second case of $c = d$, we have

$$\varphi_b\, \varphi_a^{-1} = (\lambda_b \rho_{134}^c)(\rho_{134}^{-c} \lambda_a^{-1}) = \lambda_b \lambda_a^{-1} \in \text{Int } L.$$

We can express K as

$$K = \Gamma_2 \times \langle \phi_{j_1} \rangle \times \cdots \times \langle \phi_{j_m} \rangle \times \Gamma'$$

where $\phi_{j_r} = \varphi_{j_r}$ for $1 \leq r \leq m$, $j_r \neq b$ and $\phi_b = \varphi_b\, \varphi_a^{-1}$ if $c = d$, $\phi_b = \varphi_b\, \varphi_a$ if $c + d = 3$. Repeating this process for other $\varphi_r \notin \text{Int } L$ we can express K as

$$K = \langle \varphi_a \rangle \times A$$

where $A \subset \text{Int } L$.

Hence our claim holds for any situation. ∎

Now we obtain important corollaries about the gradings on Lie algebras of type D_4. The first is a consequence of the previous theorem and Theorem 1.2.4.

Corollary 2.4.2 *For any grading by a finite group G on a Lie algebra L of type D_4, $L = \bigoplus_{g\in G} L_g$, with $G = S(G)$, we can express $f(\widehat{G})$ as*

$$f(\widehat{G}) = \langle\varphi\rangle \times B \tag{2.3}$$

where $B \subset \mathrm{Int}\ L$, $\varphi = \pi\lambda$, $\pi \in \mathrm{Aut}\ \Delta$, $\lambda \in \mathrm{Int}\ L$, the order of φ, $o(\varphi)$ is $o(\pi)^u$ where $o(\pi)$ is the order of π and u is some non-negative integer. We can also express $\widehat{G}$ as

$$\widehat{G} = \langle\chi\rangle \times \Lambda \tag{2.4}$$

where $f(\chi) = \varphi$ and $f(\Lambda) = B$.

Corollary 2.4.3 *For any grading by a finite group G of Lie algebra L of type D_4, $L = \bigoplus_{g\in G} L_g$, with $G = S(G)$ we can express $\widehat{G}$ and G as*

$$\widehat{G} = \langle\chi\rangle \times \Lambda,$$

$$G = \langle z\rangle \times A \tag{2.5}$$

where $f(\Lambda) \subset \mathrm{Int}\ L$, the order of χ and the order of z both equal $n = p^u$ where p is 2 or 3 and u is some non-negative integer u. Also $\chi(z) = \varepsilon$, ε a n^{th} primitive root of one, $\chi(g) = 1$ for all $g \in A$, $\phi(z) = 1$ for all $\phi \in \Lambda$.

It is useful to distinguish between the cases where $f(\widehat{S(G)}) \subset \text{Int } L$ and $f(\widehat{S(G)}) \not\subset \text{Int } L$. The following definition comes from [1]

Definition 2.4.4 *We call a grading by a finite group G of a Lie algebra L an* inner grading *if $f(\widehat{S(G)}) \subset \text{Int } L$. All other gradings are called* outer gradings.

Chapter 3

Matrix Gradings

3.1 Automorphisms of $K(M_8, *)$ that can be lifted to M_8.

Lie algebras of type D_4 have realisations as the skew-symmetric matrices in M_8 with respect to certain involutions. It is well known that all involutions $*$ on M_n such that $(a\,x)^* = a\,x^*$, for all $a \in \mathbb{C}$ and for all $x \in M_n$ can be expressed as $x^* = Tx^tT^{-1}$ where x^t is the transpose of $x \in M_n$ and T is symmetric or skew-symmetric. When T is symmetric, $*$ is called a *transpose involution*, otherwise it is called a *sympletic involution.* $L = K(M_8, *)$ is a Lie algebra of type D_4 if and only if $*$ is a transpose involution. We now fix a transpose involution *, a maximal toral subalgebra H of L, a corresponding root system Φ and a base Δ of Φ. Denote by Ω the subgroup of Aut L consisting of all automorphisms λ for which there exists an invertible matrix T_λ such that $\lambda(x) = T_\lambda x T_\lambda^{-1}$ for all $x \in L$. It is known [10] that Int $L \subset \Omega$, that the matrices T_λ are are orthogonal with respect to $*$ and that for $\lambda \in$ Int L the associated matrices T_λ have determinant 1. Let Ω^+ be the subgroup of

Ω that consist of all $\lambda \in \Omega$ such that $\det T_\lambda = 1$. We refer to Section 2.3 for the structure of Aut L. The index of Int L in Aut L is 6 and the index of Ω^+ in Ω is 2 which implies that the index of Int L in Ω is a multiple of 2. The index of Ω in Aut L is [Aut L : Int L]/[Ω : Int L] $= 6/(2n)$ for some positive integer n. Hence the index is either 3 or 1. It can be shown that $\rho_{134} \in \text{Aut }\Delta$ does not have an associated matrix $T_{\rho_{134}}$. Thus the index of Ω in Aut L is 3 and Aut $L = \Omega \cup \rho_{134}\Omega \cup \rho_{134}^2\Omega$. The subgroup Aut $\Delta \cap \Omega$ is a non identity group since Int L is a proper subgroup of Ω and any automorphism of L can be written as $\lambda\pi$ where $\lambda \in \text{Int}\, L$, $\pi \in \text{Aut}\,\Delta$. This gives us that Aut $\Delta \cap \Omega = \langle \sigma_{ij} \rangle$ for some distinct $i, j \in \{1, 3, 4\}$. From now on we choose a base Δ such that $\sigma_{34} \in$ Aut $\Delta \cap \Omega$ for every canonical basis. For the realisation given in Section 2.2, the associated matrix $T_{\sigma_{34}}$ is $(\sum_{l=1}^{7} e_{ii}) - e_{88}$.

In [4] the gradings on Lie algebras L of type D_l, $l > 4$, were found using the fact that $\Omega =$ Aut L. To see this, we look at the Dynkin diagram for a Lie algebra L of type D_l and note that [Aut L : Int L] $= 2$, so either $\Omega =$ Int L or $\Omega =$ Aut L. Recall that the inner automorphisms of L have associated matrices which are orthogonal with determinant 1 so $\Omega \neq$ Int L. This gives us that $\Omega =$ Aut L. We can lift the actions of Ω to a subgroup of Aut M_{2l} by setting $\lambda * X = T_\lambda X T_\lambda^{-1}$ for all $X \in M_{2l}$, $\lambda \in \Omega$. Thus we can view Ω as a subset of Aut M_{2l}. For any grading by a group G on L, $f(\widehat{G}) \subset \Omega$ and hence we can write for $R = M_{2l}$:

$$R = \bigoplus_{g \in G} R_g$$

where $R_g = \{X \in R \,|\, T_{f(\chi)} X T_{f(\chi)}^{-1} = \chi(g)X,\ \forall \chi \in \widehat{G}\}$. By finding all gradings on M_{2l} such that $K(M_{2l}, *)$, $*$ a transpose involution, is a graded subspace, we also find all gradings on $K(M_{2l}, *)$.

For the case of $l = 4$, we do not recover all gradings but we can find all matrix gradings (recall), all gradings such that $f(\widehat{G}) \subset \langle\sigma_{34}\rangle \operatorname{Int} L = \Omega$.

3.2 Matrix gradings of $K(M_8, *)$

This section is a quotation from [1] and [4]. We consider a construction of gradings on the tensor product $A \otimes B$ of two algebras A and B.

The following definition works even in the case where G is not abelian.

Definition 3.2.1 *Let $A = \bigoplus_{g\in G} A_g$ be any G-graded algebra over an algebraically closed field F. $M_n(F) = B = \bigoplus_{g\in G} B_g$ be a matrix algebra over F with an elementary grading given by an n-tuple $(g_1, \ldots, g_n) \in G^n$, that is $e_{ij} \in B_{g_i^{-1}g_j}$. Then direct computations show that $R = A \otimes B$ will be given a G-grading if one sets*

$$R_g = \operatorname{Span}\{a \otimes e_{ij} \mid a \in A_h,\ g_i^{-1}hg_j = g\}.$$

The grading just defined will be called induced.

If the support of A and the support of B commute, the induced grading above has the decomposition $R = \bigoplus_{g\in G} R_g$ where $R_g = (A \otimes B)_g = \bigoplus_{g=st\in G}(A_s \otimes B_t)$. We can now quote Theorem 3.1 from [4].

Theorem 3.2.2 *Let F be an algebraically closed field of characteristic zero. Then as G-graded algebra $R = M_n(F)$ is isomorphic to the tensor product*

$$R^{(0)} \otimes R^{(1)} \otimes \cdots \otimes R^{(k)}$$

where $R^{(0)} = M_{n_0}(F)$ *has an elementary* G*-grading,* $\operatorname{Supp} R^{(0)} = S$ *is a finite subset of* G*,* $R^{(i)} = M_{n_i}$ *has the* ε_i*-grading (recall Definition 1.1.14),* ε_i *being a primitive* n_i^{th} *root of 1,* $\operatorname{Supp} R^{(i)} = H_i \simeq \mathbb{Z}_{n_i} \otimes \mathbb{Z}_{n_i}$*,* $i = 1, \dots, k$*. Also* $H = H_1 \cdots H_k \simeq H - 1 \times \cdots \times H_k$ *and* $S \cap H = \{e\}$ *in* G*.*

We quote Lemma 4.3 and Theorem 5.1 of [4] for which we are interested in the case where G is an abelian group, the grading is over $\mathbb{C}$ and φ is a transpose involution on $M_8(\mathbb{C})$.

Lemma 3.2.3 *Let* $R = C \otimes D = \bigoplus_{g \in G} R_g$ *be a* G*-graded matrix algebra with an elementary grading on* C *and a fine grading on* D*. Let* $\varphi : R \to R$ *be an antiautomorphism on* R *preserving* G*-grading, i.e.,* $\varphi(R_g) = R_g$*. Let also* φ *act as an involution on the identity component* R_e *i.e.* $\varphi^2|_{R_e} = \mathrm{id}$*. Then*

1) $R_e = C_e \otimes I$ *is* φ*-stable where* I *is the unit element of* D *and hence* φ *induces an involution* $*$ *on* C_e*;*

2) there are subalgebras $B_1, \dots, B_k \subseteq C_e$ *such that* $C_e = B_1 \oplus \cdots \oplus B_k$*,*

$$B_1 \otimes I, \dots, B_k \otimes I \tag{3.1}$$

are φ*-stable and all* $B_1, \dots, B_k$ *are* $*$*-simple algebras, i.e.* B_i *does not contain non-trivial ideals invariant under* $*$*.*

3) φ *acts on* $R_e = C_e \otimes I$ *as* $\varphi * X = S^{-1} X^t S$ *where* $S = S_1 \otimes I + \cdots + S_k \otimes I$*,* $S_i \in B_i C B_i$ *and* $S_i = I_{p_i}$ *if* B_i *is* $p_i \times p_i$*-matrix algebra with transpose involution,* $S_i = \begin{pmatrix} 0 & I_{p_i} \\ -I_{p_i} & 0 \end{pmatrix}$ *if* B_i *is* $2p_i \times 2p_i$*-matrix algebra with symplectic* *involution or* $S_i = \begin{pmatrix} 0 & I_{p_i} \\ I_{p_i} & 0 \end{pmatrix}$ *if* $B_i \simeq M_{p_i} \oplus M_{p_i}$.

