THE SPECTRUM OF SKOLEM AND HOOKED SKOLEM SEQUENCES WITH A PRESCRIBED NUMBER OF PAIRS IN COMMON AND APPLICATIONS

DANIELA SILVESAN

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by

©Daniela Silvesan

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Department of Mathematics and Statistics
Memorial University of Newfoundland

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Abstract

A Skolem sequence of order n is a sequence $S_n = (s_1, s_2, \ldots, s_{2n})$ of $2n$ integers satisfying the conditions:

1. for every $k \in \{1, 2, \ldots, n\}$ there are exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = k$, i.e. each number in $\{1, 2 \ldots, n\}$ appear twice in the sequence, and

2. if $s_i = s_j = k$, $i < j$, then $j - i = k$, i.e. the indices i, j where k appears has $j - i = k$.

A hooked Skolem sequence of order n is a sequence $hS_n = (s_1, s_2, \ldots, s_{2n+1})$ of $2n+1$ integers satisfying conditions (1), (2) and (3): $s_{2n} = 0$. A *triple system of order v and index λ,* denoted $TS(v, \lambda)$, is a v-set of V elements, together with a collection B of 3-element subsets of V called *triples* such that each 2-subset of V appears in precisely λ triples of B. This is also called a λ-fold triple system. If λ is not specified, then $\lambda = 1$ and the triple systems are Steiner triple systems, denoted $STS(v)$. An $STS(v)$ is cyclic if it has an automorphism consisting of a single cycle of length v. A cyclic $STS(v)$ is denoted $CSTS(v)$ and a cyclic $TS(v, \lambda)$ is denoted $CTS(v, \lambda)$. We denote by Int_{S_n}=$\{k$: there exists two [hooked]Skolem sequences of order n with k pairs in common$\}$. This is *the intersection spectrum* of two [hooked] Skolem sequences of order n which gives the intersection spectrum of two $CSTS(v)$, $v \equiv 1, 3 \,(mod\ 6)$. Given a $CTS(v, \lambda)$, the *fine structure* of the system is the vector $(c_1, c_2, \ldots, c_\lambda)_c$, where c_i is the number of base blocks repeated exactly i times in the cyclic triple system.

In this thesis we prove, using [hooked] Skolem sequences of order n, that there exists two cyclic Steiner triple systems of order $6n+1$ intersecting in $0, 1, 2, \ldots, n$ base

blocks and there exists two cyclic Steiner triple systems of order $6n + 3$ intersecting in $1, 2, \ldots, n+1$ base blocks. From here we derive that a twofold cyclic triple system of order $6n + 1$ intersect in $0, 1, \ldots, n$ base blocks and a twofold cyclic triple system of order $6n + 3$ intersect in $1, 2, \ldots, n + 1$ base blocks. We also prove, with some possible exceptions, that there exists two [hooked] Skolem sequences of order n which can have $0, 1, 2, \ldots, n - 3, n$ pairs in common and, using this intersection spectrum, we determine the fine structure of a threefold cyclic triple system of order $6n + 1$.

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Chapter 1

Introduction

Combinatorial design theory is thought to have started in 1776, when Euler posed the question of constructing two orthogonal latin squares of order 6. This was known as *Euler's 36 Officers Problem.* Over the years, however, combinatorial researchers have discussed a wider range of designs. These have included: one-factorizations, Room Squares, designs based on unordered pairs, various tournament designs as well as other designs.

Informally, one may define a combinatorial design to be a way of selecting subsets from a finite set such that specific conditions are satisfied. As an example, suppose it is required to select 3-sets from the seven objects $\{a, b, c, d, e, f, g\}$, such that each object occurs in three of the 3-sets and every intersection of two 3-sets has precisely one member. The solution to such a problem is a combinatorial design. One possible example is $\{abc, ade, afg, bdf, beg, cdg, cef\}$, which is also called a Steiner Triple System of order 7 and denoted $STS(7)$.

Another subject systematically studied was triple systems and the most importantly is the celebrated Kirkman[25] *schoolgirl problem* which fascinated mathematicians for many years:

"Fifteen young ladies of a school walk out three abreast for seven days in succession: it is required to arrange them daily, so that no two shall walk twice abreast."

Without the requirement of arranging the triples in days, the configuration is a

Steiner triple system of order 15, and hence was known to Kirkman. The first one to publish a solution to the Kirkman schoolgirl problem was Cayley[10].

Triple systems were also studied by Heffter[22] who introduced his famous first and second difference problems in relation to the construction of cyclic Steiner triple systems of order $6n+1$ and $6n+3$. Skolem[47] also had an interest in triple systems; he constructed $STS(v)$ for $v = 6n+1$. He introduced the idea of a Skolem sequence of order n which is a sequence of integers which satisfies the following properties: every integer i, $1 \leq i \leq n$, occurs exactly twice and the two occurrences of i are exactly i integers apart. The sequence $(4,2,3,2,4,3,1,1)$ is a Skolem sequence of order 4. In the literature, Skolem sequences are also known as pure or perfect Skolem sequences. Skolem proved that a Skolem sequence of order n exist if and only if $n \equiv 0, 1\,(mod\ 4)$. Skolem[48] extended this idea to that of the hooked Skolem sequence, the existence of which for all admissible n, along with that of Skolem sequences, would constitute a complete solution to Heffter's first difference problem and leads to the constructions of cyclic $STS(6n+1)$. An example of a hooked Skolem sequence of order 3 is $(1,1,2,3,2,0,3)$. O'Keefe[36] proved that a hooked Skolem sequence of order n exists if and only if $n \equiv 2, 3\,(mod\ 4)$.

Rosa[39], in 1966, introduced other type of sequences called Rosa and hooked Rosa sequences and proved that a Rosa sequence of order n exists if and only if $n \equiv 0, 3(mod\ 4)$ and a hooked Rosa sequence of order n exists if and only if $n \equiv 1, 2(mod\ 4)$. These two type of sequences constitute a complete solution to Heffter's second difference problem which leads to construction of cyclic $STS(v)$ for $v = 6n+3$. Thus, the study of triple systems has grown into a major part of the study of combinatorial designs. Triple systems are natural generalizations of graphs and much of their study has a graph theoretic flavour. Connections with geometry, algebra, group theory and finite fields provide other perspectives.

One of the most important paper discussing disjoint cyclic Steiner triple systems was written by Colbourn[12]. Using graphical representations of solutions to Heffter's difference problem, he determined the size of the largest set of disjoint cyclic Steiner triple systems for small orders and also established some easy bounds.

In this thesis, we discuss cyclic Steiner triple systems and cyclic triple systems having a prescribed number of base blocks in common and their applications. We start from Rosa's paper [38] where the author discusses Steiner triple systems and their intersection spectrum and we provide similar results for the cyclic Steiner triple systems and cyclic triple systems with $\lambda = 2$ and $\lambda = 3$.

Starting from a [hooked] Skolem sequence of order n, we prove that there exists two cyclic Steiner triple systems of order $6n+1$ intersecting in $0, 1, 2, \ldots, n$ base blocks and there exists two cyclic Steiner triple systems of order $6n+3$ intersecting in $1, 2, \ldots, n+1$ base blocks. Using these results we prove that a twofold cyclic triple system of order $6n+1$ intersect in $0, 1, \ldots, n$ base blocks and a twofold cyclic triple system of order $6n+3$ intersect in $1, 2, \ldots, n+1$ base blocks. We also prove, with some possible exceptions, that there exists two [hooked] Skolem sequences of order n that have $0, 1, 2, \ldots, n-3, n$ pairs in common. For small orders, $1 \leq n \leq 9$, we provide examples of [hooked] Skolem sequences of order n which intersect in $0, \ldots, n-3, n$ pairs in Appendix A. Two Skolem sequences of order 5 can only have $0, 1, 5$ pairs in common. Then we assume inductively that there exists two [hooked] Skolem sequences of small orders which can have $0, \ldots, n-3, n$ pairs in common and we prove that this is true also for larger orders. To prove this, we construct new [hooked] Skolem sequences of order n by adjoining a [hooked] Skolem sequence of a smaller order with a [hooked] Langford sequence.

For this to be possible, the length of the [hooked] Langford sequence must be at least twice as big as the [hooked] Skolem sequence, so we split this problem into three parts. First, we determine, with few possible exceptions, the intersection of two distinct [hooked] Skolem sequences of order n in $[0, \lfloor \frac{n}{3} \rfloor]$ pairs by adjoining the same [hooked] Skolem sequence with two disjoint Langford sequences. This proves to be complicated and as a result we have to work with many cases. Second, we prove, with few possible exceptions that two distinct [hooked] Skolem sequences can have $(\lfloor \frac{n}{3} \rfloor, 2\lfloor \frac{n}{3} \rfloor)$ pairs in common. To prove this we develop new techniques. We construct a [hooked] Skolem sequence of order n from three different parts. Part A contain the largest odd and even numbers and determine a shell, Part B is a [hooked] Skolem sequence of small order, or a 2-near Skolem sequence or an extended Skolem sequence and fills the hole in Part A and, finally, Part C is a Langford sequence. Again we have to work with many cases to solve this problem. Third, we prove that two [hooked] Skolem sequences of order n can have $[2\lfloor \frac{n}{3} \rfloor, n]$ pairs in common by adjoining the same [hooked] Langford sequence with two disjoint [hooked] Skolem sequences. This proved to be the easiest case. We give a detailed list of all open cases (possible exceptions) in Appendix C.

Finally, using the intersection spectrum of two [hooked] Skolem sequences of order n, we give the fine structure of a threefold cyclic triple system of order $6n + 1$.

Chapter 2

Basic introduction to triple systems and their intersection spectrum

This chapter is intended to be a brief survey of some well known results about triple systems, disjoint triple systems, intersection properties of triple systems and disjoint triple systems with $\lambda \geq 1$. For more details, readers may consult *Triple systems*, by Colbourn and Rosa [15].

Definition A *triple system of order v and index λ*, denoted $TS(v, \lambda)$, is a set V of v elements, together with a collection B of 3-element subsets of V called *triples* such that each 2-subset of V is a subset in precisely λ triples of B. This definition permits B to contain repeated triples. An $TS(v, \lambda)$ is also known as a λ-fold triple system. If λ is not specified, then $\lambda = 1$ and the triple systems are Steiner triple systems.

Definition A set of blocks is a *parallel class* if no two blocks in the set share an element and it contains $\frac{v}{3}$ blocks. A Steiner triple system $STS(v)$ is *resolvable* if it has a partition of all blocks into parallel classes .

Kirkman [24] proved in 1847 that a necessary and sufficient condition for the existence of a Steiner triple system of order v is $v \equiv 1$ or $3 (mod\ 6)$. Therefore, in saying that a certain property concerning $STS(v)$ is true for all v, it is understood that $v \equiv 1$ or 3 $(mod\ 6)$.

Definition Two Steiner triple systems (V, B_1) and (V, B_2) are said to *intersect in k triples* if $|B_1 \cap B_2| = k$. If $k = 0$ then (V, B_1) and (V, B_2) are said to be *disjoint* (i.e. they have no blocks in common), and if $|B_1 \cap B_2| = 1$, they are said to be *almost disjoint.*

Notation We use the notation $Int(v)$=$\{k$: exists two $STS(v)$ intersecting in k triples$\}$.

2.1 Disjoint Steiner triple systems

Definition Two set systems (V_1, B_1) and (V_2, B_2) are *isomorphic* if there is a bijection (isomorphism) ϕ from V_1 to V_2 so that the number of times B_1 appears as a block in B_2 is the same as the number of times $\phi(B_1) = \{\phi(x) : x \in B_1\}$ appear as a block in B_1. An automorphism from a set system to itself is an automorphism.

We use the notation $d(2, 3, v)$ for the *maximum number of pairwise disjoint $STS(v)$* and $d^*(2, 3, v)$ for the *maximum number of pairwise disjoint and isomorphic $STS(v)$.*

Since each $STS(v)$ has $\frac{v(v-1)}{6}$ blocks, and there are altogether $\frac{v(v-1)(v-2)}{6}$ triples, we have

$$d(2, 3, v) \leq \frac{\frac{v(v-1)(v-2)}{6}}{\frac{v(v-1)}{6}} = v - 2$$

.

Definition A set of $v - 2$ $STS(v)s$, i.e. $\{(V, B_i) : i = 1, \ldots, v - 2\}$ is a *large set* if every two systems in the set are disjoint.

The earliest results on disjoint $STS(v)$ are due to Cayley [10] who showed in 1850 that $d(2, 3, 7) = 2$.

Example: The following systems are disjoint:

$$\{abc, ade, afg, bdf, beg, cdg, cef\}.$$

$$\{abd, acg, aef, bce, bfg, dcf, deg\}.$$

Kirkman [25] in the same year showed that $d(2,3,9) = 7$. However, Bays [5] was the first to show that there are exactly two non-isomorphic sets of seven disjoint $STS(9)$. One of these sets is given by the 7 square arrays

124	128	125	129	123	126	127
378	943	983	743	469	357	346
956	765	476	586	785	489	598

while the other one is given by the other 7 square arrays

139	192	127	174	148	186	163
275	745	485	865	635	395	925
486	863	639	392	927	274	748.

Each of this square array gives a $STS(v)$. The 12 triples of each system are the three rows, three columns and the six products in the expansion of the determinant of each array.

For example, one $STS(9)$ is given by the triples from the first square array: $\{124, 378, 956, 139, 275, 486, 176, 354, 928, 479, 236, 158\}$.

Doyen [19] was first to offer nontrivial lower bounds for $d(2,3,v)$. He showed that

$$d^*(2,3,6m+3) \geq \begin{cases} 4m+1 & \text{if } m \equiv 0,\ 2\ (mod\ 3), \\ 4m-1 & \text{if } m \equiv 1\ (mod\ 3). \end{cases}$$

This result was subsequently improved by Beenker, Gerards and Penning [7] to

$$d^*(2,3,6m+3) \geq 4m+2.$$

Doyen has also shown that

$$d^*(2,3,6m+1) \geq \begin{cases} \frac{1}{2}m & \text{if } m \equiv 0\ (mod\ 2), \\ m & \text{if } m \equiv 1\ (mod\ 2); \end{cases}$$

and that

$$d(2,3,2v+1) \geq d(2,3,v)+2 \text{ for } v \geq 7$$

which has as its corollary the following result:

$$d(2,3,6m+1) \geq 2m-1 \text{ for } m \equiv 1 (mod\ 2).$$

This was also improved by Bays [5] to

$$d(2,3,6m+1) \geq 3m+1 \text{ for } m \equiv 1 (mod\ 2).$$

In 1917, Bays[5] conjectured that

$$d(2,3,v) \geq \frac{v-1}{2} \text{ for all } v > 7.$$

This now has been shown true for all v except $v \equiv 1\ (mod\ 12)$. It was also conjectured by Doyen [19] and Teirlinck [49] that

$$d(2,3,v) = v-2,\ v \geq 9.$$

Teirlinck [49] also proved the inequality

$$d(2,3,3v) \geq 2v + d(2,3,v) \text{ for every } v \geq 3.$$

by providing a recursive construction whose immediate corollary is that

$$d(2,3,3^m) = 3^m - 2 \text{ for all } m \geq 1.$$

An exhaustive computer search showed that there are exactly two non-isomorphic sets of 11 disjoint $STS(13)$ that can be obtained in this way [18, 28].

Schreiber [44] and Wilson [54] have independently showed how to construct an $STS(v)$ with the property that the 3-subsets of $V = Z_{v-2} \cup \{\infty_1, \infty_2\}$ are partitioned into orbits under the action of $< \beta = (0,1,\ldots,v-3)(\infty_1)(\infty_2) >$, where β is a permutation of V, provided all prime divisors p of $v-2$ are such that the order of $-2(mod\ )p$ is congruent to $2(mod\ )4$. The first few orders for which this method works are: $v = 9, 25, 33, 49, 51, 73, 75, 81, 91, 105, 129, 153, 163, 169, 193, 201$ and for all these values of v, we have $d^*(2,3,v) = v - 2$.

Rosa [38] showed that

$$d(2,3,2v+1) \geq v + 1 + d(2,3,v), \text{ for } v \geq 7.$$

This enables us to construct large sets of disjoint $STS(2v+1)$ whenever there is a large set of disjoint $STS(v)$.

A further result on $d(2,3,v)$ is due to Teirlinck [53]: If v is the product of primes p for which the order of $-2(mod\, p)$ is congruent to $2(mod\; 4)$ and if $d(2,3,w) = w-2$, then

$$d(2,3,v(w-2)+2) = v(w-2).$$

In 1983-1984, Lu [33, 34] first determined the existence of large set of disjoint Steiner triple systems for all $v \neq 7$ with six possible exceptions. Teirlinck [52] solved the existence of the large set of disjoint Steiner triple systems for the remaining six orders. Therefore the existence spectrum for large set of disjoint Steiner triple systems has been finally completed.

Theorem 1 *(Lu[33, 34], Teirlinck [52]).*
For any integer $v \equiv 1$ or $3(mod\; 6)$ with $v > 7$, $d(2,3,v) = v - 2$.

Teirlinck also proved that every two Steiner triple systems can be made disjoint:

Theorem 2 *(Teirlinck [51]).*
If (V, B_1) and (V, B_2) are Steiner triple systems, $v \geq 7$, there exists a Steiner triple system (V, B_3) so that $B_1 \cap B_3 = \emptyset$ and $(V, B_2) \cong (V, B_3)$.

2.2 Intersection of Steiner triple systems

In [31], the sets $Int(v)$ are completely determined. Lindner and Rosa examine the 30 distinct $STS(7)$ on the same set and found that, for any fixed $STS(7)$, there are exactly eight $STS(7)$ that are disjoint from it, exactly 14 that are almost disjoint, and exactly seven that intersect it in three triples. Hence $Int(7) = \{0, 1, 3, 7\}$. Kramer and Mesner [27] showed that $Int(9) = \{0, 1, 2, 3, 4, 6, 12\}$. They show that for any given $STS(9)$, there are 192 other $STS(9)$ disjoint from it, 216 that are almost disjoint, 216 that intersect in two triples, 152 that intersect it in three, 27 that intersect it in four, and 36 that intersect it in six.

Definition A *partial triple system* $PTS(v, \lambda)$ is a set V of v elements and a collection B of triples, so that each unordered pair of elements occurs in at most λ triples of B.

Definition Two partial triple systems (V, D_1) and (V, D_2) are *mutually balanced* if any 2-subset of V is contained in a triple of D_1 if and only if it is contained in a triple of D_2. Two partial triple systems are *disjoint* if they have no blocks in common.

For example, there exist two PTSs with four triples on the set $\{1, 2, 3, 4, 5, 6\}$ that are disjoint and mutually balanced:

$$D_1 = \{134, 156, 235, 246\} \quad D_2 = \{135, 146, 234, 256\}.$$

Lindner and Rosa found the following important results:

Lemma 1 *(Lindner, Rosa [31])*
There does not exist a pair of disjoint mutually balanced PTS's each having one, two, three, or five triples.

As a consequence, they define b_v to be the number of triples in any Steiner triple system (i.e., $b_v = \frac{v(v-1)}{6}$) and I_v to be the set that contains all nonnegative integers not exceeding b_v with the exception of $\{b_v - 5, b_v - 3, b_v - 2 \text{ and } b_v - 1\}$, i.e. $I_v = \{0, 1, ..., v\} - \{b_v - 1, b_v - 2, b_v - 3, b_v - 5\}$ and obtain:

Lemma 2 *(Lindner, Rosa [31])*
For all $v \equiv 1,\ 3 \ (mod\ 6), Int(v) \subseteq I_v$.

The main tools in the proof of Theorem 3 are the $2v + 1$ and $2v + 7$ constructions given in Lemma 3 and Lemma 4.

Lemma 3 *(Lindner, Rosa [31])*
For $v \geq 13$, $Int(v) = I_v$ *implies* $Int(2v+1) = I_{2v+1}$.

Lemma 4 *(Lindner, Rosa [31])*
For $v \geq 15$, $Int(v) = I_v$ *implies* $Int(2v+7) = I_{2v+7}$.

Theorem 3 *(Lindner, Rosa [31])*
For every $v \geq 13$, $Int(v) = I_v$.

Proof: For $v \in \{13, 15, 19, 21, 25, 33\}$, Lindner and Rosa [30] proved the statement directly. They employed the proofs of Lemmas 3 and 4, and provided proofs of $STS(v)$ for certain cases. When $v \in \{27, 34\}$, Lemmas 3 and 4 apply. So suppose that $v \geq 37$. Assume inductively that $Int(w) = I_w$ for all admissible w satisfying $15 \leq w \leq v$. If $v \equiv 3,\ 7\ (mod\ 12)$, then $(v-1)/2 \equiv 1,\ 3(mod\ 6)$ and $15 \leq (v-1)/2 < v$; hence by Lemma 3, $Int(v) = I_v$. If $v \equiv 1,\ 9(mod\ 12)$, then $(v-7)/2 \equiv 1,\ 3(mod\ 6)$ and $15 \leq (v-7)/2 < v$; hence by Lemma 4, $Int(v) = I_v$. □

The pairs of Steiner triple systems in the previous theorem may or may not be isomorphic. Lindner and Rosa [30] posed the problem of determining the set $Int^*(v)$=$\{k$: exists isomorphic Steiner triple systems of order v intersecting in k triples$\}$, but stated incorrectly that $Int^*(v) \neq Int(v)$ when $v \geq 13$.

Koszarek proved that:

Theorem 4 *(Koszarek [23])*
For all $v \equiv 1, 3(mod 6)$, $Int^*(v) = Int(v)$.

Rosa [38] also posed the problem of determining $Int_R(v)$=$\{l$: exist two resolvable $STS(v)s$ sharing l triples$\}$. He showed that $Int_R(9) = \{0, 1, 2, 3, 4, 6, 12\}$. Lo Faro [32], Shen [42], and Chang and Lo Faro [11] established that $I_{15} \setminus \{26, 29\} \subseteq Int_R(15)$. Lo Faro [32] and Chang and Lo Faro [11] established that $I_{27} \setminus \{54, 61, 62\} \subseteq Int_R(27)$.

Theorem 5 *(Chang, Lo Faro [11])*
For all $n \geq 7$, $Int_R(6n+3) = I_{6n+3}$.

v	$\in Int_R(v)$?
15	$b_{15}-9, b_{15}-6$
21	$b_{21}-16, b_{21}-9, b_{21}-8$
27	$b_{27}-7, b_{27}-4$
33	$b_{33}-13, b_{33}-7, b_{33}-4$
39	$b_{39}-13, b_{39}-7, b_{39}-4$

Table 2.1: Exceptions in resolvable $STS(v)$

Chang and Lo Faro [11] also established $b_{27}-13$ can be realized by two resolvable $STS(27)s$, and leaves 13 possible exceptions as shown in Table 2.1.

There is an extension of the problem to cases with higher index.
Define $b_{v,\lambda}$ to be the number of base blocks in a $TS(v,\lambda)$, i.e. $b_{v,\lambda}=\lambda\binom{v}{2}/3$ and define $I_{v,\lambda}=\{0,1,\ldots,v\}-\{b_{v,\lambda}-1, b_{v,\lambda}-2, b_{v,\lambda}-3, b_{v,\lambda}-5\}$.

The sets $Int(v,\lambda)$ have been completely determined by Ajoodani-Namini and Khosrovshani [2] and, for $\lambda=2$, we have results due to Guo [20]. Colbourn and Rosa showed the following results:

Lemma 5 *(Colbourn, Rosa [15])*
For $\lambda=2$,

1. $Int(4,2)=\{4\}$
2. $Int(6,2)=\{0,4,6,10\}$
3. $Int(7,2)=\{2,5,8,14\}$
4. *for* $v\equiv 0,\ 1(mod\ 3), v\geq 9,\ Int(v,2)=I_{v,2}$.

Lemma 6 *(Colbourn, Rosa [15])*
For $v\equiv 5\ (mod\ 6),\ Int(v,3)=I_{v,3}$.

Lemma 7 *(Colbourn, Rosa [15])*
For $\lambda=6$,

1. $Int(8,6) = \{56\}$

2. $Int(14,6) = \{0,4,6,7,\ldots,175,176,178,182\}$

3. *for all* $v \equiv 2(mod\ 6), v > 14, Int(v,6) = I_{v,6}$.

Definition Given an $TS(v,\lambda)$, *the fine structure of a triple system of index* λ is the vector $(c_1,\ldots,c_\lambda)$, where c_i is the number of triples repeated precisely i times in the system.

Colbourn, Mathon, Rosa and Shalaby [16] focused on $\lambda = 3$ and determined necessary conditions for a vector (c_1,c_2,c_3) to be the fine structure of a threefold triple system, and proved the sufficiency of these conditions for all $v \equiv 1,3(mod\ 6),\ v \geq 19$. They used the following notations for the fine structure: (t,s) is the fine structure of a $TS(v,3)$ if $c_2 = t$ and $c_3 = \frac{1}{6}v(v-1) - s$, (hence $c_1 = 3s - 2t$). So, $(t,s) \leftrightarrow (3s-2t, t, \frac{1}{6}v(v-1) - s)$.

They defined $Adm(v) = \{(t,s) : 0 \leq t \leq s \leq \frac{1}{6}v(v-1),\ s \notin \{1,2,3,5\}, (t,s) \notin \{(1,4),(2,4),(3,4),(1,6),(2,6),(3,6),(5,6),(2,7),(5,7),(1,8),(3,8),(5,8)\}\}$.

Theorem 6 *(Colbourn, Mathon, Rosa, Shalaby [16])*
For $v \equiv 1,3(mod\ 6), v \geq 19,\ (t,s)$ *is the fine structure of a* $TS(v,3)$ *if and only if* $(t,s) \in Adm(v)$.

They used the notation $Fine(v)$ for the set of fine structures which arise in $TS(v,3)$ systems.

The above theorem asserts that for $v \geq 19, v \equiv 1,3(mod\ 6),\ Fine(v) = Adm(v)$.

They also describe the determination of the fine structure for small values of v.

Colbourn, Mathon, Shalaby [17] proved that the necessary conditions are sufficient for (c_1,c_2,c_3) to be the fine structure of a threefold triple system with $v \equiv 5(mod\ 6), v \geq 17$.

Theorem 7 *(Colbourn, Mathon, Shalaby [17])*
For $v \equiv 5(mod\ 6), v \geq 17, Fine(v) = Adm(v)$.

2.3 Disjoint triple systems with $\lambda \geq 1$

It is known that a necessary condition for the existence of a $TS(v, \lambda)$ is $\lambda \equiv 0(mod\ \lambda(v))$ where

$$\begin{aligned} \lambda(v) &= 1 \quad \text{if } v \equiv 1 \text{ or } 3(mod\ 6), \\ \lambda(v) &= 2 \quad \text{if } v \equiv 0 \text{ or } 4(mod\ 6), \\ \lambda(v) &= 3 \quad \text{if } v \equiv 5(mod\ 6), \\ \lambda(v) &= 6 \quad \text{if } v \equiv 2(mod\ 6). \end{aligned}$$

It is also known that if one does not require all elements of B to be distinct as subsets then these conditions are also sufficient.

We denote $d_\lambda(v)$ to be *the maximum number of pairwise disjoint* $TS(v, \lambda)s$.

In [39], Rosa showed that $d_\lambda(v) \leq \frac{(v-2)}{\lambda}$. Teirlinck [49] conjectured that $d_{\lambda(v)}(v) = \frac{v-2}{\lambda(v)}$, for all $v \neq 7$. Schreiber [43] proved $d_2(v) = \frac{1}{2}(v-2)$, for $v \equiv 0$ or 4 $(mod\ 12)$. Teirlinck [49] proved

$$d_2(v) = \frac{1}{2}(v-2), \text{ for all } v \equiv 0 \text{ or } 4\ (mod 6), v > 0$$

and

$$d_6(v) = \frac{1}{6}(v-2) \text{ for all } v \equiv 2(mod\ 6),$$

and so, for v even, $d_{\lambda(v)}(v)$ is completely determined.

For $\lambda(v) = 3$ we have two partial results. Kramer [26] showed $d_3(v) = \frac{1}{3}(v-2)$ whenever v is a prime power and $v \equiv 5(mod\ 6)$, and Teirlinck [53] show that if v is a product of primes p for which the order of $-2(mod\ p)$ is congruent to $2(mod\ 4)$, then $d_3(3v+2) = v$.

Chapter 3

Cyclic triple systems: Basic tools and definitions

In this chapter we discuss cyclic Steiner triple systems and cyclic triple systems. We start with basic definitions and results taken from the literature which will be used in this chapter. Then in Section 3.1, we give a short history of how the investigation of disjoint cyclic Steiner triple systems started. For small orders we provide the maximum number of disjoint Steiner triple systems and some easy bounds proved by Colbourn [12], and also some other bounds proved by Baker and Shalaby [4]. In Section 3.2, we discuss generalizations of triple systems such as Mendelsohn triple systems. The most important results here are given by C. Colbourn, M. Coulbourn [14] and Baker, Shalaby [4].

Definition An $STS(v)$ is *cyclic* if it has an automorphism consisting of a single cycle of length v, i.e. the automorphism when written in cyclic notation, consists of a single cycle of length v.

Notation: We use the notation $CSTS(v)$ for the cyclic Steiner triple systems of order v, and $CTS(v, \lambda)$ for the cyclic triple systems of order v and index λ.

Definition: The *fine structure* of a $CTS(v, \lambda)$ is the vector $(c_1, c_2, \ldots, c_\lambda)_c$, where c_i is the number of base blocks repeated exactly i times in the cyclic triple system.

[Hooked] Skolem sequences, [hooked] Rosa sequences, Langford sequences, k-extended Skolem sequences and 2-near Skolem sequences are important in the study of cyclic Steiner triple systems. We use these sequences in Chapter 4 to construct new [hooked] Skolem sequences of order n.

Definition A *Skolem sequence* of order n is a sequence $S_n = (s_1, s_2, \ldots, s_{2n})$ of $2n$ integers satisfying the conditions:

1. for every $k \in \{1, 2, \ldots, n\}$ there are exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = k$, and

2. if $s_i = s_j = k,\ i < j$, then $j - i = k$.

Skolem sequences are also written as a collection of ordered pairs $\{(a_i, b_i) : 1 \leq i \leq n,\ b_i - a_i = i\}$ with $\cup_{i=1}^{n}\{a_i, b_i\} = \{1, 2, \ldots, 2n\}$.

For example, a Skolem sequence of order 4 is: $S_4 = (1, 1, 4, 2, 3, 2, 4, 3)$ or, equivalently, the collection $\{(1, 2), (4, 6), (5, 8), (3, 7)\}$ (these are the indices where symbols $1, 2, 3, 4$ appear).

Definition A *hooked Skolem sequence* of order n is a sequence $hS_n = (s_1, s_2, \ldots, s_{2n+1})$ of $2n + 1$ integers satisfying conditions (1), (2) and (3): $s_{2n} = 0$.

For example, a hooked Skolem sequence of order 3 is: $hS_3 = (3, 1, 1, 3, 2, 0, 2)$.

Theorem 8 *(Skolem [47])*
A Skolem sequence of order n exists if and only if $n \equiv 0, 1 (mod\ 4)$.

Theorem 9 *(O'Keefe [36])*
A hooked Skolem sequence of order n exists if and only if $n \equiv 2, 3 (mod\ 4)$.

In 1957, Skolem [47], when studying Steiner triple systems, considered the possibility of distributing the numbers $1, 2, \ldots, 2n$ in n pairs (a_i, b_i) such that $b_i - a_i = i$, for $i = 1, 2, \ldots, n$. If $\{(a_i, b_i) | 1 \leq i \leq n\}$ is a Skolem sequence, the differences that we get from the base blocks $\{0, i, b_i + n\}$ are $\{i - 0, b_i + n - 0, b_i + n - i\} = \{i, b_i + n, a_i + n\}$. Thus, when we look at the set of differences we get for all base blocks, we see that $\{i, b_i + n, a_i + n\}$ gets us $1, 2, \ldots, n$ and n plus all the a_i and b_i which gives us $\{n + 1, n + 2, \ldots, 3n\}$ differences. Developing the base blocks mod $6n + 1$ gets us a $CSTS(6n + 1)$. For example, for $n = 4$ the following Skolem sequence $S_4 = (1, 1, 3, 4, 2, 3, 2, 4)$ gives the pairs: $(1, 2)$, $(5, 7)$, $(3, 6)$, $(4, 8)$. This partition gives rise to the base blocks $\{0, 1, 6\}, \{0, 2, 11\}, \{0, 3, 10\}, \{0, 4, 12\} (mod\, 25)$ or to the base blocks:

$\{0, 5, 6\}, \{0, 9, 11\}, \{0, 7, 10\}, \{0, 6, 12\} (mod\, 25)$.

Definition: Two [hooked] Skolem sequences S and S' of order n are *disjoint* if $s_i = s_j = k = s_t^{'} = s_u^{'}$ implies that $\{i, j\} \neq \{t, u\}$, for all $k = 1, \ldots, n$, i.e. for any symbol, the two locations for that symbol are different as a pair then the two locations for that symbol in the other sequence.

Definition: Given a Skolem sequence, $S = (s_1, \ldots, s_{2n})$ the reverse $\overleftarrow{S} = (s_{2n}, \ldots, s_1)$ is also a Skolem sequence. If S and S' are disjoint, then S is *reverse-disjoint.*

For example, the following two disjoint Skolem sequences of order 4 are reverse-disjoint: $S_4 = (1, 1, 4, 2, 3, 2, 4, 3)$ and $S_4 = (2, 3, 2, 4, 3, 1, 1, 4)$. Both are reverse-disjoint. Each reverse-disjoint Skolem sequence of order 4 gives four disjoint cyclic $STS(25)$. The first one, for example, gives the following 4 disjoint cyclic $STS(25)$:

1. $\{0, 1, 6\}$, $\{0, 2, 10\}$, $\{0, 3, 12\}$, $\{0, 4, 11\} (mod\, 25)$
2. $\{0, 5, 6\}$, $\{0, 8, 10\}$, $\{0, 9, 12\}$, $\{0, 7, 11\} (mod\, (25)$
3. $\{0, 1, 12\}$, $\{0, 2, 9\}$, $\{0, 3, 8\}$, $\{0, 4, 10\} (mod\, 25)$
4. $\{0, 11, 12\}$, $\{0, 7, 9\}$, $\{0, 5, 8\}$, $\{0, 6, 10\} (mod\, 25)$.

Two disjoint hooked Skolem sequences of order 7 are:

$hS_7 = (5, 7, 1, 1, 6, 5, 3, 4, 7, 3, 6, 4, 2, 0, 2)$ and

$hS_7 = (6, 1, 1, 5, 7, 2, 6, 2, 5, 3, 4, 7, 3, 0, 4)$.

In 1966, Rosa [39] introduced two types of sequences: one of them is an extended Skolem sequence with a hook in the middle, $s_{n+1} = 0$, known as a split Skolem sequence or a Rosa sequence and the other is a sequence with two hooks in the positions $n+1$ and $2n+1$ called hooked Rosa sequence.

Theorem 10 *(Rosa [39])*

1. *A Rosa sequence of order n (R_n) exists if and only if* $n \equiv 0, 3 (mod\ 4)$.

2. *A hooked Rosa sequence of order n (hR_n) exists if and only if* $n \equiv 1, 2 (mod\ 4)$.

Rosa and hooked Rosa sequences gives rise to cyclic Steiner triple systems of order $6n+3$.

For a cyclic $STS(27)$, where we have a base block $\{0, 9, 18\}$ we need other blocks to cover distances 1 to 13 except 9. A Rosa sequence of order 4 with a hook in position 5 work, since this implies a set $\{a_i, b_i\}|i = 1, \ldots, n\}$ which gives us every number in $1, \ldots, 2n+1$ except $n+1$, i.e. every number in $1, \ldots, 9$ except 5. So, the base blocks $\{0, i, b_i + n\}$ give differences $\{i, b_i + n, a_i + n\}$. As i varies from 1 to 4 this will give every number in $n+1, \ldots, 3n+1$ except $2n+1$, i.e. every number in $5, \ldots, 13$ except 9. So, along with the short orbit will get every distance in $\{1, 2, \ldots, 3n+1\}$. Developing these blocks mod $6n+3$, i.e. 27 we get $STS(2(3 \times 4) + 3)$=$STS(27)$. So, in general a Rosa sequence of order n gives a $CSTS(6n+3)$. For example, the Rosa sequence of order 4: $R_4 = (1, 1, 3, 4, 0, 3, 2, 4, 2)$ gives rise to the pairs $(a_i, b_i), i = 1, 2, 3, 4$; i.e. $\{(1, 2), (7, 9), (3, 6), (4, 8)\}$. These pairs gives the base blocks $\{0, i, b_i + 4\}$ or $\{0, a_i + 4, b_i + 4)\}$, $i = 1, 2, 3, 4$, i.e. $\{0, 1, 6\}$, $\{0, 2, 13\}, \{0, 3, 10\}$, $\{0, 4, 12\} (mod\ 27)$ or $\{0, 5, 6\}$, $\{0, 11, 13\}, \{0, 7, 10\}, \{0, 8, 12\} (mod\ 27)$. With the addition of the base block $\{0, 9, 18\} (mod\ 27)$, we get the blocks of two cyclic $STS(27)$ with one base block in common.

