ITERATED CONTRACTION MAPPINGS AND FIXED POINT THEOREMS IN METRIC SPACES

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by

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Abstract

The main object of this thesis is to study fixed point theorems for iterated contraction mappings in metric and generalized metric spaces.

In the beginning, we have discussed Banach's Contraction Principle, "A contraction mapping of a complete metric space into itself has a unique fixed point", together with its various generalizations in metric spaces.

Then, the iterated contraction mapping,

"$d(Tx,TTx) \leq kd(x,Tx)$, for all x, $Tx \in (X,d)$, $T : X \to X$, and for some constant k, $0 \leq k < 1$", has been considered. We have given the iterated contraction mapping principle, "An iterated contraction mapping, which is continuous at the limit of its sequence of iterates, of a complete metric space into itself has a fixed point", by following the procedure of the Banach Contraction Principle. Then generalizations of the iterated contraction principle have been given in metric spaces. Iterated contractive mappings, "$d(Tx,TTx) < d(x,Tx)$, for all x, $Tx \in (X,d)$, $x \neq Tx$, $T : X \to X$", and iterated nonexpansive mappings, "$d(Tx,TTx) \leq d(x,Tx)$, for all x, $Tx \in (X,d)$, $T : X \to X$", have also been considered briefly in metric spaces.

In the end, we have given some results on iterated contraction mappings in generalized metric spaces.

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Introduction

Existence theorems in analysis first appeared in the nineteenth century and since then have received much attention. These theorems were considered for some time by mathematicians such as Cauchy, Picard, Birkhoff, and Kellogg. Then in 1922, Banach developed his famous principle for contraction mappings. The principle is called Banach's Contraction Principle and is based on a geometric interpretation of Picard's method of successive approximations; it reads as follows:

"A contraction mapping of a complete metric space into itself has a unique fixed point".

Because of its usefulness in proving that certain differential and integral equations have a unique solution, Banach's principle has received much consideration. Extensions to it have been given by various mathematicians such as Boyd and Wong (1969), Chu and Diaz (1965), Edelstein (1961) and Rakotch (1962). This contraction mapping principle has also been extended to generalized metric spaces by Diaz and Margolis (1968), Edelstein (1964), Luxemburg (1958), Margolis (1968), and Monna(1961).

In 1968, Rheinboldt introduced a form of contraction mapping that he called an iterated contraction mapping. He uses this type of mapping to generalize and extend known results as well as to obtain new results for nonlinear operator equations in numerical analysis. Using this concept, he has given a general convergence theory for iterative processes which is based on nonlinear estimates for the iteration function and on the concept of majorizing sequences. (A sequence $\{x_n\}$ in a metric space X is said to be majorized by a real non-negative sequence $\{t_n\}$ if

$d(x_n, x_{n-1}) \leq t_n - t_{n-1}$ for $n = 1, 2, \ldots$). This general convergence theory reduces the study of iterative processes to that of a second order difference equation. (If x_n are defined by the process $x_n = Tx_{n-1}$, $n = 1, 2, \ldots$, then majorizing sequences can be constructed by solving a difference equation of the form $t_n - t_{n-1} = \phi(t_{n-1} - t_{n-2}, t_{n-1}, t_{n-2})$, where $\phi : Q \subset R^p \to R^1$ is nongenative and isotone and Q is a hypercube). The theory due to Rheinboldt gives better convergence results for iterative processes used in solving nonlinear operator equations of the form $Tx = 0$.

Numerical techniques for solving the equation $Tx = 0$ have been considered by many mathematicians, and one of the most central of these techniques is Newton's method:

(1) ... $x_n = x_{n-1} - (T^1(x_{n-1}))^{-1}\, Tx_{n-1}$, $n = 1, 2, \ldots,$

where $T^1(x)$ denotes the Frechet derivative of T at x. Because it eliminates difficulties connected with (1), the following method has received much attention;

(2) ... $x_n = x_{n-1} - B(x_{n-1})Tx_{n-1}$, $n = 1, 2, \ldots,$

where $B(x)$ is, for each x, a linear operator which is usually derived from $T^1(x)$.

A further generalization of (2) occurs when we consider only some sequence of linear operators B_n instead of $B(x_n)$:

(3) ... $x_n = x_{n-1} - B_{n-1}Tx_{n-1}$, $n = 1, 2, \ldots$

Convergence results for iterations of the form (1), (2), and (3) have been derived by Bartle (1955), Ben-Israel (1965 & 66), Byran (1963), Kantorovich (1949 & 52), Ortega and Rheinboldt (1967) and Zinchenko (1963).

When only metric spaces are used, one class of convergence theorems are those which begin with conditions about the initial approximation x_0 and conclude from this that starting from that x_0 the iterates converge to a solution. The best examples of this class are the Banach Contraction Principle and Kantorovich's proof for the convergence of Newton's method.

The general convergence theory that is developed by Rheinboldt gives a unified theory for convergence results of this class. Evidently, the Banach theorem and the Newton-Kantorovich theorem are special cases of this theory. Moreover, Rheinboldt constructed majorizing sequences for iterations of the form (1), (2), and (3) by extending the difference equation he constructed for iterations of the form $x_n = Tx_{n-1}$, $n = 1, 2, \ldots$. When he investigated the natural extensions of these difference equations, Rheinboldt found that his theory encompassed the individual results for processes of the form (1), (2), and (3); it also permitted, he discovered, the generalization of these results and provided an insight into many possibilities of proving numerous other similar results.

CHAPTER I

Contraction Mappings and Related Results

The aim of this chapter is to give preliminary definitions of some terms and to discuss some of the well-known theorems of contraction, contractive and non-expansive mappings of a metric space into itself.

1.1. Preliminary Definitions

Definition [1.1.1] Let X be a set and let $\mathbb{R}^+$ denote the set of positive real numbers. A distance function $d : X \times X \to \mathbb{R}^+$ is defined to be a metric if the following conditions are satisfied for all x, y, z belonging to X :

(i) $d(x,y) \geq 0$,

(ii) $d(x,y) = 0$ if and only if $x = y$,

(iii) $d(x,y) = d(y,x)$ (This is symmetry),

(iv) $d(x,z) \leq d(x,y) + d(y,z)$.

Property (iv) is referred to as the triangle inequality.

The set X with metric d is called a metric space and is denoted by the pair (X,d). Usually the metric space is represented by X with d understood.

Example (1.1.1) Let X be any arbitrary set and consider the function

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases},$$

where $x,y \in X$.

Properties (i) - (iv) are easily satisfied. This particular metric is called the trivial metric.

Example (1.1.2) Let X be the set of real numbers $\mathbb{R}$ and let $d(x,y) = |x - y|$, the usual metric, $x, y \in X$.

Properties (i) - (iv) can again be easily verified.

Definition [1.1.2] If property (ii) of definition (1.1.1) is replaced by

(ii)* $d(x,y) = 0$ if $x = y$, then (X,d) is called a pseudo-metric or semi-metric space.

Example (1.1.3) Let $X = \mathbb{R}^2$ and let the function d be defined by

$d(P,Q) = |x_1 - x_2|$ where $P = (x_1, y_1)$ and $Q = (x_2, y_2)$.

If we take two points R,S (say) with the same x-coordinate but different y-coordinates, we have

$d(R,S) = |x_1 - x_1| = 0$ where $R = (x_1, y_1)$ and $S = (x_1, y_2)$

However, $R \neq S$.

Definition [1.1.3] A sequence $\{x_n\}$ in a metric space X is said to converge to a point x belonging to X, if given an $\varepsilon > 0$, there exists a positive integer N, such that for all $n \geq N$, we have

$d(x_n, x) < \varepsilon$ or $\lim_{n\to\infty} d(x_n, x) = 0$ or $x_n \to x$ as $n \to \infty$.

It can be easily proved that a convergent sequence has a unique limit; that is, if $x_n \to x_0$ and $x_n \to y_0$, then $x_0 = y_0$. To prove this we use properties (ii) and (iv) of Definition (1.1.1).

Example (1.1.4) Let $X = [0,1]$ and let $\{x_n\} = \{\frac{1}{n}\}$. The sequence $\{x_n\}$ converges to 0 belonging to [0,1].

Definition [1.1.4] A sequence $\{x_n\}$ of points of a metric space X is called a Cauchy sequence if given an $\varepsilon > 0$ there exists a positive integer N, such that for all $n,m \geq N$ we have

$$d(x_n,x_m) < \varepsilon \text{ or } \lim_{n,m\to\infty} d(x_n,x_m) = 0.$$

Remark. A convergent sequence is always a Cauchy sequence.

Proof: For a given $\varepsilon > 0$, there exist $n,m \geq N$ such that

$$d(x_n,x) < \frac{\varepsilon}{2} \text{ and } d(x,x_m) < \frac{\varepsilon}{2}.$$

By the triangle inequality we have

$$d(x_n,x_m) \leq d(x_n,x) + d(x,x_m)$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence $\{x_n\}$ is a Cauchy sequence.

However, a Cauchy sequence need not always converge.

Example (1.1.5) Let $X = (0,1)$, $d(x,y) = |x - y|$ for all $x,y \in X$. The sequence $\{\frac{1}{n}\}$, $n = 1,2,3, \ldots$ is easily seen to be a Cauchy sequence which does not converge in X.

Definition [1.1.5] A metric space X is said to be complete if every Cauchy sequence in X converges to a point in X.

Example (1.1.6) $X = [0,1]$ is complete; $X = (0,1]$ is not complete.

Definition [1.1.6] Let T be a mapping of a set X into itself. A point $x_o \in X$ is called a fixed point of T if $Tx_o = x_o$; that is, a fixed point is one which remains invariant under the mapping.

Example (1.1.7) Let $T : [0,1] \to [0,1]$ be defined by

$$Tx = \frac{x}{5}.$$

Then $T0 = 0$ and 0 is a fixed point of T.

Definition [1.1.7] (i) A mapping T of a metric space X into a metric space Y is said to be continuous at a point $x_o \in X$ if and only if $\lim_{n\to\infty} Tx_n = Tx_o$ for all sequences in X with $\lim_{n\to\infty} x_n = x_o$. If it is true for all $x_o \in X$, then T is continuous on X.

or (ii) T is said to be continuous at $x_o \in X$ if given an $\varepsilon > 0$ there exists a $\delta > 0$ such that $d(x,x_o) < \delta \Rightarrow d(Tx,Tx_o) < \varepsilon$, $x \in X$. T is continuous on X if this is true for all $x_o \in X$.

Theorem [1.1.1] (An example of a continuous function with a fixed point). Let f be a continuous function from the closed interval $[-1,1]$ into itself. Then there must exist a point x_o in $[-1,1]$ such that $f(x_o) = x_o$.

Proof: We prove this fact by taking a function F such that $F(x) = f(x) - x$. We note that $F(-1) \geq 0$ and $F(1) \leq 0$. Therefore, by the "Weierstrass Intermediate Value Theorem", we find that there exists a point x_o in $[-1,1]$ such that $F(x_o) = 0$. This implies $f(x_o) = x_o$.

Definition [1.1.8] A mapping T of a metric space X into itself is said to satisfy Lipschitz's condition if there exists a real number k such that

$$d(Tx,Ty) \leq kd(x,y) \quad \text{for all } x,y \in X.$$

Remark. In the special case, when $0 \leq k < 1$, T is called a contraction mapping. Example (1.1.7) is an example of a contraction mapping.

Theroem [1.1.2] If T is a contraction mapping on a metric space X, then T is continuous on X.

Proof: Let $\epsilon > 0$ be given and let x_o be any point in X. Since T is a contraction mapping, we have

$$d(Tx_o,Tx) \leq kd(x_o,x) \quad \text{for all} \quad x \in X, \quad 0 \leq k < 1.$$

If $k = 0$, then $d(Tx_o,Tx) = 0 < \epsilon$ and T is continuous at x_o.

Otherwise, let $\delta = \frac{\epsilon}{k}$ and let x be any point in X such that

$$d(x_o,x) < \delta.$$

Therefore, we have

$$d(Tx_o,Tx) \leq kd(x_o,x) < k\delta = k\frac{\epsilon}{k} = \epsilon .$$

Hence T is continuous at x_o, and since x_o is an arbitrary point in X, T is continuous on X.

Remark: The converse of the above theorem is not necessarily true; that is, a continuous function need not be a contraction. For example, let $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$Tx = x + 1.$$

T is contunuous but T is not a contraction.

1.2 Contraction Mapping Principle.

The "Principle of Contraction Mappings" was formulated by S. Banach (1892-1945), a famous Polish mathematician, who is one of the founders of Functional Analysis. The principle is known as "Banach's Contraction Principle", and it is widely used to prove the existence and uniqueness of solutions of differential and integral equations.

Theorem [1.2.1] Let T be a contraction mapping of a complete metric space X into itself. Then T has a unique fixed point.

