DISCONTINUOUS GALERKIN METHODS AND CASCADING MULTIGRID METHODS FOR INTEGRO-DIFFERENTIAL EQUATIONS

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Discontinuous Galerkin Methods and Cascading Multigrid Methods for Integro-Differential Equations

by

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Abstract

In this thesis, we focus on the discontinuous Galerkin (DG) methods for the functional integro-differential equations and on the cascading multigrid (CMG) methods for the parabolic PDEs, Volterra integro-differential equations (VIDEs) and Fredholm equations.

We give both a priori and a posteriori error estimates of the DG method for linear, semilinear and nonstandard VIDEs. Furthermore the superconvergence of the mesh-dependent Galerkin method for VIDEs is also considered. The fully discretized DG method for VIDEs is also analyzed. Numerical examples are provided to compare the DG method with the continuous Galerkin (CG) method and the continuous collocation (CC) method. We study the primary discontinuities of several classes of VIDEs with time dependent delays, which include the functional VIDEs of Hale’s type, delay VIDEs with weakly singular kernels and delay VIDEs of neutral type (with weakly singular kernels). According to the regularity information established, we construct an adaptive DG method for functional VIDEs of Hale’s type.

Two new cascading multilevel algorithms are analyzed to the semi-linear parabolic PDEs and extended to the partial Volterra integro-differential equations (PVIDEs) and the parabolic PDEs with delays. More distinctly the cascading multigrid method could very well solve the Fredholm equations without dealing with the full stiffness matrix directly. Therefore we can save much more computing time. Most im-
portantly, we contribute to the *multigrid arts* by developing an abstract cascading
multigrid method in Besov spaces and a discontinuous Galerkin cascading multigrid
method. We extend these methods to evolutionary equations and PVIDEs. Finally,
we discuss briefly the future works on (partial) VIDEs with blow-up solutions and
artificial boundary methods for PVIDEs on unbounded domains.
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## Contents

Abstract i

Acknowledgements iii

1 The discontinuous Galerkin method for ODEs: an introduction 1

1.1 The DG method for ODEs 1

1.1.1 Basic description of the DG method for ODEs 1

1.1.2 A priori error estimate for ODEs 3

1.1.3 A posteriori error estimates for ODEs 6

1.2 Mesh-dependent Galerkin methods for ODEs 9

1.2.1 Mesh-dependent Galerkin methods for ODEs 9

1.2.2 Superconvergence 13

1.3 The discretized DG method for ODEs 15

1.3.1 The comparison of the discretized DG method with the collocation method 15

1.4 History of the DG methods for ODEs 17

2 The discontinuous Galerkin method for VIDEs 19

2.1 The discontinuous Galerkin method for linear VIDEs 19

2.1.1 A priori error estimates for linear VIDEs 20
2.1.2 A posteriori error estimates for linear VIDEs ........................................ 24
2.1.3 Efficiency of the a posteriori error estimator ........................................ 28
2.2 The discontinuous Galerkin method for semilinear VIDEs ......................... 29
  2.2.1 A priori error estimates for semilinear VIDEs ...................................... 29
  2.2.2 A posteriori error estimates for semilinear VIDEs ................................ 31
  2.2.3 Efficiency of the a posteriori error estimator ....................................... 38
2.3 Numerical examples ..................................................................................... 39
  2.3.1 Example for the case of constant coefficient ......................................... 39
  2.3.2 Example of a VIDE with time-dependent coefficient .............................. 43
2.4 The discontinuous Galerkin method for nonstandard VIDEs ....................... 49
  2.4.1 Preliminaries ......................................................................................... 49
  2.4.2 A posteriori error estimates for nonstandard VIDEs .............................. 50
  2.4.3 Efficiency of the a posteriori error estimator ....................................... 52
2.5 The discretized discontinuous Galerkin method for VIDEs ....................... 53
  2.5.1 The comparison with collocation method for VIDEs .............................. 53
  2.5.2 A posteriori error estimator .................................................................. 57
2.6 Superconvergence of mesh-dependent Galerkin methods for VIDEs ............ 64
  2.6.1 Superconvergence for semilinear VIDEs ............................................. 64
  2.6.2 Superconvergence of the discretized mesh-dependent Galerkin methods for VIDEs ................................................................. 72

3 The discontinuous Galerkin method for delay VIDEs .................................. 80
  3.1 Primary discontinuities of several classes of delay Volterra integro-
differential equations ................................................................. 80
    3.1.1 Delay VIDEs with weakly singular kernels ....................................... 80
3.1.2 Delay functional VIDEs of Hale's type .................. 87
3.1.3 Delay VIDEs of neutral type with smooth kernels .......... 91
3.1.4 Delay VIDEs of neutral type with weakly singular kernels . 92
3.2 The discontinuous Galerkin method for delay VIDEs ............ 94
  3.2.1 The discontinuous Galerkin method for functional VIDEs of
        Hale's type .............................................. 94
  3.2.2 The DG method for delay VIDEs of neutral type with weakly
        singular kernels ........................................... 105

4 Cascading multilevel discretization method for parabolic problems 107
  4.1 Introduction .............................................. 107
  4.2 Cascading multilevel discretization algorithms ............... 108
      4.2.1 Algorithm A ........................................ 108
      4.2.2 Algorithm B .......................................... 115
  4.3 Comparison and discussion .................................. 117
  4.4 Extensions to other parabolic problems ....................... 118
      4.4.1 Parabolic equation with delay argument ................ 118
      4.4.2 Parabolic equation with memory term .................. 124
  4.5 Application to parabolic Fredholm equation .................. 126
      4.5.1 Finite element method for parabolic Fredholm equation . 126
      4.5.2 Algorithm for parabolic Fredholm equation ............ 129

5 The abstract cascading multigrid method in Besov spaces 131
  5.1 Notations and definitions .................................. 132
      5.1.1 Bilinear form and induced norm ........................ 132
      5.1.2 Function spaces and their norms ....................... 133
5.2 The abstract cascading multigrid method .................................................. 136
  5.2.1 The cascading multigrid algorithm .................................................. 136
  5.2.2 The method for the heat equation .................................................. 141
  5.2.3 The cascading multigrid method in Besov spaces ............................. 145
  5.2.4 The method for VIDEs ................................................................. 148
5.3 Abstract cascading multigrid method with non-nested spaces and varying inner products and bilinear forms .................................................. 151
  5.3.1 The cascading multigrid algorithm .................................................. 151
  5.3.2 Application to interior penalty discontinuous Galerkin method 156
  5.3.3 Extension to VIDEs ................................................................. 160
5.4 History of cascading multigrid method .................................................. 162

6 Future works .......................................................................................... 163
  6.1 Adaptive discontinuous Galerkin time-stepping for (partial) VIDEs with blow-up solutions .................................................. 163
    6.1.1 VIDEs with blow-up solutions .................................................. 163
    6.1.2 PVIDEs with blow-up solutions .................................................. 164
  6.2 The artificial boundary method for PVIDEs on unbounded spatial domains .................................................. 165
    6.2.1 The artificial boundary method .................................................. 165
    6.2.2 The finite element method and the DG time-stepping for the reduced problems .................................................. 168
    6.2.3 Blow-up problems on unbounded spatial domains .......................... 169

Bibliography ......................................................................................... 170
List of Tables

4.1 Comparison of Algorithm A and Algorithm B ................. 117
List of Figures

2.1 The methods for Example 2.3.1.1 .............................. 46
2.2 The methods with Trapezoidal rule of lag term for Example 2.3.1.1 46
2.3 The methods with midpoint rule of lag term for Example 2.3.1.1 ... 47
2.4 Effect of stability on the convergence of methods for Example 2.3.1.1 47
2.5 The methods for Example 2.3.2.1 .............................. 48
2.6 The methods with midpoint rule of lag term for Example 2.3.2.1 ... 48
Chapter 1

The discontinuous Galerkin method for ODEs: an introduction

The discontinuous Galerkin (DG) method can be traced back to 1973 when Reed and Hill used it to solve the neutron transport problem. The DG method was first analyzed in 1974 by Lesaint and Raviart in the application to ODEs. Since DG methods assume discontinuous approximate solutions, they can be considered as generalizations of finite volume methods. What makes DG methods popular is that they are able to capture the physically relevant discontinuities of the exact solutions without producing spurious oscillations near them. A more detailed history of DG methods for ODEs and PDEs will be presented at the end of this chapter, in Section 1.4.

1.1 The DG method for ODEs

1.1.1 Basic description of the DG method for ODEs

Consider the initial value problem

\[ y'(t) = f(t, y(t)), \quad t \in I := [0, T], \quad y(0) = y_0, \]  

\[(1.1.1.1)\]
and assume that the (Lipschitz continuous) function \( f : I \times \Omega \subset \mathbb{R} \to \mathbb{R} \) is such that (1.1.1.1) possesses a unique solution \( y \in C^1(I) \) for all \( y_0 \in \Omega \). Let

\[
I_h := \{ t_n : 0 = t_0 < t_1 < \cdots < t_M < \cdots < t_N = T \}
\]

be a given mesh on \( I \), and set \( I_n := (t_{n-1}, t_n], \overline{I}_n := [t_{n-1}, t_n], h_n := t_n - t_{n-1} \) (\( n = 1, \ldots, N - 1 \)); \( h := \max \{ h_n : 1 \leq n \leq N - 1 \} \) will be called the diameter of the mesh \( I_h \). Note that we have, in rigorous notation,

\[
t_n := t_n^{(N)}, \quad I_n := I_n^{(N)}, \quad h_n := h_n^{(N)} \quad (n = 1, \ldots, N - 1), \quad h := h^{(N)}.
\]

We will usually suppress this dependence on \( N \), the number of subintervals corresponding to a given mesh \( I_h \), except occasionally in the convergence analysis where \( N \to \infty \) with \( Nh^{(N)} \) uniformly bounded. At the mesh points the left- and right-sided limits of piecewise continuous functions \( \varphi : I \to \mathbb{R} \) will be important. They are defined as follows:

\[
\varphi_n^{+} := \lim_{s \to 0^+, s > 0} \varphi(t_n + s), \quad 0 \leq n \leq N - 1; \quad \varphi_n^{-} := \lim_{s \to 0^-, s > 0} \varphi(t_n - s), \quad 1 \leq n \leq N.
\]

The jump across the mesh points is given by \([\varphi]_n := \varphi_n^{+} - \varphi_n^{-}\).

In the DG method, we are looking for an approximate solution of (1.1.1.1) in the finite space

\[
\mathcal{V}_N^{(m)} := \{ \varphi \in L^2(I) : \varphi|_{I_n} \in \mathcal{P}^{(m)}(I_n), 1 \leq n \leq N \}, \quad (1.1.1.2)
\]

where \( \mathcal{P}^{(m)}(I_n) \) denotes the space of all (real) polynomials of degree not exceeding \( m \). We define the DG method for (1.1.1.1) as: Find \( Y \in \mathcal{V}_N^{(m)} \) such that

\[
B_{DG}(Y, X) = Y_0^{-} X_0^{+}, \quad \forall X \in \mathcal{V}_N^{(m)}, \quad (1.1.1.3)
\]
where $Y_0^- = y_0$ and

$$B_{DG}(Y, X) := \sum_{n=1}^{M} \int_{I_n} (Y'(t) - f(t, Y)) X dt + \sum_{n=2}^{M} [Y]_{n-1} X^+_n + Y_0^+ X_0^+,$$  \hspace{1cm} (1.1.1.4)

Note that the exact solution $y$ of (1.1.1) satisfies

$$B_{DG}(y, X) = y_0 X_0^+, \quad \forall X \in \mathcal{V}_N^{(m)}.$$  

Hence the Galerkin orthogonality property,

$$B_{DG}(y - Y, X) = 0, \quad \forall X \in \mathcal{V}_N^{(m)},$$  \hspace{1cm} (1.1.1.5)

holds true. We remark also that the DG method (1.1.1.3) can be interpreted as a time-stepping scheme: For $n = 1, \cdots, N$, find $Y|_{I_n} \in \mathcal{P}^{(m)}(I_n)$, such that

$$\int_{I_n} (Y' - f(t, Y)) X dt + Y^+_{n-1} X^+_n = Y^-_{n-1} X^-_n, \quad \forall X \in \mathcal{P}^{(m)}(I_n).$$  \hspace{1cm} (1.1.1.6)

Here we set $Y_0^- := y_0$.

In order to make the readers capture the basic idea of DG methods for ODEs easily, we shall first introduce the DG method with piecewise constant approximation (i.e., $m = 0$ in (1.1.1.2)); it will be denoted by DG(0). If piecewise linear polynomial approximation ($m = 1$) is used, we write DG(1). For more general DG schemes with high-order polynomial approximation, we refer to Section 1.2. Here we also remark that an analogous analysis holds for systems of ODEs if the products are replaced by the corresponding inner products in $\mathbb{R}^d$ ($d$ denotes the dimension of the systems) [see for example Section 1.2]. Throughout this thesis we define the norm $\| \cdot \|_J := \sup_{t \in J} | \cdot |$, where $J$ is some compact interval.

### 1.1.2 A priori error estimate for ODEs

In this section we study the a priori error analysis of DG methods for ODEs.
Theorem 1.1.2.1. Assume that $f$ has continuous partial derivatives and there is a constant $L > 0$ such that

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|,$$  \hfill (1.1.2.1)

for all $t \in I$ and $y_1, y_2 \in \Omega$. Then there exists a constant $C$, independent of $h_n$, such that for $0 \leq M \leq N$, the error of $DG(m)$ for (1.1.1.1) satisfies

$$||e||_{0,M} \leq C \max_{n \leq M} h_n^{m+1} ||y^{(m+1)}||_{I_n},$$  \hfill (1.1.2.2)

with $C := C(L, t_M)$ and $m = 0, 1$.

Proof. If $V \in \mathcal{V}_N^{(m)}$ is determined by $V_0^- = y_0$ and, for $1 \leq n \leq M \leq N$, by

$$\int_{I_n} V'Xdt - \int_{I_n} f(t, y(t))Xdt + V_{n-1}^{+}X_{n-1}^{+} = V_{n-1}^{-}X_{n-1}^{+},$$  \hfill (1.1.2.3)

for all $X \in \mathcal{V}_N^{(m)} (m = 0, 1)$, then

$$||y - V||_{I_n} \leq Ch_n^{m+1} ||y^{(m+1)}||_{I_n}.$$  \hfill (1.1.2.4)

(See Estep [43] for the proof of (1.1.2.4)). We now compare $Y$ to $V$. Setting $\mu := y - V$ and $\phi := Y - V \in \mathcal{V}_N^{(m)}$, we have $e = \mu - \phi$. We subtract (1.1.2.3) from (1.1.1.6) and obtain

$$\int_{I_n} \phi'Xdt - \int_{I_n} (f(t, Y) - f(t, y))X(t)dt + \phi_{n-1}^{+}X_{n-1}^{+} = \phi_{n-1}^{-}X_{n-1}^{+},$$  \hfill (1.1.2.5)

for all $X \in \mathcal{V}_N^{(m)}$. Choosing $X = \phi$ leads to

$$\frac{1}{2} |\phi_{n-1}^{+}|^2 + \frac{1}{2} |\phi_{n-1}^{-}|^2 - \int_{I_n} (f(t, Y) - f(t, y))\phi(t)dt = \phi_{n-1}^{-}\phi_{n-1}^{+}.$$  \hfill (1.1.2.6)

Subsequently, we arrive at

$$\frac{1}{2} |\phi_{n-1}^{-}|^2 \leq \frac{1}{2} |\phi_{n-1}^{-}|^2 + \int_{I_n} |(f(t, Y) - f(t, y))\phi(t)| dt.$$  \hfill (1.1.2.7)
We now substitute $e = \mu - \phi$ into (1.1.2.7) to find that

$$|\phi^-|^2 \leq |\phi^-_{n-1}|^2 + L \int_{I_n} |\mu|^2 dt + 3L \int_{I_n} |\phi|^2 dt. \quad (1.1.2.8)$$

Next we choose $X = (t - t_{n-1})\phi'$ in (1.1.2.5) to obtain

$$h_n^2|\phi'|_{I_n}^2 \leq 8L^2 h_n \int_{I_n} |\mu|^2 dt + 8L^2 h_n \int_{I_n} |\phi|^2 dt. \quad (1.1.2.9)$$

Since

$$\int_{I_n} |\phi|^2 dt \leq 2h_n |\phi^-|^2 + \frac{2}{3} h_n^2 |\phi'|_{I_n}^2,$$

we find that

$$\int_{I_n} |\phi|^2 dt \leq 4h_n |\phi^-|^2 + \int_{I_n} \mu^2 dt, \quad (1.1.2.9)$$

provided that $\frac{16}{3} L^2 h_n^2 \leq \frac{1}{2}$.

By combining (1.1.2.8) and (1.1.2.9), we see that

$$|\phi^-|^2 \leq \frac{1}{1 - 12L h_n} |\phi^-_{n-1}|^2 + \frac{4L}{1 - 12L h_n} \int_{I_n} |\mu|^2 dt. \quad (1.1.2.10)$$

Now we iterate (1.1.2.10), assuming $12L h_n \leq \frac{1}{2}$. This yields

$$|\phi^-|^2 \leq C L t_N ||\mu||_{[0,t_N]}^2. \quad (1.1.2.11)$$

Theorem 1.1.2.1 follows from (1.1.2.11) directly when $m = 0$. For $m = 1$, we use equation (1.1.2.6) and the fact that $\phi \in V_N^{(m)}$ implies that $||\phi||_{I_n}^2 \leq |\phi^+_{n-1}|^2 + |\phi^-_{n-1}|^2$, to obtain

$$\frac{1}{4} ||\phi||_{I_n}^2 \leq |\phi^-_{n-1}|^2 + \frac{L}{2} \int_{I_n} \mu^2 dt + \frac{3L}{2} \int_{I_n} \phi^2 dt.$$

An analogous argument establishes Theorem 1.1.2.1 for $m = 1$. We also refer to Estep [43] for the original proof of Theorem 1.1.2.1.
Corollary 1.1.2.1. Assume \( f(t, y) := -a(t)y(t) + g(t) \) in (1.1.1.1) where \( a, g \) are continuous on each interval \( I_n \) (\( 1 \leq n \leq N \)), and set \( A := \|a\|_I \). Then there exists a constant \( C \), independent of \( h \), such that the error of \( DG(m) \) satisfies

\[
\|y - Y\|_{[0,t_M]} \leq Ch^{m+1}\|y^{(m+1)}\|_{[0,t_M]},
\]

with \( C := C(t_M, A) \) and \( m = 0, 1 \).

1.1.3 A posteriori error estimates for ODEs

The a posteriori error analysis is based on representing the error in terms of the solution of a continuous dual problem related to (1.1.1.1), which is used to determine the effects of the accumulation of errors, and in terms of the residual of the computed solution, which measures the propagation of error. After showing the stability of the dual problem, we can estimate the a posteriori error bound of \( DG(m) \). The details for the \( DG(0) \) method can be found in the book [45].

Theorem 1.1.3.1. Assume \( f(t, y) = -a(t)y(t) + g(t) \) in (1.1.1.1) where \( a, g \) are continuous on each interval \( I_n \) (\( 1 \leq n \leq N \)), and let \( m = 0, 1 \). Then the a posteriori error of \( DG(m) \) at the mesh point \( t_M \) (\( 0 \leq M \leq N \)) satisfies

\[
|y(t_M) - Y_M^-| \leq S(t_M)\|h_nR(Y)\|_{[0,t_M]},
\]

where \(|R(Y)| := \frac{|Y_{n+1} - Y_n^-|}{h_n} + |g - aY| (t \in I_n) \) and \( \| \cdot \| := \max_{1 \leq n \leq M}\{\| \cdot \|_{I_n}\} \). If \( A := \|a\|_I \), then \( S(t_M) \leq \exp(At_M) \) and if, in addition, \( a(t) \geq 0 \) for all \( t \in I \), then \( S(t_M) \leq 1 \).

Proof. The proof for \( DG(0) \) is in [45]. The analysis of a posteriori error estimate of \( DG(1) \) for (1.1.1.1) is similar to \( DG(0) \). We give the details as follows.
Firstly we consider the continuous dual problem for (1.1.1): Find \( x = x(t) \) such that
\[
\begin{aligned}
-x' + a(t)x &= 0, \quad \text{for } t_M > t \geq 0, \\
x(t_M) &= e_M^-, 
\end{aligned}
\] (1.1.3.1)
where we denote by \( e(t) := y(t) - Y(t) \) the error at the time \( t \) and set \( e_M^− := y(t_M) − Y_M^- \). Starting from the identity
\[
(e_M^-)^2 = (e_M^-)^2 + \sum_{n=1}^{M} \int_{I_n} e \cdot (-x' + a(t) x) dt,
\]
we integrate by parts over each subinterval \( I_n \) to obtain
\[
(e_M^-)^2 = \sum_{n=1}^{M} \int_{I_n} (e' + a(t)e)x dt + \sum_{n=1}^{M-1} [e_n]_n^+ + (y_0 - Y_0^+) x_0^+ .
\] (1.1.3.2)
Using \( Y_0^- = y_0 \), we can simplify (1.1.3.2) to
\[
(e_M^-)^2 = \sum_{n=1}^{M} \left( \int_{I_n} (g - aY - Y') x dt - [Y]_{n-1} x_{n-1}^+ \right).
\]
We use the Galerkin orthogonality (1.1.1.5) by choosing \( X = \bar{x} \) to be the \( L_2 \) projection into the space \( V_{N}^{(1)} \) and obtain the error representation formula:
\[
(e_M^-)^2 = \sum_{n=1}^{M} \left( \int_{I_n} (g - aY) (x - \bar{x}) dt - [Y]_{n-1} (x - \bar{x})_{n-1}^+ \right).
\]
From the error estimates of interpolation:
\[
\int_{I_n} |x - \bar{x}| dt \leq h_n \int_{I_n} |x'| dt; \quad |x - \bar{x}| \leq \int_{I_n} |x'| dt,
\]
we arrive at
\[
(e_M^-)^2 \leq \sum_{n=1}^{M} \left\{ \|g - aY\|_{I_n} \int_{I_n} |x - \bar{x}| dt + \frac{\|[Y]_{n-1}\|}{h_n} h_n |x - \bar{x}| \right\}
\leq \sum_{n=1}^{M} \left\{ h_n \| |g - aY| + \frac{\|[Y]_{n-1}\|}{h_n} \| \int_{I_n} |x'| dt \right\}
\leq \max_{1 \leq n \leq M} \left\{ h_n \| |g - aY| + \frac{\|[Y]_{n-1}\|}{h_n} \| \int_{I_n} |x'| dt \right\} \int_{t_M}^{t_M} |x'| dt
\leq S(t_M) \cdot |e_M^-| \cdot \|h_n R(Y)\|_{[0,t_M]}.
\] (1.1.3.3)
where \(|R(Y)| := |g - aY| + \frac{|Y|_{n-1}}{h_n} \). The stability factor \(S(t_M)\) is defined by

\[
S(t_M) := \frac{\int_0^{t_M} |x'| dt}{|e_M^-|}.
\]

For the estimate of \(S(t_M)\), we give the following lemma.

