COFIBRATIONS

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by

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Introduction

Many problems in topology can be characterized by using the ideas of "extending" and "lifting" a map. An important special case of the extension problem is the notion of a homotopy. Homotopy defines an equivalence relation on the set of maps between two spaces X and Y. The classification of topological spaces, up to homotopy equivalence, is a central problem of Homotopy Theory. The homotopy classification problem can easily be facilitated if one has the "Homotopy Extension Property" (HEP), or its dual, the "Homotopy Lifting Property" (HLP). Cofibrations satisfy the HEP whereas fibrations satisfy the HLP. Moreover, it is important to observe that every map factors as a composition of a cofibration followed by a homotopy equivalence (Theorem 2.2.10(c)). Thus, as far as homotopy theory is concerned, every map is a cofibration upto a homotopy equivalence, suggesting the importance of cofibrations in homotopy theory.

The material of this thesis is organized in four chapters. The first chapter contains background material for the thesis. Following the definition of a category, the notions of a pushout and pullback are introduced along with their properties. We then characterize pushouts and pullbacks in Top (the category of topological spaces and maps) as concrete examples. The latter part of this chapter is concerned with some topological and homotopical notions relevant to the thesis.

The second chapter, which is the core of this thesis, is primarily devoted to the theory of cofibrations with a discussion of the dual theory, fibrations, in context. We begin with the definition of a cofibration as a "weak pushout" and proceed to the categorical properties of cofibrations. In the second section, an attempt is made to unify the various characterizations of cofibrations scattered in the literature. Following the characterization theorem for cofibrations, we prove a number of results as immediate consequences. It should be noted that most theorems in the literature append a closedness condition on the subspace A of X and thus require the inclusion $A \to X$ to be a closed cofibration. This requirement is not a real restriction if X is a Hausdorff Space or if a suitable class of spaces such as "Compactly Generated Spaces" is assumed. However, since we are working on the category Top, we have attempted the difficult task of circumventing the closedness condition whenever possible. Finally, we conclude this chapter by providing some geometric examples of closed cofibrations and non-examples of cofibrations with the former contrasted with an example of a non-closed cofibration.

An examination of the paper, "A Union Theorem for Cofibrations" by Lillig [11] constitutes the third chapter. The theme of the chapter is to tackle the following problem: Given subspaces A and B of X such that the inclusion maps $A \to X$ and $B \to X$ are cofibrations, under what assumptions on the subspaces A and B, is $A \cup B \to X$ is a cofibration?

The final chapter is devoted to a recent theorem of Kieboom [10] and related results. After having proved Kieboom's Theorem, we proceed to develop some sophisticated machinery such as the "Glueing Theorem for Homotopy Equivalences" (Theorem 4.6) concerning homotopy equivalences and pushouts. We then conclude the Chapter by retrieving some of the well known results of Strom [15] as special cases of Kieboom's Theorem.

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CHAPTER I

Section 1: Category Theory

This chapter is divided into three sections: Category Theory, Topological Spaces and Homotopy Theory. The first section introduces the "universal constructions" of pushout and pullback, with the intention of laying the categorical foundations on which many topological constructions will stand. The second section on topological spaces treats basic properties of topological spaces and maps and provides topological examples that can be construed as pushouts and pullbacks. The final section is devoted to an outline of some of the basic principles of homotopy theory.

The material in this chapter is standard. Hence, in many instances the proofs are sketchy or otherwise the appropriate references are cited. The following list of references is the main source for the material in this chapter: (i) [8], (ii) [2], (iii) [4], (iv) [1].

Section I. Category Theory

Definition 1.1.1: A category $\underline{C}$ consists of three families of data:

(a) Objects

The objects of $\underline{C}$ will be denoted by A, B, C, ..., etc. and we write $A \in |\underline{C}|$ if A is an object of $\underline{C}$.

(b) Morphisms

To each ordered pair (A,B) of objects of $\underline{C}$ there is associated a set $\underline{C}(A,B)$, called the set of morphisms from domain A to the codomain B.

If $f \in \underline{C}(A,B)$, we write $f:A \to B$ or $A \xrightarrow{f} B$.

(c) A Law of Composition

To each ordered triple (A, B, C) of objects of $\underset{\sim}{C}$, there is associated a law of composition $\underset{\sim}{C}(A,B) \times \underset{\sim}{C}(B,C) \to \underset{\sim}{C}(A,C)$.

If $A \xrightarrow{f} B \xrightarrow{g} C$, then we write the composite $A \to C$ as $g \cdot f$ or gf.

In a category $\underset{\sim}{C}$, the following axioms must be satisfied:

C_1: $\underset{\sim}{C}(A_1,B_1) \cap \underset{\sim}{C}(A_2,B_2) = \emptyset$, unless $A_1 = A_2$ and $B_1 = B_2$.

C_2 (Associativity): If $A, B, C, D \in |\underset{\sim}{C}|$ and $f \in \underset{\sim}{C}(A,B)$, $g \in \underset{\sim}{C}(B,C)$ and $h \in \underset{\sim}{C}(C,D)$, then $(hg)f = h(gf)$.

C_3 (Existence of Identities): For all $A \in |\underset{\sim}{C}|$, $1_A \in \underset{\sim}{C}(A,A)$, called the identity morphism of A, such that if $f \in \underset{\sim}{C}(A,C)$ and $g \in \underset{\sim}{C}(B,A)$ then $f1_A = f$ and $1_A g = g$.

Remark 1.1.1: 1_A is uniquely determined. To see this, suppose $1'_A \in \underset{\sim}{C}(A,A)$ is also an identity morphism. Then, $1_A = 1_A 1'_A$, since $1'_A$ is an identity. On the other hand $1'_A = 1_A 1'_A$, since 1_A is an identity. Therefore, $1_A = 1'_A$ and so the identity is unique.

Definition 1.1.2: $f \in \underset{\sim}{C}(A,B)$ is said to be isomorphism if there exists $g \in \underset{\sim}{C}(B,A)$ such that $gf = 1_A$ and $fg = 1_B$.

Note that if f is an isomorphism, then g is uniquely determined and is itself an isomorphism. We write $g = f^{-1}$. If f_1 and f_2 are isomorphisms, then $f_1 f_2$ is an isomorphism and $(f_1 f_2)^{-1} = f_2^{-1} f_1^{-1}$.

Definition 1.1.3: Suppose $A, B \in |\underline{C}|$. Then A and B are said to be equivalent if there exists $A \xrightarrow{f} B$, where f is an isomorphism.

Definition 1.1.4: Let $f \in \underline{C}\ (A,B)$. Then f is called a monomorphism if for each pair of morphisms $g_1, g_2 \in \underline{C}\ (A',A)$, $f \cdot g_1 = f \cdot g_2 \Rightarrow g_1 = g_2$; that is, f is left cancellable.

Notation: We will denote a monomorphism by $A \overset{f}{\rightarrowtail} B$.

Definition 1.1.5: Let $f \in \underline{C}\ (A,B)$. Then f is called an epimorphism if for each pair of morphisms $h_1, h_2 \in \underline{C}\ (B,B')$, $h_1 \cdot f = h_2 \cdot f \Rightarrow h_1 = h_2$; that is, f is right cancellable.

Notation: We shall denote an epimorphism by $A \overset{f}{\twoheadrightarrow} B$.

Definition 1.1.6: A natural equivalence relation "$\sim$" on a category $\underline{C}$ is an equivalence relation "$\sim$" on the class of morphisms of $\underline{C}$ such that

(i) If $X, Y, Z \in |\underline{C}|$ and $f, g \in \underline{C}\ (X,Y)$, then $f \sim g \Rightarrow$ Domain f = Domain g and Codomain f = Codomain g.

(ii) If $X, Y, Z \in |\underline{C}|$ and $f, g \in \underline{C}(X,y)$, $f', g' \in \underline{C}(Y,Z)$, then $(f \sim g$ and $f' \sim g') \Rightarrow (f'f \sim g'g)$.

If "$\sim$" is a natural equivalence relation, then we can form the Quotient Category $\underline{C}/_\sim$ under the equivalence relation "$\sim$". $\underline{C}/_\sim$ has the same objects as $\underline{C}$, that is, $|\underline{C}/_\sim| = |\underline{C}|$, and the morphism are the equivalence classes of morphisms in $\underline{C}$; that is, $\underline{C}/_\sim\ (A,B) = \underline{C}(A,B)/_\sim$.

The composition of morphisms $[f]: A \to B$ and $[g]: B \to C$ in $\underline{C}/_\sim$ is defined by $[g] \cdot [f] = [g \cdot f]: A \to C$. The identity morphisms are $[1_A]:A \to A$.

Definition 1.1.7: Given an object A of a category $\underline{C}$, the category $\underline{C}^A = \underline{C}(A,-)$ of objects under A is defined as follows:

An object of $\underline{C}^A$, called an object under A, is a pair consisting of an object $X \in |\underline{C}|$ and a morphism $u \in \underline{C}(A,X)$, called the insertion.

If $X,Y \in \underline{C}^A$ with insertions u, v then a morphism of $\underline{C}^A$, called a morphism under A, is a morphism $f \in \underline{C}(X,Y)$ such that $fu = v$.

Note that equivalences in the category $\underline{C}^A$ are called equivalences under A, denote by "$\simeq^A$".

Dualizing the above definition we have

Definition 1.1.8: Given an object $B \in |\underline{C}|$, the category $\underline{C}_B = \underline{C}(\ ,B)$ of objects over B is defined as follows:

An object of $\underline{C}_B$, called an object over B, is a pair consisting of an object $X \in |\underline{C}|$ and a morphism $p \in \underline{C}(X,B)$, called the projection.

If X, Y are objects over B with projections p, q, then a morphism of $\underline{C}_B$, called a morphism over B, is a morphism $f:X \to Y$ of $\underline{C}$ such that $qf = p$.

Note that the equivalences of the category $\underline{C}_B$ are called equivalences over B, denoted by "$\simeq_B$".

Definition 1.1.9: Let $\{A_i\}_{i \in I}$ be a family of objects of a category $\underline{C}$ indexed by the set I. Then a product $(A;\pi_i)$ (if it exists) of the objects A_i is an object A of $\underline{C}$, together with morphisms $\pi_i \in \underline{C}(A, A_i)$, called projections with the following universal property:

Given any object $Y \in |\underline{C}|$ and morphisms $f_i \in \underline{C}(Y, A_i)$, $\exists!$ morphism $f \in \underline{C}(Y, A)$ with $\pi_i f = f_i$; that is, the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "\pi_i"] & A_i \\ Y \arrow[u, dashed, "f"] \arrow[ur, "f_i"'] & \end{tikzcd}$$

Dualizing the above definition we have

Definition 1.1.10: Let $\{A_i\}_i \in I$ be a family of objects of a category $\underline{C}$ indexed by the set I. Then a coproduct (A, μ_i) (if it exists) is an object A of $\underline{C}$ together with morphisms $\mu_i \in \underline{C}(A_i, A)$ called injections with the following universal property:

Given any object $Y \in |\underline{C}|$ and morphisms $f_i \in (A_i, Y)$, $\exists!$ morphism $f \in \underline{C}(A, Y)$ with $f\mu_i = f_i$; that is, the following diagram commutes.

$$\begin{tikzcd} A_i \arrow[d, "\mu_i"'] \arrow[dr, "f_i"] & \\ A \arrow[r, dashed, "f"'] & Y \end{tikzcd}$$

Although the existence of products and coproducts cannot always be guaranteed in $\underline{C}$, we can however guarantee their uniqueness, whenever they do exist.

Theorem 1.1.1: Products and coproducts, whenever they exist, are unique up to isomorphism.

Proof: Suppose both $(A;\pi_i)$ and $(A';\pi_i')$ are products in a category $\underline{C}$.

Now, since $(A;\pi_i)$ is a product, $\exists!$ morphism $u \in \underline{C}(A',A)$ such that $\pi_i u = \pi_i'$. And since (A',π_i') is a product there exists a unique morphism $v \in \underline{C}(A,A')$ such that $\pi_i' v = \pi_i$. Thus, $\pi_i uv = \pi_i' v = \pi_i 1_A$ and so by the universal property of products, we conclude that $uv = 1_A$.

A similar argument shows that $vu = 1_A'$ and hence $u:A' \to A$ is an isomorphism.

Similarly, one can show that coproducts, whenever they exist, are unique up to isomorphism.

We now show the existence of products and coproducts in the category Set.

Example 1.1.1:

(a) Let $\{A_i\}_{i\in I}$ be a family of sets indexed by I and let $A = \prod_{i\in I} A_i$ be the cartesian product of the family of sets (i.e. the set of all families $(a_i)_{i\in I}$, or mappings $f:I \to \bigcup_{i\in I} A_i$ such that $a_i = f(i) \in A_i$, for all $i \in I$). Associated with $\prod_{i\in I} A_i$ we have a family $(\pi_i)_{i\in I}$ of projections (surjective functions), where $\pi_i : \prod_{i\in I} A_i \to A_i$ is defined by $\pi_i((a_i)_{i\in I}) = a_i$.

We claim that $(\pi A_i, \pi_i)$ is a product in the category <u>set</u>.

Suppose $X \in |\underline{set}|$ and for each $i \in I$, $f_i \in \underline{set}\,(X, A_i)$. Define $\phi : X \to \prod_{i\in I} A_i$ by $\phi(x) = (f_i(x))_{i\in I}$.

ϕ is well defined, since f_i are functions, for all $i \in I$.

Moreover, $(\pi_i\phi)(x) = \pi_i(\phi(x)) = \pi_i\,((f_i(x))_{i\in I} = f_i(x)$, $i \in I$, and so the following diagram is commutative

$$\begin{tikzcd} \Pi A_i \arrow[r, "\pi_i"] & A_i \\ X \arrow[u, dashed, "\phi"] \arrow[ur, "f_i"'] & \end{tikzcd}$$

Suppose also $\exists\ \phi' : X \to \prod_{i\in I} A_i$ such that $\pi_i\phi' = f_i$, $i \in I$.

Given $x \in X$, let $\phi'(x) = (a_i)_{i\in I}$.

Then,

$$f_i(x) = (\pi_i\phi')(x) = \pi_i(\phi'(x)) = \pi_i((a_i)_{i\in I}) = a_i$$

$$\Rightarrow \quad \phi'(x) = (a_i)_{i\in I} = ((f_i(x))_{i\in I}) = \phi(x) \quad \text{and}$$

therefore, ϕ is unique. Furthermore, $\prod_{i\in I} A_i$ is uniquely determined up to a <u>bijection</u>.

(b) Let $\{X_i\}_{i\in I}$ be a family of pairwise disjoint sets and let $X = \bigcup_{i\in I} X_i$ (disjoint union). Associated with $\bigcup_{i\in I} X_i$ we have a family $(\mu_i)_{i\in I}$ of inclusion functions, where $\mu_i : X_i \to X$ $(i \in I)$.

We claim that $(\bigcup_{i\in I} X_i, \mu_i)$ is a coproduct in the category <u>set</u>.

Suppose $Y \in |\underline{set}|$ and for each $i \in I$, $f_i \in \underline{set}\,(X_i, Y)$.

Define $f : \bigcup_{i\in I} X_i \to Y$ by

$$f = \bigcup_{i\in I} f_i \text{ ; that is, } f\big|_{X_i} = f_i$$

Clearly, f is well defined, since $\bigcap_{i\in I} X_i = \emptyset$ and f is the unique function such that $f\mu_i(x_i) = f(x_i) = f_i(x_i)$, i.e. $f\mu_i = f_i$ so that the following diagram commutes.

$$\begin{tikzcd} X_i \arrow[d, "\mu_i"'] \arrow[dr, "f_i"] & \\ \bigcup X_i \arrow[r, dashed, "f"'] & Y \end{tikzcd}$$

Note that if the sets X_i fail to be pairwise disjoint, we can "<u>separate</u>" them. This is done by writing their elements as pairs (X_i, i) where $x_i \in X_i$ and i states explicitly which set is being considered. Thus, instead of X_i, we work with the set $X_i \times \{i\} = \{(x_i, i) \mid x_i \in X_i\}$. The sets $X_i \times \{i\}$, $i \in I$ are pairwise disjoint and so the set $X = \bigcup_{i\in I} X_i \times \{i\}$, together with the inclusions $\mu_i : X_i \times \{i\} \to X$, is a coproduct in set.

Remark 1.1.2 : In the category set, we usually refer to the coproduct as the sum of sets and denote it by $\coprod_{i \in I} X_i$. If $I = \{1, 2, \ldots, n\}$, then we write $X = \coprod_{i=1}^{n} X_i = X_1 \sqcup X_2 \sqcup \ldots \sqcup X_n$.

We now discuss the universal constructions "pushouts" and "pullbacks" which are essential to our work in later chapters.

Definition 1.1.11: A pushout of a diagram

$$\begin{tikzcd} B & \\ A \arrow[u, "f"] \arrow[r, "g"'] & C \end{tikzcd}$$

in a category $\underline{C}$ is a commutative square

$$\begin{tikzcd} B \arrow[r, "\bar{g}"] & P \\ A \arrow[u, "f"] \arrow[r, "g"'] & C \arrow[u, "\bar{f}"'] \end{tikzcd}$$

with the property that for each commuting square

$$\begin{tikzcd} B \arrow[r, "p_1"] & X \\ A \arrow[u, "f"] \arrow[r, "g"'] & C \arrow[u, "p_2"'] \end{tikzcd}$$

$\exists$ a unique morphism $h: P \to X$ with $h\bar{g} = p_1$ and $h\bar{f} = p_2$. That is, in the diagram

$$\begin{tikzcd}
 & & X \\
B \arrow[r, "\bar{g}"] \arrow[urr, bend left, "p_1"] & D \arrow[ur, dashed, "h"] & \\
A \arrow[u, "f"] \arrow[r, "g"] & C \arrow[u, "\bar{f}"'] \arrow[uur, bend right, "p_2"'] &
\end{tikzcd}$$

the resulting triangles commute. By an abuse of language, we refer to P as the <u>pushout of f and g</u>.

The dual notion to that of a pushout is that of a pullback.

<u>Definition 1.1.12</u>: A pullback of a diagram

$$\begin{tikzcd}
B \arrow[r, "f"] & A \\
 & C \arrow[u, "g"']
\end{tikzcd}$$

in a category $\underline{C}$ is a commutative square

$$\begin{tikzcd}
B \arrow[r, "f"] & A \\
Q \arrow[u, "\bar{g}"] \arrow[r, "\bar{f}"'] & C \arrow[u, "g"']
\end{tikzcd}$$

with the property that for each commuting square

$$\begin{tikzcd}
B \arrow[r, "f"] & A \\
X \arrow[u, "p_1"] \arrow[r, "p_2"'] & C \arrow[u, "g"']
\end{tikzcd}$$

$\exists$ a unique morhpism $h:X \to Q$ with $\bar{g}h = p_1$ and $\bar{f}h = p_2$. That is, in the diagram

$$\begin{tikzcd}
& B \arrow[r, "f"] & A \\
& Q \arrow[u, "\bar{g}"] \arrow[r, "\bar{f}"'] & {} \arrow[u, "g"'] \\
X \arrow[uur, bend left, "p_1"] \arrow[ur, dashed, "h"] \arrow[urr, bend right, "p_2"'] & &
\end{tikzcd}$$

the resulting triangles commute. Again, by an abuse of language, we refer to Q as the <u>pullback of f and g</u>.

<u>Theorem 1.1.2</u>: Pushouts and pullbacks, whenever they exist, are unique up to isomorphism.

<u>Proof</u>:

(a) Let P and P' be pushouts of f and g. Then we have the following commutative diagrams (pushout diagrams in $\underline{C}$).

$$\begin{tikzcd} B \arrow[r, "\bar{g}"] & P \\ A \arrow[u, "f"] \arrow[r, "g"'] & C \arrow[u, "\bar{f}"'] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\bar{g}'"] & P' \\ A \arrow[u, "f"] \arrow[r, "g"'] & C \arrow[u, "\bar{f}'"'] \end{tikzcd}$$

Now from the following commutative diagrams

$$\begin{tikzcd} & & P' \\ B \arrow[r, "\bar{g}"] \arrow[urr, bend left, "\bar{g}'"] & P \arrow[ur, dashed, "\phi"] & \\ A \arrow[u, "f"] \arrow[r, "g"'] & C \arrow[u, "\bar{f}"'] \arrow[uur, bend right, "\bar{f}'"'] & \end{tikzcd} \qquad \begin{tikzcd} & & P \\ B \arrow[r, "\bar{g}'"] \arrow[urr, bend left, "\bar{g}"] & P' \arrow[ur, dashed, "\phi'"] & \\ A \arrow[u, "f"] \arrow[r, "g"'] & C \arrow[u, "\bar{f}'"'] \arrow[uur, bend right, "\bar{f}"'] & \end{tikzcd}$$

Since P and P' are pushouts of f and g, there exist unique morphisms $\phi : P \to P'$ and $\phi' : P' \to P$ such that

$$\phi\bar{g} = \bar{g}' \qquad\qquad \phi'\bar{g}' = \bar{g}$$
$$\text{and}$$
$$\phi\bar{f} = \bar{f}' \qquad\qquad \phi'\bar{f}' = \bar{f}$$

Putting the above two diagrams together we obtain the following diagram

$$\begin{tikzcd} & & & P \\ & & P' \arrow[ur, dashed, "\phi'"] & \\ B \arrow[r, "\bar{g}"] \arrow[urr, bend left, "\bar{g}'"] \arrow[uurrr, bend left, "\bar{g}"] & P \arrow[ur, dashed, "\phi"] \arrow[uurr, bend right, "1_P"'] & & \\ A \arrow[u, "f"] \arrow[r, "g"'] & C \arrow[u, "\bar{f}"'] \arrow[uur, bend right, "\bar{f}'"'] \arrow[uuurr, bend right, "\bar{f}"'] & & \end{tikzcd}$$

Since P is a pushout of f and g, $\phi' \cdot \phi$ is a unique morphism in $\underline{C}$ such that $\phi' \cdot \phi \cdot \bar{g} = \phi'\bar{g}' = g$ and $\phi'\phi\bar{f} = \phi'\bar{f}' = \bar{f}$ (i.e. $\phi'\phi$ makes the triangles commute). But $1_P : P \to P$ also satifies the commutativity of the above diagram. Hence, by uniqueness of $\phi'\phi$, it follows that $\phi'\phi = 1_P$. Similarly, it can be shown that $\phi\phi' = 1_{P'}$ and so $\phi : P \to P'$ is an isomorphism in $\underline{C}$.

(b) The case of pullbacks, which is dual, is proved similarly.

<u>Example 1.1.2</u>: Pushouts and pullbacks exist in the category <u>set</u>.

(a) In <u>set</u> the pushout of $f : X \to Y_1$ and $g : X \to Y_2$ is obtained as follows:

Let $Y = Y_1 \sqcup Y_2$ (coproduct of Y_1 and Y_2) and let $\sim$ be the coarsest equivalence relation on $Y_1 \sqcup Y_2$ with $f(x) \sim g(x)$, for each $x \in X$. To explain the term coarsest, let R be an equivalence relation on a set A. We define a new relation $\bar{R}$ on A by $a\bar{R}b \Leftrightarrow$ there is a sequence $a_1, \ldots, a_n$ of elements of A such that

(a) $a_1 = a$, $a_n = b$

(b) $\forall i = 1, 2, \ldots, n - 1$, $a_i \, R \, a_{i+1}$ or $a_{i+1} \, R \, a_i$ or $a_i = a_{i+1}$.

It is not hard to see that $\bar{R}$ is an equivalence relation on the set A. Suppose also $\bar{R}'$ is an equivalence relation on A containing R. Let $a \bar{R}' b$, and let $a_1, \ldots, a_n$ be a sequence satisfying (a) and (b) above. Now $\bar{R}' \geq R \Rightarrow a_i \bar{R}' a_{i+1}$ for each $i = 1, 2, \ldots, n - 1$ (by (b) above). Hence, $a_1 \bar{R}' a_n$ and so $a \bar{R}' b$. Therefore, $\bar{R} \subseteq \bar{R}'$ and we call $\bar{R}$ the equivalence relation generated by R (or the coarsest equivalence relation on A).

Now let $\phi: Y_1 \cup Y_2 \to (Y_1 \cup Y_2)/\sim$ denote the quotient function and let $\mu_i: Y_i \to Y_1 \cup Y_2$ be the inclusion functions $i = 1, 2$. It is now a routine matter to check that the square

$$\begin{tikzcd}
Y_1 \arrow[r, "\phi\cdot\mu_1"] & (Y_1 \sqcup Y_2)/\sim \\
X \arrow[u, "f"] \arrow[r, "g"'] & Y_2 \arrow[u, "\phi\cdot\mu_2"']
\end{tikzcd}$$

is a pushout.

(b) To obtain the pullback of two functions $f: X_1 \to Y$ and $g: X_2 \to Y$ in <u>set</u>, set $Q = \{(x_1, x_2) \in X_1 \times X_2 \mid f(x_1) = g(x_2)\}$ and let $\pi_1: X_1 \times X_2 \to X_1$ and $\pi_2: X_1 \times X_2 \to X_2$ be the projections. It is now a routine matter to check that the square

$$\begin{tikzcd}
Q \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & X_1 \arrow[d, "f"] \\
X_2 \arrow[r, "g"'] & Y
\end{tikzcd}$$

is a pullback.

We now discuss some properties of pushouts and pullbacks.

<u>Theorem 1.1.3</u>: In any category $\underline{C}$, the composite of two pushouts (respectively pullbacks) is a pushout (respectively pullback).

Proof:

(a) Consider the following commutative diagram

$$\begin{tikzcd}
B \arrow[r, "\bar{f}"] & C \arrow[r, "\bar{g}"] & E \\
A \arrow[u, "i"] \arrow[r, "f"'] \arrow[ur, phantom, "\mathrm{I}"] & D \arrow[u, "\bar{i}"'] \arrow[r, "g"'] \arrow[ur, phantom, "\mathrm{II}"] & F \arrow[u, "\tilde{i}"']
\end{tikzcd}$$

where square I and square II are pushouts. We claim the diagram

$$\begin{tikzcd}
B \arrow[r, "\bar{g}\bar{f}"] & E \\
A \arrow[u, "i"] \arrow[r, "gf"'] & F \arrow[u, "\tilde{i}"']
\end{tikzcd}$$

is a pushout.

To see this, let $Z \in |\underline{C}|$ and let $t_1 \in \underline{C}\ (B,Z)$ and $t_2 \in \underline{C}\ (F,Z)$ be isomorphisms such that $t_1 i = t_2 gf$.

Thus, we obtain the following commutative diagram

$$\begin{tikzcd}
 & & & Z \\
B \arrow[r, "\bar{f}"] \arrow[urrr, bend left, "t_1"] & C \arrow[r, "\bar{g}"] \arrow[urr, bend left, "\phi"] & E \arrow[ur, dashed, "\psi"] & \\
A \arrow[u, "i"] \arrow[r, "f"'] & D \arrow[u, "\bar{i}"] \arrow[r, "g"'] & F \arrow[u, "\tilde{i}"] \arrow[uur, bend right, "t_2"'] &
\end{tikzcd}$$

Since square I is a pushout, $\exists$ a unique morphism $\phi \in \underline{C}(C,Z)$ such that $\phi\bar{f} = t_1$ and $\phi\bar{i} = t_2 g$. Again, since square II is a pushout, $\exists ! \psi \in \underline{C}(E,Z)$ such that $\psi\bar{g} = \phi$ and $\psi\tilde{i} = t_2$. Now, $\psi\bar{g}\,\bar{f} = \phi\bar{f} = t_1$ and $\psi\tilde{i} = t_2$. To complete the proof, we must show that ψ is the only morphism satisfying the last set of equations. So, suppose also $\exists\, \psi' \in \underline{C}(E,Z)$ such that $\psi'\bar{g}\,\bar{f} = t_1$ and $\psi'\tilde{i} = t_2$. Now, $\psi'\bar{g}\,\bar{i} = \psi'\tilde{i}g$ (by commutativity of square II)

$$= t_2 g \text{ and}$$

$$\psi'\bar{g}\,\bar{f} = t_1.$$

But ϕ is the unique morphism such that $\phi\bar{i} = t_2 g$ and $\phi\bar{f} = t_1$.
So, $\psi'\bar{g} = \phi$.
Again, ψ is the unique map such that $\psi\bar{g} = \phi$ and $\psi\tilde{i} = t_2$.
Hence $\psi' = \psi$.

(b) The proof for the case of pullbacks is analogous.

Remark 1.1.3: Composition of squares can be done vertically as well as horizontally. The above proof remains true in the case of vertical composition. When quoting Theorem 1.1.3, we shall be referring to either horizontal or vertical composition, depending on the context of the discussion.

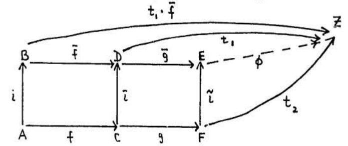
Theorem 1.1.4: Consider the following commutative diagram in a category $\underline{C}$

$$\begin{tikzcd}
B \arrow[r, "\bar{f}"] & D \arrow[r, "\bar{g}"] & E \\
A \arrow[u, "i"] \arrow[r, "f"'] & C \arrow[u, "\bar{i}"'] \arrow[r, "g"'] & {} \arrow[u, "\tilde{i}"']
\end{tikzcd}$$

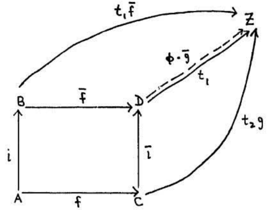
If the left square is a pushout and the composite square is a pushout, then the right square is a pushout.

Proof: Let $t_1 \in \underline{C}(D,Z)$ and $t_2 \in \underline{C}(F,Z)$ be given morphisms in $\underline{C}$ such that $t_1\bar{i} = t_2 g$.

We thus have the following commutative diagram:



Since the composite square is a pushout, $\exists ! \phi \in \underline{C}(E,Z)$ such that $\phi \cdot (\bar{g} \cdot \bar{f}) = t_1 \cdot \bar{f}$ and $\phi\tilde{i} = t_2$. We now reduce the above diagram to the following



Observe that $\phi \cdot \bar{g}$ and t_1 both make the above diagram commutative. Since the square is a pushout, we have by uniqueness that $\phi \cdot \bar{g} = t_1$. But from above, we also have $\phi \cdot \tilde{i} = t_2$.

Hence, the morphism $\phi \in \underline{C}(E,Z)$ is the required unique morphism rendering the triangles commutative in (*). Thus, the right Square is a pushout, as required.

Remark 1.1.4:

(a) Dualizing the above theorem we have the following result for pullbacks:

If the right square is a pullback and the composite square is a pullback, then the left square is a pullback.

(b) In the case of vertical composition, the results above take the following form:

(i) If the composite square is a pushout and the bottom square is a pushout, then the upper square is a pushout.

(ii) If the composite square is a pullback and the upper square is a pullback, then the bottom square is a pullback.

Section 2: The Category of Topological Spaces

We now briefly discuss some properties and results in point set topology which are relevant to our work. Many of the results will be assumed or otherwise stated with the necessary references.

Throughout our discussions, we shall denote the category of topological spaces and continuous functions by Top.

Definition 1.2.1: Let X and Y be topological spaces and let $f:X \to Y$ be a function. Then f is continous at $x_0 \in X$ iff for each neighborhood V of $f(x_0)$ in Y, there is a neighborhood U of $x_0 \in X$ such that $f(U) \subseteq V$. We say f is *continuous* on X

if f is continuous at each $x_0 \in X$.

Note that continuous functions are also called maps.

Let A and B be subspaces of a topological space. If $f:A \to Z$ and $g:B \to Z$ are functions which agree on the intersection of A and B, then we can define $f \cup g:A \cup B \to Z$ by

$$(f \cup g)(a) = f(a), \text{ for } a \in A \text{ and}$$

$$(f \cup g)(b) = g(b), \text{ for } b \in B .$$

We say that $f \cup g$ is formed by "glueing together" the functions f and g. The following result allows us, under certain conditions, to deduce the continuity of $f \cup g$ from the continuity of f and g.

Map Glueing Theorem 1.2.1: Let $X = A \cup B$, where A and B are closed in X. Let $f:A \to Y$ and $g:B \to Y$ be continuous. If $f(x) = g(x)$, for every $x \in A \cap B$, then $f \cup g:X \to Y$ is continuous.

Proof: See [12; pg. 108, Theorem 7.3]

Note that the Map Glueing Theorem remains true when A and B are both open in $A \cup B$.

Definition 1.2.1: A continuous bijection $f:X \to Y$ such that $f^{-1}:Y \to X$ is also continuous is called a homeomorphism and is denoted by $f:X \cong Y$. Two spaces X and Y are said to be homeomorphic, written $X \cong Y$, if there is a homeomorphism $f:X \cong Y$.

An equivalent definition would be to require the existence of continuous functions $f:X \to Y$, $g:Y \to X$ such that $fg = 1_Y$ and

$gf = 1_X$.

Remark 1.2.1:

(a) If $f:X \cong Y$ and $A \subseteq X$, then $f|_A : A \cong f(A)$ and $f|_{X-A} : X - A \cong Y - f(A)$.

(b) By an embedding of a space X into a space Y, we mean a map $f:X \to Y$ such that $X \cong f(X)$.

Definition 1.2.3: Suppose we are given a set X and a family $(X_\alpha)_{\alpha\in A}$ of topological spaces, together with functions $f_\alpha:X \to X_\alpha$, one for each $\alpha \in A$. A topology on X is called initial with respect to $(f_\alpha)_{\alpha\in A}$ if it has the following property: For any topological space Y, a function $k:Y \to X$ is continuous iff the composite $f_\alpha k:Y \to X_\alpha$ is continuous, for all $\alpha \in A$.

Remark 1.2.2:

(a) If X has the initial topology with respect to $(f_\alpha)_{\alpha\in A}$, then each $f_\alpha:X \to X_\alpha$ is continuous.

(b) The initial topology on X with respect to $(f_\alpha)_{\alpha\in A}$ is the smallest topology such that each f_α is continuous.

(c) The initial topology on X with respect to $(f_\alpha)_{\alpha\in A}$ exists and has subbasis the sets $f_\alpha^{-1}(U)$, for U open in X_α.

Example 1.2.1:

(a) Let A be a subspace of X and let $i:A \to X$ be the inclusion map. The initial topology on X with respect to i has as subbase the sets $i^{-1}(U)$, for U open in X. Since i is continuous, $i^{-1}(U) = U \cap A$ is open in A. Hence, the initial topology on X with respect to i is simply the relative topology on A.