4) The centralizer of $R_e = C_e \otimes I$ *in* R *can be decomposed as* $Z_1 D_1 \oplus \cdots \oplus Z_k D_k$ *where* $D_1, \ldots, D_k$ *are* φ*-stable graded subalgebras of* R *isomorphic to* D *and* $Z_i = Z_i' \otimes I$ *where* Z_i' *is the center of* B_i*;*

5) D *as a graded algebra is isomorphic to* $M_2 \otimes \cdots \otimes M_2$ *where any factor* M_2 *has the fine* (-1)*-grading.*

Theorem 3.2.4 *Let* $\varphi : X \mapsto U^{-1} X^t U$ *be an involution compatible with a grading of a matrix algebra* R, $R = \bigoplus_{g \in G} R_g$ *by a finite abelian group* G*, i.e.* $\varphi(R_g) = R_g$*. Then* $R = C \otimes D$ *where* C *has an elementary grading and* D *a fine grading. Then, after a* G*-graded conjugation, we can reduce* U *to the form*

$$U = S_1 \otimes X_{t_1} + \cdots + S_k \otimes X_{t_k} \tag{3.2}$$

where S_i *is one of the matrices* I, $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ *or* $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ *and each* X_{t_i} *is a matrix spanning* D_{t_i}, $t_i \in T$ *where* T *is the support of* D *. The defining tuple of the elementary grading on* C *should satisfy the following condition. We assume that the first* l *of summands in* (3.2) *corresponds to those* B_i *in (3.1) which are simple and the remaining* $k - l$ *to* B_i *which are not simple. Let the dimension of a simple* B_i *be equal* p_i^2 *and that of a non-simple* B_j *to* $(2p_j)^2$*. Then the defining tuple has the form*

$$\left(g_1^{(p_1)}, \ldots, g_l^{(p_l)}, (g_{l+1}')^{(p_{l+1})}, (g_{l+1}'')^{(p_{l+1})}, \ldots, (g_k')^{(p_k)}, (g_k'')^{(p_k)}\right) \tag{3.3}$$

$$g_1^2 t_1 = \cdots g_l^2 t_l = g_{l+1}' g_{l+1}'' t_{l+1} = \cdots = g_k' g_k'' t_k. \tag{3.4}$$

Additionally, if φ is a transpose involution then each S_i is symmetric (skew-symmetric) at the same time as X_{t_i}, for any $i = 1, \ldots, k$. If φ is a symplectic involution, then each S_i is symmetric (skew-symmetric) if and only if the respective X_{t_i} is skew symmetric (symmetric), $i = 1, \ldots k$.

Conversely, if we have a grading by a group G on a matrix algebra R defined by a tuple as in (3.3)*, for the component C with elementary grading, and by an elementary abelian 2-subgroup T as the support of the component D with fine grading and all of the above conditions are satisfied then* (3.2) *defines a graded involution on R.*

Remark 3.2.5 *Suppose we have an algebra R with identity 1 and with involution * such that R is the product of two of its subalgebras A and B, with 1: $R = AB = \mathrm{Span}\{ab \mid a \in A,\ b \in B\}$. For any involution we set*

$$H(S,*) = \mathrm{Span}\{s + s^* \mid s \in S\},\ \ K(S,*) = \mathrm{Span}\{s - s^* \mid s \in S\},$$

where $K(S,*)$ is equivalent to our earlier definition. We also set $a \circ b = ab + ba$ and $[a,b] = ab - ba$. Then

$$\begin{aligned} K(R,*) = \ & [K(A,*), K(B,*)] + K(A,*) \circ H(B,*) \\ & + H(A,*) \circ K(B,*) + [H(A,*), H(B,*)]. \end{aligned}$$

It is shown in [4] that for an involution compatible grading of a matrix algebra R, $R = \bigoplus_{g \in G} R_g$ by a finite abelian group G as in Lemma 3.2.3,

$$K(R,*) = \mathrm{Span}\{e_i U e_j \otimes X_u - e_j S_j^t U S_i e_i \otimes X_{t_j} X_u^{-1} X_{t_i} \mid U = e_i U e_j \in C\ , u \in T\} \tag{3.5}$$

where e_k is the unit of B_k.

For the case of $R = M_8(\mathbb{C}) = C \otimes D$ there are four choices for the dimensions of C and D since $\dim M_8(\mathbb{C}) = 64 = 4^3$, $\dim D = 4^l$ for some positive l since $D \simeq M_2 \otimes \cdots \otimes M_2$. Hence there are four choices for the full matrix algebras of C and D. The choices are

$$\begin{aligned} &C \simeq M_8(\mathbb{C}), && D \simeq \mathbb{C}, \\ &C \simeq M_4(\mathbb{C}), && D \simeq M_2(\mathbb{C}), \\ &C \simeq M_2(\mathbb{C}), && D \simeq M_4(\mathbb{C}), \\ &C \simeq \mathbb{C}, && D \simeq M_8(\mathbb{C}). \end{aligned}$$

Using Theorem 3.2.4, one can describe all transpose involutions on M_8 with the above gradings and then restrict the grading to $K(M_8, *)$ using (3.5). This gives a description, up to isomorphism, of all matrix gradings on a Lie algebra of type D_4.

An example of an involution compatible grading on $R = M_8$ is the following. Let $*$ be the regular transpose. Define a grading $R = \bigoplus_{g\in G} R_g$ by $G = \langle g_2\rangle_2 \times \langle g_3\rangle_2 \times \langle g_4\rangle_2 \times \langle a\rangle_2 \times \langle b\rangle_2$ as follows. R is graded isomorphic to $C \otimes D$ where $\otimes$ is the Kronecker product of $C = M_4$ and $D = M_2$. We set $C = M_4 = \bigoplus_{k\in K} C_k$, an elementary grading by the group $K = \langle g_2\rangle_2 \times \langle g_3\rangle_2 \times \langle g_4\rangle_2$ with tuple $\tau = (e, g_2, g_3, g_4)$ and set $D = M_2 = \bigoplus_{t\in T} D_t$ the fine (-1)-grading by the group $T = \langle a\rangle_2 \times \langle b\rangle_2$.

For $X = (x_{ij}) \in C$ and $Y \in D$ we have

$$X \otimes Y = \begin{pmatrix} x_{11}Y & x_{12}Y & x_{13}Y & x_{14}Y \\ x_{21}Y & x_{22}Y & x_{23}Y & x_{24}Y \\ x_{31}Y & x_{32}Y & x_{33}Y & x_{34}Y \\ x_{41}Y & x_{42}Y & x_{43}Y & x_{44}Y \end{pmatrix}.$$

We note that

$$(X \otimes Y)^* = \begin{pmatrix} x_{11}Y^* & x_{21}Y^* & x_{31}Y^* & x_{41}Y^* \\ x_{12}Y^* & x_{22}Y^* & x_{32}Y^* & x_{42}Y^* \\ x_{13}Y^* & x_{23}Y^* & x_{33}Y^* & x_{43}Y^* \\ x_{14}Y^* & x_{24}Y^* & x_{34}Y^* & x_{44}Y^* \end{pmatrix} = X^* \otimes Y^*.$$

It follows that if we set $R_{g_2^{k_2} g_3^{k_3} g_4^{k_4} a^{k_a} b^{k_b}} = C_{g_2^{k_2} g_3^{k_3} g_4^{k_4}} \otimes D_{a^{k_a} b^{k_b}}$ we obtain a grading on M_8 which is compatible with the regular transpose.

An example of a graded subspace of R is

$$R_{g_2a} = \left\{ \left(\begin{array}{cccc} 0 & m & 0 & 0 \\ n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \middle| \; m, n \in \mathbb{C} \right\} \otimes \left\{ \left(\begin{array}{cc} -p & 0 \\ 0 & p \end{array} \right) \middle| \; p \in \mathbb{C} \right\}$$

$$= \left\{ \left(\begin{array}{cccccccc} 0 & 0 & -m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & 0 & 0 & 0 \\ -n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \middle| \; m, n \in \mathbb{C} \right\}.$$

We obtain a grading on the Lie algebra L, $L = K(R, *) = \bigoplus_{g \in G} L_g$ by setting $L_g = L \cap R_g$. This is a matrix grading on a Lie algebra of type D_4. An example of a graded subspace of L is

$$L_{g_2a} = \left\{ \left(\begin{array}{cccccccc} 0 & 0 & -m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & 0 & 0 & 0 \\ m & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -m & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \middle| \, m \in \mathbb{C} \right\}.$$

Chapter 4

Outer gradings

4.1 General overview.

If we find all gradings on a realisation of a simple Lie algebra L of a certain type then by Observation 1.1.10 we will have found all gradings for any realisation of a Lie algebra of the same type. Let us fix a Lie algebra L of type D_4, a maximal toral subalgebra H, a corresponding root system Φ and a base Δ of Φ. All gradings by a group G, $L = \bigoplus_{g\in G} L_g$, such that $f(\widehat{G}) \subset \Omega$ have been described in the previous chapter and we also get more gradings with the help of Chapter 3.

Let $f(\widehat{G}) = \langle \varphi \rangle \times B$ where $B \subset \operatorname{Int} L$ and $\varphi = \sigma_{1j}\,\lambda$, $\lambda \in \operatorname{Int} L$, $j \in \{3,4\}$, i.e., $\sigma_{1j} \notin \Omega$. We note that $\sigma_{14} = \rho_{134}\sigma_{34}\rho_{134}^{-1}$, $\sigma_{13} = \rho_{134}^{-1}\sigma_{34}\rho_{134}$ since $(134)(34)(134)^{-1} = (134)(34)(143) = (14)$ and $(134)^{-1}(34)(134) = (143)(34)(134) = (13)$. For any $\pi \in \operatorname{Aut} L$ we can use Observation 1.1.10 to obtain a grading $L = \bigoplus_{g\in G} L'_g$ where $L'_g = \pi(L_g)$. This new grading has its own homomorphism

f' taking $\widehat{G} \to \operatorname{Aut} L$. We define f' explicitly as

$$f'(\chi)(x) = \sum_{g\in G} \chi * x'_g = \sum_{g\in G} \chi(g)x'_g$$

where $x = \sum_{g\in G} x'_g$, $x'_g \in L'_g = \pi(L_g)$. We note that

$$\begin{aligned}
\pi f(\chi)\pi^{-1}(x) &= \pi f(\chi)\pi^{-1}(\textstyle\sum_{g\in G} x'_g) \\
&= \pi f(\chi)\pi^{-1} \textstyle\sum_{g\in G} \pi(x_g), && \text{for some } x_g \in L_g \\
&= \pi f(\chi) \textstyle\sum_{g\in G}(x_g) = \pi \sum_{g\in G} \chi(g)(x_g) \\
&= \textstyle\sum_{g\in G} \chi(g)\pi(x_g) = \sum_{g\in G} \chi(g)(x'_g) \\
&= f'(\chi) \textstyle\sum_{g\in G} x'_g \\
&= f'(\chi)(x).
\end{aligned}$$

Hence $f'(\widehat{G}) = \pi f(\widehat{G})\pi^{-1}$.

If we set $\pi = \rho_{134}$ then

$$f'(\widehat{G}) = \rho_{134}(\langle\sigma_{13}\lambda\rangle \times B)\rho_{134}^{-1} = \langle\rho_{134}(\sigma_{13}\,\lambda)\rho_{134}^{-1}\rangle \times \rho_{134}B\rho_{134}^{-1} = \langle\sigma_{34}\lambda'\rangle \times B'$$

for some $\lambda' \in \operatorname{Int}\, L$ by the semidirect property of $\operatorname{Aut}\, L$, and $B' = \rho_{134}B\rho_{134}^{-1} \subset \operatorname{Int} L$.

Similarly if we set $\pi = \rho_{134}^{-1}$, then

$$f'(\widehat{G}) = \rho_{134}^{-1}(\langle\sigma_{14}\lambda\rangle \times B)\rho_{134} = \langle\sigma_{34}\lambda''\rangle \times B''$$

for some $\lambda'' \in \operatorname{Int}\, L$ and $B'' = \rho_{134}^{-1}B\rho_{134} \subset \operatorname{Int}\, L$.

