Optimized Schwarz Domain Decomposition Approaches for the Generation of Equidistributing Grids

by

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Abstract

The main purpose of this thesis is to develop and analyze iterations arising from domain decomposition methods for equidistributing meshes. Adaptive methods are powerful techniques to obtain the efficient numerical solution of physical boundary value problems (BVPs) which arise from science and engineering. If a solution of a BVP has sharp changes, equidistributed mesh can give a reasonable solution for the BVP with a fixed number of mesh points. Our concern is to solve the involved nonlinear mesh BVP using optimized domain decomposition approaches and efficiently provide a nonuniform coordinate for the original boundary value problem. We derive an implicit solution on each subdomain from the optimized Schwarz method for the mesh BVP, and then introduce an interface iteration from the Robin transmission condition, which is a nonlinear iteration. Using the theory of M-functions we provide an alternate analysis of the optimized Schwarz method on two subdomains and extend this result to an arbitrary number of subdomains. M-function theory guarantees that these iterations will converge monotonically under some restriction on p, where p is the Robin parameter. The iteration can be computed by nonlinear (block) Gauss Jacobi or Gauss Seidel methods. We conclude our study with numerical experiments.

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"*Believe in yourself and all that you are. Know that there is something inside you that is greater than any obstacle*". – Christian D. Larson.

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Chapter 1

Introduction

Adaptive mesh methods are powerful techniques to obtain the efficient numerical solution of physical partial differential equations (PDEs) which arise from science and engineering. In this study r-refinement is considered for the mesh adaptation. For time dependent problems this is known as a moving mesh method. To obtain the best possible solution r-refinement relocates mesh nodes and keeps the number of mesh nodes fixed. The equidistribution principle (EP) is a standard way to generate mesh adaptation for physical PDEs.

The equidistribution principle was first introduced by de Boor in [1]. In the last decade, EP has been generalized for multidimensional mesh adaptation in [2, 3, 4, 5]. Nowadays, EP plays an indispensable role for mesh adaptation in space and time. Suppose we are given a positive measure $M(x, u)$ of the error (which is known as a mesh density or monitor function) in the solution $u(x)$ over the physical domain. The general idea of EP is that the integral of the monitor function (or, the error in the solution) is equally distributed over all mesh elements. It is expected the error in the computed solution will be large where M is large. Essentially EP concentrates mesh points in these regions.

We would like to solve a steady state boundary value problem on an equidistributing mesh. Let us consider a general steady state boundary value problem

$$\mathcal{L}\{u\} = 0 \quad u(0) = a, \quad u(1) = b, \tag{1.1}$$

where $\mathcal{L}$ is a spatial differential operator. When the BVP has a "difficult" solution, using a uniform mesh will not give us an accurate and efficient result. We transform the physical problem in the non-uniform x-coordinate to new computational uniform ξ-coordinate within the domain $\xi \in \Omega_c = [0, 1]$, where $x(0) = 0$ and $x(1) = 1$. We choose a mesh transformation $x = x(\xi)$, and wish to use a uniform mesh

$$\xi_i = \frac{i}{N}, \quad i = 0, 1, ..., N.$$

Consider a positive measure $M(x, u)$ of the error or difficulty in the solution $u(x)$, where $x \in \Omega_p$, the physical domain, and $\xi \in \Omega_c$, the computational domain. Our goal is to apply the EP to perform a mesh adaptation in space. The equidistribution principle requires

$$\int_{x_{i-1}}^{x_i} M(\tilde{x}, u) d\tilde{x} \equiv \frac{1}{N} \int_0^1 M(\tilde{x}, u) d\tilde{x}.$$

This implies that

$$\int_0^{x(\xi_i)} M(\tilde{x}, u) d\tilde{x} = \frac{i}{N}\theta \equiv \xi_i \theta, \tag{1.2}$$

where $\theta = \int_0^1 M(\tilde{x}, u) d\tilde{x}$ is the total error in the solution. Then $\frac{1}{N}\theta$ is the average error in the solution. The portion of M is equally distributed, so the error is equally distributed. Essentially M is large where the error of the computed solution is large. The continuous version of equation (1.2) is

$$\int_0^{x(\xi)} M(\tilde{x}, u) d\tilde{x} = \xi \int_0^1 M(\tilde{x}, u) d\tilde{x}.$$

Differentiating both sides of this equation with respect to ξ, we obtain

$$M(x(\xi), u) \frac{d}{d\xi} x(\xi) = \int_0^1 M(\tilde{x}, u) d\tilde{x}.$$

Again differentiating both sides of this equation with respect to ξ gives us

$$\frac{d}{d\xi}\left(M(x(\xi),u)\frac{d}{d\xi}x(\xi)\right)=0$$

with the boundary conditions $x(0)=0$ and $x(1)=1$. The equidistributing mesh transformation can be obtained by solving this nonlinear BVP for the mesh transformation $x(\xi):\Omega_c\to\Omega_p$.

Therefore, we have to solve the coupled system,

$$\mathcal{L}\{u\}=0 \quad u(0)=a, \quad u(1)=b, \tag{1.3}$$

$$\frac{d}{d\xi}\left(M\big(x(\xi),u\big)\frac{d}{d\xi}x(\xi)\right)=0 \quad x(0)=0, \quad x(1)=1. \tag{1.4}$$

The mesh generation problem itself is a two-point nonlinear BVP (1.4), and it depends on the physical solution u which is an unknown of the original PDE. The mesh is determined by solving a mesh equation which is coupled to the physical PDE of interest. Solving this resulting coupled system of equations, namely the physical PDE and the mesh BVP, gives the required physical solution on that mesh. Recent reviews of grid generation by moving mesh methods can be found in [6, 7, 8], and grid generation for the CFD problems can be found [9, 10, 11, 12, 13], and for the meteorology problems see [14, 15, 16, 17, 18]. Computational solution of physical PDEs based on equdistribution meshes can be found in [2, 19, 20, 21].

This coupled system (1.3-1.4) can be solved in two ways, simultaneously or alternately, see more detail in [7]. For the simultaneous procedure, the coupled system is considered as one large system. The main advantage of the simultaneous procedure is that standard ordinary differential equation (ODE) solvers can be used to solve the system of ODEs. The simultaneous solution however, involves a nonlinear coupling between the mesh and the physical solution, which is a major drawback.

In the alternating solution procedure a mesh x^{n+1} at the new level is generated from the mesh and the physical solution (x^n, u^n) at the current level, and then we obtain the physical solution u^{n+1} at the new level. The advantages of the alternating procedure are: the grid generation part can be coded separately then incorporated with the physical part; we can efficiently solve each piece and as a result the solution is not tightly coupled with mesh. This is basic concept of the MP procedure, where M is stands for the mesh BVP and P is stands for physical PDE. Hence, in the MP procedure the mesh BVP is integrated followed by integration of the physical PDE. The simultaneous solution procedure is mainly used for one-dimensional problems and alternate solution procedure has been applied for the multidimensional problems [7].

In addition, we would like to take advantage of parallel computing environments to solve the mesh BVP. Domain Decomposition (DD) approaches are ideally suited for parallel computation. DD methods follow a divide-and-conquer rule: partition a domain into overlapping or nonoverlapping subdomains and solve subdomain problems in a parallel or alternating approach. Composing the subdomain solutions we obtain a global solution for the problem. In steady case, DD has been applied for nonlinear PDEs in [22, 23, 24, 25, 26, 27]. We show the subdomain problems are well-defined; that is, a solution exists and is unique, for the mesh BVP in Section 2.2.1.

In this thesis we solve the mesh equation using well-known parallel Schwarz and optimized Schwarz methods. The parallel Schwarz method (PSM) is based on Dirichlet condition at the boundaries. Overlap between two consecutive subdomains is needed to ensure convergence. The convergence rate is very slow when the overlap size is small. Lions [28] first discovered an algorithm to change the Dirichlet transmission condition, and new types of conditions to obtain a convergent nonoverlapping iteration. Recently, Japhet [29] analyzed optimized Schwarz methods on a model problem using a Fourier analysis. The optimized Schwarz method (OSM) is based on Robin boundary condition, and gives convergence results without overlap between subdomains. The combination of mesh equidistribution and a DD approach gives us a parallel mesh adaptation method. This combination of mesh equidistribution and DD has previously been presented in the papers [30, 31, 32, 33]. Also some results of Chapter 2 have previously been published by Gander and Haynes in [8].

The purpose of this study is to analyze nonlinear iterations related to optimized Robin transmission conditions for the mesh equation in a nonoverlapping domain decomposition approach. We show that the subdomain BVPs for OSM for arbitrary number of subdomains is well-posed in section 3.1.4. We derive an implicit solution for each subdomain and then introduce an interface iteration from the transmission conditions using the implicit formula on each subdomain. This gives a nonlinear iteration. Gander and Haynes in [8] have previously studied the iteration for the two subdomain case for OSM using Global Peaceman-Rachford theorem from [34, p 387]. We would like to analyze the nonlinear iteration for an arbitrary number of subdomains arising from OSM. An important tool in our analysis is the theory of M-functions.

An iteration process $x^{n+1} = Bx^n + b'$, $n = 0, 1, ...$ for the linear system $Ax = b$ is convergent if a norm condition $||B|| < 1$ is satisfied. Strong spectral properties for the iteration matrix B are needed to obtain a stronger convergence result. The well-known iterative methods, Gauss Jacobi and Gauss Seidel iterations, converge to the unique solution from any initial guess for the linear system, if A is symmetric and positive definite or an M-matrix. Now we are interested in understanding conditions which guarantee stronger convergence for a nonlinear system $Fx = b$. Bers [35] was the first to generalize and analyze Gauss Jacobi and Gauss Seidel iterations for the solution of nonlinear system of equations that arise from discrete nonlinear elliptic BVPs. In particular, the requirement that A is symmetric and positive definite in linear case has been extended to the nonlinear case by Schechter [36]. If the mapping $F : D \in \mathbb{R}^n \to \mathbb{R}^n$ defining the nonlinear system is a continuous, symmetric, and has an uniformly positive definite (Frechet) derivative on all of $\mathbb{R}^n$ then the nonlinear system has a unique solution in $\mathbb{R}^n$ for any given $b \in \mathbb{R}^n$, and the nonlinear Gauss Seidel iteration converges to a unique solution, for any initial guess in $\mathbb{R}^n$ [36]. A generalization of the M-matrix condition for linear systems to particular nonlinear systems has been given by Bers [35], Birkhoff and Kellogg [37], Ortega and Rheinboldt [38, 39], and Porsching [40]. In 1969, Ortega [41] introduced M-functions on $\mathbb{R}^n$, which contains as special cases all linear mappings induced by M-matrices. If F is a continuous M-function from $\mathbb{R}^n$ onto itself then the Gauss Seidel and the Gauss Jacobi iteration converge globally for any $b \in \mathbb{R}^n$.

The general idea in this thesis is to study the nonlinear system that arises by applying OSM to the mesh BVP (1.4). our goal is to find well-posed and convergent iterations to solve this system efficiently. We can prove this system is well-posed using M-function theory under some restriction on p, where p is the parameter used in the Robin transmission condition. Supersolutions and subsolutions are also needed. The iteration can then be computed by nonlinear (block) Gauss-Jacobi or Gauss-Seidel methods. M-function theory guarantees the iterations will converge monotonically under some restriction on p. In Section 3.2, we analyze the nonlinear interface iteration (or, recurrence relation) for two subdomains, and in Section 3.3 we analyze for three subdomains. Based on the theory of M-functions we will present new convergence results for our iterations in Chapter 3.

An outline of the thesis spread over the five chapters is as follows. **Chapter 1** (this chapter) gives the objectives and scope of the thesis, relevant literature survey, introduces the equidistribution principle (EP), and gives the model problem. In **Chapter 2** we discuss moving mesh methods as determined by the EP. We discuss how mesh equations are derived from the EP for steady state problem in a single spatial dimension, and then describe some existing solution methods for the mesh BVP. We describe domain decomposition preliminaries for the nonlinear BVPs: parallel Schwarz for an arbitrary number of subdomains and optimized Schwarz methods for two subdomains. In **Chapter 3**, certain basic theorems involving M-functions, in particular the convergence of the Gauss-Seidel and Jacobi processes for such mappings, are described. Also, we study optimized Schwarz method for many subdomains and analyze the resulting nonlinear iteration using the ideas of subsolution, supersolution and M-function theory. **Chapter 4** is devoted to the numerical results. The final chapter is **Chapter 5**, which includes some important comments and provides several useful conclusions of the present research work and future research directions.

Chapter 2

Solution Methods for Mesh BVP via the Equidistribution Principle

This chapter is devoted to introduce solution methods for the mesh BVP that arise from the equidistribution principle, which was introduced in Chapter 1. When a steady state BVP has a "difficult" solution, a uniform mesh can not provide us accurate and efficient results. It is required to transform the physical nonuniform x-coordinate to a new computational ξ-coordinate by applying the equidistribution principle. Solving the resulting coupled system of equations, namely the original problem and the mesh partial differential equation (MPDE), is a challenging task in parallel. We consider solving the involved mesh nonlinear boundary value problem using single domain and parallel domain decomposition approaches, which provide an efficient nonuniform coordinate for the original problem.

2.1 Single Domain Solution for the Mesh BVP

We have introduced the mesh equation using the EP in Chapter 1. We wish to solve the mesh equation on the computational domain Ω_c. If u is given, then from (1.4) an equidistributing mesh transformation $x(\xi) : \Omega_c \to \Omega_p$ is determined by solving the BVP

$$\frac{d}{d\xi}\left(M(x)\frac{d}{d\xi}x\right) = 0 \quad x(0) = 0, \quad x(1) = 1. \tag{2.1}$$

To discretize the mesh BVP (2.1) we use a staggered mesh with either the midpoint or trapezoidal rules, then solve the resulting system by Newton's method. In addition, we verify the order of the discretization error, rate of convergence of Newton's method, and provide a comparison between the midpoint and trapezoidal rules.

2.1.1 Discretization of the Mesh Equation

To discretize the mesh BVP (2.1) on the computational domain we use a **staggered** mesh. Let us consider

$$w(\xi, x) = M(x)\frac{d}{d\xi}x,$$

then equation (2.1) becomes

$$\frac{dw}{d\xi} = 0. \tag{2.2}$$

Let x_j approximate $x(\xi_j)$, where $\xi_j = jh, j = 0, 1, ..., N+1$, $h = \frac{1}{N+1}$, $x_0 = 0$ and $x_{N+1} = 1$. Now we discretize using a short difference, approximating $w(\xi, x)$ at $\xi_{j+\frac{1}{2}}$ and $\xi_{j-\frac{1}{2}}$ by

$$w_{j+\frac{1}{2}} = M(x_{j+\frac{1}{2}})\left(\frac{x_{j+1} - x_j}{h}\right)$$

and

$$w_{j-\frac{1}{2}} = M(x_{j-\frac{1}{2}})\left(\frac{x_j - x_{j-1}}{h}\right).$$

Using the approximation for w at the midpoints and equation (2.2), we obtain

$$\frac{1}{h}\left[w_{j+\frac{1}{2}} - w_{j-\frac{1}{2}}\right] = 0,$$

which implies

$$M(x_{j+\frac{1}{2}})(x_{j+1} - x_j) - M(x_{j-\frac{1}{2}})(x_j - x_{j-1}) \;=\; 0, \tag{2.3}$$

for $j = 1, ..., N$ and $x_0 = 0$ and $x_{N+1} = 1$.

We now can apply trapezoidal rule or midpoint technique to approximate $M(x_{j+\frac{1}{2}})$ and $M(x_{j-\frac{1}{2}})$. For the **trapezoidal** case, the short averages M at points $x_{j+\frac{1}{2}}$ and $x_{j-\frac{1}{2}}$ are

$$M(x_{j+\frac{1}{2}}) \approx \frac{M(x_{j+1}) + M(x_j)}{2}$$

and

$$M(x_{j-\frac{1}{2}}) \approx \frac{M(x_j) + M(x_{j-1})}{2}.$$

So the equation (2.3) becomes

$$\Big(M(x_{j+1}) + M(x_j)\Big)(x_{j+1} - x_j) - \Big(M(x_j) + M(x_{j-1})\Big)(x_j - x_{j-1}) = 0, \quad j = 1, 2, ..., N, \tag{2.4}$$

with the boundary conditions $x_0 = 0$ and $x_{N+1} = 1$. This is a nonlinear system of algebraic equations.

For the **midpoint** case, at point $x_{j+\frac{1}{2}}$ and $x_{j-\frac{1}{2}}$, M can be approximated as

$$M(x_{j+\frac{1}{2}}) = M\Big(\frac{x_{j+1} + x_j}{2}\Big)$$

and

$$M(x_{j-\frac{1}{2}}) = M\Big(\frac{x_j + x_{j-1}}{2}\Big).$$

So the equation (2.3) becomes

$$M\Big(\frac{x_{j+1} + x_j}{2}\Big)(x_{j+1} - x_j) - M\Big(\frac{x_j + x_{j-1}}{2}\Big)(x_j - x_{j-1}) = 0 \quad j = 1, 2, ..., N, \tag{2.5}$$

with the boundary conditions $x_0 = 0$ and $x_{N+1} = 1$. This is a nonlinear system of equations using the staggered mesh and the midpoint formula.

We can solve both systems by fixed point iteration or Newton's method. To demonstrate the approach we will use Newton's method. This system can be written as

$$G(x) = 0.$$

Newton's method is given by

$$x^{n+1} = x^n - \left(\frac{\partial G}{\partial x}(x^n)\right)^{-1} G(x^n), \quad n = 0, 1, ... \tag{2.6}$$

where x^0 is an initial guess and the Jacobian is

$$\frac{\partial G}{\partial x} = \begin{bmatrix} \frac{\partial G_1}{\partial x_1} & \frac{\partial G_1}{\partial x_2} & \cdots & \frac{\partial G_1}{\partial x_N} \\ \frac{\partial G_2}{\partial x_1} & \frac{\partial G_2}{\partial x_2} & \cdots & \frac{\partial G_2}{\partial x_N} \\ \vdots & \vdots & & \vdots \\ \frac{\partial G_N}{\partial x_1} & \frac{\partial G_N}{\partial x_2} & \cdots & \frac{\partial G_N}{\partial x_N} \end{bmatrix}.$$

We require the Jacobian matrix $\frac{\partial G}{\partial x}$ for every iteration for Newton's method. To get a better form of Newton's method, we first rewrite equation (2.6) as

$$x^{n+1} - x^n = -\left(\frac{\partial G}{\partial x}(x^n)\right)^{-1} G(x^n), \quad n = 0, 1, ...$$

and rearrange to obtain

$$\left(\frac{\partial G}{\partial x}(x^n)\right)(x^{n+1} - x^n) = -G(x^n), \quad n = 0, 1, ...$$

This implies that

$$\left(\frac{\partial G}{\partial x}(x^n)\right)\delta = -G(x^n), \quad n = 0, 1, ...$$

where $\delta = x^{n+1} - x^n$. The next iteration is obtained by $x^{n+1} = x^n + \delta$. This is a better form because we avoid the explicit calculation of the Jacobian.

Example 1 Consider a two-point nonlinear boundary value problem

$$\frac{d}{d\xi}\left((x^2+1)\frac{d}{d\xi}x\right) = 0, \quad x(0) = 0, \quad x(1) = 1.$$

Here $M(x) = x^2+1 > 0$ in the given domain. Now discretizing this BVP using a staggered mesh and the Trapezoidal rule gives

$$G_j \equiv \Big(M(x_{j+1}) + M(x_j)\Big)(x_{j+1} - x_j) - \Big(M(x_j) + M(x_{j-1})\Big)(x_j - x_{j-1}) = 0,$$
$$j = 1, 2, ..., N,$$

where $M(x_j) = x_j^2 + 1$ and the boundary conditions $x_0 = 0$ and $x_{N+1} = 1$. We will solve this nonlinear system of equation using Newton's method. Let this system be

$$G(x) = 0.$$

Due to the structure of this system, the Jacobian for this problem is

$$\frac{\partial G}{\partial x} = \begin{bmatrix} \frac{\partial G_1}{\partial x_1} & \frac{\partial G_1}{\partial x_2} & 0 & \dots & 0 \\ \frac{\partial G_2}{\partial x_1} & \frac{\partial G_2}{\partial x_2} & \frac{\partial G_2}{\partial x_3} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \frac{\partial G_N}{\partial x_{N-1}} & \frac{\partial G_N}{\partial x_N} \end{bmatrix}.$$

Where, we obtain for the first point

$$G_1 = M\Big(\frac{x_2 + x_1}{2}\Big)(x_2 - x_1) - M\Big(\frac{x_1 + x_0}{2}\Big)(x_1 - x_0).$$

Differentiating G_1 with respect to x_1 and x_2, we have

$$\frac{\partial G_1}{\partial x_1} = \frac{1}{2}M'\Big(\frac{x_2 + x_1}{2}\Big)(x_2 - x_1) - M\Big(\frac{x_2 + x_1}{2}\Big)\frac{1}{2}M'\Big(\frac{x_1 + x_0}{2}\Big)(x_1 - x_0) - M\Big(\frac{x_1 + x_0}{2}\Big)$$

and

$$\frac{\partial G_1}{\partial x_2} = \frac{1}{2}M'\Big(\frac{x_2 + x_1}{2}\Big)(x_2 - x_1) + M\Big(\frac{x_2 + x_1}{2}\Big).$$

Similarly, we can evaluate the $\frac{\partial G_j}{\partial x_{j-1}}$, $\frac{\partial G_j}{\partial x_j}$ and $\frac{\partial G_j}{\partial x_{j+1}}$ entries of Jacobian matrix for j^{th} point, here $j = 2, ..., N-1$. For the last endpoint we obtain

$$G_N = M\Big(\frac{x_{N+1} + x_N}{2}\Big)(x_{N+1} - x_N) - M\Big(\frac{x_N + x_{N-1}}{2}\Big)(x_N - x_{N-1}).$$

Differentiating G_N with respect to x_{N-1} and x_N, we have

$$\frac{\partial G_N}{\partial x_{N-1}} = -\frac{1}{2}M'\Big(\frac{x_N + x_{N-1}}{2}\Big)(x_N - x_{N-1}) + M\Big(\frac{x_N + x_{N-1}}{2}\Big),$$

and

$$\frac{\partial G_N}{\partial x_N} = \frac{1}{2}M'\Big(\frac{x_{N+1}+x_N}{2}\Big)(x_{N+1}-x_N) - M\Big(\frac{x_{N+1}+x_N}{2}\Big) - \frac{1}{2}M'\Big(\frac{x_N+x_{N-1}}{2}\Big)(x_N-x_{N-1}) - M\Big(\frac{x_N+x_{N-1}}{2}\Big).$$

Likewise, we can obtain the structure of Jacobian for midpoint approach using a similar approach.

We now show the order of the discretization error is $O(h^2)$, the rate of convergence of the Newton's method is quadratic and we provide a comparison between the midpoint and trapezoidal rules in Section 2.1.2.

2.1.2 Brief Numerical Results

2.1.2.1 Order of Discretization Error

We choose different value of step sizes and compute the error for the discretization for the mesh BVP. Assume the global error with step size h is $e = ch^q$, where c is a constant, and q is the order of the method. Now we take $\log$ of both sides of $e = ch^q$, then we obtain

$$\log(e) = \log(c) + q\log(h),$$

which is the equation of a straight line with slope q. We want to find the value of q.

Figure 2.1 also shows the order of discretization error of two point nonlinear mesh BVP. We discretized the BVP using a staggered mesh and midpoint formula with a monitor function $M(x) = x^2 + 1$. The slope of the artificial red line is 2, we compare slope of the artificial line to the computed line. We chose various value of step sizes h and compare of the error e for the discretization. We see the computed (blue) error line for midpoint is parallel to the red line and, hence, the order of discretization is $q = 2$, which is written as $O(h^2)$.

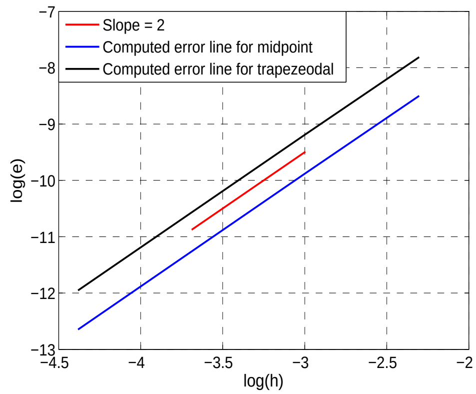


Figure 2.1: Order of discretization using midpoint and trapezoidal rule for the mesh BVP with the Dirichlet boundary conditions.

Figure 2.1 also shows a comparison between midpoint formula and the trapezoidal rules with the discretization error of nonlinear mesh BVP. We compare slope of the red line to the computed lines. The blue error line represents for the staggered mesh and the midpoint formula, and the black error line represents for the the staggered mesh and the trapezoidal rule. We can see the midpoint formula gives us a better result for the nonlinear mesh BVP, because the midpoint formula gives less error than the trapezoidal rule.

2.1.2.2 Rate of Convergence for Newton's Method

We want to compute the rate of convergence for Newton's method. First, we compute the numerical solution $\hat{x}$ for a fixed h and a Newton tolerance of 10^{-12}, then we calculate the error for each Newton step using $\hat{e}^{(k)} = ||\hat{x} - x^{(k)}||$, where $x^{(k)}$ is the numerical solution at the k-th Newton step. We assume $\hat{e}^{(k+1)} = c(\hat{e}^{(k)})^r$, where c is a constant, and r is the rate of convergence. Now taking $\log$ both sides of $\hat{e}^{(k+1)} = c(\hat{e}^{(k)})^r$, we obtain

$$\log(\hat{e}^{(k+1)}) = \log(c) + r\log(\hat{e}^{(k)}),$$

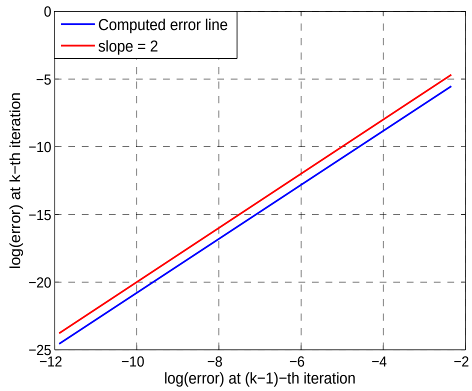


Figure 2.2: Rate of convergence of Newton's method for the mesh BVP with the Dirichlet boundary conditions.

which is the equation of a straight line with slope r. We want to find the value of r. Figure 2.2 illustrates the rate of convergence for Newton's method for Example 1. The slope of the red line is 2, and the blue line is a computed line. The two lines are parallel in this figure. Therefore, the rate of convergence for Newton's method is quadratic as expected.

2.2 Domain Decomposition Methods for the Mesh BVP

The mesh equation via the EP has been introduced in Chapter 1, and an equidistributing mesh transformation $x(\xi) : \Omega_c \to \Omega_p$ is determined by solving the BVP (2.1). Before presenting the parallel domain decomposition methods for (2.1) we introduce some preliminary results.

2.2.1 Domain Decomposition Preliminaries

In sections 2.2.2 and 2.2.3 domain decomposition methods are discussed for the solution of (2.1) with Dirichlet and Robin boundary conditions. We begin by considering (2.1) on an arbitrary subdomain $\xi \in (a,b) \subset \Omega_c = (0,1)$ with Dirichlet boundary conditions

$$\frac{d}{d\xi}\left(M(x)\frac{d}{d\xi}x\right) = 0, \quad x(a) = \zeta_a, \quad x(b) = \zeta_b. \tag{2.7}$$

This will be the subdomain problem for the nonlinear Schwarz algorithms of Sections 2.2.2 and 2.2.3. Throughout this study we consider a mesh density function $M(x)$, as

$$M(x) = \begin{cases} M(0) & \text{when} \quad x < 0 \\ M(1) & \text{when} \quad x > 1 \\ M(x) & \text{othewise,} \end{cases}$$

and assume $M(x)$ is bounded away from 0 to ∞, i.e., there exists $\check{m}$ and $\hat{m}$ such that

$$0 < \check{m} \le M(x) \le \hat{m} < \infty \quad \text{for all } x. \tag{2.8}$$

We are interested in showing that the subdomain problem is well-defined; this means the solution exists and has a unique solution. To help in this regard we use Lemmas $2.1-2.3$ from Gander and Haynes [8], which we quote below as Lemmas 2.2.1 - 2.2.3.

Lemma 2.2.1 *If M is differentiable and bounded away from 0 to ∞, i.e., satisfies (2.8), then the BVP (2.7) has a unique solution given implicitly by*

$$\int_{\zeta_a}^{x(\xi)} M(\tilde{x})d\tilde{x} = \frac{\xi - a}{b - a}\int_{\zeta_a}^{\zeta_b} M(\tilde{x})d\tilde{x}, \quad \textit{for} \quad \xi \in (a,b). \tag{2.9}$$

Proof. Integrating the differential equation (2.7) we obtain

$$M(\tilde{x})\frac{d\tilde{x}}{d\xi} = \mathcal{C}$$

where $\mathcal{C}$ is an arbitrary constant. Again integrating from a to ξ we have

$$\int_{\zeta_a}^{x(\xi)} M(\tilde{x})d\tilde{x} = \mathcal{C}(\xi - a), \quad \text{for} \quad \xi \in (a,b), \tag{2.10}$$

where the boundary condition at $\xi = a$ is satisfied, and the constant $\mathcal{C}$ is chosen to satisfy the Dirichlet boundary condition at $\xi = b$. We now want to calculate $\mathcal{C}$ using the Dirichlet boundary condition $x(b) = \zeta_b$. We obtain

$$\int_{\zeta_a}^{\zeta_b} M(\tilde{x})d\tilde{x} = \mathcal{C}(b-a),$$

this gives

$$\mathcal{C} = \frac{1}{b-a}\int_{\zeta_a}^{\zeta_b} M(\tilde{x})d\tilde{x}.$$

Substituting the value of $\mathcal{C}$ into (2.10), we arrive at the required implicit formula (2.9), that any solution of (2.7) satisfies the implicit representation.

We now want to show that there is a $x(\xi)$ satisfying the (2.9) implicit representation of the BVP (2.7). The mesh transformation $x(\xi)$ is the solution θ, of

$$\mathcal{G}(\theta) = \frac{\xi - a}{b-a}\int_{\zeta_a}^{\zeta_b} M(\tilde{x})d\tilde{x}, \tag{2.11}$$

where $\mathcal{G}(\theta)$ is defined as $\mathcal{G}(\theta) \equiv \int_{\zeta_a}^{\theta} M(\tilde{x})d\tilde{x}$. $\mathcal{G}$ is continuous since M is differentiable, and $\mathcal{G}$ is uniformly monotonic because differentiating $\mathcal{G}$ with respect to θ we obtain

$$\frac{d\mathcal{G}}{d\theta} = M(\theta) \geq \check{m} > 0.$$

Hence, by the implicit function theorem [42], there is a unique continuously differentiable solution to (2.11) and (2.9). □

Corollary 2.2.1.1 *Under the assumptions of Lemma 2.2.1, for any $\xi \in (0,1)$, the solution $x(\xi)$ which solves (2.1) satisfies the equation*

$$\int_0^{x(\xi)} M(\tilde{x})d\tilde{x} = \xi \int_0^1 M(\tilde{x})d\tilde{x}.$$

The analysis of the optimized Schwarz methods in Section 2.2.3 will require the solution of boundary value problems of the form

$$\frac{d}{d\xi}\left(M(x)\frac{d}{d\xi}x\right) = 0, \quad x(0) = 0, \quad M(x)x_\xi + px|_b = \zeta_b, \tag{2.12}$$

where p and ζ_b are constants and $b \in (0,1)$ is fixed.

Lemma 2.2.2 *Under the assumptions of Lemma 2.2.1, the BVP (2.12) has a unique solution for all $p > 0$ given implicitly by*

$$\int_0^{x(\xi)} M(\tilde{x})d\tilde{x} = \big(\zeta_b - px(b)\big)\xi, \quad \textit{for} \quad \xi \in (0, b). \tag{2.13}$$

Proof. The differential equation in (2.13) and boundary condition at $\xi = 0$ is satisfied by

$$\int_0^{x(\xi)} M(\tilde{x})d\tilde{x} = \mathcal{C}\xi, \quad \text{for} \quad \xi \in (a, b), \tag{2.14}$$

where the constant $\mathcal{C}$ is chosen to satisfy the boundary condition at $\xi = b$. Now using the Robin boundary condition $M(x)x_\xi + px|_b = \zeta_b$, we obtain

$$M(x)\frac{dx}{d\xi} + px|_b = \mathcal{C} + px(b),$$

imposing the boundary condition at $\xi = b$ gives

$$\zeta_b = \mathcal{C} + px(b),$$

which implies that

$$\mathcal{C} = \zeta_b - px(b).$$

Substituting the value of $\mathcal{C}$ into (2.14), we arrive at the implicit representation (2.13), that any solution of (2.12) satisfies the implicit representation.

We now want to show that there is a $x(\xi)$ satisfying the implicit representation (2.13) of the BVP (2.12). We first study the existence and uniqueness at the boundary $\xi = b$. Now evaluating at $\xi = b$, the boundary value $x(b)$ is the solution θ, of

$$\int_0^{\theta} M(\tilde{x})d\tilde{x} = \big(\zeta_b - p\theta\big)b,$$

or

$$\mathcal{G}(\theta) = b\zeta_b \tag{2.15}$$

where $\mathcal{G}(\theta)$ is defined as

$$\mathcal{G}(\theta) \equiv \int_0^\theta M(\tilde{x})d\tilde{x} + pb\theta.$$

$\mathcal{G}$ is continuous since M is differentiable from the assumptions of Lemma 2.2.1, and $\mathcal{G}$ is uniformly monotonic because there exists a constant $\mathcal{G}_c > 0$ such that

$$\frac{d\mathcal{G}}{d\theta} = M(\theta) + bp \geq \mathcal{G}_c.$$

Therefore, by the inverse function theorem (2.15) has a unique solution θ, which means (2.15) has a unique solution $x(b)$ at $\xi = b$. We already know $\tilde{\mathcal{G}}(\theta) = \int_0^\theta M(\tilde{x})d\tilde{x}$ is continuous and uniformly monotonic since $\frac{d\tilde{\mathcal{G}}}{d\theta} = M(\theta) \geq \check{m} > 0$ and hence has a continuously differentiable inverse by the inverse function theorem. Therefore the unique solution $x(\xi)$, for $\xi \in (0, b)$, follows by considering (2.13) for the now specified $x(b)$. $\square$

We will also be interested in solutions of Robin problems of the form

$$\frac{d}{d\xi}\left(M(x)\frac{d}{d\xi}x\right) = 0 \quad M(x)x_\xi - px|_a = \zeta_b, \quad x(1) = 1, \tag{2.16}$$

where p and ζ_a are constants and $a \in (0, 1)$ is fixed. Notice the change of sign in the boundary condition at $\xi = a$.

Lemma 2.2.3 *Under the assumptions of Lemma 2.2.1, the BVP (2.16) has a unique solution for all $p > 0$ given implicitly by*

$$\int_{x(\xi)}^1 M(\tilde{x})d\tilde{x} = \big(\zeta_a + px(a)\big)(1 - \xi), \quad \textit{for} \quad \xi \in (a, 1). \tag{2.17}$$

Proof. The differential equation in (2.16) and boundary condition at $\xi = 1$ are satisfied by

$$\int_{x(\xi)}^1 M(\tilde{x})d\tilde{x} = \mathcal{C}(1 - \xi), \quad \text{for} \quad \xi \in (a, b), \tag{2.18}$$

where $\mathcal{C}$ is chosen to satisfy the boundary condition at $\xi = a$. Now using the Robin boundary condition $M(x)x_\xi - px|_a = \zeta_a$, we obtain

$$M(x)\frac{dx}{d\xi} - px|_a = \mathcal{C} - px(a).$$

Imposing the boundary condition at $\xi = a$ gives

$$\zeta_a = \mathcal{C} - px(a),$$

which implies

$$\mathcal{C} = \zeta_a + px(a).$$

Substituting the value of $\mathcal{C}$ into (2.18), we arrive at the implicit representation (2.17), that any solution of (2.16) satisfies the implicit representation.

We now want to show that there is a $x(\xi)$ satisfying the implicit representation (2.17) of the BVP (2.16). We first study the existence and uniqueness at the boundary $\xi = a$. Now evaluating at $\xi = a$, the boundary value $x(a)$ is the solution θ, of

$$\int_\theta^1 M(\tilde{x})d\tilde{x} = \big(\zeta_a + p\theta\big)(1-a),$$

or

$$\mathcal{G}(\theta) = (1-a)\zeta_a, \tag{2.19}$$

where $\mathcal{G}(\theta)$ is defined as

$$\mathcal{G}(\theta) \equiv \int_\theta^1 M(\tilde{x})d\tilde{x} - (1-a)p\theta.$$

$\mathcal{G}$ is continuous since M is differentiable from the assumptions of Lemma 2.2.1, and $\mathcal{G}$ is uniformly monotonic because there exists a constant $\mathcal{G}_c > 0$ such that

$$\frac{d\mathcal{G}}{d\theta} = -M(\theta) - (1-a)p \leq \mathcal{G}_c < 0.$$

Therefore, by the inverse function theorem (2.19) has a unique solution θ, which means (2.19) has a unique solution $x(a)$ at $\xi = a$. We already know $\tilde{\mathcal{G}}(\theta) = \int_\theta^1 M(\tilde{x})d\tilde{x}$ is continuous and uniformly monotonic since $\frac{d\tilde{\mathcal{G}}}{d\theta} = -M(\theta) \leq -\check{m} < 0$ and has a continuously differentiable inverse. Therefore, the unique solution $x(\xi)$, for $\xi \in (a, 1)$, follows by considering (2.17) for the now specified $x(a)$.

□

Finally, we will also be interested in solutions of Robin problems of the form

$$\frac{d}{d\xi}\left(M(x)\frac{d}{d\xi}x\right) = 0 \quad M(x)x_\xi - px|_a = \zeta_a, \quad M(x)x_\xi + px|_b = \zeta_b, \tag{2.20}$$

where p, ζ_a and ζ_b are constants and $a, b \in (0,1)$ are fixed with $a < b$.

Lemma 2.2.4 *Under the assumptions of Lemma 2.2.1, the BVP (2.20) has a unique solution for all $p > 0$ given implicitly by*

$$\int_{x(a)}^{x(\xi)} M(\tilde{x})d\tilde{x} = \big(\zeta_b - px(b)\big)(\xi - a), \quad \textit{for} \quad \xi \in (a,1), \tag{2.21}$$

where $x(b) = -x(a) + \frac{1}{p}(\zeta_b - \zeta_a)$.

Proof. Integrating the differential equation (2.20), we obtain

$$M(\tilde{x})\frac{d\tilde{x}}{d\xi} = \mathcal{C}, \quad \text{for} \quad \xi \in (a,b), \tag{2.22}$$

again integrating from a to ξ, we have

$$\int_{x(a)}^{x(\xi)} M(\tilde{x})d\tilde{x} = \mathcal{C}(\xi - a), \quad \text{for} \quad \xi \in (a,b),$$

where $\mathcal{C}$ is chosen to satisfy the Robin type boundary conditions at $\xi = a$ and $\xi = b$. Using the relation $\mathcal{C} = M(x)x_\xi$ from (2.22), the Robin type boundary conditions at $\xi = a$ and $\xi = b$ can be written as

$$\mathcal{C} - px(a) = \zeta_a \quad \text{at} \quad \xi = a \tag{2.23}$$

and

$$\mathcal{C} + px(b) = \zeta_b \quad \text{at} \quad \xi = b. \tag{2.24}$$

Subtracting (2.23) from (2.24) we obtain

$$x(b) = \frac{1}{p}(\zeta_b - \zeta_a) - x(a). \tag{2.25}$$

From (2.24) we have

$$\mathcal{C} = \zeta_b - px(b).$$

Substituting the value of $\mathcal{C}$, we obtain

$$\int_{x(a)}^{x(\xi)} M(\tilde{x})d\tilde{x} = \big(\zeta_b - px(b)\big)(\xi - a), \quad \text{for} \quad \xi \in (a, 1),$$

where $x(b)$ is given in (2.25). Hence we arrive at the implicit representation (2.21), that any solution of (2.20) satisfies the implicit representation.

We now want to show that there is a $x(\xi)$ satisfying the implicit representation (2.21) of the BVP (2.20). We first study the existence and uniqueness at the boundaries. Evaluating at $\xi = b$ and substituting the value of $x(a) = \frac{1}{p}(\zeta_b - \zeta_a) - x(b)$, we have that $x(b)$ satisfies

$$\int_{\frac{1}{p}(\zeta_b - \zeta_a) - x(b)}^{x(b)} M(\tilde{x})d\tilde{x} = \big(\zeta_b - px(b)\big)(b - a). \tag{2.26}$$

Hence the boundary value $x(b)$ is the solution θ of

$$\int_{\frac{1}{p}(\zeta_b - \zeta_a) - \theta}^{\theta} M(\tilde{x})d\tilde{x} = \big(\zeta_b - p\theta\big)(b - a)$$

or

$$\mathcal{G}(\theta) = (b - a)\zeta_b \tag{2.27}$$

where $\mathcal{G}(\theta)$ is defined as

$$\mathcal{G}(\theta) \equiv \int_{\frac{1}{p}(\zeta_b - \zeta_a) - \theta}^{\theta} M(\tilde{x})d\tilde{x} + (b - a)p\theta.$$

Under the assumptions of Lemma 2.1, $\mathcal{G}$ is continuous. Moreover, $\mathcal{G}$ is uniformly monotonic; i.e., there exists a constant $\mathcal{G}_p > 0$ such that

$$\begin{aligned}\frac{d\mathcal{G}}{d\theta} &= M(\theta) - M\Big(-\theta + \frac{1}{p}(\zeta_b - \zeta_a)\Big)(-1) + (b - a)p \\ &= M(\theta) + M\Big(-\theta + \frac{1}{p}(\zeta_b - \zeta_a)\Big) + (b - a)p \geq \mathcal{G}_p > 0.\end{aligned}$$

Hence, (2.27) has a unique solution θ, which means (2.20) has a unique solution $x(b)$ at $\xi = b$. By the relation (2.25) gives a unique solution $x(a)$ at $\xi = b$. The unique, continuously differentiable solution $x(\xi)$, for $\xi \in (a, b)$, follows by considering (2.21) for the now specified $x(b)$ and noting that the map $\tilde{\mathcal{G}}(\theta) = \int_{\frac{1}{p}(\zeta_b - \zeta_a) - \theta}^{\theta} M(\tilde{x})d\tilde{x}$ is continuous and uniformly monotonic, and hence, has a continuously differentiable inverse. $\square$

2.2.2 Parallel Classical Schwarz Method

In the modern world, parallel computing environments have been used for solving complex scientific problems to reduce the computation time and improve the accuracy of the solution. We would like to take advantage of parallel computing environments for mesh generation. Domain decomposition (DD) methods are popular methods and seem ideally suited for parallel computation. In this section, we will discuss classical, parallel Schwarz iterations to solve the mesh BVP.

2.2.2.1 Parallel Classical Schwarz Method for Two Subdomains

We decompose the domain $\Omega_c = (0,1)$ into two overlapping subdomains $\Omega_1 = (0,\beta)$ and $\Omega_2 = (\alpha, 1)$ with $\alpha < \beta$,

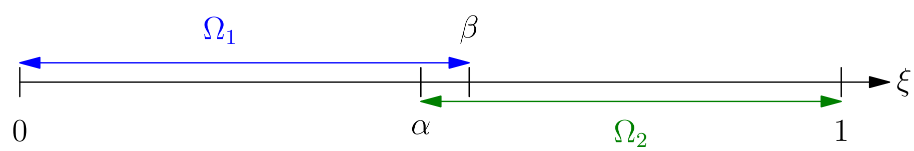


Figure 2.3: Decomposition into two overlapping subdomains.

and consider the iteration

$$
\begin{aligned}
(M(x_1^n)x_1^n{}_{,\xi})_\xi &= 0, \quad \xi \in \Omega_1, & \qquad (M(x_2^n)x_2^n{}_{,\xi})_\xi &= 0, \quad \xi \in \Omega_2,\\
x_1^n(0) &= 0, & x_2^n(\alpha) &= x_1^{n-1}(\alpha),\\
x_1^n(\beta) &= x_2^{n-1}(\beta), & x_2^n(1) &= 1.
\end{aligned}
\tag{2.28}
$$

We can solve these BVP completely independently and simultaneously. Initially, we chose arbitrary data along the artificial interfaces $\xi = \alpha$ and $\xi = \beta$. After the first iteration, they will communicate and swap the boundary data and then repeat. Now we wish to quote some useful results, Lemma 3.1 and Theorem 3.2 from Gander and Haynes [8], which we quote below as our Lemma 2.2.5 and Theorem 2.2.6.

Lemma 2.2.5 *Under the assumptions of Lemma 2.2.1, the subdomain solutions on Ω_1 and Ω_2 of (2.28) are given implicitly by the formulas*

$$\int_0^{x_1^n(\xi)} M(\tilde{x})d\tilde{x} = \frac{\xi}{\beta}\int_0^{x_2^{n-1}(\beta)} M(\tilde{x})d\tilde{x} \tag{2.29}$$

and

$$\int_{x_2^n(\xi)}^1 M(\tilde{x})d\tilde{x} = \frac{1-\xi}{1-\alpha}\int_{x_1^{n-1}(\alpha)}^1 M(\tilde{x})d\tilde{x}. \tag{2.30}$$

Proof. Simply compare the subdomain problems in (2.28) with (2.7) and use the implicit representation of the solution in (2.9). □

We will use the infinity norm defined for any function $f : (a,b) \to \mathbb{R}$ by $||f||_\infty := \sup_{x\in(a,b)} |f(x)|$.

Theorem 2.2.6 *Under the assumptions of Lemma 2.2.1, the overlapping ($\beta > \alpha$) parallel Schwarz iteration (2.28) converges for any starting values $x_1^0(\alpha)$, $x_2^0(\beta)$. Moreover, we have the linear convergence estimates*

$$||x - x_1^{2n+1}||_\infty \le \rho^n \frac{\hat{m}}{\check{m}}|x(\beta) - x_2^0(\beta)|, \quad ||x - x_2^{2n+1}||_\infty \le \rho^n \frac{\hat{m}}{\check{m}}|x(\alpha) - x_2^0(\alpha)|, \tag{2.31}$$

with contraction factor $\rho := \frac{\alpha}{\beta}\frac{1-\beta}{1-\alpha} < 1$.