Definition: A *k-extended Skolem sequence of order n* is an integer sequence $k - ext\, S_n = (s_1, s_2, \ldots, s_{2n+1})$ in which $s_k = 0$ and for each $j \in \{1, 2, \ldots n\}$, there exists a unique $i \in \{1, 2, \ldots, n\}$ such that $s_i = s_{i+j} = j$.

Such sequence exists if and only if either 1) k is odd and $n \equiv 0$ *or* 1 $(mod\ 4)$ or 2)

k is even and $n \equiv 2$ *or* 3 $(mod\ 4)$.

For example, $3 - ext\, S_5 = (5,3,0,4,3,5,2,4,2,1,1)$ is a 3-extended Skolem sequence of order 5.

Definition: A *Langford sequence* of order n and defect d, $n > d$ (also called perfect sequence) is a sequence $L_d^m = (l_1, l_2, \ldots, l_{2n})$ of $2n$ integers satisfying the conditions:

1. for every $k \in \{d, d+1, \ldots, d+n-1\}$, there exist exactly two elements $l_i, l_j \in L$ such that $l_i = l_j = k$, i.e. symbol set for L_d^m is n consecutive integers starting at d, each used twice, and

2. if $l_i = l_j = k$ with $i < j$, then $j - i = k$, i.e. between the two locations of symbol k, there are $k-1$ other symbols.

For example, a Langford sequence of order 5 and defect 3 is
$L_3^5 = (7,5,3,6,4,3,5,7,4,6)$.

Definition: A *hooked Langford sequence* of order n and defect d is a sequence $hL_d^m = (l_1, l_2, \ldots, l_{2n+1})$ of $2n+1$ integers satisfying conditions (1) and (2) of the definition above and (3): $l_{2n} = 0$.

For example, a hooked Langford sequence of order 5 and defect 2 is : $hL_2^5 = (6,4,2,5,2,4,6,3,5,0,3)$.

Definition Let m, n be positive integers, $m \leq n$. A *near-Skolem sequence of order n and defect m* is a sequence $m - near\, S_n = (s_1, s_2, \ldots, s_{2n-2})$ of integers $s_i \in \{1, 2, \ldots, m-1, m+1, \ldots, n\}$ which satisfies the following conditions:
(1) for every $k \in \{1, 2, \ldots, m-1, m+1, \ldots, n\}$, there are exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = k$, and
(2) if $s_i = s_j = k$ then $j - i = k$.

Example: $3 - near\ S_5 = (4,5,1,1,4,2,5,2)$ is a 3-near Skolem sequence of order 5.

Definition A *hooked near-Skolem sequence of order n and defect m* is a sequence $m - near\, hS_n = (s_1, s_2, \ldots, s_{2n-1})$ of integers $s_i \in \{1, 2, \ldots, m-1, m+1, \ldots, n\}$ satisfying conditions (1), (2) and the condition (3): $s_{2n-2} = 0$.

Example: $5 - near\ hS_5 = (2,3,2,6,3,7,4,1,1,6,4,0,7)$ is a hooked 5-near Skolem sequence of order 7.

We refer to near-Skolem sequences and hooked near-Skolem sequences of order n and defect m as m-near Skolem sequences and hooked m-near Skolem sequences, respectively.

Theorem 11 *(Shalaby [41])*
An m-near Skolem sequence of order n exists if and only if $n \equiv 0, 1 (mod\ 4)$ *and m is odd, or* $n \equiv 2, 3 (mod\ 4)$ *and m is even.*

Theorem 12 *(Shalaby [41])*
A hooked m-near Skolem sequence of order n exists if and only if $n \equiv 0, 1 (mod\ 4)$ *and m is even, or* $n \equiv 2, 3 (mod\ 4)$ *and m is odd.*

Some important results are given in the following theorem:

Theorem 13 *(Bermond, Brouwer, Germa [8], Linek, Mor [29], Simpson [45]) Necessary and sufficient conditions for a Langford sequence to be perfect are:*

1. $n \geq 2d - 1$, *and*

2. $n \equiv 0,\ 1 (mod\ 4)$ *for d odd,* $n \equiv 2,\ 3 (mod\ 4)$ *for d even.*

Theorem 14 *(Bermond, Brouwer, Germa [8], Linek, Mor [29], Simpson [45]) Necessary and sufficient conditions for the sequence* $(d, d+1, \ldots, d+n-1)$ *to be hooked are*

1. $n(n + 1 - 2d) + 2 \geq 2$, *and*

2. $n \equiv 2,\ 3 (mod\ 4)$ *for d odd,* $n \equiv 0, 1 (mod\ 4)$ *for d even.*

These results are important because we use these [hooked] Langford sequences to construct new [hooked] Skolem sequences of order n.

We use Skolem sequences and hooked Skolem sequences to construct $CSTS(v)$, $v \equiv 1, 3(mod\ 6)$. To construct Skolem sequences and hooked Skolem sequences of larger order we adjoin one Skolem sequence of small order with a Langford sequence. For example, the hooked Skolem sequence of order 3: $hS_3 = (3, 1, 1, 3, 2, 0, 2)$ and the Langford sequence of order 7 and defect 4: $L_4^7 = (10, 8, 6, 4, 9, 7, 5, 4, 6, 8, 10, 5, 7, 9)$ gives rise to the hooked Skolem sequence of order 10: $hS_{10} = (10, 8, 6, 4, 9, 7, 5, 4, 6, 8, 10, 5, 7, 9, 3, 1, 1, 3, 2, 0, 2)$.

From these sequence we can take the pairs: $\{(16, 17), (19, 21), (15, 18), (4, 8), (7, 12), (3, 9), (6, 13), (2, 10), (5, 14), (1, 11)\}$. These pairs gives rise to the base blocks for a cyclic $STS(61)$: $\{0, 1, 27\}, \{0, 2, 31\}, \{0, 3, 28\}, \{0, 4, 18\}, \{0, 5, 22\}, \{0, 6, 19\}, \{0, 7, 23\}, \{0, 8, 20\}, \{0, 9, 24\}, \{0, 10, 21\}(mod\ 61)$.

Now, if we take the same Langford sequence of order 7 and defect 4: $L_4^7 = (10, 8, 6, 4, 9, 7, 5, 4, 6, 8, 10, 5, 7, 9)$ with a different Skolem sequence of order 3: $hS_3 = (1, 1, 2, 3, 2, 0, 3)$, we get another Skolem sequence of order 10: $hS_{10} = (10, 8, 6, 4, 9, 7, 5, 4, 6, 8, 10, 5, 7, 9, 1, 1, 2, 3, 2, 0, 3)$ which has the first 7 pairs in common with the previous sequence.

For $n = 12$ we can take a hooked Skolem sequence of order 3 and a reversed hooked Langford sequence of order 9 and defect 4 and concatenate them such that the hooks match.

For example taking the hooked Skolem sequence of order 3: $hS_3 = (1, 1, 2, 3, 2, 0, 3)$ and the reversed hooked Langford sequence of order 9 and defect 4: $hL_4^9 = (12, 10, 8, 6, 4, 11, 9, 7, 4, 6, 8, 10, 12, 5, 7, 9, 11, 0, 5)$, we get the following Skolem sequence of order $n = 12$:

$S_{12} = (1, 1, 2, 3, 2, 5, 3, 11, 9, 7, 5, 12, 10, 8, 6, 4, 7, 9, 11, 4, 6, 8, 10, 12)$.

This is illustrated below:

(–,–,–,–,–,–,–,*,–) (hooked Skolem)

(–.*,–,–,–,–,–,–,–,–) (reversed hooked Langford)

(–,–,–,–,–,–,–,–,–,–,–,–,–,–,–,–,–,–,–) (Skolem not hooked)

Now if we take the same Langford sequence of order 9 and defect 4 and a disjoint hooked Skolem sequence of order 3, for example: $hS_3 = (3,1,1,3,2,0,2)$ and concatenate them together in the same way we get another Skolem sequence of order 12: $S_{12} = (3,1,1,3,2,5,2,11,9,7,5,12,10,8,6,4,7,9,11,4,6,8,10,12)$ which have 9 pairs in common with the previous sequence. (Therefore two cyclic Steiner triple system with $v = 73$ have 9 base blocks in common.)

3.1 Disjoint cyclic Steiner triple systems

We denote by $d_c(2,3,v)$ *the maximum number of disjoint cyclic Steiner triple systems,* $d_c^*(2,3,v)$ *the maximum number of disjoint cyclic and isomorphic Steiner triple systems* and $d_{cc}(2,3,v)$ *the size of the largest set of disjoint cyclic* $STS(v)$*s, where each have the same cyclic automorphism.*

Now we give a short introduction on what is done about $CSTS(v)$.

Heffter [21] observed that his constructions of the first and second difference problems, like Netto's constructions [35] gives cyclic Steiner triple systems. Skolem [46] developed simple techniques to construct cyclic Steiner triple systems, Bays [6] develop an extensive theory of multiplier automorphism of cyclic triple systems and Bose [9] established a new method for the direct constructions of cyclic Steiner triple systems and, for the first time, the existence problem for $\lambda = 2$ was systematically studied. Heffter [22] settled all orders less than 100 using solutions to his difference problems. Later, Peltesohn [37] solved completely Heffter's difference problems. Prior to this, Bays [6] undertook the enumeration of non-isomorphic cyclic Steiner triple systems. This research was continued by Colbourn [13]. Many other researchers studied restricted versions of Heffter's difference problems. Aleksejev [1] shows that the number of distinct cyclic Steiner triple systems tends to infinity as the order increases.

In 1966, Rosa [39] posed the problem of determining the size of the largest set of disjoint $CSTS(v)$. Colbourn [12] introduce some computational tools for determining the maximum number of disjoint $CSTS(v)$ for small orders using the solutions to Heffter's difference problems. He gave a catalogue of solutions to Heffter's difference problems using a method of finding perfect matchings in hyper graphs.

These catalogues provide exhaustive lists of distinct $CSTS(v)$. These lists are valuable in suggesting and verifying conjectures and are especially useful in determining how many disjoint $CSTS(v)$ exist.

We recall here the correspondence of cyclic Steiner triple systems with a Heffter's difference problem (HDP) [21]: suppose that $\{1, 2, \ldots, 3n\}$ can be partitioned into n triples $\{a, b, c\}$ such that $a + b \equiv c (mod\ 6n + 1)$ or $a + b + c \equiv 0\ (mod\ 6n + 1)$. Any $CSTS(6n + 1)$ yields a solution to $HDP(6k + 1)$, and any solution to $HDP(6n + 1)$ yields a cyclic $STS(6n + 1)$; in fact, it yields 2^k distinct $CSTS(6n + 1)$, since for each triple $\{a, b, c\}$ in the solution to HDP we can select either of the inequivalent starter blocks $\{0, a, a + b\}$ or $\{0, b, a + b\}$.

This transformation between $CSTS(v)$ and solutions to $HDP(v)$ will be a fundamental tool in the determination of $d_{cc}(2, 3, v)$.

The maximum number of disjoint $CSTS$ for small orders found by Colbourn [12] are:

$d_{cc}(2, 3, 7) = 2$; $d_{cc}(2, 3, 13) = 2$; $d_{cc}(2, 3, 19) = 8$;
$d_{cc}(2, 3, 25) = 15$; $21 \leq d_{cc}(2, 3, 31) \leq 26$;
$d_{cc}(2, 3, 37) = 32$.

Lemma 8 *(Colbourn [12])*
For $v \geq 1$,

$$d_{cc}(2, 3, v) = \begin{cases} 0, & v \not\equiv 1,\ 3 (mod\ 6) \\ 0, & v{=}9 \\ 1, & v \equiv 3\ (mod\ 6), v \neq 9 \end{cases}$$

Proof: A $CSTS(v)$ must exist whenever $d_{cc}(2, 3, v) \geq 1$. Since every $CSTS(6k + 3)$ contains the same short starter block, i.e. $(0, \frac{n}{3}, 2 \times \frac{n}{3})$, the last case is proved. □

We consider the case $v \equiv 1 (mod\ 6)$.

Colbourn established the following bounds:

Lemma 9 *(Colbourn [12])*
For $v \equiv 1 (mod\ 6)$, $d_{cc}(2, 3, v) \leq v - 5$.

Proof: A triple in a $CSTS(6k + 1)$ has the property that, for any two pairs of its elements, their differences are different. Thus, of the $\binom{v}{3}$ subsets of a v-set, $v(v-1)/2$ are different. The bound follows. □

Lemma 10 *(Colbourn [12])*
For $m \geq 1$, $d_{cc}(2, 3, 6m + 1) \geq 2$.

Proof: Select any solution $\{\{a_1, b_1, c_1\}, \ldots, \{a_k, b_k, c_k\}\}$ to $HDP(6k+1)$. Construct one $CSTS(6k + 1)$ by selecting starter blocks of the form $\{\{0, a_i, a_i + b_i\} | i = 1, \ldots, k\}$, and construct the second by selecting $\{\{0, b_i, a_i + b_i\} | i = 1, \ldots, k\}$. These two $CSTS(6k + 1)$ are disjoint as required. □

In 1991, Shalaby and Baker [4] showed the existence of disjoint Skolem, disjoint hooked Skolem sequences and applied these concepts to the existence problems of disjoint cyclic Steiner and Mendelsohn triple systems.

Lemma 11 *(Baker, Shalaby [4])*
The maximum number of mutually disjoint Skolem sequences of order n is at most n.

Proof: Since each [hooked] Skolem sequence must have an n in two places, n positions apart, there are only n distinct ways to select the pair of positions to contain n. □

Theorem 15 *(Baker, Shalaby [4])*
For all $n \equiv 0, 1 (mod\ 4), n \geq 4$, there exist at least four mutually disjoint Skolem sequences of order n.

Corollary 1 *(Baker, Shalaby [4])*
For all $v \geq 25, v \equiv 1$ or $7 (mod\ 24)$, $d_{cc}(2, 3, v) \geq 8$.

Theorem 16 *(Baker, Shalaby [4])*
For all $n \equiv 2, 3 (mod\ 4), n \geq 6$ there are at least three mutually disjoint Skolem sequences of order n.

Corollary 2 *(Baker, Shalaby [4])*
For all $v \geq 37, v \equiv 13$ or $19 (mod\ 24), d_{cc}(2, 3, v) \geq 6$.

3.2 Disjoint cyclic Mendelsohn triple systems

In recent years, combinatorial design researchers investigated many generalizations of triple systems. Such generalizations include twofold triple systems, directed triple systems, and Mendelsohn triple systems.

Definition A *twofold triple system* is a pair (V, B); V is a v-set, and B is a collection of 3-subsets of V, with the property that every 2-subset of elements of V appears in precisely two triples.

Definition A *directed triple system* is a pair (V, B); V is a v-set and B is a collection of edge-disjoint transitive tournaments of order 3 with vertices from V, having the property that every ordered pair of elements of V appears in precisely one of the tournaments.

Definition A *Mendelsohn triple system $MTS(v)$* differs from directed triple systems only in that B contains directed cycles of length 3.

These three types of triples are related; if we omit the directions in a Mendelsohn or directed triple system, we obtain a twofold triple system.

We use the notation $m_c(v)$ for the *maximum number of disjoint cyclic Mendelsohn triple systems* of order v.

We have the following result due to C. Colbourn and M. Colbourn:

Theorem 17 *(C. Colbourn, M. Colbourn [14])*
A cyclic $MTS(v)$ exists if and only if $v \equiv 1,\ 3 (mod\ 6),\ v \neq 9$.

They also have some bounds on $m_c(v)$. For $v \not\equiv 1, 3(mod\ 6),\ v \neq 9,\ m_c(v) = 0$ by the previous theorem. For $v \equiv 3(mod\ 6),\ v \neq 9,\ m_c(v) = 1$ since each such MTS contains the difference triple $(v/3, v/3, v/3)$. $m(v) \leq 2$, since this is an upper bound on the number of disjoint $MTS(v)$, and $m(v) \geq d_{cc}(2, 3, v)$ by using the conversion between STS and MTS.

As a consequence of the cyclic constraints, they were able to find a better upper bound.

Lemma 12 *(C. Colbourn, M. Colbourn [14])*
For $v \equiv 1, 3(mod\ 6),\ m_c(v) \leq v - 5$.

Proof: Each cyclic $MTS(v)$ is presented as $(v-1)/3$ difference triples. A difference triple of a $MTS(v)$ contains three distinct differences. There are $(v-1)(v-2)/3$ orbits of triples under the cyclic automorphism; $v - 1$ of these have differences triples with repeated differences. So, there are $(v-1)(v-5)$ difference triples. Hence the bound follows. □

In [14], there are described computational results for small orders of disjoint cyclic Mendelsohn triple systems.

For example, $m_c(7) = 2, m_c(13) = 8; 12 \leq m_c(19) \leq 14;\ 17 \leq m_c(25) \leq 20$.

C. Colbourn and M. Colbourn [14] presented the solution in terms of starter blocks: given a difference triple (a, b, c), the corresponding starter block is $(0, a, a + b)$. For example, for order 7 there are two solutions:

$$(0, 1, 2), (0, 3, 1)$$

$$(0, 1, 5), (0, 3, 2)$$

Also by Baker and Shalaby we have the following:

Corollary 3 *(Baker, Shalaby [4])*
For all $v \geq 25$ *and* $v \equiv 1,\ 7(mod\ 24), m_c(v) \geq 8$.

Corollary 4 *(Baker, Shalaby [4])*
For all $v \geq 37$ *and* $v \equiv 13,\ 19(mod\ 24), m_c \geq 6$.

Chapter 4

The intersection spectrum of two distinct Skolem sequences and two distinct hooked Skolem sequences

In this chapter we prove, with some possible exceptions, that there exists two [hooked] Skolem sequences of order n intersecting in $0, 1, 2, \ldots, n-3, n$ pairs. In Appendix A, we provide all the intersections between two [hooked] Skolem sequences of order $1 \leq n \leq 9$. Then we assume inductively that, for small orders, there exists two distinct [hooked] Skolem sequences of order n intersecting in $\{0, 1, 2, \ldots, n-3, n\}$ pairs and we prove that this is true for larger orders, with few possible exceptions. In Appendix C we give a detailed list of all the possible exceptions. For this we first prove that it is not possible for two [hooked] Skolem sequences to have exactly $n-2$ and $n-1$ pairs in common. We then split the problem of finding the intersection of two distinct [hooked] Skolem sequences of order n into three cases: first case for the pairs in the interval $[0, \lfloor \frac{n}{3} \rfloor]$, second case for the pairs in the interval $(\lfloor \frac{n}{3} \rfloor, 2\lfloor \frac{n}{3} \rfloor)$ and, finally, the third case for the pairs in the interval $[2\lfloor \frac{n}{3} \rfloor, n]$.

Lemma 13 *It is not possible for two distinct [hooked] Skolem sequences of order n to intersect in exactly $n-1$ pairs.*

Proof: If we have $n-1$ pairs in common then the last pair is also in common, so we have n pairs in common. □

Lemma 14 *It is not possible for two distinct [hooked] Skolem sequences of order n to intersect in exactly $n-2$ pairs.*

Proof: Suppose that we have two Skolem sequences that intersect in exactly $n-2$ pairs. Then we have two differences left, say a and b, and four positions, say a_1, a_2, a_3, a_4. Without loss of generality we assume that $a_1 < a_2 < a_3 < a_4$.

Case I: In the first sequence consider a in positions a_1, a_2 and b in positions a_3, a_4 In the second sequence, consider:

1)a and b in the same positions which gives n pairs in common which is impossible because these two sequences have only $n-2$ pairs in common,

2)a in positions a_1, a_3 so that we have to put b in positions a_2, a_4 which is impossible because $a_2 - a_1 = a$ in the first sequence and $a_3 - a_1 = a$ in the second sequence but $a_3 \neq a_1$,

3)a in positions a_1, a_4 so that we need to put b in positions a_2, a_3 which is impossible because $a_2 - a_1 = a$ in the first sequence and $a_4 - a_1 = a$ in the second sequence but $a_4 \neq a_2$,

4)a in positions a_3, a_4 so that we need to put b in positions a_1, a_2 which is impossible because $a_4 - a_3 = b$ in the first sequence and $a_4 - a_3 = a$ in the second sequence but $a \neq b$,

5)a in positions a_2, a_4 so that we need to put b in positions a_1, a_3 which is impossible because $a_2 - a_1 = a$ in the first sequence and $a_4 - a_2 = a$ in the second sequence but $a_4 \neq a_1$.

Case II: In the first sequence consider a in the positions a_1, a_3 and b in positions a_2, a_4

In the second sequence, consider:

1)a and b in the same positions gives n pairs in common which is impossible because these two sequences have only $n-2$ pairs in common,

2)a in positions a_1, a_2 so that we have to put b in positions a_3, a_4 which is impossible because $a_3 - a_1 = a$ in the first sequence and $a_2 - a_1 = a$ in the second sequence but $a_3 \neq a_2$,

3)a in positions a_1, a_4 so that we need to put b in positions a_2, a_3 which is impossible because $a_3 - a_1 = a$ in the first sequence and $a_4 - a_1 = a$ in the second sequence but $a_4 \neq a_3$,

4)a in positions a_2, a_4 so that we need to put b in positions a_1, a_3 which is impossible because $a_3 - a_1 = a$ in the first sequence and $a_3 - a_2 = a$ in the second sequence, but $a_1 \neq a_2$,

5)a in positions a_3, a_4 so that we need to put b in positions a_1, a_2, which is impossible because $a_3 - a_1 = a$ in the first sequence and $a_4 - a_3 = a$ in the second sequence but $a_4 \neq a_1$.

Case III: In the first sequence consider a in positions a_1, a_4 and b in positions a_2, a_3

In the second sequence, consider:

1)a and b in the same positions gives n pairs in common which is impossible because these two sequences have only $n-2$ pairs in common,

2)a in positions a_1, a_2 so that we have to put b in positions a_3, a_4 which is impossible because $a_4 - a_1 = a$ in the first sequence and $a_2 - a_1 = a$ in the second sequence but $a_4 \neq a_2$,

3)a in positions a_1, a_3 so that we need to put b in positions a_2, a_4 which is impossible because $a_4 - a_1 = a$ in the first sequence and $a_3 - a_1 = a$ in the second sequence but $a_4 \neq a_3$,

4)a in positions a_2, a_3 so that we need to put b in positions a_1, a_4 which is impossible because $a_3 - a_2 = b$ in the first sequence and $a_3 - a_2 = a$ in the second sequence but $a \neq b$,

5)a in positions a_2, a_4 so that we need to put b in positions a_1, a_4 which is impossible because $a_4 - a_1 = a$ in the first sequence and $a_4 - a_2 = a$ in the second sequence but $a_1 \neq a_2$.

Therefore, it is not possible for two [hooked]Skolem sequences of order n to have exactly $n-2$ pairs in common. □

We denote Int_{S_n}={k: there exists two [hooked]Skolem sequences of order n with k pairs in common}. This is *the intersection spectrum* of two [hooked] Skolem sequences of order n.

For small orders $1 \leq n \leq 9$ we give the intersection spectrum between two [hooked] Skolem sequences of order n in Appendix A. Using these results we have: $Int_{S_1} = \{1\}$, $Int_{S_2} = \{2\}$, $Int_{S_3} = \{0,3\}$, $Int_{S_4} = \{0,1,4\}$, $Int_{S_5} = \{0,1,5\}$, $Int_{S_6} = \{0,1,2,3,6\}$, $Int_{S_7} = \{0,1,2,3,4,7\}$, $Int_{S_8} = \{0,1,2,3,4,5,8\}$, $Int_{S_9} = \{0,1,2,3,4,5,6,9\}$.

We prove that there exists two [hooked] Skolem sequence of order n such that

$$Int_{S_n} = \{0, 1, 2, 3, \ldots, n-3, n\}.$$

We assume inductively that for [hooked] Skolem sequences of small orders this is already true and we prove that this is true for [hooked] Skolem sequences of larger orders.

We split the problem of finding the intersection spectrum of two [hooked] Skolem sequences in three cases:

1. the number of pairs between $[0, \lfloor \frac{n}{3} \rfloor]$;
2. the number of pairs between $(\lfloor \frac{n}{3} \rfloor, 2\lfloor \frac{n}{3} \rfloor)$;
3. the number of pairs between $[2\lfloor \frac{n}{3} \rfloor, n]$.

Case (I): The intersection of two distinct Skolem sequences and hooked Skolem sequences of order n in $[0, \lfloor \frac{n}{3} \rfloor]$ pairs

To find the intersection spectrum in $[0, \lfloor \frac{n}{3} \rfloor]$ pairs of two Skolem sequences of order n, we construct two new Skolem sequences of order n by adjoining a [hooked] Skolem sequence of small order with a [hooked] Langford sequence.

We construct these sequences in the following way: let k be a positive integer and repeat the process below for $k = 0, 1, \ldots, \lfloor \frac{n}{3} \rfloor - 2$ or until all the pairs are found.

Take a Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - k$ and adjoin it with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + k$ and defect $\lfloor \frac{n}{3} \rfloor - k + 1$. (If this Langford sequence does not exist for $k = 0$, take $k = 1$). If the Langford sequence is perfect, take the same Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - k$ and adjoin with the reverse Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + k$ and defect $\lfloor \frac{n}{3} \rfloor - k + 1$. The number of pairs in common between these Langford sequences and their reverse is given in Table B.1 if the Langford sequence has order $n = 4t$, Table B.6 if the Langford sequence has order $2d - 1$, Table B.2 if the Langford sequence has order $2d - 1 + 4r$, where r is a positive integer. In this way we can form two new Skolem sequences of order n with $\lfloor \frac{n}{3} \rfloor - k$ or more pairs in common. We can also adjoin a Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - k$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + k$ and defect $\lfloor \frac{n}{3} \rfloor - k + 1$. Then adjoin another Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - k$ which can have $0, 1, \ldots, \lfloor \frac{n}{3} \rfloor - k - 3, \lfloor \frac{n}{3} \rfloor - k$ pairs in common with the previous Skolem sequence and adjoin with the reverse Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + k$ and defect $\lfloor \frac{n}{3} \rfloor - k + 1$. These give two Skolem sequences of order n with $0, 1, \ldots, \lfloor \frac{n}{3} \rfloor - k - 3, \lfloor \frac{n}{3} \rfloor - k$ pairs in common. If the Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + k$ and defect $\lfloor \frac{n}{3} \rfloor - k + 1$ is hooked, fill the hook with the pair 2 which makes the sequence perfect and adjoin this Langford sequence with a 2-near Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - k$, then adjoin the previous 2-near Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - k$ with the reverse of the Langford sequence above. The number of pairs in common between the Langford sequences and their reverse is given in Table B.8 if the Langford sequence has order $4t + 2$, or in Table B.9 if the Langford sequence has order $2d + 1 + 4r$, where t and r are positive integers. In this way we can form two Skolem sequences of order n which can have $\lfloor \frac{n}{3} \rfloor - 2$ or more pairs in common. Two disjoint Skolem sequences of order n can be found in [4]. Also if the Langford sequence has order $2d - 1$ we are able to find more sequences of this kind and, in Table B.6, we have the numbers of pairs in common between these sequences or if the Langford sequence has order $2d$, *d even* we can find two different sequences and their reverse, one from Table B.1 and the other from Table B.7.

For example, to find if two Skolem sequences of order 16 intersect in $0, 1, 2, 3, 4, 5$ pairs, we have the following procedure: take a Skolem sequence of order 5 and adjoin it with a Langford sequence of order 11 and defect 6, then take the same Skolem sequence of order 5 and adjoin with the reverse Langford sequence of order 11 and defect 6. In [29], we can find more constructions for Langford sequences of order $2d - 1$. This Langford sequence and its reverse can have $0, 1, 2$ pairs in common (Table B.6). Now,

- taking those two Langford sequences that are disjoint and adjoining these two with the same Skolem sequence of order 5 gives two Skolem sequences of order 16 with 5 pairs in common;
- taking a Skolem sequence of order 5 and adjoining it with a Langford sequence of order 11 and defect 6, then taking a disjoint Skolem sequence of order 5 and adjoining it with a disjoint Langford sequence of order 11 and defect 6 gives two Skolem sequences of order 16 which are be disjoint;
- taking a Skolem sequence of order 5 and adjoining it a Langford sequence of order 11 and defect 6, then taking a Skolem sequence of order 5 which have one pair in common with the previous Skolem sequence and adjoining it with a disjoint Langford sequence of order 11 and defect 6 gives two Skolem sequences of order 16 with one pair in common;
- taking a Skolem sequence of order 5 and adjoining a Langford sequence of order 11 and defect 6, then taking a Skolem sequence of order 5 which have one pair in common with the previous Skolem sequence and adjoin with a Langford sequence of order 11 and defect 6 which have one pair in common with the previous Langford sequence gives two Skolem sequences of order 16 with two pairs in common;
- taking a Skolem sequence of order 5 and adjoining it a Langford sequence of order 11 and defect 6, then taking a Skolem sequence of order 5 which have

one pair in common with the previous Skolem sequence and adjoining it with a Langford sequence of order 11 and defect 6 which have two pairs in common with the previous Langford sequence gives two Skolem sequences of order 16 with three pairs in common;

- taking a Skolem sequence of order 4 and adjoining it a Langford sequence of order 12 and defect 5, then taking the same Skolem sequence of order 4 and adjoining it the reverse Langford sequence of order 12 and defect 5, gives two Skolem sequences of order 16 with four pairs in common.

Using this technique we prove the following theorem:

Theorem 18 *For $n \geq 1$, the necessary conditions are sufficient for the existence of two Skolem sequences of order n to intersect in $[0, \lfloor \frac{n}{3} \rfloor]$ pairs with the following possible exceptions: for $n = 12t, n \geq 72, t \equiv 0, 1(mod\ 3)$ the exceptions are $\{\lfloor \frac{n}{3} \rfloor - 5, \lfloor \frac{n}{3} \rfloor\}$, for $t \equiv 2(mod\ 3), n \neq 17$ the exceptions are $\{\lfloor \frac{n}{3} \rfloor - 5, \lfloor \frac{n}{3} \rfloor - 2, \lfloor \frac{n}{3} \rfloor\}$, for $n = 17$ the exception is $\{4\}$, for $n = 12t + 8, t \equiv 0, 2(mod\ 3)$ the exception is $\{\lfloor \frac{n}{3} \rfloor - 3\}$, for $t \equiv 1(mod\ 3), n \neq 20$ the exceptions are $\{\lfloor \frac{n}{3} \rfloor - 3, \lfloor \frac{n}{3} \rfloor\}$, for $n = 20$ the exception is $\{3\}$, for $n \equiv 9(mod\ 12), n \neq 21$ the exception is $\{\lfloor \frac{n}{3} \rfloor\}$ and for $n = 21$ the exceptions are $\{5, 7\}$.*

Proof: For $1 \leq n \leq 9$, see Appendix A. For a list of all of the possible exceptions, see Appendix C.
We divide this proof into 6 cases. Let t be a positive integer.
Case (1): $n \equiv 0(mod\ 12)$
We start with:
(a) $n = 12t, t \equiv 0, 1(mod\ 3)$
and prove that $Int_{S_n} = [0, \lfloor \frac{n}{3} \rfloor] - \{\lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{3} \rfloor - 5\}$.
Taking a 2-near Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - 1$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 1$ and defect $\lfloor \frac{n}{3} \rfloor$, then taking the same 2-near Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - 1$ with the reverse of the Langford sequence above its reverse (Table B.9) gives two Skolem sequences of order n with $\lfloor \frac{n}{3} \rfloor - 1$ pairs in common.

Taking a 2-near Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - 2$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 2$ and defect $\lfloor \frac{n}{3} \rfloor + 1$, then taking the same 2-near Skolem sequence with the reverse of the Langford sequence its reverse (Table B.8) gives two Skolem sequences of order n with $\lfloor \frac{n}{3} \rfloor - 2$ pairs in common.

Taking a Skolem sequences of order $\lfloor \frac{n}{3} \rfloor - 3$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 3$ and defect $\lfloor \frac{n}{3} \rfloor - 2$, then taking another Skolem sequence of the same order which can have $0, 1, \ldots, \lfloor \frac{n}{3} \rfloor - 6, \lfloor \frac{n}{3} \rfloor - 3$ pairs in common with the previous sequence with the reverse of the Langford sequence (Table B.2) gives two Skolem sequences of order n with $0, 1, \ldots, \lfloor \frac{n}{3} \rfloor - 6, \lfloor \frac{n}{3} \rfloor - 3$ pairs in common.

Taking a Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - 4$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 4$ and defect $\lfloor \frac{n}{3} \rfloor - 3$, then taking the same Skolem sequence with the reverse of the Langford sequence above (Table B.1) gives two Skolem sequences of order n with $\lfloor \frac{n}{3} \rfloor - 4$ pairs in common.

Then we continue with:

(b) $n = 12t, t \equiv 2 (mod\ 3), n \neq 24$

and prove that $Int_{S_n} = [0, \lfloor \frac{n}{3} \rfloor] - \{\lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{3} \rfloor - 2, \lfloor \frac{n}{3} \rfloor - 5\}$.

Taking a 2-near Skolem sequences of order $\lfloor \frac{n}{3} \rfloor - 1$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 1$ and defect $\lfloor \frac{n}{3} \rfloor$, then taking the same 2-near Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - 1$ with the reverse of the Langford sequence above (Table B.9) gives two Skolem sequences of order n with $\lfloor \frac{n}{3} \rfloor - 1$ pairs in common.

Taking a Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - 3$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 3$ and defect $\lfloor \frac{n}{3} \rfloor - 2$, then taking another Skolem sequence which can have $0, 1, \ldots, \lfloor \frac{n}{3} \rfloor - 6, \lfloor \frac{n}{3} \rfloor - 3$ pairs in common and adjoining the reverse of the Langford sequence above its reverse (Table B.2) gives two Skolem sequences of order n with $0, 1, \ldots, \lfloor \frac{n}{3} \rfloor - 6, \lfloor \frac{n}{3} \rfloor - 3$ pairs in common.

Taking a Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - 4$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 4$ and defect $\lfloor \frac{n}{3} \rfloor - 3$, then taking the same Skolem sequence with the reverse of the Langford sequence above (Table B.1) gives two Skolem sequences of order n with $\lfloor \frac{n}{3} \rfloor - 4$ pairs in common.

And we take a small case separately:

(c) $n = 24$

and we prove that $Int_{S_{24}} = [0, 8] - \{3, 6, 8\}$.

Taking a 2-near Skolem sequence of order 7 with a Langford sequence of order 17 and defect 8, then taking the same 2-near Skolem sequence of order 7 with the reverse of the Langford sequence above (Table B.9) gives two Skolem sequences of order 24 with 7 pairs in common.

Taking a Skolem sequences of order 5 with a Langford sequence of order 19 and defect 6, then taking another Skolem sequence which can have $0, 1, 5$ pairs in common with the previous sequence and adjoining the reverse of the Langford sequence above (Table B.1) gives two Skolem sequences of order 24 with $0, 1$ or 5 pairs in common.

Taking a Skolem sequence of order 4 with a Langford sequence of order 20 and defect 5, then taking the same Skolem sequence with the reverse of the Langford sequence above (Table B.1) gives two Skolem sequences of order 24 with four pairs in common.

Taking a 2-near Skolem sequences of order 3 with a Langford sequence of order 21 and defect 4, then taking the same 2-near Skolem sequences of order 3 with the reverse of the Langford sequence above (Table B.9) gives two Skolem sequences of order 24 with two pairs in common.

Case(2): $n \equiv 1(mod\ 12)$

We start with:

(a) $n \geq 25$

and we prove that $Int_{S_n} = [0, \lfloor \frac{n}{3} \rfloor]$.

Taking a perfect Langford sequences of order $n - \lfloor \frac{n}{3} \rfloor$ and defect $\lfloor \frac{n}{3} \rfloor + 1$ and adjoining a Skolem sequence of order $\lfloor \frac{n}{3} \rfloor$, then taking the reverse of the Langford sequence above and adjoining it with a different Skolem sequence of order $\lfloor \frac{n}{3} \rfloor$ which can have $0, 1, 2, \ldots, \lfloor \frac{n}{3} \rfloor - 3, \lfloor \frac{n}{3} \rfloor$ pairs in common with the previous Skolem sequence. The number of pairs in common between the Langford sequence and its reverse are given in Table B.6. Taking different combinations of these sequences gives two Skolem sequences of order n with $[0, \lfloor \frac{n}{3} \rfloor]$ pairs in common.

