MAPPING SPACES AND FIBREWISE HOMOTOPY THEORY

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Your file Votre référence
ISBN: 978-0-494-33437-9
Our file Notre référence
ISBN: 978-0-494-33437-9

MAPPING SPACES AND FIBREWISE HOMOTOPY THEORY

A thesis Presented to
the School of Graduate Studies
Memorial University of Newfoundland

by

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Supervised by Pf. Dr. Peter Booth

In Partial Fulfillment
of the Requirements for the Degree
MASTER OF SCIENCE
Mathematics

August 29, 2007

ACKNOWLEDGEMENTS

There were many people who were involved in one way or another with the writing of this thesis. I am deeply indebted to my supervisor, Pf. Dr. Peter Booth who always lead from the front. His advice, experience, knowledge in algebraic topology, and critical comments formed a very important part of myself, and this thesis. Also, I would like to thank Pf. Dr. Phillip Heath for his teaching in Nielsen Fixed Point Theory, and encouragement, Dr. Ivan Booth for his encouragement, Dr. Bruce Watson who, during my stay, served as head of the Department of Mathematics and Statistics in Memorial University, Dwayne Hart for his help in LATEX, Wanda Heath, Jackie Costello, Rosalind English, Leonce Morrissey, and my wife Elva De Leon and my daughters Jazmin and Alexandra for their understanding and encouragement.

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August 29, 2007

INTRODUCTION

If $q : Y \to B$ is a fibration and Z is a space, then the free range mapping space $Y!Z$ has a collection of partial maps from Y to Z as underlying set, namely those maps whose domains are individual fibres of q.

It is shown in [B3] that these maps have applications to several topics in homotopy theory. Three results [B3, Ths. 5.1, 6.1 and 7.1], concerning identifications, cofibrations and sectioned fibrations, are given in complete detail. The necessary topological foundations for two more complicated applications, to the cohomology of fibrations and the classification of Moore-Postnikov systems, are also given. The applications themselves are outlined in Chapters 8 and 9 of [B3].

The argument of [B3] is in the context of the usual category of all topological spaces, and this necessarily introduces some limitations. Whenever we work with exponential laws for mapping spaces in that category, we usually find that we are forced to assume that some of the spaces are locally compact and Hausdorff. These conditions detract considerably from the generality of the results obtained.

In this thesis we develop the aforementioned topological foundations in the category of compactly generated or CG-spaces, which is free of the inconvenient assumptions mentioned above. Furthermore we do not require the Hausdorff condition for CG-ification as in [S]. Thus we obtain the CG-space versions of the applications to identifications, cofibrations and sectioned fibrations, a theorem on infinite CW-complexes, and establish improved foundations for the CG-versions of the other two applications, i.e. the cohomology of fibrations and the classification theorem for Moore-Postnikov factorizations.

CONTENTS

1. COMPACTLY GENERATED SPACES

Let X be a topological space. We define kX to be the space X retopologized with the final topology (in [Br1, Pg. 92, 4.2]) relative to all incoming maps from a compact Hausdorff spaces.
Thus if

$$g : C \longrightarrow X$$

is a map, where C is compact Hausdorff, then

$$g : C \longrightarrow kX$$

is a map and, in fact kX has the finest topology for which all maps

$$g : C \longrightarrow X$$

are maps.

If kX and X have the same underlying set and the same topology, then X will be said to be a *compactly generated space* or $CG-space$, and we will write $kX = X$. We will refer to kX as the $CG-ification$ of X. For more details, concerning $CG-spaces$ in this sense see, [V]. The following alternative definition of CG-spaces is given by Steenrod.

Definition 1. *[S, 2.1] If X is a Hausdorff space, and if for each subset M and each limit point x of M there exist a compact set C in X such that x is a limit point of $M \cap C$, then X is a CG-space.*

The above definition coincides with our definition except for the Hausdorff condition which is not relevant in our theory. Kelly and other authors use the term $k-space$ for these objects. For another set of references under which the space X has to be Hausdorff as a condition for being CG-ified see, [Br1], [Br3], [K] and [S].

Theorem 1. ***Universal Property of the space*** kX**.** *Let* X*, and* Y *be spaces and* $h : X \longrightarrow Y$ *be a map. Then the composite maps* $h \circ g : C \longrightarrow X \longrightarrow Y$ *are maps for all*

$$g : C \longrightarrow X$$

with C *compact Hausdorff, if and only if*

$$h : kX \longrightarrow Y$$

is a map.

Proof. Firstly suppose that

$$h : kX \longrightarrow Y,$$

and

$$g : C \longrightarrow X$$

are maps, then

$$g : C \longrightarrow kX$$

is a map, and $g \circ h$ is a map.
Conversely, let $h \circ g$ be maps for all maps

$$g : C \longrightarrow X,$$

where C is a compact Hausdorff, we wish to prove that $h : kX \longrightarrow Y$ is a map. Let U be open in Y. Then

$$(h \circ g)^{-1}(U) = g^{-1} \circ h^{-1}(U)$$

is open implies that $h^{-1}(U)$ is open in kX, since kX has the final topology with respect to all maps

$$g : C \longrightarrow kX.$$

Proposition 1. *The identity*

$$1 : kX \longrightarrow X$$

is a map, for all spaces X*.*

Proof. From the Universal Property the identity $1 : kX \longrightarrow X$ is a map if and only if for any map

$$g : C \longrightarrow kX$$

with C a compact Hausdorff space, the composite $1 \circ g$ is a map. Since 1 and g are maps, then so is the composite $1 \circ g$ and hence so is $1 : kX \longrightarrow X$. □

Proposition 2. *If*

$$f : X \longrightarrow Y$$

is a map, and X and Y are spaces, then

$$kf : kX \longrightarrow kY$$

is a map, where $(kf)(x) = f(x)$.

Proof. Suppose that C is a compact Hausdorff space, and let

$$g : C \longrightarrow X$$

be a map. Then

$$f \circ g : C \longrightarrow Y$$

is a map. Hence

$$kf \circ g : C \longrightarrow kX \longrightarrow kY$$

is a map, for any incoming map

$$g : C \longrightarrow X.$$

It follows that

$$f \circ g : C \longrightarrow kY$$

is a map, and that kf is a map by the Universal Property. □

Proposition 3. *If X is a $CG-$space, and Y is any space, then $f : X \longrightarrow Y$ is a map if and only if*

$$f' : X \longrightarrow kY$$

is a map, where $f'(x) = f(x)$ for all $x \in X$.

Proof. Let f be a map. Then

$$f : kX = X \longrightarrow kY$$

is a map by the previous proposition.
Conversely, let

$$f' : X \longrightarrow kY$$

be a map. Then

$$f = 1 \circ f' : X \longrightarrow Y$$

is a map, where

$$1 : kY \longrightarrow Y$$

is the identity map (see Proposition 2). □

Proposition 4. *If C is a compact Hausdorff space, then a map $g : C \longrightarrow X$ is a map if and only if*

$$g : C \longrightarrow kX$$

is a map.

Proof. The *only if* part follows from the definition of kX, as was explained on page 5.
Conversely, let

$$g : C \longrightarrow kX$$

be a map. Since

$$1 : kX \longrightarrow X$$

is a map, and so the composite

$$1 \circ g : C \longrightarrow X$$

is a map. □

Proposition 5. *If C is a compact Hausdorff space, then C is a $CG-space$.*

Proof. From Proposition 1,

$$1 : kC \longrightarrow C$$

is a map. Thus the identity map

$$g : C \longrightarrow kC$$

is a map by Proposition 4. Then $kC = C$, and so C is a $CG - space$. □

Proposition 6. *If X is any space, then $kX = k(kX)$.*

Proof. The proof lies in the observation that kX, and $k(kX)$ have the final topology relative to all maps

$$g : C \longrightarrow X$$

and all maps

$$g : C \longrightarrow kX$$

respectively, and by Proposition 4 these are the same maps in each case. □

Corollary 1. *For any space X, kX is a $CG-space$.*

Proof. From the previous proposition. □

Proposition 7. *If Y has the final topology with respect to a family of maps*

$$\{f_j : X_j \longrightarrow Y\}_{j \in J},$$

where all X_j are $CG-spaces$, then Y is a $CG-space$.

Proof. Let

$$g : C \longrightarrow X_j$$

be a map, for all $j \in J$, with C a compact Hausdorff. Then

$$f_j \circ g : C \longrightarrow Y$$

is a map, and if U is open in kY, then

$$(f_j \circ g)^{-1}(U)$$

is open in C, by the definition of final topology. Thus

$$g^{-1}(f_j^{-1}(U))$$

is open in C. Hence $f_j^{-1}(U)$ is open in $kX_j = X_j$ for all $j \in J$ by the definition of final topology. Then U is open in Y, again by the definition of final topology, and so $Y = kY$ as we required. □

Corollary 2. *If*

$$f : X \longrightarrow Y$$

is an identification, and X is a $CG-space$, then Y is a $CG-space$.

Corollary 3. *If $\{X_j\}_{j\in J}$ is a family of $CG-$ spaces, then the disjoint topological sum*

$$\amalg_{j\in J} X_j$$

is a $CG-$ space.

Remark 1. *We define the n-cell as*

$$E^n = \{x \in R^n \mid |x| \le 1\},$$

the unitary (n-1)-sphere

$$S^{n-1} = \{x \in R^n \mid |x| = 1\},$$

and the unit open ball

$$B^n = \{x \in R^n \mid |x| < 1\}.$$

Let K be a not necessarily Hausdorff space. A cell structure on K is a family of maps

$$\{h_\lambda : E^{n_\lambda} \longrightarrow K\}_{\lambda\in\Lambda},$$

called characteristic maps. Let $\{D_\lambda = h_\lambda(B^{n_\lambda})\}_{\lambda\in\Lambda}$ be a family of open cells. The n-skeleton of the cell structure is

$$K^n = \bigcup_{n_\lambda \le n} D_\lambda.$$

Furthermore, the characteristic maps satisfied the following conditions:

CM1. The restriction $h_\lambda|B^{n_\lambda}$ is a bijective correspondence from B^{n_λ} to D_λ for all $\lambda \in \Lambda$.

