Hilbert Series for Free Lie Superalgebras and Related Topics

by

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Supervisors
Dr. Yuri A. Bahturin and Dr. Mikhail V. Kotchetov

Department of Mathematics and Statistics
Memorial University of Newfoundland

Memorial University of Newfoundland

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Abstract

We consider Hilbert series of ordinary Lie algebras, restricted (or p-) Lie algebras, and color Lie (p-)superalgebras. We derive a dimension formula similar to a well-known Witt's formula for free color Lie superalgebras and a certain class of color Lie p-superalgebras. A Lie (super)algebra analogue of a well-known Schreier's formula for the rank of a subgroup of finite index in a free group was found by V. M. Petrogradsky. In this dissertation, Petrogradsky's formulas are extended to the case of color Lie (p-)superalgebras. We establish more Schreier-type formulas for the ranks of submodules of free modules over free associative algebras and free group algebras. As an application, we consider Hopf subalgebras of some cocommutative Hopf algebras. Also, we apply our version of Witt and Schreier formulas to study relatively free color Lie (p-)superalgebras and to prove that the free color Lie superalgebra and its enveloping algebra have the same entropy. Y. A. Bahturin and A. Y. Olshanskii proved that the relative growth rate of a finitely generated subalgebra K of a free Lie algebra L of finite rank is strictly less than the growth rate of the free Lie algebra itself. We show that this theorem cannot be extended to free color Lie superalgebras in general. However, we establish it in a special case.

Co-Authorship

The results of Sections 5.2 and 5.3 in Chapter 5 and of Section 7.3 in Chapter 7 were obtained in collaboration with Dr. Yuri Bahturin.

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Chapter 1

Background

The following notations will be used throughout this work:

- F: the ground field,
- $\mathbb{N}$: the set of positive integers $\{1, 2, \ldots\}$,
- $\mathbb{N}_0$: the set of nonnegative integers $\{0, 1, 2, \ldots\}$,
- $\mathbb{Z}$: the set of integers,
- $\mathbb{Q}$: the set of rational numbers,
- $\mathbb{C}$: the set of complex numbers,
- $|X|$: the cardinal of the set X,
- $\subseteq$: inclusion,
- $\subset$: proper inclusion.

1.1 Introduction

Suppose that $V = \bigoplus_{k=0}^{\infty} V_k$ is a graded vector space such that all subspaces V_k are finite dimensional. The formal power series in the indeterminate t

$$\mathcal{H}(V,t) = \sum_{k=0}^{\infty} (\text{dim} V_k) t^k$$

is called the *Hilbert-Poincaré series* (also known under the name *Hilbert series*) of the graded vector space V. We will sometimes write $\mathcal{H}(V)$ for $\mathcal{H}(V,t)$.

Let $V = \bigcup_{k=1}^{\infty} V^k$ be a filtered vector space such that $\text{dim} V^k < \infty$ for all $k \in \mathbb{N}$. Set $V^0 = 0$. The Hilbert-Poincaré series of V is $\mathcal{H}(V) = \mathcal{H}(V,t) = \sum_{k=1}^{\infty} \dim \left(V^k/V^{k-1}\right) t^k$. In other words, for a filtered space V, the Hilbert-Poincaré series is the same as for the associated graded space: $\mathcal{H}(V,t) = \mathcal{H}(\text{gr} V, t)$.

Let $L = \bigoplus_{n=1}^{\infty} L_n$ be a free Lie algebra of rank r. The dimensions of homogeneous subspaces L_n are given by the well-known Witt formula (see, e.g., [3, page 73]):

$$\text{dim} L_n = \frac{1}{n} \sum_{d|n} \mu(d) r^{\frac{n}{d}},$$

where $\mu : \mathbb{N} \to \{-1, 0, 1\}$ is the Möbius function defined as follows. If n is divisible by the square of a prime number, we set $\mu(n) = 0$, and otherwise we set $\mu(n) = (-1)^k$ where k is the number of prime divisors of n (with $k = 0$ for $n = 1$, so $\mu(1) = 1$). There are similar formulas for homogeneous and multihomogeneous components of free (color) Lie superalgebras (see, e.g., [4]). In [28], Petrogradsky obtained similar dimension formulas for free Lie p-algebras. More generally, suppose Λ is a countable abelian semigroup such that every element $\lambda \in \Lambda$ can be written as a sum of other elements only in finitely many ways. Let $L = \bigoplus_{\lambda \in \Lambda} L_\lambda$ be a Λ-graded Lie algebra freely generated by $X = \bigcup_{\lambda \in \Lambda} X_\lambda$. An analogue of Witt's formula, called character formula, for the dimensions of homogeneous components $L_\lambda, \lambda \in \Lambda$, was found by Kang and Kim in [11]. In order to study Lie superalgebras, Petrogradsky added a fixed homomorphism $\kappa : \Lambda \to \mathbb{Z}_2 = \{\pm 1\}$. He obtained an analogue of Witt's formula (or character formula) in the case of free Lie superalgebras. In Chapter 2, we extend his results to free color Lie superalgebras and, in the special case $L = L_+$, to free color Lie p-superalgebras (see Definition 1.2.13).

A well-known theorem, due to Nielsen and Schreier, states that every subgroup of a free group is again free. A corresponding result for Lie algebras was later obtained independently by Shirshov [36] and Witt [42]. However, subalgebras of the free associative algebra are not necessarily free (for example, $F[x^2, x^3] \subseteq F[x]$ is not free). The Schreier index formula states that if G is a free group of rank n, and K is a subgroup of finite index in G, then the rank of K is given by

$$\operatorname{rank}(K) = (n-1)\,[G : K] + 1.$$

Kukin [14] obtained the analogue of the formula above for restricted Lie algebras. There are no straightforward analogues of Schreier index formula in the following cases:

1. The free Lie algebra, even with finitely many generators,
2. The group (monoid) where the number of generators and/or the index is infinite.

To obtain the desired formulas, one can replace numbers with power series, as follows. A *finitely graded set* is a pair (X, wt), where X is a countable set, and $\mathrm{wt} : X \to \mathbb{N}$ is a weight function such that the subsets $X_i := \{x \in X \mid \mathrm{wt}(x) = i\}$ are finite for all $i \in \mathbb{N}$. For such a set X, the generating function is defined by $\mathcal{H}(X) = \mathcal{H}(X, t) = \sum_{i=1}^{\infty} |X_i|\, t^i$. For a monomial $y = x_{i_1} \dots x_{i_r}$, $x_j \in X$, we define $\mathrm{wt}(y) = \mathrm{wt}(x_{i_1}) + \dots + \mathrm{wt}(x_{i_r})$. If A is an algebra generated by a finitely graded set X, then A has a filtration (as an algebra) $\bigcup_{i=1}^{\infty} A^i$ where A^i is spanned by all monomials of weight at most i. We denote the corresponding series by $\mathcal{H}_X(A,t)$. If A is freely generated by X, then

$$\mathcal{H}_X(A,t) = \mathcal{H}(Y,t) = \sum_{i=1}^{\infty} |Y_i|\, t^i,$$

where Y is the set of all monomials in X, which is also finitely graded. If B is a subspace of A, then the factor-space A/B also acquires a filtration:

$$(A/B)^n = (A^n + B)/B \cong A^n/(B \cap A^n).$$

In [25], Petrogradsky defined an operator $\mathcal{E}$ on $\mathbb{Z}[[t]]$ (the ring of formal power series in the indeterminate t over $\mathbb{Z}$) as follows:

$$\mathcal{E}: \sum_{i=0}^{\infty} a_i t^i \mapsto \prod_{i=0}^{\infty} \frac{1}{(1-t^i)^{a_i}}.$$

Then he introduced an analogue of Schreier's formula in terms of formal power series for free Lie algebras. He proved that if L is a free Lie algebra generated by a finitely graded set X, and K is a subalgebra of L, then there is a set of free generators Z of K such that

$$\mathcal{H}(Z) = (\mathcal{H}(X) - 1)\,\mathcal{E}\,(\mathcal{H}\,(L/K)) + 1.$$

In addition, he introduced similar formulas for free Lie superalgebras and free Lie p-algebras. In [9], a similar formula was obtained for free Lie p-superalgebras. We extend Petrogradsky's formula to *free color Lie (p-)superalgebras* in Chapter 3. Later, Petrogradsky established a more general Schreier formula for free Lie superalgebras in terms of characters (see [27]). In Chapter 4, we extend his result to free color Lie (p-)superalgebras.

In [27], Petrogradsky applied his character formulas and Schreier formulas to the study of relatively free Lie superalgebras and groups. In Chapter 5, we provide some examples of similar applications.

Let S be an algebra generated by a finite set X. We denote by $\gamma_S(n)$ the *growth function*, which is defined to be the dimension of the space spanned by all monomials in X of length not exceeding n. Let $\lambda_S(n) = \gamma_S(n) - \gamma_S(n-1)$. Then the *growth rate* (also known as *exponent* or *entropy*) of S is given by $\limsup_{n\to\infty}(\lambda_S(n))^{\frac{1}{n}}$ which is the inverse of the radius of convergence of the Hilbert series $\sum_{n=0}^{\infty} \lambda_S(n)t^n$. In Chapter 6, we show that a free color Lie superalgebra and its enveloping algebra have the same entropy. The *relative growth rate* of a finitely generated subalgebra K of a free Lie algebra L of finite rank is strictly less than the growth rate of L itself (see [7]). In Chapter 6, we show that this theorem cannot be generalized to free color Lie superalgebras, but we generalize it in a special case.

In [18], Lewin established an analogue of Schreier's formula for a submodule N of finite codimension in a finitely generated free module M over a free associative algebra of finite rank or over the group algebra of a free group of finite rank (which are examples of so-called free ideal rings, so N is again free). In [8, Section 4.5], Cohn gave a formula for the generating function of a free set of generators of an arbitrary right ideal of a free associative algebra. Using this result, an analogue of Schreier's formula for an arbitrary submodule of a free module of finite rank over a free associative algebra will be obtained in Chapter 7. In a special case, we also obtain a similar result over the free algebra of a free group, using the generalization of the Schreier formula for subgroups of infinite index in a free group established by Bahturin and Olshanskii in [5]. They also derived a Schreier formula for subactions of free actions of a free monoid, and we generalize their result. Finally, we give an application to some cocommutative bialgebras.

1.2 Preliminaries

In this section we give some definitions, notations, and results that will be used frequently throughout the thesis. Familiarity with the basic properties of groups, rings, tensor products, and Lie algebras is assumed. For all undefined terms, concepts and related results, we refer the reader to [3], [4], [19], and [38].

1.2.1 Algebras, Coalgebras, and Hopf Algebras

Definition 1.2.1. A unital associative algebra (A, m, u) is a vector space A with two linear maps, a multiplication $m : A \otimes A \to A$, and a unit $u : F \to A$ such that $m \circ (m \otimes \mathrm{id}_A) = m \circ (\mathrm{id}_A \otimes m)$ and $m \circ (u \otimes \mathrm{id}_A) = m \circ (\mathrm{id}_A \otimes u) = \mathrm{id}_A$, where we have identified $(A \otimes A) \otimes A = A \otimes A \otimes A = A \otimes (A \otimes A)$ and $A \otimes F = A = F \otimes A$.

Definition 1.2.2. A counital coassociative coalgebra (C, Δ, ϵ) is a vector space C with two linear maps $\Delta : C \to C \otimes C$ (comultiplication) and $\epsilon : C \to F$ (counit) that satisfy $(\Delta \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \Delta) \circ \Delta = \mathrm{id}_C$.

Let $(C, \Delta_C, \epsilon_C)$ and $(D, \Delta_D, \epsilon_D)$ be coalgebras. A linear map $f : C \to D$ is a coalgebra morphism if $\Delta_D \circ f = (f \otimes f) \circ \Delta_C$ and $\epsilon_C = \epsilon_D \circ f$. Now we are in a position to combine the notions of algebra and coalgebra.

Definition 1.2.3. $(H, m, u, \Delta, \epsilon)$ is called a *bialgebra* if

1. (H, m, u) is a unital associative algebra,
2. (H, Δ, ϵ) is a counital coassociative coalgebra,
3. The coalgebra structure maps Δ and ϵ are algebra morphisms (equivalently, the algebra structure maps m and u are coalgebra morphisms).

A subspace B of a bialgebra H is a subbialgebra if it is both a subalgebra and a subcoalgebra. A bialgebra H is commutative if $m \circ \tau = m$, and is cocommutative if $\tau \circ \Delta = \Delta$, where τ is the flip $a \otimes b \mapsto b \otimes a$. For the comultiplication Δ we use Sweedler notation without summation: $\Delta h = h_{(1)} \otimes h_{(2)}$.

A *monoid* is a semigroup with identity element. Let M be any monoid, and let $H = FM$ be its monoid algebra. Then Δ and ϵ defined by $\Delta : m \mapsto m \otimes m$, $\epsilon : m \mapsto 1$, for $m \in M$, make H is a cocommutative bialgebra.

Definition 1.2.4. Let H be a bialgebra. An algebra A is a (left) H-*module algebra* if A is a (left) H-module via $\varphi : H \to \mathrm{End}_F(A) : \ h \mapsto \varphi_h$, such that $\varphi_h(ab) = \varphi_{h_{(1)}}(a)\varphi_{h_{(2)}}(b)$ and $\varphi_h(1_A) = \epsilon(h)1_A$ for $h \in H$, $a, b \in A$.

Let H be a bialgebra and let A be a left H-module algebra (via $\varphi : h \mapsto \varphi_h$). Then the smash product of A and H, denoted by $A\#_\varphi H$, is a vector space $A \otimes H$ (we will write $a\#h$ for the element $a \otimes h$), together with the following operation:

$$(a\#h)(b\#k) = a\varphi_{h_{(1)}}(b)\#h_{(2)}k.$$

$A\#H$ together with the multiplication above is an associative algebra with unit $1_A\#1_H$.

Definition 1.2.5. Let $(H, m, u, \Delta, \epsilon)$ be a bialgebra. Then H is called a *Hopf algebra* if there exists a linear map $S : H \to H$ (antipode) such that $m \circ (\mathrm{id}_H \otimes S) \circ \Delta = m \circ (S \otimes \mathrm{id}_H) \circ \Delta = u \circ \epsilon$.

Let H be a Hopf algebra with antipode S, and let B be a subspace of H. Then B is a Hopf subalgebra if B is a subbialgebra, and $S(B) \subseteq B$.

Example 1.2.6.

1. Let G be any group, and let $H = FG$ be its group algebra. Then Δ, ϵ, and S defined by $\Delta : g \mapsto g \otimes g$, $\epsilon : g \mapsto 1$, and $S : g \mapsto g^{-1}$ for $g \in G$ make H a cocommutative Hopf algebra.

2. Let L be a Lie algebra, and let $H = U(L)$ be its universal enveloping algebra. Then H becomes a cocommutative Hopf algebra by defining $\Delta : x \mapsto x \otimes 1 + 1 \otimes x$, $\epsilon : x \mapsto 0$, and $S : x \mapsto -x$ for $x \in L$ (and extent Δ, ϵ, S multiplicatively/ anti-multiplicatively).

An algebra A is a left FG-module algebra if and only if the elements of G act as automorphisms on A ([19, Example 4.1.6]). Likewise, A is a left $U(L)$-module algebra if the elements of L acts as derivations of A ([19, Example 4.1.8]).

Let H be a Hopf algebra. A nonzero element $g \in H$ is *group-like* if $\Delta g = g \otimes g$. The set of group-like elements of H, denoted by $G(H)$, forms a multiplicative subgroup. An element $x \in H$ is *primitive* if $\Delta x = x \otimes 1 + 1 \otimes x$. The set of primitive elements of H, denoted by $P(H)$, forms a Lie algebra (a Lie p-algebra if $\text{char}F = p$) where H is regarded as a Lie algebra with respect to $[x, y] = xy - yx$ (a Lie p-algebra with respect to $x^{[p]} = x^p$).

Smash products arise very frequently in the theory of Hopf algebras. A classical example is the following Cartier-Kostant theorem which gives a characterization of cocommutative Hopf algebras.

Theorem 1.2.7. *If H is a cocommutative Hopf algebra over an algebraically closed field F of characteristic 0, then H is isomorphic to $U(L)\#_\varphi FG$ as an algebra, where $L = P(H)$, $G = G(H)$, and $G(H)$ acts on $P(H)$ by conjugation: $\varphi_g(x) = gxg^{-1}$ for $g \in G, x \in L$.*

Conversely, it is easy to show that the smash product $U(L)\#_\varphi FG$ (where G acts on L by φ) endowed with the tensor product coalgebra structure is a cocommutative Hopf algebra (see, e.g., [30]).

In this dissertation we mostly study primitive elements of certain Hopf algebras and their generalizations. In this way we arrive at ordinary Lie algebras, restricted (or p-) Lie algebras, color Lie superalgebras and p-superalgebras. In Chapter 7, we also turn our attention to group-like elements and smash products.

1.2.2 Graded Algebras

Definition 1.2.8. Suppose G is an abelian semigroup written multiplicatively. A *G-graded algebra* is an algebra R together with a direct sum decomposition of the form

$$R = \bigoplus_{g\in G} R_g,$$

where each R_g is a subspace and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. A nonzero element $r \in R$ is called *homogeneous* if there is $g \in G$ such that $r \in R_g$ (we write $d(r) = g$). A subspace $H \subseteq R$ is said to be homogeneous if $H = \bigoplus_{g\in G} H_g$ where $H_g = H \cap R_g$. By a *G-graded vector space* we mean a vector space V together with a direct sum decomposition $V = \bigoplus_{g\in G} V_g$. Let V and W be G-graded vector spaces. A linear map $f : V \to W$ is called *homogeneous of degree* $h \in G$ if for all $g \in G$, we have $f(V_g) \subseteq W_{hg}$. In particular, a homogeneous linear map of degree $1_G \in G$ will be called a *degree preserving map*.

If A is an algebra with two gradings, say $A = \bigoplus_{g\in G} A_g$ and $A = \bigoplus_{\lambda\in\Lambda} A_\lambda$ (G and Λ are semigroups), then the two gradings are called *compatible* if $A_g = \bigoplus_{\lambda\in\Lambda}(A_g \cap A_\lambda)$ for all $g \in G$ or, equivalently, $A_\lambda = \bigoplus_{g\in G}(A_\lambda \cap A_g)$ for all $\lambda \in \Lambda$.

Definition 1.2.9. Let A be an algebra and let $A^0 \subseteq A^1 \subseteq A^2 \subseteq \ldots$ be a chain of subspaces. This chain is called an (ascending) *filtration* of A if

$$\bigcup_{m\geq 0} A^m = A \text{ and } A^iA^j \subseteq A^{i+j}\ \forall i,j\geq 0.$$

Example 1.2.10.

1. If A is an associative algebra generated by a finite set X, then A has a filtration (as an algebra) $\bigcup_{i=1}^{\infty} A^i$ where A^i is spanned by monomials of degree $\leq i$ in A.

2. Given a $\mathbb{Z}$-graded algebra $A = \bigoplus_{n\geq 0} A_n$, then there is a corresponding filtration $\bigcup_{m=0}^{\infty} A^m$ where $A^m = \bigoplus_{k=0}^{m} A_k$.

3. Conversely, given an algebra A with filtration $A^0 \subseteq A^1 \subseteq A^2 \subseteq \cdots$, we can construct a graded algebra gr$A$ as follows: $\mathrm{gr}A = \bigoplus_{m\geq 0} (A^m/A^{m-1})$ as a vector space (set $A^{-1} = 0$) and define multiplication $A^m/A^{m-1} \times A^n/A^{n-1} \to A^{m+n}/A^{m+n-1}$ by $(a + A^{m-1})(b + A^{n-1}) = ab + A^{m+n-1}$. Note that if the filtration A^m comes from a grading ($A^m = \bigoplus_{k=0}^{m} A_k$), then $A^m/A^{m-1} \cong A_m$, and so $\mathrm{gr}A \cong A$.

Definition 1.2.11. Let $A = \bigoplus_{n\geq 0} A_n$ be a $\mathbb{Z}$-graded algebra such that $\dim A_n < \infty$ for all $n \in \mathbb{N}$. The *Hilbert series* of A is defined by

$$\mathcal{H}(A,t) = \sum_{n=0}^{\infty} (\dim A_n)\, t^n.$$

If A is an algebra with filtration $A^0 \subseteq A^1 \subseteq A^2 \subseteq \cdots$. The Hilbert series of A is defined to be

$$\mathcal{H}(A,t) = \mathcal{H}(\mathrm{gr}A, t).$$

Let $U = \bigcup_{i=1}^{\infty} U^i$ and $V = \bigcup_{i=1}^{\infty} V^i$ be filtered vector spaces with $\dim U^i, \dim V^i < \infty$

for all $i \in \mathbb{N}$. Then $U \oplus V$ and $U \otimes V$ acquire filtrations given by the subspaces

$$(U \oplus V)^n = U^n \oplus V^n$$

and

$$(U \otimes V)^n = \sum_{i=0}^{n} U^i \otimes V^{n-i},$$

respectively. It is easy to verify the following result (see [39]).

Theorem 1.2.12. $\mathcal{H}(U \oplus V, t) = \mathcal{H}(U, t) + \mathcal{H}(V, t)$ *and* $\mathcal{H}(U \otimes V, t) = \mathcal{H}(U, t)\mathcal{H}(V, t)$.

1.2.3 Color Lie Superalgebras

Color Lie (super)algebras were originally introduced by Rittenberg and Wyler ([32]) and Scheunert ([34]). They generalize Lie superalgebras (see e.g., [23]). They appear in various branches of mathematics (e.g., topology, algebraic groups, etc.) and mathematical physics (elementary particles, superstrings, etc.). Throughout this section F denotes a field of characteristic $\neq 2, 3$, and G denotes an abelian group. We call a map $\gamma : G \times G \to F^*$ ($F^* = F \setminus \{0\}$) a *skew-symmetric bicharacter* on G (analogue of skew-symmetric bilinear form for vector spaces, but written multiplicatively) if it satisfies

1. $\gamma(f, gh) = \gamma(f, g)\gamma(f, h)$ and $\gamma(gh, f) = \gamma(g, f)\gamma(h, f)$ for all $f, g, h \in G$,
2. $\gamma(f, g) = (\gamma(g, f))^{-1}$ for all $f, g \in G$.

Note that (2) implies $\gamma(g, g) = \pm 1$ $\forall g \in G$. Set $G_\pm = \{g \in G \mid \gamma(g, g) = \pm 1\}$. Then G_+ is a subgroup of G and $[G : G_+] \leq 2$. In particular, if G is finite, then either $G = G_+$ or $|G_+| = |G_-|$.

Definition 1.2.13. A (G,γ)-color Lie superalgebra is a pair $(L,[\ ,\ ])$ where $L = \bigoplus_{g\in G} L_g$ is a G-graded vector space and $[\ ,\ ]:\ L\otimes L \to L$ is a bilinear map that satisfies the following identities for any homogeneous $x, y, z \in L$

1. γ-skew-symmetry:
$$[x,y] = -\gamma(d(x),d(y))\,[y,x]\,,$$

2. γ-Jacobi identity:
$$\gamma(d(z),d(x))\,[x,[y,z]] + \gamma(d(y),d(z))\,[z,[x,y]] + \gamma(d(x),d(y))\,[y,[z,x]] = 0.$$

The positive part L_+ (respectively, negative part L_-) is defined to be $L_+ = \bigoplus_{g\in G_+} L_g$ (respectively, $L_- = \bigoplus_{g\in G_-} L_g$).

Example 1.2.14.

1. An ordinary Lie algebra is a (G,γ)-color Lie superalgebra if $G = \{1\}$ and $\gamma = 1$. A Lie algebra with a G-grading can be considered a (G,γ)-color Lie superalgebra for $\gamma = 1$.

2. A *Lie superalgebra* is a $(\mathbb{Z}_2,\gamma)$-color Lie superalgebra for
$$\gamma : \mathbb{Z}_2\times\mathbb{Z}_2 \to F^*\ :\ (i,j)\mapsto (-1)^{ij}.$$

3. Any G-graded associative algebra A, with the bracket defined by
$$[x,y]_\gamma = xy - \gamma(d(x),d(y))yx$$
for any nonzero homogeneous $x, y \in A$ (the γ-*supercommutator*), is a color Lie superalgebra, which will be denoted by $A^{(-)}$.

4. Let $A = \bigoplus_{g\in G} A_g$ be a G-graded algebra. A homogeneous linear map $D : A \to A$ of degree $h \in G$ is called a *superderivation* if

$$D(ab) = D(a)b + \gamma(h, d(a))aD(b)$$

for all homogeneous elements $a, b \in A$. By $\mathrm{Der}_h(A)$ we denote the vector space of all superderivations of A of degree $h \in G$. Then $\mathrm{Der}(A) = \bigoplus_{g\in G} \mathrm{Der}_g(A)$, with the bracket $[D_1, D_2] = D_1D_2 - \gamma(d(D_1), d(D_2))D_2D_2$, is a color Lie superalgebra.

Definition 1.2.15. In the special case $G = G_+$, we will use the term a (G, γ)- color Lie algebra.

By a *G-graded set* we mean a disjoint union $X = \bigcup_{g\in G} X_g$. If X is a G-graded set and $V = \bigoplus_{g\in G} V_g$ is a G-graded vector space, then a degree preserving map from X to V is a map $i : X \to V$ such that $i(X_g) \subseteq V_g$ $\forall g \in G$. A degree preserving linear map $\varphi : L_1 \to L_2$ between (G, γ)-color Lie superalgebras $(L_1, [\ ,\ ]_1)$ and $(L_2, [\ ,\ ]_2)$ is called a homomorphism if $\varphi([x, y]_1) = [\varphi(x), \varphi(y)]_2$, $\forall x, y \in L_1$. Free Lie algebras are well known and have been extensively studied (see e.g., [3, 31]).

Definition 1.2.16. A free color Lie superalgebra of a G-graded set X is a pair $(L, i : X \to L)$ with L a color Lie superalgebra and $i : X \to L$ a degree preserving map from X to L such that if R is any color Lie superalgebra and $j : X \to R$ is a degree preserving map, then there is a unique homomorphism of color Lie superalgebras $h : L \to R$ such that $j = h \circ i$.

For each G-graded set X, there exists a free color Lie superalgebra $L(X)$ on X, which is unique up to isomorphism. Also $L(X)$ is $\mathbb{N}$-graded by degree in X ([4]).

The *universal enveloping algebra* of a color Lie superalgebra L is a pair (U, δ), where U is a G-graded associative unital algebra, $\delta : L \to U^{(-)}$ is a homomorphism of color

Lie superalgebras, and for every G-graded associative unital algebra R and for every homomorphism $\sigma : L \to R^{(-)}$ (with the same γ) there exists a unique homomorphism $\theta : U \to R$ of G-graded associative unital algebra such that $\theta \circ \delta = \sigma$. It is clear that the universal enveloping algebras are defined uniquely (up to isomorphism of G-graded associative unital algebras). To establish the existence of (U, δ), let $T(L)$ be the tensor algebra of L, and let I be the ideal of $T(L)$ generated by the elements of the form $u \otimes v - \gamma(g, h)(v \otimes u)$, where $u \in L_g, v \in L_h$. Then the quotient $T(L)/I$ with the canonical mapping $\delta : L \to T(L)/I$ has the desired properties ([4]).
We have the following theorem.

