IDENTIFICATION AND ESTIMATION OF A FIRST ORDER BILINEAR TIME SERIES

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by

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Abstract

In this practicum, we study the properties of a special case of the general bilinear model. The general bilinear model was proposed by Granger and Andersen(1978) and Subba Rao(1981) for studying non-linear time series. Simulation studies and real life data sets are used to evaluate the performance of the theoretical results we derived. The properties we study are the mean, covariance structures, third order moments and cumulants. We find a pattern in the third-order cumulant which is useful in identifying the order of the model. This work is an extension of the result of Oyet(2001). The model is used to make forecasts on three real time series data.

Also considered are the mean and covariance structures of three other versions of the bilinear model.

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PRELIMINARIES

1.1 Introduction

A time series is a collection of observations generated sequentially over time. Examples of time series can be found in every area of human endeavor; from the daily sales of a super market, yearly enrollment in schools, yearly population of a country, to the annual gross national product of a country and so on. Due to the popularity of the subject, time series has received lots of attention in the literature. A list and discussion on recent developments in time series analysis can be found in Subba Rao(1993).

However until recently, most work on time series analysis have been based on the assumption that the series under consideration conforms to a linear model. Contrary to this assumption, recent studies have shown that some data do not conform to linear models. For example, by using tests for linearity proposed by Subba Rao and Gabr(1981), Hinich(1982), Keenan(1985) and Tsay(1986), real time series such as the lynx data and the sunspot numbers have been shown to be non-linear. Needless to say, linear models will not be the best models for analyzing these(non-linear) time series. In view of this, a number of non-linear time series models have been developed to handle the situation when linear models are inadequate. One of such models is the bilinear model proposed by Granger and Andersen(1978) and Subba Rao(1981).

This study is focussed on the first order bilinear model, which shall be called *Autoregressive Pure Bilinear Model* of order (1,1) and denoted by $APBL(1,1)$. This model is the same as the first order bilinear model, $BL(1,0,1,1)$ studied by Andersen and Granger(1978) and a special case of the Subba Rao(1981) general bilinear model. The name "Autoregressive Pure Bilinear" model reflects the fact that the model is made up of both autoregressive and pure bilinear parts. The structure of the general bilinear model, special cases and some specific bilinear models shall be given in Section 1.4. Our goal is to derive some properties of the first order bilinear model and use it for identification, estimation and forecasting. Some properties of special cases of the general bilinear model have been studied extensively by different authors- examples can be found in Oyet(2001), Subba Rao(1981), Subba Rao and Gabr(1984), Pham Dinh(1985), Liu and Brockwell(1988), etc.

Specifically, the standardized third order cumulant for the $APBL(1,1)$ model is of great interest in this study. Oyet(2001) has studied patterns in the third order cumulants of diagonal pure bilinear models and shown their usefulness in order identification. In this work we extend that result to the $APBL(1,1)$ model. The diagonal pure bilinear, $APBL(0,q)$ model is defined by

$$X_t = \sum_{j=1}^{q} \theta_j X_{t-j} e_{t-j} + e_t. \tag{1.1}$$

A summary of the pattern in the cumulants of (1.1) is given in Chapter 2. Similarly, we shall investigate if a pattern that can be used for model identification exist in the $APBL(1,q)$ model

$$X_t = \phi_1 X_{t-1} + \sum_{j=1}^{q} \theta_j X_{t-j} e_{t-j} + e_t$$

The case where $q = 1$ is investigated in this study. It is our conjecture that the cumulants of each of these models, with distinct q, have a unique pattern associated with them.

Suppose we have a series which is steadily increasing over time(i.e, shows trend) and another series which is a monthly data that is showing regular increase(peak) in certain months and decrease(trough) in some other months of the year. In both cases, it would be incorrect to assume that the observed values at each time period is representative of the mean value. Also, if the variance is not constant but, say increases as time goes on, it will be incorrect as well to believe that we can express the uncertainty around a forecasted mean level with a variance calculated based on all the data. Lastly, if the autocorrelation of one half of a series is different from that of the other half, it will be wrong to make predictions for the future using the autocorrelation of the first half. Thus, (see Vandaele(1983)) some restrictions have to be placed on the mean, variance and autocorrelation of a time series process for it to be used in making meaningful forecasts. These restrictions are summarized in what is called *stationarity*. Another restriction on time series process for forecasting is called *invertibility*. The concepts of stationarity and invertibility are discussed in Section 1.2. In Section 1.3 we discuss some methods of linear time series analysis that will be used in later chapters. Finally, the main object of this practicum, the bilinear model is introduced in Section 1.4. The method of parameter estimation for the first order bilinear model as well as the method of order selection shall also be considered in Section 1.4.

The properties of some bilinear models are studied in Chapter 2. The performance of the derived properties shall be evaluated by simulation studies in Chapter 3. Also in Chapter 3, the $APBL(1,1)$ model shall be used to make one-step-ahead forecasts for three real data. We present our findings and summary of this practicum in Chapter 4.

1.2 Stationarity and Invertibility

According to the Box-Jenkins methodology, a good time series for forecasting has to be stationary and invertible. A time series $\{X_t\}$ is said to be stationary if the expected value of $\{X_t\}$ is constant for all t and the covariance matrix $(X_{t_1},\ldots, X_{t_n})$ is the same as the covariance matrix of $(X_{t_{1+h}},\ldots, X_{t_{n+h}})$for all nonempty finite sets of indices $(t_1,t_2,\ldots,t_n)$ and all h such that $(t_1,t_2,\ldots,t_n,t_{1+h},t_{2+h},\ldots,t_{n+h})$ are contained in the index set. The time series is said to be strictly stationary if the joint distributions of $(X_{t_1},\ldots,X_{t_n})'$ and $(X_{t_{1+h}},\ldots,X_{t_{n+h}})'$ are the same for all the integers h, n and $t_1,t_2,\ldots,t_n$.

A model is said to be invertible if it is possible to estimate the e_t sequence from the given X_t values together with an exact knowledge of the generating model. In order words, if X_t are known to obey a model and the values of the parameters of the model are also known, the series is said to be invertible if good estimates of e_t can be derived from the knowledge of X_t and some start up values.

It is interesting to observe that none of the series we have used in our examples are stationary. We shall therefore discuss methods of transforming a non-stationary time series to a stationary one, while emphasis is placed on the methods used in this practicum.

By plotting the series against time, we will be able to observe if the series has a trend, seasonality, discontinuity, outliers and so on. We may then be able to decompose the data as a realization of the process as;

$$X_t = m_t + s_t + y_t$$

where m_t is the trend component, s_t the seasonal component and y_t the stationary component. The deterministic components m_t and s_t can then be estimated and extracted leaving the stationary part for modeling. Sometimes it may not be possible to decompose the series into these components, in which case other methods have to be adopted to transform the non-stationary series to a stationary one. The methods of transforming non-stationary data to stationary data described below are summaries of a few of the methods discussed in the literature. See Brockwell and Davis(1996), and Vandaele(1983) for details.

a) *Stabilization of Variance*

A useful class of variance stabilizing transformations is the Box-Cox transformation. The logarithm and square-root transformations are two useful members of this class. To stabilize the variance across a series, we can take the logarithm or the square-root of each of the observations. If the series contains non-positive observations, we need to add a number to each of the observations to make them positive before taking logarithm.

b) *Removal of the Trend and Seasonal Components*

Some methods of removing trend and seasonality discussed in the literature used in this study are;

i) *Moving Average Filter*: Let q be a non-negative integer, the trend in a series can be estimated in the absence of seasonality using the following expression,

$$\hat{m_t} = (2q+1)^{-1} \sum_{j=-q}^{q} X_{t-j}, \tag{1.2}$$

$$q+1 \leq t \leq n-q.$$

It can be observed that this equation cannot be used for $t \leq q$ or $t > n-q$, since X_t is not observed for $t \leq 0$ or $t > n$. To remedy this, it has been suggested to take $X_t = X_1$ for $t \leq 1$ and $X_t = X_n$ for $t > n$. By using these values, we will have a complete series which will make analysis much easier. However, the first q and the last q trend estimates obtained from using these values may not be as good as the remaining estimates.

ii) *Regression Models*: A regression model

$$X_t = \beta_0 + \sum_{j=1}^{q} \beta_j t^j + Y_t = m_t + Y_t,$$

can be used if the trend is assumed to be a polynomial of order q. The trend estimate is the represented by the deterministic part,

$$\hat{m}_t = \hat{\beta}_0 + \sum_{j=1}^{q} \hat{\beta}_j t^j.$$

iii) *Differencing*: This involves subtracting the values of the observations in a time series from one another in some prescribed time-dependent order. Given the time series $\{X_t\}$, the first order difference is given by;

$$\nabla X_t = X_t - X_{t-1} = (1 - B)X_t$$

where B is the backward shift operator. $B^j X_t = X_{t-j}$, i.e, $BX_t = X_{t-1}$ and the second order difference is defined as,

$$\nabla^2 X_t = \nabla(\nabla X_t) = (1 - B)(1 - B)X_t = X_t - 2X_{t-1} + X_{t-2}.$$

This can be extended to any order k. Suppose

$$X_t = m_t + Y_t,$$

where $m_t = \beta_0 + \beta_1 t$ and Y_t is stationary with mean zero. By applying the ∇ operator to the trend component the linear trend component(increasing or decreasing mean) can be stabilized or made constant as follows:

$$\nabla m_t = m_t - m_{t-1} = \beta_1.$$

In the same manner, a polynomial trend of order k can be reduced to a constant by using the operator ∇^k.

To estimate the seasonal component, the trend has to be estimated first by using appropriate moving average filter. For even period d, let $d = 2q$, then

$$\hat{m_t} = (0.5X_{t-q} + X_{t-q} + \ldots + X_{t+q-1} + 0.5X_{t+q})/d \tag{1.3}$$

$$q < t \leq n - q.$$

For odd periods we take $d = 2q + 1$ and use the moving average filter given in equation(1.2). Next we estimate the seasonal component. For each of $k = 1, 2, \ldots, d$, we compute the average w_k of the deviations

$$X_{k+jd} - \hat{m}_{k+jd}, q < k + jd \leq n - q.$$

In order for the average of the seasonal effect to be zero, the seasonal component is estimated by, $\hat{S}_k = w_k - d^{-1}\sum_{i=1}^{d} w_i$, $k = 1, 2, \ldots, d$ and $\hat{S}_k = \hat{S}_{k-d}$, $k > d$. We can then define the deseasonalized data as, $d_t = X_t - \hat{S}_t$, $t = 1, 2, \ldots, n$. Finally the trend of the deseasonalized data is estimated using any of the methods discussed earlier.

1.3 Linear Time Series Analysis

This section discusses briefly the three major linear models; autoregressive, moving average and mixed autoregressive moving average models. We discuss here stationarity and invertibility conditions and Box-Jenkins procedures for linear models.

a) Autoregressive(AR) Model

An autoregressive process of order p, denoted by $AR(p)$ is given by;

$$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + e_t \tag{1.4}$$

where, e_t is white noise distributed as $N(0, \sigma^2)$, ϕ_j are the parameters of the autoregressive model that need to be estimated, and X_s is uncorrelated with e_t for $s < t$. Finite order autoregressive processes are usually invertible by virtue of the expression (1.4). The stationarity condition for (1.4) is obtained by writing (1.4) in terms of the e_t's and seeking the condition under which the resulting infinite series will converge. Suppose the $AR(p)$ process is rewritten using the backward shift operator as

$$\Phi_p(B)X_t = e_t$$

where, $\Phi_p(B) = 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p$ and $B^j X_t = X_{t-j}$.
We now write X_t in terms of e_t as

$$X_t = \Phi_p^{-1}(B)e_t.$$

The series $\Phi_p^{-1}(B)e_t$ converges if the roots of $\Phi_p^{-1}(B) = 0$ are less than 1. In other words, the $AR(p)$ process is said to be stationary if the roots of the equation $\Phi_p(B) = 0$ lie outside of the unit circle.

For example given an $AR(1)$, $\Phi(B) = 1 - \phi_1 B$, the $AR(1)$ will be stationary if $|B| > 1$, that is when $|\phi_1| < 1$.

b) Moving Average(MA) Model

A series is said to satisfy a moving average process of order q if it can be represented as;

$$X_t = \sum_{j=0}^{q} \theta_j e_{t-j} \tag{1.5}$$

where $\theta_0 = 1$, θ_j, $j = 1, 2, \ldots, q$ are the parameters of the MA process, e_t is white noise distributed as $N(0, \sigma^2)$ and $\sum_{j=0}^{q} \theta_j < \infty$.

Similarly moving average processes are usually stationary by virtue of the expression (1.5). The condition for invertibility of the MA process is stated below. Using the backward shift operator, we have,

$$X_t = \Theta(B)e_t,$$

where $\Theta(B) = \sum_{j=0}^{q} \theta_j B^j$.

Similarly for an MA process to be invertible, the roots of $\Theta(B) = 0$, must lie outside the unit circle. That is, for an $MA(1)$, $\Theta(B) = \theta_1 B$, the condition for invertibility is that $\theta_1 < 1$.

c) Mixed Autoregressive Moving Average(ARMA) Process

The $ARMA(p, q)$, represents a process with an autoregressive term of order p and moving average of order q. It can be written using the backward shift notations as

$$\Phi_p(B)X_t = \Theta_q(B)e_t \qquad (1.6)$$

where $\Phi_p(B) = 1 - \phi_1 B - \ldots - \phi_p B^p$ and $\Theta_q(B) = 1 - \theta_1 B - \ldots - \theta_q B^q$

For the process to be invertible, the roots of $\Theta_q(B) = 0$ must lie outside of the unit circle. Likewise for the process to be stationary, the roots of the $\Phi_p(B) = 0$ must lie outside of the unit circle.