Observation 4.1.1 *The above shows that a grading on a Lie algebra L of type D_4 by a finite abelian group G, $L = \bigoplus_{g\in G} L_g$, with $G = S(G)$, such that $f(\widehat{G}) = \langle \sigma_{1j}\lambda\rangle \times B$, $\lambda \in \operatorname{Int} L$, $B \subset \operatorname{Int} L$ has a grading automorphism ρ_{134}^n, for $n \in \{1,2\}$ such that the matrix grading $L = \bigoplus_{g\in G} L'_g$, where $L'_g = \rho_{134}^n(L_g)$, $f'(\widehat{G}) = \langle \sigma_{34}\lambda'\rangle \times B'$, $\lambda' \in \operatorname{Int} L$, $B' \subset \operatorname{Int} L$ and the corresponding homomorphism f' sending $\widehat{G}$ to $\operatorname{Aut} L$ as in Section 1.2 with respect to the grading $L = \bigoplus_{g\in G} L'_g$, are isomorphic.*

The subgroup $\langle \rho_{134}\rangle$ is of index 2 in Aut Δ and hence normal in Aut L. This gives us that $\langle \rho_{134}\rangle$ Int L is a normal subgroup of Aut L, $\pi(\rho_{134}^n\lambda)\pi^{-1} \neq \sigma_{ij}^a\lambda'$ for $i,j \in \{1,3,4\}$, $i \neq j$, $a,n \in \{1,2\}$ and any λ, $\lambda' \in \operatorname{Int} L$. Hence gradings by a group G such that $f(\widehat{G}) = \langle \rho_{134}^n\lambda\rangle \times B$ where $B \subset \operatorname{Int} L$, $n \in \{1,2\}$ are not isomorphic to a matrix grading. It follows that if we have a grading by a group G such that $S(G)$ has no elements of order 3 then the grading is isomorphic to a matrix grading. In the following sections we use a different technique to find all possible gradings by $G = \langle z\rangle \times A$, with corresponding $\widehat{G}$ and $f(\widehat{G})$ as (2.5), (2.4), (2.3) in Corollaries 2.4.2, 2.4.3 with $\varphi = \sigma_{34}$, hence describing all gradings such that $\widehat{G} = \langle a\rangle_2 \times \Lambda$. Even though these gradings have now been described in Chapter 3, the technique involves looking at the natural grading by G/A and looking at the grading of $L_{\bar{e}} = \bigoplus_{a\in A} L_a$, $\bar{e}$ is the identity of G/A, which is a subalgebra of L as noted in Observation 1.1.12. For any $x \in L_{\bar{e}}$ we can express x as $x = \sum_{a\in A} x_a$, $x_a \in L_a$ and

$$\sigma_{34}(x) = \varphi(x) = \chi * x = \sum_{a\in A} \chi(a)x_a = \sum_{a\in A} x_a = x.$$

Hence $L_{\bar{e}} = L^{\sigma_{34}}$ where L^{π} is the Lie subalgebra of L that is pointwise-fixed by the map π. Similarly $L_{\bar{z}} = \bigoplus_{a\in A} L_{za} = L^{-\sigma_{34}}$ which gives us that $L = L^{\sigma_{34}} \oplus L^{-\sigma_{34}}$ is the natural grading by G/A. It turns out that $L_{\bar{e}}$ is a Lie algebra of type B_3, for which all gradings have been found in [1] and we show how we can lift an automorphism of $L_{\bar{e}}$ to an automorphism of L. It might be fruitful to generalize this technique by replacing $f(\widehat{G}) = \langle\sigma_{ij}\rangle \times B$ with $f(\widehat{G'}) = \langle\rho_{134}^n\lambda\rangle \times B'$ where $B' \subset \operatorname{Int} L$ and trying to lift the automorphisms of identity component of the natural G/A' grading on L to automorphism of L where $f(\widehat{A'}) = B'$, $G' = \langle z\rangle \times A'$ and the dual of A', $\widehat{A'}$. If $n = 1$ and $\lambda = \mathrm{id}_{\mathrm{Aut}\, L}$, then it can be shown that the identity component of the G'/A' grading is a Lie algebra of type G_2. In this work we do not explore this technique for this case because the gradings on Lie algebras of type G_2 are not well-known at the time.

4.2 Description of fixed subalgebras by certain outer automorphisms.

In this section L is a Lie algebra of type D_4. We do not specify a realisation but we do fix H, Φ, Δ. Let $L^{\sigma_{ij}}$ be a subalgebra of L which is fixed by σ_{ij}, for distinct $i, j \in \{1, 3, 4\}$.

It is known [9] that $L^{\sigma_{ij}}$ is of type B_3. Let $H^{\sigma_{ij}}$ be the subalgebra of H fixed by σ_{ij}. It is well known [8] that a maximal toral subalgebra of a Lie algebra of type B_3 is of dimension 3. $H^{\sigma_{ij}} = \operatorname{Span}\{h_{\alpha_k}, h_{\alpha_2}, h_{\alpha_i} + h_{\alpha_j}\}$, $k \in \{1,3,4\} \setminus \{i,j\}$, hence is a maximal toral subalgebra of $L^{\sigma_{ij}}$. Let

$$L^{\sigma_{ij}} = H^{\sigma_{ij}} \bigoplus_{\beta\in\Phi^{\sigma_{ij}}} L_\beta^{\sigma_{ij}}$$

be a Cartan decomposition for a root system $\Phi^{\sigma_{ij}}$ of $H^{\sigma_{ij}}$ and $\Delta^{\sigma_{ij}}$ a base of

$\Phi^{\sigma_{ij}}$ described below (see Appendix A for justification).

The base $\Delta^{\sigma_{ij}} = \{\beta_1, \beta_2, \beta_3\}$ where

$$\begin{aligned}
\beta_1 &= \alpha_k|_{H^{\sigma_{ij}}}, \\
\beta_2 &= \alpha_2|_{H^{\sigma_{ij}}}, \\
\beta_3 &= (\alpha_i + \alpha_j)/2|_{H^{\sigma_{ij}}}.
\end{aligned}$$

The positive roots of $\Phi^{\sigma_{ij}}$ are:

$$\begin{array}{lll}
\beta_2, & \beta_1, & \beta_3, \\
\beta_2 + \beta_1, & \beta_2 + \beta_3, & \beta_2 + \beta_1 + \beta_3, \\
\beta_2 + 2\beta_3, & \beta_1 + \beta_2 + 2\beta_3, & \beta_1 + 2\beta_2 + 2\beta_3.
\end{array}$$

We can obtain a canonical basis of $L^{\sigma_{ij}}$

$\{h'_{\beta_i}, x'_\beta, y'_\beta \mid \beta_i \in \Delta^{\sigma_{ij}}, \beta$ is a positive root of $\Phi^{\sigma_{ij}}\}$ such that

$$\begin{array}{ll}
x'_\beta \in L^{\sigma_{ij}}_\beta, & \text{for all } \beta \in \Phi^{\sigma_{ij}}, \\
y'_\beta \in L^{\sigma_{ij}}_{-\beta}, & \text{for all } \beta \in \Phi^{\sigma_{ij}}, \\
h'_{\beta_i} \in H^{\sigma_{ij}}, & \\
[x'_\beta, y'_\beta] = h'_\beta, & \text{for all } \beta \in \Phi^{\sigma_{ij}} \\
[h'_\beta, z'_{\beta'}] = \langle \beta, \beta' \rangle z'_{\beta'}, & \text{for all } \beta, \beta' \in \Phi^{\sigma_{ij}},\ z'_{\beta'} \in L^{\sigma_{ij}}_{\beta'}.
\end{array}$$

We express the above basis of $L^{\sigma_{ij}}$ in terms of the basis of L in the following way:

$$
\begin{aligned}
&h'_{\beta_1} = h_{\alpha_k}, && \\
&h'_{\beta_2} = h_{\alpha_2}, && \\
&h'_{\beta_3} = h_{\alpha_i} + h_{\alpha_j}, && \\
&x'_{\beta_1} = x_{\alpha_k}, && x'_{\beta_2} = x_{\alpha_2}, \\
&x'_{\beta_3} = x_{\alpha_i} + x_{\alpha_j}, && x'_{\beta_1+\beta_2} = x_{\alpha_2+\alpha_k}, \\
&x'_{\beta_2+\beta_3} = x_{\alpha_2+\alpha_i} + x_{\alpha_2+\alpha_j}, && x'_{\beta_1+\beta_2+\beta_3} = x_{\alpha_2+\alpha_k+\alpha_i} + x_{\alpha_2+\alpha_k+\alpha_j}, \\
&x'_{\beta_2+2\beta_3} = x_{\alpha_2+\alpha_i+\alpha_j}, && x'_{\beta_1+\beta_2+2\beta_3} = x_{\alpha_2+\alpha_k+\alpha_i+\alpha_j}, \\
&x'_{\beta_1+2\beta_2+2\beta_3} = x_{\alpha_2+\alpha_k+\alpha_i+\alpha_j+\alpha_2}, && \\
&y'_{\beta_1} = y_{\alpha_k}, && y'_{\beta_2} = y_{\alpha_2}, \\
&y'_{\beta_3} = y_{\alpha_i} + y_{\alpha_j}, && y'_{\beta_1+\beta_2} = y_{\alpha_2+\alpha_k}, \\
&y'_{\beta_2+\beta_3} = y_{\alpha_2+\alpha_i} + y_{\alpha_2+\alpha_j}, && y'_{\beta_1+\beta_2+\beta_3} = y_{\alpha_2+\alpha_k+\alpha_i} + y_{\alpha_2+\alpha_k+\alpha_j}, \\
&y'_{\beta_2+2\beta_3} = y_{\alpha_2+\alpha_i+\alpha_j}, && y'_{\beta_1+\beta_2+2\beta_3} = y_{\alpha_2+\alpha_k+\alpha_i+\alpha_j}, \\
&y'_{\beta_1+2\beta_2+2\beta_3} = y_{\alpha_2+\alpha_k+\alpha_i+\alpha_j+\alpha_2}. &&
\end{aligned}
\tag{4.1}
$$

Similarly let $L^{\rho_{1lm}}$ be a subalgebra of L fixed by ρ_{1lm}, $\{l,m\} = \{3,4\}$. It is known [9] $L^{\rho_{1lm}}$ is a simple Lie algebra of type G_2. Let $H^{\rho_{1lm}}$ be the subalgebra of H fixed by ρ_{1lm}. It is well known [8] that the dimension of a maximal toral subalgebra of a Lie algebra of type G_2 is two. $H^{\rho_{1lm}} = \mathrm{Span}\{h_{\alpha_2}, h_{\alpha_1} + h_{\alpha_3} + h_{\alpha_4}\}$ is of dimension two and hence is a maximal toral subalgebra of $L^{\rho_{lm}}$. Let $\Phi^{\rho_{1lm}}$ be a Cartan subalgebra of $L^{\rho_{1lm}}$ and $\Delta^{\rho_{1lm}}$ a base of $\Phi^{\rho_{1lm}}$ described below.

The base is $\Delta^{\rho_{1lm}} = \{\gamma_1, \gamma_2\}$ where $\gamma_1 = (\alpha_1 + \alpha_3 + \alpha_4)/3|_{H^{\rho_{1lm}}}$ and $\gamma_2 = \alpha_2|_{H^{\rho_{1lm}}}$.

The positive roots of $\Phi^{\rho_{1lm}}$ are

$$\begin{array}{lll} \gamma_1, & \gamma_2, & \gamma_1+\gamma_2, \\ 2\gamma_1+\gamma_2, & 3\gamma_1+\gamma_2, & 3\gamma_1+2\gamma_2. \end{array}$$

We can obtain a basis $\{h''_{\gamma_i}, x''_\gamma, y''_\gamma \mid \gamma_i \in \Delta^{\rho_{1lm}},\ \gamma \text{ is a positive root of } \Phi^{\rho_{1lm}}\}$ of $L^{\rho_{1lm}}$ such that

$$\begin{array}{ll} x''_\gamma \in L^{\rho_{ijk}}_{\gamma}, & \text{for all } \gamma, \in \Phi^{\rho_{1lm}}, \\ y''_\gamma \in L^{\rho_{ijk}}_{-\gamma}, & \text{for all } \gamma, \in \Phi^{\rho_{1lm}}, \\ h''_{\gamma_i} \in H^{\rho_{1lm}}, & \\ {[x''_\gamma, y''_\gamma] = h''_\gamma,} & \text{for all } \gamma \in \Phi^{\rho_{1lm}} \\ {[h''_\gamma, z''_{\gamma'}] = \langle\gamma,\gamma'\rangle z''_{\gamma'},} & \text{for all } \gamma,\gamma' \in \Phi^{\rho_{1lm}},\ z''_{\gamma'} \in L^{\rho_{ijk}}_{\gamma'}. \end{array}$$