(b) Let $\{X_\alpha\}_{\alpha\in A}$ be a family of topological spaces, and let X be the product of the underlying sets; that is $X = \prod_{\alpha\in A} X_\alpha$. The product topology on $X = \prod_{\alpha\in A} X_\alpha$ is the initial topology with respect to the family of projections $\pi_\beta:\Pi X_\alpha \to X_\beta$. This follows from the universal property of the product topology.

Definition 1.2.4: Given a set X, let $\{X_\alpha\}_{\alpha\in A}$ be a family of topological spaces and let $f_\alpha:X_\alpha \to X$ be a family of functions one for each $\alpha \in A$. A topology on X is said to be final with respect to the functions f_α if for any topological space Z and any function $g:X \to Z$, we have that g is continuous if and only if $gf_\alpha:X_\alpha \to Z$ is continuous, for each $\alpha \in A$.

Remark 1.2.3:

(a) If X has the final topology with respect to $(f_\alpha)_{\alpha\in A}$, then each $f_\alpha:X_\alpha \to X$ is continuous.

(b) The final topology on X with respect to $(f_\alpha)_{\alpha\in A}$ is finer than any other topology on X such that each $f_\alpha:X_\alpha \to X$ is continuous.

(c) The final topology on X with respect to $(f_\alpha)_{\alpha\in A}$ exists and is characterized by the following statement: $U \subseteq X$ is open in the final topology $\Leftrightarrow$ $f_\alpha^{-1}(U)$ is open in X_α, for each $\alpha \in A$.

Example 1.2.2:

(a) Let $X = \bigsqcup X_\alpha$ be the sum of the underlying sets of the family $\{X_\alpha\}_{\alpha\in A}$ of topological spaces, and let $i_\alpha:X_\alpha \to X$ be the inclusions. The final topology on X with respect to i_α is the sum topology.

(b) Let $X = \bigsqcup X_\alpha$ (a sum of spaces X_α). Given a set Y and functions $f_\alpha:X_\alpha \to Y$, $\alpha \in A$, let $f:X \to Y$ be the function determined by the

f_α's. Then the final topologies on Y with respect to f and $(f_\alpha)_{\alpha \in A}$ coincide.

To see this, let $i_\alpha : X_\alpha \to X_\alpha$ be the inclusions and let $g : Y \to Z$ be any function, where Z is a topological space.
Consider the following diagram:

$$\begin{tikzcd}
& \coprod X_\alpha \arrow[d, "f"] \arrow[dr, "gf"] & \\
X_\alpha \arrow[ur, "i_\alpha"] \arrow[r, "f_\alpha"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

Now, from the final topologies on Y with respect to f and $(f_\alpha)_{\alpha \in A}$ it follows that

(i) g is continuous <=> gf_α is continuous for each $\alpha \in \Lambda$.

(ii) g is continuous <=> gf is continuous.

We show that condition (i) and (ii) are equivalent.

g is continuous <=> gf is continuous (condition ii)

<=> gfi_α is continuous, for each $\alpha \in \Lambda$ (X_α has the final topology with respect to i_α)

<=> gf_α is continuous, for each $\alpha \in \Lambda$ $(gf_\alpha i_\alpha = gf_\alpha)$.

Therefore, the final topologies on Y with respect to f and with respect to $(f_\alpha)_{\alpha \in A}$ coincide.

Hence, by means of the topological sum we have reduced final topologies with respect to a family $(f_\alpha)_{\alpha\in A}$ to final topologies with respect to a single function f.

Definition 1.2.5: Assume X is a topological space, Y a set and $p:X \twoheadrightarrow Y$ a surjective function. The final topology on Y with respect to p is called the identification topology. The function $p:X \to Y$ is called an identification map.

The following is an important characterization of identification maps.

Theorem 1.2.2: Let X and B be topological spaces and $p:X \twoheadrightarrow B$ a continuous surjection. Then p is an identification map if and only if, for each space Z and each function $g:B \to Z$, $g\cdot p:X \to Z$ is continuous $\Leftrightarrow$ $g:B \to Z$ is continuous. (i.e. p has the usual universal property for final topologies.)

Proof: Follows from Definition 1.2.4.

Example 1.2.3:

(a) Let X be a topological space and let $\sim$ denote an equivalence relation on X. Then $X/\sim$ denotes the quotient set and $\pi:X \to X/\sim$ denotes the canonical projection. We equip $X/\sim$ with the final topology with respect to π. So, π is an identification map and $X/\sim$ is called the quotient space.

(b) Let A be a subspace of the topological space X. Then X with A shrunk to a point is a topological space, written X/A, which is obtained from X by identifying all of A to a single point. The elements of X/A are the equivalence classes in X under the equivalence relation generated by $x \sim y \Leftrightarrow x \in A$ and $y \in A$. The equivalence classes are therefore the sets $\{x\}$ for $x \in X - A$ and also, when $A \neq \emptyset$, the set A.

Let $\pi : X \to X/A$ be the projection;

$$\text{i.e. } \pi(x) = \begin{cases} x, & x \in X - A \\ A, & x \in A \end{cases}$$

We give X/A the final topology with respect to π and so π is an identification map. Note that if $A = \emptyset$, or consists of a single point, then X/A can be identified with X.

Let X and Y be topological spaces and denote by Y^X or $\mathrm{Map}(X,Y)$ the set of all continuous functions $X \to Y$. Define $W(K,U) = \{f \in \mathrm{Map}\ (X,Y) \mid f(K) \subseteq U\}$.

Definition 1.2.6: The compact open topology in $\mathrm{Map}(X,Y)$ is that topology having as subbasis all sets $W(K,U)$, where $K \subseteq X$ is compact and $U \subseteq Y$ is open. Note that a function $f : X \to \mathrm{Map}(Y,Z)$ induces a function $g : X \times Y \to Z$ which is defined by the rule $g(x,y) = f(x)(y)$. The most important feature of the compact open topology is the following result.

Theorem 1.2.3:

(a) If $g : X \times Y \to Z$ is continuous, then $f : X \to \mathrm{Map}(Y,Z)$ is continuous. (This is known as the proper condition).

(b) If $f : X \to \mathrm{Map}\ (Y,Z)$ is continuous and if Y is locally compact. Hausdorff, then $g : X \times Y \to Z$ is also continuous. (This is known as the admissible condition.)

Proof: See [4;page 261, Theorem 3.1].

Theorem 1.2.4:

(a) If X is locally compact and Hausdorff, the evaluation function $e:\mathrm{Map}(X,Y) \times X \to Y$, defined by $e(f,x) = f(x)$, $f \in \mathrm{Map}(X,Y)$, is continuous.

(b) Let Y be locally compact, Hausdorff. Then $\mathrm{Map}(X, \mathrm{Map}(Y,Z))$ is homeomorphic to $\mathrm{Map}(X \times Y,Z)$ the association $f \leftrightarrow g$ in Theorem 1.2.2 being the desired homeomorphism.

Proof: See [4; page 265, Theorem 5.3]

We now discuss the fundamental theorem for identification topologies in Cartesian products. Note that if $f:X \to Y$, $f':X' \to Y'$ are identification maps, it is not true in general that $f \times f':X \times X' \to Y \times Y'$ is an identification map. An example is given in [1 ; page 102]. However, under additional assumptions on X and Y' or on Y and X', the above statement holds true. We need the following preliminary result.

Lemma 1.2.1: If $p:X \to B$ is an identification map and A is locally compact Hausdorff, then $p \times 1:X \times A \to B \times A$ is also an identification map.

Proof: See [4; page 262, Theorem 4.1].

Theorem 1.2.5: Let $p:X \to B$ and $q:Y \to C$ be identification maps. Then, $p \times q:X \times Y \to B \times C$ is an identification map if either

(a) X and C are locally compact Hausdorff

or

(b) Y and B are locally compact Hausdorff.

Proof:

(a) $p \times q$ is the composite

$$X \times Y \xrightarrow{1_X \times q} X \times C \xrightarrow{p \times 1_C} B \times C$$

By Lemma 1.2.1, both $1_X \times q$ and $p \times 1_C$ are identification maps and the composite of two identification maps is an identification map.

(b) Similar to (a) above.

We now discuss some categorical properties in Top and their consequences. We are mainly interested in pullbacks and pushouts in Top.

Theorem 1.2.6: Pullbacks and pushouts exist in Top and are unique up to homeomorphism.

Proof:

(a) We show how to form a pullback in the category Top.

Consider the following diagram in Top.

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

As discussed in Example 1.1.2(b), we can form the set

$X_f \sqcap_g Y = \{(x,y) \in X \times Y \mid f(x) = g(y)\}$.

Let $\pi_1 : X_f \sqcap_g Y \to X$ and $\pi_2 : X_f \sqcap_g Y \to Y$ be the projection functions. Then the following diagram is commutative.

$$\begin{tikzcd}
X_f \sqcap_g Y \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & X \arrow[d, "f"] \\
Y \arrow[r, "g"'] & W
\end{tikzcd}$$

We now equip $X_f \sqcap_g Y$ with the initial topology with respect to the projections $\pi_1 : X_f \sqcap_g Y \to X$ and $\pi_2 : X_f \sqcap_g Y \to Y$.
We claim that diagram (*) is a pullback in <u>Top</u>.
Let Z be a topological space and let $f_1 : Z \to X$ and $f_2 : Z \to Y$ be maps such that $ff_1 = gf_2$.

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\phi"] \arrow[drr, bend left, "f_1"] \arrow[ddr, bend right, "f_2"'] & & \\
& X_f \sqcap_g Y \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & W
\end{tikzcd}$$

We require a unique map $\phi : Z \to X_f \sqcap_g Y$ such that $\pi_1\phi = f_1$ and $\pi_2\phi = f_2$.

Define $\phi : Z \to X_f \sqcap_g Y$ by

$$\phi(z) = (f_1(z), f_2(z))$$

Clearly, ϕ is unique by construction. We need to show that ϕ is a map.

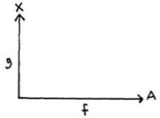
Since $X_f \sqcap_g Y$ has the initial topology with respect to π_1 and π_2, ϕ is continuous $\Leftrightarrow$ $\pi_1\phi$ and $\pi_2\phi$ are continuous. But,

$\pi_1\phi(z) = \pi_1(f_1(z), f_2(z)) = f_1(z)$ and

$\pi_2\phi(z) = \pi_2(f_1(z), f_2(z)) = f_2(z)$.

Since f_1 and f_2 are continuous functions, ϕ is continuous and diagram (*) is a pullback. The uniqueness of pullbacks in Top follows by Theorem 1.1.2.

(b) We now show how to form the pushout of the following diagram in Top.



Let $X \sqcup Y$ be the sum (coproduct) of X and Y as objects in Set.

Define "~" as the coarsest equivalence relation on $X \sqcup Y$ such that $f(a) \sim g(a)$, for all $a \in A$. We then form the quotient set $X \sqcup Y/\!\sim\; = X_f \sqcup_g Y$, whose elements are the equivalence classes of $X \sqcup Y$ under the coarsest equivalence relation generated by $\sim$ (see

Example 1.1.2(a), page 13). Hence, the equivalence classes include:

(i) pairs of points $\{(f(a),\ g(a)\}$, $a \in A$

(ii) individual points of $X - g(A)$.

(iii) individual points of $Y - f(A)$.

We now have the following sequence of function:

$$X \xrightarrow{i_X} X \sqcup Y \xrightarrow{\pi} X_f \sqcup_g Y$$

$$Y \xrightarrow{i_Y} X \sqcup Y \xrightarrow{\pi} X_f \sqcup_g Y$$

where i_X, i_Y are inclusion functions and π is the canonical projection.

Let $\bar{f} = \pi i_X : X \to X_f \sqcup_g Y$ and $\bar{g} = \pi i_Y : Y \to X_f \sqcup_g Y$. Then the following diagram is commutative.

$$\begin{tikzcd} X \arrow[r, "\bar{f}"] & X_f \sqcup_g Y \\ A \arrow[u, "g"] \arrow[r, "f"'] & Y \arrow[u, "\bar{g}"'] \end{tikzcd} \qquad (**)$$

We now equip the set $X_f \sqcup_g Y$ with the final topology with respect to $\bar{f}$ and $\bar{g}$.

We claim that diagram (**) is a pushout in Top. Let Z be any topological space and let $f_1 : X \to Z$ and $f_2 : Y \to Z$ be given maps such that $f_1 g = f_2 f$.

$$
\begin{tikzcd}
 & & Z \\
X \arrow[r, "\bar{f}"] \arrow[urr, bend left, "f_1"] & X_f \sqcup_g Y \arrow[ur, dashed, "h"] & \\
A \arrow[u, "g"] \arrow[r, "f"'] & Y \arrow[u, "\bar{g}"'] \arrow[uur, bend right, "f_2"'] &
\end{tikzcd}
$$

We require a map $h: X_f \sqcup_g Y \to Z$ such that

$$h\bar{f} = f_1$$

and

$$h\bar{g} = f_2$$

Define $h: X_f \sqcup_g Y \to Z$ by

$$h(\bar{t}) = \begin{cases} f_1(x) & \text{if } \bar{f}(x) = \bar{t} \\ f_2(y) & \text{if } \bar{g}(y) = \bar{t} \end{cases}$$

We claim that h is well defined. To see this, suppose $\bar{f}(x) = \bar{g}(y)$. Now, $\bar{f}(x) = \bar{g}(y) \Leftrightarrow \pi i_X(x) = \pi i_Y(y)$

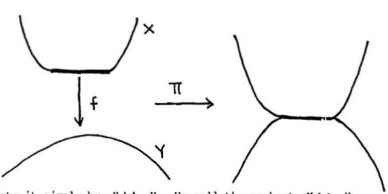
$\Leftrightarrow$ $a \in A$ such that $g(a) = x$ and $f(a) = y$.

But, $f_1 g(a) = f_1(x)$ and $f_2 f(a) = f_2(y)$ as $f_1 g = f_2 f$. Therefore, h is well defined. Clearly, h is unique. It remains to show that h is a map. Since $X_f \sqcup_g Y$ is equipped with the final topology with respect to $\bar{f}$ and $\bar{g}$, and $h\bar{f} = f_1$ and $h\bar{g} = f_2$ where f_1, f_2 are continuous, it follows that h is continuous. Hence, diagram (**) is a pushout. Again, the uniqueness of pushouts in <u>Top</u> follows from Theorem 1.1.2.

<u>Remark 1.2.4</u>:

(a) By Example 1.2.2(b), the final topology on $X_f \sqcup_g Y$ with respect to $\bar{f}: X \to X_f \sqcup_g Y$ and $\bar{g}: Y \to X_f \sqcup_g Y$ coincides with the identification topology with respect to the projection $\pi: X \quad Y \to X_f \sqcup_g Y$. A dual statement holds for the pullback space $X_f \sqcap_g Y$; i.e. the initial topology on $X_f \sqcap_g Y$ with respect to $\pi_1: X_f \sqcap_g Y \to X$ and $\pi_2: X_f \sqcap_g Y \to Y$ coincides with the initial topology with respect to the inclusion $i: X_f \sqcap_g Y \to X \times Y$ (by Example 1.2.1(a)) which is just the relative topoloogy on $X_f \sqcap_g Y$.

(b) In case A is a subspace of X and $i: A \to X$ is the inclusion, we can visualize $X_i \sqcup_f Y$ as



and denote it simply by $X \sqcup_f Y$. We call the pushout $X \sqcup_f Y$ the <u>adjunction space</u> of X to Y through f.

Example 1.2.4:

(a) If $A = \emptyset$, then $X \sqcup_{\emptyset} Y = X \sqcup Y$ (disjoint union).

(b) If $A = X$, then $X \sqcup_f Y = Y$.

(c) Suppose $X = B \cup C$ and $A = B \cap C$, where B and C are closed subspaces of X. Let $j:A \to B$ be the inclusion.

<u>We claim</u>: $X = B \cup C = B \sqcup_j C$.

We readily observe that at the set-theoretic level, the two sets are identical. The only problem here is one of topology. So it suffices to show that $X = B \cup C$ has the final topology with respect to the inclusions $B \to B \cup C$ and $C \to B \cup C$. So, let Z be any topological space and let $h:B \cup C \to Z$ be any function. Let $i_B:B \to B \cup C$ and $i_C:C \to B \cup C$ be the inclusion functions. Now, if $h:B \cup C \to Z$ is continuous, then h restricted to its subspaces B and C is continuous. That is, $h\big|_B = hi_B$ and $h\big|_C = hi_C$ are continuous. On the other hand, suppose $hi_B:B \to Z$ and $hi_C:C \to Z$ are continuous; that is, $h\big|_B$ and $h\big|_C$ are continuous. By the Map Glueing Theorem 1.2.1; $h:B \cup C \to Z$ is continuous. Therefore, by Definition 1.2.4, $X = B \cup C$ has the final topology with respect to the inclusions $i_B:B \to C$ and $i_C:C \to B \cup C$ and so $B \cup C = B \sqcup_j C$.

(d) If $f:A \to Y$ is an identification map, then so also is $\bar{f}:X \to X \sqcup_f Y$. To see this, consider the diagram

$$\begin{tikzcd} X \arrow[r, "\bar{f}"] & X \sqcup_f Y \\ A \arrow[u, "i"] \arrow[r, "f"'] & Y \arrow[u, "\bar{i}"'] \end{tikzcd}$$

Clearly, $\bar{f}: X \to X \sqcup_f Y$ is surjective. We have to prove $X \sqcup_f Y$ has the final topology with respect to $\bar{f}$. Let Z be any space and let $g: X \sqcup_f Y \to Z$ be such that $g\bar{f}: X \to Z$ is continuous.

$$\text{Now, } g\bar{f}: X \to Z \text{ is continuous} \Rightarrow g\bar{f}i \text{ is continuous}$$
$$\Rightarrow g\bar{i}f \text{ is continuous,}$$
$$(\text{as } \bar{f}i = \bar{i}).$$

But f is an identification and $g\bar{i}f$ is continuous, so $g\bar{i}$ is continuous. Since the topology on $X \sqcup_f Y$ is final with respect to $\bar{i}$ and $\bar{f}$, the continuity of $g\bar{f}$ and $g\bar{i}$ now implies that g is continuous. Therefore, $\bar{f}$ is an identification map. By way of an application, let Y be the space consisting of a single point $*$, and let $A \neq \emptyset$. Then clearly $C: A \to *$ is an identification map and so $\bar{C}: X \to X \sqcup_C Y$ is also an identification map. But $\bar{C}$ simply shrinks A to a point, and so by Example 1.2.3(b) we have that $X \sqcup_C \{*\} \cong X/A$.

We now briefly discuss Theorem 1.1.3 in the context of the category Top.

Remark 1.2.5:

(a) Given the following commutative diagram in Top.

$$\begin{tikzcd}
X \arrow[r, "\bar{g}"] \arrow[dr, phantom, "\mathrm{I}"] & X_f \sqcup_g Y \arrow[r, "\bar{h}"] \arrow[dr, phantom, "\mathrm{II}"] & (X_f \sqcup_g Y)_{\bar{f}} \sqcup_h Z \\
A \arrow[u, "f"] \arrow[r, "g"'] & Y \arrow[u, "\bar{f}"] \arrow[r, "h"'] & Z \arrow[u, "\bar{\bar{f}}"']
\end{tikzcd}$$

where square I and square II are pushouts in Top, it follows by Theorem 1.1.3 that the composite square

$$
\begin{tikzcd}
X \arrow[r, "\bar{h}\bar{g}"] & X_f \sqcup_{hg} Z \\
A \arrow[u, "f"] \arrow[r, "hg"'] & Z \arrow[u]
\end{tikzcd}
$$

is a pushout. Moreover, by Theorem 1.2.6 pushouts are unique upto a homeomorphism. Hence we can express this fact by the statement $(X_f \sqcup_g Y)_{\bar{f}} \sqcup_h Z \cong X_f \sqcup_{hg} Z$. We will refer to this fact as the Law of Horizontal Composition.

(b) In a similar manner we have the following Law of Vertical Composition in Top. $Z_h \sqcup_{\bar{g}} (X_f \sqcup_g Y) \cong Z_{hf} \sqcup_g Y$ based on the diagram

$$
\begin{tikzcd}
{} \arrow[r, "\bar{\bar{g}}"] & Z_h \sqcup_{\bar{g}} (X_f \sqcup_g Y) \cong Z_{hf} \sqcup_g Y \\
X \arrow[u, "h"] \arrow[r, "\bar{g}"] & X_f \sqcup_g Y \arrow[u, "\bar{h}"'] \\
A \arrow[u, "f"] \arrow[r, "g"'] & Y \arrow[u, "\bar{f}"']
\end{tikzcd}
$$

Theorem 1.2.7: Let Z be a locally compact space. In the following diagrams, assume that the left square is a pushout. Then the right square is a pushout.

$$
\begin{tikzcd}
X \arrow[r, "\bar{g}"] & X_f \sqcup_g Y \\
A \arrow[u, "f"] \arrow[r, "g"'] & Y \arrow[u, "\bar{f}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times Z \arrow[r] & (X_f \sqcup_g Y) \times Z \\
A \times Z \arrow[u, "f \times 1"] \arrow[r, "g \times 1"'] & Y \times Z \arrow[u, "\bar{f} \times 1"']
\end{tikzcd}
$$

Proof: Let W be any space and let $k: X \times Z \to W$ and $\ell: Y \times Z \to W$ be given maps such that $k(f \times 1) = \ell(g \times 1)$. Now consider the following diagram

$$
\begin{tikzcd}
X \times Z \arrow[r, "\bar{g} \times 1"] \arrow[rr, bend left, "k"] & (X_f \sqcup_g Y) \times Z \arrow[r, dashed, "\phi"] & W \\
A \times Z \arrow[u, "f \times 1"] \arrow[r, "g \times 1"'] & Y \times Z \arrow[u, "\bar{f} \times 1"'] \arrow[ur, bend right, "\ell"'] &
\end{tikzcd}
$$

We have to show that $\exists\,!$ map $\phi: (X_f \sqcup {}_g) \times Z \to W$ such that $\phi \cdot (\bar{g} \times 1) = k$ and $\phi \cdot (\bar{f} \times 1) = \ell$. Now, by Theorem 1.2.3(a) the maps k and ℓ determine maps $\hat{k}: X \to W^Z$ and $\hat{\ell}: Y \to W^Z$ the rules $\hat{k}(x)(z) = k(x,z)$ and $\hat{\ell}(y)(z) = \ell(y,z)$. Hence, we have the following diagram.

$$\begin{tikzcd}
X \arrow[r, "\bar{g}"] \arrow[rr, bend left, "\hat{k}"] & X_f \sqcup_g Y \arrow[r, dashed, "\hat{\phi}"] & W^Z \\
A \arrow[u, "f"] \arrow[r, "g"] & Y \arrow[u, "\bar{f}"] \arrow[ur, bend right, "\hat{\ell}"'] &
\end{tikzcd}$$

Now, $k(f \times 1) = \ell(g \times 1) \Leftrightarrow k(f \times 1)(a,z) = \ell(g \times 1)(a,z)$, for all $(a,z) \in A \times Z$

$\Leftrightarrow k(f(a),z) = \ell(g(a),z)$, for all $(a,z) \in A \times Z$

$\Leftrightarrow \hat{k}(f(a))(z) = \hat{\ell}(g(a))(z)$, for all $z \in Z$ and all $a \in A$

$\Leftrightarrow \hat{k}(f(a)) = \hat{\ell}(g(a))$, for all $a \in A$

$\Leftrightarrow \hat{k}f = \hat{\ell}g$.

Therefore, the diagram above commutes and since it is a pushout, ! map $\hat{\phi}: X_f \quad {}_g\, Y \to W^Z$ such that $\hat{\phi}\bar{g} = \hat{k}$ and $\hat{\phi}\bar{f} = \hat{\ell}$.
Since Z is locally compact, $\hat{\phi}$ induces a map $\phi: (X_f \quad {}_g\, Y) \times Z \to W$ such that $\phi \cdot (\bar{g} \times 1)(x,z) = \phi((\bar{g} \times 1)(x,z))$

$= \phi(\bar{g}(x), z)$

$= \hat{\phi}(\bar{g}(x))(z)$

$= \hat{k}(x)(z)$, as $\hat{\phi}\bar{f} = \hat{k}$

$= k(x,z)$

That is, $\phi \cdot (\bar{g} \times 1) = k$.
Similarly, $\phi \cdot (\bar{f} \times 1) = \ell$. Clearly, ϕ is unique, as $\hat{\phi}$ is unique. Therefore, diagram * is a pushout.

Remark 1.2.6: By uniqueness of the pushout object, we have that $(X_f \; {}_g Y) \times Z \cong X \times Z_{f\times 1} \; {}_{g\times 1} Y \times Z$.

Section 3: Homotopy Theory

Definition 1.3.1: Let f and g be continuous functions from X to Y. We say f is homotopic to g, written $f \simeq g$, if there is a continuous function $H: X \times I \to Y$ with $H(x,0) = f(x)$ and $H(x,1) = g(x)$, for all $x \in X$. The map H is called a homotopy from f to g.

Notation: We write $H: f \simeq g$, when H is a homotopy from f to g.

Letting $h_t(x) = H(x,t)$, for $x \in X$ and $t \in I$, the homotopy H is seen to represent a family $\{h_t | t \in I\}$ of functions from X to Y, varying continuously with t, such that $h_0 = f$ and $h_1 = g$. Depending on the situation, we will represent a homotopy either as a map H or as a family of maps $\{h_t\}_{t \in I}$, varying continuously with t.

The following results are easy consequences of the definition of homotopy.

Theorem 1.3.1:

(a) The relation "$\simeq$" is an equivalence relation.

(b) If $f, g: X \to Y$, $f', g': Y \to Z$ are maps such that $f \simeq g$ and $f' \simeq g'$, then $f'f \simeq g'g$.

(c) Let X, Y, Z be spaces. Then there exists a homotopy $H: X \times I \to Y$ from f to g <=> there exists a homotopy $G: X \times Z \times I \to Y \times Z$ from $f \times 1_Z$ to $g \times 1_Z$, for all Z.

(d) If $H: X \times I \to Y$ is a homotopy from f to g and $\phi: Y \to Z$ is a map, then $\exists$ a homotopy $G: \phi f \simeq \phi g$.

(e) If $H:X \times I \to Y$ is a homotopy from f to g and $\phi:Z \to X$ is a map, then $\exists$ a homotopy $G:f\phi \simeq g\phi$.

Proof:

(a) We leave the details to the reader.

(b) We sketch the proof.

Let $H:f \simeq g$ and $G:f' \simeq g'$.

Then, $f'H:f'f \simeq f'g$ and $G(g \times 1):f'g \simeq g'g$.

Therefore, by transitivity of the relation "$\simeq$" (see part (a)) we have that $f'f \simeq g'g$, as required.

(c) Since f and g are continuous, then so are $f \times 1$, $g \times 1:X \times Z \to Y \times Z$ (Cartesian product of maps). Also, $Q:X \times Z \times I \cong X \times I \times Z$ (commutativity). Define $G:X \times Z \times I \to Y \times Z$ by

$$G = (H \times 1_Z) \cdot Q$$

That is,
$$\begin{aligned} G(x,z,t) &= [(H \times 1_Z) \cdot Q](x,z,t) \\ &= (H \times 1_Z)(x,t,z) \\ &= (H(x,t),z) \end{aligned}$$

Then G is a homotopy from $f \times 1$ to $g \times 1$, as required. Conversely, suppose $G:f \times 1 \simeq g \times 1:X \times Z \times I \to Y \times Z$, for any space Z. Taking $Z = \{*\}$, define $H:X \times I \to Y$ by $H = p_Y G \theta$, where $\theta:X \times I \cong X \times \{*\} \times I$ and $p_Y:Y \times \{*\} \to Y$ is projection on the first factor. Then H is the required homotopy from f to g.

(d) Consider $X \times I \xrightarrow{H} Y \xrightarrow{\phi} Z$.

ϕH is continuous and $\phi H: X \times I \to Z$ is the required homotopy from ϕf to ϕg.

(e) Consider $Z \times I \xrightarrow{\phi \times 1_I} X \times I \xrightarrow{H} Y$.

Then, $H(\phi \times 1_X): Z \times I \to Y$ is continuous and is the required homotopy from $f\phi$ to $g\phi$.

Remark 1.3.1: By Theorem 1.3.1(a) and (b), the relation "$\simeq$" is a natural equivalence on the category Top. We can thus form the quotient category Top/$\simeq$, (see Definition 1.1.6) denoted by Toph. Observe that the objects of Toph are objects of Top and for all $X, Y \in |\text{Top}| = |\text{Toph}|$; Toph$(X,Y)$ is then the set of all homotopy classes of maps from X into Y, written Toph$(X,Y) = [X,Y]$. If $f: X \to Y$ is a map, we denote the homotopy class of f by $[f]$. Note that Toph is the "base category" for Algebraic Topology.

Definition 1.3.2: A continuous function $f: X \to Y$ is said to be a homotopy equivalence (or h-equivalence), if $[f]$ is an isomorphism in Toph; that is, if $\exists$ a map $g: Y \to X$ such that $gf \simeq 1_X$ and $fg \simeq 1_Y$. We then say g is a homotopy left inverse of f and f is a homotopy right inverse of g. The map g is a homotopy inverse of f if it is both a right and a left homotopy inverse of f, and f is said to be an h-equivalence if it has a homotopy inverse.

Example 1.3.1: Homeomorphisms are homotopy equivalences. A special case of h-equivalence is the notion of a space being contractible.

Definition 1.3.3: A space X is said to be contractible if it is homotopy equivalent to a point. Equivalently, X is contractible if $\exists\ x_0 \in X$ such that the map $1_X : X \to X$ is homotopic to the constant map $c_{x_0} : X \to X$ at x_0 (i.e. $c_{x_0}(x) = x_0$ for all $x \in X$).

The following are easy consequences of the definition of h-equivalence. Again, as before we give a sketch of the proofs whenever necessary.

Theorem 1.3.2:

(a) If $f : A \to B$, $g : A \to C$ and $h : B \to C$ are maps such that g and h are h-equivalences and $hf \simeq g$, then f is an h-equivalence.

(b) If $f : A \to B$ is a map and $g : B \to A$ is a map such that $gf \simeq 1_A$ and $h : B \to A$ is a map such that $fh \simeq 1_B$, then f is an h-equivalence.

Proof:

(a) g is an h-equivalence => $g' : C \to A$ such that $g'g \simeq 1_A$ and $gg' \simeq 1_C$.

But $hf \simeq g \Rightarrow g'hf \simeq g'g$ (see Theorem 1.3.1(d))

$\Rightarrow g'hf \simeq g'g \simeq 1_A$

$\Rightarrow g'hf \simeq 1_A$ (see Theorem 1.3.1(a))

That is, $g'h$ is a left homotopy inverse for f. Again, h is an h-equivalence => $h' : C \to B$ such that $h'h \simeq 1_B$ and $hh' \simeq 1_C$. Now, $h'h \simeq 1_B \Rightarrow h'hfg'h \simeq 1_B\, fg'h = fg'h$ (by Theorem 1.3.1(d) and the fact that $h'h \simeq 1_B$). Again, since $hf \simeq g$ we have that

$$h'hfg'h \simeq h'gg'h \simeq h'1_C\, h \simeq h'h \simeq 1_B.$$

So, $fg'h \simeq h'hfg'h \simeq 1_B$.

Therefore, $g'h$ is a right homotopy inverse for f. Therefore, f is an h-equivalence.

(b) Consider the following composite

$$A \xrightarrow{f} B \xrightarrow{g} A \xrightarrow{f} B$$

Now, $fh \simeq 1_B \Rightarrow fgfh \simeq fg\, 1_B = fg$.

$\Rightarrow fgfh \simeq fg$

But, $fgfh \simeq f\, 1_A\, h = fh \simeq 1_B$.

Hence, $fg \simeq fgfh \simeq 1_B$.

Since $gf \simeq 1_A$ (by hypothesis) and $fg \simeq 1_B$ from above, we have that f is an h-equivalence.

We now briefly discuss a more general concept of homotopy - that of homotopy relative to a subspace A. Here we require that the homotopy remains invariant on pts. of A.

<u>Definition 1.3.4</u>: Suppose that $A \subseteq X$ and $f,g:X \to Y$ are maps. We say that f and g are <u>homotopic relative to A</u>, denoted $f \simeq g$ (rel A) or $f \simeq_{\text{rel } A} g$, if $\exists$ a homotopy $H:f \simeq g$ such that $H(a,t) = f(a) = g(a)$ for all $a \in A$, $t \in I$.

<u>Remark 1.3.2</u>: The relation $\simeq_{\text{rel } A}$ on the set of maps from X to Y is an equivalence relation.

<u>Defintion 1.3.5</u>: A subspace A of X is a retract of X if there is a map $r:X \to A$, called a <u>retraction</u> such that $r\big|_A = 1_A$.

Definition 1.3.6: A subspace A of X is called a deformation retract (DR) of X if there is a retraction $r:X \to A$ such that $ir \simeq 1_X:X \to X$, where $i:A \to X$ is the inclusion. In other words, A is a deformation retract of X if there is a homotopy $H:X \times I \to X$ such that $H(x,0) = x$ and $H(x,1) = r(x) \in A$, for $x \in X$.

Remark 1.3.3: If A is a deformation retract of X, then A and X are homotopy equivalent.

Definition 1.3.7: A subset A of X is a strong deformation retract (SDR) if there is a retraction $r:X \to A$ such that $ir \simeq_{\text{rel } A} 1_X$.

In other words, A is a SDR of X if there is a homotopy $F:X \times I \to X$ such that $F(x,0) = x$, for all $x \in X$

$F(a,t) = a$, for all $x \in A$ and all $t \in I$

$F(x,1) = r(x) \in A$, for all $x \in X$.

Note that a SDR is, obviously, also a DR.

We now extend the definition of homotopy to the categories $\underline{Top}^A$ and $\underline{Top}_B$.

Definition 1.3.8: Let $i:A \to X$ and $i':A \to Y$ be objects of $\underline{Top}^A$. Suppose $f,g:i \to i'$ are morphisms of $\underline{Top}^A$. That is, $fi = i'$ and $gi = i'$. Then f is said to be homotopic to g under A denoted $f \simeq^A g$, if there is a homotopy $H:X \times I \to Y$ such that $H:f \simeq g$ and $H(i \times 1_I) = i' \cdot pr_A$; that is, the following diagram commutes.

$$\begin{tikzcd} X \times I \arrow[r, "H"] & Y \\ A \times I \arrow[u, "i \times 1_I"] \arrow[r, "pr_A"'] & A \arrow[u, "i'"'] \end{tikzcd}$$

Notice that the equation $H(i \times 1_I) = i' \cdot pr_A$ can be replaced by the statement $h_t i = i'$, for all $t \in I$, where $h_t: X \to Y$ is the homotopy such that $h_0 = f$ and $h_1 = g$. Therefore, a homotopy under A of f into g is a homotopy in the ordinary sense which is a map under A at each stage of the deformation.