**Lemma 1.1.3.1.** If \(A := ||a||_I\), then the solution \(x\) of (1.1.3.1) satisfies

\[
|x(t)| \leq \exp(At_M)|e_M^-|,
\]

for all \(0 \leq t \leq t_M\) and \(S(t_M) \leq \exp(At_M)\). If, in addition, \(a(t) \geq 0\) for all \(t \in I\), then \(x\) satisfies

\[
|x(t)| \leq |e_M^-|,
\]

for all \(0 \leq t \leq t_M\) and \(S(t_M) \leq 1\).

**Proof.** The proof can be found in [45].

By combining Lemma 1.1.3.1 and (1.1.3.3) we complete the proof of Theorem 1.1.3.1.

The following theorem gives a measure for the efficiency of the a posteriori estimator in Theorem 1.1.3.1.

**Theorem 1.1.3.2.** Assume \(f(t, y) = -a(t)y(t) + g(t)\) in (1.1.1.1) and that \(a, g\) are continuous on each interval \(I_n\) \((1 \leq n \leq N)\), and let \(A := ||a||_I\). Then we have

\[
|y(t_M) - Y_M^-| \leq S(t_M)||h_nR(Y)||_{[0,t_M]}
\]

\[
\leq CS(t_M)(1 + At_Me^{CAt_M})^{1/2}h||y'||_{[0,t_M]},
\]

for \(0 \leq M \leq N\).
Proof. We need to estimate $||hR(Y)||_{[0,t_M]}$ in Theorem 1.1.3.1 and Theorem 1.1.3.1:

$$||h_nR(Y)||_{[0,t_M]} := |||Y_{n-1}^+ - Y_{n-1}^-| + h_n|g - aY||_{[0,t_M]}$$

$$= ||Y_{n-1}^+ - y + y - Y_{n-1}^-||_{[0,t_M]}$$

$$+ h_n|a(t)y(t) - a(t)Y(t)||_{[0,t_M]} + h||y'||_{[0,t_M]}$$

$$\leq (2 + A)||e(t)||_{[0,t_M]} + h||y'||_{[0,t_M]}.$$  \hspace{1cm} (1.1.3.4)

Upon applying Corollary 1.1.2.1, we complete the proof.

1.2 Mesh-dependent Galerkin methods for ODEs

In this section we survey the paper Delfour and Dubeau [40] and discuss the mesh-dependent Galerkin methods (including the discontinuous Galerkin method) for ODEs and the corresponding superconvergence results.

1.2.1 Mesh-dependent Galerkin methods for ODEs

Consider the following system of ODEs,

$$y'(t) = f(t, y(t)), \quad t \in I := [0, T], \quad y(0) = y_0,$$

where $y_0 \in \Omega \subset \mathbb{R}^d$, with $d \geq 1$, and $y : [0, T] \to \Omega$ is a vector function and $f : \Omega \times [0, T] \to \Omega$ is a given map such that (1.2.1.1) possesses a unique solution for all $y_0 \in \Omega$. Before we begin the analysis, we formulate some notations and definitions.

(i) Define the inner product:

$$x \cdot y = \sum_{i=1}^{d} x_i y_i, \quad x = (x_1, \ldots, x_d), \quad y = (y_1, \ldots, y_d) \in \mathbb{R}^d.$$
(ii) $L^p([a, b]; \Omega)$ is the space of $p$-integrable ($1 \leq p < \infty$) or essentially bounded functions ($p = \infty$);

(iii) $H^k([a, b]; \Omega)$ is the Sobolev space of functions with derivatives through order $k$ in $L^2([a, b]; \Omega)$ ($k$ is a nonnegative integer);

(iv) $C([a, b]; \Omega)$ denotes the space of continuous functions;

(v) $P^{(m)}([a, b]; \Omega)$ is the space of all polynomials of degree not exceeding $m$;

(vi) $\| \cdot \|_{\infty, n} := \sup_{t \in I_n} | \cdot |$, where $| \cdot |$ is the Euclidean norm; define, for nonnegative integer $k$,

$$\| \cdot \|^2_{k, n} := \sum_{i=0}^{k} (\cdot, (\cdot)^{(i)})_n,$$

where $(\cdot, \cdot)_n$ denotes the inner product in $L^2(I_n; \Omega)$; define also

$$\| \cdot \|_{\infty} := \max \{ \| \cdot \|_{\infty, n} : n = 1, \ldots, N \}; \| \cdot \|^2_{k} := \sum_{n=1}^{N} \| \cdot \|^2_{k, n}.$$

Now we present the weak form of (1.2.1.1). On each interval $I_n$, form the inner product of (1.2.1.1) with $v_n$ in $H^1(I_n; \Omega)$ and integrate by parts:

$$y(t_n) \cdot v_n(t_n) = y(t_{n-1}) \cdot v_n(t_{n-1}) + \int_{I_n} [y \cdot v'_n + f(y) \cdot v_n] dt,$$

(1.2.1.2)

where $f(y)$ denotes the function $t \to f(t, y(t)) : [0, T] \to \Omega$. Then sum over all $n$ equations (1.2.1.2), observing (1.2.1.1) to obtain the following variational equation:

$$y(t_0) \cdot [V_0 - v_1(t_0)] + \sum_{n=1}^{N-1} y(t_n) \cdot [v_n(t_n) - v_{n+1}(t_n)] + y(t_N) \cdot v_N(t_N)$$

$$- \sum_{n=1}^{N} \int_{I_n} y \cdot v'_n dt = y_0 \cdot V_0 + \sum_{n=1}^{N} \int_{I_n} f(y) \cdot v_n dt,$$

(1.2.1.3)

which is to hold for all

$$\bar{v} := (V_0, v_1, \ldots, v_N) \in V := \Omega \times \prod_{n=1}^{N} H^1(I_n; \Omega).$$
The space $\mathcal{V}$ will be endowed with the norm
\[
||\bar{v}||_{\mathcal{V}} := \left\{ |V_0|^2 + \sum_{n=1}^{N} ||v_n||_{1,n}^2 \right\}^{1/2}
\]
This suggests the following variational problem: Find
\[
\bar{u} := (U_0, \ldots, U_N, u_1, \ldots, u_N) \in \mathcal{U} := \Omega^{N+1} \times \prod_{n=1}^{N} L^2(I_n; \Omega),
\]
such that
\[
U_0 \cdot [V_0 - v_1(t_0)] + \sum_{n=1}^{N-1} U_n \cdot [v_n(t_n) - v_{n+1}(t_n)] + U_N \cdot v_N(t_N)
- \sum_{n=1}^{N} \int_{I_n} u_n \cdot v'_n dt = y_0 \cdot V_0 + \sum_{n=1}^{N} \int_{I_n} f(u_n) \cdot v_n dt, \quad \forall \bar{u} \in \mathcal{V}. \tag{1.2.1.4}
\]
Locally, the weak form (1.2.1.4) is equivalent to finding $u_n$ in $L^2(I_n; \Omega)$ and $U_n$ in $\Omega$
such that
\[
\begin{aligned}
&U_0 = y_0, \\
&U_n \cdot v_n(t_n) = U_{n-1} \cdot v_n(t_{n-1}) + \int_{I_n} [u_n \cdot v'_n + f(u_n) \cdot v_n] dt,
\end{aligned}
\tag{1.2.1.5}
\]
for all $v_n \in H^1(I_n; \Omega)$ and $n = 1, \ldots, N$.

**Theorem 1.2.1.1.** [Delfour and Dubeau (1986)]

(a) **There exists a unique solution** $\bar{u} \in \mathcal{U}$ **to the variational equation** (1.2.1.4).

(b) **Moreover,**
\[
\bar{u} = (y(t_0), \ldots, y(t_N), y|_{I_1}, \ldots, y|_{I_N}),
\]
where $y$ is the solution of problem (1.2.1.3) and $y|_{I_n}$ denotes the restriction of the
function $y$ to the interval $I_n$.

Now we introduce the Galerkin scheme corresponding to the weak form (1.2.1.5).
Define the finite-dimensional subspaces $\mathcal{U}_h$ of $\mathcal{U}$, and $\mathcal{V}_h$ of $\mathcal{V}$ as follows:
\[
\mathcal{U}_h = \left\{ \bar{u}_h \mid \bar{u}_h = (U^h_0, \ldots, U^h_N, u^h_1, \ldots, u^h_N) \in \mathcal{U} \text{ such that } u^h_n \in \mathcal{P}^{(m)}(I_n; \Omega) \right\}
\]
subject to $J (\geq 0)$ additional conditions for $n = 1, \ldots, N$. 
\[ \mathcal{V}_h = \left\{ v_h \mid \bar{v}_h = (V_0^h, v_1^h, \ldots, v_N^h) \in \mathcal{V} \text{ such that } v_n^h \in \mathcal{P}(m+1-J)(I_n; \Omega) \text{ for } n = 1, \ldots, N. \right\}, \]

where \( m \) and \( J \) are nonnegative integers such that \( m + 1 - J \geq 0 \). Note also that

\[ \dim \mathcal{U}_h = [1 + (m + 2 - J)N] \dim \Omega = \dim \mathcal{V}_h. \]

With the above definition, the approximation scheme for (1.2.1.5) is defined to find \( \bar{u}_h \) in \( \mathcal{U}_h \) such that \( U_0 = y_0 \) and

\[ \begin{align*}
U_n^h \cdot v_n^h(t_n) - \int_{I_n} u_{n-1}^h \cdot v_n^h dt &= U_{n-1}^h \cdot v_{n-1}^h(t_{n-1}) + \int_{I_n} f(u_n^h) \cdot v_n^h dt, \\
& \text{for all } v_n^h \in \mathcal{P}(m+1-J)(I_n; \Omega) \text{ and } n = 1, \ldots, N.
\end{align*} \tag{1.2.1.6} \]

for all \( v_n^h \) in \( \mathcal{P}(m+1-J)(I_n; \Omega) \) and \( n = 1, \ldots, N \). Delfour and Dubeau [40] showed that (1.2.1.6) possesses a unique solution whenever \( h \) is small enough.

**Remark 1.2.1.1.**

(i) For \( J = 0 \) we obtain the completely discontinuous Galerkin methods;

(ii) For \( 0 < J \leq m + 1 \), and on each interval \( I_n \), the \( J \) conditions are of the form

\[ v_n^h(t_{n_l}) = U_{n_l}^h, \quad l = 1, \ldots, J, \tag{1.2.1.7} \]

where \( n_l \in \{0, \ldots, N\} \). These Galerkin methods will be referred to as nodal methods:

(1) for \( J = 1 \), i.e., \( u_n^h(t_n) = U_n^h \), \( n = 1, \ldots, N \), the nodal method coincides with the DG scheme of Lesaint and Raviart [82].

(2) for \( J = 2 \), i.e., \( u_n^h(t_{n-1}) = u_{n-1}^h(t_{n-1}) = U_{n-1}^h \), the nodal methods become the continuous Galerkin methods of Hulme [73] [74].

(3) for \( J = m + 1 \) in the nodal methods, we obtain multistep methods (see e.g., Butcher [30]).
(4) for $J < m+1$, the nodal methods reduce to hybrid methods (see e.g., Gear [50]).

(iii) for $J = 1$, on each interval $I_n$,

$$\alpha_n u_n^h(t_n) + (1 - \alpha_n) u_{n+1}^h(t_n) = U_n^h,$$

the method is called the $\alpha$-method (see Delfour, Hager and Trochu [41]).

1.2.2 Superconvergence

Now we establish the convergence results in two main theorems. The first theorem shows that if the solution of (1.2.1.1) belongs to $H^{m+1}([0, T]; \Omega)$, the $L^2$ and nodal errors are proportional to $h^{m+1}$. The second theorem states that under appropriate assumption on the function $f$ there is an asymptotic superconvergence at the mesh points proportional to $h^{2m+2-J}$ ($0 \leq J \leq m+1$). We assume that $h$ is sufficiently small, in order to guarantee the existence of a unique solution $\bar{u}$ to (1.2.1.6). $C$ will denote a generic constant independent of $h$.

**Theorem 1.2.2.1. [L^2 and Nodal Errors]**

Assume that the solution $y$ of (1.2.1.1) belongs to $H^{m+1}([0, T]; \Omega)$. For $M > 1$, assume that on the first $M - 1$ intervals the solution of (1.2.1.6) is such that

$$\max\{|U_n^h - y(t_n)| : n = 0, \ldots, M - 1\} \leq C h^{m+1}||y^{(m+1)}||_0,$$

and for $j = 0, \ldots, m + 1$,

$$\left\{ \sum_{n=1}^{M-1} ||u_n^h - y||_{j,n}^2 \right\}^{1/2} \leq C h^{m+1-j}||y^{(m+1)}||_0.$$

Thus,

$$\max\{|U_n^h - y(t_n)| : n = 0, \ldots, N\} \leq C h^{m+1}||y^{(m+1)}||_0,$$
and for \( j = 0, \ldots, m+1, \)

\[
||u^h - y||_j \leq C h^{m+1-j} ||y^{(m+1)}||_0,
\]

where \( u^h = \sum_{n=1}^N u_n \chi_{I_n} \) and \( || \cdot ||_j := \left\{ \sum_{n=1}^N || \cdot ||_{I,n}^2 \right\}^{1/2} \). Here \( \chi_{I_n} \) is the characteristic function of \( I_n \).

**Proof.** See Delfour and Dubeau [40]. It can also be found in Chapter 2 as a special case of Theorem 2.6.1.1.

**Theorem 1.2.2.2. [Superconvergence]**

Assume that the assumptions of Theorem 1.2.2.1 hold. Assume also that

(i) the matrix

\[
A(t) := (a_{i,j}(t))_{i,j=1}^d,
\]

with \( a_{i,j} := \frac{\partial f_i}{\partial y_j}(t,y) \), exists and that its columns belong to \( H^{m+1-J}([0,T];\Omega) \),

and

(ii) there exist a neighborhood \( V \) of the origin \( y \) in \( \Omega \) and a positive constant \( B \) such that

\[
|f(t,x) - f(t,y) - A(t)(x-y)| \leq B |y-x|^2,
\]

for all \( t \) and all \( x \) in \( y + V \). Then,

\[
\max\{|U_n^h - y(t_n)| : n = 0, \ldots, N\} \leq C ||u^h - y||_0 [h^{m+1-j} + ||u^h - y||_0]
\]

\[
\leq C h^{2m+2-j}
\]

**Proof.** See Delfour and Dubeau [40]. It will also be derived in Chapter 2 as a special case of Theorem 2.6.1.2.
1.3 The discretized DG method for ODEs

The discussion in this section originates from Brunner [23], which is based on Lesaint and Raviart [82].

1.3.1 The comparison of the discretized DG method with the collocation method

We recall the DG time-stepping scheme for (1.1.1.1): For \( n = 1, \ldots, N \), find \( Y|_{I_n} \in \mathcal{P}^m(I_n) \), such that

\[
\int_{I_n} (Y' - f(t, Y))X dt + Y_{n-1}^+ X_{n-1}^- = Y_{n-1}^- X_{n-1}^+, \quad \forall X \in \mathcal{P}^m(I_n).
\]

(1.3.1.1)

Here we set \( Y_0^- := y_0 \). Suppose now that the integrals in (1.3.1.1) are approximated by interpolatory \((m + 1)\)-point quadrature formulas with abscissas \( t_{n,j} := t_n + c_j h_n \) \((0 =: c_0 < c_1 < \ldots < c_m \leq 1)\) and weights \( w_j \) \((j = 0, 1, \ldots, m)\). We denote the resulting discretized DG solution in \( \mathcal{V}^{(m)}_N \) by \( \tilde{Y} \). The fully discretized version of (1.3.1.1) is then given by

\[
h_n \sum_{j=0}^{m} w_j [\tilde{Y}'(t_{n,j}) - f(t_{n,j}, \tilde{Y}(t_{n,j}))] X(t_{n,j}) + \tilde{Y}(t_n^+) X(t_n^+) - \tilde{Y}(t_n^-) X(t_n^-) = 0,
\]

(1.3.1.2)

for all \( X \in \mathcal{P}^m(I_n) \). Let

\[
\tilde{Y}_n := \tilde{Y}(t_n^-), \quad \tilde{Y}_{n,0} := \tilde{Y}(t_n^+) (= \tilde{Y}(t_{n,0}^+)), \quad \tilde{Y}_{n,j} := \tilde{Y}(t_{n,j}) \quad (j = 1, \ldots, m),
\]

and let \( L_j(v) \) be the \( j \)th Lagrange canonical polynomial (of degree \( m-1 \)) corresponding to the points \( \{c_i: i = 1, \ldots, m\} \). Moreover, denote by \( \{X_j: j = 0, 1, \ldots, m\} \) a (canonical) basis for \( \mathcal{P}^m(I_n) \) so that

\[
X_i(t_n + c_j h_n) = \delta_{i,j} \quad (i, j = 0, 1, \ldots, m).
\]
Since the restriction of $Y'$ to $I_n$ is a polynomial of degree $m - 1$ we may write

$$Y'(t_n + vh_n) = \sum_{j=1}^{m} L_j(v)Y'(t_n,j), \quad v \in (0, 1],$$

and hence

$$\tilde{Y}(t_n + vh_n) = \tilde{Y}(t_n^+) + h_n \int_{0}^{v} \tilde{Y}'(t_n + sh_n)ds, \quad v \in (0, 1]. \quad (1.3.1.3)$$

On the other hand, $(1.3.1.2)$ with $X = X_0$ yields

$$h_n w_0 [\tilde{Y}'(t_n,0) - f(t_n,0, \tilde{Y}(t_n,0))] + \tilde{Y}(t_n^+) - \tilde{Y}(t_n^-) = 0,$$

implying that

$$\tilde{Y}(t_n^+) = \tilde{Y}_n + h_n w_0 [f(t_n,0, \tilde{Y}(t_n))] - \sum_{j=1}^{m} L_j(c_0)\tilde{Y}'(t_n,j)]. \quad (1.3.1.4)$$

For $X = X_i$ $(i = 1, \ldots, m)$, with $X_i(t_n,j) = \delta_{i,j}$, we obtain from $(1.3.1.2)$ the equations

$$w_i [\tilde{Y}'(t_n,i) - f(t_n,i, \tilde{Y}(t_n,i))] = 0,$$

where $w_i \neq 0$. This result can be used in $(1.3.1.4)$ to produce

$$\tilde{Y}(t_n^+) = \tilde{Y}_n + h_n w_0 f(t_n,0, \tilde{Y}(t_n)) - h_n \sum_{j=1}^{m} w_0 L_j(c_0) f(t_n,j, \tilde{Y}(t_n,j)). \quad (1.3.1.5)$$

The identity $(1.3.1.3)$ allows us to write

$$\tilde{Y}(t_n,i) = \tilde{Y}(t_n^+) + h_n \sum_{j=1}^{m} \beta_j(c_i) f(t_n,j, \tilde{Y}(t_n,j)), \quad (1.3.1.6)$$

with

$$\beta_j(v) := \int_{0}^{v} L_j(s)ds \quad (j = 1, \ldots, m),$$

and $\beta_j(c_i) =: a_{i,j}$. Hence, setting $\tilde{Y}_{n,i} := \tilde{Y}(t_n,i)$ and recalling $(1.3.1.5)$ we obtain

$$\tilde{Y}_{n,i} = \tilde{Y}_n + h_n w_0 f(t_n,0, \tilde{Y}(t_n^+)) + h_n \sum_{j=1}^{m} [a_{i,j} - w_0 L_j(c_0)] f(t_n,j, \tilde{Y}_{n,j}) \quad (1.3.1.7)$$
The equations (1.3.1.5) and (1.3.1.7) form a system of \( m+1 \) nonlinear algebraic equations for \( \tilde{Y}_n := (\tilde{Y}(t_n^+), \tilde{Y}_{n,1}, \ldots, \tilde{Y}_{n,m})^T \in \mathbb{R}^{m+1} \): its form closely resembles the one corresponding to collocation at the points \( \{t_{n,0}, t_{n,1}, \ldots, t_{n,m}\} \).

We now show that these equations may indeed be interpreted as the stage equations of an implicit \( (m+1) \)-stage Runge-Kutta method. Let \( b_j := \beta_j(1) \ (j = 1, \ldots, m) \), and observe that

\[
\int_0^1 L_j(s)ds = \sum_{k=0}^m w_k L_j(c_k) = w_0 L_j(c_0) + w_j,
\]

because our interpolatory \( (m+1) \)-point quadrature formula is exact for polynomials of degree not exceeding \( m \). This leads to the relationship

\[
b_j - w_0 L_j(c_0) = w_j,
\]

and hence by (1.3.1.7) to

\[
\tilde{Y}_{n+1} := \tilde{Y}(t_{n+1}^-) = \tilde{Y}_n + h_n \sum_{j=0}^m w_j f(t_{n,j}, \tilde{Y}_{n,j}).
\]

We conclude that (1.3.1.7) together with (1.3.1.5) and (1.3.1.8) represents a \textit{collocation-based} \( (m+1) \)-stage \textit{implicit Runge-Kutta method} for (1.1.1.1). We summarize the above discussion as the following theorem.