Then, we continue with:

(b) $n = 13$

and we prove that $Int_{S_{13}} = [0, 4]$.

Taking two perfect Langford sequences of order 9 and defect 5 which can have $0, 1$ or 3 pairs in common(Table B.6), and adjoining it two Skolem sequence of order 4 which can have $0, 1$ or 4 pairs in common(see Appendix A).

Two disjoint Langford sequences of order 9 and defect 5 with two disjoint Skolem sequences of order 4 gives two disjoint Skolem sequences of order 13.

Two disjoint Langford sequences of order 9 and defect 5 with two Skolem sequences of order 4 with one pair in common gives two Skolem sequences of order 13 with one pair in common.

Two Langford sequences of order 9 and defect 5 with one pair in common with two Skolem sequences of order 4 with one pair in common gives two Skolem sequences of order 13 with two pairs in common.

Two Langford sequences of order 9 and defect 5 with 3 pairs in common with two disjoint Skolem sequences of order 4 gives two Skolem sequences of order 13 with three pairs in common.

Two disjoint Langford sequences of order 9 and defect 5 with two Skolem sequences of order 4 with four pairs in common gives two Skolem sequences of order 13 with four pairs in common.

Case (3): $n \equiv 4 (mod\ 12)$

We start with:

(a) $n \geq 28$

and we prove that $Int_{S_n} = [0, \lfloor \frac{n}{3} \rfloor]$.

Taking a Skolem sequence of order $\lfloor \frac{n}{3} \rfloor$ and adjoining it a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor$ and defect $\lfloor \frac{n}{3} \rfloor + 1$ and then taking another Skolem sequence of order $\lfloor \frac{n}{3} \rfloor$ which can have $\{0, 1, 2, \ldots, \lfloor \frac{n}{3} \rfloor - 3, \lfloor \frac{n}{3} \rfloor\}$ pairs in common with the previous Skolem sequence and adjoining it the reverse of the Langford sequence (Table B.6). Taking different combinations of these sequences gives two Skolem sequences of order n with $[0, \lfloor \frac{n}{3} \rfloor]$ pairs in common.

Then we continue with a small case:

(b) $n = 16$

and we prove that $Int_{S_{16}} = [0, 5]$.

Taking a Skolem sequences of order 5 and adjoin a Langford sequence of order 11 and defect 6, then taking another Skolem sequence of order 5 which can have $0, 1, 5$ pairs in common with the previous Skolem sequence (Appendix A) and adjoining it the reverse of the Langford sequence above (Table B.6). These gives two Skolem sequences of order 16 with $0, 1, 2, 3, 5$ pairs in common, next taking a Skolem sequence of order 4 and adjoining it a Langford sequence of order 12 and defect 5 and then taking the same Skolem sequence and adjoining it the reverse of the Langford sequence above. These two Langford sequences are disjoint (Table B.1). So, we get two Skolem sequences of order 16 with 4 pairs in common.

Case (4): $n \equiv 5(mod\ 12)$

We start with:

(a) $n \neq 17$

and we prove that $Int_{S_n} = [0, \lfloor \frac{n}{3} \rfloor]$.

Taking a Skolem sequence of order $\lfloor \frac{n}{3} \rfloor$ and adjoining it a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor$ and defect $\lfloor \frac{n}{3} \rfloor + 1$ and then taking a Skolem sequence of the same order as before which can have $\{0, 1, 2, \ldots, \lfloor \frac{n}{3} \rfloor - 3, \lfloor \frac{n}{3} \rfloor\}$ pairs in common with the previous Skolem sequence and adjoining it the reverse of the Langford sequence above (Table B.1 and Table B.7). Taking different combinations of these sequences gives two Skolem sequences of order n with $[0, \lfloor \frac{n}{3} \rfloor]$ pairs in common.

Then we continue with a small case:

(b) $n = 17$

and we prove that $Int_{S_{17}} = [0, 5] - \{4\}$.

Taking a Skolem sequence of order 5 and adjoin a Langford sequence of defect 6 and order 11, then taking another Skolem sequence of order 5 which can have 0, 1 or 5 pairs in common with the previous Skolem sequence (see Appendix A) and adjoining the reverse of the Langford sequence above (Table B.6). Taking different combinations of these sequences gives two Skolem sequences of order n with $\{0, 1, 2, 3, 5\}$ pairs in common.

Case (5):$n \equiv 8(mod\ 12)$

We start with:

(a) $n = 12t + 8, t \equiv 0, 2(mod\ 3)$

and we prove that $Int_{S_n} = [0, \lfloor \frac{n}{3} \rfloor] - \{\lfloor \frac{n}{3} \rfloor - 3\}$.

Taking a 2-near Skolem sequences of order $\lfloor \frac{n}{3} \rfloor$[41] and adjoining it a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor$ and defect $\lfloor \frac{n}{3} \rfloor + 1$, then taking the same 2-near Skolem sequence of order $\lfloor \frac{n}{3} \rfloor$[51] and adjoining it the reverse of the Langford sequence above (Table B.8). These constructions gives two Skolem sequences of order n with $\lfloor \frac{n}{3} \rfloor$ pairs in common.

Taking a Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - 1$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 1$ and defect $\lfloor \frac{n}{3} \rfloor$, then taking another Skolem sequence of the same order which can have $0, 1, \ldots, \lfloor \frac{n}{3} \rfloor - 4, \lfloor \frac{n}{3} \rfloor - 1$ pairs in common with the previous sequence and adjoining it the reverse of the Langford sequence(Table B.2). These constructions gives two Skolem sequences of order n with $0, 1, \ldots, \lfloor \frac{n}{3} \rfloor - 4, \lfloor \frac{n}{3} \rfloor - 1$ pairs in common.

Taking a Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - 2$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 2$ and defect $\lfloor \frac{n}{3} \rfloor - 1$, then taking the same Skolem sequence with the reverse of the Langford sequence above (Table B.1). These constructions gives two Skolem sequences of order n with $\lfloor \frac{n}{3} \rfloor - 2$ pairs in common.

Then we continue with:

(b) $n = 12t + 8, t \equiv 1(mod\ 3), n \neq 20$

and we prove that: $Int_{S_n} = [0, \lfloor \frac{n}{3} \rfloor] - \{\lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{3} \rfloor - 3\}$.

Taking a Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - 1$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 1$ and defect $\lfloor \frac{n}{3} \rfloor$, then taking another Skolem sequence which can have $0, 1, \ldots, \lfloor \frac{n}{3} \rfloor - 4, \lfloor \frac{n}{3} \rfloor - 1$ pairs in common with the previous sequence and adjoin the reverse of the Langford sequence (Table B.2)). These constructions gives two Skolem sequences of order n with $0, 1, \ldots, \lfloor \frac{n}{3} \rfloor - 4, \lfloor \frac{n}{3} \rfloor - 1$ pairs in common; the same Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - 2$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 2$ and defect $\lfloor \frac{n}{3} \rfloor - 1$ and then with its reverse (Table B.1) gives two Skolem sequences of order n with $\lfloor \frac{n}{3} \rfloor - 2$ pairs in common.

Then we prove for a small case:

(c) $n = 20$

and we prove that $Int_{S_{20}} = [0, 6] - \{3\}$.

Taking a Skolem sequence of order 5 with a Langford sequence of order 15 and defect 6, then taking another Skolem sequence of the same order which can have $0, 1, 5$ pairs in common with the previous sequence with the reverse of the Langford sequence (Table B.2). These constructions gives two Skolem sequences of order 20 with $0, 1, 5$ pairs in common; taking a Skolem sequence of order 4 with a Langford sequence of order 16 and defect 5, then taking the same Skolem sequence with the reverse of the Langford sequence (Table B.1) gives two Skolem sequences of order 20 with four pairs in common; the a 2-near Skolem sequences of order 3 with a Langford sequence of order 17 and defect 4, then taking the same 2-near Skolem sequence of order 3 with the reverse of the Langford sequence (Table B.9). These constructions gives two Skolem sequences of order 20 with two pairs in common.

Case (6): $n \equiv 9 (mod\ 12)$

We start with:

(a) $n \neq 21$

and prove that $Int_{S_n} = [0, \lfloor \frac{n}{3} \rfloor] - \{\lfloor \frac{n}{3} \rfloor\}$.

Taking a 2-near Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - 1$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 1$ and defect $\lfloor \frac{n}{3} \rfloor$, then taking the same 2-near Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - 1$ with the reverse of the Langford sequence above and its reverse (Table B.9) gives two Skolem sequences of order n with $\lfloor \frac{n}{3} \rfloor - 1$ pairs in common.

Taking a Skolem sequences of order $\lfloor \frac{n}{3} \rfloor - 2$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 2$ and defect $\lfloor \frac{n}{3} \rfloor - 1$, then taking another Skolem sequence which can have $0, 1, \ldots, \lfloor \frac{n}{3} \rfloor - 5, \lfloor \frac{n}{3} \rfloor - 2$ pairs in common with the previous sequence with the reverse of the Langford sequence (Table B.1) gives two Skolem sequences of order n with $3, 4, \ldots, \lfloor \frac{n}{3} \rfloor - 2$ pairs in common.

Taking a Skolem sequences of order $\lfloor \frac{n}{3} \rfloor - 3$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 3$ and defect $\lfloor \frac{n}{3} \rfloor - 2$, then taking another Skolem sequence which can have $0, 1, 2$ pairs in common with the previous sequence and adjoining it the reverse of the Langford sequence (Table B.2) gives two Skolem sequences of order n with $0, 1, 2$ pairs in common.

We continue with:

(b) $n = 21$

and we prove that $Int_{S_{21}} = [0, 7] - \{5, 7\}$.

Taking a 2-near Skolem sequence of order 6 with a Langford sequence of order 15 and defect 7, then taking the same 2-near Skolem sequence with the reverse of the Langford sequence above (Table B.9) gives two Skolem sequences of order 21 with 6 pairs in common.

Taking a Skolem sequences of order 5 with a Langford sequence of order 16 and defect 6, then taking another Skolem sequence of order 5 which can have $0, 1, 5$ pairs in common with the previous sequence and adjoin the reverse of the Langford sequence above (Table B.1), gives two Skolem sequences of order 21 with 3 or 4 pairs in common.

Taking a Skolem sequences of order 4 with a Langford sequence of order 17 and defect 5, then taking another Skolem sequence of order 4 which can have 0 or 1 pair in common with the previous sequence and adjoin the reverse of the Langford sequence above (Table B.2) gives two Skolem sequences of order 21 with 0 or 1 pairs in common.

Taking a 2-near Skolem sequence of order 3 with a Langford sequence of order 18 and defect 4, then taking the same 2-near Skolem sequence with the reverse of the Langford sequence above (Table B.8) gives two Skolem sequences of order 21 with two pairs in common. □

Now, we construct new hooked Skolem sequences in the same way we constructed the Skolem sequences of order n above.

To find the intersection spectrum in $[0, \lfloor\frac{n}{3}\rfloor]$ pairs of two hooked Skolem sequences of order n, form two new hooked Skolem sequences of order n, in the following way: let k be a positive integer and repeat the process below for $k = 0, 1, \ldots, \lfloor\frac{n}{3}\rfloor - 2$ or until all the pairs are found. Taking a hooked Skolem sequence of order $\lfloor\frac{n}{3}\rfloor - k$ and adjoin with a Langford sequence of order $n - \lfloor\frac{n}{3}\rfloor + k$ and defect $\lfloor\frac{n}{3}\rfloor - k + 1$. (If this Langford sequence does not exist for $k = 0$, taking $k = 1$). If the Langford sequence is perfect, taking the same hooked Skolem sequence of order $\lfloor\frac{n}{3}\rfloor - k$ and adjoin with the reverse Langford sequence of order $n - \lfloor\frac{n}{3}\rfloor + k$ and defect $\lfloor\frac{n}{3}\rfloor - k + 1$. The number of pairs in common between these Langford sequences and their reverse is given in Table B.1 if the Langford sequence has order $n = 4t$, Table B.6 if the Langford sequence has order $2d - 1$ and Table B.2 if the Langford sequence has order $2d - 1 + 4r$, where r is a positive integer. In this way we can form two hooked Skolem sequences of order n with $\lfloor\frac{n}{3}\rfloor - k$ or more pairs in common. We can also adjoin a hooked Skolem sequence of order $\lfloor\frac{n}{3}\rfloor - k$ with a Langford sequence of order $n - \lfloor\frac{n}{3}\rfloor + k$ and defect $\lfloor\frac{n}{3}\rfloor - k + 1$ and then taking another hooked Skolem sequence of order $\lfloor\frac{n}{3}\rfloor - k$ which can have $0, 1, \ldots, \lfloor\frac{n}{3}\rfloor - k - 3$, $\lfloor\frac{n}{3}\rfloor - k$ pairs in common with the previous hooked Skolem sequence and adjoin with the reverse Langford sequence of order $n - \lfloor\frac{n}{3}\rfloor + k$ and defect $\lfloor\frac{n}{3}\rfloor - k + 1$. These gives two Skolem sequences of order n with $0, 1, \ldots, \lfloor\frac{n}{3}\rfloor - k - 3$, $\lfloor\frac{n}{3}\rfloor - k$ pairs in common. If the Langford sequence of order $n - \lfloor\frac{n}{3}\rfloor + k$ and defect $\lfloor\frac{n}{3}\rfloor - k + 1$ is hooked, fill the hook with the pair 2 which makes the sequence perfect and adjoin this Langford sequence with a hooked 2-near Skolem sequence of order $\lfloor\frac{n}{3}\rfloor - k$, then adjoin the previous hooked 2-near Skolem sequence of order $\lfloor\frac{n}{3}\rfloor - k$ with the reverse of the Langford sequence above. The number of pairs in common between the Langford sequences and their reverse is given in Table 8 if the Langford sequence has order $4t+2$, or Table B.9 if the Langford sequence has order $2d+1+4r$ where t and r are positive integers. In this way we can form two hooked Skolem sequences of order n which can have $\lfloor\frac{n}{3}\rfloor - 2$ or more pairs in common. Two disjoint hooked Skolem sequences of order n can be found in [4]. Also if the Langford sequence has order $2d - 1$ we are able to find more sequences of this kind and in Table B.6 we have the numbers of pairs in common between these sequences or if the Langford sequence has order $2d$, d *even* we can find two different sequences and their reverse, one from Table 1 and the other from Table B.7.

For example, if we want to find if two Skolem sequences of order 15 intersect in $[0, 5]$ pairs, we have the following procedure:

- taking a hooked 2-near Skolem sequence of order 4[41] and adjoining it with a hooked Langford sequence of order 11 and defect 5 which we can make it perfect if we put the pair 2 to fill the hook, then taking the previous hooked 2-near Skolem sequences of order 4 and adjoining it with the reverse Langford sequence of order 11 and defect 5 above (The Langford sequence and its reverse can have 1 pair in common by Table B.9), gives 4 pairs in common.

- taking a hooked Skolem sequence of order 3 and adjoining it with a Langford sequence of order 12 and defect 4, then taking a disjoint hooked Skolem sequence of order 3 and adjoining it with the reverse of the Langford sequence above (The Langford sequence and its reverse have 1 pair in common by Table B.1), get two Skolem sequences of order 15 with 1 pair in common.

- taking a hooked Skolem sequence of order 2 with a Langford sequence of order 13 and defect 3, then taking the same hooked Skolem sequence of order 2 with the reverse Langford sequence above(Table B.2), gives two hooked Skolem sequences of order 15 with 2 pairs in common.

- to find two disjoint hooked Skolem sequences of order 15, see [4].

Therefore, two Skolem sequences of order 15 can have $0, 1, 2, 4$ pairs in common. We are not able to discover, using these constructions, if two hooked Skolem sequence of order 15 intersect in 3 or 5 pairs.

Using the same technique we prove the following result:

Theorem 19 *For $n \geq 1$, the necessary conditions are sufficient for the existence of two distinct hooked Skolem sequences of order n to intersect in $[0, \lfloor \frac{n}{3} \rfloor]$ pairs, with the* *following possible exceptions: for* $n = 12t + 2, t \equiv 0, 1, 2, 4, 5, 7, 10, 13, 14 (mod\ 15)$ $n \neq 50$ *the exceptions are* $\{\lfloor \frac{n}{3} \rfloor - 3, \lfloor \frac{n}{3} \rfloor\}$*, for* $t \equiv 6, 11, 12 (mod\ 15)$ *the exception is* $\{\lfloor \frac{n}{3} \rfloor - 3\}$*, for* $t \equiv 3, 8, 9 (mod\ 15), n \neq 38$ *the exception is* $\{\lfloor \frac{n}{3} \rfloor\}$ *and other 3 possible exceptions for the small cases* $n = 38, 50$*, for* $n \equiv 3 (mod\ 12)$ *the exceptions are* $\{\lfloor \frac{n}{3} \rfloor - 2, \lfloor \frac{n}{3} \rfloor\}$*, for* $n = 12t + 6, t \equiv 0, 1, 2, 5, 7, 10, 14 (mod\ 15), n \neq 30$ *the exceptions are* $\{\lfloor \frac{n}{3} \rfloor - 5, \lfloor \frac{n}{3} \rfloor - 2, \lfloor \frac{n}{3} \rfloor\}$*, for* $t \equiv 6, 11, 12 (mod\ 15)$ *the exceptions are* $\{\lfloor \frac{n}{3} \rfloor - 5, \lfloor \frac{n}{3} \rfloor\}$*, for* $t \equiv 8, 9 (mod\ 15)$ *exceptions are* $\{\lfloor \frac{n}{3} \rfloor - 2, \lfloor \frac{n}{3} \rfloor\}$*, for* $t \equiv 4, 13 (mod\ 15), n \neq 54$ *the exceptions are* $\{\lfloor \frac{n}{3} \rfloor - 5, \lfloor \frac{n}{3} \rfloor - 2\}$*, for* $t \equiv 3 (mod\ 15), n \neq 42$ *the exception is* $\{\lfloor \frac{n}{3} \rfloor - 2\}$*, and other 6 possible exceptions for the small cases* $n = 30, 42, 54$.

Proof: For $1 \leq n \leq 9$, see Appendix A. For a list with all the possible exceptions, see Appendix C.

We divide this proof into 6 cases. Let t be a positive integer.

Case (1): $n \equiv 2 (mod\ 12)$

We start with:

(a) $n = 12t + 2, t \equiv 0, 1, 2, 4, 5, 7, 10, 13, 14 (mod\ 15); n \neq 50$

and we prove that $Int_{S_n} = [0, \lfloor \frac{n}{3} \rfloor] - \{\lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{3} \rfloor - 3\}$.

Taking a hooked Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - 1$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 1$ and defect $\lfloor \frac{n}{3} \rfloor$, then taking another hooked Skolem sequence which can have $0, 1, \ldots, \lfloor \frac{n}{3} \rfloor - 4, \lfloor \frac{n}{3} \rfloor - 1$ pairs in common with the previous hooked Skolem sequence and adjoining it the reverse of the Langford sequence above (Table B.2) gives two hooked Skolem sequences of order n with $0, 1, \ldots, \lfloor \frac{n}{3} \rfloor - 4, \lfloor \frac{n}{3} \rfloor - 1$ pairs in common.

Taking a hooked Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - 2$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 2$ and defect $\lfloor \frac{n}{3} \rfloor - 1$, then taking the same hooked Skolem sequence and adjoining it the reverse of the Langford sequence above (Table B.2) gives two hooked Skolem sequences of order n with $\lfloor \frac{n}{3} \rfloor - 2$ pairs in common.

Then we continue with:

(b) $n = 12t + 2, t \equiv 6, 11, 12 (mod\ 15)$

and we prove that $Int_{S_n} = [0, \lfloor \frac{n}{3} \rfloor] - \{\lfloor \frac{n}{3} \rfloor - 3\}$.

Taking a hooked Skolem sequence of order $\lfloor\frac{n}{3}\rfloor - 1$ with a perfect Langford sequence of order $n - \lfloor\frac{n}{3}\rfloor + 1$ and defect $\lfloor\frac{n}{3}\rfloor$, then taking another hooked Skolem sequence which can have $0, 1, \ldots, \lfloor\frac{n}{3}\rfloor - 4, \lfloor\frac{n}{3}\rfloor - 1$ pairs in common with the previous sequence and adjoining it the reverse of the Langford sequence above (Table B.2) gives two Skolem sequences of order n with $0, 1, \ldots, \lfloor\frac{n}{3}\rfloor - 4, \lfloor\frac{n}{3}\rfloor - 1$ pairs in common.

Taking a hooked Skolem sequence of order $\lfloor\frac{n}{3}\rfloor - 2$ with a Langford sequence of order $n - \lfloor\frac{n}{3}\rfloor + 2$ and defect $\lfloor\frac{n}{3}\rfloor - 1$, then taking the same hooked Skolem sequence with the reverse Langford sequence above (Table B.1) gives two hooked Skolem sequences of order n with $\lfloor\frac{n}{3}\rfloor - 2$ pairs in common.

Taking a hooked 2-near Skolem sequence of order $\lfloor\frac{n}{3}\rfloor - 4r$ with a Langford sequence of order $n - \lfloor\frac{n}{3}\rfloor + 4r$ and defect $\lfloor\frac{n}{3}\rfloor - 4r + 1$, then taking the same hooked 2-near Skolem sequence of order $\lfloor\frac{n}{3}\rfloor - 4r$ and adjoining it the reverse of the Langford sequence above (Table B.8) $r = 1, 2, 3...$, one of these constructions gives two hooked Skolem sequences of order n with $\lfloor\frac{n}{3}\rfloor$ pairs in common.

Then we continue with:

(c) $n = 12t + 2, t \equiv 3, 8, 9 (mod\ 15), n \neq 38$

and we prove that $Int_{S_n} = [0, \lfloor\frac{n}{3}\rfloor] - \{\lfloor\frac{n}{3}\rfloor\}$.

Taking a hooked Skolem sequences of order $\lfloor\frac{n}{3}\rfloor - 1$ with a perfect Langford sequence of order $n - \lfloor\frac{n}{3}\rfloor + 1$ and defect $\lfloor\frac{n}{3}\rfloor$, then taking another hooked Skolem sequence which can have $0, 1, \ldots, \lfloor\frac{n}{3}\rfloor - 4, \lfloor\frac{n}{3}\rfloor - 1$ pairs in common with the previous sequence and adjoining it the reverse Langford sequence above (Table B.2) gives two hooked Skolem sequences of order n with $0, 1, \ldots, \lfloor\frac{n}{3}\rfloor - 4, \lfloor\frac{n}{3}\rfloor - 1$ pairs in common.

Taking a hooked Skolem sequence of order $\lfloor\frac{n}{3}\rfloor - 2$ with a Langford sequence of order $n - \lfloor\frac{n}{3}\rfloor + 2$ and defect $\lfloor\frac{n}{3}\rfloor - 1$, then taking the same hooked Skolem sequence with the reverse of the Langford sequence above (Table B.2) gives two hooked Skolem sequences of order n with $\lfloor\frac{n}{3}\rfloor - 2$ pairs in common.

Taking a hooked 2-near Skolem sequence of order $\lfloor\frac{n}{3}\rfloor - 4r$ with a Langford sequence of order $n - \lfloor\frac{n}{3}\rfloor + 4r$ and defect $\lfloor\frac{n}{3}\rfloor - 4r + 1$, then taking the same hooked 2-near Skolem sequence and adjoining it the reverse of the Langford sequence above

(Table B.8) $r = 1, 2, 3...$, one of these constructions gives two hooked Skolem sequences of order n with $\lfloor \frac{n}{3} \rfloor - 3$ pairs in common.
Then we continue with the small cases:

(d) $n = 50$

and we prove that $Int_{S_{50}} = [0, 16] - \{16\}$.
Taking a hooked Skolem sequence of order 15 with a Langford sequence of order 35 and defect 16, then taking another hooked Skolem sequence which can have $0, 1, \ldots, 12, 15$ pairs in common with the previous sequence and adjoining it the reverse of the Langford sequence above (Table B.2) gives two hooked Skolem sequences of order 50 with $0, 1, \ldots, 12, 15$ pairs in common.

Taking a hooked Skolem sequence of order 14 with a Langford sequence of order 36 and defect 15, then taking the same hooked Skolem sequence with the reverse of the Langford sequence above (Table B.1) gives two Skolem sequences of order 50 with 14 pairs in common.

Taking a hooked 2-near Skolem sequence of order 12 with a Langford sequence of order 38 and defect 13 (which is perfect if we fill the hook with 2), then taking the same hooked 2-near Skolem sequence of order 12 and adjoining it the reverse of the Langford sequence above (Table B.9) gives two hooked Skolem sequences of order 50 with 13 pairs in common.

(e) $n = 38$

and we prove that $Int_{S_{38}} = [0, 12] - \{9, 12\}$.
Taking a hooked Skolem sequences of order 11 with a Langford sequence of order 27 and defect 12, then taking another hooked Skolem sequence which can have $0, 1, \ldots, 8, 11$ pairs in common and adjoining it the reverse of the Langford sequence above (Table B.2) gives two hooked Skolem sequences of order 38 with $0, 1, \ldots, 8, 11$ pairs in common.

Taking a hooked Skolem sequence of order 10 with a Langford sequence of order 28 and defect 11, then taking the same hooked Skolem sequence with the reverse of the reverse of the Langford sequence above (Table B.1) gives two hooked Skolem sequences of order 38 with 11 pairs in common.

Case (2): $n \equiv 3(mod\ 12)$

and we prove that $Int_{S_n} = [0, \lfloor \frac{n}{3} \rfloor] - \{\lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{3} \rfloor - 2\}$.

Taking a hooked Skolem sequences of order $\lfloor \frac{n}{3} \rfloor - 2$ and adjoining it a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 2$ and defect $\lfloor \frac{n}{3} \rfloor - 1$, then taking another hooked Skolem sequence which can have $\{0, 1, 2, \ldots, \lfloor \frac{n}{3} \rfloor - 5, \lfloor \frac{n}{3} \rfloor - 2\}$ pairs in common with the previous hooked Skolem sequence and adjoin the reverse of the Langford sequence above (Table B.1). These constructions gives two hooked Skolem sequences of order n with $\{0, 1, 2, \ldots, \lfloor \frac{n}{3} \rfloor - 4, \lfloor \frac{n}{3} \rfloor - 1\}$ pairs in common. Then, taking a hooked Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - 3$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 3$ and defect $\lfloor \frac{n}{3} \rfloor - 2$ and then taking the same hooked Skolem sequence with the reverse of the Langford sequence above (Table B.2). These constructions gives two hooked Skolem sequences of order n with $\lfloor \frac{n}{3} \rfloor - 3$ pairs in common.

Case (3): $n \equiv 6(mod\ 12)$

We start with:

(a) $n = 12t + 6, t \equiv 0, 1, 2, 5, 7, 10, 14(mod\ 15), n \neq 30$

and we prove that $Int_{S_n} = [0, \lfloor \frac{n}{3} \rfloor] - \{\lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{3} \rfloor - 2, \lfloor \frac{n}{3} \rfloor - 5\}$.

Taking a hooked 2-near Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - 1$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 1$ and defect $\lfloor \frac{n}{3} \rfloor$, then taking the same hooked 2-near Skolem sequence with the reverse of the Langford sequence above (Table B.9) gives two hooked Skolem sequences of order n with $\lfloor \frac{n}{3} \rfloor - 1$ pairs in common.

Taking a hooked Skolem sequences of order $\lfloor \frac{n}{3} \rfloor - 3$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 3$ and defect $\lfloor \frac{n}{3} \rfloor - 2$, then taking another hooked Skolem sequence which can have $0, 1, \ldots, \lfloor \frac{n}{3} \rfloor - 6, \lfloor \frac{n}{3} \rfloor - 3$ pairs in common with the previous hooked Skolem sequence and adjoining it the reverse of the Langford sequence above (Table B.2) gives two hooked Skolem sequences of order n with $0, 1, \ldots, \lfloor \frac{n}{3} \rfloor - 6, \lfloor \frac{n}{3} \rfloor - 3$ pairs in common.

Taking a hooked Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - 4$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 4$ and defect $\lfloor \frac{n}{3} \rfloor - 3$, then taking the same hooked Skolem sequence with the reverse of the Langford sequence above (Table B.1) gives two hooked Skolem sequences of order n with $\lfloor\frac{n}{3}\rfloor - 4$ pairs in common.

(b) $n = 12t + 6, t \equiv 6, 11, 12 (mod\ 15)$

and we prove that $Int_{S_n} = [0, \lfloor\frac{n}{3}\rfloor] - \{\lfloor\frac{n}{3}\rfloor, \lfloor\frac{n}{3}\rfloor - 5\}$.

Taking a hooked 2-near Skolem sequence of order $\lfloor\frac{n}{3}\rfloor - 1$ with a Langford sequence of order $n - \lfloor\frac{n}{3}\rfloor + 1$ and defect $\lfloor\frac{n}{3}\rfloor$, then taking the same hooked 2-near Skolem sequence with the reverse of the Langford sequence above (Table B.9) gives two hooked Skolem sequences of order n with $\lfloor\frac{n}{3}\rfloor - 1$ pairs in common.

Taking a hooked Skolem sequence of order $\lfloor\frac{n}{3}\rfloor - 3$ with a Langford sequence of order $n - \lfloor\frac{n}{3}\rfloor + 3$ and defect $\lfloor\frac{n}{3}\rfloor - 2$, then taking another hooked Skolem sequence which can have $0, 1, \ldots, \lfloor\frac{n}{3}\rfloor - 6, \lfloor\frac{n}{3}\rfloor - 3$ pairs in common with the previous sequence and adjoin the reverse of the Langford sequence above (Table B.2) gives two hooked Skolem sequences with $0, 1, \ldots, \lfloor\frac{n}{3}\rfloor - 6, \lfloor\frac{n}{3}\rfloor - 3$ pairs in common.

Taking a hooked Skolem sequence of order $\lfloor\frac{n}{3}\rfloor - 4$ with a Langford sequence of order $n - \lfloor\frac{n}{3}\rfloor + 4$ and defect $\lfloor\frac{n}{3}\rfloor - 3$, then taking the same hooked Skolem sequence with the reverse of the Langford sequence above (Table B.1) gives two hooked Skolem sequences of order n with $\lfloor\frac{n}{3}\rfloor - 4$ pairs in common.

Taking a hooked 2-near Skolem sequence of order $\lfloor\frac{n}{3}\rfloor - 2 - 4r$ with a Langford sequence of order $n - \lfloor\frac{n}{3}\rfloor + 2 + 4r$ and defect $\lfloor\frac{n}{3}\rfloor - 4r - 1$, then taking the same hooked 2-near Skolem sequence with the reverse of the Langford sequence above (Table B.8) $r = 1, 2, 3...$,one of these constructions gives two hooked Skolem sequences of order n with $\lfloor\frac{n}{3}\rfloor - 2$ pairs in common. Then we continue with:

(c) $n = 12t + 6, t \equiv 8, 9 (mod\ 15)$

and we prove that $Int_{S_n} = [0, \lfloor\frac{n}{3}\rfloor] - \{\lfloor\frac{n}{3}\rfloor, \lfloor\frac{n}{3}\rfloor - 2\}$.

Taking a hooked 2-near Skolem sequence of order $\lfloor\frac{n}{3}\rfloor - 1$ with a Langford sequence of order $n - \lfloor\frac{n}{3}\rfloor + 1$ and defect $\lfloor\frac{n}{3}\rfloor$, then taking the same hooked 2-near Skolem sequence with the reverse of the Langford sequence above (Table B.9) gives two hooked Skolem sequences of order n with $\lfloor\frac{n}{3}\rfloor - 1$ pairs in common.

Taking a hooked Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - 3$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 3$ and defect $\lfloor \frac{n}{3} \rfloor - 2$, then taking another hooked Skolem sequence which can have $0, 1, \ldots, \lfloor \frac{n}{3} \rfloor - 6, \lfloor \frac{n}{3} \rfloor - 3$ pairs in common with the previous sequence and adjoin the reverse Langford sequence above (Table B.2) gives two hooked Skolem sequences with $0, 1, \ldots, \lfloor \frac{n}{3} \rfloor - 6, \lfloor \frac{n}{3} \rfloor - 3$ pairs in common.

Taking a hooked Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - 4$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 4$ and defect $\lfloor \frac{n}{3} \rfloor - 3$, then taking the same hooked Skolem sequence and adjoin the reverse of the Langford sequence above (Table B.1) gives two hooked Skolem sequences of order n with $\lfloor \frac{n}{3} \rfloor - 4$ pairs in common.

Taking a hooked 2-near Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - 2 - 4r$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 2 + 4r$ and defect $\lfloor \frac{n}{3} \rfloor - 4r - 1$, then taking the same hooked 2-near Skolem sequence with the reverse of the Langford sequence above (Table B.8) $r = 1, 2, 3...$ one of these constructions gives two hooked Skolem sequences of order n with $\lfloor \frac{n}{3} \rfloor - 5$ pairs in common.

And we continue with:

(d) $n = 12t + 6, t \equiv 4, 13 (mod\ 15), n \neq 54$

and we prove that $Int_{S_n} = [0, \lfloor \frac{n}{3} \rfloor] - \{\lfloor \frac{n}{3} \rfloor - 2, \lfloor \frac{n}{3} \rfloor - 5\}$.

Taking a hooked 2-near Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - 1$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 1$ and defect $\lfloor \frac{n}{3} \rfloor$, then taking the same hooked 2-near Skolem sequence with the reverse Langford sequence above (Table B.9) gives two hooked Skolem sequences of order n with $\lfloor \frac{n}{3} \rfloor - 1$ pairs in common.

Taking a hooked Skolem sequences of order $\lfloor \frac{n}{3} \rfloor - 3$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 3$ and defect $\lfloor \frac{n}{3} \rfloor - 2$, then taking another hooked Skolem sequence which can have $0, 1, \ldots, \lfloor \frac{n}{3} \rfloor - 6, \lfloor \frac{n}{3} \rfloor - 3$ pairs in common with the previous sequence and adjoin the reverse of the Langford sequence above (Table B.2) gives two hooked Skolem sequences with $0, 1, \ldots, \lfloor \frac{n}{3} \rfloor - 6, \lfloor \frac{n}{3} \rfloor - 3$ pairs in common.

Taking a hooked Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - 4$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 4$ and defect $\lfloor \frac{n}{3} \rfloor - 3$, then taking the same hooked Skolem sequence and adjoin the reverse of the Langford sequence above (Table B.1) gives two hooked

Skolem sequences of order n with $\lfloor\frac{n}{3}\rfloor - 4$ pairs in common.

Taking a hooked 2-near Skolem sequence of order $\lfloor\frac{n}{3}\rfloor - 2 - 4r$ with a Langford sequence of order $n - \lfloor\frac{n}{3}\rfloor + 2 + 4r$ and defect $\lfloor\frac{n}{3}\rfloor - 4r - 1$, then taking the same hooked 2-near Skolem sequence and adjoin the reverse of the Langford sequence above (Table B.8) $r = 1, 2, 3...$ one of these constructions gives two hooked Skolem sequences of order n with $\lfloor\frac{n}{3}\rfloor$ pairs in common.

Then, we prove also for:

<u>(e) $n = 12t + 6, t \equiv 3(mod\ 15), n \neq 42$</u>

and we prove that $Int_{S_n} = [0, \lfloor\frac{n}{3}\rfloor] - \{\lfloor\frac{n}{3}\rfloor - 2\}$.

Taking a hooked 2-near Skolem sequence of order $\lfloor\frac{n}{3}\rfloor - 1$ with a Langford sequence of order $n - \lfloor\frac{n}{3}\rfloor + 1$ and defect $\lfloor\frac{n}{3}\rfloor$, then taking the same hooked 2-near Skolem sequence with the reverse of the Langford sequence above (Table B.9) gives two hooked Skolem sequences of order n with $\lfloor\frac{n}{3}\rfloor - 1$ pairs in common.

Taking a hooked Skolem sequence of order $\lfloor\frac{n}{3}\rfloor - 3$ with a Langford sequence of order $n - \lfloor\frac{n}{3}\rfloor + 3$ and defect $\lfloor\frac{n}{3}\rfloor - 2$, then taking another hooked Skolem sequence which can have $0, 1, \ldots, \lfloor\frac{n}{3}\rfloor - 6, \lfloor\frac{n}{3}\rfloor - 3$ pairs in common with the previous sequence and adjoin the reverse of the Langford sequence above and its reverse (Table B.2) gives two hooked Skolem sequences with $0, 1, \ldots, \lfloor\frac{n}{3}\rfloor - 6, \lfloor\frac{n}{3}\rfloor - 3$ pairs in common.

