A STUDY OF THE HEURISTIC STRATEGIES USED BY GRADE SEVEN AND EIGHT STUDENTS TO SOLVE NOVEL MATHEMATICAL PROBLEMS

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by

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ABSTRACT

The purpose of this study was to analyze the heuristic strategies utilized by grade seven and eight students in attempting to solve novel mathematical problems. Also investigated was the sequencing of heuristic strategies and core procedures.

Twenty-four students of both sexes equally divided into three ability groups--high, medium and low--were involved in the study. Each student was presented with a sheet of cardboard and a marker as well as two problems to solve. These were presented one at a time and consisted of one algebra and one geometry. In addition, for one of these problems physical materials were provided. A maximum of fifteen minutes per problem was permitted. The students were videotaped individually as they used the "thinking-aloud" technique while attempting to solve the problems.

Heuristic strategies and core procedures were recorded using a coding system based on that developed by Blake (1976). The coding procedure had an intercoder reliability of .87 and a sequencing reliability of .77. From the coded protocols, an analysis of the heuristic strategies was conducted. To analyze the sequencing of patterns of heuristics a graphical representation was devised.

In the algebra problem the most commonly utilized heuristic strategy for all groups was examination of cases. The heuristics of analysis and deduction were employed by a limited number of high ability students. Systematic cases was the most commonly employed heuristic of the cases family but the manner in which systematization was applied varied among groups. High and medium ability students were able to concentrate on all aspects of the problem in their systematic approach whereas low ability students were limited to concentrating on one aspect only. It was concluded that it is not the occurrence of systematic cases but the manner in which it was utilized that was the crucial factor in problem-solving.

In the geometry problem the heuristics of random cases and symmetry were utilized with an increase in the use of the symmetry heuristic by higher ability students. It was also found that students perceived the symmetry of the problem in different ways.

Perception played a vital role in the geometry problems as some students were handicapped in the strategies they could employ by their visual perception of the problem.

In both problems high ability students approached problems differently from the medium and low ability groups.

In the algebra problem the use of physical materials was found to be most effective for low ability students. Physical materials, however, seemed to be a hindrance to the eliciting of heuristic strategies for the high ability group. In the geometry problem physical materials increased the number of students employing the heuristic strategies especially among the low ability group.

Overall, it was found that students in solving both problems were fixed in their approach. Once a general pattern or trend was established students were unwilling to change their strategy so as to explore new ideas.

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CHAPTER I

INTRODUCTION

Rationale for the Study

Problem-solving is today receiving greater emphasis from mathematics educators as they realize that insight into the processes used by children to solve mathematical problems may open up new avenues in children's thinking and mathematics learning.

Both the Cambridge Conference on School Mathematics (Goals for School Mathematics, 1963) and the National Advisory Committee on Mathematical Education (NACOME) Report (1975) stressed the importance of problem-solving.

Rosenbloom at the Conference Board of the Mathematical Sciences (1966) is quoted as saying:

> We regard problem-solving as the basic mathematical activity. Since in mathematics education, our first concern must be with what we want the students to do, we must focus our attention on this domain. (p. 130)

Hilton and Rising in summarizing the Basic Skills of the National Institute of Education (NIE) Conference agreed with the NACOME report in its emphasis on problem-solving as a basic mathematical skill.

Many educators and mathematicians feel that problem-solving is of such great importance that it is frequently cited as the goal of mathematics. In a 1976 position paper on basic skills in mathematics prepared by the National Council of Supervisors of Mathematics, it was stated that "learning to solve problems is the principal reason for studying mathematics" (p. 2).

Polya (1962), a great advocate of problem-solving in mathematics, wrote:

> What is know-how in mathematics? The ability to solve problems--not merely routine problems, but problems requiring some degree of independence, judgment, originality and creativity. (p. viii)

In a recent book on critical variables in mathematics education, Begle (1979) agreed with Polya stating:

> The real justification for teaching mathematics is that it is useful and in particular that it helps in solving many kinds of problems. (p. 143)

Despite an effort by mathematics educators to stress the vital need for problem-solving abilities in children, school textbooks tend to neglect material related to problem-solving. Exercises which give students practice in applying algorithms are often disguised as problems. These routine algorithmic exercises, however, are not true problems. They do not afford students the opportunity to reflect, to judge, to devise plans, to investigate nor to seek original solutions. These capabilities are necessary if students are to cope with diverse real-life problems which they may encounter.

This interpretation of problem-solving was also stressed by the National Council of Supervisors of Mathematics

(1977) in their policy statement:

> Problem-solving is the process of applying previously acquired knowledge to new and unfamiliar situations. Solving word problems in texts is one form of problem-solving, but students should also be faced with nontextbook problems. Problem-solving strategies involving posing questions, analyzing situations, translating results, illustrating results, drawing diagrams . . . . They [students] should be unfearful of arriving at tentative conclusions and they must be willing to subject these conclusions to scrutiny. (p. 2)

Mathematics educators are concerned with ways to increase this emphasis on problem-solving in schools. Students seem to be lacking in strategies that would equip them to become better problem-solvers. At present, the only way to assess the problem-solving procedures used by students in attacking and solving problems is by exposing them to problems and through a thinking aloud procedure to analyze the thought processes they undergo.

Purpose of the Study

The purpose of this study was to investigate the heuristic strategies utilized by grade seven and eight students in their attempt to solve novel mathematical problems. The study sought to determine the strategies of differing ability groups and the sequence of observable patterns of behaviors generated by the problem situations. Specifically, it attempted to answer the following questions.

Question 1: Without specific training in heuristics, what are the heuristic strategies employed by grade seven and eight students when solving novel mathematical problems?

Question 2: Are there any differences in the heuristic strategies utilized by students of different ability levels?

Question 3: Are there any differences in the heuristic strategies utilized by students using physical materials and those not using physical materials?

Question 4: Is there any "order" in which students tend to utilize heuristic strategies and core procedures?

Scope and Limitations

The limitations of this study arose out of the following areas:

a) The problems used.

The problems for the study were selected from a piloted series. The criteria for selection was:
i) They were problems not usually found in the mathematics curriculum of schools;
ii) They elicited the use of the specified heuristics.

b) The sample size.

The sample size was kept small due to the data collection method used. Students were videotaped as they attempted to solve novel mathematical problems using the thinking-aloud technique.

c) The method of data collection.

The thinking-aloud technique and the videotaping procedure may have caused students to commit errors they normally would not have committed. Students may, in fact, have approached the problem in a different manner than usual when asked to verbalize.

Definition of Terms

The following terms occur throughout the study and are clarified here.

Algorithm. A systematic procedure that if carried out correctly must lead to a correct solution in a finite number of steps.

Problem. A situation in which the procedure for determining the outcome by the individual is not immediately obvious. The individual . . .

must be motivated to achieve the outcome, he must become personally involved and he must combine experience, knowledge and intuition to determine that outcome. Henderson and Pingry (1953) identify the necessary conditions for the existence of a problem for an individual:

1. The individual has a clearly defined goal of which he is consciously aware and whose attainment he desires.

2. Blocking of the path toward the goal occurs, and the individual's fixed pattern of behavior or habitual responses are not sufficient for removing the block.

3. Deliberation takes place. The individual becomes aware of the

> problem, defines it more or less clearly, identifies various possible hypotheses and tests these for feasibility. (p. 230)
>
> Problems are thus defined in terms of the individual to whom it is presented and in terms of his getting from statement of the problem to the goals.

Problem-solving. The activity in which the individual is involved as he attempts to determine the outcome of a problem.

Heuristic strategies. Procedures used for the purpose of discovering insight into mathematical relationships in attempting to solve novel mathematical problems. Heuristics, while being guides to the solution, do not guarantee success as algorithms do.

Specific heuristics.

Smoothing. The changing of the problem in such a way as to establish some isomorphism between the problem and a mathematical system.

Analysis. The breaking of a problem into subproblems.

Templation. The considering of algorithms, properties, theorems or procedures to seek information that may lead to a solution.

Cases. The heuristic cases may be used in the following way:

A. Cases (all)--considering all possible cases usually if there is a small number.

B. Cases (random)--considering cases at random.

C. Cases (systematic)--considering cases in a systematic fashion. This may or may not involve looking for a pattern.

D. Cases (critical)--considering only particular, though critical cases.

E. Cases (sequential)--considering cases in a sequential format.

Deduction. The considering of the consequence implied by an assumed or given premise or set of premises.

Inverse deduction. Given a conclusion, considering what premises imply it. This is frequently referred to as "working backwards."

Invariation. The renaming of a variable as a constant or excluding a variable to attempt to solve a new problem which will lead back to the original one.

Analogy. The establishment of a relation between this problem and one previously studied. This allows us to know what questions to ask and what properties to consider.

Symmetry. The use of the interchangeability of parts. If a problem is symmetric in some ways, we may derive some profit from noticing its interchangeable parts. It may be of benefit to us to treat these parts which play the same role in the same fashion.

For a more detailed account of heuristics, together with examples, see Blake (1976, pp. 7-18).

CHAPTER II

REVIEW OF RELATED LITERATURE

Mathematics educators are today expressing an interest in problem-solving. They are realizing that if we are to improve the ability to solve problems we must begin by analyzing the thought processes students go through in attempting to solve a problem. Only after a thorough analysis may we be able to specifically recognize problem-solving characteristics that will lead to more effective teaching of problem-solving processes. In view of the limited knowledge of how students solve problems, precursors to any large scale study in the teaching of problem-solving should be clinical studies of individual subjects (Kilpatrick, 1969, p. 179).

This interest in problem-solving is not exactly new. Descartes (1596-1650), a great mathematician and philosopher, was involved in this area. He planned to devise a universal method for solving problems by developing a set of rules that could be used to reduce any problem in life to a mathematical one. Once this was completed, a mathematical problem could be reduced to an algebraic one and then to a simple equation. Though he never completed his very complex task, he did make a vital contribution to the area of mathematical problem-solving and his book, Rules for the Direction of the Mind (cited in Polya, 1962, p. 22), did have a great influence on other mathematicians in this area afterwards.

One of the best known mathematical problem-solving models is that presented by Wallas as early as 1926. He envisioned problem-solving as consisting of four stages which one must pass through in the problem-solving process.

The first stage is Preparation which involves clarifying and defining the problem. The second Incubation is an unconscious mental activity. Inspiration, the third stage, occurs when the solution suddenly appears. Checking the solution, Verification, is the final stage and leads to the conclusion of the problem-solving activity.

Other models for problem-solving process have been presented and are of interest in any study involving this vital sector of mathematics.

Dewey's analysis of reflective thinking can be taken as an analysis of the act of solving a problem (Henderson & Pingry, 1953). Dewey (1933, pp. 107-116) outlined five phases of reflective thinking which is an essential part of problem-solving.

1. Suggestion: some inhibitions of direct action resulting in conscious awareness of a "forked-road situation."

2. An intellectualization of the felt difficulty leading to the definition of the problem.

3. Hypothesizing: "the identification of various hypotheses . . . to initiate and guide observation and other operations in collection of factual material."

4. Reasoning: elaboration of each of the hypotheses by reasoning and the testing of the hypothesis.

5. Acting on the basis of the particular hypothesis selected in step four, thereby providing the ultimate test.

Johnson (1944) identified three processes which an individual proceeds through as he attempts to arrive at a solution to a novel mathematical problem. These are:

1. Orientation to the problem: the process by which the organism grasps the material of thought and keeps it available for deliberation.

2. Producing relevant materials: perception obtained directly and immediately from the existing solution or generalizations.

3. Judging: the forming and testing of hypotheses.

The greatest effect of Descartes can be seen in the work of Polya who as a mathematician and mathematics educator made a tremendous contribution to the field of problem-solving. It is basically on his work that this study and others quoted here are based.