To identify the $AR(p)$ and $MA(q)$ models presented above we use the autocorrelation function(ACF) and partial autocorrelation function(PACF). Several studies on linear time series analysis have shown that the $PACF$ of an $AR(p)$ model cuts off after lag p while the ACF decays exponentially. On the other hand, the ACF of an $MA(q)$ model cuts off after lag q and the $PACF$ tails off. Therefore we can use the $PACF$ and ACF to identify $AR(p)$ and $MA(q)$ models respectively.

For detailed discussion of linear time series analysis, interested readers could read one or more of the following books; Andersen(1971), Box and Jenkins(1970), Billinger(1975), Chatfield(1975), Pankratz(1983), Priestley(1981) and Vandaele(1983).

1.4 Bilinear Time Series Models

A time series $\{X_t\}$ is said to be bilinear if it satisfies the difference equation.

$$X_t + \sum_{j=1}^{p} a_j X_{t-j} = \sum_{j=0}^{q} c_j e_{t-j} + \sum_{i=1}^{m}\sum_{j=1}^{k} b_{ij} X_{t-i} e_{t-j} \tag{1.7}$$

where $c_0 = 1$, e_t, is a sequence of independently and identically distributed random variables, with $E(e_t) = 0$ and $E(e_t^2) = \sigma^2 < \infty$. Using the notation Subba Rao(1981), the model(1.7) can be denoted by $BL(p, q, m, k)$.

If we set $b_{ij} = 0$, for all i, j, then (1.7) reduces to $ARMA(p, q)$. Thus the bilinear model is an extension of the $ARMA$ process and the $ARMA$ process can be seen as a special case of the bilinear model. Three other special cases of the bilinear model are:

(i) If $b_{i,j} = 0$ for all $i < j$ in (1.7) we have the super-diagonal model,

(ii) if $b_{i,j} = 0$ for all $i \geq j$, we have the sub-diagonal model and

(iii) when $b_{i,j} = 0$ for all $i \neq j$, the diagonal model is obtained.

Subba Rao and Gabr(1984) have studied some properties of these models in details.

This study shall examine cases of (1.7) when $p = 1$,$q = 0$ and $m(= k) = 1, 2, 3$ and any nonnegative integer q. In what follows, a_j shall be replaced by ϕ_j, c_j shall be ommitted and b_{ij} shall be replaced by θ_j. Thus when $p = 1$, $q = 0$, $m = k = 1$, we obtain the first order bilinear model, $BL(1, 0, 1, 1)$, which shall be denoted by

$APBL(1,1)$ in this study. The expression for the $APBL(1,1)$ is given by;

$$X_t = \phi_1 X_{t-1} + \theta_1 X_{t-1} e_{t-1} + e_t. \tag{1.8}$$

This model is the object of this study. Some attention is also paid to cases, $APBL(1,2)$, $APBL(1,3)$ and $APBL(1,q)$ models, for arbitrary $q \in Z$. By using similar nomenclature as $APBL(1,1)$ these models are given below:

APBL(1,2):

$$X_t = \phi_1 X_{t-1} + \theta_{t-1} X_{t-1} e_{t-1} + \theta_{t-2} X_{t-2} e_{t-2} + e_t. \tag{1.9}$$

APBL(1,3):

$$X_t = \phi_{t-1} X_{t-1} + \theta_{t-1} X_{t-1} e_{t-1} + \theta_{t-2} X_{t-2} e_{t-2} + \theta_{t-3} X_{t-3} e_{t-3} + e_t. \tag{1.10}$$

APBL(1,q):

$$X_t = \phi_1 X_{t-1} + \sum_{j=1}^{q} \theta_j X_{t-j} e_{t-j} + e_t. \tag{1.11}$$

1.4.1 Conditions for Stationarity and Invertibility

Given the model (1.7), Pham and Tran(1981) used Markovian representation of the model to show that the condition for stationarity of the process X_t is given by $a_1^2 + \sigma^2 b_{11}^2 < 1$. This equivalent to $\phi_1^2 + \sigma^2\theta_1^2$ in the $APBL(1,1)$ model.

We state below the condition for the general time series (1.7) to be asymptotically second-order stationary according to Subba Rao(1981). Consider the bilinear model $BL(p,0,1)$ i.e:

$$X_t + \sum_{j=1}^{p} a_j X_{t-j} = e_t + \sum_{i=1}^{p} b_{i1} X_{t-i} e_{t-1} \tag{1.12}$$

Define the following matrices,

$$
A_{pxp} = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_p \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & & & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},
$$

$$
B_{pxp} = \begin{bmatrix} b_{11} & b_{21} & \dots & b_{p1} \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

where $C_{px1} = (1, 0, 0, \dots, 0)'$ and $H_{1xp} = (1, 0, 0, \dots, 0)'$.

Let $x_t = (X_t, X_{t-1}, \dots, X_{t-p+1})'$. The bilinear model (1.12) can be re-written using the above matrices in the vector form denoted by VBL(p) as,

$$
x_t = Ax_{t-1} + Bx_{t-1}e_{t-1} + Ce_t \tag{1.13}
$$

$$
X_t = H'x_t
$$

Using the above notations, Subba Rao(1981) showed that the sufficient condition for the time series X_t generated from (1.13) to be asymptotically second-order stationary is that

$$
\rho(A \bigotimes A + B \bigotimes B\sigma_e^2) < 1,
$$

where $\otimes$ is the kronecker product and $\rho(.)$ is the spectral radius.

To obtain the condition for invertibility for the bilinear model, Subba Rao(1981) made use of a more general definition of invertibility provided by Granger and Andersen(1978). They defined invertibility as follows. Suppose X_t is a time series satisfying the model,

$$X_t = f\{(X_{t-j}, e_{t-j}), j = 1, 2, \ldots, q\} + e_t \tag{1.14}$$

where $\{e_t\}$ are independent random variables. Since the random variables $\{e_t\}$ are not observable, they are "estimated" by $\hat{e}_t$ by taking the initial values of $\hat{e}_t$ to be zero. The model (1.14) is said to be invertible if,

$$\lim_{t \to \infty} E\{e_t - \hat{e}_t\}^2 \to 0$$

when the model and the parameters are known completely. And by defining the model (1.12) as $VBL(p)$ given by (1.13), the condition for invertibility can be obtained as, $H'BE(x_t \in x_t')B'H < (H'C)^2$ for the model (1.14)

1.4.2 Order Selection

To select the right order of the bilinear model to fit to a set of data, Subba Rao(1981) provided an algorithm which involves using the Akaike's criterion(AIC). Generally, what this method suggests is that we set upper bounds to p, m, k and then search over all combinations of p, m, k within these bounds. The combination with the minimum AIC is chosen as the best model. The limitation of this method is that we do not know when the minimum AIC will be obtained and thus do not know when to stop. That makes the method quite tedious to implement.

In this study, we propose a simpler method of order selection for $APBL(1,1)$ based on the standardized third order cumulants $\hat{\rho}(1,k)$. First the data is made stationary and then the plot of the $\hat{\rho}(1,k)$ is observed. The right model for the data is then selected based on the pattern in the $\hat{\rho}(1,k)$ plot as compared to that of $APBL(1,1)$.

1.4.3 Estimation of the Parameters

In order to use the model for forecasting, we need to obtain the estimates of the parameters of the model. We only state the method of estimating the parameters for the $APBL(1,1)$ model here. The method of parameters estimation for the general bilinear model $BL(p,q,m,k)$ can be found in Subba Rao(1981). We adapt the method to the first order bilinear model;

$$X_t = \phi_1 X_{t-1} + \theta_1 X_{t-1} e_{t-1} + e_t \tag{1.15}$$

where $\{e_t\}$ are independent and each of e_t is distributed $N(0,\sigma^2)$. We assume the series is invertible and we have a realization $\{X_1, X_2, \ldots, X_n\}$ on the time series $\{X_t\}$.

In obtaining the parameter estimates we used the method suggested by Subba Rao(1981), with $p = m = k = 1$ and $q = 0$ in (1.7). The likelihood function of $\{X_1, X_2, \ldots, X_n\}$ is the same as the joint density function of $\{e_2, e_3, \ldots, e_n\}$ and is given by;

$$\frac{1}{(2\pi\sigma_e^2)^{(n-1)/2}} exp\{-\frac{1}{\sigma_e^2}\sum_{t=2}^{n} e_t^2\}.$$

To obtain the parameter estimates we need to maximize the likelihood function, which is equivalent to minimizing, $Q(\theta^*)$ with respect to θ^* where $Q(\theta^*) = \sum_{t=2}^{n} e_t^2$ and $\theta^* = (\phi_1, \theta_1)$.

The values of θ^* are obtained using the Newton-Raphson iterative techniques. The partial derivative of $Q(\theta^*)$ are given by;

$$\frac{\partial^2 Q(\theta^*)}{\partial \phi_1} = 2\sum_{t=2}^{n} \frac{e_t \partial e_t}{\partial \phi_1} = 2\sum_{t=2}^{n} X_{t-1} e_t.$$

$$\frac{\partial^2 Q(\theta^*)}{\partial \theta_1} = 2\sum_{t=2}^{n} \frac{e_t \partial e_t}{\partial \theta_1} = 2\sum_{t=2}^{n} X_{t-1} e_{t-1} e_t.$$

We assumed $e_1 = 0$, thus ;

$$\frac{\partial e_1}{\partial \phi_1} = \frac{\partial e_1}{\partial \theta_1} = \frac{\partial^2 e_1}{\partial \phi_1 \theta_1} = 0.$$

Define

$$G'(\theta^*) = \begin{bmatrix} \frac{\partial Q(\theta^*)}{\partial \phi_1} & \frac{\partial Q(\theta^*)}{\partial \theta_1} \end{bmatrix},$$

$$H(\theta^*) = \begin{bmatrix} \frac{\partial^2 Q(\theta^*)}{\partial \phi_1^2} & \frac{\partial^2 Q(\theta^*)}{\partial \phi_1 \theta_1} \\ \frac{\partial^2 Q(\theta^*)}{\partial \phi_1 \theta_1} & \frac{\partial^2 Q(\theta^*)}{\partial \theta_1^2} \end{bmatrix}.$$

That is, $H(\theta^*)$ is a matrix of second order partial derivatives. Expanding near $\hat{\theta^*} = \theta^*$ in a Taylor series, we have;

$$[G(\theta^*)]_{\hat{\theta^*} = \theta^*} = 0 = G(\theta^*) + H(\theta^*)(\hat{\theta^*} - \theta^*).$$

This implies;

$$\hat{\theta^*} = \theta^* - H^{-1}(\theta^*)G(\theta^*).$$

From above, we have the Newton-Raphson iterative equation,

$$(\theta^*)^{k+1} = (\theta^*)^k - H^{-1}((\theta^*)^k)G((\theta^*)^k),$$

where $(\theta^*)^k$ is the set of estimates obtained at the k^{th} stage of iteration. It follows that by starting with some initial values for the parameters to be estimated, we can iterate to convergence using the equations above to obtain the parameters estimate of the bilinear model. In obtaining the parameter estimates in each of our examples we tried different values of initial parameters and the parameter estimates turned out to be the same.

Chapter 2

PROPERTIES OF THE $APBL$ MODELS

2.1 Introduction

This chapter, can be split into two parts. In the first part which involves the model of interest, we shall obtain expressions for the mean, the covariance structures, third order moments and third order cumulants. This research work is devoted to the model (1.8). Thus properties derived in this part form the core elements of this study. The model shall be denoted by $APBL(1,1)$ as in Chapter 1.

In the second part of this chapter, the expressions for the means and the covariance structures of some other versions of the bilinear models denoted by $APBL(1,2)$, $APBL(1,3)$ and $APBL(1,q)$ are derived. These models were given by equations (1.9) (1.10) and (1.11) respectively. The purpose of this second part is to investigate whether some pattern found in the $APBL(1,1)$ model also exist in more complicated versions of the bilinear model. For this reason, some of the results in this part are only partially derived.

In obtaining the expressions for the mean, moments and cumulants, we shall use the following assumptions and conditions:

- Stationarity and invertibility are assumed. Thus for a unique t and h, $E(X_t) = E(X_h)$, $E(X_t e_t) = E(X_h e_h)$, $E(X_t^2) = E(X_h^2)$, $E(X_t^2 e_t^2) = E(X_h^2 e_h^2)$, and so on.

- The random variable e_t is a series of independent and identically distributed Gaussian random variables. It can be shown that $E(e_t^u) = 0$, for $u = 2j+1$, $j = 0, 1, 2, \ldots$, and for any t, $E(e_t^2) = \sigma^2$, $E(e_t^4) = 3\sigma^4$, $E(e_t^6) = 15\sigma^6$, etc.

- And by expression (1.8) the random error, e_t, is independent of X_h for $h < t$, that is, $E(X_h^u e_t^r) = E(X_h^u)E(e_t^r)$, $h < t$.

Defining the third order cumulant $C(k_1, k_2)$ of a process X_t by $C(k_1, k_2) = E[(X_t - \mu)(X_{t+k_1} - \mu)(X_{t+k_2} - \mu)]$, we shall also use some symmetric relationships derived by Gabr(1988) in this chapter. Gabr(1988) has shown that the cumulants $C(k_1, k_2)$ of a real valued process X_t has the following symmetric relationship;

$$C(k_1, k_2) = C(k_2, k_1) = C(-k_1, k_2 - k_1) = C(k_1 - k_2, -k_2).$$

This shows that, once the value of $C(k_1, k_2)$ in the upper half of the quadrant is known, we can extend to the entire Euclidean plane, using the symmetry property. Thus, we shall derive the $C(k_1, k_2)$ for $k_1 = k_2 = k$ and $k_2 > k_1$ only. Oyet(2001) shows that for the diagonal pure bilinear model(1.1),
$C(k_1, k_2) = 0$ for $k_1 \leq q$, $k_2 - k_1 > q$, and $C(k_1, k_2)$ is nonzero for $k_1 > q$, and $k_2 - k_1 \geq q$ when $k_2 > k_1$. The pattern exhibited by $C(k_1, k_2)$ can be summarized as;

Table 2.1: $C(k_1, k_2)$ for arbitrary q

k_2	1	2	3	...	q	$q+1$	$q+2$	$q+3$	...
$k_1 = 1$	nz	nz	nz	...	nz	nz	0	0	

where nz denotes nonzero values. It is obvious from these patterns that $C(k_1, k_2)$ cuts off after lag $q+1$. Thus the standardized cumulant $\rho(k_1, k_2) = C(k_1, k_2)/C(0,0)$ can be used for diagonal pure bilinear model identification.

