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Research Article

Starter Labelling of k-Windmill Graphs with Small Defects

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A graph on $2n$ vertices can be starter-labelled, if the vertices can be given labels from the nonzero elements of the additive group $\mathbb{Z}_{2n+1}$ such that each label i, either i or i^{-1}, is assigned to exactly two vertices and the two vertices are separated by either i edges or i^{-1} edges, respectively. Mendelsohn and Shalaby have introduced Skolem-labelled graphs and determined the conditions of k-windmills to be Skolem-labelled. In this paper, we introduce starter-labelled graphs and obtain necessary and sufficient conditions for starter and minimum hooked starter labelling of all k-windmills.

1. Introduction

Consider $\mathbb{Z}_n$ as an additive abelian group of odd order n. A starter in $\mathbb{Z}_n$ is a partition of the nonzero elements of $\mathbb{Z}_n$ into unordered pairs $S = \{\{x_i, y_i\} : i = 1, 2, \ldots, (n-1)/2\}$ such that $\{\pm(x_i - y_i) : 1 \leq i \leq (n-1)/2\} = \mathbb{Z}_n \setminus \{0\}$. Starters were first used by Stanton and Mullin to construct Room squares [1]. Since then, starters have been widely used in several combinatorial designs such as Room cubes [2], Howell designs [3, 4], Kirkman triple systems [5], Kirkman squares and cubes [6, 7], Kotzig factorizations [8, 9], Hamilton path tournament designs [10], and optimal optical orthogonal codes [11]. A starter sequence of order n is an integer sequence; $S = (s_1, s_2, \ldots, s_{2n})$ of $2n$ integers such that, for every $r \in \{1, 2, \ldots, n\}$, we consider either r or r^{-1} such that $s_i = s_j = r$ or r^{-1}, respectively, and if $s_i = s_j = k$ with $i < j$ then $j - i = k$. When r^{-1} is the additive inverse of r in $\mathbb{Z}_{2n+1}$ and if the inverse appears in the sequence, we call it a defect. For example, the sequence 5, 3, 1, 1, 3, 5 is a starter sequence of order 3 with one defect (2^{-1}) in the group $\mathbb{Z}_7$. We notice that Skolem sequences are a special case of starter sequences when the number of defects is zero. It is well known that Skolem sequences and their generalizations have been used widely to construct several designs such as Room squares, one-factorizations, and round robin tournaments. In 1991, Mendelsohn and Shalaby [12] introduced the concept of Skolem labelling and also provided the necessary and sufficient conditions for Skolem labelling of paths and cycles. Eight years later, Mendelsohn and Shalaby [13] determined the condition for the existence of Skolem labelling for k-windmills. In 2008, Baker and Manzer [14] obtained the necessary conditions for the Skolem labelling of generalized k-windmills in which the vanes need not be of the same length and proved that these conditions are sufficient in the case where $k = 3$. In this paper, we introduce the concept of starter labelling of graphs and explore the necessary and the sufficient conditions for the existence of starter and minimum hooked starter labelling of k-windmills. Furthermore, we restate the definitions of starter and hooked starter-labelled graphs.

Definition 1. A starter-labelled graph is a pair (G, L), where

(a) $G = (V, E)$ is an undirected graph,

(b) $L : V \to \mathbb{Z}_{2n+1} \setminus \{0\}$,

(c) $L(v) = L(w) = i$ exactly once for each $i \in \{1(1^{-1}), 2(2^{-1}), \ldots, n(n^{-1})\}$,

(d) if $\widehat{G} = (V, \widehat{E})$ and $\widehat{E} \subset E$ then $(\widehat{G}, L)$ violates (c).

Definition 2. A hooked starter-labelled graph is a pair (G, L) satisfying the conditions of Definition 1 with $(\widehat{\text{b}})$ instated of (b):

$(\widehat{\text{b}})$ $L : V \to \mathbb{Z}_{2n+1}$.