Proof. Consider $\mathcal{C} := \int_0^1 M(\tilde{x})d\tilde{x}$, using Lemma 2.2.5 we can obtain

$$\begin{aligned}
\int_0^{x_1^n(\alpha)} M(\tilde{x})d\tilde{x} &= \frac{\alpha}{\beta}\int_0^{x_2^{n-1}(\beta)} M(\tilde{x})d\tilde{x} \\
&= \frac{\alpha}{\beta}\Big(\int_0^1 M(\tilde{x})d\tilde{x} - \int_{x_2^{n-1}(\beta)}^1 M(\tilde{x})d\tilde{x}\Big) \\
&= \frac{\alpha}{\beta}\Big(\mathcal{C} - \frac{1-\beta}{1-\alpha}\int_{x_1^{n-2}(\beta)}^1 M(\tilde{x})d\tilde{x}\Big) \\
&= \frac{\alpha}{\beta}\Big(\mathcal{C} - \frac{1-\beta}{1-\alpha}\Big(\mathcal{C} - \int_0^{x_1^{n-2}(\alpha)} M(\tilde{x})d\tilde{x}\Big)\Big) \\
&= \frac{\alpha}{\beta}\mathcal{C} - \frac{\alpha}{\beta}\Big(\frac{1-\beta}{1-\alpha}\Big)\Big(\mathcal{C} - \int_0^{x_1^{n-2}(\alpha)} M(\tilde{x})d\tilde{x}\Big) \\
&= \frac{\alpha}{\beta}\Big(\frac{\beta-\alpha}{1-\alpha}\Big)\mathcal{C} + \frac{\alpha}{\beta}\Big(\frac{1-\beta}{1-\alpha}\Big)\int_0^{x_1^{n-2}(\alpha)} M(\tilde{x})d\tilde{x},
\end{aligned} \tag{2.32}$$

where the third equality follows from (2.30) evaluated at $\xi = \beta$ with n replaced by $n-1$. Now defining $K_1^n = \int_0^{x_1^n(\alpha)} M(\tilde{x})d\tilde{x}$, we can obtain a linear fixed point iteration from (2.32)

$$\begin{aligned} K_1^n &= \frac{\alpha}{\beta}\Big(\frac{\beta-\alpha}{1-\alpha}\Big)\mathcal{C} + \frac{\alpha}{\beta}\Big(\frac{1-\beta}{1-\alpha}\Big)K_1^{n-2} \\ &= \frac{\alpha}{\beta}\Big(\frac{\beta-\alpha}{1-\alpha}\Big)\mathcal{C} + \rho K_1^{n-2} \end{aligned} \tag{2.33}$$

where $\rho := \frac{\alpha}{\beta}\left(\frac{\beta-\alpha}{1-\alpha}\right)$ is the contraction factor of the iteration. Clearly $\rho < 1$, therefore the iteration will converge to a limit point $K_1^* = \lim_{n\to\infty} \int_0^{x_1^n(\alpha)} M(\tilde{x})d\tilde{x}$, and K_1^* will satisfy

$$K_1^* \;=\; \frac{\alpha}{\beta}\Big(\frac{\beta-\alpha}{1-\alpha}\Big)\mathcal{C} + \frac{\alpha}{\beta}\Big(\frac{1-\beta}{1-\alpha}\Big)K_1^*,$$

or

$$\frac{1}{\beta}\Big(\frac{\beta-\alpha}{1-\alpha}\Big)K_1^* = \frac{\alpha}{\beta}\Big(\frac{\beta-\alpha}{1-\alpha}\Big)\mathcal{C},$$

which implies

$$K_1^* = \alpha\mathcal{C}. \tag{2.34}$$

Similarly, defining $K_2^n = \int_0^{x_2^n(\alpha)} M(\tilde{x})d\tilde{x}$, we can obtain a linear fixed point iteration for the second subdomain

$$\begin{aligned} K_2^n &= \Big(\frac{\beta-\alpha}{1-\alpha}\Big)\mathcal{C} + \Big(\frac{1-\beta}{1-\alpha}\Big)K_2^{n-2} \\ &= \frac{\alpha}{\beta}\Big(\frac{\beta-\alpha}{1-\alpha}\Big)\mathcal{C} + \rho K_2^{n-2}, \end{aligned} \tag{2.35}$$

where ρ is the same contraction factor as above. This iteration will also converge to a limit point $K_2^* = \lim_{n\to\infty} \int_0^{x_2^n(\alpha)} M(\tilde{x})d\tilde{x}$, and K_2^* will satisfy

$$K_2^* \;=\; \Big(\frac{\beta-\alpha}{1-\alpha}\Big)\mathcal{C} + \frac{\alpha}{\beta}\Big(\frac{1-\beta}{1-\alpha}\Big)K_2^*$$

or

$$K_2^* = \beta\mathcal{C}. \tag{2.36}$$

We can obtain from (2.34) and (2.36)

$$\lim_{n\to\infty}\int_0^{x_1^n(\alpha)} M(\tilde{x})d\tilde{x} = \alpha\int_0^1 M(\tilde{x})d\tilde{x} \quad \text{and} \quad \lim_{n\to\infty}\int_0^{x_2^n(\alpha)} M(\tilde{x})d\tilde{x} = \beta\int_0^1 M(\tilde{x})d\tilde{x}.$$

The monodomain solution x also satisfies

$$\alpha\int_0^1 M(\tilde{x})d\tilde{x} = \int_0^{x(\alpha)} M(\tilde{x})d\tilde{x} \quad \text{and} \quad \beta\int_0^1 M(\tilde{x})d\tilde{x} = \int_0^{x(\beta)} M(\tilde{x})d\tilde{x}.$$

Therefore we have convergence to the correct limit as given below

$$\lim_{n\to\infty}\int_0^{x_1^n(\alpha)} M(\tilde{x})d\tilde{x} = \int_0^{x(\alpha)} M(\tilde{x})d\tilde{x} \quad \text{and} \quad \lim_{n\to\infty}\int_0^{x_2^n(\beta)} M(\tilde{x})d\tilde{x} = \int_0^{x(\beta)} M(\tilde{x})d\tilde{x}.$$

Now it remains to prove the convergence estimate in the L^∞ norm. Subtracting (2.29) and (2.30) from the equivalent expression for $x(\xi)$ we have

$$\int_{x_1^{2n+1}(\xi)}^{x(\xi)} M(\tilde{x})d\tilde{x} = \frac{\xi}{\beta}\int_{x_1^{2n}(\beta)}^{x(\beta)} M(\tilde{x})d\tilde{x} \tag{2.37}$$

and

$$\int_{x_2^{2n+1}(\xi)}^{x(\xi)} M(\tilde{x})d\tilde{x} = \frac{1-\xi}{1-\alpha}\int_{x_2^{2n}(\alpha)}^{x(\alpha)} M(\tilde{x})d\tilde{x}. \tag{2.38}$$

Subtracting equation (2.33) from (2.34) and likewise subtracting (2.35) from (2.36) and using induction we obtain

$$\int_{x_1^{2n}(\alpha)}^{x(\alpha)} M(\tilde{x})d\tilde{x} = \rho^n\int_{x_1^{0}(\alpha)}^{x(\alpha)} M(\tilde{x})d\tilde{x} \tag{2.39}$$

and

$$\int_{x_2^{2n}(\beta)}^{x(\beta)} M(\tilde{x})d\tilde{x} = \rho^n\int_{x_2^{0}(\beta)}^{x(\beta)} M(\tilde{x})d\tilde{x}. \tag{2.40}$$

Now combining (2.40) with (2.37) and (2.39) with (2.38), we obtain

$$\int_{x_1^{2n+1}(\xi)}^{x(\xi)} M(\tilde{x})d\tilde{x} = \frac{\xi}{\beta}\rho^n\int_{x_1^{0}(\beta)}^{x(\alpha)} M(\tilde{x})d\tilde{x} \tag{2.41}$$

and

$$\int_{x_2^{2n+1}(\xi)}^{x(\xi)} M(\tilde{x})d\tilde{x} = \frac{1-\xi}{1-\alpha}\rho^n\int_{x_2^{0}(\xi)}^{x(\alpha)} M(\tilde{x})d\tilde{x}. \tag{2.42}$$

For any $a, b \in \mathbb{R}$, we have, by the boundedness of M, that there exist $\check{m}$ and $\hat{m}$ such that

$$0 < \check{m} \leq M(x) \leq \hat{m} < \infty.$$

Integrating over a to b and taking the absolute value of each term, we obtain

$$\check{m}|b-a| \leq \left| \int_a^b M(\tilde{x}) d\tilde{x} \right| \leq \hat{m}|b-a|. \tag{2.43}$$

Convergence in the interior is obtained by taking the modulus of (2.41) and using the boundedness of M. For all $\xi \in [0, \beta]$,

$$\check{m}\Big|x(\xi) - x_1^{2n+1}(\xi)\Big| \quad \leq \quad \frac{\xi}{\beta}\rho^n \hat{m}\Big|x(\beta) - x_1^0(\beta)\Big|$$

which implies

$$\Big|x(\xi) - x_1^{2n+1}(\xi)\Big| \quad \leq \quad \frac{\xi}{\beta}\rho^n \frac{\hat{m}}{\check{m}}\Big|x(\beta) - x_1^0(\beta)\Big|. \tag{2.44}$$

Similarly, for all $\xi \in [\alpha, 1]$ from (2.42),

$$\Big|x(\xi) - x_2^{2n+1}(\xi)\Big| \quad \leq \quad \frac{1-\xi}{1-\alpha}\rho^n \frac{\hat{m}}{\check{m}}\Big|x(\alpha) - x_2^0(\alpha)\Big|. \tag{2.45}$$

Taking the supremum both sides, we obtain from (2.44)

$$\text{sup}_{\xi\in[0,\beta]}\Big|x(\xi) - x_1^{2n+1}(\xi)\Big| \quad \leq \quad \text{sup}_{\xi\in[0,\beta]}\left(\frac{\xi}{\beta}\rho^n \frac{\hat{m}}{\check{m}}\Big|x(\beta) - x_2^0(\beta)\Big|\right)$$

which can be written as

$$||x(\xi) - x_1^{2n+1}(\xi)||_\infty \quad \leq \quad \rho^n \frac{\hat{m}}{\check{m}}|x(\beta) - x_2^0(\beta)|.$$

Similarly, taking the supremum of both sides on from (2.45)

$$||x(\xi) - x_2^{2n+1}(\xi)||_\infty \leq \rho^n \frac{\hat{m}}{\check{m}}|x(\alpha) - x_1^0(\alpha)|.$$

Which is the required estimate in (2.31). $\square$

We wish to present numerical results for convergence of the parallel Schwarz iteration if the overlap increases between the subdomains. Figure 2.4 shows the convergence history of the parallel Schwarz iteration (2.28) for varying amounts of overlap between the subdomains. The horizontal axis represents number of iterations and the vertical axis represents $\log$ of absolute value of DD error. Here the DD error is the infinite norm of the difference between the single domain numerical solution and the DD solution. We plot the

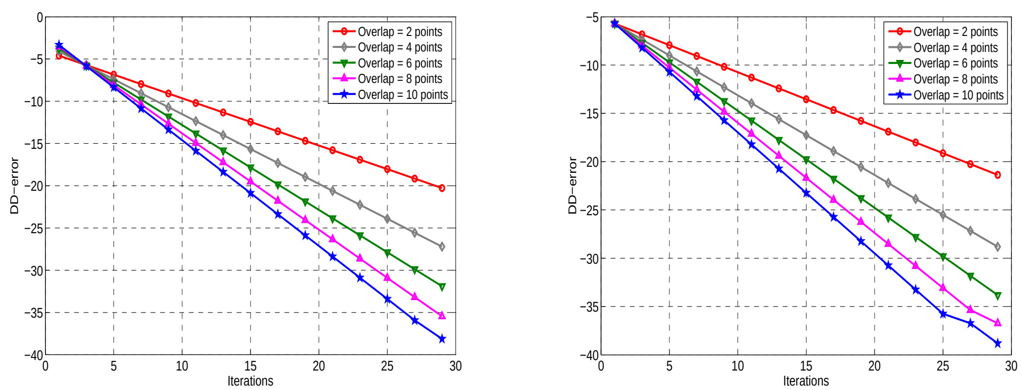


Figure 2.4: Convergence histories for parallel Schwarz iteration for different overlap on two subdomains with $M(x) = x^2 + 1$. DD error vs iterations on first subdomain (left) and right subdomain (right).

DD error at every second iterations. This figure illustrates that the convergence rate of the DD iteration improves as the overlap increases. The parallel classical Schwarz method is very slow, because it only passes Dirichlet information. This method would not converge without overlap. As a result, we are interested to build a more sophisticated transmission condition at the interface without overlap in Section 2.2.3. We will now discuss the parallel classical Schwarz on multiple subdomains in the next section.

2.2.2.2 Parallel Classical Schwarz Method for Many Subdomains

In this section we would like to extend the parallel nonlinear and classical Schwarz algorithm presented in previous section, from two subdomains to $m > 2$ overlapping subdomains. Figure 2.5 shows the decomposition of the domain into m subdomains. On the i^{th}

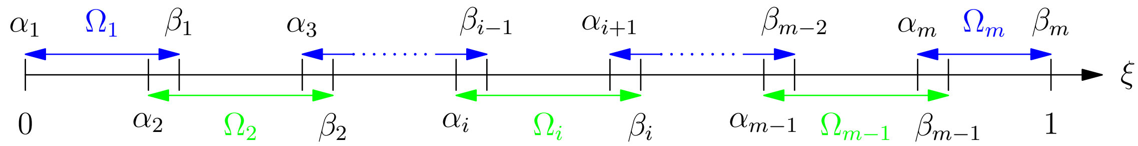


Figure 2.5: Decomposition into overlapping arbitrary number of subdomains.

subdomain, $\Omega_i = (\alpha_i, \beta_i)$, $i = 1, 2, ..., m$, $\alpha_i, \beta_i \in [0, 1]$, the boundary value problem can be written as

$$(M(x_1)x_{1,\xi})_\xi = 0, \quad x_i(\alpha_i) = x_{i-1}(\alpha_i), \quad x_i(\beta_i) = x_{i+1}(\beta_i)$$

where $\alpha_1 = 0$, $x_0(\alpha_1) = 0$, $\beta_m = 1$, and $x_{m+1}(\beta_m) = 1$. In addition we require that $\beta_i \leq \alpha_{i+2}$ for $i = 1, 2, ..., m-2$, so that there is no overlap between nonadjacent subdomains. We obtain the subdomain solution $x_i(\xi)$ on $\Omega_i = (\alpha_i, \beta_i)$ by solving the i^{th} subdomain BVP, and composing the subdomain solutions $x_i(\xi)$.

The nonlinear parallel classical Schwarz iteration can be presented as: for $n = 1, 2, ...$, solve

$$(M(x_i^n)x_i^n{}_{,\xi})_\xi = 0, \quad x_i^n(\alpha_i) = x_{i-1}^{n-1}(\alpha_i), \quad x_i^n(\beta_i) = x_{i+1}^{n-1}(\beta_i) \tag{2.46}$$

for $i = 1, 2, ..., m$, where $x_1^n(\alpha_1) \equiv 0$ and $x_{m+1}^n(\beta_m) \equiv 1$ for convenience.

This problem is studied in Gander and Haynes [8] and we quote this result in Theorem 2.2.7 below.

Theorem 2.2.7 *Under the assumptions of Lemma 2.2.1 and the restrictions on the partitioning of Ω_c detailed above, the classical Schwarz iteration (2.46) converges globally on* *an arbitrary number of subdomains.*

We wish to present a numerical experiment for the convergence of the parallel Schwarz algorithm as the number of subdomains increases. In Figure 2.6, we illustrate the convergence history of the classical parallel Schwartz iteration (2.46) for different numbers of subdomains. We plot the DD error (the infinite norm of the difference between the single

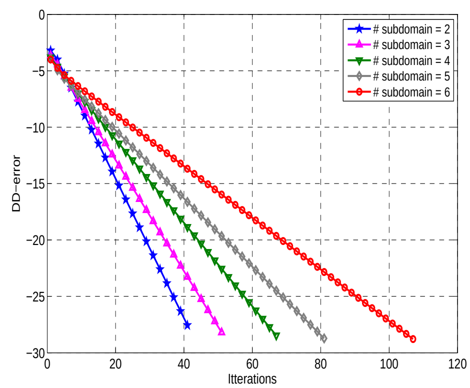


Figure 2.6: Convergence histories for parallel Schwarz iteration for different number of subdomains with $M(x) = x^2 + 1$. DD error vs iterations for 2 to 6 subdomains .

domain solution and subdomain solution) at every second iteration. This figure shows the convergence rate of the DD iteration reduces as the number of subdomains increases. This problem has been addressed in Devin Grant's B.Sc. honour's thesis [43] using a coarse correction. In the next section we will introduce the optimized Schwarz method for two subdomains.

2.2.3 Parallel Optimized Schwarz Method

The parallel classical Schwarz algorithm converges slowly and the convergence rate depends on the size of the overlap. If the overlap increases then the DD iteration converges more quickly but it is more expensive. The parallel classical Schwarz algorithm would not converge without overlap. Another way to improve convergence without overlap is to consider an alternative transmission conditions at the subdomain interfaces. In this section, we will consider nonoverlapping domain decomposition by developing a nonlinear Robin type transmission condition.

2.2.3.1 Parallel Optimized Schwarz Method for Two Subdomains

We decompose the domain $\Omega_c = [0, 1]$ into two nonoverlapping subdomains $\Omega_1 = [0, \alpha]$ and $\Omega_2 = [\alpha, 1]$ as in Figure 3.1,

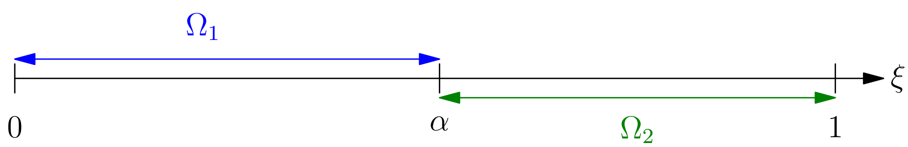


Figure 2.7: Decomposition into two nonoverlapping subdomains

and consider the parallel iteration for $n = 1, 2, ...$

$$\begin{aligned} (M(x_1^n)x_1^n, \xi)_\xi = 0, \quad \xi \in \Omega_1, \\ x_1^n(0) = 0, \\ M(x_1^n)\partial_\xi x_1^n + px_1^n|_\alpha = M(x_2^{n-1})\partial_\xi x_2^{n-1} + px_2^{n-1}|_\alpha, \end{aligned} \tag{2.47}$$

and

$$\begin{aligned} (M(x_2^n)x_2^n, \xi)_\xi = 0, \quad \xi \in \Omega_2, \\ M(x_2^n)\partial_\xi x_2^n - px_2^n|_\alpha = M(x_1^{n-1})\partial_\xi x_1^{n-1} - px_1^{n-1}|_\alpha, \\ x_2^n(1) = 1. \end{aligned} \tag{2.48}$$

Where the parameter $p > 0$ in the nonlinear Robin transmission conditions can be chosen to improve convergence. A good value of p in the transmission conditions gives quick convergence, as shown in Figure 2.8.

Lemma 2.2.8 *Under the assumptions of Lemmas 2.2.2 and 2.2.3, the subdomain solutions on Ω_1 and Ω_2 of (2.47 - 2.48) are given implicitly by the formulas*

$$\int_0^{x_1^n(\xi)} M(\tilde{x})d\tilde{x} = R_1(x_1^n(\alpha))\xi \quad \textit{and} \quad \int_{x_2^n(\xi)}^1 M(\tilde{x})d\tilde{x} = R_3(x_2^n(\alpha))(1-\xi), \tag{2.49}$$

where the operators R_1 and R_3 are given by

$$R_1(x) = \frac{1}{\alpha}\int_0^x M(\tilde{x})d\tilde{x} \quad \textit{and} \quad R_3(y) = \frac{1}{1-\alpha}\int_y^1 M(\tilde{x})d\tilde{x}. \tag{2.50}$$

The Robin conditions at the interface force the operator values to satisfy the recurrence relations:

$$R_1(x_1^n(\alpha_1)) + px_1^n(\alpha_1) = R_3(x_2^{n-1}(\alpha_1)) + px_2^{n-1}(\alpha_1) \tag{2.51}$$

and

$$R_3(x_2^n(\alpha)) - px_2^n(\alpha) = R_1(x_1^{n-1}(\alpha)) - px_1^{n-1}(\alpha). \tag{2.52}$$

Proof. For the first subdomain we integrate the nonlinear differential equation (2.47) with respect to ξ to obtain

$$\int_0^{x_1^n(\xi)} M(x_1^n)dx_1^n = C_1\xi, \quad \xi \in \Omega_1. \tag{2.53}$$

Evaluating at $\xi = \alpha$ we have

$$\int_0^{x_1^n(\alpha_1)} M(x_1^n)dx_1^n = C_1\alpha,$$

which implies

$$C_1 = R_1(x_1^n(\alpha)),$$

where $R_1(x) = \frac{1}{\alpha}\int_0^x M(\tilde{x})d\tilde{x}$. Substituting the value of C_1 into (2.53) we arrive at the implicit representation for the first subdomain.

Similarly, integrate the nonlinear differential equation (2.48) with respect to ξ for the second subdomain, we obtain

$$\int_{x_2^n(\xi)}^1 M(x_2^n)dx_2^n = C_2(1-\xi), \quad \xi \in \Omega_2. \tag{2.54}$$

Evaluating at $\xi = \alpha$ we have

$$\int_{x_2^n(\alpha)}^{1} M(x_2^n)dx_2^n = C_2(1-\alpha),$$

which gives

$$C_2 = R_3(x_2^n(\alpha)),$$

where $R_3(y) = \frac{1}{1-\alpha}\int_y^1 M(\tilde{x})d\tilde{x}$. Substituting the value of C_2 into (2.54) we arrive the implicit representation for the second subdomain on Ω_2.

Finally, we obtain the recurrence relations (2.51) and (2.52) by using the operators R_1 and R_3 to write the transmission conditions (2.47-2.48) at $\xi = \alpha$. □

The operators R_1 and R_3 defined in (2.50) are continuous and uniformly monotonic (increasing), since

$$R_1^{'}(x) = \frac{1}{\alpha_1}M(x) \geq \frac{1}{\alpha_1}\check{m} > 0 \quad \text{and} \quad -R_3^{'}(y) = \frac{1}{\alpha_1}M(y) \geq \frac{1}{1-\alpha_2}\check{m} > 0, \tag{2.55}$$

M is bounded way from 0 and ∞, as defined in (2.8). We now want to show that the iteration (2.51 - 2.52) is of the Peaceman-Rachford type; see textbook [34], and the discussion of nonlinear Peaceman-Rachford iteration in [44, 45]. This gives us a way to prove convergence of our iterations.

To derive a nonlinear Peaceman-Rachford iterations from our recurrence relations for the two subdomains, rewrite equations (2.51) at iteration $n+1$ and (2.52) as

$$\left.\begin{aligned} px_1^{n+1}(\alpha) + R_1(x_1^{n+1}(\alpha)) &= px_2^n(\alpha) + R_3(x_2^n(\alpha)) \\ px_2^n(\alpha) - R_3(x_2^n(\alpha)) &= px_1^{n-1}(\alpha) - R_1(x_1^{n-1}(\alpha)) \end{aligned}\right\}. \tag{2.56}$$

The iteration (2.56) can be written as

$$\left.\begin{aligned} p\tilde{x}^{n+1} + H\tilde{x}^{n+1} &= p\tilde{y}^n - V\tilde{y}^n \\ p\tilde{y}^n + V\tilde{y}^n &= p\tilde{x}^{n-1} - H\tilde{x}^{n-1} \end{aligned}\right\}, \tag{2.57}$$

where,

$$H = \big[R_1(x)\big], \quad V = \big[-R_3(y)\big], \quad x = x_1(\alpha) \quad \text{and} \quad y = x_2(\alpha). \tag{2.58}$$

We now present the global Peaceman-Rachford Theorem from Ortega and Rheinboldt [34] as Theorem 2.2.9.

Theorem 2.2.9 (Global Peaceman-Rachford Theorem) *Assume that the mappings $H, V : \mathbb{R}^n \to \mathbb{R}^n$ are monotone and that at least one of them is uniformly monotone. Assume further that on each compact set of $\mathbb{R}^n$ both H and V are Lipschitz continuous. Then the equation $Hx + Vx = 0$ has a unique solution x^*, and for any $x^0 \in \mathbb{R}^n$ and any $p > 0$, the sequence $\{x^k\}$ of (2.57) is well-defined and converges to x^*.*

To apply Theorem 2.2.9, we need to show H and V are monotone and at least one of them is uniformly monotone. To show H and V are monotone we will follow Theorem 2.2.10 (from [34]) below.

Theorem 2.2.10 *Let $B : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable on an open convex set $D_0 \subset D$. Then*

(a) F is monotone on D_0 if and only if $F^{'}(x)$ is positive semidefinite for all $x \in D_0$

(b) If $F^{'}(x)$ is positive definite for all $x \in D_0$, then F is strictly monotone on D_0.

(c) F is uniformly monotone on D_0 if and only if there is a $\gamma > 0$ so that $h^T F^{'} h \geq \gamma h^T h$ for all $x \in D_0$, $h \in \mathbb{R}^n$.

Theorem 2.2.10 gives us a way to prove monotonicity, strict monotonicity and uniform monotonicity. We want to verify H and V are monotone using the above theorem. To do this $H^{'}$ and $V^{'}$ need to be positive semidefinite or positive definite. Indeed, we will show that both H and V are uniformly monotone.

Lemma 2.2.11 *H and V are uniformly monotone, where H and V are defined in* (2.58).

Proof. Differentiating H with respect to x, we obtain

$$H^{'} = \left[\frac{\partial R_1}{\partial x}\right].$$

Trivially, $H^{'}$ is symmetric and $\frac{\partial R_1}{\partial x} = \frac{1}{\alpha_1}M(x) \geq \frac{\check{m}}{\alpha_1}$. So

$$h^T H^{'} h \geq \frac{\check{m}}{\alpha_1} h^T h = \gamma h^T h \quad \text{for all } h,$$

where $\gamma = \frac{\check{m}}{\alpha_1} > 0$. Hence H is uniformly monotone by Theorem 2.2.10.

Similarly, differentiating V with respect to y and we have

$$V^{'} = \left[-\frac{\partial R_3(y)}{\partial y}\right].$$

Here $V^{'}$ is also symmetric and $\frac{\partial R_3}{\partial y} = \frac{1}{1-\alpha_1}M(y) \geq \frac{\check{m}}{1-\alpha_1}$. So

$$v^T V^{'} v \geq \frac{\check{m}}{\alpha_1} v^T v = \gamma v^T v \quad \text{for all } v,$$

where $\gamma = \frac{\check{m}}{1-\alpha_1} > 0$. Hence V is uniformly monotone by Theorem 2.2.10.

□

Theorem 2.2.12 *The system* (2.57) *is well-defined and the iteration* (2.56) *converges to the unique solution for any $p > 0$.*

Proof. The assumptions of the Global Peaceman-Rachford Theorem 2.2.9 have been verified in Lemma 2.2.11. Hence, we conclude that the system (2.57) is well-defined and iteration (2.56) converges to the unique solution for any $p > 0$ by the Global Peaceman-Rachford Theorem 2.2.9. □

Theorem 2.2.13 *Under the assumptions of Lemma 2.2.1, the iteration* (2.51) *-* (2.51) *converges globally to the exact solution $x(\alpha)$ for all $p > 0$. Moreover, we have the linear* *convergence estimate*

$$||x - x_1^{2n}||_\infty \leq \frac{\hat{m}}{\check{m}} \cdot \frac{p + \frac{1}{\alpha}\hat{m}}{p + \frac{1}{\alpha}\check{m}} \rho_{robin}^n |x(\alpha) - x_1^0(\alpha)|,$$
$$||x - x_2^{2n}||_\infty \leq \frac{\hat{m}}{\check{m}} \cdot \frac{p + \frac{1}{1-\alpha}\hat{m}}{p + \frac{1}{1-\alpha}\check{m}} \rho_{robin}^n |x(\alpha) - x_2^0(\alpha)|,$$

where an estimate on the contraction factor is

$$\rho_{robin} = \sqrt{\frac{p^2 + \frac{\hat{m}^2}{(1-\alpha)^2} - 2p\frac{\check{m}}{(1-\alpha)^2}}{p^2 + \frac{\hat{m}^2}{(1-\alpha)^2} + 2p\frac{\check{m}}{(1-\alpha)^2}}} \cdot \sqrt{\frac{p^2 + \frac{\hat{m}^2}{(\alpha)^2} - 2p\frac{\check{m}}{\alpha^2}}{p^2 + \frac{\hat{m}^2}{\alpha^2} + 2p\frac{\check{m}}{\alpha^2}}}. \tag{2.59}$$

Proof. The convergence was established in Theorem 2.2.12. here we explicitly prove that the maps involved lead to the required contractions. These calculation are done generally in [34].

The iterations (2.56) can be written as

$$\big(pI + R_1\big)x_1^{n+1}(\alpha) = \big(pI + R_3\big)x_2^n(\alpha), \tag{2.60}$$
$$\big(pI - R_3\big)x_2^n(\alpha) = \big(pI - R_1\big)x_1^{n-1}(\alpha), \tag{2.61}$$

where I is the identity operator. The operators R_1 and R_3 are continuous and uniformly monotonic (increasing) from (2.55). Moreover, since $p > 0$, then $pI - R_3$ and $pI + R_1$ are also continuous, uniformly monotonic and thus, invertible. Which implies that $x_2^n(\alpha)$ and $x_1^{n+1}(\alpha)$ are well defined.

Now eliminating $x_2^n(\alpha)$ from (2.60-2.61), we obtain

$$\big(pI + R_1\big)x_1^{n+1}(\alpha) = \big(pI + R_3\big)\big(pI - R_3\big)^{-1}\big(pI - R_1\big)x_1^{n-1}(\alpha)$$
$$x_1^{n+1}(\alpha) = \big(pI + R_1\big)^{-1}\big(pI + R_3\big)\big(pI - R_3\big)^{-1}\big(pI - R_1\big)x_1^{n-1}(\alpha),$$

which gives us a recursion formula

$$x_1^{n+1}(\alpha) \equiv \mathcal{G}x_1^{n-1}(\alpha),$$

where

$$\mathcal{G} = \big(pI + R_1\big)^{-1}\big(pI + R_3\big)\big(pI - R_3\big)^{-1}\big(pI - R_1\big).$$

$\mathcal{G}$ can be written as

$$\mathcal{G} = (pI + R_1)^{-1}\mathcal{G}_2\mathcal{G}_1(pI + R_1),$$

where

$$\mathcal{G}_1 = \big(pI - R_1\big)\big(pI + R_1\big)^{-1} \quad \text{and} \quad \mathcal{G}_2 = \big(pI + R_3\big)\big(pI - R_3\big)^{-1},$$

which implies

$$x_1^{n+1}(\alpha) \equiv (pI + R_1)^{-1}\mathcal{G}_2\mathcal{G}_1(pI + R_1)x_1^{n-1}(\alpha). \tag{2.62}$$

Assume $\mathbf{D} \subset \mathbf{R}$ is a compact set. For $x, y \in \mathbf{D}$, $x \neq y$, we set $u = (pI + R_1)^{-1}x$ and $v = (pI + R_1)^{-1}y$. Then the map $\mathcal{G}_1$ satisfies

$$\begin{aligned}
\Big[\frac{||\mathcal{G}_1 x - \mathcal{G}_1 y||}{||x-y||}\Big]^2 &= \frac{(\mathcal{G}_1 x - \mathcal{G}_1 y, \mathcal{G}_1 x - \mathcal{G}_1 y)}{(x-y, x-y)} \\
&= \frac{\Big((pI - R_1)u - (pI - R_1)v, (pI - R_1)u - (pI - R_1)v\Big)}{\Big((pI + R_1)u - (pI + R_1)v, (pI + R_1)u - (pI + R_1)v\Big)} \\
&= \frac{\Big(p(u-v) + (R_1(v) - R_1(u)), p(u-v) + (R_1(v) - R_1(u))\Big)}{\Big(p(u-v) + (R_1(u) - R_1(v)), p(v-u) + (R_1(u) - R_1(v))\Big)} \\
&= \frac{p^2||u-v||^2 + 2p\big((u-v), R_1(v) - R_1(u)\big) + ||R_1(v) - R_1(u)||^2}{p^2||u-v||^2 + 2p\big((u-v), R_1(u) - R_1(v)\big) + ||R_1(u) - R_1(v)||^2} \\
&= \frac{p^2||v-u||^2 - 2p(v-u)^T\big(R_1(v) - R_1(u)\big) + ||R_1(v) - R_1(u)||^2}{p^2||u-v||^2 + 2p(u-v)^T\big(R_1(u) - R_1(v)\big) + ||R_1(u) - R_1(v)||^2} \\
&= \frac{p^2||v-u||^2 - 2p(v-u)^T\big(R_1(v) - R_1(u)\big) + ||R_1(v) - R_1(u)||^2}{p^2||u-v||^2 + 2p(u-v)^T\big(R_1(u) - R_1(v)\big) + ||R_1(u) - R_1(v)||^2} \\
&\leq \frac{(p^2 - 2p)||v-u||^2 + ||R_1(v) - R_1(u)||^2}{(p^2 + 2p)||u-v||^2 + ||R_1(u) - R_1(v)||^2} \\
&= \frac{p^2 - 2p + L}{p^2 + 2p + L} < 1,
\end{aligned}$$

where L is the Lipschitz constant of R_1. Hence $\mathcal{G}_1$ is a contraction for all $p > 0$.

Similarly, we set $u = (pI - R_3)^{-1}x$ and $v = (pI - R_3)^{-1}y$ then $\mathcal{G}_3$ is a contraction for all $p > 0$, since the operator $R_3(x)$ uniformly monotone and Lipschitz. To show this mapping $\mathcal{G}_3$ satisfies

$$
\begin{aligned}
\left[\frac{||\mathcal{G}_3 x - \mathcal{G}_3 y||}{||x-y||}\right]^2 &= \frac{(\mathcal{G}_3 x - \mathcal{G}_3 y, \mathcal{G}_3 x - \mathcal{G}_3 y)}{(x-y, x-y)} \\
&= \frac{\Big(p(u-v) + (R_3(u) - R_3(v)), p(u-v) + (R_3(u) - R_3(v))\Big)}{\Big(p(u-v) + (R_3(v) - R_3(u)), p(u-v) + (R_3(v) - R_3(u))\Big)} \\
&= \frac{p^2||u-v||^2 + 2p\big((u-v), R_3(u) - R_3(v)\big) + ||R_3(u) - R_3(v)||^2}{p^2||u-v||^2 + 2p\big((u-v), R_3(v) - R_3(u)\big) + ||R_3(v) - R_3(u)||^2} \\
&= \frac{p^2||u-v||^2 + 2p(u-v)^T\big(R_3(u) - R_3(v)\big) + ||R_3(u) - R_3(v)||^2}{p^2||u-v||^2 + 2p(v-u)^T\big(R_3(u) - R_3(v)\big) + ||R_3(u) - R_3(v)||^2} \\
&= \frac{p^2||u-v||^2 + 2p(u-v)^T\big(R_3(u) - R_3(v)\big) + ||R_3(u) - R_3(v)||^2}{p^2||u-v||^2 - 2p(u-v)^T\big(R_3(u) - R_3(v)\big) + ||R_3(u) - R_3(v)||^2} \\
&\leq \frac{(p^2 - 2p)||v-u||^2 + ||R_3(u) - R_3(v)||^2}{(p^2 + 2p)||u-v||^2 + ||R_3(u) - R_3(v)||^2}, \quad \text{since } R_3^{'}(\xi) \text{ is negative} \\
&= \frac{p^2 - 2p + L}{p^2 + 2p + L} < 1.
\end{aligned}
$$

Thus $\mathcal{G}_1$ and $\mathcal{G}_3$ are strict contractions for all $p > 0$. Hence, the iteration (2.62) written as

$$
(pI + R_1)x_1^{n+1}(\alpha) = \mathcal{G}_2\mathcal{G}_1(pI + R_1)x_1^{n-1}(\alpha), \quad \text{or} \quad z^{n+1}(\alpha) = \mathcal{G}_2\mathcal{G}_1 z^{n-1}(\alpha),
$$

where $z^n(\alpha) = (pI + R_1)x^n(\alpha)$. The iteration $z^n(\alpha) = \mathcal{G}_2\mathcal{G}_1 z^{n-2}(\alpha)$, with $z^0(\alpha) = (pI + R_1)x^0(\alpha)$, will converge. Since, $\mathcal{G} = (pI + R_1)\mathcal{G}_2\mathcal{G}_1(pI + R_1)$ and $z^{2n}(\alpha) = (pI + R_1)x^{2n}(\alpha)$, then $x_1^{2n}(\alpha)$ also converge globally for any $x_1^0(\alpha)$ to some limit $x_1^*(\alpha)$. Furthermore, since $z^{2n+1}(\alpha) = (pI + R_1)x^{2n+1}(\alpha)$, then the odd iteration $x_1^{2n+1}(\alpha)$ converges to the same limit. Similarly, the sequence $x_2^n(\alpha)$ converges globally to a limit point $x_2^*(\alpha)$. Obviously, the limit of (2.51) and (2.52) must be satisfied by the points $x_1^*(\alpha)$ and $x_2^*(\alpha)$. Adding the limits of (2.51) and (2.52) we have $x_1^*(\alpha) = x_2^*(\alpha) =: x^*(\alpha)$. Now subtracting (2.52) from (2.51) and the limit point $x^*(\alpha)$ will satisfy

$$
R_1\big(x^*(\alpha)\big) = R_3\big(x^*(\alpha)\big).
$$

This equation can be written as

$$\int_0^{x^*(\alpha)} M(\tilde{x})d\tilde{x} = \frac{\alpha}{1-\alpha}\left(\mathcal{C} - \int_0^{x^*(\alpha)} M(\tilde{x})d\tilde{x}\right)$$

where $\mathcal{C} = \int_0^1 M(\tilde{x})d\tilde{x}$. This implies that

$$\int_0^{x^*(\alpha)} M(\tilde{x})d\tilde{x} = \alpha\mathcal{C}. \tag{2.63}$$

Now we want to show $x^*(\alpha) = x(\alpha)$, where $x(\xi)$ is the global solution of the mesh BVP 2.1. From Corollary 2.2.1.1, for any $\xi \in (0,1)$, the solution $x(\xi)$ satisfies the equation

$$\int_0^{x(\xi)} M(\tilde{x})d\tilde{x} = \xi \int_0^1 M(\tilde{x})d\tilde{x}.$$

Evaluating at $\xi = \alpha$ we have

$$\int_0^{x(\alpha)} M(\tilde{x})d\tilde{x} = \alpha \int_0^1 M(\tilde{x})d\tilde{x}. \tag{2.64}$$

Hence we conclude $x^*(\alpha) = x(\alpha)$ from (2.63) and (2.64).

The contraction factor, ρ^n_{robin}, for $z^n(\alpha)$, can be found by computing the Lipschitz constant of the operator $\mathcal{G}_2\mathcal{G}_1$. The product of the Lipschitz constants of $\mathcal{G}_1$ and $\mathcal{G}_2$ is the Lipschitz constant of $\mathcal{G}_2\mathcal{G}_1$. Suppose L and $\tilde{L}$ are the Lipschitz constants for $(pI + R_1)^{-1}$ and $(pI + R_1)$, respectively, then the convergence rate of $x_1^n(\alpha)$ is related to ρ^n_{robin}, by;

$$\begin{aligned}
|x^*(\alpha) - x_1^{2n}(\alpha)| &\le L|z^*(\alpha)z_1^{2n}(\alpha)| \\
&\le L\rho^n_{robin}|z^*(\alpha) - x_1^0(\alpha)| \\
&\le L\tilde{L}\rho^n_{robin}|x^*(\alpha) - x_1^0(\alpha)|.
\end{aligned}$$

We can find that $L = (p + \frac{1}{\alpha}\check{m})^{-1}$ and $\tilde{L} = p + \frac{1}{\alpha}\hat{m}$. This together with the estimate

$$|x_1^{2n}(\xi) - x(\xi)| \le \frac{\check{m}}{\hat{m}}|x(\alpha) - x_1^{2n}(\alpha)|,$$

gives

$$|x_1^{2n}(\xi) - x(\xi)| \le \frac{\check{m}}{\hat{m}}.\frac{p + \frac{1}{\alpha}\hat{m}}{p + \frac{1}{\alpha}\check{m}}\rho^n_{robin}|x(\alpha) - x_1^0(\alpha)|.$$

Similarly, the estimate on subdomain two follows. □

We now wish to present a numerical experiment for convergence of the optimized Schwarz algorithm for different values of p. The convergence history of the optimized parallel Schwartz iteration (2.47-2.47) with $M(x) = 1 + x^2$ for different values of p is illustrated in Figure 2.8. We plot the DD error (the infinite norm of the difference between the single domain solution and subdomain solution) at every second iteration. We observed that the blue line gives less error in this figure, and the value of p is around 3. A good value of p in the transmission conditions gives quick convergence.

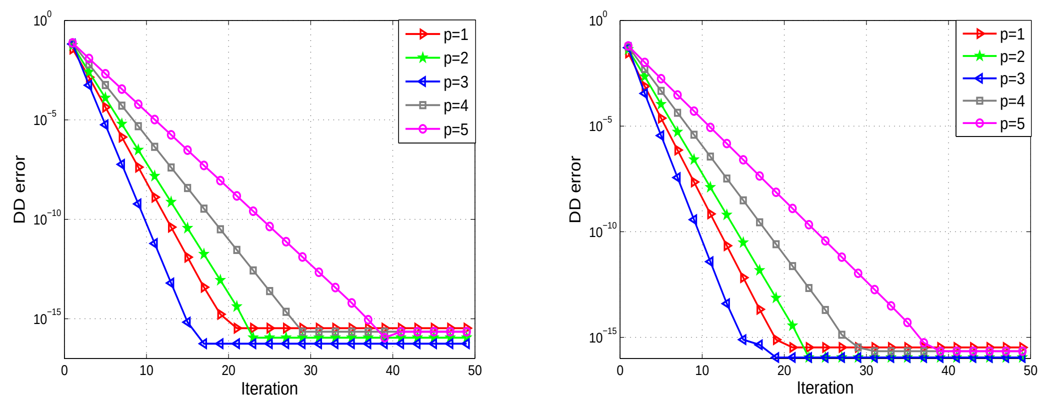


Figure 2.8: Convergence histories for parallel optimized Schwarz iteration for value of p with $M(x) = x^2 + 1$. The first subdomain results shows on the left and the second subdomain on the right.

In this chapter, we have introduced solution methods for the mesh generation problem. To discretize the mesh BVP a staggered mesh and the midpoint technique has been used and we solved the system by Newton's method. We analyzed the mesh problem for two subdomains using Parallel and optimized Schwarz method. In the next chapter we will analyze the mesh problem for the parallel optimized Schwarz method on many subdomains.

Chapter 3

Optimized Schwarz Method for an Arbitrary Number of Subdomains

This chapter is concerned with the optimized Schwarz method for an arbitrary number of subdomains. We analyze nonlinear interface iterations that arises from the optimized Schwarz method for equidistributing meshes using the theory of M-functions. The interface iterations will converge monotonically under some restriction on p, where p is used in the nonlinear Robin transmission conditions, and p can be chosen to improve convergence.

3.1 General Description

In the previous chapter, we discussed optimized Schwarz methods with nonlinear Robin transition conditions for two subdomains. We would like to extend the parallel nonlinear optimized Schwarz algorithm from two subdomains to $N > 2$ nonoverlapping subdomains. We derive an implicit interface iteration from the nonlinear Robin type transmission conditions for the optimized Schwarz iteration for an arbitrary number of subdomain. The optimized Schwarz iteration for the two subdomain case has been studied previously in

Gander and Haynes [8]. Here, we will analyze the two subdomain case in a different way, using the theory of M-functions. Then we will to extend this analysis to three and then an arbitrary number of subdomains.

In this chapter, we will use the theory of M-functions and notions of isotone and antitone maps. We begin by introducing some basic definitions in the next section.

3.1.1 Basic Definitions

Consider a nonlinear system of equations $Fx = b$, where $F : \mathbb{D} \in \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$Fx \equiv \begin{pmatrix} f_1(x_1, x_2, ..., x_n) \\ f_2(x_1, x_2, ..., x_n) \\ \vdots \\ f_n(x_1, x_2, ..., x_n) \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}. \tag{3.1}$$

Throughout this thesis, the natural partial ordering (component-wise) on the n-dimensional real linear space $\mathbb{R}^n$ is defined by

$$x \leq y, \quad x, y \in \mathbb{R}^n \quad \text{if and only if} \quad x_i \leq y_i \quad \text{for} \quad i \in \mathbb{N} = \{1, 2, ..., n\},$$

and e^i denotes the i^{th} standard basis vector in $\mathbb{R}^n$, where $i = 1, 2, ..., n$. We now begin by defining monotone, isotone and antitone mappings and then diagonally isotone, and off diagonally antitone mappings. These definitions come from [34, 46, 47, 48].

Definition 3.1.1 *A mapping $F : \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ is said to be* ***monotone*** *on $\mathbb{D}_0 \subset \mathbb{D}$ if*

$$(Fx - Fy)^T (x - y) \geq 0, \quad \forall x, y \in \mathbb{D}_0.$$

F is ***strictly monotone*** *on $\mathbb{D}_0$ if $(Fx - Fy)^T (x - y) > 0$ holds whenever $x \neq y$, and F is* ***uniformly monotone*** *if there exists a constant $\gamma > 0$, such that*

$$(Fx - Fy)^T (x - y) \geq \gamma (x - y)^T (x - y), \quad \forall x, y \in \mathbb{D}_0.$$

Definition 3.1.2 *A mapping $F : \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ is said to be* ***isotone*** *(antitone) on $\mathbb{D}$ if $x \leq y$ implies $F(x) \leq F(y)\big(F(x) \geq F(y)\big)$, for all $x, y \in \mathbb{D}$. F is* ***strictly isotone*** *(strictly antitone) on $\mathbb{D}$ if $x < y$ implies $F(x) < F(y)\big(F(x) > F(y)\big)$.*

For example, suppose $f(x) : \mathbb{D} \subset \mathbb{R} \to \mathbb{R}$ is differentiable, then f is isotone and strictly isotone if $\frac{df}{dx} \geq 0$ and $\frac{df}{dx} > 0$ respectively. Similarly, when $\frac{df}{dx} \leq 0$ and $\frac{df}{dx} < 0$ then f is antitone and strictly antitone respectively.

Definition 3.1.3 *For any fixed $x \in \mathbb{R}^n$ the n^2 functions*

$$\varphi_{ij} := t \in \mathbb{R}^1 \to \mathbb{R}^1, \quad \varphi_{ij} := f_i(x + te^j), \quad i, j \in N$$

are the ***link-functions*** *of F at x. The* ***associate network*** *$\Omega_F = \{N, \wedge_F\}$ of F consists of the set of nodes $N = 1, 2, ..., n$ and the links*

$$\wedge_F = \{(i, j) \in N \times N| \quad i \neq j, \quad \varphi_{ij} \quad \textit{not constant for some} \quad x \in \mathbb{R}^n\}.$$

A link $(i, j) \in \wedge_F$ is permanent if φ_{ij} is not constant for any $x \in \mathbb{R}^n$.

Definition 3.1.4 *A (directed) path from i to j is a sequence of links in $\wedge_F$ of the form (i, j_1), (j_1, j_2) ,..., (j_k, j), and the network is connected if any two nodes are connected by some path.*

Definition 3.1.5 *A mapping $F : \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ is said to be* ***diagonally isotone****, if for any $x \in \mathbb{D}$, the n functions $f_i(x + te^i)$, $i = 1, 2, ..., n$, are isotone when $x + te^i \in \mathbb{D}$. If the $n^2 - n$ functions $f_i(x + te^j)$, $i \neq j$, $i, j = 1, 2, ..., n$, are antitone when $x + te^i \in \mathbb{D}$, then F is called* ***off-diagonally antitone****.*

Definition 3.1.6 *Suppose $F : \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ is off-diagonally antitone. If the function $t \to f_i(\mathbf{x} + te^j)$ is strictly antitone then a link (i, j) is strict. A path $i \rightsquigarrow j$ exists if there exists a sequence of strict links $(i, j_1), (i, j_2), ..., (j_k, j)$.*

The following converse notion of isotonicity on partially ordered topological spaces was introduced by Collatz [49].

Definition 3.1.7 *A mapping $F : \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ is said to be* ***inverse isotone*** *on $\mathbb{D}$ if $F(x) \leq F(y)$ implies $x \leq y$, for any $x, y \in \mathbb{D}$.*

The following notion of an M-function was originally introduced by Ortega and developed by Rheinboldt [34].

Definition 3.1.8 *A mapping $F : \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ is said to be an M**-function** if F is* ***inverse isotone*** *and* ***off-diagonally antitone****.*

3.1.2 Iterative Methods

We will now describe, nonlinear Jacobi, nonlinear Gauss-Seidel, nonlinear Successive Over-Relaxation, and the corresponding block iterative methods to solve a nonlinear systems $Fx = b$.

The basic step of the nonlinear **Jacobi** iteration is given as :

$$\text{For} \quad k = 0, 1, ...$$
$$\begin{cases} \text{for} \quad i = 1, 2, ..., n \\ \text{solve} \quad f_i(x_1^k, ..., x_{i-1}^k, x_i, x_{i+1}^k, ..., x_n^k) = b_i \quad \text{for} \quad x_i \\ \text{and set} \quad x_i^{k+1} = x_i. \end{cases}$$

It is clear that the components of x^k are used to compute all the components x_i^{k+1} of x^{k+1} in *nonlinear Jacobi* iteration. The components $x_1^{k+1}, ..., x_{i-1}^{k+1}$ of x^{k+1} for $i > 1$ have already been computed and are expected to be better approximations to the actual solutions $x_1, ..., x_{i-1}$ than $x_1^k, ..., x_{i-1}^k$.