Remark 1.3.4: If A is a subspace of X, then $f \simeq^A g$ reduces to the special case $f \simeq_{\text{rel } A} g$.

Defintion 1.3.9 : Let $p: X \to B$ and $p': Y \to B$ be objects of $\underline{Top}_B$. Suppose $f, g: p \to p'$ are morphisms in $\underline{top}_B$; that is, $p'f = p$ and $p'g = p$. Then f is said to be <u>homotopic</u> to g <u>over</u> $\underline{B}$, denoted $f \simeq_B g$, if a homotopy $H: X \times I \to Y$ such that $H: f \simeq g$ and $p'H = p \cdot pr_X$; that is, the following diagram commutes.

$$\begin{tikzcd} X \times I \arrow[r, "H"] \arrow[d, "pr_X"'] & Y \arrow[d, "p'"] \\ X \arrow[r, "p"'] & B \end{tikzcd}$$

Therefore, as above, a homotopy over B of f into g is a homotopy in the ordinary sense which is a map over B at each stage of the deformation.

Remark 1.3.5: The relations "$\overset{A}{\simeq}$" and "$\simeq_B$" are natural equivalence relations in Top^A and Top_B. We then can form the quotient categories (see Definition 1.1.6)

$$Top^A/\overset{A}{\simeq} = Top^A h \quad \text{and} \quad Top_B/\simeq_B = Top_B h \ .$$

If $i, i' \in |Top^A| = |Top^A h|$, $Top^A h \ (i, i')$ is the set of all homotopy classes of maps X into Y under A; that is, $Top^A h \ (i, i') = [X, Y]^A$. Similarly, for $p, p' \in |Top_B| = |Top_B h|$, $Top_B h(p, p') = [X, Y]_B$. If f is a morphism in Top^A (Top_B), then we denote the homotopy class of f by $[f]^A$ $([f]_B)$.

We now extend the notion of homotopy equivalence to the categories Top^A AND Top_B.

Definition 1.3.10: A morphism f in Top^A (Top_B) is a homotopy equivalence under A (over B) if $[f]^A$ $([f]_B)$ is an isomorphism in $Top^A h$ $(Top_B h)$.

In other words, if $f: X \to Y$ is a morphism in Top^A (by an abuse of language), then f is an h-equivalence under A, if $g: Y \to X$ in Top^A such that $gf \overset{A}{\simeq} 1_X$ and $fg \overset{A}{\simeq} 1_Y$.

Remark 1.3.6:

(a) Let i and i' be maps under A. Then i is h-equivalent under A to i', if i and i' are isomorphic as objects in $Top^A h$.

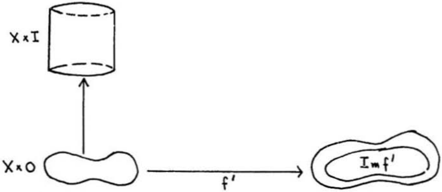
(b) Let p and p' be maps over B. Then p is h-equivalence over B to p', if p and p' are isomorphic as objects in $Top_B h$.

We conclude this section with a brief discussion of some basic properties of adjunction spaces introduced in Section 2. We begin by introducing the <u>mapping cylinder</u>, which is a special case of the adjunction space.

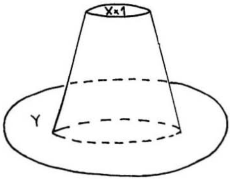
<u>Definition 1.3.11</u>: Let X and Y be topological spaces and let $f: X \to Y$ be a given map.

Define $f': X \times 0 \to Y$ by $f'(x,0) = f(x)$.

Now, $X \times 0$ is a subspace of $X \times I$ and hence the pushout of



is the adjunction space $M(f) = (X \times 1) \sqcup_{f'} Y$



which is called the <u>mapping cylinder</u> of f.

One of the important features of adjunction spaces is given by the following result.

<u>Theorem 1.3.3</u>: Consider the following pushout in <u>Top</u>

$$\begin{tikzcd} X \arrow[r, "\bar{f}"] & X \sqcup_f Y \\ A \arrow[u, "i"] \arrow[r, "f"'] & Y \arrow[u, "\bar{i}"'] \end{tikzcd}$$

where A is a closed subspace of X and i is the inclusion map. Then $\bar{i}$ is a one to one closed map and $\bar{f}|_{X-A}$ is one to one and open.

<u>Proof</u>: Clearly, $\bar{i}$ is one to one. Now let C be closed in Y and let $C' = \bar{i}(C)$. Then, $\bar{i}^{-1}(C') = \bar{i}^{-1}(\bar{i}(C)) = C$ since $\bar{i}$ is one to one. So, $\bar{i}^{-1}(C') = C$ is closed in Y. If $C' \cap (Y\ f(A)) = \phi$, then $\bar{f}^{-1}(C') = \phi$. If $C' \cap (Y - f(A)) \neq \phi$, then $\bar{f}^{-1}(C') = f^{-1}(C)$. But f is continuous and C is closed in Y. Hence, $f^{-1}(C') = f^{-1}(C)$ is closed in A and thus in X, since $A \subseteq X$ is closed. In any event, $\bar{f}^{-1}(C') \subseteq X$ is closed. Therefore, C' is closed in $X \sqcup_f Y$, as $X \sqcup_f Y$ has the final topology with respect to $\bar{f}$ and $\bar{i}$. (see Remark 1.2.3(c)). The proof that $\bar{f}|_{X-A}$ is one to one and open is similar.

Remark 1.3.7:

(a) Notice that it is immediate from above that $\bar{i}$ is a homeomorphism onto a closed subspace, and $\bar{f}|_{X-A}$ is a homeomorphism onto an open subspace of $X \sqcup_f Y$. Thus, we have that under the assumption $A \subseteq X$ is closed, Y is a closed subspace and $X \backslash A$ is an open subspace of $X \sqcup_f Y$.

(b) If X and Y are compact, then so is $X \sqcup Y$ and hence as the continuous image of a compact space, $X \sqcup_f Y$ is also compact.

(c) If $A \neq \phi$ and X and Y are path connected, then $X \sqcup_f Y$ is path connected.

(d) If X and Y are normal, then $X \sqcup_f Y$ is normal.

Lemma 1.3.1: A is a strong deformation of $A \times I$.

Proof: Clearly, $\{0\}$ is a SDR of I under the map $F: I \times I \to I$ given by $F(x,t) = (1-t)x$.

Consider $A \times I \times I \xrightarrow{1_A \times F} A \times I$.

Now, $1_A \times F$ is a map since both 1_A and F are maps.

Furthermore,

(i) for all $(a,s) \in A \times I$;

$$(1_A \times F)(a,s,0) = (a,F(s,0)) = (a,s)$$

(ii) for all $(a,0) \in A \times 0$;

$$(1_A \times F)(a,0,t) = (a,F(0,t)) = (a,0)$$

(iii) for all $(a,s) \in A \times I$;

$$(1_A \times F)(a,s,1) = (a,F(s,1)) = (a,0) \in A \times 0.$$

Therefore, $A \times 0$ is a SDR of $A \times I$.

Intuitively, the above result is obvious since the bottom of the cylinder is a SDR of the entire cylinder.

Theorem 1.3.4: Consider the following pushout diagram in Top.

$$\begin{tikzcd} Y \arrow[r, "\bar{f}"] & A \sqcup Y \\ D \arrow[u, "i"] \arrow[r, "f"'] & A \arrow[u, "\bar{i}"'] \end{tikzcd}$$

(a) If D is closed in Y and a SDR of Y, then A is a SDR of $A \sqcup_f Y$.

(b) In particular if $M(f) = A \sqcup_f D \times I$ be the mapping cylinder of the map $f: D \to A$. Then A is a SDR of $M(f)$.

Proof:

(a) Since D is a SDR of Y, $\exists$ a retraction $r: Y \to D$ and a homotopy $H: Y \times I \to Y$ such that

$$H(y,0) = y, \ y \in Y$$

$$H(d,t) = d, \ d \in D \text{ and } t \in I$$

$$H(y,1) = r(y) \in D$$

Let $X = A \sqcup_f Y$. Since I is locally compact, it follows from Theorem 1.2.7, that $X \times I \cong A \times I \sqcup_{f \times 1} Y \times I$. Consider now the following diagram.

$$\begin{tikzcd}
Y \times I \arrow[r, "\bar{f} \times 1"] \arrow[rr, bend left=40, "\bar{f}H"] & X \times I \cong A \times I \sqcup_{f \times 1} Y \times I \arrow[r, dashed, "K"] & X = A \sqcup_f Y \\
D \times I \arrow[u, "i \times 1"] \arrow[r, "f \times 1"'] & A \times I \arrow[u, "\bar{i} \times 1"'] \arrow[ur, bend right=30, "pr_X(\bar{i} \times 1)"'] &
\end{tikzcd}$$

Observe that $\bar{f}\,H(i \times 1)(d,t) = \bar{f}H(i(d),t)$

$= \bar{f}H(d,t)$

$= \bar{f}(d)$

and $pr_X(\bar{i} \times 1)(f \times 1)(d,t) = pr_X(\bar{i}f(d),t)$

$= \bar{i}f(d)$

$= \bar{f}i(d)$

$= \bar{f}(d)$

Therefore, $\bar{f}H(i \times 1) = pr_X(\bar{i} \times 1)(f \times 1)$ and so by the universal property of pushouts, $\exists!$ $K: X \times I \to X$ such that

$$K(\bar{f} \times 1) = \bar{f}H$$

and $K(\bar{i} \times 1) = pr_X(\bar{i} \times 1)$

We now show that K is the required deformation retraction.

(i) Let $\bar{x} \in X = A \sqcup_f Y$. Then either $x \in A$ or $x \in Y$.

Suppose $x \in A$.

Then, $K(\bar{x},0) = pr_X(\bar{i} \times 1)(x,0)$

$= \bar{i}(x)$

$= \bar{x}$

Suppose $x \in Y$.

Then, $K(\bar{x},0) = K(\bar{f} \times 1)(x,0)$

$= \bar{f}H(x,0)$

$= \bar{f}(x)$

$= \bar{x}$

Therefore, $K(\bar{x},0) = \bar{x}$, for all $\bar{x} \in A \sqcup_f Y$.

(ii) Let $a \in A$.

Then, $K(\bar{a},t) = pr_X(\bar{i} \times 1)(a,t)$

$= pr_X(\bar{i}(a),t)$

$= \bar{i}(a)$

$= \bar{a}$, for all $t \in I$

That is, K leaves $\bar{i}(A)$ fixed.

(iii) If $x \in A$, then

$$K(\bar{x},1) = pr_X(\bar{i}(x),1)$$

$$= \bar{i}(x) \in \bar{i}(A)$$

If $x \in Y$, then

$$K(\bar{x},1) = \bar{f}H(x,1)$$

$$= \bar{f}ir(x)$$

$$= \bar{i}f(r(x)) \in \bar{i}(A)$$

Therefore, $\bar{i}(A)$ is a SDR of $A \sqcup_f Y$. But by Theorem 1.3.3, $\bar{i}(A) \cong A$. Hence, A is a SDR of $A \sqcup_f Y$.

Chapter II

Cofibrations

This chapter is subdivided into three sections. In Section 1, we discuss the notion of HEP (Homotopy Extension Property) which is a prelude to the definition of a cofibration. The various equivalent definitions of a cofibration are discussed in detail along with some basic properties of cofibrations, some of which are categorical in nature.

The second section is the core of the chapter, where we discuss the "Characterization Theorem of Cofibrations", along with some immediate consequences of this result. An attempt is made to put together the various characterizations of cofibrations scattered in the literature.

Finally, in the third section we give some geometric examples of closed cofibrations, contrasted with examples that fail to be cofibrations, concluding with an example of a non-closed cofibration.

Section I: Definitions and Categorical Properties of Cofibrations

Definition 2.1.1: Let A be a subspace of a space X. The inclusion $i:A \to X$ has the homotopy extension property (HEP) with respect to a space Z if for all maps $f:X \to Z$, any homotopy of $f\big|_A$ extends to a homotopy of f. We say $i:A \to X$ as the HEP if the above statement is true for all spaces Z.

In other words, $i:A \to X$ is said to have the HEP with respect to Z if, given maps $f:X \to Z$ and $G:A \times I \to Z$ such that $f(a) = G(-,0)$

for $a \in A$, there is a map (not necessarily unique) $F:X \times I \to Z$ such that $F(-,0) = f(x)$ and $F\big|_{A\times I} = G$.

The existence of F is equivalent to the existence of a map represented by the dotted arrow which makes the following diagram commutative

$$\begin{tikzcd} & & Z \\ X \arrow[r] \arrow[urr, bend left, "f"] & X\times I \arrow[ur, dashed, "F"] & \\ A \arrow[u, "i"] \arrow[r] & A\times I \arrow[u, "i\times 1"'] \arrow[uur, bend right, "G"'] & \end{tikzcd} \qquad (*)$$

Thus, the HEP for $i:A \to X$ is equivalent to the condition that the square in diagram (*) above is a <u>weak pushout</u>.

<u>Definition 2.1.2</u>: A <u>cofibration</u> is a map $j:A \to X$ such that for any map $f:X \to Z$ (Z arbitrary) and any homotopy $G:A \times I \to Z$ such that $G(a,0) = fj(a)$ for all $a \in A$, there exists a homotopy $F:X \times I \to Z$ such that $F(j \times 1_I) = G$ and $F(-,0) = f(x)$ for $x \in X$. That is, there exists a map F represented by the dotted arrow making the following diagram commutative.

$$\begin{tikzcd} & & Z \\ X \arrow[r] \arrow[urr, bend left, "f"] & X\times I \arrow[ur, dashed, "F"] & \\ A \arrow[u, "j"] \arrow[r] & A\times I \arrow[u, "j\times 1_I"'] \arrow[uur, bend right, "G"'] & \end{tikzcd}$$

Thus, if A is a subspace of X, the inclusion map $i:A \to X$ is a cofibration iff the pair (X,A) has the HEP with respect to any space. In this case the pair (X,A) is called a cofibred pair or is said to possess the "Absolute Homotopy Extension Property (AHEP)".

Next, we shall show that all cofibrations are embeddings. That is, if $j:A \to X$ is a cofibration, we can without any loss of generality restrict our attention to the case A is a subspace of X and j is the inclusion. But before we do that we need the following

Lemma 2.1.1: Given a map $j:A \to X$, let $M(j)$ denote the mapping cylinder of j. Define a function $e:M(j) \to X \times I$ by

$$e[x] = (x,0),\ x \in X$$

$$e[a,t] = (j(a),t),\ (a,t) \in A \times I$$

Then (a) e is continuous

(b) j is a cofibration <=> e has a left inverse.

Proof:

(a) Consider the following commutative diagram

$$\begin{tikzcd}
 & & X \times I \\
A \times I \arrow[urr, bend left, "j \times 1"] \arrow[r, "\bar{J}"] & M(j) \arrow[ur, dashed, "e"] & \\
A \arrow[u, "\sigma_0"] \arrow[r, "j"] & X \arrow[u, "\sigma"] \arrow[uur, bend right, "k_0"'] &
\end{tikzcd}$$

where σ_0 and k_0 are inclusions at the zero level and σ, $\bar{j}$ are the canonical inclusions.

Now, $j \times 1$ and k_0 are maps and $M(j)$ is a pushout. Hence, e is continuous $\Leftrightarrow$ $j \times 1$ and k_0 are continuous (see Theorem 1.2.6).

(b) "$\Rightarrow$": Suppose j is a cofibration. We will show that e admits a left inverse. Consider the following diagram

$$\begin{tikzcd}
 & & M(j) \arrow[dl, shift left, "e"] \\
X \arrow[r, "k_0"] \arrow[urr, bend left, "\sigma"] & X \times I \arrow[ur, dashed, shift left, "\varphi"] & \\
A \arrow[u, "j"] \arrow[r, "\sigma_0"] & A \times I \arrow[u, "j \times 1"] \arrow[uur, bend right, "\bar{j}"'] &
\end{tikzcd}$$

where σ_0, k_0, σ and $\bar{j}$ are defined, as above.

Now, $\sigma_j(a) = [j(a)]$. Since $(a,0) \sim j(a)$, we have that $[j(a)] = [a,0]$. Moreover, $\bar{j}\sigma_0(a) = \bar{j}(a,0) = [a,0]$. Therefore, $\sigma j = \bar{j}\sigma_0$.

Since $j:A \to X$ is a cofibration, $\exists\ \varphi:X \times I \to M(j)$ such that $\varphi k_0 = \sigma$ and $\varphi(j \times 1) = \bar{j}$.

Now, $\varphi \cdot r[x] = \varphi(x,0) = \varphi k_0(x) = \sigma(x) = [x]$ and

$$\varphi \cdot e[a,t] = \varphi(j(a),t) = \varphi \cdot (j \times 1)(a,t) = \bar{j}(a,t) = [a,t].$$

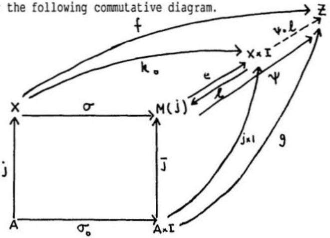
So, $\varphi \cdot e = 1_{M(j)}$ and φ is a left inverse of e.

"$\Leftarrow$":

Suppose the map $e:M(j) \to X \times I$ admits a left inverse

$\ell : X \times I \to M(j)$ such that $\ell \cdot e = 1_{M(j)}$.

Consider the following commutative diagram.



where $f : X \to Z$ and $g : A \times I \to Z$ are given maps and the other maps are defined as above. Since $M(j)$ is a pushout, there exists a unique map $\psi : M(j) \to Z$ such that $\psi\sigma = f$ and $\psi\bar{j} = g$. Composing ℓ with ψ yields a map $\psi \cdot \ell : X \times I \to Z$ with the desired properties. That is,

$$\psi \cdot \ell \cdot k_0(x) = \psi \cdot \ell \cdot e \cdot \sigma(x) = \psi \cdot \sigma(x) = f(x) \quad \text{and}$$

$$\psi \cdot \ell \cdot j \times 1(a,t) = \psi \cdot \ell \cdot e \cdot \bar{j}(a,t) = \psi \cdot \bar{j}(a,t) = g(a,t).$$

Therefore, $j : A \to X$ is a cofibration.

Theorem 2.1.1: All cofibrations are embeddings.

Proof: Let $j : A \to X$ be a cofibration.

We will show that $A \cong j(A)$.

By Lemma 2.1.1(b), the map $e : M(j) \to X \times I$ admits a left inverse.

Hence e is a homeomorphism of $M(j)$ onto $e(M(j)) = X \cup (j(A) \times 1)$.

Since $\bar{j}$ is an inclusion and $e:M(j) \cong X \cup (j(A) \times 1)$, it follows that $e\bar{j}\big|_{A\times 1}: A \times 1 \to e\bar{j}(A \times 1)$ is a homeomorphism. Hence,

$A \times 1 \cong e\bar{j}(A \times 1) = j \times 1(A \times 1) = j(A) \times 1.$

Therefore, $A \times 1 \cong j(A) \times 1$ and hence $A \cong j(A)$.

The following equivalent definition of a cofibration will be utilized whenever it is appropriate.

Definition 2.1.3: $i:A \to X$ is a cofibration if for all spaces Z and each commutative square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, dashed, "\varphi"] & Z \\
A \arrow[u, "i"] \arrow[r, "g"'] & Z^I \arrow[u, "\epsilon_0"']
\end{tikzcd}$$

where $\epsilon_0(\lambda) = \lambda(0)$, for all $\lambda:I \to Z$, is the evaluation map, the dotted arrow exists making the triangles commute.

The equivalence of Definition 2.1.2 and Definition 2.1.3 is established by considering the following two diagrams

$$\begin{tikzcd}
 & & Z \\
X \arrow[rru, bend left, "f"] \arrow[r] & X \times I \arrow[ur, dashed, "\tilde{\varphi}"] & \\
A \arrow[u, "i"] \arrow[r] & A \times I \arrow[u, "i \times 1_I"'] \arrow[uur, bend right, "\tilde{g}"'] &
\end{tikzcd}$$

Diagram (I)
(Definition 2.1.2)

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, dashed, "\varphi"] & Z \\
A \arrow[u, "i"] \arrow[r, "g"'] & Z^I \arrow[u, "\epsilon_0"']
\end{tikzcd}$$

Diagram (II)
(Definition 2.1.3)

(a) Definition 2.1.2 => Definition 2.1.3

Assume $i:A \to X$ is a cofibration in the sense of Definition 2.1.2 (i.e. consider diagram I). Now, the given map $\tilde{g}:A \times I \to Z$ determines a map $g:A \to Z^I$ defined by $g(a)(t) = \tilde{g}(a,t)$ (see Theorem 1.2.3(a).

Similarly, the existence of $\tilde{\varphi}:X \times I \to Z$ such that $\tilde{\varphi}|_X = f$ and $\tilde{\varphi}|_{A \times I} = \tilde{g}$, guarantees the existence of a map $\varphi:X \to Z^I$ such that $\varphi(x)(t) = \tilde{\varphi}(x,t)$ (see Theorem 1.2.3(a)). Now, $\epsilon_0 g(a) = g(a)(0)$ $= \tilde{g}(a,0) = \tilde{\varphi}(i \times 1_I)(a,0) = \tilde{\varphi}(i(a),0) = fi(a)$ and so outer square of diagram II commutes. Moreover, $\epsilon_0\varphi(x) = \varphi(x)(0) = \tilde{\varphi}(x,0) = f(x)$ and $\varphi(i(a))(t) = \tilde{\varphi}(i(a),t) = \tilde{\varphi}(i \times 1)(a,t) = \tilde{g}(a,t) = g(a)(t)$ for all $t \in I$ and hence $\varphi|_A = g$. Therefore, φ has the required properties.

(b) Definition 2.1.3 => Definition 2.1.2

Assume $i:A \to X$ is a cofibration in the sense of Definition 2.1.3 (i.e. consider diagram II).

Since I is locally compact and Hausdorff; $g:A \to Z^I$ is continuous $\Rightarrow \tilde{g}:A \times I \to Z$ is continuous and $\varphi:X \to Z^I$ is continuous $\Rightarrow$ $\tilde{\varphi}:X \times I \to Z$ is continuous (see Theorem 1.2.3(b)).

Now, $\tilde{g}(a,0) = g(a)(0) = \epsilon_0(g(a))$

$= fi(a)$ by commutativity of diagram II.

Moreover, $\tilde{\varphi}(x,0) = \varphi(x)(0) = \epsilon_0(\varphi(x)) = f(x)$ and

$\tilde{\varphi}(i \times 1)(a,t) = \tilde{\varphi}(i(a),t) = \varphi(i(a))(t) = g(a)(t) = \tilde{g}(a,t)$.

Thus, $\tilde{\varphi}$ has the required properties of Definition 2.1.2.

Therefore Definition 2.1.2 is equivalent to Definition 2.1.3.

The following are easy consequences of the definition of a cofibration.

Theorem 2.1.2:

(a) For any space X, (X,X) is a cofibred pair.

(b) Maps with empty domain are cofibrations.

(c) Homeomorphisms are cofibrations.

(d) Composition of cofibrations is a cofibration.

Proof: (a), (b) and (c) trivially follow from the diagram of a weak pushout. (d) similar to the proof of Theorem 1.1.3(a).

The following theorem has interesting applications for adjunction spaces and mapping cylinders.

Theorem 2.1.3: The pushout of a cofibration is a cofibration.

Proof: Let

$$\begin{tikzcd} X \arrow[r, "\bar{f}"] & Y \\ A \arrow[u, "i"] \arrow[r, "f"'] & B \arrow[u, "\bar{i}"'] \end{tikzcd}$$

be a pushout diagram where $i:A \to X$ is a cofibration. We prove that $\bar{i}:B \to Y$ is a cofibration.

Construct the following diagram

$$\begin{tikzcd} X \arrow[r, "\bar{f}"] \arrow[drr, dashed, "\varphi"] & Y \arrow[r, "\bar{g}"] \arrow[dr, dashed, "\psi"] & Z \\ A \arrow[u, "i"] \arrow[r, "f"'] & B \arrow[u, "\bar{i}"] \arrow[r, "g"'] & {} \arrow[u, "\epsilon_0"'] \end{tikzcd} \qquad (*)$$

such that Z is any space, ϵ_0 is the evaluation map and the right square commutes. Since the left square is commutative, the composite square commutes. Now, $i:A \to X$ is a cofibration implies that $\exists$ a map $\varphi:X \to Z^I$ such that $\epsilon_0\varphi = \bar{g}\,\bar{f}$ and $\varphi i = gf$. We thus obtain the following diagram

$$\begin{tikzcd}
 & & & Z \\
 & & Z^I \arrow[ur, "\epsilon_0"] & \\
X \arrow[r, "\bar{f}"] \arrow[urr, bend left, "\varphi"] \arrow[uurrr, bend left, "\bar{g}\bar{f}"] & Y \arrow[ur, dashed, "\psi"] & & \\
A \arrow[u, "i"] \arrow[r, "f"] & B \arrow[u, "\bar{i}"] \arrow[uur, bend right, "g"'] \arrow[uuurr, bend right, "\epsilon_0 g"'] & &
\end{tikzcd}$$

where $\varphi i = gf$ and $\bar{g}\,\bar{f}i = \epsilon_0 gf$.

Since the square is a pushout and $\varphi i = gf$, there exists a unique map $\psi:Y \to Z^I$ such that $\psi\bar{f} = \varphi$ and $\psi\bar{i} = g$. Now, the composite map $\epsilon_0\psi$ has the following properties:

$$(\epsilon_0\psi)\,\bar{f} = \epsilon_0(\psi\bar{f}) = \epsilon_0\varphi = \bar{g}\,\bar{f}$$

and $(\epsilon_0\psi)\,\bar{i} = \epsilon_0(\psi\bar{i}) = \epsilon_0 g = \bar{g}\,\bar{i}$

But since the square is a pushout, it follows that $\epsilon_0\psi = \bar{g}$ by uniqueness of $\epsilon_0\psi$. Hence, $\psi:Y \to Z^I$ in right square of diagram (*) has the property that $\epsilon_0\psi = \bar{g}$ and $\psi\bar{i} = g$. Therefore, $\bar{i}:B \to Y$ is a cofibration by Definition 2.1.3.

<u>Theorem 2.1.4</u>:

(a) For any A and X, the inclusions $X \to X \sqcup A$ and $A \to X \sqcup A$ are cofibrations.

(b) Suppose (X,D) is a cofibred pair. Let $A \subseteq D$ and let $f:A \to B$ be a map. Then, $(B \sqcup_f X,\ B \sqcup_f D)$ is a cofibred pair.

Proof:

(a) Consider the following diagram

$$\begin{tikzcd}
X \arrow[r, "j"] & X \sqcup_\phi A = X \sqcup A \\
\phi \arrow[u, "\phi"] \arrow[r, "\phi"'] & A \arrow[u, "i"']
\end{tikzcd}$$

where $\phi:\varphi \to X$ and $\phi:\varphi \to A$ are the empty maps and $i:A \to X \sqcup A$ and $j:X \to X \sqcup A$ are the inclusion maps.

Since $\phi:\varphi \to X$ and $\phi:\varphi \to A$ are cofibrations, it follows that $i:A \to X \sqcup A$ and $j:X \to X \sqcup A$ are cofibrations, being pushouts of cofibrations.

(b) Construct the following diagram

$$\begin{tikzcd}
{} \arrow[r, "\tilde{f}"] & B \sqcup_f X \cong (B \sqcup_f D) \sqcup_{\bar{f}} X \\
D \arrow[u, "j"] \arrow[r, "\bar{f}"] & B \sqcup_f D \arrow[u, "\bar{j} = 1_B \sqcup j"'] \\
A \arrow[u, "i"] \arrow[r, "f"'] & B \arrow[u, "\bar{i}"']
\end{tikzcd}$$

Observe that $B \sqcup_f X \cong (B \sqcup_f D) \sqcup_{\bar{f}} X$ by the "Law of Vertical Composition" (see Remark 1.2.5(b)).
Now, composite square is a pushout and square I is a pushout implies that square II is a pushout (see Remark 1.1.4(b)).
Since $j:D \to X$ is a cofibration, it follows from Theorem 2.1.3 that $\bar{j}:B \sqcup_f D \to B \sqcup_f X$ is a cofibration.

Section 2: The Characterization Theorem for Cofibrations and its Consequences

We begin this section by proving a lemma of Ström (See [16; Lemma 3]) which deserves special attention. Accordingly, we give a brief discussion of its importance.

Let $M(i)$ denote the mapping cylinder of the inclusion map $i:A \to X$; that is, $M(i) = X \sqcup_i A \times I$. Clearly, as sets, $M(i)$ can be identified with $X \times 0 \cup A \times I$. In general, however, their topologies are different. Recall that $X \sqcup_i A \times I$ has the final topology with respect to the inclusion maps $\bar{i}:A \times I \to M(i)$ and $\bar{j}:X \to M(i)$ and so C is open in $M(i) \iff \bar{i}^{-1}(C) = C \cap A \times I$ is open in $A \times I$ and $\bar{j}^{-1}(C) = C \cap (X \times 0)$ is open in $X \times 0$. The Lemma we are going to prove below is just the statement that the topology on $X \times 0 \cup A \times I$ inherited from $X \times I$ coincides with the mapping cylinder topology on $M(i)$, in the presence of retraction $X \times I \to X \times 0 \cup A \times I$. We readily observe that these two topologies are also identical if A is closed in X, even if no retraction $X \times I \to X \times 0 \cup A \times I$ exists. This is because in this situation, $A \times I \subseteq X \times I$ is closed and hence $X \times 0 \cup A \times I \subseteq X \times I$ is closed. Therefore, $C \subseteq X \times 0 \cup A \times I$ is closed $\iff C \cap (X \times 0)$ is closed in $X \times 0$ and $C \cap (A \times I)$ is closed in $A \times I$.

We now give a formal proof of the above discussion.

Lemma 2.2.1: If (X,A) is a pair such that $X \times 0 \cup A \times I$ is a retract of $X \times I$, then a subset C of $X \times 0 \cup A \times I$ is open in $X \times 0 \cup A \times I \Leftrightarrow C \cap (X \times 0)$ and $C \cap (A \times I)$ are open in $X \times 0$ and $A \times I$, respectively.

Proof: ("=>"):

Suppose $C \subseteq X \times 0 \cup A \times I$ is open in $X \times 0 \cup A \times I$. Now $X \times 0$ and $A \times I$ are subspaces of $X \times 0 \cup A \times I$. Hence, $C \cap (X \times 0)$ and $C \cap (A \times I)$ are open in the relativized topology of $X \times 0$ and $A \times I$ respectively.

("<="):

Let $C \subseteq X \times 0 \cup A \times I$ be such that $C \cap (X \times 0)$ and $C \cap (A \times I)$ are open in $X \times 0$ and $A \times I$, respectively.

Consider the following subsets of X

$$U = \{x \in X \mid (x,0) \in C\} \text{ and, for each natural number } n,$$

$$U_n = \cup\{V \mid V \text{ oepn in } X \text{ and } (V \cap A) \times [0,\tfrac{1}{n}) \subseteq C\}$$

Since $C \cap (X \times 0)$ is open in $X \times 0$ by hypothesis and U can naturally be identified with $C \cap (X \times 0)$, we have that U is an open set in X. Clearly U_n is open in X, for all n, as U_n is a union of open sets in X.

Now set $B = U \times 0 \cup \bigcup_{n=1}^{\infty} ((A \cap U_n) \times [0,\frac{1}{n}))$.

We claim that $C = (C \cap (A \times (0,1])) \cup B$ where $C \cap (A \times (0,1]$ and B are open sets in $X \times 0 \cup A \times I$ and hence C is open in

$X \times 0 \cup A \times I$.

We first show that $C \cap (A \times (0,1])$ is open in $X \times 0 \cup A \times I$. Now, $C \cap (A \times (0,1]) = (C \cap A \times I) \cap A \times (0,1]$ where $C \cap (A \times I)$ is open in $A \times I$ by hypothesis. Since $A \times (0,1]$ is a subspace of $A \times I$ it follows that $C \cap (A \times (0,1])$ is an open subset of $A \times (0,1]$. Also, $A \times (0,1] = X \times (0,1] \cap (X \times 0 \cup A \times I)$ where $X \times (0,1]$ is open in $X \times I$ and so $A \times (0,1]$ is an open subset of $X \times 0 \cup A \times I$.

Hence we have $C \cap A \times (0,1] \subseteq A \times (0,1] \subseteq X \times 0 \cup A \times I$ where $C \cap A \times (0,1]$ is open in $A \times (0,1]$. Therefore, there exists an open subset, say $W \subseteq X \times 0 \cup A \times I$, such that $C \cap (A \times (0,1])$ $= A \times (0,1] \cap W$.

Therefore, $C \cap (A \times (0,1])$ is open in $X \times 0 \cup A \times I$, as it is the intersection of two open sets in $X \times 0 \cup A \times I$. We now show that $C = (C \cap (A \times (0,1])) \cup B$.

"$\subseteq$":

Let $c \in C \subseteq (X \times 0) \cup (A \times I)$.

Then either $c = (x,0)$, for some $x \in X$, in which case $c \in U \times 0 \subseteq B$, or $c = (a,t)$, for $a \in A$ and $t \in (0,1]$, in which case $c \in C \cap (A \times (0,1])$.

In either case $c \in C \cap (A \times (0,1]) \cup B$ and so $C \subseteq (C \cap (A \times (0,1])) \cup B$.

"$\supseteq$":

Since $C \cap (A \times (0,1]) \subseteq C$, it suffices to show $B \subset C$.

Let $b \in B$.

Case (i) If $b \in U \times 0$, then $b \in C$ by definition of U.

Case (ii) If $b \in \bigcup_{n=1}^{\infty} ((A \cap U_n) \times [0,\frac{1}{n}))$, then $\exists\, n_0 \in N \ni b \in$ $(A \cap U_{n_0}) \times [0,\frac{1}{n_0})$ and so $b = (a,t)$ for some $a \in (A \cap U_{n_0})$ and some $t \in [0,\frac{1}{n_0})$.

But $a \in A \cap U_{n_0} \subseteq U_{n_0}$ implies, by definition of U_{n_0}, that $\exists\, V$, an open subset of X, such that $(a,t) \in (V \cap A) \times [0,\frac{1}{n_0}) \subseteq C$ and so $b = (a,t) \in C$.