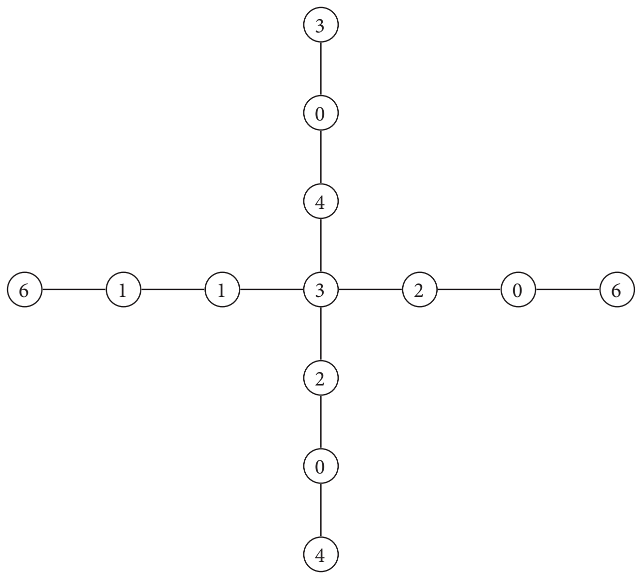

Figure 1: A hooked starter-labelled graph for 4-windmills.

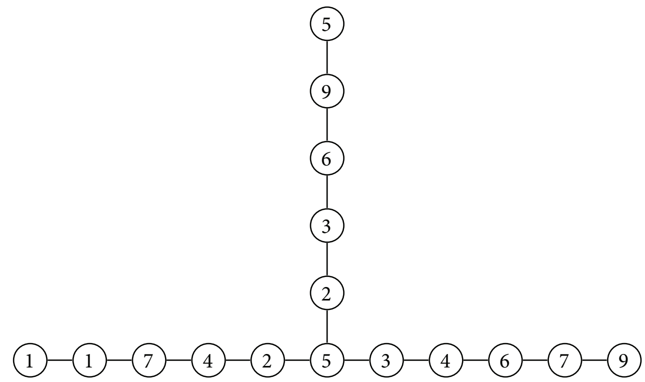

Figure 2: Weak starter-labelled 3-windmills.

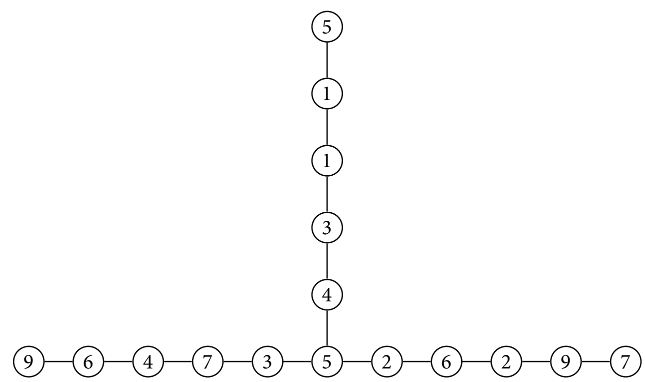

Figure 3: Strong starter-labelled 3-windmills.

Example 3. Figure 1 illustrates a hooked starter-labelled graph for 4-windmills.

According to Definition 2, a hooked starter-labelled graph can have some vertices labelled zero, but every edge is still essential. This leads us to the definition of the strong (weak) starter-labelled graph.

Definition 4. A graph on $2n$ vertices can be strongly starter-labelled if the removal of any edge destroys the starter labelling.

Definition 5. A graph on $2n$ vertices can be weakly starter-labelled if there exists at least one edge in the graph such that the removal of that edge does not destroy the starter labelling.

Example 6. Figures 2 and 3 show weak starter-labelled 3-windmills and strong starter-labelled 3-windmills, respectively.

Definition 7. A k-windmill is a tree containing k paths of equal positive length, called vanes, which share a center vertex called the pivot or the center.

2. Necessity

We notice that a tree $T = (V, E)$ can only be starter-labelled if the number of the vertices is even ($|V| = 2n$). This implies that the length of the vane must be odd and that all k-windmills where k is even cannot be starter-labelled. In addition, an obvious degeneracy condition for a starter-label (a hooked starter-label) of a tree T is that the tree must have a path of length at least $(n + 1)$. Thus, only 3-windmills can be starter-labelled.

2.1. Starter Parity

2.1. Starter Parity. Mendelsohn and Shalaby [13] defined Skolem parity and proved that it was necessary for the existence of any Skolem-labelled tree. Similarly, we establish the parity condition for starter-labelled k-windmills.