The nonlinear **Gauss-Seidel** iterative method is obtained by

$$
\text{For} \quad k = 0, 1, ...
\begin{cases}
\text{for} \quad i = 1, 2, ..., n \\
\text{solve} \quad f_i(x_1^{k+1}, ..., x_{i-1}^{k+1}, x_i, x_{i+1}^k, ..., x_n^k) = b_i \quad \text{for} \quad x_i \\
\text{and set} \quad x_i^{k+1} = x_i, \quad i = 1, 2, ...n.
\end{cases}
$$

If we set $x_i^{k+1} = x_i^k + \omega(x_i - x_i^k)$, for all values of ω, we obtain a nonlinear **Successive Over-Relaxation** (SOR) method, where ω is a relaxation parameter. Hence the nonlinear SOR iterative method is obtained by

$$
\text{For} \quad k = 0, 1, ...
\begin{cases}
\text{for} \quad i = 1, 2, ..., n \\
\text{solve} \quad f_i(x_1^{k+1}, ..., x_{i-1}^{k+1}, x_i, x_{i+1}^k, ..., x_n^k) = b_i \quad \text{for} \quad x_i \\
\text{and set} \quad x_i^{k+1} = x_i^k + \omega(x_i - x_i^k).
\end{cases}
\tag{3.2}
$$

Now we are going to introduce the corresponding block processes for a nonlinear system. Assume $n_1 + n_2 + ... + n_q = n$, $n_j \geq 1$, $q \geq 1$, let us consider $\mathbb{R}^n$ as a product-space $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times ... \times \mathbb{R}^{n_q}$ and we define $P_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$, $i = 1, 2, ..., q$ to be the corresponding natural projections. Then, for any $x \in \mathbb{R}^n$ we partition x as $x = (x^1, x^2, ..., x^q)$ where $x^i = P_i x$, $i = 1, 2, ..., q$, and, likewise, we can define block-components $F^i := \mathbb{R}^n \to \mathbb{R}^{n_j}$ of any mapping $F := \mathbb{R}^n \to \mathbb{R}^n$ by $F^i x = P_i F x$, $i = 1, 2, ..., q$.

The **block Gauss-Jacobi** iteration can be defined to solve for the partition $x^i \in \mathbb{R}^n$ of the system (3.1) as

$$
\text{For} \quad k = 0, 1, \ldots
$$
$$
\begin{cases}
\text{for} \quad i = 1, 2, \ldots, n_q \\
\text{solve} \quad F^i\Big((x^1)^k, \ldots, (x^{i-1})^k, x^i, (x^{i+1})^k, \ldots, (x^{n_q})^k\Big) = b^i \quad \text{for} \quad x_i \\
\text{and set} \quad (x^i)^{k+1} = x^i.
\end{cases}
$$

Similarly, the **block Gauss-Seidel** iteration can be defined for the partition $x^i \in \mathbb{R}^n$ as

$$
\text{For} \quad k = 0, 1, \ldots
$$
$$
\begin{cases}
\text{for} \quad i = 1, 2, \ldots, n_q \\
\text{solve} \quad F^i\Big((x^1)^{k+1}, \ldots, (x^{i-1})^{k+1}, x^i, (x^{i+1})^k, \ldots, (x^{n_q})^k\Big) = b^i \quad \text{for} \quad x_i \\
\text{and set} \quad (x^i)^{k+1} = x^i.
\end{cases} \tag{3.3}
$$

3.1.3 Fourier-Motzkin Elimination

We now wish to describe the Fourier-Motzkin elimination method to solve a system of linear inequalities. This description is primarily based on the article by Bradley and Wahi [50] and book by Dantzig and Thapa [51]. The Fourier-Motzkin Elimination method has been used for solving linear programming problems. It was proposed by Fourier [52] and reintroduced by Motzkin [53]. This elimination method is a useful part of our analysis.

Consider a system of linear inequalities

$$
\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \ldots, m. \tag{3.4}
$$

We can rewrite this system as a matrix form

$$
Ax \leq b
$$

where the dimension of the matrix is $m \times n$. We wish to know whether or not the system (3.4) is feasible; a feasible solution is a solution that satisfies all inequalities in (3.4), and the set of all possible solutions is known as feasible region, or solution space. If the system (3.4) is feasible, then we want to determine a particular feasible vector.

The Fourier-Motzkin elimination process eliminates the variable x_k by:

1. According to the coefficient of x_k we partition the m inequalities into three groups I_-, I_+, and I_0, depending on the sign of the coefficient. The groups are defined below:

$$I_- = \{r : a_{rk} < 0\}$$

$$I_+ = \{s : a_{sk} > 0\}$$

$$I_0 = \{t : a_{tk} = 0\}$$

2. Eliminate x_k, and obtain the resulting system of linear inequalities as shown below:

   (a) The inequalities in the set I_- for every component $r \in I_-$ can be written as

$$\frac{1}{a_{rk}}\big(b_r - \sum_{j\neq k}^{n} a_{rj}x_j\big) \leq x_k.$$

   (b) The inequalities in the set I_+ for every component $s \in I_+$ can be written as

$$x_k \leq \frac{1}{a_{sk}}\big(b_s - \sum_{j\neq k}^{n} a_{sj}x_j\big).$$

   (c) The inequalities in I_0 can be written as

$$\sum_{j\neq k}^{n} a_{tj}x_j \leq b_t \quad \text{for} \quad t \in I_0.$$

3. To have compatible inequalities, for every $r \in I_-$ and $s \in I_+$ we require

$$\frac{1}{a_{rk}}\big(b_r - \sum_{j\neq k}^{n} a_{rj}x_j\big) \leq \frac{1}{a_{sk}}\big(b_s - \sum_{j\neq k}^{n} a_{sj}x_j\big)$$

Therefore, the equivalent system of inequalities with x_k eliminated are

$$\left.\begin{aligned} \sum_{j\neq k}^{n}\Big(\frac{a_{sj}}{a_{sk}}-\frac{a_{rj}}{a_{rk}}\Big)x_j \leq \big(\frac{b_s}{a_{sk}}-\frac{b_r}{a_{rk}}\big), \quad \forall r \in I_- \quad and \quad \forall s \in I_+ \\ \sum_{j\neq k}^{n} a_{tj}x_j \leq b_t, \quad \forall t \in I_0. \end{aligned}\right\}$$

In general this system will be larger than the original. This new system of inequalities is a reduced system. In this system, x_k does not appear in any of the inequalities. The process is repeated on remaining variables, and finally, we derive a system of inequalities with a single unknown variable. The reduced system is feasible if and only if the original system (3.4) is feasible. We observe that each inequality in the reduced system is a nonnegative combination of inequalities in (3.4). If we start with a system $Ax \leq b$ and eliminate all variables sequentially, we will arrive at a system of inequalities of the form $0 \leq b_i^{'}$, $i = 1, ..., m^{'}$. If no $b_i^{'}$ is negative, then the final system is feasible and we can work backward to obtain a feasible solution to the original system.

3.1.4 Parallel Optimized Schwarz Method for Many Subdomains

We decompose the computational domain $\Omega_c = (0, 1)$ into $m \in \mathbb{R}$ nonoverlapping subdomains $\Omega_1 = (0, \alpha_1)$, $\Omega_2 = (\alpha_1, \alpha_2)$, $\Omega_i = (\alpha_{i-1}, \alpha_i)$, and $\Omega_m = (\alpha_{m-1}, 1)$, where $\alpha_{i-1} < \alpha_i$, $i = 2, 3, ..., m$, so there is no overlap between consecutive subdomains; see Figure 3.1.

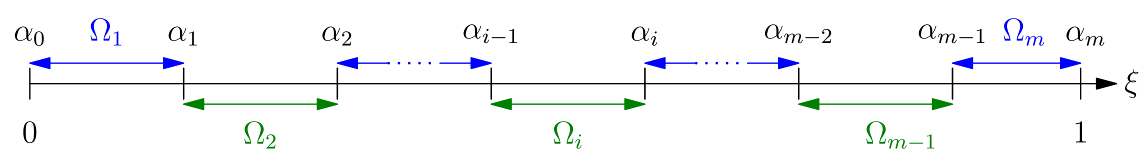


Figure 3.1: Decomposition into non-overlapping arbitrary number of subdomains.

Consider the parallel iteration

$$\left.\begin{aligned} (M(x_1^n)x_1^n{,_\xi})_\xi &= 0, \quad \xi \in \Omega_1, \\ x_1^n(0) &= 0, \\ M(x_1^n)\partial_\xi x_1^n + p x_1^n|_{\alpha_1} &= M(x_2^{n-1})\partial_\xi x_2^{n-1} + p x_2^{n-1}|_{\alpha_1}, \end{aligned}\right\} \tag{3.5}$$

$$\left.\begin{aligned} (M(x_i^n)x_i^n{,_\xi})_\xi &= 0, \quad \xi \in \Omega_i, \\ M(x_i^n)\partial_\xi x_i^n - p x_i^n|_{\alpha_{i-1}} &= M(x_{i-1}^{n-1})\partial_\xi x_{i-1}^{n-1} - p x_{i-1}^{n-1}|_{\alpha_{i-1}}, \\ M(x_i^n)\partial_\xi x_i^n + p x_i^n|_{\alpha_i} &= M(x_{i+1}^{n-1})\partial_\xi x_{i+1}^{n-1} + p x_{i+1}^{n-1}|_{\alpha_i}, \end{aligned}\right\} \tag{3.6}$$

$$i = 2, 3, ..., m-1,$$

and

$$\left.\begin{aligned} (M(x_m^n)x_m^n{,_\xi})_\xi &= 0, \quad \xi \in \Omega_m, \\ M(x_m^n)\partial_\xi x_m^n - p x_m^n|_{\alpha_{m-1}} &= M(x_{m-1}^{n-1})\partial_\xi x_{m-1}^{n-1} - p x_{m-1}^{n-1}|_{\alpha_{m-1}}, \\ x_m^n(1) &= 1. \end{aligned}\right\} \tag{3.7}$$

We observe that on the 1^{st} and m^{th} subdomains the nonlinear BVP has a Dirichlet and a Robin boundary condition, and on the inner i^{th} subdomain the BVP has a Robin boundary condition at both boundaries for $i = 2, 3, ..., m-1$. Theorem 3.1.1 tells us the subdomains problems are well-posed.

Theorem 3.1.1 *The iteration (3.5-3.7) is well-posed, that is, $x_i^n(\xi)$ exists and is unique for $i = 1, 2, ..., m$.*

Proof. Simply using Lemma 2.2.2 and Lemma 2.2.3 we conclude the 1^{st} and m^{th} subdomains BVPs (3.5) and (3.7) are well-posed. Similarly, the i^{th} subdomain problem is well-posed using Lemma 2.2.4. □

In Theorem 3.1.1, we have seen the iteration (3.5-3.7) is well-posed, that is the iterates exist and are unique. To help us study the convergence of the iteration we now derive an implicit solution on each subdomain.

Lemma 3.1.2 *The subdomain solutions on Ω_i, $i = 1, 2, ..., m$, of (3.5 - 3.7) are given implicitly by the formulae*

$$\int_0^{x_1^n(\xi)} M(\tilde{x})d\tilde{x} = R_1\big(x_1^n(\alpha_1)\big)\xi, \tag{3.8}$$

$$\int_{x_i^n(\alpha_{i-1})}^{x_i^n(\xi)} M(\tilde{x})d\tilde{x} = R_i\big(x_i^n(\alpha_{i-1}), x_i^n(\alpha_i)\big)(\xi - \alpha_{i-1}), \quad i = 2, 3, ..., m-1, \tag{3.9}$$

$$\int_{x_m^n(\xi)}^{1} M(\tilde{x})d\tilde{x} = R_m\big(x_m^n(\alpha_{m-1})\big)(1 - \xi), \tag{3.10}$$

where the operators R_1, R_i and R_m are given by

$$R_1(x) = \frac{1}{\alpha_1}\int_0^x M(\tilde{x})d\tilde{x}, \tag{3.11}$$

$$R_i(x, y) = \frac{1}{\alpha_i - \alpha_{i-1}}\int_x^y M(\tilde{x})d\tilde{x}, \quad i = 2, 3, ..., m-1, \tag{3.12}$$

and

$$R_m(x) = \frac{1}{1 - \alpha_{m-1}}\int_x^1 M(\tilde{x})d\tilde{x}. \tag{3.13}$$

Proof. For the 1^{st} subdomain we integrate the nonlinear differential equation (3.5) with respect to ξ to obtain

$$M(x_1^n)x_1^n{}_{,\xi} = C_1, \quad \xi \in \Omega_1. \tag{3.14}$$

where C_1 is an arbitrary constant. Integrating from 0 to ξ we have

$$\int_0^{x_1^n(\xi)} M(x_1^n)dx_1^n = C_1\xi, \quad \xi \in \Omega_1. \tag{3.15}$$

Evaluating at $\xi = \alpha_1$ we find

$$\int_0^{x_1^n(\alpha_1)} M(x_1^n)dx_1^n = C_1\alpha_1,$$

which implies

$$C_1 = R_1(x_1^n(\alpha_1)),$$

where $R_1(x) = \frac{1}{\alpha_1}\int_0^x M(\tilde{x})d\tilde{x}$. Substituting the value of C_1 into (3.15), we arrive the implicit representation (3.8) for the 1^{st} subdomain on Ω_1.

Secondly, for the i^{th} subdomain we integrate the nonlinear differential equation (3.6) with respect to ξ to get

$$M(x_i^n)x_i^n{}_{,\xi} = C_i, \quad \text{for} \quad \xi \in \Omega_i, \quad i = 2, 3, ..., m-1, \tag{3.16}$$

where C_i is an arbitrary constant (on each subdomain). Integrating from α_{i-1} to ξ gives

$$\int_{x_i^n(\alpha_{i-1})}^{x_i^n(\xi)} M(x_i^n)dx_i^n = C_i(\xi - \alpha_{i-1}), \quad \xi \in \Omega_i. \tag{3.17}$$

Evaluating at $\xi = \alpha_i$ we find

$$\int_{x_i^n(\alpha_{i-1})}^{x_i^n(\alpha_i)} M(x_i^n)dx_i^n = C_i(\alpha_i - \alpha_{i-1}),$$

which implies

$$C_i = R_i\big(x_i^n(\alpha_i), x_i^n(\alpha_{i-1})\big),$$

where $R_i(y, z) = \frac{1}{\alpha_i - \alpha_{i-1}}\int_y^z M(\tilde{x})d\tilde{x}$. Substituting the value of C_i into (3.17), we arrive the implicit representation (3.9) for the i^{th} subdomain on Ω_i.

Finally, for the m-th subdomain we integrate the nonlinear differential equation (3.7) with respect to ξ to obtain

$$M(x_m^n)x_m^n{}_{,\xi} = C_m, \quad \xi \in \Omega_m. \tag{3.18}$$

where C_m is an arbitrary constant. Integrating from ξ to 1 we have

$$\int_{x_m^n(\xi)}^{1} M(x_m^n)dx_m^n = C_m(1 - \xi), \quad \xi \in \Omega_m. \tag{3.19}$$

Evaluating at $\xi = \alpha_{m-1}$ we have

$$\int_{x_m^n(\alpha_{m-1})}^{1} M(x_m^n)dx_m^n = C_m(1-\alpha_{i-1}),$$

which gives

$$C_m = R_m(x_m^n(\alpha_{m-1})),$$

where $R_m(w) = \frac{1}{1-\alpha_{m-1}} \int_w^1 M(\tilde{x})d\tilde{x}$. Substituting the value of C_m into (3.19) we arrive the implicit representation (3.10) for the m^{th} subdomain on Ω_m, $\square$

Since we know the implicit solutions on each subdomain by the Lemma 3.1.2, we can build iterations at the interfaces for an arbitrary number of subdomains. We derive parrallel and alternating interface iterations. The parallel interface iteration is given by Lemma 3.1.3, and the alternating interface iteration can found in Lemma 3.1.4. The following convention will sometimes be used in this thesis,

$$R_1(x) = R_1(0, x) \quad \text{and} \quad R_m(y) = R_m(y, 1).$$

Lemma 3.1.3 (Parallel interface iteration) *The Robin conditions at the interfaces in the parallel optimized Schwarz iteration (3.5-3.7) force the operator values to satisfy the recurrence relations:*

$$R_1\big(x_1^n(\alpha_1)\big) + px_1^n(\alpha_1) = R_2\big(x_2^{n-1}(\alpha_1), x_2^{n-1}(\alpha_2)\big) + px_2^{n-1}(\alpha_1),$$

$$\left.\begin{aligned} R_i\big(x_i^n(\alpha_{i-1}), x_i^n(\alpha_i)\big) - px_i^n(\alpha_{i-1}) &= R_{i-1}\big(x_{i-1}^{n-1}(\alpha_{i-2}), x_{i-1}^{n-1}(\alpha_{i-1})\big) - px_{i-1}^{n-1}(\alpha_{i-1}) \\ R_i\big(x_i^n(\alpha_{i-1}), x_i^n(\alpha_i)\big) + px_i^n(\alpha_i) &= R_{i+1}\big(x_{i+1}^{n-1}(\alpha_i), x_{i+1}^{n-1}(\alpha_{i+1})\big) + px_{i+1}^{n-1}(\alpha_i) \end{aligned}\right\} \quad i = 2, 3, ..., m-1,$$

$$R_m\big(x_m^n(\alpha_{m-1})\big) - px_m^n(\alpha_{m-1}) = R_{m-1}\big(x_{m-1}^{n-1}(\alpha_{m-2}), x_{m-1}^{n-1}(\alpha_{m-1})\big) - px_{m-1}^{n-1}(\alpha_{m-1}), \tag{3.20}$$

where R_1, R_i, and R_m are defined in equations (3.11)-(3.13), and $x_1^n(\alpha_0) = 0$ and $x_m^n(\alpha_m) = 1$.

Proof. From equations (3.14), (3.16) and (3.18) we have

$$\begin{aligned} M(x_1^n)x_1^n{}_{,\xi} &= R_1\big(x_1^n(\alpha_1)\big), \quad \xi \in \Omega_1, \\ M(x_i^n)x_i^n{}_{,\xi} &= R_i\big(x_i^n(\alpha_i), x_i^n(\alpha_{i-1})\big), \quad \xi \in \Omega_i, \quad i = 2, ..., m, \\ M(x_m^n)x_m^n{}_{,\xi} &= R_3\big(x_m^n(\alpha_{m-1})\big), \quad \xi \in \Omega_m. \end{aligned} \tag{3.21}$$

Substituting the relations in (3.21) into the transmission conditions (3.5- 3.7) we obtain the recurrence relations in (3.20) . □

We can also obtain a sequential alternating iteration.

Lemma 3.1.4 (Alternating interface iteration) *The Robin conditions at the interfaces in the alternating Schwarz iteration force the operator values to satisfy the sequential recurrence relations:*

$$\begin{aligned} &R_1\big(x_1^{n+m-1}(\alpha_1)\big) + px_1^{n+m-1}(\alpha_1) = R_2\big(x_2^{n+m-2}(\alpha_1), x_2^{n+m-2}(\alpha_2)\big) + px_2^{n+m-2}(\alpha_1), \\ &\left.\begin{aligned} R_i\big(x_i^{n+m-i}(\alpha_{i-1}), x_i^{n+m-i}(\alpha_i)\big) - px_i^{n+m-i}(\alpha_{i-1}) =& R_{i-1}\big(x_{i-1}^{n+m-i-1}(\alpha_{i-2}), x_{i-1}^{n+m-i-1}(\alpha_{i-1})\big) \\ &- px_{i-1}^{n+m-i-1}(\alpha_{i-1}) \\ R_i\big(x_i^{n+m-i}(\alpha_{i-1}), x_i^{n+m-i}(\alpha_i)\big) + px_i^{n+m-i}(\alpha_i) =& R_{i+1}\big(x_{i+1}^{n+m-i-1}(\alpha_i), x_{i+1}^{n+m-i-1}(\alpha_{i+1})\big) \\ &+ px_{i+1}^{n+m-i-1}(\alpha_i) \end{aligned}\right\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 2, 3, ..., m-1, \\ &R_m\big(x_m^n(\alpha_{m-1})\big) - px_m^n(\alpha_{m-1}) = R_{m-1}\big(x_{m-2}^{n-1}(\alpha_{m-2}), x_{m-1}^{n-1}(\alpha_{m-1})\big) - px_{m-1}^{n-1}(\alpha_{m-1}), \end{aligned} \tag{3.22}$$

where R_1,R_i and R_m are defined in equations (3.11)-(3.13).

We now want to study recurrence relations (3.20) and (3.22) in the following section. These are nonlinear iterations and the continuous subdomain DD iterations are equivalent to the discrete interface iterations. We will show theses nonlinear iterations are well-posed and convergent under suitable restrictions. In following Section 3.2 we analyze the interface iterations (or recurrence relation) for two subdomains.

3.2 An Interface Iteration for Two Subdomains

We decompose the computational domain $\Omega_c = (0, 1)$ into two nonoverlapping subdomains $\Omega_1 = (0, \alpha_1)$, and $\Omega_2 = (\alpha_1, 1)$ as shown in Figure 3.2.

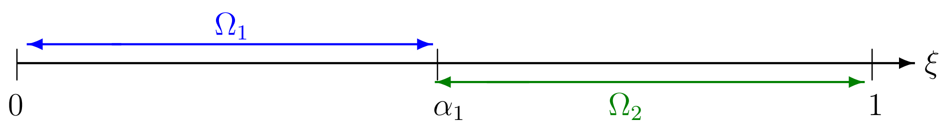


Figure 3.2: Decomposition into two nonoverlapping subdomains.

The parallel version of interface iteration on two subdomains are given from Lemma 3.1.3 as

$$R_1(x_1^n(\alpha_1)) + px_1^n(\alpha_1) = R_3(x_2^{n-1}(\alpha_1)) + px_2^{n-1}(\alpha_1), \tag{3.23}$$

$$R_3(x_2^n(\alpha_1)) - px_2^n(\alpha_1) = R_1(x_1^{n-1}(\alpha_1)) - px_1^{n-1}(\alpha_1). \tag{3.24}$$

Similarly, the alternating version of interface iteration for two subdomains from Lemma 3.1.4 gives us

$$R_1(x_1^{n+1}(\alpha_1)) + px_1^{n+1}(\alpha_1) = R_3(x_2^n(\alpha_1)) + px_2^n(\alpha_1), \tag{3.25}$$

$$R_3(x_2^n(\alpha_1)) - px_2^n(\alpha_1) = R_1(x_1^{n-1}(\alpha_1)) - px_1^{n-1}(\alpha_1). \tag{3.26}$$

The operators R_1 and R_3 are given by

$$R_1(x) = \frac{1}{\alpha}\int_0^x M(\tilde{x})d\tilde{x} \quad \text{and} \quad R_3(w) = \frac{1}{1-\alpha}\int_w^1 M(\tilde{x})d\tilde{x}. \tag{3.27}$$

We are interested in the questions of existence and uniqueness of (3.23-3.24) and (3.25-3.26) for the two subdomain case. The two subdomains case has been studied in Gander and Haynes [8], using the Peaceman-Rachford theorem, see section 2.2.3. Here we will show existence and uniqueness (well-posedness) in a different way. We wish to know if the system is well-posed for a given right-hand side of (3.23-3.24) and (3.25-3.26). Do solutions exist for the system? Are they unique? How to compute them? To show existences and uniqueness we will use Lemma 3.2.2 below. Lemma 3.2.2 has been proven by Intermediate Value Theorem, which we quote as Theorem 3.2.1 from Burden and Faires [54].

Theorem 3.2.1 (Intermediate Value Theorem) *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, and $\mu \in \mathbb{R}$ is any number between $f(a)$ and $f(b)$ then there exists a point $c \in (a, b)$ such that $f(c) = \mu$.*

To show existence and uniqueness of the solution of (3.23-3.26) we can use Lemma 3.2.2.

Lemma 3.2.2 *Assume $f : \mathbb{R} \to \mathbb{R}$ is continuous, uniformly monotonic increasing (decreasing) and*

$$\lim_{x\to\infty} f(x) = \infty(-\infty) \quad \textit{and} \quad \lim_{x\to-\infty} f(x) = -\infty(\infty) \tag{3.28}$$

then the equation $f(x) = b$ has a unique solution for any $b \in \mathbb{R}$.

Proof. First we want to show the solution exists for the equation $f(x) = b$ on $\mathbb{R}$. f is continuous by assumption.

Assume f is monotonic increasing on all of $\mathbb{R}$. Since $\lim_{x\to-\infty} f(x) = -\infty$ there exists a a so that $f(a) < b$. And since $\lim_{x\to\infty} f(x) = \infty$ there exists a c so that $f(c) > b$. By the Intermediate value theorem there exists $\hat{x} \in (a, c)$ such that $f(\hat{x}) = b$. The case where f is monotonic decreasing is handled similarly.

We now want to prove $f(x) = b$ has an unique solution. Assume the solution of $f(x) = b$ is not unique. Suppose $x^{'}$, and $x^{''}$ are two solutions of this system with $x^{'} \neq x^{''}$ with

$$f(x^{'}) = b \quad \text{and} \quad f(x^{''}) = b.$$

If $x^{'} < x^{''}$, then since f is monotonic increasing $f(x^{'}) < f(x^{''})$, which implies $b < b$, which is a contradiction. Similarly, if $x^{'} > x^{''}$, then $f(x^{'}) > f(x^{''})$, which implies $b > b$. Which is again a contradiction, hence $x^{'} = x^{''}$. Therefore $f(x) = b$ has an unique solution if f uniformly monotonic increasing. The case where f is monotonic decreasing is handled similarly.

□

3.2.1 Well-posedness of the Two Subdomain Iteration for a Given Right-Hand Side

We want to show the iterations (3.23 - 3.24) and (3.25 - 3.26) are well-defined for given right-hand side. Using Lemma 3.2.2 we will show that solution of the system (3.23 - 3.24) and (3.25 - 3.26) exists for each n. To do this for the parallel iteration we suppose that right-hand side of (3.23) and (3.24) are given. Let $\zeta_1 = R_3(x_2^{n-1}(\alpha_1)) + px_2^{n-1}(\alpha_1)$ and $\zeta_2 = -R_1(x_2^{n-1}(\alpha_1)) + px_2^{n-1}(\alpha_1)$ then (3.23) and (3.24) becomes

$$R_1(x_1^n(\alpha_1)) + px_1^n(\alpha_1) = \zeta_1 \tag{3.29}$$

and

$$-R_3(x_2^n(\alpha_1)) + px_2^n(\alpha_1) = \zeta_2. \tag{3.30}$$

In equation (3.29) and (3.30) we seek x and y that are solutions of

$$R_1(x) + px \quad = \zeta_1 \tag{3.31}$$

$$-R_3(y) + py \quad = \zeta_2. \tag{3.32}$$

For the alternating iteration we suppose that right-hand side of (3.25) and (3.26) are given. Assume $\zeta_1 = R_3(x_2^n(\alpha_1)) + px_2^n(\alpha_1)$ and $\zeta_2 = -R_1(x_2^{n-1}(\alpha_1)) + px_2^{n-1}(\alpha_1)$ then (3.25) and (3.26) becomes

$$R_1(x_1^{n+1}(\alpha_1)) + px_1^{n+1}(\alpha_1) = \zeta_1 \tag{3.33}$$

and

$$-R_3(x_2^n(\alpha_1)) + px_2^n(\alpha_1) = \zeta_2. \tag{3.34}$$

In equation (3.33) and (3.34) we seek x and y that are solutions of

$$R_1(x) + px \quad = \zeta_1 \tag{3.35}$$

$$-R_3(y) + py \quad = \zeta_2. \tag{3.36}$$

We wish to show the existence of x and y solving (3.35-3.36). This is equivalent to solving

$$\left.\begin{aligned} f_1(x) \equiv &\quad R_1(x) + px = \zeta_1 \\ f_2(y) \equiv &\ -R_3(y) + py = \zeta_2. \end{aligned}\right\} \tag{3.37}$$

This gives a system of the form $Fu = b$, where $F = (f_1, f_2)^T$ and $b = (\zeta_1, \zeta_2)^T$. It is clear that f_1 and f_2 are continuous. We notice that ζ_1 for parallel case in (3.29) and for alternating case in (3.33) are slightly different, but F has the same form. The parallel iteration (3.29-3.30) is a Gauss-Jacobi iteration for (3.37). To show the solution exists and is unique we apply Lemma 3.2.2.

Theorem 3.2.3 *The equations* (3.33) *and* (3.34) *have unique solutions for* $x_1^{n+1}(\alpha_1)$ *and* $x_2^n(\alpha_1)$ *for any* $p > 0$.

Proof. The operators $R_1(x)$ and $-R_3(y)$ are continuous and uniformly monotonic (increasing) since

$$R_1'(x) = \frac{1}{\alpha_1}M(x) \geq \frac{1}{\alpha_1}\check{m} > 0 \quad \text{and} \quad -R_3'(y) = \frac{1}{\alpha_1}M(y) \geq \frac{1}{1-\alpha_2}\check{m} > 0.$$

Therefore $f_1(x)$ and $f_2(x)$ are continuous and uniformly monotonic. Taking limits of $f_1(x)$ and $f_2(x)$ we obtain

$$\lim_{x\to\infty} f_1(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} f_1(x) = -\infty.$$

And similarly, $\lim_{y\to\infty} f_2(y) = \infty$ and $\lim_{y\to-\infty} f_2(y) = -\infty$. These properties and Lemma 3.2.2 give us existence and uniqueness of solution to equations (3.33) and (3.34).

$\square$

Theorem 3.2.3 says that the system (3.29-3.30) and (3.33-3.34) are well-posed for each iteration n for a given right-hand side. Therefore theoretically we can solve the system (3.33-3.34) and (3.33-3.34) for each iteration n for the given right-hand side. Now the question is how do we actually compute $x_1^n(\alpha_1)$ in (3.29) and $x_2^n(\alpha_1)$ from (3.30) for parallel iteration, and $x_1^{n+1}(\alpha_1)$ in (3.33) and $x_2^n(\alpha_1)$ from (3.34) for alternating iteration? In practice we can compute them using root-finding methods, for example using Matlab's *fsolve* without any restriction on p. In Theorem 3.2.5 we show that a fixed point iteration converges when applied to parallel iteration (3.29) and (3.30), in a similar manner for alternating iteration (3.33) and (3.34). To prove Theorem 3.2.5 we will need the standard Fixed-Point Theorem (from Burden and Faires [54]), which we will quote as Theorem 3.2.4. First we introduce the definition of fixed point.

Definition 3.2.1 *A fixed point for a system $x = g(x)$ is a point x^* such that $x^* = g(x^*)$.*

Theorem 3.2.4 (Fixed-Point Iteration) *Suppose $g(x)$ and $g^{'}(x)$ are continuous on a region that contains a fixed point. If the starting point is chosen sufficiently close to the fixed point and there exists a positive constant $0 < \varepsilon < 1$ such that*

$$|g^{'}(x)| \leq \varepsilon < 1 \quad for \quad all \quad x,$$

then the iteration $x^{k+1} = g(x^k), k = 0, 1, 2, ...,$ converges to the fixed point $x = x^*$.

Proof. See [54, page 173] for a proof of this theorem. □

Theorem 3.2.5 *Assume (3.31) or (3.32) is written in the form $x = g(x)$. Then sequence $x^{k+1} = g(x^k)$, $k > 0$ converges locally to the unique fixed point in $\mathbb{R}$ if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$, where $\hat{m}$ is defined in (2.8).*

Proof. Equation (3.31) can be written as

$$x = \frac{1}{p}(-R_1(x) + \zeta_1) \equiv g(x), \tag{3.38}$$

where $g(x) = \frac{1}{p}\big(- R_1(x) + \zeta_1\big)$. Clearly $g(x)$ is continuous and differentiable on $\mathbb{R}$ since the operator $-R_1(x)$ is differentiable. Now differentiate $g(x)$ with respect to x we have

$$\begin{aligned} g^{'}(x) &= \frac{1}{p}\frac{d}{dx}\big(- R_1(x) + \zeta_1\big) \\ &= \frac{1}{p}\frac{d}{dx}\big(- \frac{1}{\alpha_1}\int_0^x M(x)dx + \zeta_1\big) \\ &= -\frac{1}{p\alpha_1}M(x). \end{aligned}$$

Taking the absolute value of both sides we obtain

$$|g^{'}(x)| = \frac{1}{p\alpha_1}|M(x)|,$$

Hence, if $\frac{\hat{m}}{\alpha_1} < p$ holds then $|g^{'}(x)| < 1$. The assumptions of Theorem 3.2.4 have been verified, so we conclude the sequence $x^{k+1} = g(x^k), k = 0, 1, ...,$ will converge to the unique fixed point in $\mathbb{R}$ if $\frac{\hat{m}}{\alpha_1} < p$.

Similarly, equation (3.32) can be written as

$$x = \frac{1}{p}(R_3(x) + \zeta_2) \equiv h(x), \tag{3.39}$$

Clearly $h(x)$ is continuous and differentiable on $\mathbb{R}$ since the operator $R_3(x)$ is differentiable. Now differentiate $h(x)$ with respect to x we have

$$\begin{aligned} h^{'}(x) &= \frac{1}{p}\frac{d}{dx}\big(R_3(x)+\zeta_2\big) \\ &= \frac{1}{p}\frac{d}{dx}\big(\frac{1}{1-\alpha_1}\int_x^1 M(x)dx+\zeta_2\big) \\ &= -\frac{1}{p(1-\alpha_1)}M(x). \end{aligned}$$

Taking the absolute value of both sides we obtain

$$|h^{'}(x)| = \frac{1}{p(1-\alpha_1)}|M(x)|.$$

Thus, if $\frac{\hat{m}}{1-\alpha_1} < p$ then $|h^{'}(x)| < 1$. The assumptions of Theorem 3.2.4 has been verified, thus the sequence $x^{k+1} = h(x^k), k = 0, 1, ...,$ will converge to the unique fixed point in $\mathbb{R}$ if $\frac{\hat{m}}{1-\alpha_1} < p$. Therefore (3.38) and (3.39) converge if $\max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\} < p$.

A similar argument follows for the alternating iteration (3.35) and (3.36). □

Alternatively, the iterates for two subdomains can be computed by the bisection method [54]. The bisection technique is basically based on the Intermediate Value Theorem 3.2.1 .

Theorem 3.2.6 (Bisection Method) *Assume $f(x) = 0$, and f is continuous on the closed interval $[a, b]$ with $f(a)f(b) < 0$ then there exist some $x^* \in (a, b)$ such that $f(x^*) = 0$. Moreover, the bisection method will converge to x^*.*

Proof. Since f is a continuous on the closed interval $[a, b]$, with $f(a)$ and $f(b)$ of opposite sign; then by the Intermediate Value Theorem 3.2.1 there exist some $x^* \in (a, b)$ such that $f(x^*) = 0$.

The convergence of bisection method can be seen in reference [55]. □

Theorem 3.2.7 *The bisection algorithm applied to the equations (3.31) or (3.32) will converge for any $p > 0$.*

Proof. Equation (3.31) can be written as $f_1(x) = 0$ where $f_1(x) = R_1(x) + px - \zeta_1$. Taking the limit of $f_1(x)$, we obtain $\lim_{x\to\infty} f_1(x) = \infty$ and $\lim_{x\to-\infty} f_1(x) = -\infty$, since R_1 is continuous and uniformly monotonic increasing. f_1 is also continuous. This implies there exist at least two points $a, b \in \mathbb{R}$, with $f_1(a)$ and $f_1(b)$ of opposite sign and hence there exists some $x^* \in (a, b)$ such that $f_1(x^*) = 0$. Hence by Theorem 3.2.6 the bisection method will converge.

Again equation (3.32) gives $f_2(y) = 0$ where $f_2(y) = -R_3(y) + py - \zeta_2$. Taking the limit of $f_2(y)$, we obtain $\lim_{y\to\infty} f_2(y) = \infty$ and $\lim_{y\to-\infty} f_2(y) = -\infty$, as R_3 is continuous and uniformly monotonic decreasing. Also f_2 is continuous. This implies there exist at least two points $a, b \in \mathbb{R}$, with $f_2(a)$ and $f_2(b)$ of opposite sign and hence there exists some $y^* \in (a, b)$ such that $f_2(y^*) = 0$. Hence by Theorem 3.2.6 the bisection method will converge.

Similar argument follows for the alternating iterations (3.35) and (3.36). □

We will show that the whole system is well-posed in the next section.

3.2.2 Well-posedness of the Two Subdomain Iteration for Whole System

We now study existence of the whole system and then uniqueness of the whole system for the parallel two subdomain iteration.

Existence of solution for the Whole System

We now want to show that the whole system (3.23-3.24) is well-posed. The recurrence relations for two subdomains (3.23- 3.24) can be re-written as

$$R_1(x_1^n(\alpha_1)) + px_1^n(\alpha_1) - R_3(x_2^{n-1}(\alpha_1)) - px_2^{n-1}(\alpha_1) = 0 \tag{3.40}$$

$$-R_3(x_2^n(\alpha_1)) - px_2^n(\alpha_1) + R_1(x_1^{n-1}(\alpha_1)) - px_1^{n-1}(\alpha_1) = 0. \tag{3.41}$$

The iteration (3.40 -3.41) is a nonlinear Jacobi iteration for the system of equations

$$\left.\begin{aligned} R_1(x) - R_3(y) + px - py = 0 \\ -R_3(y) + R_1(x) + py - px = 0 \end{aligned}\right\}. \tag{3.42}$$

We denote the system (3.42) as $F(x,y) = b$, where $F = (f_1, f_2)^T$, $0 = (0,0)^T$ and

$$\left.\begin{aligned} f_1(x,y) \equiv R_1(x) - R_3(y) + px - py = 0 \\ f_2(x,y) \equiv R_1(x) - R_3(y) - px + py = 0 \end{aligned}\right\}. \tag{3.43}$$

We want to show that the whole system (3.43) is well-posed, that there is a unique solution of (3.43). To help in this regard we use Theorem 13.5.2 from Ortega and Rheinboldt [34], which we quote below as our Theorem 3.2.8. We first introduce two useful symbols $\uparrow$ and $\downarrow$ that we use in the theorem below. The condition

$$x^k \leq x^{k+1}, k = 0, 1, ..., \quad \text{and} \quad \lim_{k\to\infty} x^k = x^*$$

is denoted by $x^k \uparrow x^*$ when $k \to \infty$. Similarly,

$$x^k \geq x^{k+1}, k = 0, 1, ..., \quad \text{and} \quad \lim_{k\to\infty} x^k = x^*$$

is denoted by $x^k \downarrow x^*$ when $k \to \infty$.

Theorem 3.2.8 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuous, off-diagonally antitone, and strictly diagonally isotone, and suppose that for some $b \in \mathbb{R}^n$ there exists points $x^0, y^0 \in \mathbb{R}^n$ such that*

$$x^0 \leq y^0, \quad F(x^0) \leq b \leq F(y^0).$$

Then, for any $\omega \in (0,1]$, the successive over relaxation (SOR) iterates $\{y^k\}$ and $\{x^k\}$ given by the nonlinear SOR process (in Section 3.1.2) and starting from y^0 and x^0, respectively, are uniquely defined and satisfy

$$x^k \uparrow x^*, \quad y^k \downarrow y^*, \quad k \to \infty, \quad x^* \leq y^*, \quad Fx^* = Fy^* = b. \tag{3.44}$$

The corresponding result holds for the Jacobi iteration.

The vectors x^0 and y^0 are called a subsolution and a supersolution of $Fu = b$. Theorem 3.2.8 says that, if any continuous system is off-diagonally antitone, strictly diagonally isotone and there exists a supersolution and subsolution then the solution exists (but is not necessarily unique) and the theorem also gives us a way to solve the system. If the nonlinear SOR (or Jacobi) iteration starts from a subsolution or a supersolution then the iterations will converge to x^* or y^*.

Now we will verify the assumptions of Theorem 3.2.8 to show that solution of the system (3.43) exists. To verify the assumptions of Theorem 3.2.8, we start with Lemma 3.2.9.

Lemma 3.2.9 *Consider the system $F(x, y) = 0$ from (3.43) and the operators R_1 and R_3 as defined in (3.27). Assume M satisfies property (2.8), then $F : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous, strictly diagonally isotone, and if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$ then F is off-diagonally antitone.*

Proof. Clearly f_1 and f_2 are continuous. Now we show the system is strictly diagonally isotone. To show this, we differentiate f_1 and f_2 with respect to x and y respectively. We have

$$\frac{\partial f_1}{\partial x} = \frac{\partial R_1}{\partial x} + p = \frac{1}{\alpha_1} M(x) + p > 0$$

and

$$\frac{\partial f_2}{\partial y} = -\frac{\partial R_1}{\partial y} + p = \frac{1}{1 - \alpha_1} M(y) + p > 0.$$

This tells us f_1 and f_2 are strictly isotone with respect to x and y respectively. Therefore, F is *strictly diagonally isotone*.

We now will show that system is off-diagonally antitone. To show this, we differentiate f_1 with respect to y to obtain

$$\frac{\partial f_1}{\partial y} = -\frac{\partial R_3}{\partial y} - p = \frac{1}{1 - \alpha_1} M(y) - p.$$

If $\frac{\hat{m}}{1-\alpha_1} < p$, then f_1 is antitone with respect to y. Now differentiate f_2 with respect to x to obtain

$$\frac{\partial f_2}{\partial x} = -\frac{\partial R_1}{\partial x} - p = \frac{1}{\alpha_1}M(x) - p.$$

If $\frac{\hat{m}}{\alpha_1} < p$, then f_2 is antitone with respect to x. Given the assumption on $M(x)$ in (2.8), hence p needs to be greater than $\max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_2}\}$. Therefore, F is *off-diagonally antitone* if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$. □

To find a supersolution and a subsolution of the system (3.43), we derive an upper and lower bound of the operator $R_1(x)$ when $x \geq 0$, and the operator $R_3(y)$ when $y \leq 1$ in following lemma.

Lemma 3.2.10 *If $x \geq 0$ then $R_1(x)$ satisfies*

$$\frac{1}{\alpha_1}\check{m}x \leq R_1(x) \leq \frac{1}{\alpha_1}\hat{m}x. \tag{3.45}$$

If $y \leq 1$ then $R_3(y)$ satisfies

$$\frac{1}{1-\alpha_1}\check{m}(1-y) \leq R_3(y) \leq \frac{1}{1-\alpha_1}\hat{m}(1-y) \tag{3.46}$$

Proof. Assume $x \geq 0$. We integrate both sides of $\check{m} \leq M(x) \leq \hat{m}$ from 0 to x and multiply by $\frac{1}{\alpha_1}$ to obtain

$$\frac{1}{\alpha_1}\int_0^x \check{m}d\tilde{x} \leq \frac{1}{\alpha_1}\int_0^x M(x)d\tilde{x} \leq \frac{1}{\alpha_1}\int_0^x \hat{m}d\tilde{x}.$$

Using the definition of $R_1(x)$ we find the lower and upper bound as

$$\frac{1}{\alpha_1}\check{m}x \leq R_1(x) \leq \frac{1}{\alpha_1}\hat{m}x.$$

Similarly, assume $y \leq 1$. We integrate from y to 1 both sides of $\check{m} \leq M(x) \leq \hat{m}$ and multiply by $\frac{1}{1-\alpha_1}$. This gives

$$\frac{1}{1-\alpha_1}\int_y^1 \check{m}d\tilde{x} \leq \frac{1}{1-\alpha_1}\int_y^1 M(x)d\tilde{x} \leq \frac{1}{1-\alpha_1}\int_{y_0}^1 \hat{m}d\tilde{x}.$$

Using the definition of $R_3(y)$ we have the lower and upper bound

$$\frac{1}{1-\alpha_1}\check{m}(1-y) \leq R_3(y) \leq \frac{1}{1-\alpha_1}\hat{m}(1-y).$$

□

These lower and upper bounds of R_1 and R_3 will be useful to prove the existence of a supersolution and subsolution in Lemma 3.2.11.

Lemma 3.2.11 *If $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$ then there exists a supersolution and a subsolution for $F(x,y) = 0$ as defined in* (3.43).

Proof. We want to show that for $0 = (0,0) \in \mathbb{R}^2$, there exists $(\check{x}, \check{y}), (\hat{x}, \hat{y}) \in \mathbb{R}^2$, such that $(\check{x}, \check{y}) \leq (\hat{x}, \hat{y})$, and

$$F(\check{x}, \check{y}) \leq 0 \leq F(\hat{x}, \hat{y}). \tag{3.47}$$

That is we require

$$\left.\begin{array}{r} R_1(\check{x}) - R_3(\check{y}) + p\check{x} - p\check{y} \leq 0 \leq R_1(\hat{x}) - R_3(\hat{y}) + p\hat{x} - p\hat{y} \\ R_1(\check{x}) - R_3(\check{y}) - p\check{x} + p\check{y} \leq 0 \leq R_1(\hat{x}) - R_3(\hat{y}) - p\hat{x} + p\hat{y} \end{array}\right\}. \tag{3.48}$$

We find the region of subsolution and supersolution for this system using two different approaches as given below.

First approach: The inequalities for the subsolution are

$$\left.\begin{array}{r} R_1(\check{x}) - R_3(\check{y}) + p\check{x} - p\check{y} \leq 0 \\ R_1(\check{x}) - R_3(\check{y}) - p\check{x} + p\check{y} \leq 0 \end{array}\right\}. \tag{3.49}$$

Now using the Lemma 3.2.10, (3.49) holds if

$$\left.\begin{array}{r} \dfrac{\hat{m}}{\alpha_1}\check{x} - \dfrac{\check{m}}{1-\alpha_1}(1-\check{y}) + p\check{x} - p\check{y} \leq 0 \\ \dfrac{\hat{m}}{\alpha_1}\check{x} - \dfrac{\check{m}}{1-\alpha_1}(1-\check{y}) - p\check{x} + p\check{y} \leq 0 \\ \check{x} \geq 0 \quad \text{and} \quad \check{y} \leq 1 \end{array}\right\}.$$

This gives us

$$\left.\begin{array}{r}\Big(\dfrac{\hat{m}}{\alpha_1}+p\Big)\check{x}+\Big(\dfrac{\check{m}}{1-\alpha_1}-p\Big)\check{y}-\dfrac{\check{m}}{1-\alpha_1}\leq 0\\ \Big(\dfrac{\hat{m}}{\alpha_1}-p\Big)\check{x}+\Big(\dfrac{\check{m}}{1-\alpha_1}+p\Big)\check{y}-\dfrac{\check{m}}{1-\alpha_1}\leq 0\\ \check{x}\geq 0 \quad \text{and} \quad \check{y}\leq 1\end{array}\right\}. \tag{3.50}$$

If $\frac{\hat{m}}{\alpha_1} < p$ then we obtain the inequalities

$$\left.\begin{array}{c}\check{y}\geq\left(\dfrac{p+\frac{\hat{m}}{\alpha_1}}{p-\frac{\check{m}}{1-\alpha_1}}\right)\check{x}+\dfrac{\check{m}}{\check{m}-p(1-\alpha_1)}\\ \check{y}\leq\left(\dfrac{p-\frac{\hat{m}}{\alpha_1}}{p+\frac{\check{m}}{1-\alpha_1}}\right)\check{x}+\dfrac{\check{m}}{\check{m}+p(1-\alpha_1)}\\ \check{x}\geq 0 \quad \text{and} \quad \check{y}\leq 1\end{array}\right\}. \tag{3.51}$$

Hence if $\frac{\hat{m}}{\alpha_1} < p$ then we obtain the subsolution regions from inequalities (3.51) as shown

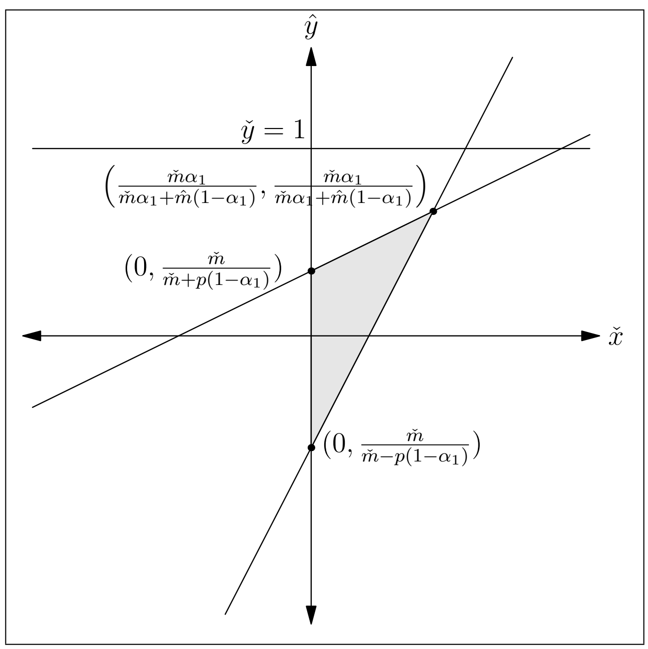


Figure 3.3: Subsolution region of the two subdomain iteration for whole system if $\frac{\hat{m}}{\alpha_1} < p$.

in Figure 3.3. So the subsolution regions exists if $\frac{\hat{m}}{\alpha_1} < p$.