**Theorem 1.3.1.1.** The fully discretized DG scheme (1.3.1.2) may lead to the \textit{collocation-based} \( (m+1) \)-stage \textit{implicit Runge-Kutta method} \{(1.3.1.7), (1.3.1.5), (1.3.1.8)\} for (1.1.1.1).

### 1.4 History of the DG methods for ODEs

In 1974, Lesaint and Raviart [82] gave the first analysis of the discontinuous Galerkin method when applied to ordinary differential equations. They showed that the
method is strongly A-stable and has order $2m + 1$ at the mesh points, and that the Gauss-Radau discretization of the DG method is also of order $2m + 1$ when piecewise polynomials of degree $m$ are used.

In 1981, Delfour, Hager, and Trochu [41] introduced a class of DG methods, the so called $\alpha$-methods, for which they proved that the global $L^2$-convergence and nodal convergence rates are $m + 1$ and $2m + 1$. It is interesting that in 1986 Delfour and Dubeau [40] (refer to Remark 1.2.1.1 in Section 1.2.1) considered the discontinuous Galerkin method based on the mesh-dependent variational framework, which includes the "completely discontinuous" Galerkin methods, the $\alpha$-methods, the continuous Galerkin methods, one-step methods of the Runge-Kutta type, hybrid and multi-step methods as special cases. It is shown that the convergence rate in the $L^2$-norm is $m + 1$. The nodal-convergence rate can go up to $2m + 2$, depending on the particular scheme under consideration.

In 1988, Johnson [77] gave an analysis of error control for the DG method for stiff ODEs and later in 1995, Estep [43] extended this analysis to general non-autonomous ODEs.

Recently Schötzau and Schwab [96] analyzed the $hp$-version of the discontinuous Galerkin methods. New a priori error bounds explicit in the time steps and in the approximation orders are derived and it is proven that the DG method gives spectral and exponential accuracy for problems with smooth and analytic time dependence, respectively. It is further shown that temporal singularities can be resolved at exponential rates of convergence if geometrically refined time steps are employed.

The readers may wish also to consult the 2000 survey paper [31] by Cockburn et al. for more applications of discontinuous Galerkin methods and for an extensive list of references.
Chapter 2

The discontinuous Galerkin method for VIDEs

The discontinuous Galerkin method for Volterra integral equations was first studied by Shaw and Whiteman [99] in 1996 extending the approach of [77] and [45]. In [99], they studied the discontinuous Galerkin method with a posteriori $L_p([0, t_l])$ error estimate for linear second-kind Volterra equations (compare [100]). Later in 1998 Larsson, Thomée, and Wahlbin [80] analyzed the discontinuous Galerkin method for linear parabolic integro-differential equations. Recently Brunner and Schözau [27] studied the $hp$-version of discontinuous Galerkin methods for parabolic Volterra integro-differential equations with weakly singular kernels.

2.1 The discontinuous Galerkin method for linear VIDEs

In this section, we consider the a priori error estimates, a posteriori estimates and superconvergence of DG method for linear Volterra integro-differential equations.
2.1.1 A priori error estimates for linear VIDEs

We study the scalar linear Volterra integro-differential equation,

\[
\begin{aligned}
    \begin{cases}
        y'(t) + a(t)y(t) = \mathcal{V}(y)(t), & t \in I = [0, T], \\
        y(0) = y_0,
    \end{cases}
\end{aligned}
\tag{2.1.1.1}
\]

where \( \mathcal{V}(y)(t) := \int_0^t k(t - s)y(s)ds \) and \( a, k \in C[0, T] \).

We use the notations introduced in Section 1.1. We define the finite-dimensional space (cf. (1.1.1.2))

\[
\mathcal{V}_N^{(m)} := \{ \varphi \in L^2(I) : \varphi|_{I_n} \in P^{(m)}(I_n), \ 1 \leq n \leq N \},
\tag{2.1.1.2}
\]

where \( P^{(m)}(I_n) \) denotes the space of all (real) polynomials of degree not exceeding \( m \). Then the DG method for (2.1.1.1) is: Find \( Y \in \mathcal{V}_N^{(m)} \) such that

\[
B_{DG}(Y, X) = Y_0^-X_0^+, \quad \forall X \in \mathcal{V}_N^{(m)},
\tag{2.1.1.3}
\]

where \( Y_0^- = y_0 \) and

\[
B_{DG}(Y, X) := \sum_{n=1}^M \int_{I_n} (y'(t) + a(t)y(t) - \mathcal{V}(y)(t))X(t)dt
\]
\[
+ \sum_{n=2}^M [Y]_{n-1}^+X_{n-1}^+ + Y_0^+X_0^+.
\tag{2.1.1.4}
\]

Note that the exact solution \( y \) of (2.1.1.1) satisfies

\[
B_{DG}(y, X) = y_0X_0^+, \quad \forall X \in \mathcal{V}_N^{(m)},
\]

hence the Galerkin orthogonality property

\[
B_{DG}(y - Y, X) = 0, \quad \forall X \in \mathcal{V}_N^{(m)},
\tag{2.1.1.5}
\]

holds true. We remark also that the DG method in (2.1.1.3) can be interpreted as a time-stepping scheme. For \( n = 1, \ldots, N \), find \( Y|_{I_n} \in P^{(m)}(I_n) \), such that

\[
\int_{I_n} (y'(t) + a(t)y(t) - \mathcal{V}(y)(t))Xdt + Y_{n-1}^+X_{n-1}^+ = Y_{n-1}^-X_{n-1}^+,
\tag{2.1.1.6}
\]
for all \( X \in \mathcal{P}(m)(I_n) \). Here we set \( Y_0^- := y_0 \). We will refer this method as DG(m).

When adopting the Picard iteration technique, we can easily prove that (2.1.1.3) has a unique solution.

**Theorem 2.1.1.1.** Assume \( A := ||a||_I, \ B := ||k||_I \). Then there is a constant \( C \), independent of \( h_n \), such that for \( 1 \leq n \leq M \leq N \) the error of DG(m) for (2.1.1.1) satisfies

\[
||e||_{[0,t_M]} \leq C \max_{n \leq M} h_n^{m+1} ||y^{(m+1)}||_{I_n},
\]

with \( C := C(t_M, A, B) \) and \( m = 0, 1 \).

**Proof.** If \( V \in \mathcal{V}_N^{(m)} \) is determined by \( V_0^- = y_0 \) and, for \( 1 \leq n \leq M \leq N \), by

\[
\int_{I_n} V'Xdt + \int_{I_n} \{a(t)y(t) - \mathcal{V}(y)(t)\}Xdt + V_{n-1}^+X_{n-1}^+ = V_{n-1}^-X_{n-1}^+,
\]

for all \( X \in \mathcal{V}_N^{(m)} \) \( (m = 0, 1) \), then

\[
||y - V||_{I_n} \leq C h_n^{m+1} ||y^{(m+1)}||_{I_n},
\]

We now compare \( Y \) to \( V \). Setting \( \mu := y - V \) and \( \phi := Y - V \in \mathcal{V}_N^{(m)} \), we have \( e = \mu - \phi \). We subtract (2.1.1.7) from (2.1.1.6) and get

\[
\int_{I_n} \phi'Xdt + \int_{I_n} \{-a(t)eX - \mathcal{V}(e)(t)X\}dt + \phi_{n-1}^+X_{n-1}^+ = \phi_{n-1}^-X_{n-1}^+,
\]

for all \( X \in \mathcal{V}_N^{(m)} \). We choose \( X = \phi \) to obtain

\[
\frac{1}{2} |\phi_{n-1}^+|^2 + \frac{1}{2} |\phi_n^-|^2 + \int_{I_n} \{-a(t)e\phi - \mathcal{V}(e)(t)\phi(t)\}dt = \phi_{n-1}^-\phi_{n-1}^+.
\]

Hence

\[
\frac{1}{2} |\phi_n^-|^2 \leq \frac{1}{2} |\phi_{n-1}^-|^2 + \int_{I_n} \{|a(t)e\phi - \mathcal{V}(e)(t)\phi(t)\}|dt.
\]
We now substitute $e = \mu - \phi$ into (2.1.1.11) and find that

\[
\left| \int_{I_n} a(t)e\phi dt \right| = \left| \int_{I_n} a(t)(\phi - \mu)\phi dt \right|
\]

\[
= \left| \int_{I_n} a(t)\phi^2 - \int_{I_n} a(t)\mu\phi dt \right|
\]

\[
\leq A \int_{I_n} \phi^2 dt + \frac{A}{2} \int_{I_n} (\mu^2 + \phi^2) dt
\]

\[
= \frac{A}{2} \int_{I_n} \mu^2 dt + \frac{3A}{2} \int_{I_n} \phi^2 dt. \quad (2.1.1.12)
\]

\[
\left| \int_{I_n} \mathcal{V}(e)(t)\phi(t) dt \right| \leq B \int_{I_n} \phi^2 dt + \frac{1}{B} \int_{I_n} (\mathcal{V}(e)(t_{n-1}))^2 dt
\]

\[
+ \frac{1}{B} \int_{I_n} \left( \int_{t_{n-1}}^t k(t-s)e(s)ds \right)^2 dt
\]

\[
\leq B \int_{I_n} \phi^2 dt + \frac{1}{B} \int_{I_n} \left( \int_{I_n} k^2(t)dt \right) \cdot \left( \int_{I_n} e^2 dt \right) dt
\]

\[
+ \frac{1}{B} \int_{I_n} \left( \int_{0}^{t_{n-1}} k^2(t-s)ds \cdot \int_{0}^{t_{n-1}} e^2(s)ds \right) dt
\]

\[
\leq B \int_{I_n} \phi^2 dt + Bh_n^2 \int_{I_n} e^2(t)dt + Bh_{n-1}n \int_{0}^{t_{n-1}} e^2(t)dt
\]

\[
\leq (B + 2Bh_n^2) \int_{I_n} \phi^2 dt + 2Bh_n^2 \int_{I_n} \mu^2 dt
\]

\[
+ Bh_{n-1}n \int_{0}^{t_{n-1}} e^2(t)dt. \quad (2.1.1.13)
\]

Combining (2.1.1.11), (2.1.1.12), and (2.1.1.13), we obtain

\[
|\phi_n^-|^2 \leq |\phi_{n-1}^-|^2 + (A + 4Bh_n^2) \int_{I_n} \mu^2 dt
\]

\[
+ (3A + 4Bh_n^2 + 2B) \int_{I_n} \phi^2 dt + 2Bh_{n-1}n \int_{0}^{t_{n-1}} e^2(t)dt. \quad (2.1.1.14)
\]
Next we choose $X = (t - t_{n-1})\phi'$ in (2.1.1.9) to obtain

$$\int_{I_n} (t - t_{n-1})(\phi')^2 dt \leq \int_{I_n} (a(t))^2 e^2 \cdot (t - t_{n-1}) dt + \frac{1}{4} \int_{I_n} (t - t_{n-1})(\phi')^2 dt$$

$$+ \int_{I_n} (\mathcal{V}(e)(t))^2 (t - t_{n-1}) dt + \frac{1}{4} \int_{I_n} (t - t_{n-1})(\phi')^2 dt$$

$$\leq A^2 h_n \int_{I_n} e^2 dt + \frac{1}{2} \int_{I_n} (t - t_{n-1})(\phi')^2 dt$$

$$+ 2h_n^3 B^2 \int_{I_n} e^2 dt + 2h_n^2 B^2 t_{n-1} \int_0^{t_{n-1}} e^2 dt$$

$$\leq 2 (A^2 h_n + 2h_n^3 B^2) \int_{I_n} (\mu^2 + \phi^2) dt$$

$$+ \frac{1}{2} \int_{I_n} (t - t_{n-1})(\phi')^2 dt + 2h_n^2 B^2 t_{n-1} \int_0^{t_{n-1}} e^2 dt.$$ 

Since $\int_{I_n} |\phi|^2 dt \leq 2h_n |\phi^n_+|^2 + \frac{2}{3} h_n^3 ||\phi'||^2_{I_n}$, we find that

$$\int_{I_n} |\phi|^2 dt \leq 4h_n |\phi^n_+|^2 + \int_{I_n} \mu^2 dt + \frac{32}{3} h_n^3 B^2 t_{n-1} \int_0^{t_{n-1}} e^2 dt,$$  \hspace{1cm} (2.1.1.15)

provided that $\frac{16}{3} h_n^2 (A^2 + 2h_n^2 B^2) \leq \frac{1}{2}$. 

By combining (2.1.1.14) and (2.1.1.15), we see that

$$|\phi^n_-|^2 \leq 2 |\phi^n_{-1}| + (8A + 4B) \int_{I_n} \mu^2 dt$$

$$+ 2(2Bt_{n-1} h_n + \frac{4}{3} h_n^2 B^2 t_{n-1}) \int_0^{t_{n-1}} e^2 dt,$$ \hspace{1cm} (2.1.1.16)

provided that $4h_n(3A + 4Bh_n^2 + 2B) \leq \frac{1}{2}$. It thus follows that

$$|\phi^n_M|^2 \leq C \left( \int_0^{t_{M-1}} B \cdot e^2 dt + [(A + B)t_M] ||\mu||^2_{[0,t_M]} \right).$$ \hspace{1cm} (2.1.1.17)

The estimate of Theorem 2.1.1.1 is obtained from (2.1.1.17) directly when $m = 0$. 

For $m = 1$, we use (2.1.1.10) and the fact that $\phi \in \mathcal{V}(m)$ implies that

$$||\phi||^2_{I_n} \leq |\phi^n_{-1}|^2 + |\phi^n_-|^2,$$
and this yields

\[
\frac{1}{4}||\phi||_{I_n}^2 \leq |\phi_{n-1}|^2 + \frac{A + 2 h_n^2 B}{2} \int_{I_n} \mu^2 dt + \frac{(3 A + 2 h_n^2 B + 2 B)}{2} \int_{I_n} \phi^2 dt
\]

\[+ \frac{1}{2} B^2 t_{n-1} h_n \int_0^{t_{n-1}} e^2 dt.\]

An analogous argument now leads to the estimate when \( m = 1 \).

2.1.2 A posteriori error estimates for linear VIDEs

We analyze the a posteriori error bound for DG(m) approximation to (2.1.1.1), by using the stability of the continuous dual problem associated with (2.1.1.1).

**Theorem 2.1.2.1.** Assume that \( A := ||a||_I, B := ||k||_I \), and let \( m = 0, 1 \). Then the DG(m) finite element solution \( Y \) for (2.1.1.1) satisfies, for \( 0 < M < N \),

\[ |y(t_M) - Y_M| \leq C||h_{n+1}^m R(Y)||_{[0,t_M]}, \]

where \( C := C(t_M, A, B) \) and \( R(Y) := ||Y|_{n-1}^I + |a(t)Y - V(Y)(t)| \) (\( t \in I_n \)).

**Proof.** We study the dual problem of (2.1.1.1),

\[
\begin{align*}
- \Delta' + a(t)z &= V^*(z)(t), \\
\quad z(t_M) &= e_M^*,
\end{align*}
\]

(2.1.2.1)

where \( V^*(z)(t) := \int_t^{t_M} k(s - t)z(s)ds \). From the definition of \( B_{DG} \) in (2.1.1.4), we find that for all piecewise continuous functions, \( x, z \in C(I) \), the exact solution of (2.1.2.1) satisfies

\[ B_{DG}(x, z) = (x_M^*, e_M^*). \]

(2.1.2.2)

If we choose \( x = e \) in (2.1.2.2) we obtain,

\[ |e_M^*|^2 = B_{DG}(e, z), \]

(2.1.2.3)
and we know from the Galerkin orthogonality (2.1.1.5) that

$$|e_{m}^{2}|^{2} = B_{DG}(e, z - X), \ \forall X \in V_{N}^{(m)}.$$  \hspace{1cm}(2.1.2.4)

Now we define for $X \in V_{N}^{(m)}$ $(m = 0, 1)$ and $1 \leq n \leq M \leq N$,

$$R_{m}(X, M; n) := S(M) (h_{n}^{m} |[X]_{n-1}| + h_{n}^{m+1} ||a(t)X - V(t)(t)||_{I_{n}}) \hspace{1cm}(2.1.2.5)$$

where $S(M) = \int_{0}^{t_{M}} |z^{(m+1)}| \, dt$, and $z$ is the exact solution of (2.1.2.1).

We now show that

$$|e_{m}^{2}|^{2} = |B_{DG}(e, z - X)| \leq \max_{n \leq M} R_{m}(Y, M; n). \hspace{1cm}(2.1.2.6)$$

Because of (2.1.1.1), and since $[y]_{n}^{0} = 0$ for all $n$, $Y_{0}^{-} = y_{0}$, and by the definition of $B_{DG}$ in (2.1.1.3), we have

$$B_{DG}(e, z - X) = - \sum_{n=1}^{M} \int_{I_{n}} \{(Y' + a(t)Y)(z - X)$$

$$- (V(Y)(t))(z - X)) \, dt - \sum_{n=1}^{M} [Y]_{n-1}(z - X)_{n-1}^{+}.\hspace{1cm}$$

Hence,

$$|B_{DG}(e, z - X)| \leq \left| \sum_{n=1}^{M} \int_{I_{n}} Y'(z - X) \, dt \right|$$

$$+ \left| \sum_{n=1}^{M} \int_{I_{n}} (a(t)Y - V(Y)(t))(z - X) \, dt \right|$$

$$+ \left| \sum_{n=1}^{M} [Y]_{n-1}(z - X)_{n-1}^{+} \right|$$

$$:= I + II + III.$$

We set $X := P(z)$, where $P(z)$ denotes the projection of $z$ onto $V_{N}^{(m)}$ and satisfies,

$$X_{n}^{-} = z(t_{n}) \text{ for } m = 0, 1; \text{ and } \int_{I_{n}} (X - z) \, dt = 0 \text{ for } m = 1. \hspace{1cm}(2.1.2.7)$$
Furthermore, we know that the estimates

$$\|z - P_0(z)\|_{I_n} \leq h_n^m \int_{I_n} |z^{(m+1)}| \, dt \quad \text{and} \quad \|z - P_0(z)\|_{I_n} \leq \max_{n \leq M} h_n^{(m+1)} \|z^{(m+1)}\|_{I_n},$$

(2.1.2.8)

hold for \( m = 0, 1 \). Thus, \( I = 0 \) and

$$II \leq \sum_{n=1}^{M} h_n \|a(t)Y - \mathcal{V}(Y)(t)\|_{I_n} \|z - P(z)\|_{I_n}$$

$$\leq \max_{n \leq M} h_n \|a(t)Y - \mathcal{V}(Y)(t)\|_{I_n} \sum_{n=1}^{M} h_n^m \int_{I_n} |z^{(m+1)}| \, dt$$

$$=: S(M) \max_{n \leq M} h_n^{(m+1)} \|a(t)Y - \mathcal{V}(Y)(t)\|_{I_n}.$$

The same kind of argument shows for \( m = 0 \) and 1,

$$III \leq S(M) \max_{n \leq M} h_n^m \|[Y]_{n-1}||.$$

Hence, (2.1.2.6) holds true.

Now we prove the stability of (2.1.2.1).

**Lemma 2.1.2.1.** If \( A := \|a\|_I \) and \( B := \|k\|_I \), then the solution \( z \) of (2.1.2.1) satisfies

$$S(M) := \int_0^t |z^{(m+1)}| \, dt \leq C |e_M|,$$

where \( C := C(t_M, A, B) \).

**Proof.** Taking \( t = t_M - s \) in (2.1.2.1) and setting \( \psi(s) = z(t_M - s) \), (2.1.2.1) can be rewritten as

$$\begin{cases}
\psi'(s) + a(t_M - s)\psi(s) = \int_0^s k(t - \nu)\psi(\nu)\,d\nu \\
\psi(0) = e_{-M}.
\end{cases} \quad (2.1.2.9)$$

Dirichlet's formula applied to (2.1.2.9) yields

$$\psi(s) = \psi(0) + \int_0^s Q(s, \nu)\psi(\nu)\,d\nu, \quad (2.1.2.10)$$
where $Q(s, \nu) := a(t_M - \nu) + \int_0^\infty k(\tau - \nu) d\tau$. Thus,

$$|\psi(s)| \leq |\psi(0)| + \int_0^s |Q(s, \nu)||\psi(\nu)| d\nu.$$  

It follows from the well-known Gronwall’s lemma [26] that

$$|\psi(s)| \leq |\psi(0)| + \int_0^s |Q(s, \nu)| \exp \left( \int_0^s |Q(\tau, \tau)| d\tau \right) |\psi(0)| d\nu,$$

and so

$$\int_0^{t_M} |z'(t)| dt = \int_0^{t_M} |\psi'(s)| ds$$

$$\leq \int_0^{t_M} \left\{ |a(t_M - s)||\psi(s)| + \int_0^s |k(\nu - s)||\psi(\nu)| d\nu \right\} ds$$

$$\leq |\psi(0)| \int_0^{t_M} \left\{ |a(t_M - s)| \left( 1 + \int_0^s |Q(s, \nu)| \exp \left( \int_0^\infty |Q(\tau, \tau)| d\tau \right) d\nu \right) \right\} ds$$

$$+ \int_0^s |k(\nu - s)| \left( 1 + \int_0^\infty |Q(\nu, \mu)| \exp \left( \int_\nu^\infty |Q(\tau, \tau)| d\tau \right) d\mu \right) d\nu \right\} ds$$

$$= C|\psi(0)|,$$

where $C := C(t_M, A, B)$. The similar argument can lead to

$$\int_0^{t_M} |z''(t)| dt \leq C|\psi(0)|.$$  

The proof of the lemma has been completed.

