EXTENDED NEAR-SKOLEM TYPE SEQUENCES, INFINITE SKOLEM SEQUENCES, AND RELATED TOPICS

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Extended near-Skolem type sequences, infinite Skolem sequences, and related topics

by
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Abstract

A *Skolem sequence* of order n is a sequence $S_n = (s_1, s_2, \ldots, s_{2n})$ of $2n$ positive integers such that for each $k \in \{1, 2, \ldots, n\}$, there exists exactly two elements $s_i, s_j \in S_n$ such that $s_i = s_j = k$ and $|j - i| = k$. Skolem sequences and their generalizations have many applications and thus any new existence results concerning these sequences and their generalizations are welcome.

In this thesis, we introduce near-λ-fold Skolem sequences and extended near-λ-fold Skolem sequences and show that the necessary conditions for their existence are also sufficient when $\lambda \geq 2$. We also discuss the case for $\lambda = 1$. We then prove that the necessary conditions are also sufficient for the existence of two new classes of near-Skolem-type sequences. The appendix also contains some computational results for near-Skolem and near-Skolem-type sequences. Finally, we discuss infinite Skolem sequences and their relationship to Beatty sequences and investigate whether or not this relationship can be extended to other generalizations of Skolem sequences. We also present a relationship between infinite Skolem sequences, Fibonacci numbers, and some restricted compositions and palindromes of n.

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Contents

Chapter 1

Introduction

A *Steiner triple system of order* v, denoted $STS(v)$, is a pair $(V, \mathcal{B})$, where V is a set of v points and $\mathcal{B}$ is a collection of triples from V such that each unordered pair in V occurs in exactly one triple in $\mathcal{B}$. For example, the blocks $\{0,1,3\}, \{1,2,4\}, \{2,3,5\}, \{3,4,6\}, \{4,5,0\}, \{5,6,1\}$ and $\{6,0,2\}$ constitute an $STS(7)$. It was Kirkman, in [12], who showed that an $STS(v)$ exists if and only if $v \equiv 1,3 \pmod 6$.

An *automorphism* π, of an $STS(v)$ with element set V is a permutation of V which preserves the block set $\mathcal{B}$, i.e., if $B \in \mathcal{B}$, then $\pi(B) \in \mathcal{B}$. The set of all such automorphisms of $STS(v)$ with the element set V form the *automorphism group of the design.* If $(V, \mathcal{B})$ is an $STS(v)$ which admits π as an automorphism, then π partitions the blocks $B \in \mathcal{B}$ into equivalence classes, called *orbits of* π, such that B_1 and B_2 belong to the same orbit if and only if $\pi^m(B_1) = B_2$, for some positive integer m. Let $(V, \mathcal{B})$ be an $STS(v)$ which admits π as an automorphism, O_i an orbit induced by π, and B_i a block in O_i. Then we can find all $B_j \in O_i$ by evaluating $\pi^k(B_i)$ for all k, where $1 \leq k \leq n$ and n is the order of π. The block B_i is called a *base block* of the $STS(v)$ with respect to π. Thus, to construct an $STS(v)$ which admits π as an automorphism, it is sufficient to construct base blocks for each of the orbits induced by π.

An $STS(v)$ is called *cyclic* if its automorphism group contains a v-cycle. It is known that an $STS(v)$ is cyclic if and only if $v \equiv 1,3 \pmod 6$, $v \neq 9$. This was shown by Peltesohn in [23]. Peltesohn did this by solving Heffter's first and second problem. *Heffter's first problem* asks: can one partition the numbers $1, 2, \ldots, 3n$ into n triples $\{a_i, b_i, c_i\}$, where $1 \leq i \leq n$, such that $a_i + b_i = c_i$ or $a_i + b_i + c_i \equiv 0 \pmod{6n+1}$? Heffter observed that a solution to this problem can be converted to a set of triples $\{0, a_i, c_i\}$ when $a_i+b_i = c_i$ or $\{0, a_i, 6n+1-c_i\}$ when $a_i + b_i + c_i \equiv 0 \pmod{6n+1}$, which are base blocks of a cyclic $STS(6n+1)$ on the element set $\mathbb{Z}_{6n+1}$. *Heffter's second problem* asks: can one partition the numbers $1, 2, \ldots, 2n, 2n+2, \ldots, 3n$ into n triples $\{a_i, b_i, c_i\}$, where $1 \leq i \leq n$, such that $a_i + b_i = c_i$ or $a_i + b_i + c_i \equiv 0 \pmod{6n+3}$? Once again, a solution to this problem can be converted to a set of triples $\{0, a_i, c_i\}$ when $a_i + b_i = c_i$ or $\{0, a_i, 6n+3-c_i\}$ when $a_i+b_i+c_i \equiv 0 \pmod{6n+3}$ which, along with the triple $\{0, 2n+1, 4n+2\}$, are base blocks of a cyclic $STS(6n+3)$ on the element set Z_{6n+3}. (The reader is referred to [11] for more information on Heffter's problems.)

It was in 1957, while studying cyclic Steiner triple systems, that Thoralf Skolem, in [36], asked if it was possible to partition the integers $1, 2, \ldots, 2n$ into n pairs (a_i, b_i) such that $\{|b_i - a_i| \mid 1 \leq i \leq n\} = \{1, 2, \ldots, n\}$. For example, for $n = 4$ we have the partition $(7,8), (2,4), (3,6), (1,5)$. Skolem called such partitions a $1, +1$ system (Skolem defines, in general, an $l, +m$ system as a system of disjoint pairs with corresponding differences $l, l+m, l+2m, \ldots$) and showed that such a partition exists if and only if $n \equiv 0, 1 \pmod 4$ (although he attributes the majority of the proof to Bang). Although C.D. Langford was the first to write similarly defined partitions as sequences, see [13], it was Nickerson, in [20], who first wrote these particular partitions of Skolem as sequences of length $2n$, placing the integer i in positions a_i and b_i of the sequence. For example, the previous partition can be written as the sequence $(4,2,3,2,4,3,1,1)$. These sequences are now known as *Skolem sequences.* Given an admissible positive integer n, Skolem took the pairs (a_i, b_i) for each i, where $1 \leq i \leq n$, and constructed the triples $\{i, a_i+n, b_i+n\}$, which form a solution to Heffter's first problem, and from these triples, constructed the base blocks $\{0, i, b_i+n\}$ of a cyclic $STS(6n+1)$(see [37]). For example, the previous partition gives us the base blocks $\{0,1,12\}, \{0,2,8\}, \{0,3,10\}, \{0,4,9\}$ of an $STS(25)$.

Skolem also asked if it was possible to partition the integers $1, 2, \ldots, 2n-1, 2n+1$ into n pairs (a_i, b_i) such that $\{|b_i - a_i| \mid 1 \leq i \leq n\} = \{1, 2, \ldots, n\}$. For example, for $n = 3$ we have the partition $(2,3), (5,7), (1,4)$. When written in Langford's format, we place a '0' or '$*$', called a "hook", into the $2n^{\text{th}}$ position of the sequence. For example, the latter partition gives us the sequence $(3,1,1,3,2,0,2)$. These sequences are known as *hooked Skolem sequences.* However, it was O'Keefe, in [22], who showed that such a sequence exists if and only if $n \equiv 2, 3 \pmod 4$. These partitions can also lead to a solution to Heffter's first problem; we convert the pairs (a_i, b_i) into the triples $\{ i, a_i+n, b_i+n \}$ for all i, where $1 \leq i \leq n$. Then, using the same construction as was used with Skolem sequences, hooked Skolem sequences can also be used to construct an $STS(6n + 1)$, for all $n \equiv 2, 3 \pmod 4$. Thus, the work of Skolem and O'Keefe combined to prove the sufficiency of the existence of cyclic $STS(6n+1)$ for all positive integers n.

The sufficiency of the existence of cyclic $STS(6n + 3)$ for all positive integers n was proven by Rosa in 1966 when he introduced Rosa and hooked Rosa sequences in [28]. A *Rosa sequence* (also known as *split-Skolem*) *of order n* is defined similarly to hooked Skolem sequences with the exception that the hook appears in position $n + 1$ of the sequence. For example, for $n = 4$ we have the sequence $(1, 1, 3, 4, 0, 3, 2, 4, 2)$. Rosa showed that these sequences exist if and only if $n \equiv 0, 3 \pmod 4$. A *hooked Rosa sequence* of order n has a hook in positions $n + 1$ and $2n + 1$ of the sequence. For example, for $n = 5$ we have the sequence $(3, 1, 1, 3, 4, 0, 5, 2, 4, 2, 0, 5)$. Rosa showed that these sequences exist if and only if $n \equiv 1, 2 \pmod 4$. Given a Rosa (or hooked Rosa) sequence of order n, Rosa used the pairs (a_i, b_i) for each i, where $1 \leq i \leq n$, which arise from the sequence in a similar fashion as Skolem sequences, to construct the triples $\{ i, a_i+n, b_i+n \}$, which form a solution to Heffter's second problem. He then used these triples to construct the blocks $\{ 0, i, b_i + n \}$ which, in addition to the triple $\{ 0, 2n + 1, 4n + 2 \}$, are the base blocks of a cyclic $STS(6n + 3)$.

These latter sequences introduced by Rosa, in addition to hooked Skolem sequences, are examples of extended Skolem sequences. *Extended Skolem sequences* may have a hook placed anywhere within the sequence. It was Abrham and Kotzig, in [1], who showed that extended Skolem sequences exist for all n, but it was Baker, in [4], who showed that these sequences exist for all admissible positions of the hook.

In addition to extended Skolem sequences, there have also been other generalizations of Skolem sequences introduced since 1957. The earliest such generalization was *Langford sequences*, first introduced by C.D. Langford in 1958 in [13], who came upon the problem after observing his son playing a game with blocks. In 1959, see [24], Priday defined a perfect Langford sequence as a sequence of $2(n - d + 1)$ positive integers, where each $k \in \{ d, d+1, \ldots, n \}$ is placed twice within the sequence such that the two k's have $k-1$ integers between them. In these sequences, n is the order of the sequence and d is the *defect*. For example, the sequence $(7, 5, 3, 6, 4, 3, 5, 7, 4, 6)$ is a Langford sequence of order 7 and defect 3. However, Langford only asked for a treatment of the problem when $d = 2$. Even though Langford gave several examples with $d = 2$, it was the work of Priday in [24] and Davies in [8] who solved the case completely for $d = 2$. The case for d in general was solved completely by Bermond, Brouwer, and Germa in [6], who completed the case for $d = 3$, $d = 4$, and $l \equiv 2d - 1 \pmod 4$, and Simpson in [35], who completed the case for $l \equiv 0 \pmod 4$, where $l = n - d + 1$.

An *extended Langford sequence of order n and defect d* is defined in a manner analogous to extended Skolem sequences. Although the problem of the existence of these sequences remains open, some cases have been solved by Jiang, Linek, and Mor in [15, 16], and hooked Langford sequences have been solved completely by Davies and Simpson (see [8, 35]). But extended Langford sequences have also been used by Priday in his concept of a looped set. He defined the set $\{ d, d+1, \dots n \}$ to be a *looped set* if there exist two extended Langford sequences of order n and defect d, one with a hook two places from the last entry of the sequence and one with a hook one place and two places from the last entry. For example, the set $\{2, 3, 4, 5\}$ gives the sequences $(2, 4, 2, 5, 3, 4, 0, 3, 5)$ and $(3, 5, 2, 3, 2, 4, 5, 0, 0, 4)$.

Another generalization, introduced by Stanton and Goulden in 1981 in [39], is *m-near-Skolem sequences of order n*, a Skolem-type sequence containing the integers $\{ 1, 2, \dots, m-1, m+1, \dots, n \}$. For example, the sequence $(4, 1, 1, 2, 4, 2)$ is a 3-near-Skolem sequence of order 4. These sequences, in addition to hooked m-near-Skolem sequences, were solved completely by Shalaby in [30]. The existence of extended m-near-Skolem sequences, which have the obvious definition, remains an open problem however.

Using notation similar to that introduced by Stanton and Goulden, Billington, in [7], introduced the pairings $P^2(1, n)/m - \{ j, k \}$, which form a partition of the integers $\{ 1, 2, \dots, 4n \}$, except j and k, into $2n - 1$ pairs such that each of the integers $\{ 1, 2, \dots, m-1, m+1, \dots, n \}$ appears exactly twice as a difference, while the integer m appears exactly once. Billington uses these partitions to construct balanced ternary designs.

A *balanced ternary design, BTD*, is a pair $(V, \mathcal{B})$, where V is a set of v elements and $\mathcal{B}$ is a collection of blocks of size k, such that each element in V occurs $0, 1$ or 2 times in each block. We define the index of the design as

$$\lambda = \sum_{m=1}^{|\mathcal{B}|} n_{im} n_{jm},$$

where $1 \leq i < j \leq v$ and n_{im} denotes the number of times the element i

occurs in block m.

In particular, Billington used the pairings of the form $P^2(1, m/2)/(m/2) - \{ m/2 + 1, 3m/2 + 1 \}$ to construct balanced ternary designs with block size 3 and index 2. She used these pairings to construct $m - 1$ triples of the form $\{ x_i, y_i, z_i \}$, where $x_i + y_i = z_i$, $x_{2i-1} = x_{2i} = i$, for $1 \leq i \leq m/2 - 1$, and $x_{m-1} = m/2$. She then used these triples to construct blocks of the form $\{ 0, x_i, z_i \}$ (mod $3m$), in addition to $\{ 0, 0, m \}$ (mod $3m$), which were base blocks of a BTD of order $V = 3m$ and index 2. For example, for $V = 12$, the pairing $P^2(1, 2)/2 - \{ 3, 7 \}$ can be used to construct the triples $\{ 1, 2, 3 \}$, $\{ 1, 5, 6 \}$, and $\{ 2, 3, 5 \}$ which gives base blocks $\{ 0, 0, 4 \}$, $\{ 0, 1, 3 \}$, $\{ 0, 1, 6 \}$ and $\{ 0, 2, 5 \}$ (mod 12).

Shortly thereafter λ-fold Skolem sequences and extended λ-fold Skolem sequences were introduced, and their existence solved, by Baker, Nowakowski, Shalaby, and Sharary in 1994 in [3]. The former sequences are similar to Skolem sequences with the exception that each integer from the set $\{ 1, 2, \ldots, n \}$ appears exactly 2λ times within the sequence. For example, the sequence $(3, 1, 1, 3, 2, 2, 2, 2, 3, 1, 1, 3)$ is a 2-fold Skolem sequence of order 3. *Extended λ-fold Skolem sequences of order n*, which are defined in the expected way, were also solved in [3].

Skolem sequences and their generalizations have numerous mathematical and practical applications. They are used in constructing group divisible designs, rotational triple systems, graph factorizations, and starters, to name just a few applications in combinatorial designs. They are also used in coding and communication networks.

Due to the importance of these sequences, in this thesis we deal with some other generalizations. In particular, looped Langford sequences and the work by Billington provide motivation for the sequences examined in this thesis. In chapter 3, we investigate m-near-λ-fold and extended m-near-λ-fold Skolem sequences. These sequences contain the integers $\{1, 2, \ldots, m-1, m+1, \ldots, n\}$ exactly 2λ times within the sequence, while they contain the integer m exactly $2\lambda - 2$ times. For example, the sequence $(4, 2, 3, 2, 4, 3, 1, 1, 4, 1, 1, 2, 4, 2)$ is a 3-near-2-fold Skolem sequence of order 4. We prove that the necessary conditions are also sufficient for the existence of these sequences when $\lambda > 1$, and discuss the case when $\lambda = 1$. In chapter 4, we prove that the necessary conditions are also sufficient for two new classes of near-Skolem-type sequences, namely $(2n - 3)$-extended m-near-Skolem sequences and hooked $(2n - 2)$-extended m-near-Skolem sequences, a near-Skolem-type sequence which contains a hook in positions $2n - 2$ and $2n - 1$ of the sequence.