Definition 8. The starter parity of a vertex u of a tree $T = (V, E)$ is the sum of the lengths of the paths from u to all the vertices of the tree (T). Thus, $P_u = \sum_{v\in V} d(u, v) \pmod 2$.

Lemma 9 (Mendelsohn and Shalaby [13]). *If T is a tree with $2n$ vertices, then the starter parity of T is independent of $u \in V$.*

Lemma 10. *If G is a starter-label k-windmill with $2n$ vertices and k vanes, then either*

(1) $n \equiv 0, 2 \pmod 4$*, and the starter parity of G is odd, or*

(2) $n \equiv 1, 3 \pmod 4$*, and the starter parity of G is even.*

Proof. Assume that G is a starter-label k-windmill with $2n$ vertices and k vanes of length m. Using the center point c to calculate the starter parity, we obtain

$$P_c = \sum_{v\in V} d(c, v) = \sum_{i=1}^{k} \frac{m(m+1)}{2} = \frac{km^2 - 1}{2} + n. \qquad (1)$$

Since G is starter-labelled, then $k = 3$ and m must be odd ($m \equiv 1$ or $3 \pmod 4$); we notice that if $m \equiv 1 \pmod 4 \Rightarrow 3m^2 \equiv 3 \pmod 4 \Rightarrow 3m^2 - 1 \equiv 2 \pmod 4$. Similarly, if $m \equiv 3 \pmod 4 \Rightarrow 3m^2 \equiv 27 \pmod 4$ and since $27 \equiv 3 \pmod 4$, then $3m^2 - 1 \equiv 2 \pmod 4$ (by the transitivity). Now we consider all the following cases of n:

(1) If $n \equiv 0 \pmod 4$, then $P_c = (1 + 2j) + (4r) \Rightarrow$ the starter parity is odd.

(2) If $n \equiv 1 \pmod 4$, then $P_c = (1 + 2j) + (1 + 4r) \Rightarrow$ the starter parity is even.

(3) If $n \equiv 2 \pmod 4$, then $P_c = (1 + 2j) + (2 + 4r) \Rightarrow$ the starter parity is odd.

(4) If $n \equiv 3 \pmod 4$, then $P_c = (1 + 2j) + (3 + 4r) \Rightarrow$ the starter parity is even.

□

2.2. The Degeneracy Condition. We saw that a graph with $2n$ vertices must have at least a path of length $(n + 1)$ in order to be starter-labelled. Therefore all windmills with more than 3 vanes cannot be labelled by a starter sequence. For a (possibly hooked) starter-label k-windmill with equal vanes of length m, the largest label is $2m$ and the maximum number of edges in the corresponding path not used in any other path is $2m$ and is covering all edges of 2 vanes. Also, labels that are bigger than m must cover parts of 2 vanes. The label m may cover the complete vane. Thus for all labels m_i with $m \le m_i \le 2m$ the maximum number of edges covered is no more than

$$2m + (2m - 1) + \cdots + m = \frac{3\left(m^2 + m\right)}{2}. \qquad (2)$$

Moreover, the labels $n_i < m$ must cover at least one edge that is covered by another label, so the total number of edges for these labels is at most

$$1 + 2 + \cdots + (m - 1) = \frac{m^2 - m}{2}. \qquad (3)$$

Therefore, the maximum number of edges is $\le (2) + (3)$ since the total number of edges in a k-windmill is km; hence $k \le 2m + 1$.

3. Sufficiency

In this section, we provide and prove the sufficient conditions for obtaining the starter-label (minimum hooked starter label) for all k-windmills, where k is the number of the vanes; we count them arbitrarily (say counterclockwise) from 1 to k. Let m indicate the length of the vane of the windmill; then each vertex v can be represented by a pair (i, j) where i is the number of the vane and j is its distance from the center, and the center point is denoted by $(0, 0)$.