Taking a hooked Skolem sequence of order $\lfloor\frac{n}{3}\rfloor - 4$ with a Langford sequence of order $n - \lfloor\frac{n}{3}\rfloor + 4$ and defect $\lfloor\frac{n}{3}\rfloor - 3$, then taking the same hooked Skolem sequence and adjoin the reverse of the Langford sequence above (Table B.1) gives two hooked Skolem sequences of order n with $\lfloor\frac{n}{3}\rfloor - 4$ pairs in common.

Taking a hooked 2-near Skolem sequence of order $\lfloor\frac{n}{3}\rfloor - 2 - 4r$ with a Langford sequence of order $n - \lfloor\frac{n}{3}\rfloor + 2 + 4r$ and defect $\lfloor\frac{n}{3}\rfloor - 4r - 1$, then taking the same hooked 2-near Skolem sequence with the reverse of the Langford sequence above (Table B.8) $r = 1, 2, 3...$ two of these constructions gives two hooked Skolem sequences of order n with $\lfloor\frac{n}{3}\rfloor$ and $\lfloor\frac{n}{3}\rfloor - 5$ pairs in common.

And finally, some small cases:

(f) $n = 42$ and we prove that $Int_{S_{42}} = [0, 14] - \{9, 12, 14\}$.

Taking a hooked 2-near Skolem sequence of order 13 with a Langford sequence of order 29 and defect 14, then taking the same hooked 2-near Skolem sequence with the reverse of the Langford sequence above and its reverse (Table B.9) gives two hooked Skolem sequences of order 42 with 13 pairs in common.

Taking a hooked Skolem sequence of order 11 with a Langford sequence of order 31 and defect 12, then taking another hooked Skolem sequence which can have $0, 1, \ldots, 8, 11$ pairs in common with the previous sequence and adjoin the reverse of the Langford sequence above (Table B.2) gives two hooked Skolem sequences of order 42 with $0, 1, \ldots, 8, 11$ pairs in common.

Taking a hooked Skolem sequence of order 10 with a Langford sequence of order 32 and defect 11, then taking the same hooked Skolem sequence with the reverse of the Langford sequence above (Table B.1) gives two hooked Skolem sequences of order 42 with 10 pairs in common.

(g) $n = 30$

and we prove $Int_{S_{30}} = [0, 10] - \{5, 8\}$.

Taking a hooked 2-near Skolem sequence of order 9 with a Langford sequence of order 21 and defect 10, then taking the same hooked 2-near Skolem sequence and adjoin the reverse of the Langford sequence above (Table B.9) gives two hooked Skolem sequences of order 30 with 9 pairs in common; taking a hooked 2-near Skolem sequence of order 8 with a Langford sequence of order 22 and defect 9, then taking the same hooked 2-near Skolem sequence with the reverse of the Langford sequence above (Table B.8) gives two hooked Skolem sequences of order 30 with 10 pairs in common.

Taking a hooked Skolem sequences of order 7 with a Langford sequence of order 23 and defect 8, then taking another hooked Skolem sequence which can have $0, 1, 2, 3, 4, 7$ pairs in common with the previous sequence and adjoin the reverse of the Langford sequence above (Table B.2) gives two hooked Skolem sequences of order 30 with $0, 1, 2, 3, 4, 7$ pairs in common.

Taking a hooked Skolem sequence of order 6 with a Langford sequence of order 24 and defect 7, then taking the same hooked Skolem sequence with the reverse of the Langford sequence above and its reverse (Table B.1) gives two hooked Skolem sequences of order 30 with 6 pairs in common.

(h) $n = 54$

and we prove that $Int_{S_{54}} = [0, 18] - \{16\}$.

Taking a hooked 2-near Skolem sequence of order 17 with a Langford sequence of order 37 and defect 18, then taking the same hooked 2-near Skolem sequence with the reverse of the Langford sequence above (Table B.9) gives two hooked Skolem sequences of order 54 with 17 pairs in common.

Taking a hooked 2-near Skolem sequence of order 16 with a Langford sequence of order 38 and defect 17, the taking the same hooked 2-near Skolem sequence with the reverse of the Langford sequence above (Table B.8) gives two hooked Skolem sequences of order 54 with 18 pairs in common.

Taking a hooked Skolem sequence of order 15 with a Langford sequence of order 39 and defect 16, then taking another hooked Skolem sequence which can have $0, 1, \ldots, 12, 15$ pairs in common and adjoin the reverse of the Langford sequence above (Table B.2) gives two hooked Skolem sequences of order 54 with $0, 1, \ldots, 12, 15$ pairs in common.

Taking a hooked Skolem sequence of order 14 with a Langford sequence of order 40 and defect 15, then taking the same hooked Skolem sequence with the reverse of the Langford sequence above (Table B.1) gives two hooked Skolem sequences of order 54 with 14 pairs in common.

Taking a hooked 2-near Skolem sequence of order 12 with a Langford sequence of order 42 and defect 13, then taking the same hooked 2-near Skolem sequence with the reverse of the Langford sequence above (Table B.8) gives two hooked Skolem sequences of order 54 with 13 pairs in common.

Case (4): $n \equiv 7 (mod\ 12)$

and we prove that $Int_{S_n} = [0, \lfloor \frac{n}{3} \rfloor]$.

Taking a hooked Skolem sequences of order $\lfloor \frac{n}{3} \rfloor$ and adjoin a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor$ and defect $\lfloor \frac{n}{3} \rfloor + 1$, then taking another hooked Skolem sequence of the same order which can have $\{0, 1, 2, \ldots, \lfloor \frac{n}{3} \rfloor - 3, \lfloor \frac{n}{3} \rfloor\}$ pairs in common with the sequence above and adjoin the reverse of the Langford sequence above (Table B.6 for the number of pairs in common between this Langford sequence and its reverse).

Case (5): $n \equiv 10(mod\ 12)$

nd we prove that $Int_{S_n} = [0, \lfloor \frac{n}{3} \rfloor]$.

Taking a hooked Skolem sequences of order $\lfloor \frac{n}{3} \rfloor$ and adjoin a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor$ and defect $\lfloor \frac{n}{3} \rfloor + 1$, then taking another hooked Skolem sequence which can have $\{0, 1, 2, \ldots, \lfloor \frac{n}{3} \rfloor - 3, \lfloor \frac{n}{3} \rfloor\}$ pairs in common with the above hooked Skolem sequence and adjoin the reverse of the Langford sequence above(Table B.1 and Table B.7).

Case (6): $n \equiv 11(mod\ 12)$

and we prove that $Int_{S_n} = [0, \lfloor \frac{n}{3} \rfloor]$.

Taking a hooked Skolem sequences of order $\lfloor \frac{n}{3} \rfloor$ and adjoin a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor$ and defect $\lfloor \frac{n}{3} \rfloor + 1$, then taking another hooked Skolem sequence which can have $\{0, 1, 2, \ldots, \lfloor \frac{n}{3} \rfloor - 3, \lfloor \frac{n}{3} \rfloor\}$ pairs in common with the previous hooked Skolem sequence and adjoin the reverse Langford sequence above (Table B.1 and Table B.7). These constructions gives two hooked Skolem sequences of order n with $[0, \lfloor \frac{n}{3} \rfloor] - \{\lfloor \frac{n}{3} \rfloor - 1\}$ pairs in common. To get two hooked Skolem sequences of order n with $\lfloor \frac{n}{3} \rfloor - 1$ pairs in common, taking a hooked Skolem sequence of order $\lfloor \frac{n}{3} \rfloor - 1$ with a Langford sequence of order $n - \lfloor \frac{n}{3} \rfloor + 1$ and defect $\lfloor \frac{n}{3} \rfloor$ and then taking the same hooked Skolem sequence above with the reverse of the Langford sequence (Table B.2). □

Case II): The intersection of two distinct Skolem sequences and two hooked Skolem sequences of order n in $(\lfloor \frac{n}{3} \rfloor, 2\lfloor \frac{n}{3} \rfloor)$ pairs

To find the intersection of two [hooked] Skolem sequences of order n in $(\lfloor \frac{n}{3} \rfloor, 2\lfloor \frac{n}{3} \rfloor)$ pairs we construct new [hooked] Skolem sequences from three different parts:

- Part A: this part is a sequence formed by even and odd numbers starting with n

and a hole in the middle of the sequence. Let t be the order of A (i.e. the number of pairs in A, t is an odd number). The sequence is: $n, n-2, ..., n-t+1, n-1, n-3, \ldots, n-t+2, \underbrace{---------}_{n-t-\lfloor \frac{t}{2} \rfloor \text{ free spaces}}, n-t+1, ..., n-2, n, n-t+2, ..., n-3, n-1.$

Example: For $n = 12$ and $t = 7$, the sequence i:

$12, 10, 8, 6, 11, 9, 7, \underbrace{--}_{2 \text{ spaces}}, 6, 8, 10, 12, 7, 9, 11.$

- Part B: this part is the space inside part A (we call this a hole). We try to fit in this space a [hooked] Skolem sequence, or a k-extended Skolem sequence, or a 2-near Skolem sequence. In the example above, since there are only 2 spaces left, we can fit a Skolem sequence of order 1: $S_1 = (1, 1)$.

- Part C: in this part we form a [hooked]Langford sequence from the elements left from part A and part B.

  Example: For $n = 12$ and $t = 7$, the elements left from part A and part B are $2, 3, 4, 5$ and we can form a Langford sequence of defect 2 and length 4: $L_2^4 = (5, 2, 4, 2, 3, 5, 4, 3)$.

So, the sequence look like this: $\underbrace{\boxed{.....A....}\underbrace{\boxed{....B....}}_{\text{hole}}\boxed{....A....}}_{\text{shell}}\boxed{....C....}$

The Skolem sequence of order 12 formed by this construction is:

$S_{12} = (\underbrace{12, 10, 8, 6, 11, 9, 7}_{A}, \underbrace{1, 1}_{B}, \underbrace{6, 8, 10, 12, 7, 9, 11}_{A}, \underbrace{5, 2, 4, 2, 3, 5, 4, 3}_{C})$

If, in part B, we need to fit a k-extended Skolem sequence in order to have a [hooked] Skolem sequence in the end, we have to fill the hole with another pair, and in this case we take the largest pair left in Part C.

Example: If $n = 25$, $t = 11$, Part A is:

$25, 23, ..., 15, 24, 22, ..., 16, \underbrace{...}_{9 \text{ spaces}}, 15, 17, ..., 25, 16, 18, ..., 24.$ Since there are 9 spaces left we can fit there a k-extended Skolem sequence of order 4. The largest pair left for Part C is 14. We put 14 at the beginning of the sequence and since $t = 11$

this takes the third place in Part B. So in Part B we are able to fit a 3-extended Skolem sequence of order 4(see [3] for construction of k-extended Skolem sequences). For Part C the following numbers are left: $5, 6, 7, 8, 9, 10, 11, 12, 13$, so we can form a Langford sequence of order 9 and defect 5. (See [8, 29, 45], for constructions of [hooked]Langford sequences.)
Another sequence that we can fit in Part B is a 2-near Skolem sequence, and in this case we add the pair $(2, 0, 2)$ to the Langford sequence in Part C; if the Langford sequence is hooked the pair $(2, 0, 2)$ fits exactly in the hole and makes the Langford sequence perfect, if the Langford sequence is hooked we add the pair $(2, 0, 2)$ either to the Langford sequence in Part C or to the sequence in Part A. For example, for $n = 38$, $t = 19$, there are $n - t - \lfloor \frac{t}{2} \rfloor = 10$ spaces left for Part B of the sequence. So, we can fit here either a Skolem sequence of order 5 or a 2-near Skolem sequence of order 6. If we fit a Skolem sequence of order 5 in Part B, in Part C we have a Langford sequence of defect 6 and length $n - t - 5 = 14$. If we fit a 2-near Skolem sequence of order 6 in Part B, then in Part C we have a Langford sequence of defect 7 and length $n - t - 6 = 13$ which is a hooked Langford. So we add the pair $(2, 0, 2)$ to this hooked Langford sequence and the Langford sequence becomes a perfect sequence.

Now, using these constructions, we can get different pairs in common between two [hooked] Skolem sequences of order n. We take the first sequence formed by Parts A, B, C in their normal positions and the second sequence by taking the reverse sequences of Part A, B or C. The reverse of Part A is always disjoint, since the even pairs are interchanged with the odd pairs and vice versa. In part B, if we have a Skolem sequence of order n, we can have different Skolem sequences of the same order with $0, 1, 2, ..., n - 3, n$ pairs in common. In part C, taking the reverse of the Langford sequence, the Langford sequence and its reverse will have pairs in common or not. To see how many pairs in common these sequences have we check Table B.1 - Table B.9.

Notations: Define $S_n + s$ *pairs* to mean that we have a [hooked] Skolem sequence of order n and s other pairs. These pairs are the largest pairs left in Part C.

Define $S_n + s$ to mean S_{n+s}.
Define $A, B, \overleftarrow{C}$ to mean that we take a Skolem sequence of order n first with Parts A, B, C in their normal positions and then with Parts A, B in their normal position and Part C is reversed. This helps us find some pairs in common between these two sequences.
Define $B*$ to mean that in Part B, we have a [hooked] Skolem sequence of order p and we understand that we can find two [hooked] Skolem sequences of order p with $0, \ldots, p-3, p$ pairs in common.
Define $hL_d^m + (2,0,2)$ to mean that we add the pair $(2,0,2)$ to the hooked Langford sequence and we end with a perfect sequence.
Define $L_d^m, (2,0,2)$ to mean that we have a perfect Langford sequence with the pair $(2,0,2)$ added to the end of sequence which make the sequence hooked.

Example: For $n = 24$, we can get a construction with $t = 11$, $B = S_4$, $C = L_5^9$. If Part A stay in its normal position we have 11 pairs in common, if we take the reverse of Part A, A and $\overleftarrow{A}$ are disjoint, therefore have 0 pairs in common. In Part B, we can find two Skolem sequences of order 4 which can be disjoint, other two Skolem sequences of order 4 which can have 1 pair in common and two Skolem sequences of order 4 which can have 4 pairs in common (Appendix A). In part C we have a Langford sequence where $m = 2d - 1$, so if we check Table B.6, we can find two perfect Langford sequences of order 9 and defect 5 with $0, 1, 3$ pairs in common. On short, we write this:
Taking $\overleftarrow{A}, B*, C$, get the pairs $9, 10, 13$.
Taking $A, B*, \overleftarrow{C}$, get the pairs $11, \ldots, 16, 18$.
Therefore, $Int_{S_{24}} = (8, 12)$.

We use this technique to prove the following theorem:

Theorem 20 *For $n \geq 1$, the necessary conditions are sufficient for the existence of two Skolem sequences of order n, to intersect in $(\lfloor \frac{n}{3} \rfloor, 2\lfloor \frac{n}{3} \rfloor)$ pairs with the following possible exceptions: for $n \equiv 0 (mod\ 12), n \geq 72$ the exceptions are $\{\lfloor \frac{n}{3} \rfloor + 3, 2\lfloor \frac{n}{3} \rfloor - 2\}$ and other 16 possible exceptions for $10 \leq n < 72$, for $n \equiv 1 (mod\ 12), n \geq 109$*

the exception is $2\lfloor \frac{n}{3} \rfloor - 4$ *and other* 14 *possible exception for* $10 \leq n < 109$, *for* $n \equiv 4(mod\ 12), n \geq 112$ *the exception is* $\{2\lfloor \frac{n}{3} \rfloor - 1\}$ *and other* 12 *possible exceptions for* $10 \leq n < 112$, *for* $n \equiv 5(mod\ 12), n \geq 77$ *the exceptions are* $\{\lfloor \frac{n}{3} \rfloor + 1, \lfloor \frac{n}{3} \rfloor - 3, \lfloor \frac{n}{3} \rfloor - 2\}$ *and other* 17 *exceptions for* $10 \leq n < 77$, *for* $n \equiv 8(mod\ 12), n \geq 92$ *the exceptions are* $\{\lfloor \frac{n}{3} \rfloor + 1, 2\lfloor \frac{n}{3} \rfloor - 3\}$ *and other* 14 *possible exceptions for* $10 \leq n < 92$, *for* $n \equiv 9(mod\ 12), n \geq 81$ *the exceptions are* $\{\lfloor \frac{n}{3} \rfloor + 3, 2\lfloor \frac{n}{3} \rfloor - 6, 2\lfloor \frac{n}{3} \rfloor - 2\}$ *and other* 25 *possible exceptions for* $10 \leq n < 81$.

Proof: For $1 \leq n \leq 9$, see Appendix A. For a list with all the possible exceptions, see Appendix C.

Let $i = 1, 2, ...$ and $t_1, t_2, ..., t_{i+1}$ be the orders of $A_1, A_2, ...A_{i+1}$

We divide this proof into 6 cases.

Case (1): $n \equiv 0(mod\ 12)$

We start with:

(a) $n \geq 72$

and we prove that $Int_{S_n} = (\lfloor \frac{n}{3} \rfloor, 2\lfloor \frac{n}{3} \rfloor) - \{\lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{3} \rfloor + 3, 2\lfloor \frac{n}{3} \rfloor - 2\}$.

Step 1) $\begin{cases} t_1 = 2\lfloor \frac{n}{3} \rfloor - 1,\ B_1 = S_1,\ C_1 = L_{d_1=2}^{m_1=\lfloor \frac{n}{3} \rfloor}; \\ t_2 = t_1 - 4;\ B_2 = B_1 + 3 = S_4,\ C_2 = L_{d_2=5}^{m_2=m_1+1} \end{cases}$

Taking $A_1, B_1, \overleftarrow{C_1}$, where C_1 and its reverse are disjoint [40], get the pair $2\lfloor \frac{n}{3} \rfloor$.

Taking $\overleftarrow{A_2}, B_2*, C_2$, get the pairs $\lfloor \frac{n}{3} \rfloor + 1, \lfloor \frac{n}{3} \rfloor + 2, \lfloor \frac{n}{3} \rfloor + 5$.

Taking $A_2, B_2*, \overleftarrow{C_2}$, where C_2 and its reverse are disjoint (Table B.2), get the pairs $2\lfloor \frac{n}{3} \rfloor - 5, 2\lfloor \frac{n}{3} \rfloor - 4, 2\lfloor \frac{n}{3} \rfloor - 1$.

Step 2) $\{\ t_3 = t_2 - 8,\ B_3 = 2 - near\ S_{11},\ C_3 = hL_{d_3=d_2+7}^{m_3=m_2+1} + (2, 0, 2)$

Taking $A_3, B_3, \overleftarrow{C_3}$, where C_3 and its reverse are disjoint(Table B.8), get the pairs $t_3 + 10$.

Step 3) $\begin{cases} t_4 = t_2 - 12,\ B_4 = B_2 + 9,\ C_4 = L_{d_4=d_2+9}^{m_4=m_2+3} \\ t_5 = t_4 - 4,\ B_5 = B_2 + 3,\ C_5 = L_{d_5=d_4+3}^{m_5=m_4+1} \end{cases}$

Taking $A_4, B_4*, \overleftarrow{C_4}$, where C_4 and its reverse have 3 pairs in common (Table B.8) and get the pairs $t_4 + 3, \ldots, t_4 + p, t_4 + p - 3$.

Taking $\overleftarrow{A_4}, B_4*, C_4$, get the pairs $m_4, \ldots, m_4 + p - 3, m_4 + p$.

Taking $A_5, B_5*, \overleftarrow{C_5}$, where C_5 and its reverse are disjoint (Table B.2), get the pairs $t_5, \ldots, t_5 + p - 3, t_5 + p$.

Taking $\overleftarrow{A_5}, B_5*, C_5$ and get the pairs $m_5, \ldots, m_5 + p - 3, m_5 + p$.

Step 4) $\begin{cases} t_i = t_{i-1} - 12,\ B_i = B_{i-1} + 9,\ C_i = L_{d_i=d_{i-1}+9}^{m_i=m_{i-1}+3} \\ t_{i+1} = t_i - 4,\ B_{i+1} = B_i + 3,\ C_{i+1} = L_{d_{i+1}=d_i+3}^{m_{i+1}=m_i+1} \end{cases}$

Taking $A_i, B_i*, \overleftarrow{C_i}$, where C_i and its reverse have 3 pairs in common (Table B.8), get the pairs $t_i + 3, \ldots, t_i + p, t_i + p - 3$.

Taking $\overleftarrow{A_i}, B_i*, C_i$, get the pairs $m_i, \ldots, m_i + p - 3, m_i + p$.

Taking $A_{i+1}, B_{i+1}*, \overleftarrow{C_{i+1}}$, where C_{i+1} and its reverse are disjoint (Table B.1), get the pairs $t_{i+1}, \ldots, t_{i+1} + p - 3, t_{i+1} + p$.

Taking $\overleftarrow{A_{i+1}}, B_{i+1}*, C_{i+1}$, get the pairs $m_{i+1}, \ldots, m_{i+1} + p - 3, m_{i+1} + p$.

Repeat Step 4) for all admissible orders $m_i \geq 2d_i + 1$ or $m_{i+1} \geq 2d_{i+1} - 1$.

Now we deal with the small orders:

(b) $10 \leq n < 72$

For $n = 12$, we prove that $Int_{S_{12}} = (4, 8) - \{6, 7\}$

$t = 7,\ B = S_1,\ C = L_2^4$

Taking $A, B, \overleftarrow{C}$, where C and its reverse are disjoint [40] and get the pair 8.

Taking $\overleftarrow{A}, B, C$, get the pair 5.

For $n = 24$, we prove that $Int_{S_{24}} = (8, 12)$

$t = 11,\ B = S_4,\ C = L_5^9$

Taking $A, B*, \overleftarrow{C}$, where C and its reverse can have $0, 1, 3$ pairs in common (Table B.6), get the pairs $11, \ldots, 16$.

Taking $\overleftarrow{A}, B*, C$ and get the pairs $9, 10, 13$.

For $n = 36$, we prove that $Int_{S_{36}} = (12, 24) - \{16, 18, 21, 22\}$

$t = 17,\ B = S_5 + 1\ pair,\ C = L_6^{13}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $14, 15, 19$.

$t = 19,\ B = S_4,\ C = L_5^{13}$

Taking $A, B*, \overleftarrow{C}$, where C and its reverse are disjoint (Table B.2), get the pairs $19, 20, 23$.

Taking $\overleftarrow{A}, B*, C$, get the pairs $13, 14, 17$.

$t = 23,\ B = S_1,\ C = L_2^{12}$

Taking $A, B, \overleftarrow{C}$, where C and its reverse are disjoint [40], get the pair 24.

Taking $\overleftarrow{A}, B, C$, get the pair 13.

For $\underline{n = 48}$, we prove that

$Int_{S_{48}} = (16, 32) - \{23, 24, 29, 30\}$

$t = 21,\ B = 5 - ext\ S_8 + 1\ pair,\ C = L_9^{18}$

Taking $\overleftarrow{A}, B, C$ get the pair 27. $t = 23,\ B = S_6 + 2\ pairs,\ C = L_7^{17}$

Taking $A, B*, \overleftarrow{C}$, where C and its reverse are disjoint [40], get the pairs $25, 26, 27, 28, 31$.

Taking $\overleftarrow{A}, B*, C$, get the pairs $19, 20, 21, 22, 25$.

$t = 27,\ B = S_4,\ C = L_5^{17}$

Taking $A, B*, \overleftarrow{C}$, where C and its reverse are disjoint(Table B.2), get the pairs $27, 28, 31$.

Taking $\overleftarrow{A}, B*, C$, get the pairs $17, 18, 21$.

$t = 31,\ B = S_1,\ C = L_2^{16}$

Taking $A, B, \overleftarrow{C}$, where C and its reverse are disjoint[40], get the pair 32.

Taking $\overleftarrow{A}, B, C$ and get the pair 17.

For $\underline{n = 60}$, we prove that $Int_{S_{60}} = (20, 40) - \{29, 30, 32, 33, 34, 36\}$

$t = 25,\ B = 10 - ext\ S_{11} + 1\ pair,\ C = L_{12}^{23}$

Taking $\overleftarrow{A}, B, C$, get the pair 35.

Taking $A, B, \overleftarrow{C}$, where C and its reverse have $0, 1, 2$ pairs in common (Table B.6), get the pairs $37, 38, 39$.

$t = 29,\ B = S_8 + 1\ pair,\ C = L_9^{22}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $23, \ldots, 28, 31$.

$t = 35,\ B = S_4,\ C = L_5^{21}$

Taking $A, B*, \overleftarrow{C}$, where C and its reverse are disjoint(Table B.2), get the pairs $35, 36, 39$.

Taking $\overleftarrow{A}, B*, C$ and get the pairs $21, 22, 25$.

$t = 39,\ B = S_1,\ C = L_2^{20}$

Taking $A, B, \overleftarrow{C}$, where C and its reverse are disjoint [40] and get the pair 40.
Taking $\overleftarrow{A}, B, C$, get the pair 21.

Case (2): $n \equiv 1(mod\ 12)$

We start with:

(a) $n \geq 109$ and we prove that

$Int_{S_n} = (\lfloor \frac{n}{3} \rfloor, 2\lfloor \frac{n}{3} \rfloor) - \{2\lfloor \frac{n}{3} \rfloor - 4\}$.

Step 1) Taking $L_{\lfloor \frac{n}{3} \rfloor+1}^{2\lfloor \frac{n}{3} \rfloor+1} + S_{\lfloor \frac{n}{3} \rfloor}$, where the Langford sequence can have $0, 1, 2$ pairs in common (Table B.6) and get the pairs: $\lfloor \frac{n}{3} \rfloor$, $\lfloor \frac{n}{3} \rfloor + 1$, $\lfloor \frac{n}{3} \rfloor + 2$.

Step 2) $\left\{ t_1 = 2\lfloor \frac{n}{3} \rfloor - 7;\ B_1 = 2 - near\ S_7;\ C_1 = hL_{d_1=8}^{m_1=\lfloor \frac{n}{3} \rfloor+1} + (2, 0, 2) \right.$

Taking $A_1, B_1, \overleftarrow{C_1}$, where C_1 and its reverse are disjoint (Table B.2), get the pair $2\lfloor \frac{n}{3} \rfloor - 1$.

Step 3) $\begin{cases} t_2 = t_1 - 4;\ B_2 = S_9;\ C_2 = L_{d_2=10}^{m_2=m_1+2=\lfloor \frac{n}{3} \rfloor+3} \\ t_3 = t_2 - 4;\ B_3 = B_2 + 3;\ C_3 = L_{d_3=d_2+3}^{m_3=m_2+1} \end{cases}$

Taking $A_2, B_2*, \overleftarrow{C_2}$, where C_2 and its reverse are disjoint (Table B.2) and get the pairs $t_2, \ldots, t_2 + p - 3, t_2 + p$.

Taking $\overleftarrow{A_2}, B_2*, C_2$, get the pairs $m_2, \ldots, m_2 + p - 3, m_2 + p$.

Taking $A_3, B_3*, \overleftarrow{C_3}$, where C_3 and its reverse are disjoint (Table B.1), get the pairs $t_3, \ldots, t_3 + p - 3, t_3 + p$.

Taking $\overleftarrow{A_3}, B_3*, C_3$, get the pairs $m_3, \ldots, m_3 + p - 3, m_3 + p$.

Step 4) $\begin{cases} t_i = t_{i-1} - 12;\ B_i = B_{i-1} + 9;\ C_i = L_{d_i=d_{i-1}+9}^{m_i=m_{i-1}+3} \\ t_{i+1} = t_i - 4;\ B_{i+1} = B_i + 3;\ C_{i+1} = L_{d_{i+1}=d_i+3}^{m_{i+1}=m_i+1} \end{cases}$

Taking $A_i, B_i*, \overleftarrow{C_i}$, where C_i and its reverse are disjoint (Table B.2) and get the pairs $t_i, \ldots, t_i + p - 3, t_i + p$.

Taking $\overleftarrow{A_i}, B_i*, C_i$, get the pairs $m_i, \ldots, m_i + p - 3, m_i + p$.

Taking $A_{i+1}, B_{i+1}*, \overleftarrow{C_{i+1}}$, where C_{i+1} and its reverse are disjoint (Table B.1), get the pairs $t_{i+1}, \ldots, t_{i+1} + p - 3, t_{i+1} + p$.

Taking $\overleftarrow{A_{i+1}}, B_{i+1}*, C_{i+1}$, get the pairs $m_{i+1}, \ldots, m_{i+1} + p - 3, m_{i+1} + p$.

Repeat Step 4) for all admissible orders $m_i \geq 2d_i - 1$ or $m_{i+1} \geq 2d_{i+1} - 1$.

And now we deal with the small orders:

(b) $10 \leq n < 109$

For $n = 13$, we prove that $Int_{S_{13}} = (4,8) - \{6\}$

$L_5^9 + S_4$, where the Langford sequence can have $0,1,3$ pairs in common (Table B.6) gives the pairs $4,5,7$.

For $n = 25$, we prove that $Int_{S_{25}} = (8,16) - \{11,12,13,15\}$

$L_9^{17} + S_8$, where Langford sequence can have $0,1,2$ pairs in common (Table B.6) gives the pairs $8,9,10$.

$t = 11,\ B = 3 - ext\ S_4 + 1\ pair,\ C = L_5^9$

Taking $\overleftarrow{A}, B, C$, get the pair 14.

Taking $A, B, \overleftarrow{C}$, where C and its reverse have $0,1,2$ pairs in common (Table B.6), get the pairs $16,17,18$.

For $n = 37$, we prove that $Int_{S_{37}} = (12,24) - \{17,18,19,21,22,23\}$

$L_{13}^{25} + S_{12}$; L_{13}^{25} gives the pairs $12,13,14$.

$t = 17,\ S_5 + 2\ pairs,\ C = L_6^{13}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $15,16,20$.

For $n = 49$, we prove that $Int_{S_{49}} = (16,32)$

$L_{17}^{33} + S_{16}$, where for the Langford sequence we can have $0,1,2,3$ pairs in common (Table B.6) gives the pairs $16,17,18,19$.

$t = 21,\ B = S_9,\ C = L_{10}^{19}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $19,\ldots,25,28$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse have $0,1,2$ pairs in common (Table B.6), get the pairs $21,\ldots,32$.

For $n = 61$, we prove that $Int_{S_{61}} = (20,40) - \{36,37\}$

$L_{21}^{41} + S_{20}$. The perfect Langford sequence can have $0,1,2$ pairs in common (Table B.6), gives the pairs $20,21,22$.

$t = 33,\ B = 2 - nearS_7,\ C = hL_8^{21} + (2,0,2)$

Taking $\overleftarrow{A}, B, C$, get the pair 28.

Taking $A, B, \overleftarrow{C}$, where C and its reverse have 1 pair in common (Table B.9), get the pair 40. $t = 29,\ B = 2 - near\ S_{10},\ C = hL_{11}^{22} + (2,0,2)$

Taking $\overleftarrow{A}, B, C$, get the pair 32.

Taking $A, B, \overleftarrow{C}$, where C and its reverse have 1 pair in common (Table B.8), get the pair 39.

$t = 29,\ B = S_9,\ C = L_{10}^{23}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $23, \ldots, 29, 32$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse are disjoint (Table B.2), get the pairs $29, \ldots, 35, 38$.

For $\underline{n = 73}$, we prove that $Int_{S_{73}} = (24, 48) - \{44\}$

$L_{25}^{49} + S_{24}$, where the perfect Langford sequence can have $0, 1, 2$ pairs in common (Table B.6), gives the pairs $24, 25, 26$.

$t = 41,\ B = 2 - near\ S_7,\ C = hL_8^{25} + (2, 0, 2)$

Taking $\overleftarrow{A}, B, C$, get the pair 32.

Taking $A, B, \overleftarrow{C}$, where C and its reverse have 1 pair in common (Table B.9), get the pair 48.

$t = 37,\ B = S_9,\ C = L_{10}^{27}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $27, \ldots, 33, 36$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse are disjoint (Table B.2), get the pairs $37, \ldots, 43, 46$.

$t = 33,\ B = S_{12},\ C = L_{13}^{28}$

Taking $\overleftarrow{A}, B*, C$ and get the pairs $28, \ldots, 37, 40$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse are disjoint (Table B.1), get the pairs $33, \ldots, 42, 45$.

$t = 31,\ B = 11 - ext\ S_{13} + 1\ pair,\ C = L_{14}^{28}$

Taking $\overleftarrow{A}, B, C$, get the pair 42.

Taking $A, B, \overleftarrow{C}$, where C and its reverse have $1, 3$ pairs in common (Table B.7 and Table B.1), get the pairs $46, 48$.

For $\underline{n = 85}$, we prove that $Int_{S_{85}} = (28, 56)$

$L_{29}^{57} + S_{28}$; The perfect Langford sequence can have $0, 1, 2, 3$ pairs in common (Table B.6) and the perfect Skolem sequence can be the same. This gives the pairs

28, 29, 30, 31.

$t = 45,\ B = S_9,\ C = L_{10}^{31}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $31, \ldots, 37, 40$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse are disjoint (Table B.2), get the pairs $45, \ldots, 51, 54$.

$t = 41,\ B = S_{12},\ C = L_{13}^{32}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $32, \ldots, 41, 43$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse are disjoint (Table B.1), get the pairs $41, \ldots, 50, 53$.

$t = 39,\ 7 - ext\ S_{13} + 1\ pair,\ C = L_{14}^{32}$

Taking $\overleftarrow{A}, B, C$, get the pair 46.

Taking $A, B, \overleftarrow{C}$, where C and its reverse have 3 pairs in common (Table B.1), get the pairs 56.

$t = 35,\ B = 15 - ext\ S_{16} + 1\ pair,\ C = L_{17}^{33}$

Taking $\overleftarrow{A}, B, C$, get the pair 50.

Taking $A, B, \overleftarrow{C}$, where C and its reverse have $0, 1, 2, 3$ pairs in common (Table B.6) and get the pairs $52, 53, 54, 55$.

For $\underline{n = 97}$, we prove that $Int_{S_{97}} = (32, 64)$

$L_{33}^{65} + S_{32}$, where the perfect Langford sequence can have $0, 1, 2$ pairs in common (Table B.6) and gives the pairs $32, 33, 34$.

$t = 57,\ B = 2 - near\ S_7,\ C = hL_8^{33} + (2, 0, 2)$

Taking $\overleftarrow{A}, B, C$,get the pair 40.

Taking $A, B, \overleftarrow{C}$, where C and its reverse have 0 pair in common (Table B.2), get the pair 63.

$t = 53,\ B = S_9,\ C = L_{10}^{35}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $35, \ldots, 41, 44$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse are disjoint (Table B.2), get the pairs $53, \ldots, 59, 62$.

$t = 49,\ B = S_{12},\ C = L_{13}^{36}$

Taking $\overleftarrow{A}, B*, C$ and get the pairs $36, \ldots, 45, 48$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse are disjoint (Table B.1), get the pairs $49, \ldots, 58, 61$.

$t = 47,\ B = 3 - ext\ S_{13} + 1\, pair,\ C = L_{14}^{36}$

Taking $A, B, \overleftarrow{C}$, where C and its reverse have 3 pairs in common (Table B.1), get the pair 64.

$t = 43,\ B = 11 - ext\ S_{16} + 1\, pair,\ C = L_{17}^{37}$ Taking $A, B, \overleftarrow{C}$, where C and its reverse have 0 pairs in common (Table B.2), get the pair 60.

Case (3): $n \equiv 4\ (mod\ 12)$

We start with:

(a) $n \geq 112$

and we prove that $Int_{S_n} = (\lfloor \frac{n}{3} \rfloor, 2\lfloor \frac{n}{3} \rfloor) - \{2\lfloor \frac{n}{3} \rfloor - 1\}$

Step 1)$L_{\lfloor \frac{n}{3} \rfloor+1}^{2\lfloor \frac{n}{3} \rfloor+1} + S_{\lfloor \frac{n}{3} \rfloor}$ and get the pairs: $\lfloor \frac{n}{3} \rfloor$, $\lfloor \frac{n}{3} \rfloor + 1$, $\lfloor \frac{n}{3} \rfloor + 2$.

Step 2) $\left\{ t = 2\lfloor \frac{n}{3} \rfloor - 7;\ B = 2 - near\ S_7;\ C = L_8^{\lfloor \frac{n}{3} \rfloor+1} + (2,0,2) \right.$

Taking $A, B, \overleftarrow{C}$, where C and its reverse have one pair in common (Table B.8), get the pair $2\lfloor \frac{n}{3} \rfloor$.