Finally, in chapter 5, we discuss infinite Skolem sequences, another concept introduced by Skolem (see [36]). As mentioned earlier, Skolem sequences were first introduced in[36] when Skolem wondered if it was possible to partition the integers 1 to $2n$ into n pairs (a_i, b_i) such that each i, where $1 \leq i \leq n$, occurs as a difference exactly once amongst the pairs. However, in the same paper, he also considered the idea of partitioning the set $\mathbb{N}$ into similar pairs. He did this by setting $a_1 = 1, b_1 = 2$ and then setting a_i, $i \geq 2$, to be the least available number not used in the partition and setting $b_i = a_i + i$. The first 10 pairs of such a partition are

$$(1,2)\,(3,5)\,(4,7)\,(6,10)\,(8,13)\,(9,15)\,(11,18)(12,20)\,(14,23)\,(16,26).$$

(The first 60 values of a_n are contained in the appendix.) In the same paper, Skolem noticed that, in this partition, the pairs (a_n, b_n) are equivalent to the pairs $([\alpha n], [\alpha^2 n])$, for all $n \in \mathbb{N}$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $[x]$ denotes the greatest integer $\leq x$. Th. Bang, in [5], also elaborates on this idea. When the partition is written as a sequence we obtain

1,1,2,3,2,4,3,5,6,4,7,8,5,9,6,10,11,7,12,8,13,14,9,15,16,10...

This was written as a sequence by Roselle in [29]. Although Roselle referred to this sequence as an "infinite version" of the Skolem partitioning problem, we call this an *infinite Skolem sequence.* In chapter 5, we try to extend this idea to some other generalizations of Skolem sequences. Chapter 5 also contains a relationship between infinite Skolem sequences, Beatty sequences, Fibonacci numbers, and some restricted compositions and palindromes of n.

The final chapter contains the conclusion, which restates the results obtained in this thesis, as well as states some open problems and conjectures related to the topics discussed. The appendix contains computational results on near-Skolem-type sequences as well as a table of values of a_n in the infinite Skolem sequence.

Chapter 2

Definitions and theorems

This chapter contains some preliminary definitions, examples, and theorems which are relevant to the development of the topics and results presented in this thesis. It is noted that the majority of the definitions presented in this chapter are the formal definitions of what was already introduced in the introduction. It also contains definitions and examples of the two new sequences introduced in this thesis, m-near-λ-fold Skolem sequences and t-extended m-near-λ-fold Skolem sequences.

2.1 Definitions and theorems

Definition 1. A *Skolem sequence of order n* is a sequence $S_n = (s_1, s_2, \dots, s_{2n})$ of $2n$ positive integers such that the following conditions hold:

1. for each $k \in \{1, 2, \dots, n\}$, there exists exactly two elements $s_i, s_j \in S_n$ such that $s_i = s_j = k$, and
2. if $s_i = s_j = k$, then $|j - i| = k$.

The sequence $(4, 2, 3, 2, 4, 3, 1, 1)$ is an example of a Skolem sequence of order 4.

Theorem 2.1 (Skolem [36]). *A Skolem sequence of order n exists if and only if* $n \equiv 0, 1 \pmod 4$.

Definition 2. A *t-extended Skolem sequence of order n* is a sequence $S_n(t) = (s_1, s_2, \dots, s_{2n+1})$ of $2n + 1$ non-negative integers such that the following conditions hold:

1. for each $k \in \{1, 2, \dots, n\}$, there exists exactly two elements $s_i, s_j \in S_n(t)$ such that $s_i = s_j = k$,
2. if $s_i = s_j = k$, then $|j - i| = k$, and
3. $s_t = 0$, for some $t \in \{1, 2, \dots, 2n + 1\}$.

The sequence $(2, 0, 2, 3, 1, 1, 3)$ is an example of a 2-extended Skolem sequence of order 3.

A *hooked Skolem sequence of order n*, hS_n, is a $(2n)$-extended Skolem sequence of order n.

Theorem 2.2 (Abrham and Kotzig [1]). *An extended Skolem sequence of order n exists for all n.*

Since the construction used by Abrham and Kotzig to prove this theorem is relatively straightforward and is needed again in chapter 5, we will reiterate it here. The construction is as follows:

Given a positive integer n, we construct the extended sequence

$$2k+1, 2k-1, \dots, 5, 3, 1, 1, 3, 5, \dots, 2k-1, 2k+1, 2l, 2l-2, \dots, 4, 2, 0, 2, 4, \dots, 2l-2, 2l$$

where $2k + 1$ and $2l$ are the largest odd and even numbers, respectively, in the sequence.

This construction gives rise to the following definitions:

Definition 3. An *even Skolem sequence of order n* is the sequence $ES_n = (2n, 2n-2, \dots, 2, 0, 2, \dots, 2n-2, 2n)$.

For example, the sequence $(6, 4, 2, 0, 2, 4, 6)$ is the even Skolem sequence of order 3.

Definition 4. An *odd Skolem sequence of order n* is the sequence $OS_n = (2n-1, 2n-3, \dots, 1, 1, \dots, 2n-3, 2n-1)$.

For example, the sequence $(5, 3, 1, 1, 3, 5)$ is the odd Skolem sequence of order 3.

Theorem 2.3 (Baker [4]). *A t-extended Skolem sequence of order n exists if and only if one of the following is true:*

1. $n \equiv 0, 1 \pmod 4$ *and t is odd*
2. $n \equiv 2, 3 \pmod 4$ *and t is even.*

Definition 5. A *(p,q)-extended Rosa sequence of order n* is a sequence $R_n(p,q) = (r_1, r_2, \ldots, r_{2n+2})$ of $2n+2$ non-negative integers such that the following conditions hold:

1. for each $k \in \{1, 2, \ldots, n\}$, there exist exactly two elements $r_i, r_j \in R_n(p,q)$ such that $r_i = r_j = k$
2. if $r_i = r_j = k$, then $|\, j - i\,| = k$, and
3. $r_p = r_q = 0$, for some $p, q \in \{1, 2, \ldots, 2n+2\}$.

The sequence $(4,0,1,1,4,2,3,2,0,3)$ is an example of a $(2,9)$-extended Rosa sequence of order 4.

A *hooked t-extended Skolem sequence of order n*, $hS_n(t)$, is a $(2n+1,t)$-extended Rosa sequence of order n.

Theorem 2.4 (Linek and Shalaby [14]). *A (p,q)-extended Rosa sequence of order n exists for all admissible positions p,q of the hook if and only if one of the following holds:*

1. $n \equiv 0, 1 \pmod 4$ *and p and q are of opposite parity*

2. $n \equiv 2, 3 \pmod 4$ *and p and q are of the same parity.*

The cases $(n,p,q) = (1,2,3)$ and $(n,p,q) = (4,5,6)$ are exceptions.

For the following definitions and theorems concerning Langford sequences, we let $l = n - d + 1$ denote the length of the sequence.

Definition 6. A *Langford sequence of order n and defect d*, $n \geq d$, is a sequence $L_n^d = (l_1, l_2, \ldots, l_{2l})$ of $2l$ positive integers such that the following conditions hold:

1. for each $k \in \{d, d+1, \ldots, n\}$, there exist exactly two elements $l_i, l_j \in L_n^d$ such that $l_i = l_j = k$, and
2. if $l_i = l_j = k$, then $|j - i| = k$.

This sequence is also known as a *perfect Langford sequence.*

The sequence $(7,5,3,6,4,3,5,7,4,6)$ is an example of a (perfect) Langford sequence of order 7 and defect 3.

Theorem 2.5 (Davies, Bermond, Brouwer, Germa, and Simpson [8, 6, 35]). *A Langford sequence of length l and defect d exists if and only if the following conditions hold:*

1. $l \geq 2d - 1$, *and*

2. $l \equiv 0, 1 \pmod 4$ *for d odd or* $l \equiv 0, 3 \pmod 4$ *for d even.*

Definition 7. A *t-extended Langford sequence of order n and defect d*, $n \geq d$, is a sequence $L_n^d(t) = (l_1, l_2, \ldots, l_{2l+1})$ of $2l + 1$ non-negative integers such that the following conditions hold:

1. for each $k \in \{d, d + 1, \ldots, n\}$, there exist exactly two elements $l_i, l_j \in L_n^d(t)$ such that $l_i = l_j = k$,

2. if $l_i = l_j = k$, then $|j - i| = k$, and

3. $l_t = 0$ for some $t \in \{1, 2, \ldots, 2l + 1\}$.

The sequence $(2, 4, 2, 5, 3, 4, 0, 3, 5)$ is an example of a 7-extended Langford sequence of order 5 and defect 2.

A *hooked Langford sequence of order n and defect d* is a $(2l)$-extended Langford sequence of order n and defect d.

We note here that the existence of t-extended Langford sequences of order n and defect d is not completely solved. To see what cases are solved, the reader is referred to Theorem 2.8 and [15] and [16]. However, the existence of hooked Langford sequences has been completed by Davies (for $d = 2$) and Simpson (for general d).

Theorem 2.6 (Davies and Simpson [8, 35]). *A hooked Langford sequence of length l and defect d exists if and only if the following conditions hold:*

1. $l(l + 1 - 2d) \geq 0$, *and*

2. $l \equiv 2, 3 \pmod 4$ *for d odd or* $l \equiv 1, 2 \pmod 4$ *for d even.*

Definition 8. A *hooked t-extended Langford sequence of order n and defect d*, $n \geq d$, is a sequence $hL_n^d(t) = (l_1, l_2, \ldots, l_{2l+2})$ of $2l+2$ non-negative integers such that the following conditions hold:

1. for each $k \in \{d, d+1, \ldots, n\}$, there exist exactly two elements $l_i, l_j \in hL_n^d(t)$ such that $l_i = l_j = k$,
2. if $l_i = l_j = k$, then $|j - i| = k$, and
3. $l_{2l+1} = l_t = 0$, for some $t \in \{1, 2, \ldots, 2l, 2l+2\}$.

For example, the sequence $(5,3,4,0,3,5,4,2,0,2)$ is a hooked 4-extended Langford sequence of order 5 and defect $d = 2$.

Theorem 2.7 (Linek and Jiang [17]). *A hooked t-extended Langford sequence of order n and defect $d = 2$ exists if and only if one of the following conditions hold:*

1. $n \equiv 0, 1 \pmod 4$ *and t is even*

2. $n \equiv 2, 3 \pmod 4$ *and t is odd.*

The cases $(n, k) = (3, 3)$ and $(n, k) = (4, 2)$ are exceptions.

Definition 9. A *looped Langford set* is a pair of sequences $(\mathcal{L}_n^d, \mathcal{K}_n^d)$ of order n and defect $d, n \geq d, \mathcal{L}_n = (l_1, l_2, \ldots, l_{2l+1})$ and $\mathcal{K}_n = (k_1, k_2, \ldots, k_{2l+2})$, satisfying conditions $(1), (2)$ of a Langford sequence such that the following conditions hold:

1. for each $k \in \{d, d+1, \ldots, n\}$, there exist exactly two elements $l_i, l_j \in \mathcal{L}_n^d$ and $k_a, k_b \in \mathcal{K}_n^d$ such that $l_i = l_j = k_a = k_b = k$,
2. if $l_i = l_j = k_a = k_b = k$ then $|\,j - i\,| = |\,b - a\,| = k$, and
3. $l_{2l-1} = k_{2l} = k_{2l+1} = 0$.

If such a set exists, we refer to the two sequences collectively as a *looped Langford sequence of order n and defect d.*

For example, the set $\{2, 3, 4, 5\}$ gives the sequences $(2,4,2,5,3,4,0,3,5)$ and $(3,5,2,3,2,4,5,0,0,4)$, a looped Langford sequence of order 5 and defect 3.

Theorem 2.8 (Linek and Jiang [17]). *A k-extended Langford sequence of order n and defect $d = 2$ exists if and only if one of the following conditions hold:*

1. $n \equiv 0, 1 \pmod 4$ *and t is odd*
2. $n \equiv 2, 3 \pmod 4$ *and t is even.*

Corollary 2.1 (Shalaby and Stuckless [33]). *A looped Langford sequence of order n and defect $d = 2$ exists if and only if $n \equiv 0, 1 \pmod 4$.*

Proof. Follows from Theorems 2.7 and 2.8. □

Definition 10. An *m-near-Skolem sequence of order n and defect m* is a sequence $m - S_n = (s_1, s_2, \ldots, s_{2n-2})$ of $2n - 2$ positive integers such that the following conditions hold:

1. for each $k \in \{1, 2, \ldots, m-1, m+1, \ldots, n\}$, there exist exactly two elements $s_i, s_j \in m - S_n$ such that $s_i = s_j = k$, and
2. if $s_i = s_j = k$, then $|j - i| = k$.

The sequence $(4, 1, 1, 2, 4, 2)$ is an example of a 3-near-Skolem sequence of order 4.

Theorem 2.9 (Shalaby [30]). *An m-near-Skolem sequence of order n exists if and only if one of the following is true:*

1. $n \equiv 0, 1 \pmod 4$ *and m is odd*
2. $n \equiv 2, 3 \pmod 4$ *and m is even.*

We note that some computational results concerning $m - S_n$ are contained in the appendix.

Definition 11. A *t-extended m-near-Skolem sequence of order n and defect m* is a sequence $m-S_n(t) = (s_1, s_2, \ldots, s_{2n-1})$ of $2n-1$ non-negative integers such that the following conditions hold:

1. for each $k \in \{1, 2, \ldots, m-1, m+1, \ldots, n\}$, there exist exactly two elements $s_i, s_j \in m - S_n(t)$ such that $s_i = s_j = k$,
2. if $s_i = s_j = k$ then $|\, j - i \,| = k$, and
3. $s_t = 0$, for some $t \in \{1, 2, \ldots, 2n-2, 2n-1\}$.

The sequence $(4, 2, 0, 2, 4, 1, 1)$ is an example of a 3-extended 3-near-Skolem sequence of order 4.

A *hooked m-near-Skolem sequence of order n*, $h(m - S_n(t))$, is a $(2n-2)$-extended m-near-Skolem sequence of order n.

Although the existence of t-extended m-near-Skolem sequences is still an open question for general t, the case has been solved for $t = n$ (see [34]) and for $t = 2n-2$ (and $t = 2$) by Shalaby:

Theorem 2.10 (Shalaby [30]). *A hooked m-near-Skolem sequence of order n exists if and only if one of the following is true:*

1. $n \equiv 0, 1 \pmod 4$ *and m is even*
2. $n \equiv 2, 3 \pmod 4$ *and m is odd.*

We note that some computational results concerning $m-S_n(t)$ for general t and for $t = 2, 3, 2n-3, 2n-2$ are contained in the appendix.