CM2. For all $\lambda, \mu \in \Lambda$, $h_\lambda(B^{n_\lambda}) \cap h_\mu(B^{m_\mu})$ is empty unless $n = m$ and $\lambda = \mu$.

CM3. For all $\lambda \in \Lambda$, $h_\lambda(S^{n_\lambda - 1}) \subset K^{n-1}$.

Definition 2. *Let $K = \bigcup_{n\ge 0} K^n$ be a not necessarily Hausdorff space, with cell structure $\{h_\lambda\}_{\lambda\in\Lambda}$, as above, with finite or infinite set Λ. The space K is said to be an infinite $CW-$complex if the following conditions are satisfied:*

CW1. *Let* $\Lambda_n = \{\lambda \in \Lambda | n_\lambda = n\}$ *and let* Λ_n *have the discrete topology. Then the map*

$$q_n : K^{n-1} \amalg (\Lambda_n \times E^n) \longrightarrow K^n$$

defined by

$$q_n(x) = \begin{cases} x, & \text{if } x \in K^{n-1}, \\ h_\lambda(e), & (\lambda, e) \in \Lambda_n \times E^n, \end{cases}$$

is an identification.

CW2. *A set* $U \subset K$ *is open if and only if* $U \cap K^n$ *is open in* K^n *for all* n. *For more details about the definition of infinite CW-complexes see [Br1, Pg. 128].*

Theorem 2. *Every infinite CW-complex* K *is a normal Hausdorff space.*

Proof. Let A and B be any two disjoint closed sets in K. We assert that there exist two open sets U and V in K such that

$$A \subset U,\ B \subset V,\ U \cap V = \varnothing.$$

For each $n \geq 0$, the intersections

$$A_n = A \cap K^n,\ \ B_n = B \cap K^n$$

are disjoint closed sets in K^n. Since K^o is normal space, then there exist two open sets U_o and V_o in K^o such that

$$A_o \subset U_o,\ B_o \subset V_o,\ \overline{U_o} \cap \overline{V_o} = \varnothing.$$

Now we wish to construct for each $n > 0$ two open sets U_n and V_n of K^n such that

$$A_n \subset U_n,\ B_n \subset V_n,\ \overline{U_n} \cap \overline{V_n} = \varnothing,$$
$$U_{n-1} = U_n \cap K^{n-1},\ \ V_{n-1} = V_n \cap K^{n-1}.$$

For this purpose, let $n > 0$ and assume that U_i and V_i have been constructed for every $i > n$. Then

$$C_n = A_n \cap \overline{U_{n-1}},\ \ D_n = B_n \cap \overline{V_{n-1}},$$

are closed sets in K^n. We know that K^n is normal space. Therefore, there exist two open sets G_n and H_n such that

$$C_n \subset G_n,\ D_n \subset H_n,\ \overline{G_n} \cap \overline{H_n} = \varnothing.$$

Since U_{n-1} and V_{n-1} are open sets in K^{n-1}, the restriction

$$q_n \mid K^{n-1} : K^{n-1} \longrightarrow K^n$$

is an embedding and so K^{n-1} is closed in K^n. The sets

$$L_n = U_{n-1} \cup K^n \setminus K^{n-1}, \;\; M_n = V_{n-1} \cup K^n \setminus K^{n-1},$$

are open in K^n.

Consider now

$$U_n = G_n \cap L_n, \;\; V_n = H_n \cap M_n.$$

Obviously U_n and V_n are open sets in K^n and satisfy the required condition, that

$$A_n \subset U_n, \; B_n \subset V_o, \; \overline{U_o} \cap \overline{V_o} = \varnothing.$$

This complete the inductive construction of the two open sets U_n and V_n for all $n \geq 0$. Now let us consider the two sets

$$U = \bigcup_{n=o}^{\infty} U_n, \;\; V = \bigcup_{n=o}^{\infty} V_n$$

in the infinite CW-complex K. Since

$$U_n = U \cap K^n, \;\; V_n = V \cap K^n.$$

for every $n \geq 0$, it follows that U and V are open in K. Furthermore, it is easy to verify that

$$A \subset U, \; B \subset V, \; U \cap V = \varnothing.$$

Hence an infinite CW-complex K is normal space. Now let x be an arbitrary point in K. Then there exist $n \geq 0$ such that $x \in K^n$, since $K = \bigcup_{n=o}^{\infty} K^n$. Since K^n is T_1 space, $\{x\}$ is a closed set in K^n. Since K^n is closed in K, this implies that $\{x\}$ is closed in K. Hence K is T_1-space. Thus K is normal Hausdorff space. □

Proposition 8. *The closed n-cell $\overline{D}_\lambda = h_\lambda(E^{n_\lambda})$ is the closure of the open n-cell D_λ in K and is a compact Hausdorff subspace of K.*

Proof. Since $E^{n_\lambda} = \overline{B^{n_\lambda}}$, by the property of maps

$$h_\lambda(E^{n_\lambda}) = h_\lambda(\overline{B^{n_\lambda}}) \subset \overline{h_\lambda(B^{n_\lambda})},$$

then $D_\lambda \subset \overline{D}_\lambda$.
As a continuous image of a compact set E^{n_λ}, $\overline{D}_\lambda = h_\lambda(E^{n_\lambda})$ is compact. Since K is a normal Hausdorff space by Theorem 2, hence $\overline{D}_\lambda$ is closed in K and so a compact Hausdorff subspace of K. □

Proposition 9. *Every infinite CW − complex K has the final topology with respect to the family of inclusion maps* $\{i_\lambda : \overline{D}_\lambda \longrightarrow K\}_{\lambda\in\Lambda}$.

Proof. Firstly we are going to prove that K has the final topology with respect to the family of inclusion maps $\{i_\lambda : \overline{D}_\lambda \longrightarrow K\}_{\lambda\in\Lambda}$, where $\overline{D}_\lambda$ are the closed cells of an infinite $CW - complex$ K.

Thus $U \subset K$ is open if and only if $i_\lambda^{-1}(U) = U \cap \overline{D}_\lambda$ is open in $\overline{D}_\lambda$ for all $\lambda \in \Lambda$.

Necessity, it is obvious, since $\overline{D}_\lambda$ is subspace of K, so has the relative topology.

For sufficiency, let U be a set in K such that $U \cap \overline{D}_\lambda$ is open in $\overline{D}_\lambda$ for every $\lambda \in \Lambda$. We now prove that $U \cap K^n$ is open in K^n for every $n \geqslant 0$.

For $n = 0$, $U \cap K^o$ is always open in K^o since the 0-skeleton $K^o = \bigcup_{0_\lambda = o} D_\lambda = \Lambda_o$, since $D_\lambda = h_\lambda(B^o) = \{\lambda\}$.

Let $n > 0$ and assume that $U \cap K^{n-1}$ is open in K^{n-1}. Consider the inverse image of the identification

$$q_n : K^{n-1} \amalg (\Lambda_n \times E^n) \longrightarrow K^n$$

with $V = q_n^{-1}(U \cap K^n)$, is open in K^{n-1} and $\Lambda_n \times E^n$. Since $V \cap K^{n-1} = U \cap K^{n-1}$ it is open in K^{n-1} by inductive hypothesis. On the other hand, Λ_n is discrete by the definition of infinite CW-complex, and so $V \cap (\lambda \times E^n)$ is open in $\Lambda_n \times E^n$ for all $\lambda \in \Lambda$. Since $\overline{D_\lambda} = h_\lambda(E^{n_\lambda}) = q_n(\lambda \times E^n)$ it follows that $V \cap (\lambda \times E^n) = q_n^{-1}(U \cap \overline{D_\lambda}) \cap (\lambda \times E^n)$. Since $U \cap \overline{D_\lambda}$ is open in $\overline{D_\lambda}$, this implies that $V \cap (\lambda \times E^n)$ is open in $\lambda \times E^n$. □

Corollary 4. *Every infinite CW − complex is a CG − space.*

Proof. The family $\{\overline{D_\lambda}\}_{\lambda\in\Lambda}$ are the closed cells of an infinite CW-complex K, and so K has the final topology with respect to the family of inclusion maps

$$\{i_\lambda : \overline{D}_\lambda \longrightarrow K\}_{\lambda\in\Lambda},$$

in which $h_\lambda(E^{n_\lambda}) = \overline{D}_\lambda$ are closed compact Hausdorff spaces since each h_λ is a map and K is Hausdorff.

We wish to prove that K is identical to kK. We know that the identity map

$$kK \longrightarrow K$$

is a map, so we just have to prove the continuity of the identity $1 : K \longrightarrow kK$. Now the $1 \circ i_\lambda$ are maps from a compact Hausdorff space into K for all $\lambda \in \Lambda$ and so, by Proposition 4, are maps. It follow by Theorem 1 that $1 : K \longrightarrow kK$ is a map, and so K is identical to kK. Hence K is a $CG - space$. □

Remark 2. *Let X carry the* initial topology *(see [Br1, Pg.153]), relative to the family of maps*

$$\{g_j : X \longrightarrow X_j\}_{j \in J}.$$

If the spaces X_j are $CG -$ spaces, it does not necessarily follow that X is a $CG -$ space. The product space $Y \times Z$ carries the initial topology relative to the projections

$$p_1 : Y \times Z \longrightarrow Y,$$

and

$$p_2 : Y \times Z \longrightarrow Z,$$

however there are well known examples (i.e. [Br2] and [D]), where Y and Z are $CW -$ complexes, yet $Y \times Z$ is not a $CG -$ space. The following result tells us that the $CG - ification$ of the initial topology in the usual sense is the appropriate model for a $CG -$ space initial topology on X, and hence is a CG-space.

Theorem 3. ***The Universal Property for*** *$CG-$sense* ***Initial Topology on*** X**.** *Let $\{X_j\}_{j \in J}$ be a family of a $CG -$ spaces, and X carry the initial topology relative to a collection of maps*

$$\{g_j : X \longrightarrow X_j\}_{j \in J}.$$

Then kX is the initial topology of X in the $CG -$ sense, that is it satisfies from the following Universal Property,

(a) each of

$$g_j : kX \longrightarrow X_j$$

are maps, and

(b) If W is a $CG-$ space and

$$h : W \longrightarrow X$$

is a map, then

$$h : W \longrightarrow kX$$

is a map if and only if the composites

$$g_j \circ h : W \longrightarrow X_j$$

are maps, for all $j \in J$.