The expression for the mean and covariance structure of some bilinear models are derived in Section 2.2 by taking expectations and using the assumptions stated above. In Section 2.3, expressions for the third order moments and third order cumulants of the $APBL(1,1)$ model are obtained.

2.2 Mean and Covariance Structure

2.2.1 Mean

For each of the four models examined, the expression for the means are presented below. Given the $APBL(1,1)$ model,

$$X_t = \phi_1 X_{t-1} + \theta_1 X_{t-1} e_{t-1} + e_t, \tag{2.1}$$

we have

$$E(X_t) = \phi_1 E(X_{t-1}) + \theta_1 E(X_{t-1} e_{t-1}).$$

Now,

$$E(X_{t-1} e_{t-1}) = E(\phi_1 X_{t-2} e_{t-1}) + \theta_1 E(X_{t-2} e_{t-2} e_{t-1}) + E(e_{t-1}^2)$$

which by assumptions in Section 2.1 yields;

$$E(X_{t-1} e_{t-1}) = E(e_{t-1}^2) = \sigma^2.$$

Thus the mean of X_t satisfying the $APBL(1,1)$ model is

$$E(X_t) = \frac{\theta_1 \sigma^2}{1-\phi_1}. \tag{2.2}$$

Following the same procedure and using the fact that $E(X_{t-2}e_{t-2}) = E(X_{t-3}e_{t-3}) = \sigma^2$, we find that the mean of X_t satisfying $APBL(1,2)$ model is

$$E(X_t) = \frac{(\theta_1+\theta_2)\sigma^2}{1-\phi_1} \tag{2.3}$$

and the mean of X_t satisfying $APBL(1,3)$ model is

$$E(X_t) = \frac{(\theta_1+\theta_2+\theta_3)\sigma^2}{1-\phi_1}. \tag{2.4}$$

The technique can be extended to the more general model $APBL(1,q)$,

$$X_t = \phi_1 X_{t-1} + \sum_{i=1}^{q} \theta_i X_{t-i} e_{t-i} + e_t \tag{2.5}$$

to obtain $E(X_{t-i}e_{t-i}) = E(e_{t-i}^2) = \sigma^2$. It follows that the mean of X_t satisfying $APBL(1,q)$ is

$$E(X_t) = \frac{\sigma^2 \sum_{i=1}^{q} \theta_i}{1-\phi_1}. \tag{2.6}$$

2.2.2 Covariance Structure

In what follows, we derive the second moment, $m(k) = E(X_t X_{t+k})$ for each of the four models studied. The expression for the covariance structure $R(k) = m(k) - \mu^2$ can then be obtained by making relevant substitution in $R(k)$ for the model in question.

APBL(1,1) Model

To obtain the second moments of the $APBL(1,1)$ model, we shall use the following expressions which can be derived easily by taking expectations and using the assumptions in Section 2.1.

$$E(X_t^2 e_t) = \frac{2\theta_1\sigma^4}{1-\phi_1} = 2\mu\sigma^2. \tag{2.7}$$

$$E(X_t^2 e_t^2) = \frac{\phi_1^2\sigma^2 E(X_{t-1}^2) + 3\sigma^4 + 4\phi_1\theta_1\theta_0\sigma^6}{1-\theta_1^2\sigma^2}. \tag{2.8}$$

where,

$$\theta_0 = \frac{\theta_1}{1-\phi_1}. \tag{2.9}$$

<u>Case 1: $k=0$</u>

When $k=0$, we have

$$\begin{aligned} m(0) &= \phi_1^2 E(X_{t-1}^2) + \theta_1^2 E(X_{t-1}^2 e_{t-1}^2) + E(e_t^2) + 2\phi_1\theta_1 E(X_{t-1}^2 e_{t-1}) \\ &+ 2\theta_1 E(X_{t-1}e_{t-1}e_t) + 2\phi_1 E(X_{t-1}e_t). \end{aligned}$$

By using the expressions above, the second moment for the $APBL(1,1)$ model when $k=0$ can be expressed as,

$$m(0) = \frac{2\theta_1^2\sigma^4 + \sigma^2 + 4\phi_1\theta_1\theta_0\sigma^4}{1-\phi_1^2-\theta_1^2\sigma^2}. \tag{2.10}$$

<u>Case 2: $k>0$</u>

When $k=1$, the second moment is given by

$$E(X_t X_{t+1}) = \phi_1 E(X_t^2) + \theta_1 E(X_t^2 e_t).$$

It follows that

$$m(1) = \phi_1 m(0) + 2\theta_1\sigma^2\mu. \tag{2.11}$$

One useful expression for obtaining the second moment of the $APBL(1,1)$ model when $k>0$ is $E(X_{t+k-1}e_{t+k-1}X_t) = E(X_t e_{t+k-1}^2) = \theta_0\sigma^4 = \sigma^2\mu$, where θ_0 is given by (2.9). When $k = 2$, it is not too difficult to verify that $m(2) = E(X_t X_{t+2}) = \phi_1 E(X_t X_{t+1}) + \theta_1 E(X_{t+1} e_{t+1} X_t)$. And by making relevant substitutions, we have

$$m(2) = \phi_1 m(1) + \theta_1 \theta_0 \sigma^4 = \phi_1 m(1) + \theta_1 \sigma^2 \mu. \tag{2.12}$$

For general k, the structure of the second moment is given in the Lemma 2.1 below.

Lemma 2.1 *For any nonnegative integer valued $k > 1$, the second moment of X_t satisfying the $APBL(1,1)$ model is given by the difference equation,*

$$m(k) = \phi_1 m(k-1) + \theta_1 \mu \sigma^2. \tag{2.13}$$

So that $R(k) = m(k) - \mu^2 = \phi_1 R(k-1)$.

The proof follows directly from using the results above. Let $\hat{\rho}(k) = \hat{R}(k)/\hat{R}(0)$ be the estimate of the autocorrelation function at lag k. One useful consequence of this result is that an initial estimate of ϕ_1 can be obtained from $\hat{\phi}_1 = \hat{\rho}(k)/\hat{\rho}(k-1)$ for iterative estimation of the parameters when dealing with a real time series. This can be seen in the results of the simulation study in Chapter 3.

APBL(1,2) Model

Given the $APBL(1,2)$ model, the expressions for the second moment of X_t can be derived by using the preliminary results below.

$$E(X_t e_t X_{t-1} e_{t-1}) = \sigma^2 E(X_{t-1} e_{t-1}) = \sigma^4. \tag{2.14}$$

$$E(X_t^2 e_t) = \frac{2(\theta_1 + \theta_2)\sigma^4}{1 - \phi_1} = 2\mu\sigma^2. \tag{2.15}$$

$$\begin{aligned} E(X_t^2 e_t^2) &= \{\phi_1^2 \sigma^2 (1-\phi_1) m(0) + 3\sigma^4 (1-\phi_1) + 4\phi_1 \theta_1^2 \sigma^6 + 2\phi_1 \theta_1 \theta_2 \sigma^6 \\ &+ 4\phi_1^2 \theta_1 \theta_2 \sigma^6 + 2\phi_1^2 \theta_2^2 \sigma^6 + 2\phi_1 \theta_2^2 \sigma^6 + 2\theta_1 \theta_2 \sigma^6\} \end{aligned}$$

$$/ \quad ((1-\phi_1)(1-\theta_1^2\sigma^2-\theta_2^2\sigma^2-2\phi_1\theta_1\sigma^2)). \tag{2.16}$$

Case 1: $k=0$

The second moment for the $APBL(1,2)$ model, when $k=0$ can be expressed as,

$$\begin{aligned} E(X_t^2) &= \phi_1^2 E(X_{t-1}^2)+(\theta_1^2+\theta_2^2)E(X_t^2e_t^2)+\sigma^2+2\phi_1\theta_1E(X_{t-1}^2e_{t-1}) \\ &+ 2\phi_1\theta_2E(X_{t-1}X_{t-2}e_{t-2})+2\theta_1\theta_2E(X_{t-1}e_{t-1}X_{t-2}e_{t-2}) \end{aligned}$$

where, $E(X_tX_{t-1}e_{t-1})=\phi_1E(X_{t-1}^2e_{t-1})+\theta_1E(X_{t-1}^2e_{t-1}^2)+\theta_2\sigma^4$.

By substituting in the preliminary results and using the assumptions in Section 2.1, it can be verified that

$$\begin{aligned} m(0) &= \{2\theta_1^2\sigma^4-2\phi_1\theta_1^2\sigma^4+4\phi_1\theta_1^4\sigma^6+4\phi_1\theta_1^3\theta_2\sigma^6+12\phi_1^2\theta_1^3\theta_2\sigma^6+12\phi_1^2\theta_1^2\theta_2^2\sigma^6 \\ &+ 8\phi_1\theta_1^2\theta_2^2\sigma^6+2\theta_2^2\sigma^4+4\phi_1\theta_1\theta_2^3\sigma^6+4\phi_1^2\theta_1\theta_2^3\sigma^6+4\phi_1^2\theta_2^4\sigma^6+4\phi_1\theta_1\theta_2\sigma^4 \\ &- 4\phi_1^2\theta_1\theta_2\sigma^4+8\phi_1^3\theta_1^2\theta_2^2\sigma^6+8\phi_1^3\theta_1\theta_2^3\sigma^6+\sigma^6-\phi_1\sigma^2-2\phi_1^2\theta_2^2\sigma^4+2\theta_1\theta_2\sigma^4 \\ &- 4\theta_1^2\theta_2^2\sigma^6-2\phi_1\theta_1\theta_2\sigma^4\} \\ &/ \quad \{(1-\phi_1)(1-\theta_1^2\sigma^2-\theta_2^2\sigma^2-2\phi_1\theta_1\theta_2\sigma^2-\phi_1^2)\}. \end{aligned} \tag{2.17}$$

Case 2: $k>0$

When $k=1$, $E(X_{t+1}X_t)=\phi_1E(X_t^2)+\theta_1E(X_t^2e_t)+\theta_2E(X_tX_{t-1}e_{t-1})$
where $E(X_tX_{t-1}e_{t-1})=\phi_1E(X_{t-1}^2e_{t-1})+\theta_1E(X_{t-1}^2e_{t-1}^2)+\theta_2E(X_{t-1}e_{t-1}X_{t-2}e_{t-2})$.
This implies that
$m(1)=\phi_1m(0)+(\theta_1+\phi_1\theta_2)E(X_t^2e_t)+\theta_1\theta_2E(X_t^2e_t^2)+\theta_2^2\sigma^4$.
Thus an expression for the second moment when $k=1$ is given by;

$$m(1)=\phi_1m(0)+\theta_1\theta_2E(X_t^2e_t^2)+\frac{2\theta_1^2\sigma^4+2\theta_1\theta_2\sigma^4+2\phi_1\theta_1\theta_2\sigma^4+2\phi_1\theta_2^2\sigma^4}{1-\phi_1}+\theta_2^2\sigma^4. \tag{2.18}$$

Using similar procedure, as for the $m(1)$, we obtained expressions for $m(2)$ and $m(3)$.

$$m(2) = \phi_1 m(1) + (\theta_1^2\sigma^4 + 3\theta_1\theta_2\sigma^4 + 2\theta_2^2\sigma^4)/(1-\phi_1). \tag{2.19}$$

$$m(3) = \phi_1 m(2) + (\theta_1^2\sigma^4 + 2\theta_1\theta_2\sigma^4 + \theta_2^2\sigma^4)/(1-\phi_1). \tag{2.20}$$

The expression for the second order moment for any nonnegative integer valued k is given in the lemma below.

Lemma 2.2 *For any nonnegative integer valued $k > 2$, the difference equation for the second moment of X_t satisfying the $APBL(1,2)$ model is*

$$m(k) = \phi_1 m(k-1) + \frac{\theta_1^2\sigma^4 + 2\theta_1\theta_2\sigma^4 + \theta_2^2\sigma^4}{1-\phi_1}. \tag{2.21}$$

Also, $R(k) = \phi_1 R(k-1)$.

Again the proof follows from the expression for $E(X_tX_{t+1})$ and the preliminary results.