Definition 12. A *(p,q)-extended m-near-Skolem sequence of order n and defect m* is a sequence $m - S_n(p,q) = (s_1, s_2, \ldots, s_{2n})$ of $2n$ non-negative integers such that the following conditions hold:

1. for each $k \in \{1, 2, \ldots, m-1, m+1, \ldots, n\}$, there exist exactly two elements $s_i, s_j \in m - S_n(p,q)$ such that $s_i = s_j = k$,
2. if $s_i = s_j = k$, then $|\, j - i \,| = k$, and
3. $s_p = s_q = 0$, for some $p, q \in \{1, 2, \ldots, 2n\}$.

For example, the sequence $(5,0,0,4,2,5,2,4,1,1)$ is a $(2,3)$-extended 3-near-Skolem sequence of order 5.

A *hooked t-extended m-near-Skolem sequence of order n and defect m*, $h(m - S_n(t))$, is $(2n-1,t)$-extended m-near-Skolem sequence of order n and defect m.

The existence of these sequences still remains an open problem. However, in this thesis we prove that the necessary conditions are also sufficient for the existence of hooked $(2n-2)$-extended m-near-Skolem sequences of order n.

Definition 13. A *λ-fold Skolem sequence of order n* is a sequence $S_n^\lambda = (s_1, s_2, \ldots, s_{2\lambda n})$ of $2\lambda n$ positive integers such that the following conditions hold:

1. for each $k \in \{1, 2, \ldots, n\}$, there exists exactly λ disjoint pairs (i_k, j_k), where $i_k, j_k \in \{1, 2, \ldots, 2\lambda n\}$, and
2. $s_{i_k},\ s_{j_k} \in S_n^\lambda$ and $s_{i_k} = s_{j_k} = k$.

The sequence $(3,1,1,3,2,2,2,2,3,1,1,3)$ is an example of a 2-fold Skolem sequence of order 3.

Theorem 2.11 (Baker, Nowakowski, Shalaby, and Sharay [3]). *A λ-fold Skolem sequence of order n exists if and only if one of the following conditions hold:*

1. $n \equiv 0, 1 \pmod 4$

2. $n \equiv 2, 3 \pmod 4$ *and λ is even.*

Definition 14. A *t-extended λ-fold Skolem sequence of order n* is a sequence $S_n^\lambda(t) = (s_1, s_2, \ldots, s_{2\lambda n+1})$ of $2\lambda n + 1$ non-negative integers such that the following conditions hold:

1. for each $k \in \{1, 2, \ldots, n\}$, there exists exactly λ disjoint pairs (i_k, j_k), where $i_k, j_k \in \{1, 2, \ldots, 2\lambda n\}$,
2. $s_{i_k},\ s_{j_k} \in S_n^\lambda(t)$ and $s_{i_k} = s_{j_k} = k$, and
3. $s_t = 0$, for some $t \in \{1, 2, \ldots, 2\lambda n + 1\}$.

The sequence $(3,1,1,3,2,2,2,2,3,1,1,3,2,0,2,3,1,1,3)$ is an example of a 14-extended 3-fold Skolem sequence of order 3.

A *hooked λ-fold Skolem sequence of order n* is a $(2\lambda n)$-extended λ-fold Skolem sequence of order n.

Theorem 2.12 (Baker, Nowakowski, Shalaby, and Sharay [3]). *A t-extended λ-fold Skolem sequence of order n exists for all admissible positions t of the hook if and only if one of the following hold*

1. $n \equiv 0, 1 \pmod 4$ *and t odd*
2. $n \equiv 2, 3 \pmod 4$ *and t and λ are of opposite parity.*

Definition 15. A *(p, q)-extended λ-fold Rosa sequence of order n* is a sequence $R_n^\lambda(p, q) = (r_1, r_2, \ldots, r_{2\lambda n+2})$ of $2\lambda n + 2$ non-negative integers such that the following conditions hold:

1. for each $k \in \{1, 2, \ldots, n\}$, there exists exactly λ disjoint pairs (i_k, j_k), where $i_k, j_k \in \{1, 2, \ldots, 2\lambda n\}$,
2. $r_{i_k}, r_{j_k} \in R_n^\lambda(p, q)$ and $r_{i_k} = r_{j_k} = k$, and
3. $r_p = r_q = 0$, for some $p, q \in \{1, 2, \ldots, 2\lambda n + 2\}$.

The sequence $(2,2,2,2,3,0,0,3,3,1,1,3,1,1)$ is an example of a $(6,7)$-extended 2-fold Skolem sequence of order 3.

Theorem 2.13 (Linek and Shalaby [14]). *A (p, q)-extended λ-fold Rosa sequence of order n exists for all admissible positions p, q of the hook if and only if one of the following holds:*

1. $n \equiv 0, 1 \pmod 4$ *and p and q are of opposite parity,*
2. $n \equiv 2, 3 \pmod 4$ *and p and q are of the same parity when λ odd,*
3. $n \equiv 2, 3 \pmod 4$ *and p and q are of opposite parity when λ even.*

We are now ready to introduce m-near-λ-fold Skolem sequences and t-extended m-near-λ-fold Skolem sequences.

Definition 16. An *m-near-λ-fold Skolem sequence of order n and defect m* is a sequence $m - S_n^\lambda = (s_1, s_2, \ldots, s_{2\lambda n-2})$ of $2\lambda n - 2$ positive integers such that the following hold:

1. for each $k \in \{1, 2, \ldots, m-1, m+1, \ldots, n\}$, there exists λ disjoint pairs $(i_k, j_k), i \in \{1, 2, \ldots, 2\lambda n - 2\}$ such that $s_{i_k}, s_{j_k} \in m - S_n^\lambda$ and $s_{i_k} = s_{j_k} = k$, and
2. there exists $\lambda - 1$ disjoint pairs (i_m, j_m) such that $s_{i_m}, s_{j_m} \in m - S_n^\lambda$ and $s_{i_m} = s_{j_m} = m$.

For example, $(4, 1, 1, 2, 4, 2, 4, 2, 3, 2, 4, 3, 1, 1)$ is a 3-near-2-fold Skolem sequence of order 4.

Definition 17. A *t-extended m-near-λ-fold Skolem sequence of order n and defect m* is a sequence $m - S_n^\lambda(t) = (s_1, s_2, \ldots, s_{2\lambda n-1})$ of $2\lambda n - 1$ non-negative integers such that the following conditions hold:

1. for each $k \in \{1, 2, \ldots, m-1, m+1, \ldots, n\}$, there exists λ disjoint pairs $(i_k, j_k), i \in \{1, 2, \ldots, 2\lambda n - 2\}$ such that $s_{i_k}, s_{j_k} \in m - S_n^\lambda$ and $s_{i_k} = s_{j_k} = k$,
2. there exist $\lambda - 1$ disjoint pairs (i_m, j_m) such that $s_{i_m}, s_{j_m} \in m - S_n^\lambda$ and $s_{i_m} = s_{j_m} = m$, and
3. $s_t = 0$, for some $t \in \{1, 2, \ldots, 2\lambda n - 1\}$.

For example, $(2, 4, 2, 3, 0, 4, 3, 1, 12, 4, 2, 1, 1, 4)$ is a 5-extended 3-near-2-fold Skolem sequence of order 4.

Definition 18. A *Beatty sequence* is a sequence $N_\mu = \{\, [\mu n] \,\}_{n=1}^{\infty}$, where μ is an irrational number and $[x]$ denotes the greatest integer $\leq x$.

Theorem 2.14 (Bang, [5]). *Let N_μ denote the Beatty sequence with μ as its irrational base. Then the sequences N_μ and N_ν are mutually disjoint with the property that $N_\mu \bigcup N_\nu = \mathbb{N}$ if and only if $\dfrac{1}{\mu} + \dfrac{1}{\nu} = 1$.*

Definition 19. Let m and l be arbitrary natural numbers. Then an $l, +m$ system is a system of disjoint pairs such that the corresponding differences are $l, l+m, l+2m, \ldots$.

For example, the pairs $(4, 6)$, $(3, 7)$, $(2, 8)$, and $(1, 9)$ constitute a $2, +2$ system.

Theorem 2.15 (Skolem, [36]). *Let m be an arbitrary natural number and $l \in \{1, \ldots, m\}$. Further, let N_1 be the set of integers of the form*

$$f(n) = \left[\frac{1}{2}\left(2 - m + \sqrt{m^2+4}\right)\left(n - \frac{m-l}{m}\right) + \frac{2(m-l)}{m}\right]$$

and N_2 the set of integers of the form

$$g(n) = \left[\frac{1}{2}\left(2 + m + \sqrt{m^2+4}\right)\left(n - \frac{m-l}{m}\right) + \frac{2(m-l)}{m}\right].$$

Then N_1 and N_2 are mutually disjoint sets with the property that $N_1 \bigcup N_2 = \mathbb{N}$, and the pairs $\big(f(n), g(n)\big)$ constitute an $l, +m$ system.

Definition 20. A *partition* of a positive integer n is a collection of **unordered** positive integers $(\lambda_1, \lambda_2, \ldots \lambda_r)$ such that $\sum_{i=1}^{r} \lambda_i = n$.

Each λ_i is called a part or summand of the partition. The case where there is only one summand, n itself, is also considered a partition of n.

For example, the collection $(1, 1, 2, 4)$ is a partition of 8.

Definition 21. A *composition* of a positive integer n is an **ordered** collection of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ such that $\sum_{i=1}^{r} \lambda_i = n$.

For example, for $n = 4$, the partitions are $(1,1,1,1)\,(1,1,2)\,(2,2)\,(1,3)\,(4)$ and the compositions are $(1,1,1,1)\,(1,1,2)\,(1,2,1)\,(2,1,1)(2,2)\,(1,3)\,(3,1)\,(4)$.

Lemma 2.1 (MacMahon, [18]). *The total number of compositions of n is equal to 2^{n-1}.*

Definition 22. A *palindrome* of a positive integer n is a composition of n such that the summands of the composition are the same when read from left to right as they are when read from right to left.

For example, for $n = 4$, the composition $(1, 2, 1)$ is also a palindrome.

Lemma 2.2 (MacMahon, [18]). *Let $N(n)$ denote the number of palindromes of n. Then $N(2k+1) = 2^k$ and $N(2k) = 2^k$.*

Definition 23. An odd summand composition (palindrome) of n is a composition (palindrome) of n containing only odd summands.

The composition $(1, 3, 1, 5)$ is an odd summand composition of 10.

We now present some theorems on certain restricted compositions and palindromes of n.

Theorem 2.16 (Grimaldi, [9]). *Let $C_n(O)$ denote the number of odd summand compositions of n. Then we have the formula*

$$C_n(O) = F_n,$$

where F_n is the n^{th} Fibonacci number.

Theorem 2.17 (Grimaldi, [9]). *Let $P_n(O)$ denote the number of odd summand palindromes of n. Then we have the formula*

$$P_n(O) = \begin{cases} F_{n/2} & \text{if } n \text{ even} \\ F_{(n+3)/2} & \text{if } n \text{ odd,} \end{cases}$$

where F_n is the n^{th} Fibonacci number.

The restricted compositions presented in the remaining theorems are self-explanatory.

Theorem 2.18 (Grimaldi, [10]). *Let $C_n(>1)$ denote the number of compositions of n without the summand 1. Then we have the formula*

$$C_n(>1) = F_{n-1},$$

where F_n is the n^{th} Fibonacci number.

Theorem 2.19 (Grimaldi, [10]). *Let $P_n(>1)$ denote the number of palindromes of n without the summand 1. Then we have the formula*

$$P_n(>1) = \begin{cases} F_{(n+2)/2} & \text{if } n \text{ even} \\ F_{(n-1)/2} & \text{if } n \text{ odd,} \end{cases}$$

where F_n is the n^{th} Fibonacci number.

Theorem 2.20 (Alladi and Hoggatt, [2]). *Let $C_n(1,2)$ denote the compositions of n containing only 1's and 2's. Then we have the formula*

$$C_n(1,2) = F_{n+1},$$

where F_n is the n^{th} Fibonacci number.

Theorem 2.21 (Alladi and Hoggatt, [2]). *Let $P_n(1,2)$ denote the palindromes of n containing only 1's and 2's. Then we have the formula*

$$P_n(1,2) = \begin{cases} F_{(n+4)/2} & \text{if } n \text{ even} \\ F_{(n-1)/2} & \text{if } n \text{ odd,} \end{cases}$$

where F_n is the n^{th} Fibonacci number.

2.2 Notation

We shall use the following notation throughout the remainder of the thesis:

S_n	=	Skolem sequence of order n
$S_n(t)$	=	t-extended Skolem sequence of order n
hS_n	=	hooked Skolem sequence of order n
$hS_n(t)$	=	hooked t-extended Skolem sequence of order n
$m - S_n$	=	m-near-Skolem sequence of order n
$m - S_n(t)$	=	t-extended m-near-Skolem sequence of order n
$h(m - S_n)$	=	hooked m-near-Skolem sequence of order n
$h(m - S_n(t))$	=	hooked t-extended m-near-Skolem sequence of order n
$R_n(p,q)$	=	(p,q)-extended Rosa sequence of order n
$R_n^\lambda(p,q)$	=	(p,q)-extended λ-fold Rosa sequence of order n
S_n^λ	=	λ-fold Skolem sequence of order n
$S_n^\lambda(t)$	=	t-extended λ-fold Skolem sequence of order n
$m - S_n^\lambda$	=	m-near-λ-fold Skolem sequence of order n
$m - S_n^\lambda(t)$	=	t-extended m-near-λ-fold Skolem sequence of order n
$m - S_n(p,q)$	=	(p,q)-extended m-near-Skolem sequence of order n
L_n^d	=	Langford sequence of order n and defect d
$L_n^d(t)$	=	t-extended Langford sequence of order n and defect d
hL_n^d	=	hooked Langford sequence of order n and defect d

$L_n^d(p,q)$	$=$	(p,q)-extended Langford sequence of order n and defect d
$(\mathcal{L}_n^d, \mathcal{K}_n^d)$	$=$	looped Langford sequence of order n and defect d
$C_n(O)$	$=$	number of odd summand compositions of n
$P_n(O)$	$=$	number of odd summand palindromes of n
$C_n(1,2)$	$=$	number of compositions of n containing only 1 and 2.
$P_n(1,2)$	$=$	number of palindromes of n containing only 1 and 2.
$C_n(>1)$	$=$	number of compositions of n without the summand 1
$P_n(>1)$	$=$	number of palindromes of n without the summand 1
F_n	$=$	the n^{th} Fibonacci number
N_μ	$=$	the Beatty sequence with irrational base μ

Chapter 3

Near-λ-fold and extended near-λ-fold Skolem sequences

In this chapter we show that the necessary conditions are also sufficient for the existence of m-near-λ-fold Skolem sequences and t-extended m-near-λ-fold Skolem sequences when $\lambda \geq 2$. (All results in this chapter can be found in [25].)

3.1 Near-λ-fold Skolem sequences

We first prove the necessary conditions for the sequence $m - S_n^\lambda$.