Proof. (a) follows from Proposition 2, (b) from Proposition 2, and the Universal Property of initial topology in the usual sense. □

Definition 3. *If X and Y are sets, then a map*

$$\alpha : W \longrightarrow X \times Y$$

is of the form $< \alpha_1, \alpha_2 >$, where

$$\alpha_1 : W \longrightarrow X,$$

and

$$\alpha_2 : W \longrightarrow Y,$$

thus $\alpha(w) =< \alpha_1, \alpha_2 > (w) = (\alpha_1(w), \alpha_2(w))$, for all $w \in W$.

If W, X and Z are spaces, then the Universal Property of products spaces asserts that α is a map if and only if α_1 and α_2 are map.

We define $X \times_k Y = k(X \times Y)$. For a $CG-$ spaces X and Y, it follows from Theorem 3 that $X \times_k Y$ is the product of X and Y in the $CG-$ sense.

Definition 4. *Given maps*

$$p : X \longrightarrow B,$$

and

$$q : Y \longrightarrow B,$$

then we will define the pullback space or fibred product space of X and Y, *to be the subspace of $X \times Y$ with underlying set*

$$X \sqcap Y = \{(x, y) | p(x) = q(y)\}.$$

In this situation

$$p^*q : X \sqcap Y \longrightarrow X,$$

and

$$q^*p : X \sqcap Y \longrightarrow Y$$

will be denote the corresponding induced projections. Let W *be a space. Then it is standard that* $X \sqcap Y$ *carries the initial topology relative to the maps* p^*q*, and* q^*p*. The typical map*

$$W \longrightarrow X \sqcap Y$$

will be denoted by $\langle h, k \rangle$*, where* $h \in M(W, X)$ *and* $k \in M(W, Y)$ *with* $ph = qk$*, thus* $\langle h, k \rangle(w) = (h(w), k(w))$ *where* $w \in W$*.*

The $CG-ification$ *of* $X \sqcap Y$ *will be denoted by* $X \sqcap_k Y$*. It follows from Theorem 3 that* $X \sqcap_k Y$ *carries the* $CG-$ *sense initial topology relative to* $k(p^*q)$*, and* $k(q^*p)$*.*

EXPONENTIAL RULES FOR A $CG-$ *spaces*

If X and Y are spaces, then $M(X, Y)$ will denote the set of all maps from X to Y. In this chapter, in cases where $M(X, Y)$ is a topological space, it will be assumed to have the compact-open topology. In the category of topological spaces, we have the following propositions.

Proposition 10. ***The Proper Condition.*** *[H, Ch. V, Lm. 3.1] Let* X*,* Y *and* Z *be an arbitrary spaces. If* $f : X \times Y \longrightarrow Z$ *is a map, then the rule* $g(x)(y) = f(x, y)$*, where* $x \in X$ *and* $y \in Y$*, determines a well defined map*

$$g : X \longrightarrow M(Y, Z).$$

Proposition 11. ***The Admissible Condition.*** *[H, Ch. V, Cr. 3.5 and Pr. 3.6] Let* X*,* Y *and* Z *be spaces. If either*

(a) Y *is locally compact and Hausdorff, or if*

(b) X *and* Y *are both first countable and Hausdorff spaces*

and if

$$g : X \longrightarrow M(Y, Z)$$

is a map, then

$$f : X \times Y \longrightarrow Z$$

is a map, where $f(x, y) = g(x)(y)$ *for all* $x \in X$*,* $y \in Y$*.*

Corollary 5. *Let Z be a space, and C be a locally compact Hausdorff space. Then*

$$e_C : M(C,Z) \times C \longrightarrow Z,$$

defined by $e_C(f,c) = f(c)$, where $f \in M(C,Z)$ and $c \in C$, is a map.

Proof. We simply apply Proposition 11 to the identity map on $M(C,Z)$, and then obtain e_C. □

Lemma 1. *If X is a $CG-$space, and C is a compact Hausdorff, then $X \times C$ is a $CG-$ space.*

Proof. We need to prove that the identity map

$$1 : X \times C \longrightarrow X \times_k C$$

is a map. The first step is to show that $X \times C$ has the final topology relative to all maps

$$h \times 1_C : K \times C \longrightarrow X \times C$$

where K is a compact Hausdorff, and $h \in M(K,X)$.

Let Z be an arbitrary space and

$$f : X \times C \longrightarrow Z$$

be a map. We will assume that

$$f \circ (h \times 1_C) : K \times C \longrightarrow Z$$

is a map for every compact Hausdorff spaces K, and all h in $M(K,X)$. It follows by the proper condition for the category of all topological spaces (Proposition 11) that there is an associated map

$$u : K \longrightarrow M(C,Z)$$

determined by the rule

$$\begin{aligned} u(y)(c) &= f \circ (h \times 1_C)(y,c) \\ &= f(h(y),c) \\ &= (gh(y))(c) \end{aligned}$$

where $y \in K$ and the map

$$g : X \longrightarrow M(C, Z)$$

corresponds to f by the rule $g(x)(c) = f(x, c)$, for $x \in K$ and $c \in C$. Hence $g \circ h = u$ is a map for all choices of K and h. The Universal Property, associated with the CG-space topology on X, implies that

$$g : X \longrightarrow M(C, Z)$$

is a map.

The admissible condition for the category of all spaces (Proposition 11) now ensures that f is a map. Hence the maps

$$h \times 1_C : K \times C \longrightarrow X \times C$$

satisfy the Universal Property associated with the final topology on $X \times C$, so $X \times C$ has that topology.

We will again assume that K is a compact Hausdorff space, and that $h : K \longrightarrow X$ is a map. Then

$$h \times 1_C : K \times C \longrightarrow X \times C$$

and

$$h \times 1_C : k(K \times C) \longrightarrow k(X \times C)$$

are maps, where $k(X \times C) = X \times C$. Now $K \times C$ is compact Hausdorff, so it is a CG-space, i.e. $k(X \times C) = X \times C$; hence

$$h \times 1_C : k(K \times C) \longrightarrow X \times_k C$$

is a map.

Now this last map is the composite

$$\begin{array}{ccc} K \times C & \xrightarrow{h \times 1_C} & X \times C \\ & \searrow^{1 \circ (h \times 1_C)} & \downarrow 1 \\ & & X \times_k C \end{array}$$

so it follows by the Universal property established earlier in this proof, that

$$1 : X \times C \longrightarrow X \times_k C$$

is a map.

Hence $X \times C = X \times_k C$, and so is a $CG-space.$ □

Theorem 4. ***The Exponential Law for a CG-spaces.*** *Let X, Y and Z be $CG-$ spaces. Then*

$$f : X \times_k Y \longrightarrow Z$$

is a map if and only if

$$g : X \longrightarrow kM(Y, Z)$$

is a map, where $f(x, y) = g(x)(y)$ for all $x \in X$, $y \in Y$, and $M(Y, Z)$ is the space of maps from Y to Z with the compact open topology.

Proof. The proof follows immediately from Propositions 11 and from Proposition 12 below. □

Proposition 12. ***The Proper Condition for a CG-spaces.*** *Let X, Y and Z be a $CG-$ spaces, and $f : X \times_k Y \longrightarrow Z$ be a map. Then the rule $g(x)(y) = f(x, y)$, where $x \in X$ and $y \in Y$, determines a well defined map*

$$g : X \longrightarrow kM(Y, Z).$$

Proof. Fixing $x \in X$, let

$$g(x) : Y \longrightarrow Z$$

be defined by $g(x)(y) = f(x, y)$ where $y \in Y$. Then $g(x)$ is clearly a well defined map. Now we need to prove that $g(x)$ is a map. If $c_x : X \longrightarrow Y$ is the constant map at value x, then

$$< c_x, 1_Y >: Y \longrightarrow X \times_k Y$$

defined by $< c_x, 1_Y > (y) = (x, y)$ is a map (see Remark 2). It follows that $g(x) = f \circ < c_x, 1_Y >$ is a map.

Let C be a compact Hausdorff space, and $\alpha \in M(C, X)$. We wish to prove that $g \circ \alpha$ is a map for all choices of α. Then, by the Universal Property associated with the CG-topology on X, g is a map.

Now if

$$\alpha \times 1_Y : C \times Y \longrightarrow X \times Y$$

and $k(\alpha \times 1_Y)$ are maps, then

$$f \circ k(\alpha \times 1_Y) : C \times Y \longrightarrow Z$$

is a map by the previous Lemma. It follows by the proper condition in the ordinary sense; (see Proposition 10), that

$$h : C \longrightarrow M(Y, Z)$$

is a map, where

$$h(c)(y) = f(\alpha(c), y) = g(\alpha(c))(y),$$

and where $c \in C$ and $y \in Y$. Hence

$$h(c) = g(\alpha(c)) = (g \circ \alpha)(c).$$

Thus $g \circ \alpha = h$ is a map for all $\alpha \in M(C, X)$, and the result follows. □

Proposition 13. *If Y and Z are $CG -$ spaces, then the map*

$$e : kM(Y, Z) \times_k Y \longrightarrow Z,$$

defined by $e(f, y) = f(y)$, for all $f \in kM(Y, Z)$ and $y \in Y$, is a map.

Proof. Given that C is compact Hausdorff, and

$$\alpha : C \longrightarrow kM(Y, Z) \times_k Y$$

is a map. We want to prove that $e \circ \alpha$ is a map, where $\alpha(c) = (\alpha_1(c), \alpha_2(c))$,

$$\alpha_1 : C \longrightarrow kM(Y, Z),$$

and

$$\alpha_2 : C \longrightarrow Y$$

are maps. Now, it follows by Proposition 4 that

$$\alpha_1 : C \longrightarrow kM(Y, Z)$$

is also a map, and

$$\alpha_2^* : M(Y, Z) \longrightarrow M(C, Z),$$

$\alpha^*(h) = h \circ \alpha_2$ is a map, where $h \in M(Y, Z)$. Now

$$e_C : M(C, Z) \times C \longrightarrow Z$$

is a map since C is compact Hausdorff (Corollary 5). Then $e \circ \alpha$ is a map because

$$e \circ \alpha \ = \ e < \alpha_1, \alpha_2 >$$

$$= e_C < \alpha_2^* \circ \alpha_1, 1_C > .$$

Hence e is a map. □

Proposition 14. ***The Admissible Condition for a CG-spaces.*** *If X, Y and Z are $CG-spaces$, and*

$$g : X \longrightarrow kM(Y, Z)$$

is a map, then

$$f : X \times_k Y \longrightarrow Z$$

is a map defined by the rule $f(x, y) = g(x)(y)$.