Step 3) $\left\{ \begin{array}{l} t_1 = 2\lfloor \frac{n}{3} \rfloor - 11;\ B_1 = S_9;\ C_1 = L_{d_1=10}^{m_1=\lfloor \frac{n}{3} \rfloor+3} \\ t_2 = 2\lfloor \frac{n}{3} \rfloor - 15;\ B_2 = B_1 + 3;\ C_2 = L_{d_2=13}^{m_2=\lfloor \frac{n}{3} \rfloor+4} \end{array} \right.$

Taking $A_1, B_1*, \overleftarrow{C_1}$, where C_1 and its reverse have 3 pairs in common (Table B.1), get the pairs $t_1 + 3, \ldots, t_1 + p, t_1 + p + 3$.

Taking $\overleftarrow{A_1}, B_1*, C_1$, get the pairs $m_1, \ldots, m_1 + p - 3, m_1 + p$.

Taking $A_2, B_2*, \overleftarrow{C_2}$, where C_2 and its reverse are disjoint (Table B.2), get the pairs $t_2, \ldots, t_2 + p - 3, t_2 + p$.

Taking $\overleftarrow{A_2}, B_2*, C_2$, get the pairs $m_2, \ldots, m_2 + p - 3, m_2 + p$.

Step 4) $\left\{ \begin{array}{l} t_i = t_{i-1} - 12;\ B_i = B_{i-1} + 9;\ C_i = L_{d_i=d_{i-1}+9}^{m_i=m_{i-1}+3} \\ t_{i+1} = t_i - 4;\ B_{i+1} = B_i + 3;\ C_{i+1} = L_{d_{i+1}=d_i+3}^{m_{i+1}=m_i+1} \end{array} \right.$

Taking $A_i, B_i*, \overleftarrow{C_i}$, where C_i and its reverse have 3 pairs in common (Table B.8), get the pairs $t_i + 3, \ldots, t_i + p, t_i + p - 3$.

Taking $\overleftarrow{A_i}, B_i*, C_i$, and get the pairs $m_i, \ldots, m_i + p - 3, m_i + p$.

Taking $A_{i+1}, B_{i+1}*, \overleftarrow{C_{i+1}}$, where C_{i+1} and its reverse are disjoint (Table B.2), get the pairs $t_{i+1}, \ldots, t_{i+1} + p - 3, t_{i+1} + p$.
Taking $\overleftarrow{A_{i+1}}, B_{i+1}*, C_{i+1}$, get the pairs $m_{i+1}, \ldots, m_{i+1} + p - 3, m_{i+1} + p$.
Repeat Step 4) for all admissible orders $m_i \geq 2d_i - 1$ or $m_{i+1} \geq 2d_{i+1} - 1$.

Now we deal with the small orders:

<u>(b) $10 \leq n < 112$.</u>

For <u>$n = 16$</u>, we prove that $Int_{s_{16}} = (5, 10)$
$L_6^{11} + S_5$, where the Langford sequences have $0, 1, 2$ pairs in common and gives the pairs $5, 6, 7$.
$t = 7;\ B = S_2 + 2\,pairs;\ C = L_3^5$
$\overleftarrow{A}, B, C$ and get the pair 9.

For <u>$n = 28$</u>, we prove that $Int_{S_{28}} = (9, 18) - \{13, 14, 16, 17\}$
$L_{10}^{19} + S_9$; The perfect Langford sequence can have $0, 1, 2$ pairs in common (Table B.6) and gives the pairs $9, 10, 11$.
$t = 13;\ B = S_4 +\ 1\,pair;\ C = L_5^{10}$
Taking $\overleftarrow{A}, B*, C$, get the pairs $11, 12, 15$.

For <u>$n = 40$</u> we prove that $Int_{S_{40}} = (13, 26) - \{18, 19, 20, 22, 24\}$
$L_{14}^{27} + S_{13}$; The perfect Langford sequence can have $0, 1, 2$ pairs in common (Table B.6) and gives the pairs $13, 14, 15$.
$t = 19;\ B = S_5 +\ 2\,pairs;\ C = L_6^{14}$
Taking $\overleftarrow{A}, B*, C$, get the pair $16, 17, 21$.
$t = 17;\ B = 6 - ext\ S_7 +\ 1\,pair;\ C = L_8^{15}$
Taking $\overleftarrow{A}, B, C$, get the pair 23.
Taking $A, B, \overleftarrow{C}$, where C and its reverse have $0, 1, 2, 3$ pairs in common (Table B.6), get the pairs $25, 26, 27, 28$.

For <u>$n = 52$</u>, we prove that $Int_{S_{52}} = (17, 34)$
$L_{18}^{35} + S_{17}$; The perfect Langford sequence can have $0, 1, 2$ pairs in common (Table B.6) and gives the pairs $17, 18, 19$.
$t = 27;\ B = 2 - near\ S_7;\ C = hL_8^{18} + (2, 0, 2)$

Taking $A, B, \overleftarrow{C}$, where C and its reverse have 1 pair in common (Table B.8), get the pair 34.

$t = 23;\ B = S_9;\ C = L_{10}^{20}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $20, \ldots, 26, 29$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse have $1, 3$ pairs in common (Table B.7 and Table B.1), get the pairs $24, \ldots, 33, 35$.

For $\underline{n = 64}$, we prove that $Int_{S_{64}} = (21, 42)$

$L_{22}^{43} + S_{21}$; The perfect Langford sequence can have $0, 1, 2$ pairs in common (Table B.6) and gives the pairs $21, 22, 23$.

$t = 35;\ B = 2 - near\ S_7;\ C = L_8^{22} + (2, 0, 2)$

Taking $A, B, \overleftarrow{C}$, where C and its reverse have 1 pair in common (Table B.8), get the pair 42.

$t = 31;\ B = 2 - near\ S_{10},\ C = L_{11}^{23} + (2, 0, 2)$

Taking $A, B, \overleftarrow{C}$, where C and its reverse have 1 pair in common (Table B.9), get the pair 41.

$t = 31;\ B = S_9;\ C = L_{10}^{24}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $24, \ldots, 30, 33$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse have 3 pairs in common (Table B.1), get the pairs $31, \ldots, 37, 40$.

$t = 27;\ B = S_{12};\ C = L_{13}^{25}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $25, \ldots, 34, 37$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse are disjoint (Table B.6), get the pairs $27, \ldots, 36, 39$.

For $\underline{n = 76}$, we prove that $Int_{S_{76}} = (25, 50) - \{49\}$

$L_{26}^{51} + S_{25}$; The perfect Langford sequence can have $0, 1, 2, 3$ pairs in common (Table B.6) and gives the pairs $25, 26, 27, 28$.

$t = 43;\ B = 2 - near\ S_7;\ C = L_8^{26} + (2, 0, 2)$

Taking $A, B, \overleftarrow{C}$, where C and its reverse have 1 pair in common (Table B.8), get the pair 50.

$t = 39$; $B = S_9$; $C = L_{10}^{28}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $28, \ldots, 34, 37$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse have 3 pairs in common (Table B.1) and get the pairs $42, \ldots, 48, 51$.

$t = 35$; $B = S_{12}$; $C = L_{13}^{29}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $29, \ldots, 38, 41$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse are disjoint (Table B.2), get the pairs $35, \ldots, 44, 47$.

For <u>$n = 88$</u>, we prove that $Int_{S_{88}} = (29, 58) - \{57\}$

$L_{30}^{59} + S_{29}$; The perfect Langford sequence can have $0, 1, 2$ pairs in common (Table B.6) and gives the pairs $29, 30, 31$.

$t = 51$; $B = 2 - near\ S_7$; $C = hL_8^{30} + (2, 0, 2)$

Taking $A, B, \overleftarrow{C}$, where C and its reverse have 1 pair in common (Table B.8), get the pair 58.

$t = 47$; $B = S_9$; $C = L_{10}^{32}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $32, \ldots, 38, 41$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse have 3 pairs in common (Table B.1), get the pairs $50, \ldots, 56, 59$.

$t = 43$; $B = S_{12}$; $C = L_{13}^{33}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $33, \ldots, 42, 45$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse are disjoint (Table B.2), get the pairs $43, \ldots, 52, 55$.

For <u>$n = 100$</u>, we prove that $Int_{S_{100}} = (33, 66) - \{65\}$

$L_{34}^{67} + S_{33}$; The perfect Langford sequence can have $0, 1, 2$ pairs in common (Table B.6) and the perfect Skolem sequence can be the same. These gives the pairs $33, 34, 35$.

$t = 59$; $B = 2 - near\ S_7$; $C = hL_8^{34} + (2, 0, 2)$

Taking $A, B, \overleftarrow{C}$, where C and its reverse have 1 pair in common (Table B.8), get the pair 66.

$t = 55$; $B = S_9$; $C = L_{10}^{36}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $36, \ldots, 42, 45$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse have 3 pairs in common (Table B.1), get the pairs $63, \ldots, 69, 72$.

$t = 51;\ B = S_{12};\ C = L_{13}^{37}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $37, \ldots, 46, 49$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse are disjoint (Table B.2), get the pairs $51, \ldots, 60, 63$.

Case (4): $n \equiv 5(mod\ 12)$

We start with:

(a) Case $n \geq 77$

and we prove that $Int_{S_n} = (\lfloor \frac{n}{3} \rfloor, 2\lfloor \frac{n}{3} \rfloor) - \{\lfloor \frac{n}{3} \rfloor + 1, 2\lfloor \frac{n}{3} \rfloor - 2, 2\lfloor \frac{n}{3} \rfloor - 3\}$.

Step 1) $\left\{\ t_1 = 2\lfloor \frac{n}{3} \rfloor - 7;\ B_1 = 2 - near\ S_6;\ C_1 = L_{d_1=7}^{m_1=\lfloor \frac{n}{3} \rfloor + 1} + (2,0,2)\right.$

Taking $A_1, B_1, \overleftarrow{C_1}$, where C_1 and its reverse have 0 pairs in common (Table B.8), get last pair $2\lfloor \frac{n}{3} \rfloor$.

Step 2) $\begin{cases} t_2 = 2\lfloor \frac{n}{3} \rfloor - 7;\ B_2 = S_5;\ C_2 = L_{d_2=6}^{m_2=m_1+1} \\ t_3 = t_2 - 4;\ B_3 = B_2 + 3;\ C_3 = L_{d_3=d_2+3}^{m_3=m_2+1} \end{cases}$

Taking $A_2, B_2*, \overleftarrow{C_2}$, where C_2 and its reverse are disjoint (Table B.2), get the pairs $t_2, \ldots, t_2 + 1, t_2 + 5$.

Taking $\overleftarrow{A_2}, B_2*, C_2$, get the pairs $m_2, m_2 + 1, m_2 + 5$.

Taking $A_3, B_3*, \overleftarrow{C_3}$, where C_3 and its reverse are disjoint (Table B.1), get the pairs $t_3, \ldots, t_3 + p - 3, t_3 + p$.

Taking $\overleftarrow{A_3}, B_3*, C_3$, get the pairs $m_3, \ldots, m_3 + p - 3, m_3 + p$.

Step 3) $\begin{cases} t_i = t_{i-1} - 12;\ B_i = B_{i-1} + 9;\ C_i = L_{d_i=d_{i-1}+9}^{m_i=m_{i-1}+3} \\ t_{i+1} = t_i - 4;\ B_{i+1} = B_i + 3;\ C_{i+1} = L_{d_{i+1}=d_i+3}^{m_{i+1}=m_i+1} \end{cases}$

Taking $A_i, B_i*, \overleftarrow{C_i}$, where C_i and its reverse are disjoint (Table B.2), get the pairs $t_i, \ldots, t_i + p - 3, t_i + p$.

Taking $\overleftarrow{A_i}, B_i*, C_i$, get the pairs $m_i, \ldots, m_i + p - 3, m_i + p$.

Taking $A_{i+1}, B_{i+1}*, \overleftarrow{C_{i+1}}$, where C_{i+1} and its reverse are disjoint (Table B.1), get the pairs $t_{i+1}, \ldots, t_{i+1} + p - 3, t_{i+1} + p$.

Taking $\overleftarrow{A_{i+1}}, B_{i+1}*, C_{i+1}$, get the pairs $m_{i+1}, \ldots, m_{i+1}+p-3, m_{i+1}+p$.
Repeat Step 3) for all admissible orders $m_i \geq 2d_i - 1$ or $m_{i+1} \geq 2d_{i+1} - 1$.

And now we deal with the small orders:

(b) $10 \leq n < 77$

For $n = 17$ we are not able to use these constructions.

For $n = 29$, we prove that $Int_{S_{29}} = (9, 18) - \{10, 17\}$

$t = 13,\ B = S_5,\ C = L_6^{11}$

Taking $\overleftarrow{A}, B*, C$, get the pairs 11, 12, 16.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse have 0, 1, 2 pairs in common (Table B.6), get the pairs $13, \ldots, 16, 18$.

For $n = 41$, we prove that $Int_{S_{41}} = (13, 26) - \{14, 23, 24, 25\}$

$t = 21,\ B = S_5,\ C = L_6^{15}$

Taking $\overleftarrow{A}, B*, C$, get the pairs 15, 16, 20.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse have 0 pairs in common (Table B.2), get the pairs 21, 22, 26.

$t = 19,\ B = S_6 + 1\ pair,\ C = L_7^{15}$

Taking $\overleftarrow{A}, B*, C$, get the pairs 16, 17, 18, 19, 22.

For $n = 53$, we prove that $Int_{S_{53}} = (17, 34) - \{18, 31, 32\}$

$t = 29,\ B = S_5,\ C = 6^{19}$

Taking $\overleftarrow{A}, B*, C$, get the pairs 19, 20, 24.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse have 0 pairs in common (Table B.2) and get the pairs 29, 30, 34.

$t = 25,\ B = S_8,\ C = L_9^{20}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $20, \ldots, 25, 28$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse have 0 pairs in common (Table B.1), get the pairs $25, \ldots, 30, 33$.

For $n = 65$, we prove that $Int_{S_{65}} = (21, 42) - \{22, 30, 31, 39\}$

$t = 27,\ B = S_5,\ C = 6^{23}$

Taking $\overleftarrow{A}, B*, C$, get the pairs 23, 24, 27.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse have 0 pairs in common (Table B.2) and get the pairs $27, 28, 32$.

$t = 33,\ B = S_8,\ C = L_9^{24}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $24, \ldots, 29, 32$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse have 0 pairs in common (Table B.1), get the pairs $33, \ldots, 38, 41$.

$t = 27,\ B = 11 - ext\ S_{12} + 1\ pair,\ C = L_{13}^{25}$

Taking $\overleftarrow{A}, B, C$, get the pairs 38.

Taking $A, B, \overleftarrow{C}$, where C and its reverse have $0, 1, 2$ pairs in common (Table B.6), get the pairs $40, 41, 42$.

Case (5): $n \equiv 8(mod\ 12)$

We start with:

(a) $n \geq 92$

and prove that $Int_{S_n} = (\lfloor\frac{n}{3}\rfloor, 2\lfloor\frac{n}{3}\rfloor) - \{\lfloor\frac{n}{3}\rfloor + 1, 2\lfloor\frac{n}{3}\rfloor - 3\}$.

Step 1) $\left\{\ t_1 = 2\lfloor\frac{n}{3}\rfloor - 5;\ B_1 = 2 - near\ S_6;\ C_1 = hL_{d_1=7}^{m_1=\lfloor\frac{n}{3}\rfloor+1} + (2, 0, 2)\right.$

Taking $A_1, B_1, \overleftarrow{C_1}$, where C_1 and its reverse have 0 pairs in common (Table B.2), get last pair $2\lfloor\frac{n}{3}\rfloor$.

Step 2) $\left\{\begin{array}{l} t_2 = 2\lfloor\frac{n}{3}\rfloor - 5;\ B_2 = S_5;\ C_2 = L_{d_2=6}^{m_2=m_1+1} \\ t_3 = t_2 - 4;\ B_3 = B_2 + 3;\ C_3 = L_{d_3=d_2+3}^{m_3=m_2+1} \end{array}\right.$

Taking $A_2, B_2*, \overleftarrow{C_2}$, where C_2 and its reverse have 3 pairs in common (Table B.1), get the pairs $t_2 + 3, t_2 + 4, t_2 + 8$.

Taking $\overleftarrow{A_2}, B_2*, C_2$, get the pairs $m_2, m_2 + 1, m_2 + 5$.

Taking $A_3, B_3*, \overleftarrow{C_3}$, where C_3 and its reverse are disjoint (Table B.2), get the pairs $t_3, \ldots, t_3 + p - 3, t_3 + p$.

Taking $\overleftarrow{A_3}, B_3*, C_3$, get the pairs $m_3, \ldots, m_3 + p - 3, m_3 + p$.

Step 3) $\left\{\begin{array}{l} t_i = t_{i-1} - 12;\ B_i = B_{i-1} + 9;\ C_i = L_{d_i=d_{i-1}+9}^{m_i=m_{i-1}+3} \\ t_{i+1} = t_i - 4;\ B_{i+1} = B_i + 3;\ C_{i+1} = L_{d_{i+1}=d_i+3}^{m_{i+1}=m_i+1} \end{array}\right.$

Taking $A_i, B_i*, \overleftarrow{C_i}$, where C_i and its reverse have 3 pairs in common (Table B.1), get the pairs $t_i + 3, \ldots, t_i + p, t_i + p + 3$.

Taking $\overleftarrow{A_i}, B_i*, C_i$, get the pairs $m_i, \ldots, m_i + p - 3, m_i + p$.

Taking $A_{i+1}, B_{i+1}*, \overleftarrow{C_{i+1}}$, where C_{i+1} and its reverse are disjoint (Table B.1), get the pairs $t_{i+1}, \ldots, t_{i+1} + p - 3, t_{i+1} + p$.

Taking $\overleftarrow{A_{i+1}}, B_{i+1}*, C_{i+1}$, get the pairs $m_{i+1}, \ldots, m_{i+1} + p - 3, m_{i+1} + p$.

Repeat Step 3) for all admissible orders $m_i \geq 2d_i - 1$ or $m_{i+1} \geq 2d_{i+1} - 1$.

And now we deal with the small orders:

(b) $10 \leq n < 92$

For $n = 20$, we prove that $Int_{S_{20}} = (6, 12) - \{7, 9\}$

$t = 9,\ B = S_3 + 1\,pair,\ C = L_4^7$

Taking $\overleftarrow{A}, B*, C$, get the pairs $8, 11$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse have $0, 1, 2$ pairs in common (Table B.6), get the pairs $10, 11, 12$.

For $n = 32$, we prove that $Int_{S_{32}} = (10, 20) - \{11, 14, 15\}$

$t = 15,\ B = S_5,\ C = L_6^{12}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $12, 13, 16$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse have $1, 3$ pairs in common (Table B.7 and Table B.1), get the pairs $16, \ldots, 19$.

For $n = 44$, we prove that $Int_{S_{44}} = (14, 28) - \{15\}$

$t = 23,\ B = S_5,\ C = L_6^{16}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $16, 17, 21$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse have 3 pairs in common (Table B.1), get the pairs $26, 27, 31$.

$t = 19,\ B = S_8,\ C = L_9^{17}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $17, \ldots, 22, 25$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse have $0, 1, 2$ pairs in common (Table B.6), get the pairs $19, \ldots, 28$.

For $n = 56$, we prove that $Int_{S_{56}} = (18, 36) - \{19, 33\}$

$t = 31,\ B = 2 - near\ S_6,\ C = hL_7^{19} + (2, 0, 2)$

Taking $A, B, \overleftarrow{C}$, where C and its reverse are disjoint (Table B.9), get the pair 36.

$t = 31,\ B = S_5,\ C = L_6^{20}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $20, 21, 25$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse have 3 pairs in common (Table B.1), get the pairs $34, 35, 39$.

$t = 27,\ B = S_8,\ C = L_9^{21}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $21, \ldots, 26, 29$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse have 0 pairs in common (Table B.2), get the pairs $27, \ldots, 32, 35$.

For $\underline{n = 68}$, we prove that $Int_{S_{68}} = (22, 44) - \{23, 34, 41\}$

$t = 39,\ B = 2 - near\ S_6,\ C = hL_7^{23} + (2, 0, 2)$

Taking $A, B, \overleftarrow{C}$, where C and its reverse are disjoint (Table B.9), get the pair 44.

$t = 39,\ B = S_5,\ C = L_6^{24}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $24, 25, 29$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse have 3 pairs in common (Table B.1), get the pairs $42, 43, 47$.

$t = 35,\ B = S_8,\ C = L_9^{25}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $25, \ldots, 33, 36$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse have 0 pairs in common (Table B.2), get the pairs $35, \ldots, 40, 43$.

$t = 33,\ B = S_9 + 1\ pair,\ C = L_{10}^{25}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $26, \ldots, 32, 35$.

For $\underline{n = 80}$, we prove that $Int_{S_{80}} = (26, 52) - \{27, 39, 40\}$

$t = 47,\ B = 2 - near\ S_6,\ C = hL_7^{27} + (2, 0, 2)$

Taking $A, B, \overleftarrow{C}$, where C and its reverse are disjoint (Table B.9), get the pair 52.

$t = 47,\ B = S_5,\ C = L_6^{28}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $28, 29, 33$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse have 3 pairs in common (Table B.1), get the pairs $50, 51, 55$.

$t = 43,\ B = S_8,\ C = L_9^{29}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $29, \dots, 44, 47$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse have 0 pairs in common (Table B.2), get the pairs $43, \dots, 48, 51$.

$t = 39,\ B = S_{10} + 2\,pairs;\ C = hL_7^{27} + (2,0,2)$

Taking $\overleftarrow{A}, B*, C$, get the pairs $30, \dots, 37, 40$.

Taking $A, B*, \overleftarrow{C}$, where C and its reverse have 0 pairs in common (Table B.9), get the pairs $41, \dots, 48, 51$.

$t = 33,\ B = 14 - ext\ S_{15} + 1\,pair,\ C = L_{16}^{31}$

Taking $\overleftarrow{A}, B, C$, get the pair 47.

Taking $A, B, \overleftarrow{C}$, where C and its reverse have $0, 1, 2$ pairs in common (Table B.6), get the pairs $49, 50, 51$.

Case (6): $n \equiv 9(mod\ 12)$

We start with:

(a) $n \geq 81$

and prove that $Int_{S_n} = (\lfloor \frac{n}{3} \rfloor, 2\lfloor \frac{n}{3} \rfloor) - \{\lfloor \frac{n}{3} \rfloor + 3, 2\lfloor \frac{n}{3} \rfloor - 2, 2\lfloor \frac{n}{3} \rfloor - 6\}$.

Step 1) $\begin{cases} t_1 = 2\lfloor \frac{n}{3} \rfloor - 1,\ B_1 = S_1,\ C_1 = L_{d_1=2}^{m_1=\lfloor \frac{n}{3} \rfloor} \\ t_2 = t_1 - 4;\ B_2 = B_1 + 3 = S_4,\ C_2 = L_{d_2=5}^{m_2=m_1+1} \end{cases}$

Taking $A_1, B_1, \overleftarrow{C_1}$, where C_1 and its reverse are disjoint [40], get the pair $2\lfloor \frac{n}{3} \rfloor$.

Taking $\overleftarrow{A_2}, B_2*, C_2$, get the pairs $\lfloor \frac{n}{3} \rfloor + 1, \lfloor \frac{n}{3} \rfloor + 2, \lfloor \frac{n}{3} \rfloor + 5$.

Taking $A_2, B_2*, \overleftarrow{C_2}$, where C_2 and its reverse are disjoint (Table B.2), get the pairs $2\lfloor \frac{n}{3} \rfloor - 5, 2\lfloor \frac{n}{3} \rfloor - 4, 2\lfloor \frac{n}{3} \rfloor - 1$.

Step 2) $\{\ t_3 = t_2 - 12,\ B_3 = 2 - near\ S_{14},\ C_3 = hL_{d_3=d_2+10}^{m_3=m_2+2} + (2,0,2)$

Taking $A_3, B_3, \overleftarrow{C_3}$, where C_3 and its reverse are disjoint(Table B.8), get the pairs $t_3 + 13$.

Step 3) $\begin{cases} t_4 = t_3,\ B_4 = B_2 + 9,\ C_4 = L_{d_4=d_2+9}^{m_4=m_2+3} \\ t_5 = t_4 - 4,\ B_5 = B_4 + 3,\ C_5 = L_{d_5=d_4+3}^{m_5=m_4+1} \end{cases}$

Taking $A_4, B_4*, \overleftarrow{C_4}$, where C_4 and its reverse have 0 pairs in common (Table B.2), get the pairs $t_4, \dots, t_4 + p - 3, t_4 + p$.

Taking $\overleftarrow{A_4}, B_4*, C_4$, get the pairs $m_4, \dots, m_4 + p - 3, m_4 + p$.

Taking $A_5, B_5*, \overleftarrow{C_5}$, where C_5 and its reverse are disjoint (Table B.2), get the pairs $t_5, \ldots, t_5 + p - 3, t_5 + p$.

Taking $\overleftarrow{A_5}, B_5*, C_5$, get the pairs $m_5, \ldots, m_5 + p - 3, m_5 + p$.

Step 4) $\begin{cases} t_i = t_{i-1} - 12,\ B_i = B_{i-1} + 9,\ C_i = L_{d_i=d_{i-1}+9}^{m_i=m_{i-1}+3} \\ t_{i+1} = t_i - 4,\ B_{i+1} = B_i + 3,\ C_{i+1} = L_{d_{i+1}=d_i+3}^{m_{i+1}=m_i+1} \end{cases}$

Taking $A_i, B_i*, \overleftarrow{C_i}$, where C_i and its reverse have 0 pairs in common (Table B.2) and get the pairs $t_i + 3, \ldots, t_i + p, t_i + p - 3$.

Taking $\overleftarrow{A_i}, B_i*, C_i$, get the pairs $m_i, \ldots, m_i + p - 3, m_i + p$.

Taking $A_{i+1}, B_{i+1}*, \overleftarrow{C_{i+1}}$, where C_{i+1} and its reverse are disjoint (Table B.2), get the pairs $t_{i+1}, \ldots, t_{i+1} + p - 3, t_{i+1} + p$.

Taking $\overleftarrow{A_{i+1}}, B_{i+1}*, C_{i+1}$, get the pairs $m_{i+1}, \ldots, m_{i+1} + p - 3, m_{I=1} + p$.

Repeat Step 4) for all admissible orders $m_i \geq 2d_i + 1$ or $m_{i+1} \geq 2d_{i+1} - 1$.

And now we deal with the small cases:

(b) $10 \leq n < 81$

For $n = 21$, we prove that $Int_{S_{21}} = (7, 14) - \{9, 10, 11, 12, 13\}$

$t = 13,\ B = S_1,\ C = L_2^7$

Taking $A, B, \overleftarrow{C}$, where C and its reverse are disjoint [40], get the pair 14.

Taking $\overleftarrow{A}, B, C$, get the pair 8.

For $n = 33$, we prove that $Int_{S_{33}} = (11, 22) - \{14, 15, 19, 20\}$

$t = 21,\ B = S_1,\ C = L_2^{11}$

Taking $A, B, \overleftarrow{C}$, where C and its reverse are disjoint [40], get the pair 22.

Taking $\overleftarrow{A}, B, C$, get the pair 12.

$t = 17,\ B = S_4,\ C = L_5^{12}$

Taking $A, B*, \overleftarrow{C}$, where C and its reverse are disjoint(Table B.1), get the pairs 17, 18, 21.

Taking $\overleftarrow{A}, B*, C$, get the pairs 12, 13, 16.

For $n = 45$, we prove that $Int_{S_{45}} = (15, 30) - \{22, 27\}$

$t = 29,\ B = S_1,\ C = L_2^{15}$

Taking $A, B, \overleftarrow{C}$, where C and its reverse are disjoint [40], get the pair 30.

Taking $\overleftarrow{A}, B, C$, get the pair 16.

$t = 25,\ B = S_4,\ C = L_5^{16}$

Taking $A, B*, \overleftarrow{C}$, where C and its reverse are disjoint(Table B.1), get the pairs $25, 26, 29$.

Taking $\overleftarrow{A}, B*, C$ and get the pairs $16, 17, 20$.

$t = 21,\ B = S_6 + 2\,pairs,\ C = L_7^{16}$

Taking $A, B*, \overleftarrow{C}$, where C and its reverse are disjoint(Table B.1), get the pairs $23, 24, 25, 26, 29$.

Taking $\overleftarrow{A}, B*, C$, get the pairs $18, 19, 20, 21, 24$.

$t = 19,\ B = 7 - ext\ S_8 + 1\ pair,\ C = L_9^{17}$

Taking $A, B, \overleftarrow{C}$, where C and its reverse have $0, 1, 2$ pairs in common (Table B.6), get the pairs $28, 29, 30$.

Taking $\overleftarrow{A}, B, C$, get the pair 26.

For $\underline{n = 57}$, we prove that

$Int_{S_{57}} = (19, 38) - \{22, 23, 25, 26, 27, 28, 29, 31, 32, 35\}$

$t = 37,\ B = S_1,\ C = L_2^{19}$

Taking $A, B, \overleftarrow{C}$, where C and its reverse are disjoint [40], get the pair 38.

$t = 33,\ B = S_4,\ C = L_5^{20}$

Taking $A, B*, \overleftarrow{C}$, where C and its reverse are disjoint(Table B.1), get the pairs $33, 34, 37$.

Taking $\overleftarrow{A}, B*, C$, get the pairs $20, 21, 24$.

$t = 27,\ B = 3 - ext\ S_8 + 1\ pair,\ C = L_9^{21}$

Taking $A, B, \overleftarrow{C}$, where C and its reverse are disjoint(Table B.2), get the pair 36.

Taking $\overleftarrow{A}, B*, C$, get the pairs 30.

For $\underline{n = 69}$, we prove that $Int_{S_{69}} = (23, 46) - \{26, 40, 43, 44\}$

$t = 45,\ B = S_1,\ C = L_2^{23}$

Taking $A, B, \overleftarrow{C}$, where C and its reverse are disjoint [40], get the pair 46.

$t = 41,\ B = S_4,\ C = L_5^{24}$

Taking $A, B*, \overleftarrow{C}$, where C and its reverse are disjoint(Table B.1), get the pairs

41, 42, 45.

Taking $\overleftarrow{A}, B*, C$, get the pairs 24, 25, 28.

$t = 29,\ B = S_{13},\ C = L_{14}^{27}$

Taking $A, B*, \overleftarrow{C}$, where C and its reverse have 0, 1, 2, 3 pairs in common (Table B.6), get the pairs 29, ..., 46.

Taking $\overleftarrow{A}, B*, C$, get the pairs 27, ..., 37, 40. □

To prove the following theorem we use the same technique as above.

Theorem 21 *For $n \geq 1$, the necessary conditions are sufficient for the existence of two hooked Skolem sequences of order n, to intersect in $(\lfloor \frac{n}{3} \rfloor, 2\lfloor \frac{n}{3} \rfloor)$ pairs with the following possible exceptions: for $n \equiv 2 (mod\ 12), n > 98, n = 98 + 60r$ the exceptions are $\{\lfloor \frac{n}{3} \rfloor + 1, \lfloor \frac{n}{3} \rfloor + 21 + 16r, \lfloor \frac{n}{3} \rfloor + 22 + 16r, \lfloor \frac{n}{3} \rfloor + 24 + 16r, \ldots, 2\lfloor \frac{n}{3} \rfloor - 4, 2\lfloor \frac{n}{3} \rfloor - 2\}$, otherwise the exceptions are $\{\lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{3} \rfloor + 1, \lfloor \frac{n}{3} \rfloor + 9 + 16r, \lfloor \frac{n}{3} \rfloor + 10 + 16r, \lfloor \frac{n}{3} \rfloor + 12 + 16r, \ldots, 2\lfloor \frac{n}{3} \rfloor - 4, 2\lfloor \frac{n}{3} \rfloor - 2\}$ and other 55 exceptions for $10 \leq n \leq 98$, for $n \equiv 3 (mod\ 12), n > 75, n = 75 + 60r$ the exceptions are $\{\lfloor \frac{n}{3} \rfloor + 3, \lfloor \frac{n}{3} \rfloor + 15 + 16r, \lfloor \frac{n}{3} \rfloor + 16 + 16r, \lfloor \frac{n}{3} \rfloor + 18 + 16r, \ldots, 2\lfloor \frac{n}{3} \rfloor - 3, 2\lfloor \frac{n}{3} \rfloor - 1\}$, otherwise the exceptions are $\{\lfloor \frac{n}{3} \rfloor + 3, \lfloor \frac{n}{3} \rfloor + 3 + 16r, \lfloor \frac{n}{3} \rfloor + 4 + 16r, \lfloor \frac{n}{3} \rfloor + 6 + 16r, \ldots, 2\lfloor \frac{n}{3} \rfloor - 3, 2\lfloor \frac{n}{3} \rfloor - 1\}$ and 41 exceptions for $10 \leq n \leq 75$, for $n \equiv 6 (mod\ 12), n > 78, n = 78 + 60r$ the exceptions are $\{\lfloor \frac{n}{3} \rfloor + 3, \lfloor \frac{n}{3} \rfloor + 15 + 16r, \lfloor \frac{n}{3} \rfloor + 16 + 16r, \lfloor \frac{n}{3} \rfloor + 18 + 16r, \ldots, 2\lfloor \frac{n}{3} \rfloor - 2\}$, otherwise the exceptions are $\{\lfloor \frac{n}{3} \rfloor + 3, \lfloor \frac{n}{3} \rfloor + 3 + 16r, \lfloor \frac{n}{3} \rfloor + 4 + 16r, \lfloor \frac{n}{3} \rfloor + 6 + 16r, \ldots, 2\lfloor \frac{n}{3} \rfloor - 2\}$ and 51 other exceptions for $10 \leq n \leq 78$, for $n \equiv 7 (mod\ 12), n \geq 67, n = 55 + 60r$ the exceptions are $\{\lfloor \frac{n}{3} \rfloor + 10 + 16r, \lfloor \frac{n}{3} \rfloor + 11 + 16r, \lfloor \frac{n}{3} \rfloor + 13 + 16r, \ldots, 2\lfloor \frac{n}{3} \rfloor - 2\}$, otherwise the exceptions are $\{\lfloor \frac{n}{3} \rfloor + 14 + 16r, \lfloor \frac{n}{3} \rfloor + 15 + 16r, \lfloor \frac{n}{3} \rfloor + 17 + 16r, \ldots, 2\lfloor \frac{n}{3} \rfloor - 2\}$ and 14 other possible exceptions for $10 \leq n < 67$, for $n \equiv 10 (mod\ 12), n \geq 70, n = 58 + 60r$ the exceptions are $\{\lfloor \frac{n}{3} \rfloor + 10 + 16r, \lfloor \frac{n}{3} \rfloor + 11 + 16r, \lfloor \frac{n}{3} \rfloor + 13 + 16r, \ldots, 2\lfloor \frac{n}{3} \rfloor\}$, otherwise the exceptions are $\{\lfloor \frac{n}{3} \rfloor + 14 + 16r, \lfloor \frac{n}{3} \rfloor + 15 + 16r, \lfloor \frac{n}{3} \rfloor + 17 + 16r, \ldots, 2\lfloor \frac{n}{3} \rfloor\}$ and other 26 exceptions for $10 \leq n < 70$, for $n \equiv 11 (mod\ 12), n > 95, n = 95 + 60r$ the exceptions are $\{\lfloor \frac{n}{3} \rfloor + 1, \lfloor \frac{n}{3} \rfloor + 21 + 16r, \lfloor \frac{n}{3} \rfloor + 22 + 16r, \lfloor \frac{n}{3} \rfloor + 24 + 16r, \ldots, 2\lfloor \frac{n}{3} \rfloor - 1\}$, otherwise the exceptions are $\{\lfloor \frac{n}{3} \rfloor + 1, \lfloor \frac{n}{3} \rfloor + 9 + 16r, \lfloor \frac{n}{3} \rfloor + 10 + 16r, \lfloor \frac{n}{3} \rfloor + 12 + 16r, \ldots, 2\lfloor \frac{n}{3} \rfloor - 1\}$ and other 56 exceptions for $10 \leq n \leq 95$, where r, t are positive integers.*

Proof: For $1 \leq n \leq 9$, see Appendix A. For a list with all the possible exceptions, see Appendix C.

We divide this proof into 6 cases. Let r, p and i be positive integers.

Case (1): $n \equiv 2(mod\ 12)$

We start with:

(a) $n > 98,\ n = 98 + 60r$

and we prove that $Int_{S_n} = (\lfloor \frac{n}{3} \rfloor, 2\lfloor \frac{n}{3} \rfloor) - \{\lfloor \frac{n}{3} \rfloor + 1, \lfloor \frac{n}{3} \rfloor + 21 + 16r, \lfloor \frac{n}{3} \rfloor + 22 + 16r, \lfloor \frac{n}{3} \rfloor + 24 + 16r, \ldots, 2\lfloor \frac{n}{3} \rfloor - 4, 2\lfloor \frac{n}{3} \rfloor - 2\}$.