We can express the above basis of $L^{\rho_{1lm}}$ in terms of the basis of L in the following way:

$$\begin{array}{ll} h''_{\gamma_2} = h_{\alpha_2}, & h''_{\gamma_1} = h_{\alpha_1} + h_{\alpha_3} + h_{\alpha_4}, \\ x''_{\gamma_2} = x_{\alpha_2}, & x''_{\gamma_1} = x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_4}, \\ x''_{\gamma_2+\gamma_1} = x_{\alpha_2+\alpha_1} + x_{\alpha_2+\alpha_3} + x_{\alpha_2+\alpha_4}, & x''_{\gamma_2+3\gamma_1} = x_{\alpha_2+\alpha_1+\alpha_3+\alpha_4}, \\ x''_{\gamma_2+2\gamma_1} = x_{\alpha_2+\alpha_1+\alpha_3} + x_{\alpha_2+\alpha_1+\alpha_4} + x_{\alpha_2+\alpha_3+\alpha_4}, & x''_{2\gamma_2+3\gamma_1} = x_{2\alpha_2+\alpha_1+\alpha_3+\alpha_4} \\ y''_{\gamma_2} = y_{\alpha_2}, & y''_{\gamma_1} = y_{\alpha_1} + y_{\alpha_3} + y_{\alpha_4}, \\ y''_{\gamma_2+\gamma_1} = y_{\alpha_2+\alpha_1} + y_{\alpha_2+\alpha_3} + y_{\alpha_2+\alpha_4}, & y''_{\gamma_2+3\gamma_1} = y_{\alpha_2+\alpha_1+\alpha_3+\alpha_4}, \\ y''_{\gamma_2+2\gamma_1} = y_{\alpha_2+\alpha_1+\alpha_3} + y_{\alpha_2+\alpha_1+\alpha_4} + y_{\alpha_2+\alpha_3+\alpha_4}, & y''_{2\gamma_2+3\gamma_1} = y_{2\alpha_2+\alpha_1+\alpha_3+\alpha_4}. \end{array}$$

4.3 Realisation of $L^{\sigma_{34}}$.

We continue on from the end of the last section. To get a better understanding of this subalgebra let us look at the realisation of Section 2.2 and note

$\sigma_{ij} = \sigma_{34}$, which sends E_{m8} to $-E_{m8}$ and leaves all other E_{ab}, $1 \le a < b \le 7$, invariant. A basis for $L^{\sigma_{34}}$ is the following:

$$
\begin{aligned}
h'_{\beta_1} &= h_{\alpha_1} = i(E_{25} - E_{36}) \\
h'_{\beta_2} &= h_{\alpha_2} = i(E_{36} - E_{47}) \\
h'_{\beta_3} &= h_{\alpha_3} + h_{\alpha_4} = 2iE_{47} \\
x'_{\beta_1} &= x_{\alpha_1} = \{E_{23} + iE_{26} + iE_{35} + E_{56}\}/2 \\
x'_{\beta_2} &= x_{\alpha_2} = \{E_{34} + iE_{37} + iE_{46} + E_{67}\}/2 \\
x'_{\beta_3} &= x_{\alpha_3} + x_{\alpha_4} = -E_{14} + iE_{17} \\
x'_{\beta_1+\beta_2} &= x_{\alpha_2+\alpha_1} = \{-E_{24} - iE_{27} - iE_{45} + E_{75}\}/2 \\
x'_{\beta_1+\beta_3} &= x_{\alpha_2+\alpha_3} + x_{\alpha_2+\alpha_4}/2 = -E_{13} + iE_{16} \\
x'_{\beta_1+\beta_2+\beta_3} &= x_{\alpha_2+\alpha_1+\alpha_3} + x_{\alpha_2+\alpha_1+\alpha_4} = E_{12} - iE_{15} \\
x'_{\beta_2+2\beta_3} &= x_{\alpha_2+\alpha_3+\alpha_4} = (-E_{34} + iE_{37} - iE_{46} + E_{67})/2 \\
x'_{\beta_1+\beta_2+2\beta_3} &= x_{\alpha_2+\alpha_1+\alpha_3+\alpha_4} = (E_{24} - iE_{27} + iE_{45} - E_{57})/2 \\
x'_{\beta_1+2\beta_2+2\beta_3} &= x_{\alpha_2+\alpha_1+\alpha_3+\alpha_4+\alpha_2} = (-E_{23} + iE_{26} - iE_{35} + E_{56})/2 \\
y'_{\beta} &= -\overline{(x'_{\beta})}^* \text{ where } \overline{x} \text{ is the conjugate of } x
\end{aligned}
$$

Notice that

$$
L^{\sigma_{34}} = \left\{ x \in M_8 \,|\, x = \sum_{1 \le a < b \le 7} c_{ab}E_{ab},\ c_{ab} \in \mathbb{C} \right\}
$$

is the embedding of the 7×7 skew-symmetric matrices over $\mathbb{C}$ in the upper left hand corner.

4.4 Certain outer gradings.

This section continues from the end of Section 4.2. Let $L = H \bigoplus_{\alpha\in\Phi} L_\alpha$ be a Cartan decomposition described in Section 2.1 with Φ a root system and

Δ a base of Φ. We choose a canonical basis of L as (2.2) of Section 2.1 and a canonical basis of $L^{\sigma_{ij}}$ as (4.1) of Section 4.2. A basis of $L^{-\sigma_{34}}$ is formed by the following elements:

$$h_{\alpha_i} - h_{\alpha_j},$$
$$x_{\alpha_i} - x_{\alpha_j}, \quad x_{\alpha_2+\alpha_i} - x_{\alpha_2+\alpha_j}, \quad x_{\alpha_2+\alpha_i+\alpha_k} - x_{\alpha_2+\alpha_j+\alpha_k},$$
$$y_{\alpha_i} - y_{\alpha_j}, \quad y_{\alpha_2+\alpha_i} - y_{\alpha_2+\alpha_j}, \quad y_{\alpha_2+\alpha_i+\alpha_k} - y_{\alpha_2+\alpha_j+\alpha_k}.$$

Theorem 4.4.1 *H is the unique maximal toral subalgebra of L containing $H^{\sigma_{ij}}$.*

Proof

Let T be a maximal toral subalgebra of L such that $H^{\sigma_{ij}} \subset T$. We can express T as $T = H^{\sigma_{ij}} \oplus \mathrm{Span}\{h\}$ for some $h \in L$ such that $h = h' + h''$, $h' \in L^{\sigma_{ij}}$ and $h'' \in L^{-\sigma_{ij}}$. Since maximal toral subalgebras are abelian, $[z, h] = 0$ for all $z \in H^{\sigma_{ij}}$. We also have

$$0 = [z, h] = [z, h'] + [z, h''] \quad \Rightarrow [z, h'] = [z, h''] = 0$$

since $[z, h'] \in L^{\sigma_{ij}}$, $[z, h''] \in L^{-\sigma_{ij}}$ and $L = L^{\sigma_{ij}} \oplus L^{-\sigma_{ij}}$. Since $L^{\sigma_{ij}}$ is a simple Lie algebra and $H^{\sigma_{ij}}$ is a maximal toral subalgebra of $L^{\sigma_{ij}}$ we have that $h' \in H^{\sigma_{ij}}$. Hence we can set $h = h'' \in L^{-\sigma_{ij}}$ and express h as

$$\begin{aligned} h = \ & a_0(h'_{\alpha_i} - h'_{\alpha_i}) + a_1(x_{\alpha_i} - x_{\alpha_j}) + a_2(x_{\alpha_2+\alpha_i} - x_{\alpha_2+\alpha_j}) \\ & + a_3(x_{\alpha_2+\alpha_i+\alpha_k} - x_{\alpha_2+\alpha_j+\alpha_k}) + a_4(y_{\alpha_i} - y_{\alpha_j}) \\ & + a_5(y_{\alpha_2+\alpha_i} - y_{\alpha_2+\alpha_j}) + a_6(y_{\alpha_2+\alpha_i+\alpha_k} - y_{\alpha_2+\alpha_j+\alpha_k}), \end{aligned}$$

$a_n \in \mathbb{C}$. Now we take the commutator of h with some elements of $H^{\sigma_{ij}}$ to determine h.

$$\begin{aligned}
0 = \; & [h_{\beta_1}, h] \\
= \; & [h_{\alpha_k}, a_0(h'_{\alpha_i} - h'_{\alpha_i}) + a_1(x_{\alpha_i} - x_{\alpha_j}) + a_2(x_{\alpha_2+\alpha_i} - x_{\alpha_2+\alpha_j}) \\
& + a_3(x_{\alpha_2+\alpha_i+\alpha_k} - x_{\alpha_2+\alpha_j+\alpha_k}) + a_4(y_{\alpha_i} - y_{\alpha_j}) \\
& + a_5(y_{\alpha_2+\alpha_i} - y_{\alpha_2+\alpha_j}) + a_6(y_{\alpha_2+\alpha_i+\alpha_k} - y_{\alpha_2+\alpha_j+\alpha_k})] \\
= \; & -a_2(x_{\alpha_2+\alpha_i} - x_{\alpha_2+\alpha_j}) + a_3(x_{\alpha_2+\alpha_i+\alpha_k} - x_{\alpha_2+\alpha_j+\alpha_k}) \\
& + a_5(y_{\alpha_2+\alpha_i} - y_{\alpha_2+\alpha_j}) - a_6(y_{\alpha_2+\alpha_i+\alpha_k} - y_{\alpha_2+\alpha_j+\alpha_k})
\end{aligned}$$

$\Rightarrow a_2 = a_3 = a_5 = a_6 = 0.$

$$\begin{aligned}
0 = \; & [h_{\beta_2}, h] \\
= \; & [h_{\alpha_2}, a_0(h'_{\alpha_i} - h'_{\alpha_i}) + a_1(x_{\alpha_i} - x_{\alpha_j}) + a_4(y_{\alpha_i} - y_{\alpha_j})] \\
= \; & -a_1(x_{\alpha_i} - x_{\alpha_j}) + a_4(y_{\alpha_i} - y_{\alpha_j})
\end{aligned}$$

$\Rightarrow a_1 = a_4 = 0.$

$\Rightarrow h = a_0(h_{\alpha_i} - h_{\alpha_j}) \in H$

$\Rightarrow T = H.$ ■

We can express L as $L = \bigoplus_{\omega \in P} L'_\omega$ for some set of weights $P \subset H^{\sigma_{34}*}$,

$$L'_\omega = \{x \in L \mid [h, x] = \omega(h)x \, \forall h \in H^{\sigma_{ij}}\}$$

since $H^{\sigma_{ij}}$ is a toral subalgebra of L. It can be easily verified that the subspaces mentioned above are:

$$\begin{array}{ll} L'_0 = H, & L'_{\pm\beta_2\pm\beta_3} = L_{\pm\alpha_2+\pm\alpha_i} \oplus L_{\pm\alpha_2\pm\alpha_j}, \\ L'_{\pm\beta_3} = L_{\pm\alpha_i} \oplus L_{\pm\alpha_j}, & L'_{\pm\beta_2\pm\beta_3\pm\beta_1} = L_{\pm\alpha_2\pm\alpha_i\pm\alpha_k} \oplus L_{\pm\alpha_2\pm\alpha_j\pm\alpha_k}, \\ L'_\beta = L^{\sigma_{ij}}_\beta, & \text{for all other } \beta. \end{array}$$

Lemma 4.4.2 *If two inner automorphisms λ and λ' of L restrictions to $L^{\sigma_{ij}}$ are equal and λ, λ' leave $L^{\sigma_{ij}}$ invariant, then $\lambda = \lambda'$.*

Proof

Let λ, $\lambda' \in \operatorname{Int} L$ be such that $\lambda|_{L^{\sigma_{ij}}} = \lambda'|_{L^{\sigma_{ij}}}$ and $\lambda(L^{\sigma_{ij}}) = L^{\sigma_{ij}}$. Since $\operatorname{Int} L = \{\exp(\operatorname{ad} x) \mid x \text{ is ad-nilpotent, } x \in L\}$ is a group there exists a $z \in L$ such that $\lambda^{-1}\lambda' = \exp(\operatorname{ad} z)$ and $(\operatorname{ad} z)^n = 0$ for some positive integer n and $(adz)^a \neq 0$ for any positive integer $a < n$. It follows that $\exp(\operatorname{ad} z)(y) = y$ for all $y \in L^{\sigma_{ij}}$. This implies that if $n \geq 2$

$$\begin{aligned} (\operatorname{ad} z)^{n-2}(y) = & \ (\operatorname{ad} z)^{n-2}(\textstyle\sum_{i=0}^{n-1}(adz)^i(y)/(i!)) \\ = & \ \textstyle\sum_{i=0}^{n-1}(adz)^{i+n-2}(y)/(i!) \\ = & \ (adz)^{n-2}(y) + (adz)^{n-1}(y) \end{aligned}$$

and hence $(adz)^{n-1}(y) = 0$ for all $y \in L^{\sigma_{ij}}$. We can now use induction on this process for $(adz)^{n-m}$, $n \geq m$, to show that $(adz)^{n-m+1}(y) = 0$ for all $y \in L^{\sigma_{ij}}$. This implies that the smallest integer t such that $(\operatorname{ad} z)^t(y) = 0$ for all $y \in L^{\sigma_{ij}}$ is $t = 1$.