Proof. The proof follows because f is the composite

$$X \times_k Y \overset{g \times_k 1_Y}{\longrightarrow} kM(Y, Z) \times_k Y \overset{e}{\longrightarrow} Z,$$

and

$$\begin{aligned} e(g \times_k 1_Y)(x, y) &= e(g(x), 1_Y(y)) \\ &= e(g(x), y) \\ &= g(x)(y) \\ &= f(x, y). \end{aligned}$$

Hence f is a map. □

Definition 5. *A map $q : Y \longrightarrow B$ in which Y and B are CG-spaces is a Hurewicz fibration in the CG-sense if, whenever we are given a CG-space A, a map $f : A \longrightarrow Y$ and homotopy $H : A \times I \longrightarrow B$ that starts with $q \circ f$, there exists a homotopy $G : A \times I \longrightarrow Y$ that starts with f and satisfies $q \circ G = H$ making commutative the following diagram:*

$$\begin{array}{ccc} A \times \{0\} & \xrightarrow{f} & Y \\ {\scriptstyle j_o}\downarrow & \overset{G}{\nearrow} & \downarrow{\scriptstyle q} \\ A \times I & \xrightarrow[H]{} & B, \end{array}$$

where $j_o : A \times \{0\} \longrightarrow A \times I$ is defined by the rule $j_o(a) = (a, 0)$.

Theorem 5. *If*

$$q : Y \longrightarrow B$$

is a Hurewicz fibration in the sense of usual category of spaces, then

$$kq : kY \longrightarrow kB$$

is a Hurewicz fibration in the CG *– sense.*

Proof. Let A be a CG – *space*, and

$$f : A \times \{0\} \longrightarrow kY,$$

and

$$F : A \times I \longrightarrow kB$$

be maps such that $F(a, 0) = kq(f(a, 0))$ for all $a \in A$. We wish to prove that there is a map

$$F^* : A \times I \longrightarrow kY$$

such that $F^*(a, 0) = f(a, 0)$ for $a \in A$, and $kp \circ G = F$.
Taking 1_Y, and 1_B to be the identity maps $kY \longrightarrow Y$, and $kB \longrightarrow B$, respectively, then

$$1_Y \circ f : A \times \{0\} \longrightarrow Y,$$

and

$$1_B \circ F : A \times I \longrightarrow B$$

are maps such that $1_B \circ F(a, 0) = q \circ (1_Y \circ f)(a, 0)$, for all $a \in A$. Then it follows from the covering homotopy property for p, that we can find a map

$$H : A \times I \longrightarrow Y$$

such that $1_B \circ F = q \circ H$ and $H(a, 0) = 1_Y f(a, 0)$, for all $a \in A$. We define

$$G : A \times I \longrightarrow kY$$

as having the same underlying map as H. Now $A \times I$ is a CG – *space* (Lemma 1), so H is a map by Proposition 3. The result follows. □

2. MAPPING SPACES AND FIBREWISE HOMOTOPY THEORY

Definition 6. *A topological space B is said to be weak Hausdorff if*

$$\triangle_B = \{(b,b) \mid b \in B\} \subset B \times B,$$

is closed in $B \times_k B$.

An alternative definition of weak Hausdorff is, that for every compact Hausdorff C, and every map $f : C \longrightarrow B$, the image $f(C)$ is closed in B. More details about this definition which is equivalent to the one above, can be found in [St].

Definition 7. *If Z is a space, we will define $Z^{\sim}$ to be the set $Z \cup \{w\}$ where $w \notin Z$. We give $Z^{\sim}$ the topology whose closed sets are $Z^{\sim}$ itself, and the closed sets of Z. Let C be a closed subspace of Y, and*

$$f : C \longrightarrow Z$$

be a map, so f is a partial map from Y to Z. Then there is an associated map

$$f^{\sim} : Y \longrightarrow Z^{\sim}$$

defined by the rule

$$f^{\sim}(y) = \begin{cases} f(y), & \text{if } y \in C; \\ w, & \text{otherwise.} \end{cases}$$

Definition 8. *Let B be a T_1 − space, and*

$$q : Y \longrightarrow B$$

be a map. We define the set

$$Y!Z = \bigcup_{b \in B} M(Y \mid b, Z)$$

where for $b \in B$, $Y \mid b = q^{-1}(b)$ *is the fibre. We also define the map*

$$q!Z : Y!Z \longrightarrow B$$

to be the function that sends all maps

$$Y \mid b \longrightarrow Z$$

to b*, for all* $b \in B$*. Since* B *is a* $T_1 -$ *space, each fibre* $q^{-1}(b) = Y \mid b$ *is closed in* Y*. It follows that if* $f \in M(Y \mid b, Z)$*, then* $i(f) = f^{\sim}$ *defines a map*

$$i : Y!Z \longrightarrow M(Y, Z^{\sim}).$$

Definition 9. *We define the modified compact-open topology on* $Y!Z$ *to be the initial topology relative to* i*, and* $q!Z$*. We call* $Y!Z$ *the free range mapping space determined by* Y*,* q *and* Z*. It has a subbase consisting of all sets of the form* $(q!Z)^{-1}(U)$*, where* U *is open in* B*, together with all sets of the form*

$$W(A, V) = \{f \in Y!Z \mid f(A \cap dom(f)) \subset V\},$$

where A *ranges over the compact subsets of* Y*, and* V *ranges over the open subsets of* Z*.*

We now introduce a $CG - version$ of the free range mapping space $Y!Z$, i.e. $k(Y!Z)$. Thus this space carries the initial topology relative to $k(q!Z)$, and $k(i)$ in the sense of $CG - spaces$, i.e. it is the CG-ification of the previously defined topology on $Y!Z$. We now remind the reader of the Fibred Exponential Law, due to P. Booth. It is followed by our CG version.

Theorem 6. ***Fibred Exponential Law.*** *[B3, Th. 3.3 and 2.4.3] Let* B *be a Hausdorff space,* Z *be a space, and*

$$p : X \longrightarrow B$$

and

$$q : Y \longrightarrow B$$

be a maps.

(a) ***Proper Condition:*** *If*

$$f^{>} : X \sqcap Y \longrightarrow Z$$

is a map, then the rule $f^{<}(x)(y) = f^{>}(x, y)$ *determines a fibrewise map*

$$f^{<} : X \longrightarrow Y!Z,$$

where $p(x) = q(y)$. *Thus* $f^{<}$ *is a map such that* $(q!Z) \circ f^{<} = p$.

(b) ***Admissible Condition:*** *Let us assume that either*

(i) *Y is locally compact and Hausdorff, or*

(ii) *X and Y are first countable and Hausdorff, or*

(iii) *W is a space,*

$$p : B \times W \longrightarrow B$$

the projection, and $Y \times W$ a CG-space.

Then, given a fibrewise map

$$f^{<} : X \longrightarrow Y!Z,$$

the above rule determines a map

$$f^{>} : X \sqcap Y \longrightarrow Z.$$

Theorem 7. ***Fibred Exponential Law for CG-spaces.*** *Let X, Y, Z and B be CG – spaces, with B weak Hausdorff space, and*

$$p : X \longrightarrow B,$$

$$q : Y \longrightarrow B$$

and

$$r : Z \longrightarrow B$$

be maps. Then there is a bijective correspondence between

(a) *maps*

$$f^{>} : X \sqcap_k Y \longrightarrow Z,$$

and

(b) *fibrewise maps*

$$f^{<} : X \longrightarrow k(Y!Z)$$

determined by the rule $f^{>}(x,y) = f^{<}(x)(y)$ *when* $p(x) = q(y)$.

Proof. There is a map

$$p \times q : X \times Y \longrightarrow B \times B,$$

and so

$$p \times_k q : X \times_k Y \longrightarrow B \times_k B$$

is a map where $X \times_k Y = k(X \times Y)$. The weak Hausdorff condition ensures that $\triangle_B$ is closed in $B \times_k B$. Hence

$$(p \times_k q)^{-1}(\triangle_B) = X \sqcap_k Y,$$

is a closed subspace of $X \times_k Y$, so it follows that our theory of partial maps from Y to Z, with closed domains, is relevant to the situation under consideration.

Let

$$f^{>} : X \sqcap_k Y \longrightarrow Z$$

be a map. Then $f^{>}$ determines a map

$$g^{>} = (f^{>})^{\sim} : X \times_k Y \longrightarrow Z^{\sim}$$

by the rule

$$g^{>}(x,y) = \begin{cases} f^{>}(x,y), & \text{for } (x,y) \in X \sqcap Y, \\ w, & \textit{otherwise.} \end{cases}$$

We know by the proper condition (Proposition 12) that there is an associated map

$$g^{<} : X \longrightarrow kM(Y, Z^{\sim})$$

defined by $g^{<}(x)(y) = g^{>}(x,y)$, where $x \in X$, and $y \in Y$. So

$$g^{<}(x)(y) = w$$

if and only if

$$p(x) \neq q(y).$$

We now define

$$f^{<} : X \longrightarrow k(Y!Z)$$

by $f^{<}(x)(y) = g^{<}(x)(y)$ for $(x, y) \in X \sqcap Y$. Then

$$\begin{aligned} f^{<}(x)(y) &= g^{<}(x)(y) \\ &= g^{>}(x, y) \\ &= f^{>}(x, y). \end{aligned}$$

However, $f^{<}(x)(y)$ is undefined when $p(x) \neq q(y)$. If $p(x) = b$, then $f^{<}(x)(y)$ is defined for all $y \in Y \mid b$. i.e. $f^{<}(x) \in Y!Z$, and $(q!Z)(f^{<}(x)) = b$. So $(q!Z) \circ f^{<} = p$, and $(q!Z) \circ f^{<}$ is a map. Also, recalling our definition of the topology on $Y!Z$, $i \circ f^{<} = g^{<}$ is a map. It follows by the Universal Property of the CG initial topology on $Y!Z$, and by Proposition 3, that $f^{<}$ is a map.