Proof: Choose any $x_o \in X$ and define the sequence $\{x_n\}$ in X inductively by

$$x_1 = Tx_o$$
$$x_2 = Tx_1 = T^2x_o$$
$$\vdots$$
$$x_n = Tx_{n-1} = T^nx_o.$$

We have to show that $\{x_n\}$ is a Cauchy sequence; that is,

$\lim\limits_{n,m\to\infty} d(x_m,x_n) = 0.$

Since T is a contraction mapping, we have

$$d(x_1,x_2) = d(Tx_o,Tx_1) \leq kd(x_o,x_1)$$
$$d(x_2,x_3) = d(Tx_1,Tx_2) \leq kd(x_1,x_2) \leq k^2d(x_o,x_1)$$
$$\vdots$$
$$d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n) \leq k^nd(x_o,x_1).$$

Also,

$$\begin{aligned} d(x_n,x_m) &= d(Tx_{n-1},Tx_{m-1}),\ m > n \\ &\leq kd(x_{n-1},x_{m-1}) \\ &\leq k^2d(x_{n-2},x_{m-2}) \\ &\ \vdots \\ &\leq k^nd(x_o,x_{m-n}). \end{aligned}$$

By the triangle inequality, we have

$$\begin{aligned} d(x_o,x_{m-n}) &\leq d(x_o,x_1) + d(x_1,x_2) + \dots + d(x_{m-n-1},x_{m-n}) \\ &\leq d(x_o,x_1) + kd(x_o,x_1) + \dots + k^{m-n-1}d(x_o,x_1) \\ &\leq d(x_o,x_1)[1 + k + k^2 + \dots + k^{m-n-1}] \\ &\leq d(x_o,x_1)\left(\frac{1}{1-k}\right). \end{aligned}$$

Therefore,

$$d(x_n,x_m) \leq k^n d(x_o,x_{m-n})$$
$$\leq k^n\left(\frac{1}{1-k}\right) d(x_o,x_1).$$

Since $k < 1$, the right hand side tends to 0 as n tends to infinity.

Hence $\{x_n\}$ is a Cauchy sequence, and since X is complete $\{x_n\}$ converges to a point $y \in X$; that is,

$$\lim_{n\to\infty} d(x_n,y) = 0 \quad \text{or} \quad \lim_{n\to\infty} x_n = y.$$

Since a contraction mapping is continuous, T is continuous and we have,

$$Ty = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = y.$$

Therefore y is a fixed point of T.

We still have to prove that y is a unique fixed point of T.

Let y and z be two fixed points of T, where $y \neq z$, i.e. $d(y,z) \neq 0$.

Then $Ty = y$ and $Tz = z$.

Thus we have $d(y,z) = d(Ty,Tz)$.

Now, since T is a contraction mapping

$$d(Ty,Tz) \leq kd(y,z), \; 0 \leq k < 1.$$

Hence

$$d(y,z) = d(Ty,Tz) \leq kd(y,z)$$

If $d(y,z) \neq 0$, then $k \geq 1$. This is a contradiction to the fact that $0 \leq k < 1$. Therefore $d(y,z) = 0$ and $y = z$.

Therefore y is a unique fixed point of T.

Remark. Both conditions of the previous theorem are necessary, as can be seen from the following examples.

(i) $T : (0,1] \to (0,1]$, defined by $Tx = \frac{x}{3}$, is a contraction mapping. However, since $X = (0,1]$ is not complete, T has no fixed point.

(ii) $T : \mathbb{R} \to \mathbb{R}$, defined by $Tx = 2x + 1$, is not a contraction mapping and has no fixed point even though $\mathbb{R}$ is complete.

Example (1.2.1)

(i) $T : [0,1] \to [0,1]$, defined by $Tx = \frac{x}{2}$, is a contraction mapping. Since $[0,1]$ is complete, T has a unique fixed point $x = 0$.

(ii) $T : \mathbb{R} \to \mathbb{R}$, defined by $Tx = \frac{1}{2}x + \frac{1}{2}$, is a contraction mapping. $\mathbb{R}$ is complete and $x = 1$ is the unique fixed point of T.

Remark. In a pseudo-metric space X (even in a complete pseudo-metric space), the uniqueness in Banach's theorem may not hold because $d(x,y) = 0$ does not imply that $x = y$ for $x, y \in X$.

The contraction mapping principle has been generalized in various ways. The following results, due to Chu and Diaz [15], are worth mentioning.

Theorem [1.2.2] If T maps a complete metric space X into itself and if T^n (for some positive integer n) is a contraction mapping of $X \to X$, then T has a unique fixed point.

Proof: Since T^n is a contraction mapping of X into itself, where X is a complete metric space, we have, by Banach's Contraction Principle, that T^n has a unique fixed point; that is, there exists a point $y \in X$ such that $T^n y = y$.

Now $Ty = TT^n y = T^n Ty$.

So Ty is a fixed point of T^n. But T^n has a unique fixed point y.

Hence $Ty = y$, and thus T has a fixed point.

This fixed point of T is unique also, since a fixed point of T is also a fixed point of T^n.

We can easily show that Theorem [1.2.2] is more general than Theorem [1.2.1].

Example (1.2.2). Let $T : [0,1] \to [0,1]$ be defined by

$$Tx = \frac{1}{2} \quad \text{if } x \in [0,\tfrac{1}{2}]$$

$$= 0 \quad \text{if } x \in (\tfrac{1}{2},1].$$

Now $X = [0,1]$ is complete and T is discontinuous at $x = \frac{1}{2}$.

Also $T^2x = T^3x = \ldots = \frac{1}{2}$ for all $x \in [0,1)$.

Hence T^r, $r = 2,3, \ldots,$ is a contraction of $[0,1]$ into $[0,1]$.

T is not a contraction, since it is not continuous at $x = \frac{1}{2}$. However, $x = \frac{1}{2}$ is a fixed point of T.

Theorem [1.2.3] Let S be any non-empty set of elements and let T be a mapping of S into itself. If, for some positive integer n, T^n has a unique fixed point, then T also has a unique fixed point.

The proof follows on the same lines as in Theorem [1.2.2].

Kannan [27] has given the following result.

Theorem [1.2.4] Let X be a metric space and let T be a mapping of X into itself. Suppose that T is continuous at a point $x_o' \in X$. If there exists a point $x \in X$ such that the sequence of iterates $\{T^n x\}$ converges to x_0, then $Tx_0 = x_0$. If, in addition,

$$d(Tx_0, Tx) \leq kd(x_0, x)\ ,\ x \in X,\ 0 \leq k < 1,$$

then x_0 is the unique fixed point of T.

Chu and Diaz [16] also gave the following result.

Theorem [1.2.5] Let T and K be two functions from a non-empty set X into itself such that K has a left-inverse (that is, a function K^{-1} such that $K^{-1}K = I$, where I is the identity mapping of X into X). Then the function T has a fixed point if and only if KTK^{-1} has a fixed point.

Proof: (i) Let T have a fixed point and let x be the fixed point of T, i.e. $Tx = x$.

Then

$$T(I(x)) = Tx = x,$$

$$T(K^{-1}K)x = x \text{ since } K^{-1}K = I .$$

Letting K act on both sides of the equation, we get

$$KT(K^{-1}K)x = Kx$$

or $\quad KTK^{-1}(Kx) = Kx.$

Therefore, Kx is a fixed point of KTK^{-1}.

(ii) Let KTK^{-1} have a fixed point y (say).

Then

$$(KTK^{-1})y = y.$$

Letting K^{-1} act on both sides of the equation, we get

$$K^{-1}(KTK^{-1})y = K^{-1}y$$

or $\quad (K^{-1}K)TK^{-1}y = K^{-1}y$

$$T(K^{-1}y) = K^{-1}y\ ,\ \text{ since } K^{-1}K = I.$$

Hence $K^{-1}y$ is a fixed point of T.

Remark. A theorem similar to Theorem [1.2.5] can be stated if the same conditions exist except that K has a right inverse; that is $KK^{-1} = I$. Then T has a fixed point if and only if $K^{-1}TK$ has a fixed point. The proof is similar to the proof of Theorem [1.2.5].

Theorem [1.2.6] Let X be a complete metric space. Let T and K be mappings of X into itself such that K has a left inverse. If KTK^{-1} is a contraction mapping of X into itself, then T has a unique fixed point.

The proof follows directly from the above theorem and the Banach Contraction Principle.

In the following theorem we are concerned with the continuity of the fixed point. We state the theorem and give a proof due to Singh [43], which is simpler than that given by Bonsall [9].

Theorem [1.2.7] Let X be a complete metric space, and let T and T_n $(n = 1,2,\ldots)$ be contraction mappings of X into itself with the same Lipschitz constant k, $0 \leq k < 1$, and with fixed points u and u_n respectively. Suppose that $\lim_{n\to\infty} T_n x = Tx$ for every $x \in X$. Then $\lim_{n\to\infty} u_n = u$.

Proof: Since $\lim_{n\to\infty} T_n x = Tx$ for every $x \in X$, there exists, for $\varepsilon > 0$, an N such that for all $n \geq N$ one has

$$d(T_n u, Tu) \leq (1 - k)\varepsilon, \quad 0 \leq k < 1.$$

$$d(u,u_n) = d(Tu,T_n u_n)$$
$$\leq d(Tu,T_n u) + d(T_n u,T_n u_n)$$
$$\leq (1-k)\varepsilon + kd(u,u_n), \text{ since } T_n \text{ is a contraction.}$$

Hence $(1-k)d(u,u_n) \leq (1-k)\varepsilon$.

Since $0 \leq k < 1$, we have

$$d(u,u_n) \leq \varepsilon \quad \text{for } n \geq N.$$

Therefore $\lim_{n\to\infty} u_n = u$.

Remark. Singh and Russell [41] have shown that the same result holds if T is any mapping of a complete metric space X into itself and T_n, $n = 1,2,\ldots,$ are contractions of X into X such that $\lim_{n\to\infty} T_n x = Tx$, for every $x \in X$.

Definition [1.2.1] A metric space X is said to be ε-chainable if for all $x,y \in X$, there exists an ε-chain joining x and y; that is, there exists a finite set of points $x = x_0, x_1, \ldots, x_n = y$ such that $d(x_{i-1},x_i) < \varepsilon$ for $i = 1,2,\ldots,n$.

Definition [1.2.2] Let X be a metric space. A mapping T of X into itself is said to be a local contraction if for every $z \in X$, there exists an ε and k ($\varepsilon > 0$, $0 \leq k < 1$), which may depend on z, such that

$$x,y \in S(z,\varepsilon) = \{w \mid d(z,w) < \varepsilon\} \text{ implies}$$
$$d(Tx,Ty) \leq kd(x,y).$$

Definition [1.2.3] A mapping T of a metric space X into itself is said to be an (ε,k)-uniform local contraction if it is a local contraction and both ε and k do not depend on z.

The following result is due to Edelstein [19].

Theorem [1.2.8] Let X be a complete-ε-chainable metric space, and let T be a mapping of X into itself such that T is an (ε,k)-uniform local contraction. Then T has a unique fixed point in X.

Rakotch gave the following in [38].

Definition [1.2.4] Let F_1 be a family of functions satisfying the following conditions:

(i) $k(x,y) = k(d(x,y))$, i.e. k depends on the distance between x and y only.

(ii) $0 \leq k(d) < 1$ for every $d > 0$.

(iii) $k(d)$ is a monotonically decreasing function of d.

Theorem [1.2.9] Let X be a complete metric space and let T be a mapping of X into itself such that

$$d(Tx,Ty) \leq k(x,y)d(x,y), \text{ for all } x,y \in X,$$

where $k(x,y) \in F_1$. Then T has a unique fixed point in X.

Remark. The above theorem can also be proved by assuming $k(x,y)$ to be monotonically increasing and $0 \leq k(x,y) < 1$ where $k(x,y) = k(d(x,y))$. It has been done by Cheema [13].

Boyd and Wong [10] and Browder [11] have both given an extension of the Banach Contraction Principle by assuming that T is a mapping from a complete metric space X into itself satisfying the following condition:

$d(Tx,Ty) \leq \Psi(d(x,y))$, where Ψ is defined on the closure of the range of d and $\Psi(r) < r$ for all $r > 0$.

Theorem [1.2.10] (Boyd and Wong). Let X be a complete metric space and let T be a mapping of X into itself such that

$$d(Tx,Ty) \leq \Psi(d(x,y)),$$

where

$\Psi : \overline{P} \rightarrow [0,\infty)$ is upper-semicontinuous from the right on $\overline{P}$ ($\overline{P}$ is the closure of $P = \{d(x,y) \mid x,y \in X\}$) and satisfies $\Psi(t) < (t)$ for all $t \in \overline{P} \setminus \{0\}$.

Then T has a unique fixed point x_o and $T^n x \to x_o$ (as $n \to \infty$) for each $x \in X$.

Remark. (1): Boyd and Wong have also shown that if X is metrically convex (that is, $x,y \neq z$ for each $x,y \in X$, $d(x,y) = d(x,z) + d(z,y)$), then the upper-semicontinuous condition can be dropped from the above theorem.

Remark. (2): If $\Psi(t) = k(t)t$, where k is decreasing and $k(t) < 1$ for $t > 0$, then we have Rokotch's theorem.