Polya (1957, 1962, 1965) provided a great quantity of information and interest in the use of "heuristics" in mathematical problem-solving. Polya's model includes a variety of procedures both general and specific for solving problems as well as a list of questions that are asked when attempting to solve mathematical problems. In his book, How To Solve It (1957), Polya gave strategies that could be used to solve problems and suggestions as to how these could be taught. Polya described four phases that a person passes through as he attempts to solve a problem, and the questions associated with those phases which will direct students when approaching novel mathematical problems. He noted that a problem-solver will not always exhibit observable behaviors as he passes through each phase. Polya's four phases (1957) include:

1. Understanding the problem.

   What is the unknown? What are the data? What is the condition? Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory? Draw a figure. Introduce suitable notation. Separate the various parts of the condition. Can you write them down?

2. Devising a plan.

   Have you seen it before? Or have you seen the same problem in a slightly different form? Do you know a related problem? Do you know a theorem that could be useful? Look at the unknown! And try to think of a familiar problem having the same or a similar unknown. Here is a problem related to yours and solved before. Could you use it? Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible? Could you restate the problem? Could you restate it still differently? Go back to definitions. If you cannot solve the proposed problem, try to solve first some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or the data, or both, if necessary, so that the new data are nearer to each other? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?

3. Carrying out the plan.

Carrying out your plan of the solution, check each step. Can you see clearly that the step is correct? Can you prove that it is correct?

4. Looking back.

Can you check the results? Can you check the argument? Can you derive the result differently? Can you see it at a glance? Can you use the result, or the method, for some other problem? (pp. 6-16)

Polya's model (1957) is based on the study of heuristic reasoning which he defined as:

> . . . reasoning not regarded as final and strict, but as provisional and plausible only, whose purpose is to discover the solution of the present problem. (p. 113)

He saw the training of intuition as an important aspect of the class of behaviors included in heuristic teaching.

> Therefore, I think that in teaching high school age youngsters, we should emphasize intuitive insight more than and long before deductive reasoning. (Polya, 1957, p. 128)

He believed that experience in solving problems and experience in watching other people solving problems, must be the basis on which heuristics are built. He further emphasized the practicality of heuristic investigation in that better understanding of the mental operations typically useful in solving problems could greatly influence mathematics teaching.

Gagné's (1966) view of problem-solving was similar to that of Polya. Gagné felt that problem-solving involved the acquisition of general principles that can be used in extended problem-solving situations. He referred to problem-solving as:

> An inferred change in human capabilities that results in the acquisition of a generalizable rule which is novel to the individual, which cannot have been established by direct recall, and which can manifest itself in applicability to the solution of a class problem. (p. 132)

Both Gagné's and Polya's models involved a search for a solution, not an immediate recall of an algorithm.

Garry and Kingsley (1970, p. 464) identified three main phases in problem-solving: (1) the search phase which involves a narrowing of the range within which a solution lies; (2) the functional solution phase as one attempts to recall past experiences which may prove useful in developing a solution, and (3) verification of the final solution which is the development of the actual solution and involves applying a mode of attack until a solution is obtained.

Polya's model of heuristics led to an investigation of problem-solving strategies by Macpherson. Macpherson's model (1970) identified three facets of mathematics which account for all procedures used, yet offers criteria for distinguishing between them. These are:

Application	Core	Discovery
Lore: statements of fact which mathematical systems are tied to the real world.	Facts: algorithms notation	Heuristics: general non-core strategy which is used for the purpose of discovering some order of mathematics generalization in a novel situation.

This concept of mathematics which Macpherson (1970) identified as discovery (containing heuristics) is the subject of this study. Upon Polya's model and modifications of it, most of the research in problem-solving has been done.

Probably the greatest contribution in recent times to the analysis of thought processes and strategies used by children in solving novel mathematicsl problems was made by Kilpatrick (1967). He studied the problem-solving processes used by above-average, eighth-grade students as they attempted twelve mathematical problems. Difficulties in analyzing the protocols led Kilpatrick (1967) to state:

> Attempts to apply the checklist to overall protocols from the pilot study demonstrated clearly that whatever merits Polya's list has for teaching problem-solving, it is of limited usefulness, as it stands, for characterizing the behavior of subjects. Many of the categories are unoccupied: subjects seemingly did not exhibit behavior even remotely resembling actions suggested by the heuristic questions. . . . Furthermore, the categories were not defined clearly enough for reliable coding. (p. 44)

Kilpatrick thus developed a checklist of process-sequence based on Polya's model to analyze the protocols of subjects in his study. He identified processes used by students such as: drew figure, made successive approximation, questioned the existence or uniqueness of an answer, used deduction, set up equations, reverted to trial and error, and checked the solution to the problem. His study showed that specific heuristics such as using successive approximation and trial and error were employed by students despite a lack of instruction. He noted, however, that these were not always the best heuristics to use, nor were the subjects always capable of using them.

Kilpatrick's work has been beneficial in the study of heuristic processes not for his results, per se, but for the coding system he devised.

Much discussion has arisen among mathematics educators as to the processes children use in their attempts to solve novel problems. Various studies have been conducted in attempts to first identify the strategies used. In order to devise methods of increasing the problem-solving abilities of students, educators first need an insight into the way children think. The focus needs to be on the processes, not the product.

> It may seem trivial to focus on the details, but it is through the study of these specific processes that a deeper understanding of the learning of mathematics will develop. (Wittrock, 1973, p. 29)

This great need for studies of the strategies of children in solving problems was seen also by Kilpatrick (1967):

> . . . the researcher . . . who chooses to investigate problem-solving in mathematics is probably best advised to undertake clinical studies of the individual subjects . . . because our ignorance in the area, demands clinical studies as precurors to larger efforts. (p. 179)

Studies aimed at exploring children's thought processes have been conducted in many areas of the mathematics curriculum. Kantowski (1974) conducted a clinical exploratory study on the processes used by eighth and ninth graders in solving novel geometry problems. Using the thinking-aloud procedure that taped protocols, she was able to conclude that most of the students with scores above the mean showed evidence of the use of heuristic processes and that the use of heuristics increased as problem-solving ability developed. Furthermore, she noted that solutions to problems were inclined to be more efficient when students used goal-oriented problem-solving processes.

A similar study of ninth graders conducted by Dalton (1974) led him to believe that in solving novel mathematical problems, students basically use two models of thinking, deduction and trial and error, with the latter proving most successful.

To further investigate the thought processes of elementary school children, Hollander (1973) conducted a study using verbal problems on sixth graders. She concluded that the successful problem-solving strategies of the subjects could be attributed to: (a) their ability to translate, (b) their ability to employ abstract analytical reasoning, and (c) their ability to reason insightfully. From this we can see the need and importance of certain reasoning processes in mathematical problem-solving.

The use of a wide range of problem-solving processes seems to make for a more successful problem-solver. This was shown in a number of clinical studies.

Webb (1975) studied problem-solving processes of second year high school algebra students. Mathematics achievement was found to be a variable with the highest relation to mathematics problem-solving ability. There was also found to be a significant relation between problem-solving ability and heuristic processes.

Grady (1975) supports the conclusion that children's thinking is an important aspect of problem-solving. He investigated the relationship between Piagetian levels of operational thought and problem-solving performance. His conclusions were:

1. That formal operational subjects used means-end processes such as pictures, diagrams and equations to solve problems.

2. That there is no significant difference between concrete and formal operational thinkers in frequency of planning processes.

3. That successful problem-solvers used heuristic strategies more frequently than unsuccessful ones.

The effect of problem context upon problem-solving processes was the subject of the study by Blake (1976). He concluded that the total number of heuristics used by subjects as well as the number of different heuristics used, account for a significant amount of the variance in the frequency of correct solution. The heuristics used therefore added to the subjects' problem-solving ability beyond their mathematical core knowledge. Problem-solving context, however, proved to be unrelated to the heuristics used.

Thinking-Aloud Technique

Most of the studies quoted above used a thinking-aloud data gathering process. Using this technique involves the subject simply verbalizing (without analyzing) his thoughts as he works and these statements are taped (audio or video). These tapes are used at a later date for analysis of the problem-solving processes used by the students.

Kilpatrick (1967) observed:

> There is one method of getting a subject to produce sequentially-linked, observable behavior that requires neither skill in self-observation nor the manipulation of mechanical devices: have the subject think aloud as he works. (p. 6)

The thinking-aloud method had been the subject of much criticism and question. Some mathematics educators believe that thoughts come and go too quickly in one's mind when solving problems to be verbalized. Subjects are also believed to remain silent during the moments of most profound thought. Some proclaim it as a serious limitation since subjects may even solve a problem differently if asked to verbalize thoughts than he would normally when thought patterns are secret (Kilpatrick, 1967; Kantowski, 1975; Days, 1978).

Evidence as to whether verbalizing and thinking interfere with or complement each other is inconclusive. Kilpatrick (1967) in his study said:

> The method of thinking aloud has special virtue of being both productive and easy to use. If the subject understands what is wanted--that he is not only to solve the problem but also to tell how he goes about finding a solution . . . and if the method is used with the awareness of its limitations, then one can obtain detailed information about thought processes. (p. 8)

Once students' thought processes are recorded they must be analyzed and classified. A general guide for coding protocols of students thinking aloud while attempting to solve mathematical problems was devised by Kilpatrick (1967). This has since been modified for various studies by Lucas (1972), Webb (1975), and Blake (1976).

A study conducted by Flaherty (1973) on 100 secondary school students solving six-word problems encountered in algebra was designed to investigate whether the thinking-aloud procedure affected problem-solving performances. Conclusions reached were that there was no significant difference between those who verbalized and those who did not verbalize on problem-solving score or time needed to complete the problems. There was a significant difference between the groups in the area of computational error. The overall result was that more difficult problems may have caused different results, but for these problems, having students think aloud did not drastically affect their performance.

Roth (1966) found no significant difference in either the correct solution or time factor criteria in subjects required to think aloud and those not required to think aloud, while solving reasoning problems.

A study conducted by Gagné and Smith (1962) found that requiring students to verbalize a rule during the problem-solving practice session, improved problem-solving performance. Furthermore, they concluded that a large amount of information about the processes being used could be obtained by having students think aloud. However, there were moments of silence in which processes used by students were not reported. Thus, the thinking-aloud technique does not provide information about all the processes used by subjects.

Zalewski (1974) had controdictory findings to those of Flaherty (1973). Zalewski's study of seventh graders found that taped thinking-aloud interviews and the associated coding system captured and classified mathematical problem-solving much better than do commercial tests. He was positive also as to the possibility of problem-solving protocols in assessing, separating and ranking seventh graders. He concluded that the thinking-aloud procedure and the related coding scheme does not classify problem-solving behaviors very well. He felt that the behavior of students during the thinking-aloud interviews, raised critical questions about the reliability and validity of the information recorded in the protocols. He noted that his recording of protocols was through audiotaping and recommended videotaping as a distinct advantage in data collection.

The increased recognition and use of the thinking-aloud procedures in research studies (Kilpatrick, 1967; Dalton, 1974; Kantowski, 1974; Hollander, 1974; Grady, 1975; Webb, 1975; Blake, 1976) provides sufficient reason to assume that the procedure is a valid one for identifying problem-solving behaviors.

CHAPTER III

DESIGN AND PROCEDURE

The purpose of this study was to investigate the heuristic strategies used by grade seven and eight students in solving novel mathematical problems. Items of interest included total heuristic strategies, heuristics related to ability levels, heuristics related to presentation of materials and the sequencing pattern of heuristic strategies and core procedures.

This chapter describes the manner in which the research was conducted. It includes a description of the sample, the pilot study, the procedure and the coding system.

Population and Sample

The population for the study consisted of students at the grade seven and eight level.

A sample of this population was selected from two elementary co-educational schools. One group was selected for the pilot study and the other for the main study. Both schools involved in the study were in the suburbs with similar populations and comparable facilities.

The sample for the main study consisted of 24 students subdivided equally by grade into three ability groups--high, medium and low. These classifications were not based on standardized testing but on student overall performance in mathematical problem-solving as determined by the mathematics teacher.