Theorem 3.1. *The sequence $m - S_n^\lambda$ exists only if one of the following conditions hold:*

1. $n \equiv 0, 1 \pmod 4$ *and* m *odd*

2. $n \equiv 2, 3 \pmod 4$ *and* m *and* λ *are of opposite parity.*

Proof. We use a proof similar to the one used in [3]. Let $m - S_n^\lambda = (s_1, s_2, \ldots, s_{2\lambda n-2})$ be an m-near-λ-fold Skolem sequence of order n and defect m. Let the $\lambda n - 1$

disjoint pairs be denoted by $\{(i_{jk}, i_{jk}+k) \mid 1 \le j \le \lambda,\ 1 \le k \le n\}$. Then

$$\sum_{\substack{k=1\\k\neq m}}^{n}\sum_{j=1}^{\lambda}(i_{jk}+k)+\sum_{j=1}^{\lambda-1}(i_{jm}+m)$$

$$= \frac{1}{2}\left[\sum_{\substack{k=1\\k\neq m}}^{n}\sum_{j=1}^{\lambda}\Big(i_{jk}+(i_{jk}+k)\Big)+\sum_{\substack{k=1\\k\neq m}}^{n}\sum_{j=1}^{\lambda}k+\sum_{j=1}^{\lambda-1}\Big(i_{jm}+(i_{jm}+m)\Big)+\sum_{j=1}^{\lambda-1}m\right]$$

$$= \frac{1}{2}\left(\sum_{i=1}^{2\lambda n-2}i+\Big(\lambda n(n+1)\Big)/2-\lambda m+(\lambda-1)m\right)$$

$$= \Big[(2\lambda n-1)(2\lambda n-2)+\lambda n(n+1)-2m\Big]/4.$$

This last expression must be an integer, and we see that this is only true when m, n, and λ satisfy the necessary conditions. □

In the proof of the following Theorem, we make use of a method incorporated in [24], where the author notes that two hooked sequences can be hooked together to form a sequence with no hooks. This is done by "hooking" the first sequence together with the reverse of the second. The last entry in the first sequence replaces the hook in the reverse of the second, and the first entry in the reverse of the second sequence replaces the hook in the first. For example, $(3,1,1,3,2,0,2)$ and $(1,1,2,0,2)$ can be hooked together to give $(3,1,1,3,2,2,2,2,1,1)$.

Theorem 3.2. *The sequence $m-S_n^\lambda$ exists if and only if one of the following conditions hold:*

1. $n \equiv 0,1 \pmod 4$ *and m odd*

2. $n \equiv 2,3 \pmod 4$ *and m and λ are of opposite parity.*

Proof. Necessity was shown in Theorem 3.2. The sufficient conditions for $\lambda = 1$ was done by Shalaby in [30]. For $\lambda > 1$, we consider three separate cases:

Case 1: Now we first consider the case for $n \equiv 0, 1 \pmod 4$ and m odd. We note that S_n and $m - S_n$ exist by Theorems 2.1 and 2.9, respectively. To construct $m - S_n^\lambda$, we simply concatenate $\lambda - 1$ copies of S_n and attach $m - S_n$ to the end.

Case 2: We now consider the case $n \equiv 2, 3 \pmod 4$, m odd and λ even. We first note that hS_n, $h(m - S_n)$ and S_n^2 exists by Theorems 2.3, 2.10 and 2.11, respectively. For $\lambda = 2$, we can hook together hS_n and $h(m - S_n)$, giving us $m - S_n^2$. For $\lambda > 2$, we can concatenate $\frac{\lambda-2}{2}$ copies of S_n^2 with $m - S_n^2$.

Case 3: We now consider the case $n \equiv 2, 3 \pmod 4$, m even and λ odd. We note that $m - S_n$ and S_n^2 exists by Theorems 2.9 and 2.11. We can string together $\frac{\lambda-1}{2}$ copies of S_n^2 and attach $m - S_n$ to the end. □

3.2 Extended near-λ-fold Skolem sequences

We now present our results on extended near-λ-fold Skolem sequences.

Theorem 3.3. *The sequence $m - S_n^\lambda(t)$ exists for all admissible positions t of the hook only if one of the following conditions hold:*

1. $n \equiv 0, 1 \pmod 4$ *and m and t have the same parity*
2. $n \equiv 2, 3 \pmod 4$, *λ even and m and t have the same parity*
3. $n \equiv 2, 3 \pmod 4$, *λ odd and m and t are of opposite parity.*

Proof. We use a proof similar to that of Theorem 3.2. Let $m - S_n^\lambda(t) = (s_1, s_2, \ldots, s_{2\lambda n-1})$ be a t-extended m-near-λ-fold Skolem sequence of order n and defect m. Let the $\lambda n - 1$ disjoint pairs be denoted by

$\{ (i_{jk}, i_{jk}+k) \mid 1 \le j \le \lambda,\ 1 \le k \le n \}$. Then

$$\begin{aligned}
&\sum_{\substack{k=1\\k\neq m}}^{n}\sum_{j=1}^{\lambda}(i_{jk}+k)+\sum_{j=1}^{\lambda-1}(i_{jm}+m)\\
=\ &\frac{1}{2}\left[\sum_{\substack{k=1\\k\neq m}}^{n}\sum_{j=1}^{\lambda}\Big(i_{jk}+(i_{jk}+k)\Big)+\sum_{\substack{k=1\\k\neq m}}^{n}\sum_{j=1}^{\lambda}k+\sum_{j=1}^{\lambda-1}\Big(i_{jm}+(i_{jm}+m)\Big)+\sum_{j=1}^{\lambda-1}m\right]\\
=\ &\frac{1}{2}\left[\sum_{i=1}^{2\lambda n-1} i-t+\Big(\lambda n(n+1)\Big)/2-\lambda m+(\lambda-1)m\right]\\
=\ &\Big[2\lambda n(2\lambda n-1)-2t+\lambda n(n+1)-2m\Big]/4.
\end{aligned}$$

This last expression must be an integer, and we see that this is only true when m, n, and λ satisfy the necessary conditions. □

Theorem 3.4. *For $\lambda \geq 2$, the sequence $m - S_n^{\lambda}(t)$ exists for all admissible positions t of the hook if and only if one of the following conditions hold:*

1. $n \equiv 0, 1 \pmod 4$ *and m and t have the same parity*
2. $n \equiv 2, 3 \pmod 4$, *λ even and m and t have the same parity*
3. $n \equiv 2, 3 \pmod 4$, *λ odd and m and t are of opposite parity.*

Proof. Necessity was done in Theorem 3.3. For sufficiency, we first note that the reverse of a t-extended m-near-λ-fold Skolem sequence of order n is a $(2\lambda n - t)$-extended m-near-λ-fold Skolem sequence of order n, so we only need to consider the cases when $t \leq \lambda n$. We shall also make use of the "hooking" method incorporated in the proof of Theorem ??. We now consider six separate cases:

Case 1: We first consider the case for $n \equiv 0, 1 \pmod 4$, m odd, and t odd. We note that S_n, $S_n(k)$ and $m - S_n$ exists for all k odd, $k \leq 2n+1$, by Theorems 2.1, 2.3 and 2.9, respectively. For $\lambda = 2$, we concatenate $S_n(t)$ with $m - S_n$ to give us $m - S_n^2(t)$. For example, for $n = 4$, $m = 3$, and $t = 5$, we have $S_4(5) = (2,4,2,3,0,4,3,1,1)$ and $3 - S_4 = (2,4,2,1,1,4)$ which, when concatenated, gives $3 - S_4^2(5) = (2,4,2,3,0,4,3,1,1;2,4,2,1,1,4)$[1]. For $\lambda > 2$, we construct $\lambda - 2$ copies of S_n and one copy each of $m - S_n$ and $S_n(k)$ such that $k = t - 2n\left\lfloor \frac{t}{2n} \right\rfloor$. We then concatenate $\left\lfloor \frac{t}{2n} \right\rfloor$ copies of S_n, followed by $m - S_n^2(k)$ and $\lambda - 2 - \left\lfloor \frac{t}{2n} \right\rfloor$ copies of S_n. For example, for $\lambda = 4$, $n = 4$, $m = 3$ and $t = 13$, we need $k = 13 - 8\left\lfloor \frac{13}{8} \right\rfloor = 5$. Hence we need 2 copies of $S_4 = (4,2,3,2,4,3,1,1)$ and one copy each of $S_4(5) = (2,4,2,3,0,4,3,1,1)$ and $3 - S_4 = (2,4,2,1,1,4)$, giving $3 - S_4^4(13) = (4,2,3,2,4,3,1,1;2,4,2,3,0,4,3,1,1;2,4,2,1,1,4;4,2,3,2,4,3,1,1)$.

Case 2: We have $n \equiv 0,1 \pmod 4$, m even, and t even. We note that S_n, $h(m - S_n)$ and $R_n(k, 2n+1)$ exist for all k even, $k \leq 2n+2$, by Theorems 2.1, 2.10 and 2.4, respectively. For $\lambda = 2$, we hook $R_n(t, 2n+1)$ together with $h(m - S_n)$ to give $m - S_n^2(t)$. For $\lambda > 2$, we construct $h(m - S_n)$, $\lambda - 2$ copies of S_n and one copy of $R_n(k, 2n+1)$ such that $k = t - 2n\left\lfloor \frac{t}{2n} \right\rfloor$. We then concatenate $\left\lfloor \frac{t}{2n} \right\rfloor$ copies of S_n with $m - S_n^2(k)$, followed by $\lambda - 2 - \left\lfloor \frac{t}{2n} \right\rfloor$ copies of S_n.

Case 3: We have $n \equiv 2,3 \pmod 4$, λ odd, m even, and t odd. We note that $m - S_n$ and $S_n^2(k)$ exists for k odd, $k \leq 4n+1$, by Theorems 2.9 and 2.12, respectively. For $\lambda = 3$, we concatenate $S_n^2(t)$ with $m - S_n$ to give $m - S_n^3(t)$. For $\lambda > 3$, we construct $m - S_n$, $\frac{\lambda-3}{2}$ copies of S_n^2 and one copy of $S_n^2(k)$ such that $k = t - 4n\left\lfloor \frac{t}{4n} \right\rfloor$. We then concatenate $\left\lfloor \frac{t}{4n} \right\rfloor$ copies of S_n^2 with $m - S_n^3(k)$, followed by $\frac{\lambda-3}{2} - \left\lfloor \frac{t}{4n} \right\rfloor$ copies of S_n^2.

Case 4: We have $n \equiv 2,3 \pmod 4$, λ odd, m odd, and t even. We first note that $h(m - S_n)$, S_n^2 and $R_n^2(k, 4n+1)$ exists for all k even, $k \leq 4n+2$, by Theorems 2.10, 2.11 and 2.13, respectively. For $\lambda = 3$, we hook $R_n^2(t, 4n+1)$ together with $h(m - S_n)$ to give $m - S_n^3(t)$. For $\lambda > 3$, we construct $h(m - S_n)$, $\frac{\lambda-3}{2}$ copies of S_n^2 and one copy of $R_n^2(k, 4n+1)$ such that $k = t - 4n\left\lfloor \frac{t}{4n} \right\rfloor$. We then concatenate $\left\lfloor \frac{t}{4n} \right\rfloor$ copies of S_n^2 with $m - S_n^3(k)$, followed by $\frac{\lambda-3}{2} - \left\lfloor \frac{t}{4n} \right\rfloor$ copies of S_n^2.

[1] the semicolon (;) within the final sequence represents the dividing line between the sequences used

Case 5: We have $n \equiv 2, 3 \pmod 4$, λ even, m odd, and t odd. We note that $h(m - S_n)$, $R_n(k, 2n+1)$ and S_n^2 exist for all k odd, $k \leq 2n+1$, by Theorems 2.10, 2.4 and 2.11, respectively. For $\lambda = 2$, we hook $R_n(t, 2n+1)$ together with $h(m - S_n)$ to give $m - S_n^2(t)$. For $\lambda > 2$, we construct $h(m - S_n)$, $\frac{\lambda-2}{2}$ copies
of S_n^2 and one copy of $R_n(k, 2n+1)$ such that $k = t - 4n \left\lfloor \frac{t}{4n} \right\rfloor$. We then concatenate $\left\lfloor \frac{t}{4n} \right\rfloor$ copies of S_n^2 with $m - S_n^2(k)$, followed by $\frac{\lambda-2}{2} - \left\lfloor \frac{t}{4n} \right\rfloor$ copies of S_n^2.

Case 6: We have $n \equiv 2, 3 \pmod 4$, λ even, m even, and t even. We note that $S_n(k)$, $m - S_n$ and S_n^2 exists for all k even, $k \leq 2n$, by Theorems 2.3, 2.9 and 2.11, respectively. For $\lambda = 2$, we concatenate $S_n(t)$ with $m - S_n$ to give us $m - S_n^2(t)$. For $\lambda > 2$, we construct $m - S_n$, $\frac{\lambda-2}{2}$ copies of S_n^2 and one copy of $S_n(k)$ such that $k = t - 4n \left\lfloor \frac{t}{4n} \right\rfloor$. We then concatenate $\left\lfloor \frac{t}{4n} \right\rfloor$ copies of S_n^2 with $m - S_n^2(k)$, followed by $\frac{\lambda-2}{2} - \left\lfloor \frac{t}{4n} \right\rfloor$ copies of S_n^2. □

Chapter 4

The existence of two new types of extended and hooked extended near-Skolem sequences

In this chapter, we consider examples of the sequences $m - S_n^\lambda(t)$ when $\lambda = 1$. In particular, we show that the necessary conditions are also sufficient for the existence of $(2n-3)$-extended m-near Skolem sequences of order n, $m - S_n(2n-3)$, and hooked $(2n-2)$-extended m-near-Skolem sequences of order n, $h(m - S_n(2n-2))$. We then present some constructions of other near-Skolem-type sequences using the latter two sequences. But first we present some simple lemmas which we need to prove the main theorems presented in this chapter. (All results in this chapter can be found in [26].)

Lemma 4.1. *If* $n \equiv 2, 7 \pmod 8$ *and* $n > 2$, *then* $4 - S_n(2n-3)$ *and* $h(4 - S_n(2n-2))$ *exist.*

Proof. Let $n = 8s + 2$. For $s = 1$ we have the sequences

$$h(4 - S_{10}(18)) = (8,6,2,9,2,5,7,6,8,10,5,3,9,7,3,1,1,0,0,10),$$
$$4 - S_{10}(17) = (3,10,8,3,5,1,1,7,9,5,8,10,6,2,7,2,0,9,6).$$

So assume $s \geq 2$. By Theorem 2.6, we can construct hL_n^5. We can then hook this sequence together with the sequences $(2,0,2,1,1,3,0,0,3)$ and $(3,0,2,3,2,0,1,1)$ to form $h(4 - S_n(2n-2))$ and $4 - S_n(2n-3)$, respectively.

Similarly, we can construct $4-S_n(2n-3)$ and $h(4-S_n(2n-2))$ for $n=8s+7$, $s\geq 1$. We simply construct hL_n^5, which exists by Theorem 2.6, and hook this sequence together with the sequences $(2,0,2,1,1,3,0,0,3)$ and $(3,0,2,3,2,0,1,1)$. For $n=7$, we have the sequences

$$4-S_7(11)=(6,7,3,1,1,3,6,5,7,2,0,2,5),$$
$$h(4-S_7(12))=(2,7,2,1,1,5,3,6,7,3,5,0,0,6).$$

□

Lemma 4.2. *If $n=8s$, then $m-S_n(2n-3)$ and $h(m-S_n(2n-2))$ exist, where $m=2t-1$, for all $1\leq t\leq s$.*

Proof. Let $n=8s$, where $s\geq 1$. For $s=t=m=1$, we can construct a $(\mathcal{L}_n^2,\mathcal{K}_n^2)$. So we can assume $s\geq t\geq 2$. We distinguish between two separate cases:

Case 1: We first consider the case $t\equiv 0 \pmod 2$. By Theorems 2.4 and 2.5, we can construct the sequences L_n^{2t+1} and $hS_{2t-2}(2t-1)$. We then place the difference $2t$ in position s_{2t-1} of $hS_{2t-2}(2t-1)$ and also at the end. Appending this newly constructed sequence to the end of L_n^{2t+1} gives $m-S_n(2n-3)$.