The following example, due to Meir and Keeler [35], shows that the condition,

"$d(Tx,Ty) \leq \Psi(d(x,y))$, where Ψ is defined on the closure of the range of d and $\Psi(r) < r$ for all $r > 0$," can be satisfied in a complete metric space without T having a fixed point.

Example (1.2.3) Let $S_n = \sum_{k=1}^{n} (1 + \frac{1}{k})$ and let $X = \{S_n\}$. Then X is complete.

Let $T(S_n) = S_{n+1}$ for all n.

Then $d(Tx,Ty) \leq \Psi(d(x,y))$ with $\Psi(1+\frac{1}{n}) = 1 + \frac{1}{n+1}$

However, T has no fixed point.

Theorem [1.2.11] (Browder) Let M be a bounded subset of a complete metric space X, and let T be a uniformly continuous mapping of M into itself. Suppose there exists a positive integer m and a monotone function $\Psi(r)$ for $r \geq 0$ with Ψ continuous on the right such that $\Psi(r) < r$ for all $r > 0$, while for all $x,y \in M$,

$$d(T^m x, T^m y) \leq \Psi(d(x,y)).$$

Then (a) For each $x \in M$, $\{T^n x\}$ converges in X to a limit point x_0, which is independent of the choice of the initial approximant $x \in M$.

(b) If T is extended continuously to a continuous map of the closure of M into itself, then x_0 is the unique fixed point of the extended mapping T in the closure of M.

1.3 Contractive Mappings.

Definition [1.3.1] A mapping T of a metric space X into itself is said to be contractive (or a strict contraction) if

$$d(Tx,Ty) < d(x,y) \quad \text{for all } x,y \in X, \quad x \neq y.$$

Remark. It is easy to see that a contractive mapping is continuous. Moreover, if a contractive mapping has a fixed point, then this fixed point is unique. However, a contractive mapping does not always have a fixed point as the following example will show.

Example (1.3.1) Let $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$Tx = \ln(1+e^x) \quad \text{for all } x \in \mathbb{R}.$$

Then T has no fixed point although T is a contractive mapping since

$$T'(x) = \frac{e^x}{1 + e^x} < 1.$$

Various mathematicians have considered contractive mappings and the conditions under which a contractive mapping will always have a fixed point.

The following known result (cf. Chu and Diaz) is given by Meir and Keeler [35].

<u>Theorem</u> [1.3.1] Let T be a contractive mapping of a complete metric space X into itself. Then, if the sequence of iterates $\{T^n x_o\}$, for any $x_o \in X$, forms a Cauchy sequence, T has a unique fixed point.

<u>Proof</u>: Let $x_n = Tx_{n-1} = T^n x_o$, x_o an arbitrary point in X and $n = 1, 2, \ldots$

Since X is complete and $\{x_n\} = \{T^n x_o\}$ forms a Cauchy sequence, $\{x_n\}$ has a limit in X; that is,

$$\lim_{n\to\infty} T^n x_o = \lim_{n\to\infty} x_n = x \in X.$$

Also T is continuous since it is contractive.

Hence

$$Tx = T \lim_{n\to\infty} T^n x_o = \lim_{n\to\infty} T^{n+1} x_o = \lim_{n\to\infty} x_{n+1} = x.$$

Thus x is a fixed point of T and is unique since T is a contractive mapping.

The following result, due to Edelstein [20], gives the sufficient conditions for a contractive mapping to have a fixed point when the space is not necessarily complete.

Theorem [1.3.2] Let T be a contractive mapping of a metric space X into itself. Suppose that the sequence of iterates $\{T^n x_0\}$, for any $x_0 \in X$, has a convergent subsequence $\{T^{n_i} x_0\}$ in X. Then T has a unique fixed point x, where $x = \lim_{n\to\infty} T^{n_i} x_0$.

Remark. A simpler proof than that due to Edelstein has been given in [23].

Definition [1.3.2] A metric space X is compact if and only if X is sequentially compact. If every sequence in X has a subsequence convergent in X, then X is sequentially compact.

The following corollary of Edelstein follows from Theorem [1.3.2] and Definition [1.3.2].

Corollary [1.3.3] If T is a contractive mapping of a metric space X into a compact metric space $Y \subset X$, then T has a unique fixed point.

Remark. Of course, if $T : X \to X$, where X is a compact metric space and T is a contractive mapping, then T has a unique fixed point.

Edelstein [20] has also considered a "uniformized" local version of the contractive mapping.

Definition [1.3.3] A mapping T from a metric space X into itself is called ε-contractive if there exists an $\varepsilon > 0$ such that

$$0 < d(x,y) < \varepsilon \text{ implies } d(Tx,Ty) < d(x,y), \quad x,y \in X.$$

Theorem [1.3.4] Let X be a metric space, and let T be an ε-contractive mapping of X into itself such that the sequence of iterates $\{T^n x_0\}$, for any $x_0 \in X$, has a subsequence $\{T^{n_i} x_0\}$ convergent in X.

Then $x = \lim_{n\to\infty} T^{n_i}x_o$ is a periodic point; that is, there exists a positive integer k such that $T^k(x) = x$.

Remark. If Definition [1.3.3] is substituted by a "non-uniformized" local version, such as

$$x \neq y,\ x,y \in S(z,\varepsilon(z)) \text{ implies } d(Tx,Ty) < d(x,y),$$

where

$$S(z,\varepsilon(z)) = \{w \mid d(z,w) < \varepsilon(z)\},$$

then Theorem [1.3.4] may not remain true. Edelstein [20] has given an example to show this.

Remark. Edelstein has also given an example to show that taking X to be an ε-chainable metric space in Theorem [1.3.4] is still too weak to guarantee the existence of a fixed point; that is, it is still too weak to guarantee that k will be equal to 1.

The following theorem of Edelstein [20] gives the sufficient conditions for the existence of a fixed point of an ε-contractive mapping of an ε-chainable metric space into itself.

Theorem [1.3.5] Let X be an ε-chainable metric space, and let T be an ε-contractive mapping of X into itself such that the sequence of iterates $\{T^n x_o\}$, for any $x_o \in X$, has a subsequence $\{T^{n_i}x_o\}$ converging in X. If $x = \lim_{n\to\infty} T^{n_i}x_o$ has a compact spherical neighbourhood $K(x,\rho)$ of radius $\rho \geq 0$, then x is the unique fixed point of T.

Remark. If X is a compact ε-chainable metric space or T maps X into a compact subspace of X in the above theorem, then such a $K(x,\rho)$, $\rho \geq 0$, always exists.

An extension of Edelstein's Theorem [1.3.2] has been given by Bailey [2].

Theorem [1.3.6] If T is a continuous mapping of a compact metric space X into itself and $0 < d(x,y)$ implies that there exists $n(x,y) \in I^+$ (the positive integers) such that

$$d(T^n x, T^n y) < d(x,y) \quad , \quad x,y \in X,$$

then T has a unique fixed point.

In [35], Meir and Keeler have considered a different form of the contractive mapping.

Definition [1.3.4] A mapping T of a metric space X into itself is called a weakly uniformly strict contraction if

given an $\varepsilon > 0$, there exists a $\delta > 0$ such that

$\varepsilon \leq d(x,y) < \varepsilon + \delta$ implies $d(Tx,Ty) < \varepsilon$, $x,y \in X$.

Theorem [1.3.7] Let X be a complete metric space, and let T be a mapping of X into itself. If T is a weakly uniformly strict contraction then T has a unique fixed point x, where $x = \lim_{n\to\infty} T^n x_0$, for any $x_0 \in X$.

Remark. Edelstein's Corollary [1.3.3] follows from this result because Meir and Keeler have proved that in a compact space, any strict contraction is a weakly uniformly strict contraction.

Singh [42] has given a theorem concerning the continuity of the fixed point of a contractive mapping.

Theorem [1.3.8] Let X be a locally compact metric space, and let $\{T_n\}$ be a sequence of contractive mappings of X into itself with fixed points u_n, $n = 1,2, \ldots$. Let $\{T_n\}$ converge pointwise to a contraction mapping T with fixed point u. Then $\{u_n\}$ converges to u.

1.4. Nonexpansive Mappings.

Definition [1.4.1] A mapping T of a metric space X into itself is said to be

(i) nonexpansive if

$$d(Tx,Ty) \le d(x,y), \quad \text{for all } x,y \in X.$$

(ii) ε-nonexpansive if

$$d(Tx,Ty) \le d(x,y), \quad \text{for all } x,y \in X$$

with $d(x,y) < \varepsilon$.

Remark. In some metric spaces ε-nonexpansive mappings are also non-expansive. This is always the case when X is ε-chainable, and the condition

$$d(x,y) = \inf_{c(x,y)} \sum_{i=1}^{n} d(x_{i-1},x_i),$$

where $c(x,y)$ denotes the collection of all ε-chains $x = x_0, x_1, \ldots, x_n = y$ (n arbitrary), $d(x_{i-1},x_i) < \varepsilon$, holds.

Proof: Since T is ε-nonexpansive, we have

$$d(Tx_{i-1},Tx_i) \le d(x_{i-1},x_i) \text{ provided } d(x_{i-1},x_i) < \varepsilon.$$

Hence

$$\begin{aligned} d(Tx,Ty) &\le \inf_{c(x,y)} d(Tx_{i-1},Tx_i) \\ &\le \inf_{c(x,y)} d(x_{i-1},x_i) \\ &= d(x,y) \quad \text{for all } x,y \in X. \end{aligned}$$

Therefore, T is nonexpansive.

Definition [1.4.2] A point $y \in Y \subset X$ is said to belong to the T-closure of Y, $y \in Y^T$, if $T(Y) \subset Y$, and there is a point $x \in Y$ and a sequence (n_i) of positive integers ($n_1 < n_2 < \ldots < n_i < \ldots$), so that

$$\lim_{i\to\infty} T^{n_i}(x) = y.$$

<u>Definition</u> [1.4.3] A sequence $\{x_i\}$ in X is said to be

(i) An isometric sequence if

$d(x_m,x_n) = d(x_{m+k},x_{n+k})$, for all $k,m,n = 1,2, \ldots$.

(ii) An ε-isometric sequence if

$d(x_m,x_n) = d(x_{m+k},x_{n+k})$, for all $k,m,n = 1,2, \ldots$, with

$d(x_m,x_n) < \varepsilon$. A point $x_o \in X$ is said to generate such an isometric (ε-isometric) sequence under T if $\{T^n x_o\}$ is such a sequence.

In [21], Edelstein has given the following theorems on nonexpansive and ε-nonexpansive mappings.

<u>Theorem</u> [1.4.1] If T is an ε-nonexpansive mapping of a metric space X into itself and $x \in X^T$, then a sequence $\{m_j\}$, $(m_1 < m_2 < \ldots)$, of positive integers exists so that

$$\lim_{i\to\infty} T^{m_i}(x) = x.$$

<u>Theorem</u> [1.4.2] If T is a nonexpansive (ε-nonexpansive) mapping of X into X, then each $x \in X^T$ generates an isometric (ε-isometric) sequence.

Cheney and Goldstein [14] have proved the following theorem for nonexpansive mappings.

<u>Theorem</u> [1.4.3] Let T be a mapping of a metric space X into itself such that

(i) $d(Tx,Ty) \le d(x,y)$, for all $x,y \in X$.

(ii) if $x \neq Tx$, then $d(Tx,T^2x) < d(x,Tx)$.

(iii) for each $x_o \in X$, the sequence of iterates $\{T^n x_o\}$ has a cluster point.

Then the sequence $\{T^n x_o\}$ converges for each x_o to a fixed point of T.

<u>Definition</u> [1.4.4] Let X be a metric space, and let the diameter of $Y \subset X$ be denoted by $\delta(Y)$, where

$$\delta(Y) = \sup\{d(x,y) / x,y \in Y\}.$$

Let T be a mapping of X into itself and, for some $x_o \in X$, let

$$0(x_o) = \{x_o, Tx_o, T^2x_o, \ldots\}.$$

If for each $x_o \in X$, $\lim_{n\to\infty} \delta(0(T^n x_o)) < \delta(0(x_o))$ whenever $\delta(0(x_o)) > 0$, then T is said to have diminishing orbital diameters on X.

Belluce and Kirk [5] proved the following result by using Theorem [1.4.2] of Edelstein.

<u>Theorem</u> [1.4.4] Let T be a nonexpansive mapping of a metric space X into itself which has diminishing orbital diameters. If for some $x_o \in X$, $\lim_{k\to\infty} T^{n_k}(x_o) = y$, then $\lim_{n\to\infty} T^n x_o = y$ and $Ty = y$.

In [6], Belluce and Kirk also gave the following result.

<u>Theorem</u> [1.4.5] Let T be a nonexpansive mapping of a metric space X into itself. For each $x_o \in X$ assume T is not an isometry on $0(x_o)$ if $\delta 0(x_o) > 0$. If for some $x_o \in X$, the sequence of iterates $\{T^n x_o\}$ has a subsequence $\{T^{n_k} x_o\}$ with limit y, then $\{T^n x_o\}$ has limit y and y is a fixed point of T.