In either case $b \in C$ and so $B \subseteq C$.

Before we can show that B is open in $X \times 0 \cup A \times I$ and hence complete the proof that C is open in $X \times 0 \cup A \times I$, we need to prove the following facts:

(a) $A \cap U = A \cap \bigcup_{n=1}^{\infty} U_n$.

"$\supseteq$":

If $x \in A \cap \bigcup_{n=1}^{\infty} U_n$ then $\exists\, n_0 \in N \ni x \in A \cap U_{n_0}$.

By definition of U_{n_0} $\exists$ an open set V in X such that $x \in A \cap V$ and $(V \cap A) \times [0,\frac{1}{n_0}) \subseteq C$. In particular, $(x,0) \in C$ and so $x \in U$. Hence, $x \in A \cap U$ and we have that $A \cap \bigcup_{n=1}^{\infty} \subseteq A \cap U$.

"$\supseteq$":

$x \in A \cap U \Rightarrow x \in U$ and $x \in A$

$\Rightarrow (x,0) \in C$ and $x \in A$

$\Rightarrow (x,0) \in C \cap (A \times I)$

But $C \cap (A \times I)$ is open in $A \times I$ by hypothesis. Hence, there exists a basic open set of the form $V' \times [0,\frac{1}{n_0}) \subseteq C \cap (A \times I)$ such that $(x,0) \in V' \times [0,\frac{1}{n_0}) \subseteq C$. Since V' is open in A, $\exists$ an open set V in X such that $V' = V \cap A$. Hence, $(x,0) \in V' \times [0,\frac{1}{n_0}) = (V \cap A) \times [0,\frac{1}{n_0}) \subseteq C$ and so $x \in U_{n_0}$. But then $x \in A \cap U_{n_0}$ and hence $x \in A \cap \bigcup_{n=1}^{\infty} U_n$ that is, $A \cap U \subseteq A \cap \bigcup_{n=1}^{\infty} U_n$.

(b) If V is an open subset of X such that $V \cap A \subseteq U_n$ then $V \subseteq U_n$.

Since V is open in X, V is a subset of U_n if V has the property that $(V \cap A) \times [0,\frac{1}{n}) \subseteq C$. Now, if $v \in V \cap A$, then $v \in U_n$ by hypothesis and so an open subset W of X such that $v \in W \cap A$ and $(W \cap A) \times [0,\frac{1}{n}) \subset C$.

In particular, we have that $\{v\} \times [0,\frac{1}{n}) \subseteq C$ and so $\bigcup_{v \in V \cap A} \{v\} \times [0,\frac{1}{n}) \subseteq C$; that is, $(V \cap A) \times [0,\frac{1}{n}) \subseteq C$.

Therefore, $V \subseteq U_n$, as required.

(c) $X - \bigcup_{n=1}^{\infty} U_n \subseteq \bar{A}$

Let $x \in X - \bigcup_{n=1}^{\infty} U_n$. Then $x \notin U_n$, for any $n \geq 1$.

Let V_x be a neighborhood of x in X such that $V_x \cap A$ $\emptyset$. Then, $(V_x \cap A) \times [0,\frac{1}{n}) = \emptyset \subseteq C$, for all n. But this implies that $x \in U_n$ for each n, a contradiction. Hence, for all neighborhoods V_x of x in X, $V_x \cap A \neq \emptyset$ and so $x \in \bar{A}$.

(d) For $t \in (0,1]$, $r(\bar{A} \times t) = A \times t$, where $r: X \times I \to X \times 0 \cup A \times I$ is a retraction (exists by hypothesis).

Now, $\bar{A}$ is closed in X and $\{t\}$ is closed in I; hence, $\bar{A} \times t = \bar{A} \times \overline{\{t\}} = \overline{A \times t}$ in $X \times I$ and so $r(\bar{A} \times t) = r(\overline{A \times t}) \subseteq \overline{r(A \times t)}$, by the continuity of r. But r is a retraction.

Hence, $r(\bar{A} \times t) = r(\overline{A \times t}) \subseteq \overline{r(A \times t)} = \overline{A \times t}$. Moreover, $\forall t \in (0,1]$; $A \times t = (X \times t) \cap (X \times 0 \cup A \times I)$ where $X \times t$ is closed in $X \times I$ and $X \times 0 \cup A \times I$ is a subspace of $X \times I$. Hence, $A \times t$ is closed in $X \times 0 \cup A \times I$.

Therefore, for all $t \in (0,1]$, $\overline{A \times t} = A \times t$ in $X \times 0 \cup A \times I$.

Consequently, $r(\bar{A} \times t) \subseteq A \times t$.

On the other hand, $A \times t = r(A \times t) \subseteq r(\bar{A} \times t)$ as $A \subseteq \bar{A}$.

Therefore, for all $t \in (0,1]$

$$r(\bar{A} \times t) = A \times t$$

Using (a), (b), (c) and (d) above we can now prove

(e) $U \subseteq \bigcup_{n=1}^{\infty} U_n$.

Let $x \in X - \bigcup_{n=1}^{\infty} U_n$. We show that $x \in X - U$.

By (c) $x \in \bar{A}$.

Let $t \in (0,1]$. Then, by (d), $r(x,t) \in A \times t$. Suppose $n \geq 1$ such that $r(x,t) \in U_n \times I$. Since $U_n \times I \subseteq X \times I$ is open, there exist basic open neighborhoods V and W of x and t such that $r(x,t) \in r(V \times W) \subseteq U_n \times I$.

Hence, $(V \cap A) \times t = r((V \cap A) \times t) \subseteq U_n \times I$. This implies that $V \cap A \subseteq U_n$ and hence by (b) above $V \subseteq U_n$. But then, $x \in U_n \subseteq \bigcup_{n=1}^{\infty} U_n$

contrary to hypothesis.

Consequently, $r\,(x,t) \in (A - \bigcup_{n=1}^{\infty} U_n) \times I$.

Now, by (a) above, $A \cap U = A \cap \bigcup_{n=1}^{\infty} U_n$ and $(A - \bigcup_{n=1}^{\infty} U_n) \times I =$

$$= [A - (A \cap \bigcup_{n=1}^{\infty} U_n)] \times I = [A - (A \cap U)] \times I$$
$$= (A - U) \times I \subseteq (X - U) \times I$$

So, $r(x,t) \in (X - U) \times I$ for all $t \in (0,1]$. Since $r(x,\frac{1}{n}) \in (X - U) \times I$, for all $n = 1, 2, \ldots$ and $(X - U) \times I$ is closed, it follows from the continuity of r that $(x,0) = r(x,0) \in (X - U) \times I$ and so $x \in (X - U)$.

Consequently, $X - \bigcup_{n=1}^{\infty} U_n \subseteq X - U$ and hence $U \subseteq \bigcup_{n=1}^{\infty} U_n$.

(f) $U = \bigcup_{n=1}^{\infty} V_n$, where $V_n = U \cap U_n$, $n = 1, 2, \ldots$

$$\bigcup_{n=1}^{\infty} V_n = \bigcup_{n=1}^{\infty} (U \cap U_n)$$
$$= U \cap \bigcup_{n=1}^{\infty} U_n$$
$$= U, \text{ since by (e) } U \subseteq \bigcup_{n=1}^{\infty} U_n.$$

(g) $A \cap U_n = A \cap V_n$ for all $n = 1, 2, \ldots$

"$\supseteq$":

The inclusion $A \cap V_n \subseteq A \cap U_n$ is clear since $V_n \subset U_n$.

"$\subseteq$":

If $x \in A \cap U_n$, then $x \in U_n$ and $x \in A$, and so an open set W in X such that $(W \cap A) \times [0,\frac{1}{n}) \subseteq C$. Since $x \in W \cap A$; it follows that, in particular, $(x,0) \in C$. Hence, $x \in U$.

Consequently, $x \in A \cap U_n \cap U = A \cap V_n$ and therefore $A \cap U_n \subseteq A \cap V_n$.

We now show that $B \subseteq X \times 0 \cup A \times I$ is open. Recall that

$B = U \times 0 \cup \bigcup_{n=1}^{\infty} ((A \cap U_n) \times [0,\frac{1}{n}))$.

From (f) and (g) we have that

$$B = (\bigcup_{n=1}^{\infty} V_n \times 0) \cup \bigcup_{n=1}^{\infty} ((A \cap V_n) \times [0,\tfrac{1}{n}))$$

$$= \bigcup_{n=1}^{\infty} V_n \times 0 \cup (\bigcup_{n=1}^{\infty} (A \times [0,\tfrac{1}{n}) \cap V_n \times [0,\tfrac{1}{n}))$$

$$= X \times 0 \cap \bigcup_{n=1}^{\infty} V_n \times [0,\tfrac{1}{n})] \cup [(A \times I) \cap \bigcup_{n=1}^{\infty} V_n \times [0,\tfrac{1}{n})]$$

$$= (X \times 0 \cup A \times I) \cap \bigcup_{n=1}^{\infty} (V_n \times [0,\tfrac{1}{n}))$$

As V_n is open in X for each n, and hence $\bigcup_{n=1}^{\infty} (V_n \times [0,\frac{1}{n}))$ is open in $X \times I$, it follows that B is open in $X \times 0 \cup A \times I$.

Characterization Theorem 2.2.2: Let A be a subspace of X. The following statements are equivalent:

(a) The inclusion $i: A \to X$ is a cofibration.

(b) For any space Y any map $X \times 0 \cup A \times I \to Y$ extends over $X \times I$.

(c) $X \times 0 \cup A \times I$ is a retract of $X \times I$.

(d) $X \times 0 \cup A \times I$ is a strong deformation retract (SDR) of $X \times I$.

(e) There exists a map $\varphi: X \to I$ such that $A \subseteq \varphi^{-1}(0)$ and a homotopy $H: X \times I \to X$ such that

$$H(x,0) = x, \text{ for all } x \in X$$
$$H(a,t) = a, \text{ for all } a \in A, \text{ for all } t \in I$$
and $H(x,t) \in A$ whenever $t > \varphi(x)$.

Proof:

(a) => (b) Let $f : X \times 0 \cup A \times I \to Y$ be any map and consider the following diagram:

$$\begin{tikzcd}
 & & Y \\
X \times 0 \arrow[r] \arrow[urr, bend left, "\varphi_1"] & X \times I \arrow[ur, dashed, "g"] & \\
A \times 0 \arrow[u] \arrow[r] & A \times I \arrow[u] \arrow[uur, bend right, "\varphi_2"'] &
\end{tikzcd}$$

where $\varphi_1 = f\big|_{X\times 0}$ and $\varphi_2 = f\big|_{A\times I}$

Since $i : A \to X$ is a cofibration, $\exists$ a map $g : X \times I \to Y$ such that $g\big|_{X\times 0} = \varphi_1$ and $g\big|_{A\times I} = \varphi_2$.

Hence, $g\big|_{X\times 0 \cup A\times I} = \varphi_1 \cup \varphi_2 = f$ and consequently g is the required extension.

(b) => (c) Suppose for any space Y, any map $f: X \times 0 \cup A \times I \to Y$ extends over $X \times I$.

Then, in particular, the identity map $1_{X\times 0 \cup A\times I}: X \times 0 \cup A \times I \to X \times 0 \cup A \times I$ extends over $X \times I$; that is, $\exists$ a map $h: X \times I \to X \times 0 \cup A \times I$ such that $h|_{X\times 0 \cup A\times I} = 1_{X\times 0 \cup A\times I}$. Therefore, $X \times 0 \cup A \times I$ is a retract of $X \times I$.

(c) => (d) By Definition 1.3.6 we have to show that $\exists$ a retraction $r: X \times I \to X \times 0 \cup A \times I$ and a homotopy $R: (X \times I) \times I \to X \times I$ such that

$$R((x,t),0) = (x,t) \qquad \forall (x,t) \in X \times I$$

$$R((x,t),s) = (x,t) \qquad \forall (x,t) \in X \times 0 \cup A \times I$$

and $$R((x,t),1) = r(x,t) \qquad \forall (x,t) \in X \times I$$

By hypothesis, $\exists$ a retraction, say $r: X \times I \to X \times 0 \cup A \times I$. Let $pr_1: X \times I \to X$ and $pr_2: X \times I \to I$ denote the projections on the first and second factors respectively. Define $R: (X \times I) \times I \to X \times I$ by

$$R((x,t),s) = (pr_1 r(x,ts),\ t(1 - s) + s\, pr_2 r(x,t))$$

Now

(i) $$R((x,t),0) = (pr_1 r(x,0),t) = (pr_1(x,0),t) = (x,t)$$

(ii) Let $(x,t) \in X \times 0 \cup A \times I$.

Then $(x,t) = (x,0)$ or $(x,t) = (a,t)$ for some $a \in A$.

$$\text{Hence, } R((x,0),s) = (pr_1 r(x,0),\ s pr_2 r(x,0)) = (pr_1(x,0),\ s pr_2(x,0)) = (x,0)$$

and

$$R((a,t),s) = (pr_1 r(a,ts),\ t(1-s) + s pr_2 r(a,t)) = (pr_1(a,ts),\ t(1-s) + s pr_2(a,t)) = (a, t(1-s) + st) = (a,t)$$

$$\text{(iii) } R((x,t),1) = (pr_1 r(x,t),\ pr_2 r(x,t)) = r(x,t)$$

Therefore, $(X \times 0) \cup (A \times I)$ is a strong deformation retract of $X \times I$.

(d) => (e) Suppose $X \times 0 \cup A \times I$ is a strong deformation retract of $X \times I$.

Let $r: X \times I \to X \times 0 \cup A \times I$ be a retraction. We define φ and H as follows:

$$\varphi(x) = \sup_{t \in I} |t - pr_2 r(x,t)|, \text{ for all } x \in X \text{ and}$$

$$H(x,t) = pr_1 r(x,t),\ x \in X,\ t \in I$$

We claim that φ is continuous.

We shall give a general proof such that the continuity of φ becomes a special case (see [1; page 237]). Let $\psi: X \times C \to \mathbf{R}$ be a map such that C is compact. Let $\omega: X \to \mathbf{R}$ be defined by

$$\omega(x) = \sup_{c \in C} \psi(x,c)$$

We show that ω is continuous.

For all $x \in X$, $x \times C$ is compact and hence $\psi(x \times C)$ is compact in $\mathbf{R}$. This implies that $\psi(x \times C)$ is a bounded subset or $\mathbf{R}$. Hence, ω is well defined.

Suppose $\omega(x) = r$ and let $N = [r - \epsilon, r + \epsilon]$ be a neighborhood of r.

Now, by definition of $\omega(x) = r$, $c \in C \Rightarrow \psi(x,c) \leq r < r + \epsilon$

$$\Rightarrow x \times C \subseteq \psi^{-1}(-\infty, r + \epsilon)$$

Since $\psi^{-1}(-\infty, r + \epsilon)$ is open in $X \times C$, there exists a basic open set $U_y \times V_y \subseteq X \times C$, such that $(x,y) \in U_y \times V_y \subseteq \psi^{-1}(-\infty, r + \epsilon)$ for all $(x,y) \in x \times C$. Then the collection $\{V_y\}_{y \in C}$ is an open cover of C. Since C is compact, $\exists$ finite subcover $V_{y_1}, \ldots, V_{y_2}, \ldots, V_{y_k}$ of $\{V_y\}_{y \in C}$ that cover C. Let U_1 be the intersection of the corresponding finite number of open sets U_{y_k}. That is, $U_1 = \bigcap_{i=1}^{k} U_{y_i}$. Now, it is easy to see that

$$x \times C \subseteq U_1 \times C \subseteq \bigcup_{i=1}^{k} (U_{y_i} \times V_{y_i}) \subseteq \psi^{-1}(-\infty, r + \epsilon).$$

Consequently, $\omega(U_1) \subseteq (-\infty, r + \epsilon]$. However, $\exists\, c \in C$ such that $\psi(x,c) \in \overset{0}{N}$ and so as above, $\exists$ an open set U_2 containing x such that $\psi(U_2 \times C) \subseteq \dot{N} \subseteq N$.

So, $y \in U_1 \cap U_2 \Rightarrow \omega(y) \leq r + \epsilon$ and $\omega(y) \geq r - \epsilon$

$$\Rightarrow \omega(y) \in [r - \epsilon, r + \epsilon] = N$$

$$\Rightarrow \omega(U_1 \cap U_2) \subseteq N$$

Therefore, ω is continuous and so φ is continuous.

The continuity of H is clear.

Now for all $a \in A$, $\varphi(a) = \sup_{t\in I} |t - pr_2 r(a,t)|$

$$= \sup_{t\in I} |t - pr_2(a,t)|$$

$$= \sup_{t\in I} |t - t|$$

$$= 0$$

Hence, $A \subseteq \varphi^{-1}(0)$.

Furthermore,

(i) for $x \in X$, $H(x,0) = pr_1 r(x,0) = pr_1(x,0) = x$

(ii) for $a \in A$ and $t \in I$,

$$H(a,t) = pr_1 r(a,t) = pr_1(a,t) = a$$

and (iii) $t > \varphi(x) \Rightarrow pr_2 r(x,t) > 0$ since $pr_2 r(x,t) = 0$ implies $t \leq \sup_{t\in I} |t - pr_2 r(a,t)| = \varphi(x)$.

Consequently, $r(x,t) \in A \times I$ and therefore $H(x,t) = pr_1 r(x,t) \in A$.

Thus, H is a homotopy of 1_X relative to A such that $H(x,t) \in A$ whenever $t > \varphi(X)$.

(e) => (a) Given φ and H define a function

$$r : X \times I \to X \times 0 \cup A \times I \text{ by}$$

$$r(x,t) = \begin{cases} (H(x,t),0) & t \leq \varphi(x) \\ (H(x,t),\ t - \varphi(x)) & t \geq \varphi(x) \end{cases}$$

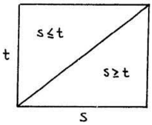
We claim that r is a retraction:

Clearly, r is well defined.

We prove that r is continuous.

Let $U = \{(s,t) \in I \times I \mid s \geq t\}$

$V = \{(s,t) \in I \times I \mid s \leq t\}$



Clearly, U and V are closed sets in $I \times I$.

Now, let $W = \{(x,t) \in X \times I \mid \varphi(x) \geq t\}$ and

$Z = \{(x,t) \in X \times I \mid \varphi(x) \leq t\}$

Then,

$$W = (\varphi \times 1_I)^{-1}(U) \quad \text{and} \quad Z = (\varphi \times 1_X)^{-1}(V).$$

Since U and V are closed sets in $I \times I$ and $\varphi \times 1_I$ is continuous it follows that W and Z are closed sets in $X \times I$. Moreover, $X \times I = W \cup Z$.

Now let $\Phi = (H|_W, 0) : W \to X \times 0$ and

$$\psi = (1_X \times \text{"}-\text{"})(H|_Z, \varphi \cdot pr_1, pr_2) : Z \to X \times I \times I \to X \times I$$

where $\psi(x,t) = (1_X \times \text{"}-\text{"})(H(x,t), \varphi(x), t)$

$= (H(x,t),\ t - \varphi(x))$

Then, $r = \Phi \cup \psi : W \cup Z = X \times I \to X \times 0 \cup A \times I$.

Since Φ and ψ are continuous, it follows that r is continuous (see Theorem 1.2.1).

Next, we prove that r is a retraction.

Since $\varphi(x) \geq 0$, $r(x,0) = (H(x,0),0) = (x,0)$, $x \in X$.
Also, since $\varphi(a) = 0$, $r(a,t) = (H(a,t),\ t - 0) = (a,t)$, $a \in A$ and $t \in I$. Therefore, r is a retraction.
Now, consider the following diagram

$$\begin{tikzcd}
 & & Z \\
 & & X\times 0 \cup A\times I \cong M(i) \arrow[u, dashed, "\varphi"] \\
X \arrow[r] \arrow[urr, bend left, "k"] \arrow[uurr, bend left, "g"] & X\times I \arrow[ur, "r"] & \\
A \arrow[u, "i"] \arrow[r] & A\times I \arrow[u, "i\times 1"'] \arrow[uur, bend right, "j"'] \arrow[uuur, bend right, "f"'] &
\end{tikzcd}$$

where f and g are arbitrary maps and j and k are inclusions.
Since $r: X \times I \to Z \times 0 \cup A \times I$ is a retraction,

$$r\Big|_{A\times I} = j: A \times I \to X \times 0 \cup A \times I$$

and $r\Big|_{X\times 0} = k: X \to X \times 0 \cup A \times I$

Also, by Lemma 2.2.1, $M(i) \cong X \times 0 \cup A \times I$. Hence, by the universal property of pushouts, there exists a unique map $\varphi: M(i) \cong X \times 0 \cup A \times I \to Z$ such that $\varphi \cdot k = g$ and $\varphi \cdot j = f$.
Now, let $\psi = \varphi \cdot r: X \times I \to Z$.
Then, $\psi\Big|_{A\times I} = \varphi \cdot r\Big|_{A\times I} = \varphi \cdot j = f$ and

$$\psi\Big|_{X\times 0} = \varphi \cdot r\Big|_{X\times 0} = \varphi \cdot k = g$$

Therefore, $i:A \to X$ is a cofibration.

Remark 2.2.1:

(i) If $i:A \to X$ is a cofibration, then $M(i) \cong X \times 0 \cup A \times I$. This is just a consequence of Lemma 2.2.1 since $i:A \to X$ is a cofibration $\Leftrightarrow$ $X \times 0 \cup A \times I$ is a retract of $X \times I$ (by Characterization Theorem 2.2.2).

(ii) If X is Hausdorff, all cofibred pairs (X,A) are closed. This follows by observing the following two facts. First, the product $X \times I$ is Hausdorff as X and I are Hausdorff. Secondly, by the Characterization Theorem, $X \times 0 \cup A \times I$ is a retract of $X \times I$ and hence $X \times 0 \cup A \times I$ is closed in $X \times I$ being a retract of a Hausdorff space. Now, $A \times I = (A \times I) \cap (X \times 0 \cup A \times I)$ where $X \times 0 \cup A \times I$ is closed in $X \times I$. Hence $A \times I \subseteq X \times I$ is closed and consequently A is closed in X.

(iii) If A is a closed subspace of X, then the map $\varphi:X \to I$ in statement (e) of the Characterization Theorem 2.2.2 has the property that $\varphi^{-1}(0) = A$. This is because if $x \in \varphi^{-1}(0)$, then $\varphi(x) = 0$ and so $H(x,\frac{1}{n}) \in A$, for all $n = 1, 2, \ldots$ But then, since A is closed, $x = H(x,0) \in A$. Therefore, $\varphi^{-1}(0) \subseteq A$. Also, in this situation, the proof (e) => (a) does not require the use of Lemma 2.2.1. It is immediate, here, that the subspace topology on $X \times 0 \cup A \times I$ coincides with the mapping cylinder topology as we have seen earlier at the beginning of Section 2.

(iv) Statement (e) of the Characterization Theorem 2.2.2 can be written in the following equivalent form:

(e'): There exists a map $\psi: X \to [0,\infty]$ such that $A \subseteq \psi^{-1}(0)$ and there exists a homotopy $k:\psi^{-1}[0,1] \times I \to X$ such that:

$K(x,0) = x$, for all $x \in \psi^{-1}[0,1]$

$K(a,t) = a$, for all $(a,t) \in A \times [0,1]$

$K(x,t) \in A$ for $t > \psi(x)$.

Clearly, (e) => (e'). Now, (e') => (e) can be obtained from the following formulas:

$$\varphi(x) = \mathrm{Min}(2\psi(x),1) \quad \text{and}$$

$$H(x,t) = \begin{cases} K(x,t) & \text{if } 2\psi(x) \leq 1 \\ K(x,t(2-2\psi(x))) & \text{if } 1 \leq 2\psi(x) \leq 2 \\ x & \text{if } \psi(x) \geq 1 \end{cases}$$

(v) In the Characterization Theorem 2.2.2 (e), $H(x,\varphi(x)) \in \bar{A}$ whenever $\varphi(x) < 1$.

This follows by observing that if $\varphi(x) < 1$, then for all $(x,t) \in x \times \langle \varphi(x),1]$ (i,e, $t > \varphi(x)$) we have that $H(x \times \langle \varphi(x),1]) \subseteq A$. Hence, $\overline{H(x \times \langle \varphi(x),1])} \subseteq \bar{A}$. Now, consider a decreasing sequence $\{t_n\} \in \langle \varphi(x),1]$ converging to $\varphi(x)$. Then, $H(x,t_n)$ converges to $H(x,\varphi(x))$ by continuity of H. Hence, $H(x,\varphi(x)) \in \overline{H(x \times \langle \varphi(x),1])} \subseteq \bar{A}$. Therefore, $H(x,\varphi(x)) \in \bar{A}$ whenever $\varphi(x) < 1$.

(vi) By choosing $U = \{x \in X \mid pr_1 r(x,1) \in A\}$

$$H = pr_1 r\big|_{U \times I} \quad \text{and}$$

$$\varphi(x) = \sup_{t \in I} |t - pr_2 r(x,t)|$$

statement (e) of the Characterization Theorem 2.2.2 for A a closed subspace of X can be written in the form:

(a) There exists a neighbourhood U of A which is deformable in X to A rel A (i.e. there exists a homotopy $H:U \times I \to X$ such that

$$H(u,0) = u, \text{ for } u \in U$$
$$H(a,t) = a, \ a \in A \text{ and } t \in I$$

and $H(u,1) \in A$, for $u \in U$,

(b) The map $\varphi:X \to I$ is such that $A = \varphi^{-1}(0)$ (as A is closed) and $\varphi(x) = 1$ for $x \in X - U$.

Note that the last remark we made is closely related to the notion of a halo (which will be defined below) and the characterization of cofibrations in terms of a halo. But first we give the following definition.

Definition 2.2.1: Let A, V be subspaces of a space X, with $A \subseteq V \subseteq X$. Then, V is a halo of A in X if there exists a map $\varphi:X \to I$ (the haloing function) such that $A \subseteq \varphi^{-1}(0)$ and $X - V \subseteq \varphi^{-1}(1)$. That is, $A \subseteq \varphi^{-1}(0) \subseteq \varphi^{-1}[0,1) \subseteq V \subseteq X$.

Remark 2.2.2:

(a) If V is a halo of A in X, then V is also a halo of $\bar{A}$ in X. This follows by observing that since $\varphi^{-1}(0)$ is closed, $A \subseteq \varphi^{-1}(0) \Rightarrow \bar{A} \subseteq \varphi^{-1}(0) \subseteq \varphi^{-1}[0,1) \subseteq V \subseteq X$.

(b) From the definition of a halo and Remark 2.2.1 (vi), the following statements are equivalent:

(i) $A \to X$ is a cofibration.

(ii) A has a halo U in X, deformable in X to A rel A via a homotopy $H:U \times I \to X$.

(iii) A has a halo V in X, deformable in X to A rel A via a homotopy $H:X \times I \to X$.

The following theorem is a consequence of the Characterization Theorem 2.2.2 and Remark 2.2.1 (v).

Theorem 2.2.3: If (X,A) is a cofibred pair, then so is (X,A).

Proof: As (X,A) is a cofibred pair, assume the existence of φ and H satisfying the properties of Characterization Theorem 2.2.2 (c).

We now define $\bar{H}(x,t) = H(x,t \wedge \varphi(x))$, where $t \wedge \varphi(x) = \text{Min}\{t,\varphi(x)\}$. Clearly, $\bar{H}$ is continuous. Now,

(a) if $\bar{a} \in \bar{A}$, let $\{a_n\} \in A$ be such that $a_n \to \bar{a}$. Since φ is continuous, $\varphi(a_n) \to \varphi(\bar{a})$. But $a_n \in A$, for all $n \in N$, and $A \subseteq \varphi^{-1}(0)$. Hence $0 \to \varphi(\bar{a})$ and $\varphi(\bar{a}) = 0$. Therefore $\bar{a} \in \varphi^{-1}(0)$ and so $\bar{A} \subseteq \varphi^{-1}(0)$.

(b)
$$\begin{aligned} \bar{H}(x,0) &= H(x,0 \wedge \varphi(x)) \\ &= H(x,0) \quad \text{as } \varphi(x) \geq 0 \\ &= x, \text{ by hypothesis.} \end{aligned}$$

(c) for all $\bar{a} \in \bar{A}$ and $t \in I$,
$$\begin{aligned} \bar{H}(\bar{a},t) &= H(\bar{a}, t \wedge \varphi(\bar{a})) \\ &= H(\bar{a},t \wedge 0) \quad \text{as } \varphi(\bar{a}) = 0 \text{ by (a) above} \\ &= H(\bar{a},0) \\ &= \bar{a} \end{aligned}$$

(d) given $t > \varphi(x)$ and hence $\varphi(x) < 1$, we have that

$$\begin{aligned}\bar{H}(x,t) &= H(x, t \wedge \varphi(x)) \\ &= H(x, \varphi(x)) \in \bar{A} \text{ by Remark 2.2.1 (v).}\end{aligned}$$

Therefore, by the Characterization Theorem 2.2.2, $(X, \bar{A})$ is a cofibred pair.

We now briefly discuss the notion of a fibration which is dual to that of cofibration. We remind the reader that not all properties we have discussed for cofibrations are dual to properties for fibrations. However, we shall record some of those properties that are genuinely dual. But first, we define the notion of homotopy lifting property (HLP) which dualizes HEP and which is the basis for the definition of a fibration.

Definition 2.2.2: A map $p:E \to B$ is said to have the homotopy lifting property (HLP) with respect to a space Z if for every map $f:Z \to E$ and homotopy $G:Z \times I \to B$ of pf, there is a homotopy $F:Z \times I \to E$ with $F(-,0) = f$ and $pF = G$ (F is said to be a lifting of G).

That is, $p:E \to B$ is said to have the HLP with respect to a space Z if, for every commutative diagram below, where $i_0(z) = (z,0)$,

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ Z \times I \arrow[r, "G"'] \arrow[ur, dashed, "F"] & B \end{tikzcd}$$

there exists a map $F:Z \times I \to E$ (dotted arrow) making the resulting triangles commute.

p is called a fibration if it has the HLP for all spaces Z. If furthermore for $x_0 \in X$, $F(x_0,t)$ is independent of t whenever $G(x_0,t)$ is, then $p:E \to B$ is called a regular fibration. We will refer to E as the total space, B as the base space and (E,p,B) as the fibre space.

We now record some of the properties of fibrations which will be needed later on in connection with cofibrations.

Remark 2.2.3 :

(a) Composition of fibrations is a fibration. (This is dual to Theorem 2.1.2 (d)).

(b) Pullback of a fibration is a fibration. (Dual to Theorem 2.1.3).

(c) Let $pr_1:B \times F \to B$ and $pr_2:B \times F \to F$ be the projections on the first and second factors. Then pr_1 and pr_2 are regular fibrations. To see this, given Z and maps $h:Z \times 0 \to B \times F$ and $H:Z \times I \to B$, define $F:Z \times I \to B \times F$ by

$$F(z,t) = (H(z,t), pr_2 h(z,0)).$$

Then $pF = H$ and $F(-,0) = h$; so pr_1 and prc_2 are fibrations. We call pr_1 and pr_2 the trivial fibrations. Note that regularity is trivially satisfied.

(d) If $p_\lambda:E_\lambda \to B_\lambda$ $(\lambda = 0,1)$ is a fibration, then $p_0 \times p_1:E_0 \times E_1 \to B_0 \times B_1$ is a fibration.

(c) The evaluation map $\epsilon_0 : X^I \to X$, defined by $\epsilon_0(\lambda) = \lambda(0)$, is a fibration. For a proof, see [14, Page 97, Theorem 2].

To have a closer look at fibrations, let (E,p,B) be a fiber space. Let $f: Z \times 0 \to E$ be given, and let $G: Z \times I \to B$ be a homotopy of pf. For every $z \in Z$, the map $t \to G(z,t)$ defines a path ψ_z in B, that is, $\psi_z : I \to B$ is such that $\psi_z(t) = G(z,t)$. The HLP is then lifting each path ψ_z in B to a path in E starting at $f(z,0)$, in such a way that the family $\{\psi_z | z \in Z\}$ is lifted "continuously" to E. This leads us to the following definition.

<u>Definition 2.2.3</u>: Let (E,p,B) be a fiber space, and let $\Omega_p \subseteq E \times B^I$ be the subspace $\Omega_p = \{(e,w) \in E \times B^I | p(e) = w(0)\}$ of the cartesian product. A <u>lifting</u> function for (E,p,B) is a map $\lambda : \Omega_p \to E^I$ such that $\lambda(e,\omega)(0) = e$ and $p\cdot\lambda(e,\omega)[t] = \omega(t)$ for all $(e,\omega) \in \Omega_p$ and $t \in I$. We say that λ is <u>regular</u> if $\lambda(e,\omega)$ is a constant path whenever ω is a constant path. Note that $\Omega_p = E_p \sqcap_{\epsilon_0'} B^I$ is the pullback defined earlier in Chapter I, and the lifting function $\lambda : \Omega_p = E_p \sqcap_{G_0'} B^I \to E^I$ has the following property shown in the diagram below:

$$\begin{tikzcd}
E^I \arrow[rrd, bend left, "\epsilon_0"] \arrow[rdd, bend right, "p^I"'] \arrow[rd, dashed, shift right, "\pi"'] & & \\
& \Omega_p = B^I \, {}_{\epsilon_0'}\!\sqcap_p E \arrow[lu, shift right, "\lambda"'] \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & E \arrow[d, "p"] \\
& B^I \arrow[r, "\epsilon_0'"] & B
\end{tikzcd}$$

where $\epsilon_0(\alpha) = \alpha(0)$, $\epsilon_0'(\omega) = \omega(0)$ and $p^I(\alpha) = p \cdot \alpha \in B^I$. Clearly, $\epsilon_0' p^I = p\epsilon_0$ and hence by the universal property of pullbacks, there exists a map $\pi: E^I \to B^I_{\epsilon_0'} \sqcap_p E$ such that $pr_2 \cdot \pi = \epsilon_0$ and $pr_1 \cdot \pi = p^I$, and consequently $\pi(\alpha) = (p \cdot \alpha, \alpha(0))$. Therefore, $\lambda: \Omega_p \to E^I$ is a lifting function iff $\pi \cdot \lambda = 1_{B^I_{\epsilon_0'} \sqcap_p E}$.

We now prove a theorem where the basic ideas of fibrations and cofibrations are jointly used to yield an important result on cofibrations. The theorem essentially asserts that "the pullback of a closed cofibration over a fibration is a closed cofibration".

Theorem 2.2.4: If (B,A) is a cofibred pair with A closed and $p:E \to B$ is a fibration, then $(E, p^{-1}(A))$ is also a closed cofibred pair.