Next, we construct $S_{2t-2}(2t)$, which exists by Theorem 2.3. We then replace the hook in this sequence with the difference $2t$ and append $0,0,2t$ to the end. Appending this newly constructed sequence to the end of L_n^{2t+1} gives $h(m-S_n(2n-2))$.

Case 2: We now consider the case $t\equiv 1 \pmod 2$. By Theorems 2.4 and 2.5, we can construct the sequences L_n^{2t} and $hS_{2t-2}(4t-4)$. Appending $hS_{2t-2}(4t-4)$ to the end of L_n^{2t} gives $h(m-S_n(2n-2))$.

Next, we construct $S_{2t-3}(4t-5)$, which exist by Theorem 2.3. Appending $S_{2t-3}(4t-5)$ to the end of L_n^{2t} gives $m-S_n(2n-3)$. □

Lemma 4.3. *For all admissible n, the sequences $m-S_n(2n-3)$ and $h(m-S_n(2n-2))$ exist, where $m=1$ or $m=n$.*

Proof. We first note that the sequence $m-S_n(2n-3)$ does not exist for $n=1$ and $(n,m)=(3,2)$, and the sequence $h(m-S_n(2n-2))$ does not exist for $n=1$ and $n=2$. For $n\equiv 0,1 \pmod 4$, $n\geq 4$ and $m=1$, we have $(\mathcal{L}_n^2,\mathcal{K}_n^2)$. For $n\equiv 1,2 \pmod 4$, $n\geq 5$, and $m=n$, we have L_{n-1}^2 with $1,1$ appended to the end. □

4.1 $(2n-3)$-extended near-Skolem sequences

We are now ready to present our results on $(2n-3)$-extended near-Skolem sequences.

Theorem 4.1. *The sequence $m - S_n(2n-3)$ exists only if one of the following is true:*

1. $n \equiv 0, 1 \pmod 4$ *and m is odd*

2. $n \equiv 2, 3 \pmod 4$ *and m is even.*

Proof. Let $m - S_n(2n-3) = (s_1, s_2, \ldots s_{2n-1})$ be the sequence in question. For each $k \in \{1, 2, \ldots, m-1, m+1, \ldots, n\}$ let the ordered pairs (i_k, j_k) be the subscripts of s_{i_k} and s_{j_k} when $s_{i_k} = s_{j_k} = k$. Then

(a)

$$\sum_{\substack{k=1,\\ k\neq m}}^{n} (i_k + j_k) = \frac{(2n)(2n-1)}{2} - (2n-3) = 2n^2 - 3n + 3, \text{ and}$$

(b)

$$\sum_{\substack{k=1,\\ k\neq m}}^{n} (j_k - i_k) = \frac{(n)(n+1)}{2} - m.$$

Adding (a) and (b) together gives us

$$\sum_{\substack{k=1,\\ k\neq m}}^{n} j_k = \frac{5n^2 - 5n - 2m + 6}{4}.$$

Since the left hand side of the equation must be an integer, the number $(5n^2 - 5n - 2m + 6)$ must be divisible by 4. When we solve for n and m, we obtain the necessary conditions. □

Theorem 4.2. *The sequence $m - S_n(2n-3)$ exists if and only if one of the following is true:*

1. $n \equiv 0, 1 (mod\ 4)$ *and m is odd*

2. $n \equiv 2, 3 (mod\ 4)$ *and m is even.*

The cases $n = 1$ and $(n, m) = (3, 2)$ are exceptions.

Proof. Necessity was shown in Theorem 4.1. For sufficiency, we first look at some cases with small m. For $n \equiv 0,1 \pmod 4$ and $m = 1$, see Lemma 4.3. For $n \equiv 1,2 \pmod 4$ and $m = n$, see Lemma 4.3. For $n \equiv 2,7 \pmod 8$ and $m = 4$, see Theorem 4.1. For $n \equiv 2,3 \pmod 4), n \geq 7$ and $m = 2$, we have L_n^3, which exists by Theorem 2.5, with $0,1,1$ appended to the end. For the remaining n and m, we distinguish eight cases. In each case, the solution is given in the form of a table, where the columns i, j denote the first and second appearance, respectively, of the difference k.

Case 1: $n \equiv 0 \pmod 8$.
Let $n = 8s, m = 2t+1$. For $m \leq 2s-1$, see Theorem 4.2. For $n > m > 2s-1$ and $n > 8$, the solution is given by the following table (ignore the lines * when $s = 2$).

i	j	k	
$1+r$	$8s-r$	$8s-2r-1$	$0 \leq r \leq 4s-t-2$
$4s-t+r$	$4s+t-r-1$	$2t-2r-1$	$0 \leq r \leq t-s-1$
$4s+t$	$4s+t+1$	1	......
$3s+r$	$11s-r$	$8s-2r$	$0 \leq r \leq 2s-1$
$8s+r+1$	$12s-r+1$	$4s-2r$	$0 \leq r \leq s-1$
$11s+1$	$13s$	$2s-1$	......
$14s-1$	$16s-1$	$2s$	......
$12s+r+2$	$14s-r-2$	$2s-4-2r$	$0 \leq r \leq s-4$
$*13s-1$	$15s-3$	$2s-2$	......
$*13s+1$	$15s-2$	$2s-3$	......
$14s+r$	$16s-r-5$	$2s-5-r$	$0 \leq r \leq s-4$
$16s-4$	$16s-2$	2	......

To complete the proof, we list below the sequence $m - S_8(2n-3)$ and all required defects:
For $n = 8$ and $m = 3,5,7$:

$(7,5,6,1,1,8,5,7,6,2,4,2,0,8,4)$
$(7,3,6,2,3,2,8,7,6,4,1,1,0,4,8)$
$(5,3,8,6,3,5,2,4,2,6,8,4,0,1,1)$

Case 2: $n \equiv 1 \pmod 8$.
Let $n = 8s+1, m = 2t+1$. For $n > m > 1$, the solution is given by the

following table (ignore the line * when $s = 1$).

i	j	k	
$1 + r$	$8s - r + 2$	$8s - 2r + 1$	$0 \le r \le 4s - t - 1$
$4s - t + r + 1$	$4s + t - r$	$2t - 2r - 1$	$0 \le r \le t - 2$
$4s + t + 1$	$4s + t + 2$	1	......
$4s + r$	$12s - r$	$8s - 2r$	$0 \le r \le 1$
$8s + 2r + 4$	$16s - 2r$	$8s - 4r - 4$	$0 \le r \le s - 2$
$*8s + 2r + 3$	$16s - 2r - 3$	$8s - 4r - 6$	$0 \le r \le s - 1$
$10s + r + 2$	$14s - r - 2$	$4s - 2r - 4$	$0 \le r \le 2s - 4$
$12s + 1$	$16s + 1$	$4s$	......
$14s$	$14s + 2$	2	......

Case 3: $n \equiv 2 \pmod 8$.
Let $n = 8s + 2, m = 2t$. For $n \ge m > 4$ and $n > 10$, the solution is given by the following table (ignore the line * when $s = 2$).

i	j	k	
$1 + r$	$8s - r + 3$	$8s - 2r + 2$	$0 \le r \le 4s - t$
$4s - t + r + 2$	$4s + t - r$	$2t - 2r - 2$	$0 \le r \le t - 4$
$4s + t + 1$	$4s + t + 2$	1	......
$4s + r - 1$	$12s - r$	$8s - 2r + 1$	$0 \le r \le 4$
$8s + 2r + 5$	$16s - 2r - 4$	$8s - 4r - 9$	$0 \le r \le s - 3$
$*8s + 2r + 4$	$16s - 2r - 7$	$8s - 4r - 11$	$0 \le r \le s - 2$
$10s + r + 1$	$14s - r - 2$	$4s - 2r - 3$	$0 \le r \le 2s - 5$
$12s + 1$	$16s$	$4s - 1$	......
$16s - 3$	$16s + 2$	5	......
$16s - 1$	$16s + 3$	4	......
$16s - 5$	$16s - 2$	3	......
$14s - 2$	$14s$	2	......

To complete the proof, we list below the sequences $m - S_2(2n - 3)$ and $m - S_{10}(2n - 3)$ and all remaining defects:

For $n = 2$: $(0, 1, 1)$

For $n = 10$ and $m = 6, 8$:
$(3, 10, 6, 3, 1, 1, 4, 7, 6, 9, 4, 10, 5, 2, 7, 2, 0, 5, 9)$
$(3, 10, 8, 3, 9, 1, 1, 2, 7, 2, 8, 10, 5, 9, 4, 7, 0, 5, 4)$

Case 4:$n \equiv 3 \pmod 8$.
Let $n = 8s+3, m = 2t$. For $n \geq m > 2$ and $n > 11$, the solution is given by the following table.

i	j	k	
$1+r$	$8s-r+3$	$8s-2r+2$	$0 \leq r \leq 4s-t$
$4s-t+r+2$	$4s+t-r$	$2t-2r-2$	$0 \leq r \leq t-3$
$4s+t+1$	$4s+t+2$	1	......
$4s+r$	$12s-r+3$	$8s-2r+3$	$0 \leq r \leq 2$
$8s+2r+5$	$16s-2r+2$	$8s-4r-3$	$0 \leq r \leq s-3$
$8s+2r+4$	$16s-2r-1$	$8s-4r-5$	$0 \leq r \leq s-2$
$10s+r+1$	$14s-r+2$	$4s-2r+1$	$0 \leq r \leq 2s-2$
$12s$	$16s+5$	$4s+5$	......
$16s+1$	$16s+4$	3	......
$14s+4$	$14s+6$	2	......

To complete the proof, we list below the sequence $m - S_{11}(2n-3)$ and all required defects:
For $n = 11$ and $m = 4, 6, 8, 10$

$(3,7,10,3,1,1,8,6,7,11,9,5,10,6,8,2,5,2,0,9,11)$
$(11,9,7,5,3,10,8,3,5,7,9,11,1,1,8,10,4,2,0,2,4)$
$(11,9,7,5,3,10,6,3,5,7,9,11,6,1,1,10,4,2,0,2,4)$
$(11,9,7,5,3,8,6,3,5,7,9,11,6,8,1,1,4,2,0,2,4)$

Case 5: $n \equiv 4 \pmod 8$.
Let $n = 8s+4, m = 2t+1$. For $n \geq m > 1$ and $n > 4$, the solution is given by the following table.

i	j	k	
$1+r$	$8s-r+4$	$8s-2r+3$	$0 \le r \le 4s-t$
$4s-t+r+2$	$4s+t-r+1$	$2t-2r-1$	$0 \le r \le t-2$
$4s+t+2$	$4s+t+3$	1	......
$4s+r+1$	$12s-r+5$	$8s-2r+4$	$0 \le r \le 1$
$8s+2r+6$	$16s-2r+6$	$8s-4r$	$0 \le r \le s-2$
$8s+2r+5$	$16s-2r+3$	$8s-4r-2$	$0 \le r \le s-1$
$10s+r+4$	$14s-r+4$	$4s-2r$	$0 \le r \le 2s-2$
$12s+3$	$16s+7$	$4s+4$	......
$14s+6$	$14s+8$	2	......

To complete the proof, we list below a sequence $3-S_4(2n-3)$:

$(1,1,4,2,0,2,4)$

Case 6: $n \equiv 5 \pmod 8$.
Let $n = 8s+5, m = 2t+1$. For $n \ge 13$ and $m = 3$, we have L_n^5, which exists by Theorem 2.5, with $1,1,4,2,0,2,4$ appended to the end. For $n \ge 16$ and $m = 5$, we have L_n^6 with $3,1,1,3,4,2,0,2,4$ appended to the end. (Both L_n^5 and L_n^6 exist by Theorem 2.5.) For $n \ge m > 5$ and $n > 13$, the solution is given by the following table (ignore the line * when $s = 2$).

i	j	k	
$1+r$	$8s-r+6$	$8s-2r+5$	$0 \le r \le 4s-t+1$
$4s-t+r+3$	$4s+t-r+2$	$2t-2r-1$	$0 \le r \le t-4$
$4s+t+3$	$4s+t+4$	1	......
$4s+r$	$12s-r+4$	$8s-2r+4$	$0 \le r \le 5$
$8s+2r+8$	$16s-2r$	$8s-4r-8$	$0 \le r \le s-3$
$*8s+2r+7$	$16s-2r-3$	$8s-4r-10$	$0 \le r \le s-2$
$10s+r+4$	$14s-r$	$4s-2r-4$	$0 \le r \le 2s-6$
$12s+5$	$16s+5$	$4s$	......
$16s+3$	$16s+9$	6	......
$16s+1$	$16s+6$	5	......
$16s+4$	$16s+8$	4	......
$16s-1$	$16s+2$	3	......
$14s+2$	$14s+4$	2	......

To complete the proof, we list below the sequences $m-S_5(2n-3)$ and

$m - S_{13}(2n-3)$ and all remaining defects:
For $n = 5$ and $m = 3$: $(4,1,1,5,4,2,0,2,5)$

For $n = 13$ and $m = 5,7,9,11$:

$(12,10,8,6,13,3,9,11,3,6,8,10,12,1,1,9,7,13,11,2,4,2,0,7,4)$
$(12,10,8,6,13,3,9,11,3,6,8,10,12,1,1,9,4,13,11,5,4,2,0,2,5)$
$(12,10,8,6,13,3,7,11,3,6,8,10,12,7,5,1,1,13,11,5,4,2,0,2,4)$
$(12,10,8,6,13,3,7,9,3,6,8,10,12,7,4,5,9,13,4,2,5,2,0,1,1)$

Case 7: $n \equiv 6 \pmod 8$.
Let $n = 8s+6, m = 2t$. For $n \geq m > 2$ and $n > 6$, the solution is given by the following table.

i	j	k	
$1+r$	$8s-r+7$	$8s-2r+6$	$0 \leq r \leq 4s-t+2$
$4s-t+r+4$	$4s+t-r+2$	$2t-2r-2$	$0 \leq r \leq t-3$
$4s+t+3$	$4s+t+4$	1	......
$4s+r+2$	$12s-r+7$	$8s-2r+5$	$0 \leq r \leq 2$
$8s+2r+9$	$16s-2r+8$	$8s-4r-1$	$0 \leq r \leq s-2$
$8s+2r+8$	$16s-2r+5$	$8s-4r-3$	$0 \leq r \leq s-1$
$10s+r+7$	$14s-r+6$	$4s-2r-1$	$0 \leq r \leq 2s-3$
$12s+8$	$16s+11$	$4s+3$	......
$16s+7$	$16s+10$	3	......
$14s+8$	$14s+10$	2	......