From (2.1.2.6), (2.1.2.7), (2.1.2.8), and Lemma 2.1.2.1, we obtain

$$|e_n| \leq \frac{S(M)}{|e_M|} \max_{n \leq M} \left\{ h_n^m ||Y||_{n-1} + h_n^{m+1} ||Y(t) - \mathcal{V}(Y)(t)||_{I_n} \right\}$$

$$\leq C h_n^{m+1} \max_{n \leq M} \left\{ \frac{||Y||_{n-1}}{h_n} + \frac{||a(t)Y - \mathcal{V}(Y)(t)||_{I_n}}{h_n} \right\}. \quad (2.1.2.11)$$

This concludes the proof of Theorem 2.1.2.1.
2.1.3 Efficiency of the a posteriori error estimator

If its upper bound is large, the a posteriori error estimator $||h^{m+1}_n R(Y)||_{[0,t_M]}$ in
Theorem 2.1.2.1 cannot efficiently indicate the error. Therefore we need to derive
a sharper upper bound of the a posteriori error estimate. This estimate is called
"efficiency of a posteriori error estimator" (see also Ainsworth and Oden [2]).

**Theorem 2.1.3.1.** Under the assumptions of Theorem 2.1.2.1 and Theorem 2.1.1.1,
we have

$$|y(t_M) - Y_{M^-}| \leq C||h^{m+1}_n R(Y)||_{[0,t_M]} \leq Ch^{m+1} (||y'(t)||_{[0,t_M]} + ||y^{(m+1)}(t)||_{[0,t_M]})$$

where $C := C(t_M, A, B)$ is independent of the mesh size $h$.

**Proof.** We only need to bound the term $||h^{m+1}_n R(Y)||_{[0,t_M]}$ in Theorem 2.1.2.1:

$$||h^{m+1}_n R(Y)||_{[0,t_M]} \leq ||h^{m}_n [Y]_{n-1}||_{[0,t_M]} + h^{m+1}_n ||a(t)Y - V(Y)(t)||_{[0,t_M]}$$

$$= ||h^{m}_n (Y^{n+}_{n-1} - y + y - Y^{n-}_{n-1})||_{[0,t_M]} + ||h^{m+1}_n (-y'(t) - a(t)y(t))$$

$$+ V(y)(t) + a(t)Y - V(Y)(t)||_{[0,t_M]}$$

$$\leq 2||h^{m}_n e||_{[0,t_M]} + ||h^{m+1}_n y'(t)||_{[0,t_M]}$$

$$+ ||h^{m+1}_n (-a(t)e(t) + V(e)(t))||_{[0,t_M]}$$

$$\leq 2||h^{m}_n e||_{[0,t_M]} + ||h^{m+1}_n y'(t)||_{[0,t_M]}$$

$$+ \mathcal{A}||h^{m+1}_n e||_{[0,t_M]} + t_M B||h^{m+1}_n e||_{[0,t_M]}.$$
2.2 The discontinuous Galerkin method for semilinear VIDEs

We now extend a priori and a posteriori error estimates of Section 2.1 to semilinear VIDEs.

2.2.1 A priori error estimates for semilinear VIDEs

We study the scalar semilinear Volterra integro-differential equation

\[
\begin{aligned}
\{ & y'(t) + a(t)y(t) = \nu_G(y)(t), \quad t \in I = [0, T] \\
& y(0) = y_0,
\end{aligned}
\]  

(2.2.1.1)

where \( \nu_G(y)(t) := \int_0^t k(t - s)G(y(s))ds \), and \( a, k \in C(I) \). Furthermore assume \( G \) satisfies

\[
|G(y_1) - G(y_2)| \leq L|y_1 - y_2|,
\]

(2.2.1.2)

for all \( y_1, y_2 \in \Omega \subset \mathbb{R} \). We begin with the definition of the DG(m) scheme to (2.2.1.1): Find \( Y \in \mathcal{V}_N^{(m)} \) such that

\[
B_{DG}(Y, X) = F_{DG}(X), \quad \forall X \in \mathcal{V}_N^{(m)},
\]

(2.2.1.3)

where

\[
B_{DG}(Y, X) := \sum_{n=1}^{M} \int_{I_n} \{Y'(t)X(t) + a(t)Y(t)X(t)
- (\nu_G(Y)(t))X(t)\}dt
+ \sum_{n=1}^{M-1} [Y]_nX^+_n + Y^+_0X^+_0,
\]

(2.2.1.4)

\[
F_{DG}(X) := y_0X^+_0.
\]

(2.2.1.5)

We note that

\[
B_{DG}(Y, X) - B_{DG}(y, X) = 0, \quad \forall X \in \mathcal{V}_N^{(m)}.
\]

(2.2.1.6)
The DG method (2.2.1.3) can again be interpreted as a time-stepping scheme: Find 
\( Y_{\mid I_n} \in P^{(m)}(I_n) \), \( n = 1, \ldots, M \), such that,

\[
\int_{I_n} \{ Y'(t)X(t) + a(t)Y(t)X(t) - \left( \int_{t_{n-1}}^t k(t-s)G(Y)ds \right) X(t) \} dt + Y_{n-1}^+X_{n-1}^+ = Y_n^-X_{n-1}^- + \int_{I_n} \left( \int_{t_{n-1}}^{t_{n-1}} k(t-s)G(Y)ds \right) X(t)dt
\]  
(2.2.1.7)

**Theorem 2.2.1.1.** Suppose that \( A := ||a||_I \), \( B := ||k||_I \). Then there is a constant \( C \), independent of \( h_n \), such that for \( 1 \leq n \leq M \leq N \), the error of \( DG(m) \) for (2.2.1.1) satisfies

\[
||e||_{[0,t_M]} \leq C \max_{n \leq M} h_n^{m+1} ||y^{(m+1)}||_{I_n},
\]

with \( C := C(t_M, L, A, B) \) and \( m = 0, 1 \).

**Proof.** If \( V \in V^{(m)}_N \) is determined by \( V_0^- = y_0 \) and by

\[
\int_{I_n} V'Xdt + \int_{I_n} \{ a(t)y(t) - V_G(Y)(t) \} X dt + V_{n-1}^+X_{n-1}^+ = V_{n-1}^-X_{n-1}^-,
\]  
(2.2.1.8)

for all \( X \in V^{(m)}_N \) (\( m = 0, 1 \)) and for \( 1 \leq n \leq M \leq N \), then it follows from the definition of the interpolant (2.1.1.8) that

\[
||y - V||_{I_n} \leq Ch_n^{m+1} ||y^{(m+1)}||_{I_n}.
\]  
(2.2.1.9)

Setting \( \mu := y - V \) and \( \phi := y - V \in V^{(m)}_N \), we have \( e = \mu - \phi \). We subtract (2.2.1.8) from (2.2.1.3) and obtain

\[
\int_{I_n} \phi'Xdt + \int_{I_n} \{ a(t)eX - \int_{t_{n-1}}^t k(t-s) (G(Y) - G(y)) dsX \} dt + \phi_{n-1}^+X_{n-1}^+ = \phi_{n-1}^-X_{n-1}^-,
\]

for all \( X \in V^{(m)}_N \). The remaining steps follow exactly those of Theorem 2.1.1.1 after we use the Lipschitz condition (2.6.1.2).
2.2.2 A posteriori error estimates for semilinear VIDEs

In this section, we derive the a posteriori estimates of the DG(m) in the mesh-point sense and the general a global posteriori error estimates of DG(0) for (2.2.1.1).

**Theorem 2.2.2.1.** Assume that $A := ||a||_I$, $B := ||k||_I$, and the function $G$ satisfies

$$|G_y(u)| \leq L, \quad \forall u \in \Omega.$$

Then the error of the DG(m) approximation to (2.2.1.1) satisfies

$$|y(t_M) - Y_M^m| \leq C||h^{m+1}R(Y)||_{[0,t_M]},$$

with $m = 0$ and 1, $C := C(t_M, A, B, L)$ and

$$R(Y) = \frac{[[Y]_{n-1}]}{h_n} + |a(t)Y - \int_0^t k(t - s)G(Y(s))ds|.$$

**Proof.** Recall (2.2.1.6):

$$B_{DG}(Y, X) - B_{DG}(y, X) = 0, \quad \forall X \in \mathcal{V}_N^{(m)}.$$

We write this as

$$\tilde{D}(e, X) = 0, \quad \forall X \in \mathcal{V}_N^{(m)}' \quad (2.2.2.1)$$

where

$$\tilde{D}(W, X) := \sum_{n=1}^{M} \int_{I_n} \{W'X + a(t)WX$$

$$- \int_0^t k(t - s) \int_0^1 G_1 (\tau Y + (1 - \tau)Y) d\tau W(s)$$

$$ds X(t)\} dt$$

$$+ \sum_{n=2}^{M} [W]_{n-1} X_{n-1}^+ + W_0^+ X_0^+. \quad (2.2.2.2)$$
We consider the linearized dual problem of (2.2.1.1):
\[
\begin{cases}
-z' + a(t)z = \int_t^{t_M} k(s-t)\bar{A}(s)z(s)ds, & t_M > t > 0, \\
z(t_M) = e_M,
\end{cases}
\tag{2.2.2.3}
\]
where \(\bar{A}(s) := \int_0^1 G_1 (ry + (1-r)Y) \, dr\). We note that, for any piecewise continuous function \(x\),
\[
\bar{D}(x, z) = x_M e_M.
\tag{2.2.2.4}
\]
Selecting \(x = e\) in (2.2.2.4), we have
\[
[e_M^-]^2 = \bar{D}(e, z).
\tag{2.2.2.5}
\]
In view of (2.2.2.1), we obtain
\[
[e_M^-]^2 = \bar{D}(e, z - X), \quad \forall X \in \mathcal{V}_N^{(m)}, \quad m = 0, 1.
\]
Similarly to the proof of Theorem 2.1.2.1, we can continue the analysis, to find
\[
\bar{D}(e, z - X) := - \sum_{n=1}^{M} \int_{I_n} \{(Y' + a(t)Y)(z - X) - (\mathcal{V}_G(Y)(t)) \cdot (z - X)\} dt - \sum_{n=1}^{M} [Y]_{n-1}(z - X)_{n-1}^+.
\]
Hence,
\[
|\bar{D}(e, z - X)| \leq \left| \sum_{n=1}^{M} \int_{I_n} Y'(z - X) dt \right|
+ \left| \sum_{n=1}^{M} \int_{I_n} (a(t)Y - \mathcal{V}_G(Y)(t)) \cdot (z - X) dt \right|
+ \left| \sum_{n=1}^{M} [Y]_{n-1}(z - X)_{n-1}^+ \right|
=: I + II + III
\leq S(M) \max_{n \leq M} \{ h_{n}^m [Y]_{n-1}^+ 
+ h_{n}^{m+1} |a(t)Y - \mathcal{V}_G(Y)(t)| \},
\tag{2.2.2.6}
\]
where $S(M)$ can be easily estimated as in Lemma 2.1.2.1,

\[ S_1(M) := \int_0^{t_M} |z^{(m+1)}| \, dt \leq C |e^-_M|. \]  

(2.2.2.7)

Here $C := C(t_M, A, B, L)$. Combining (2.2.2.5), (2.2.2.6), and (2.2.2.7), we finish the proof.

**Theorem 2.2.2.2.** Assume that $A := ||a||_I$, $\bar{B} := \int_0^{t_M} |k(t)| \, dt$, and $G$ satisfies

\[ \ell|u - v| \leq |G(u) - G(v)| \leq L|u - v|; \quad |G_1(u) - G_1(v)| \leq \bar{L}|u - v|, \]  

(2.2.2.8)

for all $u, v \in \Omega$. Then the error of the $DG(0)$ approximation to (2.2.1.1) satisfies

\[ ||e||^2_{[0,t_M]} \leq \max_{n \leq M} C \left(||Y||_{n-1} + h_n |a(t)Y - \nu_G(Y)(t)|\right)^2 \]

\[ + C \left(||Y||_{M-1}\right)^2 + C (a(t)Y + \nu_G(Y)(t))^2. \]

**Proof.** We begin the proof with the related linearized form of (2.2.2.2),

\[
D(W,X) := \sum_{n=1}^M \int_{I_n} \left\{ W'X + a(t)WX - \int_0^t k(t - s)A(s)W(s)dsX(t) \right\} dt \\
+ \sum_{n=2}^M [W]_{n-1}X^+_n + W^+_0X^+_0 \\
= \sum_{n=1}^M \int_{I_n} \left\{ -WX' + a(t)WX - \left( \int_0^t k(t - s)A(s)W(s)ds \right)X(t) \right\} dt \\
+ \sum_{n=2}^M W^-_{n-1}[X]_{n-1} + W^-_nX^-_n, \tag{2.2.2.9}
\]

where $A(s) := G_1(y(s))$.

We consider the linearized dual problem of (2.2.1.1):

\[
\begin{aligned}
-\dot{z} + a(t)z &= \int_t^{t_M} k(s - t)A(s)z(s)ds, \quad t_M > t > 0, \\
z(t_M) &= e^-_M.
\end{aligned} \tag{2.2.2.10}
\]
We note that $Z$, the DG$(m)$ approximation to $z$, solves
\[ D(X, Z) = X^{-M}e^{-M}. \] (2.2.2.11)

We claim also, for any piecewise continuous function $x$,
\[ D(x, z) = x^{-M}e^{-M}. \] (2.2.2.12)

The choice $x = e$ in (2.2.2.12) gives
\[ D(e, z) = |e^{-M}|^2 = \bar{D}(e, z) + (D - \bar{D})(e, z). \]

This, together with (2.2.2.1), yields the error representation formula
\[ |e^{-M}|^2 = \bar{D}(e, z - X) + (D - \bar{D})(e, z), \quad \forall X \in V_N^{(m)}. \] (2.2.2.13)

Now we define for $X \in V_N^{(m)}$ and $1 \leq n \leq M \leq N$,
\[ \mathcal{R}_m(X, M; n) := S(M)(|[X]_{n-1}| + h_n |a(t)X - V_C(X)(t)|), \] (2.2.2.14)

for $m = 0$ and $1$.

Note that by the assumption (2.2.2.8),
\[ |(D - \bar{D})(e, z)| = \left| \int_0^{t_M} \int_0^t k(t - s)A(s)e(s)dsz(t)dt \right| \]
\[ - \int_0^{t_M} \int_0^t k(t - s) \int_0^1 G_1 (ry + (1 - r)Y) dre(s)dsz(t)dt \left| \right| \]
\[ \leq C\bar{L} \int_0^{t_M} \int_0^t |k(t - s)|(e(s))^2 ds|z(t)|dt \]
\[ \leq C\bar{L} \left( \int_0^{t_M} \left( \int_0^t |k(t - s)(e(s))^2 ds \right)^2 dt \right)^{1/2} \left( \int_0^{t_M} z^2(t)dt \right)^{1/2} \]
\[ \leq C\bar{L} \left( \int_0^{t_M} |k(t)|dt \left( \int_0^t e^4(t)dt \right) \right)^{1/2} \left( \int_0^{t_M} z^2(t)dt \right)^{1/2} \]
\[ \leq C\bar{L}\bar{B} \left( \int_0^{t_M} e^2(t)dt \right)^{1/2} \left( \int_0^{t_M} z^2(t)dt \right)^{1/2} \cdot ||e(t)||_{[0,t_M]} \]
We define \( \varphi \) to be the solution of
\[
\begin{cases}
-\varphi' + a(t)\varphi = \int_t^{t_M} k(s-t)A(s)\varphi(s)ds, \ t_M > t > 0, \\
\varphi(t_M) = e_{-M}/|e_{-M}| 
\end{cases}
\]
That is, \( \varphi = z/|e_{-M}| \). So from (2.2.2.15)
\[
\left|(D - \tilde{D})(e, z)\right| \leq C\bar{L}\bar{B} |e_{-M}| \left( \int_0^{t_M} e^2 dt \right)^{1/2} \left( \int_0^{t_M} \varphi^2 dt \right)^{1/2} \|e(t)\|_{[0,t_M]} .
\]
Next, we conclude that there exists a \( X \in \mathcal{V}_N^{(m)} \) such that the first term of (2.2.2.13) satisfies
\[
\left| \tilde{D}(e, z - X) \right| \leq \max_{n \leq M} \mathcal{R}_m(Y, M; n).
\] (2.2.2.17)
Now we will now prove (2.2.2.17). Because of \( y' + a(t)y - \mathcal{V}_G(Y)(t) = 0, \ [y]_n = 0, \) (for all \( n \)) and \( Y_0^- = y_0 \), we find
\[
\tilde{D}(e, z - X) = - \sum_{n=1}^M \int_{I_n} (Y' + a(t)Y - \mathcal{V}_G(Y)(t)) (z - X) dt \\
- \sum_{n=1}^M [Y]_{n-1}(z - X)_{n-1}^+ .
\]
Hence,
\[
\left| \tilde{D}(e, z - X) \right| \\
\leq \left| \sum_{n=1}^M \int_{I_n} Y'(z - X) dt \right| + \left| \sum_{n=1}^M \int_{I_n} [a(t)Y - \mathcal{V}_G(Y)(t)](z - X) dt \right| \\
+ \sum_{n=1}^M |[Y]_{n-1}(z - X)_{n-1}^+| \\
=: I + II + III .
\]
The remaining lines are then the same as in the proof of (2.1.2.6).

Thus, from (2.2.2.13), (2.2.2.15), (2.2.2.17), we derive
\[
|e_{-M}| \leq \frac{\max_{n \leq M} \mathcal{R}_m(Y, M; n)}{|e_{-M}|} \\
+ C\bar{L}\bar{B}\|e\|_{[0,t_M]} \left( \int_0^{t_M} e^2 dt \right)^{1/2} \left( \int_0^{t_M} \varphi^2 dt \right)^{1/2} .
\] (2.2.2.18)
Since $G$ satisfies (2.2.2.8) and $z$ is the solution of (2.2.2.10), following the proof of Lemma 2.1.2.1 leads to

$$S(M) = \int_0^{t_M} |z'| dt \leq C |e_M^-|,$$  \hspace{1cm} (2.2.2.19)

where $C := C(t_M, A, B, L)$. Thus, combining (2.2.2.14), (2.2.2.19), and (2.2.2.19), we obtain

$$\max_{n \leq M} C (|[Y]_{n-1}| + |a(t)Y - V_G(Y)(t)| h_n)$$
$$+ C \bar{L} \bar{B} \|e\|_0 \left( \int_0^{t_M} e^2 dt \right)^{1/2} \left( \int_0^{t_M} \xi^2 dt \right)^{1/2}. \hspace{1cm} (2.2.2.20)$$

To complete the proof of the theorem, we need the following lemma.

Lemma 2.2.2.1. Let $A := \|a\|_I$, $\bar{B} := \int_0^{t_M} |k(t)| dt$, and suppose that $G$ is Lipschitz continuous, i.e.,

$$|G(u) - G(v)| \leq L|u - v|,$$

for all $u, v \in \Omega$. Then the error of $DG(0)$ to (2.2.1.1) satisfies

$$\|e\|^2_{[0,t_M]} \leq C |e_{M-1}^-|^2 + C |[Y]|_{M-1}|^2 + C (a(t)Y + V_G(Y)(t))^2,$$

where $C := C(t_M, L, A, \bar{B})$ is independent of the mesh size $h_n$.

Proof. For $m = 0$, we have the following identity on $I_n$,

$$Y' + a(t)Y - V_G(Y)(t) = a(t)Y - V_G(Y)(t).$$

Subtracting this from (2.2.1.1) leads to

$$e' + a(t)e - \left( \int_0^t k(t - s)(G(y) - G(Y))ds \right) = -a(t)Y + V_G(Y)(t). \hspace{1cm} (2.2.2.21)$$

Consequently, we have

$$e'e + a(t)e^2 - \left( \int_0^t k(t - s)(G(y) - G(Y))ds \right) e = -a(t)Ye + V_G(Y)(t)e,$$
Integrating from $t_{n-1}$ to $t$, we obtain

$$\int_{t_{n-1}}^{t} \frac{1}{2} \frac{d(e^2)}{dt} + \int_{t_{n-1}}^{t} a(t)e^2 dt - \int_{t_{n-1}}^{t} \left( \int_{0}^{t} k(t-s)(G(y)-G(Y))ds \right) e dt$$

$$= \int_{t_{n-1}}^{t} (-a(t))Ye dt + \int_{t_{n-1}}^{t} \mathcal{V}_G(Y)(t)e dt,$$

and hence

$$\frac{1}{2}(e^2) \leq \frac{1}{2}(e^2_{n-1})^2 + \int_{I_n} |a(t)Ye| dt + \int_{I_n} |\mathcal{V}_G(Y)(t) \cdot e| dt$$

$$+ \left( \int_{I_n} \left( \int_{0}^{t} k(t-s)(G(y)-G(Y))ds \right)^2 dt \right)^{1/2}$$

$$\cdot \left( \int_{I_n} e^2 dt \right)^{1/2} + \int_{I_n} |a(t)|e^2 dt$$

$$\leq \frac{1}{2}(e^2_{n-1})^2 + \int_{I_n} |a(t)Ye| dt + \int_{I_n} |\mathcal{V}_G(Y)(t) \cdot e| dt$$

$$+ \sqrt{2} \int_{0}^{t_n} |k(t)| dt \cdot \left( \int_{0}^{t_n} (G(y)-G(Y))^2 dt \right)^{1/2}$$

$$\cdot \left( \int_{I_n} e^2 dt \right)^{1/2} + \int_{I_n} |a(t)|e^2 dt$$

$$\leq \frac{1}{2}(e^2_{n-1})^2 + \int_{I_n} |a(t)Ye| dt + \int_{I_n} |\mathcal{V}_G(Y)(t) \cdot e| dt$$

$$+ \sqrt{2T\bar{B}}L||e||^2_{[0,t_n]}h_n + \int_{I_n} |a(t)|e^2 dt$$

$$\leq |e_{n-1}|^2 + |[Y]_{n-1}|^2 + \frac{h_n}{2} (a(t)Y + \mathcal{V}_G(Y)(t))^2$$

$$+ \frac{h_n}{2} ||e||^2_{I_n} + Ah_n ||e||^2_{[0,t_M]} + \sqrt{2T\bar{B}}Lh_n ||e||^2_{[0,t_M]}.$$  \hspace{1cm} (2.2.2.22)

For sufficiently small $h_n(\frac{1}{2} + A + \sqrt{2T\bar{B}}L)$ we derive that, from (2.2.2.22),

$$||e||^2_{[0,t_M]} \leq C|e_{M-1}|^2 + C|[Y]_{M-1}|^2 + C (a(t)Y + \mathcal{V}_G(Y)(t))^2.$$  \hspace{1cm} (2.2.2.23)

According to Theorem 2.2.2.1 and the procedure of the proof of Lemma 2.1.2.1, we know that, for sufficiently small $h_n$,

$$C\bar{L}\bar{B} \left( \int_{0}^{t_M} e^2 dt \right)^{1/2} \cdot \left( \int_{0}^{t_M} \varphi^2 dt \right)^{1/2} \leq \frac{1}{2}.$$  \hspace{1cm} (2.2.2.24)
Combining (2.2.2.20), Lemma 2.2.2.1, and (2.2.2.24), we arrive at
\[
||e||_{[0,t_M]}^2 \leq \max_{n \leq M} C \left( ||[Y]_{n-1} + h_n |a(t)Y - V_G(Y)(t)| ||^2 + C([Y]_{M-1})^2 + C (a(t)Y + V_G(Y)(t))^2. \right)
\]

2.2.3 Efficiency of the a posteriori error estimator

We shall use the a priori error estimates to deduce the efficiency of the a posteriori estimator described in Theorem 2.2.2.1.