<u>Remark</u>. If T is a nonexpansive mapping of a compact metric space X into itself, then the following are easily proven to be equivalent.

(1) T has diminishing orbital diameters on X.

(2) T is asymptotically regular on X, i.e.

$$\lim_{n\to\infty} d(T^n x_o, T^{n+1} x_o) = 0 \text{ for each } x_o \in X.$$

(3) T is not an isometry on $0(x_o)$ if

$$\delta 0(x_o) > 0, \quad x_o \in X.$$

CHAPTER II

Iterated Contraction (Contractive, Nonexpansive) Mappings and Fixed Point Theorems in Metric Spaces

In this Chapter, we would like to consider a form of contraction mapping, that was introduced by Rheinboldt [39] in 1968, called the iterated contraction mapping. The concept of iterated contraction, as it was developed and extended by Rheinboldt, proves to be very useful in the study of certain iterative processes and has wide applicability. Using it, as well as other concepts, Rheinboldt has developed a general convergence theory for a class of iterative processes. This theory reduces the study of iterative processes to that of a second order difference equation (for information on the solution of difference equations by the method of iteration, see Scarborough [40]). The basic purpose of this general convergence theory is to give more general and more widely usuable convergence results for iterative processes used in solving nonlinear operator equations of the form $Tx = 0$.

Various mathematicians have considered numerical techniques for solving the equation $Tx = 0$. Important among these techniques are Newton's and related methods that are mentioned in the introduction. Also, see Goldstein [25]. Many individual results have been obtained by using these techniques as can be seen by examining Bartle [4], Ben-Isreal [7], [8], Byran [12], Kantorovich [28], [29], Ortega and Rheinboldt [37], and Zincenko [45], [46].

Rheinboldt's approach gives a unified theory for convergence results, which contain as special cases the Banach Contraction theorem and the Newton-Kantorovich theorem, and which appears to encompass the other individual results mentioned above.

Hence, Rheinboldt's results illustrate the usefulness of iterated contractions in finding solutions to nonlinear operator equations and their importance in numerical analysis.

We now conduct a study of these iterated contractions, which we begin by giving some preliminaries.

2.1. Preliminaries on Iterated Contraction Mappings.

Definition [2.1.1] A mapping T of a metric space X, with metric d, into itself is said to be an iterated contraction if, for all $x, Tx \in X$, there exists a constant k, $0 \leq k < 1$, such that

$$d(Tx,TTx) \leq kd(x,Tx).$$

Remark: A contraction mapping is always an iterated contraction mapping.

Proof: Let T be a mapping of a metric space X into itself such that T is a contraction.

Then by the definition of T, we have

$d(Tx,Ty) \leq kd(x,y)$, for all $x,y \in X$, $0 \leq k < 1$.

Now if $y = Tx$ for the mapping $T : X \to X$, then

$d(Tx,TTx) \leq kd(x,Tx)$, since $Tx \in X$.

Hence

$d(Tx,TTx) \leq kd(x,Tx)$, for all $x,Tx \in X$,

and for some k, $0 \leq k < 1$. Therefore, T is an iterated contraction.

However, the converse of the above remark, is not necessarily true as the following example will show.

Example (2.1.1) Let T be a mapping from $X = [-\frac{1}{2}, \frac{1}{2}]$ into itself defined by

$$Tx = x^2 \quad \text{for all } x \in [-\frac{1}{2}, \frac{1}{2}].$$

Taking the derivative of T, we get

$$D_x(Tx) = 2x.$$

Now

$$2x \not< 1 \quad \text{for all } x \in [-\frac{1}{2}, \frac{1}{2}] \quad \text{because } 2(\frac{1}{2}) = 1.$$

Hence T is not a contraction.

However, it is easy to check that T is an iterated contraction.

We also observe that, unlike a contraction mapping which is always continuous, an iterated contraction mapping need not always be continuous, as the following example illustrates.

Example (2.1.2) Let T be a mapping of $X = \mathbb{R}$, the real numbers, into itself defined by

$$Tx = 0 \quad \text{if } x \in [0, \tfrac{1}{2})$$
$$= 1 \quad \text{if } x \in [\tfrac{1}{2}, 1].$$

Then T is not continuous at $x = \frac{1}{2}$. However, T is an iterated contraction.

Remark. From Example (2.1.2), we can easily see that an iterated contraction can have more than one fixed point. The mapping T has two fixed points for $T0 = 0$ and $T1 = 1$. Hence $x = 0$ and $x = 1$ are fixed points of T.

We can also note that a continuous function is not necessarily an iterated contraction.

Example (2.1.3) Let T be a mapping of $X = \mathbb{R}$ into itself such that T is defined by

$$Tx = 2x + 3 \quad \text{for all } x \in \mathbb{R}$$

Then T is continuous on $\mathbb{R}$.

Let us assume that T is an iterated contraction.

Then $d(Tx,TTx) \leq kd(x,Tx,$ for all $x, Tx \in X$, $0 \leq k < 1$.

Also

$$d(TTx,Tx) = 2|x + 3| \quad \text{and} \quad d(Tx,x) = |x + 3|$$

Hence

$$2|x + 3| \leq k|x + 3| \quad \text{or} \quad k \geq 2.$$

This is a contraction since $k < 1$. Therefore T is not an iterated contraction.

The following example shows that a discontinuous function need not always be an iterated contraction.

Example (2.1.4) Let T be a mapping from $X = [0,1]$ into itself such that T is defined in the following way.

$$Tx = \frac{1}{2} \text{ if } x \in [0, \tfrac{1}{2})$$
$$= 0 \text{ if } x \in [\tfrac{1}{2}, 1]$$

T is clearly discontinuous. Let us assume that T is an iterated contraction.

Now, let $x = \frac{1}{4}$. Then we have

$$d(Tx,TTx) = \left|\frac{1}{2} - 0\right| = \frac{1}{2} \quad \text{and} \quad d(x,Tx) = \left|\frac{1}{4} - \frac{1}{2}\right| = \frac{1}{4}.$$

Therefore,

$$\frac{1}{2} \leq k\frac{1}{4} \quad \text{or} \quad k \geq 2.$$

This contradiction shows that T is not an iterated contraction.

We now give an example of an iterated contraction which has no fixed point.

Example (2.1.5) Let T be a mapping of $X = \mathbb{R}$ into itself defined by

$$Tx = \frac{1}{3}x + \frac{1}{3} \quad \text{for } x \leq 0$$
$$= \frac{x}{3} \quad \text{for } x > 0.$$

Thus $\mathbb{R}$ is a complete metric space, T is an iterated contraction mapping defined on $\mathbb{R}$, and T is discontinuous at $x = 0$. We also can easily see that T has no fixed point.

The following is an example of an iterated contraction mapping having fixed points but not at the point of discontinuity.

Example (2.1.6) Let T be a mapping of $X = [0,1]$ into itself defined by

$$Tx = \frac{1}{4} \quad \text{if } x \in [0, \tfrac{1}{2})$$
$$= \frac{3}{4} \quad \text{if } x \in [\tfrac{1}{2}, 1].$$

We see that T is an iterated contraction, which is not continuous at $x = \frac{1}{2}$. Also $x = \frac{1}{2}$ is not a fixed point of T, although T has fixed points at $x = \frac{1}{4}$ and $x = \frac{3}{4}$.

We would now like to give an example of an iterated contraction, which is discontinuous at a point that is a fixed point of T.

Example (2.1.7) T is a mapping of $X = [0,1]$ into itself such that

$$Tx = 0 \quad \text{if } x \in [0, \tfrac{1}{2})$$
$$= \tfrac{1}{2} \quad \text{if } x \in [\tfrac{1}{2}, 1].$$

T is an iterated contraction which is discontinuous at $x = \frac{1}{2}$. We also note that $x = \frac{1}{2}$ is a fixed point of T.

Thus, from the preceeding examples, we can easily see that an iterated contraction mapping can be discontinuous and still have fixed points. We also observe that T being an iterated contraction mapping has no fixed point, even in a complete metric space. Therefore, we will now try to give some sufficient conditions for the existence of a fixed point of an iterated contraction mapping. We begin doing this by establishing the contraction mapping principle for an iterated contraction mapping; it will be known as the iterated contraction mapping principle.

2.2. Iterated Contraction Mappings in Metric Spaces.

Theorem [2.2.1]* Let T be a mapping of a complete metric space X into itself such that T satisfies the following condition :

(i) T is an iterated contraction mapping. Then (1) the sequence of iterates $\{x_n\}$ of T converges to $y \in X$.

and (2) if T is continuous at y then y is a fixed point of T.

Proof: Let x_o be an arbitrary point in X, and let $\{x_n\}$ be a sequence in X such that

*This theorem in $\mathbb{R}^n$ has been given by Ortega and Rheinholdt ; Iterative solution of Nonlinear equations in several variables, Academic Press, New York 1970, p. 400.

$$x_1 = Tx_o$$
$$x_2 = Tx_1 = T^2x_o$$
$$\vdots$$
$$x_n = Tx_{n-1} = T^n x_o .$$

Then we have

$$d(x_1,x_2) = d(Tx_o,Tx_1) = d(Tx_o,TTx_o), \; x_o, \; Tx_o \in X.$$
$$\leq kd(x_o,Tx_o), \quad 0 \leq k < 1,$$
$$= kd(x_o,x_1)$$

$$d(x_2,x_3) = d(Tx_1,TTx_1) \; ; \; x_1, \; Tx_1 \in X,$$
$$\leq kd(x_1,Tx_1) \quad , \quad 0 \leq k < 1,$$
$$= kd(x_1,x_2)$$
$$\leq k^2 d(x_o,x_1).$$

Continuing in this way, we have

$$d(x_n,x_{n+1}) \leq k^n d(x_o,x_1) \quad \text{for } n = 1,2, \ldots \text{ and some } k, \; 0 \leq k < 1.$$

Now, we assume that there exists a positive integer N such that $n, \; n + p \geq N$, where p is a positive integer. We have to show that $\{x_n\}$ is a Cauchy sequence; that is, $\lim_{n\to\infty} d(x_n,x_{n+p}) = 0.$

By the triangle inequality and (a), we have

$$d(x_n,x_{n+p}) \leq d(x_n,x_{n+1}) + \ldots + d(x_{n+p-1},x_{n+p})$$
$$\leq k^n d(x_o,x_1) + \ldots + k^{n+p-1} d(x_o,x_1)$$
$$= k^n d(x_o,x_1)[1 + k + k^2 + \ldots + k^{p-1}]$$
$$= k^n \left(\frac{1 - k^p}{1 - k}\right) d(x_o,x_1).$$

Since $k < 1$, the right hand side tends to 0 as n tends to infinity.

Hence $\{x_n\}$ is a Cauchy sequence.

Since X is complete $\{x_n\} = \{T^n x_o\}$ converges to a point $y \in X$; that is,

$$y = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n x_o .$$

By (2), T is continuous at y.

Hence

$$Ty = T \lim_{n\to\infty} T^n x_o = \lim_{n\to\infty} T^{n+1} x_o = \lim_{n\to\infty} x_{n+1} = y.$$

Therefore y is a fixed point of T.

Remarks. (a) From the above theorem, we conclude that a continuous iterated contraction mapping of a complete metric space into itself always has a fixed point. The mapping $T : \mathbb{R} \to \mathbb{R}$, defined by $Tx = \frac{1}{3}x + \frac{1}{3}$, is a continuous iterated contraction, which has a fixed point at $x = \frac{1}{2}$.

(b) X must be a complete metric space for suppose that $X = (0,1]$, which is not complete. Then $T : X \to X$ defined by $Tx = \frac{x}{2}$, is a contraction mapping and, therefore, an iterated contraction mapping. T is also continuous, but it has no fixed point.

(c) T has to be an iterated contraction and not just a continuous function. Let $T : X \to X$, where $X = \mathbb{R}$, which is complete, be defined by $Tx = x + 1$. T is continuous, but T is not an iterated contraction because $|(x + 1) - (x + 2)| \leq k|x - (x + 1)|$ or $k \geq 1$, which would be a contradiction to the fact that $k < 1$. We also note that T has no fixed point.

(d) Continuity of an iterated contraction mapping is sufficient but not necessary. Let $T : [0,1] \to [0,1]$ be defined by

$$Tx = 0 \quad \text{if} \quad x \in [0, \tfrac{1}{2})$$
$$= \tfrac{2}{3} \quad \text{if} \quad x \in [\tfrac{1}{2}, 1].$$

We note that $X = [0,1]$ is complete, T is not continuous, and T is an iterated contraction mapping with fixed points $x = 0$, $x = \frac{2}{3}$.

Next we will try to give some sufficient conditions for the existence of a fixed point of an iterated contraction mapping which is not necessarily continuous at some point. Then we will go even further by trying to give some sufficient conditions for the existence of a fixed point of a mapping that is not necessarily an iterated contraction.