Pilot Study

The pilot study was conducted in late March. This phase involved a series of 11 pilot problems (see Appendix A) which were presented to 12 grade seven students in a videotaped interview situation. These students were classified as high, medium and low in problem-solving abilities by their teacher.

The major aims of the pilot study were:

1. To determine if students in grade seven would exhibit problem-solving strategies while attempting to solve the given problems.
2. To determine which heuristics are most probable to be observed in this situation.
3. To select the problems to be used in the main study.
4. To familiarize the investigator with the videotaping equipment.
5. To modify the coding system if necessary.
6. To provide process-sequence samples for coding practice and revision.

A total of 11 mathematical problems were selected from the areas of algebra and geometry for the initial testing. Some of these problems incorporated the use of physical materials while others did not. The classification can be seen in Table 1.

TABLE 1

Problems Selected for the Pilot Study

Presentation	Problems* Algebra	Geometry
Using Physical Materials	# 1**	# 5,6,7**,10
Not Using Physical Materials	# 1,2,3,4,8,9,11	# 7

*Numbers in table refer to problem numbers in pilot study as listed in Appendix A.

**In the pilot study, numbers 1 and 7 were presented both with physical materials and without to investigate the differences in heuristic strategies promoted.

The researcher videotaped students on an individual basis in the school. These students were of high, medium and low abilities. Before the commencement of the testing students were introduced to the videotaping person and oriented to the entire room and equipment. An opportunity for questioning was provided. Once the student felt comfortable, the testing session began.

Each student was given two problems to solve. These were presented to him one at a time. The researcher acquainted the student with the procedure involved, instructing him that he was to attempt to solve the given problem in a maximum of fifteen minutes while thinking aloud. That is, in solving the problem he was to inform the investigator of exactly what he was thinking. This aspect of the study was vitally important and was encouraged throughout. All instructions except the problems themselves were communicated verbally by the interviewer. Each interview was designed to generate as much observable problem-solving behavior as possible.

After a student had completed the first problem or after the time had elapsed (whichever came first), he was instructed to continue to the next one. The problem was considered complete when a solution was reached, whether it was correct or not. When a prolonged period of silence was evident, subjects were encouraged to think aloud. If the student had exhausted all his possible modes of attacking the problem and had given up, hints in the form of questions to consider were presented. Examples of some of the types of questions comprising the hints were: What is the unknown? Can you draw a figure? Have you seen a problem like this before? Can you restate the problem? Can you tell me what your answer would be like?

Following the completion of this initial phase of the videotaping interviews, the taped protocols were viewed by the researcher to determine which problems would generate observable behaviors by the students. Student protocols were coded using the coding sheet (see Appendix B) and heuristic strategies utilized by each group were noted. Having examined these protocols it became interesting to experiment with what heuristics these problems would generate in slower students. Eight seemingly workable problems were selected and presented to four students who were classified by the mathematics teacher as being of low ability in problem solving. Each student was asked to solve two of these eight problems following the same format as the initial piloting.

Once the pilot was completed further analysis of the videotaped protocols was conducted. From this analysis the following recommendations for the main study emerged:

1. Two problems were selected to be presented in both physical and nonphysical milieus. These were problems numbered 1 and 7 in the pilot list (see Appendix C).

2. The problems would be presented to the students in a manner by which half the students attempted an algebra problem followed by a geometry problem and the other half vice versa. One problem attempted by each student incorporated physical materials while the other did not.

3. The sample size was extended to 24 so that each level of each grade would perform each problem twice.

4. The coding sheet was expanded to include the categories used by Blake (1976) plus two additional ones. These were the category "Uses physical materials" and the expansion of systematic cases to include "Looks for a pattern" (see Appendix D).

5. Following a prolonged period of silence by students when attempting to solve a problem, hints in the form of questions were presented. As a consequence of these hints it became interesting to examine whether any different heuristic strategies were evident or if patterns of sequencing were changed. The hints are contained in Appendix E.

6. The use of a short interview procedure for general discussion of the problems and problem-solving was incorporated into the study to follow the videotaping (see Appendix F).

Procedure

The subjects (N = 24) used in the study were subdivided into three ability groups--high, medium and low. Each was presented with two problems to solve, one algebra and one geometry as selected from the pilot study and contained in Appendix C. The problems were presented such that one problem incorporated physical materials and the other did not (see Table 2).

TABLE 2

A Breakdown of the Problems Used by Area of Mathematics and Presentation

	Area of Mathematics	
Presentation	Algebra	Geometry
Physical	Coins	Circles
Nonphysical	Coins	Circles

Having the two problems presented both in a physical and nonphysical manner in each ability grouping and in each grade resulted in 48 "problems" being analyzed. The order of problem presentation varied with half the students attempting algebra first followed by geometry and the remaining vice versa. This pattern is shown in Table 3.

TABLE 3

Problem Presentation as Related to Grade, Ability Level and Number of Students

	Grade 7	Number of Students	Grade 8	Number of Students
High	A B / C D	4	A B / C D	4
Medium	A B / C D	4	A B / C D	4
Low	A B / C D	4	A B / C D	4

A refers to Algebra-Physical with Geometry-Nonphysical
B refers to Algebra-Nonphysical with Geometry-Physical
C refers to Geometry-Physical with Algebra-Nonphysical
D refers to Geometry-Nonphysical with Algebra-Physical

Before the actual study began, students who were to be involved in the study were taken to a room in the school that had been set up in advance with the videotaping equipment. The students met with the researcher who explained the purpose of the study and what it involved. A general tour of the room, an orientation to the equipment and an opportunity for questions regarding the equipment were also provided.

Following the initial briefing, students came one at a time during class sessions to meet with the researcher in the videotaping room. Once a student felt secure in his setting and ready to begin, he was presented with a problem. Each problem was typewritten on a 20 cm x 12 cm index card. Each student was presented with a 51 cm x 33 cm sheet of cardboard and a black felt-tipped marker. He was asked to write on the cardboard as he attempted the problem so as to make all workings clearly accessible to the camera. It was explained that he was to attempt to solve the given problem in a maximum of 15 minutes while thinking aloud. After having completed the problem or after the time elapsed (whichever came first) he was instructed to continue to the next problem. The videotaping ceased when a correct solution was obtained or the time had elapsed.

If during the solution attempt, students engaged in long periods of silence with no apparent strategies or became frustrated by lack of success, a hint was given.

The researcher also questioned students concerning the number of coins or the amount of money for the algebra problem when solutions were incompatable with the given details. There were, as well, oral reminders for students to think aloud when silence became prolonged.

Following the videotaping of both problems for each student the researcher conducted a short oral interview to attempt to have students discuss the strategies that they had applied to these specific questions and to mathematical problems in general. The interview form is contained in Appendix E.

Once the data was gathered, the researcher viewed the recorded protocols of the students together with all written work and analyzed the subjects' problem-solving behaviors using a revised coding sheet (see Appendix D). This coding was conducted over a period of five weeks with the initial phase being a reliability check.

The Coding System

The coding system used in this study is based on Macpherson's model of mathematical problem-solving as employed by Blake (1976). It was designed for coding problem-solving behavior of subjects who are asked to think aloud as they solve mathematical problems.

While being videotaped subjects for this study were asked to think aloud. They were instructed to explain verbally all their writings and drawings. When periods of silence persisted, subjects were reminded of the need for verbalization.

Once the interview was completed, the subject's protocol was matched with his written work. The appropriate footage on the counter was noted by the researcher for the beginning and ending of each problem. The coding was conducted incorporating both the subject's videotaped recording and his written work.

Once the data was collected, the investigator viewed the videotapes and analyzed the thought processes, strategies, and sequences of behaviors exhibited by the students.

This coding system consisted of a coding sheet as used by Blake (1976) but modified for this study with the addition of "Uses physical materials" and "Looks for a pattern". A student's protocol for each problem was coded on the coding form. As stipulated by Blake, the procedures for the coding form are divided into five categories: core, heuristic, solution, reasonableness, and concern. As the coder identified the procedure from the subject's protocol he entered a check (✓) in the first empty column and the appropriate row.

Core

All students began with reading the problem. Thus a check was entered in the first row, first column for everyone. *Reading the problem* was also checked each time the problem or sections of it were reread. If no activity occurred between reading and rereading, it was checked only once.

Request definition of terms was checked when a student asked for a term to be defined or inquired if his interpretation of the problem was correct.

Request clarification of terms was checked when a student was unsure of terms stated in the problem or wished to get assurance of what he believed to be the problem.

Recalls same problem or *related* problem was checked if a subject indicated that he was familiar either with this problem or a similar one. If a student recalled the problem as being of a known type, *recall problem type* was coded.

If a student mentioned a fact which he believed would aid him in his solution, *recall related fact* was coded.

Each time a new figure or diagram was drawn, then the category *draw diagram* was checked. If the student drew only part of a diagram then this category was not checked. If the diagram was drawn merely as a means to indicate a heuristic, the heuristic, not draw diagram, was coded. The modification of a diagram led to *modify diagram* to be checked.

Identify variable implied that the student identified some variable in his problem attempt.

The algorithms--Algebraic and Arithmetic--were coded if subjects resorted to an approach involving an algorithm. If the algorithm was performed mentally such as 12 + 3 + 4 = 19, it was not coded.

Guessing was coded each time the subject stated a solution simply by guessing with no evidence that he planned to utilize further trials.

Heuristics

Smoothing involved the disregarding of irrelevant information in the problem.

Analysis was coded when the subject intentionally broke the problem into subproblems. His further action must indicate that it was built upon this strategy.

Templation was checked each time the student recalled relevant information. This may have been definitions, properties or theorems.

Cases (all) was coded if all possible cases were considered in the solution of the problem. Random cases was coded if subjects randomly used cases with no apparent reference to solutions already obtained. This is more commonly referred to as "trial and error." Cases (systematic) referred to the student's solution attempts being based on information previously accumulated. Sequential cases referred to the use of cases in a sequential fashion such as examining cases using one quarter then two quarters. Cases (critical) was coded when subjects looked for extreme bounds for the variables.

Deduction was coded when the subject assumed a premise and searched for all the implications of that deduction. A single logical implication of the statement did not indicate deduction. For example, saying that if I assume that there are three quarters then the number of nickels and dimes must total 16 is not deduction but rather a logical consequence. Inverse deduction referred to the strategy of examining the solution and working inversely to arrive at what was given. This is commonly called "working backwards."

Invariation was coded when one of the variables was excluded as a means of solving a simpler problem which hopefully would lead to the solution of the original one.

Analogy referred to the recalling of an analogous problem in attempting to solve the given one.

Symmetry was coded if a student's actions and comments indicated the use of the inherent or constructed symmetrical pattern of the problem. Symmetry (corners) was coded if students saw symmetry only with reference to the corner counters.

Solution

Obtain solution (correct) was checked if the subject indicated this to be a solution and it was correct. Obtain solution (incorrect) was coded if the students indicated a solution but he was in error.

Reasonableness

Checking part was coded if a subject checked only part of his workings.

Checking solution by retracing steps indicated that the student arrived at what he felt to be a solution and he checked by repeating the procedure previously used.

Checking solution by reasonable/realistic was checked if the student showed that his methods for checking his work concerned whether the answer could reasonably be correct.

Uncodable referred to subject's work which could not be classified within the framework used.

Concern

Expresses concern about method, algorithm, equation, or solution was coded whenever the subject's remarks indicated that he doubted the correctness of one of these aspects.

Work stopped--solution was coded if the subject finished work on the problem with a solution. Work stopped--no solution indicated that the subject finished with no solution as the allotted time elapsed.

Coding Reliability Check

The reliability check on the coding procedure was conducted over a period of three weeks. An alternate coder, a graduate student in mathematics education, was trained in the coding procedure. The training sessions were comprised of two stages. Stage A consisted of a discussion of the general coding procedure including definitions of terms, and criteria for classification of behaviors on the coding sheet. Stage B involved the viewing and coding by each coder separately of a sample of the pilot tapes which contained the same problems as used in the main study. This was followed by a consultation and discussion of discrepancies noted in the classification process.