Let $z = z' + z''$ where $z' \in L^{\sigma_{ij}}$ and $z'' \in L^{-\sigma_{ij}}$. Now $0 = (\operatorname{ad} z)(y) = [z' + z'', y] = y' + y''$ where $y' = [z', y] \in L^{\sigma_{ij}}$ and $y'' = [z'', y] \in L^{-\sigma_{ij}}$ for all $y \in L^{\sigma_{ij}}$ which implies $y' = y'' = 0$ since $L = L^{\sigma_{ij}} \oplus L^{-\sigma_{ij}}$. Also $z' = 0$ since $L^{\sigma_{ij}}$ is simple which implies that its center $Z(L^{\sigma_{ij}}) = \{x \in L^{\sigma_{ij}} \mid (\operatorname{ad} x)(y) = 0,\ \forall y \in L^{\sigma_{ij}}\}$ is zero. By the proof of Theorem 4.4.1, the only element of $L^{-\sigma_{ij}}$

that commutes with $H^{\sigma_{ij}} \subset L^{\sigma_{ij}}$ is in $\mathrm{Span}\{h_{\alpha_3} - h_{\alpha_4}\}$. Hence $z = 0$ since $z \in \mathrm{Span}\{h_{\alpha_3} - h_{\alpha_4}\}$ and $h_{\alpha_3} - h_{\alpha_4}$ is not ad-nilpotent. Now $\lambda^{-1}\lambda' = id_{\mathrm{Aut}\,L}$ and we are done. ■

Theorem 4.4.3 *Let L be the realisation of a Lie algebra of type D_4 as described in Section 4.3 and G a finite abelian group. For any grading of $L^{\sigma_{34}}$ by G, $L^{\sigma_{34}} = \bigoplus_{g\in G} L_g^{\sigma_{34}}$, with $G = S(G)$, there exists a unique inner grading on L by G, $L = \bigoplus_{g\in G} L_g$, such that $L_g \bigcap L^{\sigma_{34}} = L_g^{\sigma_{34}}$. Moreover $L^{-\sigma_{34}}$ is a graded subspace.*

Proof

There is a natural isomorphism ψ of $L^{\sigma_{34}}$ into $K(M_7)$ where

$$\psi\left(\left(\begin{array}{cc} B & 0 \\ 0 & 0 \end{array}\right)\right) = B$$

for all $B = K(M_7)$. It is known [10] that any automorphism λ of $K(M_7)$ is conjugation by an orthogonal matrix $T_\lambda \in M_7$. It is easy to see that for any automorphism κ of $L^{\sigma_{34}}$, $\tilde{\kappa} = \psi\kappa\psi^{-1}$ is an automorphism of $K(M_7)$ such that $\psi\kappa(x) = \tilde{\kappa}(\psi(x))$ for all $x \in L^{\sigma_{34}}$. It follows for the matrix

$$U_\kappa = \left(\begin{array}{cc} T_{\tilde{\kappa}} & 0 \\ 0 & \det T_{\tilde{\kappa}} \end{array}\right)$$

we have $\kappa(x) = U_\kappa x U_\kappa^{-1}$.

Since $\det T_\lambda = \pm 1$, U_κ is an orthogonal matrix of determinant one and it is known by [10] that conjugation by U_κ is an inner automorphism $\overline{\kappa}$ of L.

By Lemma 4.4.2, $\overline{\kappa}$ is the unique inner automorphism of L that leaves $L^{\sigma_{34}}$ invariant. By looking at $\lambda,\ \lambda' \in \operatorname{Aut} L^{\sigma_{34}}$ as conjugation by matrix we note that if $\lambda,\ \lambda'$ commute, then $\overline{\lambda}, \overline{\lambda'}$ commute as well.

Let $f : \widehat{G} \to \operatorname{Aut} L^{\sigma_{34}}$ be as in Section 1.2 with respect to the grading on $L^{\sigma_{34}}$. The above shows that there is an isomorphism $\phi : f(\widehat{G}) \to \overline{f}(\widehat{G})$ where $\overline{f}(\widehat{G}) = \{\overline{\lambda} \mid \lambda \in f(\widehat{G})\}$ is an abelian subgroup of $\operatorname{Aut} L$. As before we obtain a grading on L by G by setting

$$L_g = \{x \in L \mid \overline{f}(\chi)(x) = \chi(g)x,\ \forall \chi \in \widehat{G}\}.$$

Then $L_g \bigcap L^{\sigma_{34}} = L_g^{\sigma_{34}}$ as desired. By the 1-1 correspondence between G-gradings and $f(\widehat{G}) \subset \operatorname{Aut} L$, we see that such inner grading $L = \bigoplus_{g \in G} L_g$ is unique.

Conjugation by the matrix U_k leaves $L^{-\sigma_{34}}$ invariant since

$$L^{-\sigma_{34}} = \left\{ \left(\begin{array}{cc} 0 & y \\ -y^* & 0 \end{array} \right) \middle|\ \text{where } * \text{ is the matrix transpose } y \in M_{7,1} \right\}.$$

It follows that $L^{-\sigma_{34}}$ is left invariant by $\overline{f}(\widehat{G})$.

Since $L^{-\sigma_{34}}$ is left invariant by $\overline{f}(\widehat{G})$ it follows that $L^{-\sigma_{34}}$ is a graded subspace. ■

Observation 4.4.4 *Since* $T_{\sigma_{34}} = \left(\begin{array}{cc} I_7 & 0 \\ 0 & -1 \end{array} \right)$, σ_{34} *commutes with* $\overline{f}(\widehat{G})$ *from Theorem 4.4.3.*

Lemma 4.4.5 *If an inner automorphism λ of L and an outer automorphism φ of L restrictions to $L^{\sigma_{ij}}$ are equal and λ, φ leave $L^{\sigma_{ij}}$ and $L^{-\sigma_{ij}}$ invariant, then $\varphi = \sigma_{ij}\lambda$.*

Proof

Let $\psi = \varphi\lambda^{-1}$. Since $\varphi\lambda^{-1}(h) = h$ for all $h \in H^{\sigma_{ij}} \subset L^{\sigma_{ij}}$ $\psi(H^{\sigma_{ij}}) = H^{\sigma_{ij}}$ hence $\varphi(H) = H$ since automorphisms of L send maximal toral subalgebras to maximal toral subalgebras and by Theorem 4.4.1 H is the unique maximal toral subalgebra of L containing $H^{\sigma_{ij}}$. Since $\mathrm{Span}\{h_{\alpha_3} - h_{\alpha_4}\} = H \cap L^{-\sigma_{ij}}$, $\psi(h_{\alpha_3} - h_{\alpha_4}) = a(h_{\alpha_3} - h_{\alpha_4})$ for some non-zero $a \in \mathbb{C}$. We now try to find where ψ sends $x_{\alpha_i} - x_{\alpha_j}$ by looking at $\psi([H, x_{\alpha_i} - x_{\alpha_j}])$.

$$\begin{aligned}
\psi([h_{\alpha_k}, x_{\alpha_i} - x_{\alpha_j}]) &= \psi(0) = 0 \\
\psi([h_{\alpha_k}, x_{\alpha_i} - x_{\alpha_j}]) &= [\psi(h_{\alpha_k}), \psi(x_{\alpha_i} - x_{\alpha_j})]) \\
&= [h_{\alpha_k}, \psi(x_{\alpha_i} - x_{\alpha_j})]
\end{aligned}$$

$$\begin{aligned}
\psi([h_{\alpha_2}, x_{\alpha_i} - x_{\alpha_j}]) &= \psi(-(x_{\alpha_i} - x_{\alpha_j})) = -\psi(x_{\alpha_i} - x_{\alpha_j}) \\
\psi([h_{\alpha_2}, x_{\alpha_i} - x_{\alpha_j}]) &= [\psi(h_{\alpha_2}), \psi(x_{\alpha_i} - x_{\alpha_j})] \\
&= [h_{\alpha_2}, \psi(x_{\alpha_i} - x_{\alpha_j})]
\end{aligned}$$

$$\begin{aligned}
\psi([h_{\alpha_i} + h_{\alpha_j}, x_{\alpha_i} - x_{\alpha_j}]) &= \psi(2(x_{\alpha_i} - x_{\alpha_j})) = 2\psi(x_{\alpha_i} - x_{\alpha_j}) \\
\psi([h_{\alpha_i} + h_{\alpha_j}, x_{\alpha_i} - x_{\alpha_j}]) &= [\psi(h_{\alpha_i} + h_{\alpha_j}), \psi(x_{\alpha_i} - x_{\alpha_j})] \\
&= [h_{\alpha_i} + h_{\alpha_j}, \psi(x_{\alpha_i} - x_{\alpha_j})]
\end{aligned}$$

The above calculations show that $\psi(x_{\alpha_i} - x_{\alpha_j}) \in L'_{\beta_3} = L_{\alpha_i} \oplus L_{\alpha_j}$. We also know that $\psi(x_{\alpha_i} - x_{\alpha_j}) \in L^{-\sigma_{ij}}$ and $L'_{\beta_3} \cap L^{-\sigma_{ij}} = \mathrm{Span}\{x_{\alpha_i} - x_{\alpha_j}\}$. Hence $\psi(x_{\alpha_i} - x_{\alpha_j}) = b(x_{\alpha_i} - x_{\alpha_j})$ for some non-zero $b \in \mathbb{C}$. Now we want find where

x_{α_i} and h_{α_i} are sent.

$$\begin{aligned}
\psi(x_{\alpha_i}) &= \psi((x_{\alpha_i} - x_{\alpha_j}) + (x_{\alpha_i} + x_{\alpha_j}))/2 \\
&= (b(x_{\alpha_i} - x_{\alpha_j}) + x_{\alpha_i} + x_{\alpha_j})/2 \\
&= ((1 + b)x_{\alpha_i} + (1 - b)x_{\alpha_j})/2
\end{aligned}$$

$$\begin{aligned}
\psi(h_{\alpha_i}) &= \psi((h_{\alpha_i} - h_{\alpha_j}) + (h_{\alpha_i} + h_{\alpha_j}))/2 \\
&= (a(h_{\alpha_i} - h_{\alpha_j}) + h_{\alpha_i} + h_{\alpha_j})/2 \\
&= ((1 + a)h_{\alpha_i} + (1 - a)h_{\alpha_j})/2
\end{aligned}$$

$$\begin{aligned}
\psi([h_{\alpha_i}, x_{\alpha_i}]) &= \psi(2x_{\alpha_i}) = (1 + b)x_{\alpha_i} + (1 - b)x_{\alpha_j} \\
\psi([h_{\alpha_i}, x_{\alpha_i}]) &= [\psi(h_{\alpha_i}), \psi(x_{\alpha_i})] \\
&= [((1 + a)h_{\alpha_i} + (1 - a)h_{\alpha_j})/2, ((1 + b)x_{\alpha_i} + (1 - b)x_{\alpha_j})/2] \\
&= (1 + a)(1 + b)x_{\alpha_i}/2 + (1 - a)(1 - b)x_{\alpha_j}/2
\end{aligned}$$

Hence $(1 + a)(1 + b) = 2(1 + b)$ and $(1 - a)(1 - b) = 2(1 - b)$. If $b \neq -1$ then $a = 1$ and if $b \neq 1$ then $a = -1$. This implies $b = \pm 1$ and $a = b$. This means that either $\psi(x_{\alpha_i}) = x_{\alpha_i}$, $\psi(x_{\alpha_j}) = \alpha_j$, $\psi(h_{\alpha_i}) = h_{\alpha_i}$ or $\psi(x_{\alpha_i}) = x_{\alpha_j}$, , $\psi(x_{\alpha_j}) = x_{\alpha_i}$, $\psi(h_{\alpha_i}) = h_{\alpha_j}$ (since $\psi(x_{\alpha_i} + x_{\alpha_j}) = x_{\alpha_i} + x_{\alpha_j}$).

Also for $b = 1$, $h_{\alpha_i} = \psi(h_{\alpha_i}) = \psi([x_{\alpha_i}, y_{\alpha_i}]) = [\psi(x_{\alpha_i}), \psi(y_{\alpha_i})]$ which implies that $\psi(y_{\alpha_i}) = y_{\alpha_i}$ and similarly $\psi(y_{\alpha_j}) = y_{\alpha_j}$. The same arguments for $b = -1$ imply that $\psi(y_{\alpha_i}) = y_{\alpha_j}$ and $\psi(y_{\alpha_j}) = y_{\alpha_i}$. Therefore for $b = 1$, $\psi(z) = z$ and for $b = -1$, $\psi(z) = \sigma_{ij}(z)$ for all $z \in B = \{x_{\alpha_i}, y_{\alpha_i} \mid 1 \leq i \leq 4\}$. Since B is a generating set of L, $\psi = id|_{\mathrm{Aut}\, L}$ for $b = 1$ and $\psi = \sigma_{ij}$ for $b = -1$.