To complete the proof, we list below the sequence $m - S_6(2n-3)$ and all required defects:
For $n = 6$ and $m = 2,4$:

$(6,4,1,1,5,4,6,3,0,5,3)$
$(6,3,1,1,3,5,6,2,0,2,5)$

Case 8: $n \equiv 7 \pmod 8$.
Let $n = 8s+7, m = 2t$. For $n \geq m > 4$ and $n > 7$, the solution is given by the following table.

i	j	k	
$1+r$	$8s-r+7$	$8s-2r+6$	$0 \le r \le 4s-t+2$
$4s-t+r+4$	$4s+t-r+2$	$2t-2r-2$	$0 \le r \le t-4$
$4s+t+3$	$4s+t+4$	1	......
$4s+r+1$	$12s-r+8$	$8s-2r+7$	$0 \le r \le 4$
$8s+2r+9$	$16s-2r+6$	$8s-4r-3$	$0 \le r \le s-3$
$8s+2r+8$	$16s-2r+3$	$8s-4r-5$	$0 \le r \le s-2$
$10s+r+5$	$14s-r+6$	$4s-2r+1$	$0 \le r \le 2s-3$
$12s+3$	$16s+8$	$4s+5$	......
$16s+7$	$16s+12$	5	......
$16s+5$	$16s+9$	4	......
$16s+10$	$16s+13$	3	......
$14s+8$	$14s+10$	2	......

To complete the proof, we list below a sequence $6 - S_7(2n-3)$:

$$(7,5,3,1,1,3,5,7,4,2,0,2,4)$$

□

4.2 Hooked $(2n-2)$-extended near-Skolem sequences

In this section we present our results on hooked $(2n-2)$-extended near-Skolem sequences.

Theorem 4.3. *The sequence $h(m - S_n(2n-2))$ exists only if one of the following conditions holds:*

1. *$n \equiv 0, 1 \pmod 4$ and m is odd*
2. *$n \equiv 2, 3 \pmod 4$ and m is even.*

Proof. Let $h(m - S_n(2n-2)) = (s_1, s_2, \ldots s_{2n})$ be the sequence in question. For each $k \in \{1, 2, \ldots, m-1, m+1, \ldots, n\}$, let the ordered pairs (i_k, j_k) be the subscripts of s_{i_k} and s_{j_k} when $s_{i_k} = s_{j_k} = k$. Then

(a')

$$\sum_{\substack{k=1,\\k\neq m}}^{n}(i_k+j_k)=\frac{(2n)(2n+1)}{2}-(2n-1)-(2n-2)=2n^2-3n+3, \text{ and}$$

(b')

$$\sum_{\substack{k=1,\\k\neq m}}^{n}(j_k-i_k)=\frac{(n)(n+1)}{2}-m.$$

Adding (a') and (b') together gives us

$$\sum_{\substack{k=1,\\k\neq m}}^{n}j_k=\frac{5n^2-5n-2m+6}{4}.$$

Since the left hand side of the equation must be an integer, the number $(5n^2-5n-2m+6)$ must be divisible by 4. When we solve for n and m, we obtain the necessary conditions. □

Theorem 4.4. *The sequence $h(m-S_n(2n-2))$ exists if and only if one of the following is true:*

1. *$n\equiv 0,1 (mod\ 4)$ and m is odd*
2. *$n\equiv 2,3 (mod\ 4)$ and m is even.*

The cases $n=1$ and 2 are exceptions.

Proof. Necessity was shown in Theorem 4.3. For sufficiency, we first look at some cases with small m. For $n\equiv 0,1 \pmod 4$ and $m=1$, see Lemma 4.3. For $n\equiv 1,2 \pmod 4$ and $m=n$, see Lemma 4.3. For $n\equiv 2,7 \pmod 8$ and $m=4$, see Theorem 4.1. For $n\equiv 2,3 \pmod 4), n\geq 10$ and $m=2$, we have L_n^4, which exists by Theorem 2.5, with $1,1,3,0,0,3$ appended to the end. For the remaining n and m, we distinguish eight cases. In each case, the solution is given in the form of a table, where the columns i,j denote the first and second appearance, respectively, of the difference k.

Case 1:$n\equiv 0 \pmod 8$.
Let $n=8s, m=2t+1$. For $m\leq 2s-1$, see Theorem 4.2. For $n>m>2s-1$

and $n > 8$, the solution is given by the following table.

i	j	k	
$1+r$	$8s-r$	$8s-2r-1$	$0 \le r \le 4s-t-2$
$4s-t+r$	$4s+t-r-1$	$2t-2r-1$	$0 \le r \le t-s-1$
$4s+t$	$4s+t+1$	1	......
$3s+r$	$11s-r$	$8s-2r$	$0 \le r \le 2s-1$
$8s+r+1$	$12s-r+1$	$4s-2r$	$0 \le r \le s-1$
$11s+1$	$13s$	$2s-1$	......
$12s+r+2$	$14s-r-1$	$2s-2r-3$	$0 \le r \le s-3$
$13s+1$	$15s-1$	$2s-2$	......
$14s$	$16s$	$2s$	......
$14s+r+1$	$16s-r-3$	$2s-2r-4$	$0 \le r \le s-3$

To complete the proof, we list below the sequence $h(m - S_8(2n-2))$ and all required defects:
For $n = 8$ and $m = 3, 5, 7$:

$(6, 1, 1, 4, 8, 5, 6, 4, 7, 2, 5, 2, 8, 0, 0, 7)$
$(6, 4, 1, 1, 8, 4, 6, 3, 7, 2, 3, 2, 8, 0, 0, 7)$
$(6, 1, 1, 4, 8, 3, 6, 4, 3, 2, 5, 2, 8, 0, 0, 5)$

Case 2: $n \equiv 1 \pmod 8$.
Let $n = 8s+1, m = 2t+1$. For $n > m > 1$, the solution is given by the following table.

i	j	k	
$1+r$	$8s-r+2$	$8s-2r+1$	$0 \le r \le 4s-t-1$
$4s-t+r+1$	$4s+t-r$	$2t-2r-1$	$0 \le r \le t-2$
$4s+t+1$	$4s+t+2$	1	......
$4s+r$	$12s-r$	$8s-2r$	$0 \le r \le 1$
$8s+r+3$	$16s-r-1$	$8s-2r-4$	$0 \le r \le 2s-3$
$12s+2$	$16s+2$	$4s$	......
$10s+2r+2$	$14s-2r$	$4s-2-4r$	$0 \le r \le s-2$
$10s+2r+1$	$14s-2r-3$	$4s-4-4r$	$0 \le r \le s-2$
$14s-1$	$14s+1$	2	......

Case 3: $n \equiv 2 (mod\ 8)$.
Let $n = 8s + 2, m = 2t$. For $n > m > 4$ and $n > 10$, the solution is given by the following table (ignore the line * when $s = 2$).

i	j	k	
$1 + r$	$8s - r + 3$	$8s - 2r + 2$	$0 \leq r \leq 4s - t$
$4s - t + r + 2$	$4s + t - r$	$2t - 2r - 2$	$0 \leq r \leq t - 4$
$4s + t + 1$	$4s + t + 2$	1	......
$4s + r - 1$	$12s - r$	$8s - 2r + 1$	$0 \leq r \leq 4$
$8s + 2r + 5$	$16s - 2r - 4$	$8s - 4r - 9$	$0 \leq r \leq s - 3$
$*8s + 2r + 4$	$16s - 2r - 7$	$8s - 4r - 11$	$0 \leq r \leq s - 2$
$12s + 1$	$16s$	$4s - 1$	......
$10s + r + 1$	$14s - r - 4$	$4s - 2r - 5$	$0 \leq r \leq 2s - 6$
$16s - 1$	$16s + 4$	5	......
$16s - 3$	$16s + 1$	4	......
$16s - 5$	$16s - 2$	3	......
$14s - 2$	$14s$	2	......

To complete the proof, we list below the sequence $h(m - S_{10}(2n-2))$ and all required defects:
For $n = 10$ and $m = 6, 8$:

$$(8,4,2,9,2,4,5,7,8,10,3,5,9,3,7,1,1,0,0,10)$$
$$(6,4,2,9,2,4,6,3,7,10,3,5,9,1,1,7,5,0,0,10)$$

Case 4: $n \equiv 3$ (mod 8).
Let $n = 8s + 3, m = 2t$. For $n \geq m > 2$ and $n > 3$, the solution is given by the following table (ignore the line * when $s = 1$).

i	j	k	
$1+r$	$8s-r+3$	$8s-2r+2$	$0 \le r \le 4s-t$
$4s-t+r+2$	$4s+t-r$	$2t-2r-2$	$0 \le r \le t-3$
$4s+t+1$	$4s+t+2$	1	......
$4s+r$	$12s-r+3$	$8s-2r+3$	$0 \le r \le 2$
$8s+2r+5$	$16s-2r+2$	$8s-4r-3$	$0 \le r \le s-2$
$*8s+2r+4$	$16s-2r-1$	$8s-5-4r$	$0 \le r \le s-1$
$10s+r+3$	$14s-r$	$4s-3-2r$	$0 \le r \le 2s-4$
$12s$	$16s+1$	$4s+1$	......
$16s+3$	$16s+6$	3	......
$14s+2$	$14s+4$	2	......

To complete the proof, we list a sequence for $h(2\text{–}S_3(2n-2))$: $(1,1,3,0,0,3)$

Case 5: $n \equiv 4 \pmod 8$.
Let $n = 8s+4, m = 2t+1$. For $n \ge 11, m = 3$, we have L_n^5, which exists by Theorem 2.5, with $1,1,2,4,2,0,0,4$ appended to the end. For $n \ge m > 3$ and $n > 4$, the solution is given by the following table (ignore the lines * when $s = 1$).

i	j	k	
$1+r$	$8s-r+4$	$8s-2r+3$	$0 \le r \le 4s-t$
$4s-t+r+2$	$4s+t-r+1$	$2t-2r-1$	$0 \le r \le t-3$
$4s+t+2$	$4s+t+3$	1	......
$4s+r$	$12s-r+4$	$8s-2r+4$	$0 \le r \le 3$
$*8s+2r+6$	$16s-2r$	$8s-4r-6$	$0 \le r \le s-1$
$8s+2r+7$	$16s-2r+3$	$8s-4r-4$	$0 \le r \le s-2$
$10s+r+5$	$14s-r+1$	$4s-2r-4$	$0 \le r \le 2s-5$
$*8s+5$	$12s+5$	$4s$	......
$16s+4$	$16s+8$	4	......
$16s+2$	$16s+5$	3	......
$14s+3$	$14s+5$	2	......

To complete the proof, we list a sequence for $h(3\text{–}S_4(2n-2))$: $(1,1,2,4,2,0,0,4)$.

Case 6: $n \equiv 5 \pmod 8$.
Let $n = 8s+5, m = 2t+1$. For $n \geq m > 1$ and $n > 13$, the solution is given by the following table.

i	j	k	
$1+r$	$8s-r+4$	$8s-2r+3$	$0 \leq r \leq 4s-t$
$4s-t+r+2$	$4s+t-r+1$	$2t-2r-1$	$0 \leq r \leq t-2$
$4s+t+2$	$4s+t+3$	1	......
$4s+r+1$	$12s-r+5$	$8s-2r+4$	$0 \leq r \leq 1$
$8s+6$	$12s+6$	$4s$	......
$8s+5$	$16s+10$	$8s+5$	......
$10s+7$	$12s+7$	$2s$	......
$8s+r+7$	$16s-r+7$	$8s-2r$	$0 \leq r \leq 2s-1$
$12s+3$	$14s+7$	$2s+4$	......
$10s+r+8$	$14s-r+6$	$4s-2r-2$	$0 \leq r \leq s-4$
$11s+5$	$13s+7$	$2s+2$	......
$11s+r+6$	$13s-r+4$	$2s-2-2r$	$0 \leq r \leq s-4$
$13s+5$	$13s+9$	4	......
$13s+6$	$13s+8$	2	......

To complete the proof, we list below the sequences $h(m - S_5(2n-2))$ and $h(m - S_{13}(2n-2))$ for all remaining defects:
For $n = 5$ and $m = 3$:

$$(1,1,4,2,5,2,4,0,0,5)$$

For $n = 13$ and $m = 3,5,7,9,11$:

$$(11,9,7,5,12,10,1,1,5,7,9,11,13,6,8,10,12,4,2,6,2,4,8,0,0,13)$$
$$(11,9,7,3,12,10,3,1,1,7,9,11,13,6,8,10,12,4,2,6,2,4,8,0,0,13)$$
$$(11,9,5,3,12,10,3,5,1,1,9,11,13,6,8,10,12,4,2,6,2,4,8,0,0,13)$$
$$(11,7,5,3,12,10,3,5,7,1,1,11,13,6,8,10,12,4,2,6,2,4,8,0,0,13)$$
$$(9,7,5,3,12,10,3,5,7,9,1,1,13,6,8,10,12,4,2,6,2,4,8,0,0,13)$$

Case 7: $n \equiv 6 \pmod 8$.
Let $n = 8s+6, m = 2t$. For $n \geq m > 2$ and $n > 14$, the solution is given by the following table.

i	j	k	
$1+r$	$8s-r+7$	$8s-2r+6$	$0 \le r \le 4s-t+2$
$4s-t+r+4$	$4s+t-r+2$	$2t-2r-2$	$0 \le r \le t-3$
$4s+t+3$	$4s+t+4$	1	$\cdots\cdots$
$4s+r+2$	$12s-r+7$	$8s-2r+5$	$0 \le r \le 2$
$8s+2r+8$	$16s-2r+5$	$8s-4r-3$	$0 \le r \le s$
$8s+2r+9$	$16s-2r+8$	$8s-4r-1$	$0 \le r \le s-1$
$10s+r+9$	$14s-r+4$	$4s-2r-5$	$0 \le r \le 2s-5$
$12s+8$	$16s+7$	$4s-1$	$\cdots\cdots$
$16s+9$	$16s+12$	3	$\cdots\cdots$
$14s+6$	$14s+8$	2	$\cdots\cdots$

To complete the proof, we list below the sequences $h(m-S_6(2n-2))$ and $h(m-S_{14}(2n-2))$ for all remaining defects:
For $n=6$ and $m=2,4$:

$$(6,4,5,1,1,4,6,5,3,0,0,3)$$
$$(6,2,5,2,1,1,6,5,3,0,0,3)$$

For $n=14$ and $m=4,6,8,10,12$:

$$(14,12,10,8,6,1,1,13,11,9,6,8,10,12,14,2,7,2,9,11,13,3,5,7,3,0,0,5)$$
$$(14,12,10,8,1,1,4,13,11,9,4,8,10,12,14,2,7,2,9,11,13,3,5,7,3,0,0,5)$$
$$(14,12,10,1,1,6,4,13,11,9,4,6,10,12,14,2,7,2,9,11,13,3,5,7,3,0,0,5)$$
$$(14,12,1,1,8,6,4,13,11,9,4,6,8,12,14,2,7,2,9,11,13,3,5,7,3,0,0,5)$$
$$(14,1,1,10,8,6,4,13,11,9,4,6,8,10,14,2,7,2,9,11,13,3,5,7,3,0,0,5)$$

Case 8: $n \equiv 7 \pmod 8$.
Let $n=8s+7, m=2t$. For $n>m>4$ and $n>7$, the solution is given by the following table (ignore the line * when $s=1$).

i	j	k	
$1+r$	$8s-r+7$	$8s-2r+6$	$0\le r\le 4s-t+2$
$4s-t+r+4$	$4s+t-r+2$	$2t-2r-2$	$0\le r\le t-4$
$4s+t+3$	$4s+t+4$	1	......
$4s+r+1$	$12s-r+8$	$8s-2r+7$	$0\le r\le 4$
$8s+2r+10$	$16s-2r+7$	$8s-4r-3$	$0\le r\le s-2$
$*8s+2r+9$	$16s-2r+4$	$8s-4r-5$	$0\le r\le s-1$
$*8s+8$	$12s+9$	$4s+1$	......
$10s+r+8$	$14s-r+5$	$4s-2r-3$	$0\le r\le 2s-5$
$16s+9$	$16s+14$	5	......
$16s+6$	$16s+10$	4	......
$16s+8$	$16s+11$	3	......
$14s+7$	$14s+9$	2	......