Once a level of confidence was perceived to have been reached by both coders on the pilot study, the coding of the videotapes involved in the study began. A target level of 80% was sought by the researcher on intercoder reliability.

A five-minute section from each subject's videotaped time was randomly selected by the researcher. These sections were divided into three groups. Group one consisted of the last five minutes of eight students' protocols, group two the second five minutes of a different eight and group three up to the first five minutes of the final eight students. Problems for viewing were selected to include an equal number of algebra and geometry in both physical and nonphysical models.

Following the reliability check on each group, the researcher conducted the final coding procedure for the eight people in that group for a total of sixteen "problems."

This procedure was believed to increase the reliability by decreasing the chance of change in the coding procedure. Theoretically any change in procedure by the researcher as the coding progressed should have been reflected in a lower level of agreement on the subsequent sections.

As a measure of reliability, the percentage of heuristics coded alike were calculated using a formula described in Blake (1976, pp. 68-72). It takes into account items coded identically by both coders only once.

The number of heuristics agreed upon between coder A (the researcher) and coder B (the alternate) equals a. The number of heuristics coded by A alone equals a^1 and by B alone equals b^1.

Then $a + a^1$ = total coded by A.

Similarly $a + b^1$ = total coded by B.

Then $a + a^1 + b^1$ = total number of different heuristics coded by both A and B which equals the sum designated T.

Then $\frac{a}{T} \times 100 = y$, the percentage of heuristics coded identically by both coders is the measure of inter-coder reliability.

With reference to the measure of reliability of the sequencing coded, a table for each problem was designed and the number of agreements and disagreements of the alternate coder with the researcher were noted. The total was the number of categories used during the five-minute coding.

The number of agreements divided by the total times 100 equalled the percentage of agreement on sequencing of heuristic strategies and core procedures. An example is illistrated in Figure 1.

Analysis of Data

Following the coding of the entire videotapes, graphs were constructed of the behaviors exhibited at each ability level for each type of problem. The four students' coded protocols for the same problem, at each ability level, were placed on one graph. These graphs, twelve in all, included all the information obtained in the study. They are available in Appendix G.

A detailed analysis of these graphs was undertaken in an attempt to answer the four questions posed in Chapter I. A thorough discussion of this analysis is contained in the following chapter.

Reading problem									
Request definition of terms									
Recall same problem									
Recall related problem									
Recall problem type									
Recall related fact									
Draw diagram									
Modify diagram									
Identify variable									
Algorithm-Algebraic									
Algorithm-Arithmetic					o				
Use physical materials									
Guessing									
Smoothing									
Analysis									
Templation									
Cases (all)									
Cases (random)	o ▲			o ▲		o			
Cases (systematic)		o ▲							
Looks for a pattern									
Cases (critical)									
Cases (sequential)			▲						
Deduction									
Inverse deduction									
Invariation									
Analogy									
Symmetry									
Obtain solution (correct)									
Obtain solution (incorrect)							o ▲		
Checking part									
Checking solution									
by subst. in equation									
by retracing steps			o						
by reasonable/realistic									
Uncodable									
Exp. concern-Method									
Exp. concern-Algorithm									
Exp. concern-Equation									
Exp. concern-Solution									
Work stopped (sol'n.)									
Work stopped (no sol'n.)									

o - Information coded by the researcher

▲ - Information coded by the alternate coder

1	2	3	4	5	6	7
*	*	-	*	-	-	*

Agreements are marked by *

Disagreements are marked by -

$\frac{4}{7} = 57\%$

FIGURE 1. Example of coder error.

CHAPTER IV

ANALYSIS OF DATA

This chapter will analyze the data obtained from the coding of the videotaped protocols of the grade seven and eight students. Graphs of the coded results subdivided by ability group and problem type are found in Appendix G.

The short interview session following the videotaping provided little information concerning the strategies students used in attempting to solve problems. Students seemed to have great difficulty in verbalizing the thought processes they had used. The amount of information gathered by this technique was very restrictive and thus it was not included in the data analysis.

The measure of intercoder reliability of the heuristics observed was 87%. A reliability measure of 77% was calculated for intercoder agreement on the sequencing of the behaviors witnessed.

Research Questions and Discussion

The first three questions are discussed with direct reference to the data count. The major portion of this analytical discussion focuses on the fourth question, namely, the patterns of behaviors and heuristic strategies which students employ in solving novel mathematical problems. This search for patterns in mathematical problem-solving forms a crucial aspect of this study.

> <u>Question 1</u>: Without specific training in heuristics, what are the heuristic strategies employed by grade seven and eight students when solving novel mathematical problems?

To answer this specific question relating heuristic strategies to mathematical problems it is necessary to consider what heuristics could be employed in the particular problems used. Are students failing to utilize strategies or are these strategies not being applied as a consequence of the particular problem selected? Theoretically, it is important to consider what strategies are appropriate to the problem situations presented.

Firstly, in considering the algebra problem, four heuristic strategies seem most applicable. Students had a choice of selecting the strategies of analysis, the entire cases family, deduction, or analogy.

Secondly, in the geometry setting heuristics of direct and hypothetical deduction, inverse deduction, analogy, symmetry and the examination of cases are potential strategies.

In both problems the combination of heuristics present a variety of procedures for attacking novel mathematical problems. If in analyzing the data it is found that these strategies are not employed and the student does not solve the problem, it is not a consequence of the particular problem but rather a deficiency in the problem-solving behaviors of the students.

Algebraic Problem

Table 4 reveals that all 24 of the students sampled attempted to solve the algebraic problem by incorporating the use of the cases heuristic. Nineteen of these selected a combination of random followed by systematic, while two students proceeded immediately to a systematic approach. Only three students chose a mere random cases procedure as their only heuristic. There were three students who employed the heuristics of analysis and deduction in their problem-solving process.

TABLE 4

Total Heuristics Used for the Algebra Problem

Heuristics	Number of Students Using Each Heuristic
Cases	
a) Random only	3
b) Both Random and Systematic	19
c) Systematic only	2
Analysis	2
Deduction	1

From Table 5 it can be seen that the majority of students (21 of the 24) selected only one heuristic strategy in their problem-solving procedures. This indicates that if a student's first approach to a problem was unsuccessful, it was unlikely that he would expand his procedure and incorporate other heuristic strategies.

TABLE 5

Heuristic Combinations for the Algebra Problem

Heuristic Combinations	Number of Students Using Each Heuristic Combination
One Heuristic only	
Cases	
a) Random	3
b) Both Random and Systematic*	17
c) Systematic	1
Two Heuristics	
Analysis/Random and Systematic	1
Analysis/Systematic	1
Deduction/Random and Systematic	1

*For this table, "Both Random and Systematic" cases is coded as one heuristic.

Seventeen students used the heuristic of random and systematic only, while one student attempted the use of systematic only. (For the purpose of this discussion, random and systematic cases were considered one heuristic.) Thus it can be seen that for most students the initial response to a problem was to select random values and at best to refine these in a systematic manner in the problem-solving procedure.

The heuristics of analysis and deduction comprised the strategies of three students. All of these students used the strategy in combination with the cases family. They consisted of three pattern types--analysis plus random and systematic, analysis plus systematic, and deduction plus random and systematic. Without specific training the heuristics of analysis and deduction were within the capabilities of grade seven and eight students. The use of these strategies seemed to be limited and to follow from previously unsuccessful "cases" procedure. Special note should be given to the use of deduction plus random and systematic. In this protocol one heuristic did not follow the other but the student very capably incorporated the two heuristics together by systematically employing examination of cases based on deductions made throughout.

It should be noted also that in analyzing the protocols, the use of sequential cases was not coded. This member of the cases family is very closely linked to systematic, for to be sequential a student must also have been systematic. Due to the limited sequential cases to be observed in the particular problem, it is conceivable that students who appeared to be utilizing systematic cases were in fact employing sequential.

Geometric Problem

Table 6 presents the data relevant to the total heuristics used by the 24 grade seven and eight students in attempting to solve the novel geometry problem. The two most common heuristics employed were random cases and symmetry with both receiving equal emphasis.

TABLE 6

Total Heuristics Used for the Geometry Problem

Heuristics	Number of Students Using Each Heuristic
Cases	
Random	17
Symmetry	16

A more realistic picture of the heuristics involved can be seen in Table 7 which offers a breakdown of the heuristic combinations. Whereas 17 people used random cases, only seven of these employed random cases (trial and error) procedure as their only heuristic strategy. The remaining 10 students combined this strategy with the use of the inherent symmetry of the problem. The symmetry only heuristic was employed by six students. One student was completely unsure of what the problem was asking and employed diagram drawing without the utilization of any heuristic strategy.

TABLE 7

Heuristic Combinations for the Geometry Problem

Heuristic Combination	Number of Students Using Each Heuristic Combination*
One Heuristic only	
Cases	
Random	7
Symmetry	6
Two Heuristics	
Random and Symmetry	10

*One student used no heuristic.

With reference to this particular geometry problem one of four general approaches manifested themselves. These are graphically seen in Figure 2. First, students (▲—▲) sensed the inherent symmetry in the problem and used this as a base in seeking a solution. Second, students (●—●) did not at first see a relationship and thus used random manipulation of the circles. This led in time to the realization of the symmetry. Third, there existed students (○—○) who immediately sensed the symmetry but were unable to use it to arrive at a solution. These students physically moved the pieces or drew a diagram to exhibit evidence of the symmetry. For example, they changed the given arrangement of

```
      O                   O  O  O
    O   O              O  O  O  O
  O   O   O               O     O
O   O   O   O     to         O      .
```

They resorted to random cases but some later returned to the symmetry idea. Finally, there were those (□—□) who began by randomly moving the pieces and never developed beyond that point. About 29% of the students sampled had not developed strategies that would help them in the geometry problem beyond mere trial and error procedures.

It is significant to note that 92% of the students who employed symmetry directly or random cases leading to symmetry were successful in solving the problem. Only 9%

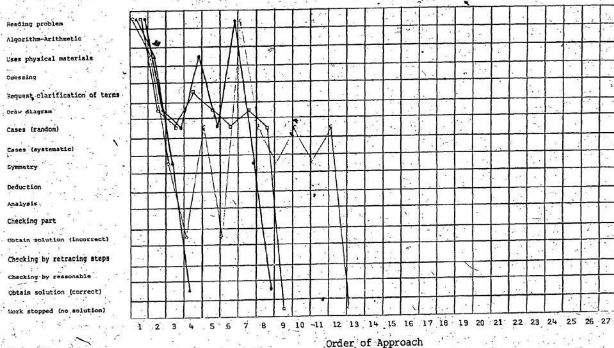


FIGURE 2. The four general approaches to the geometry problem

of those taking the symmetry-random cases-symmetry route or random cases only solved the geometry problem.

Having examined the total heuristic strategies employed by the sampled students it was of interest to investigate whether students of differing ability levels elicited distinct heuristic strategies in solving either of the two problems. Were there some strategies that students used which enabled them to become successful problem-solvers? This led to the examination of the data to seek an answer to the second question.

<u>Question 2</u>: Are there any differences in the heuristic strategies utilized by students of different abilities?

Students for this study were classified as high, medium, and low ability on problem-solving as perceived by their mathematics teachers. Table 8 reveals that the number of students solving both problems in the high ability group was significantly higher than the medium and low ability groups. The differences between low and medium groups in total problems solved correctly is minimal. Thus, teacher classification seemed to be consistent with a criterion based on correct solution.

TABLE 8

Problem Solution by Ability Level

Ability Level	Number of Students Solving Both Problems	One Problem Only	Neither
Low	1	3	4
Medium	2	3	3
High	5	2	1

Algebraic Problem

The heuristic most commonly used in attempting to solve the algebra problem was the cases family. Table 9 shows this to be true for all ability levels.