Proof: We first note that the following diagram is a pullback.

$$\begin{tikzcd}
p^{-1}(A) \arrow[r] \arrow[d, "p"'] & E \arrow[d, "p"] \\
A \arrow[r] & B
\end{tikzcd}$$

Since (B,A) is a cofibred pair, there exist maps $\varphi: B \to I$ and $H: B \times I \to B$ satisfying the properties of the Characterization Theorem 2.2.2 (e).

Now consider the following diagram

$$\begin{tikzcd}
E \arrow[r, "1_E"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\
E \times I \arrow[r, "H(p \times 1_I)"'] \arrow[ur, dashed, "\overline{H}"] & B
\end{tikzcd}$$

Since $p:E \to B$ is a fibration, there exists a map $\bar{H}:E \times I \to E$ such that $p\bar{H} = H(p \times 1_I)$ and $\bar{H}\Big|_E = 1_E$.

Define a map $\psi:E \to I$ by $\psi = \varphi p:E \to I$.

Then $\psi^{-1}(0) = (\varphi p)^{-1}(0) = p^{-1}\varphi^{-1}(0) = p^{-1}(A)$ (see Remark 2.2.1 (iii)).

Also, define $\tilde{H}:E \times I \to E$ by

$$\tilde{H}(e,t) = \bar{H}(e,t \wedge \varphi p(e)), \text{ where } t \wedge \varphi p(e) = \text{Min}\{t,\varphi p(e)\}$$

$\tilde{H}$ is continuous and

(i) $\tilde{H}(e,0) = \bar{H}(e,0 \wedge \varphi p(e))$

$= \bar{H}(e,0)$ as $\varphi p(e) \geq 0$

$= 1_E(e)$

$= e$ for all $e \in E$

(ii) Let $e \in p^{-1}(A)$. Then $p(e) \in A$

So, $\tilde{H}(e,t) = \bar{H}(e,t \wedge \varphi p(e))$

$= \bar{H}(e,t \wedge 0)$ as $p(e) \in A$ and $A = \varphi^{-1}(0)$.

$= \bar{H}(e,0)$

$= e$ for all $e \in p^{-1}(A)$.

(iii) Because A is closed, $H(b,\varphi(b)) \in A$ whenever $\varphi(b) < 1$. (see Remark 2.2.1 (v)).

Suppose $t \in I$ and $t > \psi(e)$. That is, $t > \varphi p(e)$ and so $\varphi p(e) < 1$.

Then, $\tilde{H}(e,t) = \bar{H}(e,t \wedge \varphi p(e))$

$= \bar{H}(e,\varphi p(e))$, as $\varphi p(e) < 1$ and $t \in I$

and therefore, $p\tilde{H}(e,t) = p\bar{H}(e,\varphi p(e))$

$= H(p \times 1_I)(e,\varphi p(e))$

$= H(p(e),\varphi p(e)) \in A$ as

A is closed and $\varphi(p(e)) = \psi(e) < 1$.
Hence, $p\tilde{H}(e,t) \in A$ and therefore, $\tilde{H}(e,t) \in p^{-1}(A)$.
Therefore, by the Characterization Theorem 2.2.2 (c), $(E,p^{-1}(A))$ is a closed cofibred pair.

The closedness condition on A can be circumvented by requiring that the fibration $p:E \to B$ of Theorem 2.2.4 be regular. Hence, we can reformulate Theorem 2.2.4 as

Theorem 2.2.5: The pullback of a cofibration over a regular fibration is a cofibration.

Proof: Let $\varphi:B \to I$ and $H:B \times I \to B$ be maps satisfying the properties of the Characterization Theorem 2.2.2 (e). Since $p:E \to B$ is a regular fibration, there exists $\lambda:\Omega_p \to E^I$, a regular lifting function for p.

Set $\psi = \varphi p:E \to I$, as before and define $\tilde{H}:E \times I \to E$ by $\tilde{H}(e,t) = \lambda(e,H_{p(e)})(t)$, where $H_{p(e)}(t) = H(p(e),t)$.

Then

(i) $\tilde{H}(e,0) = \lambda(e,H_{p(e)}(0)$
$= e$

(ii) Let $e \in p^{-1}(A)$.
Then $p(e) \in A$ and $\tilde{H}(e,t) = \lambda(e,H_{p(e)})(t)$.
But $H_{p(e)}(t) = H(p(e),t)$
$= p(e)$, as $p(e) \in A$ and $H(a,t) = a$, for all $a \in A$ and $t \in I$.
Hence, $H_{p(e)}(t)$ is the constant path and so by the regularity of λ, it follows that $\lambda(e,H_{p(e)})(t) = e$.

Therefore, $\tilde{H}(e,t) = \lambda(e,H_{p(e)})(t) = e$, for all $e \in p^{-1}(A)$.

(ii) Suppose $t \in I$ and $t > \psi(e) = \varphi(p(e))$.

Then, $p\tilde{H}(e,t) = p\lambda(e,H_{p(e)})(t)$

$= H_{p(e)}(t)$

$= H(p(e),t) \in A$, as $p(e) \in B$ and $t > \varphi(p(e))$.

Hence, $\tilde{H}(e,t) \in p^{-1}(A)$ whenever $t > \psi(e)$.

Therefore, by the Characterization Theorem 2.2.2 (e) $(E,p^{-1}(A))$ is a cofibred pair.

We now prove a theorem which states that if a composite map is a cofibration and the second map is a cofibration, the first map is a cofibration. But before we do that, we need to prove the following lemma which in simple terms asserts that global HEP => local HEP.

Lemma 2.2.6: Let $i:A \to B$ be an inclusion of topological spaces with the HEP and let $V \subseteq B$ be such that a continuous function $\tau:B \to [0,1]$ with $\bar{A} \cap V \subseteq \tau^{-1}(0,1] \subseteq V$. Then the restriction $i_V:A \cap V \to V$ has the HEP.

Proof: Since $i:A \to B$ is a cofibration, take φ and H as in Characterization Theorem 2.2.2 (e). We define functions $\psi:V \to [0,\infty]$ and $K:\psi^{-1}[0,1] \times I \to V$ as in Remark 2.2.1 (iv).

Let $\tilde{\tau}:B \to [0,1]$ be defined by

$$\tilde{\tau}(b) = \text{Min}\,\{\tau H(b,t) \mid 0 \le t \le 1\}.$$

Clearly, $\tilde{\tau}$ is continuous.

Now, for all $a \in A$, $\tilde{\tau}(a) = \text{Min}\{\tau H(a,t) \mid 0 \le t \le 1\}$

$= \text{Min}\{\tau(a) \mid 0 \le t \le 1\}$, as $H(a,t) = a$

$= \tau(a)$

Hence, $\tilde{\tau}\big|_A = \tau\big|_A$.

Let $\bar{a} \in \bar{A}$. Then $\{a_n\} \in A$ such that $\bar{a} = \lim a_n$.

So, $\tilde{\tau}(\bar{a}) = \tilde{\tau}(\lim a_n) = \lim \tilde{\tau}(a_n)$

$= \lim \tau(a_n)$, as $\tilde{\tau}\big|_A = \tau\big|_A$

$= \tau(\lim a_n)$

$= \tau(\bar{a})$

Therefore, $\tilde{\tau}\big|_{\bar{A}} = \tau\big|_{\bar{A}}$.

But by hypothesis, $\bar{A} \cap V \subseteq \tau^{-1}(0,1] \subseteq V$. Moreover, $\bar{A} \cap V \subseteq \bar{A}$.

Hence, $\tilde{\tau}\big|_{\bar{A}\cap V} = \tau\big|_{\bar{A}\cap V} > 0$.

Since $V \subseteq B$ and by hypothesis $H(b,t) \in A$, for all $t > \varphi(b)$, it follows that $H(v,t) \in A$ for all $t > \varphi(v)$. Now, if $v \in V$ and $\varphi(v) = 0$, then $H(v,t) \in A$ for all $t > 0$. But, $H)v,\varphi(v)) \in \bar{A}$ as $\varphi(v) = 0 < 1$ (Remark 2.2.1 (v)); that is, $v = H(v,0) \in \bar{A}$. Consequently, $v \in \bar{A} \cap V$ and so $\tilde{\tau}(v) > 0$. Therefore, the functions φ and $\tilde{\tau}$ have no common zeros in V. Thus, the function $\psi: V \to [0,\infty]$ defined $\psi(v) = \dfrac{\varphi(v)}{\tilde{\tau}(v)}$ is well defined.

Moreover, ψ is continuous. Now, let $x_0 \in A \cap V$. Then $x_0 \in A$ and hence $\phi(x_0) = 0$. But then $\tilde{\tau}(x_0) \ne 0$ as ϕ and $\tilde{\tau}$ have no common zeros in V. Therefore, $\psi(x_0) = 0$ and so $A \cap V \subseteq \psi^{-1}(0)$. Let $v \in V$ be such that $\psi(v) \le 1$.

Then, $\psi(v) \le 1 \Rightarrow \tilde{\tau}(v) > 0$

$\Rightarrow \tau H(v,t) > 0$, for all $t \in I$

$\Rightarrow H(v,t) \in \tau^{-1}(0,1] \subseteq V$

$\Rightarrow H(v,t) \in V$, for all $t \in I$

Thus, we can define $K:\psi^{-1}[0,1] \times I \to V$ by

$$K(v,t) = H(v,t).$$

Clearly, K is continuous.

Now, (i) for all $v \in \psi^{-1}[0,1] \subseteq V \subseteq E$, $K(v,0) = H(v,0) = v$

(ii) for all $a \in A \cap V$ and $t \in I$, $K(a,t) = H(a,t) = a$

(iii) for all $v \in \psi^{-1}[0,1]$, $\psi(v) \le 1$ and hence $\tilde{\tau}(v) > 0$. That is, $\frac{1}{\tilde{\tau}(v)} \ge 1$, which implies $\frac{\phi(v)}{\tilde{\tau}(v)} \ge \phi(v)$ and consequently, $\psi(v) \ge \phi(v)$.

Now, suppose $t > \psi(v)$. Then, from above, $t > \phi(v)$ and so by hypothesis $H(v,t) \in A$. Therefore, $K(v,t) = H(v,t) \in A \cap V$. Hence, $K(v,t) \in A \cap V$, whenever $t > \psi(v)$ and $\psi(v) \le 1$. Therefore, by Remark 2.2.1 (iv), $i_V:A \cap V \to V$ has the HEP.

We now are in a position to prove the following theorem.

<u>Theorem 2.2.7</u>: If $j:B \to A$ and $i:A \to X$ are maps such that i and ij are cofibrations, then j is also a cofibration.

<u>Proof</u>: Since $i:A \to X$ and $ij:B \to X$ are cofibrations, we can assume without any loss of generality that i and ij are inclusions (Theorem 2.2.1) and hence j is also an inclusion. Since $i:A \to X$ is a cofibration, it follows from Remark 2.2.2 (b) (ii) that $\exists$ a halo U around A in X together with a retraction $r:U \to A$ such that $A \subseteq \phi^{-1}(0) \subseteq \phi^{-1}[0,1) \subseteq U \subseteq X$. Since $B \subseteq A$,

it follows that U is also a halo around B in X.
So, by Lemma 2.2.6, $j_U:B \cap U \to U$, that is, $J_U:B \to U$ is a cofibration.
Now, for an arbitrary topological space Y, and maps $F:B \to Y^I$ and $f:A \to Y$, consider the following commutative diagram:

(1)
$$\begin{tikzcd} B \arrow[r, "F"] \arrow[d, "j"'] & Y^I \arrow[d, "\epsilon_0"] \\ A \arrow[r, "f"'] \arrow[ur, dashed, "H"] & Y \end{tikzcd}$$

where $\epsilon_0(\omega) = \omega(0)$ is the evaluation map. We claim that diagram (1) admits a diagonal $H:A \to Y^I$ such that the resulting triangles commute. Now the diagram

$$\begin{tikzcd} B \arrow[r, "F"] \arrow[d, "j_U"'] & Y^I \arrow[d, "\epsilon_0"] \\ U \arrow[r, "fr"'] \arrow[ur, dashed, "G"] & Y \end{tikzcd}$$

is also commutative since $frj_U = fj$ where $r:U \to A$ is a retraction (see Remark 2.2.2(b)) $\qquad = \epsilon_0 F$, from diagram (1).
Since $j_U:B \to U$ is a cofibration, diagram (2) admits a diagonal $G:U \to Y^I$ such that $\epsilon_0 G = fr$ and $Gj_u = F$.
Now, let $H = G|_A$. Then, $H = G|_A:A \to Y^I$ is a map such that

(i) $\epsilon_0 H(a) = \epsilon_0 G(a) = fr(a) = f(a)$, for all $a \in A$. That is, $\epsilon_0 H = f$, and

(ii) $Hj(b) = Gj_u(b) = F(b)$ for all $b \in B$, as $B \subseteq A \subseteq U$; that is, $Hj = F$.

Therefore, by Definition 2.1.3, $j:B \to A$ is a cofibration.

The following theorem is an application of the pullback theorem and the composition theorem we have proved above.

Theorem 2.2.8: Given the commutative diagram

$$\begin{tikzcd} X_0 \arrow[r, "p_0"] \arrow[d, "i_{x_0}"'] & B_0 \arrow[d, "i_{b_0}"] & Y_0 \arrow[l, "q_0"'] \arrow[d, "i_{y_0}"] \\ X \arrow[r, "p"'] & B & Y \arrow[l, "q"] \end{tikzcd}$$

(a) if p, p_0, q, q_0 are fibrations and (X,X_0), (B,B_0), (Y,Y_0) are closed cofibred pairs, then $(X \sqcap Y, X_0 \sqcap Y_0)$ is a closed cofibred pair. (See [6;Proposition 1.7])

(b) if p, p_0, q, q_0 are regular fibrations and (X,X_0), (B,B_0), (Y,Y_0) are cofibred pairs, then $(X \sqcap Y, X_0 \sqcap Y_0)$ is a cofibred pair.

Proof:

(a) Let $\tilde{X}_0 = p^{-1}(B_0)$ and $\tilde{Y}_0 = q^{-1}(B_0)$.

Then the following diagrams are pullbacks.

$$\begin{tikzcd}
\tilde{X}_0 = p^{-1}(B_0) \arrow[r] \arrow[d, "p"'] & X \arrow[d, "p"] \\
B_0 \arrow[r] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{Y}_0 = q^{-1}(B_0) \arrow[r] \arrow[d, "q"'] & Y \arrow[d, "q"] \\
B_0 \arrow[r] & B
\end{tikzcd}$$

where $B_0 \to B$ is a closed cofibration and $p:X \to B$ and $q:Y \to B$ are fibrations. Hence, by Theorem 2.2.4, it follows that $\tilde{X}_0 = p^{-1}(B_0) \to X$ and $\tilde{Y}_0 = q^{-1}(B_0) \to Y$ are closed cofibrations. Now, for all $x_0 \in X$, $p(x_0) = pi_{X_0}(x_0) = i_{b_0}p_0(x_0) \in B$. Hence, $X_0 \to \tilde{X}_0$ is an inclusion. Similarly, $Y_0 \to \tilde{Y}_0$ is an inclusion. Thus, we have the following two compositions

$$X_0 \to \tilde{X}_0 \to X \quad \text{and} \quad Y_0 \to \tilde{Y}_0 \to Y,$$

where the composite inclusions are closed cofibrations and the second inclusions are closed cofibrations. Therefore, by Theorem 2.2.7, the first inclusions $X_0 \to \tilde{X}_0$ and $Y_0 \to \tilde{Y}_0$ are closed cofibrations.

Now, consider the following diagrams. For convenience we drop the double subscript notation on the pullback symbol.

$$\begin{tikzcd} X_0 \sqcap Y_0 \arrow[r, "i_{X_0}\times 1_Y"] \arrow[d, "pr_{X_0}"'] & \tilde{X}_0 \sqcap Y_0 \arrow[r, "pr_{Y_0}"] \arrow[d, "pr_{\tilde{X}_0}"] & Y_0 \arrow[d, "q_0"] \\ X_0 \arrow[r] & \tilde{X}_0 \arrow[r] & B_0 \end{tikzcd} \qquad \begin{tikzcd} \tilde{X}_0 \sqcap Y_0 \arrow[r, "1_{\tilde{X}_0}\times i_{Y_0}"] \arrow[d, "pr_{Y_0}"'] & \tilde{X}_0 \sqcap \tilde{Y}_0 \arrow[r, "pr_{\tilde{X}_0}"] \arrow[d, "pr_{\tilde{Y}_0}"] & \tilde{X}_0 \arrow[d, "p"] \\ Y_0 \arrow[r] & \tilde{Y}_0 \arrow[r] & B \end{tikzcd}$$

In each of the above two diagrams, the right hand squares are pullbacks and the outer squares are pullbacks. Hence in both cases the left hand squares are pullbacks (see Remark 1.1.4(a)). Since q_0 and p are fibrations and pullbacks of fibrations are fibrations, it follows that $pr_{\tilde{X}_0}:\tilde{X}_0 \sqcap Y_0 \to \tilde{X}_0$ and $pr_{\tilde{Y}_0}:\tilde{X}_0 \sqcap \tilde{Y}_0 \to \tilde{Y}_0$ are fibrations.

Now, let $X \sqcap Y$ be the pullback of the diagram

$$\begin{tikzcd} & X \arrow[d, "p"] \\ Y \arrow[r, "q"'] & B \end{tikzcd}$$

That is, $X \sqcap Y = \{(x,y) \in X \times Y | p(x) = q(y)\}$. Since p and q are fibrations, the composite $q\cdot pr_Y:X \sqcap Y \to B$ is a fibration. Consider the following diagram

$$\begin{tikzcd} & X \sqcap Y \arrow[d, "q\,pr_Y"] \\ B_0 \arrow[r] & B \end{tikzcd}$$

The pullback of this diagram is $\{(x,y) | qpr_Y(x,y) = ppr_X(x,y) \in B_0\}$

$= \{(x,y) | q(y) = p(x) \in B_0\}$

$= \{(x,y) | x \in p^{-1}(B_0)$ and $y \in q^{-1}(B_0)\}$

$= \tilde{X}_0 \sqcap \tilde{Y}_0$

That is,

$$\begin{tikzcd}
\tilde{X}_0 \sqcap \tilde{Y}_0 \arrow[r] \arrow[d, "q\,pr_{\tilde{Y}} = p\,pr_{\tilde{X}}"'] & X \sqcap Y \arrow[d, "q\,pr_Y = p\,pr_X"] \\
B_0 \arrow[r] & B
\end{tikzcd}$$

So, we have the following three pullback diagrams

$$\begin{tikzcd}
X_0 \sqcap Y_0 \arrow[r] \arrow[d, "pr_X"'] & \tilde{X}_0 \sqcap Y_0 \arrow[d, "pr_{\tilde{X}_0}"] \\
X_0 \arrow[r] & \tilde{X}_0
\end{tikzcd}
\quad
\begin{tikzcd}
\tilde{X}_0 \sqcap Y_0 \arrow[r] \arrow[d, "pr_{Y_0}"'] & \tilde{X}_0 \sqcap \tilde{Y}_0 \arrow[d, "pr_{\tilde{Y}_0}"] \\
Y_0 \arrow[r] & \tilde{Y}_0
\end{tikzcd}
\quad
\begin{tikzcd}
\tilde{X}_0 \sqcap \tilde{Y}_0 \arrow[r] \arrow[d, "q\,pr_{\tilde{Y}}"'] & X \sqcap Y \arrow[d, "q\,pr_Y"] \\
B_0 \arrow[r] & B
\end{tikzcd}$$

Since the right hand vertical maps are fibrations and the bottom horizontal maps are closed cofibrations, it follows from Theorem 2.2.4 that each of the inclusions $X_0 \sqcap Y_0 \to \tilde{X}_0 \sqcap Y_0 \to \tilde{X}_0\ \tilde{Y}_0 \to X \sqcap Y$ are closed cofibrations. Since the composite of closed cofibrations is a closed cofibration, it follows that $X_0 \sqcap Y_0 \to X \sqcap Y$ is a closed cofibration.

(b) The proof is analogous, except that we use the fact that a pullback of a cofibration along a regular fibration is a cofibration.

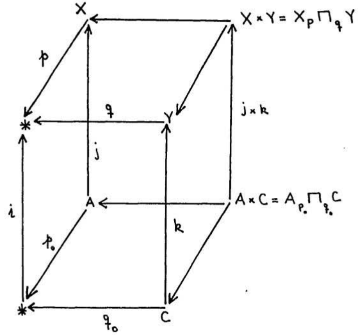
As an application of the above theorem we have the following result.

Corollary 2.2.9: If (X,A) and (Y,C) are ("closed") cofibred pairs, then $(X \times Y,\ A \times C)$ a ("closed") cofibred pair.

Proof:

Case 1: Suppose (X,A) and (Y,C) are closed cofibred pairs.

We construct the following diagram



where $X_0 = A$, $Y_0 = C$ and $B = B_0 = *$ in the theorem above. The inclusions i, j and k are closed cofibrations. Clearly, p, q, p_0 and q_0 are fibrations. Hence, by Theorem 2.2.8 (a), $A \times C \to X \times Y$ is a closed cofibration.

Case 2: Suppose (X,A) and (Y,C) are cofibred pairs.

Then, clearly $p:X \to *$, $q:Y \to *$, $p_0:A \to *$ and $q_0:C \to *$ are regular fibrations. Hence, by Theorem 2.2.8 (b) $A \times C \to X \times Y$ is a cofibration.

Finally, to conclude this section we have the following important results which will be applied in Chapter IV.

Theorem 2.2.10: Let $f:D \to A$ be any map and let $M(f)$ denote the mapping cylinder of f.
Then

(a) the inclusion $\bar{i}:A \to M(f)$ is a closed cofibration.

(b) the composite map

$$i_D:D \cong D \times 1 \to D \times I \to M(f)$$

is a closed cofibration.

(c) the map f factors through i_D; more precisely, $f = r_f \cdot i_D$ where r_f is an h-equivalence.

(d) $f:D \to A$ is an h-equivalence $\Longleftrightarrow$ $i_D:D \to M(f)$ is an h-equivalence.

Proof:

(a) Consider the following diagram

$$\begin{tikzcd} D \times I \arrow[r, "\bar{f}"] & M(f) \\ D \times 0 \cong D \arrow[u, "i"] \arrow[r, "f"'] & A \arrow[u, "\bar{i}"'] \end{tikzcd}$$

It will be shown in the next section (see Example 2.3.1 that the inclusion $\{0\} \to I$ is a cofibration. Hence, by Corollary 2.2.9,

$i: D \times 0 \to D \times I$ is a cofibration. Therefore, $\bar{I}: A \to M(f)$ is also a cofibration by Theorem 2.1.3.

(b) Construct the following diagram

$$\begin{tikzcd}
D \cong D\times 1 \arrow[r, "k"] & D\times I \arrow[r, "\bar{f}"] & M(f) \\
D \sqcup D \cong D\times\dot{I} \arrow[r, "\tilde{f}"] \arrow[u, "j"] & A\,{}_f\!\sqcup_\ell (D\times\dot{I}) \cong A\sqcup D \arrow[u, "\bar{J}"'] \\
D \cong D\times 0 \arrow[r, "f"] \arrow[u, "\ell"] & A \arrow[u, "\bar{\ell}"']
\end{tikzcd}$$

observe that $D \times \dot{I} = D \times 0 \cup D \times 1 \cong D \sqcup D$ (disjoint union).

Consider the following diagram

$$\begin{tikzcd}
D \arrow[r] & D \sqcup D \arrow[r] & A\,{}_f\!\sqcup_\ell (D\times\dot{I}) \cong A\sqcup D \\
\phi \arrow[r] \arrow[u] & D \arrow[r, "f"'] \arrow[u, "\ell"] & A \arrow[u]
\end{tikzcd}$$

By horizontal composition (see Remark 1.2.5 (a)) it follows that $A \sqcup_f (D \times \dot{I}) \cong A \sqcup D$.

Now, bottom square of diagram (*) is a pushout and composite square is a pushout. Hence, by Remark 1.1.4 (b) (ii), it follows that upper square of diagram (*) is a pushout.

Now, $\dot{I} \to I$ is a cofibration (see Example 2.3.1)

=> $D \times \dot{I} \to D \times I$ is a cofibration (Corollary 2.2.9)

=> $\bar{J}: A \quad D \to M(f)$ is a cofibration (Theorem 2.1.3)

Now, $i_D = \bar{f} \cdot k = \bar{f}j\big|_{D\times 1} = \bar{J} \cdot \tilde{f}\big|_{D\times 1}$.

From above, $\bar{J}: A \quad D \to M(f)$ is a cofibration and $\tilde{f}\big|_{D\times 1}$ is the inclusion $D \to A \sqcup D$ which is a cofibration (Theorem 2.1.4 (a)). Therefore, the composite $i_D = \bar{f} \cdot k : D \times 1 \to M(f)$ is a cofibration. Clearly, $D \subseteq M(f)$ is closed.

(c) Construct the following diagram

$$\begin{tikzcd}
D\times I \arrow[r, "\bar{f}"] \arrow[rr, bend left, "f\cdot pr_D"] & M(f) \arrow[r, dashed, "r_f"] & A \\
D\times 0 \arrow[u, "i"] \arrow[r, "f"] & A \arrow[u, "\bar{\iota}"] \arrow[ur, bend right, "1_A"'] &
\end{tikzcd}$$

Now, $(f \cdot pr_D) \cdot i(d,0) = fpr_D(d,0)$
$= f(d)$

and $1_A\, f(d,0) = f(d)$. Therefore, diagram commutes and since the square is a pushout, there exists a unique map $r_f : M(f) \to A$ such that $r_f\bar{f} = f \cdot pr_D$ and $f_f\bar{I} = 1_A$. Hence, we have the following diagram

$$\begin{tikzcd}
D \cong D\times 1 \arrow[rr, "i_D"] \arrow[dr, "f"'] && M(f) \arrow[dl, shift left, "r_f"] \\
& A \arrow[ur, dashed, shift left, "\bar{\iota}"] &
\end{tikzcd}$$

Now for all $d \in D$, $r_f\, i_D(d) = r_f(\bar{f}k(d,1))$ (see part (b) above)
$= r_f(\bar{f}(d,1))$
$= fpr_D(d,1)$
$= f(d)$

Therefore, $r_f \cdot i_D = f$.

We now show that r_f is an h-equivalence. We already have from above, that $r_f\bar{I} = 1_A$. We need only show that $\bar{I}r_f \simeq 1_{M(f)}$. So, we define a homotopy $H: M(f) \times I \to M(f)$ as follows

$$H([x,t],s) = [x,(1-s)t], \quad (x,t) \in D \times I$$

$$H([a],s) = [a], \quad a \in A$$

Then,

(a) $H([x,t],0) = [x,t]$

$H[a],0) = [a]$

(b) $H([x,t],1) = [x,0]$

$H([a],1) = [a]$

But, $\bar{i}r_f([x,t]) = \bar{i}(f \cdot pr_D(x,t))$

$= \bar{i}f(x)$

$= \bar{f}i(x,0)$

$= [x,0]$

and $\bar{i}r_f[a] = \bar{i}r_f\bar{i}(a)$

$= \bar{i}(a)$, since $r_f\bar{i} = 1_A$

$= [a]$

Therefore, $\bar{i}r_f \simeq 1_{M(f)}$ and r_f is a homotopy equivalence.

(d) From part (c) above, we have the following commutative diagram

$$\begin{tikzcd}
D \arrow[rr, "i_D"] \arrow[dr, "f"'] & & M(f) \arrow[dl, "r_f"] \\
& A &
\end{tikzcd}$$

where r_f is an h-equivalence, i_D a closed cofibration and $f = r_f i_D$.

"<=":

If $i_D: D \to M(f)$ is an h-equivalence, then so is the composite

$r_f \cdot i_D$ an h-equivalence. Therefore, $f:D \to A$ is an h-equivalence.

"$\Rightarrow$":

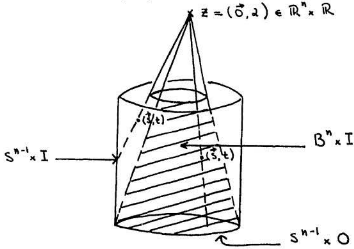
Suppose $f:D \to A$ is an h-equivalence. Since $f = r_f i_D$ and r_f is an h-equivalence, it follows from Theorem 1.3.2 (a) that i_D is an h-equivalence.

Section III: Examples and Non-Examples of Cofibrations

The following are examples of closed cofibrations:

Examples 2.3.1: The inclusion $i:S^{n-1} \to B^n$ (n - 1 dimensional sphere into the n-dimensional ball) is a closed cofibration. By Theorem 2.2.2 (c) it is sufficient to show that $B^n \times 0 \cup S^n \times I$ is a retract of $B \times I$. Clearly S^{n-1} is closed in B^n.

Consider the following figure



Geometrically, the required retraction is obtained by projecting $B^n \times I$ onto $B^n \times I \cup S^{n-1} \times I$ via the radial projection from $z = (\vec{0},2) \in \mathbf{R}^n \times \mathbf{R}$. An explicit description of the retraction is obtained as follows:

The vector equation of the line in $\mathbf{R}^n \times \mathbf{R}$ passing through $(\vec{0},1) \in \mathbf{R}^n \times \mathbf{R}$ and $(\vec{s},t) \in B^n \times I \subseteq \mathbf{R}^n \times \mathbf{R}$ is given by:

$$(*) \quad (\vec{y}_1, y_2) = (\vec{0},2) + \lambda(\vec{s}, t - 2), \text{ where } \lambda \geq 0$$

We want the point on the line through $(\vec{0},2)$ for which $y_2 = 0$.

$$\text{Now, } y_2 = 0 \iff 2 + \lambda(t - 2) = 0$$
$$\iff \lambda = \frac{2}{2 - t}$$

Therefore, the point on the line through $(\vec{0},2)$ for which $y = 0$ is $\frac{2}{2 - t}(\vec{s},0)$.

Now, observe that when $\frac{2}{2 - t} = \frac{1}{\|\vec{s}\|}$, that is, $\|s\| = 1 - \frac{t}{2}$, the point $\frac{2}{2 - t}(\vec{s},0)$ belongs to $S^{n-1} \times 0$. Hence, for $\|\vec{s}\| \leq 1 - \frac{t}{2}$, we have that for all such $(\vec{s},t) \in B^n \times I$, $\frac{2}{2 - t}(\vec{s},0) \in B^n \times 0$.

Consider again equation (*). Suppose we want $\vec{y}_1 \in S^{n-1}$. Then, $\vec{y} \in S^{n-1} \iff \|y_1\| = 1 \iff \|\lambda\vec{s}\| = 1 \iff \lambda = \frac{1}{\|\vec{s}\|}$, since $\lambda \geq 0$.

Hence, the point on the line through $(\vec{0},2)$ for which $\vec{y}_1 \in S^{n-1}$ is given by $(\vec{0},2) + \frac{1}{\|\vec{s}\|}(\vec{s}, t - 2) = \frac{1}{\|\vec{s}\|}(\vec{s}, 2\|\vec{s}\| + t - 2)$

$$= \left(\frac{\vec{s}}{\|\vec{s}\|},\ 2 - \frac{2 - t}{\|\vec{s}\|}\right)$$

So define $r: B^n \times I \to B^n \times 0 \cup S^{n-1} \times I$ by

$$r(\vec{s},t) = \begin{cases} \dfrac{2}{2-t}(\vec{s},0), & \|\vec{s}\| \le 1 - \dfrac{t}{2} \\ \left(\dfrac{\vec{s}}{\|\vec{s}\|},\ 2 - \dfrac{2-t}{\|\vec{s}\|}\right), & \|\vec{s}\| \ge 1 - \dfrac{t}{2} \end{cases}$$

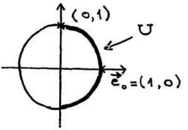
Then, r is the retraction described geometrically above and so the inclusion $r: S^{n-1} \to B^n$ is a closed cofibration. Notice that if $n = 1$, then $i:\{-1,1\} \to [-1,1]$ is a closed cofibration. Since $\{-1,1\} \cong \{0,1\}$ and $[-1,1] \cong [0,1]$ and homeomorphisms are cofibrations, it follows that $\dot{I} = \{0,1\} \to I$ is a closed cofibration.

<u>Example 2.3.2</u>: The inclusion of the base point $\vec{e}_0 = (1,\ldots,0) \in S^n$ is a closed cofibration.

We use the Characterization Theorem for <u>closed</u> cofibrations (see Remark 2.2.1 (vi)).

Write $\vec{e}_0 = (1,0) \in R \times R^n$.

Let $U = \{(x,\vec{y}) \in S^n \mid x \ge 0,\ \vec{y} \in R^n\}$. For the case $n = 1$, see diagram below.



Define $H: U \times I \to S^n$ by $H(\vec{x},t) = \dfrac{(1-t)\vec{x} + t\vec{e}_0}{\|(1-t)\vec{x} + t\vec{e}_0\|}$

Then, (i) $H(\vec{u},0) = \vec{u}$ as $\|\vec{u}\| = 1$

(ii) $H(\vec{e}_0,t) = \vec{e}_0$ as $\|\vec{e}_0\| = 1$

(iii) $H(\vec{u},1) = \vec{e}_0$

So, U is deformable in S^n to $\vec{e}_0$ rel $\vec{e}_0$.

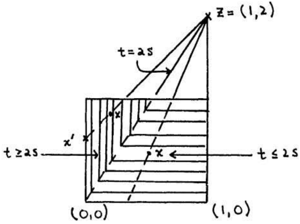
Next, define $\phi: S^n \to I$ by

$$\phi((x,\vec{y})) = \begin{cases} 1 & , \text{ if } x \le 0 \\ \sqrt{1-x^2} & , \text{ if } x \ge 0 \end{cases}$$

Clearly, ϕ is well defined and continuous. Now, $\phi(\vec{c}_0) = \phi(1,0) = 0$ and so $\vec{e}_0 \in \phi^{-1}(0)$. On the other hand, suppose $\phi(x,\vec{y}) = 0$ where $(x,\vec{y}) \in S^n$. Then, $\sqrt{1-x^2} = 0$ whee $x \ge 0$ and so $x = 1$. But then $\vec{y} = 0$ and hence $(x,\vec{y}) = \vec{e}_0 = (1,0)$. That is, $\phi^{-1}(0) = \vec{c}_0$. Moreover, $\phi(x,\vec{y}) = 1$, for all $(x,\vec{y}) \in S^n - U$ since $x < 0$. Therefore $i:\vec{e}_0 \to S^n$ is a closed cofibration.