Since $\varphi \notin \mathrm{Int}\, L$, $\psi \notin \mathrm{Int}\, L$ which implies that $\psi \neq \mathrm{id}_{\mathrm{Aut}\, L}$, so $\psi = \sigma_{ij}$ and $\varphi = \sigma_{ij}\lambda$. ■

In the next theorem we use the inner gradings on L, $L = \bigoplus_{g\in G} L_g$, from Theorem 4.4.3 to get outer gradings on L by refining the grading using σ_{34}, i.e., decomposing the graded subspaces L_g further as $L_g = (L_g)^{\sigma_{34}} \oplus (L_g)^{-\sigma_{34}}$.

Theorem 4.4.6 *Let L be the realisation of a Lie algebra of type D_4 as described in Section 4.3 and G a finite abelian group. For any grading $L^{\sigma_{34}} = \bigoplus_{g\in G} L_g^{\sigma_{34}}$ by a finite abelian group G, with $G = S(G)$, there exists a unique grading by $J = \langle z\rangle_n \times G$, on L, $L = \bigoplus_{s\in J} \widetilde{L}_s$, such that $\widetilde{L}_s = L_s^{\sigma_{34}}$ for $s \in G$ and $\widetilde{L}_s \subseteq L^{-\sigma_{34}}$ for $s \notin G$ and the natural grading by $J/\langle z\rangle$ on L is inner. Moreover, this grading is an outer matrix grading and $n = 2$.*

Proof

The existence follows from Theorem 4.4.3 and Observation 4.4.4. We construct the inner grading by G, $L = \bigoplus_{g\in G} L_g$ with $L_g \bigcap L^{\sigma_{34}} = L_g^{\sigma_{34}}$, and observe that the spaces are σ_{34} invariant for all $g \in G$. We can decompose $L_g = (L_g)^{\sigma_{34}} \oplus (L_g)^{-\sigma_{34}}$ and set $\widetilde{L} = (L_g)^{\sigma_{34}}$ and $\widetilde{L}_{zg} = (L_g)^{-\sigma_{34}}$. This defines a grading on L by $J = \langle z\rangle_2 \times G$.

Suppose we have a grading on L by J, $L = \bigoplus_{s\in J} \widetilde{L}_s$, that satisfies the conditions of the theorem. The dual group of J is $\widehat{J} = \langle\eta\rangle_n \times \widehat{G}$ where $\eta(g) = 1$, $\eta(z) = e^{2\pi i/n}$, $\chi(z) = 1$, for all $g \in G$, $\chi \in \widehat{G}$. Let $F : \widehat{J} \to \operatorname{Aut} L$ be the group homomorphism defined by $F(\chi)(x_s) = \chi(s)x_s$ for all $\chi \in \widehat{J}$, $s \in J$, $x_s \in \widetilde{L}_s$. Then $L^{\sigma_{34}} = \bigoplus_{g\in G} \widetilde{L}_g$ which implies $F(\eta)|_{L^{\sigma_{34}}} = id_{\operatorname{Aut} L^{\sigma_{34}}}$. Also $\widetilde{L}_s = L_s^{\sigma_{34}}$ for $s \in G$ and $\widetilde{L}_s \subseteq L^{-\sigma_{34}}$ for $s \notin G$ imply that $L^{-\sigma_{34}}$ is a graded subspace. By Lemma 4.4.2 and Lemma 4.4.5 either $F(\eta)$ is the identity of $\operatorname{Aut} L$ or σ_{34}. Thus $F(\eta)(x) = -x$ for all $x \in L^{-\sigma_{34}}$. We conclude that $F(\eta) = \sigma_{34}$.

Since the natural grading by $J/\langle z\rangle$ is inner, $F(\widehat{G})$ is inner. By Theorem 4.4.3 the grading by $J/\langle z\rangle$ is unique which gives the uniqueness of the grading of L by J and thus $n = 2$. ■

With Observation 1.1.10, Observation 4.4.4 and the description of all gradings on Lie algebras of type B_3 in [1], it follows that for a of a Lie algebra L of type D_4, all matrix gradings $L = \bigoplus_{g\in G} L_g$ by finite abelian groups G, $G = S(G)$ and $f(\widehat{G}) = \langle\sigma_{ij}\rangle_2 \times B$, $B \subset \operatorname{Int} L$ are described in this section.

4.5 Example of a non-matrix grading

We construct an example of a grading by a group $G = \langle z\rangle_3 \times A$ on the realisation L from Section 2.2 such that $f(\widehat{G}) = \langle\rho_{134}\rangle_3 \times B$, $A \simeq B$, $B \subset \operatorname{Int} L$ and the grading by $G/\langle z\rangle_3 \simeq A$ induced by an elementary grading on M_8, $M_8 = R = \bigoplus_{g\in A} R_g$, i.e., $L = \bigoplus_{g\in A} L_g$ where $L_g = R_g \cap L$. It then follows [1] that the tuple $(g_1, \ldots, g_8)$ associated with the elementary grading on M_8 has the property that $g_1^2 = \cdots = g_8^2$. The idea is to find an elementary grading on M_8 that is transpose invariant and that $\rho_{134}(L_g) = L_g$. The requirement that $\rho_{134}(L_g) = L_g$ further decomposes L_g as $L_g = L_g^{\rho_{134}} \oplus L_g^{\varepsilon\rho_{134}} \oplus L_g^{\varepsilon^2\rho_{134}}$, $\varepsilon = e^{2\pi i/3}$ so that we have a grading on L, $L = \bigoplus_{g'\in G} \widetilde{L}_{g'}$ where $\widetilde{L}_{z^n g} = L_g^{\varepsilon^n\rho_{134}}$, $n \in \{0, 1, 2\}$.

From the calculations in Appendix B it follows that ρ_{134} leaves invariant the following subspaces.

$$
\begin{aligned}
V_1 &= \mathrm{Span}\{E_{12}, E_{34}, E_{58}, E_{67}\},\\
V_2 &= \mathrm{Span}\{E_{13}, E_{24}, E_{57}, E_{68}\},\\
V_3 &= \mathrm{Span}\{E_{14}, E_{23}, E_{56}, E_{78}\},\\
V_4 &= \mathrm{Span}\{E_{15}, E_{28}, E_{37}, E_{46}\},\\
V_5 &= \mathrm{Span}\{E_{16}, E_{27}, E_{38}, E_{45}\},\\
V_6 &= \mathrm{Span}\{E_{17}, E_{26}, E_{35}, E_{48}\},\\
V_7 &= \mathrm{Span}\{E_{18}, E_{25}, E_{36}, E_{47}\}.
\end{aligned}
$$

We notice that these subspaces are also maximal toral subalgebras. We can decompose V_i as $V_i = V_i^{\rho_{134}} \oplus V_i^{\varepsilon\rho_{134}} \oplus V_i^{\varepsilon^2\rho_{134}}$ which is also a grading by $\langle z \rangle_3$ on V_i for $1 \le i \le 7$. For example

$$
\begin{aligned}
V_7^{\rho_{134}} &= \mathrm{Span}\{E_{36} - E_{47}, E_{25} + E_{47}\},\\
V_7^{\varepsilon\rho_{134}} &= \mathrm{Span}\{E_{25} - E_{36} + \varepsilon^2(E_{47} - E_{18}) + \varepsilon(E_{47} + E_{18})\},\\
V_7^{\varepsilon^2\rho_{134}} &= \mathrm{Span}\{E_{25} - E_{36} + \varepsilon(E_{47} - E_{18}) + \varepsilon^2(E_{47} + E_{18})\}.
\end{aligned}
$$

Our next step is to choose $(g_1, \dots, g_8)$ in such that a way that $g_i g_j = g_k g_l$ for all $E_{ij}, E_{kl} \in V_m$ for distinct i, j, k, l, $1 \le i, j, k, l \le 8$ and $1 \le m \le 7$. This step ensures that ρ_{134} commutes with $f(\widehat{A})$. Now let $A = \langle g_2 \rangle_2 \times \langle g_3 \rangle_2 \times \langle g_5 \rangle_2$. The tuple $\tau = (e, g_2, g_3, g_2g_3, g_5, g, g_3g_8, g_3g_5, g_2g_5)$, $g = g_2g_3g_5$ satisfies the above requirements. The elementary grading on M_8 with associated tuple τ

can be visualized with the help of the following matrix.

$$\begin{pmatrix} e & g_2 & g_3 & g_2g_3 & g_5 & g & g_3g_5 & g_2g_5 \\ g_2 & e & g_2g_3 & g_3 & g_2g_5 & g_3g_5 & g & g_5 \\ g_3 & g_2g_3 & e & g_2 & g_3g_5 & g_2g_5 & g_5 & g \\ g_2g_3 & g_3 & g_2 & e & g & g_5 & g_2g_5 & g_3g_5 \\ g_5 & g_2g_5 & g_3g_5 & g & e & g_2g_3 & g_3 & g_2 \\ g & g_3g_5 & g_2g_5 & g_5 & g_2g_3 & e & g_2 & g_3 \\ g_3g_5 & g & g_5 & g_2g_5 & g_3 & g_2 & e & g_2g_3 \\ g_2g_5 & g_5 & g & g_3g_5 & g_2 & g_3 & g_2g_3 & e \end{pmatrix}$$

We set $\widehat{G} = \langle\zeta\rangle_3 \times \langle\chi_2\rangle_2 \times \langle\chi_3\rangle_2 \times \langle\chi_5\rangle_2$ where $\chi_i(g_i) = -1$, $\chi_i(g_j) = \chi_i(z) = 1$, $\zeta(z) = e^{2\pi i/3}$ and $\zeta(g_i) = 1$ for distinct i, j such that $i, j \in \{2, 3, 5\}$. It then follows that $f(\widehat{G}) \doteqdot \langle\rho_{134}\rangle_3 \times B$ where $B \subset \operatorname{Int} L$.

Bibliography

[1] Y. Bahturin, I. Shestakov, M. Zaicev, Gradings on simple Jordan and Lie algebras, Journal of Algebra 283 (2005), no. 2, 849-868.

[2] Y. Bahturin, M. Zaicev, Group gradings on matrix algebras, Canadian Mathematical Bulletin, 45 (2002), no. 4, 499-508.

[3] Y. Bahturin, M. Zaicev, Group gradings on Lie Algebras of type "A", Journal of Lie Theory, to appear.

[4] Y. Bahturin, M. Zaicev, Automomorphisms and antiautomorphisms of graded matrix algebras, Preprint.

[5] J. Patera, H. Zassenhaus, On Lie gradings. I, Linear Algebra and its Applications, 112 (1989), 87-159.

[6] M. Havlvcek, J. Patera, E. Pelantova, On Lie gradings. II, Linear Algebra and its Applications, 277 (1998), no. 1-3, 97-125.

[7] M. Havlvcek, J. Patera, E. Pelantova On Lie gradings. III. Gradings of the real forms of classical Lie algebras, Linear Algebra and its Applications, 314 (2000), no. 1-3, 1-47.

[8] J. Humphreys, "Introduction to Lie algebras and representation theory", Springer-Verlag, New York, 1972.

[9] A. Onishchik, E. Vinberg, "Lie Groups and Lie algebras III", Springer-Verlag, Berlin Heidelberg New York, 1994.

[10] N. Jacobson, "Lie algebras", Dover Publications, 1979.

Appendix A

Justification for the claim that $L^{\sigma_{ij}}$ is a Lie algebra of type B_3

We show $L^{\sigma_{ij}}$ is a Lie algebra of type B_3 for distinct $i, j \in \{1, 3, 4\}$. A Cartan subalgebra of $L^{\sigma_{ij}}$ is $H^{\sigma_{ij}} = Span\{h_{\alpha_k}, h_{\alpha_2}, h_{\alpha_i} + h_{\alpha_j}\}$.