APBL(1,3) Model

In obtaining the expression for the second moment of the $APBL(1,3)$ model, we shall use the following preliminary results;

$$E(X_{t+k}e_{t+k}X_{t+k-1}e_{t+k-1}) = \sigma^4.$$

$$E(X_t^2e_t) = \frac{2(\theta_1+\theta_2+\theta_3)\sigma^4}{1-\phi_1} = 2\mu\sigma^2,$$

and $E(X_{t+k-i}e_{t+k-i}X_t) = E(X_te_{t+k-i}^2) = \mu\sigma^2$, for $i < k$. Also we shall denote the moment of the model when $k = 0$ by $m(0)$. When $k = 1$, the second moment is obtained as follows;

$E(X_{t+1}X_t) = \phi_1 E(X_t^2) + \theta_1 E(X_{t-1}^2e_t) + \theta_2 E(X_tX_{t-1}e_{t-1}) + \theta_3 E(X_tX_{t-2}e_{t-2})$

From the preliminary results, we have;

$E(X_tX_{t-1}e_{t-1}) = \phi_1 E(X_{t-1}^2e_{t-1}) + \theta_1 E(X_t^2e_t^2) + (\theta_2+\theta_3)\sigma^4$ and

$E(X_tX_{t-2}e_{t-2}) = \phi_1 E(X_{t-1}X_{t-2}e_{t-2}) + \theta_1\sigma^4 + \theta_2 E(X_{t-2}^2 e_{t-2}^2) + \theta_3\sigma^4.$

And by making substitutions of previous results, we have

$$\begin{aligned} m(1) &= \phi_1 m(0) + 2(\theta_1 + \phi_1\theta_2 + \phi_1^2\theta_3)\mu\sigma^2 + (\theta_1\theta_2 + \phi_1\theta_1\theta_3 + \theta_2\theta_3)E(X_t^2 e_t^2) \\ &+ (\theta_2^2 + \theta_2\theta_3 + \phi_1\theta_2\theta_3 + \phi_1\theta_3^2 + \theta_1\theta_3 + \theta_3^2)\sigma^4. \end{aligned} \tag{2.22}$$

Similarly, an expression for $m(2)$ and $m(3)$ can be obtained as;

$$m(2) = \phi_1 m(1) + (\theta_1 + 2\theta_2 + 2\phi_1\theta_3)\mu\sigma^2 + \theta_1\theta_3 E(X_{t-1}^2 e_{t-1}^2) + (\theta_2\theta_3 + \theta_3^2)\sigma^4, \tag{2.23}$$

and

$$m(3) = \phi_1 m(2) + \theta_1\mu\sigma^2 + \theta_2\mu\sigma^2 + 2\theta_3\mu\sigma^2 \tag{2.24}$$

respectively. Using similar procedure as $m(1)$ above, we obtained an expression for the second moment of any nonnegative integer valued k in the lemma below.

Lemma 2.3 *For any nonnegative integer valued $k > 3$ the expression of $m(k)$ for X_t satisfying the $APBL(1,3)$ model can be obtained as,*

$$m(k) = \phi_1 m(k-1) + (\theta_1 + \theta_2 + \theta_3)\mu\sigma^2. \tag{2.25}$$

The proof of the lemma follows easily from the results above.

APBL(1,q) Model

First we derive some of the preliminary results that will be used to obtain the expression for the second moment of the $APBL(1,q)$ when $k = 0$.

$$\begin{aligned} X_{t-i}e_{t-i}X_{t-j}e_{t-j} &= \phi_1 X_{t-i-1}X_{t-j}e_{t-j}e_{t-i} \\ &+ \sum_{k=1}^{q} \theta_i X_{t-i-k}e_{t-i-k}X_{t-j}e_{t-j}e_{t-i} + X_{t-j}e_{t-j}e_{t-i}^2. \end{aligned}$$

$E(X_{t-i}e_{t-i}X_{t-j}e_{t-j}) = E(X_{t-j}e_{t-j}e_{t-i}^2) = \sigma^4$.

Also,

$$\begin{aligned} X_t^2 e_t &= \phi_1^2 X_{t-1}^2 e_t + \sum_{i=1}^{q} \theta_i^2 X_{t-i}^2 e_{t-i}^2 e_t + e_t^3 + 2\phi_1 X_{t-1} \sum_{i=1}^{q} \theta_i X_{t-i} e_{t-i} e_t + 2\phi_1 X_{t-1} e_t^2 \\ &+ 2\sum_{i<}^{q}\sum_{j}^{q} \theta_i \theta_j X_{t-i} e_{t-i} X_{t-j} e_{t-j} e_t + 2\sum_{i=1}^{q} \theta_i X_{t-i} e_{t-i} e_t^2. \end{aligned}$$

Thus, $E(X_t^2 e_t) = 2\phi_1 E(X_{t-1}e_t^2) + 2\sum_{i=1}^{q} \theta_i E(X_{t-i}e_{t-i}e_t^2)$.

This can be simplified as;

$$E(X_t^2 e_t) = \frac{2\sigma^4 \sum_{i=1}^{q} \theta_i}{1-\phi_1} = 2\mu\sigma^2.$$

Also by using similar procedure as above the following expression can be easily obtained.

$$E(X_{t-1}X_{t-i}e_{t-i}) = 2\phi_1^{i-1}\mu\sigma^2 + \sigma^4 \sum_{k=1}^{i-1} \phi_1^{i-(k+1)} \sum_{j\neq k}^{q} \theta_j + \sum_{j=1}^{i-1} \phi_1^{i-(j+1)} \theta_j E(X_t^2 e_t^2). \quad (2.26)$$

Another useful expression is that of the $X_t^2 e_t^2$. For the $APBL(1,q)$ model, we have,

$$\begin{aligned} X_t^2 e_t^2 &= \phi_1^2 X_{t-1} e_t^2 + \sum_{i=1}^{q} \theta_i^2 X_{t-i}^2 e_{t-i}^2 e_t^2 + e_t^4 + 2\phi_1 X_{t-1} \sum_{i=1}^{q} \theta_i X_{t-i} e_{t-i} e_t^2 \\ &+ 2\phi_1 X_{t-1} e_t^3 + 2\sum_{i<}^{q}\sum_{j}^{q} \theta_i\theta_j X_{t-i}e_{t-i}X_{t-j}e_{t-j}e_t^2 + 2\sum_{i=1}^{q} \theta_i X_{t-i} e_{t-i} e_t^3. \end{aligned}$$

And,

$$\begin{aligned} E(X_t^2 e_t^2) &= \phi_1^2 \sigma^2 E(X_{t-1}^2) + \sum_{i=1}^{q} \theta_i^2 \sigma^2 E(X_{t-i}^2 e_{t-i}^2) + 3\sigma^4 \\ &+ 2\phi_1\sigma^2 \sum_{i=1}^{q} \theta_i E(X_{t-i}e_{t-i}X_{t-1}) + 2\sigma^6 \sum_{i<}^{q}\sum_{j}^{q} \theta_i\theta_j. \end{aligned}$$

This simplifies to,

$$
\begin{aligned}
E(X_t^2 e_t^2) &= \{\phi_1^2\sigma^2 m(0) + 3\sigma^4 + 2\sigma^6 \sum_{i<}\sum_{j} \theta_i\theta_j + 4\sigma^4\mu \sum_{k=1}^{q} \phi_1^k\theta_k \\
&+ 2\sum_{i=2}^{q} \theta_i(\phi_1^{i-1} \sum_{k\neq 1} \theta_k + \phi_1^{i-2} \sum_{k\neq 2} \theta_k + \ldots + \phi_1 \sum_{k\neq(i-1)} \theta_k)\sigma^6\} \\
&/ \ (1 - \sum_{i=1}^{q} \theta_i^2\sigma^2 - 2\sigma^2 \sum_{i=1}^{q}\sum_{j=1}^{i-1} \phi_1^{i-j}\theta_j).
\end{aligned}
$$

Using the results above we can obtain an expression for $m(0)$ as follows

$$
\begin{aligned}
E(X_t^2) &= \phi_1^2 E(X_{t-1}^2) + \sum_{i=1}^{q} \theta_i^2 E(X_{t-i}^2 e_{t-i}^2) + E(e_t^2) + 2\phi_1 \sum_{i=1}^{q} \theta_i E(X_{t-1}X_{t-i}e_{t-i}) \\
&+ 2\sum_{i<}^{q}\sum_{j}^{q} \theta_i\theta_j E(X_{t-i}e_{t-i}X_{t-j}e_{t-j}).
\end{aligned}
$$

$$
\begin{aligned}
(1-\phi_1)m(0) &= \sum_{i=1}^{q} \theta_i^2 E(X_{t-i}^2 e_{t-i}^2) + \sigma^2 + 2\sigma^4 \sum_{i<}^{q}\sum_{j}^{q} \theta_i\theta_j \\
&+ 2\phi_1 \sum_{i=2}^{q} \theta_i\{2\phi_1^{i-1}\mu\sigma^2 + (\phi_1^{i-2} \sum_{k\neq 1} \theta_k + \phi_1^{i-3} \sum_{k\neq 2} \theta_k + \ldots \\
&+ \sum_{k\neq(i-1)} \theta_k)\sigma^4) + \sum_{k=1}^{i-1} \phi_1^{i-k-1}\theta_k E(X_t^2 e_t^2)\}.
\end{aligned}
$$

Thus an expression for the second moment of the $APBL(1,q)$ model when $k=0$ is given by;

$$
\begin{aligned}
m(0) &= \{(\sum_{i=1}^{q} \theta_i^2 + 2\sum_{i=2}^{q} \theta_i \sum_{k=1}^{i-1} \theta_k) E(X_t^2 e_t^2) + \sigma^2 + 2\sigma^4 \sum_{i<}^{q}\sum_{j}^{q} \theta_i\theta_j + 4\mu\sigma^2 \sum_{i=1}^{q} \phi_1^i\theta_i \\
&+ 2\sum_{i=2}^{q} \theta_i(\phi_1^{i-1} \sum_{k\neq 1} \theta_k + \phi_1^{i-2} \sum_{k\neq 2} \theta_k + \ldots + \phi_1 \sum_{k\neq(i-1)} \theta_k)\sigma^4)\}/(1-\phi_1^2). \quad (2.27)
\end{aligned}
$$

Similarly, the second moment for the model $APBL(1,q)$ when $k=1$ is given below.

$$\begin{aligned} m(1) &= \phi_1 m(0) + \sum_{i=1}^{q}\theta_1(2\phi_1^{i-1}\mu\sigma^2) + \sum_{i=2}\theta_i(\sum_{k=1}^{i-1}\phi_1^{i-k-1}\theta_k)E(X_t^2e_t^2) \\ &+ \sum_{i=2}^{q}\theta_i(\phi_1^{i-2}\sum_{k\neq 1}\theta_k + \phi_1^{i-3}\sum_{k\neq 2}\theta_k + \ldots + \sum_{k\neq(i-1)}\theta_k)\sigma^4. \end{aligned} \tag{2.28}$$

For any nonnegative integer valued $k>1$, the expression for the second moment of X_t satisfying $APBL(1,q)$ can be summarized in the following lemma.

Lemma 2.4 *If X_t is a time series satisfying the $APBL(1,q)$ model, then the second order moment $m(k)$ for any nonnegative integer valued k, can be obtained from the expression (2.29).*

$$\begin{aligned} m(k) &= \phi_1 m(k-1) + (\sum_{i=1}^{k-1}\theta_i + 2\sum_{i=k}^{q}\phi_1^{i-k})\mu\sigma^2 + \sum_{i=k+1}^{q}\theta_i\{\sum_{j=1}^{i-k}\phi_1^{i-j-k}\theta_j E(X_t^2e_t^2) \\ &+ (\phi_1^{i-k-1}\sum_{j\neq 1}\theta_j + \phi_1^{i-k-2}\sum_{j\neq 2}\theta_j + \ldots + \sum_{j\neq(i-k)}\theta_j)\sigma^4.\} \end{aligned} \tag{2.29}$$

The proof of the lemma can be obtained by using the preliminary results and following similar procedure for $m(1)$. Observe from (2.29) that for $k>q$, we have $m(k) = \phi_1 m(k-1) + \sum_{i=1}^{q}\theta_i\mu\sigma^2$. It follows that for APBL(1,q) model we have that $R(k) = \phi_1 R(k-1)$ for $k>q$. Thus, in general, an initial estimate of ϕ_1 for the APBL(1,q) model is $\hat{\phi}_1 = \hat{\rho}(k)/\hat{\rho}(k-1)$, for any $k>q$. Where $\hat{\rho}(k)$ is an estimate of the autocorrelation function at lag k.

2.3 Third Order Moment And Cumulants

In this section, we derive the third-order moments and cumulants for the $APBL(1,1)$ model only.

2.3.1 Third Order Moments

By definition, the third-order moment of a nonnegative integer valued process X_t is given by

$$m(k_1, k_2) = E(X_t X_{t+k_1} X_{t+k_2}). \tag{2.30}$$

To obtain $m(k_1, k_2)$ of the model $APBL(1,1)$, for any nonnegative integer valued k_1 and k_2, we shall use some of the previous results and the results below.

$$E(X_{t+k-1}e_{t+k-1}X_t) = E(X_t e_{t+k-1}^2) = \sigma^2\mu. \tag{2.31}$$

$$E(X_{t+1}^2 X_t e_{t+1}) = 2\phi_1\sigma^2 m(0) + 4\theta_1\sigma^4\mu. \tag{2.32}$$

And for any $k > 1$ we have,

$$E(X_{t+k}^2 X_t e_{t+k}) = 2\phi_1\sigma^2 m(k-1) + 2\theta_1\sigma^4\mu. \tag{2.33}$$

Also,

$$E(X_{t+1}^2 X_t e_{t+1}^2) = \phi_1^2\sigma^2 m(0,0) + \theta_1^2\sigma^2 E(X_t^3 e_t^2) + 3\mu\sigma^4 + 2\phi_1\theta_1\sigma^2 E(X_t^3 e_t). \tag{2.34}$$

For $k > 1$ we have,

$$\begin{aligned} E(X_{t+k}^2 X_t e_{t+k}^2) &= \phi_1^2\sigma^2 \sum_{r=0}^{k-1} (\theta_1\sigma)^{2r} m(k-r-1, k-r-1) + (\theta_1\sigma)^{2k} E(X_t^3 e_t^2) \\ &+ 2\phi_1\theta_1^{2k-1}\sigma^{2k} E(X_t^2 e_t) + 4\phi_1^2\theta_1\sigma^4 \sum_{r=0}^{k-2} (\theta_1\sigma)^{2k} m(k-r-2) \\ &+ \theta_1^{2(k-2)}\sigma^{2k}(3 + 3\theta^2\sigma^2 + 8\phi_1\theta_1^2\sigma^2)\mu \\ &+ 4\phi_1\theta_1\sigma^6\mu \sum_{r=0}^{k-3} (\theta_1\sigma)^{2r}. \end{aligned} \tag{2.35}$$

$$E(X_t^3 e_t) = 3\sigma^4 + 3\phi_1^2\sigma^2 m(0) + 3\theta_1^2\sigma^2 E(X_t^2 e_t^2) + 12\phi_1\theta_1\mu\sigma^4. \tag{2.36}$$

$$E(X_t^3e_t^2) = \frac{\phi_1^3\sigma^2 m(0,0) + \theta_1^3\sigma^2 E(X_t^3e_t^3) + 3\phi_1^2\theta_1\sigma^2 E(X_t^3 e_t) + 9\phi_1\sigma^4\mu + 9\theta_1\sigma^6}{1 - 3\phi_1\theta_1^2\sigma^2}. \tag{2.37}$$

$$E(X_t^3e_t^3) = 15\sigma^6 + 9\phi_1^2\sigma^4 m(0) + 9\theta_1^2\sigma^4 E(X_{t-1}^2 e_{t-1}^2) + 36\phi_1\theta_1\mu\sigma^6. \tag{2.38}$$