Similarly, we find a region of supersolutions for our system. The inequalities for the supersolution are

$$\left.\begin{aligned} R_1(\hat{x}) - R_3(\hat{y}) + p\hat{x} - p\hat{y} \geq 0 \\ R_1(\hat{x}) - R_3(\hat{y}) - p\hat{x} + p\hat{y} \geq 0 \end{aligned}\right\}. \tag{3.52}$$

Using the Lemma 3.2.10, (3.52) holds if

$$\left.\begin{aligned} \frac{\check{m}}{\alpha_1}\hat{x} - \frac{\hat{m}}{1-\alpha_1}(1-\hat{y}) + p\hat{x} - p\hat{y} \geq 0 \\ \frac{\check{m}}{\alpha_1}\hat{x} - \frac{\hat{m}}{1-\alpha_1}(1-\hat{y}) - p\hat{x} + p\hat{y} \geq 0 \\ \hat{x} \geq 0 \quad \text{and} \quad \hat{y} \leq 1. \end{aligned}\right\}.$$

This implies

$$\left.\begin{aligned} \left(\frac{\check{m}}{\alpha_1} + p\right)\hat{x} + \left(\frac{\hat{m}}{1-\alpha_1} - p\right)\hat{y} - \frac{\hat{m}}{1-\alpha_1} \geq 0 \\ \left(\frac{\check{m}}{\alpha_1} - p\right)\hat{x} + \left(\frac{\hat{m}}{1-\alpha_1} + p\right)\hat{y} - \frac{\hat{m}}{1-\alpha_1} \geq 0 \\ \hat{x} \geq 0 \quad \text{and} \quad \hat{y} \leq 1. \end{aligned}\right\}. \tag{3.53}$$

If $\frac{\hat{m}}{1-\alpha_1} < p$ this gives us the inequalities

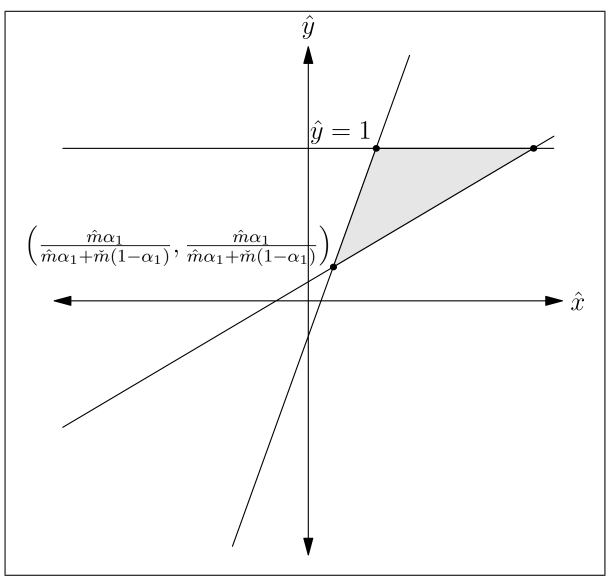


Figure 3.4: Supersolution region of the two subdomain iteration for whole system if $\frac{\hat{m}}{1-\alpha_1} < p$.

$$\left.\begin{aligned} \hat{y} &\leq \left(\frac{p+\frac{\check{m}}{\alpha_1}}{p-\frac{\hat{m}}{1-\alpha_1}}\right)\hat{x} + \frac{\hat{m}}{\hat{m}-p(1-\alpha_1)} \\ \hat{y} &\geq \left(\frac{p-\frac{\check{m}}{\alpha_1}}{p+\frac{\hat{m}}{1-\alpha_1}}\right)\hat{x} + \frac{\hat{m}}{\hat{m}+p(1-\alpha_1)} \\ &\hat{x} \geq 0 \quad \text{and} \quad \hat{y} \leq 1. \end{aligned}\right\}. \tag{3.54}$$

Hence if $\frac{\hat{m}}{1-\alpha_1} < p$ then we obtain the subsolution regions from inequalities (3.54) as shown in Figure 3.4. So the subsolution regions exists if $\frac{\hat{m}}{1-\alpha_1} < p$.

Therefore, we can conclude that supersolution and subsolution exist for the system (3.25-3.26) if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$.

Second approach: We now try to show the existence of a subsolution using Fourier-Motzkin elimination. The inequalities for the subsolution from (3.50) can be written as

$$\left.\begin{aligned} \Big(p+\frac{\hat{m}}{\alpha_1}\Big)\check{x} - \Big(p-\frac{\check{m}}{1-\alpha_1}\Big)\check{y} - \frac{\check{m}}{1-\alpha_1} &\leq 0 \\ -\Big(p-\frac{\hat{m}}{\alpha_1}\Big)\check{x} + \Big(p+\frac{\check{m}}{1-\alpha_1}\Big)\check{y} - \frac{\check{m}}{1-\alpha_1} &\leq 0 \\ -\check{x} \leq 0 \quad \text{and} \quad \check{y} &\leq 1 \end{aligned}\right\}. \tag{3.55}$$

Now we first eliminate the variable $\check{x}$. To do this we choose $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$ then $p-\frac{\hat{m}}{\alpha_1}$ and $p-\frac{\check{m}}{1-\alpha_1}$ are positive, now partition the inequalities in (3.55) into three groups, I_-, I_+ and I_0, according to the coefficient of $\check{x}$: whether it is negative or positive respectively:

$$I_- : \begin{cases} -\Big(p-\frac{\hat{m}}{\alpha_1}\Big)\check{x} + \Big(p+\frac{\check{m}}{1-\alpha_1}\Big)\check{y} - \frac{\check{m}}{1-\alpha_1} & \leq 0 \\ -\check{x} & \leq 0, \end{cases}$$

$$I_+ : \quad \Big(p+\frac{\hat{m}}{\alpha_1}\Big)\check{x} - \Big(p-\frac{\check{m}}{1-\alpha_1}\Big)\check{y} - \frac{\check{m}}{1-\alpha_1} \quad \leq 0,$$

$$I_0 : \qquad \check{y} \leq 1.$$

We now make the coefficient of $\check{x}$ for the inequalities in I_- to be -1, and coefficient of $\check{x}$

for the inequalities in I_+ to be $+1$, this gives

$$I_- : \begin{cases} -\check{x} + \left(\frac{p+\frac{\check{m}}{1-\alpha_1}}{p-\frac{\hat{m}}{\alpha_1}}\right)\check{y} - \left(\frac{\frac{\check{m}}{1-\alpha_1}}{p-\frac{\hat{m}}{\alpha_1}}\right) & \leq 0 \\ -\check{x} & \leq 0, \end{cases}$$

and

$$I_+ : \ \check{x} - \left(\frac{p-\frac{\check{m}}{1-\alpha_1}}{p+\frac{\hat{m}}{\alpha_1}}\right)\check{y} - \left(\frac{\frac{\check{m}}{1-\alpha_1}}{p+\frac{\hat{m}}{\alpha_1}}\right) \ \leq 0.$$

Isolating the variable $\check{x}$ in each group gives

$$I_- : \begin{cases} \left(\frac{p+\frac{\check{m}}{1-\alpha_1}}{p-\frac{\hat{m}}{\alpha_1}}\right)\check{y} - \left(\frac{\frac{\check{m}}{1-\alpha_1}}{p-\frac{\hat{m}}{\alpha_1}}\right) & \leq \check{x} \\ 0 & \leq \check{x}, \end{cases}$$

and

$$I_+ : \ \check{x} \ \leq \left(\frac{p-\frac{\check{m}}{1-\alpha_1}}{p+\frac{\hat{m}}{\alpha_1}}\right)\check{y} + \left(\frac{\frac{\check{m}}{1-\alpha_1}}{p+\frac{\hat{m}}{\alpha_1}}\right).$$

This implies that

$$\left\{ \begin{matrix} \left(\frac{p+\frac{\check{m}}{1-\alpha_1}}{p-\frac{\hat{m}}{\alpha_1}}\right)\check{y} & - & \left(\frac{\frac{\check{m}}{1-\alpha_1}}{p-\frac{\hat{m}}{\alpha_1}}\right) \\ & 0 & \end{matrix} \right\} \leq \check{x} \leq \left\{ \left(\frac{p-\frac{\check{m}}{1-\alpha_1}}{p+\frac{\hat{m}}{\alpha_1}}\right)\check{y} + \left(\frac{\frac{\check{m}}{1-\alpha_1}}{p+\frac{\hat{m}}{\alpha_1}}\right) \right\}. \tag{3.56}$$

Now eliminating $\check{x}$ from (3.56) we obtain

$$\begin{cases} \left(\frac{p+\frac{\check{m}}{1-\alpha_1}}{p-\frac{\hat{m}}{\alpha_1}}\right)\check{y} - \left(\frac{\frac{\check{m}}{1-\alpha_1}}{p-\frac{\hat{m}}{\alpha_1}}\right) & \leq & \left(\frac{p-\frac{\check{m}}{1-\alpha_1}}{p+\frac{\hat{m}}{\alpha_1}}\right)\check{y} + \left(\frac{\frac{\check{m}}{1-\alpha_1}}{p+\frac{\hat{m}}{\alpha_1}}\right) \\ 0 & \leq & \left(\frac{p-\frac{\check{m}}{1-\alpha_1}}{p+\frac{\hat{m}}{\alpha_1}}\right)\check{y} + \left(\frac{\frac{\check{m}}{1-\alpha_1}}{p+\frac{\hat{m}}{\alpha_1}}\right), \end{cases}$$

this implies

$$\begin{cases} \left(\frac{p+\frac{\check{m}}{1-\alpha_1}}{p-\frac{\hat{m}}{\alpha_1}} - \frac{p-\frac{\check{m}}{1-\alpha_1}}{p+\frac{\hat{m}}{\alpha_1}}\right)\check{y} & \leq & \left(\frac{\frac{\check{m}}{1-\alpha_1}}{p+\frac{\hat{m}}{\alpha_1}} + \frac{\frac{\check{m}}{1-\alpha_1}}{p-\frac{\hat{m}}{\alpha_1}}\right) \\ -\left(\frac{p-\frac{\check{m}}{1-\alpha_1}}{p+\frac{\hat{m}}{\alpha_1}}\right)\check{y} & \leq & \left(\frac{\frac{\check{m}}{1-\alpha_1}}{p+\frac{\hat{m}}{\alpha_1}}\right). \end{cases}$$

We rewrite these inequalities as

$$\begin{cases} \left(\frac{2p(\frac{\hat{m}}{\alpha_1}+\frac{\check{m}}{1-\alpha_1})}{(p-\frac{\hat{m}}{\alpha_1})(p+\frac{\hat{m}}{\alpha_1})}\right)\check{y} & \leq & \frac{2p\left(\frac{\check{m}}{1-\alpha_1}\right)}{\left(p-\frac{\hat{m}}{\alpha_1}\right)\left(p+\frac{\hat{m}}{\alpha_1}\right)} \\ -\left(p - \frac{\check{m}}{1-\alpha_1}\right)\check{y} & \leq & \frac{\check{m}}{1-\alpha_1}, \end{cases}$$

which implies

$$\begin{cases} \left(\frac{\hat{m}}{\alpha_1} + \frac{\check{m}}{1-\alpha_1}\right)\check{y} & \leq & \frac{\check{m}}{1-\alpha_1} \\ \left(\frac{\check{m}}{1-\alpha_1} - p\right)\check{y} & \leq & \frac{\check{m}}{1-\alpha_1}. \end{cases}$$

This gives us

$$\begin{cases} \left[\hat{m}(1-\alpha_1) + \check{m}\alpha_1\right]\check{y} & \leq & \check{m}\alpha_1 \\ -\left[p(1-\alpha_1) - \check{m}\right]\check{y} & \leq & \check{m}. \end{cases} \tag{3.57}$$

Therefore, including I_0 we obtain

$$\begin{cases} \left[\hat{m}(1-\alpha_1) + \check{m}\alpha_1\right]\check{y} & \leq & \check{m}\alpha_1 \\ -\left[p(1-\alpha_1) - \check{m}\right]\check{y} & \leq & \check{m} \\ \check{y} & \leq & 1. \end{cases} \tag{3.58}$$

Rearranging this system we arrive at the requirement

$$\left\{ -\frac{\check{m}}{p(1-\alpha_1) - \check{m}} \right\} \leq \check{y} \leq \left\{ \begin{array}{c} \frac{\check{m}\alpha_1}{\hat{m}(1-\alpha_1)+\check{m}\alpha_1} \\ 1 \end{array} \right\}, \tag{3.59}$$

and eliminating $\check{y}$ gives

$$\begin{cases} -\frac{\check{m}}{p(1-\alpha_1)-\check{m}} & \leq & \frac{\check{m}\alpha_1}{\hat{m}(1-\alpha_1)+\check{m}\alpha_1} \\ -\frac{\check{m}}{p(1-\alpha_1)-\check{m}} & \leq & 1. \end{cases} \tag{3.60}$$

Hence we observe the resulting system (3.60) does not involve the variable $\check{x}$, and the value of the expression of the left hand sides of (3.60) is negative and right sides is positive if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$. Hence the inequalities (3.60) are always true, so the resulting system (3.60) is feasible for $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$. Hence, the original system (3.55) is feasible if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$. Therefore we can find a subsolution from inequalities (3.55).

Similarly, we will find a supersolution. The inequalities for the supersolution from

(3.53) are

$$\left.\begin{array}{r}\Big(p+\dfrac{\check{m}}{\alpha_1}\Big)\hat{x}-\Big(p-\dfrac{\hat{m}}{1-\alpha_1}\Big)\hat{y}-\dfrac{\hat{m}}{1-\alpha_1}\geq 0\\ -\Big(p-\dfrac{\check{m}}{\alpha_1}\Big)\hat{x}+\Big(p+\dfrac{\hat{m}}{1-\alpha_1}\Big)\hat{y}-\dfrac{\hat{m}}{1-\alpha_1}\geq 0\\ \hat{x}\geq 0 \quad \text{and} \quad -\hat{y}\geq -1\end{array}\right\}. \tag{3.61}$$

Since $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$ then $p - \frac{\check{m}}{\alpha_1}$ and $p - \frac{\hat{m}}{1-\alpha_1}$ are positive. Partition the inequalities in (3.61) into two groups, I_- and I_+, according to the coefficient of $\hat{x}$:

$$\begin{array}{ll} I_-: & -\Big(p-\frac{\check{m}}{\alpha_1}\Big)\hat{x}+\Big(p+\frac{\hat{m}}{1-\alpha_1}\Big)\hat{y}-\frac{\hat{m}}{1-\alpha_1} \quad \geq 0,\\ I_+: & \begin{cases}\Big(p+\frac{\check{m}}{\alpha_1}\Big)\hat{x}-\Big(p-\frac{\hat{m}}{1-\alpha_1}\Big)\hat{y}-\frac{\hat{m}}{1-\alpha_1} \quad \geq 0\\ \hat{x} \quad \geq 0,\end{cases}\\ I_0: & -\hat{y}\geq -1.\end{array}$$

We now make the coefficient of $\check{x}$ for the inequalities in I_- to be -1, and coefficient of $\check{x}$ for the inequalities in I_+ to be $+1$, this gives

$$I_-: \quad -\hat{x}+\left(\frac{p+\frac{\hat{m}}{1-\alpha_1}}{p-\frac{\check{m}}{\alpha_1}}\right)\hat{y}-\left(\frac{\frac{\hat{m}}{1-\alpha_1}}{p-\frac{\check{m}}{\alpha_1}}\right) \quad \geq 0,$$

and

$$I_+: \begin{cases}\hat{x}-\left(\frac{p-\frac{\hat{m}}{1-\alpha_1}}{p+\frac{\check{m}}{\alpha_1}}\right)\hat{y}-\left(\frac{\frac{\hat{m}}{1-\alpha_1}}{p+\frac{\check{m}}{\alpha_1}}\right) \quad \geq 0\\ \hat{x} \quad \geq 0.\end{cases}$$

Isolating the variable $\hat{x}$ in each group we have

$$I_-: \quad \left(\frac{p+\frac{\hat{m}}{1-\alpha_1}}{p-\frac{\check{m}}{\alpha_1}}\right)\hat{y}-\left(\frac{\frac{\hat{m}}{1-\alpha_1}}{p-\frac{\check{m}}{\alpha_1}}\right) \quad \geq \hat{x},$$

and

$$I_+: \begin{cases}\hat{x} \quad \geq \left(\frac{p-\frac{\hat{m}}{1-\alpha_1}}{p+\frac{\check{m}}{\alpha_1}}\right)\hat{y}+\left(\frac{\frac{\hat{m}}{1-\alpha_1}}{p+\frac{\check{m}}{\alpha_1}}\right)\\ \hat{x} \quad \geq 0,\end{cases}$$

which implies that

$$\left\{\begin{array}{c}\left(\frac{p-\frac{\hat{m}}{1-\alpha_1}}{p+\frac{\check{m}}{\alpha_1}}\right)\hat{y} \quad + \quad \left(\frac{\frac{\hat{m}}{1-\alpha_1}}{p+\frac{\check{m}}{\alpha_1}}\right)\\ 0\end{array}\right\} \leq \hat{x} \leq \left\{\left(\frac{p+\frac{\hat{m}}{1-\alpha_1}}{p-\frac{\check{m}}{\alpha_1}}\right)\hat{y}-\left(\frac{\frac{\hat{m}}{1-\alpha_1}}{p-\frac{\check{m}}{\alpha_1}}\right)\right\}. \tag{3.62}$$

Now eliminating $\hat{x}$ from (3.62) we obtain the requirement

$$\begin{cases} \left(\frac{p-\frac{\hat{m}}{1-\alpha_1}}{p+\frac{\check{m}}{\alpha_1}}\right)\hat{y} + \left(\frac{\frac{\hat{m}}{1-\alpha_1}}{p+\frac{\check{m}}{\alpha_1}}\right) & \leq & \left(\frac{p+\frac{\hat{m}}{1-\alpha_1}}{p-\frac{\check{m}}{\alpha_1}}\right)\hat{y} - \left(\frac{\frac{\hat{m}}{1-\alpha_1}}{p-\frac{\check{m}}{\alpha_1}}\right) \\ 0 & \leq & \left(\frac{p+\frac{\hat{m}}{1-\alpha_1}}{p-\frac{\check{m}}{\alpha_1}}\right)\hat{y} - \left(\frac{\frac{\hat{m}}{1-\alpha_1}}{p-\frac{\check{m}}{\alpha_1}}\right), \end{cases}$$

which gives

$$\begin{cases} \left(\frac{-2p(\frac{\check{m}}{\alpha_1}+\frac{\hat{m}}{1-\alpha_1})}{(p+\frac{\check{m}}{\alpha_1})(p-\frac{\check{m}}{\alpha_1})}\right)\hat{y} & \leq & \frac{-2p(\frac{\hat{m}}{1-\alpha_1})}{(p+\frac{\check{m}}{\alpha_1})(p-\frac{\check{m}}{\alpha_1})} \\ -\left(\frac{p+\frac{\hat{m}}{1-\alpha_1}}{p-\frac{\check{m}}{\alpha_1}}\right)\hat{y} & \leq & -\left(\frac{\frac{\hat{m}}{1-\alpha_1}}{p-\frac{\check{m}}{\alpha_1}}\right). \end{cases}$$

We can rewrite these inequalities as

$$\begin{cases} \left(\frac{\check{m}}{\alpha_1}+\frac{\hat{m}}{1-\alpha_1}\right)\hat{y} & \geq & \frac{\hat{m}}{1-\alpha_1} \\ \left(p+\frac{\hat{m}}{1-\alpha_1}\right)\hat{y} & \geq & \left(\frac{\hat{m}}{1-\alpha_1}\right), \end{cases}$$

which implies

$$\begin{cases} \left[\check{m}(1-\alpha_1)+\hat{m}\alpha_1\right]\hat{y} & \geq & \hat{m}\alpha_1 \\ \left[p(1-\alpha_1)+\hat{m}\right]\hat{y} & \geq & \hat{m}. \end{cases}$$

Therefore, adding the inequalities from I_0 we have

$$\begin{cases} \left[\check{m}(1-\alpha_1)+\hat{m}\alpha_1\right]\hat{y} & \geq & \hat{m}\alpha_1 \\ \left[p(1-\alpha_1)+\hat{m}\right]\hat{y} & \geq & \hat{m} \\ -\hat{y} & \geq & -1. \end{cases} \tag{3.63}$$

Rearranging this system we obtain

$$\left\{\begin{matrix} \frac{\hat{m}\alpha_1}{\check{m}(1-\alpha_1)+\hat{m}\alpha_1} \\ \frac{\hat{m}}{p(1-\alpha_1)+\hat{m}} \end{matrix}\right\} \leq \hat{y} \leq \{1\},$$

and eliminating $\hat{y}$ gives us

$$\begin{cases} \frac{\hat{m}\alpha_1}{\check{m}(1-\alpha_1)+\hat{m}\alpha_1} & \leq & 1 \\ \frac{\hat{m}}{p(1-\alpha_1)+\hat{m}} & \leq & 1. \end{cases} \tag{3.64}$$

Hence we observe the resulting system (3.63) does not involve the variable $\hat{x}$, and the value of the expression of the left hand sides of (3.64) is less then 1 if $p > \max\{\frac{\check{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$.

Thus the inequalities (3.60) are always true, so the resulting system (3.64) is feasible for $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$. Therefore we obtain a supersolution form the inequalities (3.61). Hence we have obtain a supersolution and a subsolution for the system (3.25-3.26) if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$. □

Example 1 To illustrate the subsolution and supersolution, let us consider a monitor function

$$M(x) = 1 + \beta_1 \exp^{(x-x_0)} + \beta_2 \exp^{(x-x_n)} \tag{3.65}$$

where $\beta_1 = 10$, $\beta_2 = 5$, $x_0 = 0$ and $x_n = 1$. The lower-bound is $\check{m} = 12.83939$ and the upper-bound is $\hat{m} = 33.18282$ on interval $[0, 1]$. We need to show $F(\check{x}, \check{y}) \leq 0$ when $\check{x}$, $\check{y}$ is chosen from the subsolution region, on the other hand, if $\hat{x}$ and $\hat{y}$ are chosen from supersolution region then $F(\hat{x}, \hat{y}) \geq 0$. We choose some points on the boundaries of super and subsolution region and choose $p = 68$ that satisfies the condition on p that $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$.

A notation BL_{sub} in Table 3.1 is used for the boundary lines of the subsolution region, and BL_{sup} in Table 3.2 is used for the boundary lines of the supsolution region. Numerical

Table 3.1: Subsolution for two subdomains optimized Schwarz interface iteration for $p = 68$ with $M(x) = 1 + \beta_1 \exp^{x} + \beta_2 \exp^{(x-1)}$.

$\check{x}$	$\check{y}$ on BL_{sub}	$f_1(\check{x}, \check{y})$	$f_2(\check{x}, \check{y})$	$\check{y}$ on BL_{sub}	$f_1(\check{x}, \check{y})$	$f_2(\check{x}, \check{y})$
0.00	-0.606760	-17.008048	-99.527343	0.274115	-53.310956	-16.031270
0.05	-0.448014	-19.012292	-86.742260	0.274988	-48.627310	-18.028988
0.10	-0.289269	-20.954291	-73.894932	0.275860	-43.881395	-19.964437
0.15	-0.130524	-22.830853	-60.982169	0.276732	-39.070020	-21.834427
0.20	0.028221	-24.629106	-47.991095	0.277605	-34.189830	-23.635601
0.25	0.186966	-25.933173	-34.505835	0.278477	-29.237298	-25.364432

results in Table 3.1 shows that the function value are negative in columns $3, 4, 6$, and 7, hence subsolution exists if we chose any value from the shaded region in Figure 3.3.

Table 3.2: Supersolution for two-subdomain optimized Schwarz interface iteration for $p = 68$ with $M(x) = 1 + \beta_1 \exp^x + \beta_2 \exp^{(x-1)}$.

$\breve{x}$	$\breve{y}$ on BL_{sup}	$f_1(\breve{x}, \breve{y})$	$f_2(\breve{x}, \breve{y})$	$\breve{y}$ on BL_{sup}	$f_1(\breve{x}, \breve{y})$	$f_2(\breve{x}, \breve{y})$
0.83	0.762867	22.786543	13.656455	0.755344	22.902502	12.749245
0.86	0.804714	25.948373	18.429477	0.764793	26.511094	13.562917
0.89	0.846561	29.253301	23.345597	0.774242	30.174593	14.431495
0.94	0.916306	35.094537	31.872153	0.789990	36.406746	16.005447
0.97	0.958153	38.808523	37.197331	0.799440	40.224503	17.028284
1.00	1.000000	42.686842	42.686842	0.808889	44.103184	18.112044

Similarly, the numerical results in Table 3.2 shows that the function values are positive in columns $3, 4, 6$, and 7, so supersolution exists if we choose any value from the shaded region in Figure 3.4.

Theorem 3.2.12 *Solutions of the system* (3.43) *exists if* $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$.

Proof. The assumptions of Theorem 3.2.8 have been verified in Lemmas 3.2.9 and 3.2.11. Hence the system has a solution. □

Now we arrive at the following result.

Theorem 3.2.13 *If* $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$ *then the nonlinear Gauss Jacobi iteration (3.40-3.41) for* $Fu = b$ *defined in* (3.43) *will converge to a solution if the iteration starts at a supersolution or a subsolution.*

Proof. The assumptions of Theorem 3.2.8 have been verified in Lemmas 3.2.9 and 3.2.11. Theorem 3.2.8 guarantees that if the iteration starts at a supersolution or a subsolution then the nonlinear Gauss Jacobi iteration (3.40-3.41) will converge to a solution. $\square$

Note, the Theorem 3.2.13 also guarantee the convergence of nonlinear Gauss Seidel (SOR, $\omega = 1$) for system (3.43).

Uniqueness of the Whole System for Two Subdomain Iteration

The following lemmas are useful to show that the system (3.43) has a unique solution.

Lemma 3.2.14 *Consider a system $F(x) = b$. If F is an M-function, then F is inverse isotone and if the solution exists, it is unique for a given right-hand side b.*

Proof. Inverse isotonicity of F holds by definition of an M-function, as given in Definition 3.1.8. Assume x^* and y^* are two solutions of $F(x) = b$, we have $F(x^*) = F(y^*) = b$. Since $F(x^*) \leq F(y^*)$, then the inverse isotonicity of F gives us $x^* \leq y^*$. On the other hand, we have $F(y^*) \leq F(x^*)$ which implies $y^* \leq x^*$. Therefore, $x^* = y^*$, hence the solution is unique. $\square$

Thus, by the Lemma 3.2.14, to show uniqueness it is sufficient to show our F is an M-function. To do this we use Theorem 5.1 from Rheinboldt [47], which we quote below as our Theorem 3.2.15.

Theorem 3.2.15 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be off-diagonally antitone, and suppose that there exists a diagonal M-function $H : D \subset \mathbb{R}^n \to \mathbb{R}^n$ such that $F(\mathbb{R}^n) \subset D$ and that, for any $x \in \mathbb{R}^n$, the function*

$$Q : \mathbb{R}^1 \to \mathbb{R}^n \qquad q_i(t) = \sum_{j=1}^{n} h_j(f_j(x + te^i)), \quad i = 1, ..., n, \tag{3.66}$$

is isotone. Let

$$\varphi : \mathbb{R}^1 \to \mathbb{R}^n, \quad \varphi_i(t), i = 1, ..., n,$$

$$\psi : \mathbb{R}^1 \to \mathbb{R}^n, \quad \psi_i(t), i = 1, ..., n$$

be isotone mappings such that $\psi + \varphi$ is strictly isotone, and assume that for every node i in the associate network Ω_F (see Definition 3.1.3) there exists a node $l = l(i)$, which is strictly connected to i and for which there is strict isotonicity either of φ_i or of q_i for any $x \in \mathbb{R}$. Then

$$\hat{F} : \mathbb{R}^n \to \mathbb{R}^n, \qquad \hat{f}_i(t) = \varphi_i(x_i) + \psi_j(f_j(x + te^i)), \quad i = 1, ..., n,$$

is an M-function.

If $\varphi = 0$ and $\psi = I$, we have in this theorem a result about the mapping F itself. Theorem 3.2.15 gives us a way to prove F is an M-function: if F is off-diagonally antitone and q_i is isotone for every i with $h_j(y) = y$. In the following Theorem 3.2.16, we will show that F in system (3.43) is an M-function if p is big enough.

Theorem 3.2.16 *If $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$ then F as defined in (3.43) is an M-function.*

Proof. We have F is off-diagonally antitone, if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$ from Lemma 3.2.9. If we can show q_i is isotone for every i, then we are done. Now we will build the functions $q_i(t)$ using the $f_1(x, y)$ and $f_2(x, y)$ as defined in (3.43). From the theorem statement

$$q_i(t) = \sum_{j=1}^{2} f_j(X + te^i), \quad 1 \le i \le 2, \quad e^i \in \mathbb{R}^2 \text{ (i^{th} standard basis vector).}$$

Specifically,

$$q_1(t) = f_1(x + t, y) + f_2(x + t, y) = 2R_1(x + t) - 2R_3(y)$$

and

$$q_2(t) = f_1(x, y + t) + f_2(x, , y + t) = 2R_1(x) - 2R_3(y + t).$$

Now differentiating q_1 and q_2, we have $\frac{dq_1}{dt} = \frac{2}{\alpha_1}M(t) > 0$ and $\frac{dq_2}{dt} = \frac{2}{1-\alpha_1}M(t) > 0$, since $M(x)$ is bounded away from 0 to ∞ for all x. Therefore, the functions q_i are strictly isotone. Hence, F is an M-function if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$. □

So we arrive at the following uniqueness result.

Theorem 3.2.17 *Let $F(x, y) = b$ be a system defined in (3.43). This system has an unique solution if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$.*

Proof. Theorem 3.2.16 guarantees F is an M-function if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$. Then uniqueness follows from Lemma 3.2.14. □

Remark: Uniqueness was shown in Theorem 3.2.17. Hence, Theorem 3.2.13 guarantees convergence of nonlinear Jacobi to the unique solution.

Theorem 3.2.18 *The nonlinear Jacobi iteration (3.40-3.41) converges to the unique solution of (3.43) starting from a supersolution or subsolution if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$.*

Proof. Theorem 3.2.13 has already shown convergence to a solution from a supersolution or subsolution initial guess. Uniqueness was shown in Theorem 3.2.17, hence convergence to the unique solution from a supersolution or subsolution follows. □

Note, the Theorem 3.2.13 also guarantee the convergence of nonlinear Gauss Seidel (SOR, $\omega = 1$) for system (3.43). We have shown convergence to the unique solution of (3.43), using a nonlinear Jacobi iteration (3.40-3.41) starting from a supersolution or subsolution. Now we want to generalize this result of convergence for any initial guess. Theorem 3.2.19 below (which is Theorem 13.5.9 from Ortega and Rheinboldt [47]), guarantees convergence from any start value if F is a continuous and onto M function.

Theorem 3.2.19 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuous M-function from $\mathbb{R}^n$ onto itself. Then for any $b \in \mathbb{R}^n$, any starting point $x^0 \in \mathbb{R}^n$, and any $\omega \in (0, 1]$, the SOR iteration (3.2), as well as the Jacobi iteration, converges to the unique solution x^* of $Fu = b$.*

Hence, for any initial guess, the SOR and nonlinear Jacobi iteration for a continuous system will converge to the unique solution if the system F is an onto (or surjective) M-function. The next lemma gives us a way to prove surjectivity.

Lemma 3.2.20 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuous, off-diagonally antitone, and strictly isotone. Assume $F(\mathbf{x}) = b$ has a supersolution and a subsolution for any $b \in \mathbb{R}^n$. Then F is onto.*

Proof. To prove F is onto, we need to show F is continuous, off-diagonally antitone, strictly isotone, and $F(\mathbf{x}) = b$ has a solution for any $b \in \mathbb{R}^n$. If $F(\mathbf{x}) = b$ has a subsolution or supersolution for any $b \in \mathbb{R}^n$, then there existence of a solution is guaranteed by Theorem 3.2.8. □

Lemma 3.2.21 *Let $F(x, y) = b$ be the system (3.43), where $b \in \mathbb{R}^2$, then F is onto if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$.*

Proof. F is continuous, strictly isotone, and off-diagonally antitone if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$ from Lemma 3.2.9. Now we want to show that for any $b = (b_1, b_2) \in \mathbb{R}^2$, there exists $(\check{x}, \check{y}), (\hat{x}, \hat{y}) \in \mathbb{R}^2$, such that $(\check{x}, \check{y}) \leq (\hat{x}, \hat{y})$, and

$$F(\check{x}, \check{y}) \leq b \leq F(\hat{x}, \hat{y}).$$

That is we require

$$\left.\begin{array}{r} R_1(\check{x}) - R_3(\check{y}) + p\check{x} - p\check{y} \leq b_1 \leq R_1(\hat{x}) - R_3(\hat{y}) + p\hat{x} - p\hat{y} \\ R_1(\check{x}) - R_3(\check{y}) - p\check{x} + p\check{y} \leq b_2 \leq R_1(\hat{x}) - R_3(\hat{y}) - p\hat{x} + p\hat{y} \end{array}\right\}. \tag{3.67}$$

We now will find the region for subsolutions and supersolutions of our system. The inequalities for the subsolution are

$$\left.\begin{array}{r} R_1(\check{x}) - R_3(\check{y}) + p\check{x} - p\check{y} \leq b_1 \\ R_1(\check{x}) - R_3(\check{y}) - p\check{x} + p\check{y} \leq b_2 \end{array}\right\}. \tag{3.68}$$

To show that the subsolution exists for any b_1 and b_2, we assume $\check{x} \leq 0$ and $\check{y} \leq 1$ then we obtain

$$R_1(\check{x}) = \frac{1}{\alpha_1}\int_0^{\check{x}} \check{m}dx = \frac{\check{m}}{\alpha_1}\check{x}$$

and

$$\frac{1}{1-\alpha_1}\check{m}(1-\check{y}) \leq R_3(\check{y}) \leq \frac{1}{1-\alpha_1}\hat{m}(1-\check{y}).$$

Using these values of $R_1(\check{x})$ and $R_3(\check{y})$ in (3.68) holds if

$$\left.\begin{array}{c} \frac{\check{m}}{\alpha_1}\check{x} - \frac{\check{m}}{1-\alpha_1}(1-\check{y}) + p\check{x} - p\check{y} \leq b_1 \\ \frac{\check{m}}{\alpha_1}\check{x} - \frac{\check{m}}{1-\alpha_1}(1-\check{y}) - p\check{x} + p\check{y} \leq b_2 \\ \text{with} \quad \check{x} \leq 0 \quad \text{and} \quad \check{y} \leq 1 \end{array}\right\},$$

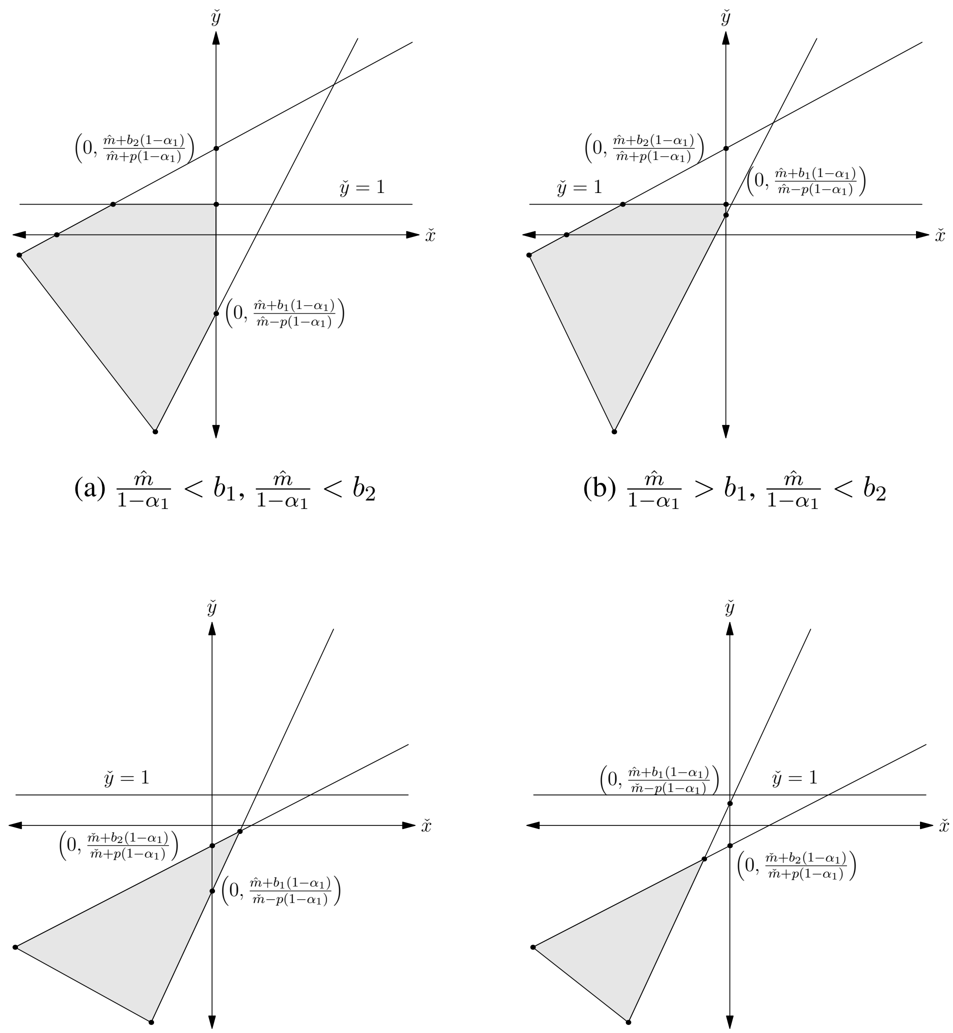
which implies

$$\left.\begin{array}{c} \left(\frac{\check{m}}{\alpha_1} + p\right)\check{x} + \left(\frac{\check{m}}{1-\alpha_1} - p\right)\check{y} - \frac{\check{m}}{1-\alpha_1} \leq b_1 \\ \left(\frac{\check{m}}{\alpha_1} - p\right)\check{x} + \left(\frac{\check{m}}{1-\alpha_1} + p\right)\check{y} - \frac{\check{m}}{1-\alpha_1} \leq b_2 \\ \text{with} \quad \check{x} \leq 0 \quad \text{and} \quad \check{y} \leq 1 \end{array}\right\}. \tag{3.69}$$

If $\frac{\hat{m}}{\alpha_1} < p$ then we obtain following inequalities

$$\left.\begin{array}{c} \check{y} \geq \left(\frac{p + \frac{\check{m}}{\alpha_1}}{p - \frac{\check{m}}{1-\alpha_1}}\right)\check{x} + \frac{\check{m} + b_1(1-\alpha_1)}{\check{m} - p(1-\alpha_1)} \\ \check{y} \leq \left(\frac{p - \frac{\check{m}}{\alpha_1}}{p + \frac{\check{m}}{1-\alpha_1}}\right)\check{x} + \frac{\check{m} + b_2(1-\alpha_1)}{\check{m} + p(1-\alpha_1)} \\ \text{with} \quad \check{x} \leq 0 \quad \text{and} \quad \check{y} \leq 1 \end{array}\right\}. \tag{3.70}$$

There are the four cases depending on the values of b_1 and b_2 as shown in Figure 3.5. We now obtain the subsolution regions from inequalities (3.70) as shown in Figure 3.5 when b_1 and b_2 satisfy strict inequalities. The existence of a subsolution region is also guaranteed when b_1 and b_2 equals $\frac{\hat{m}}{1-\alpha_1}$. Hence the subsolution regions exists for any values of b_1 and b_2 if $\frac{\hat{m}}{1-\alpha_1} < p$.



(a) $\frac{\hat{m}}{1-\alpha_1} < b_1$, $\frac{\hat{m}}{1-\alpha_1} < b_2$

(b) $\frac{\hat{m}}{1-\alpha_1} > b_1$, $\frac{\hat{m}}{1-\alpha_1} < b_2$

(c) $\frac{\hat{m}}{1-\alpha_1} < b_1$, $\frac{\hat{m}}{1-\alpha_1} > b_2$

(d) $\frac{\hat{m}}{1-\alpha_1} > b_1$, $\frac{\hat{m}}{1-\alpha_1} > b_2$

Figure 3.5: Subsolution region of the two subdomain iteration if $\frac{\hat{m}}{1-\alpha_1} < p$ with $\check{x} \leq 0$ and $\check{x} \leq 1$.

Similarly, we will find a supersolution region for our system. The inequalities for the supersolution are

$$\left.\begin{aligned} R_1(\hat{x}) - R_3(\hat{y}) + p\hat{x} - p\hat{y} \geq b_1 \\ R_1(\hat{x}) - R_3(\hat{y}) - p\hat{x} + p\hat{y} \geq b_2 \end{aligned}\right\}. \tag{3.71}$$

We wish to find a supersolution for any b_1 and b_2. Consider $\hat{x} \geq 0$ and $\hat{y} \geq 1$ then we obtain

$$\frac{\check{m}}{\alpha_1}\hat{x} \leq R_1(\hat{x}) \leq \frac{\hat{m}}{\alpha_1}\hat{x}$$

and

$$R_3(\hat{y}) = \frac{1}{1-\alpha_1}\int_{\hat{y}}^{1} \hat{m}dx = \frac{\hat{m}}{\alpha_1}(1-\hat{y}).$$

Using these values of $R_1(\hat{x})$ and $R_3(\hat{y})$ in (3.71) holds if

$$\left.\begin{array}{c} \dfrac{\hat{m}}{\alpha_1}\hat{x} - \dfrac{\hat{m}}{1-\alpha_1}(1-\hat{y}) + p\hat{x} - p\hat{y} \geq b_1 \\ \dfrac{\hat{m}}{\alpha_1}\hat{x} - \dfrac{\hat{m}}{1-\alpha_1}(1-\hat{y}) - p\hat{x} + p\hat{y} \geq b_2 \\ \text{with} \quad \hat{x} \geq 0 \quad \text{and} \quad \hat{y} \geq 1 \end{array}\right\}.$$

Rewriting this system gives

$$\left.\begin{array}{c} \Big(\dfrac{\hat{m}}{\alpha_1} + p\Big)\hat{x} + \Big(\dfrac{\hat{m}}{1-\alpha_1} - p\Big)\hat{y} - \dfrac{\hat{m}}{1-\alpha_1} \geq b_1 \\ \Big(\dfrac{\hat{m}}{\alpha_1} - p\Big)\hat{x} + \Big(\dfrac{\hat{m}}{1-\alpha_1} + p\Big)\hat{y} - \dfrac{\hat{m}}{1-\alpha_1} \geq b_2 \\ \text{with} \quad \hat{x} \geq 0 \quad \text{and} \quad \hat{y} \geq 1 \end{array}\right\}. \tag{3.72}$$

If $\frac{\hat{m}}{1-\alpha_1} < p$ then we obtain

$$\left.\begin{array}{c} \hat{y} \leq \left(\dfrac{p + \frac{\hat{m}}{\alpha_1}}{p - \frac{\hat{m}}{1-\alpha_1}}\right)\hat{x} + \dfrac{\hat{m} + b_1(1-\alpha_1)}{\hat{m} - p(1-\alpha_1)} \\ \hat{y} \geq \left(\dfrac{p - \frac{\hat{m}}{\alpha_1}}{p + \frac{\hat{m}}{1-\alpha_1}}\right)\hat{x} + \dfrac{\hat{m} + b_2(1-\alpha_1)}{\hat{m} + p(1-\alpha_1)} \\ \text{with} \quad \hat{x} \geq 0 \quad \text{and} \quad \hat{y} \geq 1 \end{array}\right\}. \tag{3.73}$$

There are the four cases depending on the values of b_1 and b_2 as shown in Figure 3.6. We now obtain the supersolution regions from inequalities (3.73) as shown in Figure 3.6 when b_1 and b_2 satisfy strict inequalities. The existence of a supersolution region is also guaranteed when b_1 and b_2 equals $\frac{\hat{m}}{1-\alpha_1}$. Hence the supersolution regions exists for any b_1 and b_2, if $\frac{\hat{m}}{1-\alpha_1} < p$.

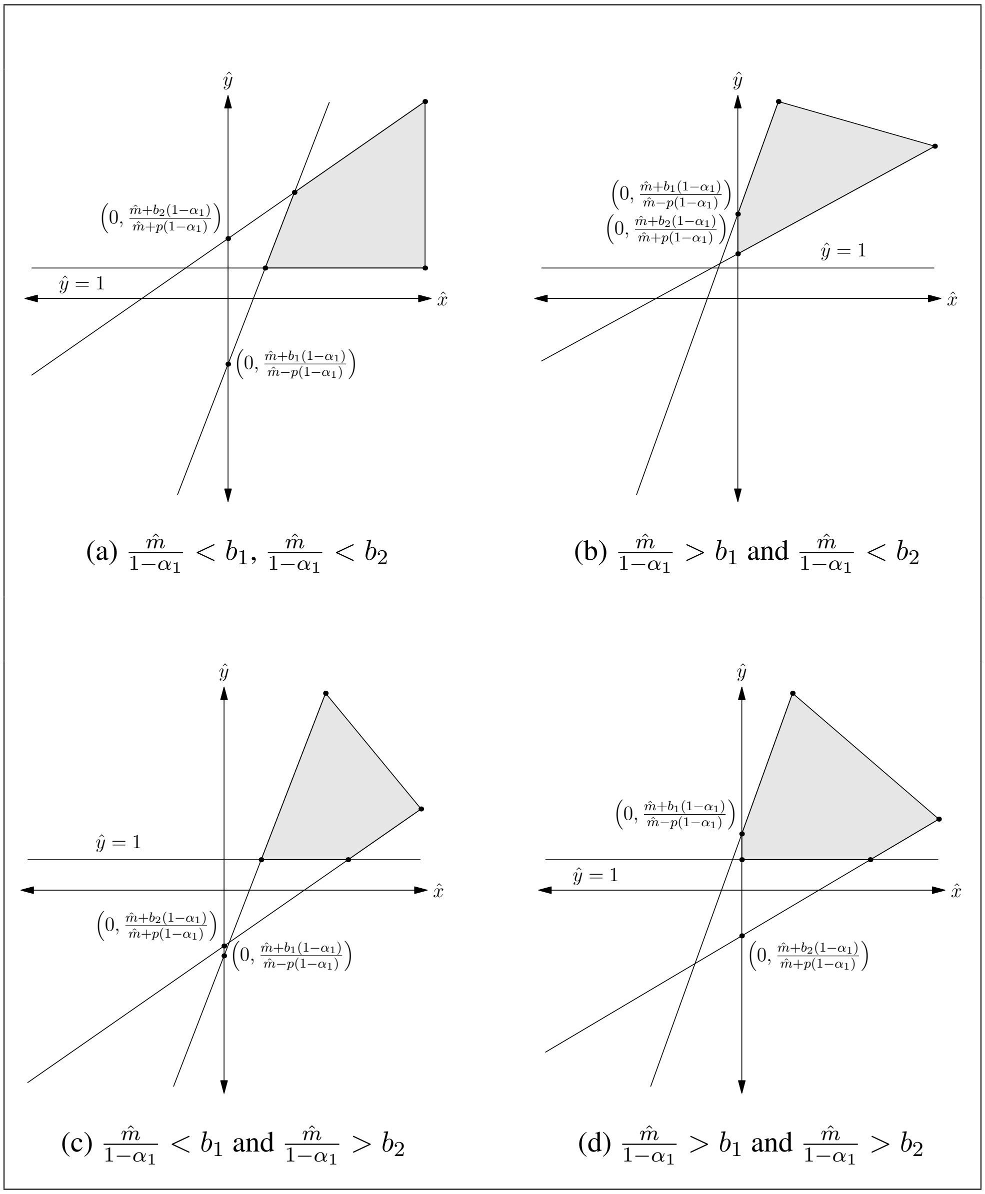


Figure 3.6: Supersolution region for the two subdomain iteration if $\frac{\hat{m}}{1-\alpha_1} < p$ with $\hat{x} \geq 0$ and $\hat{y} \geq 1$.