Note that the step that uses the continuity of $g^{<}$ to establish the continuity of $f^{<}$ the admissible condition for CG-spaces, is the place where we use the fact that X and Y are CG-spaces. The argument is reversible, and so the proof is complete. □

Definition 10. *If X and Y are spaces, then $[X, Y]$ will denote the set of homotopy classes of maps from X to Y. If X and Y are based spaces, then $M^o(X, Y)$ denotes the set of based maps from X to Y, with the CG-ified compact-open topology. In this case $[X, Y]^o$ will denote the corresponding set of based homotopy classes. If Y and B are based spaces, and*

$$q : Y \longrightarrow B$$

is a map, the set of based sections to q, i.e.

$$Sec^o(q) = \{f \in M^o(B, Y) \mid q \circ f = 1_B\}$$

is equipped with the $CG-ified$ compact-open topology.
In addition, if B and Z have basepoints $b_o \in B$ and $z_o \in Z$, then the constant map

$$c_{z_o} : Y \mid b_o \longrightarrow Z$$

is defined to be $c_{z_o}(y) = z_o$. We take c_{z_o} as basepoint for $Y!Z$. The space $M(X, A; Y, B)$ denotes the set of maps from X to Y for which $f(A) \subseteq B$, again with the CG-ified compact-open topology, and $[X, A; Y, B]$ the corresponding set of homotopy classes.

Definition 11. ***Vertical Homotopy.*** *Let*

$$q : Y \longrightarrow B$$

be a map, and ℓ_o *and* ℓ_1 *be sections to* q*. A homotopy*

$$F : B \times I \longrightarrow Y$$

such that

$$F_t = F(-, t) : B \longrightarrow Y$$

is a section to q*, for all* $t \in I$*, is said to be a vertical homotopy. The sections* ℓ_o*, and* ℓ_1 *are said to be vertically homotopic if there is a vertical homotopy from* ℓ_o *to* ℓ_1*.*

Corollary 6. ***Section Rule.*** *[B3, Cr. 3.4] Let* B *be a Hausdorff space, and*

$$q : Y \longrightarrow B$$

be a map.

(a) *If*

$$l : Y \longrightarrow Z$$

is a map, then the rule

$$l^\bullet(b) = l|(Y|b) : Y|b \longrightarrow Z$$

where $b \in B$*, defines a section* $l^\bullet$ *to* $q!Z$*. Equivalently, we may define* $l^\bullet$ *by* $l^\bullet(b)(y) = l(y)$*, where* $q(y) = b$*.*

(b) *If* Y *is CG-space and* $l^\bullet$ *is a section to* $q!Z$*, then the rule stated in (a) determines a map*

$$l : Y \longrightarrow Z.$$

Let (Z, z_o), and (B, b_o) be based spaces, B being weak Hausdorff sapace, and

$$q : Y \longrightarrow B$$

be a map.

If

$$\ell : (Y, Y|b_o) \longrightarrow (Z, z_o)$$

is a map, then we define

$$\ell^\bullet(b) : Y|b \longrightarrow Z$$

to be the restriction of ℓ to $Y|b$.

Corollary 7. *Section Rule for CG-spaces.* *Let X, Y and Z be CG-spaces. There is a bijective correspondence*

(a) $\theta : M(Y, Y|b_o; Z, z_o) \longrightarrow Sec^o(q!Z)$,
defined by $\theta(\ell) = \ell^\bullet$, *for all* $\ell \in M(Y, Y|b_o; Z, z_o)$.

(b) Let $\ell_o, \ell_1 \in M(Y, Y|b_o; Z, z_o)$. *Then* $\ell_o \simeq \ell_1$ *via a homotopy*

$$F : (Y \times I; (Y|b_o) \times I) \longrightarrow (Z, z_o),$$

if and only if $\ell_o^\bullet \simeq \ell_1^\bullet$ *via a based vertical homotopy.*

(c) The rule $[\ell] \rightsquigarrow [\ell^\bullet]$ *defines a bijection*

$$\lambda : [Y, Y|b_o; Z, z_o] \longrightarrow \pi_o(Sec^o(q!Z)),$$

where $\pi_o(Sec^o(q!Z))$ *denotes the set of based vertical homotopy classes of based sections to* $q!Z$, *(or equivalently the path components of* $Sec^o(q!Z)$ *with the compact open topology).*

Proof.

(a) The domain of $\ell^\bullet(b)$ is $Y|b$ so $q \circ \ell = 1_B$. Also $\ell^\bullet(b_o) = \ell|(Y|b_o) = c_{z_o}$, so $\ell^\bullet$ is base point preserving. If $B \sqcap Y$ is the pullback of 1_B, and

$$q : Y \longrightarrow B,$$

then the projection

$$\pi : B \sqcap Y \longrightarrow Y$$

is a homeomorphism. Thus we have a bijective correspondence between maps

$$\ell : Y \longrightarrow Z$$

and maps

$$\ell \circ \pi : B \sqcap Y \longrightarrow Z.$$

The map

$$\ell^\bullet : B \longrightarrow Y!Z,$$

defined earlier, is the image of $\ell \circ \pi$ under the Exponential correspondence, so $\ell^\bullet(b)(y) = \ell \circ \pi(b, y) = \ell(y)$, where $q(y) = b$. The above argument is reversible, so the result follows.

(b) It follows by arguments similar to those in the proof of (a), that

$$F : (Y \times I; (Y|b_o) \times I) \longrightarrow (Z, z_o)$$

is a map, if and only if

$$G : (B \times I, \{b_o\} \times I) \longrightarrow (Y!Z, c_{z_o})$$

is a map, where $F(y,t) = G(b,t)(y)$, for all $y \in Y$, $t \in I$ and $b = q(y)$. Moreover, $F(Y|b_o \times I) = z_o$ if and only if $G(b_o \times I)(y) = z_o$ for all $y \in Y|b_o$. Now

$$(B \sqcap Y) \times I \cong (B \times I) \sqcap Y,$$

and so $\ell_o \simeq \ell_1$ if and only if $\ell_o^\bullet \simeq \ell_1^\bullet$, as required.

(c) This follows easily from (a) and (b). □

Comparison 1. Note that the inconvenient assumptions built into the admissible condition of Theorem 6 are avoided in the CG-version of Theorem 7. In the same way in the inconvenient assumption of Corollary 6 of [B3] are avoided in Corollary 7.

Example Let

$$q : Y \longrightarrow B$$

be a map, Z a space and $z_o \in Z$. Then there is a map

$$\sigma_{z_o} : B \longrightarrow Y!Z,$$

defined by the rule $\sigma_{z_o}(b)(y) = z_o$ for all $y \in Y|b$ where

$$\sigma_{z_o}(b) : Y|b \longrightarrow Z$$

is the constant map with value z_o.
Now σ_{z_o} corresponds, via Corollary 6, part(a), to the constant map

$$Y \longrightarrow Z$$

valued at z_o. Hence σ_{z_o} is a map. It is easily seen that it is also a section to $q!Z$.

Definition 12. *We now introduce some fibrewise terminology. Fibrewise spaces in the free sense are simply maps of spaces into* B*. Let*

$$p : X \longrightarrow B$$

and

$$q : Y \longrightarrow B$$

be fibrewise spaces in the free sense. Then a fibrewise map from

$$p : X \longrightarrow B$$

to

$$q : Y \longrightarrow B$$

in the free sense is a map

$$f : X \longrightarrow Y$$

such that $q \circ f = p$*.*

A fibrewise space in the based sense is a pair (p, s)*, where*

$$p : X \longrightarrow B$$

is a map, and

$$s : B \longrightarrow X$$

is a section to p*. The reader can observe that if* B *is a point* $*$*, then*

$$s : * \longrightarrow X$$

is essentially just the point $s(*) \in X$*, so* $(p : X \longrightarrow *, s : * \longrightarrow X)$ *is essentially just the based space* $(X, s(*))$*.*

If (p, s) and (q, t) are fibrewise based spaces, then $(p \sqcap q, \langle s, t \rangle)$ is also a fibrewise based space.

A fibrewise map in the based sense, from (p, s) to (q, t) is a map

$$f : X \longrightarrow Y$$

such that $q \circ f = p$ and $f \circ s = t$. The set of based maps of this sort will be denoted by $M_B(X, Y)$.

Definition 13. *If $f, g \in M_B(X, Y)$, then a fibrewise based homotopy from f to g is both a fibrewise map*

$$F : X \times I \longrightarrow Y$$

and a based homotopy such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, for all $x \in X$.

Thus a fibrewise based homotopy from f to g is just a homotopy in the ordinary sense from f to g, which is a fibrewise based map at each stage of the deformation. We write $f \simeq_B g$.

Definition 14. *The fibrewise tertiary system (q, s, m) consists of a fibrewise based space Y over B, i.e. a based map*

$$q : Y \longrightarrow B,$$

a based section

$$s : B \longrightarrow Y$$

to q, and a fibrewise based map

$$m : Y \sqcap Y \longrightarrow Y.$$

The map m is a fibrewise multiplication. Thus m is fibrewise in the sense that $q \circ m = q \sqcap q$,

$$q \sqcap q : Y \sqcap Y \longrightarrow B$$

is define by $(q \sqcap q)(y_1, y_2) = q(y_1) = q(y_2)$ where $(y_1, y_2) \in Y \sqcap Y$

Definition 15. *The fibrewise multiplication m is fibrewise homotopy commutative if $m \simeq_B m \circ \tau$, where τ is the switching fibrewise homeomorphism*

$$\tau : Y \sqcap Y \longrightarrow Y \sqcap Y$$

defined by $\tau(y, y') = (y', y)$, for $(y, y') \in Y \sqcap Y$.