Case 1: $k_1 = k_2 = k$

When $k_1 = k_2 = k = 0$, it can be shown that

$$\begin{aligned}(1-\phi_1^3)E(X_t^3) &= \theta_1^3 E(X_{t-1}^3 e_{t-1}^3) + 3\phi_1^2\theta_1 E(X_{t-1}^3 e_{t-1}) + 3\phi_1\theta_1^2 E(X_{t-1}^3 e_{t-1}^2) \\ &+ 3\phi_1^2 E(X_{t-1}^2 e_t) + E(e_t^3) + 3\theta_1^2 E(X_{t-1}^2 e_{t-1}^2 e_t) + 3\phi_1 E(X_{t-1} e_t^2) \\ &+ 3\theta_1 E(X_{t-1} e_{t-1} e_t^2) + 6\phi_1\theta_1 E(X_{t-1}^2 e_{t-1} e_t).\end{aligned}$$

Thus by using the results above, the third order moment of the X_t when $k = 0$ is given by;

$$\begin{aligned}m(0,0) &= \{\theta_1^3 E(X_t^3 e_t^3) + 3\phi_1^2\theta_1 E(X_t^3 e_t) + 18\phi_1^2\theta_1^2\mu\sigma^4 + 18\phi_1\theta_1^3\sigma^6 \\ &+ 3\phi_1\mu\sigma^2 + 3\theta_1\sigma^4\}/((1-\phi_1^3)(1-3\phi_1\theta_1^2\sigma^2) - 3\phi_1^4\theta_1^2\sigma^2). \end{aligned} \tag{2.39}$$

When $k_1 = k_2 = 1$, we have $m(1,1) = E(X_{t+1}^2 X_t) = \phi_1^2 E(X_t^3) + \theta_1^2 E(X_t^3 e_t^2) + E(X_t e_{t+1}^2) + 2\phi_1\theta_1 E(X_t^3 e_t) + 2\phi_1 E(X_t^2 e_{t+1}) + 2\theta_t^2 e_{t+1})$.
From above, it can be shown quite easily that;

$$m(1,1) = \phi_1^2 m(0,0) + \theta_1^2 E(X_t^3 e_t^2) + \sigma^2\mu + 2\phi_1\theta_1 E(X_t^3 e_t). \tag{2.40}$$

For $k_1 = k_2 = 2$, we find that

$$\begin{aligned}m(2,2) &= \phi_1^2 E(X_{t+1}^2 X_t) + 2\phi_1 E(X_{t+1} e_{t+2} X_t) + \theta_1^2 E(X_{t+1}^2 e_{t+1}^2 X_t) + E(X_t e_{t+2}^2) \\ &+ 2\phi_1\theta_1 E(X_{t+1}^2 e_{t+1} X_t) + 2\theta_1 E(X_{t+1} e_{t+1} e_{t+2} X_t).\end{aligned}$$

Which simplifies into,

$$\begin{aligned} m(2,2) &= \phi_1^2 m(1,1) + \phi_1^2\theta_1^2\sigma^2 m(0,0) + \theta^4\sigma^2 E(X_t^3 e_t^2) + 3\theta_1^2\mu\sigma^4 \\ &+ 2\phi_1\theta_1^3\sigma^2 E(X_t^3 e_t) + \sigma^2\mu + 4\phi_1^2\theta_1\sigma^2 m(0) + 8\phi_1\theta_1^2\sigma^4\mu. \end{aligned} \quad (2.41)$$

For $k_1 = k_2 = 3$, we obtained the third order moment, m(3,3) as;

$m(3,3) = \phi_1^2 E(X_{t+2}^2 X_t) + \theta_1^2 E(X_{t+2}^2 e_{t+2}^2 X_t) + E(X_t e_{t+3}^2) + 2\phi_1\theta_1 E(X_{t+2}^2 e_{t+2} X_t)$

By making relevant substitutions we have

$$\begin{aligned} m(3,3) &= \phi_1^2 m(2,2) + \phi_1^2\theta_1^2\sigma^2 m(1,1) + \phi_1^2\theta_1^4\sigma^4 m(0,0) + \theta_1^6\sigma^4 E(X_t^3 e_t^2) \\ &+ 2\phi_1\theta_1^5\sigma^4 E(X_t^3 e_t) + 4\phi_1^2\theta_1^3\sigma^4 m(0) + \theta_1^2\sigma^4\mu\{3 + 3\theta_1^2\sigma^2 + 8\phi_1\theta_1^2\sigma^2\} \\ &+ \sigma^2\mu + 4\phi_1^2\theta_1\sigma^2 m(2) + 4\phi_1\theta_1^2\sigma^4\mu. \end{aligned} \quad (2.42)$$

Using similar procedure, the $m(k,k)$ for any real $k > 2$ is presented in the lemma below. The proof of the lemma follows accordingly.

Lemma 2.5 *For any real-valued $k > 2$, the third moment of the of X_t satisfying $APBL(1,1)$ model is given by*

$$\begin{aligned} m(k,k) &= \phi_1^2 m(k-1,k-1) + \phi_1^2\theta_1^2\sigma^2 \sum_{r=0}^{k-2} (\theta_1\sigma)^{2r} m(k-r-2, k-r-2) \\ &+ \theta_1^{2k}\sigma^{2(k-1)} E(X_t^3 e_t^2) + 2\phi_1\theta_1^{2k-1)}\sigma^{2(k-1)} E(X_t^2 e_t) \\ &+ 4\phi_1^2\theta_1^3\sigma^4 \sum_{r=0}^{k-3} (\theta_1\sigma)^{2r} m(k-r-3) + \theta_1^{2(k-2)}\sigma^{2(k-1)}\mu\{3 + 3\theta_1^2\sigma^2 + 8\phi_1\theta_1^2\sigma^2\} \\ &+ 4\phi_1\theta_1^3\sigma^6\mu \sum_{r=0}^{k-4} (\theta_1\sigma)^{2r} + \sigma^2\mu + 4\phi_1^2\theta_1\sigma^2 m(k-2) + 4\phi_1\theta_1\sigma^4. \end{aligned} \quad (2.43)$$

where $E(X_{t+k-1}^2 e_{t+k-1}^2 X_t)$ and $E(X_{t+k-1}^2 e_{t+k-1} X_t)$ are as in the preliminary results above.

Case 2: $k_2 > k_1$

When $k_1 = 1$ and $k_2 = 2$, we have;

$E(X_{t+2}X_{t+1}X_t) = m(1,2) = \phi_1 E(X_{t+1}^2 X_t) + \theta_1 E(X_{t+1}^2 e_{t+1} X_t)$.

Simplifying this expression using previous results, we obtain,

$$m(1,2) = \phi_1 m(1,1) + 2\phi_1\theta_1\sigma^2 m(0) + 4\theta_1^2\mu\sigma^4. \tag{2.44}$$

Also when $k_1 = 1$ and $k_2 = 3$;

$m(1,3) = \phi_1 m(1,2) + \theta_1 E(X_{t+2}e_{t+2}X_{t+1}X_t)$, which we simplify to obtain

$$m(1,3) = \phi_1 m(1,2) + \theta_1\sigma^2 m(1). \tag{2.45}$$

Using similar procedures as for m(1,3), we obtained the following expressions for third order moments:

$$m(1,4) = \phi_1 m(1,3) + \theta_1\sigma^2 m(1). \tag{2.46}$$

$$m(1,5) = \phi_1 m(1,4) + \theta_1\sigma^2 m(1). \tag{2.47}$$

$$m(2,3) = \phi_1 m(2,2) + 2\phi_1\theta_1\sigma^2 m(1) + 2\theta_1^2\mu\sigma^4. \tag{2.48}$$

$$m(2,4) = \phi_1 m(2,3) + \theta_1\sigma^2 m(2). \tag{2.49}$$

$$m(2,5) = \phi_1 m(2,4) + \theta_1\sigma^2 m(2). \tag{2.50}$$

$$m(3,4) = \phi_1 m(3,3) + 2\phi_1\theta_1\sigma^2 m(2) + 2\theta_1^2\mu\sigma^4. \tag{2.51}$$

$$m(3,5) = \phi_1 m(3,4) + \theta_1\sigma^2 m(3). \tag{2.52}$$

$$m(4,5) = \phi_1 m(4,4) + 2\phi_1\theta_1\sigma^2 m(3) + 2\theta_1^2\mu\sigma^4. \tag{2.53}$$

Below we present expressions for the third order moments of X_t following $APBL(1,1)$ model for cases when $k_2 - k_1 = 1$ and when $k_2 - k_1 > 1$.

Lemma 2.6 *For a time series $\{X_t\}$ that satisfies the $APBL(1,1)$ model, the third* *order moment $m(k_1,k_2)$, when $k_2-k_1=1$ and when $k_2-k_1>1$ are given by (2.54) and (2.55) respectively for any real-valued k_1, k_2, where $k_2>2$.*

$$m(k_1,k_2)=\phi_1^2 m(k_1,k_1)+2\phi_1\theta_1\sigma^2 m(k_1-1)+2\theta_1^2\sigma^4\mu \tag{2.54}$$

$$m(k_1,k_2)=\phi_1 m(k_1,k_2-1)+\theta_1\sigma^2 m(k_1) \tag{2.55}$$

The proof of lemma follows from above.

2.3.2 Third Order Cumulants

As stated earlier, the third order cumulant $C(k_1,k_2)$ of a real-valued process X_t is defined by $C(k_1,k_2)=E[(X_t-\mu)(X_{t+k_1}-\mu)(X_{t+k_2}-\mu)]$. This can be simplified as $C(k_1,k_2)=E(X_tX_{t+k_1}X_{t+k_2})-\mu[R(k_1)+R(k_2)+R(k_2-k_1)]-\mu^3$, where $\mu=E(X_t)$ and $R(k)=E(X_tX_{t+k})-\mu^2$.

Case 1: $k_1=k_2=k$

When $k_1=k_2=k$ we have, $C(k,k)=m(k,k)-\mu(R(0)+2R(k))-\mu^3$.

It follows that for $k=1$, $C(1,1)=m(1,1)-\mu[R(0)+2R(1)]-\mu^3$, which can be simplified into,

$$\begin{aligned} C(1,1) &= \phi_1^2C(0,0)+\theta_1^2E(X_t^3e_t^2)+2\phi_1\theta_1E(X_t^3e_t)+(3\phi_1^2-1)\mu R(0)+\phi_1^2\mu^3 \\ &- 2R(1)+\sigma^2\mu-\mu^3. \end{aligned} \tag{2.56}$$

And when $k=2$ the third order cumulant can be obtained as,

$$\begin{aligned} C(2,2) &= \phi_1^2C(1,1)+\phi_1^2\theta_1^2\sigma^2C(0,0)+\theta_1^4\sigma^2E(X_t^3e_t)+2\phi_1\theta_1^3E(X_t^3e_t) \\ &+ (3\phi_1^2\theta_1^2\sigma^2+\phi_1^2-1)\mu R(0)+2\phi_1^2\mu R(1)-2\mu R(2)+\phi_1^2\theta_1^2\mu^3+\phi_1^2\mu^3 \\ &+ 3\theta_1^2\mu\sigma^4+\mu\sigma^4+4\phi_1^2\theta_1\sigma^2m(0)+8\phi_1\theta_1^2\sigma^4\mu-\mu^3. \end{aligned} \tag{2.57}$$

We can obtain the third order cumulant, $C(k,k)$ for $k > 1$ as shown below. By defining $C(k,k) = m(k,k) = \mu[R(0) - 2R(k)] - \mu^3$ we have,

$$\begin{aligned}
C(k,k) &= \phi_1^2 m(k-1,k-1) + \phi_1^2\theta_1^2\sigma^2 \sum_{r=0}^{k-2} (\theta_1\sigma)^{2r} m(k-r-2, k-r-2) \\
&+ \theta_1^{2k}\sigma^{2(k-1)} E(X_t^3 e_t^2) + 2\phi_1\theta_1^{2k-1)}\sigma^{2(k-1)} E(X_t^2 e_t) \\
&+ 4\phi_1^2\theta_1^3\sigma^4 \sum_{r=0}^{k-3} (\theta_1\sigma)^{2r} m(k-r-3) + \theta_1^{2(k-2)}\sigma^{2(k-1)}\mu\{3 + 3\theta_1^2\sigma^2 + 8\phi_1\theta_1^2\sigma^2\} \\
&+ 4\phi_1\theta_1^3\sigma^6\mu \sum_{r=0}^{k-4} (\theta_1\sigma)^{2r} + \sigma^2\mu + 4\phi_1^2\theta_1\sigma^2 m(k-2) + 4\phi_1\theta_1\sigma^4 \\
&- \mu[R(0) + 2R(k)] - \mu^3.
\end{aligned}$$

This can be simplified into

$$C(k,k) = \phi_1^2 C(k-1,k-1) + \phi_1^2\theta_1^2\sigma^2 \sum_{r=0}^{k-2} (\theta_1\sigma)^{2r} C(k-r-2, k-r-2) + SB \quad (2.58)$$

where

$$\begin{aligned}
SB &= \phi_1^2\theta_1^2\sigma^2 \sum_{r=0}^{k-2} (\theta_1\sigma)^{2r} \mu\{m(0) + 2R(k-r-2)\} + SB \\
&+ \mu\{2R(k) + 2\phi_1^2 R(k-1) + R(0) + \phi_1^2 m(0) - \mu^2\} \\
&+ \theta_1^{2k}\sigma^{2(k-1)} E(X_t^3 e_t^2) + 2\phi_1\theta_1^{2k-1)}\sigma^{2(k-1)} E(X_t^2 e_t) \\
&+ 4\phi_1^2\theta_1^3\sigma^4 \sum_{r=0}^{k-3} (\theta_1\sigma)^{2r} m(k-r-3) + \theta_1^{2(k-2)}\sigma^{2(k-1)}\mu\{3 + 3\theta_1^2\sigma^2 + 8\phi_1\theta_1^2\sigma^2\} \\
&+ 4\phi_1\theta_1^3\sigma^6\mu \sum_{r=0}^{k-4} (\theta_1\sigma)^{2r} + \sigma^2\mu + 4\phi_1^2\theta_1\sigma^2 m(k-2) + 4\phi_1\theta_1\sigma^4.
\end{aligned}$$

This is a form of Yuke-Walker type of difference equation for the cumulants of APBL(1,1).