<u>Theorem</u> [2.2.2] Let X be a complete metric space, and let T be an iterated contraction mapping of X into itself such that T^r, for some positive integer r, satisfies the following conditions:

(i) T^r is an iterated contraction mapping of X into itself; that is, there exists a positive integer r, and a constant k, $0 \leq k < 1$, such that

$$d(T^r x, T^r T^r x) \leq k d(x, T^r x), \quad \text{for all } x, T^r x \in X.$$

(ii) T^r is continuous at a point y, where $y = \lim_{n\to\infty} T^{rn} x_0$, $n = 1, 2, \ldots,$ and x_0 is an arbitrary point in X.

Then T has a fixed point at y.

<u>Proof</u>: Since X is a complete metric space and T^r is an iterated contraction that is continuous at y, we have that T^r has a fixed point at y, by the previous theorem.

Hence, $T^r y = y$, where $y = \lim_{n\to\infty} T^{rn} x_0$, $n = 1, 2, \ldots, x_0$ is any point in

X, and $\{x_n\} = \{T^r x_{n-1}\} = \{T^{rn} x_0\}$ is the sequence of iterates of T^r.

We now have to show that y is also a fixed point of T.

$$d(y,Ty) = d(T^r y, TT^r y), \quad \text{since } T^r y = y.$$
$$= d(TT^{r-1}y, TTT^{r-1}y), \quad \text{for } T^{r-1}y,\ TT^{r-1}y \in X.$$
$$\leq k\,d(T^{r-1}y, TT^{r-1}y), \quad \text{for some } k,\ 0 \leq k < 1, \text{ since } T \text{ is an iterated contraction.}$$
$$= k\,d(TT^{r-2}y, TTT^{r-2}y), \quad T^{r-2}y,\ TT^{r-2}y \in X.$$
$$\leq k^2 d(T^{r-2}y, TT^{r-2}y), \quad 0 \leq k < 1.$$
$$\vdots$$
$$\leq k^r d(y,Ty), \quad 0 \leq k < 1.$$

So if $d(y,Ty) \neq 0$, then $k^r \geq 1$. This is a contradiction since $k < 1$ implies $k^r < 1$, where r is a positive integer.

Thus

$$d(y,Ty) = 0 \ ; \text{ that is, } Ty = y.$$

Therefore T has a fixed point at y.

We now give an example to show that Theorem [2.2.2] is more general than Theorem [2.2.1].

Example (2.2.1) Let T be a mapping of $X = [0,1]$ into itself defined by

$$Tx = \tfrac{1}{4} \quad \text{if } x \in [0, \tfrac{1}{4}]$$
$$= 0 \quad \text{if } x \in (\tfrac{1}{4}, \tfrac{1}{2})$$
$$= \tfrac{1}{2} \quad \text{if } x \in [\tfrac{1}{2}, 1].$$

We can easily check that T is an iterated contraction.

Now,

$$T^2x = T^3x = \dots = \frac{1}{4} \quad \text{if} \quad x \in [0, \tfrac{1}{2})$$
$$= \frac{1}{2} \quad \text{if} \quad x \in [\tfrac{1}{2}, 1].$$

Hence T^r, for some $r = 2, 3, \dots$, is an iterated contraction which is continuous at $x = \frac{1}{4}$ and $T^r\frac{1}{4} = \frac{1}{4}$.

Moreover, $T\frac{1}{4} = \frac{1}{4}$.

Therefore, $x = \frac{1}{4}$ is a fixed point of T, but T is not continuous at $x = \frac{1}{4}$.

Theorem [2.2.3] Let T be a mapping of a complete metric space X into itself such that T^r, for some $r = 2, 3, \dots$, is an iterated contraction mapping of X into itself satisfying the following conditions:

(i) there exists some constant k, $0 \leq k < 1$, such that

$d(T^r x, T^{r+1}x) \leq k^r d(x, Tx)$, for all $x, Tx \in X$.

(ii) T^r is continuous at y, where $y = \lim_{n\to\infty} T^{rn}x_o$, $n = 1, 2, \dots$, x_o is an arbitrary point in X.

Then T has a fixed point.

Proof: Let $x_n = T^r x_{n-1} = T^{rn}x_o$, for any $x_o \in X$, $n = 1, 2, \dots$. Then by Theorem [2.2.1], T^r has a fixed point at y, since it is an iterated contraction which is continuous at y, i.e. $T^r y = y$.

Also, we have

$$d(y, Ty) = d(T^r y, T^{r+1}y) \leq k^r d(y, Ty), \quad \text{by (i)}.$$

Now, if $d(y, Ty) \neq 0$, then $k^r \geq 1$. This is a contradiction since $k < 1$ implies $k^r < 1$.

Hence $d(y,Ty) = 0$ and T has a fixed point at y.

Remark (1). If T is an iterated contraction, then condition (i) of the above theorem is always satisfied as can be seen from the proof of Theorem [2.2.2]. However, if condition (i) is satisfied, then T is not necessarily an iterated contraction as Example (2.2.2) illustrates.

Remark (2). The theorem also holds, and the proof is exactly the same if condition (i) is replaced by

(i)* there exists some constant k, $0 \leq k < 1$, such that

$d(T^r x, T^{r+1} x) \leq kd(x,Tx)$, for all x, $Tx \in X$.

Of course, T can satisfy (i)* without being an iterated contraction as Example (2.2.2) also illustrates.

Example (2.2.2) Let $T : [0,1] \to [0,1]$, where $X = [0,1]$ is complete, be a mapping defined by

$$\begin{aligned} Tx &= \tfrac{1}{4} \quad \text{if } x \in [0, \tfrac{1}{4}] \\ &= 0 \quad \text{if } x \in (\tfrac{1}{4}, \tfrac{1}{2}) \\ &= \tfrac{1}{2} \quad \text{if } x \in [\tfrac{1}{2}, \tfrac{3}{4}] \\ &= \tfrac{3}{4} \quad \text{if } x \in (\tfrac{3}{4}, 1]. \end{aligned}$$

Since $\left|\frac{7}{8} - \frac{6}{8}\right| k \geq \left|\frac{6}{8} - \frac{4}{8}\right|$ implies $k \geq 2$, T is not an iterated contraction.

Now

$$\begin{aligned} T^2x = T^3x = \ldots &= \tfrac{1}{4} \quad \text{if } x \in [0, \tfrac{1}{2}) \\ &= \tfrac{1}{2} \quad \text{if } x \in [\tfrac{1}{2}, 1]. \end{aligned}$$

Therefore, $T^2, T^3, \ldots$ are all iterated contractions. Also $T^2, T^3, \ldots$ satisfy $d(T^r x, T^{r+1}x) \le kd(x,Tx)$ for any $k \ge 0$.

Therefore $T^2, T^3, \ldots$ satisfy (i) of the theorem for $k^2, k^3, \ldots$, respectively. $T^2, T^3, \ldots$ are also continuous at $x = \frac{1}{4}$ but T is not. We note that T has a fixed point at $x = \frac{1}{4}$.

Theorem [2.2.4] Let T be a mapping of a complete metric space X into itself. Also let K be a mapping of X into X such that K has a left inverse; that is, there exists a K^{-1} such that $K^{-1}K = I$, where I is the identity mapping of X into X. KTK^{-1} is also a mapping of X into itself such that the following conditions are satisfied:

(i) $d((KTK^{-1})x, (KTK^{-1})(KTK^{-1})x) \le kd(x, (KTK^{-1})x)$, for some constant k, $0 \le k < 1$ and all $x, (KTK^{-1})x \in X$; that is, KTK^{-1} is an iterated contraction mapping.

(ii) KTK^{-1} is continuous at y, where $y = \lim_{n\to\infty} (KTK^{-1})^n x_o$, $n = 1, 2, \ldots, x_o$ is any point in X.

Then T has a fixed point.

Proof: Let $x_n = (KTK^{-1})x_{n-1} = (KTK^{-1})^n x_o$, where x_o is an arbitrary point of X and $n = 1, 2, \ldots$.

Since KTK^{-1} is an iterated contraction mapping which is continuous at y, KTK^{-1} has a fixed point at y; that is,

$$(KTK^{-1})y = y, \text{ where } y = \lim_{n\to\infty} (KTK^{-1})^n x_o$$

or

$$KT(K^{-1}y) = y \qquad \text{(a)}$$

Letting K^{-1} act on both sides of (a), we have

$$K^{-1}KT(K^{-1}y) = K^{-1}y$$

or $$T(K^{-1}y) = K^{-1}y \text{ , since } K^{-1}K = I.$$

Hence $K^{-1}y$ is a fixed point of T.

Remark. A similar result can be obtained if K has a right inverse and $K^{-1}TK$ satisfies conditions (i) and (ii) of the above theorem.

Theorem [2.2.5] Let T be a mapping of a metric space X into itself such that the following conditions are satisfied:

(i) X with metric d is complete while X with metric δ is not complete, where d and δ are two distinct metrics defined on X.

(ii) $d(x,Tx) \leq \delta(x,Tx)$, for all $x, Tx \in X$.

(iii) $\delta(Tx,TTx) \leq k\delta(x,Tx)$, for some constant k, $0 \leq k < 1$, and all $x, Tx \in X$; that is, T is an iterated contraction with respect to δ.

Then, for some $x_o \in X$, $x_n = T^n x_o$ converges to $y \in X$.

Let T be continuous at y.

Then y is a fixed point of T.

Proof: Let $x_n = Tx_{n-1} = T^n x_o$, $n = 1, 2, \ldots$, x_o is some point in X.

By using (iii), we can show that $\{x_n\}$ is a Cauchy sequence in X with respect to δ; that is, there exists a positive integer N such that

$$\delta(x_n,x_{n+p}) < \varepsilon \text{ , for } \varepsilon > 0,\ n,\ n+p \geq N \quad (p \text{ is a positive integer})$$

$$\delta(x_i,x_{i+1}) \leq k^i\, \delta(x_o,x_1), \quad i = 1, 2, \ldots, n.$$

Then, by (a) and the triangle inequality, we have

$$d(x_n,x_{n+p}) \le d(x_n,x_{n+1}) + \dots + d(x_{n+p-1},x_{n+p})$$

$$= d(x_n,Tx_n) + \dots + d(x_{n+p-1},Tx_{n+p-1})$$

$$\le \delta(x_n,Tx_n) + \dots + \delta(x_{n+p-1},Tx_{n+p-1}), \quad \text{by (ii)},$$

$$= \delta(x_n,x_{n+1}) + \dots + \delta(x_{n+p-1},x_{n+p})$$

$$\le \frac{k^n(1-k^p)}{1-k}\,\delta(x_o,x_1) \quad \to 0 \text{ as } n \to \infty.$$

Therefore, $d(x_n,x_{n+p}) < \varepsilon$, for $n, n+p \ge N$, $p > 0$, $\varepsilon > 0$.

Hence $\{x_n\}$ is a Cauchy sequence in X with respect to metric d.

Therefore, the sequence $\{x_n\}$ converges to a point $y \in X$, since X is complete with metric d.

i.e. $y = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n x_o$

Since T is continuous at y, we have

$$Ty = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = y.$$

Hence y is a fixed point of T.

Theorem [2.2.6] Let T and K be mappings of a metric space X into itself such that K has a right inverse $K^{-1}(KK^{-1} = I)$. $K^{-1}TK$ is also a mapping of X into X such that the following conditions are satisfied:

(i) X is complete with d while X with δ is not complete, where d and δ are two distinct metrics defined on X.

(ii) $d(x,(K^{-1}TK)x) \le \delta(x,(K^{-1}TK)x)$, for all $x, (K^{-1}TK)x \in X$.

(iii) $\delta((K^{-1}TK)x, (K^{-1}TK)(K^{-1}TK)x) \le k\delta(x,(K^{-1}TK)x)$, for some constant k, $0 \le k < 1$, and all $x, (K^{-1}TK)x \in X$; that is $K^{-1}TK$ is an iterated contraction mapping with respect to δ.

Then for some $x_0 \in X$, $x_n = (K^{-1}TK)^n x_0$ converges to $y \in X$.

Let T be continuous at y.

Then T has a fixed point.

Proof: Let $x_n = (K^{-1}TK)x_{n-1} = (K^{-1}TK)^n x_0$, $n = 1, 2, \ldots,$ and x_0 is some point in X.

Then by Theorem [2.2.5], $K^{-1}TK$ has a fixed point at y, where $y = \lim_{n\to\infty} (K^{-1}TK)^n x_0$.

i.e. $(K^{-1}TK)y = y$.

Letting K act on both sides of the equation, we have

$$K(K^{-1}TK)y = Ky$$

or $$KK^{-1}T(Ky) = Ky$$

$$T(Ky) = Ky, \quad \text{since } KK^{-1} = I.$$

Hence Ky is a fixed point of T.

We would now like to give the following result, which is an extension to iterated contraction mappings of a result due to Rakotch [38] for contraction mappings.