Hence, we can conclude that supersolution and subsolution exists for the system (3.25-3.26) if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$. Therefore we can conclude F is surjective (or onto) from Lemma 3.2.20. $\square$

Theorem 3.2.22 *Iteration* (3.40) *or* (3.41) *converge to the unique solution for any initial* *guess if* $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$.

Proof. The assumptions of Theorem 3.2.19 have been verified in Theorem 3.2.16 and Lemma 3.2.21 if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$. Hence the SOR and nonlinear Jacobi iteration converge to the unique solution from any initial guess. □

We proved that our iteration converges without conditions on p in Theorem 2.2.13 using the Global Peaceman-Rachford Theorem 2.2.9. Here we get a condition on p. The general question is why this happened? If we start with a supersolution or subsolution then the M function criterion guarantees convergence will be monotonic. Global Peaceman-Rachford does not guarantee monotonicity. Monotonicity is a stronger requirement, which places a restriction on p.

3.2.3 Alternative Approach to Show Well-posedness for the Two Subdomains Iteration

The iteration (3.40 -3.41) is a nonlinear Jacobi iteration for the system of equations

$$R_1(x) - R_3(y) + px - py = 0$$
$$-R_3(y) + R_1(x) + py - px = 0.$$

Adding the equations in system (3.42) we obtain

$$R_1(x) = R_3(y), \tag{3.74}$$

and subtracting the second equation from the first equation in system (3.42) gives us

$$x = y. \tag{3.75}$$

Substituting the value of y from equation (3.75) in equation (3.75) we obtain

$$R_1(x) = R_3(x),$$

which is equivalent to

$$\frac{1}{\alpha_1}\int_0^x M(x)dx = \frac{1}{1-\alpha_1}\int_x^1 M(x)dx. \tag{3.76}$$

We now want to know if equation (3.76) has a unique solution? Let $\int_0^1 M(x)dx = \mathcal{C}$. Then from (3.76) we obtain

$$\frac{1}{\alpha_1}\int_0^x M(x)dx = \frac{1}{1-\alpha_1}\Big(\mathcal{C} - \int_0^x M(x)dx\Big), \tag{3.77}$$

this can be written as

$$\frac{1}{\alpha_1(1-\alpha_1)}\int_0^x M(x)dx = \frac{1}{1-\alpha_1}\mathcal{C},$$

or

$$\frac{1}{\alpha_1}\int_0^x M(x)dx = \mathcal{C},$$

which implies

$$R_1(x) = \mathcal{C}. \tag{3.78}$$

Since $R_1(x)$ is continuous and uniformly monotonic increasing, it is onto, and hence, a unique solution exists for equation (3.78).

Similarly, we have from (3.76)

$$\frac{1}{1-\alpha_1}\int_x^1 M(x)dx = \frac{1}{\alpha_1}\Big(\mathcal{C} - \int_x^1 M(x)dx\Big).$$

This gives us

$$\frac{1}{\alpha_1(1-\alpha_1)}\int_x^1 M(x)dx = \frac{1}{\alpha_1}\mathcal{C},$$

or

$$\frac{1}{1-\alpha_1}\int_x^1 M(x)dx = \mathcal{C}.$$

This is equivalent to

$$R_3(x) = \mathcal{C}. \tag{3.79}$$

Since $R_3(x)$ is continuous and uniformly monotonic decreasing, it is onto and therefore, a unique solution exists for equation (3.79).

3.3 An Interface Iteration for Three Subdomains

We decompose the computational domain $\Omega_c = (0,1)$ into three nonoverlapping subdomains $\Omega_1 = (0,\alpha_1)$, $\Omega_2 = (\alpha_1,\alpha_2)$, and $\Omega_3 = (\alpha_2, 1)$. The parallel version of interface

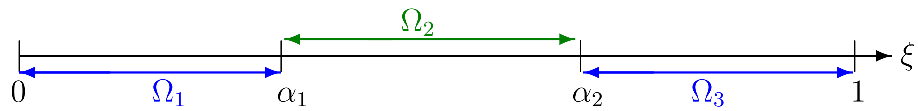


Figure 3.7: Decomposition into three nonoverlapping subdomains

iteration from (3.20) on three subdomains is given by

$$R_1(x_1^n(\alpha_1)) + px_1^n(\alpha_1) = R_2(x_2^{n-1}(\alpha_1), x_2^{n-1}(\alpha_2)) + px_2^{n-1}(\alpha_1) \tag{3.80}$$

$$\left.\begin{aligned} R_2(x_2^n(\alpha_1), x_2^n(\alpha_2)) - px_2^n(\alpha_1) &= R_1(x_1^{n-1}(\alpha_1)) - px_1^{n-1}(\alpha_1) \\ R_2(x_2^n(\alpha_1), x_2^n(\alpha_2)) + px_2^n(\alpha_2) &= R_3(x_3^{n-1}(\alpha_2)) + px_3^{n-1}(\alpha_2) \end{aligned}\right\} \tag{3.81}$$

and

$$R_3(x_3^n(\alpha_2)) - px_3^n(\alpha_2) = R_2(x_2^{n-1}(\alpha_1), x_2^{n-1}(\alpha_2)) - px_2^{n-1}(\alpha_2). \tag{3.82}$$

The operators R_1, R_2 and R_3 are given by

$$\begin{aligned} R_1(x) &= \frac{1}{\alpha_1}\int_0^x M(\tilde{x})d\tilde{x} \\ R_2(y,z) &= \frac{1}{\alpha_2-\alpha_1}\int_y^z M(\tilde{x})d\tilde{x} \\ \text{and} \quad R_3(w) &= \frac{1}{1-\alpha_2}\int_w^1 M(\tilde{x})d\tilde{x}. \end{aligned} \tag{3.83}$$

We are interested in the following questions: Is the system (3.80-3.82) well-posed? Can we solve for $x_1^n(\alpha_1)$, $x_2^n(\alpha_1)$, $x_2^n(\alpha_2)$, and $x_3^n(\alpha_2)$ for the parallel iteration? How to compute them? Does iteration (3.80-3.82) converge?

In the next section we wise to prove the iteration (3.80-3.82) is well-posed and give a way of solving the equations for given right-hand side.

3.3.1 Well-posedness of the Three Subdomain Iteration for a Given Right-Hand Side

To prove the iteration (3.80-3.82) is well defined for a given right-hand side, let $\zeta_1 = R_2(x_2^{n-1}(\alpha_1), x_2^{n-1}(\alpha_2)) + px_2^{n-1}(\alpha_1)$, $\zeta_2 = R_1(x_1^{n-1}(\alpha_1)) - px_1^{n-1}(\alpha_1)$, $\zeta_3 = R_3(x_3^{n-1}(\alpha_2)) + px_3^{n-1}(\alpha_2)$, and $\zeta_4 = -R_2(x_2^{n-1}(\alpha_1), x_2^{n-1}(\alpha_2)) + px_2^{n-1}(\alpha_2)$, then the system (3.80-3.82) becomes

$$\begin{aligned} R_1(x_1^n(\alpha_1)) + px_1^n(\alpha_1) &= \zeta_1, \\ \left.\begin{aligned} R_2(x_2^n(\alpha_1), x_2^n(\alpha_2)) - px_2^n(\alpha_1) &= \zeta_2 \\ R_2(x_2^n(\alpha_1), x_2^n(\alpha_2)) + px_2^n(\alpha_2) &= \zeta_3 \end{aligned}\right\} &\text{[Inner subdomain iteration]} \\ R_3(x_3^n(\alpha_2)) - px_3^n(\alpha_2) &= \zeta_4. \end{aligned} \tag{3.84}$$

We wish to consider existence and uniqueness of solutions for the system

$$\begin{aligned} R_1(x) + px &= \zeta_1, \\ \left.\begin{aligned} -R_2(y,z) + py &= \zeta_2 \\ R_2(y,z) + pz &= \zeta_3 \end{aligned}\right\} &\text{[Coupled system]} \\ -R_3(w) + pw &= \zeta_4. \end{aligned} \tag{3.85}$$

The first and the last equations arise from the boundary subdomains of system (3.85), and they are separated and independent. However, the coupled equations of the system (3.85) arise from the inner subdomain. We first study boundary subdomains then we analyze the inner subdomain.

To show the boundary subdomain equations are well defined we take first and last equations from the system (3.85). This gives

$$\begin{aligned} R_1(x) + px &= \zeta_1 \\ -R_3(w) + pw &= \zeta_4. \end{aligned}$$

We wish to consider the existence and uniqueness of solutions for these equations. This is equivalent to solving

$$f_1(x) \equiv R_1(x) + px = \zeta_1 \tag{3.86}$$

$$f_2(w) \equiv -R_3(w) + pw = \zeta_4. \tag{3.87}$$

This gives a decoupled system of the form $F(x, w) = b$, where $F = (f_1, f_2)^T$ and $b = (\zeta_1, \zeta_4)$. To show the solution exists and is unique we apply Lemma 3.2.2.

Theorem 3.3.1 *The equations* (3.86) *and* (3.87) *have unique solutions for any* $p > 0$.

Proof. The system (3.86-3.87) is the equivalent system of (3.33 - 3.34), thus Theorem 3.2.3 gives us existence and uniqueness of solution for the equations (3.86-3.87). ☐

We now wish to show the inner subdomain system is well defined. The coupled equations can be written as

$$\left.\begin{aligned} -R_2(y,z) + py = \zeta_2 \\ R_2(y,z) + pz = \zeta_3 \end{aligned}\right\}. \tag{3.88}$$

We wish to show the existence of y and z solving (3.88). This is equivalent to solving $F(y, z) = b$, where $F = (f_1, f_2)^T$, $b = (\zeta_2, \zeta_3)^T$ and

$$f_1(y, z) \equiv -R_2(y, z) + py = \zeta_2 \tag{3.89}$$

$$f_2(y, z) \equiv R_2(y, z) + pz = \zeta_3. \tag{3.90}$$

To show the solutions exists for this system we will apply Theorem 3.2.8.

Lemma 3.3.2 *Consider the coupled (inner) system* $F(y, z) = b$ *from* (3.88) *and the operator* R_2 *as defined in* (3.83). *Then* F *is continuous, strictly diagonally isotone and off-diagonally antitone.*

Proof. The operator R_2 is continuous and strictly increasing in z with y fixed and strictly decreasing in y with z fixed. Thus F is continuous. Now we want to show that this coupled system is strictly diagonally isotone. To show this we differentiate f_1 with respect to y to obtain

$$\frac{\partial f_1}{\partial y} = -\frac{\partial R_2}{\partial y} + p = \frac{1}{\alpha_2 - \alpha_1} M(y) + p > 0.$$

This tells us f_1 is strictly isotone with respect to y. We differentiate f_2 with respect to z, we have

$$\frac{\partial f_2}{\partial z} = \frac{\partial R_2}{\partial z} + p = \frac{1}{\alpha_2 - \alpha_1} M(z) + p > 0.$$

This implies that f_2 is strictly isotone with respect to z. Therefore, F is *strictly diagonally isotone*.

We will show that the system (3.88) is off-diagonally antitone. To show this, we differentiate f_1 with respect to z, we get

$$\frac{\partial f_1}{\partial z} = -\frac{\partial R_2}{\partial z} = -\frac{1}{\alpha_2 - \alpha_1} M(z) < 0,$$

which implies f_1 is antitone with respect to z. We differentiate f_2 with respect to y, we have

$$\frac{\partial f_2}{\partial y} = \frac{\partial R_2}{\partial y} = -\frac{1}{\alpha_2 - \alpha_1} M(y) < 0.$$

This implies that f_2 is antitone with respect to y. Therefore, F is *off-diagonally antitone*.

□

We wish to derive an upper and lower bound of the operator R_2 using the definition of R_2 and bounds on M in following lemma.

Lemma 3.3.3 *If $y \leq z$ then $R_2(y,z)$ satisfies*

$$\frac{1}{\alpha_2-\alpha_1}\check{m}(z-y) \leq R_2(y,z) \leq \frac{1}{\alpha_2-\alpha_1}\hat{m}(z-y), \tag{3.91}$$

$$and \quad -\frac{1}{\alpha_2-\alpha_1}\hat{m}(z-y) \leq -R_2(y,z) \leq -\frac{1}{\alpha_2-\alpha_1}\check{m}(z-y). \tag{3.92}$$

If $y \geq z$ then $R_2(y,z)$ satisfies

$$\frac{1}{\alpha_2-\alpha_1}\check{m}(y-z) \leq -R_2(y,z) \leq \frac{1}{\alpha_2-\alpha_1}\hat{m}(y-z). \tag{3.93}$$

$$and \quad -\frac{1}{\alpha_2-\alpha_1}\hat{m}(y-z) \leq R_2(y,z) \leq -\frac{1}{\alpha_2-\alpha_1}\check{m}(y-z) \tag{3.94}$$

Proof. We now use the definition of R_2 and bounds on M to derive an inequality to bound the operator $R_2(y,z)$ when $y \leq z$.

Assume $y \leq z$ then we wish to find upper and lower bounds on $R_2(y,z)$. To do this we integrate both sides of $\check{m} \leq M(x) \leq \hat{m}$ from y to z and multiply by $\frac{1}{\alpha_2-\alpha_1}$, with $\alpha_2-\alpha_1 > 0$. This gives

$$\frac{1}{\alpha_2-\alpha_1}\int_y^z \check{m}dx \leq \frac{1}{\alpha_2-\alpha_1}\int_y^z M(x)dx \leq \frac{1}{\alpha_2-\alpha_1}\int_y^z \hat{m}dx,$$

Using the definition of R_2 we have

$$\frac{1}{\alpha_2-\alpha_1}\check{m}(z-y) \leq R_2(y,z) \leq \frac{1}{\alpha_2-\alpha_1}\hat{m}(z-y).$$

Multiplying both sides of this inequality by -1, we find

$$-\frac{1}{\alpha_2-\alpha_1}\hat{m}(z-y) \leq -R_2(y,z) \leq -\frac{1}{\alpha_2-\alpha_1}\check{m}(z-y).$$

Hence we obtain a upper and lower bound the operator $R_2(y,z)$ as required when $y \leq z$.

Similarly, if $y \geq z$, then we want to bound the operator $R_2(y,z)$. We integrate both sides of $\check{m} \leq M(x) \leq \hat{m}$ from z to y and multiply by $\frac{1}{\alpha_2-\alpha_1}$. This gives us

$$\frac{1}{\alpha_2-\alpha_1}\int_z^y \check{m}dx \leq \frac{1}{\alpha_2-\alpha_1}\int_z^y M(x)dx \leq \frac{1}{\alpha_2-\alpha_1}\int_y^z \hat{m}dx$$

Using the definition of R_2 we find

$$\frac{1}{\alpha_2-\alpha_1}\check{m}(y-z) \le R_2(z,y) \le \frac{1}{\alpha_2-\alpha_1}\hat{m}(y-z),$$

which is equivalent to

$$\frac{1}{\alpha_2-\alpha_1}\check{m}(y-z) \le -R_2(y,z) \le \frac{1}{\alpha_2-\alpha_1}\hat{m}(y-z). \tag{3.95}$$

Multiply both sides of this inequality by -1, we have

$$-\frac{1}{\alpha_2-\alpha_1}\hat{m}(y-z) \le R_2(y,z) \le -\frac{1}{\alpha_2-\alpha_1}\check{m}(y-z). \tag{3.96}$$

Therefore we get upper and lower bounds on the operator $R_2(y,z)$ as required for $y \geq z$. □

These upper and lower bounds of R_2 will be useful to prove the existence of a supersolution and subsolution in Lemma 3.3.4.

Lemma 3.3.4 *For any given $b \in \mathbb{R}^2$, there exists a supersolution and subsolution for the system $F(x,y) = b$ from* (3.88)*.*

Proof. We want to show that for any $b \in \mathbb{R}^2$, there exists $\check{\mathbf{x}}, \hat{\mathbf{y}} \in \mathbb{R}^2$, such that $\check{\mathbf{x}} \le \hat{\mathbf{y}}$, where $\check{\mathbf{x}} = (\check{x}_1, \check{x}_2)$ and $\hat{\mathbf{y}} = (\hat{y}_1, \hat{y}_2)$, and

$$F(\check{x}_1, \check{x}_2) \le b \le F(\hat{y}_1, \hat{y}_2).$$

That is we require

$$\left.\begin{aligned} -R_2(\check{x}_1,\check{x}_2) + p\check{x}_1 \le b_1 \le -R_2(\hat{y}_1,\hat{y}_2) + p\hat{y}_1, \\ R_2(\check{x}_1,\check{x}_2) + p\check{x}_2 \le b_2 \le R_2(\hat{y}_1,\hat{y}_2) + p\hat{y}_2. \end{aligned}\right\}. \tag{3.97}$$

We find the region for subsolution and supersolution satisfying (3.97) using two different approaches as given below.

First approach: To find a subsolution, we have to satisfy the

$$\left.\begin{aligned} -R_2(\check{x}_1, \check{x}_2) + p\check{x}_1 \leq b_1 \\ R_2(\check{x}_1, \check{x}_2) + p\check{x}_2 \leq b_2 \end{aligned}\right\}. \tag{3.98}$$

There are two cases: $\check{x}_1$ can satisfy $\check{x}_1 \leq \check{x}_2$ or $\check{x}_1 \geq \check{x}_2$. Let us first consider

$$\check{x}_1 \leq \check{x}_2. \tag{3.99}$$

Using Lemma 3.3.3 the system (3.98) holds if

$$\left.\begin{aligned} -\frac{1}{\alpha_2 - \alpha_1}\check{m}(\check{x}_2 - \check{x}_1) + p\check{x}_1 \leq b_1 \\ \frac{1}{\alpha_2 - \alpha_1}\hat{m}(\check{x}_2 - \check{x}_1) + p\check{x}_2 \leq b_2 \end{aligned}\right\}. \tag{3.100}$$

Hence if $\check{x}_1 \leq \check{x}_2$ then the inequalities for the subsolution are

$$\left.\begin{aligned} \check{x}_2 \geq \frac{\check{m} + (\alpha_2 - \alpha_1)p}{\check{m}}\check{x}_1 - \frac{(\alpha_2 - \alpha_1)b_1}{\check{m}} \\ \check{x}_2 \leq \frac{\hat{m}}{\hat{m} + (\alpha_2 - \alpha_1)p}\check{x}_1 + \frac{(\alpha_2 - \alpha_1)b_2}{\hat{m} + (\alpha_2 - \alpha_1)p} \end{aligned}\right\}. \tag{3.101}$$

There are the four cases depending on the sign of b_1 and b_2 as shown in Figure 3.8. If $\check{x}_1 \leq \check{x}_2$ then we obtain the subsolution regions from inequalities (3.101) as shown in Figure 3.8 when b_1 and b_2 satisfy strict inequalities. The existence of a subsolution region is also guaranteed when b_1 and b_2 equals 0. Hence the subsolution exists for all b_1 and b_2 if $\check{x}_1 \leq \check{x}_2$.

Let us consider the case when

$$\check{x}_1 \geq \check{x}_2.$$

Using Lemma 3.3.3 the system (3.98) holds if

$$\left.\begin{aligned} \frac{1}{\alpha_2 - \alpha_1}\hat{m}(\check{x}_1 - \check{x}_2) + p\check{x}_1 \leq b_1 \\ -\frac{1}{\alpha_2 - \alpha_1}\check{m}(\check{x}_1 - \check{x}_2) + p\check{x}_2 \leq b_2 \end{aligned}\right\}. \tag{3.102}$$

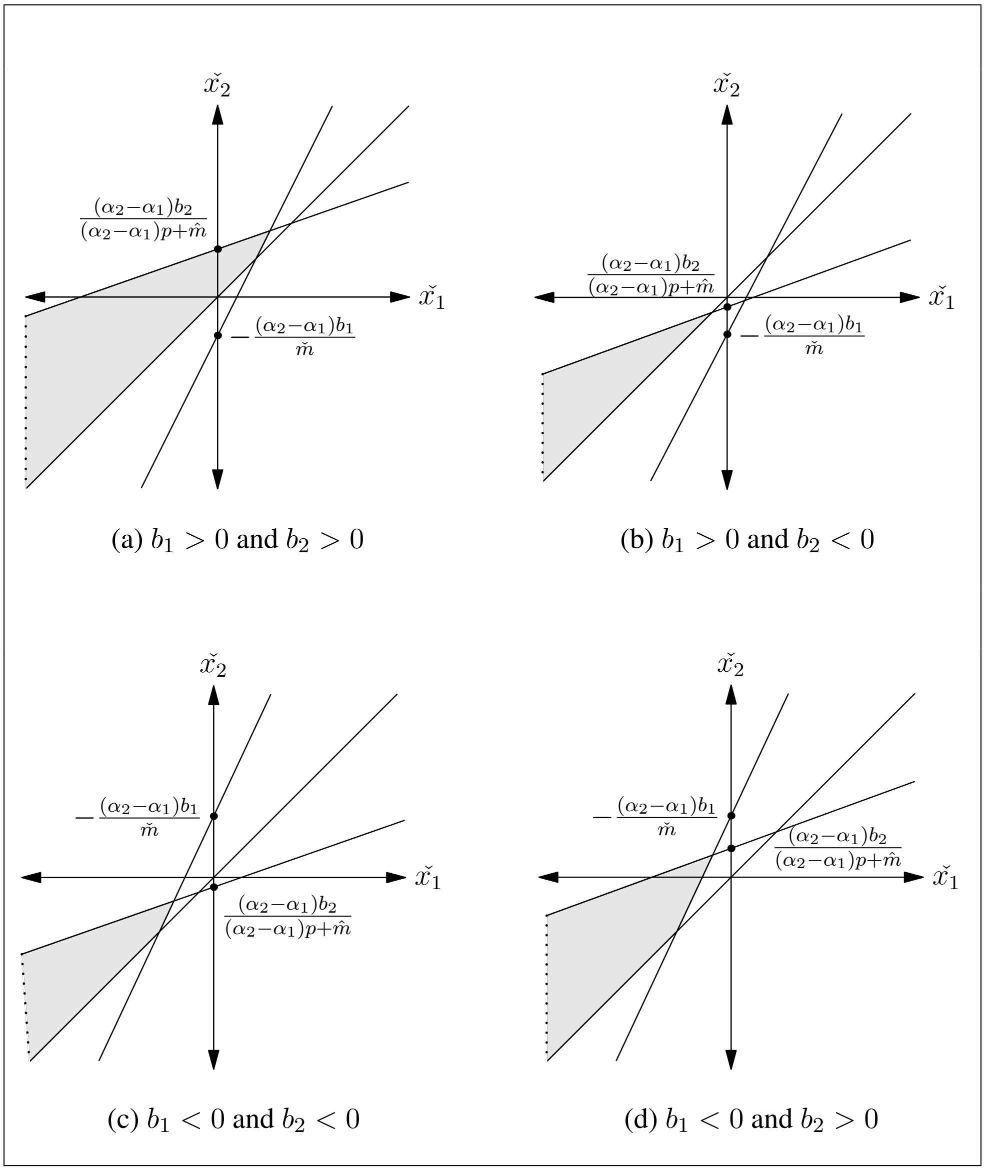


Figure 3.8: Subsolution region for the coupled (inner) system in the three subdomain iteration when $\check{x}_1 \leq \check{x}_2$.

Hence if $\check{x}_1 \geq \check{x}_2$ then the inequalities for the subsolution are

$$\left.\begin{aligned} \check{x}_2 &\geq \frac{\hat{m}+(\alpha_2-\alpha_1)p}{\hat{m}}\check{x}_1 - \frac{(\alpha_2-\alpha_1)b_1}{\hat{m}} \\ \check{x}_2 &\leq \frac{\check{m}}{\check{m}+(\alpha_2-\alpha_1)p}\check{x}_1 + \frac{(\alpha_2-\alpha_1)b_2}{\check{m}+(\alpha_2-\alpha_1)p} \end{aligned}\right\}. \tag{3.103}$$

These are the four cases depending on the sign of b_1 or b_2 as shown in Figure 3.9. If

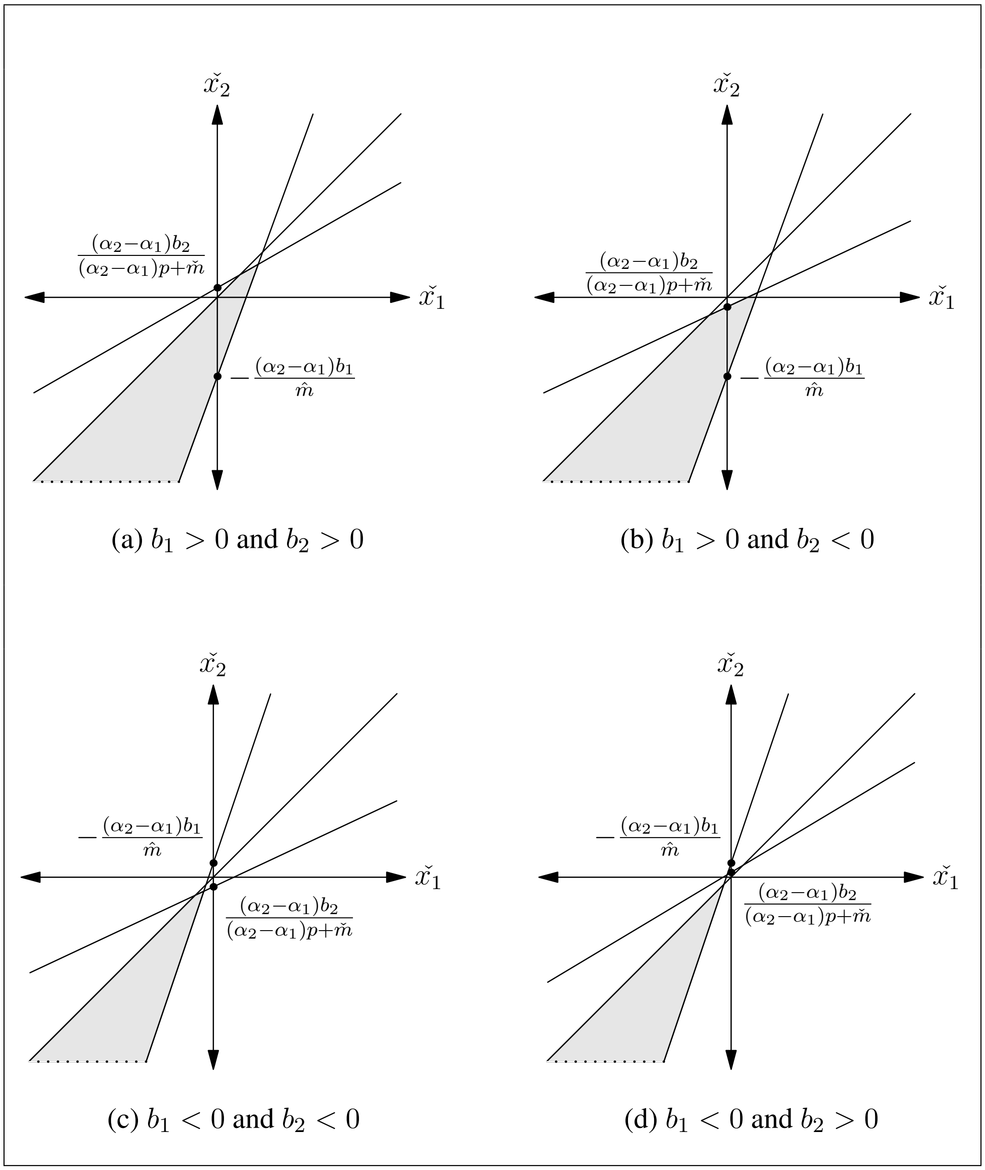


Figure 3.9: Subsolution region for the coupled (inner) system in the three subdomain iteration when $\check{x}_1 \geq \check{x}_2$.

$\check{x}_1 \geq \check{x}_2$ then we obtain the subsolution regions from inequalities (3.103) as shown in Figure 3.9 when b_1 and b_2 satisfy strict inequalities. The existence of a subsolution region is also guaranteed when b_1 and b_2 equals 0. Hence if $\check{x}_1 \geq \check{x}_2$ then solution sets exist for the subsolution for all b_1 and b_2. Therefore, the subsolution exists for the coupled system.

Similarly, to find the supersolution, we have to satisfy

$$\left.\begin{aligned} b_1 &\le -R_2(\hat{y}_1,\hat{y}_2)+p\hat{y}_1,\\ b_2 &\le R_2(\hat{y}_1,\hat{y}_2)+p\hat{y}_2 \end{aligned}\right\}. \tag{3.104}$$

There are two cases for supersolution: $\hat{y}_1$ can satisfy $\hat{y}_1 \le \hat{y}_2$ or $\hat{y}_1 \ge \hat{y}_2$. Let us first consider $\hat{y}_1 \le \hat{y}_2$. Using Lemma 3.3.3 the system (3.104) holds if

$$\left.\begin{aligned} b_1 &\le -\frac{1}{\alpha_2-\alpha_1}\hat{m}(\hat{y}_2-\hat{y}_1)+p\hat{y}_1\\ b_2 &\le \frac{1}{\alpha_2-\alpha_1}\check{m}(\hat{y}_2-\hat{y}_1)+p\hat{y}_2 \end{aligned}\right\}. \tag{3.105}$$

Hence the inequalities for the supersolution are

$$\left.\begin{aligned} \hat{y}_2 &\le \frac{\hat{m}+(\alpha_2-\alpha_1)p}{\hat{m}}\hat{y}_1-\frac{(\alpha_2-\alpha_1)b_1}{\hat{m}}\\ \hat{y}_2 &\ge \frac{\check{m}}{\check{m}+(\alpha_2-\alpha_1)p}\hat{y}_1+\frac{(\alpha_2-\alpha_1)b_2}{\check{m}+(\alpha_2-\alpha_1)p} \end{aligned}\right\}, \tag{3.106}$$

when $\hat{y}_1 \le \hat{y}_2$. These are the four cases depending on the sign of b_1 and b_2 as shown in Figure 3.10. If $\hat{y}_1 \ge \hat{y}_2$ then we obtain the supersolution regions from inequalities (3.106) as shown in Figure 3.10 when b_1 and b_2 satisfy strict inequalities. The existence of a supersolution region is also guaranteed when b_1 and b_2 equals 0. Hence supersolution region exists for all b_1 and b_2 when $\hat{y}_1 \le \hat{y}_2$.

Now let us consider the case where $\hat{y}_1 \ge \hat{y}_1$. Using Lemma 3.3.3 the system (3.104) holds if

$$\left.\begin{aligned} b_1 &\le \frac{1}{\alpha_2-\alpha_1}\check{m}(\hat{y}_1-\hat{y}_2)+p\hat{y}_1\\ b_2 &\le -\frac{1}{\alpha_2-\alpha_1}\hat{m}(\hat{y}_1-\hat{y}_2)+p\hat{y}_2 \end{aligned}\right\}. \tag{3.107}$$

Hence, the inequalities for the supersolution are

$$\left.\begin{aligned} \hat{y}_2 &\le \frac{\check{m}+(\alpha_2-\alpha_1)p}{\check{m}}\hat{y}_1-\frac{(\alpha_2-\alpha_1)b_1}{\check{m}}\\ \hat{y}_2 &\ge \frac{\hat{m}}{\hat{m}+(\alpha_2-\alpha_1)p}\hat{y}_1+\frac{(\alpha_2-\alpha_1)b_2}{\hat{m}+(\alpha_2-\alpha_1)p} \end{aligned}\right\}. \tag{3.108}$$

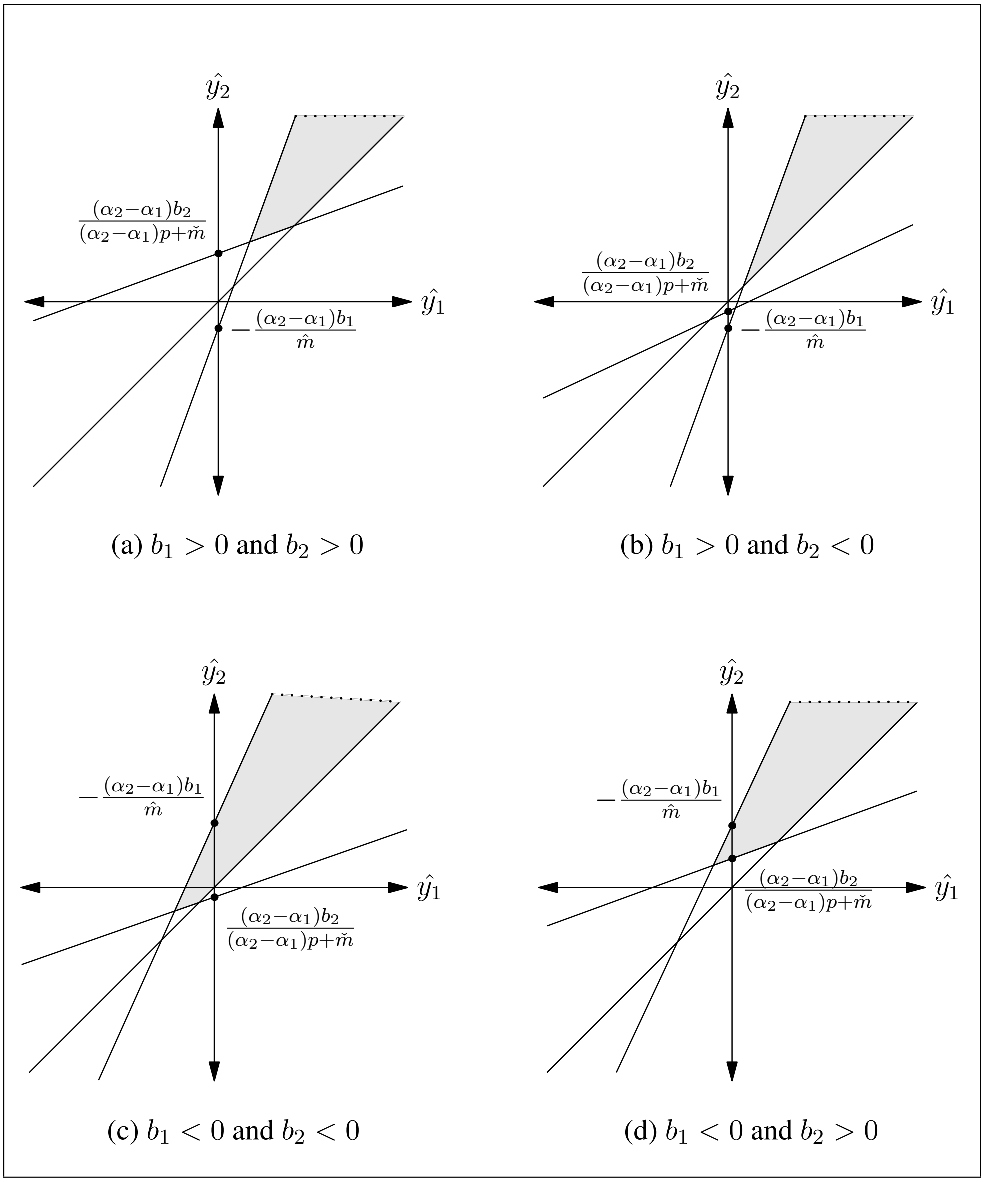


Figure 3.10: Supersolution region for the coupled (inner) system in the three subdomain iteration when $\hat{y}_1 \leq \hat{y}_2$

when $\hat{y}_1 \geq \hat{y}_2$. There are the four cases depending on the sign of b_1 and b_2 as shown in Figure 3.11. If $\hat{y}_1 \geq \hat{y}_2$ then we obtain the supersolution regions from inequalities (3.108) as shown in Figure 3.11 when b_1 and b_2 satisfy strict inequalities. The existence of a supersolution region is also guaranteed when b_1 and b_2 equals 0. Hence we have a

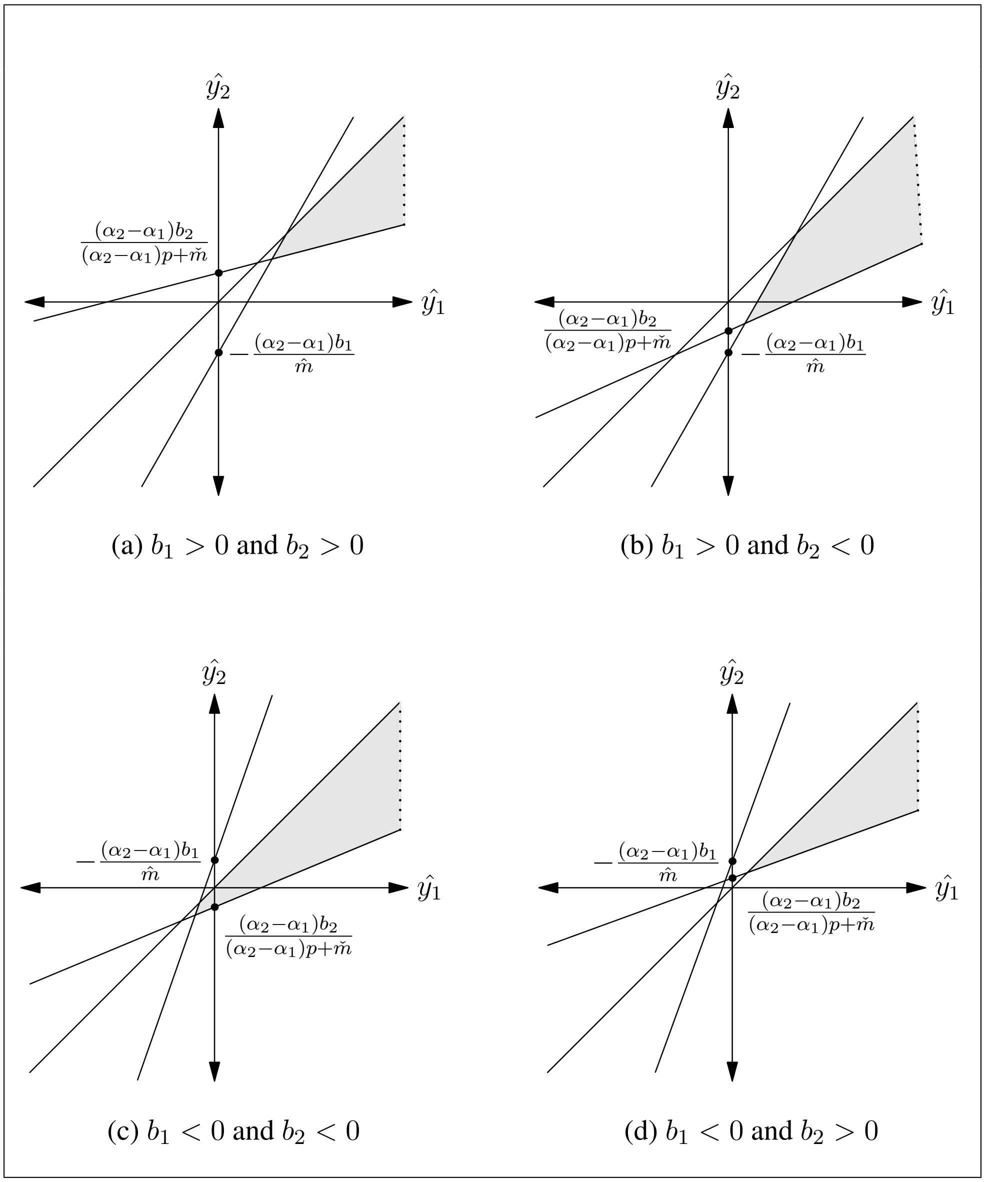


Figure 3.11: Supersolution region for the coupled (inner) system in the three subdomain iteration when $\hat{y}_1 \geq \hat{y}_2$

supersolution region if $\hat{y}_1 \geq \hat{y}_2$, and for all b_1 and b_2. Then a supersolution exists for the coupled (inner) system when $\hat{y}_1 \geq \hat{y}_2$. As shown in Figures (3.8-3.11), we can conclude supersolution and subsolution exists for the coupled (inner) system in the three subdomain iteration for all b_1 and b_2.

We observed from Figures 3.8-3.11 that the supersolution is greater then the subsolution.

Second approach: Here we show the existence of a subsolution and supersolution for the coupled (inner) system using Fourier-Motzkin elimination. From (3.100) the inequalities for the subsolution are

$$\left.\begin{aligned} \left(p+\frac{\check{m}}{\alpha_2-\alpha_1}\right)\check{x}_1-\frac{\check{m}}{\alpha_2-\alpha_1}\check{x}_2\le b_1 \\ -\frac{\hat{m}}{\alpha_2-\alpha_1}\check{x}_1+(p+\frac{\hat{m}}{\alpha_2-\alpha_1})\check{x}_2\le b_2 \end{aligned}\right\}. \tag{3.109}$$

We first eliminate the variable $\check{x}_1$. To do this partition the inequalities (3.109) and $\check{x}_1 \le \check{x}_2$ into two groups I_- and I_+, according to the coefficient of $\check{x}_1$. We have

$$I_-:\quad -\tfrac{\hat{m}}{\alpha_2-\alpha_1}\check{x}_1+(p+\tfrac{\hat{m}}{\alpha_2-\alpha_1})\check{x}_2 \quad\le\quad b_2,$$

and

$$I_+:\left\{\begin{aligned} \left(p+\tfrac{\check{m}}{\alpha_2-\alpha_1}\right)\check{x}_1-\tfrac{\check{m}}{\alpha_2-\alpha_1}\check{x}_2 &\quad\le\quad b_1 \\ \check{x}_1-\check{x}_2 &\quad\le\quad 0. \end{aligned}\right.$$

We make the coefficient of $\check{x}_1$ a -1 for the inequalities in I_- and the coefficient of $\check{x}_1$ a $+1$ for the inequalities in I_+ . This gives us

$$I_-:\quad -\check{x}_1+\left(\frac{p+\frac{\hat{m}}{\alpha_2-\alpha_1}}{\frac{\hat{m}}{\alpha_2-\alpha_1}}\right)\check{x}_2 \quad\le\quad \frac{b_2}{\frac{\hat{m}}{\alpha_2-\alpha_1}}$$

and

$$I_+:\left\{\begin{aligned} \check{x}_1-\left(\frac{\frac{\check{m}}{\alpha_2-\alpha_1}}{p+\frac{\check{m}}{\alpha_2-\alpha_1}}\right)\check{x}_2 &\quad\le\quad \frac{b_1}{p+\frac{\check{m}}{\alpha_2-\alpha_1}} \\ \check{x}_1-\check{x}_2 &\quad\le\quad 0. \end{aligned}\right.$$

Isolating the variable $\check{x}_1$ in each group gives

$$I_-:\quad \left(\frac{p+\frac{\hat{m}}{\alpha_2-\alpha_1}}{\frac{\hat{m}}{\alpha_2-\alpha_1}}\right)\check{x}_2-\frac{b_2}{\frac{\hat{m}}{\alpha_2-\alpha_1}} \quad\le\quad \check{x}_1,$$

and

$$I_+:\left\{\begin{aligned} \check{x}_1 &\quad\le\quad \left(\frac{\frac{\check{m}}{\alpha_2-\alpha_1}}{p+\frac{\check{m}}{\alpha_2-\alpha_1}}\right)\check{x}_2+\frac{b_1}{p+\frac{\check{m}}{\alpha_2-\alpha_1}} \\ \check{x}_1 &\quad\le\quad \check{x}_2. \end{aligned}\right.$$

This implies that

$$\left\{\left(\frac{p+\frac{\hat{m}}{\alpha_2-\alpha_1}}{\frac{\hat{m}}{\alpha_2-\alpha_1}}\right)\check{x}_2-\frac{b_2}{\frac{\hat{m}}{\alpha_2-\alpha_1}}\right\}\le\check{x}_1\le\left\{\begin{array}{l}\left(\frac{\frac{\check{m}}{\alpha_2-\alpha_1}}{p+\frac{\check{m}}{\alpha_2-\alpha_1}}\right)\check{x}_2+\frac{b_1}{p+\frac{\check{m}}{\alpha_2-\alpha_1}}\\ \check{x}_2\end{array}\right\},$$

and eliminating $\check{x}_1$ we obtain

$$\left.\begin{array}{l}\left(\frac{p+\frac{\hat{m}}{\alpha_2-\alpha_1}}{\frac{\hat{m}}{\alpha_2-\alpha_1}}\right)\check{x}_2-\frac{b_2}{\frac{\hat{m}}{\alpha_2-\alpha_1}}\le\left(\frac{\frac{\check{m}}{\alpha_2-\alpha_1}}{p+\frac{\check{m}}{\alpha_2-\alpha_1}}\right)\check{x}_2+\frac{b_1}{p+\frac{\check{m}}{\alpha_2-\alpha_1}}\\ \left(\frac{p+\frac{\hat{m}}{\alpha_2-\alpha_1}}{\frac{\hat{m}}{\alpha_2-\alpha_1}}\right)\check{x}_2-\frac{b_2}{\frac{\hat{m}}{\alpha_2-\alpha_1}}\le\check{x}_2\end{array}\right\}.$$

This implies

$$\left.\begin{array}{rl}\left[p^2+\frac{p}{\alpha_2-\alpha_1}(\hat{m}+\check{m})\right]\check{x}_2 & \le b_2\left(p+\frac{\check{m}}{\alpha_2-\alpha_1}\right)+\frac{\hat{m}b_1}{\alpha_2-\alpha_1}\\ p\check{x}_2 & \le b_2\end{array}\right\}.$$

This equivalent to

$$\left.\begin{array}{rl}\check{x}_2 & \le\frac{b_2\left(p+\frac{\check{m}}{\alpha_2-\alpha_1}\right)+\frac{\hat{m}b_1}{\alpha_2-\alpha_1}}{p^2+\frac{p}{\alpha_2-\alpha_1}(\hat{m}+\check{m})}\\ \check{x}_2 & \le\frac{b_2}{p}.\end{array}\right\}.\tag{3.110}$$

The system (3.110) does not involve the variable $\check{x}_1$, and we can find $\check{x}_2$ for any value of b_1 and b_2, and $\check{x}_2\le\min\left(\frac{b_2}{p},\frac{b_2\left(p+\frac{\check{m}}{\alpha_2-\alpha_1}\right)+\frac{\hat{m}b_1}{\alpha_2-\alpha_1}}{p^2+\frac{p}{\alpha_2-\alpha_1}(\hat{m}+\check{m})}\right)$, so this system is feasible. Hence the original system (3.100) is feasible when $\check{x}_1\le\check{x}_2$ and for all b_1 and b_2.

Now let us consider the case when

$$\check{x}_1\ge\check{x}_2.\tag{3.111}$$

From (3.102) the inequalities for the subsolution are

$$\left.\begin{array}{r}\left(p+\frac{\check{m}}{\alpha_2-\alpha_1}\right)\check{x}_1-\frac{\hat{m}}{\alpha_2-\alpha_1}\check{x}_2\le b_1\\ -\frac{\check{m}}{\alpha_2-\alpha_1}\check{x}_1+(p+\frac{\check{m}}{\alpha_2-\alpha_1})\check{x}_2\le b_2\end{array}\right\}.$$

Now eliminating as above we obtain

$$\left.\begin{array}{rl}\left[p^2+\frac{p}{\alpha_2-\alpha_1}(\hat{m}+\check{m})\right]\check{x}_2 & \le b_2\left(p+\frac{\hat{m}}{\alpha_2-\alpha_1}\right)+\frac{\check{m}b_1}{\alpha_2-\alpha_1}\\ p\check{x}_2 & \le b_1\end{array}\right\}.$$

This implies that

$$\left.\begin{array}{ll} \check{x}_2 & \leq \frac{b_2\left(p+\frac{\hat{m}}{\alpha_2-\alpha_1}\right)+\frac{\check{m}b_1}{\alpha_2-\alpha_1}}{p^2+\frac{p}{\alpha_2-\alpha_1}(\hat{m}+\check{m})} \\ \check{x}_2 & \leq \frac{b_1}{p} \end{array}\right\}. \tag{3.112}$$

We observe that the system (3.110) does not involve the variable $\check{x}_1$, and we can find $\check{x}_2$ for any value of b_1 and b_2, and $\check{x}_2 \leq \min\left(\frac{b_1}{p}, \frac{b_2\left(p+\frac{\hat{m}}{\alpha_2-\alpha_1}\right)+\frac{\check{m}b_1}{\alpha_2-\alpha_1}}{p^2+\frac{p}{\alpha_2-\alpha_1}(\hat{m}+\check{m})}\right)$, so this system is also feasible. Hence the system (3.102) is feasible when $\check{x}_1 \geq \check{x}_2$ and for all b_1 and b_2. Therefore, a subsolution exists for the coupled system.