<u>Case 2: $k_2 > k_1$</u>

When $k_1 = 1$ and $k_2 = 2$, the third order cumulant is derived as shown below;

$$\begin{aligned}
C(1,2) &= m(1,2) - \mu(2R(1) + R(2)) - \mu^3 \\
&= \phi_1 m(1,1) + 2\phi_1\theta_1\sigma^2 m(0) - \mu(2R(1) + R(2)) - \mu^3 + 4\theta_1^2\sigma^4\mu \\
&= \phi_1\{m(1,1) - \mu[R(0) + 2R(1)] - \mu^3\} + \phi_1\mu[R(0) + 2R(1)] + \phi_1\mu^3 \\
&+ 2\phi_1\theta_1\sigma^2 m(0) + 4\theta_1^2\sigma^4\mu - \mu(2R(1) + R(2)) - \mu^3 \\
&= \phi_1 C(1,1) + \mu\{\phi_1 R(0) + 2(\phi_1 - 1)R(1) - R(2)\} + (\phi_1 - 1)\mu^3 \\
&+ 2\phi_1\theta_1\sigma^2 m(0) + 4\theta_1^2\sigma^4\mu \\
&= \phi_1 C(1,1) + \phi_1 m(0)\mu - \phi_1\mu^3 + 2\phi_1^2 m(0)\mu + 6\phi_1\mu^3 - 4\phi_1^2\mu^3 \\
&- 2\phi_1^2 m(0)\mu - 2\mu^3 - \phi_1^2 m(0)\mu - \phi_1\mu^3 + 2\phi_1^2\mu^3 + 2\phi_1 m(0)\mu \\
&- 2\phi_1^2 m(0)\mu + 4\mu^3 - 8\phi_1\mu^3 + 4\phi_1^2\mu^3 + \phi_1\mu^3 - \mu^3 \\
&= \phi_1 C(1,1) + \phi_1 m(0)\mu - 3\phi_1\mu + \mu^3 - \phi_1^2 m(0)\mu + 2\phi_1^2\mu^3 \\
&= \phi_1 C(1,1) + \phi_1 m(0)\mu(1 - \phi_1) + \mu^3(1 - 3\phi_1 + 2\phi_1^2) \\
&= \phi_1 C(1,1) + \phi_1 m(0)\mu(1 - \phi_1) + \mu^3\{(1 - \phi_1)(1 - 2\phi_1)\} \\
&= \phi_1 C(1,1) + \mu(1 - \phi_1)\{\phi_1(m(0) - \mu^2) + \mu^2(1 - \phi_1)\} \\
&= \phi_1 C(1,1) + \mu(1 - \phi_1)\{\phi_1 R(0) + \theta_1\sigma^2\mu\} \\
&= \phi_1 C(1,1) + \theta_1\sigma^2(\phi_1 R(0) + \theta_1\sigma^2\mu). \qquad (2.59)
\end{aligned}$$

Proceeding as for C(1,2), we obtained expressions for the third order cumulant when $k = 3$.

$$C(1,3) = \phi_1^2 C(1,1) + \phi_1\theta_1\sigma^2(\phi_1 R(0) + \theta_1\sigma^2\mu) = \phi_1 C(1,2). \qquad (2.60)$$

Similarly an expression for the third order cumulant for a real-valued k, when $k_1 = 1$ and $k_2 = k$ can be shown to be;

$$C(1,k) = \phi_1^{k-1} C(1,1) + \phi_1^{k-2}\theta_1\sigma^2\{\phi_1 R(0) + \theta_1\sigma^2\mu\} = \phi_1^{k-2} C(1,2) \qquad (2.61)$$

USing similar procedures, it is easy to show that for a real-valued $k_2 = k$, the third order cumulant of X_t following the $APBL(1,1)$ model when $k_1 = 2$ can be obtained usiug the expression;

$$C(2,k) = \phi_1^{k-2}\{C(2,2) + \theta_1\sigma^2(\phi_1 R(0) + \theta_1\sigma^2\mu)\} = \phi_1^{k-3}C(2,3) \tag{2.62}$$

From these results we found that, if $\phi_1 < 1$ the $C(1,k)$ and $C(2,k)$ will decrease exponentially, otherwise, it will increase exponentially. It is also observed that the standardized cumulants $\rho(k_1, k_2)$, follow the same pattern as the $C(k_1, k_2)$. We shall use the exponential pattern observed in the $C(1, k_2)$ or alternatively $\rho(k_1, k_2)$ for model identification

Chapter 3

MODEL APPLICATIONS

3.1 Introduction

In Chapter 2, we derived some basic properties of some versions of the bilinear model. In Section 3.2 of this chapter we shall present the results of simulation studies used to examine the performance of the derived properties of the $APBL(1,1)$ and the $APBL(1,2)$ models.

One of the important uses of time series models is to provide forecasts for the future. Therefore in Section 3.3 we shall investigate the usefulness of the $APBL(1,1)$ model, by using it to make one-step-forecasts on three real life data. For each of the data, we use the $C(1,k)$ derived in Chapter 2 to ensure that the $APBL(1,1)$ is a suitable model for the data before any estimation is done. We shall use the method of parameter estimation described in Chapter 1 to estimate the parameters of the bilinear models. The results of the forecasts shall then be compared to similar forecasts using appropriate linear models where applicable. Linear model identification procedures were discussed in Chapter 1.

In this chapter, firstly, we shall use simulated data to study the pattern in $\rho(1,k_2)$ derived in Chapter 2 for the $APBL(1,1)$ model. Secondly, we shall transform the three data sets studied to stationary forms, investigate the pattern of the $\rho(1, k_2)$, then fit the $APBL(1, 1)$ model to them. Thirdly, the ACF and PACF shall be used to determine the order of appropriate linear models for comparison.

3.2 SIMULATION STUDIES

From the $APBL(1, 1)$ and $APBL(1, 2)$ models, we generated 1000 observations for three distinct values of ϕ_1, θ_1, θ_2, and σ^2. The simulated random variable e_t, $t \in z$, are mutually independent and identically distributed as N(0,σ^2), for each generated set of observations.

The sample mean, variance and autocorrelation were calculated for each of the data in the 1000 simulations for the two models with fixed parameters. While the standardized third order cumulants, $\rho(k_1, k_2)$ for $k_1 = 1, \&2$, and $k_2 = 1, 2, \ldots, 30$ are calculated for the $APBL(1, 1)$ model only. The reported results are the averages of the means, variances, autocorrelation values and the standardized third order cumulants. The $\rho(k_1, k_2)$ will be used for model identification as we noted in Chapter 2.

The expressions for the theoretical mean, the covariance structure for both the $APBL(1, 1)$ and $APBL(1, 2)$ models and the cumulants of the $APBL(1, 1)$ model were given in Chapter 2. The theoretical results in all our tables are computed from these expressions.

According to Brockwell and Davis(1996), we can estimate the mean, autocovariance and third order cumulant as follows. Suppose $x_1, x_2, \ldots, x_n$ are observations of a time series. The sample mean of $x_1, x_2, \ldots, x_n$ is estimated by;

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

The sample autocovariance function for the observed data $x_1, x_2, \ldots, x_n$ is estimated by;

$$\hat{R}(k) = \frac{\sum_{t=1}^{n-k}(X_{t+k} - \bar{x})(X_t - \bar{x})}{n-k}$$

and the sample autocorrelation function is;

$$\hat{\rho}(k) = \frac{\hat{R}(k)}{\hat{R}(0)}$$

The third order cumulants are estimated by;

$$\hat{C}(k_1, k_2) = \frac{\sum_{t=1}^{n-k_1-k_2}(X_t - \bar{x})(X_{t+k_1} - \bar{x})(X_{t+k_2} - \bar{x})}{n - k_1 - k_2} \tag{3.1}$$

while the standardized third order cumulant(See Oyet(2001)) and the sample version are given by;

$$\rho(k_1, k_2) = \frac{C(k_1, k_2)}{C(0,0)} \tag{3.2}$$

$$\hat{\rho}(k_1, k_2) = \frac{\hat{C}(k_1, k_2)}{\hat{C}(0,0)} \tag{3.3}$$

respectively.

3.2.1 Result of Simulation Studies

The parameters used for the $APBL(1,1)$ models for the three simulations are given below.

First Simulation; $\phi_1 = 0.70$, $\theta_1 = 0.50$, $\theta_2 = 0.20$, $\sigma^2 = 1.0$.

Second Simulation; $\phi_1 = 0.50$, $\theta_1 = 0.20$, $\theta_2 = 0.05$, $\sigma^2 = 1.1$.

Third Simulation; $\phi_1 = 0.40$, $\theta_1 = 0.35$, $\theta_2 = 0.20$, $\sigma^2 = 1.15$.

Table 3.1 presents the mean, variance and the $C(0,0)$ for the $APBL(1,1)$ model. In all the tables TH and ET denotes theoretical(from derived properties) and estimated values respectively. The table shows that the estimated values of the mean, variance and $C(0,0)$ are quite close to their theoretical values for the $APBL(1,1)$ model.

Table 3.1: Mean, Variance and $C(0,0)$ Using $APBL(1,1)$

	Simulation 1		Simulation 2		Simulation 3	
	TH	ET	TH	ET	TH	ET
Mean	1.6667	1.6552	0.4840	0.4840	0.7715	0.7706
Variance	11.9658	11.6510	1.9912	1.9839	2.8302	2.8286
$C(0,0)$	390.8152	411.5872	1.3785	1.3950	5.4691	5.5987

Tables 3.2, 3.3 and 3.4 present ten values of the autocorrelation, standardized $C(1,k)$ and standardized $C(2,k)$ respectively using the $APBL(1,1)$ model. The theoretical values compare perfectly well with the estimated values in each of the tables except for $k \geq 8$. These confirm the accuracy of the derived properties of the $APBL(1,1)$ model. It is important to note that as k increases, $R(k)$, $C(1,k)$ and $C(2,k)$ approaches zero for the $APBL(1,1)$ model. This behavior is a feature of the ACF of bilinear models. In fact, for the diagonal pure bilinear model, the ACF cuts off after lag $q+1$.

Table 3.2: Autocorrelation Using $APBL(1,1)$

Lag	Simulation 1		Simulation 2		Simulation 3	
k	TH	ET	TH	ET	TH	ET
1	0.7696	0.7657	0.5588	0.5556	0.5262	0.5235
2	0.5387	0.5341	0.2794	0.2743	0.2105	0.2068
3	0.3771	0.3699	0.1397	0.1336	0.0842	0.0793
4	0.2640	0.2514	0.0699	0.0641	0.0337	0.0290
5	0.1848	0.1662	0.0349	0.0304	0.0135	0.0086
6	0.1294	0.1099	0.0175	0.0143	0.0054	0.0011
7	0.0905	0.0722	0.0087	0.0046	0.0022	-0.0011
8	0.0634	0.0472	0.0044	-0.0006	0.0009	-0.0024
9	0.0444	0.0307	0.0022	-0.0023	0.0003	-0.0035
10	0.0311	0.0189	0.0011	-0.0030	0.0001	-0.0036

We note that these results satisfy the property $\hat{\phi}_1 = \hat{\rho}(k)/\hat{\rho}(k-1)$ derived in Chapter 2 for all the ten values of k in the first 2 simulations and up to when $k = 8$ in all the third simulation. In a real time series, we can use this result to obtain an initial estimate of ϕ_1. All we need to do is to estimate $\hat{\rho}(k)$ from the data.