Step 1) $\begin{cases} t_1 = 2\lfloor \frac{n}{3} \rfloor - 5,\ B_1 = S_5,\ C_1 = L_{d_1=6}^{m_1=\lfloor \frac{n}{3} \rfloor + 2} \\ t_2 = t_1 - 4;\ B_2 = B_1 + 3 = S_8,\ C_2 = L_{d_2=d_1+3}^{m_2=m_1+1} \end{cases}$

Taking $\overleftarrow{A_1}, B_1*, C_1$, get the pairs:

$\lfloor \frac{n}{3} \rfloor + 2,\ \lfloor \frac{n}{3} \rfloor + 3,\ \lfloor \frac{n}{3} \rfloor + 7$.

Taking $\overleftarrow{A_2}, B_2*, C_2$, get the pairs: $\lfloor \frac{n}{3} \rfloor + 3, \ldots, \lfloor \frac{n}{3} \rfloor + 8, \lfloor \frac{n}{3} \rfloor + 11$.

Step 2) $\begin{cases} t_3 = t_2 - 8;\ B_3 = 2 - near\ S_{15};\ C_3 = L_{d_3=d_2+7}^{m_3=m_2+1}, (2,0,2) \\ t_4 = t_3 - 4;\ B_4 = 2 - near\ S_{18};\ C_4 = L_{d_4=d_3+3}^{m_4=m_3+1}, (2,0,2) \end{cases}$

Taking $A_3, B_3, \overleftarrow{C_3}, (2,0,2)$, where C_3 and its reverse have one pair in common (Table B.1), get the pair: $2\lfloor \frac{n}{3} \rfloor - 1$

Taking $A_4, B_4, \overleftarrow{C_4}, (2,0,2)$, where C_4 and $\overleftarrow{C_4}$ are disjoint (Table B.2), get the pair: $2\lfloor \frac{n}{3} \rfloor - 3$.

Step 3) $\begin{cases} t_5 = t_4;\ B_5 = B_2 + 9;\ C_5 = L_{d_5=d_2+9}^{m_5=m_2+3} \\ t_6 = t_5 - 4;\ B_6 = B_5 + 3;\ C_6 = L_{d_6=d_5+3}^{m_6=m_5+1} \end{cases}$

Taking $\overleftarrow{A_5}, B_5*, C_5$, get the pairs: $m_5, \ldots, m_5 + p - 3, m_5 + p$.

Taking $\overleftarrow{A_6}, B_6*, C_6$, get the pairs: $m_6, \ldots, m_6 + p - 3, m_6 + p$.

If $r = 1$, take $i = 7$ and go to Step 5), otherwise, take $i = 7$ and go to Step 4).

Step 4) $\begin{cases} t_i = t_{i-1} - 12;\ B_i = B_{i-1} + 9;\ C_i = L_{d_i=d_{i-1}+9}^{m_i=m_{i-1}+3} \\ t_{i+1} = t_i - 4;\ B_{i+1} = B_i + 3;\ C_{i+1} = L_{d_{i+1}=d_i+3}^{m_{i+1}=m_i+1} \end{cases}$

Taking $\overleftarrow{A_i}, B_i*, C_i$, get the pairs:

$m_i, \ldots, m_i + p - 3, m_i + p$.

Taking $\overleftarrow{A_{i+1}}, B_{i+1}*, C_{i+1}$, get the pairs:

$m_{i+1}, \ldots, m_{i+1}+p-3, m_{i+1}+p$.

Repeat Step 4) $r-1$ times, then take $i = 5+2r$ and go to Step 5).

Step 5) $t_i = t_{i-1} - 12;\ B_i = B_{i-1} + 9;\ C_i = L^{m_i = m_{i-1}+3}_{d_i = d_{i-1}+9}$

Taking $\overleftarrow{A}_i, B_i*, C_i$, get the pairs:

$m_i, \ldots, m_i + p - 3, m_i + p$.

Then we continue with:

(b) $n > 98, n = 50+60r,\ n = 62+60r,\ n = 74+60r,\ n = 86+60r$

and we prove that $Int_{S_n} = (\lfloor \frac{n}{3} \rfloor, 2\lfloor \frac{n}{3} \rfloor) - \{\lfloor \frac{n}{3} \rfloor + 1, \lfloor \frac{n}{3} \rfloor + 9 + 16r, \lfloor \frac{n}{3} \rfloor + 10 + 16r, \lfloor \frac{n}{3} \rfloor + 12 + 16r, \ldots, 2\lfloor \frac{n}{3} \rfloor - 4, 2\lfloor \frac{n}{3} \rfloor - 2\}$.

Taking Step 1),2),3) and 4) above without Step 5).

And now we taking the small cases:

(c) $10 \leq n \leq 98$

For $n = 14$ and $n = 26$ these constructions cannot be applied.

For $n = 38$, we prove that

$Int_{S_{38}} = (12, 24) - \{13, 16, 17, 18, 20, 22, 23\}$.

$t = 19,\ B = S_5,\ C = L^{14}_6$

Taking $\overleftarrow{A}, B*, C$, get the pairs 14, 15, 19.

$t = 17,\ B = 4 - ext\ S_6 + 1\ pair,\ C = L^{14}_7$

Taking $\overleftarrow{A}, B, C$, get the pair 21.

For $n = 50$, we prove that $Int_{S_{50}} = (16, 32) - \{17, 25, 26, 28, 30\}$.

$t = 27,\ B = S_5,\ C = L^{18}_6$

Taking $\overleftarrow{A}, B*, C$, get the pairs 18, 19, 23.

$t = 23,\ B = S_8,\ C = L^{19}_9$

Taking $\overleftarrow{A}, B*, C$, get the pairs $19, \ldots, 24, 27$.

$t = 21,\ B = 7 - ext\ S_9 + 1\ pair,\ C = L^{19}_{10}$

Taking $\overleftarrow{A}, B, C$, get the pair 29.

Taking $A, B, \overleftarrow{C}$, where C and its reverse can have 0, 1, 2 pairs in common (Table B.6), get the pairs 31, 32.

For $\underline{n=62}$, we prove that

$Int_{S_{62}} = (20,40) - \{21,29,30,32,34,35,36,37,38\}$.

$t = 35,\ B = S_5,\ C = L_6^{22}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $22, 23, 27$.

$t = 31,\ B = S_8,\ C = L_9^{23}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $23, \ldots, 28, 31$.

$t = 29,\ B = 3 - ext\ S_9 + 1\ pair,\ C = L_{10}^{23}$

Taking $\overleftarrow{A}, B, C$, get the pair 33.

Taking $A, B, \overleftarrow{C}$, where C and its reverse are disjoint (Table B.2), get the pair 39.

For $\underline{n=74}$, we prove that

$Int_{S_{74}} = (24,48) - \{25,34,36,37,38,39,40,42,43,44,45,47\}$.

$t = 43,\ B = S_5,\ C = L_6^{26}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $26, 27, 31$.

$t = 39,\ B = S_8,\ C = L_9^{27}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $27, \ldots, 32, 35$.

$t = 33,\ B = 7 - ext\ S_{12} + 1\ pair,\ C = L_{13}^{28}$

Taking $\overleftarrow{A}, B, C$ and get the pairs 41.

Taking $A, B, \overleftarrow{C}$, where C and its reverse are disjoint (Table B.1), get the pair 46.

For $\underline{n=86}$, we prove that

$Int_{S_{86}} = (28,56) - \{29,37,38,40,41,42,43,44,46,48,50,51,52,53\}$.

$t = 51,\ B = S_5,\ C = L_6^{30}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $30, 31, 35$.

$t = 47,\ B = S_8,\ C = L_9^{31}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $31, \ldots, 36, 39$.

$t = 41,\ B = 3 - ext\ S_{12} + 1\ pair,\ C = L_{13}^{32}$

Taking $\overleftarrow{A}, B, C$, get the pair 45.

Taking $A, B, \overleftarrow{C}$, where C and its reverse are disjoint (Table B.1), get the pair 54.

$t = 39,\ B = 2 - near\ S_{15},\ C = L_{16}^{32}, (2,0,2)$

Taking $\overleftarrow{A}, B, C, (2,0,2)$, get the pair 47.

Taking $A, B, \overleftarrow{C}, (2,0,2)$, where C and its reverse have one pair in common (Table B.1), get the pair 55.

$t = 37,\ B = 12 - ext\ S_{15} + 1\ pair,\ C = L_{16}^{33}$

Taking $\overleftarrow{A}, B, C$, get the pair 49.

For $\underline{n = 98}$, we prove that

$Int_{S_{98}} = (32, 64) - \{33, 53, 54, 56, 58, 59, 60\}$.

$t = 59,\ B = S_5,\ C = L_6^{34}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $34, 35, 39$.

$t = 55,\ B = S_8,\ C = L_9^{35}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $35, \ldots, 40, 43$.

$t = 47,\ B = 2 - near\ S_{15},\ C = L_{16}^{36} + (2,0,2)$

Taking $\overleftarrow{A}, B, C, (2,0,2)$, get the pair 51.

Taking $A, B, \overleftarrow{C}, (2,0,2)$, where C and its reverse have one pair in common (Table B.1), get the pair 63.

$t = 43,\ B = 2 - near\ S_{18},\ C = L_{19}^{37} + (2,0,2)$

Taking $\overleftarrow{A}, B, C, (2,0,2)$, get the pair 55.

Taking $A, B, \overleftarrow{C}, (2,0,2)$, where C and its reverse have $0, 1, 2$ pairs in common (Table B.6), get the pairs $61, 62, 63$.

$t = 43,\ B = S_{17},\ C = L_{18}^{38}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $38, \ldots, 52, 55$

$t = 41,\ B = 16 - ext\ S_{18} + 1\ pair,\ C = L_{19}^{38}$

Taking $\overleftarrow{A}, B, C$, get the pair 57.

Case (2): $n \equiv 3 (mod\ 12)$

We start with:

(a) $n > 75,\ n = 75 + 60r$

and we prove that $Int_{S_n} = (\lfloor \frac{n}{3} \rfloor, 2\lfloor \frac{n}{3} \rfloor) - \{\lfloor \frac{n}{3} \rfloor + 3, \lfloor \frac{n}{3} \rfloor + 15 + 16r, \lfloor \frac{n}{3} \rfloor + 16 + 16r, \lfloor \frac{n}{3} \rfloor + 18 + 16r, \ldots, 2\lfloor \frac{n}{3} \rfloor - 3, 2\lfloor \frac{n}{3} \rfloor - 1\}$.

Step 1) $\begin{cases} t_1 = 2\lfloor \frac{n}{3} \rfloor - 1,\ B_1 = S_1,\ C_1 = L_{d_1=2}^{m_1=\lfloor \frac{n}{3} \rfloor} \\ t_2 = t_1 - 4;\ B_2 = B_1 + 3 = S_4,\ C_2 = L_{d_2=d_1+3}^{m_2=m_1+1} \end{cases}$

Taking $\overleftarrow{A_1}, B_1, C_1$, get the pair $\lfloor\frac{n}{3}\rfloor + 1$.

Taking $\overleftarrow{A_2}, B_2*, C_2$, get the pairs $\lfloor\frac{n}{3}\rfloor + 1, \lfloor\frac{n}{3}\rfloor + 2, \lfloor\frac{n}{3}\rfloor + 5$.

Step 2) $\left\{ \; t_3 = t_2 - 8, \; B_3 = 2 - near\ S_{11}, \; C_3 = L_{d_3=d_2+7}^{m_3=m_2+1}, (2,0,2) \right.$

Taking $A_3, B_3, \overleftarrow{C_3}, (2,0,2)$, C_3 and its reverse are disjoint (Table B.2), get the pair $2\lfloor\frac{n}{3}\rfloor - 2$.

Step 3) $\left\{ \begin{array}{l} t_4 = t_3 - 4; \; B_4 = B_2 + 9; \; C_4 = L_{d_4=d_2+9}^{m_4=m_2+3} \\ t_5 = t_4 - 4; \; B_5 = B_4 + 3; \; C_5 = L_{d_5=d_4+3}^{m_5=m_4+1} \end{array} \right.$

Taking $\overleftarrow{A_4}, B_4*, C_4$, get the pairs:

$m_4, \ldots, m_4 + p - 3, m_4 + p$.

Taking $\overleftarrow{A_5}, B_5*, C_5$, get the pairs:

$m_5, \ldots, m_5 + p - 3, m_5 + p$.

If $r = 1$, take $i = 6$ and go to Step 5), otherwise, take $i = 6$ and go to Step 4).

Step 4) $\left\{ \begin{array}{l} t_i = t_{i-1} - 12; \; B_i = B_{i-1} + 9; \; C_i = L_{d_i=d_{i-1}+9}^{m_i=m_{i-1}+3} \\ t_{i+1} = t_i - 4; \; B_{i+1} = B_i + 3; \; C_{i+1} = L_{d_{i+1}=d_i+3}^{m_{i+1}=m_i+1} \end{array} \right.$

Taking $\overleftarrow{A_i}, B_i*, C_i$, and get the pairs:

$m_i, \ldots, m_i + p - 3, m_i + p$.

Taking $\overleftarrow{A_{i+1}}, B_{i+1}*, C_{i+1}$, get the pairs:

$m_{i+1}, \ldots, m_{i+1} + p - 3, m_{i+1} + p$.

Repeat Step 4) $r - 1$ times, then take $i = 4 + 2r$ and go to Step 5).

Step 5) $t_i = t_{i-1} - 12; \; B_i = B_{i-1} + 9; \; C_i = L_{d_i=d_{i-1}+9}^{m_i=m_{i-1}+3}$

Taking $\overleftarrow{A_i}, B_i*, C_i$, get the pairs:

$m_i, \ldots, m_i + p - 3, m_i + p$.

Then we continue with:

(b) $n > 75, n = 27 + 60r, n = 39 + 60r, n = 51 + 60r, n = 63 + 60r$

and prove that $Int_{S_n} = (\lfloor\frac{n}{3}\rfloor, 2\lfloor\frac{n}{3}\rfloor) - \{\lfloor\frac{n}{3}\rfloor + 3, \lfloor\frac{n}{3}\rfloor + 3 + 16r, \lfloor\frac{n}{3}\rfloor + 4 + 16r, \lfloor\frac{n}{3}\rfloor + 6 + 16r, \ldots, 2\lfloor\frac{n}{3}\rfloor - 3, 2\lfloor\frac{n}{3}\rfloor - 1\}$.

Take Step 1), 2), 3) 4) without Step 5) above.

And now we take the small cases:

(c) $10 \leq n \leq 87$

For $\underline{n=15}$, we prove $Int_{S_{15}} = (5,10) - \{7,8,9\}$.

$t = 9,\ B = S_1,\ C = L_2^5$

Taking $\overleftarrow{A}, B, C$, get the pair 6.

For $\underline{n=27}$, we prove that $Int_{S_{27}} = (9,18) - \{12,13,15,16,17\}$.

$t = 17,\ B = S_1,\ C = L_2^9$

Taking $\overleftarrow{A}, B, C$, get the pair 10.

$t = 13,\ B = S_4,\ C = L_5^{10}$

Taking $\overleftarrow{A}, B*, C$, get the pairs 10, 11, 14.

For $\underline{n=39}$, we prove that $Int_{S_{39}} = (13,26) - \{17,19,21,22,23,24,25\}$.

$t = 21,\ B = S_4,\ C = L_5^{14}$

Taking $\overleftarrow{A}, B*, C$, get the pairs 14, 15, 18.

$t = 19,\ B = 2 - near\ S_6 + 1\ pair,\ C = L_7^{13}, (2,0,2)$

Taking $\overleftarrow{A}, B, C, (2,0,2)$, get the pair 20.

Taking $A, B, \overleftarrow{C}, (2,0,2)$, where C and its reverse can have 0, 1, 2 pairs in common (Table B.6), get the pairs 26, 27, 28.

$t = 19,\ B = S_5 + 1\ pair,\ C = L_6^{14}$

Taking $\overleftarrow{A}, B*, C$, get the pairs 15, 16, 20.

For $\underline{n=51}$, we prove that

$Int_{S_{51}} = (17,34) - \{20,21,23,24,25,26,27,29,30,31,32,33\}$.

$t = 29,\ B = S_4,\ C = L_5^{18}$

Taking $\overleftarrow{A}, B*, C$, get the pairs 18, 19, 22.

$t = 23,\ B = 5 - ext\ S_8 + 1\ pair,\ C = L_9^{19}$

Taking $\overleftarrow{A}, B, C$, get the pair 28.

For $\underline{n=63}$, we prove that $Int_{S_{63}} = (21,42) - \{30,31,33,35,37,38\}$.

$t = 37,\ B = S_4,\ C = L_5^{22}$

Taking $\overleftarrow{A}, B*, C$, get the pairs 22, 23, 26.

$t = 31,\ B = S_8 + 1\ pair,\ C = L_9^{23}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $24, \ldots, 29, 32$.

$t = 29,\ B = 2 - near\ S_{11} + 1,\ C = L_{12}^{23}, (2,0,2)$

Taking $\overleftarrow{A}, B, C, (2,0,2)$, get the pair 34.

Taking $A, B, \overleftarrow{C}, (2,0,2)$, where C and its reverse can have $0,1,2$ pairs in common (Table B.6), get the pairs $40, 41, 42$.

$t = 27,\ B = 8 - ext\ S_{11} + 1\,pair,\ C = L_{12}^{24}$

Taking $\overleftarrow{A}, B, C$, get the pair 36.

Taking $A, B, \overleftarrow{C}$, where C and its reverse can have $0,1$ pairs in common (Table B.7 and Table B.1), get the pairs $39, 40$.

For $\underline{n = 75}$, we prove that

$Int_{S_{75}} = (25, 50) - \{28, 41, 43, 44, 45, 46, 47, 49\}$.

$t = 45,\ B = S_4,\ C = L_5^{26}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $26, 27, 30$.

$t = 37,\ B = 2 - near\ S_{11},\ C = L_{12}^{27}, (2,0,2)$

Taking $\overleftarrow{A}, B, C, (2,0,2)$, get the pair 38.

Taking $A, B, \overleftarrow{C}, (2,0,2)$, where C and its reverse are disjoint (Table B.2), get the pair 48.

$t = 35,\ B = 4 - ext\ S_{11} + 1\,pair,\ C = L_{12}^{28}$

Taking $\overleftarrow{A}, B, C$, get the pair 40.

Taking $A, B, \overleftarrow{C}$, where C and its reverse have one pair in common (Table B.1), get the pair 48.

$t = 33,\ B = S_{13},\ C = L_{14}^{29}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $29, \ldots, 39, 42$.

Case (3): $n \equiv 6(mod\ 12)$

We start with:

(a) $n > 78, n = 78 + 60r$

and we prove that $Int_{S_n} = (\{\lfloor \frac{n}{3} \rfloor, 2\lfloor \frac{n}{3} \rfloor) - \{\lfloor \frac{n}{3} \rfloor + 3, \lfloor \frac{n}{3} \rfloor + 15 + 16r, \lfloor \frac{n}{3} \rfloor + 16 + 16r, \lfloor \frac{n}{3} \rfloor + 18 + 16r, \ldots, 2\lfloor \frac{n}{3} \rfloor - 2\}$.

The proof for this case is similar with Case (2)(a) with the only difference that in Step 2) C_3 and its reverse have one pair in common and get the pair $2\lfloor \frac{n}{3} \rfloor - 1$. Then we continue with:

(b) $n > 78, n = 30 + 60r, n = 42 + 60r, n = 54 + 60r, n = 66 + 60r$

and we prove that:

$Int_{S_n} = (\{\lfloor\frac{n}{3}\rfloor, 2\lfloor\frac{n}{3}\rfloor) - \{\lfloor\frac{n}{3}\rfloor+3, \lfloor\frac{n}{3}\rfloor+3+16r, \lfloor\frac{n}{3}\rfloor+4+16r, \lfloor\frac{n}{3}\rfloor+6+16r, \ldots, 2\lfloor\frac{n}{3}\rfloor-2\}$.

The proof for this case is similar with Case (2)(b) with the only difference that in Step 2) C_3 and its reverse have one pair in common and gives the pair $2\lfloor\frac{n}{3}\rfloor - 1$.

Now we take the small cases:

(c) $10 \leq n \leq 78$

For $n = 18$,we prove that:

$Int_{S_{18}} = (6, 12) - \{8, 9, 10, 11\}$.

$t = 11,\ B = S_1,\ C = L_2^6$

Taking Taking $\overleftarrow{A}, B, C$, get the pair 7.

For $n = 30$, we prove that $Int_{S_{30}} = (10, 20) - \{13, 14, 16, 18\}$.

$t = 15,\ B = S_4,\ C = L_5^{11}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $11, 12, 15$.

$t = 13,\ B = 13 - ext\ S_5 + 1\ pair,\ C = L_6^{11}$

Taking $\overleftarrow{A}, B, C$, get the pair 17.

Taking $A, B, \overleftarrow{C}$, where C and its reverse have $0, 1, 2$ pairs in common (Table B.6), get the pairs $19, 20, 21$.

For $n = 42$, we prove that:

$Int_{S_{42}} = (14, 28) - \{17, 18, 20, 21, 22, 23, 24, 25, 26, 27\}$.

$t = 23,\ B = S_4,\ C = L_5^{15}$

Taking $\overleftarrow{A}, B*, C$ and get the pairs $15, 16, 19$.

For $n = 54$, we prove that

$Int_{S_{54}} = (18, 36) - \{21, 22, 24, 25, 26, 27, 28, 30, 31, 32, 33, 35\}$.

$t = 31,\ B = S_4,\ C = L_5^{19}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $19, 20, 23$.

$t = 25,\ B = 3 - ext\ S_8 + 1\ pair,\ C = L_9^{20}$

Taking $\overleftarrow{A}, B, C$, get the pair 29.

Taking $A, B, \overleftarrow{C}$, where C and its reverse are disjoint (Table B.1), get the pair 34.

For $\underline{n = 66}$, we prove that:

$Int_{S_{66}} = (22, 44) - \{25, 26, 28, 29, 30, 31, 32, 33, 34, 36, 38, 39, 40, 41\}$.

$t = 39,\ B = S_4,\ C = L_5^{23}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $23, 24, 27$.

$t = 31,\ B = 2 - near\ S_{11},\ C = L_{12}^{24}, (2, 0, 2)$

Taking $\overleftarrow{A}, B, C$, get the pair 35.

Taking $A, B, \overleftarrow{C}$, where C and its reverse can have $0, 1$ pairs in common (Table B.1 and Table B.7), get the pairs $42, 43$.

$t = 29,\ B = 8 - ext\ S_{11} + 1\ pair,\ C = L_{12}^{25}$

Taking $\overleftarrow{A}, B, C$, get the pair 37.

For $\underline{n = 78}$, we prove that:

$Int_{S_{78}} = (26, 52) - \{29, 42, 44, 46, 47, 48\}$.

$t = 47,\ B = S_4,\ C = L_5^{27}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $27, 28, 31$.

$t = 39,\ B = 2 - near\ S_{11},\ C = L_{12}^{28}, (2, 0, 2)$

Taking $\overleftarrow{A}, B, C, (2, 0, 2)$, get the pair 39.

Taking $A, B, \overleftarrow{C}, (2, 0, 2)$, where C and its reverse can have one pair in common (Table B.1), get the pair 51.

$t = 37,\ B = 4 - ext\ S_{11} + 1\ pair,\ C = L_{12}^{29}$

Taking $\overleftarrow{A}, B, C$, get the pair 41.

$t = 35,\ B = 2 - near\ S_{14},\ C = L_{15}^{29}, (2, 0, 2)$

Taking $\overleftarrow{A}, B, C, (2, 0, 2)$, get the pair 43.

Taking $A, B, \overleftarrow{C}, (2, 0, 2)$, where C and its reverse can have $0, 1, 2$ pairs in common (Table B.6), get the pairs $49, 50, 51$.

$t = 35,\ B = S_{13},\ C = L_{14}^{30}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $30, \ldots, 40, 43$.

$t = 33,\ B = 12 - ext\ S_{14} + 1\ pair,\ C = L_{15}^{30}$

Taking $\overleftarrow{A}, B, C$, get the pair 45.

Case (4): $n \equiv 7(mod\ 12)$

We start with:

(a) $n \geq 67, n = 55 + 60r$

and we prove that $Int_{S_n} = (\lfloor \frac{n}{3} \rfloor, 2\lfloor \frac{n}{3} \rfloor) - \{\lfloor \frac{n}{3} \rfloor + 10 + 16r, \lfloor \frac{n}{3} \rfloor + 11 + 16r, \lfloor \frac{n}{3} \rfloor + 13 + 16r, \ldots, 2\lfloor \frac{n}{3} \rfloor - 2\}$.

Step 1) $\left\{ L_{\lfloor \frac{n}{3} \rfloor + 1}^{2\lfloor \frac{n}{3} \rfloor + 1} + S_{\lfloor \frac{n}{3} \rfloor} \right.$

Table 3.6 gives two perfect Langford sequences $L_{\lfloor \frac{n}{3} \rfloor + 1}^{2\lfloor \frac{n}{3} \rfloor + 1}$ with $0, 1, 2$ pairs in common. These constructions gives the pairs $\lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{3} \rfloor + 1, \lfloor \frac{n}{3} \rfloor + 2$.

Step 2) $\begin{cases} t_1 = 2\lfloor \frac{n}{3} \rfloor - 7,\ B_1 = 2 - near\ S_7,\ C_1 = L_{d_1=8}^{m_1=\lfloor \frac{n}{3} \rfloor + 1}, (2,0,2) \\ t_2 = t_1 - 4;\ B_2 = 2 - near\ S_{10},\ C_2 = L_{d_2=d_1+3}^{m_2=m_1+1}, (2,0,2) \end{cases}$

Taking $A_1, B_1, \overleftarrow{C_1}, (2,0,2)$, where C_1 and its reverse are disjoint (Table B.2), get the pair $2\lfloor \frac{n}{3} \rfloor$.

Taking $A_2, B_2, \overleftarrow{C_2}, (2,0,2)$, where C_2 and its reverse are disjoint (Table B.1), get the pair $2\lfloor \frac{n}{3} \rfloor - 1$.

Step 3) $\begin{cases} t_3 = t_2;\ B_3 = S_9;\ C_3 = L_{d_3=10}^{m_3=m_2+1} \\ t_4 = t_3 - 4;\ B_4 = B_3 + 3;\ C_4 = L_{d_4=d_3+3}^{m_4=m_3+1} \end{cases}$

Taking $\overleftarrow{A_3}, B_3*, C_3$, get the pairs:

$m_3, \ldots, m_3 + p - 3, m_3 + p$.

Taking $\overleftarrow{A_4}, B_4*, C_4$, get the pairs:

$m_4, \ldots, m_4 + p - 3, m_4 + p$.

For $r = 1$, take $i = 5$ and go to Step 5), otherwise, take $i = 5$ and go to Step 4).

Step 4) $\begin{cases} t_i = t_{i-1} - 12;\ B_i = B_{i-1} + 9;\ C_i = L_{d_i=d_{i-1}+9}^{m_i=m_{i-1}+3} \\ t_{i+1} = t_i - 4;\ B_{i+1} = B_i + 3;\ C_{i+1} = L_{d_{i+1}=d_i+3}^{m_{i+1}=m_i+1} \end{cases}$

Taking $\overleftarrow{A_i}, B_i*, C_i$, get the pairs:

$m_i, \ldots, m_i + p - 3, m_i + p$.

Taking $\overleftarrow{A_{i+1}}, B_{i+1}*, C_{i+1}$, get the pairs:

$m_{i+1}, \ldots, m_{i+1} + p - 3, m_{i+1} + p$.

Repeat Step 4) $r - 1$ times, then take $i = 5 + 2r$ and go to Step 5).

Step 5) $t_i = t_{i-1} - 12;\ B_i = B_{i-1} + 9;\ C_i = L_{d_i=d_{i-1}+9}^{m_i=m_{i-1}+3}$

Taking $\overleftarrow{A_i}, B_i*, C_i$, get the pairs:

$m_i, \ldots, m_i + p - 3, m_i + p$.

Then we continue with:

(b) $n \geq 67, n = 67 + 60r, n = 79 + 60r, n = 91 + 60r, n = 103 + 60r$

and we prove that $Int_{S_n} = [\lfloor \frac{n}{3} \rfloor, 2\lfloor \frac{n}{3} \rfloor] - \{\lfloor \frac{n}{3} \rfloor + 14 + 16r, \lfloor \frac{n}{3} \rfloor + 15 + 16r, \lfloor \frac{n}{3} \rfloor + 17 + 16r, \ldots, 2\lfloor \frac{n}{3} \rfloor - 2\}$.

The proof is similar with Case (3)(b), the only difference is that Step 4) are repeated r times.

And now we prove for small cases:

(c) $10 \leq n < 67$

For $n = 19$, we prove that $Int_{S_{19}} = (6, 12) - \{9, 10, 11\}$.

$L_7^{13} + S_6$.

Taking two Langford sequences L_7^{13} with $0, 1, 2$ pairs in common (Table B.6), get the pairs $6, 7, 8$.

For $n = 31$, we prove that $Int_{S_{31}} = (10, 20) - \{14, 15, 17, 18, 19\}$.

$L_{11}^{21} + S_{10}$.

Taking two Langford sequences L_{11}^{21} with $0, 1, 2, 3$ pairs in common (Table B.6), get the pairs $10, \ldots, 13$.

$t = 15,\ B = S_4 + 1\ pair,\ C = L_5^{11}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $12, 13, 16$.

For $n = 43$, we prove that $Int_{S_{43}} = (14, 28) - \{17, 18, 19, 20, 21, 23, 25\}$.

$L_{15}^{29} + S_{14}$.

Taking two Langford sequences L_{15}^{29} with $0, 1, 2$ pairs in common (Table B.6), get the pairs $14, 15, 16$.

$t = 19,\ B = 2 - near\ S_7,\ C = L_8^{15}, (2, 0, 2)$

Taking $\overleftarrow{A}, B, C, (2, 0, 2)$, get the pair 22.

Taking $A, B, \overleftarrow{C}, (2, 0, 2)$, where C and its reverse can have $0, 1, 2, 3$ pairs in common (Table B.6), get the pairs $26, 27, 28$.

$t = 19,\ B = 4 - ext\ S_7 + 1\ pair,\ C = L_8^{16}$

Taking $\overleftarrow{A}, B, C$, get the pair 24.

Taking $A, B, \overleftarrow{C}$, where C and its reverse can have $0, 1$ pairs in common (Table B.7 and Table B.1), get the pairs $27, 28$.

For $\underline{n = 55}$, we prove that $Int_{S_{55}} = (18, 36) - \{28, 29, 31, 33\}$.

$L_{19}^{37} + S_{18}$.

Taking two Langford sequences L_{19}^{37} with $0, 1, 2$ pairs in common (Table B.6), get the pairs $18, 19, 20$.

$t = 29,\ B = 2 - near\ S_7,\ C = L_8^{19}, (2, 0, 2)$

Taking $\overleftarrow{A}, B, C, (2, 0, 2)$, get the pair 26.

Taking $A, B, \overleftarrow{C}, (2, 0, 2)$, where C and its reverse are disjoint (Table B.2), get the pair 36.

$t = 25,\ B = S_9,\ C = L_{10}^{21}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $21, \ldots, 27, 30$.

$t = 23,\ B = 8 - ext\ S_{10} + 1\ pair,\ C = L_{11}^{21}$

Taking $\overleftarrow{A}, B, C$, get the pair 32.

Taking $A, B, \overleftarrow{C}$, where C and its reverse can have $0, 1, 2, 3$ pairs in common (Table B.6), get the pairs $34, 35, 36, 37$.

Case (5): $n \equiv 10 (mod\ 12)$

We start with:

(a) $n \geq 70, n = 58 + 60r$

and prove that $Int_{S_n} = (\lfloor \frac{n}{3} \rfloor, 2\lfloor \frac{n}{3} \rfloor) - \{\lfloor \frac{n}{3} \rfloor + 10 + 16r, \lfloor \frac{n}{3} \rfloor + 11 + 16r, \lfloor \frac{n}{3} \rfloor + 13 + 16r, \ldots, 2\lfloor \frac{n}{3} - 1\}$.

The proof for this case is similar with Case(4)(a) with the only difference that in Step 2) C_1 and its reverse have one pair in common (Table B.1) and gives the pair $2\lfloor \frac{n}{3} \rfloor + 1$.

Then we continue with:

(b) $n \geq 70, n = 70 + 60r, n = 82 + 60r, n = 94 + 60r, n = 106 + 60r$

and we prove that $Int_{S_n} = (\lfloor \frac{n}{3} \rfloor, 2\lfloor \frac{n}{3} \rfloor) - \{\lfloor \frac{n}{3} \rfloor + 14 + 16r, \lfloor \frac{n}{3} \rfloor + 15 + 16r, \lfloor \frac{n}{3} \rfloor + 17 + 16r, \ldots, 2\lfloor \frac{n}{3} \rfloor - 1\}$.

The proof is similar with Case(4)(b), with the only difference that in Step 2) C_1 and its reverse have one pair in common (Table B.1) and gives the pair $2\lfloor \frac{n}{3} \rfloor + 1$.

And now, we prove for small cases:

<u>(c) $10 \leq n < 70$</u>

For <u>$n = 22$</u>, we prove that $Int_{S_{22}} = (7, 14) - \{11, 12, 13\}$.

$L_8^{15} + S_7$.

Taking two Langford sequences L_8^{15} with $0, 1, 2, 3$ pairs in common (Table B.6), get the pairs $7, \ldots, 10$.

For <u>$n = 34$</u>, we prove that $Int_{S_{34}} = (11, 22) - \{14, 15, 16, 17, 18, 20, 21\}$.

$L_{12}^{23} + S_{11}$.

Taking two Langford sequences L_{12}^{23} with $0, 1, 2$ pairs in common (Table B.6), get the pairs $11, 12, 13$.

$t = 15,\ B = 3 - ext\ S_5 + 2\, pairs,\ C = L_6^{12}$

Taking $\overleftarrow{A}, B, C$, get the pair 19.

Taking $A, B, \overleftarrow{C}$, where C and its reverse can have $1, 3$ pairs in common (Table B.1 and Table B.7), get the pairs $23, 25$.

For <u>$n = 46$</u>, we prove that

$Int_{S_{46}} = (15, 30) - \{18, 19, 20, 21, 22, 24, 26, 27, 28, 29\}$.

$L_{16}^{31} + S_{15}$.

Taking two Langford sequences L_{16}^{31} with $0, 1, 2$ pairs in common (Table B.6), get the pairs $15, 16, 17$.

$t = 23,\ B = 2 - near\ S_7,\ C = L_8^{16}, (2, 0, 2)$

Taking $\overleftarrow{A}, B, C, (2, 0, 2)$, get the pair 23.

Taking $A, B, \overleftarrow{C}, (2, 0, 2)$, where C and its reverse are disjoint (Table B.7), get the pairs 30.

$t = 21,\ B = 4 - ext\ S_7 + 1\, pair,\ C = L_8^{17}$

Taking $\overleftarrow{A}, B, C$, get the pair 25.

For <u>$n = 58$</u>, we prove that $Int_{S_{58}} = (19, 38) - \{29, 30, 32, 34, 35, 36\}$.

$L_{20}^{39} + S_{19}$.

Taking two Langford sequences L_{20}^{39} with $0, 1, 2, 3$ pairs in common (Table B.6), get the pairs $19, 20, 21, 22$.

$t = 31,\ B = 2 - near\ S_7,\ C = L_8^{20}, (2, 0, 2)$

Taking $\overleftarrow{A}, B, C, (2, 0, 2)$, get the pair 27.

Taking $A, B, \overleftarrow{C}, (2, 0, 2)$, where C and its reverse have one pair in common (Table B.1), get the pair 39.

$t = 27,\ B = 2 - near\ S_{10},\ C = L_{11}^{21}, (2, 0, 2)$

Taking $\overleftarrow{A}, B, C, (2, 0, 2)$, get the pair 31.

Taking $A, B, \overleftarrow{C}, (2, 0, 2)$, where C and its reverse have $0, 1, 2, 3$ pairs in common (Table B.6), get the pairs $37, 38, 39, 40$.

$t = 27,\ B = S_9,\ C = L_{10}^{22}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $22, \ldots, 28, 31$.

$t = 25,\ B = 8 - ext\ S_{10} + 1\,pair,\ C = L_{11}^{22}$

Taking $\overleftarrow{A}, B, C$, get the pair 33.