3.1. 3-Windmills

Lemma 11. *All 3-windmills with $m \equiv 1, 3, 5, 7 \pmod 8$ have a starter labelling, except for the case $m = 1$.*

Table 1

$b_{i,j}$	$a_{i,j}$	$\le r \le$	Label
$(2, (m-1)/2 + r + 1)$	$(2, (m-1)/2 - r)$	$0 \le r \le \dfrac{m-1}{2}$	$2r + 1$
$(3, r)$	$(1, r)$	$1 \le r \le m$	$2r$

Table 2

$a_{i,j}$	$b_{i,j}$	$\le r \le$	Label
$(2, m/2 - r)$	$(2, m/2 + r + 1)$	$0 \le r \le \dfrac{m}{2} - 1$	$2r + 1$
$(3, m)$	$(0, 0)$	—	m
$(3, r)$	$(1, r)$	$1 \le r \le \dfrac{m}{2} - 1$	$2r$
$(3, r)$	$(1, r + 1)$	$\dfrac{m}{2} \le r \le m - 1$	$2r + 1$

Proof. The required construction is shown in Table 1, where $a_{i,j}$ and $b_{i,j}$ represent the two positions in the windmill of the label k. We notice that the number of the defects is $\lfloor m/4 \rfloor$ in case that $m \equiv 1, 5 \pmod 8$ and $\lceil m/4 \rceil$ in case that $m \equiv 3, 7 \pmod 8$. □

Lemma 12. *For all 3-windmills with vane length $m \equiv 0, 2, 4, 6 \pmod 8$ there is a minimum hooked starter labelling with exactly one hook.*

Proof. The solution is given by Table 2, where the number of the defects is $\lfloor m/4 \rfloor$. □

3.2. 4-Windmills. All 4-windmills have an odd number of vertices, so there is no starter labelling. The minimum hooked starter labelling in this case has at least three hooks.

Lemma 13. *All 4-windmills with $m \ge 2$ have a minimum hooked starter labelling with exactly three hooks.*

Proof. We divide the proof into two cases.

Case 1 (m is odd). The solution is given by Table 3.

Case 2 (m is even). The solution is given by Table 4.

Table 5 provides us with the construction of the pairs $a_{i,j}$ and $b_{i,j}$ for a weak starter labelling of 4-windmills. □

Remark 14. We can construct a hooked starter labelling with zero defects (Skolem labelling) and one hook for all 4-windmills. Tables 6 and 7 provide such a required construction.

Case 1. $m \equiv 0 \pmod 2$ is given by Table 6.

Case 2. $m \equiv 1 \pmod 2$ is given by Table 7.

Table 3

$a_{i,j}$	$b_{i,j}$	$\leq r \leq$	Label
$(3, r)$	$(1, r)$	$1 \leq r \leq m$	$2r$
$(2, m)$	$(0, 0)$	—	m
$(4, r+1)$	$(2, r)$	$1 \leq r < m - \left(\dfrac{m+1}{2}\right)$	$2r+1$
$(4, m-(m+1)/2+1)$	$(4, m-(m+1)/2+2)$	—	1
$(4, r+2)$	$(2, r-1)$	$m - \dfrac{m+1}{2} < r \leq m-2$	$2r+1$

Table 4

$a_{i,j}$	$b_{i,j}$	$\leq r \leq$	Label
$(3, r)$	$(1, r)$	$1 \leq r \leq m$	$2r$
$(4, 1)$	$(2, m)$	—	$m+1$
$(4, r+1)$	$(2, r)$	$1 \leq r < \dfrac{m}{2}$	$2r+1$
$(4, m/2+2)$	$(4, m/2+1)$	—	1
$(4, r+2)$	$(2, r-1)$	$\dfrac{m}{2} < r \leq m-2$	$2r+1$

Table 5

$a_{i,j}$	$b_{i,j}$	$\leq r \leq$	Label
$(3, r)$	$(1, r)$	$1 \leq r \leq m$	$2r$
$(4, r+1)$	$(2, r)$	$0 \leq r \leq m-2$	$2r+1$

Table 6

$a_{i,j}$	$b_{i,j}$	$\leq r \leq$	Label
$(2, r)$	$(1, r)$	$1 \leq r \leq m$	$2r$
$(3, 1)$	$(3, m)$	—	$m-1$
$(4, m/2)$	$(4, m/2-1)$	—	1
$(3, r)$	$(4, r+1)$	$\dfrac{m}{2} \leq r \leq m-1$	$2r+1$
$(3, r+1)$	$(4, r)$	$1 \leq r \leq \dfrac{m}{2} - 2$	$2r+1$

3.3. K-Windmills, $K > 4$. In this case there is no starter labelling; thus the only possibility is a minimum hooked starter labelling.