We now repeat for the supersolution for the case $\hat{y}_1 \leq \hat{y}_2$. Using (3.105) the inequalities for the supersolution are

$$\left.\begin{array}{rl} \left(p+\frac{\hat{m}}{\alpha_2-\alpha_1}\right)\hat{y}_1 - \frac{\hat{m}}{\alpha_2-\alpha_1}\hat{y}_2 & \geq b_1 \\ -\frac{\check{m}}{\alpha_2-\alpha_1}\hat{y}_1 + \left(p+\frac{\check{m}}{\alpha_2-\alpha_1}\right)\hat{y}_2 & \geq b_2 \end{array}\right\}. \tag{3.113}$$

We now eliminate the variable $\hat{y}_1$. To do this partition the inequalities (3.113) and $\hat{y}_1 \leq \hat{y}_2$ into two groups I_- and I_+, according to the coefficient of $\check{x}_1$. This gives

$$I_- : \left\{\begin{array}{rcl} -\frac{\check{m}}{\alpha_2-\alpha_1}\hat{y}_1 + \left(p+\frac{\check{m}}{\alpha_2-\alpha_1}\right)\hat{y}_2 & \geq & b_2, \\ -\hat{y}_1+\hat{y}_2 & \geq & 0. \end{array}\right.$$

$$I_+ : \quad \left(p+\frac{\hat{m}}{\alpha_2-\alpha_1}\right)\hat{y}_1 - \frac{\hat{m}}{\alpha_2-\alpha_1}\hat{y}_2 \quad \geq \quad b_1$$

Now make the coefficient of $\hat{y}_1$ a -1 for the inequalities in I_- and the coefficient of $\hat{y}_1$ a $+1$ for the inequalities in I_+. This gives

$$I_- : \left\{\begin{array}{rcl} -\hat{y}_1 + \left(\frac{p+\frac{\check{m}}{\alpha_2-\alpha_1}}{\frac{\check{m}}{\alpha_2-\alpha_1}}\right)\hat{y}_2 & \geq & \frac{b_2}{\frac{\check{m}}{\alpha_2-\alpha_1}} \\ -\hat{y}_1+\hat{y}_2 & \geq & 0, \end{array}\right.$$

and

$$I_+ : \quad \hat{y}_1 - \left(\frac{\frac{\hat{m}}{\alpha_2-\alpha_1}}{p+\frac{\hat{m}}{\alpha_2-\alpha_1}}\right)\hat{y}_2 \quad \geq \quad \frac{b_1}{p+\frac{\hat{m}}{\alpha_2-\alpha_1}}.$$

Isolating the variable $\hat{y}_1$ in each group gives us

$$I_- : \begin{cases} \left(\frac{p+\frac{\check{m}}{\alpha_2-\alpha_1}}{\frac{\check{m}}{\alpha_2-\alpha_1}}\right)\hat{y}_2 - \frac{b_2}{\frac{\check{m}}{\alpha_2-\alpha_1}} & \geq & \hat{y}_1 \\ \hat{y}_2 & \geq & \hat{y}_1, \end{cases}$$

and

$$I_+ : \ \hat{y}_1 \ \geq \left(\frac{\frac{\hat{m}}{\alpha_2-\alpha_1}}{p+\frac{\hat{m}}{\alpha_2-\alpha_1}}\right)\hat{y}_2 + \frac{b_1}{p+\frac{\hat{m}}{\alpha_2-\alpha_1}}.$$

This implies that

$$\left\{\left(\frac{p+\frac{\hat{m}}{\alpha_2-\alpha_1}}{\frac{\hat{m}}{\alpha_2-\alpha_1}}\right)\hat{y}_2 + \frac{b_1}{\frac{\hat{m}}{\alpha_2-\alpha_1}}\right\} \leq \hat{y}_1 \leq \left\{\begin{array}{l} \left(\frac{\frac{\check{m}}{\alpha_2-\alpha_1}}{p+\frac{\check{m}}{\alpha_2-\alpha_1}}\right)\hat{y}_2 - \frac{b_2}{p+\frac{\check{m}}{\alpha_2-\alpha_1}} \\ \hat{y}_2 \end{array}\right\}.$$

Eliminating $\hat{y}_1$ we obtain

$$\left.\begin{array}{l} \left(\frac{p+\frac{\hat{m}}{\alpha_2-\alpha_1}}{\frac{\hat{m}}{\alpha_2-\alpha_1}}\right)\hat{y}_2 + \frac{b_1}{\frac{\hat{m}}{\alpha_2-\alpha_1}} \leq \left(\frac{\frac{\check{m}}{\alpha_2-\alpha_1}}{p+\frac{\check{m}}{\alpha_2-\alpha_1}}\right)\hat{y}_2 - \frac{b_2}{p+\frac{\check{m}}{\alpha_2-\alpha_1}} \\ \left(\frac{p+\frac{\hat{m}}{\alpha_2-\alpha_1}}{\frac{\hat{m}}{\alpha_2-\alpha_1}}\right)\hat{y}_2 + \frac{b_1}{\frac{\hat{m}}{\alpha_2-\alpha_1}} \leq \hat{y}_2, \end{array}\right\}.$$

This implies

$$\left.\begin{array}{rl} \left[p^2 + \frac{p}{\alpha_2-\alpha_1}(\hat{m}+\check{m})\right]\hat{y}_2 & \leq b_2\left(p + \frac{\hat{m}}{\alpha_2-\alpha_1}\right) + \frac{\check{m}b_1}{\alpha_2-\alpha_1} \\ p\hat{y}_2 & \leq b_1 \end{array}\right\}.$$

This equivalent to

$$\left.\begin{array}{ll} \hat{y}_2 & \frac{\leq b_2\left(p+\frac{\hat{m}}{\alpha_2-\alpha_1}\right)+\frac{\check{m}b_1}{\alpha_2-\alpha_1}}{p^2+\frac{p}{\alpha_2-\alpha_1}(\hat{m}+\check{m})} \\ \hat{y}_2 & \leq \frac{b_1}{p} \end{array}\right\}. \tag{3.114}$$

The system (3.114) does not involve the variable $\hat{y}_1$, and we can find $\hat{y}_2$ for any value of b_1 and b_2, and $\check{x}_2 \leq \min\left(\frac{b_1}{p}, \frac{\leq b_2\left(p+\frac{\hat{m}}{\alpha_2-\alpha_1}\right)+\frac{\check{m}b_1}{\alpha_2-\alpha_1}}{p^2+\frac{p}{\alpha_2-\alpha_1}(\hat{m}+\check{m})}\right)$, so this system is feasible. Hence the original system (3.105) is feasible when $\hat{y}_1 \leq \hat{y}_2$ and for all b_1 and b_2.

Now consider $\hat{y}_1 \geq \hat{y}_2$. Using (3.107) the inequalities for the supersolution are

$$\left.\begin{array}{rl} \left(p + \frac{\check{m}}{\alpha_2-\alpha_1}\right)\hat{y}_1 - \frac{\check{m}}{\alpha_2-\alpha_1}\hat{y}_2 & \geq b_1 \\ -\frac{\hat{m}}{\alpha_2-\alpha_1}\hat{y}_1 + \left(p + \frac{\hat{m}}{\alpha_2-\alpha_1}\right)\hat{y}_2 & \geq b_2 \end{array}\right\}.$$

Now eliminating as above we obtain

$$\left.\begin{aligned} \left[p^2 + \tfrac{p}{\alpha_2-\alpha_1}(\hat{m}+\check{m})\right]\hat{y}_2 &\geq b_2\left(p + \tfrac{\check{m}}{\alpha_2-\alpha_1}\right) + \tfrac{\hat{m}b_1}{\alpha_2-\alpha_1} \\ p\hat{y}_2 &\geq b_2 \end{aligned}\right\}.$$

This implies that

$$\left.\begin{aligned} \hat{y}_2 &\geq \frac{b_2\left(p+\frac{\check{m}}{\alpha_2-\alpha_1}\right)+\frac{\hat{m}b_1}{\alpha_2-\alpha_1}}{p^2+\frac{p}{\alpha_2-\alpha_1}(\hat{m}+\check{m})} \\ \hat{y}_2 &\geq \frac{b_2}{p} \end{aligned}\right\}. \tag{3.115}$$

We observe that the system (3.115) does not involve the variable $\hat{y}_1$, and we can find $\hat{y}_2$ for any value of b_1 and b_2, and $\check{x}_2 \leq \min\left(\frac{b_2}{p}, \frac{b_2\left(p+\frac{\check{m}}{\alpha_2-\alpha_1}\right)+\frac{\hat{m}b_1}{\alpha_2-\alpha_1}}{p^2+\frac{p}{\alpha_2-\alpha_1}(\hat{m}+\check{m})}\right)$, thus this system is also feasible. Hence our original system (3.107) is feasible when $\hat{y}_1 \geq \hat{y}_2$ and for all b_1 and b_2. Therefore, supersolution exists for the coupled system. Hence we can conclude that supersolution and subsolution exists for the coupled system of the three subdomain iteration for all b_1 and b_2.

□

Theorem 3.3.5 *Consider the coupled (inner) system $F(y,z) = b$ from (3.88) where the operator R_2 is defined in (3.83). This system has (at least one) solution and moreover nonlinear SOR (or Jacobi) will converge starting from a supersolution or a subsolution.*

Proof. Lemmas 3.3.2 and 3.3.4 show that F is a continuous, off-diagonally antitone, and strictly diagonally isotone and there exists a supersolution and subsolution. Therefore by Theorem 3.2.8 a solution exists and nonlinear SOR (or Jacobi) will converge starting from a supersolution or a subsolution. □

Uniqueness for Inner Subdomain Iteration of the Three Subdomains

We are interested in question of uniqueness for the system of (3.88) for a given right-hand side. We first will show that F defined in system (3.89 - 3.90) is an M-function, and then we apply Theorem 3.2.15.

Theorem 3.3.6 *Consider the coupled (inner) system $F(y,z) = b$ from (3.88) and the operator R_2 is defined in (3.83). $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a continuous M-function for all $p > 0$.*

Proof. The system (3.88) can be written as $F(y,z) = b$, where, $b = (0,0)^T$ and

$$\begin{aligned} f_1(y,z) &= -R_2(y,z) + py - \zeta_2, \\ f_2(y,z) &= R_2(y,z) + pz - \zeta_3. \end{aligned} \tag{3.116}$$

Lemma 3.3.2 proves F is off-diagonally antitone. Using Theorem 3.2.15 we now build the functions $q_i(t)$. Choosing $h_j = 1$ in (3.66), we construct the functions $q_i(t)$ as

$$q_i(t) = \sum_{j=1}^{2} f_j(\mathbf{x} + te^i), \quad 1 \le i \le 2,$$

where e^i denotes the i^{th} standard basis vector in R^2. When $i = 1$, we have

$$\begin{aligned} q_1(t) &= f_1(y+t,z) + f_2(y+t,z) \\ &= -R_2(y+t,z) + p(y+t) - \zeta_2 + R_2(y+t,z) + pz - \zeta_3 \\ &= p(y+t) + pz - \zeta_2 - \zeta_3. \end{aligned}$$

Differentiating with respect to t we have

$$\frac{dq_1}{dt} = p > 0.$$

Again, when $i = 2$ then

$$\begin{aligned} q_2(t) &= f_1(y,z+t) + f_2(y,z+t) \\ &= -R_2(y,z+t) + py - \zeta_2 + R_2(y,z+t) + p(z+t) - \zeta_3 \\ &= py + p(z+t) - \zeta_2 - \zeta_3. \end{aligned}$$

Differentiate with respect to t we obtain

$$\frac{dq_2}{dt} = p > 0.$$

Therefore $\frac{dq_1}{dt}$ and $\frac{dq_2}{dt}$ are strictly positive, hence the functions q_i are strictly isotone. Therefore by Theorem 3.2.15 the coupled system (3.88) defines an M-function for all $p > 0$. □

Theorem 3.3.7 *Consider the coupled (inner) system* $F(y, z) = b$ *from* (3.88) *with the operator* R_2 *is defined in* (3.83)*. This system has a unique solution.*

Proof. The assumptions of the Lemma 3.2.14 has been verified by the Lemma 3.3.6. Hence the system (3.88) has a unique solution. □

Lemma 3.3.8 *Consider the coupled (inner) system* $F(y, z) = b$ *from* (3.88) *where the operator* R_2 *is defined in* (3.83)*. The function* $F : \mathbb{R}^2 \to \mathbb{R}^2$ *is onto.*

Proof. The assumptions of Lemma 3.2.20 has been verified by Lemmas 3.3.2 and 3.3.4. Hence F is onto by Lemma 3.2.20. □

Theorem 3.3.9 *Consider the coupled (inner) system* $F(y, z) = b$ *from* (3.88)*. Nonlinear Jacobi (or SOR) will converge to the unique solution for any starting value.*

Proof. The assumptions of Theorem 3.2.19 has been verified by Theorem 3.3.6 and Lemma 3.3.8. Hence by Theorem 3.2.19 we can conclude the nonlinear Jacobi (or SOR) will converge to the unique solution for any starting value.

□

Hence Gauss-Jacobi or Gauss-Seidel iteration for the coupled (inner) system of 3.85 converge to a unique solution by the Theorem 3.3.9. In the next section we will show the whole three subdomain system is well-posed.

3.3.2 Well-posedness of the Three Subdomain Iteration for Whole System

In this section we wish to study the existence and uniqueness of solutions for the whole system (3.80 - 3.82). We rewrite the recurrence relations (3.80 - 3.82) for the three subdomains as

$$\begin{aligned}
R_1(x_1^n(\alpha_1)) - R_2(x_2^{n-1}(\alpha_1), x_2^{n-1}(\alpha_2)) + p(x_1^n(\alpha_1) - x_2^{n-1}(\alpha_1)) = 0 \\
\left.\begin{aligned}
R_1(x_1^{n-1}(\alpha_1)) - R_2(x_2^n(\alpha_1), x_2^n(\alpha_2)) + p(x_2^n(\alpha_1) - x_1^{n-1}(\alpha_1)) = 0 \\
-R_3(x_3^{n-1}(\alpha_2)) + R_2(x_2^n(\alpha_1), x_2^n(\alpha_2)) + p(x_2^n(\alpha_2) - x_3^{n-1}(\alpha_2)) = 0
\end{aligned}\right\} \\
-R_3(x_3^n(\alpha_2)) + R_2(x_2^{n-1}(\alpha_1), x_2^{n-1}(\alpha_2)) + p(x_3^n(\alpha_2) - x_2^{n-1}(\alpha_2)) = 0.
\end{aligned}$$

If the above iteration converges then the limit points must satisfy

$$R_1(x) - R_2(y,z) + p(x-y) = 0 \tag{3.117}$$

$$\left.\begin{aligned}
R_1(x) - R_2(y,z) + p(y-x) = 0 \\
-R_3(w) + R_2(y,z) + p(z-w) = 0
\end{aligned}\right\} \tag{3.118}$$

$$-R_3(w) + R_2(y,z) + p(w-z) = 0. \tag{3.119}$$

We wish to show the existence of x, y, z and w solving system (3.117-3.119). This is equivalent to solving the system

$$\begin{aligned}
f_1(x,y,z,w) &\equiv R_1(x) - R_2(y,z) + p(x-y) = 0 \\
f_2(x,y,z,w) &\equiv R_1(x) - R_2(y,z) + p(y-x) = 0 \\
f_3(x,y,z,w) &\equiv -R_3(w) + R_2(y,z) + p(z-w) = 0 \\
f_4(x,y,z,w) &\equiv -R_3(w) + R_2(y,z) + p(w-z) = 0.
\end{aligned} \tag{3.120}$$

This gives a system of the form $F(x,y,z,w) = b$, where $F = (f_1, f_2, f_3, f_4)^T$ and $b = (0,0,0,0)^T$.

We wish to show the existence and uniqueness by showing the system (3.120) is a onto M-function. We now show the Jacobian of (3.120) has a required sign pattern.

Lemma 3.3.10 *Consider the system $F(x,y,z,w) = b$ from (3.120). Then F is a continuous, strictly diagonally isotone and if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{\alpha_2-\alpha_1}, \frac{\hat{m}}{1-\alpha_2}\}$ then F is off-diagonally antitone.*

Proof. It is clear that f_1, f_2, f_3 and f_4 are continuous. Now we want to show that system is strictly diagonally isotone. To show this, we differentiate f_1 with respect to x to obtain

$$\frac{\partial f_1}{\partial x} = \frac{\partial R_1}{\partial x} + p = \frac{1}{\alpha_1}M(x) + p > 0.$$

Differentiating f_2 with respect to y to find

$$\frac{\partial f_2}{\partial y} = -\frac{\partial R_2}{\partial y} + p = \frac{1}{\alpha_2 - \alpha_1}M(y) + p > 0.$$

Differentiating f_3 with respect to z to obtain

$$\frac{\partial f_3}{\partial z} = \frac{\partial R_2}{\partial z} + p = \frac{1}{\alpha_2 - \alpha_1}M(z) + p > 0.$$

And finally differentiate f_4 with respect to w to find

$$\frac{\partial f_4}{\partial w} = \frac{\partial R_3}{\partial w} + p = \frac{1}{1 - \alpha_2}M(w) + p > 0.$$

This tells us f_1, f_2, f_3 and f_4 are strictly isotone with respect to x, y, z, and w for all $p > 0$. Therefore, F is strictly diagonally isotone.

We now will show that the F is off-diagonally antitone. To show this, we differentiate f_1 with respect to y, z, w, we obtain

$$\begin{aligned}
\frac{\partial f_1}{\partial y} &= -\frac{\partial R_2}{\partial y} - p = \frac{1}{\alpha_2 - \alpha_1}M(y) - p,\\
\frac{\partial f_1}{\partial z} &= -\frac{\partial R_2}{\partial z} = -\frac{1}{\alpha_2 - \alpha_1}M(z),\\
\frac{\partial f_1}{\partial w} &= 0.
\end{aligned}$$

Hence f_1 is antitone with respect to y, z and w, if p satisfies $p > \frac{\hat{m}}{\alpha_2 - \alpha_1}$. So, f_1 is antitone if $p > \frac{\hat{m}}{\alpha_2 - \alpha_1}$.

Now differentiate f_2 with respect to x, z and w, we obtain

$$\begin{aligned}
\frac{\partial f_2}{\partial x} &= -\frac{\partial R_1}{\partial x} - p = \frac{1}{\alpha_1}M(x) - p,\\
\frac{\partial f_2}{\partial z} &= -\frac{\partial R_2}{\partial z} = -\frac{1}{\alpha_2 - \alpha_1}M(z),\\
\frac{\partial f_2}{\partial w} &= 0.
\end{aligned}$$

Thus f_2 is antitone with respect to x, z and w, if p satisfies $\frac{\hat{m}}{\alpha_1} < p$.

Differentiating f_3 with respect to x, y and w, we have

$$\begin{aligned}
\frac{\partial f_3}{\partial x} &= 0, \\
\frac{\partial f_3}{\partial y} &= \frac{\partial R_2}{\partial y} = -\frac{1}{\alpha_2 - \alpha_1} M(y), \\
\frac{\partial f_3}{\partial w} &= -\frac{\partial R_3}{\partial w} - p = \frac{1}{\alpha_2 - \alpha_1} M(w) - p.
\end{aligned}$$

Hence f_3 is antitone with respect to x, y and w, if p satisfies $p > \frac{\hat{m}}{1-\alpha_2}$.

Similarly, differentiate f_4 with respect to x, y and z, we obtain

$$\begin{aligned}
\frac{\partial f_4}{\partial x} &= 0, \\
\frac{\partial f_4}{\partial y} &= -\frac{\partial R_2}{\partial y} = -\frac{1}{\alpha_2 - \alpha_1} M(y), \\
\frac{\partial f_4}{\partial z} &= -\frac{\partial R_2}{\partial z} - p = \frac{1}{\alpha_2 - \alpha_1} M(z) - p.
\end{aligned}$$

Therefore f_4 is antitone with respect to x, y and z, if p satisfies $p > \frac{\hat{m}}{\alpha_2-\alpha_1}$.

In conclusion, p needs to be greater than $\max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{\alpha_2-\alpha_1}, \frac{\hat{m}}{1-\alpha_2}\}$ to satisfy all the above conditions on p. Hence, F is off-diagonally antitone if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{\alpha_2-\alpha_1}, \frac{\hat{m}}{1-\alpha_2}\}$. $\square$

We now wish to show the function F that arises from the three subdomain system is onto. To show this the mean value theorem for integrals from [56] is necessary, and the theorem is stated below.

Theorem 3.3.11 (Mean Value Theorem for Integrals (MVTI)) *If f is continuous on a closed interval $[a, b]$, there exists at least one point c on the interval (a, b) such that*

$$\int_a^b f(x)dx = f(c)(b - a).$$

To show F is surjective for we use the flowing idea due to Felix Kwok [57]. We modify system (3.120) for any right-hand side vector. Then we will show the modified system has a supersolution and a subsolution. So, for any right-hand side vector the system the system (3.120) written as

$$\begin{aligned} x_1 &= 0 \\ R_1(x_1,y_1) - R_2(x_2,y_2) + p(y_1 - x_2) &= \zeta_1 \\ R_1(x_1,y_1) - R_2(x_2,y_2) + p(x_2 - y_1) &= \zeta_2 \\ -R_3(x_3,y_3) + R_2(x_2,y_2) + p(y_2 - x_3) &= \zeta_3 \\ -R_3(x_3,y_3) + R_2(x_2,y_2) + p(x_3 - y_2) &= \zeta_4, \\ y_3 &= 1 \end{aligned} \tag{3.121}$$

where

$$R_i(x_i,y_i) = \frac{1}{\alpha_i - \alpha_{i-1}} \int_{x_i}^{y_i} M(x)dx, \quad \text{for} \quad i = 1,2,3, \tag{3.122}$$

with $\alpha_i < \alpha_{i-1}$, $\alpha_0 = 0$ and $\alpha_3 = 1$.

Adding and subtracting equations from each other except the first and last equations in the system (3.121) gives

$$\left.\begin{aligned} x_1 &= 0 \\ R_1(x_1,y_1) - R_2(x_2,y_2) &= \gamma_1 \\ x_2 - y_1 &= \gamma_2 \\ R_2(x_2,y_2) - R_3(x_3,y_3) &= \gamma_3 \\ x_3 - y_2 &= \gamma_4 \\ y_3 &= 1 \end{aligned}\right\}, \tag{3.123}$$

where $\gamma_1 = \frac{\zeta_1+\zeta_2}{2}$, $\gamma_2 = \frac{\zeta_2-\zeta_1}{2}$, $\gamma_3 = \frac{\zeta_3+\zeta_4}{2}$ and $\gamma_4 = \frac{\zeta_4-\zeta_3}{2}$. We want to study this equivalent system to original system. We denote this system as $G(x_1,y_1,x_2,y_2,x_3,y_3) = b'$, where

$G = (g_1, g_2, g_3, g_4, g_5, g_6)^T$, $b' = (0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, 1)^T$, where

$$\left.\begin{aligned} g_1(x_1, y_1, x_2, y_2, x_3, y_3) &= x_1 \\ g_2(x_1, y_1, x_2, y_2, x_3, y_3) &= R_1(x_1, y_1) - R_2(x_2, y_2) \\ g_3(x_1, y_1, x_2, y_2, x_3, y_3) &= x_2 - y_1 \\ g_4(x_1, y_1, x_2, y_2, x_3, y_3) &= R_2(x_2, y_2) - R_3(x_3, y_3) \\ g_5(x_1, y_1, x_2, y_2, x_3, y_3) &= x_3 - y_2 \\ g_6(x_1, y_1, x_2, y_2, x_3, y_3) &= y_3 \end{aligned}\right\}. \tag{3.124}$$

For any $s \in \mathbb{R}$, consider the following recipe:

$$\text{set} \quad x_1 = 0 \tag{3.125}$$

$$\text{solve} \quad R_1(x_1, y_1) = s \quad \text{for} \quad y_1, \tag{3.126}$$

$$\text{set} \quad x_2 = y_1 + \gamma_2 \tag{3.127}$$

$$\text{solve} \quad R_2(x_2, y_2) = s - \gamma_1 \quad \text{for} \quad y_2, \tag{3.128}$$

$$\text{set} \quad x_3 = y_2 + \gamma_4 \tag{3.129}$$

$$\text{solve} \quad R_3(x_3, y_3) = s - \gamma_1 - \gamma_3 \quad \text{for} \quad y_3. \tag{3.130}$$

Which is derived based on the (3.123). We want to show this recipe is well-defined.

Lemma 3.3.12 *For any $s \in \mathbb{R}$ the recipe (3.125-3.130) is well-defined.*

Proof. Clearly x_1, x_2, and x_3 are unique from (3.125), (3.127) and (3.129). We know $R_i(x_i, y_i), i = 1, 2, 3$ are continuous and uniformly monotonically increasing or decreasing with respect to y_i and x_i respectively. Since x_1 is unique and R_1 is monotonic, hence $R_1(x_1, y_1) = s$ can be solved for y_1 uniquely. As well as, we can solve $R_2(x_2, y_2) = s - \gamma_1$ and $R_3(x_3, y_3) = s - \gamma_1 - \gamma_3$ for y_2 and y_3 uniquely. Thus, x_1, x_2, x_3, y_1, y_2 and y_3 are unique and the recipe (3.125-3.130) is well-defined.

□

Lemma 3.3.13 *Assume $G(x_1, y_1, x_2, y_2, x_3, y_3) = b^{'}$ is the system from (3.123) where $b^{'} = (0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, 1) \in \mathbb{R}^6$ and the operators are defined in (3.122), then G is onto.*

Proof. We need to prove for any $b^{'} = (0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, 1) \in \mathbb{R}^6$, there exist points $\check{\mathbf{x}}, \hat{\mathbf{y}} \in \mathbb{R}^6$, such that $\check{\mathbf{x}} \leq \hat{\mathbf{y}}$, where $\check{\mathbf{x}} = (\check{x_1}, \check{y_1}, \check{x_2}, \check{y_2}, \check{x_3}, \check{y_3})$ and $\hat{\mathbf{y}} = (\hat{x_1}, \hat{y_1}, \hat{x_2}, \hat{y_2}, \hat{x_3}, \hat{y_3})$, and

$$G(\check{x_1}, \check{y_1}, \check{x_2}, \check{y_2}, \check{x_3}, \check{y_3}) \leq b^{'} \leq G(\hat{x_1}, \hat{y_1}, \hat{x_2}, \hat{y_2}, \hat{x_3}, \hat{y_3}).$$

The existence of a solution will then be obtained by continuity. That is we require

$$\left.\begin{array}{c} \check{x_1} \leq 0 \leq \hat{x_1} \\ R_1(\check{x_1}, \check{y_1}) - R_2(\check{x_2}, \check{y_2}) \leq \gamma_1 \leq R_1(\hat{x_1}, \hat{y_1}) - R_2(\hat{x_2}, \hat{y_2}) \\ \check{x_2} - \check{y_1} \leq \gamma_2 \leq \hat{x_2} - \hat{y_1} \\ R_2(\check{x_2}, \check{y_2}) - R_3(\check{x_3}, \check{y_3}) \leq \gamma_3 \leq R_2(\hat{x_2}, \hat{y_2}) - R_3(\hat{x_3}, \hat{y_3}) \\ \check{x_3} - \check{y_2} \leq \gamma_4 \leq \hat{x_3} - \hat{y_2} \\ \check{y_3} \leq 1 \leq \hat{y_3} \end{array}\right\} . \qquad (3.131)$$

We now use the MVTI in (3.126), (3.128) and (3.130). From $R_1(x_1, y_1) = s$ we have $\frac{m_1^*}{\alpha_1 - \alpha_0}(y_1 - x_1) = s$. Rearranging and substituting $x_1 = 0$ into this equation gives

$$y_1 = \frac{1}{m_1^*}(\alpha_1 - \alpha_0)s.$$

$R_2(x_2, y_2) = s - \gamma_1$ can be written as $\frac{m_2^*}{\alpha_2 - \alpha_1}(y_2 - x_2) = s - \gamma_1$ using MVTI. Substituting $x_2 = y_1 + \gamma_2$ and rewriting the resulting equation gives

$$\begin{aligned} y_2 &= \frac{1}{m_2^*}(\alpha_2 - \alpha_1)(s - \gamma_1) + y_1 + \gamma_2 \\ &= \frac{1}{m_2^*}(\alpha_2 - \alpha_1)(s - \gamma_1) + \frac{1}{m_1^*}(\alpha_1 - \alpha_0)s + \gamma_2. \end{aligned}$$

Similarly, from $R_3(x_3, y_3) = s - \gamma_1 - \gamma_3$ we obtain $\frac{m_3^*}{\alpha_3 - \alpha_2}(y_3 - x_3) = s - \gamma_1 - \gamma_3$. Rearranging and substituting $x_3 = y_2 + \gamma_4$ into this equation gives us

$$\begin{aligned} y_3 &= \frac{1}{m_3^*}(\alpha_3 - \alpha_2)(s - \gamma_1 - \gamma_3) + y_2 + \gamma_4 \\ &= \frac{1}{m_3^*}(\alpha_3 - \alpha_2)(s - \gamma_1 - \gamma_3) + \frac{1}{m_2^*}(\alpha_2 - \alpha_1)(s - \gamma_1) + \frac{1}{m_1^*}(\alpha_1 - \alpha_0)s + \gamma_2 + \gamma_4. \end{aligned}$$

Hence we have the resulting system

$$
\begin{aligned}
x_1 =& 0\\
y_1 =& \frac{1}{m_1^*}(\alpha_1-\alpha_0)s\\
x_2 =& y_1+\gamma_2\\
y_2 =& \frac{1}{m_2^*}(\alpha_2-\alpha_1)(s-\gamma_1)+\frac{1}{m_1^*}(\alpha_1-\alpha_0)s+\gamma_2\\
x_3 =& y_2+\gamma_4\\
y_3 =& \frac{1}{m_3^*}(\alpha_3-\alpha_2)(s-\gamma_1-\gamma_3)+\frac{1}{m_2^*}(\alpha_2-\alpha_1)(s-\gamma_1)+\frac{1}{m_1^*}(\alpha_1-\alpha_0)s+\gamma_2+\gamma_4,
\end{aligned}
$$

where m_i^*, $i=1,2,3$ are values obtained form the MVTI. This is an equivalent system to (3.125-3.130). If it has an unique solution then so does the system (3.125-3.130). It is clear that this system has a unique solution.

We now assume $(\check{x}_1,\check{y}_1,\check{x}_2,\check{y}_2,\check{x}_3,\check{y}_3)\in\mathbb{R}^6$. For a subsolution we require $F(\check{x}_1,\check{y}_1,\check{x}_2,\check{y}_2,\check{x}_3,\check{y}_3)\le b$. Obviously $\frac{1}{m_i^*}\le\frac{1}{\check{m}}$ for $i=1,2,3$.

If $(\alpha_1-\alpha_0)s>0$ then we obtain

$$\check{y}_1\le\frac{1}{\check{m}}(\alpha_1-\alpha_0)s.$$

If in addition $(\alpha_2-\alpha_1)(s-\gamma_1)>0$ then we have

$$\check{y}_2\le\frac{1}{\check{m}}(\alpha_2-\alpha_1)(s-\gamma_1)+\frac{1}{\check{m}}(\alpha_1-\alpha_0)s+\gamma_2,$$

Similarly if in addition $(\alpha_3-\alpha_2)(s-\gamma_1-\gamma_3)>0$ then we find

$$\check{y}_3\le\frac{1}{\check{m}}(\alpha_3-\alpha_2)(s-\gamma_1-\gamma_3)+\frac{1}{\check{m}}(\alpha_2-\alpha_1)(s-\gamma_1)+\frac{1}{\check{m}}(\alpha_1-\alpha_0)s+\gamma_2+\gamma_4.$$

Thus the resulting inequalities are

$$
\begin{aligned}
&\check{x}_1 = 0\\
&\check{y}_1 \le \frac{1}{\check{m}}(\alpha_1 - \alpha_0)s\\
&\check{x}_2 = y_1 + \gamma_2\\
&\check{y}_2 \le \frac{1}{\check{m}}(\alpha_2 - \alpha_1)(s - \gamma_1) + \frac{1}{\check{m}}(\alpha_1 - \alpha_0)s + \gamma_2\\
&\check{x}_3 = y_2 + \gamma_4\\
&\check{y}_3 \le \frac{1}{\check{m}}(\alpha_3 - \alpha_2)(s - \gamma_1 - \gamma_3) + \frac{1}{\check{m}}(\alpha_2 - \alpha_1)(s - \gamma_1) + \frac{1}{\check{m}}(\alpha_1 - \alpha_0)s + \gamma_2 + \gamma_4.
\end{aligned}
\tag{3.132}
$$

Again, chose a value $\check{s} \in \mathbb{R}$, we set $\check{x}_1 = 0$ and $R_1(\check{x}_1, \check{y}_1) = \check{s}$. If $\check{s}$ satisfies the following inequalities

$$
\left.
\begin{aligned}
\frac{1}{\check{m}}(\alpha_1 - \alpha_0)\check{s} \le \alpha_1 - \alpha_0\\
\frac{1}{\check{m}}(\alpha_2 - \alpha_1)(\check{s} - \gamma_1) + \gamma_2 \le \alpha_2 - \alpha_1\\
\frac{1}{\check{m}}(\alpha_3 - \alpha_2)(\check{s} - \gamma_1 - \gamma_3) + \gamma_4 \le \alpha_3 - \alpha_2
\end{aligned}
\right\},
\tag{3.133}
$$

then we obtain from (3.134)

$$
\begin{aligned}
\check{y}_3 &\le \alpha_3 - \alpha_2 + \alpha_2 - \alpha_1 + \alpha_1 - \alpha_0\\
&= \alpha_3 - \alpha_0\\
&= 1.
\end{aligned}
$$

This inequality confirms a subsolution exists. Hence we have a subsolution for (3.123) if $\check{s}$ satisfies the inequalities in (3.133).

Similarly, we assume $(\hat{x}_1, \hat{y}_1, \hat{x}_2, \hat{y}_2, \hat{x}_3, \hat{y}_3) \in \mathbb{R}^6$. For a supersolution $F(\hat{x}_1, \hat{y}_1, \hat{x}_2, \hat{y}_2, \hat{x}_3, \hat{y}_3) \ge b$ needs to be satisfied. Clearly $\frac{1}{m_i^*} \ge \frac{1}{\hat{m}}$ for $i = 1, 2, 3$.

If $(\alpha_1 - \alpha_0)s > 0$ then we obtain

$$
\hat{y}_1 \ge \frac{1}{\hat{m}}(\alpha_1 - \alpha_0)s.
$$

If in addition $(\alpha_2 - \alpha_1)(s - \gamma_1) > 0$ then we have

$$\hat{y}_2 \geq \frac{1}{\hat{m}}(\alpha_2 - \alpha_1)(s - \gamma_1) + \frac{1}{\hat{m}}(\alpha_1 - \alpha_0)s + \gamma_2.$$

Likewise if $(\alpha_3 - \alpha_2)(s - \gamma_1 - \gamma_3) > 0$ then

$$\hat{y}_3 \geq \frac{1}{\hat{m}}(\alpha_3 - \alpha_2)(s - \gamma_1 - \gamma_3) + \frac{1}{\hat{m}}(\alpha_2 - \alpha_1)(s - \gamma_1) + \frac{1}{\hat{m}}(\alpha_1 - \alpha_0)s + \gamma_2 + \gamma_4.$$

Hence we obtain the inequalities

$$\begin{aligned}
\hat{x}_1 &= 0 \\
\hat{y}_1 &\geq \frac{1}{\hat{m}}(\alpha_1 - \alpha_0)s \\
\hat{x}_2 &= y_1 + \gamma_2 \\
\hat{y}_2 &\geq \frac{1}{\hat{m}}(\alpha_2 - \alpha_1)(s - \gamma_1) + \frac{1}{\hat{m}}(\alpha_1 - \alpha_0)s + \gamma_2 \\
\hat{x}_3 &= y_2 + \gamma_4 \\
\hat{y}_3 &\geq \frac{1}{\hat{m}}(\alpha_3 - \alpha_2)(s - \gamma_1 - \gamma_3) + \frac{1}{\hat{m}}(\alpha_2 - \alpha_1)(s - \gamma_1) + \frac{1}{\hat{m}}(\alpha_1 - \alpha_0)s + \gamma_2 + \gamma_4.
\end{aligned} \tag{3.134}$$

We now choose a value $\hat{s} \in \mathbb{R}$ and set $\hat{x_1} = 0$ and $R_1(\hat{x}_1, \hat{y}_1) = \hat{s}$. If $\hat{s}$ satisfies the following conditions

$$\left.\begin{aligned}
\frac{1}{\hat{m}}(\alpha_1 - \alpha_0)\hat{s} &\geq \alpha_1 - \alpha_0 \\
\frac{1}{\hat{m}}(\alpha_2 - \alpha_1)(\hat{s} - \gamma_1) + \gamma_2 &\geq \alpha_2 - \alpha_1 \\
\frac{1}{\hat{m}}(\alpha_3 - \alpha_2)(\hat{s} - \gamma_1 - \gamma_3) + \gamma_4 &\geq \alpha_3 - \alpha_2,
\end{aligned}\right\}, \tag{3.135}$$

then from (3.134) we have

$$\begin{aligned}
\hat{y}_3 &\geq \alpha_3 - \alpha_2 + \alpha_2 - \alpha_1 + \alpha_1 - \alpha_0 \\
&= \alpha_3 - \alpha_0 \\
&= 1.
\end{aligned} \tag{3.136}$$

And hence, we have a supersolution for (3.123) if $\hat{s}$ satisfies the inequalities in (3.135). In conclusion, a supersolution and subsolution exists for the system. Moreover by continuity there exist $\bar{s} \in [\check{s}, \hat{s}]$, so that $y_3 = 1$, and hence we have solution for (3.123). Thus G is onto. $\square$

Lemma 3.3.14 *Consider the system $F(x, y, z, w) = b$ from (3.120) and the operators R_i, $i = 1, 2, 3$ as defined in (3.83). The function $F : \mathbb{R}^4 \to \mathbb{R}^4$ is onto.*

Proof. In Lemma 3.3.13 we show that a system $G(x_1, y_1, x_2, y_2, x_3, y_3) = b'$, which is an equivalent system to the original system $F(x, y, z, w) = b$ is onto. Hence F is onto. $\square$

Theorem 3.3.15 *Consider the system $F(x, y, z, w) = b$ from (3.120) and the operators R_i, $i = 1, 2, 3$ as defined in (3.83). Then $F : \mathbb{R}^4 \to \mathbb{R}^4$ is a continuous onto M-function if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{\alpha_2-\alpha_1}, \frac{\hat{m}}{1-\alpha_2}\}$.*

Proof. F is onto from Lemma 3.3.14.

Lemma 3.3.10 proves F is off-diagonally antitone when $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{\alpha_2-\alpha_1}, \frac{\hat{m}}{1-\alpha_2}\}$. Now using Theorem 3.2.15 we now build the functions $q_i(t)$. Choosing $h_j = 1$ in (3.66), we construct the functions $q_i(t)$ as

$$q_i(t) = \sum_{j=1}^{4} f_j(X + te^i), \quad 1 \le i \le 4,$$

where e^i denote the i^{th} standard basis vector in R^4.

If $i = 1$, then we obtain

$$\begin{aligned} q_1(t) &= f_1(x+t, y, z, w) + f_2(x+t, y, z, w) + f_3(x+t, y, z, w) + f_4(x+t, y, z, w) \\ &= R_1(x+t) - R_2(y, z) + p(x+t-y) + R_1(x+t) - R_2(y, z) + p(y-x-t) - R_3(w) \\ &+ R_2(y, z) + p(z-w) - R_3(w) + R_2(y, z) + p(w-z) \\ &= 2R_1(x+t) - 2R_3(w). \end{aligned}$$

Differentiating with respect to t we have

$$\frac{dq_1}{dt} = 2\frac{dR_1(x+t)}{dt} = \frac{2}{\alpha_1}M(t) > 0.$$

Again when $i = 2$, then

$$\begin{aligned}
q_2(t) &= f_1(x, y+t, z, w) + f_2(x, y+t, z, w) + f_3(x, y+t, z, w) + f_4(x, y+t, z, w) \\
&= R_1(x) - R_2(y+t, z) + p(x-y-t) + R_1(x) - R_2(y+t, z) + p(y+t-x) - R_3(w) \\
&+ R_2(y+t, z) + p(z-w) - R_3(w) + R_2(y+t, z) + p(w-z) \\
&= 2R_1(x) - 2R_3(w).
\end{aligned}$$

Differentiating with respect to t we obtain

$$\frac{dq_2}{dt} = 0.$$

Similarly if $i = 3$, then we obtain

$$\begin{aligned}
q_3(t) &= f_1(x, y, z+t, w) + f_2(x, y, z+t, w) + f_3(x, y, z+t, w) + f_4(x, y, z+t, w) \\
&= R_1(x) - R_2(y, z+t) + p(x-y) + R_1(x) - R_2(y, z+t) + p(y-x) - R_3(w) \\
&+ R_2(y, z+t) + p(z+t-w) - R_3(w) + R_2(y, z+t) + p(w-z-t) \\
&= 2R_1(x) - 2R_3(w).
\end{aligned}$$

Differentiating with respect to t we find

$$\frac{dq_3}{dt} = 0.$$

And finally when $i = 4$, then we have

$$\begin{aligned}
q_4(t) &= f_1(x, y, z, w+t) + f_2(x, y, z, w+t) + f_3(x, y, z, w+t) + f_4(x, y, z, w+t) \\
&= R_1(x) - R_2(y, z) + p(x-y) + R_1(x) - R_2(y, z) + p(y-x) - R_3(w+t) \\
&+ R_2(y, z) + p(z-w-t) - R_3(w+t) + R_2(y, z) + p(w+t-z) \\
&= 2R_1(x) - 2R_3(w+t).
\end{aligned}$$

Differentiating with respect to t we find

$$\frac{dq_4}{dt} = \frac{dR_3(w+t)}{dt} = \frac{2}{1-\alpha_2}M(t) > 0.$$

Therefore $\frac{dq_1}{dt}$ and $\frac{dq_4}{dt}$ are strictly positive, hence the functions q_1 and q_4 are strictly isotone. But $\frac{dq_2}{dt}$ and $\frac{dq_3}{dt}$ are not strictly positive, hence the functions q_2 and q_3 are not strictly isotone. Therefore we need to show that a path $k \rightsquigarrow i$ exists for $i = 2, 3$ where the functions q_k are strictly isotone. Possible paths can be

$$\begin{array}{cccc} 1 \rightsquigarrow 2 & 1 \rightsquigarrow 2 & 1 \rightsquigarrow 3 & 4 \rightsquigarrow 2 \\ 1 \rightsquigarrow 3 & 4 \rightsquigarrow 3 & 4 \rightsquigarrow 2 & 4 \rightsquigarrow 3 \end{array}$$

We now try to find out the strict links. From the definition of a strict link we know that if the function $t \to f_i(\mathbf{x} + te^j)$ is strictly antitone then a link (i, j) is a strict. We have

$$\begin{aligned} \frac{\partial}{\partial y} f_1(\mathbf{x} + te^2) &= \frac{\partial}{\partial y} f_1(x, y+t, z, w) \\ &= \frac{\partial}{\partial y}[R_1(x) - R_2(y+t, z) + p(x-y-t)] \\ &= \frac{M(y+t)}{\alpha_2 - \alpha_1} - p, \end{aligned}$$

which implies $(1, 2)$ is strict link if $p > \frac{\hat{m}}{\alpha_2 - \alpha_1}$. Similarly we obtain

$$\begin{aligned} \frac{\partial}{\partial z} f_1(\mathbf{x} + te^3) &= -\frac{M(z+t)}{\alpha_2 - \alpha_1} < 0, \\ \frac{\partial}{\partial x} f_2(\mathbf{x} + te^3) &= -\frac{M(z+t)}{\alpha_2 - \alpha_1} - p < 0, \\ \frac{\partial}{\partial z} f_3(\mathbf{x} + te^2) &= -\frac{M(y+t)}{\alpha_2 - \alpha_1} < 0, \\ \frac{\partial}{\partial w} f_3(\mathbf{x} + te^4) &= \frac{M(w+t)}{\alpha_2 - \alpha_1} - p, \\ \frac{\partial}{\partial y} f_4(\mathbf{x} + te^2) &= -\frac{M(y+t)}{\alpha_2 - \alpha_1} < 0, \\ \frac{\partial}{\partial z} f_4(\mathbf{x} + te^3) &= \frac{M(z+t)}{\alpha_2 - \alpha_1} - p. \end{aligned}$$

Thus $(1,3)$, $(2,3)$, $(3,2)$, and $(4,2)$ are strict links. Here $(3,4)$ and $(4,3)$ will be strict links if $p > \frac{\hat{m}}{\alpha_2-\alpha_1}$. Hence we have a possible path $1 \rightsquigarrow 2$ and $4 \rightsquigarrow 3$ where q_1 and q_4 are strictly isotone. So, all assumptions has been satisfied of Theorem 3.2.15. Hence F is an M-function if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{\alpha_2-\alpha_1}, \frac{\hat{m}}{1-\alpha_2}\}$. □

Theorem 3.3.16 *Consider the system $F(x,y,z,w) = b$ from (3.120) and the operators R_i, $i = 1,2,3$ as defined in (3.83). This system has a unique solution if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{\alpha_2-\alpha_1}, \frac{\hat{m}}{1-\alpha_2}\}$.*

Proof. The assumptions of Lemma 3.2.14 have been verified by Theorem 3.3.15, hence system (3.88) has a unique solution. □

Theorem 3.3.17 *Consider the system $F(x,y,z,w) = b$ from (3.120) and the operators R_i, $i = 1,2,3$ as defined in (3.83). Nonlinear Jacobi (or SOR) will converge to the unique solution for any starting value, when $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{\alpha_2-\alpha_1}, \frac{\hat{m}}{1-\alpha_2}\}$.*

Proof. The assumptions of Theorem 3.2.19 has been verified by Theorem 3.3.15 and Lemma 3.3.14 if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{\alpha_2-\alpha_1}, \frac{\hat{m}}{1-\alpha_2}\}$. Hence by the Theorem 3.2.19 we can conclude the nonlinear Jacobi (or SOR) will converge to the unique solution for any starting value.

□

Hence Gauss-Jacobi or Gauss-Seidel iterations will converge to a unique solution for the system (3.120) if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{\alpha_2-\alpha_1}, \frac{\hat{m}}{1-\alpha_2}\}$. Our original parallel iteration (3.80-3.82) however, is not a nonlinear Gauss-Jacobi or Gauss-Seidel iteration. Instead it is a block Gauss-Jacobi iteration. Block Gauss-Jacobi and block Gauss-Seidel processes have been analyzed in Rheinboldt [48].

An implicit iterative process for a nonlinear system $Fx = b$ is given by

$$G(x^n, x^{n-1}) = b, \quad n = 0, 1, 2, ... \tag{3.137}$$

To analyze iteration of the form (3.137) we first introduce a regular iteration function from [48].

Definition 3.3.1 *A mapping $G : \mathbb{D}_0 \times \mathbb{D}_0 \subset \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a regular iteration function for $F : \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ on a subset $\mathbb{D}_0$ of $\mathbb{D}$ if*

$$G(x, x) = Fx, \quad \textit{for any} \quad x \in \mathbb{D}_0, \tag{3.138}$$

$$G(., x) := \mathbb{D}_0 \to \mathbb{R}^n, \quad \textit{is inverse isotone, for any fixed} \quad x \in \mathbb{D}_0,$$

$$and \quad G(y, .) := \mathbb{D}_0 \to \mathbb{R}^n, \quad \textit{is antitone, for any fixed} \quad y \in \mathbb{D}_0.$$

We quote Theorems 3.3.18 and 3.3.19 from Rheinboldt [48]; these theorems gives us a way to prove the block Gauss-Jacobi and Gauss-Seidel process converge globally.

Theorem 3.3.18 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuous, inverse isotone, and onto (surjective). Suppose, further, that $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a regular iteration function for F on $\mathbb{R}^n$ with the property that $G(., x) : \mathbb{R}^n \to \mathbb{R}^n$ is surjective for any fixed $x \in \mathbb{R}^n$. Then, for any $b \in \mathbb{R}^n$ and any initial point $x_0 \in \mathbb{R}^n$, the process (3.137) converges to a unique solution $x^* \in \mathbb{R}^n$ of $Fx = b$.*

We observed that with

$$G := \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \tag{3.139}$$

$$P_i G(y, x) := F^i(y^1, ..., y^i, x^{i+1}, ..., x^n), \quad i = 1, ..., q,$$

the block Gauss-Seidel process (3.3) assume the general form (3.137). The following result, Theorem 3.3.19, ensures the applicability of Theorem 3.2.8 and 3.3.18 to the block Gauss-Seidel iteration (3.3) for M-functions F. Also if G is a regular iteration function for $Fx = b$ then Theorem 3.3.19 gives a way to prove $G(., x)$ is onto for any fixed $x \in \mathbb{R}^n$ when F is an $M-$function.