The number of different heuristic strategies utilized by low and medium ability students was the same. For the high ability students there appeared a slight decrease in the "one heuristic only" category in favor of a combination of two heuristic strategies. Three students in the high ability group paired examination of cases with the higher heuristic of analysis and deduction. For example, one student involved in systematic cases used deduction relating the oddness or evenness of the number of quarters to the number of nickels. This ability to utilize two heuristics was only employed by students classified in this study as high ability.

TABLE 9

Heuristic Combinations by Ability Levels for the Algebra Problem

Heuristic Combinations	Number of Students Using Each Heuristic		
	Low	Medium	High
One Heuristic only			
Cases			
a) Random	2	1	-
b) Both Random and Systematic	6	7	4
c) Systematic	-	-	1
Two Heuristics			
Analysis/Random and Systematic	-	-	1
Analysis/Systematic	-	-	1
Deduction/Random and Systematic	-	-	1

The most prominent difference in strategies as related to ability groups is derived from the method of attacking a problem. Whereas high ability students consciously evoked the use of systematic cases, the lower ability students stumbled on systematization more by chance than by a predetermined effort. This difference in approach can be clearly evidenced in the accompanying graphs. Figure 3 is taken from the coded protocols of three low ability students. It can be clearly seen that none of these students

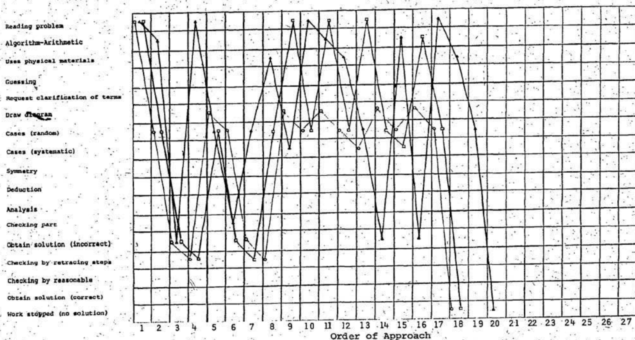


FIGURE 3. Examples of the use of systematic cases by low ability students.

had decided to employ systematic cases early in their protocol. It appears more likely that the students became engaged in a systematic design for a short period of time without a conscious effort to do so. Figure 4 is quite different. This figure depicts the planned attempt of high ability students to utilize systematic cases in the solution attempt.

Geometric Problem

An analysis of the heuristic strategies used by the three ability levels in attempting to solve the geometry problem shows that no difference exists in the types of heuristics employed. Table 10 reveals that all three levels applied the heuristic of examination of cases and symmetry. The greatest difference that appeared was with respect to the increased use of symmetry among the high ability group. Four of the eight low performance students and five of the eight medium ability group used symmetry. There were seven of the eight high ability class, however, who saw the crucial role of symmetry in seeking the solution to the given geometry problem. The high performance group also experienced a decrease in the use of "random cases only."

In combining the data from the algebra and geometry problem to consider the number of different heuristics employed by subjects at each ability level, it becomes evident that the majority utilize only one heuristic in attempting to solve the problem (see Table 11). Students

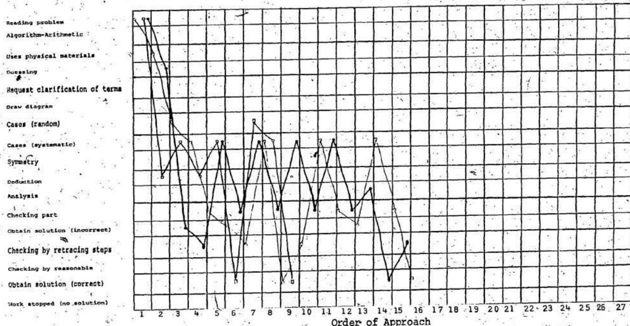


FIGURE 4. Examples of the use of systematic cases by high ability students.

TABLE 10

Heuristic Combinations by Ability Level for the Geometry Problem

Heuristic Combinations	Number of Students Using Each Heuristic* Low	Medium	High
One Heuristic only			
Cases			
Random	3	3	1
Symmetry	1	2	3
Two Heuristics			
Symmetry/Random	3	3	4

*One student utilized no heuristics.

TABLE 11

Number of Different Strategies Utilized by Students of Differing Ability Levels

Ability Level	Number of Different Heuristics Used 2	1	0
Low	3	12	1
Medium	3	13	-
High	7	9	-

appeared unable to expand their heuristic strategies to investigate new ideas. Only three of the low ability students and an equal number of medium ability students were capable of investigating two strategies for the problem solution. In the high ability group 7 of the 16 students extended to two, the heuristic strategies employed.

Overall there is very little difference in the specific heuristics employed at each level. A more thorough analysis of the ways in which heuristic strategies were used by differing ability groups will follow in answering question four.

> Question 3: Are there any differences in the heuristic strategies utilized by subjects using physical materials and those not using them?

As discussed in Chapter III, each student was presented with two novel mathematical problems--one algebra and one geometry. These problems were further subdivided such that each student was asked to solve one problem for which physical materials were provided and one which was presented nonphysically. Among the 24 students both problems were presented in each manner 12 times. The question to be investigated is whether the availability of physical materials elicited different heuristic strategies.

To discuss the answer to this question the problems (algebra and geometry) will be considered separately.

Algebraic Problem

The majority of students in both the physical and nonphysical presentation attacked the problem via examination of cases (see Table 12). It is interesting to note that while only three students of the 24 expanded their heuristic strategies beyond cases into analysis and deduction, all

TABLE 12

Heuristic Combinations Utilized in the Algebra Problem as Related to Problem Presentation

Heuristic Combinations	Number of Students Using Each in the Plysical Mode	Number of Students Using Each in the Nonphysical Mode
One Heuristic only		
Cases		
a) Random	2	1
b) Both Random and Systematic*	10	8
c) Systematic	-	-
Two Heuristics		
Analysis/Random and Systematic	-	1
Analysis/Systematic	-	1
Deduction/Random and Systematic	-	1

*For this table, "Both Random and Systematic" cases is coded as one heuristic.

three did so when the algebraic situation was presented nonphysically. From the videotapes it seemed that the physical materials elicited a great amount of simple counting while using cases. This may have hindered thought processes that could extend to other heuristics. It is noteworthy that the three students who incorporated the use of analysis and deduction into their solution procedure were all of the high ability group. All four students at this level who were asked to solve the algebra problem physically employed the same heuristics as the low and medium ability groups. Thus, it seems to indicate that for the high ability students the presentation of the algebra problem nonphysically may be most beneficial with respect to eliciting additional heuristics.

Geometric Problem

The heuristics evoked by the physical and nonphysical presentation of the geometry problem were identical. As shown in Table 13 no difference was found in the heuristics strategies employed in these situations. A difference in the number of students employing each heuristic is clearly indicated. Whereas six students used random cases as their only strategy in the nonphysical mode, only one student resorted to this mere trial and error procedure in the physical mode. With reference to the heuristic combination of symmetry and random cases,

TABLE 13

Heuristic Combinations Utilized in the Geometry Problem as Related to Problem Presentation

Heuristic Combinations	Number of Students Using Each in the Physical Mode	Number of Students Using Each in the Nonphysical Mode
One Heuristic only		
Cases		
Random	1	6
Symmetry	3	3
Two Heuristics		
Symmetry/Random	8	2

eight students presented with materials were successful in applying these strategies. Without the materials only two students could extend beyond "one heuristic only." Physical materials seemed to reduce the occurrence of random cases as the only stretegy and increase the use of a combination heuristic or random cases with symmetry.

In examining closely the students in each group using symmetry either alone or in combination with random, it appeared that physical materials elicited to a greater extent the symmetry heuristic--11 as compared to 5. If these data are further subdivided by ability group the effects of physical materials become evident.

Table 14 reveals that the physical presentation of the problem does not greatly affect the use of the symmetry heuristic at the medium and high ability levels. A great difference is seen, however, at the low ability level. Whereas no student of low ability employed the strategy of symmetry while attempting the problem nonphysically, all four of this class used it either alone or in combination with random in the physical presentation. This seems to indicate that for the low ability students sampled the introduction of physical materials was an asset in enabling them to utilize the symmetry heuristic.

TABLE 14

Use of Symmetry in the Geometry Problems as Related to Ability Groups, and Presentation of Problem

Ability Level	Physical Presentation	Nonphysical Presentation
Low	4	-
Medium	3	2
High	4	3

This concludes the analysis of the data with respect to the first three questions. An analysis of the protocols indicates that the examination of total data gives limited information. There are other relevant and crucial factors concerning the thought processes used by students that need more indepth consideration.

Analysis is required that serves to answer such vital questions as: What general patterns are observed in the protocols of students? Do these differ by ability levels? How does the presentation of materials affect patterns, if at all? Are there patterns in the behaviors exhibited by students immediately after reading a problem? These and other interesting questions are the subject of careful examination and analysis in the following section. Specific reference to Appendix G will provide additional information.

<u>Question 4:</u> Is there any "order" in which students tend to utilize heuristic strategies and core procedures?

<u>Algebraic Problem</u>

Differences in patterns of procedure can be observed by analyzing the graphs for various problems and levels. One such pattern involves the systematic examination of cases approach. For the purpose of discussion, three stages of systematization were identified from the protocols.

Level One occurred when students used information from a previous trial to systematically adjust one aspect of the problem--the money or the coins--but had difficulty in considering the two together. For example, a student selected the following number of each coin: 4 quarters, 4 dimes, and 8 nickels. He computed the value and discovered he had too much money. He decided to reduce the dimes by 2, leaving him with the correct money. No consideration was given to the coins for the original 16 was now reduced to 14, whereas the problem required a total of 19. Though systematization occurred, it was limited to one aspect of the problem.

Level Two involved systematically considering the amount of money relevant to the number of coins. The student was unable, however, to carry this procedure through to the end result. For example, consider the following sequence that occurred during the problem-solving procedure. One student considered 1 quarter, 13 dimes, and 1 nickel which he computed to total the exact amount of money involved. He realized that he had only 15 coins so he proceeded to adjust the coins by an exchange procedure. This enabled him to increase the coins but to leave the money constant. He now had 1 quarter, 11 dimes, and 5 nickels which totaled 17 coins. He was unable to continue this pattern and left this systematic trend to investigate 10 dimes and 12 nickels.

Level Three was an extension of the previous one for students were now able to extend their thought processes in a systematic fashion to the final conclusion. For example, consider the following sequence: A student has reached the point where he is investigating whether 3 quarters, 3 dimes, and 11 nickels is the correct solution. He calculated the money to be $1.60 but the coins to be only 17. He decided that a simple exchange of one dime for two nickels will increase the number of coins without changing the money value. This left him with 3 quarters, 2 dimes, and 13 nickels totalling 18 coins. He proceeded with the previous procedure for even exchange and arrived at a correct solution of 3 quarters, 1 dime, and 15 nickels which met both the money and coin criteria.

All three of these classifications occurred throughout the coded protocols with varied degress of frequency related to ability groupings. Levels two and three were most predominant among medium and high ability groups. This suggests that these students are better able to extend their thought processes to concentrate on more than one aspect of a problem. While there is evidence of systematic cases among all three ability levels, it appears that it is not the occurrence of systematization but the level at which it is used that must be considered.

Table 15 illustrates the relation between using systematic cases and obtaining a correct solution. There are no differences in the number of students using systematic cases among the levels but an increase is evidenced in the number of students who obtained a correct solution by employing this heuristic--two for the low ability group as compared to seven for the high ability. This supports the idea that it is not only the occurrence of systematic cases but the level of systematization used that is important.