Definition 16. *The fibrewise multiplication m is fibrewise homotopy associative if*

$$m(m \sqcap 1_Y) \simeq_B m(1_Y \sqcap m)$$

where

$$Y \sqcap Y \sqcap Y \xrightarrow{1_B \sqcap m} Y \sqcap Y \xrightarrow{m} Y,$$

and

$$Y \sqcap Y \sqcap Y \xrightarrow{m \sqcap 1_B} Y \sqcap Y \xrightarrow{m} Y.$$

Definition 17. *The fibrewise multiplication m has a fibrewise homotopy identity, or satisfies the fibrewise Hopf condition if*

$$m(1_Y \sqcap (s \circ q))\triangle \simeq_B 1_Y \simeq_B m((s \circ q) \sqcap 1_Y)\triangle,$$

in which $\triangle : Y \longrightarrow Y \sqcap Y$ denotes the diagonal map, and where

$$Y \xrightarrow{\triangle} Y \sqcap Y \xrightarrow{1_Y \sqcap (s \circ q)} Y \sqcap Y \xrightarrow{m} Y,$$

and

$$Y \xrightarrow{\triangle} Y \sqcap Y \xrightarrow{(s \circ q) \sqcap 1_Y} Y \sqcap Y \xrightarrow{m} Y.$$

Definition 18. *The fibrewise based map*

$$\mu : Y \longrightarrow Y$$

is a fibrewise homotopy inversion for the fibrewise multiplication m if

$$m(1_Y \sqcap \mu)\triangle \simeq_B s \circ q \simeq_B m(\mu \sqcap 1_Y)\triangle,$$

where

$$Y \xrightarrow{\triangle} Y \sqcap Y \xrightarrow{1_Y \sqcap \mu} Y \sqcap Y \xrightarrow{m} Y,$$

and

$$Y \xrightarrow{\triangle} Y \sqcap Y \xrightarrow{\mu \sqcap 1_Y} Y \sqcap Y \xrightarrow{m} Y.$$

Definition 19. *A homotopy associative fibrewise tertiary system satisfying the fibrewise Hopf condition, and for which the fibrewise multiplication admits an inversion, is called a fibrewise H-group. If a fibrewise H-group is fibrewise homotopy commutative, then it will be said to be fibrewise homotopy Abelian. More details concerning fibrewise homotopy are given in a locus classicus [J].*

Proposition 15. *Let Y, Z and B be a CG-spaces, Z an $H-group$, B weak Hausdorff space, and a*

$$q : Y \longrightarrow B$$

a map. Then there is a fibrewise map

$$n : Y!Z \sqcap Y!Z \longrightarrow Y!Z,$$

$n(f,g) = m(f \times g)\triangle_b$, where $b \in B$, $f, g \in M(Y|b, Z)$, m denotes the operation on Z, $\triangle_b$ is the diagonal map for $Y|b$, and $m(f \times g)\triangle_b$ is the following composite of maps

$$Y|b \xrightarrow{\triangle_b} Y|b \times Y|b \xrightarrow{f \times g} Z \times Z \xrightarrow{m} Z.$$

Then, defining σ_e as in the latest example, the tertiary system

$$(q!Z, \sigma_{(e)}, n)$$

is a fibrewise H-group. Further, if Z is homotopy Abelian, then $(q!Z, \sigma_e, n)$ is fibrewise homotopy Abelian.

Proof. If Y is a space and Z is an $H-group$, then the operation

$$n : M(Y,Z) \times M(Y,Z) \longrightarrow M(Y,Z),$$

defined by $n(f,g) = n \circ (f \times g) \circ \triangle_Y$, together with the identity map $c_e : Y \longrightarrow Z$, makes $M(Y,Z)$ into an $H-group$. In fact, the proof of this preliminary result is as follows. There are maps

$$M(Y,Z) \times M(Y,Z) \times Y \longrightarrow M(Y,Z) \times Y \times M(Y,Z) \times Y,$$

$(f,g,y) \longrightarrow (f,y,g,y)$, where $f,g \in M(Y,Z)$ and $y \in Y$, and

$$M(Y,Z) \times Y \times M(Y,Z) \times Y \longrightarrow Z \times Z,$$

$(f,y,g,y) \longrightarrow (f(y),g(y))$ where again $f,g \in M(Y,Z)$ and $y \in Y$. The former is obviously a map, the continuity of the latter follows from Proposition 11. Composing these two maps with the operation on Z, that is

$$Z \times Z \longrightarrow Z,$$

$(z_1,z_2) \longrightarrow z_1 \cdot z_2$ where $z_1, z_2 \in Z$, we obtain a composite map

$$M(Y,Z) \times M(Y,Z) \times Y \longrightarrow Z,$$

$(f,g,y) \longrightarrow f(y) \cdot g(y)$. If we now apply the Proper Condition (Proposition 10), we obtain the map

$$M(Y,Z) \times M(Y,Z) \longrightarrow M(Y,Z),$$

$(f,g) \longrightarrow (y \longrightarrow f(y) \cdot g(y))$ namely the map n defined above. It is now routine to verify that $(M(Y,Z), n)$, with the constant map value the identity of Z as identity, is an H-group. If Z is homotopy Abelian, then so also is $M(Y,Z)$.

The proof of Proposition 15 is a direct generalization of that argument just given, using the fibred exponential law of Theorem 7, rather than the usual exponential law for spaces. □

Proposition 16. *If* $(q : Y \longrightarrow B, t : B \longrightarrow Y, m : Y \sqcap Y \longrightarrow Y)$ *is a fibrewise homotopy Abelian* $H-$*group, where* B *and* Y *are CG-spaces, then* $Sec^o(q)$ *is a homotopy Abelian* $H-$*group. Thus if* $t_1, t_2 \in Sec^o(q)$*, the operation* $+_B$ *on* $Sec^o(q)$ *is defined by*

$$t_1 +_B t_2 = m \circ \langle t_1, t_2 \rangle,$$

and the identity for $Sec^o(q)$ *is* t.

Proof. There is a map

$$Sec^o(q) \times Sec^o(q) \times B \longrightarrow Y,$$

given by $(t_1, t_2, b) \longrightarrow m(t_1(b), t_2(b))$ where $t_1, t_2 \in Sec^o(q)$ and $b \in B$. Its continuity can be verified by and argument similar to that used in the proof of Proposition 13. Applying the Proper Condition (Proposition 10), we obtain a map

$$Sec^o(q) \times Sec^o(q) \longrightarrow M(B, Y),$$

given by $(t_1, t_2) \longrightarrow (b \longrightarrow m(t_1(b), t_2(b)))$, where $t_1, t_2 \in Sec^o(q)$ and $b \in B$. Now $q \circ m(t_1(b), t_2(b))) = q(t_1(b)) = b$, for all $b \in B$. So $b \longrightarrow m(t_1(b), t_2(b))$, which is the map $t_1 +_B t_2$, is a section to q, and $(t_1, t_2) \longrightarrow t_1 +_B t_2$ is a map from

$$Sec^o(q) \times Sec^o(q) \longrightarrow Sec^o(q)$$

as required.

It is then routine to verify that $(Sec^o(q), +_B)$ with identity t, is a homotopy Abelian H-group.

□

Corollary 8. *Let* (Z, z_o) *be a CG-space and an Abelian* $H-$*group,* (B, b_o) *a based weak Hausdorff space, and* $q : Y \longrightarrow B$ *a map. Then* $Sec^o(q!Z)$ *is an Abelian* $H-$*group, and* $\pi_o(Sec^o(q!Z))$ *an Abelian group.*

Theorem 8. *Let* (Z, z_o) *be a CG-space and an Abelian* $H-$*group,* (B, b_o) *a based weak Hausdorff space, and* $q : Y \longrightarrow B$ *a map. Then*

(a) the set

$$[Y, Y|b_o; Z, z_o]$$

carries an Abelian group structure, defined by pointwise addition of homotopy classes, and *(b) the bijection of Corollary 7, part (c), i.e.*

$$\lambda : [Y, Y|b_o; Z, z_o]^o \approx \pi_o(Sec^o(q!Z))$$

defined by $\lambda([\ell]) = [\ell^\bullet]$, *is an isomorphism of Abelian groups.*

Proof. The proof of part (a) is routine. (b) The two group structures are both induced by the H-group structure on Z; it is routine to verify that, as expected, λ is an isomorphism of Abelian groups. □

Theorem 9. *[B3, Th. 4.1] Let B, Y and Z be spaces, where B is Hausdorff and Y is locally compact Hausdorff. If $q : Y \longrightarrow B$ is a Hurewicz fibration, then $q!Z$ is also a Hurewicz fibration.*

Theorem 10. *Let Y, Z and B be CG-spaces, B weak Hausdorff space, and*

$$q : Y \longrightarrow B$$

a Hurewicz fibration in CG-sense. Then

$$q!Z : Y!Z \longrightarrow B$$

is also a Hurewicz fibration in the CG-sense.

Proof. This is just the argument that proves Theorem 4.1 of [B3], but reinterpreted in the CG-context. We assume that

$$F : A \times I \longrightarrow B$$

is a homotopy and the restriction $F \mid A \times 0$ is denoted by F_o. We then have pullback spaces $(A \times I) \sqcap Y$, and $(A \times 0) \sqcap Y$, induced by the homotopy F and the map F_o, respectively, and associate projections

$$F^*q : (A \times I) \sqcap Y \longrightarrow A \times I,$$

$$(F_o)^*q : (A \times 0) \sqcap Y \longrightarrow A \times I,$$

$$q^*F : (A \times I) \sqcap Y \longrightarrow Y,$$
$$q^*F_o : (A \times 0) \sqcap Y \longrightarrow Y,$$

and such that

$$q \circ (q^*F) = F \circ (F^*q)$$

and

$$q \circ (q^* F_o) = F_o \circ (F_o)^* q.$$

We recall that $(A \times 0) \sqcap Y$ is a retract of $(A \times I) \sqcap Y$. The proof of this, in the usual topological context, is given in [B3, Lm. 4.2]; the CG proof is similar. Let

$$k^< : A \times 0 \longrightarrow Y!Z$$

be a map such that $(q!Z) \circ k^< = F_o$. It follows, by the Fibred Exponential Law for CG-spaces, that there is an associated map

$$k^> : (A \times 0) \sqcap Y \longrightarrow Z$$

defined by $k^>(a, 0, y) = k^<(a, 0)(y)$ where $(a, 0, y) \in (A \times 0) \sqcap Y$. So $(A \times 0) \sqcap Y$ is indeed a retract of $(A \times I) \sqcap Y$.