To complete the proof we, list below the $h(m-S_7(2n-2))$ and all remaining defects:
For $n=7$ and $m=2,6$:

$(7,5,1,1,6,3,5,7,3,4,6,0,0,4)$
$(7,5,3,1,1,3,5,7,2,4,2,0,0,4)$ □

We note that some computational results concerning $h(m-S_n(2n-2))$ are contained in the appendix.

The sequences presented in the following Corollaries are simple consequences of Theorems 4.2 and 4.4. We simply reverse the sequences presented in the theorems.

Corollary 4.1. *The sequence $m-S_n(3)$ exists if and only if one of the following is true:*

1. $n\equiv 0,1 (mod\ 4)$ and m is odd

2. $n\equiv 2,3 (mod\ 4)$ and m is even.

The cases $n=1$ and $(n,m)=(3,2)$ are exceptions.

Corollary 4.2. *The sequence $m-S_n(2,3)$ exists if and only if one of the following is true:*

1. $n\equiv 0,1 (mod\ 4)$ and m is odd

2. $n\equiv 2,3 (mod\ 4)$ and m is even.

The cases $n=1$ or 2 are exceptions.

4.3 Constructions

In this section, we show how the sequences $m - S_n(2n-3)$ and $h(m - S_n(2n-2))$ can be used to construct other generalizations of Skolem sequences.

Corollary 4.3. *When* $n \equiv 0, 1 \pmod 4$, *the existence of* $m - S_n(2n-3)$ *and/or* $h(m - S_n(2n-2))$ *implies the existence of* $m - S_n^2$.

Proof. The sequence $hS_n(2n)$, which exists by Theorem 2.7, can be hooked together with $m - S_n(2n-3)$ to form $m - S_n^2$, and the sequence hS_n, which exists by Theorem 2.3, can be hooked together with $h(m - S_n(2n-2))$ to form $m - S_n^2$. □

Corollary 4.4. *For admissible n and k, the existence of* $m - S_n(2n-3)$ *implies the existence of*

1. $m - S_n^2(k, 4n-2)$
2. $m - S_n^2(3, 4n+1-k)$
3. $m - S_n^2(2n-3, 2n-1+k)$
4. $m - S_n^2(2n+2-k, 2n+4)$
5. $m - S_n^2(4n-3)$
6. $m - S_n^2(3)$
7. $m - S_n^2(2n-3)$
8. $m - S_n^2(2n+3)$.

Proof. Given $m - S_n(2n-3)$, we can construct $S_n(k)$, which exists by Theorem 2.3, and append this to the beginning [end], giving (1) [(3)]. We can then reverse the resulting sequence to give (2) [(4)]. Or we can construct S_n, which exists by Theorem 2.1, and append it to the beginning [end] of $m - S_n(2n-3)$ to get (5) [(7)]. Reversing the resulting sequence then gives (6) [(8)]. □

Corollary 4.5. *For admissible n and k, the existence of $h(m - S_n(2n-2))$ implies the existence of*

1. $m - S_n^2(k\,, 4n-1\,, 4n)$
2. $m - S_n^2(2\,, 3\,, 4n+2-k)$
3. $m - S_n^2(2n-2\,, 2n-1\,, 2n+k)$
4. $m - S_n^2(2n+2-k\,, 2n+3\,, 2n+4)$
5. $m - S_n^2(4n-2\,, 4n-1)$
6. $m - S_n^2(2\,, 3)$
7. $m - S_n^2(2n-2\,, 2n-1)$
8. $m - S_n^2(2n+2\,, 2n+3)$.

Proof. Given $h(m - S_n(2n-2))$, we can construct $S_n(k)$, which exists by Theorem 2.3, and append this to the beginning [end], giving (1) [(3)]. We can then reverse the resulting sequence to give (2) [(4)]. Or we can construct S_n, which exists by Theorem 2.1, and append it to the beginning [end] of $h(m - S_n(2n-2))$ to get (5) [(7)]. Reversing the resulting sequence then gives (6) [(8)]. □

Chapter 5

Infinite Skolem sequences

In this chapter, we investigate a relationship between infinite Skolem sequences and Beatty sequences, as well as present a relationship between infinite Skolem sequences, Fibonacci numbers and restricted compositions and palindromes of n (all results in this chapter can be found in [27]).

5.1 Infinite Skolem sequences and Beatty sequences

When an infinite Skolem sequence is generated according to the method-presented in the introduction to this thesis, Skolem noticed that for each positive integer n, the pair (a_n, b_n) is given by the formula $([\alpha n], [\alpha^2 n])$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $[x]$ denotes the greatest integer $\leq x$. In fact, it is well known that, given irrational numbers μ and ν, such that

$$\frac{1}{\mu} + \frac{1}{\nu} = 1, \tag{5.1}$$

the pair of sequences N_μ and N_ν are mutually disjoint and partitions the set $\mathbb{N}$. (See Theorem 2.14 as well as [40] exercise 9, p. 98, for more information on the relationship given by equation (5.1).) These sequences are examples of Beatty sequences. (For more examples of Beatty sequences, the reader is referred to [38].)

Given a pair of Beatty sequences with μ and ν as the irrational base, we can generate another infinite Skolem-type sequence by setting $s_{[\mu n]} = [\nu n] - [\mu n] = s_{[\nu n]}$, where s_k denotes the k^{th} position of the sequence.

With this idea in mind, in this chapter we pose the question, "Do there exist irrational numbers μ and ν which generate infinite generalizations of Skolem sequences using the above method?" Thus, throughout the remainder of the thesis, when we refer to an infinite Skolem-type sequence, we are referring to one constructed using this method.

Now we first notice that for $\mu = \sqrt{2}$ and $\nu = 2 + \sqrt{2}$ (note that these values are obtained from Theorem 2.15 by setting $l = 2$ and $m = 2$), we have the pairs

$$(1,3)\,(2,6)\,(4,10)\,(5,13)\,(7,17)\ldots$$

which gives us the sequence

2,4,2,6,8,4,10,12,14,6,16,18,8,20,22,24,10...

We call this an *infinite even Skolem sequence.*

We make a here that, interestingly, any finite even Skolem sequence contains a hook, while the infinite version does not. For example, the sequence $(8,6,4,2,0,2,4,6,8)$ contains a hook in the middle of the sequence.

So we now ask: for what irrational numbers μ and ν, if any, do there exist sequences N_μ and N_ν which generate an infinite odd Skolem sequence?

Theorem 5.1. *There are no irrational numbers μ and ν such that the sequences N_μ and N_ν generates an infinite odd sequence.*

Proof. Assume that such a μ and ν exists. So we have the sequence

1,1,3,5,7,3,9,11,5,...

Consider the Beatty sequence N_ν. Then we have $[\nu] = 2$ and $[2\nu] = 6$. Therefore, we need $\left[\frac{6+x}{2}\right] = 2$, where $0 < x < 1$. However, this is not possible since $\frac{6}{2} = 3 \Rightarrow \left[\frac{6+x}{2}\right] \neq 2$. Contradiction. □

However, it is possible to obtain expressions which generate disjoint pairs $\big(f(n), g(n)\big)$ whose union is $\mathbb{N}$ and which generate an infinite odd Skolem sequence by setting $f(n) = g(n) - f(n) = g(n)$. We do this by using Theorem 2.15 and setting $l = 1$ and $m = 2$. These expressions are as follows:

$$f(n) = \left[\sqrt{2}\left(n - \frac{1}{2}\right) + 1\right]$$
$$g(n) = \left[\left(2 + \sqrt{2}\right)\left(n - \frac{1}{2}\right) + 1\right].$$

Now, we also notice that for $\mu = \sqrt{3}$, $\nu = \frac{1}{2}(3 + \sqrt{3})$, we have the pairs $(1,2)\,(3,4)\,(5,7)\,(6,9)\,(8,11)\,(10,14)\,(12,16)\,(13,18)\,(15,21)$, which generates the sequence

1,1,1,1,2,3,2,3,3,4,3,4,5,4,6,4,6,5,7,8,6,...

This sequence is similar to an infinite 2-fold Skolem sequence, with the exception that some differences only appear once amongst the pairs (e.g. 2 and 5). So we naturally ask ourselves: for what irrational numbers μ and ν, if any, do there exist sequences N_μ and N_ν which generate an infinite 2-fold Skolem sequence?

This is settled by the following theorem:

Theorem 5.2. *There are no irrational numbers μ and ν such that the sequences N_μ and N_ν generate an infinite 2-fold Skolem sequence.*

Proof. Assume that such a μ and ν exist. Then we have the sequence

1,1,1,1,2,2,2,2,3,3,4,3,3,...

The first few terms of the Beatty sequence N_μ are

$$[\mu] = 1, [2\mu] = 3, [3\mu] = 5, [4\mu] = 6, [5\mu] = 9\ldots$$

Since $[5\mu] = 9$ and $[4\mu] = 6$, we need $\left[\frac{4(9+x)}{5}\right] = 6$, where $0 < x < 1$. Thus we need $\left[\frac{36}{5} + \frac{4x}{5}\right] = 6$, which is impossible. Contradiction. □

In fact, we can prove that no such irrational numbers μ and ν exist which give the sequences N_μ and N_ν which generate an infinite λ-fold Skolem sequence, for all λ.

Theorem 5.3. *There are no irrational numbers μ and ν such that the sequences N_μ and N_ν generate an infinite λ-fold Skolem sequence.*

Proof. We consider two cases, λ even and λ odd.

For λ even: Assume that such μ and ν exist. Then we have the sequence

$$\overbrace{1,1,\ldots,1,1}^{2\lambda},\overbrace{2,2,\ldots,2,2}^{2\lambda},3,\ldots$$

Consider the Beatty sequence N_μ:

$$[\mu]=1,[2\mu]=3,\ldots,[2\lambda\mu]=4\lambda-2,[(2\lambda+1)\mu]=4\lambda+1,\ldots$$

So we need $\left[2\lambda\left(\frac{4\lambda+1+x}{2\lambda+1}\right)\right]=4\lambda-2$, where $0<x<1$. However,

$\frac{2\lambda(4\lambda+1)}{2\lambda+1}=4\lambda-1+\frac{1}{2\lambda+1}>4\lambda-1$. Therefore $\left[2\lambda\left(\frac{4\lambda+1+x}{2\lambda+1}\right)\right]\neq 4\lambda-2$.

Contradiction.

For λ odd: Assume that such μ and ν exist. Then we have the sequence

$$\overbrace{1,1,\ldots,1,1}^{2\lambda},2,2,2,2\ldots$$

This time we consider the Beatty sequence N_ν:

$$[\nu]=2,[2\nu]=4,\ldots,[(\lambda+1)\nu]=2\lambda+3,[(\lambda+2)\nu]=2\lambda+4,\ldots$$

Then we have $[(\lambda+2)\nu]=2\lambda+4$ and
$[(\lambda+1)\nu]=2\lambda+3$, which means we need $\left[\frac{(2\lambda+4+x)(\lambda+1)}{\lambda+2}\right]=2\lambda+3$, where $0<x<1$. However, $\left[\frac{(2\lambda+4+x)(\lambda+1)}{\lambda+2}\right]=\left[2\lambda+2+\frac{x(\lambda+1)}{\lambda+2}\right]=2\lambda+2$. Contradiction. □

Another generalization which we could investigate is Langford sequences. Does there exist irrational numbers μ and ν which gives us the sequences N_μ and N_ν and generates an infinite Langford sequence? This is settled by the following theorem:

Theorem 5.4. *There are no irrational numbers μ and ν such that the sequences N_μ and N_ν generate an infinite Langford sequence.*

Proof. Assume that such a μ and ν exists. Then we have the sequence

$$d, d+1, d+2, \ldots 2d-1, d, 2d, d+1, \ldots$$

Let's consider the Beatty sequence N_ν. Then we have $[\nu] = d+1$ and $[2\nu] = d+3$. Thus we need $[\frac{d+3+x}{2}] = d+1$, where $0 < x < 1$. But $[\frac{d+3+x}{2}] < [\frac{d+4}{2}] < d+1$. Contradiction. □

The reader is referred to [27] for more information on infinite Langford sequences.

5.2 Infinite Skolem sequences and restricted compositions of n

We now present an interesting relationship between infinite Skolem sequences, Fibonacci numbers, and restricted compositions and palindromes of n.

Theorem 5.5. *Let (a_n, b_n) denote the positions of the positive integer n in the infinite Skolem sequence, and let $\beta = (1-\sqrt{5})/2$. Then we have the following identities:*

1. $a_n = \left[\left(\beta^n + \sqrt{5}F_n\right)^{\frac{1}{n}} n\right]$

2. $a_n = \left[\left(\beta^n + \sqrt{5}C_n(O)\right)^{\frac{1}{n}} n\right]$

3. $a_n = \begin{cases} \left[\left(\beta^n + \sqrt{5}P_{2n}(O)\right)^{\frac{1}{n}} n\right] & n \text{ is even} \\ \left[\left(\beta^n + \sqrt{5}P_{2n-3}(O)\right)^{\frac{1}{n}} n\right] & n \text{ is odd} \end{cases}$

4. $a_n = \left[\left(\beta^n + \sqrt{5}C_{n-1}(1,2)\right)^{\frac{1}{n}} n\right]$

5. $a_n = \begin{cases} \left[\left(\beta^n + \sqrt{5}P_{2n-4}(1,2)\right)^{\frac{1}{n}} n\right] & n \text{ is even} \\ \left[\left(\beta^n + \sqrt{5}P_{2n+1}(1,2)\right)^{\frac{1}{n}} n\right] & n \text{ is odd} \end{cases}$

6. $a_n = \left[(\beta^n + \sqrt{5}C_{n+1}(>1))^{\frac{1}{n}} n\right]$

7. $a_n = \begin{cases} \left[(\beta^n + \sqrt{5}P_{2n-2}(>1))^{\frac{1}{n}} n\right] & n \text{ is even} \\ \left[(\beta^n + \sqrt{5}P_{2n+1}(>1))^{\frac{1}{n}} n\right] & n \text{ is odd.} \end{cases}$

Proof. Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, then we know that $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$, for all $n \in \mathbb{N}$. Solving this equation for α, we have

$$\alpha = (\beta^n + \sqrt{5}F_n)^{\frac{1}{n}}.$$

Now, using the fact that $a_n = [\alpha n]$, for each n in the infinite Skolem sequence, and using the theorems on compositions and palindromes in chapter 2, our results follow. □

We also note that, for n sufficiently large, the β^n term in each of the formulas in Theorem 5.5 is negligible and wecan compute a_n without having to calculate β^n.

Chapter 6

Conclusions and further research

In this thesis, we first proved that the necessary conditions are also sufficient for the existence of m-near-λ-fold Skolem sequences and extended m-near-λ-fold sequences. We then showed that the necessary conditions are also sufficient for the existence of $(2n-3)$-extended m-near-Skolem sequences and hooked $(2n-2)$-extended m-near-Skolem sequences. It is hoped that these sequences may be used in the construction of group divisible designs, rotational triple systems, and graph factorizations, or any other designs, and we are now investigating to see whether or not this can, in fact, be done.