Theorem [2.2.7] Let T be a mapping of a complete metric space X into itself such that the following conditions are satisfied:

(i) $d(Tx,TTx) \leq k(x,Tx)d(x,Tx)$, for all $x, Tx \in X$

and some $k \in F_1$, where F_1 is a family of functions satisfying the following:

(a) $k(x,Tx) = k(d(x,Tx))$; that is, k depends on the distance between x and Tx only.

(b) $0 \leq k(d) < 1$ for every $d > 0$.

(c) $k(d)$ is a monotonically decreasing function of d ($k(d) < d < 1$.

Then, for some $x_o \in X$, $x_n = T^n x_o$ converges to $y \in X$.

Let T be continuous at y.

Then T has a fixed point.

Proof: Let $x_n = Tx_{n-1} = T^n x_o$, $n = 1, 2, \ldots$, x_o is some point in X.

Now

$$d(x_1,x_2) = d(Tx_o,TTx_o) \leq k(x_o,Tx_o)d(x_o,Tx_o) \text{ by (i).}$$
$$= k(x_o,x_1)d(x_o,x_1).$$

Hence

$$d(x_1,x_2) < d(x_o,x_1) \text{ since } k(x_o,x_1) < 1.$$

Continuing in this way, we have

$$(1) \quad d(x_o,x_1) > d(x_1,x_2) > d(x_2,x_3) > \ldots > d(x_{n-1},x_n) > \ldots$$

Since k is monotonically decreasing on d, we have

$$(2) \quad k(x_o,x_1) < k(x_1,x_2) < k(x_2,x_3) < \ldots < k(x_{n-1},x_n) < \ldots$$

Now

$$d(x_n,x_{n+1}) = d(Tx_{n-1},TTx_{n-1})$$
$$\leq k(x_{n-1},Tx_{n-1})d(x_{n-1},Tx_{n-1}) \quad \text{by (i)}$$
$$\leq k(x_{n-1},Tx_{n-1})k(x_{n-2},Tx_{n-2})d(x_{n-2},Tx_{n-2})$$
$$\vdots$$
$$\leq k(x_{n-1},Tx_{n-1})k(x_{n-2},Tx_{n-2}) \ldots k(x_o,Tx_o)d(x_o,Tx_o)$$
$$= k(x_{n-1},x_n)k(x_{n-2},x_{n-1}) \ldots k(x_o,x_1)d(x_o,x_1)$$
$$< [k(x_o,x_1)]^n d(x_o,x_1), \quad \text{by (2) above} \quad \ldots\ldots \quad (3)$$

We have to show that $\{x_n\}$ ~~is a Cauchy sequence in~~ X i.e.

$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ (p is a positive integer).

By the triangle inequality, we have

$$d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + \ldots + d(x_{n+p-1}, x_{n+p})$$

Hence by (3), we have

$$d(x_n, x_{n+p}) < [k(x_{n-1}, x_n)]^n d(x_o, x_1) + [k(x_n, x_{n+1})]^{n+1} d(x_o, x_1) + \ldots + [k(x_{n+p-2}, x_{n+p-1})]^{n+p-1} d(x_o, x_1).$$

Then by (2), we have

$$d(x_n, x_{n+p}) < [k(x_o, x_1)]^n d(x_o, x_1) + [k(x_o, x_1)]^{n-1} d(x_o, x_1) + \ldots + [k(x_o, x_1)]^{n+p-1} d(x_o, x_1).$$

$$d(x_n, x_{n+p}) < [k(x_o, x_1)]^n d(x_o, x_1) \ldots\ldots [k(x_o, x_1)]$$

$$< [k(x_o, x_1)]^n \frac{1 - [k(x_o, x_1)]}{1 - k(x_o, x_1)} d(x_o, x_1)$$

As n tends to infinity, the right hand side tends to 0 since

$k(x_o, x_1) < 1$ by (b) of the theorem.

Hence $\{x_n\}$ is a Cauchy sequence.

Since X is complete $\{x_n\}$ has a limit $y \in X$; that is.

$y = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n x_o$.

T is continuous at $y = \lim_{n\to\infty} T^n x_o$ implies

$$Ty = T \lim_{n\to\infty} T^n x_o = \lim_{n\to\infty} T^{n+1} x_o = \lim_{n\to\infty} x_{n+1} = y.$$

Therefore y is a fixed point of T.

Remark. A similar result can be obtained if k is taken to be a monotonically increasing function instead of a monotonically decreasing function as in Theorem [2.2.7]. In this case, (2) in the proof of Theorem [2.2.7] will be replaced by

$$(2)^{*} \quad k(x_o,x_1) > k(x_1,x_2) > \dots > k(x_{n-1},x_n) > \dots$$

Then we would have

$$d(x_n,x_{n+p}) < [k(x_o,x_1)]^n \left(\frac{1-[k(x_o,x_1)]^p}{1-k(x_o,x_1)}\right) d(x_o,x_1),$$

where $0 \leq k(x_o,x_1) < 1$.

As n tends to infinity, the right side again tends to 0. Hence $\{x_n\}$ is again a Cauchy sequence, and the result follows as in Theorem [2.2.7].

We would now like to give the following result for a metric space which is not necessarily complete.

Theorem [2.2.8] Let T be a mapping of a metric space X into itself such that the following conditions hold:

(i) T is an iterated contraction.

(ii) T is continuous on X,

(iii) For any point $x_o \in X$, the sequence of iterates $\{T^n x_o\}$ has a subsequence $\{T^{n_i} x_o\}$ converging to $y \in X$.

Then T has a fixed point at y.

Proof: Let $x_n = Tx_{n-1} = T^n x_o$, for some $x_o \in X$, $n = 1, 2, \ldots$. Then by using (i), we can show that $\{x_n\}$ is a Cauchy sequence as in Theorem [2.2.1].

Since $\{x_n\} = \{T^n x_o\}$ is a Cauchy sequence and by (iii) has a subsequence $\{T^{n_i} x_o\}$ converging to $y \in X$, $\{x_n\}$ converges to $y \in X$.

i.e. $\lim_{n\to\infty} x_n = y$.

The continuity of T at y implies

$$Ty = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = y.$$

Therefore y is a fixed point of T.

In [26] Kannan has given the following result.

Theorem [2.2.9] Let T be a mapping of a complete metric space X into itself such that the following condition is satisfied:

(i) $d(Tx,Ty) \leq k[d(x, Tx) + d(y,Ty)]$ for all $x,y \in X$ and some constant k, $0 \leq k < 1/2$.

Then T has a unique fixed point.

Kannan [27] has also given the following result.

Theorem [2.2.10] Let T be a mapping of a metric space X into itself such that the following are satisfied:

(i) $d(Tx,Ty) \leq k[d(x,Tx) + d(y,Ty)]$, for all $x,y \in X$ and some constant k, $0 \leq k < \frac{1}{2}$.

(ii) T is continuous on X.

(iii) There exists an arbitrary point $x_o \in X$ such that the sequence of iterates $\{T^n x_o\}$ has a subsequence $\{T^{n_i} x_o\}$ converging to y. Then T has a unique fixed point.

Remark. If we consider Kannan's results in the light of an iterated contraction mapping, then we get no new result. This is because condition (i) of Theorems [2.2.9] and [2.2.10] would read as follows

$$\text{(i)* } d(Tx,TTx) \leq k[d(x,Tx) + d(Tx,TTx)], \text{ for all } x, Tx \in X,$$

and some constant k, $0 \leq k < \frac{1}{2}$.

From (i)*, we immediately get

$$\text{(ii)** } d(Tx,TTx) \leq \frac{k}{1-k} d(x,Tx), \text{ where } 0 \leq \frac{k}{1-k} < 1.$$

Hence by considering (i)**, we can easily see that if T satisfies (i)*, then T is an iterated contraction. Therefore, if we consider (i)* along with the condition that T is continuous at $y = \lim_{n\to\infty} T^n x_o$, $x_o \in X$, then we immediately get Theorem [2.2.1].

On the other hand, if we consider (i)* along with (ii) and (iii) of Theorem [2.2.10], we immediately get Theorem [2.2.8]. Hence we get no new result in either case.

We will now try to obtain a result when we have, not just one mapping, but two distinct mappings from a complete metric space into itself.

Theorem [2.2.11]. Let X be a complete metric space, and let $T_1, T_2 (T_1 \neq T_2)$ be two mappings of X into itself such that the following conditions are satisfied:

(i) (a) $d(T_1x, T_2T_1x) \leq k \cdot d(x,T_1x)$

(b) $d(T_2x, T_1T_2x) \leq kd(x,T_2x)$,

for some constant k, $0 \leq k < 1$ and all x, T_1x, $T_2x \in X$.

(ii). T_1 and T_2 are both continuous on X.

Then T_1 and T_2 have a common fixed point.

Proof: Let x_o be an arbitrary point in X and let the sequence $\{x_n\}$ be defined in the following way.

$x_1 = T_1(x_o)$, $x_2 = T_2(x_1)$, $x_3 = T_1(x_2)$, $x_4 = T_2(x_3)$, and so on.

i.e. $x_{2n} = T_2(x_{2n-1})$ and $x_{2n+1} = T_1(x_{2n})$, for $n = 1, 2, \ldots$

Now, we have

$$\begin{aligned} d(x_1,x_2) &= d(T_1x_o,T_2x_1) = d(T_1x_o,T_2T_1x_o) \\ &\leq kd(x_o,T_1x_o) \quad \text{by (i) (a)} \\ &= kd(x_o,x_1). \end{aligned}$$

$$\begin{aligned} d(x_2,x_3) &= d(T_2x_1,T_1T_2x_1) \leq kd(x_1,T_2x_1) \quad \text{by (i) (b)} \\ &= kd(x_1,x_2) \\ &\leq k^2d(x_o,x_1). \end{aligned}$$

$$\begin{aligned} d(x_3,x_4) &= d(T_1x_2,T_2T_1x_2) \leq kd(x_2,T_1x_2) = kd(x_2,x_3) \\ &\leq k^3d(x_o,x_1). \end{aligned}$$

$$\begin{aligned} d(x_4,x_5) &= d(T_2x_3,T_1T_2x_3) \leq kd(x_3,T_2x_3) = kd(x_3,x_4) \\ &\leq k^4d(x_o,x_1). \end{aligned}$$

Continuing in this way, we get

$$d(x_n,x_{n+1}) \leq k^n d(x_o,x_1) \quad \ldots\ldots \quad (A),$$

And by the triangle inequality, we have for some integer $p > 0$

$$d(x_n,x_{n+p}) \leq \sum_{i=n}^{n+p-1} d(x_i,x_{i+1})$$

$$\leq \sum_{i=n}^{n+p-1} k^i d(x_0,x_1) \quad \text{by} \quad (A)$$

$$\leq k^n \frac{1-k^p}{1-k} d(x_0,x_1) .$$

Since $k < 1$, the right hand side tends to 0 as n tends to ∞.

Hence $\{x_n\}$ is a Cauchy sequence, and since X is complete, $\{x_n\}$ has a limit in X, i.e. $\lim_{n\to\infty} x_n = y \in X$.

Thus we have

$$\lim_{n\to\infty} x_{2n-1} = y \quad \text{and} \quad \lim_{n\to\infty} x_{2n} = y .$$

Since T_1 and T_2 are continuous on X, we have

$$T_1 y = T_1 \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} T_1 x_{2n} = \lim_{n\to\infty} x_{2n+1} = y$$

$$T_2 y = T_2 \lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} T_2 x_{2n-1} = \lim_{n\to\infty} x_{2n} = y .$$

Hence $T_1 y = y = T_2 y$. Therefore y is a common fixed point of T_1 and T_2.

<u>Remark</u>. If $T = T_1 = T_2$, then (i)(a) and (i)(b) of the theorem are identical, and T is an iterated contraction. The theorem becomes the iterated contraction mapping principle.

In [26] Kannan gave the following result.

<u>Theorem</u> [2.2.12] If T_1 and T_2 are two distinct mappings of a complete metric space X into itself such that, for some constant k, $0 \leq k < \frac{1}{2}$, and all $x, y \in X$, we have

$$d(T_1 x, T_2 y) \leq k[d(x,T_1 x) + d(y,T_2 y)],$$

then T_1 and T_2 have a common unique fixed point.

Remark. If we try to extend Kannan's theorem to one that is similar to Theorem [2.2.11], then we have the following result: "T_1 and T_2 are two distinct mappings of a complete metric space X into itself such that, for some constant k, $0 \leq k < \frac{1}{2}$, and all x, $T_1x, T_2x \in X$, we have

(i)* $d(T_1x, T_2T_1x) \leq k[d(x,T_1x) + d(T_1x,T_2T_1x)]$

(ii)* $d(T_2x, T_1T_2x) \leq k[d(x,T_2x) + d(T_2x,T_1T_2x)]$

(iii)* T_1 and T_2 are continuous on X.

Then T_1 and T_2 have a common fixed point".

However, we note that (i)* and (ii)* are easily seen to be equivalent to (i)(a) and (i)(b) of Theorem [2.2.11]. Hence we get no new result.

2.3 Iterated Contractive and Iterated Nonexpansive Mappings.

We will now give a few results on iterated contractive and iterated nonexpansive mappings.