TABLE 15

Correct Solutions as Related to Use of Systematic Cases and Ability Levels

Ability Level	Number of Students Using Systematic Cases	Number of Students Obtaining Correct Solution
Low	5	2
Medium	6	4
High	8	7

With reference also to systematic cases it is significant to note the patterns developed by students once they have refined their problem-solving procedure. Figure 5 provides evidence for the observation that once students become systematic in their problem-solving strategy, they

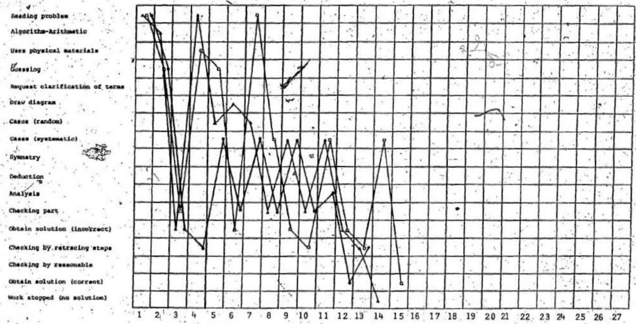


FIGURE 5. Examples of pattern trends involving systematic cases for high and medium ability students.

are inclined to continue in this vein even after breaks in the thought process for checking part, incorrect solution or retracing steps. The figure illustrates the coded protocols of three students. These students began their problem-solving processes by attempting an algorithm or guessing an answer. They eventually refined their procedure to an examination of systematic cases. It was found that students who entered a trend of systematic cases, even though they may break the thought pattern for checking, rereading or requesting clarification of terms, generally returned to a systematized approach incorporating any information gained into their next effort. This pattern was most pronounced for the high ability students and for the medium ability group in the nonphysical setting. There were students who wavered from this pattern by becoming systematic but later aborting it in favor of a random cases approach or an algorithm. These were generally low ability students who more accidently than purposely became systematic at all (see Figure 6).

The pattern in checking solution is also relevant to this discussion. Low ability students tended to limit their checking procedure. Generally it followed the obtaining of a correct solution. Low ability students also halted their strategies very infrequently to check part of the problem. This was especially true for the

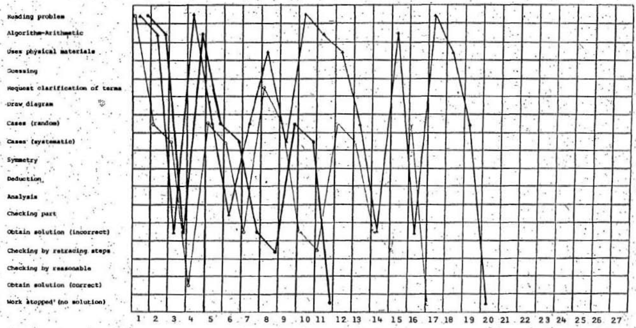
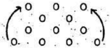


FIGURE 6. Examples of pattern trends involving systematic cases for low ability students.

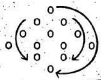
algebra-nonphysical problem. Incorporating checking solution or part of solution with systematic cases in a complementary fashion increased with ability levels and was most dominant at the high ability level with the nonphysical problem.

Geometric Problem

Patterns in groups was difficult to establish due to a lack of varied strategies used. From viewing the videotapes, it was observed that there was more activity in all groups when materials were presented. Students seemed more apt to experiment with ideas when objects could be manipulated. For example, a student presented with materials who felt that the bottom two circles should move to the first row like this



would physically move them, evaluate this idea and decide on his next move. Students who were presented the problem nonphysically tended to be inhibited about the illustration of ideas. If, for example, a student sensed that the top three circles belonged at the bottom



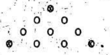
he often simply looked at the given diagram and performed no action. When the researcher inquired about his line of thought, he verbally explained his approach but did not illustrate it. Some students felt insecure and presented only diagrams they thought were correct.

From the graphs of the coded behaviors, it is significant to note that low ability students seemed to continue attempting to solve a problem in a constant procedure until they had reached a solution. They then decided on its correctness and if incorrect they would resort to a random approach. This was especially true for the physical presentation. The medium and high ability groups would experiment with ideas but appeared to be able to envision where this would lead them. This would promote a change in procedure without having to continue until a solution was found.

Symmetry was the key heuristic used in the geometry problem. Two perceptions of symmetry were evident in the protocols. One involved the student sensing the symmetrical design of the structure and attempting to answer the question by reconstructing the pattern. For example, the original pattern

```
      o                   o   o   o
    o   o               o   o   o   o
  o   o   o                 o   o
o   o   o   o   became        o       .
```

The other manifestation of symmetry involved a student sensing the symmetrical pattern evolving around the corner circles. A student would thus move the indicated circles



The positioning of these circles was often in doubt by students. Of the 16 students using symmetry the two perceptions were fairly equally divided.

With reference to the symmetry heuristic, it was noticed that every student (13 in all) who obtained the correct solution used symmetry. Furthermore, there were only three students who used symmetry and were unsuccessful in solving the problem. All three of these sensed the symmetry early in their attempt but were unable to reconstruct it to aid in obtaining a correct solution.

The concept of fixation effect is also interesting to note. Students seemed to have their own method for attempting a problem. Once a pattern was begun students followed it through without any major changes in strategy. This held true for both algebra and geometry in all three ability groups.

This fixation effect was also evident after oral presentation of hints. When students became confused or frustrated by lack of success in their approach, the investigator provided a hint. Generally these hints failed to elicit any heuristics different from the ones previously used. There were three exceptions. Two people, in attempting the algebra problem, changed their strategy from an algorithmic approach to random cases. One of these later incorporated systematic cases. In the geometry problem, one person who used only random cases before the hint revised his strategy to random followed by symmetry. The symmetry heuristic had not been evident before.

Furthermore, of greater importance is the fact that no changes were evident in sequencing patterns of heuristic strategies as a result of the presentation of the hints.

The reasons for the ineffectiveness of the hints can only be speculated. Firstly, the answer may be the lack of experience with this approach. Students who were not exposed to being asked thought-provoking questions during their problem attempt may have found the hints interruptive and unprofitable. Secondly, students may have not been able to incorporate the answer to the hint in the solution to the original problem. Even students who could answer the hint question correctly were unable to relate this to their problem situation. For example, a student who used the algorithm $19\overline{)1.60}$ for the algebra problem solved the simplified hint using random and systematic cases but returned to an algorithm for the original problem. Thirdly, the hints themselves may not have encouraged the utilization of various strategies. Hints of another type may have proved more beneficial. Fourthly, the timing of the presentation of the hints may have been inappropriate. The researcher had to judge when to give the hints. It was often difficult to decide if students had exhausted all ideas or if they were internally applying a strategy. For example, a student might have appeared to be simply looking at the diagram of the circles with no strategy under consideration and thus the researcher provided a hint. The student may, however, have been deciding on the results of a particular move without physically exhibiting any behavior. The hint may have interfered with his thought process. Given at another time it might have proven more beneficial.

Students' initial reaction to a problem immediately after reading it is interesting to examine in analyzing sequences of behaviors exhibited in the solving of mathematical problems. Tables 16 and 17 show the distribution of initial responses by ability group for algebra and geometry problems, respectively. No major differences occurred except for the absence of an algorithmic approach among high ability students. This algorithmic aspect is worthy of further investigation.

TABLE 16

Initial Response to the Algebra Problem by Ability Groups

Ability Group	Initial Responses			
	Algorithm	Guessing	Random Cases	Deduction
Low	3	1	4	-
Medium	3	-	5	-
High	-	2	5	1

TABLE 17

Initial Response to the Geometry Problem by Ability Groups

Ability Group	Initial Responses		
	Diagram*	Random	Symmetry
Low	3	3	2
Medium	2	3	3
High	1	4	3

*The initial response for the physical representation follows "uses physical materials" and for the nonphysical it follows draws first diagram as the nature of the problems elicit these responses.

Although the algebra problem could not be solved by application of an algorithm, three of the eight low ability students selected it as their initial approach. One other chose to resort to an algorithm in the latter stages. All three of the eight who began with an algorithm returned to it later in their protocol. In the medium ability group, four students selected an algorithmic procedure. Of these, three chose it early in their protocol and one in a more advanced stage. Two students, both in the physical setting, decided to return to an algorithm again later in their process. Only two of the high ability chose to use an algorithm and both of these occurred late in the protocol after other procedures had led them to incorrect solutions.

This suggests that while low and medium ability students tended to conceive of an algorithm as their initial reaction to this problem, high ability students seemed to realize that this was not a realistic procedure but utilized heuristic strategies instead. Some students resorted to an algorithmic approach when frustrated by unsuccessful attempts at other procedures.

An analysis of the use of a heuristic also reveals that students who applied a heuristic to one particular problem did not always utilize it in another problem. It is noted that 21 of the 24 people used systematic cases in the algebra problem whereas not one used it in the geometry one. This is in agreement with the findings of Webb (1975) that heuristics are problem-dependent.

The role of perception is seen as crucial in solving the geometry problem especially in the nonphysical mode. Students experienced difficulty both in copying the given diagram onto their cardboard and in visualizing what the answer would look like. Of the 24 students taped, eight clearly exhibited difficulty with the visual aspects of the problem. The majority of these were low ability students. Evidence of this difficulty could be seen by the misconstrued representation of the given triangle, for example,

```
  O
O   O
O   O.
```

Other students were unable to visualize either the problem or the answer. One medium ability student had a problem placing the circles back to their original position after randomly moving them. He placed the circles in this manner

```
      O
   O     O
    O  O  O
   O   O   O      O    and said,
```

"That's back in place.". These visual perception problems must definitely affect any heuristic approach that the student might have considered.

This concludes the analysis of the data pertaining to the heuristic strategies utilized by the sampled grade seven and eight students in solving novel mathematical problems. The chapter which follows will draw conclusions from the analysis and will make recommendations for further research.

CHAPTER V

SUMMARY AND CONCLUSIONS

The purpose of this chapter is to summarize the research that was conducted, to draw conclusions from the analysis of the data and to make recommendations for future research.

Summary

Due to advances in science and technology, interest in problem-solving is increasing among educators in general and mathematics educators in particular. This interest is evidenced by the National Council of Teachers of Mathematics dedication of their 1980 Yearbook to Problem-Solving in School Mathematics which features Polya's How To Solve It on its inside covers. The ability to approach novel mathematical problems with a set of strategies is indeed an asset.

In schools today emphasis on the development of mathematical strategies is limited. It is not surprising that many students experience difficulty in attacking mathematical problems as much of the mathematics in school is concerned with the application of algorithms.

This study was designed to investigate the strategies that grade seven and eight students actually do use in attempting to solve novel mathematical problems. The study sought answers to the four following questions.

1. Without specific training in heuristics, what are the heuristic strategies employed by grade seven and eight students when solving novel mathematical problems?

2. Are there any differences in the heuristic strategies utilized by the students of different abilities?

3. Are there any differences in the heuristics utilized by students using physical materials and those not using them?

4. Is there any "order" in which students tend to utilize heuristic strategies and core procedures?

Investigations also involved other interesting aspects including the levels of systematic cases, the different aspects of symmetry and the role of visual perception.

The sample for the study consisted of 24 grade seven and eight students enrolled in an elementary school in the suburbs of the city. The number of students was divided equally by grade level and ability group--low, medium and high--as categorized by the mathematics teachers.

A section of the school had been set up in advance as an area for conducting the research. Each student individually came to this room to partake in the study. Here he was presented with a mathematical problem of the type--algebra with physical materials, geometry without materials or algebra without physical materials and geometry with materials. A maximum of 15 minutes per problem was permitted. When the correct solution was obtained or the designated time had elapsed (whichever came first), the student was presented with a second problem of the alternate type to the first. Again a 15-minute restriction was imposed. During this problem-solving time the students' behaviors were videotaped by the investigator.

Following the videotaping sessions, the researcher analyzed the protocols using a modified version of the coding system used by Blake (1976). A reliability check on both the heuristics observed and the sequencing of behaviors was conducted with the assistance of a graduate student in mathematics education. The coefficients were found to be .87 and .77, respectively.

Limitations

The conclusions reached in this chapter must be qualified and limited by the sample size, the problems selected and the data gathering techniques.