**Theorem 2.2.3.1.** Under the assumptions of Theorem 2.2.2.2 and Theorem 2.2.1.1, we have
\[
|y(t_M) - Y^-_M| \leq C ||h^{m+1} R(Y)||_{[0,t_M]} \leq C h^{m+1} (||y'(t)||_{[0,t_M]} + ||y^{(m+1)}(t)||_{[0,t_M]})
\]
where $C := C(t_M, L, A, B)$ is independent of the mesh size $h$.

**Proof.** We only need to bound the term $||h^{m+1} R(Y)||_{[0,t_M]}$ in Theorem 2.2.2.1:
\[
||h^{m+1} R(Y)||_{[0,t_M]} \leq ||h^{m} [Y]_{n-1}||_{[0,t_M]} + ||h^{m+1}(a(t)Y - V_G(Y)(t))||_{[0,t_M]}
\]
\[
= ||h^{m} (Y_{n-1}^+ - y + y - Y_{n-1}^-)||_{[0,t_M]} + ||h^{m+1}(-y'(t) - a(t)y(t))||_{[0,t_M]}
\]
\[
+ V_G(y)(t) + a(t)Y - V_G(Y)(t)||_{[0,t_M]}
\]
\[
\leq 2 ||h^{m} e(t)||_{[0,t_M]} + ||h^{m+1} y'(t)||_{[0,t_M]} + ||h^{m+1}[-a(t)e(t)]||_{[0,t_M]}
\]
\[
+ \int_0^t k(t - s)(G(y(s)) - G(Y(s)))ds||_{[0,t_M]}
\]
\[
\leq 2h^{m} ||e(t)||_{[0,t_M]} + h^{m+1} ||y'(t)||_{[0,t_M]}
\]
\[
+ (Ah^{m+1} + Lh^{m+1} t_M)B ||e(t)||_{[0,t_M]}
\]

Then we combine this with the result of Theorem 2.2.1.1 and complete the proof.
2.3 Numerical examples

In this section, we compare the accuracy and stability of DG(0), CG(1) and the collocation method using piecewise linear polynomial approximation (denoted by CC(1)) by means of numerical examples. The effect of quadrature on the total error is considered, too.

2.3.1 Example for the case of constant coefficient

Example 2.3.1.1. We consider the linear scalar Volterra integro-differential equation

\[ y' + ay = \int_0^t \exp(-(t-s))y(s)ds, \quad t \in I = [0,1], \quad y(0) = 1, \]  

(2.3.1.1)

where \( a \) is a constant.

The exact solution of (2.3.1.1) is

\[
 y(t) = \exp(-\frac{a+1}{2}t) \cosh \left( \sqrt{1 - a + \frac{(a+1)^2}{4}t} \right) \\
+ \frac{1 - a}{\sqrt{a^2 - 2a + 5}} \exp(-\frac{a+1}{2}t) \sinh \left( \sqrt{1 - a + \frac{(a+1)^2}{4}t} \right) .
\]

(2.3.1.2)

We take uniform meshes: \( \{t_i : t_i = ih, \ i = 0,1,\ldots,n\} \), where \( h \) is the mesh size, and the initial value is \( Y_0^- = y(0) = 1 \).

DG(0):

\[
(ah - h + 2 - \exp(-h))Y_n^- = Y_{n-1}^- \\
+ \sum_{i=1}^{n-1} Y_i^- [\exp(-(t_n - t_i)) + \exp(-(t_n - t_{i-1}))) \\
+ \exp(-(t_{n-1} - t_i)) - \exp(-(t_{n-1} - t_{i-1}))].
\]

(2.3.1.3)
\( CG(1): \)

\[
Y_n - Y_{n-1} + \int_{I_n} a(t)Y(t)dt = \int_{I_n} \int_0^t k(t - s)Y(s)dsdt, \tag{2.3.1.4}
\]

Taking \( Y|_{I_n} = \frac{t-a}{k}Y_{n-1} + \frac{t-a-1}{k}Y_n \) in (2.3.1.4), we obtain the exact \( CG(1) \) and \( CG(1) \) with quadrature scheme for Example 2.3.1.1 as follows.

**Exact CG(1):**

\[
\begin{align*}
(2 - h/2 + ah/2 + \frac{1}{h}\exp (-h) - 1/h)Y_n &= (1 - ah/2 + h/2 + \exp (-h) + \frac{1}{h}\exp (-h) - 1/h)Y_{n-1} \\
+ \sum_{i=1}^{n-1} Y_{i-1}[\exp(-(t_n-t_{i-1})) - \frac{1}{h}\exp(-(t_n-t_i)) + \frac{1}{h}\exp(-(t_n-t_{i-1})) \\
- \exp(-(t_{n-1}-t_{i-1})) + \frac{1}{h}\exp(-(t_{n-1}-t_i)) - \frac{1}{h}\exp(-(t_{n-1}-t_{i-1}))] \\
+ \sum_{i=1}^{n-1} Y_i[\exp(-(t_n-t_i)) + \frac{1}{h}\exp(-(t_n-t_i)) - \frac{1}{h}\exp(-(t_n-t_{i-1})) \\
+ \exp(-(t_{n-1}-t_i)) - \frac{1}{h}\exp(-(t_{n-1}-t_i)) \\
+ \frac{1}{h}\exp(-(t_{n-1}-t_{i-1}))]. \tag{2.3.1.5}
\end{align*}
\]

From [23], we know that the collocation method using piecewise linear polynomial, i.e., \( CC(1) \) to (2.1.1.1), has the form

\[
(1 + ha(t_{n+1}) - h^2 \int_0^1 k(t_{n+1} - (t_n + sh))ds)Y_{n+1} = (1 + h^2 \int_0^1 k(t_{n+1} - (t_n + sh))(1 - s)ds)Y_n + hF_n(t_{n+1}),
\]

where \( F_n(t_{n+1}) = \sum_{i=0}^{n-1} h \int_0^1 k(t_{n+1} - (t_i + sh))[(1 - s)Y_i + sY_{i+1}]ds. \)
Exact CC(1):

\[
(2 + ah - h - \exp(-h))Y_{n+1} = (2 - h \exp(-h) - \exp(-h))Y_n + h \sum_{i=0}^{n-1} \left[ \exp\left(-(t_{n+1} - t_i)\right) \left(\exp(h) - \exp(h)/h + 1/h\right)Y_{i+1} \right.
+ \left. \left[- \exp\left(-(t_{n+1} - t_i)\right) + \frac{1}{h} \exp\left(-(t_{n+1} - t_i)\right)(\exp(h) - 1)\right]Y_i \right].
\] (2.3.1.6)

We take the end-point rule for the inner product and the trapezoidal rule for the memory term, and call it “Quadrature Scheme 1”.

**Quadrature Scheme 1 for DG(0):**

\[
[ah + 1 - h^2/2 - h^2 \exp(-h)/2]Y_n^- = Y_{n-1}^- + \sum_{i=1}^{n-1} Y_i^- h^2 \left[ \exp(-t_n - t_i) + \exp(-(t_n - t_{i-1})) \right].
\] (2.3.1.7)

**Quadrature Scheme 1 for CG(1):**

\[
(1 + ah - h^2/2)Y_n = (1 + h^2/2 \exp(-h))Y_{n-1} + \sum_{i=1}^{n-1} Y_i^- h^2 \exp(-t_n - t_i).
\] (2.3.1.8)

**Quadrature Scheme 1 for CC(1):**

\[
(1 + ah - h^2/2)Y_{n+1} = (1 + h^2 \exp(-h)/2)Y_n + h \sum_{i=0}^{n-1} \frac{h}{2} \exp(-(t_{n+1} - t_i))Y_i
+ h \sum_{i=0}^{n-1} \frac{h}{2} \exp(-(t_{n+1} - t_i - h))Y_{i+1}.
\] (2.3.1.9)

The “Quadrature Scheme 2” is defined by taking the end point rule for the inner product and the mid-point rule for the memory term.

**Quadrature Scheme 2 for DG(0):**

\[
[ah + 1 - h^2 \exp(-h/2)]Y_n^- = Y_{n-1}^- + \sum_{i=1}^{n-1} Y_i^- h^2 \exp(-t_n - \frac{t_i + t_{i-1}}{2})).
\] (2.3.1.10)
Quadrature Scheme 2 for CG(1):

\[
[1 + ah - h^2 \exp(-h/2)/2] Y_n \\
= [h^2 \exp(-h/2)/2 + 1] Y_{n-1} + \frac{h^2}{2} \sum_{i=1}^{n-1} Y_{i-1} \exp\left(-\left(t_n - \frac{t_i + t_{i-1}}{2}\right)\right) \\
+ \frac{h^2}{2} \sum_{i=1}^{n-1} Y_i \exp\left(-\left(t_n - \frac{t_i + t_{i-1}}{2}\right)\right). \tag{2.3.1.11}
\]

Quadrature Scheme 2 for CC(1):

\[
(1 + ah - h^2 \exp(-h/2)/2) Y_{n+1} \\
= (1 + \frac{h^2}{2} \exp(-h/2)) Y_n + h^2/2 \sum_{i=0}^{n-1} Y_i \exp\left(-\left(t_{n+1} - t_i - h/2\right)\right) \\
+ h^2/2 \sum_{i=0}^{n-1} Y_{i+1} \exp\left(-\left(t_{n+1} - t_i - h/2\right)\right). \tag{2.3.1.12}
\]

First of all, we compare the accuracy of DG(0), CG(1) and CC(1) through Example 2.3.1.1 with \(a = 7\), using the same method as [44]. Other values of \(a\) lead to similar results. We assume that the maximum norm of the error in \(I\) is proportional to \(h^p\), that is, \(|\text{error}| \approx Ch^p\) with the constant of \(C\) independent of meshsize \(h\). To determine the order \(p\) experimentally, we take logarithms:

\[
\log(|\text{error}|) \approx \log(C) + p \log(h),
\]

By doing so we can determine \(p\) as the slope of a line that passes through the points \((\log(h), \log(|\text{error}|))\). We plot the logarithms of the errors versus the logarithms of the corresponding time steps for exact scheme in Figure 2.1. The slopes of the lines are 0.9837 for DG(0), 0.9836 for CC(1) and 2.0004 for CG(1). Correspondingly in Figure 2.2 for Quadrature Scheme 1, the line slope for DG(0) is 0.9861, for CG(1), 0.9414, for CC(1), 0.9836, which reveals that they have the same order of convergence. However, the distance between the lines indicates that the constant
\( C \) in the representation of the error, \(|\text{error}| \approx C h^p\), for Quadrature Scheme 1 is different from the exact DG(0). The same explanation holds for Quadrature Scheme 2 in Figure 2.3. Figure 2.4 illustrates that the stability of DG(0) and CC(1) is better than that of CG(1).

2.3.2 Example of a VIDE with time-dependent coefficient

**Example 2.3.2.1.** We consider the linear scalar Volterra integro-differential equation

\[
y' + a(t)y = \int_0^t \exp(-(t-s))y(s)ds, \quad t \in I = (0,1), \quad y(0) = 1,
\]

where \( a(t) = r + \frac{1}{1-r} - \frac{1}{1-r} \exp((r-1)t) \) and \( r \neq 1 \) is a positive constant.

The exact solution of (2.3.2.1) is

\[
y(t) = \exp(-rt).
\]

We take uniform meshes: \( \{t_i : t_i = ih, \ i = 0,1,\ldots,n\} \) and choose the initial value \( y_0^- = y(0) = 1 \).

We easily formulate the schemes of DG(0), CG(1) and CC(1) for Example 2.3.2.1, as follows.

**Exact DG(0):**

\[
\begin{align*}
(h(r + \frac{1}{1-r}) + \frac{1}{(r-1)^2}[\exp((r-1)t) &- \exp((r-1)t_{n-1})] - h + 2 - \exp(-h)Y_n^- \\
= & \ Y_{n-1}^- + \sum_{i=1}^{n-1} Y_i^-[-\exp(-(t_n - t_i)) + \exp(-(t_n - t_{i-1})) \\
+ & \ \exp(-(t_{n-1} - t_i)) - \exp(-(t_{n-1} - t_{i-1}))].
\end{align*}
\]
Exact CG(1):

\[
(2 - \frac{h}{2} + (r + \frac{1}{1 - r}) \frac{h}{2} + \frac{1}{(r - 1)^2} \exp ((r - 1)t_n) - \frac{1}{h} \frac{1}{(r - 1)^3} \exp ((r - 1)t_n)
\]
\[
+ \frac{1}{h} \frac{1}{(r - 1)^3} \exp ((r - 1)t_{n-1}) + \frac{1}{h} \exp (-h) - 1/h)Y_n
\]
\[
= [1 - ((r + \frac{1}{1 - r}) \frac{h}{2} - \frac{1}{(r - 1)^2} \exp ((r - 1)t_{n-1}) + \frac{1}{h} \frac{1}{(r - 1)^3} \exp ((r - 1)t_n)
\]
\[
- \frac{1}{h} \frac{1}{(r - 1)^3} \exp ((r - 1)t_{n-1})] + \frac{h}{2} + \exp (-h) + \frac{1}{h} \exp (-h) - 1/h]Y_{n-1}
\]
\[
+ \sum_{i=1}^{n-1} Y_{i-1} \{ \exp(-(t_n - t_{i-1})) - \frac{1}{h} \exp(-(t_n - t_i)) + \frac{1}{h} \exp(-(t_n - t_{i-1}))
\]
\[
- \exp(-(t_{n-1} - t_{i-1})) + \frac{1}{h} \exp(-(t_{n-1} - t_i)) - \frac{1}{h} \exp(-(t_{n-1} - t_{i-1}))\}
\]
\[
+ \sum_{i=1}^{n-1} Y_i \{ \exp(-(t_n - t_i)) + \frac{1}{h} \exp(-(t_n - t_i)) - \frac{1}{h} \exp(-(t_n - t_{i-1}))
\]
\[
+ \exp(-(t_{n-1} - t_i)) - \frac{1}{h} \exp(-(t_{n-1} - t_i))
\]
\[
+ \frac{1}{h} \exp(-(t_{n-1} - t_{i-1}))]. \quad (2.3.2.4)
\]

Exact CC(1):

\[
(2 + h(r + \frac{1}{1 - r} - \frac{1}{1 - r} \exp ((r - 1)t_{n+1})) - h - \exp(-h))Y_{n+1}
\]
\[
= (2 - h \exp(-h) - \exp(-h))Y_n + h \sum_{i=0}^{n-1} \{ \exp(-(t_{n+1} - t_i))
\]
\[
\cdot (\exp(h) - \exp(h)/h + 1/h)Y_{i+1} + [- \exp(-(t_{n+1} - t_i))
\]
\[
+ \frac{1}{h} \exp(-(t_{n+1} - t_i))(\exp(h) - 1)]Y_i\}. \quad (2.3.2.5)
\]

Now we consider the “Quadrature Scheme 2” for DG(0), CG(1) and CC(1).

Quadrature Scheme 2 for DG(0):

\[
[h(r + \frac{1}{1 - r} - \frac{1}{1 - r} \exp ((r - 1)t_n)) + 1 - h^2 \exp(-h/2)]Y_n^-
\]
\[
= Y_{n-1}^- + \sum_{i=1}^{n-1} Y_i^- h^2 \exp(-(t_n - \frac{t_i + t_{i-1}}{2})). \quad (2.3.2.6)
\]
Quadrature Scheme 2 for CG(1):

\[
[1 + h\left(\frac{1}{1 - r} - \frac{1}{1 - r} \exp((r - 1)t_n)\right) - h^2 \exp(-h/2)/2]Y_n
\]

\[
= \left[h^2 \exp(-h/2)/2 + 1\right]Y_{n-1} + \frac{h^2}{2} \sum_{i=1}^{n-1} Y_{i-1} \exp\left(-\left(t_n - \frac{t_i + t_{i-1}}{2}\right)\right)
\]

\[
+ \frac{h^2}{2} \sum_{i=1}^{n-1} Y_i \exp\left(-\left(t_n - \frac{t_i + t_{i-1}}{2}\right)\right). \tag{2.3.2.7}
\]

Quadrature Scheme 2 for CC(1):

\[
(1 + h\left(\frac{1}{1 - r} - \frac{1}{1 - r} \exp((r - 1)t_{n+1})\right) - h^2 \exp(-h/2)/2)Y_{n+1}
\]

\[
= \left(1 + \frac{h^2}{2} \exp(-h/2)\right)Y_n + h^2/2 \sum_{i=0}^{n-1} Y_i \exp\left(-\left(t_{n+1} - t_i - h/2\right)\right)
\]

\[
+ h^2/2 \sum_{i=0}^{n-1} Y_{i+1} \exp\left(-\left(t_{n+1} - t_i - h/2\right)\right). \tag{2.3.2.8}
\]

When we take \(r = 6\), we obtain the numerical results shown in Figure 2.5 and Figure 2.6, whose explanation is the same as for Example 2.3.1.1. Other values of \(r\) lead to similar results.
Figure 2.1: The methods for Example 2.3.1.1

Figure 2.2: The methods with Trapezoidal rule of lag term for Example 2.3.1.1
The distance of Lines:
Quad. DG(0) - Exact DG(0) = 0.0184
Quad. CC(1) - Exact DG(0) = 0.0349
Quad. CC(1) - Exact DG(0) = 0.0349

Figure 2.3: The methods with midpoint rule of lag term for Example 2.3.1.1

Effect of a(t) to the convergence of exact DG(0), exact CC(1) and exact CC(1)

Figure 2.4: Effect of stability on the convergence of methods for Example 2.3.1.1
Figure 2.5: The methods for Example 2.3.2.1

Figure 2.6: The methods with midpoint rule of lag term for Example 2.3.2.1
2.4 The discontinuous Galerkin method for non-standard VIDEs

2.4.1 Preliminaries

In this section, we study the nonstandard Volterra integro-differential equation,

\[
\begin{align*}
& y' + a(t)y = V_G^N(y)(t), \quad t \in I := [0,T], \\
& y(0) = y_0,
\end{align*}
\]

where \( V_G^N(y)(t) := \int_0^t k(t-s)G(y(t),y(s))ds \) and \( a, k \in C(I) \). Assume that the (Lipschitz continuous) function \( G : \Omega \times \Omega \to \mathbb{R} (\Omega \subset \mathbb{R}) \) is such that (2.4.1.1) possesses a unique solution \( y \in C^1(I) \) for all \( y_0 \in \Omega \).