Definition [2.3.1] A mapping T of a metric space X into itself is said to be iterated contractive if

$$d(Tx,TTx) < d(x,Tx), \quad \text{for all } x, Tx \in X, \quad x \neq Tx.$$

The following is an attempt to give some sufficient conditions for the existence of a fixed point of an iterated contractive mapping.

Theorem [2.3.1] Let T be an iterated contractive mapping of a metric space X into itself such that the following conditions are satisfied:

(i) ~~T is continuous on X.~~

(ii) for an arbitrary element $x_0 \in X$, the sequence of iterates of T, $\{T^n x_0\}$, has a subsequence $\{T^{n_i} x_0\}$ converging to y in X.

Then y is a fixed point of T.

<u>Proof</u>: Let $x_n = Tx_{n-1} = T^n x_o$, $n = 1, 2, \ldots$, and x_o is an arbitrary point of X.

Then we have

$$d(T^n x_o, T^{n+1} x_o) = d(TT^{n-1} x_o, TTT^{n-1} x_o)$$
$$< d(T^{n-1} x_o, TT^{n-1} x_o)$$
$$< d(T^{n-2} x_o, TT^{n-2} x_o),$$

Continuing this process, we get

$$d(T^n x_o, T^{n+1} x_o) < d(x_o, Tx_o).$$

Thus $\{d(T^n x_o, T^{n+1} x_o)\}$ is a non-increasing sequence of real numbers, bounded below by 0, and therefore has a limit. Since $\{T^{n_i} x_o\}$ converges to $y \in X$, and since T is continuous on X, the sequence $\{T^{n_i+1} x_o\}$ converges to Ty and the sequence $\{T^{n_i+2} x_o\}$ converges to $T^2(y) = TTy$.

Hence, we get

$$d(y, Ty) = \lim_{i\to\infty} d(T^{n_i} x_o, T^{n_i+1} x_o)$$
$$= \lim_{i\to\infty} d(T^{n_i+1} x_o, T^{n_i+2} x_o)$$
$$= d(Ty, TTy) \quad \ldots \ldots \quad (A)$$

If $y \neq Ty$, then $d(Ty, TTy) < d(y, Ty)$, since T is an iterated contractive mapping. From (A), we get

$d(y, Ty) = d(Ty, TTy) < d(y, Ty)$. This is impossible. From this contradiction, we conclude that $Ty = y$; thus y is a fixed point of T.

Corollary [2.3.2] If T is a continuous iterated contractive mapping of a compact metric space X into itself, then T has a fixed point.

Definition [2.3.2] A mapping T of a metric space X into itself is said to be iterated nonexpansive if

$$d(Tx,TTx) \leq d(x,Tx), \quad \text{for all } x, Tx \in X.$$

We now try to give some sufficient conditions for the existance of a fixed point of such a mapping.

Theorem [2.3.3] Let T be an iterated nonexpansive mapping of a metric space X into itself such that the following conditions are satisfied:

(i) T is continuous on X.

(ii) if $x \neq Tx$, then $d(Tx,TTx) < d(x,Tx)$.

(iii) for some point $x_o \in X$, the sequence of iterates $\{T^n x_o\}$ of T has a subsequence $\{T^{n_i} x_o\}$ converging to $y \in X$.

Then T has a fixed point at y.

Proof: Let, $x_n = Tx_{n-1} = T^n x_o$, $n = 1, 2, \ldots$, x_o is an arbitrary point in X.

Then

$$d(T^n x_o, T^{n+1} x_o) = d(TT^{n-1} x_o, TTT^{n-1} x_o)$$

$$\leq d(T^{n-1} x_o, TT^{n-1} x_o)$$

$$\leq d(x_o, Tx_o).$$

Hence $\{d(T^n x_o, T^{n+1} x_o)\}$ is again a non-increasing sequence of real numbers that is bounded below by 0, and thus has a limit.

Since $\{T^{n_i}x_o\}$ converges to y and since T is continuous on X, we get

$$d(y,Ty) = \lim_{i\to\infty} d(T^{n_i}x_o, T^{n_i+1}x_o)$$
$$= \lim_{i\to\infty} d(T^{n_i+1}x_o, T^{n_i+2}x_o)$$
$$= d(Ty, TTy).$$

This contradicts (ii) of the statement of the theorem, unless $y = Ty$.

Hence y is a fixed point of T.

We get the following corollary as a result of the above theorem.

Corollary [2.3.4] Let T be a continuous iterated nonexpansive mapping from a compact metric space X into itself such that if $x \neq Tx$, then $d(Tx,TTx) < d(x,Tx)$. Then T has a fixed point.

This corollary is true because in a compact metric space condition (iii) of Theorem [2.2.15] is always satisfied.

Next, we will discuss iterated contraction mappings of a generalized metric space into itself.

CHAPTER III

Iterated Contraction Mappings and Fixed Point Theorems in Generalized Metric Spaces

The aim of this chapter is to give known results for "contraction" mappings and to prove some for "iterated contraction" mappings in generalized metric spaces. Luxemburg [31], [32] considered contraction mappings in generalized metric spaces and applied the results to ordinary differential equations. Monna[36] has generalized Luxemburg's result, which was for one mapping, to a result for a suitable family of mappings and has given applications in differential equations. Monna's result was later completed by Edelstein [22]. Further generalizations were given by Margolis [33]. Diaz and Margolis [18] have also given a general theorem by considering one mapping of a generalized metric space into itself. Covitz and Nadler, Jr. [17] have since given similar results for multi-valued contraction mappings in generalized metric spaces.

3.1 Preliminaries on Generalized Metric Spaces.

Definition [3.1.1] Let X be any nonempty abstract set of which the elements are denoted by $x, y, \ldots,$ and assume that on the Cartesian product $X \times X$ a distance function $d(x,y)$ is defined such that, for all $x, y, z \in X$, the following conditions are satisfied:

(i) $0 \leq d(x,y) \leq \infty$.

(ii) $d(x,y) = 0$ if and only if $x = y$.

(iii) $d(x,y) = d(y,x)$ (This is symmetry).

(iv) $d(x,y) \leq d(x,z) + d(z,y)$ (This is the triangle inequality).

The set X with the distance function d is called a generalized metric space, and d is called the generalized metric. We usually write X for the generalized metric space with d understood.

Definition [3.1.2] A generalized metric space X with metric d, is called complete with respect to d if every d-Cauchy sequence in X is d-convergent in X; that is, if $\lim\limits_{n,m\to\infty} d(x_n,x_m) = 0$ for a sequence $\{x_n\}$ in X ($n = 1, 2, \ldots$), then there exists an element $y \in X$ such that $\lim\limits_{n\to\infty} d(x_n,y) = 0$ or $\lim\limits_{n\to\infty} x_n = y$ with respect to d.

Remark. (1) The difference between a metric space and a generalized metric space is that in a metric space the distance function can take only finite values. As an example, $[0,\infty)$ is a metric space and $[0,\infty]$ is a generalized metric space with the usual metric.

Remark. (2) The limit of a sequence in a generalized metric space is unique.

Proof: Let $\lim\limits_{n\to\infty} d(x_n,y_1) = 0$ and $\lim\limits_{n\to\infty} d(x_n,y_2) = 0$.

Then by (iv) of Definition [3.1.1], we have

$$\begin{aligned} d(y_1,y_2) &\le d(y_1,\lim_{n\to\infty} x_n) + d(\lim_{n\to\infty} x_n,y_2) \\ &\le \lim_{n\to\infty} d(y_1,x_n) + \lim_{n\to\infty} d(x_n,y_2) \\ &\le \quad 0 \quad + \quad 0 \end{aligned}$$

$d(y_1,y_2) \le 0$ implies $d(y_1,y_2) = 0$ by (i) and

$y_1 = y_2$ by (ii) of Definition [3.1.1].

Remark. (3) A generalized metric space can be "remetrized" with a genuine metric by taking the minimum of the generalized metric and the real number one. The topology is preserved by this "remetrization", but the Lipschitz structure is changed. Therefore, since we are dealing with contractions, the generalized metric space structure cannot be replaced by a metric space structure.

3.2 Iterated Contractions in Generalized Metric Spaces.

We first give some results of Luxemburg for complete generalized metric spaces.

The following theorem was given by Luxemburg [31].

Theorem [3.2.1] Let T be a mapping of a generalized complete metric space X into itself such that the following conditions hold:

(i) There exists a constant k, $0 \leq k < 1$, such that $d(Tx,Ty) \leq kd(x,y)$, for all $x,y \in X$ with $d(x,y) < \infty$.

(ii) For every sequence of successive approximations $x_n = Tx_{n-1}$, $n = 1,2,\ldots$, where x_0 is an arbitrary element of X, there exists an index $N(x_0)$ such that

$$d(x_N, x_{N+\ell}) < \infty, \quad \text{for all } \ell = 1, 2, \ldots,$$

(iii) If x and y are two fixed points of T, i.e. $Tx = x$, $Ty = y$, then $d(x,y) < \infty$.

Then T has a unique fixed point which is the limit of the sequence of successive approximations.

In [32], Luxemburg has given the following localization of the above theorem.

Theorem [3.2.2] Let X be a generalized complete metric space, and let T be a mapping of X into itself such that the following conditions are satisfied:

(i) There exists a constant $C > 0$ such that for all $x, y \in X$ with $d(x,y) \leq C$, we have

$$d(Tx,Ty) \leq kd(x,y), \quad 0 \leq k < 1.$$

(ii) For every sequence $x_n = Tx_{n-1}$, $n = 1, 2, \ldots$, where x_0 is an arbitrary element of X, there exists an index $N(x_0)$ such that

$$d(x_n,x_{n+\ell}) \leq C, \quad \text{for all } n \geq N \text{ and } \ell = 1, 2, \ldots,$$

(iii) If x and y are two fixed points of T, then

$$d(x,y) \leq C.$$

Then the successive approximations defined by $x_n = Tx_{n-1}$ converge in distance to a unique fixed point of T.

We now modify these results for an iterated contraction mapping.

Theorem [3.2.3] Let X be a generalized complete metric space, and let T be a mapping of X into itself such that the following are satisfied:

(i) There exists a constant k, $0 \leq k < 1$, such that

$$d(Tx,TTx) \leq kd(x,Tx), \quad \text{for all } x, Tx \in X \text{ with } d(x,Tx) < \infty.$$

(ii) For every sequence of successive approximations $x_n = Tx_{n-1}$, $n = 1, 2, \ldots$, where x_0 is an arbitrary element of X, there exists an index $N(x_0)$ such that

$$d(x_N,x_{n+\ell}) < \infty, \text{ for all } \ell = 1, 2, \ldots.$$

(iii) T is continuous.

Then the successive approximations of T converge in distance to a fixed point of T.

Proof: Let x_o be an arbitrary point in X and let

$$x_n = Tx_{n-1} = T^n x_o, \quad n = 1, 2, \ldots .$$

By (ii) of the theorem, there exists an index $N(x_o)$ such that

$$d(x_N, x_{N+\ell}) < \infty \quad , \quad \ell = 1, 2, \ldots .$$

Then (i) implies

$$d(x_{N+1}, x_{N+2}) = d(Tx_N, TTx_N)$$
$$\leq kd(x_N, Tx_N) < \infty$$
$$d(x_{N+2}, x_{N+3}) = d(Tx_{N+1}, TTx_{N+1})$$
$$\leq kd(x_{N+1}, Tx_{N+1})$$
$$\leq k^2 d(x_N, Tx_N) < \infty .$$

Generally, we have

$$d(x_n, x_{n+1}) \leq k^{n-N} d(x_N, Tx_N) \quad \text{for all } n \geq N.$$

Using this and the triangle inequality, we get

$$d(x_n, x_{n+\ell}) \leq \sum_{i=1}^{\ell} d(x_{n+i-1}, x_{n+i})$$
$$\leq \sum_{i=1}^{\ell} k^{n+i-1-N} d(x_N, Tx_N)$$
$$\leq k^{n-N} \left(\frac{1 - k^{\ell}}{1 - k} \right) d(x_N, Tx_N) \quad \text{for all } n \geq N$$

and $\ell = 1, 2, \ldots$.

Since $k < 1$ and $d(x_N, Tx_N) < \infty$, the right hand side tends to 0 as n tends to ∞.

Hence $\{x_n\} = \{T^n x_o\}$ is a Cauchy sequence, and since X is complete

$\lim_{n\to\infty} d(x_n, y) = 0$ or $\lim_{n\to\infty} x_n = y$.

Since T is continuous at y, we have

$$Ty = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = y.$$

Hence y is a fixed point of T.

Theorem [3.2.4] Let T be a mapping of a generalized complete metric space X into itself such that the following conditions hold:

(i) There exists a constant $C > 0$ such that for all $x, Tx \in X$ with $d(x,Tx) \leq C$, we have

$$d(Tx,TTx) \leq kd(x,Tx), \quad 0 \leq k < 1.$$

(ii) For every sequence $x_n = Tx_{n-1}$, $n = 1, 2, \ldots$, where x_o is an arbitrary element of X, there exists an index $N(x_o)$ such that

$$d(x_n, x_{n+\ell}) \leq C, \text{ for all } n \geq N \text{ and } \ell = 1, 2, \ldots .$$

(iii) T is continuous.