Let

$$R : (A \times I) \sqcap Y \longrightarrow (A \times 0) \sqcap Y$$

be a retraction. Then the composite

$$k^> \circ R : (A \times I) \sqcap Y \longrightarrow Z$$

corresponds, via the fibred exponential law, to

$$K : A \times I \longrightarrow Y!Z,$$

where $K(a, t)(y) = (k^> \circ R)(a, t, y)$, and $(a, t, y) \in (A \times I) \sqcap Y$. Then K is fibrewise over B, i.e. $(q!Z)K(a, t) = F(a, t)$, so $(q!Z) \circ K = F$. Also if $(a, 0, y) \in (A \times 0) \sqcap Y$,

$$K(a, 0)(y) = (k^> \circ R)(a, 0, y) = k^>(a, 0, y) = k^<(a, 0)(y).$$

So $K(a, 0) = k^<(a, 0)$ for $a \in A$. i.e. K extends $k^<$. Thus K lifts F and extends $k^<$, and $q!Z$ is a Hurewicz fibration. $\square$

3. APPLICATIONS TO HOMOTOPY THEORY

In this chapter we wish to give the CG results which are analogous of ones given in [B3], which there require the rather specific, and inconvenient assumptions that the CG approach seeks to eliminate. We also wish to sketch the proofs. Now the initial results (Theorem 15, 16 and 17) depend on the CG analogues of three theorems from [B3] (Theorems 5.1, 6.1 and 7.1 of that reference). Our Theorem 11, 12 and 13 below, indicate how the CG approach eliminates these specific conditions. **In this chapter we do not assume that spaces are CG-spaces, unless we specifically say so.**

The following Theorem is the CG-version of the corresponding result in [B3, Th. 8.1(b)].

Theorem 11. *Let*

$$q : Y \longrightarrow B$$

be a fibration in the CG-sense, and B be a weak Hausdorff space. Then there is a canonical bijection:

$$\theta : H^m(Y, Y|b; G) \longrightarrow \pi_o(Sec^o(q!K(G, m)))$$

where the map θ is determined by the rule $\theta[l] = [l^\bullet]$, where $l^\bullet(b)(y) = l(y)$ and $q(y) = b$.

Proof. This is the proof of [B3, Th. 8.1(b)] except that we substitute our Theorem 7 and Corollary 7 for Theorem 3.3 and Corollary 3.4 of that paper. □

If we follow the $\Omega -$ *spectrum* approach to cohomology theory which uses Eilenberg-MacLane spaces in its definition, then the associated cohomology groups are defined by

$$H^m(Y, Y|b; G) = [Y, Y|b_o; K(G, m), e],$$

(for more details concerning this spectra the reader can see [Ma, Section 8.4]).

We now give the promised analogues of the theorems from [B3], which are needed in the proofs of our main theorems. We will not state the corresponding CG-results, but rather indicate their existence in remark 4 below.

Theorem 12. *[B3, Th. 5.1] Let A be a CG-space and B be a Hausdorff space. If*

$$q : Y \longrightarrow B$$

is an identification and

$$f : A \longrightarrow B$$

is a map, then

$$f^*p : Y \sqcap A \longrightarrow A$$

is an identification.

Theorem 13. *[B3, Th. 6.1] Let*

$$q : Y \longrightarrow B$$

be a Hurewicz fibration, where B is a Hausdorff space and Y is locally compact Hausdorff. If

$$A \longrightarrow B$$

is a closed cofibration, then

$$Y|A \longrightarrow Y$$

is also a closed cofibration.

Remark 3. *Let*

$$q : Y \longrightarrow B$$

be a map and

$$t : B \longrightarrow Y$$

be a section to q. If

$$f : A \longrightarrow B$$

is a map, then

$$\sigma : A \longrightarrow Y \sqcap A,$$

defined by $\sigma(a) = (tf(a), a)$, for all $a \in A$, is a section to the projection $f^*q : Y \sqcap A \longrightarrow A$.

Theorem 14. *[B3, Th. 7.1] Let*

$$q : Y \longrightarrow B$$

be a Hurewicz fibration, with closed cofibration section t, B be Hausdorff and Y locally compact Hausdorff. If

$$f : A \longrightarrow B$$

is a map, then

$$f^*q : Y \sqcap A \longrightarrow A$$

is a Hurewicz fibration with a closed cofibration section σ.

Remark 4. *The last two Theorems depend on our Theorem 6 and Corollary 6 (Theorem 3.3 and Corollary 3.4 of [B3]). If we modify those results and proofs by assuming that all spaces are CG-spaces, that B is weak Hausdorff space, and replacing the use of Theorem 6 and Corollary 6 in the proof by our Theorem 7 and Corollary 7, we obtain the following analogies of the results above due to P. Booth.*

Theorem 15. *Let Y, A and B be CG-spaces, B weak Hausdorff space,*

$$q : Y \longrightarrow B$$

an identification and

$$f : A \longrightarrow B$$

a map. Then

$$f^*q : Y \sqcap A \longrightarrow A$$

is an identification.

Theorem 16. *Let Y, A and B be CG-spaces,*

$$q : Y \longrightarrow B$$

a Hurewicz fibration in the CG-sense, B weak Hausdorff space, and

$$A \longrightarrow B$$

a closed cofibration. Then

$$Y|A \longrightarrow Y$$

is also a closed cofibration.

Theorem 17. *Let Y, A and B be CG-spaces, B weak Hausdorff space,*

$$q : Y \longrightarrow B$$

Hurewicz fibration in the CG-sense with closed cofibration section t, and

$$f : A \longrightarrow B$$

a map. Then

$$f^*q : Y \sqcap A \longrightarrow A$$

is a Hurewicz fibration in the CG-sense with a closed cofibration section σ.

Comparison 2. The reader will notice that the locally compact Hausdorff assumption of Theorems 13 and 14 have been eliminated in Theorems 16 and 17.

The final application, given in Ch.9 of [B3], concerns the classification of 3-stage Postnikov towers, see [Ba] and [St1].

Let G and H be Abelian groups and m and n be integers with $1 < m < n$. Then

$$q_1 : PK(G, m+1) \longrightarrow K(G, m+1)$$

and

$$q_2 : PK(H, n+1) \longrightarrow K(H, n+1)$$

will denote the path fibrations over the Eilenberg-MacLane spaces $K(G, m+1)$ and $K(H, n+1)$ respectively (see [Sp, Pgs. 75 and 99]). Let (B, b_o) be a space with a basepoint. Then a 3-stage Postnikov tower $\tau(k_1, k_2) = p_1 \circ p_2$, over B with fibres $K(G, m)$ and $K(H, n)$, consists of principal fibrations

$$p_1 : E_1 \longrightarrow B$$

and

$$p_2 : E_2 \longrightarrow E_1$$

with fibres $K(G, m)$ and $K(H, n)$ respectively. So p_1 is induced from q_1 by a first k-invariant

$$k_1 : B \longrightarrow K(G, m+1),$$

as shown in the following pullback diagram:

$$\begin{array}{ccc} E_1 & \longrightarrow & PK(G,m+1) \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle q_1} \\ B & \xrightarrow[k_1]{} & K(G,m+1), \end{array}$$

i.e. $E_1 = B \sqcap PK(G,m+1)$, $p_1 = k_1^* q_1$. Now p_2 is induced by q_2 from a second k-invariant

$$k_2 : E_1 \longrightarrow K(H,n+1),$$

as shown in the following pullback diagram:

$$\begin{array}{ccc} E_2 & \longrightarrow & PK(H,n+1) \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle q_2} \\ E_1 & \xrightarrow[k_2]{} & K(H,n+1), \end{array}$$

i.e. $E_2 = E_1 \sqcap PK(H,n+1)$ and $p_2 = k_2^* q_2$.

We picture a 3-stage Postnikov tower $\tau(k_1,k_2) = p_1 \circ p_2$, over B as a diagram of the form:

$$\begin{array}{ccccc} K(G,n) & \xrightarrow[i_2]{} & E_2 & & \\ & & {\scriptstyle p_2}\downarrow & & \\ K(H,m) & \xrightarrow[i_1]{} & E_1 & \xrightarrow[k_2]{} & K(H,n+1) \\ & & {\scriptstyle p_1}\downarrow & & \\ & & B & \xrightarrow[k_1]{} & K(G,m+1). \end{array}$$

Using the principal fibration

$$q_1 : PK(G,m+1) \longrightarrow K(G,m+1)$$

we define $PK(G,m+1)!K(H,n+1)$. The path component of this space that contains the maps from fibres of q_1 to $K(H,n+1)$ will be denoted by M_∞. The main idea in this approach is that the second k-invariant

$$k_2 : B \sqcap PK(G,m+1) \longrightarrow K(H,n+1)$$

should correspond, via a fibred exponential law, to a map

$$k_2^< : B \longrightarrow M_\infty.$$

Further, $k_2^<$ should then act as a classifying map for the tower $\tau(k_1, k_2)$. However, if we work with Theorem 6 in the category of all spaces, we have to deal with the difficulty that $PK(G, m+1)$ is not likely to be locally compact. In that case part (b) of Theorem 6 can not be applied and the anticipated argument breaks down. To get round this, we work in the CG-context, we assume that B is a weak Hausdorff CG-space, and recall that the CW-complexes $K(G, m+1)$ and $K(H, n+1)$ are CG-spaces. We then use the path fibrations

$$kq_1 : kPK(G, m+1) \longrightarrow K(G, m+1)$$

and

$$kq_2 : kPK(H, n+1) \longrightarrow K(H, n+1)$$

and form pullbacks in the CG-sense. We can now use Theorem 7 rather than Theorem 6, and the question of whether spaces are locally compact is then no longer relevant.

We recall the concept of fibre homotopy equivalence [DL].