We give the a posteriori error estimates of \( \text{DG}(m) \) to (2.4.1.1). As in the above sections, we write the DG scheme as

\[
B_{DG}(Y,X) := F_{DG}(X), \quad \forall X \in \mathcal{V}_N^{(m)},
\]

where

\[
B_{DG}(Y,X) := \sum_{n=1}^N \int_{I_n} \{ Y'(t)X(t) + a(t)Y(t)X(t) \} dt - \int_{I_n} V_G^N(Y)(t)X(t) \}
- \sum_{n=1}^M \{ [Y]_n X_n^+ + Y_0^+ X_0^+ \},
\]

\[
F_{DG}(X) := y_0 X_0^+.
\]

To show that (2.4.1.2) has a unique solution \( Y|_{I_n} \in \mathcal{P}^{(m)}(I_n) \) we define that for \( \tilde{Y} \in \mathcal{P}^{(m)}(I_n), Y = \tilde{T}\tilde{Y} \in \mathcal{P}^{(m)}(I_n) \) as the solution of

\[
\begin{align*}
\int_{I_n} \{ Y'(t)X(t) + a(t)Y(t)X(t) \} dt \\
+ \int_{I_n} k(t-s)G(\tilde{Y}(t),\tilde{Y}(s))ds)X(t) \}
\end{align*}
\]

for all \( X(t) \in \mathcal{P}^{(m)}(I_n) \). If the operator \( \tilde{T} \) is a contraction on \( \mathcal{P}^{(m)}(I_n) \) for all sufficiently small \( h_n \), the assertion follows from Banach’s fixed point theorem.
2.4.2 A posteriori error estimates for nonstandard VIDEs

Theorem 2.4.2.1. Let \( A := ||a||_I, \ B := ||k||_I, \) and let \( m = 0, 1. \) Assume that \( G \) satisfies

\[ |\nabla G(u, v)| \leq L, \quad \forall u, v \in \Omega. \]

Then the error of the DG(m) approximation to (2.4.1.1) satisfies

\[ |y(t_M) - Y_M^e| \leq C||h_0^{m+1} R(Y)||_{[0,t_M]}, \]

where \( C := C(t_M, A, B, L) \) and \( R(Y) := |V_G^{N-1}| + |a(t)Y - V_G^N(Y)(t)| \quad (t \in I_n). \)

Proof. From (2.4.1.2), we know that

\[ B_{DG}(Y, X) - B_{DG}(y, X) = 0, \quad \forall X \in \mathcal{V}_N^{(m)}. \]

We write this as

\[ \tilde{D}(e, X) = 0, \quad \forall X \in \mathcal{V}_N^{(m)}, \quad (2.4.6) \]

where

\[ \tilde{D}(W, X) := \sum_{n=1}^{M} \int_{I_n} \left\{ W'X + a(t)WX - \int_{0}^{t} k(t - s) \right\} \]

\[ \cdot \left\{ \int_{0}^{1} \nabla G(ry(t) + (1 - r)Y(t), ry(s) + (1 - r)Y(s))dr \right\} \]

\[ \cdot (W(t), W(s))ds \cdot X(t) \right\} dt + \sum_{n=1}^{M-1} [W]_n X_n^+ + W_0^+ X_0^+. \]

We know that for all piecewise continuous \( w \)

\[ \tilde{D}(w, z) = w_M^e z_M^e, \quad (2.4.8) \]

where \( z \) is the solution of the continuous linearized dual problem of (2.4.1.1), defined
as
\[
\begin{aligned}
&\left\{
\begin{align*}
-z' + [a(t) + \int_0^t k(t-s) \int_0^1 G_1(ry(t) + (1-r)Y(t), ry(s)) \, dr \, ds]z = \int_t^{t_M} k(t-s) \tilde{A}(s) z(s) \, ds, \quad t_M > t > 0, \\
z(t_M) = e_{-M}^r.
\end{align*}
\end{aligned}
\]
\]
(2.4.2.9)

Here, \( \tilde{A}(s) := \int_0^1 G_2(ry(t) + (1-r)Y(t), ry(s) + (1-r)Y(s)) \, dr, \nabla G := (G_1, G_2) \).

Selecting \( w = e := y - Y \) in (2.4.2.8), we get
\[
\begin{aligned}
\left[ e_{-M}^r \right]^2 &= \tilde{D}(e, z) = D(e, z - X) \\
&= B_{DG}(y, z - X) - B_{DG}(Y, z - X) \\
&= - \sum_{n=1}^{M} \int_{I_n} Y'(t)(z - X) \, dt - \sum_{n=1}^{M} \{ a(t) Y(t)(z - X) \\
&- \nabla_G(Y)(t)(z - X) \} \, dt - \sum_{n=1}^{M} [Y]_{n-1}(z - X)_{n-1}^+ \sum_{n=1}^{M} |Y|_{n-1}(z - X)_{n-1} \\
&=: I + II + III,
\end{aligned}
\]
(2.4.2.10)

From the definition of (2.1.2.7), we easily derive
\[
\begin{aligned}
I &= 0, \quad |III| \leq S(M) \max_{n \leq M} \|[Y]_{n-1}\|, \\
|II| &:= \left| \sum_{n=1}^{M} \left[ a(t) Y(t) - \nabla_G(Y)(t) \right] (z - X) \, dt \right|, \\
&\leq \sum_{n=1}^{M} h_n \left\| a(t) Y(t) - \nabla_G(Y)(t) \right\|_{I_n} \left\| z - X \right\|_{I_n}, \\
&\leq \max_{n \leq M} h_n \left\| a(t) Y(t) - \nabla_G(Y)(t) \right\|_{I_n} \sum_{n=1}^{M} h_n^{m+1} \int_{I_n} |z'| \, dt, \\
&= S(M) \max_{n \leq M} h_n \left\| a(t) Y(t) - \nabla_G(Y)(t) \right\|_{I_n},
\end{aligned}
\]
(2.4.2.12)

where \( S(M) := \int_0^{t_M} |z^{(m+1)}| \, dt \). We need the stability of (2.4.2.9),
\[
S(M) = \int_0^{t_M} |z^{(m+1)}| \, dt \leq C|e_{-M}^r|,
\]
(2.4.2.13)
where $C := C(t_M, A, B, L)$. Because (2.4.2.9) is the linearized dual problem, the
proof of (2.4.2.13) is very similar to that of Lemma 2.1.2.1.

Combining the estimates (2.4.2.11), (2.4.2.12), and (2.4.2.13) with (2.4.2.10), we
complete the proof of Theorem 2.4.2.1.

2.4.3 Efficiency of the a posteriori error estimator

We shall first derive the a priori error estimate of (2.4.1.1). This is then used to
prove the efficiency of the a posteriori error estimator in Theorem 2.4.2.1.

**Theorem 2.4.3.1.** Define $A := ||a||_I$, $B^* := (\int_{I_n} |k(t)| dt)^2$, and assume $G$ satisfies

$$|\nabla G(u, v)| \leq L, \quad \forall u, v \in \Omega.$$ 

Then the error of the DG($m$) approximation to (2.4.1.1) satisfies

$$||e(t)||_{[0,t_M]} \leq C \max_{n \leq M} h_n^{m+1} ||y^{(m+1)}||_{I_n},$$

with $m = 0, 1$, and $C := C(t_M, L, A, B^*)$.

**Proof.** The proof is very similar to that of Theorem 2.2.1.1 and is thus left to the
reader.

**Theorem 2.4.3.2.** Under the assumptions of Theorem 2.4.2.1 and Theorem 2.4.3.1,
we have

$$|y(t_M) - Y_{t_M}^*| \leq C h^{m+1} ||y^{m+1}(0, t) - Y^{m+1}(0, t)||_{[0,t_M]} + ||y^{m+1}(t)||_{[0,t_M]},$$

where $C := C(t_M, L, A, B)$ is independent of the mesh size $h$. 
Proof. We only need to bound the term $||h_n^{m+1}R(Y)||_{[0,t_M]}$ in Theorem 2.4.2.1:

$$
||h_n^{m+1}R(Y)||_{[0,t_M]} \leq ||h_n^m[Y]_{n-1}||_{[0,t_M]} + ||h_n^{m+1}[a(t)Y - \mathcal{V}_G^N(Y)(t)]||_{[0,t_M]}
$$

$$
= [h_n^m[Y^+_n - Y + Y^-_n]||_{[0,t_M]} + ||h_n^{m+1}(-y(t) - a(t)y(t)) + \mathcal{V}_G^N(y)(t) + a(t)Y - \mathcal{V}_G^N(Y)(t)||_{[0,t_M]}
$$

$$
\leq 2||h_n^m e(t)||_{[0,t_M]} + ||h_n^{m+1}y'(t)||_{[0,t_M]} + ||h_n^{m+1}[-a(t)e(t)] + \int_0^t k(t-s)(G(y(t), y(s)) - G(Y(t), Y(s)))ds||_{[0,t_M]}
$$

$$
\leq 2h^m||e(t)||_{[0,t_M]} + h^{m+1}||y'(t)||
$$

$$
+ Ah^{m+1}||e(t)||_{[0,t_M]} + \sqrt{2}Lh^{m+1}t_M B||e(t)||_{[0,t_M]}.
$$

The proof is completed by recalling Theorem 2.4.3.1.

2.5 The discretized discontinuous Galerkin method for VIDEs

In this section we consider the discontinuous Galerkin methods with quadrature for the memory term and for the inner product for linear Volterra integro-differential equations. The readers are suggested to compare this section with Brunner [23].

2.5.1 The comparison with collocation method for VIDEs

We recall the DG time-stepping scheme for (2.1.1.1): For $n = 1, \cdots, N$, find $Y|_{I_n} \in \mathcal{P}^m(I_n)$, such that

$$
\int_{I_n} (Y'(t) + a(t)Y(t) - \mathcal{V}(Y)(t))X dt + Y_{n-1}^+X_{n-1}^+ = Y_{n-1}^-X_{n-1}^+.
$$

(2.5.1.1)

for all $X \in \mathcal{P}^m(I_n)$. Here we set $Y_0^- = y_0$. Suppose now that the integrals in (2.5.1.1) are approximated by interpolatory $(m+1)$-point quadrature formulas with abscissas $t_{n,j} := t_n + c_jh_n$ ($0 =: c_0 < c_1 < \cdots < c_m \leq 1$) and weights $w_j$ ($j = 1, \cdots, m+1$).
0, 1, ..., m). We denote the resulting discretized DG(m) solution in $V^{(m)}_N$ by $\tilde{Y}$. The semi-discretized version of (2.5.1.1) is then given by

$$\sum_{j=0}^{m} w_j [\tilde{Y}'(t_{n,j}) + a(t_{n,j})\tilde{Y}(t_{n,j}) - Z(t_{n,j})] X(t_{n,j}) + \tilde{Y}(t_{n,j}^+) X(t_{n,j}^+) - \tilde{Y}(t_{n,j}^-) X(t_{n,j}^-) = 0,$$

for all $X \in P^{(m)}(I_n)$, where

$$Z(t_{n,j}) := \mathcal{V}(\tilde{Y})(t_{n,j})$$

$$= \int_0^{t_{n,j}} k(t_{n,j} - s)\tilde{Y}(s)ds$$

$$= F_n + \int_{t_{n,j}}^{t_{n,j}} k(t_{n,j} - s)\tilde{Y}(s)ds.$$

We denote the discretized version of $Z(t_{n,j})$ by

$$\tilde{Z}(t_{n,j}) := \tilde{F}_n + h_n \sum_{l=0}^{j} w_{n,l} k(t_{n,j} - t_{n,l})\tilde{Y}(t_{n,l}).$$

The fully discretized version of (2.5.1.1) is then defined by

$$\sum_{j=0}^{m} w_j [\tilde{Y}'(t_{n,j}) + a(t_{n,j})\tilde{Y}(t_{n,j}) - \tilde{Z}(t_{n,j})] X(t_{n,j}) + \tilde{Y}(t_{n,j}^+) X(t_{n,j}^+) - \tilde{Y}(t_{n,j}^-) X(t_{n,j}^-) = 0,$$

for all $X \in P^{(m)}(I_n)$.

The above fully discretized DG method (2.5.1.4) is called:

(i) an extended fully discretized DG method if the lag term formula for $F_n$ is given by

$$\tilde{F}_n := h_n \sum_{\ell=0}^{n-1} \sum_{j=1}^{m} w_{n,j} k(t_{n} - t_{\ell} + c_j h)\tilde{Y}_{\ell,j}, \quad n = 1, \ldots, N - 1;$$

(ii) a mixed fully discretized DG method if the lag term formula is defined by

$$\tilde{F}_n := h_n \sum_{\ell=0}^{n-1} w_{n,\ell} k(t_{n} - t_{\ell})\tilde{Y}_{\ell}, \quad n = 1, \ldots, N - 1.$$
Let
\[ \tilde{Y}_n := \tilde{Y}(t_n^-), \]
\[ \tilde{Y}_{n,0} := \tilde{Y}(t_n^+) (= \tilde{Y}(t_{n,0}^+)), \]
\[ \tilde{Y}_{n,j} := \tilde{Y}(t_{n,j}) (j = 1, \ldots, m), \]
\[ \tilde{Z}_{n,j} := \tilde{Z}(t_{n,j}) (j = 1, \ldots, m). \]

and let \( L_j(v) \) be the \( j \)th Lagrange canonical polynomial (of degree \( m-1 \)) corresponding to the points \( \{c_i : i = 1, \ldots, m\} \). Moreover, denote by \( \{X_j : j = 0, 1, \ldots, m\} \) a (canonical) basis for \( \mathcal{P}^m(I_n) \) so that
\[ X_i(t_n + c_j h_n) = \delta_{i,j} \quad (i, j = 0, 1, \ldots, m). \]

Since the restriction of \( \tilde{Y}' \) to \( I_n \) is a polynomial of degree \( m-1 \) we may write
\[ \tilde{Y}'(t_n + \nu h_n) = \sum_{j=1}^{m} L_j(v) \tilde{Y}'(t_{n,j}), \quad \nu \in (0, 1], \]

and hence
\[ \tilde{Y}(t_n + \nu h_n) = \tilde{Y}(t_n^+) + h_n \int_{0}^{\nu} \tilde{Y}'(t_n + s h_n) ds, \quad \nu \in (0, 1]. \]

On the other hand, (2.5.1.4) with \( X = X_0 \) yields
\[ h_n w_0 [\tilde{Y}'(t_{n,0}) + a(t_{n,0}) \tilde{Y}(t_{n,0}) - \tilde{Z}(t_{n,0})] + \tilde{Y}(t_n^+) - \tilde{Y}(t_n^-) = 0, \]

implying that
\[ \tilde{Y}(t_n^+) = \tilde{Y}_n + h_n w_0 [\tilde{Z}(t_{n,0}) - a(t_{n,0}) \tilde{Y}(t_{n,0}^+) - \sum_{j=1}^{m} L_j(c_0) \tilde{Y}'(t_{n,j})]. \]

For \( X = X_i \) \((i = 1, \ldots, m)\), with \( X_i(t_{n,j}) = \delta_{i,j} \), we obtain from (2.5.1.4) the equations
\[ w_i [\tilde{Y}'(t_{n,i}) + a(t_{n,i}) \tilde{Y}(t_{n,i}) - \tilde{Z}(t_{n,i})] = 0, \]
where $w_i \neq 0$. This result can be used in (2.5.1.8) to produce
\[
\tilde{Y}(t_n^+) = \tilde{Y}_n + h_n w_0 [-a(t_n,0)\tilde{Y}(t_n,0) + \tilde{Z}(t_n,0)] + h_n \sum_{j=1}^{m} w_0 L_j(c_0)[-a(t_n,j)\tilde{Y}(t_n,j) + \tilde{Z}(t_n,j)].
\]
(2.5.1.9)

The identity (2.5.1.7) allows us to write
\[
\tilde{Y}(t_{n,i}) = \tilde{Y}(t_n^+) + h_n \sum_{j=1}^{m} \beta_j(c_i)[-a(t_n,j)\tilde{Y}(t_n,j) + \tilde{Z}(t_n,j)],
\]
(2.5.1.10)
with
\[
\beta_j(v) := \int_0^v L_j(s)ds \quad (j = 1, \ldots, m),
\]
and $\beta_j(c_i) =: a_{i,j}$. Hence, setting $\tilde{Y}_{n,i} := \tilde{Y}(t_{n,i})$ and recalling (2.5.1.9) we obtain
\[
\tilde{Y}_{n,i} = \tilde{Y}_n + h_n w_0 [-a(t_n,0)\tilde{Y}(t_n,0) + \tilde{Z}(t_n,0)]
+ h_n \sum_{j=1}^{m} [-a_{i,j} + w_0 L_j(c_0)] [a(t_n,j)\tilde{Y}(t_n,j) - \tilde{Z}(t_n,j)]
\]
(2.5.1.11)

$(i = 1, \ldots, m)$. The equations (2.5.1.9) and (2.5.1.11) form a system of $m+1$ nonlinear algebraic equations for $\tilde{Y}_n := (\tilde{Y}(t_n^+), \tilde{Y}_{n,1}, \ldots, \tilde{Y}_{n,m})^T \in \mathbb{R}^{m+1}$: its form closely resembles the one corresponding to collocation at the points $\{t_n,0, t_n,1, \ldots, t_n,m\}$. We now show that these equations may indeed be interpreted as the stage equations of an implicit $(m+1)$-stage Volterra-Runge-Kutta (VRK) method. Let $b_j := \beta_j(1)$ $(j = 1, \ldots, m)$, and observe that
\[
b_j = \int_0^1 L_j(s)ds = \sum_{k=0}^{m} w_k L_j(c_k) = w_0 L_j(c_0) + w_j,
\]
because our interpolatory $(m+1)$-point quadrature formula is exact for polynomials of degree not exceeding $m$. This leads to the relationship
\[
b_j - w_0 L_j(c_0) = w_j,
\]
and hence by (2.5.1.7) to

\[ \tilde{Y}_{n+1} := \tilde{Y}(t_{n+1}^-) = \tilde{Y}_n + h_n \sum_{j=0}^{m} w_j [-a(t_{n,j})\tilde{Y}(t_{n,j}) + \tilde{Z}(t_{n,j})]. \]  

(2.5.1.12)

We conclude that (2.5.1.12) together with (2.5.1.9) and (2.5.1.10) represents a collocation-based \((m + 1)\)-stage implicit VRK method for (2.1.1.1). We summarize the above presentation as the following theorem.

**Theorem 2.5.1.1.** The fully discretized DG scheme (2.5.1.4) may lead to the collocation-based \((m + 1)\)-stage implicit VRK method \{(2.5.1.12), (2.5.1.9), (2.5.1.10)\} for (2.1.1.1).

**Remark 2.5.1.1.** We see that the discussion is exactly the same in Section 1.3 for \(k = 0\).

### 2.5.2 A posteriori error estimator

If the memory term \(\mathcal{V}(y)(t)\) in (2.1.1.1) is computed approximately, then the resulting quadrature error also contributes to the total error. We consider the quadrature error as the perturbation of the DG(m) approximation to (2.1.1.1). The total error can be estimated by using the triangle inequality.

DG approximation to (2.1.1.1) is described by: Find \(Y \in \mathcal{V}_N^{(m)}\) such that

\[ B_{DG}(Y, X) = F_{DG}(X), \quad \forall X \in \mathcal{V}_N^{(m)}, \]  

(2.5.2.1)

where

\[ B_{DG}(Y, X) := \sum_{n=1}^{M} \int_{I_n} \{ Y'(t)X(t) + a(t)Y(t)X(t) - \mathcal{V}(Y)(t)X(t) \} dt \]

\[ + \sum_{n=1}^{M-1} [Y]_{n}X_{n}^{+} + Y_{0}^{+}X_{0}^{+}, \]

\[ F_{DG}(X) := y_{0}X_{0}^{+}. \]
Thus we define the discretized DG(m) with quadrature for the memory term as:

Find \( \bar{Y} \in \mathcal{V}_N^{(m)} \),

\[
\tilde{B}_{DG}(\bar{Y}, X) = \tilde{F}_{DG}(X), \quad \forall X \in \mathcal{V}_N^{(m)},
\]

(2.5.2.2)

where

\[
\begin{align*}
\tilde{B}_{DG}(\bar{Y}, X) & := \sum_{n=1}^{M} \int_{I_n} \{ \bar{Y}'(t)X(t) + a(t)\bar{Y}(t)X(t) - \bar{V}(\bar{Y})(t)X(t) \} dt \\
+ & \sum_{n=1}^{M-1} [\bar{Y}]_n X_n^+ + \bar{Y}_0^+ X_0^+,
\end{align*}
\]

\[
\tilde{F}_{DG}(X) := y_0 X_0^+.
\]

Thus the total error of the DG approximation to (2.1.1.1) with quadrature for the memory term can be written as

\[
\bar{e} := \bar{Y} - y = (Y - y) + (\bar{Y} - Y) := e + Q.
\]

(2.5.2.3)

We analyze \( e := Y - y \) as in the discussion in the above sections. The remaining work consists in estimating the term \( Q := \bar{Y} - Y \). By subtracting (2.5.2.1) from (2.5.2.2), we get

\[
\tilde{B}_{DG}(\bar{Y}, X) - B_{DG}(Y, X) = 0,
\]

(2.5.2.4)

that is,

\[
\begin{align*}
& \sum_{n=1}^{M} \int_{I_n} \{ Q'(t)X(t) + a(t)Q(t)X(t) - (\bar{V}(\bar{Y})(t) - V(Y(t)))X(t) \} dt \\
+ & \sum_{n=1}^{M-1} [Q]_n X_n^+ + Q_0^+ X_0^+ \\
= & \sum_{n=1}^{M} \int_{I_n} \{ Q'(t)X(t) + a(t)Q(t)X(t) - V(Q(t))X(t) \} dt \\
+ & \sum_{n=1}^{M-1} [Q]_n X_n^+ + Q_0^+ X_0^+ - \sum_{n=1}^{M} \int_{I_n} (\bar{V}(\bar{Y})(t) - V(\bar{Y})(t))X(t) dt \\
=: & I + II = 0,
\end{align*}
\]

(2.5.2.5)
where

\[ I := \sum_{n=1}^{M} \int_{I_n} \{Q'(t)X(t) + a(t)Q(t)X(t) - \nu(Q(t))X(t)\} dt \]

\[ + \sum_{n=1}^{M-1} [Q]_n X_n^+ + Q_0^+ X_0^+ , \]

\[ II := \sum_{n=1}^{M} \int_{I_n} (-\tilde{\nu}(\tilde{Y})(t) + \nu(\tilde{Y})(t))X(t) dt. \]

These observations allow us to establish the following theorem. It focuses on VIDEs with completely monotonic kernels, due to their importance in many applications (see e.g., Gripenberg, Londen and Staffans [53, Ch. 5]).