We need to define elements h_{β_1}, h_{β_3} and h_{β_3} that span $H^{\sigma_{ij}}$ and a corresponding root system $\Phi^{\sigma_{ij}}$ with base $\Delta^{\sigma_{ij}} = \{\beta_1, \beta_2, \beta_3\}$. We are going to construct a basis $L^{\sigma_{ij}}$ by defining

$$\begin{aligned}
&\beta_1 = \alpha_k|_{H^{\sigma_{ij}}}, && \beta_2 = \alpha_2|_{H^{\sigma_{ij}}}, && \beta_3 = (\alpha_i + \alpha_j)/2|_{H^{\sigma_{ij}}}, \\
&h_{\beta_1} = h_{\alpha_k}, && h_{\beta_2} = h_{\alpha_2}, && h_{\beta_3} = (h_{\alpha_i} + h_{\alpha_j}), \\
&x_{\beta_1} = x_{\alpha_k}, && x_{\beta_2} = x_{\alpha_2}, && x_{\beta_3} = (x_{\alpha_i} + x_{\alpha_j}), \\
&y_{\beta_1} = y_{\alpha_k}, && y_{\beta_2} = y_{\alpha_2}, && y_{\beta_3} = (y_{\alpha_i} + y_{\alpha_j})
\end{aligned}$$

and verifying that $(\langle\beta_k,\beta_l\rangle)_{kl}$ is the Cartan matrix of a Lie algebra of type B_3 and the relations

$$\beta_k(h_{\beta_l}) = \langle\beta_k,\beta_l\rangle,\ [x_{\beta_k},y_{\beta_k}] = h_{\beta_k},\ [h_{\beta_k},x_{\beta_k}] = 2x_{\beta_k},\ [h_{\beta_k},y_{\beta_k}] = -2y_{\beta_k}$$

hold. We will need the Cartan matrix of a Lie algebra of type D_4 in order to find $\langle\beta_k,\beta_l\rangle$. The Cartan matrices of Lie algebras of type D_4 and B_3 are

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

respectively. The root system $\Phi^{\sigma_{ij}}$ has positive roots

$$\begin{array}{lll} \beta_2, & \beta_1, & \beta_3, \\ \beta_2+\beta_1, & \beta_2+\beta_3, & \beta_2+\beta_1+\beta_3, \\ \beta_2+2\beta_3, & \beta_1+\beta_2+2\beta_3, & \beta_1+2\beta_2+2\beta_3. \end{array}$$

We now verify that the matrix $(\langle\beta_k,\beta_l'\rangle)_{kl}$ corresponds to the Cartan matrix of B_3.

$\langle \beta_1, \beta_2 \rangle = -1$

$\langle \beta_1, \beta_2 \rangle = \langle \alpha_k, \alpha_2 \rangle = -1$

$\langle \beta_1, \beta_3 \rangle = 0$

$\langle \beta_1, \beta_3 \rangle = \langle \alpha_k, (\alpha_i + \alpha_j)/2 \rangle = 2\frac{(\alpha_k,(\alpha_i+\alpha_j)/2)}{((\alpha_i+\alpha_j)/2,(\alpha_i+\alpha_j)/2)}$

$= 4\frac{(\alpha_k,\alpha_i)+(\alpha_k,\alpha_j)}{(\alpha_i,\alpha_i)+(\alpha_i,\alpha_j)+(\alpha_j,\alpha_i)+(\alpha_j,\alpha_j)} = 4\frac{(\alpha_k,\alpha_i)+(\alpha_k,\alpha_j)}{2(\alpha_i,\alpha_i)} = \langle \alpha_k, \alpha_i \rangle + \langle \alpha_k, \alpha_j \rangle = 0$

$\langle \beta_2, \beta_1 \rangle = -1$

$\langle \beta_2, \beta_1 \rangle = \langle \alpha_2, \alpha_k \rangle = -1$

$\langle \beta_2, \beta_3 \rangle = -2$

$\langle \beta_2, \beta_3 \rangle = \langle \alpha_2, (\alpha_i + \alpha_j)/2 \rangle = 2\frac{(\alpha_2,(\alpha_i+\alpha_j)/2)}{((\alpha_i+\alpha_j)/2,(\alpha_i+\alpha_j)/2)}$

$= 4\frac{(\alpha_2,\alpha_i)+(\alpha_2,\alpha_j)}{2(\alpha_i,\alpha_i)} = \langle \alpha_2, \alpha_i \rangle + \langle \alpha_2, \alpha_j \rangle = -2$

$$\langle\beta_3, \beta_1\rangle = 0$$
$$\langle\beta_3, \beta_1\rangle = \langle(\alpha_i + \alpha_j)/2, \alpha_k\rangle = (\langle\alpha_i, \alpha_k\rangle + \langle\alpha_j, \alpha_k\rangle)/2 = (0+0)/2 = 0$$

$$\langle\beta_3, \beta_2\rangle = -1$$
$$\langle\beta_3, \beta_2\rangle = \langle(\alpha_i + \alpha_j)/2, \alpha_2\rangle = \tfrac{1}{2}(\langle\alpha_i, \alpha_2\rangle + \langle\alpha_j, \alpha_2\rangle) = -1$$

This justifies our initial choice for β_1, β_2 and β_3. Now to verify equations $\beta_m(h_{\beta_l}) = \langle\beta_m, \beta_l\rangle$, $[x_{\beta_m}, y_{\beta_m}] = h_{\beta_m}$, $[h_{\beta_m}, x_{\beta_m}] = 2x_{\beta_m}$, $[h_{\beta_m}, y_{\beta_m}] = -2y_{\beta_m}$.

$$\beta_1(h_{\beta_1}) = \langle\beta_1, \beta_1\rangle = 2$$
$$\beta_1(h_{\beta_1}) = \alpha_k(h_{\alpha_k}) = 2$$

$$\beta_2(h_{\beta_2}) = \langle\beta_2, \beta_2\rangle = 2$$
$$\beta_2(h_{\beta_2}) = \alpha_2(h_{\alpha_2}) = 2$$

$$\beta_3(h_{\beta_3}) = \langle\beta_3, \beta_3\rangle = 2$$
$$\beta_3(h_{\beta_3}) = (\alpha_i + \alpha_j)((h_{\alpha_i} + h_{\alpha_j}))/2$$
$$= (\alpha_i(h_{\alpha_i}) + \alpha_i(h_{\alpha_j}) + \alpha_j(h_{\alpha_i}) + \alpha_j(h_{\alpha_j}))/2 = \tfrac{2+0+2+0}{2} = 2$$

$$[x_{\beta_1}, y_{\beta_1}] = h_{\beta_1}$$
$$[x_{\beta_1}, y_{\beta_1}] = [x_{\alpha_k}, y_{\alpha_k}] = h_{\alpha_k} = h_{\beta_1}$$

$$[x_{\beta_2}, y_{\beta_2}] = h_{\beta_2}$$
$$[x_{\beta_2}, y_{\beta_2}] = [x_{\alpha_2}, y_{\alpha_2}] = h_{\alpha_2} = h_{\beta_2}$$

$$[x_{\beta_3}, y_{\beta_3}] = h_{\beta_3}$$
$$[x_{\beta_3}, y_{\beta_3}] = [(x_{\alpha_i} + x_{\alpha_j}), (y_{\alpha_i} + y_{\alpha_j})]$$
$$= ([x_{\alpha_i}, y_{\alpha_i}] + [x_{\alpha_i}, y_{\alpha_j}] + [x_{\alpha_j}, y_{\alpha_j}] + [x_{\alpha_j}, y_{\alpha_j}])$$
$$= (h_{\alpha_i} + 0 + 0 + h_{\alpha_j}) = h_{\beta_3}$$

$$[h_{\beta_1}, x_{\beta_1}] = 2x_{\beta_1}$$
$$[h_{\beta_1}, x_{\beta_1}] = [h_{\alpha_k}, x_{\alpha_k}] = 2x_{\alpha_k} = 2x_{\beta_1}$$

$$[h_{\beta_2}, x_{\beta_2}] = 2x_{\beta_2}$$
$$[h_{\beta_2}, x_{\beta_2}] = [h_{\alpha_2}, x_{\alpha_2}] = 2x_{\alpha_2} = 2x_{\beta_2}$$

$$[h_{\beta_3}, x_{\beta_3}] = 2x_{\beta_3}$$
$$[h_{\beta_3}, x_{\beta_3}] = [(h_{\alpha_i} + h_{\alpha_j}), (x_{\alpha_i} + x_{\alpha_j})]$$
$$= ([h_{\alpha_i}, x_{\alpha_i}] + [h_{\alpha_i}, x_{\alpha_j}] + [h_{\alpha_j}, x_{\alpha_j}] + [h_{\alpha_j}, x_{\alpha_j}])$$
$$= (2x_{\alpha_i} + 0 + 0 + 2x_{\alpha_j}) = 2x_{\beta_3}$$

$$[h_{\beta_1}, y_{\beta_1}] = -2y_{\beta_1}$$
$$[h_{\beta_1}, y_{\beta_1}] = [h_{\alpha_k}, y_{\alpha_k}] = -2y_{\alpha_k} = -2y_{\beta_1}$$

$$[h_{\beta_{-2}}, y_{\beta_{-2}}] = -2y_{\beta_{-2}}$$
$$[h_{\beta_{-2}}, y_{\beta_{-2}}] = [h_{\alpha_{-2}}, y_{\alpha_{-2}}] = -2y_{\alpha_{-2}} = -2y_{\beta_{-2}}$$

$$[h_{\beta_3}, y_{\beta_3}] = -2y_{\beta_3}$$
$$[h_{\beta_3}, y_{\beta_3}] = [(h_{\alpha_i} + h_{\alpha_j}), (y_{\alpha_i} + y_{\alpha_j})]$$
$$= ([h_{\alpha_i}, y_{\alpha_i}] + [h_{\alpha_i}, y_{\alpha_j}] + [h_{\alpha_j}, y_{\alpha_j}] + [h_{\alpha_j}, y_{\alpha_j}])$$
$$= (-2y_{\alpha_i} + 0 + 0 + -2y_{\alpha_j}) = -2y_{\beta_3}$$

Our claim that $L^{\sigma_{ij}}$ is a simple Lie algebra of type B_3, with a canonical basis, is now justified.

Appendix B

Action of ρ_{134} on the realisation of Section 2.2

Here verify that $V_1, \ldots, V_7$ defined in Section 4.5 are invariant under ρ_{134}.