Lemma 15. *For any k-windmill, the condition $k+1 < 2m$ is sufficient for a minimum hooked starter labelling.*

Proof. Fix m and consider separate cases for k.

Case 1 (the number of vanes is even ($k = 2t$)). Label the vanes $L_1, L_k, L_2, L_{k-1}, \ldots, L_t, L_{t+1}$, and the solution is given by Table 8.

Table 7

$a_{i,j}$	$b_{i,j}$	$\leq r \leq$	Label
$(2, r)$	$(1, r)$	$1 \leq r \leq m$	$2r$
$(0, 0)$	$(3, m)$	—	m
$(3, r)$	$(4, r+1)$	$\dfrac{m+1}{2} \leq r \leq m-1$	$2r+1$
$(3, r+1)$	$(4, r)$	$1 \leq r \leq \dfrac{m-1}{2} - 1$	$2r+1$
$(4, (m-1)/2)$	$(4, (m-1)/2+1)$	—	1

Table 8

$a_{i,j}$	$b_{i,j}$	$\leq r \leq$	Label
(k, m)	$(1, m)$	—	$2m$
$(k-r+1, m-r)$	(r, m)	$2 \leq r \leq t$	$2m-r$
$(k-r+2, m-r)$	$(k-r+2, m)$	$3 \leq r \leq t+1$	r
$(k, r-1)$	$(1, r+1)$	$t+2 \leq 2r \leq 2m-t-1$	$2r$
$(k-1, r-1)$	$(2, r+2)$	$t+2 \leq 2r+1 \leq 2m-t-1$	$2r+1$
$(3, 2)$	$(3, 1)$	—	1
$(4, 2)$	$(4, 0)$	—	2

Case 2 ($k = 2t+1$, $t > 2$). Label the vanes $L_1, L_k, L_2, L_{k-1}, \ldots, L_t, L_{k+1-t}, L_{k-t}$. The required construction is shown in Table 9.

Case 3 ($k = 5$). Label the vanes $L_1, L_2, \ldots, L_5$. The required construction is demonstrated by Table 10. □

4. Future Research

Open questions include

(1) finding the necessary and sufficient conditions for starter labelling of trees,

(2) finding the necessary and sufficient conditions for starter labelling of generalized k-windmills, where $k \geq 3$.

Table 9

$a_{i,j}$	$b_{i,j}$	$\leq r \leq$	Label
(k, m)	$(1, m)$	—	$2m$
$(k - r + 1, m - r)$	(r, m)	$2 \leq r \leq t$	$2m - r$
$(3, m - t - 1)$	$(k - t, m)$	—	$2m - t - 1$
$(k - r + 2, m - r)$	$(k - r + 2, m)$	$3 \leq r \leq t + 1$	r
$(k, r - 1)$	$(1, r + 1)$	$t + 2 \leq 2r < 2m - t - 1$	$2r$
$(k - 1, r - 1)$	$(2, r + 2)$	$t + 2 \leq 2r + 1 < 2m - t - 1$	$2r + 1$
$(4, 1)$	$(0, 0)$	—	1
$(4, 4)$	$(4, 2)$	—	2

Table 10

$a_{i,j}$	$b_{i,j}$	$\leq r \leq$	Label
$(5, m)$	$(1, m)$	—	$2m$
$(4, m - 2)$	$(2, m)$	—	$2m - 2$
$(1, m - 2)$	$(3, m - 1)$	—	$2m - 3$
$(4, m - 3)$	$(4, m)$	—	3
$(5, r)$	$(1, r)$	$4 \leq 2r < 2m - 4$	$2r$
$(4, r - 1)$	$(2, r + 2)$	$4 \leq 2r + 1 < 2m - 4$	$2r + 1$
$(3, m - 3)$	$(1, m - 1)$	—	$2m - 4$
$(5, m - 2)$	$(5, m - 1)$	—	1
$(3, m - 2)$	$(3, m)$	—	2

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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