Theorem 3.3.19 *Let $F := \mathbb{R}^n \to \mathbb{R}^n$ be an M-function; then the mapping $G := \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defined by (3.139) is a regular iteration function for F on $\mathbb{R}^n$. If, in addition, F is continuous and surjective, then $G(., x) := \mathbb{R}^n \to \mathbb{R}^n$ is surjective for any fixed $x \in \mathbb{R}^n$.*

Hence we can conclude the global convergence of the block Gauss-Jacobi and Gauss-Seidel process from Theorems 3.3.18 and 3.3.19 for continuous onto $M-$functions.

Now back to our original parallel iteration (3.80-3.82) which is a block Gauss-Jacobi iteration. The theorem below guarantees the parallel iteration converges to the unique solution.

Theorem 3.3.20 *Consider the system* $F(x, y, z, w) = b$ *from* (3.120). *Nonlinear block Gauss-Jacobi converge to a unique solution for any starting value if* $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{\alpha_2-\alpha_1}, \frac{\hat{m}}{1-\alpha_2}\}$.

Proof. We wish to verify assumptions of Theorem 3.3.18. The assumptions of the Theorem are that F is continuous, inverse isotone, and surjective, and the regular iteration function $G(., x)$ is surjective for any fixed $x \in \mathbb{R}^4$.

Clearly, F is continuous. The surjectivity of F has been verified by the Lemma 3.3.14 if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{\alpha_2-\alpha_1}, \frac{\hat{m}}{1-\alpha_2}\}$, and F is an M- function by Theorem 3.3.15. By the definition of M-function implies that F is inverse isotone. All assumptions of Theorem 3.3.19 have been verified, thus $G(., x)$ is surjective for any fixed $x \in \mathbb{R}^4$ by the Theorem 3.3.19.

Hence, the assumptions of Theorem 3.3.18 has been verified so, the nonlinear block Gauss-Jacobi iteration (or implicit iteration) (3.80-3.82) converges to the unique solution for any starting value if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{\alpha_2-\alpha_1}, \frac{\hat{m}}{1-\alpha_2}\}$. □

Therefore the block Gauss-Jacobi iterations (3.80-3.82) for the system (3.120) will converge monotonically to a unique solution if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{\alpha_2-\alpha_1}, \frac{\hat{m}}{1-\alpha_2}\}$, since the system is a onto M-function.

3.3.3 Alternative Approach to Show the Well-posedness of the Three Subdomain Iteration

An equivalent system for the interface iteration for three subdomains is given as

$$R_1(x) - R_2(y,z) + p(x-y) = 0 \tag{3.140}$$

$$R_1(x) - R_2(y,z) + p(y-x) = 0 \tag{3.141}$$

$$-R_3(w) + R_2(y,z) + p(z-w) = 0 \tag{3.142}$$

$$-R_3(w) + R_2(y,z) + p(w-z) = 0. \tag{3.143}$$

Adding (3.140) and (3.141), and adding (3.142) and (3.143) we obtain

$$R_1(x) = R_2(y,z) \quad \text{and} \quad R_3(w) = R_2(y,z). \tag{3.144}$$

Now subtracting (3.140) from (3.141), and subtracting (3.142) from (3.143) we have

$$x = y \quad \text{and} \quad z = w. \tag{3.145}$$

From (3.144) and (3.145) we obtain

$$R_1(x) = R_2(x,w) = R_3(w).$$

If we can show that x and w are unique then we are done. We know the operator $R_1(x)$ is uniformly continuous and monotnic increasing, and the operator $R_3(w)$ is uniformly continuous and monotonic deceasing, then from $R_1(x) = R_3(w)$ we can conclude that x and w are unique.

3.4 An Interface Iteration for an Arbitrary Number of Subdomains

We decompose the computational domain $\Omega_c = (0,1)$ into an arbitrary number of nonoverlapping subdomains $\Omega_1 = (\alpha_0, \alpha_1)$, $\Omega_2 = (\alpha_1, \alpha_2)$, ..., and $\Omega_m = (\alpha_{m-1}, \alpha_m)$, with $\alpha_0 = 0$

and $\alpha_m = 1$, where $\alpha_{i-1} < \alpha_i$, $i = 2, 3, ..., m$, so there is no overlap between consecutive subdomains as shown in Figure 3.1. The parallel version of interface iteration on an arbitrary number subdomains is given from Lemma 3.1.3 as

$$R_1\big(0, x_1^n(\alpha_1)\big) + px_1^n(\alpha_1) = R_2\big(x_2^{n-1}(\alpha_1), x_2^{n-1}(\alpha_2)\big) + px_2^{n-1}(\alpha_1) \tag{3.146}$$

$$\left.\begin{aligned} R_i\big(x_i^n(\alpha_{i-1}), x_i^n(\alpha_i)\big) - px_i^n(\alpha_{i-1}) &= R_{i-1}\big(x_{i-1}^{n-1}(\alpha_{i-2}), x_{i-1}^{n-1}(\alpha_{i-1})\big) - px_{i-1}^{n-1}(\alpha_{i-1}) \\ R_i\big(x_i^n(\alpha_{i-1}), x_i^n(\alpha_i)\big) + px_i^n(\alpha_i) &= R_{i+1}\big(x_{i+1}^{n-1}(\alpha_i), x_{i+1}^{n-1}(\alpha_{i+1})\big) + px_{i+1}^{n-1}(\alpha_i) \end{aligned}\right\} \tag{3.147}$$

$$i = 2, 3, ..., m-1,$$

and

$$R_m\big(x_m^n(\alpha_{m-1}, 1)\big) - px_m^n(\alpha_{m-1}) = R_{m-1}\big(x_{m-2}^{n-1}(\alpha_{m-2}), x_{m-1}^{n-1}(\alpha_{m-1})\big) - px_{m-1}^{n-1}(\alpha_{m-1}) \tag{3.148}$$

with $x_1 = 0$, $y_m = 1$, and $\alpha_i < \alpha_{i-1}$, $\alpha_0 = 0$, and $\alpha_m = 1$ where

$$R_i(x_i, y_i) = \frac{1}{\alpha_i - \alpha_{i-1}} \int_{x_i}^{y_i} M(x)dx, \quad i = 1, 2, ..., m. \tag{3.149}$$

If the above iteration converges then the limit points must satisfy

$$R_1\big(x_1, y_1\big) + py_1 = R_2\big(x_2, y_2\big) + px_2,$$

$$\left.\begin{aligned} R_i\big(x_i, y_i\big) - px_i &= R_{i-1}\big(x_{i-1}, y_{i-1}\big) - py_{i-1} \\ R_i\big(x_i, y_i\big) + py_i &= R_{i+1}\big(x_{i+1}, x_{i+1}\big) + px_{i+1} \end{aligned}\right\},$$

$$i = 2, 3, ..., m-1,$$

and

$$R_m\big(x_m, y_m)\big) - px_m = R_{m-1}\big(x_{m-2}, y_{m-1})\big) - py_{m-1},$$

Rewriting this system gives us

$$R_1\big(x_1, y_1\big) - R_2\big(x_2, y_2\big) + p(y_1 - x_2) = 0, \tag{3.150}$$

$$\left.\begin{aligned} R_{i-1}(x_{i-1}, y_{i-1}) - R_i(x_i, y_i) + p(x_i - y_{i-1}) = 0 \\ R_i(x_i, y_i) - R_{i+1}(x_{i+1}, x_{i+1}) + p(y_i - x_{i+1}) = 0 \end{aligned}\right\}, \qquad i = 2, 3, ..., m-1, \tag{3.151}$$

and

$$R_{m-1}(x_{m-1}, y_{m-1})) - R_m(x_m, y_m)) + p(x_m - y_{m-1}) = 0. \tag{3.152}$$

We wish to study the existence of y_1, x_i, y_i, $i = 2, ..., m-1$, and x_m solving system (3.150-3.152), where x_1 and y_m are given. Hence this system has $2m-2$ equations and $2m-2$ unknowns. This is equivalent to solving the system

$$\begin{aligned} &f_1(y_1, x_2, y_2) \equiv R_1(0, y_1) - R_2(x_2, y_2) + p(y_1 - x_2) = 0, \\ &\left.\begin{aligned} f_{2i-2}(x_{i-1}, y_{i-1}, x_i, y_i) &\equiv R_{i-1}(x_{i-1}, y_{i-1}) - R_i(x_i, y_i) + p(x_i - y_{i-1}) = 0 \\ f_{2i-1}(x_i, y_i, x_{i+1}, y_{i+1}) &\equiv R_i(x_i, y_i) - R_{i+1}(x_{i+1}, x_{i+1}) + p(y_i - x_{i+1}) = 0 \end{aligned}\right\}, \\ &\qquad i = 2, 3, ..., m-1, \\ &f_{2m-2}(x_{m-1}, y_{m-1}, x_m) \equiv R_{m-1}(x_{m-1}, y_{m-1})) - R_m(x_m, 1)) + p(x_m - y_{m-1}) = 0. \end{aligned} \tag{3.153}$$

This gives a system of the form $F(y_1, x_2, y_2, ..., x_{m-1}, y_{m-1}, x_m) = 0$, where $F = (f_1, f_2, ..., f_{2m-2})^T$ and $0 = (0, 0, ..., 0)^T$.

We wish to show the existence and uniqueness by showing the system (3.153) is a onto M-function. We now show the Jacobian of (3.153) has a required sign pattern.

Lemma 3.4.1 *Consider the system $F(y_1, x_2, y_2, ..., x_{m-1}, y_{m-1}, x_m) = b$ from* (3.153) *with the operators R_i, $i = 1, 2, ..., m$ as defined in* (3.149)*. The function $F : \mathbb{R}^{2m-2} \to \mathbb{R}^{2m-2}$ is continuous, strictly diagonally isotone, and if $p > \max\{\frac{\hat{m}}{\alpha_1 - \alpha_0}, \frac{\hat{m}}{\alpha_2 - \alpha_1}, ..., \frac{\hat{m}}{\alpha_m - \alpha_2}\}$ then F is off-diagonally antitone.*

Proof. The operators $R_i(x_i, y_i)$ are continuous and strictly increasing in y_i with x_i fixed and strictly decreasing in x_i with y_i fixed. Hence f_1, f_2, ..., f_{2m-2} are continuous. Now we want to show that system is strictly diagonally isotone. To show this, we differentiate f_1 with respect to y_1, we find

$$\frac{\partial f_1}{\partial y_1} = \frac{\partial R_1}{\partial y_1} + p = \frac{1}{\alpha_1 - \alpha_0} M(y_1) + p > 0.$$

Differentiating f_{2i-2} with respect to x_i, for $i = 2, 3, ..., m-1$, we have

$$\frac{\partial f_{2i-2}}{\partial x_i} = -\frac{\partial R_i}{\partial x_i} + p = \frac{1}{\alpha_i - \alpha_{i-1}} M(x_i) + p > 0.$$

Differentiating f_{2i-1} with respect to y_i, for $i = 2, 3, ..., m-1$, gives us

$$\frac{\partial f_{2i-1}}{\partial y_i} = \frac{\partial R_i}{\partial y_i} + p = \frac{1}{\alpha_{i+1} - \alpha_i} M(y_i) + p > 0.$$

And finally differentiating f_{2m-2} with respect to x_m we obtain

$$\frac{\partial f_(2m-2)}{\partial x_m} = \frac{\partial R_{2m-2}}{\partial x_m} + p = \frac{1}{1 - \alpha_{m-1}} M(x_m) + p > 0.$$

This tells us f_1, f_2, ..., f_{2m-2} are strictly isotone with respect to y_1, x_i, and y_i, for $i = 2, ..., m-1$, and x_m respectively for all $p > 0$. Therefore, F is strictly diagonally isotone.

Now we will show that F is off-diagonally antitone. To show this, we differentiate f_1 with respect to x_i, y_i, for $i = 2, ..., m-1$, and x_m, we obtain

$$\begin{aligned}
&\frac{\partial f_1}{\partial x_2} = -\frac{\partial R_2}{\partial x_2} - p = \frac{1}{\alpha_2 - \alpha_1} M(x_2) - p, \\
&\frac{\partial f_1}{\partial y_2} = -\frac{\partial R_2}{\partial y_2} = -\frac{1}{\alpha_2 - \alpha_1} M(y_2), \\
&\frac{\partial f_1}{\partial x_j} = 0 \quad \text{and} \quad \frac{\partial f_1}{\partial y_j} = 0 \quad \text{for} \quad j = 3, 4, ..., m-1, \\
&\frac{\partial f_1}{\partial x_m} = 0.
\end{aligned}$$

If p satisfies $\frac{\hat{m}}{\alpha_2 - \alpha_1} < p$ then f_1 is antitone with respect to x_2. Already, f_i is antitone with respect to remaining variables (or unknowns). Hence f_1 is antitone if $p > \frac{\hat{m}}{\alpha_2 - \alpha_1}$.

Now differentiate f_2 with respect to all variables except x_2, we obtain

$$
\begin{aligned}
&\frac{\partial f_2}{\partial y_1} = \frac{\partial R_1}{\partial y_1} - p = \frac{1}{\alpha_1 - \alpha_0} M(y_1) - p,\\
&\frac{\partial f_2}{\partial y_2} = -\frac{\partial R_2}{\partial y_2} = -\frac{1}{\alpha_2 - \alpha_1} M(y_2),\\
&\frac{\partial f_2}{\partial x_j} = 0 \quad \text{and} \quad \frac{\partial f_2}{\partial y_j} = 0 \quad \text{for} \quad j = 3, 4, ..., m-1,\\
&\frac{\partial f_2}{\partial x_m} = 0.
\end{aligned}
$$

If p satisfies $\frac{\hat{m}}{\alpha_1 - \alpha_0} < p$ then f_2 is antitone with respect to y_1, f_2 is already antitone with respect to all other variables as we seen above. Therefore f_2 is antitone when $p > \frac{\hat{m}}{\alpha_1 - \alpha_0}$.

Differentiating f_{2i-2} with respect to y_1, and x_{i-1}, y_{i-1}, y_i, for $i = 3, 4, ..., m-1$, and x_m gives us

$$
\begin{aligned}
&\frac{\partial f_{2i-2}}{\partial y_1} = 0,\\
&\frac{\partial f_{2i-2}}{\partial x_{i-1}} = \frac{\partial R_{i-1}}{\partial x_{i-1}} = -\frac{1}{\alpha_{i-1} - \alpha_{i-2}} M(x_{i-1}),\\
&\frac{\partial f_{2i-2}}{\partial y_{i-1}} = \frac{\partial R_{i-1}}{\partial y_{i-1}} - p = \frac{1}{\alpha_{i-1} - \alpha_{i-2}} M(y_{i-1}) - p,\\
&\frac{\partial f_{2i-2}}{\partial y_i} = -\frac{\partial R_i}{\partial y_i} = -\frac{1}{\alpha_i - \alpha_{i-1}} M(y_i),\\
&\frac{\partial f_{2i-2}}{\partial x_m} = 0.
\end{aligned}
$$

If p satisfies $\frac{\hat{m}}{\alpha_{i-1} - \alpha_{i-2}} < p$ then f_{2i-2} is antitone with respect to y_{i-1}. f_{2i-2} is already antitone with respect to all other variables as we see above. Therefore f_{2i-2} is antitone when $p > \frac{\hat{m}}{\alpha_{i-1} - \alpha_{i-2}}$.

Differentiating f_{2i-1} with respect to y_1, and x_i,x_{i+1},y_{i+1}, $i = 2, 3, ..., m-2$, and x_m, gives us

$$
\begin{aligned}
&\frac{\partial f_{2i-1}}{\partial y_1} = 0,\\
&\frac{\partial f_{2i-1}}{\partial x_i} = -\frac{\partial R_i}{\partial x_i} = -\frac{1}{\alpha_i - \alpha_{i-1}} M(x_i) < 0,\\
&\frac{\partial f_{2i-1}}{\partial x_{i+1}} = -\frac{\partial R_{i+1}}{\partial x_{i+1}} = \frac{1}{\alpha_{i+1} - \alpha_i} M(x_{i+1}) - p,\\
&\frac{\partial f_{2i-1}}{\partial y_{i+1}} = \frac{\partial R_{i+1}}{\partial y_{i+1}} = -\frac{1}{\alpha_{i+1} - \alpha_i} M(y_{i+1}) < 0,\\
&\frac{\partial f_{2i-1}}{\partial x_m} = 0.
\end{aligned}
$$

If p satisfies $\frac{\hat{m}}{\alpha_{i+1}-\alpha_i} < p$ then f_{2i-1} is antitone with respect to x_{i+1}. f_{2i-1} is already antitone with respect to all other variables as we see above. Therefore f_{2i-1} is antitone when $p > \frac{\hat{m}}{\alpha_{i+1}-\alpha_i}$.

Differentiating f_{2m-3} with respect to all variables except y_{m-1}, we have

$$
\begin{aligned}
&\frac{\partial f_{2m-3}}{\partial y_1} = 0,\\
&\frac{\partial f_{2m-3}}{\partial x_j} = 0 \quad \text{and} \quad \frac{\partial f_{2m-3}}{\partial y_j} = 0 \quad \text{for} \quad j = 2, 3, ..., m-2,\\
&\frac{\partial f_{2m-3}}{\partial x_{m-1}} = \frac{\partial R_{m-1}}{\partial x_{m-1}} = -\frac{1}{\alpha_{m-1} - \alpha_{m-2}} M(x_{m-1}) < 0,\\
&\frac{\partial f_{2m-3}}{\partial x_m} = -\frac{\partial R_m}{\partial x_m} - p = \frac{1}{\alpha_m - \alpha_{m-1}} M(x_m) - p,
\end{aligned}
$$

If p satisfies $\frac{\hat{m}}{\alpha_m-\alpha_{m-1}} < p$ then f_{2m-3} is antitone with respect to x_m. f_{2m-3} is already antitone with respect to all other variables as we see above. Hence f_{2m-3} is antitone if $p > \frac{\hat{m}}{\alpha_m-\alpha_{m-1}}$.

Similarly, differentiate f_{2m-2} with respect to all variables except x_m, then we obtain

$$
\begin{aligned}
&\frac{\partial f_{2m-2}}{\partial y_1} = 0,\\
&\frac{\partial f_{2m-2}}{\partial x_j} = 0 \quad \text{and} \quad \frac{\partial f_{2m-2}}{\partial y_j} = 0 \quad \text{for} \quad j = 2, 3, ..., m-2,\\
&\frac{\partial f_{2m-2}}{\partial x_{m-1}} = \frac{\partial R_{m-1}}{\partial x_{m-1}} = -\frac{1}{\alpha_{m-1} - \alpha_{m-2}} M(x_{m-1}) < 0,\\
&\frac{\partial f_{2m-2}}{\partial y_{m-1}} = \frac{\partial R_{m-1}}{\partial y_{m-1}} - p = \frac{1}{\alpha_{m-1} - \alpha_{m-2}} M(y_{m-1}) - p.
\end{aligned}
$$

If p satisfies $\frac{\hat{m}}{\alpha_{m-1}-\alpha_{m-2}} < p$ then f_{2m-2} is antitone with respect to x_m. f_{2m-2} is already antitone with respect to all other variables as we seen above. Hence f_{2m-2} is antitone if $p > \frac{\hat{m}}{\alpha_{m-1}-\alpha_{m-2}}$.

In conclusion, p need to be greater than $\max\{\frac{\hat{m}}{\alpha_1-\alpha_0}, \frac{\hat{m}}{\alpha_2-\alpha_1}, ..., \frac{\hat{m}}{\alpha_m-\alpha_{m-1}}\}$ to satisfy all of the above conditions on p. Hence, F is off-diagonally antitone if $p > \max\{\frac{\hat{m}}{\alpha_1-\alpha_0}, \frac{\hat{m}}{\alpha_2-\alpha_1}, ..., \frac{\hat{m}}{\alpha_m-\alpha_{m-1}}\}$. □

We now wish to show to show F is surjective, using the flowing idea due to Felix Kwok [57]. We modify system (3.153) for any right-hand side vector. So, for any right-hand side vector the system (3.153) written as

$$
\begin{aligned}
x_1 &= 0,\\
R_1\big(x_1, y_1\big) - R_2\big(x_2, y_2\big) + p(y_1 - x_2) &= \zeta_1,\\
\left.\begin{aligned}
R_{i-1}\big(x_{i-1}, y_{i-1}\big) - R_i\big(x_i, y_i\big) + p(x_i - y_{i-1}) &= \zeta_{2i-2}\\
R_i\big(x_i, y_i\big) - R_{i+1}\big(x_{i+1}, x_{i+1}\big) + p(y_i - x_{i+1}) &= \zeta_{2i-1}
\end{aligned}\right\} &, \quad i = 2, 3, ..., m-1,\\
R_{m-1}\big(x_{m-1}, y_{m-1})\big) - R_m\big(x_m, y_m)\big) + p(x_m - y_{m-1}) &= \zeta_{2m-2},\\
y_m &= 1.
\end{aligned}
\tag{3.154}
$$

where R_i is defined on (3.149). Now adding and subtracting of the equations in system (3.154) we have

$$
\begin{aligned}
x_1 &= 0\\
R_1(x_1, y_1) - R_2(x_2, y_2) &= \gamma_1\\
x_2 - y_1 &= \gamma_2\\
\left.\begin{aligned}
R_i\big(x_i, y_i\big) - R_{i+1}\big(x_{i+1}, x_{i+1}\big) &= \gamma_{2i-1}\\
x_{i+1} - y_i &= \gamma_{2i}
\end{aligned}\right\} &, i = 2, ..., m-1,\\
y_m &= 1
\end{aligned}
\tag{3.155}
$$

where $\gamma_{2m-1} = \frac{\zeta_i+\zeta_{i+1}}{2}$, $\gamma_{2i} = \frac{\zeta_{i+1}-\zeta_i}{2}$, $i = 1, 2, ..., m-1$. This system is equivalent to orginal system. We want to show this system has a supersolution and a subsolution. We denote this system as $G(x_1, y_1, ..., x_m, y_m) = b'$, where $F = (g_1, g_2, ..., g_{2m})^T$, $b' = (0, \gamma_1, \gamma_2, ..., \gamma_{2m-2}, 1)^T$, where

$$
\begin{aligned}
&g_1(x_1, y_1, ..., x_m, y_m) = x_1\\
&g_2(x_1, y_1, ..., x_m, y_m) = R_1(x_1, y_1) - R_2(x_2, y_2)\\
&g_3(x_1, y_1, ..., x_m, y_m) = x_2 - y_1\\
&\left.\begin{aligned}
g_{2i-2}(x_1, y_1, ..., x_m, y_m) &= R_i(x_i, y_i) - R_{i+1}(x_{i+1}, x_{i+1})\\
g_{2i-1}(x_1, y_1, ..., x_m, y_m) &= x_{i+1} - y_i
\end{aligned}\right\}, i = 2, ..., m-1,\\
&g_{2m-2}(x_1, y_1, ..., x_m, y_m) = y_m.
\end{aligned} \tag{3.156}
$$

For any $s \in \mathbb{R}$, consider the following recipe:

$$x_1 = 0 \tag{3.157}$$

$$\text{solve} \quad R_1(x_1, y_1) = s \quad \text{for} \quad y_1, \tag{3.158}$$

$$x_2 - y_1 = \gamma_2 \tag{3.159}$$

$$\text{solve} \quad R_2(x_2, y_2) = s - \gamma_1 \quad \text{for} \quad y_2, \tag{3.160}$$

$$x_{i+1} - y_i = \gamma_{2i} \tag{3.161}$$

$$\text{solve} \quad R_{i+1}(x_{i+1}, y_{i+1}) = s - \sum_{j=1}^{i} \gamma_{2j-1} \quad \text{for} \quad y_{i+1}, \tag{3.162}$$

$$i = 2, 3, ..., m-1.$$

This recipe has been derived based on the system (3.155). We prove this system has solution exist and is unique in lemma below.

Lemma 3.4.2 *For any $s \in \mathbb{R}$ the recipe (3.157-3.162) is well-defined.*

Proof. From (3.157), (3.159) and (3.161) we have x_1, x_2, and x_{i+1} are unique once y_i's are known. We know that $R_i(x_i, y_j)$, $i = 1, 2, ..., m$ are continuous and uniformly monotonically increasing or decreasing with respect to y_i and x_i respectively. Since R_1 is monotonic and x_1 is unique, hence $R_1(x_1, y_1) = s$ can be solved for y_1 uniquely. Similarly we can solve $R_2(x_2, y_2) = s - \gamma_1$ and $R_{i+1}(x_{i+1}, y_{i+1}) = s - \sum_{j=1}^{i} \gamma_{2j-1}$ for y_2 and y_{i+1} uniquely. Therefore $x_1, y_1, ..., x_m, y_m$ are unique and the recipe (3.157-3.162) is well-defined.

□

Lemma 3.4.3 *Assume* $G(x_1, y_1, ..., x_m, y_m) = b^{'}$ *is the system from* (3.155)*, where* $b = (0, \gamma_1, \gamma_2, ..., \gamma_{2m-2}, 1) \in \mathbb{R}^{2m}$ *and the operators* R_i*,* $i = 1, 2, ..., m$ *are defined in* (3.149)*. Then* G *is onto.*

Proof. We wish to show that for any $b = (0, \gamma_1, \gamma_2, ..., \gamma_{2m-2}, 1) \in \mathbb{R}^{2m}$, there exist points $\check{\mathbf{x}}, \hat{\mathbf{y}} \in \mathbb{R}^{2m}$, such that $\check{\mathbf{x}} \leq \hat{\mathbf{y}}$, where $\check{\mathbf{x}} = (\check{x}_1, \check{y}_1, \check{x}_2, \check{y}_2, ..., \check{x}_m, \check{y}_m)$ and $\hat{\mathbf{y}} = (\hat{x}_1, \hat{y}_1, \hat{x}_2, \hat{y}_2, ..., \hat{x}_m, \hat{y}_m)$, and

$$G(\check{x}_1, \check{y}_1, \check{x}_2, \check{y}_2, ..., \check{x}_m, \check{y}_m) \leq b^{'} \leq G(\hat{x}_1, \hat{y}_1, \hat{x}_2, \hat{y}_2, ..., \hat{x}_m, \hat{y}_m).$$

The existence of a solution will then be obtained by continuity.

We will apply the MVTI in (3.158), (3.160) and (3.162). The equation $R_1(x_1, y_1) = s$ can be written as $\frac{m_1^*}{\alpha_1 - \alpha_0}(y_1 - x_1) = s$ using the MVTI. Substituting $x_1 = 0$ into this equation and rearranging gives

$$y_1 = \frac{1}{m_1^*}(\alpha_1 - \alpha_0)s.$$

The equation $R_2(x_2, y_2) = s - \gamma_1$ can be written as $\frac{m_2^*}{\alpha_2 - \alpha_1}(y_2 - x_2) = s - \gamma_1$ using the MVTI. Rewriting and substituting $x_2 = y_1 + \gamma_2$ into this equation we obtain

$$y_2 = y_1 + \frac{1}{m_2^*}(\alpha_2 - \alpha_1)(s - \gamma_1) + \gamma_2.$$

Likewise, the equation $R_{i+1}(x_{i+1}, y_{i+1}) = s - \sum_{j=1}^{i} \gamma_{2j-1}$ can be written as $\frac{m^*_{m+1}}{\alpha_{i+1}-\alpha_i}(y_{i+1} - x_{i+1}) = s - \sum_{j=1}^{i} \gamma_{2j-1}$. Rearranging and substituting $x_{i+1} = y_i + \gamma_{2i}$ into this equation, we have

$$y_{i+1} = y_i + \frac{1}{m^*_{i+1}}(\alpha_{i+1} - \alpha_i)(s - \sum_{j=1}^{i} \gamma_{2j-1}) + \gamma_{2i}.$$

Hence we have the resulting system

$$\begin{aligned}
x_1 =& 0,\\
y_1 =& \frac{1}{m^*_1}(\alpha_1 - \alpha_0)s,\\
x_2 =& y_1 + \gamma_2,\\
y_2 =& \frac{1}{m^*_2}(\alpha_2 - \alpha_1)(s - \gamma_1) + y_1 + \gamma_2,\\
x_{i+1} =& y_i + \gamma_{2i},\\
y_{i+1} =& \frac{1}{m^*_{i+1}}(\alpha_3 - \alpha_2)(s - \gamma_1 - \gamma_3) + y_i + \gamma_{2i}, \quad i = 2, ..., m-1.
\end{aligned}$$

where m^*_i, $i = 1, 2, ..., m$ are values obtained form the MVTI. This is an equivalent system to (3.157-3.162). If it has an unique solution then so does the system (3.157-3.162).

We now assume $(\check{x}_1, \check{y}_1, ..., \check{x}_m, \check{y}_m) \in \mathbb{R}^{2m}$. We need to satisfy $F(\check{x}_1, \check{y}_1, ..., \check{x}_m, \check{y}_m) \leq b$ for a subsolution. Clearly we have, $\frac{1}{m^*_j} \geq \frac{1}{\check{m}}, \quad \text{for} \quad j = 1, 2, ..., m.$

If $(\alpha_1 - \alpha_0)s > 0$ then we obtain

$$\check{y}_1 \leq \frac{1}{\check{m}}(\alpha_1 - \alpha_0)s.$$

If in addition $(\alpha_2 - \alpha_1)(s - \gamma_1) > 0$ then we have

$$\check{y}_2 \leq \frac{1}{\check{m}}(\alpha_2 - \alpha_1)(s - \gamma_1) + y_1 + \gamma_2.$$

Similarly, in addition $(\alpha_{i+1} - \alpha_i)(s - \sum_{j=1}^{i} \gamma_{2j-1}) > 0$ then we obtain

$$\check{y}_{i+1} \leq \frac{1}{\check{m}}(\alpha_{i+1} - \alpha_i)(s - \sum_{j=1}^{i} \gamma_{2j-1}) + y_i + \gamma_{2i} \quad \text{for} \quad i = 2, 3, ..., m-1.$$

Hence we have the system of inequalities

$$\begin{aligned}
\check{x}_1 &= 0 \\
\check{y}_1 &\leq \frac{1}{\check{m}}(\alpha_1 - \alpha_0)s \\
\check{x}_2 &= y_1 + \gamma_2 \\
\check{y}_2 &\leq \frac{1}{\check{m}}(\alpha_2 - \alpha_1)(s - \gamma_1) + y_1\gamma_2 \\
\check{x}_{i+1} &= y_i + \gamma_{2i} \\
\check{y}_{i+1} &\leq \frac{1}{\check{m}}(\alpha_{i+1} - \alpha_i)(s - \sum_{j=1}^{i}\gamma_{2j-1}) + y_i + \gamma_{2i}, \quad i = 2, ..., m-1.
\end{aligned} \tag{3.163}$$

We now choose $\check{s} \in \mathbb{R}$, and we set $\check{x_1} = 0$ and $R_1(\check{x_1}, \check{y_1}) = \check{s}$ where $\check{s}$ satisfies the following inequalities

$$\begin{aligned}
\frac{1}{\check{m}}(\alpha_1 - \alpha_0)\check{s} &\leq \alpha_1 - \alpha_0 \\
\frac{1}{\check{m}}(\alpha_2 - \alpha_1)(\check{s} - \gamma_1) + \gamma_2 &\leq \alpha_2 - \alpha_1 \\
\frac{1}{\check{m}}(\alpha_{i+1} - \alpha_i)(s - \sum_{j=1}^{i}\gamma_{2j-1}) + y_i + \gamma_{2i} &\leq \alpha_{i+1} - \alpha_i,
\end{aligned} \tag{3.164}$$

where $i = 2, 3, ..., m-1$ then we have from (3.165)

$$\begin{aligned}
\check{y}_m &\leq \alpha_m - \alpha_{m-1} + ... + \alpha_2 - \alpha_1 + \alpha_1 - \alpha_0 \\
&= \alpha_m - \alpha_0 \\
&= 1.
\end{aligned}$$

This inequality gives us a confirmation for a subsolution. Hence, we have a subsolution for (3.155) if $\check{s}$ satisfies the inequalities in (3.164).

Similarly, we assume $(\hat{x}_1, \hat{y}_1, ..., \hat{x}_m, \hat{y}_m) \in \mathbb{R}^{2m}$, this to be a supersolution we need to satisfy $F(\hat{x}_1, \hat{y}_1, ..., \hat{x}_m, \hat{y}_m) \geq b$. Clearly, we have $\frac{1}{m_j^*} \geq \frac{1}{\hat{m}}, \quad \text{for} \quad j = 1, 2, ..., m.$

If $(\alpha_1 - \alpha_0)s > 0$ then we have

$$\hat{y}_1 \geq \frac{1}{\hat{m}}(\alpha_1 - \alpha_0)s.$$

If in addition $(\alpha_2 - \alpha_1)(s - \gamma_1) > 0$ then we find

$$\hat{y}_2 \geq \frac{1}{\hat{m}}(\alpha_2 - \alpha_1)(s - \gamma_1) + y_1 + \gamma_2.$$

Likewise, if in addition $(\alpha_{i+1} - \alpha_i)(s - \sum_{j=1}^{i} \gamma_{2j-1}) > 0$, for $i = 2, 3, ..., m-1$, then

$$\hat{y}_{i+1} \geq \frac{1}{\hat{m}}(\alpha_{i+1} - \alpha_i)(s - \sum_{j=1}^{i} \gamma_{2j-1}) + y_i + \gamma_{2i}.$$

Hence we have the resulting system of inequalities

$$\begin{aligned}
\hat{x}_1 &= 0 \\
\hat{y}_1 &\geq \frac{1}{\hat{m}}(\alpha_1 - \alpha_0)s \\
\hat{x}_2 &= y_1 + \gamma_2 \\
\hat{y}_2 &\geq \frac{1}{\hat{m}}(\alpha_2 - \alpha_1)(s - \gamma_1) + y_2 + \gamma_2 \\
\hat{x}_{i+1} &= y_i + \gamma_{2i} \\
\hat{y}_{i+1} &\geq \frac{1}{\hat{m}}(\alpha_3 - \alpha_2)(s - \sum_{j=1}^{i} \gamma_{2j-1}) + y_i + \gamma_{2i}. \quad i = 2, ..., m-1.
\end{aligned} \tag{3.165}$$

We now choose $\hat{s} \in \mathbb{R}$, and we set $\hat{x_1} = 0$ and $R_1(\hat{x_1}, \hat{y_1}) = \hat{s}$. If $\hat{s}$ satisfies the following inequalities

$$\begin{aligned}
\frac{1}{\hat{m}}(\alpha_1 - \alpha_0)\hat{s} &\geq \alpha_1 - \alpha_0 \\
\frac{1}{\hat{m}}(\alpha_2 - \alpha_1)(\hat{s} - \gamma_1) + \gamma_2 &\geq \alpha_2 - \alpha_1 \\
\frac{1}{\hat{m}}(\alpha_{i+1} - \alpha_i)(\hat{s} - \sum_{j=1}^{i} \gamma_{2j-1}) &\geq \alpha_{i+1} - \alpha_i
\end{aligned} \tag{3.166}$$

where $i = 2, ..., m-1$, then we have from (3.165)

$$\begin{aligned}
\hat{y}_m &\geq \alpha_m - \alpha_{m-1} + ... + \alpha_2 - \alpha_1 + \alpha_1 - \alpha_0 \\
&= \alpha_m - \alpha_0 \\
&= 1.
\end{aligned}$$

This inequality gives us a confirmation of a supersolution. Hence we have a supersolution of (3.155) if $\hat{s}$ satisfies the inequalities in (3.166).

Therefore we conclude subsolution and supersolution exists for the system. Moreover by continuity there exist $\bar{s} \in [\check{s}, \hat{s}]$, so that $y_m = 1$, and hence we have solution for (3.155). Thus G is onto. □

Lemma 3.4.4 *Consider the system $F(y_1, x_2, y_2, ..., x_{m-1}, y_{m-1}, x_m) = b$ from (3.153) and the operators R_i, $i = 1, 2, ..., m$ as defined in (3.149). The function $F : \mathbb{R}^{2m-2} \to \mathbb{R}^{2m-2}$ is onto.*

Proof. In Lemma 3.4.3 we show that a system $G(x_1, y_1, ..., x_m, y_m) = b'$, which is an equivalent system to the original system $F(y_1, x_2, y_2, ..., x_{m-1}, y_{m-1}, x_m) = b$ is onto. Hence F is onto. □

Theorem 3.4.5 *Assume $F(y_1, x_2, y_2, ..., x_{m-1}, y_{m-1}, x_m) = b$ is the system of (3.153) and the operators R_i, $i = 1, 2, ..., m$, as defined in (3.149). Then $F : \mathbb{R}^{2m-2} \to \mathbb{R}^{2m-2}$ is a continuous onto M-function if $p > \max\{\frac{\hat{m}}{\alpha_1-\alpha_0}, \frac{\hat{m}}{\alpha_2-\alpha_1}, ..., \frac{\hat{m}}{\alpha_m-\alpha_2}\}$.*

Proof. F is onto from Lemma 3.4.4.

Lemma 3.4.1 proves F is off-diagonally antitone when $p > \max\{\frac{\hat{m}}{\alpha_1-\alpha_0}, \frac{\hat{m}}{\alpha_2-\alpha_1}, ..., \frac{\hat{m}}{\alpha_m-\alpha_2}\}$. Now using Theorem 3.2.15 we now build the functions $q_i(t)$. Choosing $h_j = 1$ in (3.66), we construct the functions $q_i(t)$ as

$$q_i(t) = \sum_{j=1}^{2m-2} f_j(X + te^i), \quad 1 \leq i \leq 2m-2.$$

Here e^i denotes the i^{th} standard basis vector in R^{2m-2}. When $i = 1$, then

$$\begin{aligned} q_1(t) &= f_1(y_1 + t, x_2, y_2) + f_2(y_1 + t, x_2, y_2, x_3) + ... + f_{2m-2}(x_{m-1}, y_{m-1}, x_m) \\ &= 2R_1(0, y_1 + t) - 2R_m(x_m, 1). \end{aligned}$$

Differentiating with respect to t we have

$$\begin{aligned}\frac{dq_1}{dt} &= 2\frac{dR_1(y_1+t)}{dt}\\ &= \frac{2}{\alpha_1}M(t) > 0.\end{aligned}$$

Again, when $i = 2$, then

$$\begin{aligned}q_2(t) &= f_1(y_1, x_2+t, y_2) + f_2(y_1, x_2+t, y_2, x_3) + ... + f_{2m-2}(x_{m-1}, y_{m-1}, x_m)\\ &= 2R_1(0, y_1) - 2R_m(x_m, 1).\end{aligned}$$

And differentiating with respect to t we find

$$\frac{dq_2}{dt} = 0.$$

When $i = k$, where $k = 3, 4, ..., 2m-3$ then

$$q_k(t) = 2R_1(0, y_1) - 2R_m(x_m, 1).$$

Differentiate with respect to t we have

$$\frac{dq_k}{dt} = 0.$$

Finally, if $i = 2m-2$, then we obtain

$$\begin{aligned}q_{2m-2}(t) &= f_1(y_1, x_2, y_2) + f_2(y_1, x_2+t, y_2, x_3) + ... + f_{2m-2}(x_{m-1}, y_{m-1}, x_m+t)\\ &= 2R_1(0, y_1) - 2R_m(x_m+t, 1).\end{aligned}$$

Differentiating with respect to t we find

$$\begin{aligned}\frac{dq_{2m-2}}{dt} &= \frac{dR_m(w+t)}{dt}\\ &= \frac{2}{\alpha_m - \alpha_{m-1}}M(t) > 0.\end{aligned}$$

Therefore, $\frac{dq_1}{dt}$ and $\frac{dq_{2m-2}}{dt}$ are strictly positive, hence the functions q_1 and q_{2m-2} are strictly isotone. However $\frac{dq_k}{dt}$, $k = 2, ..., 2m-3$ are not strictly positive, and hence the q_k is not strictly isotone. We need to show that, a path $k \rightsquigarrow i$ exists for $i = 2, 3, ..., 2m-3$ where the functions q_k are strictly isotone. Possible paths for the k-th node include

$$1 \rightsquigarrow 2 \qquad 2 \rightsquigarrow 3 \qquad ... \qquad (k-1) \rightsquigarrow k$$

or

$$(2m-2) \rightsquigarrow (2m-3) \qquad (2m-3) \rightsquigarrow (2m-4) \qquad ... \qquad k+1 \rightsquigarrow k.$$

Notice that the our system has $2m-2$ unknowns, x_i is the $(2i-2)^{th}$ unknown and y_i is the $(2i-1)^{th}$ unknown for $i = 2, 3, ..., m-1$. The first and last unknowns are y_1 and x_m respectively.

We know from the definition of strict link that if the function $t \to f_i(\mathbf{x} + te^j)$ is strictly antitone then a link (i, j) is strict. We have

$$\begin{aligned}
\frac{\partial}{\partial x_2} f_1(\mathbf{x} + te^2) &= \frac{\partial}{\partial x_2} f_1(y_1, x_2 + t, y_2) \\
&= \frac{\partial}{\partial x_2} [R_1(y_1) - R_2(x_2 + t, y_2) + p(y_1 - x_2 - t)] \\
&= \frac{M(x_2 + t)}{\alpha_2 - \alpha_1} - p
\end{aligned}$$

which is less than zero if p is big enough, and hence $(1, 2)$ is a strict link if $\frac{\hat{m}}{\alpha_2 - \alpha_1} < p$. Similarly we obtain for $i = 2, ...m-1$,

$$\begin{aligned}
\frac{\partial}{\partial y_i} f_{2i-2}(\mathbf{x} + te^{2i-1}) &= -\frac{M(y_i + t)}{\alpha_i - \alpha_{i-1}} < 0 \quad \Longrightarrow \ (2i-2, 2i-1) \text{ is a strict link,} \\
\frac{\partial}{\partial x_{i+1}} f_{2i-1}(\mathbf{x} + te^{2i}) &= -\frac{M(x_{i+1} + t)}{\alpha_i - \alpha_{i-1}} < 0 \quad \Longrightarrow \ (2i-1, 2i) \text{ is a strict link,} \\
\frac{\partial}{\partial y_{i+1}} f_{2i-1}(\mathbf{x} + te^{2i+1}) &= -\frac{M(y_{i+1} + t)}{\alpha_{i+1} - \alpha_i} < 0 \quad \Longrightarrow \ (2i-1, 2i+1) \text{ is a strict link.}
\end{aligned}$$

And

$$\frac{\partial}{\partial y_{m-1}} f_{2m-2}(\mathbf{x} + te^{2m-3}) = \frac{M(y_m + t)}{\alpha_{m-1} - \alpha_{m-2}} - p$$

which implies $(2m-2, 2m-3)$ is a strict link if $\frac{\hat{m}}{\alpha_{m-1} - \alpha_{m-2}} < p$. Finally we obtain at strict links $(1, 2)$, $(2, 3)$, ..., $((k-1), k)$ for the path $1 \rightsquigarrow k$, and the strict links $(k, k+1)$,

$(k+1, k+2)$, ..., $(2m-3, 2m-2)$ for the path $k \rightsquigarrow (2m-2)$, where q_1 and q_{2m-2} are strictly isotone. All assumptions of Theorem 3.2.15 have been satisfied. Hence F is an M-function if $p > \max\{\frac{\hat{m}}{\alpha_1-\alpha_0}, \frac{\hat{m}}{\alpha_2-\alpha_1}, ..., \frac{\hat{m}}{\alpha_m-\alpha_2}\}$. □

Theorem 3.4.6 *Consider the system* $F(y_1, x_2, y_2, ..., x_{m-1}, y_{m-1}, x_m) = b$ *from* (3.153) *and the operators* R_i*,* $i = 1, 2, ..., m$ *as defined in* (3.149)*. This system has a unique solution.*

Proof. The assumptions of Lemma 3.2.14 have been verified by Theorem 3.4.5, hence system (3.153) has a unique solution. □

Theorem 3.4.7 *Consider the system* $F(y_1, x_2, y_2, ..., x_{m-1}, y_{m-1}, x_m) = b$ *from* (3.153) *and the operators* R_i*,* $i = 1, 2, ..., m$ *as defined in* (3.149)*. Nonlinear Jacobi (or SOR) will converge to the unique solution for any starting value if*

$$p > \max\{\frac{\hat{m}}{\alpha_1-\alpha_0}, \frac{\hat{m}}{\alpha_2-\alpha_1}, ..., \frac{\hat{m}}{\alpha_m-\alpha_{m-1}}\}.$$

Proof. The assumptions of Theorem 3.2.19 has been verified by Theorem 3.4.5 and Lemma 3.4.4, if $p > \max\{\frac{\hat{m}}{\alpha_1-\alpha_0}, \frac{\hat{m}}{\alpha_2-\alpha_1}, ..., \frac{\hat{m}}{\alpha_m-\alpha_{m-1}}\}$. Hence by the Theorem 3.2.19 we can conclude the nonlinear Jacobi (or SOR) will converge to the unique solution for any starting value. □

Hence Gauss-Jacobi or Gauss-Seidel iterations will converge to a unique solution for the system (3.153) if $p > \max\{\frac{\hat{m}}{\alpha_1-\alpha_0}, \frac{\hat{m}}{\alpha_2-\alpha_1}, ..., \frac{\hat{m}}{\alpha_m-\alpha_{m-1}}\}$. Our original parallel iteration (3.146-3.148), however, is not a nonlinear Gauss-Jacobi or Gauss-Seidel iteration. It is actually a block Gauss-Jacobi iteration. The theorem below guarantees the parallel iteration converges to the unique solution.

Theorem 3.4.8 *Consider the system* $F(y_1, x_2, y_2, ..., x_{m-1}, y_{m-1}, x_m) = b$ *from* (3.153) *and the operators* R_i*,* $i = 1, 2, ..., m$ *as defined in* (3.149)*. Nonlinear block Gauss-Jacobi converge to a unique solution for any starting value if* $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{\alpha_2-\alpha_1}, ..., \frac{\hat{m}}{1-\alpha_2}\}$*.*

Proof. We wish to verify assumptions of Theorem 3.3.18. The assumptions of the Theorem are that F is continuous, inverse isotone, and surjective, and the regular iteration function $G(.,x)$ is surjective for any fixed $x \in \mathbb{R}^m$.

Clearly, F is continuous. The surjectivity of F has been verified by the Lemma 3.4.4 if $p > \max\{\frac{\hat{m}}{\alpha_1-\alpha_0}, \frac{\hat{m}}{\alpha_2-\alpha_1}, ..., \frac{\hat{m}}{\alpha_m-\alpha_{m-1}}\}$, and F is an M- function by Theorem 3.4.5. By the definition of M-function implies that F is inverse isotone. All assumptions of Theorem 3.3.19 has been verified, thus $G(.,x)$ is surjective for any fixed $x \in \mathbb{R}^m$ by the Theorem 3.3.19.

Hence the assumptions of Theorem 3.3.18 has been verified so, the nonlinear block Gauss-Jacobi iteration (or implicit iteration) (3.146-3.148) converges to a unique solution for any starting value if $p > \max\{\frac{\hat{m}}{\alpha_1-\alpha_0}, \frac{\hat{m}}{\alpha_2-\alpha_1}, ..., \frac{\hat{m}}{\alpha_m-\alpha_{m-1}}\}$. $\square$

Therefore the block Gauss-Jacobi iterations (3.146-3.148) for the system (3.153) will converge monotonically to a unique solution if $p > \max\{\frac{\hat{m}}{\alpha_1-\alpha_0}, \frac{\hat{m}}{\alpha_2-\alpha_1}, ..., \frac{\hat{m}}{\alpha_m-\alpha_{m-1}}\}$, since the system is a onto M-function.

In this chapter, the tools for nonlinear analysis, such as isotone, antinote, strictly diagonally isotone, off-diagonally antitone, inverse isotone, link-function, $M-$function, subsolution, supersolution, and iterative methods has been introduced. We have analyzed the nonlinear parallel iteration that arisen from optimized Schwarz method for an arbitrary number of subdomains. In the next chapter we will show numerical results for mesh BVP using the parallel optimized Schwarz method.