Definition 20. *Let F and B be spaces, where F has just two non-zero homotopy groups, i.e. G in dimension m and H in dimension n. Then $FHE(F, B)$ will denoted the set of all fibre homotopy equivalence classes of 3-stage Postnikov Towers over B, with fibre of the homotopy type of F.*

Theorem 18. *Let the CG-space B have the homotopy type of a CW-complex, and m and n be integers greater than 1. If $\tau(k_1, k_2)$ denotes a 3-stage Postnikov tower over B with distinguished fibre $K(G, m) \times K(H, n)$, then there is an associated based map*

$$k_2^< : B \longrightarrow M_\infty$$

defined by the rule $k_2^<(b)(y) = k_2(b, y)$, where $b \in B$, $y \in kPK(G, m+1)$, and $k_1^<(b) = kq_1(y)$. Further, there is a bijection from $FHE(K(G, m) \times K(H, n) : B)$ to an orbit set of $[B, kM_\infty]^o$ determined by and action of the group of homotopy classes of self-homotopy equivalences of $K(G, m) \times K(H, n)$, where $[\tau(k_1, k_2)]$ denotes the fibre homotopy equivalence class of $\tau(k_1, k_2)$, and $[k_2^<]$ the based homotopy class of the map $k_2^<$.

The last Theorem is the subject of current research by P. Booth. Its proof is beyond the scope of this thesis, we wish to make some remarks which we hope will clarify some remarks made by Booth in [B3].

We quote the said comments, together with the paragraph preceding it ([B3; pp.428/429]), then comment on how they are fulfilled here in this work. In fact the quotation bellow follows a paragraph which is similar to our last one above, so that the thoughts of the quotation follow naturally after the thought of the latter paragraph.

"Applying Theorem 3.3(a) to $k_2 : A \sqcap PK(G, m+1) \longrightarrow K(H, n+1)$, we obtain a fibrewise map, i.e. $k : A \longrightarrow PK(G, m+1)!K(H, n+1)$ such that $(q!K(H, n+1)) \circ k = k_1$. So k determines both k_1 and k_2; the former by composition with $q!K(H, n+1)$, the latter by the fibred exponential law.

However, this argument is best given in a convenient category context, as $PK(G, m+1)!K(H, n+1)$ can then shown to act as a *classifying space*, and k as a *classifying map*, for such *3-stage* tower."

Since the basic methodology of the proof of Theorem 18, is at the very heart of this thesis, we define the classifying bijection on representatives, and indicate precisely why the ordinary topological version fails to go through. According let $\tau(k_1, k_2)$ be a 3-stage Postnikov tower. As mentioned earlier the proper condition of the fibred exponential law applied to k_2, gives a map of the bijection in Theorem 18.

The point of course is that the admissible condition does not hold in the ordinary topological category. In particular the function defined above by the proper condition is not a bijection, and the classification fails. On the other hand the CG context gives a map

$$k_2^{<} : B \longrightarrow kM_\infty,$$

and in this context both the proper and the admissible conditions hold.

Comparison 3. We have already explained that the arguments just refereed to, work well in the CG-space situation, but not with topological spaces generally. We conclude by discussing the question of how 3-stage Postnikov towers in the CG sense relate to the corresponding construction in the category of all topological spaces.

Definition 21. *If X and Y are spaces, a map $f : X \longrightarrow Y$ is said to be a* *weak homotopy equivalence if*

$$f_* : \pi_o(X) \longrightarrow \pi_o(Y)$$

is a one to one correspondence, and

$$f_* : \pi_r(X, x) \longrightarrow \pi_r(Y, f(x))$$

is an isomorphism for all $r \geq 1$.

More details concerning to this important idea are given in [W].

The following result is well known, for details concerning a proof (see either [Ma, Th.7.5.4] or [Sp, Cor.7.6.24]).

Theorem 19. Whitehead's Theorem. *If*

$$f : K \longrightarrow L$$

is a weak homotopy equivalence of $CW-complexes$*, then* f *is a homotopy equivalence.*

Corollary 9. *If*

$$f : K \longrightarrow L$$

is a weak homotopy equivalence, and K, L *have the same homotopy type of CW-complexes, then* f *is a homotopy equivalence.*

Proposition 17. *Let* X *be a space.*

(a) The identity map

$$1 : kX \longrightarrow X$$

is a weak homotopy equivalence.

(b) If X *has the homotopy type of a CW-complex, then*

$$1 : kX \longrightarrow X$$

is a homotopy equivalence.

Proof. The proof of part (a) cames from the observation that the induced homomorphisms of homotopy groups

$$1_* : \pi_r(kX, x) \longrightarrow \pi_r(X, x)$$

are isomorphisms for all $r \geq 1$, and

$$1_* : \pi_o(kX, x) \longrightarrow \pi_o(X, x)$$

is a one to one correspondence. Hence

$$1 : kX \longrightarrow X$$

is a weak homotopy equivalence. For part (b), there are maps

$$h : X \longrightarrow K$$

and

$$\widetilde{h} : K \longrightarrow X$$

such that $h \circ \widetilde{h} \simeq 1_X$ and $\widetilde{h} \circ h \simeq 1_K$ where 1_X and 1_K are the identity maps and K is a CW-complex, since X has the same homotopy type of a CW-complex. By Proposition 2 and Corollary 4,

$$kh : kX \longrightarrow K$$

and

$$\widetilde{kh} : K \longrightarrow kX$$

are maps such that $kh \circ \widetilde{kh} \simeq 1_{kX}$ and $\widetilde{kh} \circ kh \simeq 1_K$, then kX has the same homotopy type of a CW-complex. Hence by (a) and Corollary 9

$$1 : kX \longrightarrow X$$

is a homotopy equivalence. □

Let A, D and E be spaces and maps $f : D \longrightarrow A$ and $r : E \longrightarrow A$. Then we can define the pullback space $D \sqcap E$, and use the maps kf and kr to define the pullback space $kD \sqcap kE$.

Proposition 18. *If*

$$p : E \longrightarrow B$$

and

$$q : D \longrightarrow B$$

are maps, then

$$k(D \sqcap E) = k(kD \sqcap kE).$$

Proof. Firstly suppose that C is a compact Hausdorff space, and let

$$g : C \longrightarrow D \sqcap E$$

be a map. By the Universal Property for final topologies, f is a map if and only if, both composites

$$(f^*r) \circ g : C \longrightarrow D$$

and

$$(r^*f) \circ g : C \longrightarrow E$$

are maps, where (f^*r) and (r^*f) are the projections over D and E respectively. So by Propositions 3 and 5

$$(f^*r) \circ g : C \longrightarrow kD$$

and

$$(r^*f) \circ g : C \longrightarrow kE$$

are maps. Hence

$$C \longrightarrow kD \sqcap kE$$

is a map since $k(f^*r)$ and $k(r^*f)$ are a map. The above argument is reversible, so it follows by the definition of kX, that $k(D \sqcap E) = k(kD \sqcap kE)$. □

Proposition 19. *Let $\tau(k_1, k_2)$ be a 3-stage Postnikov tower*

$$\begin{array}{ccccc}
K(G,n) & \xrightarrow[i_2]{} & E_2 & & \\
 & & \big\downarrow{\scriptstyle p_2} & & \\
K(H,m) & \xrightarrow[i_1]{} & E_1 & \xrightarrow[k_2]{} & K(H,n+1) \\
 & & \big\downarrow{\scriptstyle p_1} & & \\
 & & B & \xrightarrow[k_1]{} & K(G,m+1),
\end{array}$$

in the sense of the category of topological spaces with k-invariants

$$k_1 : B \longrightarrow K(G, m+1)$$

and

$$k_2 : E_1 \longrightarrow K(H, n+1).$$

Then $\tau(kk_1, kk_2)$ *is the 3-stage tower*

$$\begin{array}{ccccc} K(G,n) & \xrightarrow[i_2]{} & kE_2 & & \\ & & \Big\downarrow{\scriptstyle kp_2} & & \\ K(H,m) & \xrightarrow[i_1]{} & kE_1 & \xrightarrow[kk_2]{} & K(H,n+1) \\ & & \Big\downarrow{\scriptstyle kp_1} & & \\ & & kB & \xrightarrow[kk_1]{} & K(G,m+1), \end{array}$$

in the CG-sense with k-invariants

$$kk_1 : kB \longrightarrow K(G, m+1)$$

and

$$kk_2 : kE_1 \longrightarrow K(H, n+1).$$

Proof. This follows using Proposition 17 twice. We have a commutative diagram:

$$\begin{array}{ccc} kE_2 & \xrightarrow[1_{E_2}]{} & E_2 \\ {\scriptstyle kp_2}\Big\downarrow & & \Big\downarrow{\scriptstyle p_2} \\ kE_1 & \xrightarrow[1_{E_1}]{} & E_1 \\ {\scriptstyle kp_1}\Big\downarrow & & \Big\downarrow{\scriptstyle p_1} \\ kB & \xrightarrow[1_B]{} & B, \end{array}$$

where the underlying maps of 1_B, 1_{E_1} and 1_{E_2} are the identity maps on B, E_1 and E_2, respectively. □

Proposition 20. *If* B *has the homotopy type of a CW-complex, then* 1_B, 1_{E_1} *and* 1_{E_2} *are homotopy equivalences.*

Proof. In fact 1_B is a homotopy equivalence by corollary 9. Now the fibre $p_1^{-1}(b_o) = K(G,m)$ and B have the same homotopy type of CW-complexes. Thus by Theorem 2 of [RS], E_1 has the same type of a CW-complex and so 1_{E_1} is a homotopy equivalence by Corollary 9. For the proof of 1_{E_2} the argument is the same. □

Note: Let us assume that B is a CG-space. If would be useful if we could now show that 1_{E_1} must be a fibre homotopy equivalence between the 3-stage Postnikov towers $\tau(k_1, k_2)$ and $\tau(kk_1, kk_2)$. The obvious way to do this would be to use Theorem 6.1 of [DL]. However, $\tau(kk_1, kk_2)$ is only known to have the covering homotopy property with respect to CG-spaces, so that theorem is not applicable. In particular E_2 may not be a CG-space, $\tau(kk_1, kk_2)$ may not have the covering homotopy property relative to E_2, and an attempted proof of the required result breaks down over that issue.

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