Then T has a fixed point and the sequence of successive approximations converges in distance to this fixed point.

Proof: Let x_o be an arbitrary element in X, and let

$x_n = Tx_{n-1} = T^n x_o$, $n = 1, 2, \ldots$, be the sequence of successive approximations of T.

From (ii) of the theorem, it follows that there exists an index $N(x_o)$ such that

$$d(x_n, x_{n+\ell}) \leq C, \text{ for } n \geq N \text{ and } \ell = 1, 2, \ldots$$

Hence by (i), we have

$$d(x_{N+1},x_{N+2}) = d(Tx_N,TTx_N) \leq kd(x_N,Tx_N).$$

And generally for $n \geq N$

$$d(x_n,x_{n+1}) \leq k^{n-N}d(x_N,Tx_N) = k^{n-N}d(x_N,x_{N+1})$$

$$\leq k^{n-N}C \quad , \quad \text{by (ii).}$$

By using this and the triangle inequality, we get

$$d(x_n,x_{n+\ell}) \leq \sum_{i=n}^{n+\ell-1} d(x_i,x_{i+1})$$

$$\leq \sum_{i=n}^{n+\ell-1} k^{i-N}C$$

$$\leq k^{n-N}\left(\frac{1-k^{\ell}}{1-k}\right)C.$$

The right hand side tends to 0 as n tends to ∞ since $k < 1$.

Hence $\{x_n\}$ is a Cauchy sequence, and since X is complete there exists a $y \in X$ such that

$$\lim_{n\to\infty} d(x_n,y) = 0.$$

Now, by the triangle inequality

$$d(y,Ty) \leq d(y,\lim_{n\to\infty} x_n) + d(\lim_{n\to\infty} x_n,Ty).$$

And since T is continuous at y,

$$d(y,Ty) \leq d(y,\lim_{n\to\infty} x_n) + d(T\lim_{n\to\infty} x_{n-1},Ty)$$

$$\leq 0 + 0 .$$

Hence $d(y,Ty) = 0$ and y is a fixed point of T.

The following result was given by Diaz and Margolis [18].

Theorem [3.2.5] Let X be a generalized complete metric space, and let the mapping T of X into itself satisfy the following condition: There exists a constant k, $0 \leq k < 1$, such that whenever $d(x,y) < \infty$ one has

$$d(Tx,Ty) \leq kd(x,y),$$

Let x_0 be an arbitrary point in X and consider the sequence of successive approximations $\{T^n x_0\}$, $n = 1, 2, \ldots$.

Then the following alternative holds: either

(A) for every integer $\ell = 0, 1, 2, \ldots$, one has

$$d(T^{\ell} x_0, T^{\ell+1} x_0) = \infty \quad \text{or}$$

(B) the sequence of successive approximations $\{T^n x_0\}$ converges to a fixed point of T.

In [18] Diaz and Margolis also gave the following localization of the above theorem.

Theorem [3.2.6] Let X be a generalized complete metric space, and let T be a mapping of X into itself such that T satisfies the following condition: There exists a constant k, $0 \leq k < 1$ and a positive constant C, such that whenever $d(x,y) \leq C$ one has

$$d(Tx,Ty) \leq kd(x,y).$$

Let x_0 be an arbitrary element in X, and consider the sequence of successive approximations $\{T^n x_0\}$, $n = 1, 2, \ldots$.

Then the following alternative holds: either

(A) for every integer $\ell = 0, 1, 2, \ldots$, one has

$$d(T^{\ell}x_0, T^{\ell+1}x_0) > C \quad \text{or}$$

(B) the sequence of successive approximations $\{T^n x_0\}$ converges to a fixed point of T.

We now try to give these results for an iterated contraction mapping.

<u>Theorem</u> [3.2.7] Let X be a generalized complete metric space, and let T be a mapping of X into itself such that the following conditions hold:

(i) There exists a constant k, $0 \leq k < 1$, such that whenever $d(x, Tx) < \infty$ one has

$$d(Tx, TTx) \leq kd(x, Tx).$$

(ii) T is continuous,

and

$\{T^n x_0\}$, $n = 1, 2, \ldots$, is the sequence of successive approximations of T.

Then the following alternative holds: either

(A) for every integer $j = 0, 1, 2, \ldots$, one has

$$d(T^j x_0, T^{j+1} x_0) = \infty \quad \text{or}$$

(B) the sequence of successive approximations $\{T^n x_0\}$, $n = 1, 2, \ldots$, converges to a fixed point of T.

<u>Proof</u>: Consider the sequence of numbers $d(x_0, Tx_0)$, $d(Tx_0, T^2x_0), \ldots$, $d(T^j x_0, T^{j+1} x_0), \ldots$, "the sequence of distances between successive neighbours" of the sequence of successive approximations of T, where

x_0 is an arbitrary point of X and can be considered as a point of the sequence.

Then there are two mutually exclusive possibilities: either,

(a) for every integer $j = 0, 1, 2, \ldots,$ one has

$d(T^j x_0, T^{j+1} x_0) = \infty$, which is the alternative (A) of the conclusion of the theorem, or

(b) for some integer $j = 0, 1, 2, \ldots,$ one has

$$d(T^j x_0, T^{j+1} x_0) < \infty .$$

In order to complete the proof, it remains only to show that (b) implies Alternative (B) of the conclusion of the theorem.

In case (b) holds, let $N = N(x_0)$ denote the smallest of all the integers, $j = 0, 1, 2, \ldots,$ such that

$$d(T^j x_0, T^{j+1} x_0) < \infty .$$

Then $d(T^N x_0, T^{N+1} x_0) < \infty$. So by (i) it follows that

$$\begin{aligned} d(T^{N+1} x_0, T^{N+2} x_0) &= d(TT^N x_0, TTT^N x_0) \\ &\leq kd(T^N x_0, TT^N x_0), \quad 0 \leq k < 1 \\ &< \infty, \text{ since } k < 1. \end{aligned}$$

And

$$\begin{aligned} d(T^{N+2} x_0, T^{N+3} x_0) &= d(TT^{N+1} x_0, TTT^{N+1} x_0) \\ &\leq kd(T^{N+1} x_0, TT^{N+1} x_0) \\ &\leq k^2 d(T^N x_0, T^{N+1} x_0) < \infty \end{aligned}$$

Hence by induction, we have

$$\begin{aligned} d(T^{N+j} x_0, T^{N+j+1} x_0) &\leq k^j d(T^N x_0, T^{N+1} x_0) \\ &< \infty \quad \text{for every integer } j = 0, 1, 2, \ldots . \end{aligned}$$

In

other words, it has just been proved that if n is any integer such that $n \geq N$, then

$$d(T^n x_0, T^{n+1} x_0) \leq k^{n-N} d(T^N x_0, T^{N+1} x_0) < \infty$$

Now, using this and the triangle inequality, we have, for all $n \geq N$ and any $\ell = 1, 2, \ldots,$ that

$$\begin{aligned} d(T^n x_0, T^{n+\ell} x_0) &\leq \sum_{i=1}^{\ell} d(T^{n+i-1} x_0, T^{n+i} x_0) \\ &\leq \sum_{i=1}^{\ell} k^{n+i-1-N} d(T^N x_0, T^{N+1} x_0) \\ &\leq k^{n-N} \frac{1-k^{\ell}}{1-k} d(T^N x_0, T^{N+1} x_0). \end{aligned}$$

Since $k < 1$ and $d(T^N x_0, T^{N+1} x_0) < \infty$, the right hand side tends to 0 as n tends to infinity.

Hence $\{T^n x_0\}$ forms a Cauchy sequence. Since X is complete, $\lim_{n \to \infty} T^n x_0 = y \in X$.

Now $d(y, Ty) = d(\lim_{n\to\infty} T^n x_0, T \lim_{n\to\infty} T^n x_0)$.

Since T is continuous at y,

$$\begin{aligned} d(y, Ty) &= d(\lim_{n\to\infty} T^n x_0, \lim_{n\to\infty} T^{n+1} x_0) \\ &= d(y, y) = 0. \end{aligned}$$

Hence $y = Ty$; therefore y is a fixed point of T.

<u>Remarks</u> (i) We note that Theorem [2.2.1] is a special case of the above theorem, because if X is a complete metric space, then $d(x, Tx) < \infty$ for all $x, Tx \in X$. Hence condition (i) of Theorem [3.2.7] holds for all $x, Tx \in X$, and T is, therefore, just the iterated contraction as it was defined for metric spaces. Thus we see that alternative (A) is

excluded from the statement of Theorem [3.2.7] and alternative (B) gives us Theorem [2.2.1].

(2) Theorem [3.2.3] is also a special case of Theorem [3.2.7], because, if condition (ii) of Theorem [3.2.3] is added to the statement of Theorem [3.2.7], then alternative (A) of Theorem [3.2.7] is excluded, and alternative (B) gives the conclusion of Theorem [3.2.3].

We may also give a localization of Theorem [3.2.7] as follows.

Theorem [3.2.8] Let X be a generalized complete metric space, and let T be a mapping of X into itself such that the following conditions are satisfied:

(i) There exists a constant k, $0 \leq k < 1$, and a positive constant C, such that whenever $d(x,Tx) \leq C$, one has

$$d(Tx,TTx) \leq kd(x,Tx).$$

(ii) T is continuous,

and

$\{T^n x_0\}$, $n = 1, 2, \ldots$, is a sequence of successive approximations of T.

Then the following alternative holds: either

(A) for every integer $\ell = 0, 1, 2, \ldots$, one has

$$d(T^\ell x_0, T^{\ell+1} x_0) > C \quad \text{or}$$

(B) the sequence of successive approximations $\{T^n x_0\}$ converges to a fixed point of T, i.e. y is a fixed point of T.

The proof can be completed by following the proofs of Theorems [3.2.4] and [3.2.7].

When X is a generalized metric space, which is not necessarily complete, we may give the following results.

Theorem [3.2.9] Let T be a mapping of a generalized metric space X into itself satisfying the following conditions:

(i) There exists a constant k, $0 \leq k < 1$, such that for all $x, Tx \in X$ with $d(x,Tx) < \infty$ we have $d(Tx,TTx) \leq kd(x,Tx)$.

(ii) For the sequence of successive approximations $x_n = Tx_{n-1}$, $n = 1, 2, \ldots, x_o$ is an arbitrary point in X, there exists an index $N(x_o)$ such that

$$d(x_N, x_{N+\ell}) < \infty \quad \text{for} \quad \ell = 1, 2, \ldots$$

(iii) The sequence of iterates (successive approximations) of T, $\{T^n x_o\}$, has a subsequence $\{T^{n_i} x_o\}$ converging to $y \in X$.

(iv) T is continuous at y.

Then y is a fixed point of T.

Proof: Let x_o be an arbitrary point in X and form the sequence of successive approximations

$$x_n = Tx_{n-1} = T^n x_o, \quad n = 1, 2, \ldots$$

Using (i) and (ii) and following the proof of Theorem [3.2.3], we can show that $\{x_n\}$ is a Cauchy sequence.

Then by (iii), $\{x_n\}$ has a convergent subsequence.

Since $\{T^{n_i} x_o\}$ converges to y and $\{T^n x_o\}$ is a Cauchy sequence, $\{T^n x_o\}$ converges to y.

Thus, since T is continuous we have

$$Ty = T \lim_{n\to\infty} T^n x_o = \lim_{n\to\infty} T^{n+1} x_o = y.$$

Hence y is a fixed point of T.

Remark. If X is a metric space, then condition (i) of the above theorem is true for all $x, Tx \in X$, because in a metric space $d(x,Tx) < \infty$ for all $x, Tx \in X$. Thus, in a metric space, condition (ii) will also be always true. Hence, if X is a metric space, we get Theorem [2.2.9].

We can also state a localization of Theorem [3.2.9] in the following manner.

Theorem [3.2.10] Let X be a generalized metric space, and let T be a mapping of X into itself such that the following conditions are satisfied:

(i) There exists a constant k, $0 \leq k < 1$, and a positive constant C, such that whenever $d(x,Tx) \leq C$, one has

$$d(Tx,TTx) \leq kd(x,Tx).$$

(ii) Let x_o be an arbitrary point in X. For the sequence of successive approximations

$$x_n = Tx_{n-1} = T^n x_o, \quad n = 1, 2, \ldots,$$

there exists an index $N(x_o)$ such that

$$d(x_n, x_{n+\ell}) \leq C \text{ for all } n \geq N \text{ and } \ell = 1, 2, \ldots .$$

(iii) The sequence $(T^n x_o)$ has a subsequence $(T^{n_i} x_o)$ converging to y in X.

(iv) T is continuous at y.

Then T has a fixed point at y.

The proof follows that of Theorems [3.2.9] and [3.2.4].

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