Theorem 1.2.17 ([4, p.47])**.** *The universal enveloping algebra of the free color Lie superalgebra $L(X)$ is the free associative algebra of X.*

The following theorem is a version of the Poincaré-Birkhoff-Witt (PBW) Theorem for color Lie superalgebras.

Theorem 1.2.18 ([4, p.16])**.** *Let $L = L_+ \oplus L_-$ be a (G, γ)-color Lie superalgebra, B a basis of L_+, C is a basis of L_-, and $\leq$ is a total order on $B \cup C$. Then the set D, formed by 1 and by all monomials of the form*

$$e_1^{s_1} \cdots e_n^{s_n},$$

where $e_i \in B \cup C$, $e_1 < \cdots < e_n, 0 \leq s_i$ for $e_i \in B$ and $s_i = 0, 1$ for $e_i \in C$, is a basis of the universal enveloping algebra, $U(L)$, of the Lie color superalgebra L.

Corollary 1.2.19. *Let $L = L_+ \oplus L_-$ be a (G, γ)-color Lie superalgebra, $S(L_+)$ the symmetric algebra of the vector space L_+, and $\Lambda(L_-)$ the Grassmann algebra of the vector space L_-. The universal enveloping algebra of L_+, $U(L_+)$, is an associative* *subalgebra of $U(L)$ generated by L_+, and we have the vector space isomorphisms*

$$U(L_+) \cong S(L_+) \text{ and } U(L) \cong U(L_+) \otimes \Lambda(L_-).$$

1.2.4 Restricted Color Lie Superalgebras

Let $L = \bigoplus_{g \in G} L_g$ be a color Lie superalgebra over a field of characteristic $p \neq 0, 2, 3$. For a homogeneous element $a \in L$, we consider a mapping

$$\mathrm{ad}a : L \to L \ : \ b \mapsto [a, b].$$

Definition 1.2.20. A color Lie superalgebra $L = \bigoplus_{g \in G} L_g$ is called *restricted* (or *p-superalgebra*) if for any $g \in G_+$ and for any homogeneous component L_g, there is a map

$$^{[p]} \ : L_g \to L \ : x \mapsto x^{[p]},$$

satisfying

- $(\mathrm{ad}x)^p = \mathrm{ad}\left(x^{[p]}\right)$ for all $x \in L_g$,
- $(\alpha x)^{[p]} = \alpha^p x^{[p]}$ for all $\alpha \in F, x \in L_g$,
- $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$ where y is homogeneous of the same degree as x, and $is_i(x, y)$ is the coefficient of λ^{i-1} in $\mathrm{ad}\,(\lambda x + y)^{p-1}\,(x)$.

If $G = \{1\}$, then we get a standard definition of a restricted Lie algebra. In this case, we can allow any prime characteristic p.

Example 1.2.21. Let A be a G-graded associative algebra. Then the map

$$(\ \ )^p : A_g \to A \ : \ x \mapsto x^p,$$

$\forall g \in G_+$, makes $A^{(-)}$ into a color Lie p-superalgebra (denoted by $[A]^p$).

The *restricted universal enveloping algebra* of a color Lie p-superalgebra L is a pair (u, δ) where u is a G-graded associative unital algebra, $\delta : L \to [u]^p$ is a homomorphism of color Lie p-superalgebra, and for any G-graded associative unital algebra R, and for any homomorphism $\sigma : L \to [R]^p$ of color Lie p-superalgebra there exists a unique homomorphism $\theta : u \to R$ of G-graded associative unital algebra such that $\theta \circ \delta = \sigma$. The restricted enveloping algebra of a color Lie p-superalgebra L exists (and will be denoted by $u(L)$), and it is unique up to an isomorphism of G-graded associative unital algebra ([4, p.87]).

There is a version of Poincaré-Birkhoff-Witt (PBW) Theorem for the color Lie p-superalgebra stating that if $L = L_+ \oplus L_-$ is a color Lie p-superalgebra, B is a basis of L_+, C is a basis of L_-, and $\leq$ is a total order on $B \cup C$, then the set of all monomials of the form

$$b_1^{s_1} \cdots b_n^{s_n},$$

where $b_i \in B \cup C$, $b_1 < \cdots < b_n, 0 \leq s_i \leq p-1$ for $b_i \in B$ and $b_i = 0, 1$ for $b_i \in C$, is a basis of the restricted universal enveloping algebra, $u(L)$, of the Lie p-superalgebra L ([4, Theorem 2.5]).

Given a nonempty G-graded set $X = \bigcup_{g \in G} X_g$, a free color Lie p-superalgebra on X over F is a color Lie p-superalgebra L^p over F, together with a degree one mapping $i : X \to L^p$, with the following universal property: for every degree one mapping $j : X \to R$ where R is a color Lie p-superalgebra (with the same G and γ), there exists a unique color Lie p-superalgebra homomorphism $h : L^p \to R^p$ such that $j = h \circ i$. For any nonempty G-graded set, such a color Lie p-superalgebra exists and unique up to isomorphism ([4]). By $L^p(X) = \bigoplus_{g \in G} L^p_g(X)$ we denote the free color Lie p-superalgebra over a field F of characteristic p with the set of free generators X.

Let X be a G-graded set and $F\langle X\rangle$ be the free associative algebra on X over a field of characteristic $p > 0$. Then the restricted enveloping algebra of a free color Lie p-superalgebra generated by X, $u(L^p(X))$, is $F\langle X\rangle$.

Throughout the thesis, all (G, γ)-color Lie superalgebras $L = L_+ \oplus L_-$ will be over a field F of characteristic different from 2 or 3. If L is an ordinary Lie algebra, we do not impose any restrictions on F.

Chapter 2

On Character Formulas for Color Lie Superalgebras

Let Λ be a countable additive abelian semigroup such that every element $\lambda \in \Lambda$ can be written as a sum of other elements only in finitely many ways (the *finiteness condition*). In order to study (color) Lie (p-)superalgebras, we fix a homomorphism $\kappa : \Lambda \to \mathbb{Z}_2 = \{\pm 1\}$. This implies that Λ can be partitioned as

$$\Lambda = \Lambda_+ \cup \Lambda_-,$$

where

$$\Lambda_\pm = \{\lambda \in \Lambda \mid \kappa(\lambda) = \pm 1\}.$$

In this chapter (and also in Chapter 5) we consider Λ-graded color Lie superalgebras $L = \bigoplus_{\lambda\in\Lambda} L_\lambda$, where for each $g \in G_+$ (respectively, $g \in G_-$), we have $L_g = \bigoplus_{\lambda\in\Lambda_+}(L_\lambda \cap L_g)$ (respectively, $L_g = \bigoplus_{\lambda\in\Lambda_-}(L_\lambda \cap L_g)$). The main purpose in this chapter is to derive a dimension formula for the homogeneous subspaces of the free color Lie superalgebras. Also, we will obtain similar results for a certain case of color Lie p-superalgebras.

2.1 Characters of Color Lie Superalgebras

Let $U = \bigoplus_{\lambda\in\Lambda} U_\lambda$ be a Λ-graded space. The character of U is defined by

$$\mathrm{ch}_\Lambda U = \sum_{\lambda\in\Lambda} (\mathrm{dim} U_\lambda) e^\lambda.$$

It is an element in $\mathbb{Q}[[\Lambda]]$, the completion of the semigroup algebra $\mathbb{Q}[\Lambda]$, whose basis consists of symbols e^λ, $\lambda \in \Lambda$ with the multiplication $e^\lambda e^\mu = e^{\lambda+\mu}$ for all $\lambda, \mu \in \Lambda$. Gradings $U = \bigoplus_{\lambda\in\Lambda} U_\lambda$ and $V = \bigoplus_{\lambda\in\Lambda} V_\lambda$ induce gradings on the spaces $U \oplus V$ and $U \otimes V$:

$$(U \oplus V)_\lambda = U_\lambda \oplus V_\lambda;\ (U \otimes V)_\lambda = \sum_{\lambda=\mu+\nu} (U_\mu \otimes V_\nu).$$

By finiteness condition the sum above is finite. The following theorem obviously holds.

Theorem 2.1.1. $\mathrm{ch}_\Lambda(U \oplus V) = \mathrm{ch}_\Lambda U + \mathrm{ch}_\Lambda V$, *and* $\mathrm{ch}_\Lambda(U \otimes V) = \mathrm{ch}_\Lambda U \mathrm{ch}_\Lambda V$.

An important special case is $\Lambda = \mathbb{N}$ where $\mathbb{Q}[[\Lambda]]$ is the algebra of formal power series in one variable (without constant term).

2.1.1 Characters of Color Lie Superalgebras and Their Enveloping Algebras

Let $L = L_+ \oplus L_-$ be a free color Lie superalgebra generated by X where $L_\pm = \bigoplus_{\lambda\in\Lambda_\pm} L_\lambda$, with $\mathrm{dim} L_\lambda < \infty$, $\forall\lambda \in \Lambda$ over F. In [12], the authors considered a particular case of our grading, the grading by $\Lambda = \Gamma \times G$, where Γ is a countable additive abelian semigroup satisfying the following condition: every element $(\alpha, g) \in \Gamma \times G$ can be presented as a sum of other elements only in finitely many ways, and also $\Lambda_+ = \Gamma \times G_+$ and $\Lambda_- = \Gamma \times G_-$. As before, the character of L with respect to Λ-grading is

$$\mathrm{ch}_\Lambda L = \sum_{\lambda \in \Lambda} (\mathrm{dim} L_\lambda) e^\lambda,\ \mathrm{dim} L_\pm = \sum_{\lambda \in \Lambda_\pm} (\mathrm{dim} L_\lambda) e^\lambda.$$

Note that the universal enveloping algebra is graded by $\bar{\Lambda} = \Lambda \cup \{0\}$. We shall give here the proof of the following formula, established in [27], which relates the characters of Lie color superalgebra to that its enveloping algebra.

Lemma 2.1.2. *Let $L = L_+ \oplus L_-$ be a Λ-graded color Lie superalgebra. Then*

$$\mathrm{ch}_{\bar{\Lambda}} U(L) = \frac{\prod_{\lambda \in \Lambda_-} (1 + e^\lambda)^{\mathrm{dim} L_\lambda}}{\prod_{\lambda \in \Lambda_+} (1 - e^\lambda)^{\mathrm{dim} L_\lambda}}.$$

Proof. Let $\{e_\lambda \mid \lambda \in \Lambda\}$ be a basis of the positive part L_+ and $\{f_\mu \mid \mu \in \Lambda\}$ be a basis of the negative part L_-. By Corollary 1.2.19, $U(L)$, as a vector space, is the tensor product of the polynomial algebra $F[\dots, e_\lambda, \dots]$ and the Grassmann algebra $\Lambda[\dots, f_\mu, \dots]$. Now, the result follows from Theorem 2.1.1. □

The superdimension of the homogeneous subspace L_λ is defined by

$$\mathrm{sdim} L_\lambda = \kappa(\lambda) \mathrm{dim} L_\lambda,\ \lambda \in \Lambda.$$

Note that

$$\mathrm{ch}_\Lambda L = \sum_{\lambda \in \Lambda} (\mathrm{sdim} L_\lambda) E^\lambda \in \mathbb{Q}[[\Lambda]],$$

where $E^\lambda = \kappa(\lambda) e^\lambda$. It is convenient to define the following operation, called the *twisted dilation*, on $\mathbb{Q}[[\bar{\Lambda}]]$:

$$^{[m]}\ : \sum_{\lambda \in \bar{\Lambda}} f_\lambda E^\lambda \to \sum_{\lambda \in \bar{\Lambda}} f_\lambda E^{m\lambda},\ m \in \mathbb{N}.$$

We have the following properties (see [26, Lemma 1.1]).

Lemma 2.1.3.

1. $f^{[1]} = f$,

2. *the dilation* $f \mapsto f^{[m]}$ *is an endomorphism of the algebra* $\mathbb{Q}[[\bar{\Lambda}]]$,

3. $(f^{[m]})^{[n]} = (f^{[n]})^{[m]} = f^{[mn]}$ *for all* $m, n \in \mathbb{N}$.

Let us define the following two operators over formal series:

$$\mathcal{E} : \mathbb{Q}[[\Lambda]] \to 1 + \mathbb{Q}[[\Lambda]] \quad : \quad f \mapsto \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} f^{[m]}\right),$$
$$\mathcal{L} : 1 + \mathbb{Q}[[\Lambda]] \to \mathbb{Q}[[\Lambda]] \quad : \quad f \mapsto \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \ln f^{[n]}.$$

The following lemma, proved by Petrogradsky in [26], shows that the operators above are similar to the exponential and logarithm.

Lemma 2.1.4.

1. *The mappings* $\mathcal{E}$ *and* $\mathcal{L}$ *are well defined and mutually inverse,*

2. $\mathcal{E}(f_1 + f_2) = \mathcal{E}(f_1)\mathcal{E}(f_2)$, $f_1, f_2 \in \mathbb{Q}[[\Lambda]]$,

3. $\mathcal{L}(f_1 f_2) = \mathcal{L}(f_1) + \mathcal{L}(f_2)$, $f_1, f_2 \in 1 + \mathbb{Q}[[\Lambda]]$.

Lemma 2.1.2 was used by Petrogradsky to prove the following theorem.

Theorem 2.1.5 ([26]). *Let* $L = \bigoplus_{\lambda \in \Lambda} L_\lambda$ *be a* Λ*-graded color Lie superalgebra and* $U(L)$ *be its enveloping algebra. Then*

1. $\mathrm{ch}_{\bar{\Lambda}} U(L) = \mathcal{E}(\mathrm{ch}_\Lambda L)$,

2. $\mathrm{ch}_\Lambda L = \mathcal{L}(\mathrm{ch}_{\bar{\Lambda}} U(L))$.

2.1.2 G-Characters of Color Lie Superalgebras and Their Enveloping Algebra

Assume that the G-grading on L is determined by the Λ-grading in the sense that: there exists a homomorphism $\kappa_G : \Lambda \to G$ such that $L_g = \bigoplus_{\substack{\lambda\in\Lambda \\ \kappa_G(\lambda)=g}} L_\lambda$. Define $\upsilon : G \to \mathbb{Z}_2 = \{\pm 1\}$ by $\upsilon(g) = 1$ (respectively, -1) if $g \in G_+$ (respectively, $g \in G_-$). In this case, we can define the G-character of $L = \bigoplus_{\lambda\in\Lambda} L_\lambda$, where $\dim L_\lambda < \infty$ for all $\lambda \in \Lambda$, as follows

$$\mathrm{ch}^G_\Lambda L = \sum_{\lambda\in\Lambda} (\dim L_\lambda)\kappa_G(\lambda)e^\lambda \in \mathbb{Q}[G][[\Lambda]],$$

where $\mathbb{Q}[G]$ is the group algebra of G with coefficients in $\mathbb{Q}$ and $\mathbb{Q}[G][[\Lambda]]$ is the completion of the semigroup algebra $\mathbb{Q}[G][\Lambda]$. For $\lambda \in \Lambda$, we set $\mathrm{sdim} L_\lambda = \upsilon(\kappa_G(\lambda))\dim L_\lambda$ and color superdimension $\mathrm{csdim} L_\lambda = \kappa_G(\lambda)\mathrm{sdim} L_\lambda$. Now, the twisted dilation is defined by

$$^{[m]} : \sum_{\lambda\in\bar{\Lambda}} r_\lambda g_\lambda E^\lambda \to \sum_{\lambda\in\bar{\Lambda}} r_\lambda g_\lambda^m E^{m\lambda},\ r_\lambda \in \mathbb{Q}, \lambda g_\lambda \in G,\ \text{and } m \in \mathbb{N},$$

where $E^\lambda = \upsilon(\kappa_G(\lambda))\kappa_G(\lambda)e^\lambda$. The character of L can be also written as

$$\mathrm{ch}^G_\Lambda L = \sum_{\lambda\in\Lambda} (\mathrm{sdim} L_\lambda)E^\lambda.$$

We have the following properties of the twisted dilation operator.

Lemma 2.1.6.

1. *The dilation $f \mapsto f^{[m]}$ is an endomorphism of the algebra $\mathbb{Q}[G][[\bar{\Lambda}]]$,*
2. *$(f^{[m]})^{[n]} = (f^{[n]})^{[m]} = f^{[mn]}$ for all $m, n \in \mathbb{N}$,*
3. *$\left(\sum_{\lambda\in\Lambda} r_\lambda g_\lambda e^\lambda\right)^{[m]} = \sum_{\lambda\in\Lambda} r_\lambda g_\lambda^m (\nu(\kappa_G(\lambda)))^{m+1} e^{m\lambda}$, $r_\lambda \in \mathbb{Q}, g_\lambda \in G$.*

Proof. It is clear that the first two properties hold. Hence it remains to prove the last claim.

$$\begin{aligned}\left(\sum_{\lambda\in\Lambda} r_\lambda g_\lambda e^\lambda\right)^{[m]} &= \left(\sum_{\lambda\in\Lambda} r_\lambda g_\lambda \nu(\kappa_G(\lambda))(\kappa_G(\lambda))^{-1}E^\lambda\right)^{[m]}\\ &= \sum_{\lambda\in\Lambda} r_\lambda \nu(\kappa_G(\lambda)) g_\lambda^m (\kappa_G(\lambda))^{-m}E^{m\lambda}\\ &= \sum_{\lambda\in\Lambda} r_\lambda \nu(\kappa_G(\lambda)) g_\lambda^m (\kappa_G(\lambda))^{-m}\nu(\kappa_G(m\lambda))\kappa_G(m\lambda)e^{m\lambda}\\ &= \sum_{\lambda\in\Lambda} r_\lambda (\nu(\kappa_G(\lambda)))^{m+1} g_\lambda^m e^{m\lambda}.\end{aligned}$$

□

We introduce the following two operators over formal power series:

$$\begin{aligned}\mathcal{E}_G : \mathbb{Q}[G][[\Lambda]] \to 1+\mathbb{Q}[G][[\Lambda]] \quad &: \quad f \mapsto \exp\left(\sum_{m=1}^{\infty}\frac{1}{m}f^{[m]}\right),\\ \mathcal{L}_G : 1+\mathbb{Q}[G][[\Lambda]] \to \mathbb{Q}[G][[\Lambda]] \quad &: \quad f \mapsto \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\ln f^{[n]}.\end{aligned}$$

We can easily prove the following lemma.

Lemma 2.1.7.

1. *The mappings $\mathcal{E}_G$ and $\mathcal{L}_G$ are well defined and are mutually inverse,*

2. $\mathcal{E}_G(f_1+f_2) = \mathcal{E}_G(f_1)\mathcal{E}_G(f_2)$, $f_1, f_2 \in \mathbb{Q}[G][[\Lambda]]$,

3. $\mathcal{L}_G(f_1 f_2) = \mathcal{L}_G(f_1) + \mathcal{L}_G(f_2)$, $f_1, f_2 \in 1+\mathbb{Q}[G][[\Lambda]]$.

Theorem 2.1.8. *Let $L = \bigoplus_{\lambda\in\Lambda} L_\lambda$ be a Λ-graded color Lie superalgebra and $U(L)$ be its enveloping algebra. Then*

1. $\mathrm{ch}_\Lambda^G U(L) = \mathcal{E}_G(\mathrm{ch}_\Lambda^G L)$,

2. $\mathrm{ch}_\Lambda^G L = \mathcal{L}_G(\mathrm{ch}_\Lambda^G U(L))$.

Proof. According to PBW-Theorem, we have

$$\mathrm{ch}_{\Lambda}^{G} U(L) = \prod_{\lambda\in\Lambda} (1 - E^{\lambda})^{-\mathrm{sdim} L_{\lambda}}.$$

Then, we see that

$$\mathrm{ch}_{\Lambda}^{G} U(L) = \exp\left(- \sum_{\lambda\in\Lambda} (\mathrm{sdim} L_{\lambda})(1 - E^{\lambda}) \right).$$

Using $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$, we obtain

$$\mathrm{ch}_{\Lambda}^{G} U(L) = \exp\left(\sum_{\lambda\in\Lambda} (\mathrm{sdim} L_{\lambda}) \sum_{m=1}^{\infty} \frac{E^{m\lambda}}{m} \right).$$

Then,

$$\begin{aligned} \mathrm{ch}_{\Lambda}^{G} U(L) &= \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} \sum_{\lambda\in\Lambda} (\mathrm{sdim} L_{\lambda}) E^{m\lambda} \right) \\ &= \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} \left(\mathrm{ch}^{G}\right)^{[m]} L \right) \\ &= \mathcal{E}_G \mathrm{ch}_{\Lambda}^{G} L. \end{aligned}$$

To prove the second relation, note that

$$\mathrm{ch}_{\Lambda}^{G} L = \mathcal{L}_G \mathcal{E}_G (\mathrm{ch}_{\Lambda}^{G} L) = \mathcal{L}_G (\mathcal{E}_G (\mathrm{ch}_{\Lambda}^{G} L)) = \mathcal{L}(\mathrm{ch}_{\Lambda}^{G} U(L)).$$

2.1.3 Character Formula of Free Color Lie Superalgebras

By a Λ-graded set we mean a disjoint union $X = \bigcup_{\lambda \in \Lambda} X_\lambda$. If in addition, we have $|X_\lambda| < \infty$ for all $\lambda \in \Lambda$, then we define its character

$$\mathrm{ch}_\Lambda X = \sum_{\lambda \in \Lambda} |X_\lambda| \, e^\lambda \in \mathbb{Q}[[\Lambda]].$$

For an element $x \in X_\lambda \subseteq X$, we say Λ-weight of x is λ, and we write $\mathrm{wt}_\Lambda x = \lambda$. We call such a set Λ-*finitely graded* (if $\Lambda = \mathbb{N}$, then we say X is a finitely graded set). For any monomial $y = x_1 \dots x_n$, where $x_j \in X$, we set $\mathrm{wt}_\Lambda y = \mathrm{wt}_\Lambda x_1 + \dots + \mathrm{wt}_\Lambda x_n$. Suppose Y is a set of all monomials (associative, Lie, $\dots$) in X. We denote

$$Y_\lambda = \{ y \in Y \mid \mathrm{wt}_\Lambda y = \lambda \}.$$

Also, the Λ-generating function of Y is

$$\mathrm{ch}_\Lambda (Y) = \sum_{\lambda \in \Lambda} |Y_\lambda| \, e^\lambda \in \mathbb{Q}[[\Lambda]].$$

Lemma 2.1.9 ([26]). *Let $X = \bigcup_{\lambda \in \Lambda} X_\lambda$ be a Λ-graded set with $|X_\lambda| < \infty, \lambda \in \Lambda$, and let $F \langle X \rangle$ be the free associative algebra generated by X. Then*

$$\mathrm{ch}_{\bar{\Lambda}} F \langle X \rangle = \sum_{n=0}^{\infty} (\mathrm{ch}_\Lambda X)^n = \frac{1}{1 - \mathrm{ch}_\Lambda X}.$$

The following theorem was obtained by Petrogradsky in [26] in the setting of Lie superalgebras. Theorems 1.2.17 and 2.1.5 help us to extend it to color Lie superalgebra case.

Theorem 2.1.10. *Let $L = L(X)$ be the free color Lie superalgebra generated by a*

Λ-graded set $X = \bigcup_{\lambda\in\Lambda} X_\lambda$, with $|X_\lambda| < \infty$ for all $\lambda \in \Lambda$. Then

$$\mathrm{ch}_\Lambda L(X) = -\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \ln(1 - \mathrm{ch}_\Lambda^{[n]} X).$$

Proof. The universal enveloping algebra $U(L)$ is isomorphic to the free associative algebra $F\langle X\rangle$ generated by X (Theorem 1.2.17). Thus

$$\mathrm{ch}_{\bar\Lambda} U(L) = \frac{1}{1 - \mathrm{ch}_\Lambda X}.$$

Applying Theorem 2.1.5, we have

$$\mathrm{ch}_\Lambda L = \mathcal{L}(\mathcal{E}(\mathrm{ch}_\Lambda L)) = \mathcal{L}\left(\frac{1}{1 - \mathrm{ch}_\Lambda X}\right) = -\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \ln(1 - \mathrm{ch}_\Lambda^{[n]} X),$$

as desired. □

We are going to discuss several corollaries of the above result.
If $|G| = r$, then we can make any finite set X a Λ-graded set for $\bar\Lambda = \mathbb{N}_0^r$. Write $G = G_+ \cup G_-$ where $G_+ = \{g_1, \ldots, g_k\}$ and $G_- = \{g_{k+1}, \ldots, g_r\}$ (of course, $|G| = |G_+|$ or $|G_+| = |G_-|$) is an abelian group, and L is a free color Lie superalgebra freely generated by a set $X = X_{g_1} \cup \cdots \cup X_{g_r}$ with $|X_{g_i}| = s_i \geq 1$, $i = 1, \ldots, r$. Consider the case $\Lambda = \mathbb{N}_0^r$. We define a weight function

$$\mathrm{wt} : X \to \mathbb{N}_0^r \ : x \mapsto \lambda_i, \text{ for } i = 1, \ldots, r \text{ and } x \in X_{g_i},$$

where $\lambda_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the ith place. We define the homomorphism $\kappa : \mathbb{N}_0^r \to \mathbb{Z}_2 = \{\pm 1\}$ by $\kappa(\lambda_i) = 1$ for $1 \leq i \leq k$ and $\kappa(\lambda_i) = -1$ for $k+1 \leq i \leq r$. We denote $t_i = e^{\lambda_i}$, so the algebra $\mathbb{Q}[[\bar\Lambda]]$ turns into the formal power series ring $\mathbb{Q}[[\mathbf{t}]] = \mathbb{Q}[[t_1, \ldots, t_r]]$. In this case, the character of a Λ-graded Lie superalgebra, L, is the multivariable Hilbert-Poincaré series, $\mathcal{H}(L,\mathbf{t})=\mathcal{H}(L;t_1,\dots,t_r)$, of L. We have the following result.

Corollary 2.1.11. *Suppose that $G=G_+\cup G_-$ is an abelian group, where $G_+=\{g_1,\dots,g_k\}$ and $G_-=\{g_{k+1},\dots,g_r\}$ ($r=k$ or $r=2k$), and L is a free color Lie superalgebra freely generated by a set $X=X_{g_1}\cup\cdots\cup X_{g_r}$ with $|X_{g_i}|=s_i\geq 1$, $i=1,\dots,r$. Then*

$$\mathcal{H}(L;t_1,\dots,t_k,t_{k+1},\dots,t_r)=-\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\ln\left(1-\sum_{i=1}^{k}s_it_i^n+\sum_{j=k+1}^{r}s_j(-t_j)^n\right).$$

Proof. In this case $\operatorname{ch}_\Lambda X=\sum_{i=1}^r s_it_i$, and so $\operatorname{ch}^{[n]}X=\sum_{i=1}^k s_it_i^n-\sum_{j=k+1}^r s_j(-t_j)^n$. The formula follows from Theorem 2.1.10. □

The weight function $\operatorname{wt}:X\to\mathbb{N}_0^r$ defines the multidegree $\alpha=(\alpha_1,\dots,\alpha_r)\in\mathbb{N}_0^r$ for elements of L, and the degree $|\alpha|=\alpha_1+\cdots+\alpha_r$. Also, we write $|\alpha|_+=\alpha_1+\cdots+\alpha_k$ and $|\alpha|_-=\alpha_{k+1}+\cdots+\alpha_r$. By $n|\alpha$ we denote that n divides all components α_i of α. Then we have the following result.