The sample for this study consisted of only 24 students. This limited sample was necessary because of the type of analysis conducted. Care must be taken with generalizations to other than seventh and eighth graders. The results also can only be generalized to students with mathematical backgrounds similar to those of the subjects used.

The problems selected were an obvious limitation. Each student was asked to solve the same two problems;

although the presentation of physical materials differed. A student may have exhibited different heuristic strategies had he been presented with a different set of problems.

The thinking-aloud technique is the source of another limitation on the conclusions. By having students think aloud as they attempt to solve problems, the researcher gains a large amount of information about the processes being used. However, there are periods when students' thoughts are not verbalized and the thinking-aloud technique provides no information concerning these processes.

The use of videotaping equipment in the data gathering process may also have effected the problem-solving strategies of students and thus placed a limitation on the results. It appeared that students in this study, after the initial orientation and questioning, were not disturbed by the presence of the camera and other equipment. This possibility must be considered, however, in interpreting any conclusions.

Conclusions

This study involved the analysis of student protocols regarding problem-solving behaviors. Conclusions were drawn from an analysis of the graphs and general observations noted during the coding procedure.

The most commonly utilized heuristic used in solving the algebra problem was examination of cases, particularly systematic cases. Students used this strategy in various degrees according to the ability groupings. High and medium ability students were better able to extend their thought processes to concentrate on more than one aspect of a problem in their systematic approach. Low ability students who utilized systematic cases were capable of centering on only one aspect of the problem. It is concluded that it is not the occurrence of systematic cases alone in the problem-solving process but the manner to which it is utilized that is the crucial factor.

Students of high ability also utilized the heuristics of analysis and deduction in their problem-solving strategy.

Consistency was evidenced among students who employed systematic cases. Once a student began a systematic trend he continued in this fashion despite breaks for checking and rereading.

Heuristic strategies did not differ greatly by ability groups. While most students utilized examination of cases, a limited number of the high ability students incorporated the heuristics of analysis and deduction in their overall procedure. Differences in methods of attacking problems were evidenced between groups. High ability students utilized a strategy that was planned and definite. The lower ability groups appeared to be searching for a procedure to employ and usually engaged in haphazard rambling.

Physical materials had a mixed effect on heuristic strategies, depending on the problem situation and the ability group involved. For the low ability students in solving the algebra problem, physical materials were perceived to be of great assistance, while they tended to be a hindrance in the eliciting of heuristic strategies for the high ability group. For the geometry problem, physical materials did not promote different heuristics but they did increase the number of students employing the heuristic strategies. A greater increase in the use of the symmetry heuristic was evidenced for the low ability group with the physical materials than for any other group. Physical materials also increased the activity conducted by students related to experimenting with ideas.

The procedure of checking or "looking back" was found to be an integral aspect of the strategy of high ability students. This aspect of a student's strategy decreased with levels of ability.

Fixation was evidenced in the protocols of students. They fixed their strategy on one idea and were unwilling to change their approach even after the presentation of hints.

The algorithmic approach to a problem as the initial response was found to be used more often by low and medium ability groups than by the higher ability students.

In the geometry problems the heuristics of random cases and symmetry were most often employed by students. Though there were no differences in the heuristics used, the number of occurrences of the symmetry heuristic was significantly greater for the high ability group. Students also sensed the symmetry component in the problem in different ways.

Perception played a vital role in the utilization by students of heuristic strategies in the geometry problem. Many students were handicapped by their visual perception of the problem and this interfered with their mathematical strategies.

Students' behavior in this study supported the belief by some mathematics educators that heuristics are problem-dependent. Students who utilized a particular heuristic strategy in one problem did not use it in the other problem.

Overall, the heuristic strategies utilized by low and medium ability students were comparable. There are difficulties in making classification distinctions between the two. The high ability group engaged in heuristic procedures superior to the other groups. The difference lies in both the approach to a problem and the heuristic strategies employed during the solution procedure with the quality of the high ability being distinct from the others.

Recommendations for Future Research

The focus of this study was the problem-solving process. It is believed that if success with problem-solving is to increase then it is necessary to explore and attempt to understand the processes students use in solving mathematical problems. This requires carefully designed research.

One practical difficulty involving such studies on process is the method of determining the procedures used. The method of data collection used in this study was the "thinking-aloud" procedure via videotaping. There exists some controversy concerning the reliability of this technique (Roth, 1966; Kilpatrick, 1967; Flaherty, 1973; Zalewski, 1974). Researchers are concerned about whether students attempt problems differently when asked to think aloud. Further research is needed to verify the validity of this procedure.

Research such as that conducted by Webb (1975) has found heuristics to be problem-dependent. Future research should focus on studies on a similar nature to the present one with a different set of problems. These may possibly involve open-ended problems to investigate whether different heuristics are elicited.

In this study students were categorized as high, medium and low ability. Only two distinct groups seemed to emerge in the analysis. Further research should investigate the differences in strategies employed by those classified as good problem-solvers versus all other mathematics students.

An instructional aspect was not involved in this study as it was designed to observe the heuristics being used by seventh and eighth graders without specific heuristic training. The next step is to undertake research which would include instruction in heuristics to determine if this would increase the use of heuristics. The format of the study could involve a taping session of students solving selected problems and an analysis of the results. This would be followed by an instructional phase in which students could be taught heuristics through working problems with the guidance of the researcher. A unit of heuristics could be developed for use at a particular grade level.

Two approaches are possible for this instructional package. Students could be exposed to a wide range of different strategies to increase student's problem-solving ability in general. An alternate approach would involve the teaching of a few heuristics to a group of students. Research could examine whether students can be taught to use these specific heuristics in solving a wide range of problems.

This study involved an analysis of heuristic strategies and the sequencing of behaviors utilized to solve mathematical problems. The study used a graphical structure of coded behaviors to give the researcher a visual representation of sequence and patterns. This procedure though tedious to construct proved to be most beneficial in the analysis. The analysis of sequencing of behaviors provided new insights into the pattern students used to approach problems. Further research on sequencing and pattern styles needs to be undertaken.

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APPENDIX A

Problems Used in the Pilot

Problems Used in the Pilot

1. John got $1.60 a week for his allowance. This week his mother paid him 19 coins. He got nickels, dimes and quarters. Show how many of each coin he could have gotten.

2. A younger brother said to the older, "Give me 8 walnuts and then we will each have the same number." But the older brother said to the younger, "Give me 8 walnuts and then I will have twice as many as you do."
How many walnuts did each have?

3. There are 4 people in a room. Everyone shook hands with everyone else once. How many handshakes were there?
If there were 5 people in the room, how many handshakes would there be?
If there were 8 people in the room, how many handshakes would there be?

4. A fireman stood on the middle step of a ladder, directing water into a burning building. As the smoke gets less, he climbed up three steps and continued his work. The fire got worse, so he had to go down five steps. Later he climbed up the last six steps and was at the top of the ladder. How many steps are there?

5. Five boxes are arranged in a row. The red one is next to the blue one, the green is next to the yellow, the purple is next to the red one and the blue is next to the green one. Which box is in the middle? (Students are given the blocks)

6. Subjects are given a ruled piece of cardboard (25cm by 25cm), divided into 25 congruent squares. Find the total number of rectangles on this board.

7. Can you move three circles and turn the triangle upside down?

O

O O

O O O

O O O O

8. There are some rabbits and some cages. When one rabbit is put in each cage, one rabbit will have no cage. When two rabbits are put in each cage, there are two empty cages.
How many rabbits and how many cages are there?

9.

1 1

1 2 1

1 3 3 1

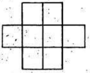
1 4 6 4 1

1 5 10 - - -

Find the sum of the elements in the 50th row.

10. Find the maximum possible area of a rectangle with a perimeter of 24 centimeters. (Given a geoboard)

11. Use the numerals 1-8 once to fill in the small squares such that no two numbers that follow in order (such as 4,5) are in the squares that touch.



APPENDIX B

Original Coding Sheet

Reading problem									
Request definition of terms									
Draw diagram									
Modify diagram									
Use physical materials									
Algorithm--Arithmetic									
Guessing									
Smoothing									
Analysis									
Templation									
Cases (all)									
Cases (random)									
Cases (systematic)									
Looks for a pattern									
Cases (critical)									
Cases (sequential)									
Deduction									
Inverse deduction									
Invariation									
Analogy									
Symmetry									
Obtain solution (correct)									
Obtain solution (incorrect)									
Uncodable									
Work stopped (sol'n)									
Work stopped (no sol'n)									

APPENDIX C

Problems Used in the Study

Problems Used in the Study

Algebra:

John received $1.60 a week for his allowance. This week his mother paid him 19 coins. He got nickels, dimes and quarters. Show how many of each he could have gotten.

Physical Presentation: Students were given cardboard circles of differing sizes to represent nickels, dimes and quarters.

Geometry:

Show how you could move three circles and turn the triangle upside down.

```
      O
    O   O
  O   O   O
O   O   O   O
```

Physical Presentation: Students were given red plastic circles.

APPENDIX D

Revised Coding Sheet

CODING SHEET

Tape # ________ Student Level ______

Counter # ________ Student Grade ______

Problem ________

Reading problem									
Request definition of terms									
Recall same problem									
Recall related problem									
Recall problem type									
Recall related fact									
Draw diagram									
Modify diagram									
Identify variable									
Algorithm-Algebraic									
Algorithm-Arithmetic									
Use physical materials									
Guessing									
Smoothing									
Analysis									
Templation									
Cases (all)									
Cases (random)									
Cases (systematic)									
Looks for a pattern									
Cases (critical)									
Cases (sequential)									
Deduction									
Inverse deduction									
Invariation									
Analogy									
Symmetry									
Obtain solution (correct)									
Obtain solution (incorrect)									
Checking part									
Checking solution									
by subst. in equation									
by retracing steps									
by reasonable/realistic									
Uncodable									
Exp. concern-Method									
Exp. concern-Algorithm									
Exp. concern-Equation									
Exp. concern-Solution									
Work stopped (sol'n.)									
Work stopped (no sol'n.)									

APPENDIX E

Problem Hints

Problem Hints

Algebra Problem:

Support you had 45¢ and six coins, how many nickels and dimes could you have?

Geometry Problem:

1) What will the triangle look like after you have solved the problem?

2) Can you see any row that might remain the same?

APPENDIX F

Interview Questions

Interview Questions

Name: ____________________ Grade: ______ Level: ______

1. What were you thinking about as you tried to solve:

   (a) Coins problem

   (b) Circles problem

2. Have you seen problems like these before? Where?

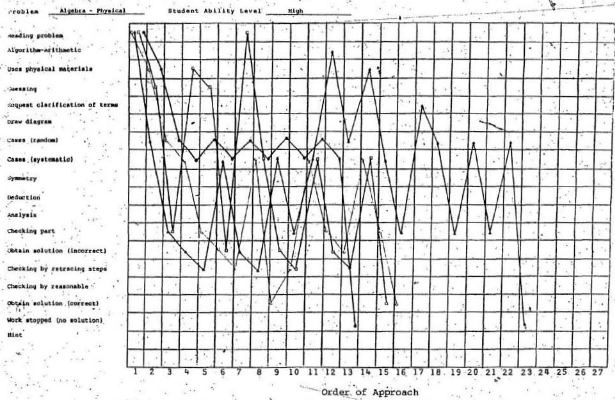
3. Do you consider (a) coins and (b) circles problems to be mathematics problems? Why or why not?

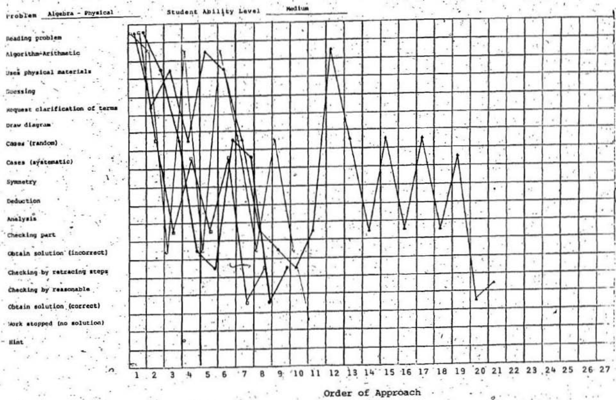
4. Do you do many mathematics problems in school?