Case (6): $n \equiv 11 (mod\ 12)$

We start with:

(a) $n > 95, n = 95 + 60r$

and we prove that $Int_{S_n} = (\lfloor \frac{n}{3} \rfloor, 2\lfloor \frac{n}{3} \rfloor) - \{\lfloor \frac{n}{3} \rfloor + 1, \lfloor \frac{n}{3} \rfloor + 21 + 16r, \lfloor \frac{n}{3} \rfloor + 22 + 16r, \lfloor \frac{n}{3} \rfloor + 24 + 16r, \ldots, 2\lfloor \frac{n}{3} \rfloor - 1\}$.

The proof for this case is similar with Case (1)(a), with the only difference that C_3 and its reverse are disjoint (Table B.2) and gives the pair $2\lfloor \frac{n}{3} \rfloor - 2$.

Then we continue with:

(b) $n > 95, n = 47 + 60r, n = 59 + 60r, n = 71 + 60r, n = 83 + 60r$

and prove that $Int_{S_n} = (\lfloor \frac{n}{3} \rfloor, 2\lfloor \frac{n}{3} \rfloor) - \{\lfloor \frac{n}{3} \rfloor + 1, \lfloor \frac{n}{3} \rfloor + 9 + 16r, \lfloor \frac{n}{3} \rfloor + 10 + 16r, \lfloor \frac{n}{3} \rfloor + 12 + 16r, \ldots, 2\lfloor \frac{n}{3} \rfloor - 1\}$.

The proof is similar with Case (1)(b), with the only difference that C_3 and its reverse are disjoint (Table B.2) and gives the pair $2\lfloor \frac{n}{3} \rfloor - 2$.

And now we prove for small cases:

(c) $10 \le n \le 95$

For $\underline{n = 23}$ these construction cannot be applied.

For $\underline{n = 35}$, we prove that

$Int_{S_{35}} = (11, 22) - \{12, 15, 16, 17, 19, 21\}$.

$t = 17,\ B = S_5,\ C = L_6^{13}$

Taking $\overleftarrow{A}, B*, C$, get the pairs 13, 14, 18.

$t = 15,\ B = 4 - ext\ S_6 + 1\ pair,\ C = L_7^{13}$

Taking $\overleftarrow{A}, B, C$, get the pair 20.

Taking $A, B, \overleftarrow{C}$, where C and its reverse have 0, 1, 2 pairs in common (Table B.6) and get the pairs 22, 23, 24.

For $\underline{n = 47}$, we prove that

$Int_{S_{47}} = (15, 30) - \{16, 24, 25, 27, 28, 29\}$.

$t = 25,\ B = S_5,\ C = L_6^{17}$

Taking $\overleftarrow{A}, B*, C$, get the pairs 17, 18, 23.

$t = 21,\ B = S_8,\ C = L_9^{18}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $18, \ldots, 23, 26$.

For $\underline{n = 59}$, we prove that

$Int_{S_{59}} = (19, 38) - \{20, 28, 29, 31, 33, 34, 35, 36, 37\}$.

$t = 33,\ B = S_5,\ C = L_6^{21}$

Taking $\overleftarrow{A}, B*, C$, get the pairs 21, 22, 26.

$t = 29,\ B = S_8,\ C = L_9^{22}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $22, \ldots, 27, 30$.

$t = 27,\ B = 5 - ext\ S_9 + 1\ pair,\ C = L_{10}^{22}$

Taking $\overleftarrow{A}, B, C$, get the pair 32.

For $\underline{n = 71}$, we prove that

$Int_{S_{71}} = (23, 46) - \{24, 35, 37, 38, 39, 41, 42, 43, 44, 45\}$.

$t = 41,\ B = S_5,\ C = L_6^{25}$

Taking $\overleftarrow{A}, B*, C$, get the pairs 25, 26, 30.

$t = 37,\ B = S_8,\ C = L_9^{26}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $26, \ldots, 31, 34$.

$t = 35,\ B = S_9 + 1\ pair,\ C = L_{10}^{26}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $27, \ldots, 33, 36$.

$t = 31,\ B = 9 - ext\ S_{12} + 1\ pair,\ C = L_{13}^{27}$

Taking $\overleftarrow{A}, B, C$, get the pair 40.

For $\underline{n = 83}$, we prove that

$Int_{S_{83}} = (27, 54) - \{28, 36, 37, 39, 40, 41, 42, 43, 45, 47, 49, 50\}$.

$t = 49,\ B = S_5,\ C = L_6^{29}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $29, 30, 34$.

$t = 45,\ B = S_8,\ C = L_9^{30}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $30, \ldots, 35, 38$.

$t = 39,\ B = 5 - ext\ S_{12} + 1\ pair,\ C = L_{13}^{31}$

Taking $\overleftarrow{A}, B, C$, get the pair 44.

$t = 37,\ B = 2 - near\ S_{15},\ C = L_{16}^{31}, (2, 0, 2)$

Taking $\overleftarrow{A}, B, C, (2, 0, 2)$, get the pair 46.

Taking $A, B, \overleftarrow{C}, (2, 0, 2)$, where C and its reverse have $0, 1, 2$ pairs in common (Table B.6), get the pairs $52, 53, 54$.

$t = 35,\ B = 12 - ext\ S_{15} + 1\ pair,\ C = L_{16}^{32}$

Taking $\overleftarrow{A}, B, C$, get the pair 48.

Taking $A, B, \overleftarrow{C}$, where C and its reverse have $0, 1$ pairs in common (Table B.7 and Table B.1), get the pairs $51, 52$.

For $\underline{n = 95}$, we prove that

$Int_{S_{95}} = (31, 62) - \{32, 52, 55, 56, 57, 58, 61\}$.

$t = 57,\ B = S_5,\ C = L_6^{33}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $33, 34, 38$.

$t = 53,\ B = S_8,\ C = L_9^{34}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $34, \ldots, 39, 42$.

$t = 45,\ B = 2 - near\ S_{15},\ C = L_{16}^{35}, (2, 0, 2)$

Taking $\overleftarrow{A}, B, C, (2, 0, 2)$, get the pair 50.

Taking $A, B, \overleftarrow{C}, (2, 0, 2)$, where C and its reverse are disjoint (Table B.2), get the pair

60.

$t = 41,\ B = 2 - near\ S_{18},\ C = L_{19}^{36}, (2, 0, 2)$

Taking $\overleftarrow{A}, B, C, (2, 0, 2)$, get the pair 54.

Taking $A, B, \overleftarrow{C}, (2, 0, 2)$, where C and its reverse are disjoint (Table B.1), get the pair 59.

$t = 41,\ B = S_{17},\ C = L_{18}^{37}$

Taking $\overleftarrow{A}, B*, C$, get the pairs $37, \ldots, 51, 54$. □

Case (III): The intersection of two Skolem sequences of order n and two hooked Skolem sequences of order n in $[2\lfloor\frac{n}{3}\rfloor, n]$ pairs

We take the same Langford sequence and adjoin it first with a [hooked] Skolem sequence of order n and second with a disjoint [hooked] Skolem sequence of order n. For example, taking two hooked Skolem sequences of order 15, these intersect in 11, 12 or 15 pairs:

- for 15 pairs in common take the same hooked Skolem sequence of order 15 twice [4].
- for 12 pairs in common, take a Langford sequence of order 12 and defect 4 [45] (which is a perfect sequence) and adjoin it a hooked Skolem sequence of order 3 [4], then take the same Langford sequence of order 12 and defect 4 and adjoin it a disjoint [hooked]Skolem sequence of order 3.
- for the 11 pairs in common, take a Langford sequence of order 11 and defect 5 [45] (which is a hooked sequence) and adjoin it with a Skolem sequences of order 4 [4], then take the same hooked Langford sequence of order 11 and defect 5 and adjoin a disjoint Skolem sequence of order 4.

Theorem 22 *The necessary conditions are sufficient for two [hooked] Skolem sequences of order $n \equiv 0$ or $1 (mod\ 3)$ to intersect in $\{2\lfloor\frac{n}{3}\rfloor + 1, \ldots, n-3, n\}$ pairs.*

Proof: For $1 \le n \le 9$, see Appendix A.

For n pairs in common, take the same Skolem sequence of order n twice [4].

For $n-3$ pairs in common, take a Langford sequence of order $n-3$ and defect 4 [45], and adjoin this sequence first with a hooked Skolem sequence of order 3, and second with a disjoint hooked Skolem sequence of order 3.

For $n-4$ pairs in common, take a Langford sequence of order $n-4$ and defect 5 and adjoining it first a Skolem sequence of order 4, and second with a disjoint Skolem sequence of order 4. Continue this process until $n < 2d-1$. This gives the entire spectrum. □

Theorem 23 *The necessary conditions are sufficient for two [hooked] Skolem sequences of order* $n \equiv 2(mod\ 3)$ *to intersect in* $\{2\lfloor\frac{n}{3}\rfloor+2,\ldots,n-3,n\}$ *pairs.*

Proof: For $1 \leq n \leq 9$, see Appendix A.

For n pairs in common, take the same Skolem sequence of order n twice.

For $n-3$ pairs in common, take a Langford sequence of order $n-3$ and defect 4 [45], and adjoining it this sequence first with a hooked Skolem sequences of order 3 and second with a disjoint hooked Skolem sequence of order 3.

For $n-4$ pairs in common, take a Langford sequence of order $n-4$ and defect 5 and adjoin this sequence first with a Skolem sequence of order 4, and second with a disjoint Skolem sequence of order 4. Continue this process until $n < 2d-1$. This gives the entire spectrum. □

Theorems 18 – 23 gives the primarily result of this thesis which we state in the following two theorems.

Theorem 24 Main Theorem

The necessary conditions are sufficient for two Skolem sequences of order n to intersect in $\{0,\ldots,n-3,n\}$ *pairs, with the following possible exceptions: for* $n = 12t, n \geq 72, t \equiv 0,1(mod\ 3)$ *the exceptions are* $\{\lfloor\frac{n}{3}\rfloor-5,\lfloor\frac{n}{3}\rfloor\}$*, otherwise the exceptions are* $\{\lfloor\frac{n}{3}\rfloor-5,\lfloor\frac{n}{3}\rfloor-2,\lfloor\frac{n}{3}\rfloor\}$ *and other* 23 *exceptions for* $10 \leq n \leq 60$*, for* $n \equiv 1(mod\ 12), n \geq 109$ *the exceptions are* $\{2\lfloor\frac{n}{3}\rfloor-4, 2\lfloor\frac{n}{3}\rfloor\}$ *and other* 16 *possible exceptions for* $10 \leq n \leq 97$*, for* $n \equiv 4(mod\ 12), n \geq 112$ *the exception is* $\{2\lfloor\frac{n}{3}\rfloor-1\}$ *and other* 14 *exceptions for* $10 \leq n \leq 100$*, for* $n \equiv 5(mod\ 12), n \geq 77$

the exceptions are $\{\lfloor\frac{n}{3}\rfloor, \lfloor\frac{n}{3}\rfloor + 1, 2\lfloor\frac{n}{3}\rfloor - 3, 2\lfloor\frac{n}{3}\rfloor - 2, 2\lfloor\frac{n}{3}\rfloor + 1\}$ *and other* 24 *exceptions for* $10 \leq n \leq 65$, *for* $n \geq 92, n = 12t + 8, t \equiv 0, 2(mod\ 3)$ *the exceptions are* $\{\lfloor\frac{n}{3}\rfloor - 3, \lfloor\frac{n}{3}\rfloor + 1, 2\lfloor\frac{n}{3}\rfloor - 3, 2\lfloor\frac{n}{3}\rfloor + 1\}$, *otherwise the exceptions are* $\{\lfloor\frac{n}{3}\rfloor - 3, \lfloor\frac{n}{3}\rfloor, \lfloor\frac{n}{3}\rfloor + 1, 2\lfloor\frac{n}{3}\rfloor - 3, 2\lfloor\frac{n}{3}\rfloor + 1\}$ *and other* 28 *exceptions for* $10 \leq n \leq 80$, *for* $n \equiv 9(mod\ 12), n \geq 81$ *the exceptions are* $\{\lfloor\frac{n}{3}\rfloor, \lfloor\frac{n}{3}\rfloor + 3, 2\lfloor\frac{n}{3}\rfloor - 6, 2\lfloor\frac{n}{3}\rfloor - 2\}$ *and other* 31 *possible exceptions for* $10 \leq n \leq 69$.

Proof: For $1 \leq n \leq 9$, see Appendix A. For a list with all the possible exceptions, see Appendix C. Otherwise, apply Lemma 13, Lemma 14, Theorem 18, Theorem 20, Theorem 22 and Theorem 23. □

Theorem 25 Main Theorem

The necessary conditions are sufficient for two hooked Skolem sequences of order n *to intersect in* $\{0, \ldots, n-3, n\}$ *pairs, with the following possible exceptions: for* $n = 12t + 2, n > 98, n = 98 + 60r, t \equiv 0, 1, 2, 4, 5, 7, 10, 13, 14(mod\ 15)$ *the exceptions are* $\{\lfloor\frac{n}{3}\rfloor - 3, \lfloor\frac{n}{3}\rfloor, \lfloor\frac{n}{3}\rfloor + 1, \lfloor\frac{n}{3}\rfloor + 21 + 16r, \lfloor\frac{n}{3}\rfloor + 22 + 16r, \lfloor\frac{n}{3}\rfloor + 24 + 16r, \ldots, 2\lfloor\frac{n}{3}\rfloor - 4, 2\lfloor\frac{n}{3}\rfloor, 2\lfloor\frac{n}{3}\rfloor + 1\}$ *(1), for* $t \equiv 6, 11, 12(mod\ 15)$ *the exceptions are* $\{$*(1)*-$\{\lfloor\frac{n}{3}\rfloor\}\}$, *for* $t \equiv 3, 8, 9(mod\ 15)$ *the exceptions are* $\{$*(1)*-$\{\lfloor\frac{n}{3}\rfloor - 3\}\}$, *for* $n = 12t + 2, n > 98, n = 50+60r, 62+60r, 74+60r, 86+60r, t \equiv 0, 1, 2, 4, 5, 7, 10, 13, 14(mod\ 15)$ *the exceptions are* $\{\lfloor\frac{n}{3}\rfloor - 3, \lfloor\frac{n}{3}\rfloor, \lfloor\frac{n}{3}\rfloor + 1, \lfloor\frac{n}{3}\rfloor + 9 + 16r, \lfloor\frac{n}{3}\rfloor + 10 + 16r, \lfloor\frac{n}{3}\rfloor + 12 + 16r, \ldots, 2\lfloor\frac{n}{3}\rfloor - 4, 2\lfloor\frac{n}{3}\rfloor - 2, 2\lfloor\frac{n}{3}\rfloor, 2\lfloor\frac{n}{3}\rfloor + 1\}$ *(2), for* $t \equiv 6, 11, 12(mod\ 15)$ *the exceptions are* $\{$*(2)*-$\{\lfloor\frac{n}{3}\rfloor\}\}$, *for* $t \equiv 3, 8, 9(mod\ 15)$ *the exceptions are* $\{$*(2)*-$\{\lfloor\frac{n}{3}\rfloor - 3\}\}$ *and other* 89 *exceptions for* $10 \leq n \leq 98$, *for* $n \equiv 3(mod\ 12), n = 75+60r, n > 75$ *the exceptions are* $\{\lfloor\frac{n}{3}\rfloor - 2, \lfloor\frac{n}{3}\rfloor, \lfloor\frac{n}{3}\rfloor + 3, \lfloor\frac{n}{3}\rfloor + 15 + 16r, \lfloor\frac{n}{3}\rfloor + 16 + 16r, \lfloor\frac{n}{3}\rfloor + 18 + 16r, \ldots, 2\lfloor\frac{n}{3}\rfloor - 3, 2\lfloor\frac{n}{3}\rfloor - 1, 2\lfloor\frac{n}{3}\rfloor\}$, *otherwise the exceptions are* $\{\lfloor\frac{n}{3}\rfloor - 2, \lfloor\frac{n}{3}\rfloor, \lfloor\frac{n}{3}\rfloor + 3, \lfloor\frac{n}{3}\rfloor + 3 + 16r, \lfloor\frac{n}{3}\rfloor + 4 + 16r, \lfloor\frac{n}{3}\rfloor + 6 + 16r, \ldots, 2\lfloor\frac{n}{3}\rfloor - 3, 2\lfloor\frac{n}{3}\rfloor - 1, 2\lfloor\frac{n}{3}\rfloor\}$ *and other* 52 *pairs for* $10 \leq n \leq 75$, *for* $n = 12t+6, n > 78, n = 78+60r, t \equiv 0, 1, 2, 5, 7, 10, 14(mod\ 15)$ *the exceptions are* $\{\lfloor\frac{n}{3}\rfloor - 5, \lfloor\frac{n}{3}\rfloor - 2, \lfloor\frac{n}{3}\rfloor, \lfloor\frac{n}{3}\rfloor + 3, \lfloor\frac{n}{3}\rfloor + 15 + 16r, \lfloor\frac{n}{3}\rfloor + 16 + 16r, \lfloor\frac{n}{3}\rfloor + 18 + 16r, \ldots, 2\lfloor\frac{n}{3}\rfloor - 2\}$ *(3), for* $t \equiv 8, 9(mod\ 15)$ *the exceptions are* $\{$*(3)*-$\{\lfloor\frac{n}{3}\rfloor - 5\}\}$, *for* $t \equiv 4, 13(mod\ 15)$ *the exceptions are* $\{$*(3)*-$\{\lfloor\frac{n}{3}\rfloor\}\}$, *for* $t \equiv 3(mod\ 15)$ *the exceptions are* $\{$*(3)*-$\{\lfloor\frac{n}{3}\rfloor - 5, \lfloor\frac{n}{3}\rfloor\}\}$, *for* $n = 12t + 6, n > 78, n = 30 + 60r, 42 + 60r, 54 + 60r, 66 + 60r$

$t \equiv 0,1,2,5,7,10,14 (mod\, 15)$ the exceptions are $\{\lfloor\frac{n}{3}\rfloor-5, \lfloor\frac{n}{3}\rfloor-2, \lfloor\frac{n}{3}\rfloor, \lfloor\frac{n}{3}\rfloor+3, \lfloor\frac{n}{3}\rfloor+3+16r, \lfloor\frac{n}{3}\rfloor+4+16r, \lfloor\frac{n}{3}\rfloor+6+16r, \ldots, 2\lfloor\frac{n}{3}\rfloor-2\}$ *(4), for $t \equiv 8,9 (mod\, 15)$ the exceptions are* $\{$*(4)*-$\{\lfloor\frac{n}{3}\rfloor-5\}\}$, *for $t \equiv 4,13 (mod\, 15)$ the exceptions are* $\{$*(4)*-$\{\lfloor\frac{n}{3}\rfloor\}\}$, *for $t \equiv 3 (mod\, 15)$ the exceptions are* $\{$*(4)*- $\{\lfloor\frac{n}{3}\rfloor-5, \lfloor\frac{n}{3}\rfloor\}\}$, *and other 63 exceptions for $10 \leq n \leq 78$, for $n \equiv 7 (mod\, 12), n \geq 67, n = 55+60r$ the exceptions are* $\{\lfloor\frac{n}{3}\rfloor+10+16r, \lfloor\frac{n}{3}\rfloor+11+16r, \lfloor\frac{n}{3}\rfloor+13+16r, \ldots, 2\lfloor\frac{n}{3}\rfloor-2\}$, *otherwise the exceptions are* $\{\lfloor\frac{n}{3}\rfloor+14+16r, \lfloor\frac{n}{3}\rfloor+15+16r, \lfloor\frac{n}{3}\rfloor+17+16r, \ldots, 2\lfloor\frac{n}{3}\rfloor-2\}$ *and other 21 exceptions for $10 \leq n \leq 55$, for $n \equiv 10 (mod\, 12), n \geq 70, n = 58+60r$ the exceptions are* $\{\lfloor\frac{n}{3}\rfloor+10+16r, \lfloor\frac{n}{3}\rfloor+11+16r, \lfloor\frac{n}{3}\rfloor+13+16r, \ldots, 2\lfloor\frac{n}{3}\rfloor\}$, *otherwise the exceptions are* $\{\lfloor\frac{n}{3}\rfloor+14+16r, \lfloor\frac{n}{3}\rfloor+15+16r, \lfloor\frac{n}{3}\rfloor+17+16r, \ldots, 2\lfloor\frac{n}{3}\rfloor\}$ *and other 27 possible exceptions for $10 \leq n \leq 58$, for $n \equiv 11 (mod\, 12), n > 95, n = 95+60r$ the exceptions are* $\{\lfloor\frac{n}{3}\rfloor, \lfloor\frac{n}{3}\rfloor+1, \lfloor\frac{n}{3}\rfloor+21+16r, \lfloor\frac{n}{3}\rfloor+22+16r, \lfloor\frac{n}{3}\rfloor+24+16r, \ldots, 2\lfloor\frac{n}{3}\rfloor-1, 2\lfloor\frac{n}{3}\rfloor, 2\lfloor\frac{n}{3}\rfloor+1\}$, *otherwise the exceptions are* $\{\lfloor\frac{n}{3}\rfloor, \lfloor\frac{n}{3}\rfloor+1, \lfloor\frac{n}{3}\rfloor+9+16r, \lfloor\frac{n}{3}\rfloor+10+16r, \lfloor\frac{n}{3}\rfloor+12+16r, \ldots, 2\lfloor\frac{n}{3}\rfloor-1, 2\lfloor\frac{n}{3}\rfloor, 2\lfloor\frac{n}{3}\rfloor+1\}$ *and other 75 possible exceptions for $10 \leq n \leq 95$, where r, t are positive integers.*

Proof: For $1 \leq n \leq 10$, see Appendix A. For a list with all the possible exceptions, see Appendix C. Otherwise, apply Lemma 13, Lemma 14, Theorem 19, Theorem 21, Theorem 22 and Theorem 23. □

Chapter 5

Applications of the spectrum of [hooked]Skolem sequences with a prescribed number of pairs in common to cyclic triple systems

In this chapter, we prove that there exists two cyclic Steiner triple systems of order $6n + 1$ intersecting in $0, 1, 2, \ldots, n$ base blocks and there exists two cyclic Steiner triple systems of order $6n + 3$ intersecting in $1, 2, \ldots, n + 1$ base blocks. From here, we derive that a twofold triple system of order $6n+1$ has $0, 1, 2, \ldots, n$ base blocks in common and a twofold triple system of order $6n+3$ has $1, 2, \ldots, n+1$ base blocks in common.

We also prove that the necessary conditions are sufficient for the vector $(c_1, c_2, c_3)_c$ to be the fine structure of a $CTS(v, 3)$ for $v \equiv 1(mod\ 6)$. This is similar with the results obtained by Colbourn, Mathon, Rosa, Shalaby [16] for the non-cyclic triple systems. Their results are given in Chapter 2.

We denote $Int_c(v) = \{k :$ there exist two cyclic Steiner triple systems of order v intersecting in k base blocks$\}$.

Theorem 26 $Int_c(6n+1) = \{0, 1, 2, \ldots, n\}$.

Proof: Case (1): $n \equiv 0,\ 1(mod\ 4)$

Let S_n be a Skolem sequence of order n. Take the base blocks of the form:

1. $\{\{0, a_i + n, b_i + n\}(mod\ 6n+1)|i = 1, \ldots, j\}$ together with the base blocks $\{\{(0, i, b_i + n\}(mod\ 6n+1)|i = j+1, \ldots, n\}$
2. $\{\{(0, a_i + n, b_i + n\}(mod\ 6n+1)|i = 1, \ldots, n\}$.

Repeating this process for $j = 1, 2, \ldots, n$ we get two cyclic Steiner triple systems of order $6n+1$ with j base blocks in common. Taking the base blocks of the form:

1. $\{\{0, a_i + n, b_i + n\}(mod\ 6n+1)|i = 1, \ldots, n\}$
2. $\{\{(0, i, b_i + n\}(mod\ 6n+1)|i = 1, \ldots, n\}$

we get two $CSTS(6n+1)$ which are disjoint.

We can replace the base block $\{0, a_i + n, b_i + n\}$ by the base block $\{0, i, b_i + n\}$, $i = 1, \ldots, n$ because these blocks gives the same differences, and therefore the same pairs.

Case (2): $n \equiv 2,\ 3(mod\ 6n+1)$

Let hS_n be a hooked Skolem sequence of order n. Take the base blocks of the form:

1. $\{\{0, a_i+n, b_i+n\}(mod\ 6n+1)|i = 1, \ldots, j\}$ together with the base blocks $\{\{0, i, b_i+n\}(mod\ 6n+1)|i = j+1, \ldots, n\}$
2. $\{\{0, a_i + n, b_i + n\}(mod\ 6n+1)|i = 1, \ldots, n\}$.

Repeating this process for $j = 1, 2, \ldots, n$ we get two cyclic Steiner triple systems of order $6n+1$ with j base blocks in common. Taking the following systems:

1. $\{\{0, a_i + n, b_i + n\}(mod\ 6n+1)|i = 1, \ldots, n\}$
2. $\{\{(0, i, b_i + n\}(mod\ 6n+1)|i = 1, \ldots, n\}$

we get two $CSTS(6n+1)$ which are disjoint. □

Corollary 5 *A twofold triple system of order* $6n+1$ *intersect in* $0, 1, \ldots, n$ *base blocks.*

Proof: Two $CSTS(6n+1)$ from the above theorem gives a twofold triple system of order $6n+1$ which intersect in $0, 1, \ldots, n$ base blocks. □

Theorem 27 $Int_c(6n+3) = \{1, 2, \ldots, n+1\}$

Proof: Case (1): $n \equiv 0,\ 3(mod\ 4)$

Let R_n be a Rosa sequence of order n. Take the following two systems:

1. $\{\{0, a_i+n, b_i+n\}(mod\ 6n+3)|i = 1, \ldots, j\}$ together with the base blocks $\{\{0, i, b_i+n\}(mod\ 6n+1)|i = j+1, \ldots, n\}$ and $\{0, 2n+1, 4n+2\}(mod\ 6n+3)$

2. $\{\{0, a_i+n, b_i+n\}(mod\ 6n+3)|i = 1, \ldots, n\}$ together with $\{0, 2n+1, 4n+2\}(mod\ 6n+3)$

Repeating this process for $j = 1, \ldots, n$ we get two cyclic $STS(6n+3)$ with $j+1$ base blocks in common.

Taking the following systems:

1. $\{\{0, a_i+n, b_i+n\}(mod\ 6n+3)|i = 1, \ldots, n\}$ together with the base block $\{0, 2n+1, 4n+2\}(mod\ 6n+3)$

2. $\{\{0, i, b_i+n\}(mod\ 6n+1)|i = 1, \ldots, n\}$ together with the base block $\{0, 2n+1, 4n+2\}(mod\ 6n+3)$

we get two $CSTS(6n+3)$ with one base block in common.

Case (2): $n \equiv 1, 2(mod\ 4)$

Let hR_n be a hooked Rosa sequence of order n. Take the following two systems: 1. $\{\{0, a_i+n, b_i+n\}(mod\ 6n+3)|i = 1, \ldots, j\}$ together with the base blocks $\{\{0, i, b_i+n\}(mod\ 6n+1)|i = j+1, \ldots, n\}$ and $\{0, 2n+1, 4n+2\}(mod\ 6n+3)$

2. $\{\{0, a_i+n, b_i+n\}(mod\ 6n+3)|i = 1, \ldots, n\}$ together with $\{0, 2n+1, 4n+2\}(mod\ 6n+3)$

Repeating this process for $j = 1, \ldots, n$ we get two $CSTS(6n+3)$ with j base block in common. Taking the following systems:

1. $\{\{0, a_i+n, b_i+n\}(mod\ 6n+3)|i = 1, \ldots, n\}$ together with the base block $\{0, 2n+1, 4n+2\}(mod\ 6n+3)$

2. $\{\{0, i, b_i+n\}(mod\ 6n+1)|i = 1, \ldots, n\}$ together with the base block $\{0, 2n+1, 4n+2\}(mod\ 6n+3)$

we get two $CSTS(6n+3)$ with one base block in common. □

Corollary 6 *A twofold triple system of order* $6n+3$ *intersect in* $1, 2, \ldots, n+1$ *base* *blocks.*

Proof: Two $CSTS(6n+3)$ from the above theorem gives a twofold triple system of order $6n+3$ which intersect in $1, 2, \ldots, n+1$ base blocks. □

In [16], it is proven that the necessary conditions are sufficient for a vector (c_1, c_2, c_3) to be the fine structure of a threefold triple system of order $v \equiv 1, 3(mod\ 6)$, $v \geq 19$. Also, in [17] it is proven that the necessary conditions are sufficient for a vector (c_1, c_2, c_3) to be the fine structure of a threefold triple system of order $v \equiv 5(mod\ 6), v \geq 17$.

In this thesis we give similar results for the fine structure of a threefold cyclic triple system of order $v \equiv 1(mod\ 6)$.

We use next the intersection spectrum of two [hooked] Skolem sequences to determine the fine structure of a cyclic triple system of order $v \equiv 1(mod\ 6)$ with $\lambda = 3$.

Since we proved with some possible exceptions that two [hooked] Skolem sequences of order n intersect in $\{0, 1, \ldots, n-3, n\}$ pairs, to determine the fine structure of a $CTS(v, 3)$, $v \equiv 1(mod\ 6)$, we first find the fine structure of a $CTS(v, 3)$ when two [hooked] Skolem sequences of order n intersect in n pairs (take the same [hooked] Skolem sequence twice), then we determine the fine structure of a $CTS(v, 3)$ when two [hooked] Skolem sequence intersect in p pairs and finally, we determine the fine structure of a $CTS(v, 3)$ when two [hooked] Skolem sequences intersect in $0 \leq p \leq n-3$ pairs. Using these we determine in the Main Theorem 31 the fine structure of a $CTS(v, 3)$, when two [hooked] Skolem sequences intersect in $\{0, \ldots, n-3, n\}$ pairs with some possible exceptions. These exceptions are given by the exceptions in the Main Theorems 24 and 25.

Theorem 28 *For $v \equiv 1(mod\ 6), v \geq 7$ the necessary conditions are sufficient for the vector $(n-j, n-j, j)_c, j = 0, \ldots, n$ to be the fine structure of a $CTS(v, 3)$, where $n = \frac{v-1}{6}$.*

Proof: Let S_n be a [hooked] Skolem sequence of order n and let $(a_i, b_i), 1 \leq i \leq n$, be the pairs determined by this [hooked] Skolem sequence. Then, if we take the same sequence twice, these pairs gives the base blocks of four $CSTS(6n+1)$.

1) $(0, a_i+n, b_i+n)(mod\ 6n+1), i = 1, \ldots, n$

2) $(0, i, b_i+n)(mod\ 6n+1), i = 1, \ldots, n$

3) $(0, a_i+n, b_i+n)(mod\ 6n+1), i = 1, \ldots, n$

4) $(0, i, b_i+n)(mod\ 6n+1), i = 1, \ldots, n$.

We use only the first three systems because these gives the fine structure for a three-fold cyclic triple system. It is known that the first two systems are disjoint. Taking the first three systems together, we get a $CTS(v,3)$ with the fine structure $(n,n,0)_c$. Take the base blocks:

1) $(0, a_i+n, b_i+n), i = 1\ldots, n$

2) $(0, a_i+n, b_i+n), i = 1, \ldots, j$

$(0, i, b_i+n), i = j+1, \ldots, n$

3)$(0, a_i+n, b_i+n), i = 1, \ldots, n$.

Repeat this process for $j = 1, \ldots, n$. This gives a $CTS(v,3)$ with the fine structure $(n-j, n-j, j)_c, j = 1, \ldots, n$. □

Theorem 29 *For $v \equiv 1(mod\ 6), v \geq 19$ the necessary conditions are sufficient for the vector $(3n-2(p-i+j)-3i, p-i+j, i)_c, i = 0, \ldots, p,\ j = 0, \ldots, n-i$ to be the fine structure of a $CTS(v,3)$, where $n = \frac{v-1}{6}$ and p is the number of pairs in common between two [hooked] Skolem sequences of order n.*

Proof: Let S_n and S'_n to be two [hooked] Skolem sequences of order n with p pairs in common, $0 \leq p \leq n-3$, and let $(a_i, b_i), 1 \leq i \leq n$, and $(\alpha_i, \beta_i), 1 \leq i \leq n$, be the pairs determined by these two [hooked] Skolem sequences. Without lost of generality, assume that the first p pairs are in common. Then, these gives the base blocks of four $CSTS(6n+1)$.

1) $(0, a_i+n, b_i+n)(mod\ 6n+1), i = 1, \ldots, n$

2) $(0, i, b_i+n)(mod\ 6n+1), i = 1, \ldots, n$

3) $(0, a_i+n, b_i+n)(mod\ 6n+1), i = 1, \ldots, p$

$(0, \alpha_i+n, \beta_i+n)(mod\ 6n+1), i = p+1, \ldots, n$

4) $(0, i, \beta_i+n)(mod\ 6n+1), i = 1, \ldots, n$

We use only the first three systems.

Take

1) $(0, a_i + n, b_i + n)(mod\ 6n+1), i = 1, \ldots, n$

2) $(0, a_i + n, b_i + n)(mod\ 6n+1), i = 1, \ldots, j$

$(0, i, b_i + n)(mod\ 6n+1), i = j+1, \ldots, n$

For $i = 1, \ldots, j$ take the base blocks

$(0, a_i + n, b_i + n), i = p+1, \ldots, k$

$(0, i, b_i + n), i = k+1, \ldots, n.$

Repeat this process for $k = p+1, \ldots, n$.

3) $(0, a_i + n, b_i + n)(mod\ 6n+1), i = 1, \ldots, p$

$(0, \alpha_i + n, \beta_i + n)(mod\ 6n+1), i = p+1, \ldots, n$

Repeat this process for $j = 1, \ldots, n$.

This gives a $CTS(v, 3)$ with a fine structure $(3n - 2(p-i+j) - 3i, p-i+j, i)_c, i = 0, \ldots, p,\ j = 0, \ldots, n-i$. □

Theorem 30 *For $v \equiv 1(mod\ 6), v \geq 19$ the necessary conditions are sufficient for the vector $(3n-3i-2j, j, i)_c, i = 0, \ldots, n-3,\ j = 0, \ldots, n-i$ to be the fine structure of a $CTS(v, 3)$, where $n = \frac{v-1}{6}$.*

Proof: Apply Theorem 29 for $p = 0, \ldots, n-3$. □

Theorem 31 Main Theorem

For $v \equiv 1(mod\ 6), v \geq 19, v \neq 31$, the necessary conditions are sufficient for the vector $(3n-2j-3i, j, i)_c, i = 0, \ldots, n-3,\ j = 0, \ldots, n-i$, and the vector $(n-j, n-j, j)_c, j = 0, \ldots, n$, with the possible exception of the vector $(3n-2i-3p, i, p)_c, i = 0, \ldots, n-4-p$, to be the fine structure of a $CTS(v, 3)$, where $n = \frac{v-1}{6}$, and p is the number of pairs in common between two [hooked] Skolem sequences of order n.

Proof: Theorem 30 gives the fine structure of a $CTS(v, 3)$ when two [hooked] Skolem sequences intersect in $0, \ldots, n-3, n$ pairs and Theorem 29 gives the fine structure of a $CTS(v, 3)$ when two [hooked] Skolem sequences intersect in p pairs. If two [hooked]

Skolem sequences do not intersect in p pairs (see Theorems 24, 25 and Appendix C for the exceptions), then taking the difference between the fine structure given in Theorem 30 and Theorem 29, we get that $(3n-2i-3p, i, p)_c, i = 0, \ldots, n-4-p$, is not a fine structure for a $CTS(v, 3)$. □

Small cases

For $v = 7$, we can use a Skolem sequence of order 1 to get a cyclic Steiner triple system of order 7. Since there is only one possibility for the Skolem sequence of order 1, we take the same Skolem sequence twice and, by Theorem 28, $(1,1,0)_c$ and $(0,0,1)_c$ is the fine structure of $CTS(7, 3)$.

For $v = 13$, we can use a hooked Skolem sequence of order 2 to get a cyclic Steiner triple system of order 13. Since there is only one possibility for the hooked Skolem sequence of order 2, we take the same hooked Skolem sequence twice. By Theorem 28, $(2,2,0)_c, (1,1,1)_c, (0,0,2)_c$ is the fine structure of a $CTS(13, 3)$.