However, we also note that the existence of t-extended m-near-Skolem sequences and (p,q)-extended m-near-Skolem sequences, for all admissible positions of the hooks, is still open. It is not hard to see that the necessary conditions for the existence of these sequences are:

Corollary 6.1. *A t-extended m-near-Skolem sequence of order n exists only if one of the following is true:*

1. $n \equiv 0, 1 \pmod 4$ *m and t are of the same parity*
2. $n \equiv 2, 3 \pmod 4$ *and m and t are of opposite parity.*

Proof. These conditions follow from Theorem 4.1 with $\lambda = 1$. □

Theorem 6.1. *A (p,q)-extended m-near-Skolem sequence of order n exists only if one of the following is true:*

1. $n \equiv 0, 1 \pmod 4$, *m even, and p and q are of the same parity*

2. $n \equiv 0, 1 \pmod 4$, *m odd, and p and q are of opposite parity*

3. $n \equiv 2, 3 \pmod 4$, *m odd, and p and q are of the same parity*

4. $n \equiv 2, 3 \pmod 4$, *m even, and p and q are of opposite parity.*

Proof. Let $m{-}S_n(p, q) = (s_1, s_2, \ldots, s_{2n})$ be a (p, q)-extended m-near-Skolem sequence of order n and defect m. For each $k \in \{1, 2, \ldots, m{-}1, m{+}1, \ldots, n\}$, let the ordered pairs (i_k, j_k) denote the subscripts of s_{i_k} and s_{j_k} when $s_{i_k} = s_{j_k} = k$. Then

(a)

$$\sum_{\substack{k=1,\\ k\neq m}}^{n} (i_k + j_k) = \frac{(2n)(2n+1)}{2} - p - q = 2n^2 + n - p - q, \text{ and}$$

(b)

$$\sum_{\substack{k=1,\\ k\neq m}}^{n} (j_k - i_k) = \frac{(n)(n+1)}{2} - m.$$

Adding (a) and (b) together gives us

$$\sum_{\substack{k=1,\\ k\neq m}}^{n} j_k = \frac{5n^2 + 3n - 2m - 2p - 2q}{4}.$$

Since the left hand side of the equation must be an integer, the number $(5n^2 + 3n - 2m - 2p - 2q)$ must be divisible by 4. When we solve for n, m, p, and q, we obtain the necessary conditions. □

Another Skolem generalization similar to the ones presented in this thesis is (m_1, m_2)-near-Skolem sequences, which contain two defects in the sequence as opposed to the one in the traditional m-near-Skolem sequences. For example, the sequence $(5, 3, 1, 1, 3, 5)$ is an example of a $(2, 4)$-near-Skolem sequence of order 5. Again, it is not hard to see that the necessary conditions for the existence of these sequences are :

Theorem 6.2. *An* (m_1, m_2)-near-Skolem sequence of order n *exists only if one of the following is true:*

1. $n \equiv 0, 1 \pmod 4$ *and* m_1 *and* m_2 *are of the same parity*

2. $n \equiv 2, 3 \pmod 4$ *and* m_1 *and* m_2 *are of opposite parity.*

Proof. Let $(m_1, m_2) - S_n = (s_1, s_2, \ldots, s_{2n-2})$ be an (m_1, m_2)-near-Skolem sequence of order n and defects m_1 and m_2. For each $k \in \{1, 2, \ldots, n\} - \{m_1, m_2\}$, let the ordered pairs (i_k, j_k) denote the subscripts of s_{i_k} and s_{j_k} when $s_{i_k} = s_{j_k} = k$. Then

(a)

$$\sum_{\substack{k=1,\\ k \neq m_1, m_2}}^{n} (i_k + j_k) = \frac{(2n-4)(2n-3)}{2} = 2n^2 - 7n + 6, \text{ and}$$

(b)

$$\sum_{\substack{k=1,\\ k \neq m_1, m_2}}^{n} (j_k - i_k) = \frac{(n)(n+1)}{2} - m_1 - m_2.$$

Adding (a) and (b) together gives us

$$\sum_{\substack{k=1,\\ k \neq m_1 . m_2}}^{n} j_k = \frac{5n^2 - 13n - 2m_1 - 2m_2 + 12}{4}.$$

Since the left hand side of the equation must be an integer, the number $(5n^2 - 13n - 2m_1 - 2m_2 + 12)$ must be divisible by 4. When we solve for n, m_1, and m_2 we obtain the necessary conditions. □

The existence of all three of these sequences is still an open question and we reiterate some conjectures which have been mentioned through conversation:

1. The necessary conditions for the existence of a k-extended m-near-Skolem sequence of order n are also sufficient.

2. The necessary conditions for the existence of a (p, q)-extended m-near-Skolem sequence of order n are also sufficient.

3. The necessary conditions for the existence of an (m_1, m_2)-near-Skolem sequence of order n are also sufficient.

The construction of these sequences seem difficult at this point and require methods other than those presented in the thesis. We do note, however, that the necessary conditions have been shown to be sufficient for the existence of t-extended m-near-Skolem sequences when $t = 2, 3, n, 2n-3, 2n-2$ (see [34, 30, 26]).

Finally, in this thesis we also discussed a relationship between infinite Skolem sequences and Beatty sequences and proved that a similar relationship could not be extended to infinite λ-fold Skolem sequences and infinite Langford sequences. We also showed that the relationship could not be extended to infinite odd Skolem sequences despite the fact that it could be extended to infinite even Skolem sequences. We did, however, produce expressions which could generate an infinite odd Skolem sequence using Theorem 2.15. (The reader is referred to [27] for similar expressions involving infinite Langford sequences.)

A natural question to ask is whether there exists irrational numbers α_1, α_2, ..., α_n which generate n mutually disjoint sets whose union is $\mathbb{N}$ via the sequences N_{μ_1}, N_{μ_2}, ..., N_{μ_n}. Skolem proves in [36] that this is impossible, but he showed that it is possible to get expressions which satisfy this property. For example, the three expressions

$$[\alpha[\alpha n]], \qquad [\alpha[\alpha^2 n]], \qquad [\alpha^2 n]$$

generate three mutually disjoint sequences which have $\mathbb{N}$ as their union.

In this thesis, we also presented a relationship between infinite Skolem sequences, Fibonacci numbers and restricted compositions and palindromes of n.

and the columns represent the defects), as well as some values of a_n in the infinite Skolem sequence.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\mathbf{a_n}$	1	3	4	6	8	9	11	12	14	16	17	19	21	22	24
n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$\mathbf{a_n}$	25	27	29	30	32	33	35	37	38	40	42	43	45	46	48
n	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
$\mathbf{a_n}$	50	51	53	55	56	58	59	61	63	64	66	67	69	71	72
n	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
$\mathbf{a_n}$	74	76	77	80	82	84	85	87	88	90	92	93	95	97	98

Table 1: Values of a_n in the infinite Skolem sequence

n|m	1	2	3	4	5	6	7	8	9	10	11	12	13
2	0	1											
3	0	1	0										
4	2	0	2	0									
5	2	0	4	0	6								
6	0	6	0	8	0	10							
7	22	0	24	0	38	0							
8	52	0	88	0	108	0	128	0					
9	300	0	340	0	416	0	480	0	504				
10	0	1444	1760	0	2004	0	2352	0	2656				
11	0	7052	0	8784	0	10,012	0	11,472	0	12,704	0		
12	35,288	0	43,296	0	50,936	0	59,384	0	66,720	0	72,976	0	
13	216,288	0	260,296	0	305,840	0	353,344	0	398,104	0	434,992	0	455,936

Table 2: Computational results for m-near-Skolem sequences

n|m	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0												
2	1	0											
3	1	0	1										
4	0	1	0	2									
5	0	2	0	4	0								
6	6	0	10	0	8	0							
7	18	0	24	0	28	0	38						
8	0	66	0	84	0	114	0	124					
9	0	304	0	368	0	392	0	504	0				
10	1,348	0	1,492	0	1,728	0	2,112	0	2,392	0			
11	6,824	0	7,456	0	8,944	0	10,488	0	11,624	0	12,808		
12	0	38,396	0	46,032	0	53,004	0	60,704	0	67,848	0	72,648	
13	0	233,216	0	276,152	0	319,912	0	364,960	0	405,016	0	439,048	0

Table 3: Computational results for t-extended m-near-Skolem sequences for $t = 2$ and $t = 2n - 2$.

n|m	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0												
2	1	0											
3	0	0	0										
4	1	0	1	0									
5	2	0	2	0	4								
6	0	6	0	10	0	8							
7	0	20	0	22	0	32	0						
8	62	0	76	0	80	0	98	0					
9	260	0	304	0	348	0	452	0	452				
10	0	1,396	0	1,504	0	1,724	0	2,112	0	102,316			
11	0	6,992	0	7,660	0	8,848	0	10,361	0	11,404	0		
12	35,880	0	39,400	0	46,560	0	53,152	0	60,376	0	66,352	0	
13	223,352	0	239,240	0	281,176	0	323,960	0	364,072	0	405,136	0	433,920

Table 4: Computational results for t-extended m-near-Skolem sequences for $t = 3$ and $t = 2n - 3$.

n|m	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0												
2	1	2											
3	2	2	2										
4	6	2	6	6									
5	10	8	16	12	22								
6	22	32	36	48	34	48							
7	76	120	108	152	136	204	160						
8	354	368	540	484	624	612	736	636					
9	1,876	1,800	2,340	2,272	2,848	2,544	3,416	3,104	3,556				
10	8,316	10,816	10,196	13,144	12,256	15,136	14,408	17,792	16,348	19,488			
11	46,768	58,704	56,480	71,632	67,952	82,448	79,664	96,072	88,336	105,136	95,872		
12	320,208	312,776	385,104	375,816	457,296	436,936	529,312	504,328	597,696	560,480	651,216	594,320	
13	2,127,544	2,071,232	2,517,040	2,460,624	2,960,120	2,848,336	3,413,576	3,264,672	3,857,808	3,649,232	4,233,968	3,923,744	4,459,88

Table 5: Computational results for extended m-near-Skolem sequences

n|m	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0												
2	1	0											
3	0	1	0										
4	1	0	1	0									
5	2	0	2	0	4								
6	0	6	0	6	0	12							
7	0	24	0	18	0	24	0						
8	74	0	68	0	88	0	110	0					
9	288	0	308	0	372	0	428	0	492				
10	0	1,416	0	1,528	0	1,788	0	2,056	0	2,412			
11	0	7,404	0	7,772	0	9,128	0	10,272	0	11,916	0		
12	37,224	0	40,552	0	47,520	0	54,944	0	61,312	0	69,352	0	
13	227,896	0	248,096	0	287,688	0	331,272	0	369,880	0	409,320	0	445,952

Table 6: Computational results for (p,q)-extended m-near-Skolem sequences for $(p,q)=(2,3)$ and $(p,q)=(2n-2,2n-1)$.

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and the columns represent the defects), as well as some values of a_n in the infinite Skolem sequence.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a_n	1	3	4	6	8	9	11	12	14	16	17	19	21	22	24
n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
a_n	25	27	29	30	32	33	35	37	38	40	42	43	45	46	48
n	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
a_n	50	51	53	55	56	58	59	61	63	64	66	67	69	71	72
n	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
a_n	74	76	77	80	82	84	85	87	88	90	92	93	95	97	98

Table 1: Values of a_n in the infinite Skolem sequence

n|m	1	2	3	4	5	6	7	8	9	10	11	12	13
2	0	1											
3	0	1	0										
4	2	0	2	0									
5	2	0	4	0	6								
6	0	6	0	8	0	10							
7	22	0	24	0	38	0							
8	52	0	88	0	108	0	128	0					
9	300	0	340	0	416	0	480	0	504				
10	0	1444	1760	0	2004	0	2352	0	2656				
11	0	7052	0	8784	0	10,012	0	11,472	0	12,704	0		
12	35,288	0	43,296	0	50,936	0	59,384	0	66,720	0	72,976	0	
13	216,288	0	260,296	0	305,840	0	353,344	0	398,104	0	434,992	0	455,936

Table 2: Computational results for m-near-Skolem sequences

n|m	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0												
2	1	0											
3	1	0	1										
4	0	1	0	2									
5	0	2	0	4	0								
6	6	0	10	0	8	0							
7	18	0	24	0	28	0	38						
8	0	66	0	84	0	114	0	124					
9	0	304	0	368	0	392	0	504	0				
10	1,348	0	1,492	0	1,728	0	2,112	0	2,392	0			
11	6,824	0	7,456	0	8,944	0	10,488	0	11,624	0	12,808		
12	0	38,396	0	46,032	0	53,004	0	60,704	0	67,848	0	72,648	
13	0	233,216	0	276,152	0	319,912	0	364,960	0	405,016	0	439,048	0

Table 3: Computational results for t-extended m-near-Skolem sequences for $t = 2$ and $t = 2n - 2$.

n|m	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0												
2	1	0											
3	0	0	0										
4	1	0	1	0									
5	2	0	2	0	4								
6	0	6	0	10	0	8							
7	0	20	0	22	0	32	0						
8	62	0	76	0	80	0	98	0					
9	260	0	304	0	348	0	452	0	452				
10	0	1,396	0	1,504	0	1,724	0	2,112	0	102,316			
11	0	6,992	0	7,660	0	8,848	0	10,361	0	11,404	0		
12	35,880	0	39,400	0	46,560	0	53,152	0	60,376	0	66,352	0	
13	223,352	0	239,240	0	281,176	0	323,960	0	364,072	0	405,136	0	433,920

Table 4: Computational results for t-extended m-near-Skolem sequences for $t = 3$ and $t = 2n - 3$.

n|m	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0												
2	1	2											
3	2	2	2										
4	6	2	6	6									
5	10	8	16	12	22								
6	22	32	36	48	34	48							
7	76	120	108	152	136	204	160						
8	354	368	540	484	624	612	736	636					
9	1,876	1,800	2,340	2,272	2,848	2,544	3,416	3,104	3,556				
10	8,316	10,816	10,196	13,144	12,256	15,136	14,408	17,792	16,348	19,488			
11	46,768	58,704	56,480	71,632	67,952	82,448	79,664	96,072	88,336	105,136	95,872		
12	320,208	312,776	385,104	375,816	457,296	436,936	529,312	504,328	597,696	560,480	651,216	594,320	
13	2,127,544	2,071,232	2,517,040	2,460,624	2,960,120	2,848,336	3,413,576	3,264,672	3,857,808	3,649,232	4,233,968	3,923,744	4,459,88

Table 5: Computational results for extended m-near-Skolem sequences

n|m	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0												
2	1	0											
3	0	1	0										
4	1	0	1	0									
5	2	0	2	0	4								
6	0	6	0	6	0	12							
7	0	24	0	18	0	24	0						
8	74	0	68	0	88	0	110	0					
9	288	0	308	0	372	0	428	0	492				
10	0	1,416	0	1,528	0	1,788	0	2,056	0	2,412			
11	0	7,404	0	7,772	0	9,128	0	10,272	0	11,916	0		
12	37,224	0	40,552	0	47,520	0	54,944	0	61,312	0	69,352	0	
13	227,896	0	248,096	0	287,688	0	331,272	0	369,880	0	409,320	0	445,952

Table 6: Computational results for (p,q)-extended m-near-Skolem sequences for $(p,q) = (2,3)$ and $(p,q) = (2n-2, 2n-1)$.

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