Table 3.3: Standardized $C(1, k)$ Using $APBL(1, 1)$

Lag	Simulation 1		Simulation 2		Simulation 3	
k	TH	ET	TH	ET	TH	ET
1	0.8152	0.7897	1.1096	1.0829	0.9223	0.8688
2	0.5824	0.5559	0.7501	0.7249	0.4950	0.4591
3	0.4077	0.4128	0.3751	0.3591	0.1980	0.1755
4	0.2854	0.2658	0.1875	0.1780	0.0792	0.0673
5	0.1998	0.1565	0.0938	0.0868	0.0317	0.0258
6	0.1398	0.0733	0.0469	0.0403	0.0127	0.0092
7	0.0979	0.0334	0.0234	0.0161	0.0051	0.0018
8	0.0685	0.0131	0.0117	0.0017	0.0020	-0.0002
9	0.0480	0.0028	0.0059	-0.0065	0.0008	-0.0045
10	0.0336	-0.0013	0.0029	-0.0089	0.0003	-0.0059

Table 3.4: Standardized $C(2, k)$ Using $APBL(1,1)$

Lag	Simulation 1		Simulation 2		Simulation 3	
k	TH	ET	TH	ET	TH	ET
1	0.6259	0.5559	0.7501	0.7249	0.4950	0.4591
2	0.5012	0.6033	0.5437	0.5388	0.4525	0.4319
3	0.3225	0.4328	0.3695	0.3567	0.2314	0.2117
4	0.1612	0.2962	0.1848	0.1769	0.0926	0.0808
5	0.0806	0.1794	0.0924	0.0862	0.0370	0.0321
6	0.0403	0.1065	0.0462	0.0411	0.0148	0.0124
7	0.0202	0.0511	0.0231	0.0158	0.0059	0.0045
8	0.0101	0.0214	0.0115	0.0034	0.0024	0.0011
9	0.0050	0.0080	0.0058	-0.0029	0.0009	-0.0007
10	0.0025	0.0012	0.0029	-0.0078	0.0004	-0.0040

Table 3.5 shows the mean and variance computed from the three simulations using the $APBL(1,2)$ model. Again we find that the theoretical values compare closely with the estimated values. Table 3.6 is the table of the first ten autocorrelation using the $APBL(1,2)$ model. The theoretical results also compare closely with the estimated values, except for when $k \geq 4$.

Table 3.5: Mean, Variance and $C(0,0)$ Using $APBL(1,2)$

	Simulation 1		Simulation 2		Simulation 3	
	TH	ET	TH	ET	TH	ET
Mean	2.3333	2.3087	0.6050	0.6047	1.212	1.2122
Variance	39.3787	49.5405	3.2623	2.1395	6.276	4.8375

Table 3.6: Autocorrelation Using $APBL(1,2)$

Lag	Simulation 1		Simulation 2		Simulation 3	
k	TH	ET	TH	ET	TH	ET
1	0.9385	0.8222	0.6062	0.5946	0.7708	0.6839
2	0.6214	0.5733	0.2941	0.3108	0.3037	0.3351
3	0.4172	0.3803	0.1134	0.1517	0.0832	0.1301
4	0.2742	0.2314	0.0230	0.0729	-0.0051	0.0496
5	0.1742	0.1229	-0.0221	0.0346	-0.0403	0.0182
6	0.1042	0.0578	-0.0447	0.0160	-0.0545	0.0061
7	0.0551	0.0209	-0.0560	0.0053	-0.0601	0.0010
8	0.0208	0.0000	-0.0617	-0.0005	-0.0624	-0.0019
9	-0.0032	-0.0081	-0.0645	-0.0024	-0.0633	-0.0039
10	-0.0200	-0.0116	-0.0659	-0.0031	-0.0636	-0.0048

The plots of the estimated values of all the properties studied for both the $APBL(1,1)$ and the $APBL(1,2)$ models are given below. For each of the plots, the theoretical values are overlaid on the estimated for comparison. The pattern of exponential decay derived in Chapter 2 is closely modeled by the plots in Figure 3.1. We note that the $\hat{\rho}(1,k)$ in $X_t = \sum_{j=1}^{q} \theta_j X_{t-j} e_{t-j} + e_t$ cuts off after $k = q+1$, a pattern which can be used for identification of a diagonal pure bilinear model. Thus if $\hat{\rho}(1,k)$ does not cut off after $k = q+1$, but decays exponentially, the model is most likely to be a $APBL(1,1)$. These distinct patterns in different versions of bilinear models can be used to determine the order q of the model.

It is worth mentioning that in practice plots of standardized cumulants computed from real data sets may not be as smooth as the plots in Figures 3.1 and 3.2 due to presence of noise and other components in the data that may distort the behavior slightly. The plot should however, exhibit the general pattern shown here. See for instance plots of the cumulants in Section 3.3.

Figure 3.1: Plots for First Simulation

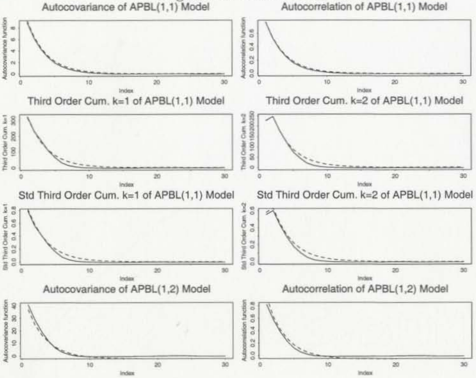


Figure 3.2: Plots for Second Simulation

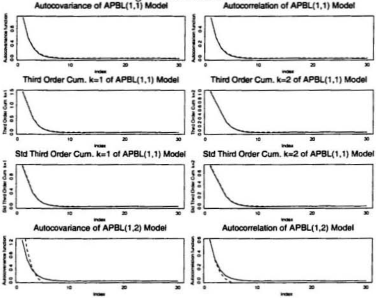
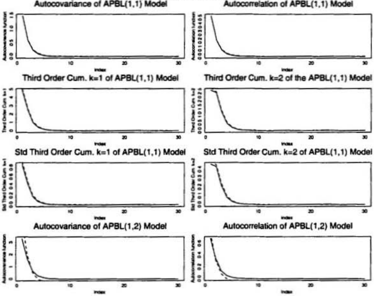


Figure 3.3: Plots for Third Simulation



3.3 APPLICATIONS TO REAL DATA

In order to investigate the performance of the $APBL(1,1)$ models as compared to "best" linear models, the mean absolute deviation(MAD) of each of the forecasts from the original values are calculated using the equation,

$$MAD = \frac{\sum_{t=1}^{n} |Y_t - \tilde{Y}_t|}{n}$$

where Y_t is original value at time t and $\tilde{Y}_t$ is the predicted value at time t.

3.3.1 International Airline Passengers

Here, we modeled data on international airline passengers. The totals(in thousands) of international airline passengers data from January 1949 to December 1960 is given in Table 1 of the Appendix. The data was quoted by Brown(1962) and has been analyzed by Box and Jenkins(1970) and many others.

A plot of the data and the ACF are given in Figures 3.4 and 3.5 respectively. The series shows a marked seasonal pattern and a bit of an upward trend. The seasonal pattern could be attributed to the fact that more people travel during late summer months as reflected by the plot. Specifically, the plot reveals that the series exhibits a periodic behavior with $d = 12$ months. We also note that the variability across the time plot is not constant. These and the time plot features of the airline data indicate the need for some transformation on the data.

Figure 3.4: Plot of Airline Passengers

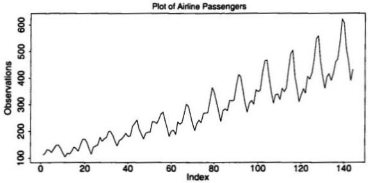
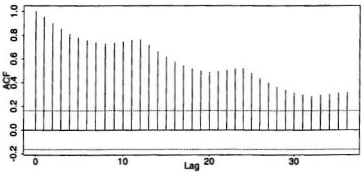


Figure 3.5: Plot ACF of Airline Passengers



In analyzing this data, we took logarithm to reduce variability across the series. The seasonal effect was estimated by a 12-month moving-average as described in Chapter 1. Finally, the trend component was estimated by linear regression. Thus given the time series $\{X_t\}$, with estimated seasonal component $\hat{S}_t$ and trend component $\hat{M}_t$, we can estimate the stationary component Y_t by;

$$Y_t = X_t - \hat{M}_t - \hat{S}_t$$

A plot of the stationary component, the autocorrelation, partial autocorrelation, and the $\hat{\rho}(1, k)$ are given in Figure 3.6. We note that the pattern in the $\hat{\rho}(1, k)$ suggests a general pattern of exponential decay. Based on the plots we fit a $APBL(1, 1)$ model to $\hat{Y}_t$.

Figure 3.6: Transformed Data, ACF, PACF and $\hat{\rho}(1, k)$ for Airline Data

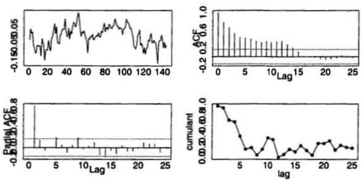
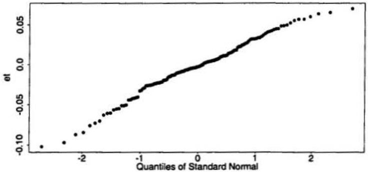


Figure 3.7: Plot of Airline Passengers e_t's



To judge the performance of the $APBL(1,1)$ model(i.e validation of model), we removed the last k, $k = 1, 2, \ldots, 10$ observations from the total observations $n = 144$, then fitted the model to the first $n - k$ observations and predicted the $(n - k + 1)th$

observation removed initially. That is, we obtained a one-step-ahead forecast.The predicted observations were then compared to the original values from the data. In a similar fashion as for the $APBL(1,1)$ model above , we obtained a one-step-ahead forecast using a linear model. Suppose $Y(t)$ is the series of interest, when at time $t = t_0$, we want to forecast a future value $Y(t_0 + h)$ given $\{Y(h), -\infty < h \leq t_0\}$. Let this predicted values be denoted by $\hat{Y_{t_0}}(h)$. We use the fact that ;

$E[Y(t_0 + h) - \hat{Y_{t_0}}(h)]^2$ is minimum if and only if, $\hat{Y_{t_0}}(h) = E[Y(t_0 + h)|Y(h), h \leq t_0]$. The $Y_{t_0}(1)$ values for the ten observations are obtained separately using both linear and $APBL(1,1)$ models. One of the ten fitted linear and $APBL(1,1)$ models on the Y_t are given below.

Autoregressive Model(AR)

Using the PACF plot in Figure 3.7, we fitted AR(1) models to the stationary component, Y_t when k observations are removed. We fitted the following model when the last observation was removed,

$$Y_t = 0.7841 Y_{t-1} + e_t.$$

$APBL(1,1)$ Model

Similarly, the following bilinear model was fitted on Y_t, with the last observation removed,

$$Y_t = 0.0561 Y_{t-1} + 2.657 Y_{t-1} e_{t-1} + e_t.$$

The estimated X_t are then obtained using the $Y_{t_0}(1)$'s and re-transforming. The original and re-transformed predicted values of X_t at time $t = 135, 135, \ldots, 144$, using both linear and $APBL(1,1)$ models are shown in Table 3.7. A Q-Q plot of the $\hat{e_t}$'s is used to examine the assumption for normality. From the plot shown in Figure 3.7, the assumption of normality seems plausible.

Table 3.7: Original and Predicted Values for $APBL(1,1)$ and Linear Models

Original Values	Predicted Values	
	Bilinear	Linear
432	451	474
390	406	418
461	454	478
508	534	547
606	625	627
622	609	626
535	541	554
472	469	486
461	421	480
419	464	499

The mean absolute deviation of one-step-ahead forecast errors for the ten values of the $APBL(1,1)$ model is 19.4 and for the linear model is 28.3. This result shows that the $APBL(1,1)$ model is quite better for the airline passengers data than the linear model.

3.3.2 Annual Wolfer Sunspot Number(1700-1988)

The annual Wolfer sunspot numbers data is given in the Table 2 of the Appendix. It is a series that measures the extent of the visible surface of the sun that covered by sunspots. This series has been studied by several researchers using different methods. A few of previous work on this data set can be found in Box and Jenkins(1970), Granger and Andersen(1978) and Tong(1990) books. A plot of the data and the ACF are given by Figures 3.8 and 3.9 respectively.

Figure 3.8: Plot of Sunspot Numbers

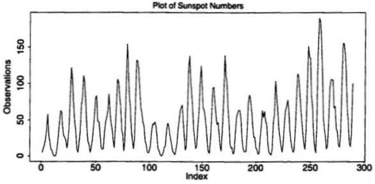
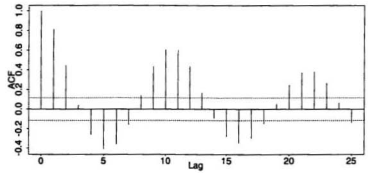


Figure 3.9: Plot ACF of Sunspot Numbers



From the plots of the data and the ACF, it is obvious that we need to transform the data to be able to apply the $APBL(1,1)$ model. To reduce variability across the series, we took logarithm, while adding one to each of the observations as there are zeros in the data set. Next we differenced the data three times to make the series stationary. The plot of the transformed data set, the ACF, PACF and the $\hat{\rho}(1,k)$ are given by Figure 3.10. A normal plot of the e's was used to investigate the assumption of normality of the errors, e_t. Figure 3.11 shows that the e_t's are approximately normal.

Figure 3.10: Transformed Data, ACF, PACF and $\hat{\rho}(1,k)$ for Sunspot Numbers

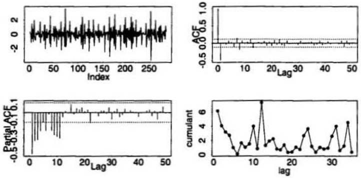
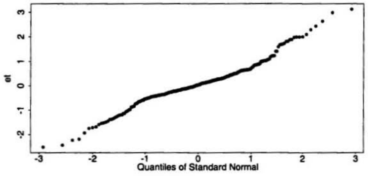


Figure 3.11: Plot of Sunspot Numbers e_t's



Using similar procedure as for the international airline passengers data, we made a one-step forecast using both $APBL(1,1)$ and "best" linear models. Then predicted values were re-transformed back, so that they could be compared to the original observations and those obtained using linear models. One of the ten fitted linear and $APBL(1,1)$ models are given below.

Moving Average Model(MA)

Using the ACF plot in Figure 3.10, we fitted an $MA(1)$ model to the stationary component, Y_t when the last observation is removed;

$$Y_t = 0.9571e_{t-1} + e_t.$$

$APBL(1,1)$ Model

The following bilinear model is fitted on Y_t with the last observation removed;

$$Y_t = -0.4611Y_{t-1} - 0.095Y_{t-1}e_{t-1} + e_t.$$

The estimated X_t are then obtained using the $Y_{t_0}(1)$'s and re-transforming. The original and re-transformed predicted values of X_t at time $t = 280, 281, \ldots, 289$ using both linear and $APBL(1,1)$ models are shown in Table 3.8.