**Theorem 2.5.2.1.** Consider the discretized DG(m) with quadrature for the memory term ((2.5.2.2) with \(m = 0\) and \(1\)) for equation (2.1.1.1). We suppose that \(k \in C(\bar{I})\), \(k \in C^d(\mathbb{R}_+)\), and \(k\) is completely monotonic: \((-1)^j k^{(j)}(t) \geq 0\) \((t \geq 0, 0 \leq j \leq d)\), and we take the quadrature form as

\[ \tilde{\nu}(\tilde{Y})(t) = \sum_{i=0}^{M-1} w_{M_1} k(t - t_i) \tilde{Y}(t_i^-) \]

\[ + \sum_{i=1}^{M-1} \bar{w}_{M_1} k(t - t_i) \tilde{Y}(t_i^+) + w_{MM} k(0) t \tilde{Y}(t). \] (2.5.2.6)

Then

\[ |\tilde{e}_M| := |\tilde{Y}_M - y(t_M)| \leq C ||h_n^{m+1} R(\tilde{Y})||_{A_{t_M}}, \] (2.5.2.7)

where \(C := C(t_M, A, B)\) and \(A, B\) are as in Theorem 2.1.2.1, and

\[ R(\tilde{Y}) := \left| \frac{[\tilde{Y}]_{n-1}}{h_n} + |a(t)\tilde{Y} - \nu(\tilde{Y})(t)| \right| (t \in I_n). \]

**Proof.** First we prove (2.5.2.7) with \(m = 0\). The well-known Peano theorem for
quadrature [44] enables us to write the error as

\[
\mathcal{E}(\tilde{Y})(t) := \mathcal{V}(\tilde{Y})(t) - \tilde{\mathcal{V}}(\tilde{Y})(t) = \sum_{i=0}^{M-1} \int_{I_i} K_q(s) \partial_s^{(q)}[k(t-s)\tilde{Y}(s)]ds + \int_{t_{n-1}}^{t} K_q(s) \partial_s^{(q)}[k(t-s)\tilde{Y}(s)]ds,
\]

where the Peano kernel is given by

\[
K_q(s) = \frac{1}{(q-1)!} \int_0^T (t-s)^{q-1} dt - \frac{1}{(q-1)!} \sum_{i=1}^n w_{ni} (t_i - s)^{q-1},
\]

with \( q \geq 2 \). From (2.5.2.5),

\[
\int_{I_n} \{ Q'(t)X(t) + a(t)Q(t)X(t) \} dt + Q_{n-1}^+X_{n-1}^+
\]

for all \( X \in \mathcal{P}(m)(I_n), n = 1, \cdots, M. \) Selecting \( X(t) = Q(t) \) in (2.5.2.9)

\[
\frac{1}{2} [Q_n^-]^2 + \frac{1}{2} [Q_{n-1}^+]^2 + \int_{I_n} aQ^2 dt
\]

\[
= Q_{n-1}^-Q_{n-1}^+ - \int_{I_n} ( \int_0^t k(t-s)Q(s)ds )Q(t)dt
\]

\[
- \int_{I_n} \mathcal{E}(\tilde{Y})(t)X(t)dt.
\]

Since \( Q_{n-1}^-Q_{n-1}^+ \leq \frac{1}{2} [Q_{n-1}^-]^2 + \frac{1}{2} [Q_{n-1}^+]^2 \), we obtain

\[
\frac{1}{2} [Q_n^-]^2 + \int_{I_n} aQ^2 dt
\]

\[
\leq \frac{1}{2} [Q_{n-1}^-]^2 + \int_{I_n} ( \int_0^t |k(t-s)||Q(s)||Q(t)|ds )|Q(t)|dt
\]

\[
+ \int_{I_n} |\mathcal{E}(\tilde{Y})(t)||Q(t)||dt,
\]

(2.5.2.11)
We notice that for $m = 0$,

\[
\frac{1}{2}[Q_n^{-}]^2 + \int_{I_n} a(t)dt [Q_n^{-}]^2 \\
\leq \frac{1}{2}[Q_{n-1}^{-}]^2 + \int_{I_n} \int_0^{t_{n-1}} |k(t-s)| ds dt \sum_{i=1}^{n-1} |Q_i^{-}| |Q_{n}^{-}|
\]

\[
+ \frac{h_n}{2} [Q_n^{-}]^2 + \frac{1}{2} \int_{I_n} [E(\bar{Y})(t)]^2 dt,
\]

we obtain, if $h_n$ is sufficiently small, $0 < \beta_1 \leq \frac{1}{2}$,

\[
\left[ \frac{1}{2} - \frac{h_n}{2} \right] - \frac{3}{2} \int_{I_n} \int_0^t |k(t-s)| ds dt [Q_n^{-}]^2 \\
\leq \frac{1}{2}[Q_{n-1}^{-}]^2 + \frac{1}{2} \sum_{i=1}^{n-2} (\int_{I_n} \int_0^{t_{i-1}} |k(t-s)| ds dt) |Q_i^{-}|^2
\]

\[
+ \frac{1}{2} \int_{I_n} [E(\bar{Y})(t)]^2 dt.
\]

We abbreviate (2.5.2.13) as

\[
\beta_1 [Q_n^{-}]^2 \leq \beta_2 [Q_{n-1}^{-}]^2 + \frac{1}{2} \sum_{i=1}^{n-2} (\int_{I_n} \int_0^{t_{i-1}} |k(t-s)| ds dt) |Q_i^{-}|^2 + \frac{1}{2} \int_{I_n} [E(\bar{Y})(t)]^2 dt,
\]

where $\beta_1$, $\beta_2$ are obvious. Using discrete Gronwall lemma [26], we get

\[
[Q_n^{-}]^2 \leq C \int_{I_n} (E(\bar{Y})(t))^2 dt.
\]

The estimate

\[
|\bar{e}_n^M| \leq |e_n^M| + |Q_n^{-}| \leq C(t_M, A, B) ||h_n R(Y)||_{[0, t_M]} + |Q_n^{-}|
\]
now follows from (2.5.2.3) and Theorem 2.1.2.1. Because \( Y = \tilde{Y} - Q \),

\[
\| h_n R(Y) \|_{[0,t_M]} = \max_{1 \leq n \leq M} \| h_n R(Y) \|_{I_n}
\]

\[
= \max_{1 \leq n \leq M} \left\{ h_n \left[ \| \tilde{Y} \|_{n-1} - \| Q \|_{n-1} \right] \right\}
\]

\[
+ h_n \| a(t) \tilde{Y} - a(t) Q_n^- - \int_0^t k(t-s) \tilde{Y} - Q_n^- ds \|_{I_n}
\]

\[
\leq \max_{1 \leq n \leq M} \left\{ h_n \left[ \| \tilde{Y} \|_{n-1} + \| Q \|_{n-1} \right] \right\}
\]

\[
+ h_n \| a(t) \tilde{Y} - \int_0^t k(t-s) \tilde{Y} ds \|_{I_n}
\]

\[
+ h_n \| a(t) Q_n^- \|_{I_n} + h_n \| \int_0^t k(t-s) Q_n^- ds \|_{I_n}
\]

\[
\leq \| h_n R(\tilde{Y}) \| + \max_{1 \leq n \leq M} \left\{ \| Q \|_{n-1} \right\}
\]

\[
+ \| a(t) Q_n^- \|_{I_n} + \| \int_0^t k(t-s) Q_n^- ds \|_{I_n},
\]

(2.5.2.16)

Combining (2.5.2.14), (2.5.2.15), (2.5.2.16), and noting that

\[
\max_{1 \leq n \leq M} \left\{ \| Q \|_{n-1} + \| a(t) Q_n^- \|_{I_n} + \| \int_0^t k(t-s) Q_n^- ds \|_{I_n} \right\} = O(h^q),
\]

with \( q \geq 2 \), we obtain (2.5.2.7) with \( m = 0 \).

We shall now prove (2.5.2.7) with \( m = 1 \). Recall (2.5.2.10):

\[
\frac{1}{2} [Q_n^-]^2 + \frac{1}{2} [Q_n^+]^2 + \int_{I_n} aQ^2 dt
\]

\[
= Q_{n-1} Q_{n-1}^+ - \int_{I_n} \left( \int_0^t k(t-s) Q(s) ds \right) Q(t) dt
\]

\[
- \int_{I_n} \epsilon(\tilde{Y})(t) Q(t) dt.
\]

(2.5.2.17)

Since \( Q_{n-1} Q_{n-1}^+ \leq \epsilon^2 [Q_{n-1}]^2 + \frac{1}{\epsilon} [Q_{n-1}^+]^2 \) and for \( m = 1 \), \( \| Q \|^2_{I_n} \leq |Q_{n-1}^+|^2 + |Q_n^-|^2 \),
we obtain

\[
[Q_n^-]^2 + [Q_{n-1}^+]^2 \leq C \int_{I_n} \left( \mathcal{E}(\tilde{Y}) \right)^2 dt + C[Q_{n-1}^-]^2 \\
+ C \sum_{i=1}^{n-2} h_i \left( [Q_i^-]^2 + [Q_i^+]^2 \right).
\] (2.5.2.18)

Using again discrete Gronwall lemma [26], we reach

\[
[Q_n^-]^2 + [Q_{n-1}^+]^2 \leq C \int_{I_n} \left( \mathcal{E}(\tilde{Y}) \right)^2 dt.
\] (2.5.2.19)

Because \( Y = \tilde{Y} - Q \),

\[
\left\| h_n^2 R(Y) \right\|_{[0,t_M]} = \max_{1 \leq n \leq M} \left\| h_n^2 R(Y) \right\|_{I_n} \\
= \max_{1 \leq n \leq M} \left\{ h_n^2 \left[ [\tilde{Y}]_{n-1} - [Q]_{n-1} \right] \right\} \\
+ h_n^2 \left| a(t)\tilde{Y} - a(t)Q - \int_0^t k(t-s)[\tilde{Y} - Q] ds \right|_{I_n} \\
\leq \max_{1 \leq n \leq M} \left\{ h_n^2 \left[ [\tilde{Y}]_{n-1} + \left| [Q]_{n-1} \right| \right] \right\} \\
+ h_n^2 \left| a(t)\tilde{Y} - \int_0^t k(t-s)\tilde{Y}(s) ds \right|_{I_n} \\
+ h_n^2 \left| a(t)Q \right|_{I_n} + h_n^2 \left| \int_0^t k(t-s)Q ds \right|_{I_n} \\
\leq |h_n^2 R(\tilde{Y})| + \max_{1 \leq n \leq M} \left\{ h_n \left| [Q]_{n-1} \right| \right\} \\
+ h_n^2 \left| a(t)Q \right|_{I_n} + h_n^2 \left| \int_0^t k(t-s)Q ds \right|_{I_n}.
\] (2.5.2.20)

Combining (2.5.2.19), (2.5.2.15), (2.5.2.20), an observing that

\[
\max_{1 \leq n \leq M} \left\{ \left| [Q]_{n-1} \right| + \left| a(t)Q \right|_{I_n} + \left| \int_0^t k(t-s)Q ds \right|_{I_n} \right\} = \mathcal{O}(h^n),
\]

with \( q \geq 2 \), we obtain (2.5.2.7) with \( m = 1 \).
2.6 Superconvergence of mesh-dependent Galerkin methods for VIDEs

In this section we extend the mesh-dependent Galerkin methods (including the discontinuous Galerkin method) of Section 1.2 to Volterra integro-differential equations, including the semilinear case. Furthermore, theorems on superconvergence are proven.

2.6.1 Superconvergence for semilinear VIDEs

Consider again the semilinear Volterra integro-differential equation

\[ \begin{array}{l}
y'(t) + a(t)y(t) = \mathcal{V}_G(y)(t),
\quad t \in I = [0, T],
\end{array} \]

\[ \begin{array}{l}
y(0) = y_0,
\end{array} \]

(2.6.1.1)

where \( \mathcal{V}_G(y)(t) := \int_0^t k(t - s)G(u(s))ds \). Assume that \( a, k \in C(I) \), and \( G \) is Lipschitz continuous, i.e.,

\[ |G(y_1) - G(y_2)| \leq L|y_1 - y_2|, \]

(2.6.1.2)

for all \( y_1, y_2 \in \Omega \subseteq \mathbb{R} \).

As in Section 1.2.1, we introduce the mesh-dependent weak form of (2.6.1.1):

Find \( \bar{u} := (U_0, \ldots, U_N, u_1, \ldots, u_N) \in \mathcal{U} := \Omega^{N+1} \times \prod_{n=1}^{N} L^2(I_n; \Omega) \),

such that

\[ \begin{align*}
U_0[V_0 - v_1(t_0)] + &\sum_{n=1}^{N-1} U_n[v_n(t_n) - v_{n+1}(t_n)] + U_N v_N(t_N) \\
- \sum_{n=1}^{N} \int_{I_n} u_n v'_n dt &= y_0 V_0 + \sum_{n=1}^{N} \int_{I_n} [-a(t)u_n] \\
+ \int_0^t k(t - s)G(u(s))ds |v_n dt,
\end{align*} \]

(2.6.1.3)
for all
\[ \bar{v} := (V_0, v_1, \ldots, v_N) \in \mathcal{V} := \Omega \times \prod_{n=1}^{N} H^1(I_n; \Omega), \]
where \( u = \sum_{i=1}^{n} u_i \chi_{I_i} \), with \( \chi_{I_i} \) denoting the characteristic function of \( I_i \).

The local meaning of (2.6.1.3) is to find \( u_n \) in \( L^2(I_n; \Omega) \) and \( U_n \) in \( \Omega \) such that
\[
\begin{align*}
U_n v_n(t_n) &= U_{n-1} v_n(t_{n-1}) + \int_{I_n} [u_n v'_n + (-a(t) u_n(t) + \int_{0}^{t} k(t-s) G(u(s)) ds) v_n] dt, \\
U_0 &= y_0,
\end{align*}
\]

for all \( v_n \in H^1(I_n; \Omega) \) and \( n = 1, \ldots, N \).

Thus the mesh-dependent Galerkin scheme for (2.6.1.4) is to find \( \tilde{u}_h \in U_h \) such that \( U_0 = y_0 \) and
\[
\begin{align*}
\tilde{u}_h &\in U_h \quad \text{such that} \\
U_n v_n(t_n) &= U_{n-1} v_n(t_{n-1}) + \int_{I_n} [u_n v'_n + (-a(t) u_n(t) + \int_{0}^{t} k(t-s) G(u'(s)) ds) v_n] dt, \\
U_0 &= y_0,
\end{align*}
\]

for all \( v_n \in H^1(I_n; \Omega) \) and \( n = 1, \ldots, N \). Here \( u_h = \sum_{i=1}^{n} u_i \chi_{I_i} \) and
\[
U_h = \left\{ \tilde{u}_h \left| \begin{array}{l}
\tilde{u}_h = (U_{h_0}^0, \ldots, U_{h_N}^0, u_1^h, \ldots, u_N^h) \in U_h \text{ such that } u_n^h \in \mathcal{P}^{(m)}(I_n; \Omega) \\
\text{subject to } J (\geq 0) \text{ additional conditions for } n = 1, \ldots, N.
\end{array} \right. \right\},
\]

We estimate the \( L^2 \)- and nodal error of the mesh-dependent Galerkin scheme (2.6.1.5) for VIDE (2.6.1.1) in the following theorem.

**Theorem 2.6.1.1.** Assume that the solution \( y \) of (2.6.1.1) belongs to \( H^{m+1}([0, T]; \Omega) \).

For \( M > 1 \), assume that on the first \( M - 1 \) intervals the solution of (2.6.1.5) is such that
\[
\max\{|U_n^h - y(t_n)| : n = 0, \ldots, M - 1\} \leq Ch^{m+1} ||y^{(m+1)}||_0,
\]

(2.6.1.6)
and for $j = 0, \ldots, m + 1$,

$$\left\{ \sum_{n=1}^{M-1} \|u_n^h - y\|_{j_n}^2 \right\}^{1/2} \leq C h^{m+1-j} \|y^{(m+1)}\|_0. \quad (2.6.1.7)$$

Subsequently, we have that for sufficiently small $h$,

$$\max\{|U_n^h - y(t_n)| : n = 0, \ldots, N\} \leq C h^{m+1} \|y^{(m+1)}\|_0, \quad (2.6.1.8)$$

and for $j = 0, \ldots, m + 1$,

$$\|u^h - y\|_j \leq C h^{m+1-j} \|y^{(m+1)}\|_0, \quad (2.6.1.9)$$

where $u_h = \sum_{n=1}^{N} u_n^h \chi_{I_n}$ and $\| \cdot \|_j := \left\{ \sum_{n=1}^{N} \| \cdot \|_{j_n}^2 \right\}^{1/2}$.

Proof. Since $y$ solves (2.6.1.5), we have

$$(U_n^h - y(t_n))v_n^h(t_n) = [U_{n-1}^h - y(t_{n-1})]v_n^h(t_{n-1}) + \int_{I_n} (u_n^h - y)(v_n^h)' dt - \int_{I_n} a(t)(u_n^h - y)v_n^h dt + \int_{I_n} \int_0^t k(t - s)|G(u^h(s)) - G(y(s))| ds v_n^h dt. \quad (2.6.1.10)$$

Let $v_n^h$ be the solution of

$$(v_n^h)'(t) = -\varphi_J(u_n^h - \bar{u}_n^h)(t), \quad t \in I_n, \quad v_n^h(t_n) = 0,$$

where $\bar{u}_n^h$ is the Lagrange interpolating polynomial of degree $m$, such that

$$\bar{u}_n^h(t_{n_l}) = y(t_{n_l}), \quad l = 1, \ldots, J.$$

$\varphi_J$ denotes the $L^2$-projector of $L^2(I_n; \Omega)$ onto $P^{m-J}(I_n; \Omega)$. We substitute $v_n^h$ into
(2.6.1.10) and obtain

\[
\int_{I_n} (u_n^h - \bar{u}_n^h)[\varphi_J(u_n^h - \bar{u}_n^h)(t)]dt
\]
\[
= [U_{n-1}^h - y(t_{n-1})] \int_{I_n} \varphi_J(u_n^h - \bar{u}_n^h)(t)dt + \int_{I_n} (\bar{u}_n^h - y) [-\varphi_J(u_n^h - \bar{u}_n^h)](t)dt
\]
\[
- \int_{I_n} a(t)(u_n^h - \bar{y}) \int_{t}^{t_n} \varphi_J(u_n^h - \bar{u}_n^h)(\nu)d\nu dt + \int_{I_n} \{ \int_{t}^{t_n} k(t - s)[G(u^h(s)) - G(y(s))]ds \int_{t}^{t_n} \varphi_J(u_n^h - \bar{u}_n^h)(\nu)d\nu \} dt.
\]

Hence we arrive at

\[
\|\varphi_J(u_n^h - \bar{u}_n^h)(t]\|_{0,n} \leq h_n^{1/2}|U_{n-1} - y(t_{n-1})| + h_n^{1/2}|a(t)||_{0,n}||u_n^h - \bar{u}_n^h||_{0,n}
\]
\[
+ [1 + h_n^{1/2}|a(t)||_{0,n}]||\bar{u}_n^h - y||_{0,n}
\]
\[
+ Lh_n^{1/2} \int_{I_n} \int_{0}^{t} |k(t - s)||u^h(s) - y(s)||ds| dt. \quad (2.6.1.11)
\]

We need Lemma 2.6.1.1 whose proof can be found in Delbour and Dubeau [40].