Corollary 2.1.12. *Suppose $G=G_+\cup G_-$ and $L=L(X)$ as in Corollary 2.1.11. Then*

$$\dim L_\alpha=\frac{(-1)^{|\alpha|_-}}{|\alpha|}\sum_{n|\alpha}\mu(n)\frac{\left(\frac{|\alpha|}{n}\right)!(-1)^{\frac{|\alpha|_-}{n}}}{\left(\frac{\alpha_1}{n}\right)!\cdots\left(\frac{\alpha_r}{n}\right)!}s_1^{\frac{\alpha_1}{n}}\dots s_r^{\frac{\alpha_r}{n}}.$$

In particular, if L is a free Lie algebra, then we get the classical Witt's formula.

Proof. We apply the formula for $\mathcal{H}(L;t_1,\dots,t_r)$ from the corollary above. We have

$$\begin{aligned}\mathcal{H}(L;\mathbf{t}) &= -\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\ln\left(1-\sum_{i=1}^{k}s_it_i^n+\sum_{j=k+1}^{r}s_j(-t_j)^n\right)\\ &= \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\sum_{s=1}^{\infty}\frac{(s_1t_1^n+\cdots+s_kt_k^n-s_{k+1}(-t_{k+1})^n-\cdots-s_r(-t_r)^n)^s}{s}.\end{aligned}$$

Applying the multinomial formula, we get

$$\mathcal{H}(L;\mathbf{t}) = \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\sum_{s=1}^{\infty}\frac{1}{s}\sum_{|\beta|=s}\frac{|\beta|!}{\beta_1!\dots\beta_r!}(s_1t_1^n)^{\beta_1}\dots(s_kt_k^n)^{\beta_k}((-s_{k+1})(-t_{k+1}^n)^{\beta_{k+1}}\dots((-s_r)(-t_r^n))^{\beta_r}.$$

Hence,

$$\begin{aligned}\mathcal{H}(L;\mathbf{t}) &= \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\sum_{s=1}^{\infty}\frac{1}{s}\sum_{|\beta|=s}\frac{|\beta|!(-1)^{(n+1)|\beta|_-}}{\beta_1!\dots\beta_r!}s_1^{\beta_1}\dots s_r^{\beta_r}t_1^{n\beta_1}\dots t_r^{n\beta_r}\\ &= \sum_{\alpha\in\mathbb{N}_0^r\setminus\{0\}}\frac{1}{|\alpha|}\sum_{n|\alpha}\mu(n)\frac{\left(\frac{|\alpha|}{n}\right)!\,(-1)^{|\alpha|_-+\frac{|\alpha|_-}{n}}}{\left(\frac{\alpha_1}{n}\right)!\cdots\left(\frac{\alpha_r}{n}\right)!}s_1^{\frac{\alpha_1}{n}}\dots s_r^{\frac{\alpha_r}{n}}t_1^{\alpha_1}\dots t_r^{\alpha_r}.\end{aligned}$$

On the other hand, $\mathcal{H}(L;\mathbf{t}) = \sum_{\alpha\in\mathbb{N}_0^r\setminus\{0\}}\dim L_\alpha\mathbf{t}^\alpha$. Therefore

$$\dim L_\alpha = \frac{(-1)^{|\alpha|_-}}{|\alpha|}\sum_{n|\alpha}\mu(n)\frac{\left(\frac{|\alpha|}{n}\right)!(-1)^{\frac{|\alpha|_-}{n}}}{\left(\frac{\alpha_1}{n}\right)!\cdots\left(\frac{\alpha_r}{n}\right)!}s_1^{\frac{\alpha_1}{n}}\dots s_r^{\frac{\alpha_r}{n}},$$

as desired. □

Let X be a finite generating set of the free color Lie superalgebra $L(X)$ with the weight function

$$\mathrm{wt}: X\to\mathbb{N}^2,$$

defined by

$$x\mapsto(1,0) \text{ if } x\in X_+ \text{ and } x\mapsto(0,1) \text{ if } x\in X_-.$$

If we denote $t_+ = e^{(1,0)}$ and $t_- = e^{(0,1)}$, then the algebra $\mathbb{Q}[[\mathbb{N}_0^2]]$ is the formal power series ring $\mathbb{Q}[[t_+,t_-]]$. We have the following corollary.

Corollary 2.1.13. *Let $L = L(X)$ be a free color Lie superalgebra freely generated by the set $X = X_+\cup X_-$, where $X_+ = \{x_1,\dots,x_k\}$ and $X_- = \{x_{k+1},\dots,x_r\}$. Then*

1. $\mathcal{H}(L;t_+,t_-) = -\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\ln\left(1-kt_+^n+(r-k)(-t_-)^n\right)$.

2. $\mathcal{H}(L,t)=\mathcal{H}(L;t_+,t_-)|_{t_+=t_-=t}=-\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\ln\left(1-(k-(-1)^n(r-k))t^n\right)$.

Corollary 2.1.14. *Let $L=L(X)$ be a free color Lie superalgebra freely generated by the set $X=X_+\cup X_-$, where $X_+=\{x_1,\ldots,x_k\}$ and $X_-=\{x_{k+1},\ldots,x_r\}$. Consider the weight function* $\mathrm{wt}:X\to\mathbb{N};x\mapsto 1$. *Then*

$$\dim L_n=\frac{1}{n}\sum_{m|n}\mu(m)(k-(-1)^m(r-k))^{\frac{n}{m}}.$$

Let us return to the general setting. Let Λ and Γ be two additive abelian semigroups satisfying the finiteness condition, $\kappa:\Lambda\to\mathbb{Z}_2$ and $\kappa':\Gamma\to\mathbb{Z}_2$ are homomorphisms. Suppose that $\varphi:\Lambda\to\Gamma$ is a semigroup homomorphism such that $\kappa=\kappa'\circ\varphi$ and for each $\gamma\in\Gamma$ the set $\{\lambda\in\Lambda\mid\varphi(\lambda)=\gamma\}$ is finite. Let $L=L_+\oplus L_-$ be a free Λ-graded algebra generated by $X=\bigcup_{\lambda\in\Lambda}X_\lambda$. Using the homomorphism φ, we can also regard L as Γ-graded. Then

$$\mathrm{ch}_\Gamma L=\sum_{\gamma\in\Gamma}\dim L_\gamma e^\gamma=\sum_{\gamma\in\Gamma}\left(\sum_{\substack{\lambda\in\Lambda\\ \varphi(\lambda)=\gamma}}\dim L_\lambda\right)e^\gamma. \tag{2.1}$$

Now, we consider the case where $\Lambda=\mathbb{N}\times G$. Such a situation can be obtained from the grading given in Theorem 2.1.11 by taking the grading

$$\mathrm{wt}:X\to\mathbb{N}\times G\ :\ x\mapsto(1,g_i)\text{ for }x\in X_{g_i}.$$

For such gradings, we will use superscripts instead of subscripts. As a result, we have the following corollary.

Corollary 2.1.15. $\dim L^{(n,g)}=\sum_{\substack{\alpha_1+\cdots\alpha_r=n\\ g_1^{\alpha_1}\cdots g_r^{\alpha_r}=g}}\dim L_{(\alpha_1,\ldots,\alpha_r)}$.

Proof. The result is the formula 2.1 applied to

$$\varphi : \mathbb{N}^r \to \mathbb{N} \times G \ : \ \lambda_i \mapsto (1, g_i).$$

□

Example 2.1.16. Consider the free $(\mathbb{Z}_2 \oplus \mathbb{Z}_2, \gamma)$-color Lie superalgebra $L = L(X)$ over the field $F = \mathbb{C}$ where

$$\gamma : (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \times (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \to \mathbb{C}^* \ : \ ((a_1, a_2), (b_1, b_2)) \mapsto (-1)^{(a_1+a_2)(b_1+b_2)}.$$

Hence, $G_+ = \{(0,0), (1,1)\}$ and $G_- = \{(0,1), (1,0)\}$. Let $g_1 = (0,0), g_2 = (1,1), g_3 = (0,1)$, and $g_4 = (1,0)$, and let $|X_{g_1}| = 1$, $|X_{g_2}| = 2$, $|X_{g_3}| = |X_{g_4}| = 1$. According to the above theorem, we have

$$\text{dim} L^{(3,(1,1))} = \text{dim} L_{(0,3,0,0)} + \text{dim} L_{(2,1,0,0)} + \text{dim} L_{(1,0,1,1)}.$$

Now, if we apply the formula given in Corollary 2.1.12, we have

$$\text{dim} L_{(0,3,0,0)} = \frac{(-1)^0}{3} \left(\mu(1) \frac{(3!)(-1)^0}{3!} 2^3 + \mu(3) \frac{1!(-1)^0}{1!} 2^1 \right) = 2.$$

Similarly, we obtain $\text{dim} L_{(2,1,0,0)} = 2$, and $\text{dim} L_{(1,0,1,1)} = 2$. Hence $\text{dim} L^{(3,(1,1))} = 2 + 2 + 2 = 6$.

2.2 Characters of Free Restricted Color Lie Superalgebras

Let $L = L_+ \oplus L_-$ be a free color restricted Lie superalgebra generated by X where $L_\pm = \bigoplus_{\lambda \in \Lambda_\pm} L_\lambda$ with $\dim L_\lambda < \infty\ \forall \lambda \in \Lambda$ over a field F. Now we deduce the following formula which relates the character of Lie color p-superalgebra to that of its restricted enveloping algebra.

Lemma 2.2.1. *Let $L = L_+ \oplus L_-$ be a Λ-graded color Lie p-superalgebra. Then*

$$\mathrm{ch}_{\bar{\Lambda}} u(L) = \prod_{\lambda \in \Lambda_-} (1 + e^{\lambda})^{\dim L_\lambda} \prod_{\lambda \in \Lambda_+} (1 + \cdots + e^{(p-1)\lambda})^{\dim L_\lambda}.$$

Proof. As in the case of Lemma 2.1.2, one has to use PBW-theorem for color Lie p-superalgebras. The details are omitted. □

In the remaining part of this section we consider a Λ-graded color Lie p-superalgebra satisfying $G = G_+$; note that the ordinary restricted Lie algebra is a particular case. (Recall that such color Lie p-superalgebras are called color Lie p-algebras.)

Petrogradsky ([28]) has defined functions $1_p, \mu_p : \mathbb{N} \to \mathbb{N}$ by:

$$1_p(n) = \begin{cases} 1, & \text{if } (p, n) = 1 \\ 1 - p, & \text{if } (p, n) = p, \end{cases}$$

and

$$\mu_p(n) = \begin{cases} \mu(n), & \text{if } (p, n) = 1 \\ \mu(m)(p^s - p^{s-1}), & \text{if } n = mp^s, (p, m) = 1, s \geq 1. \end{cases}$$

Recall that a function $f : \mathbb{N} \to \mathbb{N}$ is multiplicative if $f(nm) = f(n)f(m)$ for any coprime n, m. One can easily show that 1_p and μ_p are multiplicative functions. Also we have the following property.

Lemma 2.2.2 ([28]). $\sum_{ab=n} 1_p(b)\mu_p(a) = 0$ *for all* $n > 1$.

Proof. We fill the details of the proof in [28]. First, we assume n is not divisible by p. Let $a, b \in \mathbb{N}$ with $ab = n$. Then a and b are not divisible by p. Hence $1_p(b) = 1$ and $\mu_p(a) = \mu(a)$. Now, the statement follows from the property of the Möbius function. Next, we suppose n is divisible by p. Write $n = n'p^k$, $k \geq 1$, where n' is not divisible by p. For all $a, b \in \mathbb{N}$ with $ab = n$, we write accordingly $a = a'p^r$ and $b = b'p^s$ where $r + s = k$. Then

$$\begin{aligned}
\sum_{ab=n} 1_p(b)\mu_p(a) &= \sum_{a'b'=n'} 1_p(b')\mu_p(a') \sum_{r+s=k} 1_p(p^s)\mu_p(p^r) \\
&= \sum_{a'b'=n'} \mu(a') \left(1(p^k - p^{k-1}) + (1-p)(p^{k-1} - p^{k-2}) + \cdots + (1-p)1\right) \\
&= \sum_{a'b'=n'} \mu(a') \left((p^k - p^{k-1}) + (1-p)(p^{k-1} - 1) + (1-p)\right) \\
&= 0,
\end{aligned}$$

where in the first line we used the fact that 1_p and μ_p are multiplicative functions. □

We introduce the following two operators on formal series, which were defined by Petrogradsky [28] in the case of $\bar{\Lambda} = \mathbb{N}_0^m$.

$$\begin{aligned}
\mathcal{E}_p : \mathbb{Q}[[\Lambda]] \to 1 + \mathbb{Q}[[\Lambda]] \quad &: \quad f \mapsto \exp\left(\sum_{m=1}^{\infty} \frac{1_p(m)}{m} f^{[m]}\right), \\
\mathcal{L}_p : 1 + \mathbb{Q}[[\Lambda]] \to \mathbb{Q}[[\Lambda]] \quad &: \quad f \mapsto \sum_{n=1}^{\infty} \frac{\mu_p(n)}{n} \ln f^{[n]}.
\end{aligned}$$

Now we show that these operators are similar to the exponential and logarithm.

Lemma 2.2.3.

1. *The maps* $\mathcal{E}_p$ *and* $\mathcal{L}_p$ *are well defined and mutually inverse,*

2. $\mathcal{E}_p(f_1+f_2)=\mathcal{E}_p(f_1)\mathcal{E}_p(f_2), f_1, f_2 \in \mathbb{Q}[[\Lambda]]$,

3. $\mathcal{L}_p(f_1f_2)=\mathcal{L}_p(f_1)+\mathcal{L}_p(f_2), f_1, f_2 \in 1+\mathbb{Q}[[\Lambda]]$.

Proof. It follows from the finiteness condition of Λ that $\mathcal{E}_p$ and $\mathcal{L}_p$ are well defined. Let $f \in \mathbb{Q}[[\Lambda]]$. Then

$$\begin{aligned}
\mathcal{L}_p(\mathcal{E}_p(f)) &= \mathcal{L}_p\left(\exp\left(\sum_{m=1}^{\infty}\frac{1_p(m)}{m}f^{[m]}\right)\right) \text{ (definition of } \mathcal{E}_p)\\
&= \sum_{n=1}^{\infty}\frac{\mu_p(n)}{n}\ln\left(\exp\left(\sum_{m=1}^{\infty}\frac{1_p(m)}{m}f^{[m]}\right)\right)^{[n]} \text{ (definition of } \mathcal{L}_p)\\
&= \sum_{n=1}^{\infty}\frac{\mu_p(n)}{n}\ln\left(\prod_{m=1}^{\infty}\exp\left(\frac{1_p(m)}{m}f^{[m]}\right)\right)^{[n]}\\
&= \sum_{n=1}^{\infty}\frac{\mu_p(n)}{n}\ln\left(\prod_{m=1}^{\infty}\exp\left(\frac{1_p(m)}{m}f^{[m]}\right)^{[n]}\right) \text{ (Lemma 2.1.3)}\\
&= \sum_{n=1}^{\infty}\frac{\mu_p(n)}{n}\sum_{m=1}^{\infty}\frac{1_p(m)}{m}(f^{[m]})^{[n]}\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{f^{[mn]}}{mn}1_p(m)\mu_p(n) \text{ (Lemma 2.1.3)}\\
&= \sum_{k=1}^{\infty}\frac{f^{[k]}}{k}\sum_{mn=k}1_p(m)\mu_p(n)\\
&= f^{[1]} \text{ (Lemma 2.2.2)}\\
&= f.
\end{aligned}$$

In a similar way, we can prove $\mathcal{E}_p(\mathcal{L}_p(f))=f, f \in 1+\mathbb{Q}[[\Lambda]]$. The relations (2) and (3) are clear. □

Theorem 2.2.4. *Let $L=\bigoplus_{\lambda\in\Lambda}L_\lambda$ be a Λ-graded color Lie p-algebra ($G=G_+$) and $u(L)$ be its restricted enveloping algebra. Then*

1. $\mathrm{ch}_{\bar{\Lambda}}u(L)=\mathcal{E}_p(\mathrm{ch}_\Lambda L)$,

2. $\mathrm{ch}_\Lambda L=\mathcal{L}_p(\mathrm{ch}_{\bar{\Lambda}}u(L))$.

Proof.

1. By Lemma 2.2.1, we have

$$\mathrm{ch}_{\bar{\Lambda}} u(L) = \prod_{\lambda \in \Lambda} (1 + e^{\lambda} + \ldots + e^{(p-1)\lambda})^{\dim L_{\lambda}}.$$

Now, as $(1 - e^{p\lambda}) = (1 - e^{\lambda})(1 + e^{\lambda} + \ldots + e^{(p-1)\lambda})$, $\mathrm{ch}_{\bar{\Lambda}} u(L)$ can be written as:

$$\mathrm{ch}_{\bar{\Lambda}} u(L) = \prod_{\lambda \in \Lambda} \left(\frac{1 - e^{p\lambda}}{1 - e^{\lambda}} \right)^{\dim L_{\lambda}}.$$

Therefore,

$$\mathrm{ch}_{\bar{\Lambda}} u(L) = \exp\left(\sum_{\lambda \in \Lambda} \dim L_{\lambda} \left((-\ln(1 - e^{\lambda}) + \ln(1 - e^{p\lambda})) \right) \right).$$

Using $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$, we obtain

$$\mathrm{ch}_{\bar{\Lambda}} u(L) = \exp\left(\sum_{\lambda \in \Lambda} \dim L_{\lambda} \left(\sum_{n=1}^{\infty} \frac{e^{n\lambda}}{n} - \sum_{n=1}^{\infty} \frac{e^{pn\lambda}}{n} \right) \right).$$

Then, we see that

$$\begin{aligned}
\mathrm{ch}_{\bar{\Lambda}} u(L) &= \exp\left(\sum_{\lambda \in \Lambda} \dim L_{\lambda} \left(\sum_{n=1}^{\infty} \frac{e^{n\lambda}}{n} - \sum_{n=1}^{\infty} \frac{e^{pn\lambda}}{n} \right) \right) \\
&= \exp\left(\sum_{\lambda \in \Lambda} \dim L_{\lambda} \left(\sum_{n=1, p \nmid n}^{\infty} \frac{e^{n\lambda}}{n} + \sum_{n=1}^{\infty} \left(\frac{e^{np\lambda}}{np} - \frac{e^{np\lambda}}{n} \right) \right) \right) \\
&= \exp\left(\sum_{\lambda \in \Lambda} \dim L_{\lambda} \left(\sum_{n=1, p \nmid n}^{\infty} \frac{e^{n\lambda}}{n} + \sum_{n=1}^{\infty} \frac{e^{np\lambda} - pe^{np\lambda}}{np} \right) \right) \\
&= \exp\left(\sum_{\lambda \in \Lambda} \dim L_{\lambda} \sum_{n=1}^{\infty} e^{n\lambda} \frac{1_p(n)}{n} \right) \\
&= \exp\left(\sum_{n=1}^{\infty} \frac{1_p(n)}{n} \sum_{\lambda \in \Lambda} \dim L_{\lambda} e^{n\lambda} \right)
\end{aligned}$$

$$
\begin{aligned}
&= \exp\left(\sum_{n=1}^{\infty} \frac{1_p(n)}{n} (\mathrm{ch}_\Lambda L)^{[n]}\right) \\
&= \mathcal{E}_p(\mathrm{ch}_\Lambda L).
\end{aligned}
$$

2. This relation follows directly from Lemma 2.2.3 and (1):

$$\mathrm{ch}_\Lambda L = \mathcal{L}_p \mathcal{E}_p(\mathrm{ch}_\Lambda L) = \mathcal{L}_p(\mathcal{E}_p(\mathrm{ch}_\Lambda L)) = \mathcal{L}_p(\mathrm{ch}_{\bar{\Lambda}} u(L)).$$

□

Remark 2.2.5. One can also extend the definition of $\mathcal{E}_p$ to the general case $\Lambda = \Lambda_+ \cup \Lambda_-$ as follows:

$$\mathcal{E}_p : \mathbb{Q}[[\Lambda]] \to 1 + \mathbb{Q}[[\Lambda]] \ : \ f = f_+ + f_- \mapsto \exp\left(\sum_{m=1}^{\infty} \frac{1_p(m)}{m} f_+^{[m]}\right) \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} f_-^{[n]}\right).$$

Again, $\mathcal{E}_p$ is a well defined operator. Also, it is easy to see that

1. $\mathcal{E}_p(f_1 + f_2) = \mathcal{E}_p(f_1)\mathcal{E}_p(f_2)$, $f_1, f_2 \in \mathbb{Q}[[\Lambda]]$,
2. $\mathrm{ch}_{\bar{\Lambda}} u(L) = \mathcal{E}_p(\mathrm{ch}_\Lambda L)$.

Theorem 2.2.6. *Let $L = L(X)$ be the free color Lie p-algebra ($G = G_+$) generated by a Λ-graded set $X = \bigcup_{\lambda \in \Lambda} X_\lambda$, with $|X_\lambda| < \infty$ for all $\lambda \in \Lambda = \Lambda_+$. Then*

$$\mathrm{ch}_\Lambda L(X) = -\sum_{n=1}^{\infty} \frac{\mu_p(n)}{n} \ln(1 - \mathrm{ch}_\Lambda^{[n]} X).$$

Proof. For the restricted color Lie superalgebra $L = L(X)$, we denote the restricted enveloping algebra of L by $u(L)$. Let $F\langle X\rangle$ be the free associative algebra on X. It is well known (see e.g., [4]) that $u(L(X))$ is isomorphic to $F\langle X\rangle$. Thus,

$$\mathrm{ch}_{\bar{\Lambda}} u(L) = \frac{1}{1 - \mathrm{ch}_\Lambda X}.$$

Using Theorem 2.2.4, we get

$$\mathrm{ch}_\Lambda L = \mathcal{L}_p \mathrm{ch}_{\bar{\Lambda}} u(L) = \mathcal{L}_p \left(\frac{1}{1 - \mathrm{ch}_\Lambda X} \right) = -\sum_{n=1}^{\infty} \frac{\mu_p(n)}{n} \ln(1 - \mathrm{ch}_\Lambda^{[n]} X).$$

□

Corollary 2.2.7. *Let $L = L(X)$ be the free color Lie p-algebra generated by at most countable set $X = \{x_i \mid i \in I\}$. Then*

$$\mathcal{H}(L, t_i \mid i \in I) = -\sum_{n=0}^{\infty} \frac{\mu_p(n)}{n} \ln \left(1 - \sum_{i \in I} t_i^n \right).$$

In particular, if L is generated by $X = \{x_1, \ldots, x_r\}$, then

$$\mathcal{H}(L; t_1, \ldots, t_r) = -\sum_{n=1}^{\infty} \frac{\mu_p(n)}{n} \ln(1 - t_1^n - \cdots - t_r^n).$$

Consider the special case $\Lambda = \mathbb{N}$ and $\mathrm{wt} : X \to \mathbb{N} : x \mapsto 1$. Then we have the following result.

Corollary 2.2.8. *Let L be a free color Lie p-algebra freely generated by $X = \{x_1, \ldots, x_r\}$. Then*

$$\mathcal{H}(L, t) = -\sum_{n=1}^{\infty} \frac{\mu_p(n)}{n} \ln(1 - rt^n).$$

Suppose that L is a free color Lie p-superalgebra generated by $X = \{x_1, \ldots, x_r\}$, and is multihomogeneous with respect to the set X. For elements of L we introduce the multidegree $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}_0^r$, and the degree $|\alpha| = \alpha_1 + \cdots + \alpha_r$. We have the following analogue of the Witt formula for the dimension of the multihomogeneous components of L.

Corollary 2.2.9. *Suppose L is a free color Lie p-algebra generated by $X = \{x_1, \ldots, x_r\}$.*

Then

$$\begin{aligned}
\dim L_n &= \frac{1}{n}\sum_{m|n}\mu_p(m)r^{\frac{n}{m}},\\
\dim L_\alpha &= \frac{1}{|\alpha|}\sum_{m|\alpha}\mu_p(m)\frac{(|\alpha|/m)!}{(\alpha_1/m)!\cdots(\alpha_r/m)!}.
\end{aligned}$$

In particular, if L is the ordinary free Lie p-algebra, then we get Petrogradsky's formulas ([28]).

The following theorem was originally proved by Petrogradsky in [27, 26] for Lie superalgebras case.

Theorem 2.2.10. *Let $L = L(X) = \bigoplus_{n=1}^{\infty} L_n$ be a free color Lie p-algebra ($G = G_+$) generated by a Λ-graded set $X = \bigcup_{\lambda\in\Lambda} X_\lambda$. Then*

$$\mathrm{ch}_\Lambda L_n = \frac{1}{n}\sum_{k|n}\mu_p(k)(\mathrm{ch}_\Lambda^{[k]}X)^{\frac{n}{k}}.$$

Proof. We consider the new semigroup

$$\Lambda' = \Lambda\times\mathbb{N}.$$

Define a weight function

$$\mathrm{wt} : X \to \Lambda' :\ x \mapsto (\lambda, 1),\ x\in X_\lambda.$$

Then, we consider L as a Λ'-graded. If we denote $t = e^{(0,1)}$ and $e^\lambda = e^{(\lambda,0)}$, then

$$\begin{aligned}
\mathrm{ch}_{\Lambda'}X &= \sum_{(\lambda,i)\in\Lambda'}|X_{(\lambda,i)}|e^{(\lambda,i)}\\
&= \sum_{\lambda\in\Lambda}|X_{(\lambda,1)}|e^{(\lambda,1)}
\end{aligned}$$

$$\begin{aligned} &= \sum_{\lambda\in\Lambda} |X_{(\lambda,1)}| e^{(\lambda,0)} e^{(0,1)} \\ &= t\, \mathrm{ch}_\Lambda X. \end{aligned}$$

Using Theorem 2.2.6 and the operator of dilation, we see that

$$\begin{aligned} \mathrm{ch}_{\Lambda'} L &= -\sum_{k=1}^{\infty} \frac{\mu_p(k)}{k} \ln\left(1 - \mathrm{ch}_{\Lambda'}^{[k]} X\right) \\ &= -\sum_{k=1}^{\infty} \frac{\mu_p(k)}{k} \ln\left(1 - t^k \mathrm{ch}_{\Lambda}^{[k]} X\right). \end{aligned}$$

By the expansion of the logarithm, we have

$$\mathrm{ch}_{\Lambda'} L = \sum_{k=1}^{\infty} \frac{\mu_p(k)}{k} \sum_{m=1}^{\infty} \frac{t^{mk} (\mathrm{ch}_{\Lambda}^{[k]} X)^m}{m}.$$

Therefore,

$$\mathrm{ch}_{\Lambda'} L = \sum_{n=1}^{\infty} \frac{t^n}{n} \sum_{k|n} \mu_p(k) \left(\mathrm{ch}_{\Lambda}^{[k]} X\right)^{\frac{n}{k}}.$$

On the other hand, it is clear that

$$\mathrm{ch}_{\Lambda'} L = \sum_{n=1}^{\infty} \mathrm{ch}_\Lambda L_n t^n.$$

Hence,

$$\mathrm{ch}_\Lambda L_n = \frac{1}{n} \sum_{k|n} \mu_p(k) (\mathrm{ch}_{\Lambda}^{[k]} X)^{\frac{n}{k}},$$

as desired. □

Suppose that $G = G_+ = \{g_1, \ldots, g_r\}$ is an abelian group, and L is a free color Lie p-superalgebra freely generated by a set $X = X_{g_1} \cup \cdots \cup X_{g_r}$ with $|X_{g_i}| = s_i \geq 1$ $i = 1, \ldots, r$. We define a weight function

$$\mathrm{wt} : X \to \mathbb{N}^r \ : x \mapsto \lambda_i, \text{ for } i = 1, \ldots, r \text{ and } x \in X_{g_i},$$

where $\lambda_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the ith place. Again, we denote $t_i = e^{\lambda_i}$, and so we have the following result.