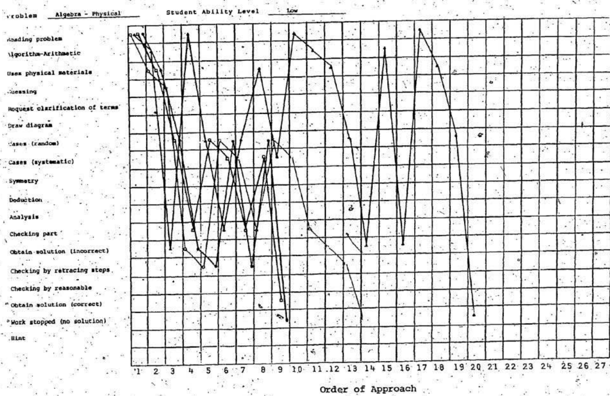
5. After reading the problem, what thoughts first come to mind when you attempt to solve a problem in mathematics?

APPENDIX G

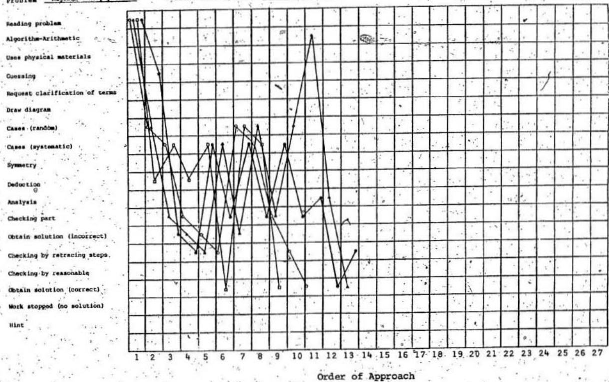
Graphs

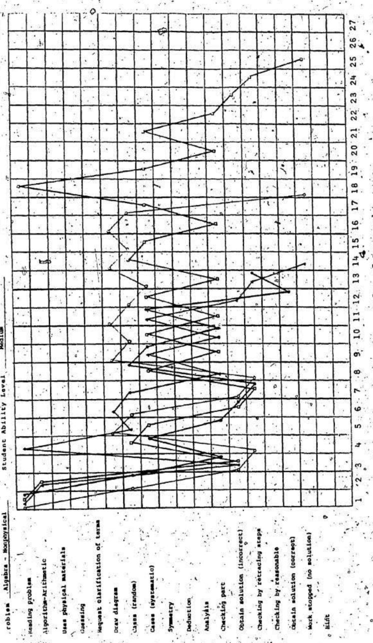

Problem Algebra - Physical
Student Ability Level High
Reading problem
Algorithm-arithmetic
Uses physical materials
Guessing
Request clarification of terms
Draw diagram
Cases (random)
Cases (systematic)
Symmetry
Deduction
Analysis
Checking part
Obtain solution (incorrect)
Checking by retracing steps
Checking by reasonable
Obtain solution (correct)
Work stopped (no solution)
Hint
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27
Order of Approach


Problem Algebra - Physical
Student Ability Level Medium
Reading problem
Algorithm-Arithmetic
Uses physical materials
Guessing
Request clarification of terms
Draw diagram
Cases (random)
Cases (systematic)
Symmetry
Deduction
Analysis
Checking part
Obtain solution (incorrect)
Checking by retracing steps
Checking by reasonable
Obtain solution (correct)
Work stopped (no solution)
Hint
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27
Order of Approach

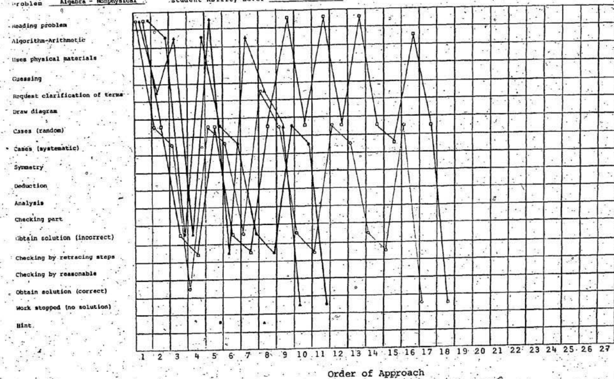

Problem Algebra - Physical
Student Ability Level Low
Reading problem
Algorithm-Arithmetic
Uses physical materials
Guessing
Request clarification of terms
Draw diagram
Cases (random)
Cases (systematic)
Symmetry
Deduction
Analysis
Checking part
Obtain solution (incorrect)
Checking by retracing steps
Checking by reasonable
Obtain solution (correct)
Work stopped (no solution)
Hint
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27
Order of Approach

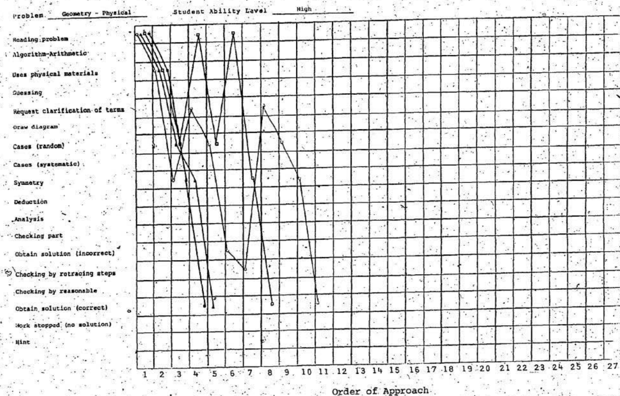
Problem Algebra - Nonphysical   Student Ability Level High



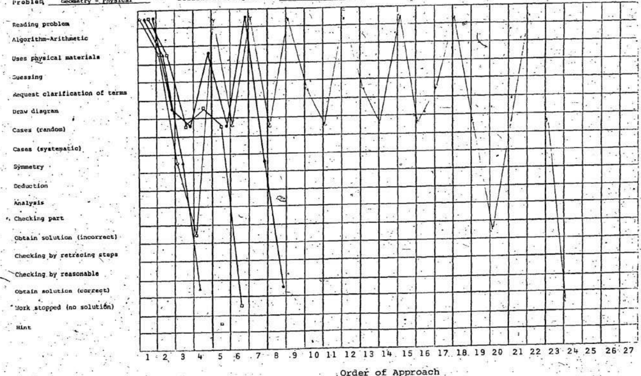

roblem Algebra - Nonphysical
Student Ability Level Medium
Reading problem
Algorithm-Arithmetic
Uses physical materials
Guessing
Request clarification of terms
Draw diagram
Cases (random)
Cases (systematic)
Symmetry
Deduction
Analysis
Checking part
Obtain solution (incorrect)
Checking by retracing steps
Checking by reasonable
Obtain solution (correct)
Work stopped (no solution)
Hint
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27
Order of Approach

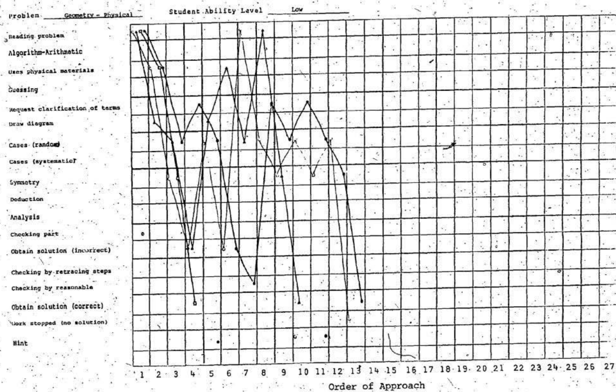
Problem Algebra - Nonphysical Student Ability Level Low

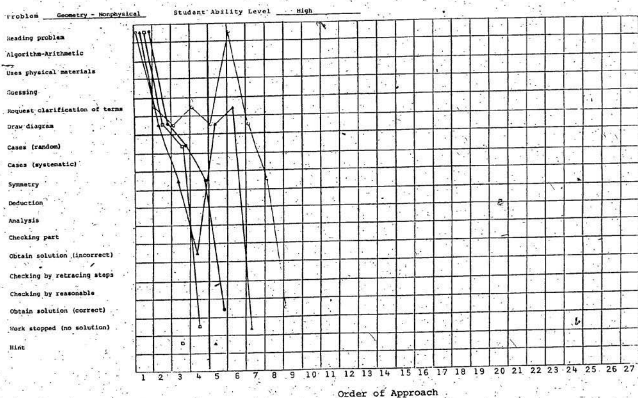



Problem Geometry - Physical
Student Ability Level High
Reading problem
Algorithm-Arithmetic
Uses physical materials
Guessing
Request clarification of terms
Draw diagram
Cases (random)
Cases (systematic)
Symmetry
Deduction
Analysis
Checking part
Obtain solution (incorrect)
Checking by retracing steps
Checking by reasonable
Obtain solution (correct)
Work stopped (no solution)
Hint
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27
Order of Approach

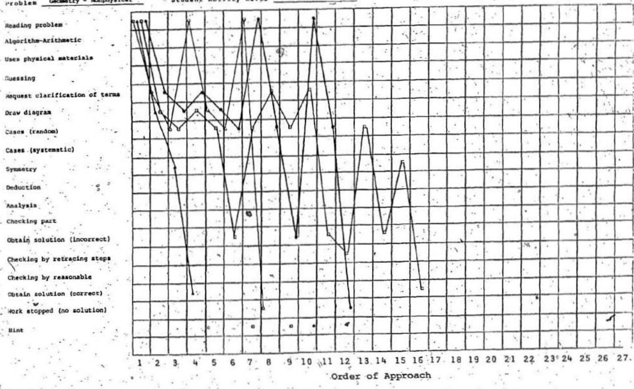
Problem Geometry - Physical Student Ability Level Medium

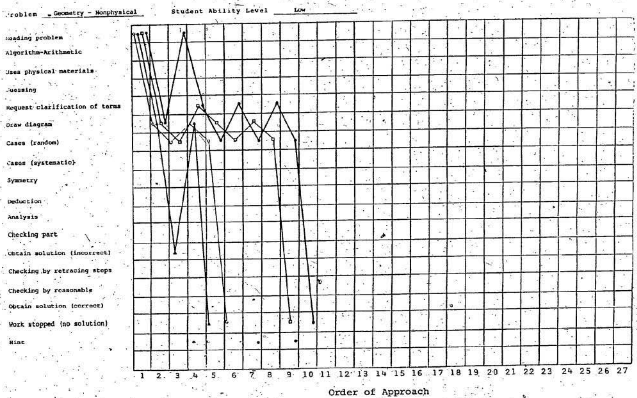



Problem Geometry - Physical
Student Ability Level Low
Reading problem
Algorithm-Arithmetic
Uses physical materials
Guessing
Request clarification of terms
Draw diagram
Cases (random)
Cases (systematic)
Symmetry
Deduction
Analysis
Checking part
Obtain solution (incorrect)
Checking by retracing steps
Checking by reasonable
Obtain solution (correct)
Work stopped (no solution)
Hint
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27
Order of Approach


Problem Geometry - Nonphysical
Student Ability Level High
Reading problem
Algorithm-Arithmetic
Uses physical materials
Guessing
Request clarification of terms
Draw diagram
Cases (random)
Cases (systematic)
Symmetry
Deduction
Analysis
Checking part
Obtain solution (incorrect)
Checking by retracing steps
Checking by reasonable
Obtain solution (correct)
Work stopped (no solution)
Hint
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27
Order of Approach

Problem Geometry - Nonphysical   Student Ability Level Medium




Problem Geometry - Nonphysical
Student Ability Level Low
Reading problem
Algorithm-Arithmetic
Uses physical materials
Guessing
Request clarification of terms
Draw diagram
Cases (random)
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Order of Approach