Finally, observe that each inclusion $e_0 \to S^n$ and $S^n \to B^{n+1}$ is a closed cofibration. Hence, the composite $e_0 \to B^{n+1}$ is also a closed cofibration. Consequently, when $n = 0$, the inclusion $\{1\} \to [-1,1]$ is a closed cofibration. Now, composing with homeomorphisms, the inclusion $\{0\} \to I$ is a closed cofibration. Let us give a geometric proof of this last statement.

<u>Example 2.3.2</u>: $\{0\} \to I$ is a closed cofibration. We show that $I \times 0 \cup 0 \times I$ is a retract of $I \times I$. Take $z = (1,2) \in \mathbb{R}^2$ and consider $I \times I \subseteq \mathbb{R}^2$. Let $x' \in (0 \times I) \cup (I \times 0)$. Now, consider the following diagram



As before, $r: x \to x'$ is the required retraction. Using similar techniques as in Example 2.3.1, the required retraction $r: I \times I \to 0 \times I \cup I \times 0$ is defined by

$$r(s,t) = \begin{cases} \left(0, 2 + \frac{1}{1-s}(t-2)\right), & t \geq 2s \\ \left(1 + \frac{2}{2-t}(s-1), 0\right), & t \leq 2s \end{cases}$$

Therefore $\{0\} \to I$ is a (closed) cofibration by Theorem 2.2.2 (c).

<u>Example 2.3.4</u>: The inclusions $A \to A \vee B$ and $B \to A \vee B$ where $A \vee B$ is the "<u>wedge</u>" of two spaces A and B, are cofibrations.

The wedge $A \vee B$ is defined by: $A \vee B = (A \times \{b_0\}) \cup (\{a_0\} \times B)$. Consider the commutative diagram

$$\begin{tikzcd}
A \vee B \arrow[r] \arrow[rr, bend left, "g = h \vee k"] & (A \vee B) \times I \arrow[r, dashed, "G"] & Y \\
A \arrow[u, "i"] \arrow[r] & A \times I \arrow[u] \arrow[ur, "F"'] &
\end{tikzcd}$$

where g and F are given maps such that $g = h$ on $A \times \{b_0\}$, $g = k$ on $\{a_0\} \times B$.

Now define $G:X \to Y$ by

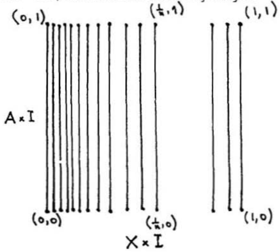
$$G = F \cup C_{g|\{a_0\}\times B}$$

$= F \cup C_k$ where $C_k:B \times I \to Y$ is the constant homotopy to k; that is, $C_k(b,t) = k(a_0,b)$ for all $t \in I$. This shows that $i:A \to A \vee B$ is a cofibration. Similary for $j:B \to A \vee B$. Examples 2.3.1 - 2.3.3 are particular cases of the general statement that "Inclusions of subcomplexes in CW complexes are cofibrations. (See [13, page 28, Theorem 1.4.12)

The following fail to be cofibrations.

<u>Example 2.3.5</u>: Let $X = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$ be topologized as a subspace of $\mathbf{R}$ and let $A = \{0\}$. We show that $i:A \to X$ is not a cofibration.

Suppose, on the contrary, $i:A \to X$ is a cofibration. Then, by Characterization Theorem 2.2.2 (c) $\exists$ a retraction $r:X \times I \to X \times 0 \cup A \times I$. Now, consider the following diagram



Since r is continuous and the points $(\frac{1}{n},0)$ for all $n = 1, 2, \ldots$ are left fixed by r, r then collapses $\{\frac{1}{n}\} \times I$ to $(\frac{1}{n},0)$ for each n by connectness. On the other hand, $r(0,t) = (0,t)$ for all $t \in I$. That is, $0 \times I$ is left fixed pointwise by r. Now, $(\frac{1}{n}, 1)$ converges to $(0,1)$, but $r(\frac{1}{n},1) = (\frac{1}{n}, 0)$ does not converge to $(0,1)$.

But this contradicts the continuity of r at $(0,1)$. Therefore, there exists no retraction $r:X \times I \to X \times 0 \cup A \times I$ and hence $i:A \to X$ is not a cofibration.

Example 2.3.6: Let M be an uncountable set.

Let $X = I^M$ with the product topology and $A = 0^M$.
We claim $0^M \to I^M$ is not a cofibration. Suppose that $i:0^M \to I^M$ is a cofibration. Since 0 is closed in I, it follows that 0^M is closed in I^M. Hence, by Remark 2.2.1 (vi), $\exists$ a map $u:I^M \to I$ such that $u^{-1}(0) = 0^M$.

Now, $0 = \bigcap_{n=1}^{\infty} [0,\frac{1}{n})$ and hence $u^{-1}(0) = \bigcap_{n=1}^{\infty} u^{-1}[0,\frac{1}{n})$.
Since u is continuous, for each $n \in u^{-1}[0,\frac{1}{n})$ is an open neighbourhood of 0^M in I^M. Thus, for each n, there exists a basic open set $B = \prod_{m\in M} B_m$ with $0^M \in B \subseteq u^{-1}[0,\frac{1}{n})$, where B_m is open in I for all $m \in M$ and $B_m = I$ for all but finitely many m, say, $m_1, m_2, \ldots, m_n$. Let $E_n = \{m_1, m_2, \ldots, m_n\} \subseteq M$. Then E_n is a finite set in M and $0^{E_n} \times I^{M-E_n} \subseteq u^{-1}[0,\frac{1}{n})$.
Now let $M' = \bigcup_{n=1}^{\infty} E_n$. Then M' is a countable set with

$0^{M'} \times I^{M-M'} \subseteq \bigcap_{n=1}^{\infty} u^{-1}[0,\frac{1}{n}) = u^{-1}(0) = 0^{M}$. But $M - M' \neq \emptyset$, as M is an uncountable set. This is impossible and hence i cannot be a cofibration.

The following is an example of a cofibration which is not closed.

Example 2.3.7: Let $X = \{a,b\}$ and $T_X = \{\phi, X, \{a\}\}$ be a topology on X. Let $A = \{a\}$. Clearly, A is not closed in X. We claim $i:A \to X$ is a cofibration.

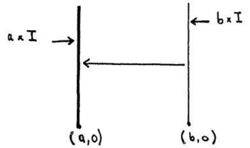
Now $A \times I = \{(a,t) \mid t \in I\}$

$X \times 0 = \{(a,0)\} \cup \{(b,0)\}$

$X \times I = \{(a,t) \mid t \in I\} \cup \{(b,t) \mid t \in I\}$

Define $r:X \times I \to X \times 0 \cup A \times I$ by

$$r(x,t) = \begin{cases} (x,t) & \text{if } t = 0 \\ (a,t) & \text{if } t > 0 \end{cases}$$



It is easy to check that r is continuous and obviously $r|_{X\times 0\cup A\times I} = 1_{X\times 0\cup A\times I}$. Hence, r is the required retraction and so $i:A \to X$ is a non-closed cofibration.

CHAPTER III

Lillig's Union Theorems

This chapter is entirely devoted to a paper of Lillig "A Union Theorem for Cofibrations" [11]. The gist of the problem is the following: Given subspaces A and B of a space X such that the inclusion maps $i:A \to X$ and $j:B \to X$ have the H.E.P. with respect to Z, under what conditions does $A \cup B \to X$ have the H.E.P. with respect to Z and consequently is a cofibration?

In the presentation of this chapter, theorems will be stated and proved for the case of H.E.P. with respect to Z and then reformulated for cofibrations, as a consequence. Before we prove our first result on the HEP, we need the following two lemmas.

Lemma 3.1: If $i:A \times I \to X \times I$ has the H.E.P. with respect to Z, then $(A \times I) \cup (X \times \dot{I}) \to X \times I$ and $(A \times I) \cup (X \times 0) \to X \times I$ have the H.E.P with respect to Z.
Here $(A \times I) \cup (X \times \dot{I})$ is not considered as a subspace of $X \times I$, but as a quotient space of the topological sum $(A \times I) \cup (X \times \dot{I})$ obtained by identifying $(a,0)$ with $i(a,0)$ and $(a,1)$ with $i(a,1)$. Similarly for $(A \times I) \cup (X \times 0)$.

Proof: Assume we are given the following commutative diagram

(*)

$$\begin{tikzcd}
X\times I\times 0 \arrow[r] \arrow[rr, bend left, "g"] & X\times I\times I \arrow[r, dashed, "\Phi"] & Z \\
((A\times I)\cup(X\times\dot{I}))\times 0 \arrow[u] \arrow[r] & ((A\times I)\cup(X\times\dot{I}))\times I \arrow[u] \arrow[ur, bend right, "\psi"'] &
\end{tikzcd}$$

where g and ψ are given maps such that

$$g\Big|_{((A\times I)\cup(X\times\dot{I}))\times 0} = \psi\Big|_{((A\times I)\cup(X\times\dot{I}))\times 0}$$

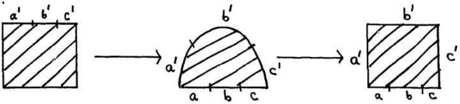
We have to show $\exists$ a map $\Phi: X\times I\times I \to Z$ such that

$$\Phi\Big|_{X\times I\times 0} = g \quad\text{and}\quad \Phi\Big|_{((A\times I)\cup(X\times\dot{I}))\times I} = \psi$$

Let $Q: I\times I \to I\times I$ be a homeomorphism such that

$$Q((I\times 0)\cup(\dot{I}\times I)) = I\times 0.$$

The existence of such a homeomorphism is illustrated by the diagram below.



Now, $Q^{-1}(I\times 0) = (I\times 0)\cup(\dot{I}\times I)$.

Hence, we have the following map

$$X \times I \times 0 \xrightarrow{1_X \times Q^{-1}} X \times (I \times 0 \cup \dot{I} \times I) = X \times I \times 0 \cup X \times \dot{I} \times I$$

Define maps $g_Q' = g \cdot (1_X \times Q^{-1})\big|_{X \times Q(I \times 0)}$ (1)

$$\text{i.e.} \quad X \times Q(I \times 0) \xrightarrow{1_X \times Q^{-1}} X \times I \times 0 \xrightarrow{g} Z$$

and $\Psi_Q' = \Psi \cdot (1 \times Q^{-1})\big|_{A \times I \times I \cup X \times Q(\dot{I} \times I)}$ (2)

$$\text{i.e.} \quad A \times I \times I \cup X \times Q(\dot{I} \times I) \xrightarrow{1 \times Q^{-1}} A \times I \times I \cup X \times \dot{I} \times I \xrightarrow{\Psi} Z$$

Now define $\Psi_Q : A \times I \times I \to Z$ by

$$\Psi_Q = \Psi_Q'\big|_{A \times I \times I} \tag{3}$$

On the other hand, $X \times I \times 0 = X \times Q[(I \times 0) \cup (\dot{I} \times I)]$

$$= X \times Q(I \times 0) \cup X \times Q(\dot{I} \times I)$$

So, define $g_Q : X \times I \times 0 \to Z$ as follows:

$$g_Q\big|_{X \times Q(I \times 0)} = g_Q' \quad \text{and} \tag{4}$$

$$g_Q\big|_{X \times Q(\dot{I} \times I)} = \Psi_Q'\big|_{X \times Q(\dot{I} \times I)}$$

That is,

$$g_Q(x,t,0) = \begin{cases} g_Q'(x,t,0) & \text{if} \quad (t,0) \in Q(I \times 0) \\ \Psi_Q'(x,t,0) & \text{if} \quad (t,0) \in Q(\dot{I} \times I) \end{cases} \quad \text{(by eq. (4))}$$

$$= \begin{cases} g_Q'(x,t,0) & \text{if } (t,0) = Q(t',0),\ (t',0) \in I \times 0 \\ \psi_Q'(x,t,0) & \text{if } (t,0) = Q(s,s'),\ (s,s') \in \dot{I} \times I \end{cases}$$

$$= \begin{cases} g(1 \times Q^{-1})(x,Q(t',0)) & \text{(by eq. (1))} \\ \psi(1 \times Q^{-1})(x,Q(s,s')) & \text{(by eq. (2))} \end{cases}$$

$$= \begin{cases} g(x,t',0) & \text{if } (t,0) = Q(t',0) \\ \psi(x,s,s') & \text{if } (t,0) = Q(s,s') \end{cases} \qquad (5)$$

Now, if $(t,0) \in Q(I \times 0)$, i.e. $(t,0) = Q(t',0)$,

then, $g_Q(a,t,0) = g(a,t',0)$

$= \psi(a,t',0)$ by commutativity of (*)

$= \psi(a,Q^{-1}(t,0))$

$= \psi \cdot (1 \times Q^{-1})(a,t,0)$

$= \psi_Q'(a,t,0)$ (by eq. (2))

$= \psi_Q(a,t,0)$ (by eq. (3))

Again, if $(t,0) \in Q(\dot{I} \times I)$, i.e. $(t,0) = Q(s,s')$, then

$g_Q(a,t,0) = \psi(a,s,s')$ (by eq. (5))

$= \psi(a,Q^{-1}(t,0))$

$= \psi \cdot (1 \times Q^{-1})(a,t,0)$

$= \psi_Q'(a,t,0)$ (by eq. (2))

$= \psi_Q(a,t,0)$ (by eq. (3))

Thus we have the following commutative diagram

$$\begin{tikzcd}
X\times I\times 0 \arrow[r] \arrow[rr, bend left, "g_Q"] & X\times I\times I \arrow[r, dashed, "\Phi_Q"] & Z \\
A\times I\times 0 \arrow[u] \arrow[r] & A\times I\times I \arrow[u] \arrow[ur, bend right, "\psi_Q"'] &
\end{tikzcd}$$

Since $A \times I \to X \times I$ has the H.E.P. with respect to Z, a map $\Phi_Q: X \times I \times I \to Z$ such that

$$\Phi_Q\Big|_{X\times I\times 0} = g_Q \tag{6}$$

and

$$\Phi_Q\Big|_{A\times I\times I} = \psi_Q$$

Now define a map $\Phi: Z \times I \times I \to Z$ by

$$\Phi = \Phi_Q \cdot (1_X \times Q)$$

We claim that Φ is the required map completing the diagram (*). First,

$$\begin{aligned}
\Phi(x,t,0) = \Phi_Q(1_X \times Q)(x,t,0) &= \Phi_Q(x,Q(t,0)) \\
&= g_Q(x,Q(t,0)) \quad \text{(by eq. (6) as } Q(t,0) \in I \times 0) \\
&= g_Q'(x,Q(t,0)) \quad \text{(by eq. (4))}
\end{aligned}$$

$$= g \cdot (1_X \times Q^{-1})(x, Q(t,0)) \quad \text{(by eq. (1))}$$

$$= g(x,t,0)$$

Therefore, $\Phi\big|_{X \times I \times 0} = g\big|_{X \times I \times 0}$

Now, let $(a,t,s) \in A \times I \times I$.

Then $\Phi(a,t,s) = \Phi_Q(1_X \times Q)(a,t,s,) = \Phi_Q(a,Q(t,s))$

$$= \Psi_Q(a,Q(t,s)) \quad \text{(by eq. (6) as } Q(t,s) \in I \times 0 \subseteq I \times I)$$

$$= \Psi_Q'(a,Q(t,s)) \quad \text{(by eq. (3))}$$

$$= \Psi \cdot (1_X \times Q^{-1})(a,Q(t,s)) \quad \text{(by eq. (2))}$$

$$= \Psi(a,t,s)$$

Again, let $(x,t,s) \in X \times \dot{I} \times I$; that is, $(t,s) \in (\dot{I} \times I)$.

Then $\Phi(x,t,s) = \Phi_Q(1 \times Q)(x,t,s) = \Phi_Q(x,Q(t,s))$

$$= g_Q(x,Q(t,s)) \quad \text{(by eq. (6) since } Q(t,s) \in \dot{I} \times 0)$$

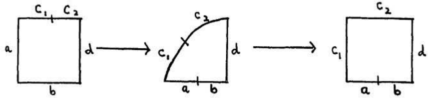
$$= \Psi_Q'(x,Q(t,s)) \quad \text{(by eq. (4))}$$

$$= \Psi \cdot (1 \times Q^{-1})(x,Q(t,s)) \quad \text{(by eq. (2))}$$

$$= \Psi(x,t,s)$$

Therefore, $\Phi|_{A\times I\times I\cup X\times \dot{I}\times I} = \psi$ and so $(A \times I) \cup (X \times \dot{I}) \to X \times I$ has the H.E.P. with respect to Z.

Similarly, one can show that $(A \times I) \cup (X \times 0) \to X \times I$ has the H.E.P. with respect to Z. Notice that in this case we use a homeomorphism $P: I \times I \to I \times I$ with the property that $P((I \times 0) \cup (0 \times I)) = I \times 0$. Such a homeomorphism P can be illustrated diagramatically as follows:



We leave the details of the proof to the reader.

Before we proceed to the second lemma, we need the following definition.

Definition 3.1: A subspace A of a space X is called a Nullstellen set if there exists a continuous map $u: X \to I$ with $u^{-1}(0) = A$. By Remark 2.2(ii), if (X, A) is a closed cofibred pair, then A is a Nullstellen set.

Lemma 3.2: Let $A \subseteq X$ be a Nullstellen set. Let $f, g: X \to Z$ be continuous maps with $\Phi: f \simeq g$ rel A. Then there exists a homotopy $\bar{\Phi}$: $f \simeq g$ rel A with $\bar{\Phi}(x,t) = \bar{\Phi}(x,u(x)) = \Phi(x,1)$, for all $x \in X$ and $t \geq u(x)$.

Proof: Since $A \subseteq X$ is a Nullstellen set, there exists a map $u: X \to I$ such that $u^{-1}(0) = A$.

Now, $\Phi: f \simeq g$ rel A means that $\Phi: X \times I \to Z$ is a map such that

$$\Phi(x,0) = f$$

$$\Phi(x,1) = g$$

and $\Phi(a,t) = f(a) = g(a)$, $a \in A$ and $t \in I$.

Define $\bar{\Phi}: X \times I \to Z$ by

$$\bar{\Phi}(x,t) = \begin{cases} \Phi(x,1), & \text{for } t \geq u(x) \\ \Phi(x,t), & \text{for } u(x) = 0 \\ \Phi(x,\frac{t}{u(x)}), & \text{for } t \leq u(x) \text{ and } u(x) \neq 0 \end{cases}$$

If $t = u(x)$, then $\Phi(x,\frac{u(x)}{u(x)}) = \Phi(x,1)$.

If $u(x) = 0$, then $x \in A$ and $\Phi(x,t) = \Phi(x,1)$, for all $t \in I$.

Hence, $\bar{\Phi}$ is well-defined.

Let $F = \{(x,t) \in X \times I | t \geq u(x)\}$ and

$G = \{(x,t) \in X \times I | t \leq u(x)\}$

Now, $\bar{\Phi}|_F = \Phi(x,1)$ and hence $\bar{\Phi}|_F$ is continuous.

We now show that $\bar{\Phi}|_G$ is continuous.

Case 1: Let $x \in X - A$. Then $u(x) \neq 0$ and so $\bar{\Phi}|_G (x,t) = \Phi(x,\frac{t}{u(x)})$

Hence, $\bar{\Phi}$ is continuous at (x,t).

Case 2: Let $a \in A$. Then $u(a) = 0$ and so $(a,0) \in G$. We claim $\bar{\Phi}|_G$ is continuous at $(a,0)$ for all $(a,0) \in A \times 0$.

Now, $\bar{\Phi}|_G (a,0) = \Phi(a,0) = f(a)$.

Let V be a neighbourhood of $f(a)$ in Z.

Since Φ is continuous at (a,t), $\exists$ neighbourhoods U_t of a in X

and R_t of $t \in I$ such that $\Phi(U_t \times R_t) \subseteq V$.

Since I is compact, there exist finitely many $t_0, t_1, \dots, t_m \in I$) such that $I = \bigcup_{k=0}^{m} R_{t_k}$.

Let U be the intersection of the corresponding finite number of neighbourhoods U_{t_k}; that is, $U = \bigcap_{k=0}^{m} U_{t_k}$. Then, U is a neighbourhood of a in X such that, for all $(a,t) \in U \times I$, $\Phi(U \times I) \subseteq V$.

Now, if $(a,t) \in (U \times I) \cap G$, then $t = 0$ and $\bar{\Phi}(a,0) = \Phi(a,0) \in \Phi(U \times I) \subseteq V$.

Therefore, $(U \times I) \cap G$ is a neighbourhood of $(a,0)$ in G such that $\bar{\Phi}|_G((U \times I) \cap G) \subseteq V$.

Therefore, $\bar{\Phi}|_G$ is continuous at $(a,0)$, for all $(a,0) \in A \times 0$.

Hence, combining cases (1) and (2) we have that $\bar{\Phi}|_G$ is continuous.

Now, $\bar{\Phi}$ is continuous on each of the closed sets F and G, and on their intersection where $t = u(x)$, $\bar{\Phi}(x,t)$ has the unique value $\Phi(x,1)$. Thus Φ is continuous by Theorem 1.2.1.

Finally, $\bar{\Phi}(x,0) = \Phi(x,0) = f$

$$\bar{\Phi}(x,1) = \Phi(x,1) = g$$

$$\bar{\Phi}(a,t) = \Phi(a,t) = f(a) = g(a), \; a \in A, \; t \in I$$

Also, for $t \geq u(x)$,

$$\bar{\Phi}(x,t) = \Phi(x,1) = \bar{\Phi}(x,u(x)), \text{ as required.}$$

<u>Theorem 3.1</u>: Assume $A \to X$ has the H.E.P. with respect to Z and let B be a subspace of X. Assume also that there exists a map $u: X \to I$ with $A \subseteq u^{-1}(0)$ and $\left(u\big|_B\right)^{-1}(0) = A \cap B$. If $B \times I \to X \times I$ and $(A \cap B) \times I \to B \times I$ have the H.E.P. with respect to Z, then $A \cup B \to X$ has the H.E.P. with respect to Z.

<u>Proof</u>: Given the commutative diagram

(*)

$$\begin{tikzcd}
 & & Z \\
X \times 0 \arrow[r] \arrow[urr, bend left, "f"] & X \times I \arrow[ur, dashed] & \\
(A \cup B) \times 0 \arrow[r] \arrow[u] & (A \cup B) \times I = (A \times I) \cup (B \times I) \arrow[u] \arrow[uur, bend right, "\varphi"'] &
\end{tikzcd}$$

Construct the following diagram

$$\begin{tikzcd}
 & & Z \\
X \times 0 \arrow[r] \arrow[urr, bend left, "f"] & X \times I \arrow[ur, dashed, "\Phi"] & \\
A \times 0 \arrow[r] \arrow[u] & A \times I \arrow[u] \arrow[uur, bend right, "\varphi|_{A \times I}"'] &
\end{tikzcd}$$

Since the diagram (*) commutes, it follows that $f\big|_{A\times 0} = \varphi\big|_{A \times 0}$.

Now, since $i:A \to X$ has the H.E.P. with respect to Z, there exists a map $\Phi:X \times I \to Z$ such that

$$\Phi\big|_{X\times 0} = f \quad\text{and}\quad \Phi\big|_{A\times I} = \varphi\big|_{A\times I}$$

Define maps $\Phi':X \times I \times 0 \to Z$ by

$$\Phi'(x,s,0) = \Phi(x,s)$$

$$\varphi':B \times I \times 1 \to Z \quad\text{by}$$

$$\varphi'(b,s,1) = \varphi(b,s)$$

$$F:X \times 0 \times I \to Z \quad\text{by}$$

$$F(x,0,t) = f(x,0)$$

and $\psi:(A \cap B) \times I \times I \to Z$ by

$$\psi(a,s,t) = \psi(a,s,0) = \varphi(a,s).$$

Now construct the following commutative diagram

$$\begin{tikzcd}
 & & Z \\
B \times 0 \times I \arrow[urr, bend left, "F"] \arrow[r] & B \times I \times I \arrow[ur, dashed, "\Psi"] & \\
(A\cap B)\times 0\times I \cup B\times 0\times \dot{I} \arrow[u] \arrow[r] & (A\cap B)\times I\times I \cup B\times I\times 0 \cup B\times I\times 1 \arrow[u] \arrow[uur, bend right, "\psi\cup\Phi'\cup\varphi'"'] &
\end{tikzcd}$$

By Lemma 3.1, there exists $\Psi: B \times I \times I \to Z$ such that

$$\Psi(b,s,0) = \Phi'(b,s,0) = \Phi(b,s)$$

$$\Psi(b,s,1) = \varphi'(b,1,s) = \varphi(b,t)$$

$$\Psi(b,0,t) = F(b,0,t) = f(b,0)$$

$$\Psi(a,s,t) = \psi(a,s,t) = \psi(a,s,0) = \varphi(a,s).$$

This implies that $\Psi: \Phi \simeq \varphi$ rel $A \cap B$

Now define $u': B \times I \to I$ by

$$u'(b,s) = u(b)$$

Then $(u')^{-1}(0) = \left(u\big|_B\right)^{-1}(0) \times I = (A \cap B) \times I$ and so $(A \cap B) \times I$

is a Nullstellan set.

Hence, by Lemma 3.2, we can deform ψ to $\tilde{\psi}$ such that

$\tilde{\Psi}: \Phi \simeq \varphi$ rel $A \cap B$ with

$$\tilde{\Psi}(b,s,t) = \tilde{\Psi}(b,s,u'(b,s))$$

$$= \tilde{\Psi}(b,s,u(b)) = \Psi(b,s,1), \text{ for } (b,s) \in B \times I$$

$$\text{and } t \geq u'(b,s) = u(b)$$

Now construct the following commutative diagram

$$\begin{tikzcd}
 & & Z \\
X \times I \times 0 \arrow[rr] \arrow[urr, bend left, "\Phi'"] & & X \times I \times I \arrow[u, dashed, "\Omega"] \\
{[(B\times I)\cup(X\times 0)]\times 0} \arrow[u] \arrow[rr] & & {[(B\times I)\cup(X\times 0)]\times I} \arrow[u] \arrow[uu, bend right, "\tilde{\Psi}\cup F"']
\end{tikzcd}$$

Since $B \times I \to X \times I$ has the H.E.P. with respect to Z, it follows by Lemma 3.1 that $(B \times I) \cup (X \times 0) \to X \times I$ has the H.E.P. with respect to Z and so there exists a map $\Omega: X \times I \times I \to Z$ such that

$$\Omega(x,s,0) = \Phi'(x,s,0) = \Phi(x,s)$$

$$\Omega(x,0,t) = F(x,0,t) = f(x,0)$$

and $\Omega(b,s,t) = \tilde{\psi}(b,s,t)$, $b \in B$

Finally, define $H: X \times I \to Z$ by

$$H(x,s) = \Omega(x,s,u(x))$$

Then,

$$\begin{aligned} H(x,0) &= \Omega(x,0,u(x)) \\ &= F(x,0,u(x)) \\ &= f(x,0) \end{aligned}$$

and
$$\begin{aligned} H(b,s) &= \Omega(b,s,u(b)) \\ &= \tilde{\psi}(b,s,u(b)) \\ &= \psi(b,s,1) \\ &= \varphi(b,s) \end{aligned}$$

Also,

$$\begin{aligned} H(a,s) &= \Omega(a,s,u(a)) \\ &= \Omega(a,s,0) \end{aligned}$$

$$= \Phi(a,s)$$

$$= \varphi(a,s)$$

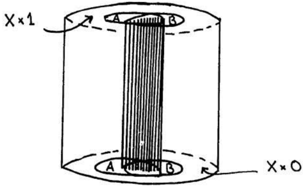
Therefore, $H: X \times I \to Z$ as defined above, is the required map making (*) commute.

Therefore, $(A \cup B) \to X$ has the H.E.P. with respect to Z.

Given two subspaces A and B of X, we define an equivalence relation $\sim$ in $X \times I$ by identifying (x,t) and $(x,0)$ for $t \in I$ and $x \in A \cap B$.

That is,

$$[(x,t)] = \begin{cases} (x,t) & \text{if } x \notin A \cap B \\ \{(x,t) \mid t \in I\} & \text{if } x \in A \cap B \end{cases}$$



Let $\tilde{X} = X \times I/\sim$. Observe that $\tilde{X}$ is the pushout of the diagram

$$\begin{array}{ccc} X \times I & & \\ \uparrow & & \\ (A \cap B) \times I & \longrightarrow & A \cap B \end{array}$$

Let $\pi:\tilde{X} \to X$ be the projection map; i.e. $\pi[x,t] = x$.

Definition 3.2: Let $\sim$ be the equivalence relation defined above. We call two subspaces A and B of X separated if there exists a continuous map $j:X \to \tilde{X}$ such that $\pi \cdot j = 1x$ and $j(x) = [x,0]$ for $x \in A$, $j(x) = [x,1]$ for $x \in B$.

We now give several criteria for the separation of two subspaces of a space X and eventually show that closed cofibrations are separated.

Lemma 3.3:

(a) Given subspaces A and B of X and a map $u:X - (A \cap B) \to I$ with $A - (A \cap B) \subseteq u^{-1}(0)$ and $B - (A \cap B) \subseteq u^{-1}(1)$, then A and B are separated.

(b) (i) If A and B are Nullstellen sets, then a map u exists satisfying the hypothesis in (a). In particular, if $A \to X$ and $B \to X$ are closed cofibrations, then A and B are separated.

(ii) If $\bar{A}$ and $\bar{B}$ are Nullstellen sets and if $\mathrm{Fr}A \cap \mathrm{Fr}B \subseteq A \cap B$, then a map u exists satisfying the hypothesis in (a). In particular, if $A \to X$ and $B \to X$ are cofibrations and if $\mathrm{Fr}A \cap \mathrm{Fr}B \subseteq A \cap B$, then A and B are separated. (Here, $\mathrm{Fr}A$ denotes the frontier of A; i.e. $\mathrm{Fr}A = \bar{A} \cap \overline{(X - A)}$.)

Proof:

(a) Define $j:X \to \tilde{X}$ by

$$j(x) = \begin{cases} [x,u(x)], & \text{for } x \notin (A \cap B) \\ [x,0] = [x,t], & \text{for } x \in (A \cap B),\ t \in I \end{cases}$$

We claim j is continuous.

Define $\bar{u}:\overline{X - (A \cap B)} \to I$ by

$$\bar{u} = u \text{ on } X - (A \cap B) \text{ and if}$$

$\lim x_\lambda = x \in A \cap B$ where $(x_\lambda) \in X - (A \cap B)$ is a net, then $\bar{u}(x) = \lim u(x_\lambda) = \lim \bar{u}(x_\lambda)$. Hence $j|_{\overline{X-(A\cap B)}}$ is the following composite $\overline{X - (A \cap B)} \xrightarrow{(i,\bar{u})} X \times I \xrightarrow{q} \tilde{X}$ which is continuous.

Moreover, $j|_{\overline{A\cap B}}:\overline{A \cap B} \to \tilde{X}$ is continuous since clearly $j|_{A\cap B}$ is continuous and if $\lim x_\lambda = x \in X - (A \cap B)$ where $(x_\lambda) \in A \cap B$ is a net, then

$$\begin{aligned} \lim j(x_\lambda) = \lim [x_\lambda, 0] &= \lim [x_\lambda, u(x)] \\ &= [x,u(x)] \\ &= j(x) \end{aligned}$$

Since $X = \overline{X - A \cap B} \cup \overline{A \cap B}$ and $j|_{\overline{A\cap B}}$ and $j|_{\overline{X-(A\cap B)}}$ are continuous, it follows by the Map Glueing Theorem (Theorem 1.2.1) that j is continuous.

Now,

$$\pi \cdot j(x) = \begin{cases} \pi([x,u(x)]), & x \notin A \cap B \\ \pi([x,0]), & x \in A \cap B \end{cases}$$

$$= x \text{ for all } x \in X$$

That is, $\pi \cdot j = 1_X$.

Suppose $x \in A$.

<u>Case 1</u>: $X \in A \cap B$

Then $j(x) = [x,0]$ by definition of j.

<u>Case 2</u>: $x \in A - (A \cap B)$

Then $j(x) = [x,u(x)]$

But $x \in A - (A \cap B) \Rightarrow u(x) = 0$ as $A - A \cap B \subseteq u^{-1}(0)$

Therefore $j(x) = [x,0]$.

In either case, $j(x) = [x,0]$ for $x \in A$.

Suppose $x \in B$.

<u>Case 1</u>: $x \in A \cap B$

Then $j(x) = [x,0] = [x,t]$, $t \in I$

In particular, $j(x) = x,1]$.

<u>Case 2</u>: $x \in B - (A \cap B)$

Then $j(x) = [x,u(x)]$, as $x \notin (A \cap B)$

$= [x,1]$, as $B - (A \cap B) \subseteq u^{-1}(1)$.

In either case, $j(x) = [x,1]$, for $x \in B$.

Thus, A and B are separated.

(b) (i) This is a special case of case (ii). To see this, A and B are Nullstellen sets implies that there exist maps $u:X \to I$ and $v:X \to I$ such that $u^{-1}(0) = A$ and $v^{-1}(0) = B$. Since $u^{-1}(0)$ and $v^{-1}(0)$ are closed in X, we have that $A = \overline{A}$ and $B = \overline{B}$ and so $\overline{A}$ and $\overline{B}$ are Nullstellen sets. Now, $Fr(A) = \overline{A} \cap \overline{(X - A)} \subseteq \overline{A} = A$ and $Fr(B) = \overline{B} \cap \overline{(X - B)} \subseteq \overline{B} = B$ and so $Fr(A) \cap Fr(B) \subseteq$

$A \cap B$. Hence, by case (ii) there exists a map u satisfying the hypothesis in (a). Therefore, A and B are separated.

Now, if $A \to X$ and $B \to X$ are closed cofibrations, then by Remark 2.2 (ii), A and B are Nullstellen sets and hence from above it follows that A and B are separated. Therefore, closed cofibrations are separated.

(ii) $\bar{A}$ and $\bar{B}$ are Nullstellen sets => maps $\lambda, \mu : X \to I$ such that $\bar{A} = \lambda^{-1}(0)$ and $\bar{B} = \mu^{-1}(0)$.

Define $u : X - (A \cap B) \to I$ by

$$u(x) = \begin{cases} \dfrac{\lambda(x)}{\lambda(x) + \mu(x)}, & \text{for } x \notin \bar{A} \cap \bar{B} \\ 1, & \text{for } x \in (\bar{A} - A) \cap \dot{B} \\ 0, & \text{for } x \in (\bar{B} - B) \cap \dot{A} \end{cases}$$

We claim that u is continuous.