Theorem 2.2.11. *Suppose that $G = G_+ = \{g_1, \ldots, g_r\}$ is an abelian group, and L is a free color Lie p-algebra freely generated by a set $X = X_{g_1} \cup \cdots \cup X_{g_r}$ with $|X_{g_i}| = s_i \geq 1$ $i = 1, \ldots, r$. Then*

1. $\mathcal{H}(L; t_1, \ldots, t_r) = -\sum_{n=1}^{\infty} \frac{\mu_p(n)}{n} \ln\left(1 - \sum_{i=1}^{r} s_i t_i^n\right)$,
2. $\dim L_\alpha = \frac{1}{|\alpha|} \sum_{n|\alpha} \mu_p(n) \frac{\left(\frac{|\alpha|}{n}\right)!}{\left(\frac{\alpha_1}{n}\right)! \cdots \left(\frac{\alpha_r}{n}\right)!} s_1^{\frac{\alpha_1}{n}} \ldots s_r^{\frac{\alpha_r}{n}}$.
3. $\dim L^{(n,g)} = \sum_{\substack{\alpha_1 + \cdots \alpha_r = n \\ g_1^{\alpha_1} \ldots g_r^{\alpha_r} = g}} \dim L_{(\alpha_1, \ldots, \alpha_r)}$.

Chapter 3

Schreier Formula for Free (Restricted) Color Lie Superalgebras in Terms of Power Series

A well-known theorem, due to Nielsen and Schreier, states that every subgroup of a free group is again free. Moreover, if F is a free group of rank n and K is a subgroup in F of finite index t, then the rank of K is given by

$$\mathrm{rank}(K) = t\,(n-1)+1.$$

Kukin [14] obtained the analogues of these results for restricted Lie algebras: every p-subalgebra of a free Lie p-algebra is again free and, moreover, if L is a free Lie p-algebra of rank n and K is a p-subalgebra in L of finite codimension t, then the rank of K is given by

$$\operatorname{rank}(K) = p^t\,(n-1) + 1.$$

In this chapter we aim to establish an analogue of Schreier formula for subalgebras of free (G, γ)-color Lie (p-)superalgebras in terms of formal power series. Throughout this work, the term *subalgebra* means a G-homogeneous subalgebra; all generators will also be G-homogeneous. In order to define Hilbert-Poincaré series, the free generators will be assigned degrees in $\mathbb{N}$, so the free color Lie (p-)superalgebra will also be $\mathbb{N}$-graded. The term *homogeneous* will refer to this grading.

3.1 The Case of Restricted Color Lie Superalgebra

Let G be an abelian group with $|G| < \infty$. Mikhalev obtained the following theorem.

Theorem 3.1.1 ([22]). *A subalgebra of a free color Lie p-superalgebra is free.*

Mikhalev also proved the following analogue of Schreier's formula for free color Lie p-superalgebras.

Theorem 3.1.2 ([22]). *Let $L = L^p(X) = L_+ \oplus L_-$ be a free color Lie p-superalgebra on X where $\operatorname{rank}(L) = |X| < \infty$, and let $K = K_+ \oplus K_-$ be a p-subalgebra of L with $\dim(L_+/K_+) = t < \infty$ and $\dim(L_-/K_-) = s < \infty$. Then*

$$\operatorname{rank}(K) = 2^s p^t(|X| - 1) + 1.$$

However, there is no straightforward analogue of Schreier index formula for the free color Lie superalgebras in general. To obtain the desired generalization, we can replace numbers with power series. Recall that a *finitely graded set X* is a countable set with a weight function $\mathrm{wt} : X \to \mathbb{N}$ such that the subsets $X_i := \{x \in X \mid \mathrm{wt}(x) = i\}$

are finite for all $i \in \mathbb{N}$. Let $G = \{g_1, \ldots, g_r\}$ and let L be a free color Lie superalgebra freely generated by a finitely graded set $X = \bigcup_{g\in G} X_g$. We define the color series for L as follows:

$$\mathcal{H}_X(L; t_{g_1}, \ldots, t_{g_r}) = \sum_{j=1}^{\infty} \sum_{i=1}^{r} a_j^{g_i} t_{g_i}^j \in \mathbb{Q}[[t_{g_1}]] \oplus \cdots \oplus \mathbb{Q}[[t_{g_r}]],$$

where

$$a_j^{g_i} = \dim \{u \in L_{g_i} \mid \mathrm{wt}(u) = j\},\ j \in \mathbb{N}.$$

This is analogous to the G-character of L as in Subsection 2.1.2 for $\Lambda = \mathbb{N} \times G$, $\kappa_G : (n, g) \mapsto g$, if we identify $\mathbb{Q}[G][[t]]$ with $\bigoplus_{i=1}^{r} \mathbb{Q}[[t_{g_i}]]$ by means of the mapping $g_i t \mapsto t_{g_i}$. Let us assume that $G_+ = \{g_1, \ldots, g_k\}$ and $G_- = \{g_{k+1}, \ldots, g_r\}$ (of course, in general $|G| = |G_+|$ or $|G_+| = |G_-|$). Now, we define an operator $\mathcal{E}_p$ on the color series $\phi(t_{g_1}, \ldots, t_{g_r}) = \sum_{j=1}^{\infty} \sum_{i=1}^{r} a_j^{g_i} t_{g_i}^j$

$$\mathcal{E}_p :\ \sum_{g\in G} \sum_{j=1}^{\infty} a_j^g t_g^j \mapsto \prod_{j=1}^{\infty} \left(1 + t^j\right)^{\zeta_{i,-}} \prod_{j=1}^{\infty} \left(1 + t^j + t^{2j} + \ldots + t^{(p-1)j}\right)^{\zeta_{i,+}} \in \mathbb{Q}[[t]],$$

where $\zeta_{i,+} = \sum_{i=1}^{k} a_j^{g_i}$ and $\zeta_{i,-} = \sum_{i=k+1}^{r} a_j^{g_i}$. Then it is easy to verify that the operator $\mathcal{E}_p$ is multiplicative (that is $\mathcal{E}_p(0) = 1$ and $\mathcal{E}_p(\varphi_1(t_{g_1}, \ldots, t_{g_r}) + \varphi_2(t_{g_1}, \ldots, t_{g_r})) = \mathcal{E}_p(\varphi_1(t_{g_1}, \ldots, t_{g_r}))\mathcal{E}_p(\varphi_2(t_{g_1}, \ldots, t_{g_r})))$.

Remark 3.1.3. It can be shown that $\mathcal{E}_p$ defined here is the composition of the operator defined in Remark 2.2.5 (for $\Lambda = \mathbb{N} \times G$) and the homomorphism from $\mathbb{Q}[[\mathbb{N}_0 \times G]]$ to $\mathbb{Q}[[\mathbb{N}_0]] = \mathbb{Q}[[t]]$ defined by $(n, g) \mapsto n$.

Let $L^p(X)$ be a free Lie p-algebra ($G = \{1\}$) generated by a finitely graded set X. In this case,

$$\mathcal{E}_p : \sum_{n=0}^{\infty} a_n t^n \mapsto \prod_{n=1}^{\infty} (1 + t^n + \cdots + t^{(p-1)n})^{a_n}.$$

Petrogradsky (see [25]) introduced an analogue of Schreier's formula in terms of formal power series for free Lie p-algebras. He proved that if L is a free Lie p-algebra generated by a finitely graded set X, and K is a subalgebra of L, then there is a set of free generators Y of K such that

$$\mathcal{H}(Y,t) = (\mathcal{H}(X,t) - 1)\,\mathcal{E}_p\left(\mathcal{H}\left(L/K,t\right)\right) + 1.$$

Now we are going to extend Petrogradsky's formula to the case of color Lie p-superalgebras. We denote by $ad'y$ the operator $L \to L: \ x \mapsto [x,y]$, so $x\,(ad'y)^m = [\ldots[x,\underbrace{y],\ldots,y]}_{m-\text{times}}$. We will require the following theorem.

Theorem 3.1.4. *Let $L^p(X)$ be a free color Lie p-superalgebra freely generated by a set X.*

1. *If $z \in X_+$, then the set*

$$\left\{z^p, y\,(ad'z)^m \ \mid y \in X \setminus \{z\}\,, m = 0, 1, \ldots, p-1\right\}$$

is a a free generating set of a color Lie p-subsuperalgebra of $L^p(X)$.

2. *If $z \in X_-$, then the set*

$$\{x, [x,z] \mid x \in X \setminus \{z\}\} \cup \{[z,z]\}$$

is a free generating set of a color Lie p-subsuperalgebra of $L^p(X)$.

Proof. See [4, page 71]. ☐

Lemma 3.1.5. *Let $G = \{g_1, \ldots, g_r\}$ be an abelian group, $L = L^p(X) = \bigoplus_{g\in G} L_g$ be a free color Lie p-superalgebra generated by a finitely graded set X, and $K \subseteq L$ be a* *homogeneous p-subalgebra such that* $L = K \oplus \langle Z \rangle_F$, *where* $Z \subseteq X$, $|Z| < \infty$ *and* $\langle Z \rangle_F$ *denotes the* F*-vector space spanned by* Z. *If* Y *is a free generating set for* K, *then*

$$\mathcal{H}(Y,t) = (\mathcal{H}(X,t) - 1)\, \mathcal{E}_p \left(\mathcal{H}\left(L/K; t_{g_1}, \ldots, t_{g_r}\right)\right) + 1.$$

Proof. We use the induction on $|Z|$. Let $Z = \{z\}$, $\mathrm{wt}(z) = a$. We have two cases

- $d(z) = g_k \in G_+$ $(1 \le k \le r)$: then $\mathcal{H}\left(L/K\right) = t_{g_k}^a$, and so $\mathcal{E}_p(\mathcal{H}(L/K, t_{g_1}, t_{g_2}, \ldots, t_{g_r})) = 1 + t^a + \cdots + t^{(p-1)a}$. By the theorem above K is freely generated by the set

$$Y = \left\{z^p, y\,(ad'z)^m \mid y \in X \setminus \{z\}\,, m = 0, 1, \ldots, p-1\right\}.$$

  Also,

$$\begin{aligned}
\mathcal{H}(Y) &= \mathcal{H}(X \setminus \{z\})(1 + t^a + t^{2a} + \ldots + t^{(p-1)a}) + t^{pa} \\
&= (\mathcal{H}(X) - 1 + (1 - t^a))(1 + t^a + t^{2a} + \ldots + t^{(p-1)a}) + t^{pa} \\
&= (\mathcal{H}(X) - 1)(1 + t^a + t^{2a} + \ldots + t^{(p-1)a}) \\
&\quad + (1 - t^a)(1 + t^a + t^{2a} + \ldots + t^{(p-1)a}) + t^{pa} \\
&= (\mathcal{H}(X) - 1)(1 + t^a + t^{2a} + \ldots + t^{(p-1)a}) + 1 - t^{pa} + t^{pa} \\
&= (\mathcal{H}(X) - 1)(1 + t^a + t^{2a} + \ldots + t^{(p-1)a}) + 1 \\
&= (\mathcal{H}(X) - 1)\mathcal{E}_p(\mathcal{H}(L/K)) + 1.
\end{aligned}$$

- $d(z) = g_k \in G_-$ $(1 \le k \le r)$: then $\mathcal{E}_p\left(\mathcal{H}\left(L/K, t_{g_1}, t_{g_2}, \ldots, t_{g_r}\right)\right) = 1 + t^a$. Also K is freely generated by the following set

$$Y = \{x, [x, z] \mid x \in X \setminus \{z\}\} \cup \{[z, z]\}\,.$$

Now,

$$\begin{aligned}\mathcal{H}(Y,t) &= \mathcal{H}\left(X\setminus\{z\}\right)(1+t^a)+t^{2a}\\ &= \left(\mathcal{H}(X,t)-1+(1-t^a)\right)(1+t^a)+t^{2a}\\ &= 1+\left(\mathcal{H}(X,t)-1\right)\mathcal{E}_p\left(\mathcal{H}\left(L/K;t_{g_1},\ldots,t_{g_r}\right)\right).\end{aligned}$$

Now suppose $|Z|>1$. Pick out $z\in Z$, and let $Z'=Z\setminus\{z\}$. Then $K'=K\oplus\langle Z'\rangle_F$ is a subalgebra, and so it is free. Let Y' be a basis of K'. Now apply the inductive hypothesis to the inclusions $K\subset K'$, $K'\subset L$, to get

$$\mathcal{H}(Y',t)-1=\left(\mathcal{H}(X,t)-1\right)\mathcal{E}_p\left(L/K';t_{g_1},\ldots,t_{g_r}\right),$$

and

$$\mathcal{H}(Y,t)-1=\left(\mathcal{H}(Y',t)-1\right)\mathcal{E}_p\left(K'/K;t_{g_1},\ldots,t_{g_r}\right).$$

Therefore,

$$\mathcal{H}(Y,t)-1=\left(\mathcal{H}(X,t)-1\right)\mathcal{E}_p\left(L/K';t_{g_1},\ldots,t_{g_r}\right)\mathcal{E}_p\left(K'/K;t_{g_1},\ldots,t_{g_r}\right).$$

This implies that

$$\mathcal{H}(Y,t)-1=\left(\mathcal{H}(X,t)-1\right)\mathcal{E}_p\left(\mathcal{H}\left(L/K',t_{g_1},\ldots,t_{g_r}\right)+\mathcal{H}\left(K'/K,t_{g_1},\ldots,t_{g_r}\right)\right),$$

and so

$$\mathcal{H}(Y,t)=\left(\mathcal{H}(X,t)-1\right)\mathcal{E}_p\left(\mathcal{H}\left(L/K;t_{g_1},t_{g_2},\ldots,t_{g_r}\right)\right)+1.$$

□

Lemma 3.1.6. *Let $G=\{g_1,\ldots,g_r\}$ be an abelian group, L be a free color Lie* *p-superalgebra freely generated by a finitely graded set* $X = \bigcup_{g \in G} X_g$. *Suppose that* $K = \bigoplus_{i=1}^{\infty} K_i$ *is a homogeneous p-subalgebra of* L. *Then we can find a homogeneous set of free generators* Y *for* K *such that*

$$\mathcal{H}(Y,t) = (\mathcal{H}(X,t) - 1)\mathcal{E}_p\left(\mathcal{H}\left(L/K; t_{g_1}, \ldots, t_{g_r}\right)\right) + 1.$$

Proof. We break the proof into three steps: 1) we construct a homogeneous free generating set Y for K using a recursive procedure, 2) for each step of the recursion, we express the Hilbert series of the free generating set in terms of its predecessor, and 3) derive the desired expression for $\mathcal{H}(Y,t)$.

Step 1: We represent K as the intersection of the decreasing sequence

$$L = K^{(0)} \supseteq K^{(1)} \supseteq K^{(2)} \supseteq \cdots \supseteq K^{(s)} \supseteq \cdots \supseteq K^{(\infty)} = K,$$

where $K^{(s)} = K_1 \oplus \cdots \oplus K_s \oplus L_{s+1} \oplus \cdots \ \forall s = 1, 2, \ldots$. Suppose we have constructed a free homogeneous generating set $Y^{(s)}$ for any $K^{(s)}$ $(s = 1, 2, \ldots)$ such that $Y_i^{(s)} = Y_i^{(s-1)} \ \forall i = 1, 2, \ldots, s-1$. Then $Y = \bigcup_{i=1}^{\infty} Y_i^{(i)}$ is a free generating set for K. Indeed, a homogeneous element of degree s in K can be expresses as a linear combination of monomials in the elements of $Y^{(s)}$, and these elements actually belong to Y since their degrees do not exceed s. It follows that Y generates K and hence we have a surjection $L(Y) \to K$, whose kernel must in fact be zero. Indeed, otherwise we would have a relation in K among the generators that does not hold in the free algebra $L(Y)$ and, since this relation involves only finitely many generators, it can be regarded as a relation among the elements of $Y^{(s)}$ for a sufficiently large s, contradicting the fact that $Y^{(s)}$ is a free generating set of $K^{(s)}$. Now, our goal is to construct a free generating set for $K^{(s)}$ that includes the free generators of $K^{(s-1)}$ of degree at most $s-1$.

Suppose that we have a subalgebra K with $K_t = L_t \ \forall t \neq s$, and for $t = s$

we have $L_s = K_s \oplus L'_s$ as a vector space. If we put $X_t = X \cap L_t$ $t = 1, 2, \ldots$, then our purpose is to build a free homogeneous generating set Y for K such that $Y_1 = X_1, \ldots, Y_{s-1} = X_{s-1}$. Let R be a homogeneous subalgebra in L generated by $X_1 \cup \cdots \cup X_{s-1}$. As X is a free generating set, one can get $L_s = \langle X_s \rangle_F \oplus R_s$ and $R_s \subseteq K_s$. Also, by applying the modularity law, we get $K_s = (\langle X_s \rangle_F \cap K_s) \oplus R_s$. Moreover, there exists a subspace $L''_s \subseteq L_s$ such that $\langle X_s \rangle_F = (\langle X_s \rangle_F \cap K_s) \oplus L''_s$. Thus one may choose a basis $\bar{X}_s = X'_s \cup X''_s$ where X'_s and X''_s are bases for the vector spaces $\langle X_s \rangle_F \cap K_s$ and L''_s, respectively. Putting $\bar{X}_t = X_t$ $\forall t \neq s$, then $\bar{X} = \bigcup_{i=1}^{\infty} \bar{X}_i$ is a free generating set for L.

Step 2: According to our construction in the first step, we observe that $K^{(s-1)} = K^{(s)} \oplus \langle Z \rangle_F$ where Z is a subset of the free generators of $K^{(s-1)}$. It follows from Lemma 3.1.5,

$$\mathcal{H}(Y_{(s)}) = (\mathcal{H}(Y_{(s-1)}) - 1)\mathcal{E}_p(\mathcal{H}(K^{(s-1)}/K^{(s)})) + 1,$$

for all $s = 1, 2, \ldots$.

Step 3: By induction, one can easily prove that

$$\mathcal{H}(Y_i) = (\mathcal{H}(X) - 1)\mathcal{E}_p(\mathcal{H}(L/K^{(i)}) + 1,$$

for all $i = 0, 1, \ldots$. According to our construction above, we have that for each $i \in \mathbb{N}$

$$\mathcal{H}(Y) = \mathcal{H}(Y_i) \bmod (t^{i+1}),$$

and

$$\mathcal{H}(L/K) = \mathcal{H}(L/K^{(i)}) \bmod (t^{i+1}).$$

Hence, we can take the limit as $i \to \infty$, so that we have

$$\mathcal{H}(Y) = (\mathcal{H}(X) - 1)\mathcal{E}_p(\mathcal{H}(L/K)) + 1.$$

□

If K is an arbitrary subalgebra of L (not necessarily homogeneous), then the Hilbert-Poincaré series is defined to be

$$\mathcal{H}(K) = \mathcal{H}(\mathrm{gr}K) \text{ and } \mathcal{H}(L/K) = \mathcal{H}(L/\mathrm{gr}K),$$

where the filtrations of K and L/K are imposed by the grading on L. Now we are ready to prove the main theorem of this section.

Theorem 3.1.7. *Let $G = \{g_1, \ldots, g_r\}$ be an abelian group and $L = L_{g_1} \oplus \cdots \oplus L_{g_r}$ be a free color Lie p-superalgebra generated by a finitely graded set $X = \bigcup_{i=1}^{r} X_{g_i}$. If $K = K_{g_1} \oplus \cdots \oplus K_{g_r}$ is a subalgebra of L, then there is a set of free generators Y such that*

$$\mathcal{H}(Y,t) = (\mathcal{H}(X,t) - 1)\, \mathcal{E}_p(\mathcal{H}(L/K; t_{g_1}, \ldots, t_{g_r}) + 1.$$

Proof. The subspace K has a filtration

$$0 = K^0 \subseteq K^1 \subseteq \cdots$$

where

$$K^i = \{k \in K \mid \mathrm{wt}(k) \leq i\} \quad i \in \mathbb{N}.$$

By Lemma 3.1.6, we can find a set of free generators $\tilde{Y}$ for $\mathrm{gr}K$ such that

$$\mathcal{H}(\tilde{Y}, t) = (\mathcal{H}(X,t) - 1)\mathcal{E}_p\left(\mathcal{H}\left(L/\mathrm{gr}K; t_{g_1}, \ldots, t_{g_r}\right)\right) + 1.$$

The elements of $\tilde{Y}$ have the form $y_n + K^{n-1}$ for some n, where $y_n \in K^n$. Now, for every element $\tilde{y} = y_n + K^{n-1} \in \tilde{Y}$, we can choose an element of the form $y = z + y_n$ $z \in K^{n-1}$. Let Y denotes the set of these forms. For $y = z + y_n \in Y$, we write $\mathrm{wt}(y) = n$. Clearly, $\mathcal{H}(Y,t) = \mathcal{H}(\tilde{Y},t)$. Now, Y is a free generating set of K. Indeed, any linear dependence among the linear basis of $L(Y)$ corresponds with a linear dependence among the linear basis in $L(\tilde{Y})$. Now $\mathcal{H}(L/K,t) = \mathcal{H}(L/\mathrm{gr}K)$. Then,

$$\begin{aligned}
\mathcal{H}(Y,t) &= \mathcal{H}(\tilde{Y},t) \\
&= (H(X,t)-1)\mathcal{E}_p(\mathcal{H}(L/\mathrm{gr}K,t)+1 \\
&= (H(X,t)-1)\mathcal{E}_p(\mathcal{H}(L/K,t)+1.
\end{aligned}$$

□

Let L be a color Lie p-superalgebra. Suppose that $K \subseteq L$ is a subspace of L closed under Lie bracket. A p-hull of K is defined to be

$$\langle K \rangle_p = \left\langle k^{p^n} \mid k \in K_g, g \in G_+, n \in \mathbb{N} \right\rangle_F \oplus K_-.$$

Clearly, $\langle K \rangle_p$ is a color Lie p-subsuperalgebra (indeed it is the minimal p-subalgebra containing K).

Remark 3.1.8. Let $L = L^p(X)$ be a free color Lie p-superalgebra freely generated by a set X with $|X| \geq 2$. Suppose that X is totally ordered in such a way that if $x \in X_+$ and $y \in X_-$, then $y > x$. Then the set Y of all monomials of the form

$$[[x_1, \dots, x_{s-1}], x_s],$$

where $x_i \in X$, $s \geq 2$, $x_1 > x_2 \leq x_3 \leq \cdots \leq x_s$, $x_i \neq x_{i+1}$ for $x \in X_-$, and also if $x_1, \ldots, x_s \in X_-$, then $x_1 \geq x_2 < x_3 < \cdots < x_s$, is a basis of the derived algebra $L' = [L, L]$ ([4, page 61]). Hence it is a generating set for the p-hull of L' ($\langle L' \rangle_p$). Indeed, it is a basis of $\langle L' \rangle_p$. For, any free generating set, $\bar{Y}$, for L'_p is, in fact, a free generating set for L'.

Remark 3.1.9. Let $L = L^p(X)$ be a free color Lie p-superalgebra freely generated by a finitely graded set X. Suppose that K is a homogeneous p-subalgebra of L freely generated by Y. Then

$$\mathcal{H}(Y, t) = \mathcal{H}(K/[K, K] + \langle K_+ \rangle_p).$$

As a result, any homogeneous set of free generators for K satisfies the Schreier's formula given in Theorem 3.1.7.

Example 3.1.10. Let $G = G_+ \cup G_-$, where $G_+ = \{g_1, \ldots, g_k\}$ and $G_- = \{h_1, \ldots, h_m\}$ ($m = 0$ or $m = k$), be an abelian group. Let $L = L^p(X)$ be a free color Lie p-superalgebra freely generated by $X = \bigcup_{g \in G} X_g = X_+ \cup X_-$ ($X_\pm = \bigcup_{g \in G_\pm} X_g$) with $|X_{g_i}| = s_i, i = 1, \ldots, k$, $|X_{h_j}| = q_j, j = 1, \ldots, m$, $|X_+| = s_1 + \cdots + s_k = s$, and $|X_-| = q_1 + \cdots + q_m = q$. Let $\langle L' \rangle_p$ be the p-hull of the derived algebra $L' = [L, L]$. Then

$$L / \langle L' \rangle_p = \left\langle x^{p^n} \mid x \in X_g, g \in G_+, n \in \mathbb{N} \right\rangle_F \oplus \langle y \mid y \in X_h, h \in G_- \rangle_F,$$

and so

$$\mathcal{H}(L / \langle L' \rangle_p ; t_1, \ldots, t_k, u_1, \ldots, u_m) = s_1(t_1 + t_1^p + \cdots) + \cdots + s_k(t_k + t_k^p + \cdots) + q_1 u_1 + \cdots + q_k u_k.$$

This means that

$$\begin{aligned}
\mathcal{E}_p(\mathcal{H}(L/\langle L'\rangle_p)) &= \Big((1+t+\cdots+t^{p-1})^s(1+t^p+\cdots+t^{p(p-1)})^s\cdots\Big)(1+t)^q \\
&= \left(\frac{1-t^p}{1-t}\right)^s\left(\frac{1-t^{p^2}}{1-t^p}\right)^s\cdots(1+t)^q \\
&= \frac{(1+t)^q}{(1-t)^s}.
\end{aligned}$$

According to Theorem 3.1.7 and Remark 3.1.9, for every free generators Y of $\langle L'\rangle_p$, we have

$$\begin{aligned}
\mathcal{H}(Y,t) &= ((s+q)t-1)\frac{(1+t)^q}{(1-t)^s}+1 \\
&= \frac{(1+t)^q(-1+(s+q)t)}{(1-t)^s}+1.
\end{aligned}$$

In particular, if $s=t=1$, we get $\mathcal{H}(Y,t)=\frac{(1+t)(2t-1)}{1-t}+1=\frac{2t^2}{1-t}$.