Table 3.8: Original and Predicted Values for $APBL(1,1)$ and Linear Models

Original Values	Predicted Values	
	Bilinear	Linear
17.9	30.845427	38.038485
45.9	13.389178	26.362325
66.6	117.859525	86.793594
115.9	88.775325	116.392467
140.5	102.370354	91.250176
154.7	132.795550	132.389079
155.4	385.783023	479.218803
92.5	140.747714	150.845456
27.5	27.960497	18.878941
12.6	3.858637	2.915898

The mean absolute deviation of one-step-ahead forecast errors for the ten values using the bilinear $APBL(1,1)$ model, is 47.17 and for the linear model is 53.24. Although the $APBL(1,1)$ model seems to make better prediction than the $MA(1,1)$ model, from the predicted values we note that the difference between most of the predicted values and the original values are large. Thus the $APBL(1,1)$ and the $MA(1,1)$ are not suitable for analyzing this data.

3.3.3 IBM Common Stock Closing Prices

The daily IBM stock prices during a period of 18 May 1961 to 30 March 1962 is given in Table 3 of the Appendix. Usually the stock market closes on weekends and holidays, leading to missing observations. To avoid any complications that this may cause, we treat partial observations as full. Other time series analysts who have analyzed this data in a similar way are Box and Jenkins(1970) and Tong(1990). An alternative approach would be to use imputation techniques to estimate the missing observations before modelling the data. We have not done that here because the emphasis of this practicuum is on using the patterns in the third order cumulants for modeling. A plot of the data and the ACF are given in Figures 3.12 and 3.13 respectively.

Figure 3.12: A Plot of IBM data

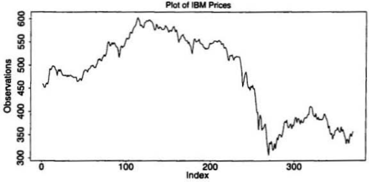
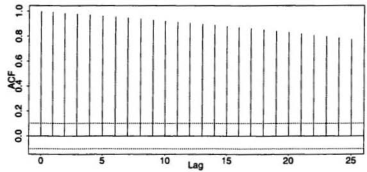


Figure 3.13: Plot ACF of IBM data



In analyzing the data, we took the logarithm and differenced once in an attempt to stabilize the mean and variance.The ACF plot does not decay very fast suggesting some problem with the data arising from the trend. The plot of the transformed data set, the ACF, PACF and the $\hat{\rho}(1,k)$ are given by Figure 3.14. A normality plot of the random error e_t (Figure 3.15) shows that the normality of the e_t can be assumed.

Figure 3.14: Transformed Data, ACF, PACF and $C(1,k)$ for IBM data

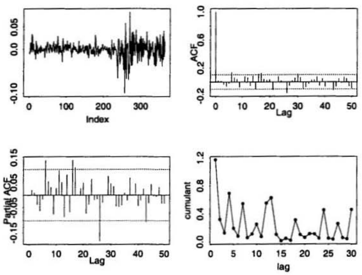
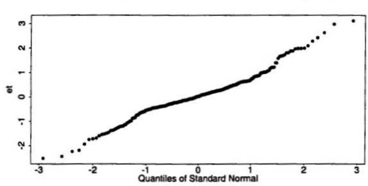


Figure 3.15: Plot of Airline Passengers e_t's



The ACF and PACF plots appear to suggest that the series Y_t is white noise. In order to verify this, we obtained the ACF plot of Y_t^2. It is well known that if Y_t is a white noise then Y_t^2 should also be a white noise. However, the ACF plot of Y_t^2 which we do not display here indicates that an ARMA model is more appropriate. This is a typical behavior of a bilinear series which has a masking effect on the ACF. For this reason we only fitted the bilinear model to the data and compared the result with the original values in a similar fashion as was done for the international passengers data. One of the fitted ten bilinear models is given below.

$APBL(1,1)$ Model

The following $APBL(1,1)$ model is fitted to Y_t with the last observation removed;

$$Y_t = 0.0561Y_{t-1} + 2.657Y_{t-1}e_{t-1} + e_t.$$

The estimated X_t are then obtained using the $Y_{t_0}(1)$'s and re-transforming. The original, re-transformed predicted values of X_t at time $t = 360, 361, \ldots, 369$. using the bilinear models are shown in Table 3.9.

Table 3.9: Original and Predicted Values for $APBL(1,1)$ and Linear Models

Original Values	Predicted Values
357	352.2045
352	346.3503
346	351.6824
352	345.4097
345	332.9585
331	338.6820
339	339.6675
340	331.0940
330	342.9945
343	347.5481

The mean absolute deviation of errors of one-step-ahead forecast errors for the ten values of the $APBL(1,1)$ model is 6.96. This value is quite small. This shows that the predicted and original values are very close. The appropriateness of the bilinear model is for the IBM data is evident in the predicted values.

Chapter 4

CONCLUSION

The general bilinear model is a time series with a number of special cases which can be studied. Several analysts have studied a variety of special cases of this model. See for instance Granger and Andersen(1978), Gabr(1988), Subba Rao (1981). This study is devoted to the $APBL(1,1)$ model with little extension to more complicated versions of the bilinear model; $APBL(1,2)$, $APBL(1,3)$ and $APBL(1,q)$.

We studied the mean, covariance structure, third order moments and third order cumulants of the $APBL(1,1)$ model. Simulation studies to check the performance of the derived properties yielded commendable results - see Tables 3.1-3.4. One major goal of this study was to investigate the pattern in the third order cumulants of the model in order to use it for bilinear model identification. In his study, Oyet showed that the $\rho(1,k)$ of the diagonal pure bilinear model $DPBL(q)(1.1)$ cuts off after lag $q+1$. This study showed that the $\rho(1,k)$ of the $APBL(1,1)$ decays exponentially as the lags increase. This result was confirmed by simulation studies. Thus from the foregoing, given a time series whose underlying model is unknown but is thought to follow either the $DPBL(q)$ or $APBL(1,1)$ model, the methods outlined in this study can be used to identify the right model depending on whether the $\rho(1,k)$ cuts off after lag $q+1$ or decays exponentially.

Another useful results of this work are the difference equations for the second order moments and third order cumulants of X_t satisfying the $APBL(1,q)$ model for any real valued q. As can be seen in Chapter 2, remarkable patterns can be observed in the properties of the different versions of the bilinear model. We found that the ACF estimates can be used to obtain an initial estimate of ϕ_1. Our results also show that for an arbitrary q, the mean of a bilinear model $APBL(1,q)$ can be expressed as ,

$$\mu = \frac{\sigma^2 \sum_{i=1}^{q} \theta_i}{1-\phi_1}.$$

For example, when $q = 1\&2$ the mean of X_t satisfying $APBL(1,1)$ and $APBL(1,2)$ models are given by,

$$\frac{\theta_1\sigma^2}{1-\phi_1}$$

and

$$\frac{(\theta_1+\theta_2)\sigma^2}{1-\phi_1}$$

respectively. Similar patterns for the second and third order moments are given by Lemmas (2.1-2.4) and Lemmas (2.5 & 2.6) respectively. Simulation studies using the $APBL(1,2)$ model showed that the results are influenced by the chosen ϕ_1, θ_1 and θ_2 values used. This may be due to the violation of the stationarity and invertibility conditions for these models.

The $APBL(1,1)$ was used to make one-step-ahead forecast on three real data. This model was identified for these data based on the exponential decay observed in the plot of their $\rho(1,k)$ (see Figures 3.7,& 3.11). For the international passengers data the $APBL(1,1)$ model produced better forecasts than their corresponding "best" linear models (see Table 3.7). We found that both the $APBL(1,1)$ and $MA(1)$ models were not appropriate for the sunspot numbers based on their forecasting ability. For the IBM Prices data, no appropriate linear model could be identified from the ACF and PACF plots(see Figure 3.18). Further work on the data revealed that it is non-linear in nature. And since the general pattern on the $\hat{\rho}(1, k)$ plot of the data indicates exponential decay, we fitted the $APBL(1,1)$ model to the data. The predicted result on the IBM Prices also turned out to be very close to the original values(see Table 3.9).

This study and other studies in the literature have revealed that non-linear time series exist in all fields; business, economics, science, etc. It is therefore hoped that similar studies will be carried out on more complicated versions of the bilinear model.

Appendix A

Data Sets

Table A.1: International Airline Passengers Data

112	118	132	129	121	135	148	148	136	119	104	118
115	126	141	135	125	149	170	170	158	133	114	140
145	150	178	163	172	178	199	199	184	162	146	166
171	180	193	181	183	218	230	242	209	191	172	194
196	196	236	235	229	243	264	272	237	211	180	201
204	188	235	227	234	264	302	293	259	229	203	229
242	233	267	269	270	315	364	347	312	274	237	278
284	277	317	313	318	374	413	405	355	306	271	306
315	301	356	348	355	422	465	467	404	347	305	336
340	318	362	348	363	435	491	505	404	359	310	337
360	342	406	396	420	472	548	559	463	407	362	405
417	391	419	461	472	535	622	606	508	461	390	432

Table A.2: Sunspot Numbers Data

5.0	11.0	16.0	23.0	36.0	58.0	29.0	20.0	10.0	8.0	3.0	0.0
0.0	2.0	11.0	27.0	47.0	63.0	60.0	39.0	28.0	26.0	22.0	11.0
21.0	40.0	78.0	122.0	103.0	73.0	47.0	35.0	11.0	5.0	16.0	34.0
70.0	81.0	111.0	101.0	73.0	40.0	20.0	16.0	5.0	11.0	22.0	40.0
60.0	80.9	83.4	47.7	47.8	30.7	12.2	9.6	10.2	32.4	47.6	54.0
62.9	85.9	61.2	45.1	36.4	20.9	11.4	37.8	69.8	106.1	100.8	81.6
66.5	34.8	30.6	7.0	19.8	92.5	154.4	125.9	84.8	68.1	38.5	22.8
10.2	24.1	82.9	132.0	130.9	118.1	89.9	66.6	60.0	46.9	41.0	21.3
16.0	6.4	4.1	6.8	14.5	34.0	45.0	43.1	47.5	42.2	28.1	10.1
8.1	2.5	0.0	1.4	5.0	12.2	13.9	35.4	45.8	41.1	30.1	23.9
15.6	6.6	4.0	1.8	8.5	16.6	36.3	49.6	64.2	67.0	70.9	47.8
27.5	8.5	13.2	56.9	121.5	138.3	103.2	85.7	64.6	36.7	24.2	10.7
15.0	40.1	61.5	98.5	124.7	96.3	66.6	64.5	54.1	39.0	20.6	6.7
4.3	22.7	54.8	93.8	95.8	77.2	59.1	44.0	47.0	30.5	16.3	7.3
37.6	74.0	139.0	111.2	101.6	66.2	44.7	17.0	11.3	12.4	3.4	6.0
32.3	54.3	59.7	63.7	63.5	52.2	25.4	13.1	6.8	6.3	7.1	35.6
73.0	85.1	78.0	64.0	41.8	26.2	26.7	12.1	9.5	2.7	5.0	24.4
42.0	63.5	53.8	62.0	48.5	43.9	18.6	5.7	3.6	1.4	9.6	47.4
57.1	103.9	80.6	63.6	37.6	26.1	14.2	5.8	16.7	44.3	63.9	69.0
77.8	64.9	35.7	21.2	11.1	5.7	8.7	36.1	79.7	114.4	109.6	88.8
67.8	47.5	30.6	16.3	9.6	33.2	92.6	151.6	136.3	134.7	83.9	69.4
31.5	13.9	4.4	38.0	141.7	190.2	184.8	159.0	112.3	53.9	37.5	27.9
10.2	15.1	47.0	93.8	105.9	105.5	104.5	66.6	68.9	38.0	34.5	15.5
12.6	27.5	92.5	155.4	154.7	140.5	115.9	66.6	45.9	17.9	13.4	29.2
100.2											

Table A.3: IBM Prices Data

460	457	452	452	459	462	459	463	479	493	490	492	498	499
497	496	490	489	478	487	491	487	491	487	482	479	478	479
477	479	475	479	476	478	479	477	476	475	473	474	474	474
465	466	467	471	471	467	473	481	488	490	489	489	485	491
492	494	499	498	500	497	494	495	500	504	513	511	514	510
509	515	519	523	519	523	531	547	551	547	541	545	549	545
549	547	543	540	539	532	517	527	540	542	538	541	541	547
553	559	557	557	560	571	571	569	575	580	584	585	590	599
603	599	596	585	587	585	581	583	592	592	596	596	595	598
598	595	595	592	588	582	576	578	589	585	580	579	584	581
581	577	577	578	580	586	583	581	576	571	575	575	573	577
582	584	579	572	577	571	560	549	556	557	563	564	567	561
559	553	553	553	547	550	544	541	532	525	542	555	558	551
551	552	553	557	548	547	545	545	539	539	535	537	535	536
537	543	548	546	547	548	549	553	553	552	551	550	553	554
551	551	545	547	547	537	539	538	533	525	513	510	521	521
521	523	516	511	518	517	520	519	519	519	518	513	499	485
454	462	473	482	486	475	459	451	453	446	455	452	457	449
450	435	415	398	399	361	383	393	385	360	364	365	370	374
359	335	323	306	333	330	336	328	316	320	332	320	333	344
339	350	351	350	345	350	359	375	379	376	382	370	365	367
372	373	363	371	369	376	387	387	376	385	385	380	373	382
377	376	379	386	387	386	389	394	393	409	411	409	408	393
391	388	396	387	383	388	382	384	382	383	383	388	395	392
386	383	377	364	369	355	350	353	340	350	349	358	360	360
366	359	356	355	367	357	361	355	348	343	330	340	339	331
345	352	346	352	357									

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