Observe that
$$\begin{aligned} \bar{A} \cap \bar{B} &= (\dot{A} \cup \mathrm{Fr}A) \cap (\dot{B} \cup \mathrm{Fr}B) \\ &= (\dot{A} \cap \dot{B}) \cup (\dot{A} \cap \mathrm{Fr}B) \cup (\dot{B} \cap \mathrm{Fr}A) \cup (\mathrm{Fr}A \cap \mathrm{Fr}B) \end{aligned}$$

Since $\dot{A} \cap \dot{B} \subseteq A \cap B$ and $\mathrm{Fr}A \cap \mathrm{Fr}B \subseteq A \cap B$, it follows that
$$\begin{aligned} (\bar{A} \cap \bar{B}) - (A \cap B) &= [(\dot{A} \cap \mathrm{Fr}B) - (A \cap B)] \cup [(\dot{B} \cap \mathrm{Fr}A) - (A \cap B)] \\ &= \dot{A} \cap (\bar{B} - B) \cup \dot{B} \cap (\bar{A} - A) \end{aligned}$$

Therefore, $X - (A \cap B) = X - (\bar{A} \cap \bar{B}) \cup (\bar{A} - A) \cap \dot{B} \cup (\bar{B} - B) \cap \dot{A}$.
Now, $X - (\bar{A} \cap \bar{B}),\ \dot{A},\ \dot{B} \subseteq X$ are open. So, $X - (\bar{A} \cap \bar{B}),\ \dot{A} - (A \cap B)$, $\dot{B} - (A \cap B) \subseteq X - (A \cap B)$ are open and $X - (A \cap B) = X - (\bar{A} \cap \bar{B}) \cup (\dot{A} - A \cap B) \cup (\dot{B} - A \cap B)$. Now define $\eta : X - (\bar{A} \cap \bar{B}) \to I$ by

$$\eta(x) = \frac{\lambda(x)}{\lambda(x) + \mu(x)}$$

$$C_0:\dot{A} - A \cap B \to I \quad \text{by} \quad C_0(x) = 0, \text{ for all } x.$$

$$C_1:\dot{B} - A \cap B \to I \quad \text{by} \quad C_1(x) = 1, \text{ for all } x.$$

Clearly, η, C_0 and C_1 are continuous and $\eta \cup C_0 \cup C_1 : X - A \cap B \to I$ is continuous since the maps η, C_0 and C_1 agree on overlaps of open sets (see comment following Theorems 1.2.1). For example if $x \in [X - (\bar{A} \cap \bar{B})] \cap \dot{A} - (A \cap B)$ then $x \in \dot{A}$, $x \notin \bar{B}$, i.e. $x \in \bar{A}$, $x \notin \bar{B}$, so $\eta(x) = \dfrac{0}{0 + \mu(x)} = 0$ and $C_0(x) = 0$. Note that $(\dot{A} - A \cap B) \cap (\dot{B} - A \cap B) = \emptyset$. Finally, the continuity of u follows by observing that $u = \mu \cup C_0 \cup C_1$.

We now show that the map $u:X - (A \cap B) \to I$ satisfies the hypothesis in (a).

Let $x \in B - (A \cap B)$. Then, $x \in B$ and $x \notin A$. So, $x \in (\bar{A} - A) \cap \dot{B}$ or $X \notin (\bar{A} \cap \bar{B})$.

If $x \in (\bar{A} - A) \cap \dot{B}$, then $u(x) = 1$ by definition of u.

If $x \notin (\bar{A} \cap \bar{B})$, then since $x \in B$ and hence $x \in \bar{B}$, it follows that $x \notin \bar{A}$. Consequently, $\lambda(x) \neq 0$ and $\mu(x) = 0$. Therefore, $u(x) = \dfrac{\lambda(x)}{\lambda(x) + \mu(x)} = \dfrac{\lambda(x)}{\lambda(x) + 0} = 1$. In either case, $u(x) = 1$. That is, $B - (A \cap B) \subseteq u^{-1}(1)$. A similar argument works for $A - (A \cap B)$, that is, $A - (A \cap B) \subseteq u^{-1}(0)$. Hence, the map u defined above satisfies the hypothesis in (a) and so A and B are separated.

Suppose now that $A \to X$ and $B \to X$ are cofibrations such that $\mathrm{Fr}A \cap \mathrm{Fr}B \subseteq A \cap B$. Then by Theorem 2.6, $\bar{A} \to X$ and $\bar{B} \to X$ are closed cofibrations and so by Remark 2.2(ii), $\bar{A}$ and $\bar{B}$ are Nullstellen sets with $\mathrm{Fr}A \cap \mathrm{Fr}B \subseteq A \cap B$. Therefore by Lemma 3.3(b)(ii) above, A and B are separated.

<u>Lemma 3.4</u>: Let A be a subspace of X such that $A \times I \subseteq X \times I$ has the H.E.P. with respect to Z. Let $K, L: X \times I \to Z$ be homotopies with $K_0 = K(-,0) = L(-,0) = L_0$ and $K|_{A\times I} = L|_{A\times I}$. Then there exists a homotopy $\Phi: K \simeq L$ rel $(A \times I) \cup (X \times 0)$.

<u>Proof</u>: Define $g: (X \times I \times \dot{I}) \cup (X \times 0 \times I) \to Z$ by

$$g(x,s,t) = \begin{cases} K(x,s) & \text{for } t = 0 \text{ or } s = 0 \\ L(x,s) & \text{for } t = 1 \end{cases}$$

and $\psi: A \times I \times I \to Z$ by

$$\psi(a,s,t) = K(a,s) = L(a,s), \text{ for al } t \in I.$$

Since g is defined by continuous maps on closed subspaces, and on the overlaps $X \times 0 \times 0$ and $X \times 0 \times 1$ these maps agree, that is, $g(x,0,0) = K(x,0)$ and $g(x,0,1) = K(x,0) = L(x,0)$, g is continuous by the Map Glueing Theorem (see Theorem 1.2.1). Clearly, ψ is continuous.

Consider now the following commutative diagram

$$\begin{tikzcd}
 & & \mathcal{Z} \\
X \times 0 \times I \arrow[r] \arrow[urr, bend left, "g|_{X\times 0 \times I}", "Z" near end] & X \times I \times I \arrow[ur, dashed] & \\
A \times 0 \times I \cup X \times 0 \times \dot{I} \arrow[u] \arrow[r] & A \times I \times I \cup X \times I \times \dot{I} \arrow[u] \arrow[uur, bend right] &
\end{tikzcd}$$

Since $A \times I \to X \times I$ has the H.E.P. with respect to $\mathcal{Z}$, Lemma 3.1 implies the existence of a map $\Phi: X \times I \times I \to Z$ such that $\Phi|_{X\times 0\times I} = g|_{X\times 0\times I}$ and $\Phi|_{A\times I\times I\cup X\times I\times \dot{I}} = \psi \cup g|_{X\times I\times \dot{I}}$, that is, $\Phi|_{A\times I\times I} = \psi$ and $\Phi|_{X\times I\times \dot{I}\cup X\times 0\times I} = g$. We show that $\Phi: K \simeq L$ rel $(A \times I) \cup (X \times 0)$.

First, $\Phi(x,s,0) = g(x,s,0) = K(x,s)$ and

$\Phi(x,s,1) = g(x,s,1) = L(x,s)$>

Now, let $(a,s) \in A \times I$. Then,

$$\Phi(a,s,t) = \psi(a,s,t) = K(a,s) = L(a,s),\ t \in I.$$

Let $(x,o)\ \rho\ X \times 0$. Then,

$$\Phi(x,o,t) = g(x,o,t)$$

$$= K(x,o) = L(x,o)$$

Therefore, $\Phi: K \simeq L$ rel $(A \times I) \cup (X \times 0)$, as required.

Theorem 3.2: Assume that $A \to X$, $B \to X$ have the H.E.P. with respect to Z. If $(A \cap B) \times I \to X \times I$ has the H.E.P. with respect to Z and A and B are separated, then $A \cup B \to X$ has the H.E.P. with respect to Z.

Proof: Let

$$\begin{tikzcd}
 & & Z \\
X \times 0 \arrow[r] \arrow[urr, bend left] & X \times I \arrow[ur, dashed] & \\
(A \cup B) \times 0 \arrow[u] \arrow[r] & (A \cup B) \times I \arrow[u] \arrow[uur, bend right] &
\end{tikzcd} \qquad (*)$$

be a commutative diagram, where f and φ are given maps such that $\varphi\big|_{(A\cup B)\times 0} = f\big|_{(A\cup B)\times 0}$. But $A \to X$ and $B \to X$ have the H.E.P. with respect to Z; that is, the following diagrams are commutative,

$$\begin{tikzcd}
 & & Z \\
X \times 0 \arrow[r] \arrow[urr, bend left] & X \times I \arrow[ur, dashed, "\Phi_A"] & \\
A \times 0 \arrow[u] \arrow[r] & A \times I \arrow[u] \arrow[uur, bend right, "\varphi|_{A\times I}"'] &
\end{tikzcd} \qquad
\begin{tikzcd}
 & & Z \\
X \times 0 \arrow[r] \arrow[urr, bend left] & X \times I \arrow[ur, dashed, "\Phi_B"] & \\
B \times 0 \arrow[u] \arrow[r] & B \times I \arrow[u] \arrow[uur, bend right, "\varphi|_{B\times I}"'] &
\end{tikzcd}$$

and hence maps $\Phi_A : X \times I \to Z$ and $\Phi_B : X \times I \to Z$ such that $\Phi_A\big|_{A\times I} = \varphi\big|_{A\times I}$, $\Phi_A(-,0) = f$ and $\Phi_B\big|_{B\times I} = \varphi\big|_{B\times I}$, $\Phi_B(-,0) = f$.

Now, $A \cap B \subseteq A$ and $A \cap B \subseteq B$ and hence

$$\Phi_A\Big|_{(A\cap B)\times I} = \varphi\Big|_{(A\cap B)\times I} = \Phi_B\Big|_{(A\cap B)\times I}.$$

Since $(A \cap B) \times I \to X \times I$ has the H.E.P. with respect to Z, there exists, by Lemma 3.4, a homotopy $\psi:\Phi_A \simeq \Phi_B$ rel $((A \cap B) \times I) \cup (X \times 0))$. By hypothesis, we have a continuous map $j:X \to \tilde{X}$ with

$j(x) = [x,0]$, $x \in A$ and

$j(x) = [x,1]$, $x \in B$.

Now consider the following diagram

$$\begin{tikzcd}
X \times I \times I \arrow[r, "1_X \times T"] \arrow[d, "p \times 1_X"'] & X \times I \times I \arrow[r, "\psi"] & Z \\
\tilde{X} \times I \arrow[urr, dashed, "\Omega"] & &
\end{tikzcd}$$

where $p:X \times I \to \tilde{X}$ is the identification map and $T:I \times I \to I \times I$ switches the factors. The map $\psi \cdot (1_X \times T)$ factors through $p \times 1_I$ and hence by the universal property of quotients, it induces a map $\Omega:\tilde{X} \times I \to Z$ such that

$$\Omega \cdot (p \times 1_I) = \psi \cdot (1_X \times T).$$

Now define

$$\Omega \cdot (j \times id):X \times I \to Z$$

Then,
$$\begin{aligned}
\Omega \cdot (j \times id)(x,0) &= \Omega(j(x),0) \\
&= \Omega([x,t],0) \\
&= \Omega(p(x,t),0)
\end{aligned}$$

$$
\begin{aligned}
&= \Omega((p \times 1_I)(x,t,0)) \\
&= \psi \cdot (1_X \times T)(x,t,0) \\
&= \psi(x,T(t,0)) \\
&= \psi(x,0,t) \\
&= \Phi_A(x,0) = \Phi_B(x,0) \\
&= f
\end{aligned}
$$

Similarly, $\Omega \cdot (j \times id)(x,t) = \varphi(x,t)$, $x \in A \cup B$. Therefore, $\Omega \cdot (j \times id)$ is the required diagram filler in (*) and so $A \cup B \to X$ has the H.E.P. with respect to Z.

We now reformulate Theorem 3.1 and Theorem 3.2 in terms of cofibrations and obtain the following important results on cofibrations.

Theorem 3.3: (Union Theorems) Let $A \to X$ and $B \to X$ be cofibrations. Suppose

either (a) $A \cap B \to B$ is a cofibration and $\bar{A} \cap B = A \cap B$

or (b) $A \cap B \to X$ is a cofibration and A, B are separated.

Then $A \cup B \to X$ is a cofibration.

Proof:

(a) Since $A \to X$ is a cofibration, $\bar{A} \to X$ is a closed cofibration by Theorem 2.6, and so by Remark 2.2(ii), there exists a map $\varphi: X \to I$ with $\bar{A} = \varphi^{-1}(0)$.

Consequently, $\left(\varphi\big|_B\right)^{-1}(0) = \bar{A} \cap B$
$= A \cap B$, by hypothesis.

Now, $B \to X$ and $A \cap B \to B$ are cofibrations imply that $B \times I \to X \times I$ and $(A \cap B) \times I \to B \times I$ are cofibrations by

Corollary 2.9 and hence have the H.E.P. with respect to every space Z. Therefore, by Theorem 3.1, $A \cup B \to X$ is a cofibration.

(b) Since $A \to X$ and $B \to X$ are cofibrations, A and B have the H.E. P. with respect to every space Z. Moreover, $A \cap B \to X$ is a cofibration implies that $(A \cap B) \times I \to X \times I$ is a cofibration and hence has the H.E.P. with respect to every space Z. By hypothesis, A and B are separated and so it follows from Theorem 3.2 that $A \cup B \to X$ has the H.E.P. with respect to every space Z. Therefore, $A \cup B \to X$ is a cofibration.

The following are easy consequences of the Union Theorems for cofibrations:

Theorem 3.4:

(a) If $A \to X$ and $B \to X$ are closed cofibrations and if $A \cap B \to X$ is a cofibration, then $A \cup B \to X$ is a cofibration.

(b) Let $A_1 \to X, \ldots, A_n \to X$ be closed cofibrations. For each subset $\sigma \subseteq \{1, 2, \ldots, n\}$, let $A_\sigma = \bigcap_{\ell \in \sigma} A_\ell \subseteq X$ be a cofibration. The, $\bigcup_{\ell=1}^{m} A_\ell \to X$ is a cofibration.

Proof:

(a) If $A \to X$ and $B \to X$ are closed cofibrations, then A and B are separated by Lemma 3.3(b)(i). Since $A \cap B \to X$ is a cofibration by hypothesis, it follows that $A \cup B \to X$ is a cofibration by Theorem 3.3(b).

(b) Follows by induction.

<u>Remark 3.1</u>: Theorem 3.4(b) does not hold in general for countably many cofibrations. To see this, let $X = I$, and $A_\ell = \{0, 1/\ell\}$. $\ell = 1, 2, \ldots$ and $A = \bigcup_{\ell=1}^{\infty} A_\ell = \{0\} \cup \left\{\frac{1}{\ell} \middle| \ell = 1, 2, \ldots\right\}$.

Clearly, the set A is closed in X. Now, for each $\ell \in \{1, 2, \ldots\}$, $A_\ell = \{0, 1/\ell\} \cong \{0, 1\} \to [0,1] = X$ and hence the inclusion maps $A_\ell \to X$ are closed cofibrations by Example (2.1). But we have seen in Example (2.6) that $A \to X$ is not a cofibration.

<u>Example 3.1</u>: If $a_0 \in A$ and $b_0 \in B$ are non-degenerate base points (i.e. $\{a_0\} \to A$ and $\{b_0\} \to B$ are closed cofibrations) and $A \vee B = A \times \{b_0\} \cup \{a_0\} \times B$ (called the <u>wedge</u> of A and B), then $A \vee B \to A \times B$ is a cofibration. This follows from Theorem 3.4(a) since $A \times \{b_0\} \to A \times B$, $\{a_0\} \times B \to A \times B$ and $A \times \{b_0\} \cap \{a_0\} \times B = \{a_0\} \times \{b_0\} \to A \times B$ are closed cofibrations by Corollary 2.9.

CHAPTER IV

Further Results on Cofibrations

This chapter is devoted to a recent theorem of Kieboom (see [10]) and related results. In particular, some of the well known classical results of Ström [15] are retrieved as special cases of Kieboom's Theorem, thus avoiding the technicalities of local arguments given by Strom. But first, we give a preliminary definition which is essential to Kieboom's Theorem.

Definition 4.1: A map $i:p_A \to p_X$ in Top_B is said to be cofibration over B

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr, "p_A"'] & & X \arrow[dl, "p_X"] \\ & B & \end{tikzcd}$$

if there exists a fibre retraction of the canonical inclusion

$$\sigma(i):\ \{M_i = X \times 0 \sqcup_i A \times I \xrightarrow{q(i)} B\} \to \{X \times I \xrightarrow{p_X pr_X} B\}$$

of the mapping cylinder over B; that is the dotted arrow exists in the diagram

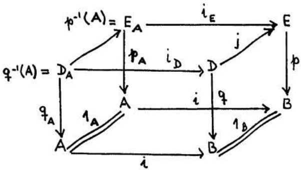
$$\begin{tikzcd} X \times 0 \sqcup_i A \times I \arrow[rr] \arrow[d, "\sigma(i)"'] & & X \times 0 \sqcup_i A \times I \arrow[d, "q(i)"] \\ X \times I \arrow[rr, "p_X pr_X"'] \arrow[urr, dashed] & & {} \end{tikzcd}$$

such that the resulting triangles commute.

Remark 4.1: If $i:A \to X$ is a closed cofibration in Top and if further p_A and p_X are Hurewicz Fibrations, then $i:p_A \to p_X$ is a cofibration over B (see [7; Theorem 1.3]).

We now prove the main theorem in this chapter. It is due to Kieboom (see [10; Theorem 1]).

Kieboom's Theorem 4.1: Consider the following diagram in Top.



in which i and j are inclusions, $D_A = q^{-1}(A)$ and $E_A = p^{-1}(A)$. The other maps i_E, i_D, j_A, p_A and q_A are induced by i, j, p and q, respectively.

If

(a) i is a closed cofibration
p is a fibration and
j is a cofibration over B

OR

(b) i is a cofibration
p is a regular fibration and
j is a closed cofibration over B

then $E_A \cup D \to E$ is a cofibration.

Proof:

(a) $j:D \to E$ is a cofibration over $B \Rightarrow \exists$ a retraction $r:E \times I \to E \times 0 \cup D \times I$ over B, that is, the following triangles commute

$$\begin{tikzcd}
E \times 0 \sqcup_i D \times I \arrow[rr] \arrow[d, "\sigma(j)"'] & & E \times 0 \sqcup_i D \times I \arrow[d, "q(j)"] \\
E \times I \arrow[rr, "p\,pr_E"'] \arrow[urr, dashed, "r"] & & B
\end{tikzcd}$$

and hence $q(j)r(e,t) = p(e)$ for all $(e,t) \in E \times I$. Note that,

$d \in D \cap E_A \Leftrightarrow d \in D$ and $d \in E_A = p^{-1}(A)$

$\Leftrightarrow d \in D$ and $p(d) = q(d) \in A$

$\Leftrightarrow d \in q^{-1}(A) = D_A$

and so $D \cap E_A = D_A$.

Now, for all $(e,t) \in E_A \times I$, $q(j)r(e,t) = p(e) \in A$ and so

$r(e,t) \in (E \cap E_A) \times 0 \cup (D \cap E_A) \times I$

$\Rightarrow r(e,t) \in E_A \times 0 \cup D_A \times I$

$\Rightarrow r$ restricts to a retraction $r_A:E_A \times I \to E_A \times 0 \cup D_A \times I$

$\Rightarrow j_A:D_A \to E_A$ is a cofibration.

Now, consider the following diagram

$$\begin{tikzcd}
p^{-1}(A) = E_A \arrow[r, "i_E"] \arrow[d, "p_A"'] & E \arrow[d, "p"] \\
A \arrow[r, "i"'] & B
\end{tikzcd}$$

where $i:A \to B$ is a closed cofibration and $p:E \to B$ is a fibration. By Theorem 2.2.4, it follows that $i_E:E_A \to E$ is a closed cofibration. But $i_E j_A = j i_D$ from (*) and $i_E j_A$ is a cofibration (composition of cofibrations) and so $j i_D$ is also a cofibration. Since j is a cofibration by hypothesis and $j i_D$ is a cofibration, it follows from Theorem 2.2.7 that $i_D:D_A \to D$ is also a cofibration.

Since $p:E \to B$ is continuous and A is a closed subspace of B, it follows that $E_A = p^{-1}(A)$ is a closed subspace of E.

Consequently, $\overline{E}_A = E_A$ and $\overline{E}_A \cap D = E_A \cap D$.

We now have
$$\left.\begin{array}{r} E_A \to E \\ D \to E \\ E_A \cap D = D_A \to D \end{array}\right\} \text{ are all cofibrations}$$
and $\overline{E}_A \cap D = E_A \cap D$

Therefore, by Lillig's Theorem 3.3(a), $E_A \cup D \to E$ is a cofibration.

(b) Since $p:E \to B$ is a regular fibration and $i:A \to B$ is a cofibration, it follows from Theorem 2.2.5 that $i_E:E_A \to E$ is a cofibration. Now, as in (a) above we have
$$\left.\begin{array}{r} D \to E \\ E_A \to E \\ D \cap E_A = D_A \to E_A \end{array}\right\} \text{ are all cofibrations}$$
and $\overline{D} \cap E_A = D \cap E_A$ (since j is a closed cofibration).

Therefore, $D \cup E_A \to E$ is a cofibration by Lillig's Theorem 3.3(a).

<u>Corollary 4.1</u>: If in diagram (*) of Theorem 4.1, i and j are closed cofibrations and p and q are fibrations, then $E_A \cup D \to E$ is a closed cofibration.

Proof: By Remark 4.1, j is a cofibration over B. Hence, by Theorem 4.1(a), $E_A \cup D$ is a closed cofibration.

The following theorem is a modified version of Ström's Theorem (see [16;13]).

Theorem 4.2: Let $i:A \to B$ be a closed cofibration, and $p:E \to B$ a fibration with $s:B \to E$ a section of p (i.e. $p.s. = 1_B$) such that $s(B) \to E$ is a closed cofibration. Then $E_A \cup s(B) \to E$ is a closed cofibration.

Proof: Consider the following diagram

$$\begin{tikzcd}
 & E_A \arrow[rr, "i_E"] \arrow[dd] & & E \arrow[dd, "p"] \\
D_A \arrow[ur] \arrow[rr, "i_D"] \arrow[dd, "q_A"'] & & s(B) \arrow[ur, "j"] \arrow[dd] & \\
 & A \arrow[rr, "i"] & & B \\
A \arrow[ur, equal, "1_A"] \arrow[rr, "i"'] & & B \arrow[ur, equal, "1_B"] &
\end{tikzcd}$$

Since $s:B \to E$ is a section of p, it follows that

$$q = p\big|_{s(B)} : s(B) \to B \text{ is a homemorphism}$$

and therefore $q = p\big|_{s(B)}$ is a fibration.

Now, $q^{-1}(A) = s(B) \cap E_A$. Hence, by Corollary 4.1, $E_A \cup s(B) \to E$ is a closed cofibration.

We now see how Theorem 4.1 is applied to retrieve Ström's Product Theorem (see [16; Theorem 6]).

Theorem 4.3: If (X,A) and (Y,B) are cofibred pairs with either A or B closed, then the product pair $(X,A) \times (Y,B) = (X \times Y, X \times B \cup A \times Y)$ is also cofibred.

Proof: Let $j:A \to X$ and $i:B \to Y$ be inclusions. Assume without any loss of generality that A is a closed subspace of X.

We construct the following commutative diagram

$$\begin{tikzcd}
 & p_Y^{-1}(B) = X \times B \arrow[rr, "1_X \times i"] \arrow[dd] & & X \times Y \arrow[dd, "p_Y"] \\
q_Y^{-1}(B) = A \times B \arrow[ur, "j \times 1_B"] \arrow[rr, "1_A \times i"] \arrow[dd] & & A \times Y \arrow[ur, "j \times 1_Y"] \arrow[dd, "q_Y"] & \\
 & B \arrow[rr, "i"] & & Y \\
B \arrow[ur, equal, "1_B"] \arrow[rr, "i"] & & Y \arrow[ur, equal, "1_Y"] &
\end{tikzcd}$$

By hypothesis, $i:B \to Y$ is a cofibration and p_Y is a regular fibration being the trivial fibration. Since $j:A \to X$ is a closed cofibration, it follows from Theorem 2.2.9 that $j \times 1_Y:A \times Y \to X \times Y$ is a closed cofibration. As p_Y and q_Y are Hurewicz Fibrations, it follows from Remark 4.1, that $j \times 1_Y$ is a closed cofibration over Y. Thus, by Theorem 4.1(b) we have that $X \times B \cup A \times Y \to X \times Y$ is a cofibration. The case that B is a closed subspace of Y is a consequence of Theorem 4.1(a). The verification is left for the reader.

Note that $(X \times Y,\ X \times B \cup A \times Y)$ need not be cofibred if neither A nor B is closed. To see this we consider the following example (see [2; page 81, Example 3.23]).

<u>Example 4.1</u>: Take $X = \{a,b\}$ where $a \neq b$ and $A = \{a\}$. Topologize X by taking ϕ, A and X as the open sets. Clearly, A is not a closed subspace of X and we have seen that $A \to X$ is a cofibration by Example 2.3.7.

Now take $B = A$ and $Y = X$. We will show that $C = (X \times A) \cup (A \times X) \to X \times X$ is not a cofibration.

Suppose $C \to X \times X$ is a cofibration. Then, by Remark 2.2.2(b), C has a halo V in $X \times X$ and a retraction $\sigma : V \to C$. Again, by Remark 2.2.2(a), V is also a halo of $\overline{C}$ in X. Since $\overline{A} = X$, it follows that $\overline{C} = X \times X$ and so we take $V = X \times X$.

Now, $b \in \overline{A} = \overline{\{a\}} \Rightarrow (b,b) \in \overline{\{a\}} \times \overline{\{b\}} = \overline{\{a\} \times \{b\}} = \overline{\{(a,b)\}}$

$\Rightarrow \sigma(b,b) \in \overline{\{\sigma(a,b)\}}$ by continuity of σ.

Now, $(a,b) \in C$ and $\sigma : V \to C$ is a retraction. Hence $\sigma(a,b) = (a,b)$ and so $\sigma(b,b) \in \overline{\{(a,b)\}}$. Notice that $\overline{\{(a,b)\}} = \overline{\{a\}} \times \overline{\{b\}} = \overline{\{a\}} \times \overline{\{b\}}$ as $\{b\}$ is closed in X and consequently, $\sigma(b,b) \in \overline{\{a\}} \times \{b\}$. Thus, $pr_X\, \sigma(b,b) = b$. But $\sigma(b,b) \in C$ and so we have $\sigma(b,b) = (a,b)$. By a symmetric argument, we obtain $\sigma(b,b) = (b,a)$ which then implies $a = b$ contrary to hypothesis. Therefore, $C = (A \times X) \cup (X \times A) \to X \times X$ is not a cofibration.

<u>Remark 4.2</u>: (X,A) is cofibred $\Rightarrow (X \times I,\ X \times 0 \cup A \times I \cup X \times 1)$ is cofibred. By Example 2.3.1 (the case $n = 1$), $\dot{I} \to I$ is a closed cofibration and hence by Theorem 4.3 $(X,A) \times (I,\dot{I}) = (X \times I,\ X \times 0 \cup A \times I \cup X \times 1)$ is a cofibred pair.

The following theorem is a type of converse to the product rule (Theorem 4.3).

Theorem 4.4: Suppose that for $A \subseteq X$, there exists a continuous function $\sigma: X \to I$ with $A \subseteq \sigma^{-1}(0)$, and that there exists a point $x_0 \in X - A$ such that $\sigma(x_0) \neq 0$. Then if (Y,B) is a pair such that $(X \times Y, X \times B \cup A \times Y)$ is cofibred, (Y,B) itself is cofibred.

Proof: Let $\eta: X \times Y \to I$ and $F: X \times Y \times I \to X \times Y$ be maps for $(X \times Y, X \times B \cup A \times Y)$ as described in the Characterization Theorem 2.2.2(e).

That is, $X \times B \cup A \times Y \subseteq \eta^{-1}(0)$ and

$F(x,y,0) = (x,y)$ for all $(x,y) \in X \times Y$

$F(r,s,t) = (r,s)$, $(r,s) \in X \times B \cup A \times Y$, $t \in I$

$F(x,y,t) \in X \times B \cup A \times Y$ whenever $t > \eta(x,y)$.

Let $\sigma(x_0) = \epsilon$, where $0 < \epsilon \leq 1$; and define $\psi: I \to I$ by $\psi(t) = t/\epsilon$. Then $\psi\sigma: X \to I$ is a map such that $\psi\sigma(a) = \psi(0) = 0$ and $\psi\sigma(x_0) = 1$. Hence, we may assume that $\sigma(x_0) = 1$. We now define functions $G: Y \times I \to Y$ and $\psi: Y \to I$ by $G(y,t) = pr_Y F(x_0,y,t)$ and $\psi(y) = \text{Max}(\eta(x_0,y), 1 - \inf_{t \in I} \sigma pr_X F(x_0,y,t)$.

Clearly, G is continuous as F is continuous and pr_Y is continuous. In the case of ψ, notice that $\eta\big|_{\{x_0\} \times Y}: Y \to I$ is continuous and the continuity of $\inf_{t \in I} \sigma pr_X F(x_0,y,t)$ is analogous to the continuity of the function σ defined in Characterization Theorem 2.2.2(e). So, ψ being the maximum of two continuous real valued functions is continuous. Furthermore,

$$\begin{aligned}\psi(b) &= \max(\eta(x_0,b),\ 1 - \operatorname*{Inf}_{t\in I} \sigma pr_X F(x_0,b,t)) \\ &= \max(\eta(x_0,b),\ 1 - \operatorname*{Inf}_{t\in I} \sigma pr_X (x_0,b)) \\ &= \max(\eta(x_0,b),\ 1 - \operatorname*{Inf}_{t\in I} \sigma(x_0)) \\ &= \max(\eta(x_0,b),0) \\ &= \eta(x_0,b), \text{ as } \eta(x_0,b) \geq 0 \\ &= 0, \text{ as } (x_0,b) \in X \times B \text{ and } X \times B \cup A \times Y \subseteq \eta^{-1}(0)\ .\end{aligned}$$

Therefore,

$$B \subseteq \psi^{-1}(0)\ .$$

Next, $G(y,0) = pr_Y F(x_0,y,0) = pr_Y(x_0,y) = y$.

$G(b,t) = pr_Y F(x_0,b,t) = pr_Y(x_0,b) = b$.

Suppose, $\eta(x_0,y) \geq 1 - \operatorname*{Inf}_{t\in I} \sigma pr_X F(x_0,y,t)$.

Then, $F(x_0,y,t) \in A \times Y \Rightarrow \eta(x_0,y) = 1$. Hence, if $t > \psi(y) = \eta(x_0,y)$, then $\eta(x_0,y) < 1$ and so $G(y,t) = pr_Y F(x_0,y,t) \in B$, since $F(x_0,y,t) \in X \times B$. A similar argument holds true for the case $1 - \operatorname*{Inf}_{t\in I} \sigma pr_X F(x_0,y,t) \geq \eta(x_0,y)$. Therefore, (Y,B) is cofibred by the Characterization Theorem 2.2.2(e).

<u>Corollary 4.2</u>: $(X.A)$ is cofibred $\Leftrightarrow$ $(X \times I,\ X \times 0 \cup A \times I)$ is cofibred.

Proof:

"=>": Suppose (X,A) is cofibred.

By Example 2.3.3, $(I, \{0\})$ is a closed cofibred pair. Hence, by Theorem 4.3, $X \times 0 \cup A \times I \to X \times I$ is a cofibration.

"<=": Suppose $X \times 0 \cup A \times I \to X \times I$ is a cofibration. Put $A = \{0\}$, $Y = X$, $B = A$ and $X = I$ in Theorem 4.4, and observe that since $0 \to I$ is a closed cofibration, there exists a map $\sigma : I \to I$ with $\sigma^{-1}(0) = 0$, as $\{0\}$ is closed in I and so there exists $x_0 \in I - \{0\}$, (i.e. $x_0 \neq 0$) such that $\sigma(x_0) \neq 0$. Therefore, by Theorem 4.4, (X,A) is cofibred.

Theorem 4.5: Let $i : A \to X$ be a cofibration. Then, i is a homotopy equivalence iff A is a strong deformation retract of X.

Proof: Suppose $i : A \to X$ is a homotopy equivalence. Then there exists $f : X \to A$ such that $fi \simeq 1_A$ and $if \simeq 1_X$. Consider the following diagram

$$\begin{tikzcd}
 & & A \\
X \arrow[urr, bend left, "f"] \arrow[r, "\beta_0"] & X \times I \arrow[ur, dashed, "\sigma"] & \\
A \arrow[u, "i"] \arrow[r, "\alpha_0"'] & A \times I \arrow[u, "i \times 1_I"'] \arrow[uur, bend right, "F"'] &
\end{tikzcd}$$

where $F:A \times I \to A$ is the homotopy fi to 1_A; that is, $F(-,0) = fi$ and $F(-,1) = 1_A$.

So, $F\alpha_0(a) = F(a,0) = fi(a)$; and therefore, $fi = F\alpha_0$. Since $i:A \to X$ is a cofibration, there exists $\sigma:X \times I \to A$ such that $\sigma\beta_0(x) = \sigma(x,0) = f(x)$ and $\sigma(i(a),t) = F(a,t)$. Define $r:X \to A$ by $r(x) = \sigma(x,1)$. Then, for all $a \in A$, $ri(a) = \sigma(i(a),1)$ $= F(a,1) = 1_A(a) = a$.

$\Rightarrow$ r is a retraction of X onto A and $\sigma:X \times I \to A$ is such that $\sigma(x,0) = f(x)$ and $\sigma(x,1) = r(x)$.

$\Rightarrow$ $f \simeq r$ (i.e. f is homotopic to a retract).

$\Rightarrow$ $if \simeq ir$

$\Rightarrow$ $1_X \simeq ir$

So, let $G:X \times I \to X$ be a homotopy from 1_X to ir. That is, $G(x,0) = 1_X$ and $G(x,1) = ir$.

Since $I \to I$ and $A \to X$ are cofibrations, so is their product $(X \times I,\ X \times 0 \cup A \times I \cup X \times 1)$ a cofibred pair by Remark 4.2.