$$\begin{aligned}
\rho_{134}(E_{12}) = {} & \rho_{134}((x_{\alpha_2+\alpha_1+\alpha_3} + x_{\alpha_2+\alpha_1+\alpha_4} - y_{\alpha_2+\alpha_1+\alpha_3} - y_{\alpha_2+\alpha_1+\alpha_4})/4) \\
= {} & (x_{\alpha_2+\alpha_3+\alpha_4} + x_{\alpha_2+\alpha_3+\alpha_1} - y_{\alpha_2+\alpha_3+\alpha_4} - y_{\alpha_2+\alpha_3+\alpha_1})/4 \\
= {} & (E_{12} - E_{34} - E_{58} + E_{67})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{34}) = {} & \rho_{134}((x_{\alpha_2} - x_{\alpha_2+\alpha_3+\alpha_4} - y_{\alpha_2} + y_{\alpha_2+\alpha_3+\alpha_4})/4) \\
= {} & (x_{\alpha_2} - x_{\alpha_2+\alpha_4+\alpha_1} - y_{\alpha_2} + y_{\alpha_2+\alpha_4+\alpha_1})/4 \\
= {} & (-E_{12} + E_{34} - E_{58} + E_{67})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{58}) = \;& \rho_{134}(-(x_{\alpha_2+\alpha_1+\alpha_3} - x_{\alpha_2+\alpha_1+\alpha_4} - y_{\alpha_2+\alpha_1+\alpha_3} + y_{\alpha_2+\alpha_1+\alpha_4})/4) \\
= \;& -(x_{\alpha_2+\alpha_3+\alpha_4} - x_{\alpha_2+\alpha_3+\alpha_1} - y_{\alpha_2+\alpha_3+\alpha_4} + y_{\alpha_2+\alpha_3+\alpha_1})/4 \\
= \;& (E_{12} + E_{34} - E_{58} - E_{67})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{67}) = \;& \rho_{134}((x_{\alpha_2} + x_{\alpha_2+\alpha_3+\alpha_4} - y_{\alpha_2} - y_{\alpha_2+\alpha_3+\alpha_4})/4) \\
= \;& (x_{\alpha_2} + x_{\alpha_2+\alpha_4+\alpha_1} - y_{\alpha_2} - y_{\alpha_2+\alpha_4+\alpha_1})/4 \\
= \;& (E_{12} + E_{34} + E_{58} + E_{67})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{13}) = \;& \rho_{134}(-(x_{\alpha_2+\alpha_3} + x_{\alpha_2+\alpha_4} - y_{\alpha_2+\alpha_3} - y_{\alpha_2+\alpha_4})/4) \\
= \;& -(x_{\alpha_2+\alpha_4} + x_{\alpha_2+\alpha_1} - y_{\alpha_2+\alpha_4} - y_{\alpha_2+\alpha_1})/4 \\
= \;& (E_{13} + E_{68} + E_{24} + E_{57})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{24}) = \;& \rho_{134}(-(x_{\alpha_2+\alpha_1} - x_{\alpha_2+\alpha_1+\alpha_3+\alpha_4} - y_{\alpha_2+\alpha_1} + y_{\alpha_2+\alpha_1+\alpha_3+\alpha_4})/4) \\
= \;& -(x_{\alpha_2+\alpha_3} - x_{\alpha_2+\alpha_3+\alpha_1+\alpha_4} - y_{\alpha_2+\alpha_3} + y_{\alpha_2+\alpha_3+\alpha_4+\alpha_1})/4 \\
= \;& (E_{13} + E_{24} - E_{57} - E_{68})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{57}) = \;& \rho_{134}(-(x_{\alpha_2+\alpha_1} + x_{\alpha_2+\alpha_1+\alpha_3+\alpha_4} - y_{\alpha_2+\alpha_1} - y_{\alpha_2+\alpha_1+\alpha_3+\alpha_4})/4) \\
= \;& -(x_{\alpha_2+\alpha_3} + x_{\alpha_2+\alpha_3+\alpha_4+\alpha_1} - y_{\alpha_2+\alpha_3} - y_{\alpha_2+\alpha_3+\alpha_4+\alpha_1})/4 \\
= \;& (E_{13} - E_{24} + E_{57} - E_{68})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{68}) = \;& \rho_{134}((x_{\alpha_2+\alpha_3} - x_{\alpha_2+\alpha_4} - y_{\alpha_2+\alpha_3} + y_{\alpha_2+\alpha_4})/4) \\
= \;& (x_{\alpha_2+\alpha_4} - x_{\alpha_2+\alpha_1} - y_{\alpha_2+\alpha_4} + y_{\alpha_2+\alpha_1})/4 \\
= \;& (-E_{13} + E_{24} + E_{57} - E_{68})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{14}) &= \rho_{134}(-(x_{\alpha_3}+x_{\alpha_4}-y_{\alpha_3}-y_{\alpha_4})/4) \\
&= -(x_{\alpha_4}+x_{\alpha_1}-y_{\alpha_4}-y_{\alpha_1})/4) \\
&= (E_{14}-E_{23}-E_{56}+E_{78})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{23}) &= \rho_{134}((x_{\alpha_1}-x_{\alpha_2+\alpha_1+\alpha_3+\alpha_4+\alpha_2}-y_{\alpha_1}+y_{\alpha_2+\alpha_1+\alpha_3+\alpha_4+\alpha_2})/4) \\
&= (x_{\alpha_3}-x_{\alpha_2+\alpha_1+\alpha_3+\alpha_4+\alpha_2}-y_{\alpha_3}+y_{\alpha_2+\alpha_1+\alpha_3+\alpha_4+\alpha_2})/4 \\
&= (-E_{14}+E_{23}-E_{56}+E_{78})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{56}) &= \rho_{134}((x_{\alpha_1}+x_{\alpha_2+\alpha_1+\alpha_3+\alpha_4+\alpha_2}-y_{\alpha_1}-y_{\alpha_2+\alpha_1+\alpha_3+\alpha_4+\alpha_2})/4) \\
&= (x_{\alpha_3}+x_{\alpha_2+\alpha_3+\alpha_4+\alpha_1+\alpha_2}-y_{\alpha_3}-y_{\alpha_2+\alpha_3+\alpha_4+\alpha_1+\alpha_2})/4 \\
&= (-E_{14}-E_{23}+E_{56}+E_{78})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{78}) &= \rho_{134}((x_{\alpha_3}-x_{\alpha_4}-y_{\alpha_3}+y_{\alpha_4})/4) \\
&= (x_{\alpha_4}-x_{\alpha_1}-y_{\alpha_4}+y_{\alpha_1})/4 \\
&= (-E_{14}-E_{23}-E_{56}-E_{78})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{15}) &= \rho_{134}(i(x_{\alpha_2+\alpha_1+\alpha_3}+x_{\alpha_2+\alpha_1+\alpha_4}+y_{\alpha_2+\alpha_1+\alpha_3}+y_{\alpha_2+\alpha_1+\alpha_4})/4) \\
&= i(x_{\alpha_2+\alpha_3+\alpha_4}+x_{\alpha_2+\alpha_3+\alpha_1}+y_{\alpha_2+\alpha_3+\alpha_4}+y_{\alpha_2+\alpha_3+\alpha_1})/4 \\
&= (E_{15}+E_{28}-E_{37}+E_{46})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{28}) &= \rho_{134}(i(x_{\alpha_2+\alpha_1+\alpha_3}-x_{\alpha_2+\alpha_1+\alpha_4}+y_{\alpha_2+\alpha_1+\alpha_3}-y_{\alpha_2+\alpha_1+\alpha_4})/4) \\
&= i(x_{\alpha_2+\alpha_3+\alpha_4}-x_{\alpha_2+\alpha_3+\alpha_1}+y_{\alpha_2+\alpha_3+\alpha_4}-y_{\alpha_2+\alpha_3+\alpha_1})/4 \\
&= (-E_{15}-E_{28}-E_{37}+E_{46})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{37}) = &\ \rho_{134}(-i(x_{\alpha_2} + x_{\alpha_2+\alpha_3+\alpha_4} + y_{\alpha_2} + y_{\alpha_2+\alpha_3+\alpha_4})/4) \\
= &\ -i(x_{\alpha_2} + x_{\alpha_2+\alpha_4+\alpha_1} + y_{\alpha_2} + y_{\alpha_2+\alpha_4+\alpha_1})/4 \\
= &\ (-E_{15} + E_{28} + E_{37} + E_{46})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{46}) = &\ \rho_{134}(-i(x_{\alpha_2} - x_{\alpha_2+\alpha_3+\alpha_4} + y_{\alpha_2} - y_{\alpha_2+\alpha_3+\alpha_4})/4) \\
= &\ -i(x_{\alpha_2} - x_{\alpha_2+\alpha_4+\alpha_1} + y_{\alpha_2} - y_{\alpha_2+\alpha_4+\alpha_1})/4 \\
= &\ (E_{15} - E_{28} + E_{37} + E_{46})
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{16}) = &\ \rho_{134}(-i(x_{\alpha_2+\alpha_3} + x_{\alpha_2+\alpha_4} + y_{\alpha_2+\alpha_3} + y_{\alpha_2+\alpha_4})/4) \\
= &\ -i(x_{\alpha_2+\alpha_4} + x_{\alpha_2+\alpha_1} + y_{\alpha_2+\alpha_4} + y_{\alpha_2+\alpha_1})/4 \\
= &\ (E_{16} - E_{38} - E_{45} - E_{27})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{27}) = &\ \rho_{134}(i(x_{\alpha_2+\alpha_1} + x_{\alpha_2+\alpha_1+\alpha_3+\alpha_4} + y_{\alpha_2+\alpha_1} + y_{\alpha_2+\alpha_1+\alpha_3+\alpha_4})/4) \\
= &\ i(x_{\alpha_2+\alpha_3} + x_{\alpha_2+\alpha_1+\alpha_3+\alpha_4} + y_{\alpha_2+\alpha_3} + y_{\alpha_2+\alpha_1+\alpha_3+\alpha_4})/4) \\
= &\ (-E_{16} + E_{27} - E_{38} - E_{45})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{38}) = &\ \rho_{134}(-i(x_{\alpha_2+\alpha_3} - x_{\alpha_2+\alpha_4} + y_{\alpha_2+\alpha_3} - y_{\alpha_2+\alpha_4})/4) \\
= &\ -i(x_{\alpha_2+\alpha_4} - x_{\alpha_2+\alpha_1} + y_{\alpha_2+\alpha_4} - y_{\alpha_2+\alpha_1})/4 \\
= &\ (E_{16} + E_{27} - E_{38} + E_{45})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{45}) = &\ \rho_{134}(i(x_{\alpha_2+\alpha_1} - x_{\alpha_2+\alpha_1+\alpha_3+\alpha_4} + y_{\alpha_2+\alpha_1} - y_{\alpha_2+\alpha_1+\alpha_3+\alpha_4})/4) \\
= &\ i(x_{\alpha_2+\alpha_3} - x_{\alpha_2+\alpha_3+\alpha_4+\alpha_1} + y_{\alpha_2+\alpha_3} - y_{\alpha_2+\alpha_3+\alpha_4+\alpha_1})/4 \\
= &\ (-E_{16} - E_{27} - E_{38} + E_{45})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{17}) = &\ \rho_{134}(-i(x_{\alpha_3} + x_{\alpha_4} + y_{\alpha_3} + y_{\alpha_4})/4) \\
= &\ -i(x_{\alpha_4} + x_{\alpha_1} + y_{\alpha_4} + y_{\alpha_1})/4 \\
= &\ (E_{17} + E_{26} + E_{35} - E_{48})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{26}) = &\ \rho_{134}(-i(x_{\alpha_1} + x_{\alpha_2+\alpha_1+\alpha_3+\alpha_4+\alpha_2} + y_{\alpha_1} + y_{\alpha_2+\alpha_1+\alpha_3+\alpha_4+\alpha_2})/4) \\
= &\ -i(x_{\alpha_3} + x_{\alpha_2+\alpha_1+\alpha_3+\alpha_4+\alpha_2} + y_{\alpha_3} + y_{\alpha_2+\alpha_1+\alpha_3+\alpha_4+\alpha_2})/4 \\
= &\ (E_{17} + E_{26} - E_{35} + E_{48})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{35}) = &\ \rho_{134}(-i(x_{\alpha_1} - x_{\alpha_2+\alpha_1+\alpha_3+\alpha_4+\alpha_2} + y_{\alpha_1} - y_{\alpha_2+\alpha_1+\alpha_3+\alpha_4+\alpha_2})/4) \\
= &\ -i(x_{\alpha_3} - x_{\alpha_2+\alpha_1+\alpha_3+\alpha_4+\alpha_2} + y_{\alpha_3} - y_{\alpha_2+\alpha_1+\alpha_3+\alpha_4+\alpha_2})/4 \\
= &\ (E_{17} - E_{26} + E_{35} + E_{48})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{48}) = &\ \rho_{134}(-i(x_{\alpha_3} - x_{\alpha_4} + y_{\alpha_3} - y_{\alpha_4})/4) \\
= &\ -i(x_{\alpha_4} - x_{\alpha_1} + y_{\alpha_4} - y_{\alpha_1})/4 \\
= &\ (E_{17} - E_{26} - E_{35} - E_{48})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{18}) = &\ \rho_{134}(i(h_{\alpha_3} - h_{\alpha_4})/2) = i(h_{\alpha_4} - ih_{\alpha_1})/2 \\
= &\ (-E_{18} + E_{25} - E_{36} - E_{47})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{25}) = &\ \rho_{134}(-i(2h_{\alpha_1} + 2h_{\alpha_2} + h_{\alpha_3} + h_{\alpha_4})/2) \\
= &\ -i(2h_{\alpha_3} + 2h_{\alpha_2} + h_{\alpha_4} + h_{\alpha_1})/2 \\
= &\ (-E_{18} + E_{25} + E_{36} + E_{47})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{36}) = &\ \rho_{134}(-i(2h_{\alpha_2} + h_{\alpha_3} + h_{\alpha_4})/2) \\
= &\ -i(2h_{\alpha_2} + h_{\alpha_4} + h_{\alpha_1})/2 \\
= &\ (E_{18} + E_{25} + E_{36} - E_{47})/2
\end{aligned}$$

$$\begin{aligned}
\rho_{134}(E_{47}) = &\ \rho_{134}(-i(h_{\alpha_3} + h_{\alpha_4})/2) = -i(h_{\alpha_4} + h_{\alpha_1})/2 \\
= &\ (E_{18} + E_{25} - E_{36} + E_{47})/2
\end{aligned}$$