Remark 3.1.11. Suppose that $G=G_+\cup G_-$ is an abelian group. Consider a free color Lie p-superalgebra $L=L_+\oplus L_-$ ($L_\pm=\bigoplus_{g\in G_\pm}L_g$) freely generated by $X=\{x_1,\ldots,x_k\}$, $\mathrm{wt}(x_i)=1$ $i=1,\ldots,k$. Suppose $K=K_+\oplus K_-\subseteq L$ be a G-homogeneous subalgebra with $\dim(L_+/K_+)=t<\infty$ and $\dim(L_-/K_-)=s<\infty$. Assume Y is a free homogeneous generating set for K with $\mathcal{H}(Y,t)=\sum_{i=1}^{\infty}d_it^i$. Now, $\mathcal{H}(L/K)=\sum_{n=1}^{\infty}\sum_{g\in G_+}b_n^g t_g^n+\sum_{n=1}^{\infty}\sum_{g'\in G_-}c_n^{g'}t_{g'}^n$. Obviously, only finitely many coefficients b_n^g and $c_n^{g'}$ are nonzero, $\sum_{n=1}^{\infty}b_n^g=t$, and $\sum_{n=1}^{\infty}c_n^{g'}=s$. According to the theorem above there exists a free generating set Y for K such that

$$\mathcal{H}(Y,t)=(\mathcal{H}(X,t)-1)\,\mathcal{E}_p(\mathcal{H}(L/K))+1.$$

Next, we replace t by 1, to get

$$\text{rank}K = |Y| = 2^s p^t(|X| - 1) + 1.$$

This is the formula of Mikhalev (Theorem 3.1.2).

In particular, we obtain the Kukin's result mentioned at the beginning of this chapter.

Corollary 3.1.12. *Suppose that K is a subalgebra of the Lie p-algebra $L^p(X)$ of rank $|X| < \infty$ such that $\dim(L^p/K) = t < \infty$. Then*

$$\text{rank}K = p^t(|X| - 1) + 1.$$

3.2 The Case of Color Lie Superalgebra

A subalgebra of a free color Lie superalgebra is free [20]. The following analogue of Schreier's formula for free color Lie superalgebra also holds.

Theorem 3.2.1 ([21]). *Let $L = L_+ \oplus L_-$ ($L_\pm = \bigoplus_{g\in G_\pm} L_g$) be a free color Lie superalgebra on X, where $\text{rank}(L) = |X| < \infty$, and let $K = K_+ \oplus K_-$ be a subalgebra with $K_+ = L_+$ and $\dim(L_-/K_-) = s < \infty$. Then*

$$\text{rank}(K) = 2^s (\text{rank}(L) - 1) + 1.$$

Let us assume that $G_+ = \{g_1, \ldots, g_k\}$ and $G_- = \{g_{k+1}, \ldots, g_r\}$ (of course $|G_+| = |G_-|$ or $|G_-| = 0$). We define an operator $\mathcal{E}$ on the color series $\phi(t_{g_1}, \ldots, t_{g_r}) = \sum_{j=1}^{\infty} \sum_{i=1}^{r} a_j^{g_i} t_{g_i}^j$ by

$$\mathcal{E} : \sum_{g\in G} \sum_{i=1}^{\infty} a_i^g t_g^i \mapsto \frac{\prod_{j=1}^{\infty} (1 + t^j)^{\zeta_{i,-}}}{\prod_{j=1}^{\infty} (1 - t^j)^{\zeta_{i,+}}}.$$

where $\zeta_{i,+} = \sum_{i=1}^{k} a_j^{g_i}$ and $\zeta_{i,-} = \sum_{i=k+1}^{r} a_j^{g_i}$. We note that $\mathcal{E}$ is a multiplicative operator.

Remark 3.2.2. It can be shown that $\mathcal{E}$ defined here is the composition of the operator defined in Subsection 2.1.1 (for $\Lambda = \mathbb{N} \times G$) and the homomorphism from $\mathbb{Q}[[\mathbb{N}_0 \times G]]$ to $\mathbb{Q}[[\mathbb{N}_0]] = \mathbb{Q}[[t]]$ defined by $(n, g) \mapsto n$.

Let $L(X)$ be a free Lie superalgebra ($G = \mathbb{Z}_2$) generated by a finitely graded set $X = X_0 \cup X_1$ where

$$X_0 = \{x \in X \mid d(x) = 0\} \text{ and } X_1 = \{x \in X \mid d(x) = 1\}.$$

In this case,

$$\mathcal{E} : \sum_{n=0}^{\infty} \left(b_n t_0^n + b_n^{'} t_1^n \right) \mapsto \prod_{n=1}^{\infty} \frac{(1+t^n)^{b_n^{'}}}{(1-t^n)^{b_n}},$$

Petrogradsky (see [25]) introduced an analogue of Schreier's formula in terms of formal power series for free Lie superalgebras. He proved that if L is a free Lie superalgebra generated by a finitely graded set $X = X_0 \cup X_1$, and $K = K_0 \oplus K_1$ is a subalgebra of L, then there is a set of free generators Y of K such that

$$\mathcal{H}(Y, t) = (\mathcal{H}(X, t) - 1)\, \mathcal{E}\left(\mathcal{H}\left(L/K, t_0, t_1\right)\right) + 1.$$

Now we are going to extend Petrogradsky's formula to the case of color Lie superalgebras.

Theorem 3.2.3. *Let $G = \{g_1, \ldots, g_r\}$ be an abelian group and $L = L_{g_1} \oplus \cdots \oplus L_{g_r}$ be a free color Lie superalgebra generated by a finitely graded set $X = \bigcup_{i=1}^{r} X_{g_i}$. If $K = K_{g_1} \oplus \cdots \oplus K_{g_r}$ is a subalgebra of L, then it has a free generating set Y with*

$$\mathcal{H}(Y, t) = (\mathcal{H}(X, t) - 1)\, \mathcal{E}(\mathcal{H}(L/K; t_{g_1}, \ldots, t_{g_r}) + 1.$$

Proof. We proceed as in the proof of Theorem 3.1.7. The only difference appears in the proof of Lemma 3.1.5 when considering the following situation: $K = \bigoplus_{g\in G} K_g$ is a homogeneous subalgebra of L with $L = K \oplus \langle z \rangle_F$ where $z \in X$ and $d(z) = g_k \in G_+$. Assume $\text{wt}(z) = a$. Then $\mathcal{H}(L/K) = t_{g_k}^a$, and so $\mathcal{E}(\mathcal{H}(L/K, t_{g_1}, t_{g_2}, \dots, t_{g_r})) = \frac{1}{1-t^a}$. It is known that K is freely generated by the following set (see [4, page 56])

$$Y = \left\{ y, y\,(ad'z)^m \ \mid y \in X \setminus \{z\}, m \in \mathbb{N} \right\},$$

where $y\,(ad'z) = [y, z]$, so $y\,(ad'z)^m = [\dots[y, \underbrace{z], \dots, z]}_{m-\text{times}}$. Then

$$\begin{aligned}
\mathcal{H}(Y,t) &= \mathcal{H}\left(X \setminus \{z\}\, t\right)\left(1 + t^a + t^{2a} + \cdots + t^{ma} + \cdots\right) \\
&= \frac{(\mathcal{H}(X,t) - t^a)}{1-t^a} \\
&= \frac{(\mathcal{H}(X,t) - 1 + (1-t^a))}{1-t^a} \\
&= 1 + \frac{(\mathcal{H}(X,t) - 1)}{1-t^a} \\
&= 1 + (\mathcal{H}(X,t) - 1)\,\mathcal{E}\left(\mathcal{H}\left(L/K; t_{g_1}, \dots, g_{g_r}\right)\right).
\end{aligned}$$

□

Remark 3.2.4. Consider the case where K is a homogeneous subalgebra of L and under the same assumptions as Theorem 3.2.3, we note that $\mathcal{H}(Y) = \mathcal{H}(K/[K,K])$. As a result, any homogeneous set of free generators for K satisfies the Schreier's formula.

In the special case that L is a Lie algebra, we obtain the following formula first obtained by Petrogradsky [25].

Corollary 3.2.5. *Let L be a free Lie algebra generated by a finitely graded set X,* *and let K be a subalgebra. Then, there is a free generators Y of K such that*

$$\mathcal{H}(Y) = (\mathcal{H}(X) - 1)\,\mathcal{E}\left(\mathcal{H}\left(L/K\right)\right) + 1.$$

Remark 3.2.6. Let $G = \{g_1, \ldots, g_r\}$ be an abelian group. Consider a free color Lie superalgebra $L = L_+ \oplus L_-$ ($L_\pm = \oplus_{g \in G_\pm} L_g$) freely generated by $X = \{x_1, \ldots, x_k\}$, $\mathrm{wt}(x_i) = 1$ $i = 1, \ldots, k$. Suppose $K = K_+ \oplus K_- \subseteq L$ be a subalgebra with $K_+ = L_+$ and $\dim\left(L_-/K_-\right) = s < \infty$. Assume Y is a free homogeneous generating set for K with $\mathcal{H}(Y, t) = \sum_{i=1}^{\infty} d_i t^i$. Now, $\mathcal{H}(L/K; t_{g_1}, \ldots, t_{g_r}) = \sum_{g \in G_-} \sum_{i=1}^{\infty} c_{g,i} t_g^i$ where only finitely many coefficients $c_{g,i}$ are nonzero for all $g \in G_-$, and $\sum_{g \in G_-} \sum_{i=1}^{\infty} c_{g,i} = \dim(L_-/K_-) = s$. According to Theorem 3.2.3 and by substituting $t = 1$, we have

$$|Y| = 2^s(k-1) + 1.$$

This is the formula due to Mikhalev (Theorem 3.2.1).

Corollary 3.2.7. *Let $L = L_+ \oplus L_-$ be a free color Lie superalgebra freely generated by $X = X_+ \cup X_-$ where $|X| = r < \infty$ (with $r > 1$). Suppose that $K = K_+ \oplus K_-$ is a proper subalgebra of L such that $K_+ \neq L_+$ and $\dim(L/K) = s < \infty$. Then $\mathrm{rank} K = \infty$.*

Proof. Suppose $\mathrm{rank} K$ is finite. By Theorem 3.2.3, there exists a free generating set Y for K such that

$$\mathcal{H}(Y, t) = (\mathcal{H}(X, t) - 1)\,\mathcal{E}(\mathcal{H}(L/K; t_{g_1}, \ldots, t_{g_r})) + 1.$$

Since $\dim(L/K) = s < \infty$, we have

$$\mathcal{E}(\mathcal{H}(L/K; t_{g_1}, \ldots, t_{g_r})) = \frac{(1+t)^{a_1}(1+t^2)^{a_2}\cdots(1+t^k)^{a_k}}{(1-t)^{c_1}(1-t^2)^{c_2}\cdots(1-t^j)^{c_j}},$$

where $a_1 + \cdots + a_k = \dim(L_-/K_-)$ and $c_1 + \cdots + c_j = \dim(L_+/K_+)$. As rank$K$ is finite, we have $\mathcal{H}(Y,t) = b_1 t + \cdots + b_m t^m$. Therefore,

$$b_1 t + \cdots + b_m t^m - 1 = \frac{(rt-1)(1+t)^{a_1}(1+t^2)^{a_2}\cdots(1+t^k)^{a_k}}{(1-t)^{c_1}(1-t^2)^{c_2}\cdots(1-t^j)^{c_j}}.$$

This contradicts with the fact that $(1-t)$ does not divide the numerator. ☐

Corollary 3.2.8. *Let $L = L_+ \oplus L_-$ be a free color Lie superalgebra freely generated by $X = X_+ \cup X_-$ where $|X| > 1$. Suppose that $K = K_+ \oplus K_-$ is a nonzero ideal of L such that $K_+ \neq L_+$. Then* rank$K = \infty$.

Proof. Choose an element $a \in L_+ \backslash K_+$. As K is an ideal and $a \in L_+$, $E = K \oplus \langle a \rangle_F$ is a subalgebra of L and $[E, E] \subseteq K$. Then apply Corollary 3.2.7. ☐

Chapter 4

Schreier Formula for Free (Restricted) Color Lie Superalgebras in Terms of Characters

In this chapter we generalize the Shreier formulas obtained in Chapter 3 by replacing $\mathbb{N}$-gradings and Hilbert series in $\mathbb{Q}[[t]]$ with, respectively, Λ-gradings and characters in $\mathbb{Q}[[\Lambda]]$ introduced in Chapter 2.

4.1 The Setup

Let G be an abelian group and $\gamma : G \times G \to F^*$ be a skew-symmetric bicharacter. Let Λ be a countable additive abelian semigroup satisfying the following conditions:

1. finiteness condition: each element $\lambda \in \Lambda$ can be written as a sum of other elements only in finitely many ways,

2. Λ is well ordered by $\leq$ such that if $\lambda < \mu$, then $\lambda+\gamma < \mu+\gamma$ for all $\lambda, \mu, \gamma \in \Lambda$, and also $\lambda + \mu > \lambda$ for all $\lambda, \mu \in \Lambda$ (such semigroups are called positive),

3. There exists a homomorphism, $\kappa_G : \Lambda \to G$, from Λ onto G. In this case, Λ can be partitioned as

$$\Lambda = \Lambda_+ \cup \Lambda_-,$$

where

$$\Lambda_\pm = \{\lambda \in \Lambda \mid \kappa_G(\lambda) \in G_\pm\}.$$

We consider Λ-graded color Lie (p-)superalgebras $L = \bigoplus_{\lambda\in\Lambda} L_\lambda$, with $\dim L_\lambda < \infty$ $\forall \lambda \in \Lambda$, where the G-grading is determined by the Λ-grading through κ_G in the sense that $L_g = \bigoplus_{\substack{\lambda\in\Lambda \\ \kappa_G(\lambda)=g}} L_\lambda$. Let $K \subseteq L$ be a not necessarily homogeneous subalgebra of L. For $k \in K$, we consider its decomposition into homogeneous components $k = k_{\lambda_1} + \cdots + k_{\lambda_n}$, $\lambda_i \in \Lambda$, $\lambda_1 \leq \cdots \leq \lambda_n$, $k_{\lambda_n} \neq 0$. In this case we write $\mathrm{wt}_\Lambda k = \lambda_n$. Also, K has a filtration (as an algebra) $\bigcup_{\lambda\in\Lambda} K^\lambda$ where

$$K^\lambda = K \cap (\bigoplus_{\theta\leq\lambda} L_\theta), \lambda \in \Lambda.$$

Also, the factor-space L/K acquires a factor-filtration given by

$$(L/K)^\lambda = (L^\lambda + K)/K \cong L^\lambda/(K \cap L^\lambda).$$

Denote

$$K^{\lambda-} = K \cap (\bigoplus_{\theta<\lambda} L_\theta), \lambda \in \Lambda.$$

Then, one can construct a graded algebra gr$K$ as follows: set $\mathrm{gr}K = \bigoplus_{\lambda\in\Lambda\cup\{0\}} K^{\lambda}/K^{\lambda-}$ as a vector space (set $K^0 = \{0\}$), and define multiplication as:

$$K^{\lambda}/K^{\lambda-} \times K^{\theta}/K^{\theta-} \to K^{\lambda+\theta}/K^{(\lambda+\theta)-} : \ (a + K^{\lambda-})(b + K^{\theta-}) \mapsto ab + K^{(\lambda+\theta)-}.$$

Then, in the nonhomogeneous case we define characters as:

$$\mathrm{ch}_{\Lambda}(K) = \mathrm{ch}_{\Lambda}(\mathrm{gr}K) \text{ and } \mathrm{ch}_{\Lambda}(L/K) = \mathrm{ch}_{\Lambda}(L/\mathrm{gr}K).$$

4.2 Schreier Formulas

Let L be a free Lie superalgebra ($G = \mathbb{Z}_2$) generated by a Λ-graded set $X = \bigcup_{\lambda\in\Lambda} X_{\lambda}$ where $|X_{\lambda}| < \infty$ for all $\lambda \in \Lambda$. Petrogradsky ([27]) showed that if K is a subalgebra of L, then there exists a set of free generators Y of K such that

$$\mathrm{ch}_{\Lambda}Y = (\mathrm{ch}_{\Lambda}X - 1)\mathcal{E}(\mathrm{ch}_{\Lambda}L/K) + 1,$$

where

$$\mathcal{E} : \sum_{\lambda\in\Lambda} h_{\lambda}e^{\lambda} \mapsto \frac{\prod_{\lambda\in\Lambda_-}(1+e^{\lambda})^{h_{\lambda}}}{\prod_{\lambda\in\Lambda_+}(1-e^{\lambda})^{h_{\lambda}}},$$

Moreover, if K is homogeneous, then any homogeneous set of free generators satisfies this equality.

In the following theorem, we will extend his result to color Lie p-superalgebras case. Let us introduce the operator $\mathcal{E}_p$ on $\mathbb{Q}[[\Lambda]]$,

$$\mathcal{E}_p : \sum_{\lambda\in\Lambda} h_{\lambda}e^{\lambda} \mapsto \prod_{\lambda\in\Lambda_-}(1+e^{\lambda})^{h_{\lambda}} \prod_{\lambda\in\Lambda_+}(1+\cdots+e^{(p-1)\lambda})^{h_{\lambda}}.$$

The definition of $\mathcal{E}_p$ is motivated by the fact $\mathrm{ch}_{\bar{\Lambda}} u(L) = \mathcal{E}_p(\mathrm{ch}_\Lambda L)$ (Remark 2.2.5). Recall that $\mathcal{E}_p$ is multiplicative (Remark 2.2.5), that is,

1. $\mathcal{E}_p(0) = 1$,

2. $\mathcal{E}_p(f+g) = \mathcal{E}_p(f)\mathcal{E}_p(g), f, g \in \mathbb{Q}[[\Lambda]]$.

Theorem 4.2.1. *Let L be a free color Lie p-superalgebra freely generated by $X = \bigcup_{\lambda\in\Lambda} X_\lambda$ with $|X_\lambda| < \infty$ for all $\lambda \in \Lambda$. Suppose that K is a p-subalgebra of L. Then there is a set of free generators Y of K such that*

$$\mathrm{ch}_\Lambda Y = (\mathrm{ch}_\Lambda X - 1)\mathcal{E}_p(\mathrm{ch}_\Lambda L/K) + 1.$$

Proof. We use the same method as in the proof of Theorem 3.1.7. We break the proof into three steps.

Step 1: Suppose that K is a homogeneous subalgebra of L with $L = K \oplus \langle Z\rangle_F$, $Z \subseteq X$, and $|Z| < \infty$. We prove our statement by induction on $|Z|$. Let $Z = \{z\}$, $z \in X_\lambda$. There are two cases. First, $\lambda \in \Lambda_+$. Then $\mathrm{ch}_\Lambda L/K = e^\lambda$, and so $\mathcal{E}_p(\mathrm{ch}_\Lambda L/K) = 1 + e^\lambda + \ldots + e^{(p-1)\lambda}$. According to Theorem 3.1.4, the subalgebra K is freely generated by

$$\left\{z^p, y\,(ad'z)^m \ \mid y \in X \setminus \{z\}, m = 0, 1, 2, \ldots, p-1\right\},$$

where $y\,(ad'z) = [y, z]$. Then

$$\begin{aligned}
\mathrm{ch}_\Lambda(Y) &= \mathrm{ch}_\Lambda(X \setminus \{z\})(1 + e^\lambda + \ldots + e^{(p-1)\lambda}) + e^{p\lambda} \\
&= (\mathrm{ch}_\Lambda(X) - 1 + (1 - e^\lambda))(1 + e^\lambda + \ldots + e^{(p-1)\lambda}) + e^{p\lambda} \\
&= (\mathrm{ch}_\Lambda(X) - 1)(1 + e^\lambda + e^{2\lambda} + \ldots + e^{(p-1)\lambda}) + 1 - e^{p\lambda} + e^{p\lambda} \\
&= (\mathrm{ch}_\Lambda(X) - 1)(1 + e^\lambda + e^{2\lambda} + \ldots + e^{(p-1)\lambda}) + 1
\end{aligned}$$

$$= \quad (\mathrm{ch}_\Lambda(X) - 1)\mathcal{E}_p(\mathrm{ch}_\Lambda(L/K)) + 1,$$

and the desired formula is valid for this particular case.

Second, let $\lambda \in \Lambda_-$. Then K is freely generated by the homogeneous elements ([4])

$$Y = \{x, [x, z] \mid x \in X \setminus \{z\}\} \cup \{[z, z]\}.$$

Now,

$$\begin{aligned} \mathrm{ch}_\Lambda(Y) &= \mathrm{ch}_\Lambda(X \setminus \{z\}) \left(1 + e^\lambda\right) + e^{2\lambda} \\ &= \left(\mathrm{ch}_\Lambda(X) - 1 + (1 - e^\lambda)\right)(1 + e^\lambda) + e^{2\lambda} \\ &= 1 + (\mathrm{ch}_\Lambda(X) - 1)\, \mathcal{E}_p\left(\mathrm{ch}_\Lambda(L/K)\right). \end{aligned}$$

Now suppose that $|Z| > 1$. Then take $z \in Z$, and let $Z' = Z \setminus \{z\}$. Then $K' = K \oplus \langle Z' \rangle_F$ is a subalgebra of codimension one. Also, of course it is free. Let Y' be a basis of K'. Now apply the inductive hypothesis to the inclusions $K \subset K'$, $K' \subset L$, to get

$$\begin{aligned} \mathrm{ch}_\Lambda(Y') - 1 &= (\mathrm{ch}_\Lambda(X) - 1)\mathcal{E}_p(\mathrm{ch}_\Lambda(L/K'), \\ \mathrm{ch}_\Lambda(Y) - 1 &= (\mathrm{ch}_\Lambda(Y') - 1)\mathcal{E}_p(\mathrm{ch}_\Lambda(K'/K). \end{aligned}$$

Therefore

$$\mathrm{ch}_\Lambda(Y) - 1 \quad = \quad (\mathrm{ch}_\Lambda(X) - 1)\mathcal{E}_p(\mathrm{ch}_\Lambda(L/K')\mathcal{E}_p(\mathrm{ch}_\Lambda(K'/K).$$

Using the multiplicative property of $\mathcal{E}_p$, we see that

$$\begin{aligned}\mathrm{ch}_\Lambda(Y)-1 &= (\mathrm{ch}_\Lambda(X)-1)\mathcal{E}_p(\mathrm{ch}_\Lambda(L/K')+\mathrm{ch}_\Lambda(K'/K))\\ &= (\mathrm{ch}_\Lambda(X)-1)\mathcal{E}_p(\mathrm{ch}_\Lambda(L/K)).\end{aligned}$$

Step 2: Suppose that $K=\bigoplus_{\lambda\in\Lambda}K_\lambda$ is a homogeneous subalgebra. Then $K_\lambda = K\cap L_\lambda$, $\lambda\in\Lambda$. As in [27], we can prove our formula for a specially constructed homogeneous free generating set Y. It follows from the finiteness condition of Λ and assumption $|X_\lambda|<\infty$, $\lambda\in\Lambda$ that $\dim L_\lambda<\infty$ for all $\lambda\in\Lambda$. We represent K as the intersection of the decreasing sequence

$$K^{(\lambda)}=\left(\bigoplus_{\theta\le\lambda}K_\theta\right)\bigoplus\left(\bigoplus_{\theta>\lambda}L_\theta\right),\lambda\in\Lambda.$$

Suppose that we have constructed for any $K^{(\lambda)}$ a free homogeneous generating set $Y^{(\lambda)}$ such that for any $\theta<\lambda$ one has $Y^{(\lambda)}_\sigma=Y^{(\theta)}_\sigma,\sigma\le\theta$. Then, as in the proof of Lemma 3.1.6, one shows that $Y=\bigcup_{\lambda\in\Lambda}Y^{(\lambda)}_\lambda$ is a free generating set for K.
Let us introduce the series of subalgebras

$$K^{(\lambda-)}=\left(\bigoplus_{\theta<\lambda}K_\theta\right)\bigoplus\left(\bigoplus_{\theta\ge\lambda}L_\theta\right),\lambda\in\Lambda.$$

Now, our goal is to construct $Y^{(\lambda)}$ by transfinite induction. Suppose that for any $\sigma<\lambda$ these sets are constructed. We have one of the following two cases. First, there exists a maximal θ among $\{\sigma\in\Lambda\mid\sigma<\lambda\}$, then $K^{(\lambda-)}=K^{(\theta)}$ and this algebra is generated by $Y^{(\theta)}$. Second, when such maximal element does not exist, we consider all $Y^{(\lambda)}$, $\sigma<\lambda$. We obtain $Y^{(\sigma)}$ by deleting some elements from $Y^{(\sigma-)}_\sigma$ and adding some elements in components of type $\sigma_1+m\sigma, m\ge 1$. Now, take arbitrary $\tau\ge\lambda$, by finiteness condition, there exists $\delta(\tau)=\max\{\sigma\mid\tau=\sigma+\sigma_1,\sigma<\lambda\}$. If we choose θ

with $\delta(\tau) < \theta < \lambda$, then on the θ-th step changes cannot occur in the τ-th component. Hence, after λ, we have the following stabilization in components $Y_\tau^{(\delta(\tau))} = Y_\tau^{(\theta)}$ for $\delta(\tau) \leq \theta < \lambda$. We set

$$Y^{(\lambda-)} = \bigcup_{\sigma<\lambda} Y_\sigma^{(\sigma)} \bigcup_{\tau\geq\lambda} Y_\tau^{(\delta(\tau))}.$$

Suppose we have a finite subset $\tilde{Y} \subset Y^{(\lambda-)}$ lying in components $\sigma_1 < \cdots < \sigma_s < \lambda < \sigma_{s+1} < \cdots < \sigma_{s+t}$. Then for $\nu = \max\{\sigma_s, \delta(\sigma_{s+1}), \ldots, \delta(\sigma_{s+t})\}$ we have $\tilde{Y} \subset Y^{(\nu)}$. This shows that $Y^{(\lambda-)}$ is the free generating set for $K^{(\lambda-)}$.

We temporarily consider that $L = K^{(\lambda-)}$, $K = K^{(\lambda)}$. Then $K_\sigma = L_\sigma, \sigma \neq \lambda$, and $L_\lambda = K_\lambda \oplus \bar{L}_\lambda$ where $\bar{L}_\lambda$ is the complement of the of the vector space. Our goal now is to construct a free generating set Y for K with $Y_\sigma = X_\sigma, \sigma < \lambda$. Let R be the homogeneous subalgebra in K generated by $\bigcup_{\sigma<\lambda} X_\sigma$. As X is a free generating set, one can get $L_\lambda = \langle X_\lambda \rangle_F \oplus R_\lambda$ and $R_\lambda \subseteq K_\lambda$. Also, by applying the modularity law, we get $K_\lambda = (\langle X_\lambda \rangle_F \cap K_\lambda) \oplus R_\lambda$. Moreover, there exists a linear basis $L''_\lambda \subseteq L_\lambda$ such that $\langle X_\alpha \rangle_F = (\langle X_\lambda \rangle_F \cap K_\lambda) \oplus L''_\lambda$. Thus one may choose a basis $\bar{X}_\lambda = X'_\lambda \cup X''_\lambda$ where X'_λ and X''_λ are bases for the vector spaces $\langle X_\lambda \rangle_F \cap K_\lambda$ and L''_λ, respectively. Putting $\bar{X}_\sigma = X_\sigma \ \forall \sigma \neq \lambda$, then $\bar{X} = \bigcup_{\sigma\in\Lambda}^{\infty} \bar{X}_\sigma$ is a free generating set for L.