Now, define a homotopy $H_*:(X \times 0 \cup A \times I \cup X \times 1) \times I \to X$ by the following equations

$$H_*((x,0),s) = x$$
$$H_*((a,t),s) = G(a,(1-s)t)$$
$$H_*((x,1),s) = G(r(x),1-s)$$

Now, for all $a \in A$,

$H_*((a,0),s) = a = G(a,0)$ by the first two equations and

$$H_*((a,1),s) = G(a,1-s)$$
$$= G(r(a),1-s) \text{ by}$$

the last two equations. Hence, H_* is well defined. We claim that H_* is continuous.

Since (X,A) is cofibred and $(I,\dot I)$ is a cofibred pair by Example 2.3.1 (the case $n = 1$), it follows by Theorem 2.2.9 that $(X \times I \times \dot I,\ A \times I \times 0)$ is cofibred and hence, by the Characterization Theorem 2.2.2(c) and Remark 2.2.2(a), we have that $X \times I \times \dot I \cup A \times I \times I \cong X \times \dot I \times I \cup A \times I \times I$ has the final topology with respect to the inclusions of the subspaces $X \times \dot I \times I$ and $A \times I \times I$. But the restrictions of H_* to each of the subspaces $X \times \dot I \times I$ and $A \times I \times I$ is clearly continuous. Hence, globally H_* is continuous.

Consider now the following diagram

$$\begin{tikzcd}
 & & X \\
X \times I \arrow[r] \arrow[urr, bend left, "G"] & X \times I \times I \arrow[ur, dashed, "H"] & \\
X \times 0 \cup A \times I \cup X \times 1 \arrow[r] \arrow[u] & (X \times 0 \cup A \times I \cup X \times 1) \times I \arrow[u] \arrow[uur, bend right, "H_*"'] &
\end{tikzcd}$$

Since
$$H_*(x,0,0) = x = G(x,0)$$
$$H_*(a,t,0) = G(a,t)$$
$$H_*(x,1,0) = G(r(x),1) = ir(r(x)) = ir(x) = G(x,1)$$

the diagram commutes and hence there exists a map $H: X \times I \times I \to X$ such that $H|_{X \times I} = G$ and $H|_{(X \times 0 \cup A \times I \cup X \times 1) \times I} = H_*$.

Define $\tilde{H}:X \times I \to X$ by

$$\tilde{H}(x,t) = H(x,t,1):X \times I \to X$$

Then (i) $\tilde{H}(x,0) = H(x,0,1) = H_*(x,0,1) = x$

(ii) $\tilde{H}(x,1) = H(x,1,1) = G(r(x),0) = r(x) \in A$

(iii) $\tilde{H}(a,t) = H(a,t,1) = H_*(a,t,1) = G(a,0) = a$

Therefore, A is a SDR of X.

"<=": Clearly, $i:A \to X$ is a SDR of $X \Rightarrow i$ is a homotopy equivalence.

<u>Corollary 4.3</u>: $f:D \to A$ is an h-equivalence $\Leftrightarrow$ the map $i_D:D \to M(f)$ defined by $i_D(x) = [x,1]$ is an h-equivalence $\Leftrightarrow$ D is a SDR of $M(f)$ via i_D.

<u>Proof</u>: We have already proved that $f:D \to A$ is an h-equivalence $\Leftrightarrow i_D:D \to M(f)$ is an h-equivalence (See Theorem 2.2.10(d)). Furthermore, by Theorem 2.2.10(b) $i_D:D \to M(f)$ is a closed cofibration. Hence, by Theorem 4.5, D is a SDR of $M(f)$ via $i_D \Leftrightarrow i_D$ is an h-equivalence.

We now prove the Glueing Theorem for Homotopy Equivalences. There are serveral proofs of this theorem in the literature. For example [1;7.57]. However, the proof given here is due to R. Piccinini and R. Fritsch (see [5]). But before we do that we need the following result.

<u>Lemma 4.1</u>: If $f:D \to A$ is an h-equivalence and $i:D \to Y$ is a cofibration, then the induced map $\bar{f}:Y \to A \sqcup_f Y$ is an h-equivalence.

<u>Proof</u>: Given the following diagram

$$\begin{tikzcd} Y \arrow[r, "\bar{f}"] & A \sqcup_f Y \\ D \arrow[u, "i"] \arrow[r, "f"'] & A \arrow[u, "\bar{i}"'] \end{tikzcd}$$

We need to show that $\bar{f}: Y \to A \sqcup_f Y$ is an h-equivalence. Since $f: D \to A$ is an h-equivalence, it follows from Corollary 4.3 that D is a SDR of $M(f)$ via i_D. Now consider the following two diagrams

$$\begin{tikzcd} M(f) \arrow[r] & Y \sqcup_i M(f) \\ D \arrow[u, "i_D"] \arrow[r, "i"'] & Y \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r] & Y \sqcup_{i_D} M(f) \\ D \times 1 \arrow[u, "i"] \arrow[r, "i_D"'] & M(f) \arrow[u] \end{tikzcd}$$

By Theorem 1.3.4(a), Y is a SDR of $Y \sqcup_i M(f) \cong M(f) \sqcup_{i_D} Y$ as D is a a SDR of $M(f)$. We now compute $A \sqcup_f (D \times I \cup Y \times 1)$ by considering the following diagram

$$\begin{tikzcd} D \times I \cup Y \times 1 \arrow[r] & A \sqcup_f (D \times I \cup Y \times 1) \cong M(f) \sqcup_{f^*} (D \times I) \cup Y \times 1 \\ D \times I \arrow[u] \arrow[r, "f^*"] \arrow[ur, phantom, "\mathrm{II}"] & M(f) \arrow[u] \\ D \arrow[u] \arrow[r, "f"'] \arrow[ur, phantom, "\mathrm{I}"] & A \arrow[u] \end{tikzcd}$$

Square I is a pushout and out square is a pushout. Hence, by Remark 1.1.4(b)(i), square II is also a pushout. So, by vertical composition (see Remark 1.2.5(b)); we have that $A \sqcup_f (D \times I \cup Y \times 1)$ $\cong M(f) \ _{\bar{f}} (D \times I \cup Y \times 1)$. Again, we consider the following diagram

$$\begin{tikzcd}
Y \times 1 \arrow[r] & D \times I \cup Y \times 1 \arrow[r, "\overline{f^*}"] & M(f) \sqcup_{f^*} (D \times I \cup Y \times 1) \cong M(f) \sqcup_{i_D} Y \times 1 \\
D \arrow[u] \arrow[r] \arrow[ur, phantom, "\mathrm{I}"] & D \times I \arrow[u] \arrow[r, "f^*"'] \arrow[ur, phantom, "\mathrm{II}"] & M(f) \arrow[u]
\end{tikzcd}$$

Since $i: D \to D \times I$ is a cofibration, it follows that $M(i) \cong D \times I \cup Y \times 1$ (see Remark 2.2.2(a)). Hence, square I is a pushout. Since outer square is a pushout, we have that square II is a pushout (see Theorem 1.1.4). Therefore, by Horizontal Composition, $M(f) \sqcup_{\bar{f}} (D \times I \cup Y \times 1) \cong M(f) \sqcup_{i_D} Y$. Thus, combining the results we have obtained so far, we have the following: $A \sqcup_f (D \times I \cup Y \times 1) \cong M(f) \sqcup_{i_D} Y$

$$\cong M(f) \sqcup_{\bar{f}^*} (D \times I \cup Y \times 1)$$

We now consider the following two diagrams.

$$\begin{tikzcd}
Y \times I \arrow[r] & \overline{M(f)} = (M(f) \sqcup_{i_D} Y) \sqcup_{\tilde{f}} Y \times I \cong A \sqcup_f (Y \times I) \\
D \times I \cup Y \times 1 \arrow[u] \arrow[r] & M(f) \sqcup_{i_D} Y \cong A \sqcup_f (D \times I \cup Y \times 1) \arrow[u] \\
D \arrow[u] \arrow[r, "f"'] & A \arrow[u]
\end{tikzcd}$$

By a similar argument as above, $\overline{M(f)} = (M(f) \sqcup_{i_D} Y) \sqcup_{\bar{f}} Y \times I$

$$\cong A \sqcup_f Y \times I$$

Again, by considering the diagram

$$\begin{tikzcd}
Y \times I \arrow[r] & M(\bar{f}) = (A \sqcup_f Y) \sqcup_{\bar{f}} (Y \times I) \cong A \sqcup_f Y \times I \\
Y \arrow[u] \arrow[r, "\bar{f}"] & A \sqcup_f Y \arrow[u] \\
D \arrow[u] \arrow[r, "f"] & A \arrow[u]
\end{tikzcd}$$

We get $M(\bar{f}) = A \sqcup_f Y \times I$. Combining the result obtained above,

$\overline{M(f)} \cong A \sqcup_f Y \times I$

$\cong M(\bar{f})$

Since $i: d \to Y$ is a cofibration, it follows by Characterization Theorem 2.2.2(d), that $D \times I \cup Y \times 1$ is a SDR of $Y \times I$ (where $Y \times 1$ is identified by Y). We now consider the following diagram

$$\begin{tikzcd}
Y \times I \arrow[r] & A \sqcup_f (Y \times I) \cong \overline{M(f)} \\
D \times I \cup Y \times 1 \arrow[u, "\dot{j}"] \arrow[r] & A \sqcup_f (D \times I \cup Y \times 1) \cong M(f) \sqcup_{i_D} Y \arrow[u, "1 \sqcup \dot{j}"']
\end{tikzcd}$$

Since $j: D \times I \cup Y \times I \to Y \times I$ is a SDR, there exists a homotopy

$F: Y \times I \times I \to Y \times I$ such that

$F(-,-,0) = 1_{Y \times I}$

$F(-,-,1) = r: Y \times I \to D \times I \cup Y \times I$

Now, let $p_A: A \times I \to A$ be the projection map and observe that $(A \sqcup_f (Y \times I)) \times I \cong A \times I \sqcup_{f\times 1} (Y \times I \times I)$. We now consider the function $A \times I \sqcup_{f\times 1} (Y \times I \times I) \xrightarrow{p_A \sqcup F} A \sqcup_f (Y \times I)$ defined by:

$[(a,t)] \to [a]$ and

$[(y,u,v)] \to [F(y,u,v)]$.

Since $A \times I \sqcup_{f\times 1} (Y \times I \times I)$ is a pushout, it follows that $p_A \sqcup F$ is continuous. It is now an easy matter to show that $A \sqcup_F (D \times I \cup Y \times 1)$ is a SDR of $A \sqcup_f (Y \times I)$ under the homotopy $p_A \sqcup F: (A \sqcup_f (Y \times I)) \times I \cong A \times I \sqcup_{f\times 1} (Y \times I \times I)$ $\to A \sqcup_f (Y \times I)$. Therefore, $M(f) \sqcup_{i_D} Y \cong A \sqcup_f (D \times I \cup Y \times 1)$ $\xrightarrow{SDR} A \sqcup_f (Y \times I) \cong \overline{M(f)}$. Thus, $Y \xrightarrow{SDR} M(f) \sqcup_{i_D} Y \xrightarrow{SDR} \overline{M(f)} \cong M(\bar{f})$

$\Rightarrow Y \xrightarrow{SDR} M(\bar{f})$

$\Rightarrow \bar{f}: Y \to A \sqcup_f Y$ is an h-equivalence by Corollary 4.3.

The Glueing Theorem 4.6: Let

$$\begin{tikzcd}
Y \arrow[d, "h_Y"'] & D \arrow[l, "i"'] \arrow[r, "f"] \arrow[d, "h_D"] & A \arrow[d, "h_A"] \\
Y' & D' \arrow[l, "i'"] \arrow[r, "f'"'] & A'
\end{tikzcd}$$

be a commutative diagram in which i, i' are closed cofibrations, and h_Y, h_D, h_A are h-equivalences. Then $A \sqcup_f Y \simeq A' \sqcup_{f'} Y'$.

Proof: Let $X = A \sqcup_f Y$ and $X' = A' \sqcup_{f'} Y'$. Consider the following diagram

$$\begin{tikzcd}
 & & X' \\
Y \arrow[r, "\bar{f}"] \arrow[urr, bend left, "\overline{f'}h_Y"] & X \arrow[ur, dashed, "h"] & \\
D \arrow[u, "i"] \arrow[r, "f"'] & A \arrow[u, "\bar{\imath}"'] \arrow[uur, bend right, "\overline{i'}h_A"'] &
\end{tikzcd}$$

The universal property of pushouts yields a unique map $h:X \to X'$ such that

$$h\bar{\imath} = \overline{i'}h_A \quad \text{and} \quad h\bar{f} = \overline{f'}h_Y \tag{1}$$

Four different cases will be discussed:

Case 1: D closed and a SDR of Y and D' closed and a SDR of Y'.

Then by Theorem 1.3.4 (a), A is a SDR of X and A' is a SDR of X'. So, $\bar{i}$ and $\overline{i'}$ are homotopy equivalences. Hence,

$$\bar{i}^*: X \to A \text{ and } \overline{i'}^*: X \to A' \text{ such that } \begin{array}{l} \bar{i}\,\bar{i}^* \simeq 1_X \\ \bar{i}^*\,\bar{i} \simeq 1_A \end{array} \text{ and } \begin{array}{l} \overline{i'}\,\overline{i'}^* \simeq 1_{X'} \\ \overline{i'}^*\,\overline{i'} \simeq 1_{A'} \end{array}$$

Similarly, since h_A is a h.e. $h_A^*: A' \to A$ such that

$$h_A h_A^* \simeq 1_{A'} \quad \text{and} \quad h_A^* h_A \simeq 1_A$$

$$\begin{aligned} \text{Now, } h(\bar{i}h_A^*\,\overline{i'}^*) &= (h\bar{i})h_A^*\overline{i'}^* \\ &= (\overline{i'}h_A)h_A^*\overline{i'}^* \text{ ... by eq. (1)} \\ &\simeq \overline{i'}\,\overline{i'}^* \simeq 1_{X'} \end{aligned}$$

Again, $\bar{i}h_A^*\overline{i'}^*h\bar{i}\;\bar{i}^* \simeq \bar{i}h_A^*\overline{i'}^*h$

$\Rightarrow \quad \bar{i}h_A^*\overline{i'}^*\overline{i'}h_A\bar{i}^* \simeq \bar{i}h_A^*\overline{i'}^*h$

$\Rightarrow \quad 1_X \simeq (\bar{i}h_A^*\overline{i'}^*)h$

Therefore, h is a homotopy equivalence.

Case 2: Suppose f and f' are homotopy equivalences. Then by Lemma 4.1, $\bar{f}$ and $\overline{f'}$ are homotopy equivalences. Now using the equality $h\cdot\bar{f} = \overline{f'}\cdot h_Y$ from eq. (1) and using the same kind of techniques as in Case 1, we conclude that h is a homotopy equivalence.

Case 3: The map f' is a cofibration.

We then construct the following commutative diagram below stepwise.

$$\begin{tikzcd}
Y \arrow[rrr, "\bar{f}"] \arrow[dr, "h_Y"] \arrow[ddr, "\bar{h}_D"'] & & & X = A \sqcup_f Y \arrow[dl, "h"'] \arrow[ddl, "\bar{h}_A"] \\
& Y' \arrow[r, "\bar{f'}"] \arrow[dr, phantom, "3"] & X' & \\
& Y'' \arrow[r, "\tilde{f}"] \arrow[u, "g"] \arrow[dr, phantom, "2"] & X'' \arrow[u, "\bar{g}"'] \arrow[ddr, phantom, "4"] & \\
& D' \arrow[r, "f'"] \arrow[u, "i''"] \arrow[uu, bend right=60] & A' \arrow[u, "\bar{i}''"'] \arrow[uu, bend left=60] & \\
D \arrow[uuuu, "i"] \arrow[ur, "h_D"] \arrow[rrr, "f"] \arrow[uur, phantom, "1"] & & & A \arrow[uuuu, "\bar{i}"'] \arrow[ul, "h_A"']
\end{tikzcd}$$

Step 1: Construct trapezoid 1 as a pushout.

Since $i:D \to Y$ is a cofibration and $h_D:D \to D'$ is a homotopy equivalence, it follows that $\bar{h}_D: Y \to Y''$ is a homotopy equivalence by Lemma 41.

By commutativity of (*) we have that $h_Y i = i' h_D$. Hence, $\exists! g:Y'' \to Y'$ such that

$$g \cdot i'' = i' \quad \text{and} \quad g \cdot \bar{h}_D = h_Y \quad \ldots \quad \text{eq. (2)}$$

since $\bar{h}_D$ and h_Y are homotopy equivalences, it follows that g is a h.e.

Step 2: Construct square 2 as a pushout.

Hence, $X'' = A' \sqcup_{f'} Y''$

Since f' is a cofibration and square 2 is a pushout it follows by Theorem 2.1.3 that $\tilde{f}$ is a cofibration.

Now, $\overline{f'} g i'' = \overline{f'} i'$ by eq. (2)

$= \overline{i'} f'$ since $X' = A' \sqcup_f Y'$ (i.e. square commutes)

Since square 2 is a pushout and larger square is a pushout i.e. $X' = A' \sqcup_{f'} Y'$, it follows that square (3) is a pushout and hence $X'' \sqcup_{\tilde{f}} Y' \cong X'$.

Also, by the universal property of pushouts applied to square 2, $\bar{g}$ is the unique map such that

$$\overline{g}\tilde{f} = \overline{f'}g \quad \text{and} \quad \bar{g}\,\overline{i''} = \overline{i'} \; \ldots \text{ eq. } (3)$$

Now, $\tilde{f}$ is a cofibration and g is a h.e.; hence by Lemma 4.1, $\bar{g}$ is a h.e.

Step 3: Consider the outer rectangle where $X = A \sqcup_f Y$. Now, $\tilde{f}\bar{h}_D : Y \to X''$ and $\overline{i''}h_A : A \to X''$ are maps such that

$\tilde{f}\bar{h}_D i = \tilde{f}i''h_D$ (commutativity of trap. 1)

$= \overline{i''}f'h_D$ (commutativity of square 2)

$= \overline{i''}h_A f$ (commutativity of *)

Hence, $!\bar{h}_A : X \to X''$ such that

$$\bar{h}_A\bar{i} = \overline{i''}h_A \quad \text{and} \quad \bar{h}_A \cdot \bar{f} = \tilde{f} \cdot \bar{h}_D \quad \text{eq. } (4)$$

Now the maps $h : X \to X'$ and $\bar{g}\,\bar{h}_A : X \to X'$ are such that

$h\bar{f} = \overline{f'}\,h_Y$ and

$h\bar{i} = \overline{i'}h_A$.

But, $\bar{g}\,\bar{h}_A\,\bar{f} = \bar{g}\,\tilde{f}\,\bar{h}_D$ by eq. (3)

$= \overline{f'}g\bar{h}_D$ Communitivity of square (3)

$= \overline{f'}h_Y$ equation (2)

and $\bar{g}\,\bar{h}_A\,\bar{i} = \bar{g}\,\overline{i''}h_A$ equation (4)

$= \overline{i'}h_A$ equation (3)

Therefore, by uniqueness of h, $h = \bar{g} \cdot \bar{h}_A$ ... equation (5).
But both $\bar{g}$ and $\bar{h}_A$ are homotopy equivalences.
Therefore, h is a homotopy equivalence.

Case 4: General Case:

Consider the mapping cylinders $M(f)$, $M(f')$ of the maps f and f' respectively.

Let $j:D \to D \times I$ and $j':D' \to D' \times I$ denote the embeddings at the 0th level.

Then j and j' are closed cofibrations and D and D' are SDR's of $D \times I$ and $D' \times I$ respectively.

Case 1 applied to the diagram

$$\begin{tikzcd}
D \times I \arrow[d, "h_D \times 1_I"'] & D \arrow[l, "j"'] \arrow[r, "f"] \arrow[d, "h_D"] & A \arrow[d, "h_A"] \\
D' \times I & D' \arrow[l, "j'"] \arrow[r, "f'"'] & A'
\end{tikzcd}$$

implies the existence of a homotopy equivalence

$$h_M:M(f) \to M(f') \text{ such that}$$

$$h_M\bar{j} = \overline{j'}h_A \text{ and } h_M \cdot \bar{f} = \overline{f'} \cdot (h_D \times 1_I) \quad \ldots \text{ eq. (6)}$$

Since D and D' are SDR's of $D \times I$ and $D' \times I$ respectively, it follows by Theorem 1.3.4 (b) that A and A' are SDR's of $M(f)$ and (Mf') respectively.

Let $r_f:M(f) \to A$ and $r_f':M(f') \to A'$ be the respective deformation retracts such that $r_f\bar{f} = f \cdot pr_1$ and $r_f'\overline{f'} = f' \cdot pr_1$ (see

Theorem 2.2.10).

Now, consider the following diagram

$$\begin{tikzcd}
D\times I \arrow[r, "\bar{f}"] \arrow[rrr, bend left, "r_f'\overline{f'}(h_D\times 1_I)"] & M(f) \arrow[r, dashed, "h_M"] \arrow[rr, bend right, "h_A r_f"'] & M(f') \arrow[r, "r_f'"] & A' \\
D \arrow[u, "j"] \arrow[r, "f"] & A \arrow[u, "\bar{j}"] \arrow[urr, bend right, "h_A"'] & &
\end{tikzcd}$$

Now, $r_f' h_M \bar{f}(d,t) = r_f' \overline{f'}(h_D \times 1_I)(d,t)$

$= r_f' \overline{f'}(H_D(d),t)$

$= f' pr_1(h_D(d),t)$ (See

$= f' h_D(d)$

$= h_A f(d)$ by commitivity of (*)

and $h_A r_f \bar{f}(d,t) = h_A f pr_1(d,t)$

$= h_A f(d)$

$\Rightarrow r_f' h_M \bar{f} = h_A r_f \bar{f}$

Similarly, $r_f' h_M \bar{j} = h_A = h_A r_f \bar{j}$. Hence, by uniqueness of $r_f' h_M$, it follows that

$$r_f' h_M = h_A r_f \qquad (7)$$

Now consider the following diagram

$$\begin{tikzcd}
Y \arrow[d, "h_Y"'] & D \arrow[l, "i"'] \arrow[r, "i_D"] \arrow[d, "h_D"] & M(f) \arrow[d, "h_M"] \\
Y' & D' \arrow[l, "i'"] \arrow[r, "i_{D'}"'] & M(f')
\end{tikzcd}$$

where i, i', i_D and i_D' are cofibrations and h_Y, h_D and h_M are h-equivalences. From (*), $h_Y i = i' h_D$. On the other hand, $h_M i_D(d) = h_M[d,1]$

$$
\begin{aligned}
&= h_M \bar{f}(d,1) \\
&= \overline{f'}(h_D \times 1_I)(d,1) \\
&= \overline{f'}(h_D(d),1) \\
&= [h_D(d),1] \\
&= i_{D'} h_D(d)
\end{aligned}
$$

Therefore, the above diagram is commutative. Applying case (3) to the above diagram, we obtain an h-equivalence

$\tilde{h}: M(f) \sqcup Y \to M(f') \sqcup Y'$ such that

$$\tilde{h}\,\bar{i} = \overline{i'} h_M \tag{8}$$

Finally, consider the following diagram

$$
\begin{tikzcd}
M(f) \sqcup_{i_D} Y \arrow[d, "\tilde{h}"'] & M(f) \arrow[l, "i"'] \arrow[r, "r_f"] \arrow[d, "h_M"] & A \arrow[d, "h_A"] \\
M(f') \sqcup_{i_{D'}} Y' & M(f') \arrow[l, "\overline{i'}"] \arrow[r, "r_{f'}"'] & A'
\end{tikzcd}
$$

By equation (8), $\tilde{h}\,\bar{i} = \overline{i'} h_M$ and by equation (7), $h_A r_f = r_{f'} h_M$. Hence, the diagram is commutative; $\bar{i}$, $\overline{i'}$ are cofibrations, r_f, $r_{f'}$, h_A, h_M and $\tilde{h}$ are h-equivalences. Hence, case (2) applied to the diagram above gives rise to an h-equivalence

$$A \sqcup_{r_f} (M(f) \sqcup_{i_D} Y) \to A' \sqcup_{r_{f'}} (M(f') \sqcup_{i_{D'}} Y')$$

But by Theorem 2.2.10, $f = r_f\, i_D$ and $f' = r_{f'}\, i_{D'}$ and hence applying horizontal composition, we get

$$A \sqcup_{r_f i_D} Y \simeq A' \sqcup_{r_{f'} i_{D'}} Y'$$

$$\Rightarrow \quad A \sqcup_f Y \simeq A' \sqcup_{f'} Y'$$

Theorem 4.7: Suppose in addition to the hypothesis in Theorem 4.1 (a), j is closed and a homotopy equivalence over B. Then $E_A \cup D$ is a SDR of E.

Proof: $j: D \to E$ is an h-equivalence over $B \Rightarrow$ there exists a map $m: E \to D$ over B and homotopies $H: m \cdot j \simeq 1_D$ over B and $K: j \cdot m \simeq 1_E$ over B.

Clearly, m restricts to a map $m_A: E_A \to D_A$ and similarly H and K restrict to $H_A: m_A \cdot j_A \simeq 1_{D_A}$ over A and $K_A: j_A \cdot m_A \simeq 1_{E_A}$ over A. Therefore, j_A is an h-equivalence (over A).

Now consider the following diagram

$$\begin{tikzcd}
D \arrow[d, "j"'] & D_A \arrow[l, "i_D"'] \arrow[r, "j_A"] \arrow[d, "j_A"] & E_A \arrow[d, "1_{E_A}"] \\
E & E_A \arrow[l, "i_E"] \arrow[r, "1_{E_A}"'] & E_A
\end{tikzcd}$$

where i_D and i_E are closed cofibrations and the vertical maps are h-equivalences. Hence, by Theorem 4.6, we have that $E_A \sqcup_{j_A} D \simeq E_A \sqcup_{1_{E_A}} E$. Since $D \cap E_A = D_A$, it follows that $E_A \sqcup_{j_A} D = E_A \cup D$, and $E_A \sqcup_{1_{E_A}} E = E$. Therefore, $D \cup E_A \to E$ is an h-equivalence. Now, by Theorem 4.1 (a), $D \cup E_A \to E$ is also a cofibration. Therefore, by Theorem 4.5, it follows that $D \cup E_A \to E$ is a SDR.

Lemma 4.2: Let $p:E \to B$ and $q:E' \to B$ be maps to a fixed space B. Let $\phi:E \to E'$ be a map such that $q\phi \simeq p$. If q is a fibration, then $\phi \simeq \psi$ for some $\psi:E \to E'$ over B.

Proof: $q\phi \simeq p \Rightarrow$ there exists a map $H:E \times I \to B$ such that $H(-,0)$ $q\phi$ and $H(-,1) = p$.

Consider the following commutative diagram

$$\begin{tikzcd}
E \times 0 \arrow[r, "\phi"] \arrow[d, "i"'] & E' \arrow[d, "q"] \\
E \times I \arrow[r, "H"'] \arrow[ur, dashed, "F"] & B
\end{tikzcd}$$

Since q is a fibration, there exists a map $F:E \times I \to E'$ such that $qF = H$ and $F|_{E \times 0} = \phi$.

Let $F(-,1) = \psi:E \to E'$. Then $F:\phi \simeq \psi$ and

$q\psi = qF(-,1) = H(-,1) = p$ and so ψ is a map over B.

Lemma 4.3: Let $p:E \to B$ be a fibration. Let $\sigma:E \to E$ be a map over B, and suppose that $\sigma \simeq 1_E$. Then there exists a map $\sigma':E \to E$ over B such that $\sigma\sigma' \simeq_B 1_E$.

Proof: Consider the following diagram

$$\begin{tikzcd} E \arrow[rr, "\sigma"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

Since σ is a map over B, $p\sigma = p$. Now, $\sigma \simeq 1_E \Rightarrow$ there exists $F:E \times I \to E$ such that $F(-,0) = \sigma$ and $F(-,1) = 1_E$.

Now, $pF:E \times I \to B$ is a map such that

$pF(-,0) = p\sigma = p$ and

$pF(-1,) = p1_E = p$

$\Rightarrow$ $pF:p \simeq p$ and so $pF(e,t) = p(e)$

Now consider the following commutative diagram

$$\begin{tikzcd} E \arrow[r, "1_E"] \arrow[d, "\alpha_0"'] & E \arrow[d, "p"] \\ E \times I \arrow[r, "pF"'] \arrow[ur, dashed, "K"] & B \end{tikzcd}$$

Since p is a fibration there exists a map $K:E \times I \to E$ such that the resulting triangles commute.

Let $\phi = K(-,1)$. Then, $\phi \simeq_B 1_E$. We now consider the following diagram

$$\begin{tikzcd} E \times 0 \arrow[r, "\sigma"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ E \times I \arrow[r, "pF"'] \arrow[ur, dashed, "G"] & B \end{tikzcd}$$

$pF(-,0) = p\sigma = p = p\phi$ (as ϕ is a map over B). Hence, diagram commutes.

Since $p:E \to B$ is a fibration, there exists $G:E \times I \to E$ such that $pG = pF$ and $G\big|_{EX0} = \Phi$.

Let $\sigma' = G(-,1):E \to E$.

We claim $\sigma\sigma' \simeq_B 1_E$.

Define $H:E \times I \to E$ by

$$H(e,s) = \begin{cases} \sigma G(e,1-2s), & e \in E \text{ and } 0 \le s \le 1/2 \\ F(e,2s-1), & e \in E \text{ and } 1/2 \le s \le 1 \end{cases}$$

Then, $H(-,0) = \sigma G(-,1) = \sigma\sigma'$ and

$H(-,1) = F(-,1) = 1_E$

Hence, $H:\sigma\sigma' \simeq 1_E$. We still need to show $\sigma\sigma' \simeq_B 1_E$.

Observe that

$$pH(e,1-s) = \begin{cases} pG(e,2s-1), & 1/2 \le s \le 1 \\ pF(e,1-2s), & 0- \le s \le 1/2 \end{cases}$$

But from above, $pG = pF$.

Hence,

$$pH(e,1-s) = \begin{cases} pF(e,2s-1), & 1/2 \le s \le 1 \\ pF(e,1-2s), & 0 \le s \le 1/2 \end{cases}$$

On the other hand,

$$pH(e,s) = \begin{cases} p\sigma G(e,1-2s), & 0 \le s \le 1/2 \\ pF(e,2s-1), & 1/2 \le s \le 1 \end{cases}$$

$$= \begin{cases} pF(e,1-2s), & 0 \le s \le 1/2 \\ pF(e,2s-1), & 1/2 \le s \le 1 \end{cases}$$

Therefore, $pH(e,s) = pH(e,1-s)$.

We now define $\Phi : E \times I \times I \to B$ by

$$\Phi(e,s,t) = \begin{cases} pF(e,1-2s(1-t)), & 0 \le s \le 1/2 \\ pF(e,1-2(1-s)(1-t)), & 1/2 \le s \le 1 \end{cases}$$

Then,

$$\Phi(e,s,0) = \begin{cases} pF(e,1-2s), & 0 \le s \le 1/2 \\ pF(e,2s-1), & 1/2 \le s \le 1 \end{cases}$$

$$= pH(e,s)$$

and $\Phi(e,0,t) = \Phi(e,s,1) = \Phi(e,1,t) = p(e)$. So, the following diagram commutes.

$$\begin{tikzcd} E \times I \times 0 \arrow[r, "H"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ E \times I \times I \arrow[r, "\Phi"'] \arrow[ur, dashed, "\tilde{\Phi}"] & B \end{tikzcd}$$

Since $p:E \to B$ is a fibration, there exists a map $\tilde{\Phi}:E \times I \times I \to E$ such that $p\tilde{\Phi} = \Phi$ and $\Phi(e,s,0) = H(e,s)$.
We now define $\tilde{\Phi}_{(s,t)}:E \to E$ by

$$\tilde{\Phi}_{(s,t)}(e) = \tilde{\Phi}(e,s,t)$$

Then, $\sigma\sigma' = H(-,0) = \tilde{\Phi}_{(0,0)} \simeq_B \tilde{\Phi}_{(0,1)} \simeq_B \tilde{\Phi}_{(1,1)} \simeq_B \tilde{\Phi}_{(1,0)} = H(-,1) = 1_E$
$\Rightarrow \sigma\sigma' \simeq_B 1_E$.

Theorem 4.8: Let $p:E \to B$ and $q:E' \to B$ be fibrations. Let $\phi:E \to E'$ be a map over B. Suppose that ϕ, as an ordinary map, is an h-equivalence. Then, ϕ is an h-equivalence over B.

Proof: Let $\psi:E' \to E$ be a homotopy inverse of ϕ, as an ordinary map. Then, $p\psi = q\phi\psi \simeq q$. Hence, by Lemma 4.2, $\psi \simeq \psi'$ for some ψ' over B. Since $\phi\psi' \simeq 1_{E'}$ and $\phi\psi'$ is over B, there exists by Lemma 4.3 a map $\psi'':E' \to E'$ over B such that $\phi\psi'\psi'' \simeq_B 1_{E'}$. Thus, ϕ admits a homotopy right inverse $\phi' = \psi'\psi''$ over B.

Now, ϕ' is an h-equivalence, since ϕ is an h-equivalence, and so the same argument applied to ϕ' instead of ϕ, shows that ϕ' admits a homotopy right inverse ϕ'' over B. Thus, ϕ' admits both a homotopy left inverse ϕ over B and a homotopy right inverse ϕ'' over B. Hence, ϕ' is an h-eqivalence over B and so ϕ itself is an h-equivalence over B.

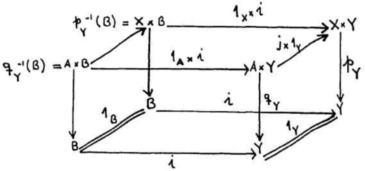
Theorem 4.9: If in diagram (*) of Theorem 4.1, i and j are closed cofibrations, p and q are fibrations and $j:iD \to E$ is also an h-equivalence, then $E_A \cup D$ is a SDR of E.

Proof: By Remark 4.1, j is a closed cofibration over B and by Theorem 4.8, j is an h-equivalence over B. Therefore, by Theorem 4.7, it follows that $E_A \cup D$ is a SDR of E.

Finally, by way of application of the above theorem, we have the following result of Ström on SDR (see [16;Theorem 6]

Corollary 4.4: Let (X,A) and (Y,B) be closed cofibred pairs. If in addition, A (or B) is a SDR of $X(Y)$, then $X \times B \cup A \times Y$ is a SDR of $X \times Y$.

Proof: We consider the diagram used in Theorem 4.3



and assume without loss of generality that A is a SDR of X. Then $Z \to X$ is an h-equivalence and so $A \times Y \to X \times Y$ is an h-equivalence. Therefore, by Theorem 4.9, $X \times B \cup A \times Y$ is a SDR of $X \times Y$.

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