For $v = 31$, we can use a Skolem sequence of order 5 to get a $CSTS(31)$. We showed in Appendix A, that two Skolem sequences of order 5 can have $0, 1, 5$ pairs in common. Since in Theorem 30, we proved that $(3n-3i-2j, j, i)_c, i = 0, \ldots, n-3, \ j = 0, \ldots, n-i$ to be the fine structure of a $CTS(v, 3)$ using the fact that two [hooked] Skolem sequences intersect in $0, \ldots, n-3, n$ pairs, we can use this result here with the only exceptions that two Skolem sequences of order 5 intersect in $0, 1, 5$ pairs. Therefore, $(3n-3i-2j, j, i)_c, i = 0, 1, \ j = 0, \ldots, 5-i$ and $(5-j, 5-j, j)_c, j = 0, \ldots, 5$ is the fine structure of a $CTS(31, 3)$.

These results can also be applied for $\lambda = 4$ by taking all four systems given by two Skolem or two hooked Skolem sequences of order n in the previous theorems.

Chapter 6

Conclusions and open questions

In this thesis, we discussed triple systems and their intersection spectrum and then gave similar results for cyclic triple systems and their intersection spectrum. In Chapter 1, we gave a short introduction to this field. In Chapter 2, we discussed triple systems, disjoint triple systems and their intersection spectrum. In Chapter 3, we discussed Steiner triple systems and Mendelsohn triple systems and, in Chapter 4, we proved, with some possible exceptions, that there exists two Skolem sequences and there exists two hooked Skolem sequences of order n which can have $0, 1, 2, \ldots, n-3, n$ pairs in common. The exceptions are listed in Appendix C. We proved then in Chapter 5, using Skolem sequences of order n and hooked Skolem sequences of order n, that there exists two cyclic Steiner triple systems of order $6n+1$ intersecting in $0, 1, 2, \ldots, n$ base blocks and there exists two cyclic Steiner triple systems of order $6n+3$ intersecting in $1, 2, \ldots, n+1$ base blocks. Using these we proved that a twofold cyclic triple system of order $6n+1$ can have $0, 1, \ldots, n$ base blocks in common and a twofold cyclic triple system of order $6n+3$ can have $1, 2, \ldots, n+1$ base blocks in common.

Also in Chapter 5, using the intersection spectrum between two Skolem sequences and two hooked Skolem sequences of order n, we proved that the necessary conditions are sufficient for the vector $(3n-2j-3i, j, i)_c, i = 0, \ldots, n-3,\ j = 0, \ldots, n-i$, and the vector $(n-j, n-j, j)_c, j = 0, \ldots, n$, with the exception of the vector

$(3n-2i-3p, i, p)_c, i = 0, \ldots, n-4-p$, to be the fine structure of a cyclic threefold triple system of order $6n+1$. We now present some problems available for future work.

Open questions:

1. Solve all the remaining possible exceptions for the intersection spectrum of two Skolem sequences of order n.

2. Solve all the remaining possible exceptions for the intersection spectrum of two hooked Skolem sequences of order n.

3. Find the intersection spectrum of two Rosa sequences of order n.

4. Find the intersection spectrum of two hooked Rosa sequences of order n.

5. Find the fine structure of a $CTS(v, \lambda)$, for $v \equiv 1 (mod\ 6)$ with $\lambda = 4$.

6. Find the fine structure of a $CTS(v, \lambda)$, for $v \equiv 3 (mod\ 6)$ with $\lambda = 3$ or $\lambda = 4$.

7. Extend the result in chapter 5 to Mendelsohn cyclic triple systems and to directed cyclic triple systems.

8. Find more applications of these intersection spectrum of two Skolem sequences of order n and hooked Skolem sequences of order n.

Appendix A

The intersection of Skolem and hooked Skolem sequences for small orders

In this Appendix we list all the small $1 \leq n \leq 9$ cases necessary for Theorems 24 and 25.

There is only one Skolem sequence of order 1: $S_1 = (1, 1)$. Taking the same sequence twice we have $Int_{S_1} = \{1\}$.

There exist only one hooked Skolem sequence of order 2: $hS_2 = (1, 1, 2, 0, 2)$. So, $Int_{S_2} = \{2\}$.

There are two disjoint hooked Skolem sequences of order 3:

$hS_3 = (1, 1, 2, 3, 2, 0, 3)$

$hS_3 = (3, 1, 1, 3, 2, 0, 2)$

Taking these two disjoint sequences or taking the same sequence twice will give us: $Int_{S_3} = \{0, 3\}$.

Two Skolem sequences of order 4 can have $0, 1$ or 4 pairs in common:

0 pairs in common:

$S_4 = (4, 1, 1, 3, 4, 2, 3, 2)$

$S_4 = (2, 3, 2, 4, 3, 1, 1, 4)$

1 pair in common:

$S_4 = (2, 3, 2, 4, 3, 1, 1, 4)$

$S_4 = (1, 1, 3, 4, 2, 3, 2, 4)$

For 4 pairs in common we will take the same sequence twice. So, $Int_{S_4} = \{0, 1, 4\}$.

Two Skolem sequences of order 5 can have $0, 1$ or 5 pairs in common.

0 pairs in common:

$S_5 = (1, 1, 5, 2, 4, 2, 3, 5, 4, 3)$

$S_5 = (5, 2, 4, 2, 3, 5, 4, 3, 1, 1)$

1 pair in common:

$S_5 = (5, 1, 1, 3, 4, 5, 3, 2, 4, 2)$

$S_5 = (1, 1, 5, 2, 4, 2, 3, 5, 4, 3)$

For 5 pairs in common we will take the same sequence twice. So, $S_5 = \{0, 1, 5\}$.

Two hooked Skolem sequences of order 6 can have $0, 1, 2, 3$ or 6 pairs in common.

0 pairs in common:

$hS_6 = (1, 1, 2, 5, 2, 4, 6, 3, 5, 4, 3, 0, 6)$

$hS_6 = (3, 4, 6, 3, 2, 4, 2, 5, 6, 1, 1, 0, 5)$

1 pair in common:

$hS_6 = (1, 1, 2, 6, 2, 3, 4, 5, 3, 6, 4, 0, 5)$

$hS_6 = (3, 4, 6, 3, 2, 4, 2, 5, 6, 1, 1, 0, 5)$

2 pairs in common:

$hS_6 = (1, 1, 2, 5, 2, 4, 6, 3, 5, 4, 3, 0, 6)$

$hS_6 = (1, 1, 2, 6, 2, 3, 4, 5, 3, 6, 4, 0, 5)$

3 pairs in common:

$hS_6 = (6, 3, 5, 2, 3, 2, 6, 5, 4, 1, 1, 0, 4)$

$hS_6 = (6, 3, 1, 1, 3, 5, 6, 2, 4, 2, 5, 0, 4)$

For 6 pairs in common take the same sequence twice, so, $Int_{S_6} = \{0, 1, 2, 3, 6\}$

Two hooked Skolem sequences of order 7 can have $0, 1, 2, 3, 4$ or 7 pairs in common.

0 pairs in common:

$hS_7 = (6, 4, 2, 5, 2, 4, 6, 7, 5, 3, 1, 1, 3, 0, 7)$

$hS_7 = (2, 3, 2, 4, 3, 7, 5, 4, 6, 1, 1, 5, 7, 0, 6)$

1 pair in common:

$hS_7 = (4, 5, 1, 1, 4, 7, 5, 3, 6, 2, 3, 2, 7, 0, 6)$

$hS_7 = (3, 1, 1, 3, 6, 7, 2, 4, 2, 5, 6, 4, 7, 0, 5)$

2 pairs in common:

$hS_7 = (3, 1, 1, 3, 4, 5, 6, 7, 4, 2, 5, 2, 6, 0, 7)$

$hS_7 = (2, 3, 2, 5, 3, 4, 6, 7, 5, 4, 1, 1, 6, 0, 7)$

3 pairs in common:

$hS_7 = (1, 1, 2, 5, 2, 6, 4, 7, 5, 3, 4, 6, 3, 0, 7)$

$hS_7 = (4, 5, 1, 1, 4, 6, 5, 7, 2, 3, 2, 6, 3, 0, 7)$

4 pairs in common:

$hS_7 = (1, 1, 3, 6, 4, 3, 5, 7, 4, 6, 2, 5, 2, 0, 7)$

$hS_7 = (4, 1, 1, 6, 4, 3, 5, 7, 3, 6, 2, 5, 2, 0, 7)$

For 7 pairs in common take the same sequence twice, so $Int_{S_7} = \{0, 1, 2, 3, 4, 7\}$. Two Skolem sequences of order 9 can have $0, 1, 2, 3, 4, 5$ or 8 pairs in common:

0 pairs in common:

$S_8 = (5, 7, 2, 6, 2, 5, 4, 8, 7, 6, 4, 3, 1, 1, 3, 8)$

$S_8 = (3, 1, 1, 3, 6, 4, 8, 5, 7, 4, 6, 2, 5, 2, 8, 7)$

1 pair in common:

$S_8 = (2, 5, 2, 6, 7, 3, 5, 8, 3, 6, 4, 7, 1, 1, 4, 8)$

$S_8 = (1, 1, 3, 7, 4, 3, 6, 8, 4, 5, 7, 2, 6, 2, 5, 8)$

2 pairs in common:

$S_8 = (3, 4, 6, 3, 5, 4, 7, 8, 6, 5, 1, 1, 2, 7, 2, 8)$

$S_8 = (5, 1, 1, 6, 4, 5, 7, 8, 4, 6, 2, 3, 2, 7, 3, 8)$

3 pairs in common:

$S_8 = (3, 1, 1, 3, 4, 6, 7, 8, 4, 5, 2, 6, 2, 7, 5, 8)$

$S_8 = (1, 1, 2, 5, 2, 6, 7, 8, 5, 3, 4, 6, 3, 7, 4, 8)$

4 pairs in common:

$S_8 = (1, 1, 2, 5, 2, 6, 7, 8, 5, 3, 4, 6, 3, 7, 4, 8)$

$S_8 = (1, 1, 2, 6, 2, 5, 7, 8, 4, 6, 5, 3, 4, 7, 3, 8)$

5 pairs in common:

$S_8 = (3, 1, 1, 3, 6, 7, 8, 2, 5, 2, 6, 4, 7, 5, 8, 4)$

$S_8 = (3, 6, 2, 3, 2, 7, 8, 6, 5, 1, 1, 4, 7, 5, 8, 4)$

For 8 pairs in common take the same sequence twice, so, $Int_{S_8} = \{0, 1, 2, 3, 4, 5, 8\}$.

Two Skolem sequences of order 9 intersect in $0, 1, 2, 3, 4, 5, 6$ or 9 pairs.

0 pairs in common:

$S_9 = (9, 5, 3, 1, 1, 3, 5, 6, 8, 9, 7, 4, 2, 6, 2, 4, 8, 7)$

$S_9 = (8, 9, 7, 4, 2, 6, 2, 4, 8, 7, 9, 6, 5, 3, 1, 1, 3, 5)$

1 pair in common:

$S_9 = (9, 5, 8, 4, 1, 1, 5, 4, 7, 9, 8, 6, 2, 3, 2, 7, 3, 6)$

$S_9 = (9, 7, 5, 8, 6, 1, 1, 5, 7, 9, 6, 8, 4, 2, 3, 2, 4, 3)$

2 pairs in common:

$S_9 = (9, 7, 8, 3, 1, 1, 3, 6, 7, 9, 8, 4, 5, 6, 2, 4, 2, 5)$

$S_9 = (9, 6, 8, 5, 7, 1, 1, 6, 5, 9, 8, 7, 4, 2, 3, 2, 4, 3)$

3 pairs in common:

$S_9 = (9, 7, 8, 2, 3, 2, 6, 3, 7, 9, 8, 5, 6, 4, 1, 1, 5, 4)$

$S_9 = (9, 7, 8, 3, 1, 1, 3, 6, 7, 9, 8, 4, 5, 6, 2, 4, 2, 5)$

4 pairs in common:

$S_9 = (9, 7, 8, 4, 2, 6, 2, 4, 7, 9, 8, 6, 5, 3, 1, 1, 3, 5)$

$S_9 = (9, 7, 8, 4, 1, 1, 6, 4, 7, 9, 8, 5, 6, 2, 3, 2, 5, 3)$

5 pairs in common:

$S_9 = (9, 7, 8, 4, 1, 1, 6, 4, 7, 9, 8, 5, 6, 2, 3, 2, 5, 3)$

$S_9 = (9, 7, 8, 2, 3, 2, 6, 3, 7, 9, 8, 5, 6, 4, 1, 1, 5, 4)$

6 pairs in common:

$S_9 = (9, 7, 5, 8, 6, 1, 1, 5, 7, 9, 6, 8, 4, 2, 3, 2, 4, 3)$

$S_9 = (9, 7, 5, 8, 6, 1, 1, 5, 7, 9, 6, 8, 3, 4, 2, 3, 2, 4)$

For 9 pairs in common,take the same sequence twice, so $Int_{S_9} = \{0, 1, 2, 3, 4, 5, 6, 9\}$.

Appendix B

Tables giving the intersection between a Langford sequence and its reverse

In this Appendix we give the number of pairs in common between a Langford sequence and its reverse.

In Simpson's paper [45], all the Langford sequences of order $n = 4t$ are perfect sequences. Table B.1 gives the intersection between a Langford sequence of order $n = 4t$ and defect d and its reverse. For example, taking the Langford sequence of order $n = 8$ and defect $d = 4$ ($t = 2$ and $s = 1$) from [45] the pairs are:
$(t-2s+1, 3t+s+1); (5t-s+1, 7t+2s); (3t-s+2, 5t-s+2); (2t, 4t+1);$
$(t-s+1, 5t-s+3); (2t-3s+2, 6t-s+3); (2t+1, 6t-s+2); (2t+s, 6t+3s)$.
These pairs gives the following Langford sequence of order 8 and defect 4: $L_4^8 = (7, 10, 11, 5, 8, 9, 4, 7, 5, 6, 4, 10, 8, 11, 9, 6)$.
Its reverse is another perfect Langford of order 8 and defect 4:
$L_4^8 = (6, 9, 11, 8, 10, 4, 6, 5, 7, 4, 9, 8, 5, 11, 10, 7)$.
These two Langford sequences have one pair in common(i.e. 11 is in the same position in both sequences).
To see that these sequences are disjoint or have some pairs in common, we check that

all the values (i, j) and their new positions $(2n+1-j, 2n+1-i)$ are distinct or not. For example, $(t-2s+1, 3t+s+1)$ becomes $(8t+1-(3t+s+1), 8t+1-(t-2s+1))$ in the reverse sequence and the two pairs are distinct.

In [8], we can find other perfect Langford sequences of order $n \equiv 2d-1 (mod\ 4)$. The number of pairs in common between these sequences and their reverse is given in table B.2.

Table 0a)[29], gives perfect Langford sequences of order $2d-1$ and defect d. The number of pairs in common between these sequences and their reverse are given in table B.3.

Table 0B[29], gives other perfect Langford sequences of order $2d-1$ and defect d. The number of pairs in common between these sequences and their reverse are given in table B.4.

Table 0C[29], also gives other perfect Langford sequences of order $2d-1$ and defect d. The number of pairs in common between these sequences and their reverse are given in table B.5.

In table 0a)[29], the pair $n = 2d-1$ can be in the first or last position, therefore we can have two different Langford sequences of order $2d-1$ and defect d. For example, from the following Langford sequence of order 5 and defect 3:
$L_3^5 = (6, 7, 3, 4, 5, 3, 6, 4, 7, 5)$ we can get the following Langford sequence of order 5 and defect 3 by changing 5 from the last position to the first position: $L_3^5 = (5, 6, 7, 3, 4, 5, 3, 6, 4, 7)$.
Now if we take the Langford sequence of order $2d-1$ and defect d and its reverse from[7], another Langford sequence of order $2d-1$ and defect d and its reverse from Table0a)[29] and the Langford sequence and its reverse obtained from the previous sequences by moving the pair $2d-1$ from the last position to the first position, we will have six different Langford sequences of order $2d-1$ and defect d. The number of pairs in common between these sequences and their reverse are given in table B.6.

Another perfect Langford sequence of order $n = 2d$ and defect d for d even is given in Table 1d[29]. The number of pairs in common between these Langford and their reverse are given in table B.7. A Langford sequence of order $4t + 2$ or order $2d + 1 + 4r$ [29] is a hooked Langford sequence. If the Langford sequence is hooked we will fill the hook with the pair $(2, 0, 2)$ and this will make the sequence perfect. Table B.8 will give the number of pairs in common between such Langford sequence of order $n = 4t + 2$ and defect d and their reverse.

Table B.9 will give the number of pairs in common between a hooked Langford sequence of order $2d + 1 + 4r$ and defect d and its reverse.

n	d	Conditions	Pairs in common
n=4t	d=4s	$s \geq 1, t \geq 2s$	1
	d=4s+2	$s \geq 1, t \geq 2s + 1$	3
	d=4s-1	$s \geq 1, t \geq 2s$	0
	d=4s+1	$s \geq 1, t \geq 2s + 1$	0

Table B.1: Intersection of Langford sequences of order $n = 4t$ and defect d with their reverse

$d \geq 2$, d even	the sequences are disjoint
$d \geq 3$, d odd	if $n \equiv 2d - 1, n \neq 2d - 1$ the sequences are disjoint if $n = 2d - 1$ the sequences have one pair in common

Table B.2: Intersection of Langford sequences of order $n \equiv 2d - 1(mod\ 4)$ and defect d with their reverse

$d \equiv 2(mod\ 3); d \geq 1$	the sequences are disjoint
$d \equiv 0, 1(mod\ 3); d \geq 1$	the sequences have one pair in common

Table B.3: Intersection of Langford sequences of order $2d - 1$ and defect d with their reverse

$d \equiv 1, 2(mod\, 4); d \geq 4$	the sequences are disjoint
$d \equiv 0, 3(mod\, 4); d \geq 4$	the sequences have one pair in common

Table B.4: Intersection of Langford sequences of order $2d - 1$ and defect $d(2)$ with their reverse

$d \equiv 0, 2(mod\, 3), d \geq 8$	the sequences have one pair in common
$d \equiv 1(mod\, 3), d \geq 8$ $d = 7$	the sequences are disjoint the sequences have one pair in common

Table B.5: Intersection of Langford sequences of order $2d - 1$ and defect $d(3)$ with their reverse

$d = 2; n = 3$	disjoint
$d = 3; n = 5$	0 or 1 pairs in common
$d = 4; n = 7$	0, 1 or 2 pairs in common
$d = 5; n = 9$	0,1 or 3 pairs in common
$d \equiv 0, 1(mod\, 3); d \geq 6$	0, 1 or 2 pairs in common
$d \equiv 2(mod\, 3); d \geq 6$	0,1,2 or 3 pairs in common

Table B.6: Intersection of Langford sequences of order $2d - 1$ and defect $d(4)$ with their reverse

$n = 2d, d = 2t$	disjoint if t is even 1 pair if t is odd

Table B.7: Intersection of Langford sequences of order $n = 2d$ and defect d with their reverse

defect d	Conditions	Pairs in common with its reverse
$d = 4s$	$t - 2s = r \geq 0$	1 pair if $d = 8$, 0 pairs otherwise
$d = 4s + 2$	$t - 2s - 1 = r \geq 0$	2 pairs if $s \equiv 2(mod\, 3)$ 0 pairs if $s \equiv 0, 1(mod\, 3); s \neq 1$ 1 pair if $s = 1$
$d = 4s + 3$	$t - 2s - 1 = r \geq 0$	0 pairs if $s \equiv 1(mod\, 3)$ 1 pair if $s \equiv 0, 2(mod\, 3); s \neq 0$ 0 pairs if $s = 0$
$d = 4s + 1$	$t - 2s \geq 0$	0 pairs if $d = 5$ 3 pairs if $d = 9$ 2 pairs if $d = 13$ $s - 1$ pairs if $s \equiv 1(mod\, 3)$ s pairs if $s \equiv 0, 2(mod\, 3)$

Table B.8: Intersection of Langford sequences of order $n = 4t + 2$ and defect d with their reverse

$d = 3$	$r = 0$ $r \geq 1$	1 pair 0 pairs
$d = 4$	$r = 0$ $r \geq 1$	2 pairs 0 pairs
d even, $d \geq 6$	$r = 0, 1$ $r \geq 2$	1 pair 0 pairs
d odd, $d \geq 5$	$r = 0$ $r \geq 1$	1 pair 0 pairs

Table B.9: Intersection of Langford sequences of order $n \equiv 2d + 1(mod\, 4)$ and defect d with their reverse

Appendix C

The possible exceptions in Theorems 24 and 25 that are still open

Below are the possible exceptions of the intersection spectrum of two Skolem sequences.

<u>$n \equiv 0(mod\ 12)$</u>

$n = 12$, the possible exceptions are $4, 6, 7$.

$n = 24$, the possible exceptions are $3, 6, 8$.

$n = 36$, the possible exceptions are $12, 16, 18, 21, 22$.

$n = 48$, the possible exceptions are $16, 23, 24, 29, 30$.

$n = 60$, the possible exceptions are $20, 29, 30, 32, 33, 34, 36$.

$n = 12t, n \geq 72, t \equiv 0, 1(mod\ 3)$, the possible exceptions are $\lfloor \frac{n}{3} \rfloor - 5$, $\lfloor \frac{n}{3} \rfloor$, otherwise, the possible exceptions are $\lfloor \frac{n}{3} \rfloor - 5$, $\lfloor \frac{n}{3} \rfloor - 2$, $\lfloor \frac{n}{3} \rfloor$.

<u>$n \equiv 1(mod\ 12)$</u>

$n = 13$, the possible exceptions are $6, 8$.

$n = 25$, the possible exceptions are $11, 12, 13, 15$.

$n = 37$, the possible exceptions are $17, 18, 19, 21, 22, 23, 24$.

$n = 61$, the possible exception is $36, 37$.

$n = 73$, the possible exception is 44.

$n \geq 109$, the possible exceptions are: $2\lfloor\frac{n}{3}\rfloor - 4, 2\lfloor\frac{n}{3}\rfloor$.

$\underline{n \equiv 4(mod\ 12)}$

$n = 16$, the possible exception is 10.

$n = 28$, the possible exceptions are $13, 14, 16, 17, 18$.

$n = 40$, the possible exceptions are $18, 19, 20, 22, 24$.

$n = 76$, the possible exception is 49.

$n = 88$, the possible exception is 57.

$n = 100$, the possible exception is 65.

$n \geq 112$, the possible exception is $2\lfloor\frac{n}{3}\rfloor - 1$.

$\underline{n \equiv 5(mod\ 12)}$

$n = 17$, the possible exceptions are $5, 10, 11$.

$n = 29$, the possible exceptions are $9, 10, 17, 19$.

$n = 41$, the possible exceptions are $13, 14, 23, 24, 25, 27$.

$n = 53$, the possible exceptions are $17, 18, 31, 32, 35$.

$n = 65$, the possible exceptions are $21, 22, 30, 31, 39, 43$.

$n \geq 77$, the possible exceptions are $\lfloor\frac{n}{3}\rfloor, \lfloor\frac{n}{3}\rfloor + 1, 2\lfloor\frac{n}{3}\rfloor - 3, 2\lfloor\frac{n}{3}\rfloor - 2, 2\lfloor\frac{n}{3}\rfloor + 1$.

$\underline{n \equiv 8(mod\ 12)}$

$n = 20$, the possible exceptions are $3, 6, 7, 9, 13$.

$n = 32$, the possible exceptions are $10, 11, 14, 15, 20, 21$.

$n = 44$, the possible exceptions are $14, 15, 29$.

$n = 56$, the possible exceptions are $18, 19, 33, 37$.

$n = 68$, the possible exceptions are $22, 23, 34, 41, 45$.

$n = 80$, the possible exceptions are $26, 27, 39, 40, 53$.

$n \geq 92, n = 12t+8, t \equiv 0, 2(mod\ 3)$, the possible exceptions are $\lfloor\frac{n}{3}\rfloor - 3, \lfloor\frac{n}{3}\rfloor + 1, 2\lfloor\frac{n}{3}\rfloor - 3, 2\lfloor\frac{n}{3}\rfloor + 1$, otherwise, the possible exceptions are $\lfloor\frac{n}{3}\rfloor - 3, \lfloor\frac{n}{3}\rfloor, \lfloor\frac{n}{3}\rfloor + 1, 2\lfloor\frac{n}{3}\rfloor - 3, 2\lfloor\frac{n}{3}\rfloor + 1$.

$\underline{n \equiv 9(mod\ 12)}$

$n = 21$, the possible exceptions are $5, 7, 10, 11, 12, 13, 14$.

$n = 33$, the possible exceptions are:$11, 14, 15, 19, 20$.

$n = 45$, the possible exceptions are $15, 22, 27$.

$n = 57$, the possible exceptions are $19, 22, 23, 25, 26, 27, 28, 29, 31, 32, 35$.

$n = 69$, the possible exceptions are $23, 26, 40, 43, 44$.

$n \equiv 9(mod\ 12), n \geq 81$, the possible exceptions are $\lfloor\frac{n}{3}\rfloor, \lfloor\frac{n}{3}\rfloor+3, 2\lfloor\frac{n}{3}\rfloor-6, 2\lfloor\frac{n}{3}\rfloor-2$.

Below are the possible exceptions of the intersection of two hooked Skolem sequences.

<u>$n \equiv 2(mod\ 12)$.</u>

$n = 14$, the possible exceptions are $1, 4, 5, 6, 7, 8, 9$.

$n = 26$, the possible exceptions are $5, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17$.

$n = 38$, the possible exceptions are $9, 12, 13, 16, 18, 20, 22, 23, 24, 25$.

$n = 50$, the possible exceptions are $16, 17, 25, 26, 28, 30, 33$.

$n = 62$, the possible exceptions are $17, 20, 21, 29, 30, 32, 34, 35, 36, 37, 38, 40, 41$.

$n = 74$, the possible exceptions are $21, 24, 25, 34, 36, 39, 40, 42, 43, 44, 45, 47, 48$.

$n = 86$, the possible exceptions are

$25, 28, 29, 37, 38, 40, 41, 42, 43, 44, 46, 48, 50, 51, 52, 53, 56, 57$.

$n = 98$, the possible exceptions are $32, 33, 53, 54, 56, 58, 59, 60, 64, 65$.

$n = 12t+2, n > 98, n = 98+60r, t \equiv 0, 1, 2, 4, 5, 7, 10, 13, 14(mod\ 15)$: $\lfloor\frac{n}{3}\rfloor-3, \lfloor\frac{n}{3}\rfloor, \lfloor\frac{n}{3}\rfloor+1, \lfloor\frac{n}{3}\rfloor+21+16r, \lfloor\frac{n}{3}\rfloor+22+16r, \lfloor\frac{n}{3}\rfloor+24+16r, \ldots, 2\lfloor\frac{n}{3}\rfloor-4, 2\lfloor\frac{n}{3}\rfloor, 2\lfloor\frac{n}{3}\rfloor+1$ (1), $t \equiv 6, 11, 12(mod\ 15)$: (1)-$\{\lfloor\frac{n}{3}\rfloor\}$, $t \equiv 3, 8, 9(mod\ 15)$: (1)-$\{\lfloor\frac{n}{3}\rfloor-3\}$, $n = 12t+2, n > 98, n = 50+60r, 62+60r, 74+60r, 86+60r, t \equiv 0, 1, 2, 4, 5, 7, 10, 13, 14(mod\ 15)$: $\lfloor\frac{n}{3}\rfloor-3, \lfloor\frac{n}{3}\rfloor, \lfloor\frac{n}{3}\rfloor+1, \lfloor\frac{n}{3}\rfloor+9+16r, \lfloor\frac{n}{3}\rfloor+10+16r, \lfloor\frac{n}{3}\rfloor+12+16r, \ldots, 2\lfloor\frac{n}{3}\rfloor-4, 2\lfloor\frac{n}{3}\rfloor-2, 2\lfloor\frac{n}{3}\rfloor, 2\lfloor\frac{n}{3}\rfloor+1$ (2), $t \equiv 6, 11, 12(mod\ 15)$: (2)-$\{\lfloor\frac{n}{3}\rfloor\}$, $t \equiv 3, 8, 9(mod\ 15)$: (2)-$\{\lfloor\frac{n}{3}\rfloor-3\}$

<u>$n \equiv 3(mod\ 12)$.</u>

$n = 15$, the possible exceptions are $3, 5, 7, 8, 9, 10$.

$n = 27$, the possible exceptions are $7, 9, 12, 13, 15, 17, 18$.

$n = 39$, the possible exceptions are $11, 13, 17, 19, 21, 22, 23, 24, 25$.

$n = 51$, the possible exceptions are $15, 17, 20, 21, 23, 27, 29, 30, 31, 32, 33, 34$.

$n = 63$, the possible exceptions are $19, 21, 30, 31, 33, 37, 38$.

$n = 75$, the possible exceptions are $23, 25, 28, 41, 43, 44, 45, 46, 47, 49, 50$.

$n \equiv 3(mod\ 12), n = 75 + 60r, n > 75$: $\lfloor\frac{n}{3}\rfloor - 2, \lfloor\frac{n}{3}\rfloor, \lfloor\frac{n}{3}\rfloor + 3, \lfloor\frac{n}{3}\rfloor + 15 + 16r, \lfloor\frac{n}{3}\rfloor + 16 + 16r, \lfloor\frac{n}{3}\rfloor + 18 + 16r, \ldots, 2\lfloor\frac{n}{3}\rfloor - 3, 2\lfloor\frac{n}{3}\rfloor - 1, 2\lfloor\frac{n}{3}\rfloor$, otherwise: $\lfloor\frac{n}{3}\rfloor - 2, \lfloor\frac{n}{3}\rfloor, \lfloor\frac{n}{3}\rfloor + 3, \lfloor\frac{n}{3}\rfloor + 3 + 16r, \lfloor\frac{n}{3}\rfloor + 4 + 16r, \lfloor\frac{n}{3}\rfloor + 6 + 16r, \ldots, 2\lfloor\frac{n}{3}\rfloor - 3, 2\lfloor\frac{n}{3}\rfloor - 1, 2\lfloor\frac{n}{3}\rfloor$.

$\underline{n \equiv 6(mod\ 12)}$.

$n = 18$, the possible exceptions are $6, 8, 9, 10, 11, 12$.

$n = 30$, the possible exceptions are $5, 8, 13, 14, 16, 18$.

$n = 42$, the possible exceptions are $9, 12, 14, 17, 18, 20, 21, 22, 23, 24, 25, 26, 27, 28$.

$n = 54$, the possible exceptions are $15, 21, 22, 24, 25, 26, 27, 28, 30, 31, 32, 33, 35, 36$.

$n = 66$, the possible exceptions are

$22, 25, 26, 28, 29, 30, 31, 32, 33, 34, 35, 38, 39, 40, 41, 44$.

$n = 78$, the possible exceptions are $26, 29, 42, 44, 46, 47, 48$.

$n = 12t + 6, n > 78, n = 78 + 60r, t \equiv 0, 1, 2, 5, 7, 10, 14(mod\ 15)$: $\lfloor\frac{n}{3}\rfloor - 5, \lfloor\frac{n}{3}\rfloor - 2, \lfloor\frac{n}{3}\rfloor, \lfloor\frac{n}{3}\rfloor + 3, \lfloor\frac{n}{3}\rfloor + 15 + 16r, \lfloor\frac{n}{3}\rfloor + 16 + 16r, \lfloor\frac{n}{3}\rfloor + 18 + 16r, \ldots, 2\lfloor\frac{n}{3}\rfloor - 2$ (3), $t \equiv 8, 9(mod\ 15)$: (3)-$\{\lfloor\frac{n}{3}\rfloor - 5\}$, $t \equiv 4, 13(mod\ 15)$: (3)-$\{\lfloor\frac{n}{3}\rfloor\}$, $t \equiv 3(mod\ 15)$:(3)-$\{\lfloor\frac{n}{3}\rfloor - 5, \lfloor\frac{n}{3}\rfloor\}$, $n = 12t + 6, n > 78, n = 30 + 60r, 42 + 60r, 54 + 60r, 66 + 60r$ $t \equiv 0, 1, 2, 5, 7, 10, 14(mod\ 15)$: $\lfloor\frac{n}{3}\rfloor - 5, \lfloor\frac{n}{3}\rfloor - 2, \lfloor\frac{n}{3}\rfloor, \lfloor\frac{n}{3}\rfloor + 3, \lfloor\frac{n}{3}\rfloor + 3 + 16r, \lfloor\frac{n}{3}\rfloor + 4 + 16r, \lfloor\frac{n}{3}\rfloor + 6 + 16r, \ldots, 2\lfloor\frac{n}{3}\rfloor - 2$(4), $t \equiv 8, 9(mod\ 15)$: (4)-$\{\lfloor\frac{n}{3}\rfloor - 5\}$, $t \equiv 4, 13(mod\ 15)$: (4)-$\{\lfloor\frac{n}{3}\rfloor\}$, $t \equiv 3(mod\ 15)$: (4)- $\{\lfloor\frac{n}{3}\rfloor - 5, \lfloor\frac{n}{3}\rfloor\}$.

$\underline{n \equiv 7(mod\ 12)}$.

$n = 19$, the possible exceptions are $9, 10, 11, 12$.

$n = 31$, the possible exceptions are $14, 15, 17, 18, 19, 20$.

$n = 43$, the possible exceptions are $17, 18, 19, 20, 21, 23, 25$.

$n = 55$, the possible exceptions are $28, 29, 31, 33$.

$n \equiv 7(mod\ 12), n \geq 67, n = 55 + 60r$: $\lfloor\frac{n}{3}\rfloor + 10 + 16r, \lfloor\frac{n}{3}\rfloor + 11 + 16r, \lfloor\frac{n}{3}\rfloor + 13 + 16r, \ldots, 2\lfloor\frac{n}{3}\rfloor - 2$, otherwise: $\lfloor\frac{n}{3}\rfloor + 14 + 16r, \lfloor\frac{n}{3}\rfloor + 15 + 16r, \lfloor\frac{n}{3}\rfloor + 17 + 16r, \ldots, 2\lfloor\frac{n}{3}\rfloor - 2$.

$\underline{n \equiv 10(mod\ 12)}$.

$n = 22$, the possible exceptions are $11, 12, 13, 14$.

$n = 34$, the possible exceptions are $14, 15, 16, 17, 18, 20, 21, 22$.

$n = 46$, the possible exceptions are $18, 19, 20, 21, 22, 24, 26, 27, 28, 29$.

$n = 58$, the possible exceptions are $29, 32, 34, 35, 36$.

$n \equiv 10(mod\ 12), n \geq 70, n = 58 + 60r$: $\lfloor \frac{n}{3} \rfloor + 10 + 16r, \lfloor \frac{n}{3} \rfloor + 11 + 16r, \lfloor \frac{n}{3} \rfloor + 13 + 16r, \ldots, 2\lfloor \frac{n}{3} \rfloor$, otherwise: $\lfloor \frac{n}{3} \rfloor + 14 + 16r, \lfloor \frac{n}{3} \rfloor + 15 + 16r, \lfloor \frac{n}{3} \rfloor + 17 + 16r, \ldots, 2\lfloor \frac{n}{3} \rfloor$.

$n \equiv 11(mod\ 12)$.

$n = 23$, the possible exceptions are $7, 8, 9, 10, 11, 12, 13, 14$.

$n = 35$, the possible exceptions are $11, 12, 15, 16, 17, 19, 21$.

$n = 47$, the possible exceptions are $15, 16, 24, 25, 27, 28, 29, 30$.

$n = 59$, the possible exceptions are $19, 20, 28, 29, 31, 33, 34, 35, 36, 37, 38$.

$n = 71$, the possible exceptions are $23, 24, 35, 37, 38, 39, 41, 42, 43, 44, 45, 46$.

$n = 83$, the possible exceptions are $27, 28, 36, 37, 39, 40, 41, 42, 43, 45, 47, 49, 50$.

$n = 95$, the possible exceptions are $31, 32, 52, 53, 55, 56, 57, 58, 61, 62$.

$n \equiv 11(mod\ 12), n > 95, n = 95 + 60r$: $\lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{3} \rfloor + 1, \lfloor \frac{n}{3} \rfloor + 21 + 16r, \lfloor \frac{n}{3} \rfloor + 22 + 16r, \lfloor \frac{n}{3} \rfloor + 24 + 16r, \ldots, 2\lfloor \frac{n}{3} \rfloor - 1, 2\lfloor \frac{n}{3} \rfloor, 2\lfloor \frac{n}{3} \rfloor + 1$, otherwise: $\lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{3} \rfloor + 1, \lfloor \frac{n}{3} \rfloor + 9 + 16r, \lfloor \frac{n}{3} \rfloor + 10 + 16r, \lfloor \frac{n}{3} \rfloor + 12 + 16r, \ldots, 2\lfloor \frac{n}{3} \rfloor - 1, 2\lfloor \frac{n}{3} \rfloor, 2\lfloor \frac{n}{3} \rfloor + 1$.

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