According to our construction, we observe that $K^{(\lambda-)} = K^{(\lambda)} \oplus \langle Z \rangle_F$ where Z is a subset of the free generators of $K^{(\lambda-)}$. It follows from the first step,

$$\mathrm{ch}_\Lambda Y^{(\lambda)} = 1 + (\mathrm{ch}_\Lambda Y^{(\lambda-)} - 1)\mathcal{E}_p(\mathrm{ch}_\Lambda(K^{(\lambda-)}/K^{(\lambda)})).$$

Using the inductive assumption, one can prove that (see [27])

$$\mathrm{ch}_\Lambda Y^{(\lambda)} = 1 + (\mathrm{ch}_\Lambda X - 1)\mathcal{E}_p(L/K^{(\lambda)}),$$

and then

$$\mathrm{ch}_\Lambda Y = 1 + (\mathrm{ch}_\Lambda X - 1)\mathcal{E}_p(\mathrm{ch}_\Lambda(L/K)).$$

Step 3: Suppose that K is a nonhomogeneous subalgebra. In this case we consider the associated graded subspace gr$K$. Then the second step yields a homogeneous free generating set $\bar{Y}$ for gr$K$ with

$$\mathrm{ch}_\Lambda \bar{Y} = (\mathrm{ch}_\Lambda X - 1)\mathcal{E}_p(\mathrm{ch}_\Lambda(L/\mathrm{gr}K)) + 1.$$

For $\bar{y} \in \bar{Y}$, we can choose one element of the form $y = y_1 + \ldots + y_n$, $y_i \in K_{\lambda_i}$, $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, $\lambda_n \neq 0$; in this case we write deg$y = \lambda_n$ and gr$y = y_n$. Hence, there exists $Y \subset K$ such that $\bar{Y} = \mathrm{gr}Y$ $(\mathrm{gr}Y = \{\mathrm{gr}y \mid y \in Y\})$, and also we have a bijection between Y and $\bar{Y}$. Then one can prove that Y is a free generating set for K (see [27]). Clearly, $\mathrm{ch}_\Lambda Y = \mathrm{ch}_\Lambda \bar{Y}$. Finally, we have

$$\begin{aligned}\mathrm{ch}_\Lambda Y - 1 &= \mathrm{ch}_\Lambda \bar{Y} - 1\\ &= (\mathrm{ch}_\Lambda X - 1)\mathcal{E}_p(\mathrm{ch}_\Lambda(L/\mathrm{gr}K))\\ &= (\mathrm{ch}_\Lambda X - 1)\mathcal{E}_p(\mathrm{ch}_\Lambda(L/K)).\end{aligned}$$

□

In particular, if L is a p-Lie algebra (i.e., $G = \{1\}$), then Λ_- is an empty set, so that we have the following result.

Corollary 4.2.2. *Let L be a free Lie p-algebra generated by a finitely Λ-graded set $X = \bigcup_{\lambda\in\Lambda} X_\lambda$. Suppose K is a restricted subalgebra. Then it has a free generator Y with*

$$\mathrm{ch}_\Lambda Y = (\mathrm{ch}_\Lambda X - 1)\mathcal{E}_p(\mathrm{ch}_\Lambda(L/K)) + 1,$$

where

$$\mathcal{E}_p : \sum_{\lambda\in\Lambda} h_\lambda e^\lambda \mapsto \prod_{\lambda\in\Lambda}(1+e^\lambda+\ldots+e^{(p-1)\lambda})^{h_\lambda}.$$

Let $G=\{g_1,\ldots,g_r\}$, where $G_+=\{g_1,\ldots,g_k\}$ and $G_-=\{g_{k+1},\ldots,g_r\}$ (of course $|G_+|=|G|$ or $|G_+|=|G_-|$), and $X=\bigcup_{g\in G} X_g$ be a G-graded set. Suppose that for all g_i, $i=1,\ldots,r$, X_g is a finitely graded set with $\mathrm{wt}_g : X_g \to \mathbb{N}$. In this case, we can consider a weight function $\mathrm{wt} : X \to \mathbb{N}^r$ defined by

$$\mathrm{wt} : x \mapsto (0,\ldots,\mathrm{wt}_{g_s}(x_{g_s}),\ldots,0),\ x\in X_{g_s}.$$

We define the homomorphism $\kappa_G : \mathbb{N}_0^r \to G$ by $\kappa(\lambda_i)=g_i$ for $1\le i\le r$ where $\lambda_i=(0,\ldots,0,1,0,\ldots)$ with unit in the ith place. We denote $t_i=e^{\lambda_i}$, and so the algebra $\mathbb{Q}[[\mathbb{N}_0^r]]$ turns into the formal power series ring $\mathbb{Q}[[\mathbf{t}]]=\mathbb{Q}[[t_1,\ldots,t_r]]$. We have the following result.

Corollary 4.2.3. *Let $G=\{g_1,\ldots,g_r\}$ be an abelian group and $L=L_{g_1}\oplus\cdots\oplus L_{g_r}$ be a free color Lie p-superalgebra generated by a finitely graded set $X=\bigcup_{i=1}^r X_{g_i}$. If $K=K_{g_1}\oplus\cdots\oplus K_{g_r}$ is a p-subalgebra of L, then there is a set of free generators Y such that*

$$\mathcal{H}(Y;t_{g_1},\ldots,t_{g_r})=(\mathcal{H}(X;t_{g_1},\ldots,t_{g_r})-1)\,\mathcal{E}_p(\mathcal{H}(L/K;t_{g_1},\ldots,t_{g_r})+1,$$

where

$$\mathcal{E}_p :\ \sum_{g\in G}\sum_{i=1}^{\infty} a_i^g t_g^i \mapsto \prod_{\substack{g\in G_-\\ i\ge 1}}\left(1+t_g^i\right)^{a_i^g}\prod_{\substack{g\in G_+\\ i\ge 1}}\left(1+t_g^i+t_g^{2i}+\ldots+t_g^{(p-1)i}\right)^{a_i^g}.$$

Theorem 4.2.4. *Let L be a free color Lie superalgebra generated by a finitely Λ-graded set $X=\bigcup_{\lambda\in\Lambda} X_\lambda$. Suppose K is a subalgebra of L. Then it has a free generators*

Y with

$$\mathrm{ch}_\Lambda Y = (\mathrm{ch}_\Lambda X - 1)\mathcal{E}(\mathrm{ch}_\Lambda L/K) + 1.$$

If K is homogeneous, then any homogeneous set of free generators satisfies this equality.

Proof. Similar to Theorem 4.2.1. □

Chapter 5

Application to Varieties of (Restricted) Color Lie Superalgebras

Let F be a field of characteristic $p \neq 2, 3$ ($p \neq 0$ in the case of p-superalgebras), G an abelian group, $\gamma : G \times G \to F^*$ a skew-symmetric bicharacter. Let us consider the free (G, γ)-color Lie (p-)superalgebra $L = L(X)$ (or $L^p(X)$), with a countable free basis $X = \{x_1, x_2, \ldots\}$. We call the elements of L Lie (p-)polynomials in the free variables in the set X. Given a set $V \subset L$, consider all (G, γ)-color (p-)superalgebras P satisfying identical relations (or identities) $v(x_1, \ldots x_n) = 0$ in V. This means that for any homomorphism $\varphi : L \to P$ one has $v(\varphi(x_1), \ldots, \varphi(x_n)) = 0$. We say that P belongs to the *variety* $\mathfrak{V}$ of (G, γ)-color (p-)superalgebras defined by the set of identities (with right hand sides) from the set V. If P is not necessarily in $\mathfrak{V}$, consider the (p-)ideal $V(P)$ generated by all values of Lie polynomials in the set V, as above, when the variables are replaced by arbitrary elements of P. Then $V(P)$ is the smallest (p-)ideal of P such that $P/V(P) \in \mathfrak{V}$. One calls $V(P)$ the *verbal ideal* (or *T-ideal*) of P defined by the set V. For example, if $V = \{[x_1, x_2]\}$ then $V(P) = [P, P] = P^{(1)}$, the commutator subalgebra of P and $\mathfrak{V} = \mathfrak{A}$ is the variety of all abelian (G, γ)-color (p-)superalgebras. If $V = \{x^{[p]}\}$ then $\mathfrak{V}$ consists of superalgebras with zero p-map.

If Y is a nonempty set, then among all superalgebras in $\mathfrak{V}$ which can be generated by a set Y there is one, denoted by $L_{\mathfrak{V}}(Y)$ called a *free algebra in* $\mathfrak{V}$ with *free basis* Y (or relatively free Lie superalgebra). This has the same universal property as L but only for the superalgebras in $\mathfrak{V}$, namely, $L_{\mathfrak{V}}(Y)$ is an algebra in $\mathfrak{V}$ generated by Y and such that any map $\varphi : Y \to P \in \mathfrak{V}$ uniquely extend to a homomorphism of Lie (p-)superalgebras $\varphi : L_{\mathfrak{V}}(Y) \to P \in \mathfrak{V}$. One can easily prove that $L_{\mathfrak{V}}(Y) \cong L/V(L)$.

Since the growth of $L_{\mathfrak{V}}(Y)$ majorizes the growth of any algebra in $\mathfrak{V}$ that can be generated by a set of the same cardinality as Y, it is important to try to determine the growth of relatively free (G, γ) color Lie (p-)superalgebras, at least for the most common varieties. Since relatively free superalgebras over an infinite field are graded, we may be able to compute their Hilbert series, which contains information about the growth (see Chapter 6). Many verbal ideals appear in L by iterating the operations of taking the commutators of previously defined verbal ideals. For instance, if I is a verbal ideal, one can take the commutator ideal $[I, I]$ for I, and this is again a verbal ideal. If I and J are two verbal ideals then $[I, J]$ is a verbal ideal. In this way we obtain the terms of the derived series $L^{(n)} = [L^{(n-1)}, L^{(n-1)}]$ where $L^{(0)} = L$. The verbal ideal $L^{(n)}$, called the n-th *derived algebra* of L, defines the variety $\mathfrak{S}_n$ of solvable Lie superalgebras of solvability class at most n. In these algebras we have $P^{(n)} = \{0\}$. Another set of verbal ideals $L^{c+1} = [L^c, L]$, $L_1 = L$, forms a descending chain of ideals of L called the *lower central series* of L. The verbal ideal L^{c+1} defines the variety of nilpotent Lie superalgebras of nilpotent class at most c. This variety is denoted by $\mathfrak{N}_c$. We have $\mathfrak{N}_1 = \mathfrak{A} = \mathfrak{S}_1$.

If $\mathfrak{U}$ and $\mathfrak{V}$ are two varieties defined to the sets of Lie polynomials U and V of L, consider a Lie superalgebra $V(L)$ and the verbal ideal $W(L) = U(V(L))$. This defines a variety $\mathfrak{W}$ called the *product of varieties* $\mathfrak{U}$ and $\mathfrak{V}$. We write $\mathfrak{W} = \mathfrak{U}\mathfrak{V}$. Thus a variety $\mathfrak{S}_n$ of all solvable superalgebras of class at most n is the product $\mathfrak{A}^n$ of n copies of the variety $\mathfrak{A}$ of abelian Lie superalgebras. Solvable algebras of class at most 2 are called metabelian. The variety of metabelian Lie superalgebras is the square $\mathfrak{A}^2$ of the variety of abelian Lie superalgebras. The product variety of the form $\mathfrak{N}_{c_k}\mathfrak{N}_{c_{k-1}}\cdots\mathfrak{N}_{c_1}$ is called the variety of polynilpotent algebras relative to the sequence $c_k, c_{k-1}, \ldots, c_1$. Its verbal ideal equals $(\ldots((L^{c_1+1})^{c_2+1})\ldots)^{c_k+1}$. If all c_i equal 1, we come back to $\mathfrak{A}^k$.

Another important operation is the commutator of two varieties $[\mathfrak{U}, \mathfrak{V}]$. Its verbal ideal is the commutator of verbal ideals $U(L)$ and $V(L)$. An algebra P is in the commutator $[\mathfrak{U}, \mathfrak{V}]$ if $[U(P), V(P)] = \{0\}$, in other words, any value in P of a Lie polynomial of U commutes with any value of a Lie polynomial in V. For example, if $\mathfrak{E}$ is the trivial variety consisting of zero superalgebra only (its verbal ideal is the whole of L) then $[\mathfrak{E}, \mathfrak{E}] = \mathfrak{A}$ and $[\mathfrak{N}_c, \mathfrak{E}] = \mathfrak{N}_{c+1}$, for any $c = 2, 3, \ldots$. An important variety is the commutator $[\mathfrak{A}^2, \mathfrak{E}]$ of *centre-by-metabelian* Lie superalgebras. A superalgebra P is in $[\mathfrak{A}^2, \mathfrak{E}]$ if the second commutator ideal $P^{(2)}$ is in the centre of P, that is $[P^{(2)}, P] = \{0\}$.

Using his original Schreier formula for Lie algebra and then superalgebras, Petrogradsky was able to calculate Hilbert series for a large number of relatively free Lie algebras and superalgebras [25, 27]. We will generalize his results to color Lie superalgebras. The main point is that if we have a Hilbert series for a superalgebra

$P = L(X)/Q$ ($L(X)$ is a free color Lie superalgebra), where Q is an ideal of $L(X)$, one can find the Hilbert series for a free basis Y of Q hence a Hilbert series for a linear basis $Y + [Q,Q]$ for $Q/[Q,Q]$, hence a Hilbert series for $L/[Q,Q]$. If P is a relatively free algebra in a variety $\mathfrak{V}$ defined by Lie polynomials V, that is, $Q = V(L)$ then $L/[Q,Q] = L/[V(L),V(L)]$ is a relatively free algebra in the variety $\mathfrak{A}\mathfrak{V}$, and we have determined a Hilbert series of this algebra. Moreover, if we know the Hilbert series for Y, as above, we have the Hilbert series for the whole of Q, hence Witt's formula for the terms of the factors Q^c/Q^{c+1}. This allows us to find the Hilbert series of Q/Q^{c+1}, as the sum of Hilbert series for $Q/Q^2, \ldots, Q^c/Q^{c+1}$. Since Q^{c+1} is the verbal ideal of Q, corresponding the variety $\mathfrak{N}_c$, adding this to the Hilbert series of P we have determined the Hilbert series of P/Q^{c+1}, which is a relatively free superalgebra in the product variety $\mathfrak{N}_c\mathfrak{V}$. In this way one can determine the Hilbert series for any relatively free polynilpotent Lie superalgebra.

5.1 Characters and Hilbert Series for Solvable (Color) Lie Superalgebras

We shall consider gradings by a semigroup Λ satisfying the conditions given in Section 4.1. Using the same arguments as in [26, Corollary 2.1] we can prove the following lemma.

Lemma 5.1.1. *Let $L = L(X)$ be a free (color) Lie superalgebra freely generated by $X = \bigcup_{\lambda\in\Lambda} X_\lambda$. Let $L = L^1 \supset L^2 \supset \cdots$ be the lower central series and $L = \bigoplus_{n=1}^{\infty} L_n$ the respective gradation. Then*

$$\mathrm{ch}_\Lambda^G L_n = \frac{1}{n}\sum_{k|n} \mu(k) \left(\left(\mathrm{ch}_\Lambda^G\right)^{[k]} X\right)^{\frac{n}{k}},$$

where $\mu(k)$ is the Möbius function.

The following theorem generalizes results from [25, 27] to the color case.

Theorem 5.1.2. *Let $L_{\mathfrak{N}_{c_q}\mathfrak{N}_{c_{k-1}}\cdots\mathfrak{N}_{c_1}}(X)$ be the free polynilpotent (color) Lie superalgebra freely generated by a finitely Λ-graded set $X = \bigcup_{\lambda\in\Lambda} X_\lambda$. We define G-characters $\psi_i^G, \varphi_i^G \in \mathbb{Q}[G][[\Lambda]], i = 0, 1, \ldots, q,$ by setting $\psi_0^G = 0, \varphi_0^G = \mathrm{ch}_\Lambda^G X$, and*

$$\begin{aligned} \psi_i^G = \psi_{i-1}^G + \sum_{m=1}^{c_i} \frac{1}{m} \sum_{k|m} \mu(k)((\varphi_{i-1}^G)^{[k]})^{\frac{m}{k}} \quad &, \quad 1 \le i \le q, \\ \varphi_i^G = 1 + (\mathrm{ch}_\Lambda^G X - 1)\mathcal{E}_G(\psi_i^G) \quad &, \quad 1 \le i \le q. \end{aligned}$$

Then $\mathrm{ch}_\Lambda^G L_{\mathfrak{N}_{c_q}\mathfrak{N}_{c_{k-1}}\cdots\mathfrak{N}_{c_1}}(X) = \psi_q^G$.

Proof. Suppose that $L(X)$ is the free color Lie superalgebra generated by X. We consider the series of ideals

$$L_{(0)} = L(X), L_{(n)} = (L_{(n-1)})^{c_n},\ n = 1, \ldots, q.$$

Suppose that $Y_{(n)}, n = 0, 1, \ldots, q$ is the free homogeneous generating set for $L_{(n)}$. We prove by induction that

$$\mathrm{ch}_\Lambda^G Y_{(n)} = \varphi_n^G, \mathrm{ch}_\Lambda^G(L_{(0)}/L_{(n)}) = \psi_n^G,\ n = 0, 1, \ldots, q.$$

If $n = 0$, this is obvious. For $n > 0$. Using Lemma 5.1.1 and the inductive assumption we obtain

$$\begin{aligned} \mathrm{ch}_\Lambda^G(L_{(0)}/L_{(n)}) &= \mathrm{ch}_\Lambda^G(L_{(0)}/L_{(n-1)}) + \mathrm{ch}_\Lambda^G(L_{(n-1)}/L_{(n)}) \\ &= \psi_{n-1}^G + \sum_{m=1}^{c_n} \frac{1}{m} \sum_{k|m} \mu(k)((\varphi_{n-1}^G)^{[k]})^{\frac{m}{k}} = \psi_n^G. \end{aligned}$$

By Corollary 4.2.4,

$$\begin{aligned}\mathrm{ch}^G_\Lambda Y_{(n)} &= 1 + (\mathrm{ch}^G_\Lambda X - 1)\mathcal{E}_G(\mathrm{ch}^G_\Lambda(L_{(0)}/L_{(n)}))\\ &= 1 + (\mathrm{ch}^G_\Lambda X - 1)\mathcal{E}_G(\psi^G_n) = \varphi^G_n.\end{aligned}$$

In particular, if $n = q$, then $L_{\mathfrak{N}_{c_q}\mathfrak{N}_{c_{k-1}}\cdots\mathfrak{N}_{c_1}}(X) \cong L(X)/L_{(q)}$, and we obtain the desired formula. □

Remark 5.1.3. We can define characters $\psi_i, \varphi_i \in \mathbb{Q}[[\Lambda]], i = 0, 1, \ldots, q$, by the same recursive formulas (as was done in [27] for Lie superalgebras), so φ_i and ψ_i are the images of, respectively, φ^G_i and ψ^G_i under the homomorphism $\mathbb{Q}[G][[\Lambda]] \to \mathbb{Q}[[\Lambda]]$ induced by the augmentation map $\mathbb{Q}[G] \to \mathbb{Q}$. Then Theorem 5.1.2 implies that $\mathrm{ch}_\Lambda L_{\mathfrak{N}_{c_q}\mathfrak{N}_{c_{k-1}}\cdots\mathfrak{N}_{c_1}}(X) = \psi_q$.

As a possible application of the above theorem, we would like to mention the study of growth of finitely generated free polynilpotent color Lie superalgebras, as was done in [13] in the case of Lie superalgebras. Using similar ideas, one can obtain a version of Schreier formula for exponential generating functions, which can be applied to study the so-called codimension growth of varieties of (color) Lie superalgebras (see [24]).

Example 5.1.4. Let $G_+ = \{g_1, \ldots, g_k\}$ and $G_- = \{h_1, \ldots, h_m\}$ ($k = m$ or $m = 0$). Suppose that $L = L(X) = \bigoplus_{g\in G} L_g$ is a free color Lie superalgebra freely generated by $X = \bigcup_{g\in G} X_g = X_+ \cup X_-$, where $|X_{g_i}| = s_i, i = 1, \ldots, k$ and $|X_{h_j}| = q_j, j = 1, \ldots, m$. Consider the grading by the semigroup $\Lambda = \mathbb{N}_0^r \setminus \{0\}$ where $\kappa : \mathbb{N}_0^r \to G \ : \ \lambda_i \mapsto g_i$ and $\nu : G \to \mathbb{Z}_2$ satisfies $\nu(g_i) = 1$ for $1 \le i \le k$ and $\nu(h_j) = -1$ for $1 \le j \le m$. We denote $t_i = e^{\lambda_i}$, $i = 1, \ldots, k$, and $u_j = e^{\lambda_j}$, $j = 1, \ldots, m$. Then, the algebra $\mathbb{Q}[G][[\bar{\Lambda}]]$ turns into the formal power series ring $\mathbb{Q}[G][[t_1, \ldots, t_k, u_1, \ldots, u_m]]$. Let $M(X)$ be a free metabelian on X. Then

$$\mathcal{H}(M(X)) = \mathcal{H}(L/L^{(1)}) + \mathcal{H}(L^{(1)}/L^{(2)}).$$

Now, we have $\mathcal{H}(L/L^{(1)}) = \mathcal{H}(X) = s_1g_1t_1 + \cdots + s_kg_kt_k + q_1h_1u_1 + \cdots + c_mh_mu_m$. The second term is obtained by our Schreier's formula (Corollary 4.2.4) for the free basis Y of the commutator subalgebra $L^{(1)}$:

$$\mathcal{H}(Y) = \frac{(s_1g_1t_1 + \cdots + s_kg_kt_k + q_1h_1u_1 + \cdots + c_mh_mu_m - 1)(1+h_1u_1)^{q_1}\dots(1+h_mu_m)^{q_m}}{(1-g_1t_1)^{s_1}\dots(1-s_kt_k)^{s_k}} + 1.$$

5.2 Special Universal Enveloping Algebra

Let G be a finite abelian group. A (G,γ)-color Lie superalgebra L is called *special* (or *SPI*) if there exists an associative G-graded PI-algebra A such that $L \subset A^{(-)}$, where $A^{(-)}$ is the color Lie superalgebra obtained from A by taking the γ-supercommutator for the bracket (see Section 1.2.3). For the abelian Lie superalgebras, the universal enveloping algebra is color supercommutative, hence a PI-algebra. On the other hand, if $\operatorname{char} F = 0$ then Latyshev proved that the universal enveloping algebra of a Lie algebra is PI if and only if this Lie algebra is abelian. But we can also choose A equal to L as a graded vector space and set the product zero. Thus L is embedded in an associative algebra satisfying the identity $x_1x_2 = 0$. This observation was extended in [2] to the case of Lie algebras in the variety $\mathfrak{N}_c\mathfrak{A}$, which consists of the Lie algebras whose commutator subalgebras are nilpotent of class at most c. The construction of the associative PI-envelope A for L is standard, which leads us to the notion of a *special enveloping algebra* for the algebras in $\mathfrak{N}_c\mathfrak{A}$. In the remainder of this section we generalize the result of [2] to the case of (G,γ)-superalgebras and calculate the Hilbert series for the special enveloping algebras of (G,γ) superalgebras in $\mathfrak{N}_c\mathfrak{A}$.

Theorem 5.2.1. *For any (G,γ)-Lie superalgebra L whose commutator subalgebra $N = L^2$ is nilpotent there is an associative G-graded algebra A, with two-sided graded nilpotent ideal B such that A/B is γ-commutative and L is a subalgebra of $A^{(-)}$.*

Proof. Set $N^0 = L$ and assume $N^{c+1} = \{0\}$. Then choose a graded basis $E = E_0 \cup E_1 \cup \ldots \cup E_c$ so that for each $i = 0, 1, 2, \ldots, c$ the set $E_i \cup \ldots \cup E_c$ is a basis of N^i. Let us assign weight i to the elements of E_i and totally order E in such a way that for any $x, y \in E$ if $\mathrm{wt}(x) \leq \mathrm{wt}(y)$ then $x \leq y$. Now consider the universal enveloping algebra $U(L)$ for L. If $m = e_1 e_2 \cdots e_m$ is an element of the PBW-basis of $U(L)$, we set $\mathrm{wt}(m) = \mathrm{wt}(e_1) + \mathrm{wt}(e_2) + \cdots + \mathrm{wt}(e_m)$. If $u \in U(L)$ is such that $u = \sum_s \alpha_s m_s$, with all $\alpha_s \neq 0$ then we set $\mathrm{wt}(u) = \min_s\{\mathrm{wt}(m_s)\}$.

Since in L, we have $[N^i, N^j] \subset N^{i+j}$, for all $i, j = 0, 1, \ldots, c$, we have that $\mathrm{wt}[x,y]_\gamma \geq \mathrm{wt}(x) + \mathrm{wt}(y)$, for all $x, y \in L$. This enables us to conclude that if $w = f_1, f_2, \ldots, f_m \in E$ then $\mathrm{wt}(f_1 f_2 \cdots f_m) \geq n = \mathrm{wt}(f_1) + \mathrm{wt}(f_2) + \cdots + \mathrm{wt}(f_m)$. Indeed, this is clear if $f_1 \leq f_2 \leq \cdots \leq f_m$. Otherwise, if say, $f_1 > f_2$, then $f_1 f_2 = \gamma(g_1, g_2) f_2 f_1 + [f_1, f_2]_\gamma$. Applying this procedure sufficiently many times, and considering $\mathrm{wt}([f_1, f_2]_\gamma) \geq \mathrm{wt}(f_1) + \mathrm{wt}(f_2)$ we finally arrive to the linear combination of ordered monomials each of which has the same form as w and weight at least n. Then the minimum of weights is at least n and so our claim comes true. An immediate corollary of this is the fact that for any $u, v \in U(L)$ one has $\mathrm{wt}(uv) \geq \mathrm{wt}(u) + \mathrm{wt}(v)$.

Finally, one can consider the set U_s which includes 0 and all elements $u \in U(L)$ satisfying $\mathrm{wt}(u) \geq s$. By what we have just proved, U_s is a two-sided ideal of $U(L)$, for any $s \geq 1$. Since U_s is spanned by the products of graded elements (in E) it is graded. It follows from $N^{c+1} = \{0\}$ that L has no elements of weight greater or equal than $c+1$. Therefore, $L \cap U_{c+1} = \{0\}$. Let us set $A = U(L)/U_{c+1}$ and the consider the natural (graded) homomorphism of associative algebras $\nu : U(L) \to A$. Its restriction is a graded homomorphism of Lie superalgebras $\nu : L \to A^{(-)}$. Since $\ker(\nu) \cap L = U_{c+1} \cap L = \{0\}$, we have that ν isomorphically embeds L into $A^{(-)}$, as graded Lie superalgebras.

If $I = U_1$ then $I^{c+1} \subset U_{c+1}$. Therefore, $J = \nu(I)$ is a nilpotent ideal of $A = \nu(U(L)$. Now A/J is the span of the elements of the form $e + J$ where $e \in E_0$. The color commutators of $e, f \in E_0$ is in L^2 hence is in the linear span of $E_1 \cup \ldots \cup E_c$. Thus A/J is a supercommutative algebra, proving our theorem. □

The construction from the above theorem allows one to introduce a special universal enveloping algebra $SU(L)$ for the algebras in the variety $\mathfrak{N}_c\mathfrak{A}$ by setting $SU(L) = U(L)/U_{c+1}$. As in the setting of Chapter 2, assume that L is graded by semigroup Λ, with the same conditions as requested there. Consider the Λ-character of L, $\mathrm{char}_\Lambda L$, and $\bar{\Lambda}$-character of $SU(L)$, $\mathrm{char}_{\bar{\Lambda}} SU(L)$. to establish the connection between these characters one needs a version of PBW-Theorem for $SU(L)$.