Chapter 4

Numerical Implementation and Results

This chapter focuses on numerical implementation results for the $1-D$ mesh BVP using the parallel optimized Schwarz domain decomposition method. We describe how the nonlinear Robin transmission condition is implemented in the code. Additionally, numerical results for the implicit interface iteration that arises from the parallel optimized Schwarz iteration are presented. We show how monotonic convergence is obtained for large values of p.

4.1 Discretization and Implementation of the Robin Conditions

The parallel optimized Schwarz method is based on the Robin transmission condition. In this section, we describe how to implement the Robin transmission condition for the mesh equation.

4.1.1 Discretization of the Mesh BVP with Robin Boundary Conditions

We discretize the mesh BVP using a staggered mesh and the midpoint technique in Chapter 2. Recall the discrete system from (2.5), $G(i)$ is given as

$$G(i) \equiv M\Big(\frac{x_{j+1}+x_j}{2}\Big)(x_{j+1}-x_j) - M\Big(\frac{x_j+x_{j-1}}{2}\Big)(x_j-x_{j-1}) = 0. \tag{4.1}$$

In Chapter 2, we studied the second order accuracy of the discretization of the mesh BVP with Dirichelet Boundary conditions. An optimized Schwarz iteration for every subdomain problem has Robin boundary conditions at the left, right, or both endpoints. The boundary conditions are given by

$$M(x)\partial_\xi x - px|_{\alpha_i} = b_l \tag{4.2}$$

$$M(x)\partial_\xi x + px|_{\alpha_{i+1}} = b_r \tag{4.3}$$

where $\alpha_i < \alpha_{i+1}$ and p is the Robin parameter. Assume a subdomain has N mesh points. To preserve second order accuracy of discretization, we use centered differences with the "ghost" value technique for imposing the nonlinear Robin transmission condition. Let us impose the Robin condition at the left endpoint. Discretizing (4.2) by centered differences gives us

$$M(x_1)\Big(\frac{x_2-x_0}{2h}\Big) - px_1 = b_l$$

where h is grid spacing and x_0 is the "ghost" point. Rearranging this system gives

$$x_0 = x_2 + \frac{2h}{M(x_1)}(px_1+b_l).$$

Assuming x_0 is a function of x_1 and x_2, and differentiating x_0 with respect to x_1 and x_2, we obtain

$$\frac{\partial x_0}{\partial x_1} = -2h\Big(\frac{pM(x_1)-(px_1+b_l)M'(x_1)}{M(x_1)^2}\Big) \quad \text{and} \quad \frac{\partial x_0}{\partial x_2} = 1.$$

For the first point, we obtain from (4.1)

$$G(1) = M\Big(\frac{x_2 + x_1}{2}\Big)(x_2 - x_1) - M\Big(\frac{x_1 + x_0}{2}\Big)(x_1 - x_0).$$

Differentiate $G(1)$ with respect to x_1 and x_2, we have

$$\begin{aligned}\frac{\partial G(1)}{\partial x_1} =& \frac{1}{2}M'\Big(\frac{x_2 + x_1}{2}\Big)(x_2 - x_1) - M\Big(\frac{x_2 + x_1}{2}\Big) - \\ & \frac{1}{2}\Big(1 + \frac{\partial x_0}{\partial x_1}\Big)M'\Big(\frac{x_1 + x_0}{2}\Big)(x_1 - x_0) - \Big(1 - \frac{\partial x_0}{\partial x_1}\Big)M\Big(\frac{x_1 + x_0}{2}\Big),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial G(1)}{\partial x_2} =& \frac{1}{2}M'\Big(\frac{x_2 + x_1}{2}\Big)(x_2 - x_1) + M\Big(\frac{x_2 + x_1}{2}\Big) - \\ & \frac{1}{2}\frac{\partial x_0}{\partial x_2}M'\Big(\frac{x_1 + x_0}{2}\Big)(x_1 - x_0) + \frac{\partial x_0}{\partial x_2}M\Big(\frac{x_1 + x_0}{2}\Big).\end{aligned}$$

Substitute the value of $\frac{\partial x_0}{\partial x_1}$ and $\frac{\partial x_0}{\partial x_2}$ into these equations we evaluate the first two entries of Jacobian matrix.

We now wish to impose the Robin condition at the right endpoint. Likewise, discretizing (4.3) by centered differences we have

$$M(x_N)\Big(\frac{x_{N+1} - x_{N-1}}{2h}\Big) + px_N = b_r$$

where x_{N+1} is the "ghost" value. Rearranging this system we obtain

$$x_{N+1} = x_{N-1} + \frac{2h}{M(x_N)}(b_r - px_N).$$

Considering x_{N+1} as a function of x_{N-1} and x_N, we differentiate x_{N+1} with respect to x_{N-1} and x_N to obtain

$$\frac{\partial x_{N+1}}{\partial x_{N-1}} = 1 \quad \text{and} \quad \frac{\partial x_{N+1}}{\partial x_N} = -2h\Bigg(\frac{pM(x_N) + (b_r - px_N)M'(x_N)}{M(x_N)^2}\Bigg).$$

For the last endpoint we obtain from (4.1)

$$G(N) = M\Big(\frac{x_{N+1} + x_N}{2}\Big)(x_{N+1} - x_N) - M\Big(\frac{x_N + x_{N-1}}{2}\Big)(x_N - x_{N-1}).$$

Differentiate $G(N)$ with respect to x_{N-1} and x_N, we have

$$\begin{aligned}\frac{\partial G(N)}{\partial x_{N-1}} =&\frac{1}{2}\frac{\partial x_{N+1}}{\partial x_{N-1}}M'\Big(\frac{x_{N+1}+x_N}{2}\Big)(x_{N+1}-x_N)+\frac{\partial x_{N+1}}{\partial x_{N-1}}M\Big(\frac{x_{N+1}+x_N}{2}\Big)-\\ &\frac{1}{2}M'\Big(\frac{x_N+x_{N-1}}{2}\Big)(x_N-x_{N-1})+\frac{\partial x_{N+1}}{\partial x_{N-1}}M\Big(\frac{x_N+x_{N-1}}{2}\Big),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial G(N)}{\partial x_N} =&\frac{1}{2}\Big(1+\frac{\partial x_{N+1}}{\partial x_N}\Big)M'\Big(\frac{x_{N+1}+x_N}{2}\Big)(x_{N+1}-x_N)-\Big(1-\frac{\partial x_{N+1}}{\partial x_N}\Big)M\Big(\frac{x_{N+1}+x_N}{2}\Big)-\\ &\frac{1}{2}M'\Big(\frac{x_N+x_{N-1}}{2}\Big)(x_N-x_{N-1})-M\Big(\frac{x_N+x_{N-1}}{2}\Big).\end{aligned}$$

Substituting the value of $\frac{\partial x_{N+1}}{\partial x_{N-1}}$ and $\frac{\partial x_{N+1}}{\partial x_{N-1}}$ into these equations gives the last two entries of Jacobian matrix.

4.1.2 Implementation of the Robin Conditions

Assume the Robin boundary conditions at $\alpha_i \in (0,1)$ are given by

$$\begin{aligned}b_{i,l}^n &= M(x_i^n)\partial_\xi x_i^n - p x_i^n|_{\alpha_i}\\ b_{i,r}^n &= M(x_i^n)\partial_\xi x_i^n + p x_i^n|_{\alpha_i}\end{aligned}$$

where the first subscript of $b_{i,r}^n$ and $b_{i,l}^n$, indicate the i-th subdomain, and r denotes a boundary condition at the right-boundary of the i-th subdomain and l denotes a boundary condition at the left-boundary of the i-th subdomain. The superscript n denotes the iteration number.

The Robin transmission conditions at the i-th interface α_i is given by

$$\begin{aligned}M(x_{i+1}^n)\partial_\xi x_{i+1}^n - p x_{i+1}^n|_{\alpha_i} &= M(x_i^{n-1})\partial_\xi x_i^{n-1} - p x_i^{n-1}|_{\alpha_i},\\ &= \big(b_{i,r}^{n-1} - p x_i^{n-1}\big) - p x_i^{n-1}|_{\alpha_i}\\ &= b_{i,r}^{n-1} - 2p x_i^{n-1}(\alpha_i),\end{aligned}$$

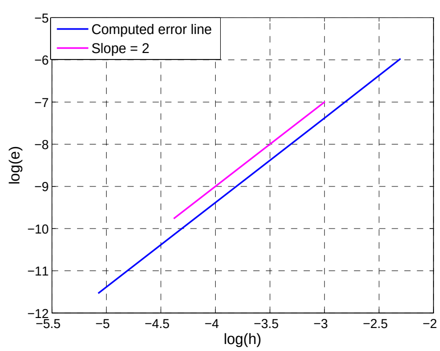
and

$$\begin{aligned}
M(x_i^n)\partial_\xi x_i^n + px_i^n|_{\alpha_i} &= M(x_{i+1}^{n-1})\partial_\xi x_{i+1}^{n-1} + px_{i+1}^{n-1}|_{\alpha_i}, \\
&= \left(b_{i+1,l}^{n-1} + px_{i+1}^{n-1}\right) + px_{i+1}^{n-1}|_{\alpha_i} \\
&= b_{i+1,l}^{n-1} + 2px_{i+1}^{n-1}(\alpha_i).
\end{aligned}$$

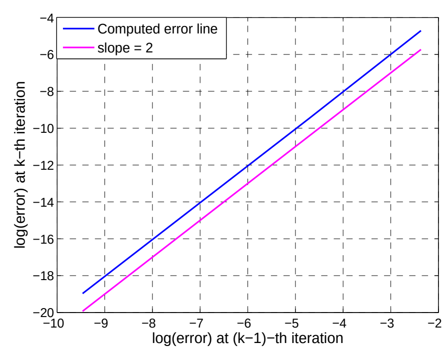
Hence, we may update the interface condition at the i-the interface α_i by

$$b_{i+1,l}^n = b_{i,r}^{n-1} - 2px_i^{n-1}(\alpha_i) \quad \text{and} \quad b_{i,r}^n = b_{i+1,l}^{n-1} + 2px_{i+1}^{n-1}(\alpha_i).$$

The nonlinear mesh BVP with Robin boundary conditions has been discretized above. We now wish to verify the order of discretization error and rate of convergence of Newton's method. We compute the order of accuracy for discretization as discussed in Chapter 2.



(a) Order of discretization error for the mesh BVP.



(b) Rate of convergence of Newton's method.

Figure 4.1: The order of discretization and the rate of convergence of Newton's method for the mesh BVP with the Robin boundary conditions and $M(x) = 1 + x^2$.

In Figure 4.1, the slope of the artificial magenta lines are 2, we compare slope of the artificial lines with the computed lines. We choose various values of step sizes h and compute the error e for the discretization of the mesh BVP with a monitor function $M(x) =$

$1 + x^2$. The blue line is our computed line for different h in Figure 4.1a. The computed line is parallel to the artificial magenta line. Hence we find second order accuracy of the discretization.

In Figure 4.1b the blue line gives the computed error for Newton's iteration, we compute the rate of convergence for Newton's method as we discussed in Chapter 2. It is parallel to the artificial magenta line. Hence the rate of convergence of the Newton's method is quadratic.

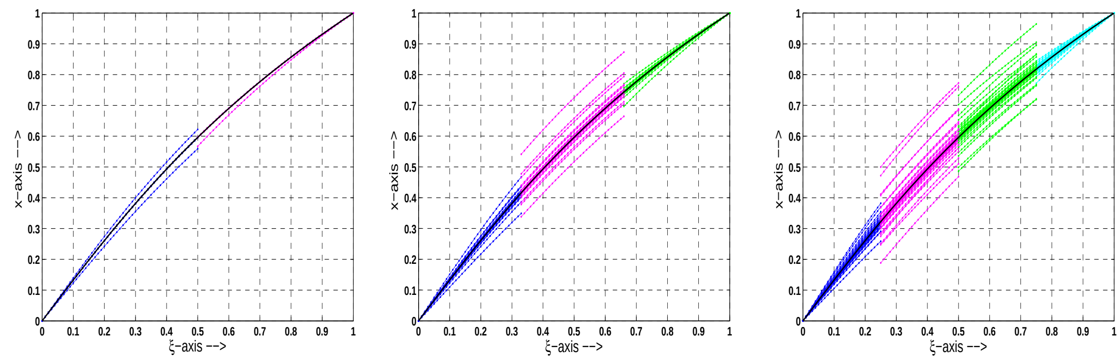
4.2 Numerical Results of Optimized Schwarz Iteration

In this section we present some numerical results how the DD iterations converge to the global solution, and the interface iteration converges monotonically with different monitor function.

4.2.1 DD Solution for an Arbitrary Number of Subdomains

The optimized Schwarz algorithm is applied to (2.1) with the monitor function $M(x) = 1 + x^2$, and using the parallel iteration iteration (3.5)-(3.7) we obtain plots in Figure 4.2. The plots illustrates how the DD iterations converge to the global solution for two, three, and four subdomains. The smooth black lines are the single domain solutions in the figure. The tolerance for DD iteration is 10^{-12}, a step size $h = 0.01$ and $p = 3$ have been used in the matlab script. After 12, 29, and 53 DD iterations we find desired solution. So, when the number of subdomains increases then the optimized Schwarz takes more DD iterations to obtain the required solution. We now want to observe the effect on convergence for the parallel optimized Schwarz iteration for varying values of p.

Table 4.1 shows the number of DD iterations required for convergence as a function



(a) Solution on 2-subdomain (b) Solution on 3-subdomain. (c) Solution on 4-subdomain.

Figure 4.2: DD solution for varying numbers of subdomains using OSM for $p = 3$.

of the number of subdomains (#SD) and the value of p. For each case we use a total of 101 mesh points and distribute these mesh points into each subdomain equally. This table

Table 4.1: The number of DD iterations as a function of the number of subdomains and the Robin parameter p

#SD \ p	0.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0
2	73	16	$\boxed{11}$	12	16	19	22	25	28	31
3	203	51	38	29	$\boxed{20}$	$\boxed{20}$	21	28	32	37
4	340	81	66	53	44	36	32	31	$\boxed{30}$	39

illustrates that the optimal value of p for two, three, and four subdomain optimized Schwarz iterations are 2.5, 3.5, and 5.5 respectively. Hence, to obtain quick convergence the value of p needs to be increased with an increase in the number of subdomains.

4.2.2 An Interface Iteration for Two Subdomains Converges Monotonically

We studied the nonlinear system that arises from the optimized Schwarz iteration for two subdomains in Theorems 3.2.16 and 3.2.17. We now show the two subdomain nonlinear iteration (3.23-3.24) converges monotonically at the interface under the condition $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$ as presented by the theory.

The operators R_1 and R_3 are implemented in matlab script by

$$R_1(0,y) \equiv R_1(y) = \frac{1}{\alpha_1}\int_0^y M(x)dx$$
$$= \frac{1}{\alpha_i} \times \begin{cases} (y-0)\check{m} & \text{if} \quad y < 0, \\ [adM(y) - adM(0)] & \text{if} \quad 0 \le y \le 1, \\ [adM(1) - adM(0) + (y-1)\hat{m}] & \text{if} \quad 1 < y, \end{cases}$$

and

$$R_3(x,1) \equiv R_3(x) = \frac{1}{1-\alpha_1}\int_x^1 M(x)dx$$
$$= \frac{1}{1-\alpha_1} \times \begin{cases} [(0-x)\check{m} + adM(1) - adM(0)] & \text{if} \quad x < 0, \\ [adM(1) - adM(x)] & \text{if} \quad 0 \le x \le 1, \\ (1-x)\hat{m} & \text{if} \quad 1 < x, \end{cases}$$

where $adM(x)$ is anti-derivative of $M(x)$.

Consider a monitor function

$$M(x) = 1 + \beta_1 \exp^{(x-x_0)} + \beta_2 \exp^{(x-x_n)}, \tag{4.4}$$

where β_1, β_2 are constant. We choose $\beta_1 = 10$, $\beta_2 = 5$, $x_0 = 0$, $x_n = 1$, and $\alpha_1 = 0.5$. The tolerance for consecutive iterations is 10^{-12}. The calculated values of the lower bound and the upper bound of $M(x)$ are $\check{m} = 12.8394$ and $\hat{m} = 32.1828$.

Table 4.2: The number of DD iterations for two subdomains interface iteration for varying values of the Robin parameter p with $M(x) = 1 + \beta_1 \exp^{(x-x_0)} + \beta_2 \exp^{(x-x_n)}$, where $\beta_1 = 10$ and $\beta_2 = 5$.

p	1	5	10	20	30	40	45	$\boxed{46}$	47	50	60	70	80	90	100	150
#Iter	501	120	60	29	18	11	7	6	8	11	14	18	21	24	27	41

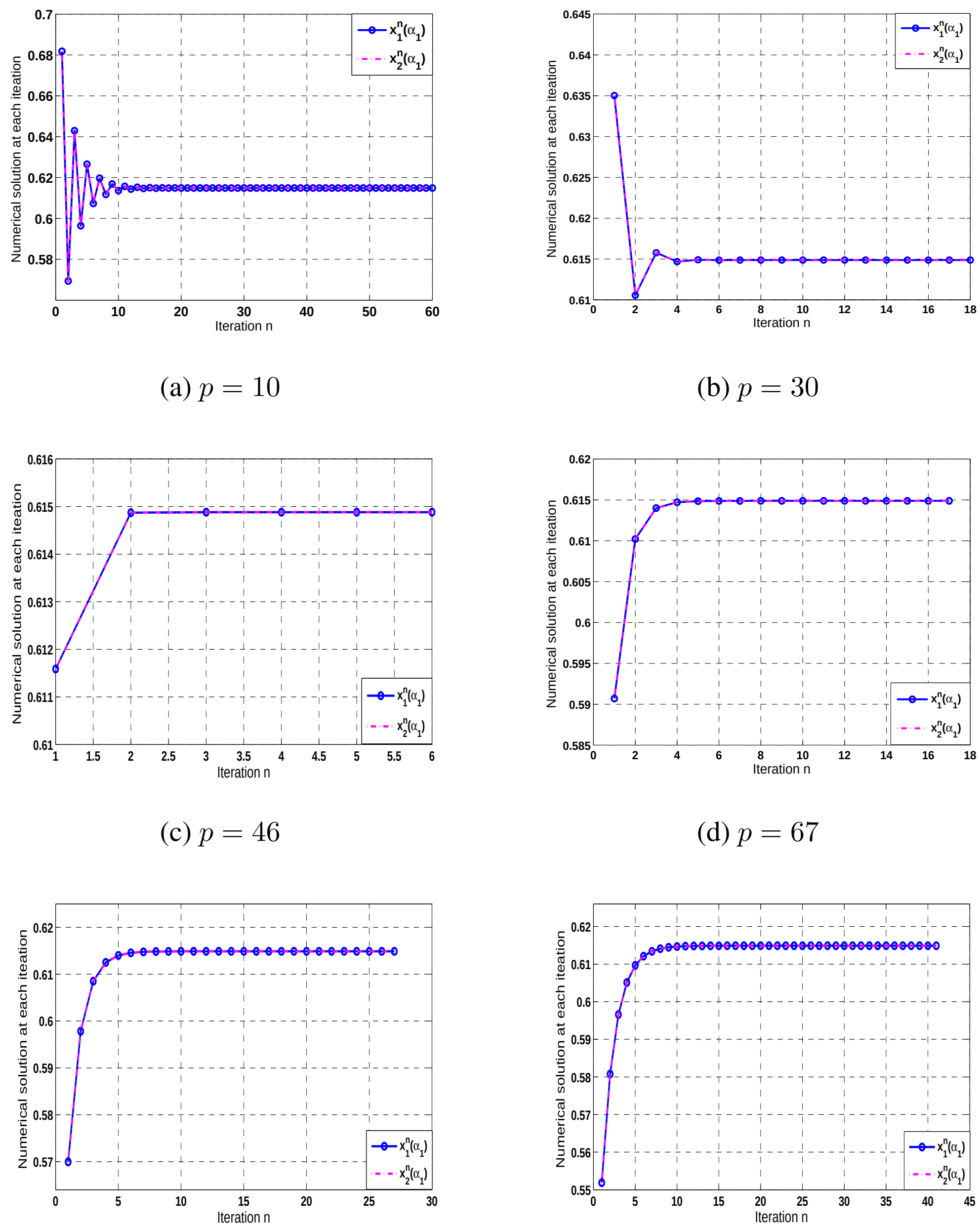
Table 4.2 presents the number of DD iteration required for the two subdomain interface iteration (3.80-3.82) for varying values of the Robin parameter p. The iteration converges for all values of $p > 0$, and the optimal value of p is around 46.

Theorem 3.2.13 guarantees the system is an $M-$function if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$. In this case p needs to be larger than

$$\max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\} = \max\{\frac{33.1828}{0.5}, \frac{33.1828}{1-0.5}\} = 66.3656,$$

to guarantee the system is an $M-$function. This value is greater then the optimal value of p.

Figure 4.3 shows a plot of the numerical solution as a function of iteration number for $x_1^n(\alpha_1)$, and $x_2^n(\alpha_1)$ for the interface nonlinear iteration (3.23-3.24) for $p = 10, 30, 46, 67, 100,$ and 150. The iteration give monotonic convergence results for $p = 46$, 67, 100 and 150, where as for $p = 10$ and 30 the iteration does not convergence monotonically in Figure 4.3. Hence if p is satisfies the required condition $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{1-\alpha_1}\}$ in Theorem 3.2.13 then the iteration (3.23-3.24) gives monotonic convergence to the unique solution. It is interesting that the optimal value of p found experimentally is close to the bound on p which guarantees monotonic convergence.



(a) $p = 10$ (b) $p = 30$

(c) $p = 46$ (d) $p = 67$

(e) $p = 100$ (f) $p = 150$

Figure 4.3: Numerical solutions of the two subdomain interface iteration for $p = 10, 30, 67,$ and 100 with a monitor function $M(x) = 1 + \beta_1 \exp^{(x-x_0)} + \beta_2 \exp^{(x-x_n)}$, where $\beta_1 = 10$ and $\beta_2 = 5$.

4.2.3 An Interface Iteration for Three Subdomains with an Easy Monitor Function

We now show the interface iteration (3.80-3.82) for three subdomains converges monotonically under the condition $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{\alpha_2-\alpha_1}, \frac{\hat{m}}{1-\alpha_2}\}$. The operators $R_i(x_i, y_i)$ in the iteration (3.80-3.82) are implemented in matlab script as

$$
\begin{aligned}
R_i(x_i, y_i) &= \frac{1}{\alpha_i - \alpha_{i-1}} \int_{x_i}^{y_i} M(x)dx \\
&= \frac{1}{\alpha_i - \alpha_{i-1}} \times \begin{cases} (y_i - x_i)\check{m} & \text{if} \quad x_i, y_i < 0, \\ [adM(y_i) - adM(x_i)] & \text{if} \quad 0 \le x_i, y_i \le 1, \\ [adM(1) - adM(x_i) + (y_i - 1)\hat{m}] & \text{if} \quad 0 \le x_i \le 1, 1 < y_i, \\ [(0 - x_i)\check{m} + adM(y_i) - adM(0)] & \text{if} \quad x_i < 0, 0 \le y_i \le 1, \\ [(0 - x_i)\check{m} + adM(1) - adM(0)] & \text{if} \quad x_i < 0, 1 \le y_i, \\ (y_i - x_i)\hat{m} & \text{if} \quad 1 < x_i, y_i, \end{cases}
\end{aligned}
\tag{4.5}
$$

where $adM(x)$ is anti-derivative of $M(x)$, and $\alpha_i, \alpha_{i-1} \in (0, 1)$ with $\alpha_{i-1} < \alpha_i$. For this experiment the monitor function $M(x)$ is defined in (4.4), and we choose $\beta_1 = 10$, $\beta_2 = 5$, $x_0 = 0$, $x_n = 1$, $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{2}{3}$ and the tolerance for consecutive iterations is 10^{-12}.

Table 4.3 presents the number of DD iterations required of interface iteration (3.80-3.82) for the three subdomain for varying values of p. The optimal value of p is 59 (approximately), and the iteration converges for all values of $p > 0$.

Theorem 3.2.13 guarantees the system is an $M-$function if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{\alpha_2-\alpha_1}, \frac{\hat{m}}{1-\alpha_2}\}$. Now we wish to check this theorem with numerical results. The calculated values of the lower bound and the upper bound of $M(x)$ are $\check{m} = 12.83939720$ and $\hat{m} = 33.18281828$.

Table 4.3: The number of DD iterations required for three subdomain interface iteration for varying values of p with $M(x) = 1 + \beta_1 \exp^{(x-x_0)} + \beta_2 \exp^{(x-x_n)}$, where $\beta_1 = 10$ and $\beta_2 = 5$.

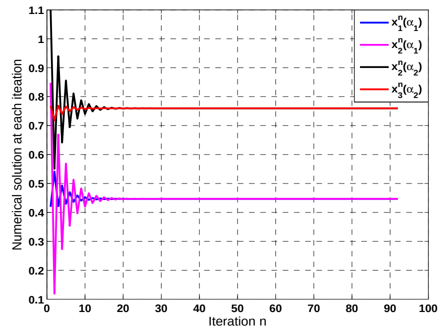
p	1	10	20	40	58	$\boxed{59}$	60	80	100	150	200	250	300	400	500
#Iter	981	92	44	20	11	9	10	12	18	29	38	48	57	75	93

So p needs to be greater than

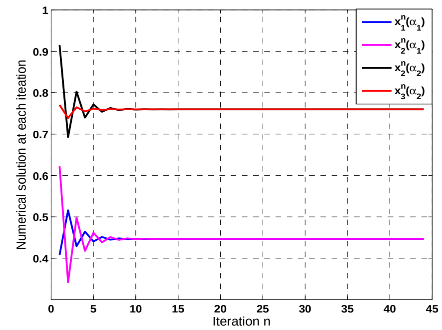
$$\max\{\frac{32.18281828}{1/3}, \frac{32.18281828}{2/3 - 1/3}, \frac{32.18281828}{1 - 1/3}\} = 99.54845485,$$

to guarantee the system is an $M-$function. Which is greater then the optimal value of the Robin parameter.

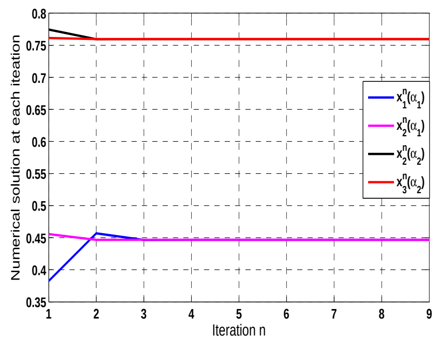
Figure 4.4 shows a plot of the numerical solution as a function of the iteration number for $x_1^n(\alpha_1)$, $x_2^n(\alpha_1)$, $x_2^n(\alpha_2)$, and $x_3^n(\alpha_2)$ for the interface iteration (3.80-3.82) for $p = 10, 20, 59, 100, 150$, and 200. The iteration gives monotonic convergence results for $p = 100, 150$ and 200 whereas for $p = 10$ and 20 the iteration does not converge monotonically in the figure. If p satisfies the required condition in Theorem 3.2.13 then the iteration (3.80-3.82) gives monotonic convergence to the required solution. It is interesting that the optimal value of p found experimentally is close to the bound on p which guarantees monotonic convergence.



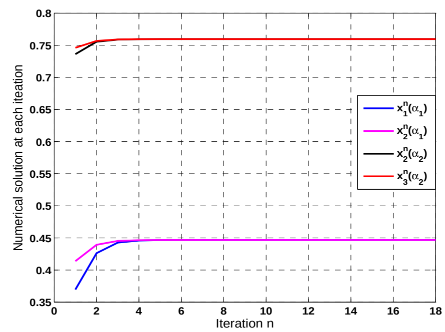
(a) $p = 10$



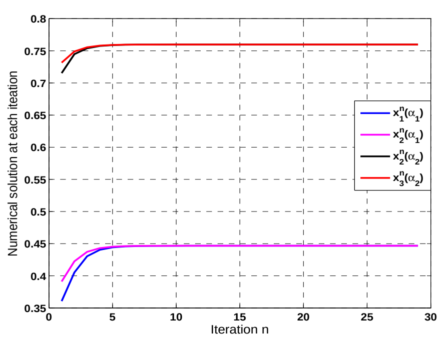
(b) $p = 20$



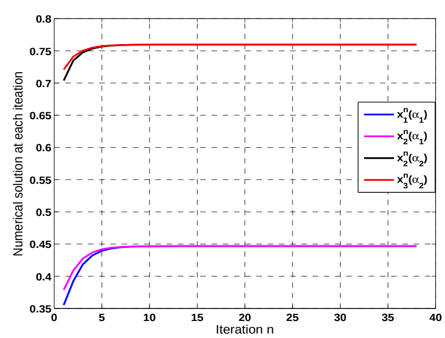
(c) $p = 59$



(d) $p = 100$



(e) $p = 150$



(f) $p = 200$

Figure 4.4: Numerical solutions of the three subdomain interface iteration for $p = 10$, 20, 59, 100, 150, and 200 with a monitor function $M(x) = 1 + \beta_1 \exp^{(x-x_0)} + \beta_2 \exp^{(x-x_n)}$, where $\beta_1 = 10$, and $\beta_2 = 5$.

4.2.4 An Interface Iteration for Three Subdomains with a Difficult Monitor Function

We now presents numerical solutions of interface iteration (3.80-3.82) for three subdomains for varying values of p, with a difficult monitor function

$$M(x) = 1 + \beta_1 \exp^{\left(\frac{x-x_0}{\xi_1}\right)} + \beta_2 \exp^{\left(\frac{x-x_n}{\xi_2}\right)}.$$

For this experiment we choose $\beta_1 = 10$, $\beta_2 = 5$, $\xi_1 = 0.12$, $\xi_2 = 0.1$, $x_0 = 0$, $x_n = 1$, $\alpha_1 = \frac{1}{3}$, and $\alpha_2 = \frac{2}{3}$ and the tolerance for consecutive iterations is 10^{-12}.

In Table 4.4, we show the number of DD iterations required for the three subdomain interface iteration (3.80-3.82) for varying values of p. The optimal value of the Robin parameter p is around 60000.

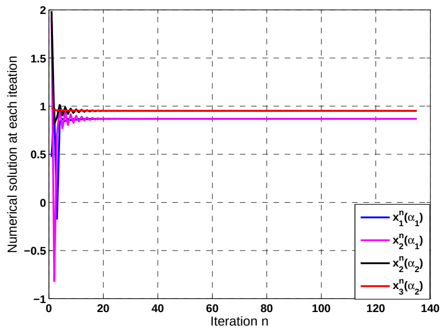
Table 4.4: The number of DD iterations required for the three subdomain interface iteration for varying values of p with a difficult monitor function $M(x) = 1 + \beta_1 \exp^{\left(\frac{x-x_0}{\xi_1}\right)} + \beta_2 \exp^{\left(\frac{x-x_n}{\xi_2}\right)}$, where $\beta_1 = 10$, $\beta_2 = 5$, $\xi_1 = 0.12$, and $\xi_2 = 0.1$.

p	500	1000	5000	59000	60000	61000	100000	124826	130000	150000
#Iter	1350	675	135	16	15	16	28	35	37	43

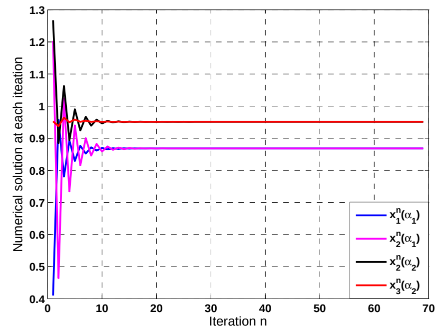
We now wish to check Theorem 3.2.13 with numerical results; if $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{\alpha_2-\alpha_1}, \frac{\hat{m}}{1-\alpha_2}\}$ then the system is an $M-$function . The calculated values of the lower bound and the upper bound of $M(x)$ are $\check{m} = 11.00022699$ and $\hat{m} = 41608.62005375$. Thus, p needs to be greater than

$$\max\{\frac{41608.62005375}{1/3}, \frac{41608.62005375}{2/3-1/3}, \frac{41608.62005375}{1-1/3}\} = 124825.86016125,$$

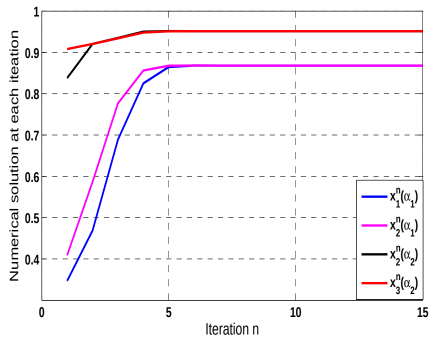
to guarantee the system is an $M-$function.



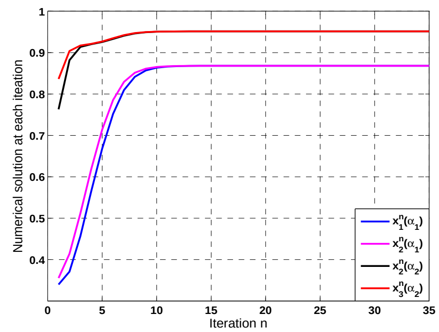
(a) $p = 5000$



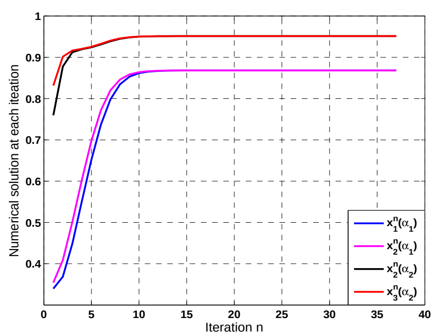
(b) $p = 10000$



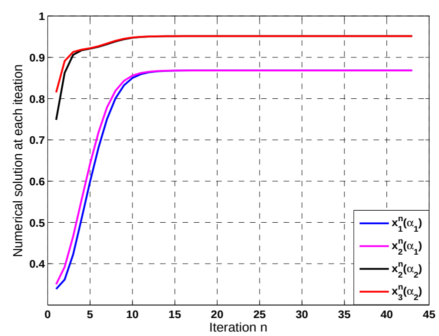
(c) $p = 60000$



(d) $p = 124826$



(e) $p = 130000$



(f) $p = 150000$

Figure 4.5: Numerical solutions for the three subdomain interface iteration for $p = 5000$, 10000, 30000, 124826, 130000, and 150000 with a monitor function $M(x) = 1 + \beta_1 \exp^{\left(\frac{x-x_0}{\xi_1}\right)} + \beta_2 \exp^{\left(\frac{x-x_n}{\xi_2}\right)}$, where $\beta_1 = 10$, $\beta_2 = 5$, $\xi_1 = 0.12$, and $\xi_2 = 0.1$.

Figure 4.5 gives a plot of the numerical solution for $x_1^n(\alpha_1)$, $x_2^n(\alpha_1)$, $x_2^n(\alpha_2)$, and $x_3^n(\alpha_2)$ for the interface nonlinear iteration (3.80-3.82) for $p = 5000, 10000, 60000, 124826, 130000,$ and 150000. The iteration gives monotonic convergence for $p = 60000, 124826, 130000$ and 150000, whereas for $p = 5000$ and 10000 the iteration does not give monotonic converge. It is interesting that the optimal value of p found experimentally is close to the bound on p which guarantees monotonic convergence. Thus if p satisfies $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{\alpha_2-\alpha_1}, \frac{\hat{m}}{1-\alpha_2}\}$ then the iteration (3.80-3.82) gives monotonic convergence.

In conclusion, these numerical results do agree with the theory. The M-function theory guarantees that the parallel nonlinear optimized Schwarz iteration will converge monotonically when $p > \max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{\alpha_2-\alpha_1}, ..., \frac{\hat{m}}{1-\alpha_m}\}$ for an arbitrary number of subdomains, where p is used in the nonlinear Robin transmission condition. And these experiments also suggest that the optimal value of the Robin parameter should be in between 0 to $\max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{\alpha_2-\alpha_1}, ..., \frac{\hat{m}}{1-\alpha_m}\}$. The M function criteria guarantees convergence will be monotonic. Monotonicity is a stronger requirement which places a restriction on p.

Chapter 5

Concluding Remarks and Future Work

This chapter includes a summary of this thesis, the important comments and useful conclusions of the present research work and future research directions.

In **Chapter 1**, we discussed the objectives of the thesis, relevant literature survey, introduced the equidistribution principle (EP), and presented the model problem that arises from the nonlinear parallel optimized Schwarz iteration. Our concern was to solve the involved nonlinear mesh BVP using parallel optimized domain decomposition approach and provide a nonuniform coordinate for the original physical PDE of interest.

Chapter 2 focused on moving mesh methods as determined by the EP. We showed how the mesh equations are derived from the EP for steady state problems in one space dimension. Additionally, we described some existing solution methods for the mesh BVP. We presented domain decomposition preliminaries for the nonlinear BVP including parallel Schwarz for an arbitrary number of subdomains and optimized Schwarz methods.

In **Chapter 3**, we derived an implicit solution on each subdomain for the optimized Schwarz iteration for the nonlinear mesh BVP. We introduced an interface iteration from the transmission condition, which is a nonlinear iteration. The continuous subdomain DD iteration is equivalent to the discrete interface iteration. Some basic theorems involving

M-functions, in particular the convergence of the Gauss-Seidel and Jacobi processes for such mappings was described. Using the theory of M-functions we provided an analysis of the parallel optimized Schwarz method on two subdomains and extended this result to an arbitrary number of subdomains. This is the first known analysis of optimized Schwarz on many subdomains for this class of problems. M-function theory guarantees that these iterations will converge monotonically when p is greater than $\max\{\frac{\hat{m}}{\alpha_1}, \frac{\hat{m}}{\alpha_2-\alpha_1}, ..., \frac{\hat{m}}{1-\alpha_m}\}$, where p is the Robin parameter. The iteration was computed by nonlinear (block) Gauss Jacobi or Gauss Seidel methods.

Chapter 4 focused some numerical results, which confirm the theory from Chapter 3.

The main purpose of this thesis was to develop and analyze nonlinear iterations arising from an optimized Schwarz domain decomposition method. Numerically we see that the optimized Schwarz iteration converges for all $p > 0$. Our theory explains convergence for p large enough. This gap will be the subject of future work. Also it would be nice to understand if the transition to monotonic convergence occurs at the optimal value of p.

Bibliography

[1] Carl de Boor. *Good approximation by splines with variable knots. II.* In Conference on the Numerical Solution of Differential Equations, volume 363. Springer-Verlag, 1974.

[2] Weizhang Huang. Practical aspects of formulation and solution of moving mesh partial differential equations. *Journal of Computational Physics*, 171(2):753–775, 2001.

[3] Weizhang Huang and Robert D Russell. A high dimensional moving mesh strategy. *Applied Numerical Mathematics*, 26(1):63–76, 1998.

[4] Weizhang Huang and David M Sloan. A simple adaptive grid method in two dimensions. *SIAM Journal on Scientific Computing*, 15(4):776–797, 1994.

[5] MJ Baines. Least squares and approximate equidistribution in multidimensions. *Numerical Methods for Partial Differential Equations*, 15(5):605–615, 1999.

[6] Chris J Budd, Weizhang Huang, and Robert D Russell. Adaptivity with moving grids. *Acta Numerica*, 18:111–241, 2009.

[7] Weizhang Huang and Robert D. Russell. *Adaptive Moving Mesh Methods*, volume 174. Springer, New York, 2011.

[8] Martin J Gander and Ronald D Haynes. Domain decomposition approaches for mesh

generation via the equidistribution principle. *SIAM Journal on Numerical Analysis*, 50(4):2111–2135, 2012.

[9] Yana Di, Ruo Li, Tao Tang, and Pingwen Zhang. Moving mesh finite element methods for the incompressible Navier-Stokes equations. *SIAM Journal on Scientific Computing*, 26(3):1036–1056, 2005.

[10] Tao Tang. Moving mesh methods for computational fluid dynamics. *Contemporary Mathematics*, 383:141–174, 2005.

[11] Marsha J Berger and Joseph Oliger. Adaptive mesh refinement for hyperbolic partial differential equations. *Journal of Computational Physics*, 53(3):484–512, 1984.

[12] Joe F Thompson. Grid generation techniques in computational fluid dynamics. *AIAA Journal*, 22(11):1505–1523, 1984.

[13] Krzysztof J Fidkowski and David L Darmofal. Review of output-based error estimation and mesh adaptation in computational fluid dynamics. *AIAA Journal*, 49(4):673–694, 2011.

[14] Jörn Behrens, Natalja Rakowsky, Wolfgang Hiller, Dörthe Handorf, Matthias Läuter, Jürgen Päpke, and Klaus Dethloff. amatos: Parallel adaptive mesh generator for atmospheric and oceanic simulation. *Ocean Modelling*, 10(1):171–183, 2005.

[15] Christiane Jablonowski. *Adaptive grids in weather and climate modeling*. PhD thesis, The University of Michigan, 2004.

[16] William Skamarock, Joseph Oliger, and Robert L Street. Adaptive grid refinement for numerical weather prediction. *Journal of Computational Physics*, 80(1):27–60, 1989.

[17] Hilary Weller, Todd Ringler, Matthew Piggott, and Nigel Wood. Challenges facing adaptive mesh modeling of the atmosphere and ocean. *Bulletin of the American Meteorological Society*, 91(1):105–108, 2010.

[18] Gary S Dietachmayer. Application of continuous dynamic grid adaption techniques to meteorological modeling. Part ii: Efficiency. *Monthly Weather Review*, 120(8):1707–1722, 1992.

[19] G Beckett, JA Mackenzie, A Ramage, and DM Sloan. On the numerical solution of one-dimensional PDEs using adaptive methods based on equidistribution. *Journal of Computational Physics*, 167(2):372–392, 2001.

[20] G Beckett, JA Mackenzie, A Ramage, and DM Sloan. Computational solution of two-dimensional unsteady PDEs using moving mesh methods. *Journal of Computational Physics*, 182(2):478–495, 2002.

[21] Weizhang Huang, Yuhe Ren, and Robert D Russell. Moving mesh partial differential equations (MMPDE) based on the equidistribution principle. *SIAM Journal on Numerical Analysis*, 31(3):709–730, 1994.

[22] Igor P Boglaev. Iterative algorithms of domain decomposition for the solution of a quasilinear elliptic problem. *Journal of Computational and Applied Mathematics*, 80(2):299–316, 1997.

[23] Xiao-Chuan Cai and Maksymilian Dryja. Domain decomposition methods for monotone nonlinear elliptic problems. *Contemporary Mathematics*, 180:21–21, 1994.

[24] Pierre-Louis Lions. On the Schwarz alternating method. I. In *First International Symposium on Domain Decomposition Methods for Partial Differential Equations*, pages 1–42. Paris, France, 1988.

[25] Shiu-Hong Lui. On monotone and Schwarz alternating methods for nonlinear elliptic PDEs. *ESAIM: Mathematical Modelling and Numerical Analysis*, 35(01):1–15, 2001.

[26] SH Lui. On linear monotone iteration and Schwarz methods for nonlinear elliptic PDEs. *Numerische Mathematik*, 93(1):109–129, 2002.

[27] Xue-Cheng Tai and Magne Espedal. Rate of convergence of some space decomposition methods for linear and nonlinear problems. *SIAM Journal on Numerical Analysis*, 35(4):1558–1570, 1998.

[28] Pierre-Louis Lions. On the Schwarz alternating method. III: a variant for nonoverlapping subdomains. In *Third International Symposium on Domain Decomposition Methods for Partial Differential Equations*, volume 6, pages 202–223. SIAM Philadelphia, PA, 1990.

[29] Caroline Japhet. Optimized Krylov-ventcell method. Application to convection-diffusion problems. In *Proceedings of the 9th International Conference on Domain Decomposition Methods*, pages 382–389, 1998.

[30] Ronald D. Haynes. *Recent advances in Schwarz waveform relaxation moving mesh methods - a new moving subdomain method. Domain decomposition methods in science and engineering XIX, Lect. Notes Comput. Sci. Eng.*, volume 78. Springer, Berlin, 2011.

[31] Ronald D Haynes and Robert D Russell. A Schwarz waveform moving mesh method. *SIAM Journal on Scientific Computing*, 29(2):656–673, 2007.

[32] Weizhang Huang Ronald D. Haynes and Robert D. Russell. *A moving mesh method for time-dependent problems based on Schwarz waveform relaxation. In Domain De-*

composition Methods in Science and Engineering XVII, Lect. Notes Comput. Sci. Eng, volume 60. Springer, Berlin, 2008.

[33] Ronald D. Haynes and Alexander J.M. Howse. Alternating Schwarz methods partial for differential equation-based mesh generation. *Electron. Trans. Numer. Anal.*, 92:349–376, 2015.

[34] James M. Ortega and Werner C. Rheinboldt. *Iterative solution of nonlinear equations in several variables*, volume 30 of Classics Appl. Math. Siam, Philadelphia, 2000.

[35] Lipman Bers. On mildly nonlinear partial difference equations of elliptic type. *J. Res. Nat. Bur. Standards*, 51(5):229–236, 1953.

[36] Samuel Schechter. Iteration methods for nonlinear problems. *Transactions of the American Mathematical Society*, pages 179–189, 1962.

[37] G. Birkhoff and B. Kellogg. *Solution of equilibrium equations in thermal networks, Proc. Symp. on Generalized Networks*. Brooklyn Polytechnic Press, 1966.

[38] James M Ortega and Werner C Rheinboldt. On a class of approximate iterative processes. *Archive for Rational Mechanics and Analysis*, 23(5):352–365, 1967.

[39] James M Ortega and Werner C Rheinboldt. Local and global convergence of generalized linear iterations. *SIAM Symp. on Num. Sol. of Nonlinear Problems*, 23:352–365, 1968.

[40] TA Porsching. Jacobi and gauss-seidel methods for nonlinear network problems. *SIAM Journal on Numerical Analysis*, 6(3):437–449, 1969.

[41] James M Ortega. Notes on monotone convergence. *Unpublished personal note*, 1969.

[42] Oswaldo de Oliveira et al. The implicit and inverse function theorems: Easy proofs. *Real Analysis Exchange*, 39(1):207–218, 2013.

[43] Devin Grant. *Acceleration Techniques for Mesh Gereration via Domain Decomposition Methods*. M.Sc thesis, Memorial University of Newfoundland, St. John's, NL, Canada, 2015.

[44] R. B. Kellogg. On classes of n-dimensional nonlinear mappings generalizing several types of matrices. *Math. Comp.*, 23:23–27, 1969.

[45] Pierre-Louis Lions. On the schwarz alternating method. i. In *First International Symposium on Domain Decomposition Methods for Partial Differential Equations*, pages 1–42. Paris, France, 1988.

[46] Ronald D Haynes and Felix Kwok. Discrete analysis of domain decomposition algorithms for grid generation via the equidistribution principle. *Accepted AMS Mathematics of Computation*, 2015.

[47] Werner C Rheinboldt. On M-functions and their application to nonlinear Gauss-Seidel iterations and to network flows. *Journal of Mathematical Analysis and Applications*, 32(2):274–307, 1970.

[48] Werner C Rheinboldt. On classes of n-dimensional nonlinear mappings generalizing several types of matrices. *AMS*, II:501–546, 1971.

[49] L. Collatz. Aufgaben monotoner art. *Arch. Math.*, 3:366–376, 1952.

[50] Gordon H Bradley and Pran N Wahi. An algorithm for integer linear programming: a combined algebraic and enumeration approach. *Operations Research*, 21(1):45–60, 1973.

[51] George B Dantzig and Mukund N Thapa. *Linear Programming 1: Introduction*, volume 1. Springer Science & Business Media, 1997.

[52] Joseph Fourier. Histoire de l'académie, partie mathématique (1824). In *Mémoires de l'Académie des sciences de l'Institut de France*. Gauthier-Villars, 1824.

[53] Theodore Motzkin. Beiträge zur theorie der linearen ungleichungen (inaugural dissertation basel). Azriel, Jerusalem, 1936.

[54] Richard L. Burden and J. Douglas Faires. *Numerical Analysis*. Richard Stratton, 9th edition, 2013.

[55] Doron Levy. Introduction to numerical analysis. *Department of Mathematics and Center for Scientific Computation and Mathematical Modeling, CSCAMM, University of Maryland*, 2010.

[56] Samuel Daniel Conte, Carl de Boor, Endre Süli, and David F Mayers. *Elementary Numerical Analysis: An Algorithmic Approach*. McGraw-Hill Higher Education, 3rd edition, 1980.

[57] Felix Kwok. Personal communication. 2015.