Proposition 5.2.2. *Let $E = E_0 \cup \ldots \cup E_c$ be a totally ordered basis of L chosen in the same way as in the proof of Theorem 5.2.1. Let t be an independent variable. Consider $\mathcal{E}(\sum_{i=0}^{c} t^i ch(N^i/N^{i+1}))$ where the operators of dilation $^{[n]}$ act also on this variable. Discard all terms of degree $c+1$ in t and plug $t = 1$ in the expression obtained. This is* $\mathrm{char}_{\bar{\Lambda}} SU(L)$*, the character of $SU(L)$.*

We illustrate this proposition by finding the special enveloping algebra for the free metabelian Lie algebras.

Theorem 5.2.3. *Let X be a finitely graded set, $L = L(X)$ the free color Lie superalgebra with free basis X, $M = M(X)$ free metabelian Lie color Lie superalgebra, with the same free basis, $SU(M)$ the special universal enveloping algebra for M. Then*

$$\mathrm{char}_{\bar{\Lambda}}(SU(M)) = \mathcal{E}(\mathrm{char}_{\Lambda}(X))((\mathrm{char}_{\Lambda}(X) - 1)\mathcal{E}(\mathrm{char}_{\Lambda}(X)) + 1).$$

Proof. Using the construction of $SU(M)$ described earlier, we obtain the following.

$$\begin{aligned}\mathrm{char}_{\bar{\Lambda}}(SU(M)) &= \mathcal{E}(\mathrm{char}_{\Lambda}(M/M^2))\mathrm{char}_{\Lambda}(M^2) = \mathcal{E}(\mathrm{char}_{\Lambda}(L/L^2))\mathrm{char}_{\Lambda}(L^2/[L^2, L^2]) \\ &= \mathcal{E}(\mathrm{char}_{\Lambda}(X))\mathrm{char}_{\Lambda}(Y),\end{aligned}$$

where Y is a free basis for L^2. This latter can be computed using Schreier formula in Chapter 5. We have the following

$$\mathrm{char}_{\Lambda}(Y) = (\mathrm{char}_{\Lambda}(X) - 1)\mathcal{E}(\mathrm{char}_{\Lambda}(L/L^2)) + 1 = (\mathrm{char}_{\Lambda}(X) - 1)\mathcal{E}(\mathrm{char}_{\Lambda}(X)) + 1.$$

Combining the last two equations, we obtain the formula in the statement of the theorem. □

5.3 Hilbert Series for Centre-by-Metabelian Lie Algebras

Centre-by-metabelian Lie algebras became of interest after Professor Kanta Gupta found a 2-torsion in the free centre-by-metabelian groups. Earlier it was believed that free groups in the varieties given by commutator identical relations, in particular, in the variety of centre-by-metabelian groups given by $[[[x, y], [x, t]], u] = 1$, are torsion-free. In [15, 41] the authors studied free centre-by-metabelian Lie rings and algebras.

We use a basis from [41] to prove the following.

Theorem 5.3.1. *Over any field of characteristic different from* 2 *the Hilbert series of the free centre-by-metabelian Lie algebra* P *of rank* 2 *has the following form:*

$$\mathcal{H}(P,t) = 2t + \frac{t^2}{(1-t)^2} + \frac{2t^5}{1-t^2}.$$

Proof. Let L be a free Lie algebra of rank 2, with free generators x, y. Then $P = L/[L^{(2)}, L]$ where $L^{(2)}$ is the second commutator subalgebra of L. We have the following graded vector space isomorphism of P:

$$P \cong (L/L^2) \oplus (L^2/L^{(2)}) \oplus (L^{(2)}/[L^{(2)}, L]).$$

Thus

$$\mathcal{H}(P,t) = \mathcal{H}(L/L^2, t) \oplus \mathcal{H}(L^2/L^{(2)}, t) \oplus \mathcal{H}(L^{(2)}/[L^{(2)}, L], t).$$

Now we have $\mathcal{H}(L/L^2, t) = \mathcal{H}(\{x, y\}, t) = 2t$. The second term is obtained using Petrogradsky's Schreier series formula for the free basis Y of the commutator subalgebra L^2:

$$\mathcal{H}(Y,t) = (\mathcal{H}(\{x,y\},t) - 1)\mathcal{E}(\mathcal{H}(L/L^2,t)) + 1 = (2t-1)\mathcal{E}(2t) + 1 = \frac{t^2}{(1-t)^2}.$$

Finally, to determine $\mathcal{H}(L^{(2)}/[L^{(2)}, L], t)$, we will use the linear basis of the second commutator of the free centre-by-metabelian algebra found by Kuzmin [15] (we use their form suggest by Umirbaev [41]). All commutators are left normed, that is, $[u, v, w] = [[u, v], w]$. Consider the set E of commutators

$$[[x_j, x_i], [x_l, x_k, x_{i_1}, x_{i_2}, \ldots, x_{i_r}]] \tag{5.1}$$

where $r+4 \geq n+1$, $j > i$, $l > k$, $j \geq l$, $k \leq i \leq i_1 \leq i_2 \leq \ldots \leq i_r$ and, moreover, if $(j,i) = (l,k)$ then r is odd. Then E is a basis of $P^{(2)} \cap P^{n+1}$.

In the case where we have two generators x, y, with $x > y$, the condition if $(j,i) = (l,k)$ is always satisfied, hence the basis is formed by monomials (5.1) of odd degree only. The least degree of monomials is 5 and if we use notation $[u, v^m]$ for $[u, \underbrace{v, \ldots, v}_{m}]$ then the basis is formed by $[[x,y],[x,y^a,x^b]]$ for arbitrary natural $a > 0$, $b \geq 0$, where $a+b$ is an even number. As a result, in the Hilbert series $\mathcal{H}(L^{(2)}/[L^{(2)},L],t)$ the coefficients for t^{2k} are equal zero while the coefficient of t^{2k+1} equals $2k-2$, $k = 2, 3, \ldots$. Hence, we have the series

$$\mathcal{H}(L^{(2)}/[L^{(2)},L],t) = \sum_{k=2}^{\infty}(2k-2)t^{2k+1} = \frac{2t^5}{1-t^2}.$$

□

5.4 Characters and Hilbert Series for Solvable Restricted (Color) Lie Superalgebras

In this section Λ denotes a countable additive abelian semigroup satisfying the conditions given in Section 4.1.

Proposition 5.4.1. *Assume $\Lambda = \Lambda_+$ and let $B = \bigcup_{\lambda\in\Lambda} B_\lambda$ be a finitely Λ-graded set. Let V be a vector space with basis B over a field F of characteristic $p > 0$. Consider a vector space $\tilde{V}$ with basis $\tilde{B} = \left\{b^{[p^n]} \mid b \in B, n = 0, 1, \ldots\right\}$, where $\deg b^{[p^n]} = p^n \deg b$, for each $b \in B$. Then*

$$\mathcal{E}_p(\mathrm{ch}_\Lambda \tilde{V}) = \mathcal{E}(\mathrm{ch}_\Lambda V),$$

where $\mathcal{E}_p$ is defined in Remark 2.2.5.

Proof. Let us view V as an abelian Lie algebra and let $U(V)$ be its universal enveloping algebra. Then $\mathrm{ch}_{\bar{\Lambda}}U(V) = \mathcal{E}(\mathrm{ch}_\Lambda V)$ (Theorem 2.1.5). Inside $U(V)$, which is actually isomorphic to the polynomial algebra $S(V)$, we consider the elements b^{p^n}, for $b \in B$, $n = 0, 1, 2, \ldots$. These elements are linearly independent and they form an abelian Lie p-algebra, which we can identify with $\tilde{V}$. The character of a Lie p-algebra and that of restricted enveloping algebra are related by the formula $\mathrm{ch}_{\bar{\Lambda}}u(\tilde{V}) = \mathcal{E}_p(\mathrm{ch}_\Lambda \tilde{V})$ (Remark 2.2.5). Since $u(\tilde{V}) \cong U(V)$, we have $\mathcal{E}_p(\mathrm{ch}_\Lambda \tilde{V}) = \mathcal{E}(\mathrm{ch}_\Lambda V)$, as claimed. □

It is convenient to define a new operator $\mathcal{J}_p : \mathbb{Q}[[\Lambda]] \to \mathbb{Q}[[\Lambda]]$ as follows. If $\lambda \in \Lambda_+$, then we set $\mathcal{J}_p(e^\lambda) = e^\lambda + e^{\lambda p} + e^{\lambda p^2} + \cdots$. If $\lambda \in \Lambda_-$ then we set $\mathcal{J}_p(e^\lambda) = e^\lambda$. So, if we consider a vector space V with basis $B_+ \cup B_-$ and a vector space $\tilde{V}$ with basis $\tilde{B}_+ \cup B_-$, then $\mathrm{ch}_\Lambda \tilde{V} = \mathcal{J}_p(\mathrm{ch}_\Lambda B_+) + \mathrm{ch}_\Lambda B_- = \mathcal{J}_p(\mathrm{ch}_\Lambda V)$.

Corollary 5.4.2. *For V and $\tilde{V}$ as above, we have*

1. $\mathrm{ch}_\Lambda V = \mathcal{L}(\mathcal{E}_p(\mathrm{ch}_\Lambda \tilde{V}))$,
2. $\mathrm{ch}_\Lambda \tilde{V} = \mathcal{J}_p(\mathrm{ch}_\Lambda V)$.

Theorem 5.4.3. *Let L be the relatively free (color) Lie p-superalgebra in the variety of Lie p-superalgebras $\mathfrak{N}_{c_q}\mathfrak{N}_{c_{k-1}}\cdots\mathfrak{N}_{c_1}$ freely generated by a finitely Λ-graded set $X = \bigcup_{\lambda\in\Lambda} X_\lambda$. Let the characters $\varphi_0, \ldots, \varphi_q, \psi_0, \ldots, \psi_q$ be defined as in Remark 5.1.3. Then*

$$\mathrm{ch}_\Lambda L = \mathcal{J}_p(\psi_q).$$

Proof. This follows directly from Theorem 5.1.2 and Corollary 5.4.2. □

Corollary 5.4.4. *Let L as in the above theorem, $X = \{x_i \mid i \in I\}$, $I = \bigcup_{g\in G} I_g$ and $X_g = \left\{x_g^i \mid i \in I_g\right\}$, where I is finite or countable. Let $I_+ = \bigcup_{g\in G_+} I_g$ and $I_- =$*

$\bigcup_{g\in G_-} I_g$. *Define* $\psi_i(\boldsymbol{t}), \varphi_i(\boldsymbol{t}) \in \mathbb{Q}[[\boldsymbol{t}]]$ *as in [27, Corollary 4.1]. Then*

$$\mathcal{H}(L, \boldsymbol{t}) = \mathcal{J}_p(\psi_q(\boldsymbol{t})).$$

Let us consider the case of free metabelian Lie p-superalgebra $[M(X)]^p$. For a free metabelian Lie superalgebra $M(X)$ we have the following: the character for $L(X)/L(X)^{(1)}$ is $\mathrm{ch}_\Lambda X$. The free basis Y for $L(X)^{(1)}$, hence the vector space basis for $L(X)^{(1)}/L(X)^{(2)}$, is given by Schreier's formula

$$\mathrm{ch}_\Lambda Y - 1 = (\mathrm{ch}_\Lambda X - 1)\mathcal{E}(\mathrm{ch}_\Lambda X).$$

So the character of $M(X)$ equals

$$\mathrm{ch}_\Lambda M(X) = \mathrm{ch}_\Lambda X + (\mathrm{ch}_\Lambda X - 1)\mathcal{E}(\mathrm{ch}_\Lambda X) + 1.$$

Example 5.4.5. Consider a free metabelian Lie p-superalgebra generated by X with $|X| = 2$ in three cases:

- Case 1: $X = X_+$ with natural grading by $\mathbb{N}$. Then $\mathrm{ch}X = 2t_0$, and $\mathrm{ch}Y = \frac{2t_0-1}{(1-t_0)^2} + 1 = \frac{t_0^2}{(1-t_0)^2}$, and then $\mathrm{ch}M(X) = 2t_0 + \frac{t_0^2}{(1-t_0)^2}$. Therefore, $\mathrm{ch}[M(X)]^p = \mathcal{J}_p\left(\frac{2t_0-3t_0^2+2t_0^3}{(1-t_0)^2}\right)$. This case includes the free metabelian p-algebra, whose basis was founded by Artamonov [1].

- Case 2: $X = X_-$ with natural grading by $\mathbb{N}$. Then $\mathrm{ch}X = 2t_1$, and $\mathrm{ch}Y = (2t_1 - 1)(1+t_1)^2 + 1 = 3t_1^2 + 2t_1^3$, and so $\mathrm{ch}M(X) = 2t_1 + 3t_1^2 + 2t_1^3$. Hence, $\mathrm{ch}[M(X)]^p = 3\mathcal{J}_p(t_1^2) + 2t_1 + 2t_1^3 = 3(t_1^2 + t_1^{2p} + t_1^{2p^2} + \cdots) + 2t_1 + 2t_1^3$.

- Case 3: $|X_+| = |X_-| = 1$ with natural grading by $\mathbb{N}\times\mathbb{N}$. Then $\mathrm{ch}X = t_0 + t_1$, and $\mathrm{ch}Y = (t_0 + t_1 - 1)\frac{1+t_1}{1-t_0} + 1$, so that after some calculations, we can get

$$\mathrm{ch}M(X) = \frac{t_0+t_1^2-t_0^2}{1-t_0} + \frac{t_1}{1-t_0}. \text{ Hence, } \mathrm{ch}[M(X)]^p = \mathcal{J}_p\left(t_0 + \frac{t_1^2}{1-t_0}\right) + \frac{t_1}{1-t_0}.$$

Indeed, in the latter case the basis of $[M(X)]^p$ is given by (where $X_+ = \{x_1\}$ and $X_- = \{x_2\}$)

$$x_1^{[p^n]}, [x_2, x_2, \underbrace{x_1, x_1, \ldots, x_1}_{k-\text{times}}]^{[p^n]} \text{ and } [x_2, \underbrace{x_1, \ldots, x_1}_{l-\text{times}}], n, l, k = 0, 1, 2, \ldots.$$

Chapter 6

Relative Growth Rate of Subalgebras of Color Lie Superalgebras

Hilbert series can be used to study the growth rate of a finitely generated algebra. Let S be an associative or a Lie algebra generated by a finite subset X, and consider the filtration associated to X, namely, let $S_n(X)$ be the F-subspace of S spanned by all monomials of length less than or equal to n in the elements of X. The *growth function* of S with respect to X is defined by

$$\gamma_S(n) = \dim S_n(X).$$

In the case of groups, if $X = \{x_1, \ldots, x_k\}$ is a generating set of a group H, then the growth function of H is defined to be the number of elements of H that can be written as words of length at most n in terms of the generators and their inverses; it is the same as the growth function of the group algebra FH, $\gamma_{FH}(n)$, with respect to the generating set $X \cup X^{-1}$, where $X^{-1} = \left\{x_1^{-1}, \dots, x_k^{-1}\right\}$. The limit

$$\lim_{n\to\infty} (\gamma_S(n))^{\frac{1}{n}}$$

always exists and does not depend on X (see e.g., [37]); it is called the *growth rate* (or *exponent* or *entropy*) of S. We will denote it by α_S.

1. If $\lim_{n\to\infty} (\gamma_S(n))^{\frac{1}{n}} > 1$, then S has *exponential growth*, otherwise, S has *subexponential growth*.

2. If there exists a polynomial p with $\gamma_S(n) \leq p(n)$ for all sufficiently large n, then S has *polynomially bounded growth*.

3. If S has subexponential and not polynomially bounded growths, then S has *intermediate growth* (that is, S lies between polynomial and exponential).

Let $\lambda_S(n) = \gamma_S(n) - \gamma_S(n-1)$. It is known that S has subexponential growth if and only if $\lim\sup_{n\to\infty} (\lambda_S(n))^{\frac{1}{n}} \leq 1$ ([37]).

The *relative growth rate* of a subalgebra A of S is defined by

$$\alpha_A = \text{limsup}_{n\to\infty}(\gamma_A(n) - \gamma_A(n-1))^{\frac{1}{n}},$$

where $\gamma_A(n) = \dim(A \cap S_n(X))$. The relative growth rate of A could also be defined as $\lim_{n\to\infty}\gamma_A(n)^{\frac{1}{n}}$, but the former definition is preferred as it allows us to compute the relative growth rate as the inverse of the radius of convergence of the Hilbert series of the subalgebra A.

6.1 Equality of the Growth Rate of the Free Color Lie Superalgebra and Its Enveloping Algebra

Let $L = L(X)$ be a free color Lie superalgebra freely generated by the set $X = X_+ \cup X_-$, where $|X_+| = r$ and $|X_-| = s$. Recall that the dimension of the homogeneous subspaces L_n is given by Witt formula (Corollary 2.1.14)

$$\mathrm{dim} L_n = \frac{1}{n} \sum_{m|n} \mu(m)(r - (-1)^m s)^{\frac{n}{m}}.$$

Theorem 6.1.1. *The free color Lie superalgebra L and its enveloping algebra have the same growth rate.*

Proof. If $m \geq 2$, then $|\mu(m)(r-(-1)^m s|^{\frac{n}{m}} \leq (r+s)^{\frac{n}{2}}$. Since $\lim_{n\to\infty} \sum_{m=2}^{n} \frac{1}{(r+s)^{\frac{n}{2}}} = 0$, we have

$$\lim_{n\to\infty} \frac{n\mathrm{dim} L_n}{(r+s)^n} = 1.$$

Hence, for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\left| \frac{n\mathrm{dim} L_n}{(r+s)^n} - 1 \right| < \epsilon,$$

that is

$$\frac{(r+s)^n}{n}(1-\epsilon) \leq \mathrm{dim} L_n \leq \frac{(r+s)^n}{n}(1+\epsilon).$$

As $\lim_{n\to\infty} \left(\frac{(r+s)^n}{n}(1-\epsilon) \right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{(r+s)^n}{n}(1+\epsilon) \right)^{\frac{1}{n}} = r+s$, we have

$$\lim_{n\to\infty} (\lambda_L(n))^{\frac{1}{n}} = r+s.$$

On the other hand, the universal enveloping algebra of L is the free associative algebra generated by X, and so its growth rate is $r+s$. $\square$

6.2 Growth of Finitely Generated Subalgebras of Free Color Lie Superalgebras

The relative growth rate of a finitely generated subalgebra K of a free Lie algebra L of finite rank is strictly less than the growth rate of the free Lie algebra itself ([7, Theorem 1]). The following example shows that this theorem cannot be generalized to the case of free color Lie superalgebras.

Example 6.2.1. Let L be a free color Lie superalgebra freely generated by $X = X_+ \cup X_-$ where $X_+ = \{x_1, \ldots, x_r\}$ and $X_- = \{y\}$. Let us consider the subalgebra K of L where $L = K \oplus \langle\{y\}\rangle_F$. It is well known that K is freely generated by the following set (see [4, page 56])

$$Y = \{x, [x, y] \mid x \in X \setminus \{y\}\} \cup \{[y, y]\}.$$

Therefore, $\mathrm{rank}K = 2r + 1$. According to the Schreier formula

$$\mathrm{rank}(K) = 2^s\,(\mathrm{rank}(L) - 1) + 1,$$

we have $\dim L/K = 1$; that is K is a subspace of L of finite codimension. Consequently, the growth rate of L/K is 0. Therefore, the growth rate of K equals to the growth rate of L.

Let L be a color Lie superalgebra generated by X, and $K \subseteq L$ be a subalgebra, not necessarily homogeneous with respect to the natural $\mathbb{N}$-grading of L. We note that the relative growth of K is the same as the relative growth of the associated graded subspace $\mathrm{gr}K$, the linear span of the leading parts (LP) of nonzero elements of K. In the following result we show that the theorem above extends to the case $L = L_+$.

Theorem 6.2.2. *Let $L = L_+$ be a free color Lie superalgebra freely generated by $X = \{x_1, \ldots, x_r\}$, and K be a finitely generated proper subalgebra of L. Then the relative growth rate of K is strictly less than the growth rate of the free color Lie superalgebra itself.*

Proof. In our proof, we follow [7]. We break the proof into the following two steps:

Step 1: Assume K is homogeneous. Let $\{z_1, \ldots, z_k\}$ be a free homogeneous basis of K. As K is proper, the intersection $K \cap L_1$ is a proper subspace of L_1. Let Z be the set of all z_i that have degrees 1. This is a basis of the space $K \cap L_1$, which can be extended to a basis Z' of the space L_1. Hence, Z' is a new free basis of L that properly includes Z. Replacing X with Z', we may assume without loss of generality that the elements of degree 1 in the free homogeneous basis $\{z_1, \ldots, z_k\}$ of K are contained in X, and that x_1 is not in K. Let us consider L as a subalgebra in the free associative algebra $A = A\langle X\rangle$. We can write $z_1, \ldots, z_k$ as linear combination of finitely many monomials $u_1, \ldots, u_t$ in A. Let B be a subalgebra of A generated by $\{u_1, \ldots, u_t\}$, so $K \subseteq B$, and it is sufficient to to show that the exponent of the growth of B is less than r, which is also the exponent of L and also of A according Theorem 6.1.1. As $x_1 \notin K$, we have $x_1 \notin \{u_1, \ldots, u_t\}$. Also, none of $u_1, \ldots, u_t$ is a power of x_1, since no such monomial can appear while writing the elements of the free color Lie superalgebra $L(X)$. (Note that this statement is not true for general free color Lie superalgebras.) Let $m = \max\{\deg u_1, \ldots, \deg u_t\}$ (the degree with respect to $x_1, \ldots, x_r$), and choose $d \geq 2m$. Since each of the monomials $u_1, \ldots, u_t$ contains as a factor a letter different from x_1, no product of these monomials can contain a subword x_1^d. Indeed, the maximum length of uninterrupted string of x_1 in the product

u_iu_j is $\deg u_i + \deg u_j - 2 \leq 2m - 2 < d$. According to [6, Lemma 8], there exists $C, \epsilon > 0$ such that the number of words of length n which do not have x_1^d as a subword is bounded by $C(r-\epsilon)^n$. That is, the exponent of B is strictly less than r, as desired.

Step 2: the general case. We will use the associated graded algebra gr$K$, which is a proper homogeneous subalgebra of L. Following [4], we will say that a subset M of G-homogeneous elements in $L(X)$ is called reduced if the leading part of any of its elements does not belong to the subalgebra generated by the leading parts of the remaining elements of M. By [4, Lemma 3.12], there exists a reduced subset M in K such that M generates K. Also, by [4, Theorem 3.15], M is independent, and so M is a free generating set of K. Let us denote

$$M' = \{\mathrm{LP}(m) \mid m \in M\},$$

the set of leading parts of the elements in M. As in [7, Lemma 5], one shows that M' is a free generating set of gr$K$. According the first step, the growth rate of gr$K$ is strictly less than r. □

Chapter 7

Schreier Formulas for Other Objects

Schreier techniques have been applied to a variety of objects, notably to absolutely free, free commutative and free anticommutative algebras [29], to free modules over free associative algebras and over group algebras of free groups [18], and to free actions by free monoids [5]. Here we will generalize some of these results and discuss some applications.

7.1 Actions by Free Monoids

Let X be a nonempty set (will be called an alphabet and its elements will be called letters). Then the set of all words over X including the empty words denoted by 1, X^*, is a monoid under the juxtaposition (concatenation)

$$(u, v) \mapsto uv.$$

A generating subset B of a monoid S is a base if and whenever

$$b_1 \dots b_n = c_1 \dots c_m$$

with $b_1, \dots, b_n, c_1, \dots, c_m \in B$, then $n = m$ and $b_i = c_i$ $\forall i = 1, \dots, n$. A monoid M is free if and only if there exists a base B such that $B^* = M$. B is called the base of M. The cardinality of B depends only on S and it is called the rank of S.
A monoid S acts on a nonempty set M if there is a map $\mu : M \times S \to M$ (we write $\mu(m, s) = ms$ for $s \in S$ and $m \in M$) satisfying the following conditions:

1. $m(st) = (ms)t$,

2. $m = m1$,

for all $s, t \in S$ and $m \in M$. In this case we say M is a (right) act over S. A subset B of M is a generating set if $M = BS$. Given an S-act M with a nonempty generating subset B, we say that B is a basis of M if whenever $b_1 s_1 = b_2 s_2$ ($b_1, b_2 \in B$, and $s_1, s_2 \in S$), then $b_1 = b_2$, and $s_1 = s_2$. In this case M is free. Also, if $|B| = r$, then M is free of rank r and we write rank$M = r$.

For an element $s = s_1 \dots s_k, s_i \in S$ of a free monoid S^*, the number k is the weight of s. Also, if M is a free (right) act with basis A over S^*, and M is a finitely graded set, then the weight of ms, $m \in M$ and $s \in S^*$, is the sum of the weights of m and s. Bahturin and Olshanskii obtained the following analogue of Schreier's formula for the free monoids.

Theorem 7.1.1 ([5]). *Let M be a free (right) act with basis A over a free monoid S with a basis X, and let P be a subact. Then P is free. Moreover, if A and X are* *finitely graded sets, and B is a basis of P, then*

$$\mathcal{H}(B,t) = \mathcal{H}(A,t) + \mathcal{H}(M \setminus P,t)(\mathcal{H}(X,t) - 1).$$

Note that in this formula the factor space is replaced by a set theoretic complement $M \setminus P$.

Now we establish a more general result.

Theorem 7.1.2. *Let Λ be a countable additive abelian semigroup with finiteness condition. Suppose that P is a subact of a free (right) act M with a basis A over a free monoid S with a basis X. If A and X are Λ-finitely graded set and B is a basis of P, then*

$$\mathrm{ch}_\Lambda B = \mathrm{ch}_\Lambda A + \mathrm{ch}_\Lambda (M \setminus P)(\mathrm{ch}_\Lambda X - 1).$$

Proof. The characters for M and P are equal to

$$\mathrm{ch}_\Lambda M = \mathrm{ch}_\Lambda A \mathrm{ch}_\Lambda S \text{ and } \mathrm{ch}_\Lambda P = \mathrm{ch}_\Lambda B \mathrm{ch}_\Lambda S,$$

respectively. The first claim follows from the definition, the second follows from the freeness of P. Obviously, $\mathrm{ch}_\Lambda M = \mathrm{ch}_\Lambda P + \mathrm{ch}_\Lambda (M \setminus P)$. So

$$\mathrm{ch}_\Lambda (A) \mathrm{ch}_\Lambda (S) = \mathrm{ch}_\Lambda (B) \mathrm{ch}_\Lambda (S) + \mathrm{ch}_\Lambda (M \setminus P). \tag{7.1}$$

According to Lemma 2.1.9, we have

$$\mathrm{ch}_\Lambda S = \frac{1}{1 - \mathrm{ch}_\Lambda X}.$$

Now, applying 7.1 we obtain the desired formula

$$\begin{aligned} \mathrm{ch}_\Lambda B &= \mathrm{ch}_\Lambda A - \mathrm{ch}_\Lambda (M \setminus P)(1 - \mathrm{ch}_\Lambda X) \\ &= \mathrm{ch}_\Lambda A + \mathrm{ch}_\Lambda (M \setminus P)(\mathrm{ch}_\Lambda X - 1). \end{aligned}$$

□

Corollary 7.1.3. *Let Λ be a countable additive abelian semigroup with finiteness condition. Let X be an alphabet with Λ-character $\mathrm{ch}_\Lambda X$, and let v be a word in X with weight λ relative to Λ ($\mathrm{wt}_\Lambda v = \lambda$). Suppose that W_v is the set of words which do not contain v as a prefix. Then*

$$\mathrm{ch}_\Lambda W_v = \frac{1 - e^\lambda}{1 - \mathrm{ch}_\Lambda X}.$$

Proof. Consider X^* as free right X^* act with one generator 1, and P as subact vX^* (the set of words over X that contain prefix v). The free basis B of P is v. Hence $\mathrm{ch}_\Lambda B = e^\lambda$. Since $W_v = X^* \setminus P$, we have $\mathrm{ch}_\Lambda W_v = \mathrm{ch}_\Lambda (X^* \setminus P)$. By Theorem 7.1.2, we obtain

$$e^\lambda = 1 + \mathrm{ch}_\Lambda W_v (\mathrm{ch}_\Lambda X - 1).$$

Hence

$$\mathrm{ch}_\Lambda W_v = \frac{1 - e^\lambda}{1 - \mathrm{ch}_\Lambda X}.$$

□

Corollary 7.1.4. *Consider the grading by $\Lambda = \mathbb{N}$. Let $X = \{x_1, \ldots, x_r\}$, $\mathrm{wt} x_i = 1, i = 1, \ldots, r$. If v is a word in X with $\mathrm{wt}_\Lambda v = m$, then*

$$\mathcal{H}(W_v, t) = \frac{t^m - 1}{rt - 1},$$

where W_v is the set of words which do not contain v as a prefix.

7.2 Actions by Free Associative Algebras

Cohn ([8, Section 4.5]) obtained the following formula for the generating function of the set of generators of an arbitrary right ideal of free associative algebras.

Theorem 7.2.1. *Let X be a finitely graded set, $A(X)$ a free associative algebra with free generating set X, and I an arbitrary right ideal of $A(X)$. Then I is a free right $A(X)$-module and, for any basis B of I (as a right $A(X)$-module), we have*

$$\mathcal{H}(B,t) = \mathcal{H}((A(X)/I,t))(\mathcal{H}(X,t)-1)+1.$$

This result in fact holds more generally.

Theorem 7.2.2. *Let $A = A(X)$ be a free associative algebra freely generated by a finitely graded set X. Suppose N is a submodule of free left (right) A-module M of rank s. Then, for any basis B of N, we have*

$$\mathcal{H}(B,t) = \mathcal{H}(M/N,t)(\mathcal{H}(X,t)-1)+s.$$

Proof. $M \cong A_1 \oplus \cdots \oplus A_s$ where $A_i \cong A, i = 1,\ldots,s$. The claim is clearly true if $s = 1$. So we proceed by induction. For $s > 1$. Then $N \cap A_1$ is a submodule of A_1 (rank$A_1 = 1$). Also, by the second isomorphism theorem $N/(N \cap A_1)$ is isomorphic to $(N + A_1)/A_1$ which is isomorphic to a submodule (say Q) of $A_2 \oplus \cdots \oplus A_s$. Therefore, $N \cong P \oplus Q$ where N is a submodule of A_1 and Q is a submodule of $A_2 \oplus \cdots \oplus A_s$. Hence, any base B of N is the union of a basis B_1 of P and a basis B_2 of Q. Using the inductive hypothesis, we have

$$\begin{aligned}\mathcal{H}(B,t) &= \mathcal{H}(B_1,t)+\mathcal{H}(B_2,t)\\ &= \mathcal{H}(A_1/P,t)(\mathcal{H}(X,t)-1)+1+\mathcal{H}(A_2/Q,t)(\mathcal{H}(X,t)-1)+s-1\\ &= \mathcal{H}(M/N,t)(\mathcal{H}(X,t)-1)+s.\end{aligned}$$

□

7.3 Schreier Transversals and Schreier Free Bases of the Commutator Subgroup of the Free Group

In [5] the authors have suggested the generalization of Schreier formula for an even subgroup H (that is, generated by elements of even length) of a free group F_r of finite rank r. In their formula the rank of the subgroup does not need to be a finite number. Instead, the authors have considered certain power series for a specially constructed free basis of H and for the set F_r/H of right cosets of H. The notion of degree for an element w in the case of groups is replaced by the minimal length of word representing w. The notion of degree for a right coset Hu is replaced by the minimal length of elements in the coset Hu.

One of the basic notions is that of Schreier transversal T for a subgroup H of a free group F_r, with a symmetric group alphabet $X = \{x_1^{\pm 1}, \ldots, x_r^{\pm 1}\}$. This is the set of reduced words (without subwords of the form yy^{-1}, $y \in X$) such that, for any $u, v \in T$, $Hu = Hv$ implies $u = v$, $F_r = \cup_{u \in T} Hu$ and, finally, any prefix of a word in T is itself in T. If $w \in F_r$ is such that $Hw = Hu$, where $u \in T$, then we write $u = \overline{w}$. Given a Schreier transversal T for H in F_r one can construct a Schreier free basis B for H as follows. If $u \in T$ and $y \in X$, consider $uy\overline{uy}^{-1}$. It is well-known [17, Section 1.3] that the set of these latter elements different from e, forms a free basis of the subgroup H. A Schreier transversal is called geodesic if the length of $u \in T$ is minimal among the lengths of all elements in Hu. It is proven in [5] that the lengths of the elements in the Schreier basis B for H and the lengths of the elements of the geodesic Schreier transversal T for H, in the case where H is even are related by the following Schreier-type formula. Let b_n be the number of elements of length n in the (symmetric) Schreier basis B for H, c_n the number of elements of length n in the Schreier transversal T, $\mathcal{H}(B,t) = \sum_{n=1}^{\infty} b_n t^n$, $\mathcal{H}(F_r/H,t) = \sum_{n=0}^{\infty} c_n t^n$. Then

$$\mathcal{H}(B,t) = 2\left(\frac{2rt^2}{t^2+1} - 1\right)\mathcal{H}(F_r/H,t^2) + 2. \tag{7.2}$$

In this section we compute the Hilbert series for a free basis of the commutator subgroup F_2' of the free group, based on the fact that the factor group F_2/F_2' is a free abelian group of rank 2. We will compare this with the basis of F_2' constructed with the help of the Schreier transversal T for F_2'.

It is convenient to use the geometric approach to the set of right cosets of a subgroup H in F_r, as follows. We consider a graph $\Gamma(F_r/H)$ whose vertices are the cosets Hu, $u \in F_r$. We connect Hu with Hv with an edge e if there is $y \in X$ such that $Huy = Hv$. We assign to e the label y. The inverse edge e^{-1} goes from Hv to Hu and its label is y^{-1}. The length of Hu is the length of the shortest path on the graph from the origin H to Hu. If we multiply consecutively all labels of edges of the minimal path, then we arrive at a reduced word of shortest length in Hu. Thus the number c_n defined earlier is the number of vertices in $\Gamma(F_r/H)$ in the distance n from the origin. One calls the set of vertices at the distance at most n from the origin the ball of radius n and denote $B_n(F_r/H)$. The set $S_n(F_r/H) = B_n(F_r/H) \setminus B_{n-1}(F_r/H)$ is called the sphere of radius n in $\Gamma(F_r/H)$. To obtain a Schreier transversal for a subgroup H one has to consider in $\Gamma(F_r/H)$ a maximal subtree $\mathcal{T}$ with root at the origin. If $u \in F_r$ then one has to take a vertex Hu of $\Gamma(F_r/H)$. Because $\mathcal{T}$ is maximal, there is a unique path π on the tree from the root to Hu. If we multiply the labels of all edges of π in order, we obtain $\overline{u} \in T$. If now $y \in X$ then one has to consider an edge e from Hu to Huy, with label y. If e is not on $\mathcal{T}$ then we consider the loop λ from the origin to Hu to Huy and back to the origin. The consecutive product of the labels of edges of λ, also called the label of λ, is a nontrivial element of H, one of the elements of the free Schreier basis B of H. It easily follows that the elements of B are in one-one correspondence with the non-tree edges of $\Gamma(F_r/H)$.

In the case of $F_2 = F(x, y)$, $H = F_2'$, $\Gamma(F_2/H)$ is the set if integral points on the real plane, the edges being oriented horizontal and vertical unit intervals. Each point with coordinates (k, l) represents the right coset $Hx^k y^l$. There is an edge from (k, l) to $(k, l + 1)$ with label y, etc. Altogether, four edges start from each point (k, l) and four edges end in (k, l), with labels $x^{\pm 1}$ and $y^{\pm 1}$. The sphere of each radius n in this graph is the rhombus $|k| + |l| = n$. The set $T = \{x^k y^l \mid k, l \in \mathbb{Z}\}$ is therefore a geodesic Schreier transversal for $H = F_2'$. The number of elements in each sphere S_n (here rhombus) is $4n$, except S_0, which has one point. The Hilbert series of this set is thus

$$\mathcal{H}(F_r/H, t) = 1 + \sum_{n=1}^{\infty} 4nt^n = \frac{(1+t)^2}{(1-t)^2}.$$

Plugging this series in (7.2), we have the following

$$\mathcal{H}(B, t) = 2\left(\frac{4t^2}{t^2+1} - 1\right)\frac{(1+t^2)^2}{(1-t^2)^2} + 2 = 2\frac{(3t^2-1)(1+t^2)}{(1-t^2)^2} + 2 = \frac{8t^4}{(1-t^2)^2}.$$

To confirm this formula by direct computation, we will find the Schreier geodesic basis arising from the above geodesic Schreier transversal. Note that geometrically, the corresponding maximal tree consists of the horizontal line $l = 0$ as a "trunk" and the vertical lines $k = c$ as branches. In each point (k, l) with $l \neq 0$ there are two non-tree (horizontal) edges, with labels x and x^{-1}. They go to the vertices $(k+1, l)$ $(k-1, l)$. On the loop λ going on the tree from $(0, 0)$ to (k, l), then, say, to $(k+1, l)$, then on the tree back to $(0, 0)$, we read the label $x^k y^l x y^{-l} x^{-k-1}$, which is one of the elements of the free basis of F_2'. If we follow the opposite direction from (k, l), then we obtain another generator $x^k y^l x^{-1} y^{-l} x^{-k+1}$. We do this for every vertex (k, l), $l \neq 0$. If $k \neq 0$, then one of the generators has length $2|k| + 2|l| + 2$, while the other $2|k| + 2|l|$. If $k = 0$, both generators have length $2|k| + 2|l| + 2 = 2|l| + 2$.

As a result, all, except four points on the rhombus $|k| + |l| = n$ ($4n - 4$, in total) produce $4n - 4$ free generators of F_2' of length $2n + 2$ and $4n - 4$ free generators of length $2n$. The vertices $(0, n)$ and $(0, -n)$ produce 4 free generators of length $2n + 2$. The vertices $(n, 0)$ and $(-n, 0)$ do not produce free generators. Altogether, from this rhombus we have $4n$ free generators of length $2n + 2$ and $4n - 4$ free generators of length $2n$. Hence the total number of free generators of length $2n$ is equal to $4(n - 1) + (4n - 4) = 8n - 8$. This allows us to write the Hilbert series for this particular free basis of F_2' in the form $\sum_{n=1}^{\infty}(8n - 4)t^{2n} = \frac{8t^4}{(1-t^2)^2}$. It should be noted that this series generates the *symmetric* free basis. Indeed, the label of the loop λ^e built using a point (k, l) going "eastward", that is, with the help of label x is an inverse element for the loop λ^w built using a point (k, l) going "westward", that is, with the help of label x^{-1}. This explicit construction confirms the series we have obtained using Bahturin - Olshanskii's formula (7.2).

We remark that the number of geodesic Schreier transversals for F_2' is uncountable, because we can draw uncountably many maximal subtrees in the graph $\Gamma(F_2/F_2')$, which is an integral lattice on the real plane. This is an inductive process: once we have drawn a maximal subtree in the sphere S_n of radius n (rhombus $|k| + |l| = n$) we arbitrarily connect each point of S_{n+1} with exactly one point in S_n. By induction, this process leads to a maximal subtree $\mathcal{T}$ of Γ. To make sure the tree is geodesic, the branches must be shortest paths from the root.

As a result of our calculations, we have obtained a free geodesic Schreier basis of F_2' is the form

$$\{w_{k,l} = x^{k-1}y^l x y^{-l} x^k, \text{ where } k, l \in \mathbb{Z}, \ l \neq 0\}.$$

There are a number of other known bases of F_2', which are not Schreier, for example $\{u_{k,l} = [x^k, y^l] \mid k, l \in \mathbb{Z}, \ k, l \neq 0\}$. Many bases or various terms of the lower central or derived series of free groups are found in [10]. One, for F_2' consists of the commutators $[x, \underbrace{y^{\pm 1}, ..., y^{\pm 1}}_{k}, \underbrace{x^{\pm 1}, ..., x^{\pm 1}}_{l}]$. Clearly, the series for various bases can be very different. Actually, even for F_2 an application of a Tietze transformation produces a free basis $\{x, yx\}$. The Hilbert series for the standard basis is $2t$ while for the modified one $t + t^2$.

7.4 Subgroups of Infinite Index in Free Groups

Suppose X is a finitely graded set. Set $X_0 = \emptyset$. Then, we define a formal power series

$$\tilde{\mathcal{H}}(X, t) = \sum_{n=1}^{\infty} a(n) t^n,$$

where, for each $n \geq 1$,

$$a(n) = \frac{1}{4}|X_{2n-2}| + \frac{1}{2}|X_{2n-1}| + \frac{1}{4}|X_{2n}|.$$

The following theorem is a generalization of the Schreier index theorem in the case of subgroups of infinite index (see [5]).

Theorem 7.4.1. *Let F_r be a free group of rank $r \geq 1$, and let H be a subgroup of F_r. Then there exists a symmetric basis B of H such that*

$$\tilde{\mathcal{H}}(B,t) = \left(rt - \frac{t+1}{2}\right)\mathcal{H}\left(F_r/H,t\right) + \frac{t+1}{2}.$$

Let X be a finitely graded set, and consider the formal power series $\tilde{\mathcal{H}}(X,t)$. We could write

$$\begin{aligned}\sum_{n=1}^{\infty} |X_{2n}|\, t^n &= \sum_{n=1}^{\infty} |X_{2n}| \left(\sqrt{t}\right)^{2n} \\ &= \frac{1}{2}\left(\mathcal{H}(X,\sqrt{t})\right) + \left(\mathcal{H}(X,-\sqrt{t})\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{n=1}^{\infty} |X_{2n-2}|\, t^n &= \left(\sqrt{t}\right)^2 \sum_{n=1}^{\infty} |X_{2n-2}| \left(\sqrt{t}\right)^{2n-2} \\ &= \frac{\left(\sqrt{t}\right)^2}{2}\left(\mathcal{H}(X,\sqrt{t})\right) + \left(\mathcal{H}(X,-\sqrt{t})\right).\end{aligned}$$

Also notice that

$$\begin{aligned}\sum_{n=1}^{\infty} |X_{2n-1}|\, t^n &= \sqrt{t}\sum_{n=1}^{\infty} |X_{2n-1}| \left(\sqrt{t}\right)^{2n-1} \\ &= \frac{\sqrt{t}}{2}\left(\mathcal{H}(X,\sqrt{t})\right) - \left(\mathcal{H}(X,-\sqrt{t})\right).\end{aligned}$$

It follows then that

$$4\tilde{\mathcal{H}}(X,t) = \sum_{n=1}^{\infty} \left(|X_{2n-2}| + 2\,|X_{2n-1}| + |X_{2n}|\right) t^n$$

$$= \left(\frac{1 + \left(\sqrt{t}\right)^2}{2} \right) \left(\mathcal{H}(X, \sqrt{t}) + \mathcal{H}(X, -\sqrt{t}) \right) + \sqrt{t} \left(\mathcal{H}(X, \sqrt{t}) - \mathcal{H}(X, -\sqrt{t}) \right)$$

$$= \frac{\left(1 + \sqrt{t}\right)^2}{2} \mathcal{H}(X, \sqrt{t}) + \frac{\left(1 - \sqrt{t}\right)^2}{2} \mathcal{H}(X, -\sqrt{t}).$$

As a result, we have proven the following lemma.

Theorem 7.4.2. *Let X be a finitely graded set, and consider the formal power series $\tilde{\mathcal{H}}(X, t)$. Then, with some calculations and derivations, we may write $\tilde{\mathcal{H}}(X, t)$ as*

$$\tilde{\mathcal{H}}(X, t) = \frac{\left(1 + \sqrt{t}\right)^2}{8} \mathcal{H}(X, \sqrt{t}) + \frac{\left(1 - \sqrt{t}\right)^2}{8} \mathcal{H}(X, -\sqrt{t}).$$

7.5 Actions by Free Group Algebras

According to ([18]), we have the following theorem.

Theorem 7.5.1 (**Lewin**)**.** *Let G be a free group of rank r, M is a free FG-module of rank k, and N is a submodule of M of finite codimension. Then N is free with*

$$\text{rank} N = (r - 1)\dim(M/N) + k.$$

The following lemma allows us to obtain a generalization of the Lewin's formula in a special case.

Lemma 7.5.2 ([16, page 20])**.** *Let G be a free group, $R = FG$ be its group algebra, and K is a subgroup of G on a set of free generators $Y = \{y_j \mid j \in J\}$. Then $I_K = \{(k-1)R \mid k \in K\}$ is a right ideal of R. Moreover, it is free as R-module with the generators $\tilde{Y} = \{y_j - 1 \mid j \in J\}$.*

We will consider the degree function given by Lewin ([18]). Let $R = FG$ be the group algebra of a free group G freely generated by $X = \{x_i \mid i \in I\}$. Let M be a free right R-module with basis $B = \{e_\lambda \mid \lambda \in \Lambda\}$. If $w = x_{i_1}^{\epsilon_1} \dots x_{i_n}^{\epsilon_n}$ is an element of G, then the element $m = e_\lambda w$ is called a monomial of M. An element $ce_\lambda w$ ($c \in F$) is called a monomial term of M. Recall that the notion of degree for an element $w \in G$ is replaced by the minimal length of word representing w. Then, The length function on G induces a degree function d on R. If we assign degree 0 to the e_λ's, then we also have, in the obvious way, a degree again called d on the module M. Indeed, if $m = \alpha_1 e_{j_1} w_1 + \dots + \alpha_k e_{j_k} w_k$, $\alpha_i \in F, e_{j_i} \in B, w_i \in G, i = 1, \dots, k$ where $d(w_1) \le d(w_2) \le \dots \le d(w_k)$, then $d(m) = d(w_k)$. In particular, if $M = R$ and $N = I_K$ (K is a subgroup of G, then the Hilbert series of R/I_K equals the Hilbert series of the quotient group G/K. Thus, using Theorem 7.4.1, we have the following result.

Theorem 7.5.3. *Let G be a free group of rank r, and K be a subgroup of G. Then there exists a set of free generators Y of I_K as R-module with*

$$\tilde{\mathcal{H}}(Y, t) = \left(rt - \frac{t+1}{2}\right) \mathcal{H}\left(R/I_K, t\right) + \frac{t+1}{2}.$$

Moreover, if K is an even subgroup, then

$$\mathcal{H}(Y, t) = 2\left(\frac{2rt^2}{t^2+1} - 1\right) \mathcal{H}\left(R/I_K, t^2\right) + 2.$$

Let G be a free group and K be a subgroup of G such that $[G : K]$ is finite. Then the set Y (in the theorem above) and G/K are finite. Also, one can easily get

$$\tilde{\mathcal{H}}(Y, t) = \frac{1}{2}\mathcal{H}(Y, t) = \text{rank} I_K.$$

Therefore, if we replace t by 1 in $\tilde{\mathcal{H}}(Y,t)$ and $\mathcal{H}\left(R/I_K,t\right)$, we get

$$\text{rank} I_K = (rt-1)[R:I_K]+1,$$

which is the Lewin's formula in this case ($\text{rank}R=1$).

The Schreier-Lewin formula (Theorem 7.5.1) tells us that in a free group algebra of finite rank, any right ideal I of finite codimension is finitely generated. We will show below that the theorem holds more generally.

Theorem 7.5.4. *Let G be a group generated by r elements, and let $R = FG$ be its group algebra. Then any right ideal I of finite codimension m in the group algebra $R = FG$ can be generated by $(r-1)m+1$ elements.*

Proof. Write $\mathcal{F}$ for the free group algebra of rank r, so that $R \cong \mathcal{F}/\mathtt{n}$ for some ideal $\mathtt{n}$ of $\mathcal{F}$. The right ideal I of R corresponds to a right ideal J of $\mathcal{F}$ containing $\mathtt{n}$ with $R/I \cong \mathcal{F}/J$ as right F-spaces and $\dim(\mathcal{F}/J) = m$. According to Theorem 7.5.1, J is free as right $\mathcal{F}$-module of rank $(r-1)m+1$. Hence $I = J/\mathtt{n}$ can be generated by $(r-1)m+1$ elements. □

7.6 Schreier Formulas for Cocommutative Hopf Algebras

Any cocommutative Hopf algebra H over a field F of characteristic 0 is, as an algebra, a smash product $U(P(H))\#_{\varphi}FG(H)$ (Theorem 1.2.7). So the structure is completely determined by $P(H)$, $G(H)$, and the action of $G(H)$ on $P(H)$ by automorphisms. If K is a Hopf subalgebra of H, then K is completely defined by $P(K)$, which is a subalgebra of $P(H)$ invariant under the action of $G(K)$, which is a subgroup of $G(H)$.

The action of $G(K)$ on $P(K)$ is just the restriction of the action of $G(H)$ on $P(H)$. If $P(H)$ is a free Lie algebra with free basis X and $G(H)$ a free group with free basis A then also $P(K)$ is a free Lie algebra, with some free basis Y, and $G(K)$ is a free group. Our versions of Schreier formula allow us to find the Hilbert series, each in its own sense, for X and B.

If F has characteristic p, we consider the smash product H of the free associative algebra $F\langle X\rangle$ and group ring of the free group FG, where G is of finite rank. Then $P(H)$ is the free restricted Lie algebras. Consider Hopf subalgebras of the form $K = u(D)\#FN$, where D is a p-subalgebra of $M = P(H) = L^p(X)$ and N is a subgroup of G. Then if you have codimension M/D and index of N in G, you can find finite bases for D and N, hence an upper bound for the number of generators in K.

Let V be a vector space, which is a free module of rank s over a free group algebra FG of a free group G of rank r. Let U be a submodule of V. We know that this is a free FG-module. Let us generate by V a free Lie algebra $L(V)$ and consider in $L(V)$ the Lie subalgebra $L(U)$ generated by U. If $x_1, \dots, x_r$ is a free basis of G, and $e_1, \dots, e_s$ is a free basis of V, then as in [18], we have the degree defined on V in terms of length of words, with respect to $x_1, \dots, x_r$. By our generalization of Lewin's formula the Hilbert series of the free basis B of U is determined by the Hilbert series of V/U. The basis A of $L(V)$ as a free Lie algebra is the basis of V as a vector space. Thus we have a finitely graded free basis A of $L(V)$. Likewise, we have a finitely graded basis of $L(U)$. Schreier formula for free Lie algebras (Corollary 3.2.5) allows one to find the Hilbert series of $L(V)/L(U)$. In particular, if V is a free (right) FG-module of rank 1, H is a subgroup of G, and I_H is the (right) submodule of V (see Section 2.5), then we know the Hilbert series for the natural free basis of I_H (Lemma

7.5.2). In case H is even this is given by the Schreier formula from [5]. Then, by applying Petrogradsky's formula (Corollary 3.2.5), one can find the Hilbert series for $L(V)/L(I_H)$.

Let H be an even subgroup of a free group G of rank r, with free Schreier basis Y built by a Schreier transversal T. Any $y \in Y$ can be written a reduced word $y = a(y)b(y)^{-1}$ where $l(a(y)) = l(b(y))$ (since the length of y is even). Then the (right) ideal I_H is freely generated as FG-module by the elements $v(y) = a(y) - b(y)$, for $y \in Y$ (Lemma 7.5.2). Also, we have

Lemma 7.6.1. *The Hilbert series for the submodule $v(y)FG$ equals $t^k\mathcal{H}(FG,t)$, where $y \in Y$, $l(y) = 2k$.*

Proof. Since $y = a(y)b(y)^{-1}$ is reduced, the last letters of $a(y)$ and $b(y)^{-1}$ are not mutually inverse, so that for any reduced word c, we have $(a(y) - b(y))c$ has length $k + l(c)$. For different c, we obtain different elements of length $k + l(c)$. As a result, the Hilbert series for this submodule is $t^k\mathcal{H}(FG,t)$, as needed. □

Now the Hilbert series for I_H can be found as follows. We know the Hilbert series for the set Y of free generators of H given by

$$\mathcal{H}(Y,t) = 2\left(\frac{2rt^2}{t^2+1} - 1\right)\mathcal{H}\left(T,t^2\right) + 2 = \sum_{k=1}^{\infty} \alpha_{2k}t^{2k}.$$

If we write each $y \in Y$ as $y = a(y)b(y)^{-1}$, then the Hilbert series for $Z = \{a(y) - b(y)^{-1} \mid y \in Y\}$ will be $\sum_{k=1}^{\infty} \alpha_{2k}t^k$. Then the Hilbert series for I_H will be

$$\begin{aligned}
\mathcal{H}(I_H,t) &= \left(\sum_{k=1}^{\infty} \alpha_{2k}t^k\right)\mathcal{H}(FG,t) \\
&= \mathcal{H}(Y,\sqrt{t})\frac{1+t}{1-(2r-1)t}
\end{aligned}$$

$$
\begin{aligned}
&= \left(2\left(\frac{2rt}{t+1}-1\right)\mathcal{H}\left(T,t\right)+2\right)\left(\frac{1+t}{1-(2r-1)t}\right) \\
&= -2\mathcal{H}(T,t)+\frac{2(1+t)}{1-(2r-1)t} \\
&= 2(\mathcal{H}(FG,t)-\mathcal{H}(T,t)).
\end{aligned}
$$

Thus, by Corollary 3.2.5,

$$
\frac{2(\mathcal{H}(FG,t)-\mathcal{H}(T,t))-1}{\mathcal{H}(FG,t)-1}=\mathcal{E}(\mathcal{H}(L(V)/L(I_H))),
$$

and so

$$
\mathcal{H}(L(V)/L(I_H))=\mathcal{L}\left(\frac{2(\mathcal{H}(FG,t)-\mathcal{H}(T,t))-1}{\mathcal{H}(FG,t)-1}\right).
$$

Example 7.6.2. According to the above formula and our calculations in Section 7.3, we can compute $\mathcal{H}(L(FG)/(L(I_{FG'})))$, where G is a free group of rank 2 and G' is its commutator subgroup, as follows

$$
\mathcal{H}(L(FG)/(L(I_{FG'}))) \;=\; \mathcal{L}\left(\frac{2\left(\frac{1+t}{1-3t}-\frac{(1+t)^2}{(1-t)^2}\right)-1}{\frac{1+t}{1-3t}-1}\right).
$$

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