**Lemma 2.6.1.1.** The map \( J_n \) defined by

\[
u \mapsto J_n \nu = (\varphi_J \nu, u(t_n), \ldots, u(t_{n_j})) \colon \mathcal{P}^{(m)}(I_n; \Omega) \to \mathcal{P}^{(m-J)}(I_n; \Omega) \times \Omega^J
\]

is an isomorphism, and there exist two constants \( \beta_1 \) and \( \beta_2 \) (independent of \( h \) and the points \( \{t_n\}_{i=1}^{J} \)) such that

\[
\beta_1 \|\nu\|_{0,n} \leq \|J_n \nu\| \leq \beta_2 \|\nu\|_{0,n}.
\]

It follows from Lemma 2.6.1.1 and (2.6.1.11) that

\[
[eta_1 - h_n^{1/2}|a(t)||_{0,n} - Lh_n^{3/2}|k(t)||_{0,n}]||u_n^h - \bar{u}_n^h||_{0,n}
\]
\[
\leq h_n^{1/2}|U_{n-1}^h - y(t_{n-1})| + [1 + h_n^{1/2}|a(t)||_{0,n}]||\bar{u}_n^h - y||_{0,n}
\]
\[
+ Lh_n^{1/2}|k(t)||_{0,n}||\bar{u}_n^h(t) - y(t)||_{0,n} + h_n^{1/2} \sum_{i=1}^{J} |u_n^h(t_n) - \bar{u}_n^h(t_n)|
\]
\[
+ Lh_n^{1/2} \int_{I_n} \sum_{j=1}^{n-1} \int_{I_j} |k(t - s)||u_j^h(s) - y(s)||ds| dt.
\]
Hence,

\[
||u_n^h - y||_{0,n} \leq ||u_n^h - \bar{u}_n^h||_{0,n} + ||\bar{u}_n^h - y||_{0,n}
\]

\[
\leq Ch_n^{1/2}(|U_{n-1}^h - y(t_{n-1})| + \sum_{i=1}^J |u_n^h(t_{ni}) - \bar{u}_n^h(t_{ni})|)
\]

\[
+ C||\bar{u}_n^h(t) - y(t)||_{0,n}
\]

\[
+ Ch_n^{1/2} \int_{I_n} \sum_{j=1}^{n-1} \int_{I_{j}} |k(t-s)||u_j(s) - y(s)| ds dt. \quad (2.6.1.12)
\]

Substitute \( u_n^h = U_n^h - y(t_n) \) into (2.6.1.10):

\[
|U_n^h - y(t_n)| \leq |U_{n-1}^h - y(t_{n-1})| + |a(t)||u_n^h - y||_{0,n} \quad (2.6.1.13)
\]

\[
+ Lh_n^{3/2}||k(t)||_{0,n}||u_n^h(t) - y(t)||_{0,n}
\]

\[
+ Lh_n^{1/2} \int_{I_n} \left( \sum_{j=1}^{n-1} \int_{I_{j}} |k(t-s)||u_j^h(s) - y(s)| ds \right) dt.
\]

We therefore obtain

\[
|U_n^h - y(t_n)| \leq \sum_{i=1}^{n} ||a(t)||_{0,i}||u_n^h - y||_{0,i} + \sum_{i=1}^{n} Lh_i^{3/2}||k(t)||_{0,i}||u_i^h - y||_{0,i}
\]

\[
+ \sum_{i=1}^{n} Lh_i^{1/2} \int_{I_i} \sum_{j=1}^{n-1} \int_{I_{j}} |k(t-s)||u_j^h(s) - y(s)| ds dt. \quad (2.6.1.14)
\]

We note that for \( J \leq M \) and \( u_n^h(t_{ni}) = U_n^h \), inequality (2.6.1.12) can be rearranged to read

\[
||u_n^h - y||_{0,n} \leq Ch_n^{1/2} \sum_{i=1}^{M} |U_{n-i}^h - y(t_{n-i})| + Ch^{m+1}||y^{(m+1)}||_{0,n}
\]

\[
+ Ch_n^{1/2} \int_{I_n} \sum_{j=1}^{n-1} \int_{I_{j}} |k(t-s)||u_j^h(s) - y(s)| ds dt, \quad (2.6.1.15)
\]

where \( || \cdot ||_{0,n} \) is the \( L^2 \)-norm over \([t_{n-M}, t_n] \). Set \( \alpha_n := |U_n^h - y(t_n)|, \quad n = 1, \ldots, N, \) and

\[
\beta_j := ||a(t)||_{0,j}||u_j^h - y||_{0,j} + Lh_j||k(t)||_{0,j}||u_j^h - y||_{0,j}
\]

\[
+ Lh_j^{1/2} \int_{I_j} \sum_{m=1}^{n-1} \int_{I_m} |k(t-s)||u_m^h(s) - y(s)| ds dt.
\]
for $j = 1, \ldots, N$. If we abbreviate (2.6.1.14) by setting $\alpha_j \leq \sum_{j=1}^{n} \beta_j$, then

$$\sum_{i=1}^{M} |U_{n-i}^{h} - y(t_{n-i})| = \sum_{i=1}^{M} \alpha_{n-i} \leq \sum_{i=1}^{M} \sum_{j=1}^{n-i} \beta_j \leq M \sum_{j=1}^{n-1} \beta_j.$$ 

Since

$$\beta_j \leq h_j^{1/2} A ||u_j^{h} - y||_{0,j} + Lh_j^{3/2} B ||u_j^{h} - y||_{0,j}$$

$$+ LBM_j^{3/2} \sum_{m=1}^{n-1} h_m^{1/2} ||u_m^{h}(s) - y(s)||_{0,m},$$

we have

$$\sum_{i=1}^{M} ||U_{n-i}^{h} - y(t_{n-i})| \leq LBT M \sum_{m=1}^{n-1} (h_m^{1/2} + \frac{A h_m^{1/2}}{LBT} + \frac{h_m^{3/2}}{T}) ||u_m^{h} - y||_{0,m}.$$ 

Recalling (2.6.1.15), we then derive the bound

$$||u_n^{h} - y||_{0,n} \leq Ch_n^{1/2} LBT M \sum_{m=1}^{n-1} (h_m^{1/2} + \frac{A h_m^{1/2}}{LBT} + \frac{h_m^{3/2}}{T}) ||u_m^{h} - y||_{0,m}$$

$$+ CBh_n \sum_{m=1}^{n-1} h_m^{1/2} ||u_m^{h} - y||_{0,m} + Ch_n^{m+1} ||y^{(m+1)}||_{0,n}$$

$$\leq Ch_n^{1/2} \sum_{m=1}^{n-1} h_m^{1/2} ||u_m^{h} - y||_{0,m} + Ch_n^{m+1} ||y^{(m+1)}||_{0,n},$$

and Gronwall’s lemma leads to

$$||u_n^{h} - y||_{0,n} \leq Ch_n^{m+1} \exp(C h_n^{1/2} \sum_{m=1}^{n-1} h_m^{1/2}) ||y^{(m+1)}||_{0,n}$$

$$\leq Ch_n^{m+1} ||y^{(m+1)}||_{0,n}.$$ 

Thus (2.6.1.7) and (2.6.1.9) hold true when $j = 0$. From (2.6.1.13) and Gronwall’s lemma we also obtain (2.6.1.6) and (2.6.1.8). The inequalities (2.6.1.9) ($1 \leq j \leq m + 1$) are obtained by using the estimates

$$||u_n^{h} - y||_{j,n} \leq ||u_n^{h} - \bar{u}_n^{h}||_{j,n} + ||\bar{u}_n^{h} - y||_{j,n}$$
and
\[ ||u^h_n - \bar{u}^h_n||_{j,n} \leq Ch^{-j}||u^h_n - \bar{u}^h_n||_{0,n}. \]

Now we describe the superconvergence property of the mesh-dependent Galerkin method (2.6.1.5) for the VIDE (2.6.1.1).

**Theorem 2.6.1.2.** Assume that the assumptions of Theorem 2.6.1.1 hold. Then for all sufficiently small \( h > 0 \) we have

\[ \max\{|U^h_n - y(t_n)| : n = 0, \ldots, N\} \leq Ch^{2m+2-J}. \]

**Proof.** Choose \( \tilde{v}^h = (V_0, v_1^h, \ldots, v_N^h) \) such that \( v_1^h(t_0) = V_0 \) and \( v_n^h(t_n) = v_{n+1}^h(t_n), \; n = 1, \ldots, N - 1 \). Substitute that \( \tilde{v}^h \) in (2.6.1.10) and sum over \( j = 1, \ldots, n \). This yields

\[ (U_n^h - y(t_n))v_n^h(t_n) = \int_0^{t_n} (u^h - y)(v^h) dt - \int_0^{t_n} a(t)(u^h - y)v^h dt \]
\[ + \int_0^{t_n} \int_0^t k(t-s)[G(u^h(s)) - G(y(s))] ds v^h dt. \]

Let \( w \) in \( H^{m+2-J}(0, t; \Omega) \) be the solution of

\[ \begin{align*}
  w' - a(t)w + \int_t^{t_n} k(s-t) & \int_0^1 G_1(ru^h + (1-r)y) dr \cdot w(s) ds = 0, \\
  w(t_n) = U_n^h - y(t_n),
\end{align*} \]

for \( t \in [0, t_n] \). Let \( w^h \) be a continuous piecewise interpolating polynomial of degree \( m+1-J \) of \( w \) such that \( w^h(t_n) = U_n^h - y(t_n) \). It follows from Sobolev interpolation theory [22] that

\[ ||w - w^h||_1 \leq Ch^{m+1-J}||w^{m+2-J}||_0. \]

Now we shall use the following Lemma 2.6.1.2 to express the norm of \( w^{m+2-J} \) in terms of \( U_n^h - y(t_n) \).
Lemma 2.6.1.2. Suppose that $g$ is such that $g|_{I_n} \in H^{m+1-J}(I_n; \Omega)$ ($n = 1, \ldots, N$), and that $w$ in $H^1([0,T]; \Omega)$ satisfies the equation

$$w'(t) - a(t)w(t) + \int_t^{t_n} k(s-t)\tilde{A}(s)w(s)ds = g(t), \quad (2.6.1.18)$$

for $t \in [0,t_n]$. Then there exists a constant $C$, independent of $g$ and $t_n$, such that

$$\|w\|_{m+2-J} \leq C\{\|w(t_n)\| + \|g\|_{m+1-J}\}.$$

Proof. We differentiate (2.6.1.18) $m+1-J$ times to express $w^{(m+2-J)}$ in terms of

$$\{w, \mathcal{V}_q^*w(t) (q = 0, \ldots, m+1-J), \ g, \ g^{(1)}, \ldots, g^{(m+1-J)}\},$$

where

$$\mathcal{V}_q^*w(t) := \int_t^{t_n} k_t^{(q)}(s-t)\tilde{A}(s)w(s)ds.$$

Then we replace $w$ by the identity

$$w(t) = R(t,t_n)w(t_n) + \int_t^{t_n} R(t,s)g(s)ds, \quad (2.6.1.19)$$

where the resolvent kernel $R$ has the form

$$R(t,s) = 1 + \int_t^s r(t,u)du, \quad (t,s) \in S := \{(t,s) : 0 \leq t \leq s \leq t_n \leq T\},$$

with $r$ satisfying

$$r(t,s) = Q(t,s) + \int_t^s Q(t,\tau)r(\tau,s)d\tau,$$

and with

$$Q(t,s) := a(s) + \int_t^s k_t^{(q)}(u-s)du, \quad (t,s) \in S.$$

We refer to ([26] or [23]) for the proof of (2.6.1.19). The proof of Lemma 2.6.1.2 is now complete.

We then have

$$\|w - w^h\|_1 \leq Ch^{m+1-J}|U_n^h - y(t_n)|.$$
Set $v^h = w^h$ in (2.6.1.17):

$$
|U_n^h - y(t_n)|^2 \leq ||u^h - y||_0 ||(w^h)' - a(t)w^h + \int_t^{t_n} k(s-t)\bar{A}(s)w^h(s)ds||_0
$$

$$
\leq ||u^h - y||_0 ||(w^h)' - a(t)w^h + \int_t^{t_n} k(s-t)\bar{A}(s)w^h(s)ds
$$

$$
- [w' - a(t)w + \int_t^{t_n} k(s-t)\bar{A}(s)w(s)ds]|_0
$$

$$
\leq ||u^h - y||_0 \{|| (w^h)' - w'||_0 + A||w^h - w||_0 + TLB||w^h - w||_0\}
$$

$$
\leq ||u^h - y||_0 \{Ch^{m+1-J} |U_n^h - y(t_n)|
$$

$$
+ C(A + TLB)h^{m+2-J} |U_n^h - y(t_n)|\}
$$

Hence,

$$
|U_n^h - y(t_n)| \leq Ch^{m+1-J} ||u^h - y||_0.
$$

Combining this equation with (2.6.1.9) we arrive at the desired estimate (2.6.1.16).

### 2.6.2 Superconvergence of the discretized mesh-dependent Galerkin methods for VIDEs

Consider, for ease of exposition, the linear VIDE

$$
\begin{cases}
  y'(t) + a(t)y(t) = \mathcal{V}(y)(t), & t \in I = [0, T], \\
  y(0) = y_0,
\end{cases}
$$

where $\mathcal{V}(y)(t) := \int_0^t k(t-s)y(s)ds$ and $a, k \in C(I)$.

As in Section 1.2.1, we introduce the mesh-dependent weak form of (2.6.1.1): To find

$$
\tilde{u} := (U_0, \ldots, U_N, u_1, \ldots, u_N) \in U := \Omega^{N+1} \times \prod_{n=1}^N L^2(\Omega; \Omega),
$$
such that

\[
U_0[V_0 - v_1(t_0)] + \sum_{n=1}^{N-1} U_n[v_n(t_n) - v_{n+1}(t_n)] + U_Nv_N(t_N)
\]

\[
- \sum_{n=1}^{N} \int_{I_n} u_n v'_n dt = y_0V_0 + \sum_{n=1}^{N} \int_{I_n} [-a(t)u_n + \mathcal{V}(u)(t)]v_n dt,
\]

\[
\left\{ \begin{array}{l}
U_n v_n(t_n) = U_{n-1} v_n(t_{n-1}) + \int_{I_n} [u_n v'_n + (-a(t)u_n(t))v_n] dt \\
U_0 = y_0,
\end{array} \right.
\]

(2.6.2.2)

where \( u = \sum_{i=1}^{n} u_i \chi_{I_i} \).

The local form of (2.6.2.2) is: Find \( u_n \) in \( L^2(I_n; \Omega) \) and \( U_n \) in \( \Omega \) such that

\[
\left\{ \begin{array}{l}
U_n v_n(t_n) = U_{n-1} v_n(t_{n-1}) + \int_{I_n} [u_n v'_n + (-a(t)u_n(t))v_n] dt \\
J \text{ additional conditions on } u_n,
\end{array} \right.
\]

(2.6.2.3)

for all \( v_n \in H^1(I_n; \Omega) \) and \( n = 1, \ldots, N \).

Thus the approximation scheme for (2.6.2.3) consists in finding \( \tilde{u}_h \) in \( \mathcal{U}_h \) such that \( U_0 = y_0 \) and

\[
\left\{ \begin{array}{l}
U^h v^h(t_n) = U^h v^h(t_{n-1}) + \int_{I_n} [u^h v'^h + (-a(t)u^h)v^h] dt \\
J \text{ additional conditions on } u^h,
\end{array} \right.
\]

(2.6.2.4)

for all \( v^h_n \in \mathcal{P}^{(m+1-j)}(I_n; \Omega) \) and \( n = 1, \ldots, N \). Here \( u^h = \sum_{i=1}^{n-1} u^h_i \chi_{I_i} \) and

\[
\mathcal{U}_h = \left\{ \tilde{u}_h \mid \tilde{u}_h = (U^h_0, U^h_N, u^h_1, \ldots, u^h_N) \in \mathcal{U} \text{ such that } u^h_n \in \mathcal{P}^{(m)}(I_n; \Omega) \text{ subject to } J \geq 0 \text{ additional conditions for } n = 1, \ldots, N. \right\}
\]

When we apply the numerical quadrature to the memory term, we obtain the semidiscretized DG scheme for (2.6.2.3): Find \( \tilde{\tilde{u}}_h \) in \( \tilde{\mathcal{U}}_h \) such that \( \tilde{U}_0 = y_0 \) and

\[
\left\{ \begin{array}{l}
\tilde{U}^h \tilde{v}^h(t_n) = \tilde{U}^h \tilde{v}^h(t_{n-1}) + \int_{I_n} [\tilde{u}^h \tilde{v}'^h + (-a(t)\tilde{u}^h)\tilde{v}^h] dt \\
J \text{ additional conditions on } \tilde{u}^h_n,
\end{array} \right.
\]

(2.6.2.5)
for all $\tilde{v}_n^h$ in $\mathcal{P}^{(m+1-J)}(I_n; \Omega)$ and $n = 1, \ldots, N$. Here $\tilde{u}_n^h = \sum_{i=1}^{n-1} \tilde{u}_i^h \chi_i$ and

$$\tilde{U}_h = \left\{ \tilde{u}_h \bigg| \tilde{u}_h = (\tilde{U}_0^h, \ldots, \tilde{U}_N^h, \tilde{v}_1^h, \ldots, \tilde{v}_N^h) \in \tilde{U} \right\},$$

subject to $J \geq 0$ additional conditions for $n = 1, \ldots, N$.

where we will use the interpolatory quadrature approximation (for example, Newton-Cotes formulas [44]) for the memory term:

$$\tilde{V}(\tilde{u}_n^h)(t) := \sum_{i=0}^{n-1} \omega_{ni} k(t_{n-1} - t_i) \tilde{U}_i^h + \omega_{nn} k(0) t \tilde{u}_n^h(t). \quad (2.6.2.6)$$

To make the error of the quadrature formula (2.6.2.6) be $O(h^m)$ for all $t \in (0, T]$, we adapt the old mesh (with meshsize $h := \max\{h_n, n = 1, \ldots, N\}$) by choosing

$$\max\{h_i : 1 \leq i \leq m\} := h^m.$$

We describe the $L^2$ and nodal error of the semi-discretized DG scheme (2.6.2.5) for (2.6.2.1) in the following theorem.

**Theorem 2.6.2.1.** Assume that the solution $y$ of (2.6.2.1) belongs to $H^{m+1}([0, T]; \Omega)$. For $M > 1$, assume that on the first $M - 1$ intervals the solution of (2.6.2.5) is such that

$$\max\{|\tilde{U}_n^h - y(t_n)| : n = 0, \ldots, M - 1\} \leq C h^{m+1} |y^{(m+1)}|_0, \quad (2.6.2.7)$$

and for $j = 0, \ldots, m + 1$,

$$\left\{ \sum_{n=1}^{M-1} |\tilde{u}_n^h - y|_{j,n}^2 \right\}^{1/2} \leq C h^{m+1-j} |y^{(m+1)}|_0. \quad (2.6.2.8)$$

Hence, we have that for sufficiently small $h > 0$,

$$\max\{|\tilde{U}_n^h - y(t_n)| : n = 0, \ldots, N\} \leq c h^{m+1} |y^{(m+1)}|_0, \quad (2.6.2.9)$$

and for $j = 0, \ldots, m + 1$,

$$||\tilde{u}_n^h - y||_j \leq C h^{m+1-j} |y^{(m+1)}|_0, \quad (2.6.2.10)$$

where $\tilde{u}_h = \sum_{n=1}^{N} \tilde{u}_n^h \chi_n$ and $|| \cdot ||_j := \left\{ \sum_{n=1}^{N} || \cdot ||_{j,n}^2 \right\}^{1/2}$.
Proof. Since \( y \) satisfies (2.6.2.5), we have

\[
(\tilde{U}_n^h - y(t_n))\tilde{v}_n^h(t_n) = [\tilde{U}_{n-1}^h - y(t_{n-1})]\tilde{v}_n^h(t_{n-1}) + \int_{I_n} (\tilde{u}_n^h - y)(\tilde{v}_n^h)'dt
\]

\[
- \int_{I_n} a(t)(\tilde{u}_n^h - y)\tilde{v}_n^h dt + \int_{I_n} [\tilde{V}(\tilde{u}_n^h)(t)
- \mathcal{V}(y)(t)]\tilde{v}_n^h dt. \tag{2.6.2.11}
\]

Let \( \tilde{v}_n^h \) be the solution of

\[
(\tilde{v}_n^h)'(t) = -\varphi_J(\tilde{u}_n^h - \tilde{u}_n^h)(t), \quad t \in I_n, \quad \tilde{v}_n^h(t_n) = 0,
\]

where \( \tilde{u}_n^h \) is the Lagrange interpolating polynomial of degree \( m \),

\[
\tilde{v}_n^h(t_{nl}) = y(t_{nl}), \quad l = 1, \ldots, J.
\]

\( \varphi_J \) is the \( L^2 \)-projector of \( L^2(I_n; \Omega) \) onto \( P^{(m-J)}(I_n; \Omega) \). Then we substitute \( \tilde{v}_n^h \) into (2.6.2.11) and obtain

\[
\int_{I_n} (\tilde{u}_n^h - \tilde{v}_n^h)[\varphi_J(\tilde{u}_n^h - \tilde{u}_n^h)(t)]dt
= [\tilde{U}_{n-1}^h - y(t_{n-1})]\int_{I_n} \varphi_J(\tilde{u}_n^h - \tilde{u}_n^h)(t)dt + \int_{I_n} (\tilde{u}_n^h - y)[-\varphi_J(\tilde{u}_n^h - \tilde{u}_n^h)(t)]dt
- \int_{I_n} a(t)(\tilde{u}_n^h - y)\int_t^{t_n} \varphi_J(\tilde{u}_n^h - \tilde{u}_n^h)(\nu)d\nu dt + \int_{I_n} [\tilde{V}(\tilde{u}_n^h)(t)
- \mathcal{V}(y)(t)]\int_t^{t_n} \varphi_J(\tilde{u}_n^h - \tilde{u}_n^h)(\nu)d\nu dt.
\]

So we can arrive at

\[
||\varphi_J(\tilde{u}_n^h - \tilde{u}_n^h)(t)||_{0,n} \leq h_n^{1/2}|\tilde{U}_{n-1} - y(t_{n-1})| + h_n^{1/2}|a(t)||_{0,n}||\tilde{u}_n^h - \tilde{u}_n^h||_{0,n}
+ [1 + h_n^{1/2}|a(t)||_{0,n}]||\tilde{u}_n^h - y||_{0,n}
+ Lh_n^{1/2}\int_{I_n} |\tilde{V}(\tilde{u}_n^h)(t) - \mathcal{V}(y)(t)|dt. \tag{2.6.2.12}
\]
We again use Lemma 2.6.1.1, where we have now replaced \( u \) by \( \tilde{u} \). Recalling (2.6.2.12), we obtain

\[
[\beta_1 - h_n^{1/2}||a(t)||_{0,n} - Ch_n^{3/2}||\tilde{u}_n^h - u_n^h||_{0,n} \\
\leq h_n^{1/2}|\tilde{U}_n^h - y(t_{n-1})| + [1 + h_n^{1/2}||a(t)||_{0,n}]||a_n^h - y||_{0,n} \\
+ Ch_n^{3/2}||\tilde{u}_n^h(t) - y(t)||_{0,n} + h_n^{1/2}\sum_{l=1}^{j} |\tilde{u}_n^h(t_{n_l}) - \tilde{u}_n^h(t_{n_l})| \\
+ Ch_n^{3/2} [||\tilde{V}(\tilde{u}_n^h(t_{n-1}) - \tilde{V}(y)(t_{n-1}))|| + ||\tilde{V}(y)(t_n) - V(y)(t_n)||].
\]

Hence

\[
||\tilde{u}_n^h - y||_{0,n} \leq ||\tilde{u}_n^h - \tilde{u}_n^h||_{0,n} + ||\tilde{u}_n^h - y||_{0,n} \\
\leq Ch_n^{1/2} [||\tilde{U}_n^h - y(t_{n-1})|| + \sum_{l=1}^{j} |\tilde{u}_n^h(t_{n_l}) - \tilde{u}_n^h(t_{n_l})||] \\
+ C||\tilde{u}_n^h(t) - y(t)||_{0,n} + Ch_n^{3/2}\sum_{\ell=1}^{n-1} |\tilde{U}_\ell^h - y(t_{\ell})| \\
+ Ch_n^{3/2}||\tilde{V}(y)(t_n) - V(y)(t_n)||.
\]  

(2.6.2.13)

Substitute \( \tilde{v}_n^h = \tilde{U}_n^h - y(t_n) \) into (2.6.2.11)

\[
|\tilde{U}_n^h - y(t_n)| \leq |\tilde{U}_{n-1}^h - y(t_{n-1})| + ||a(t)||_{0,n}||\tilde{u}_n^h - y||_{0,n} \\
+ Ch_n^{3/2}||\tilde{u}_n^h(t) - y(t)||_{0,n} + Ch_n^{3/2}\sum_{\ell=1}^{n-1} |\tilde{U}_\ell^h - y(t_{\ell})| \\
+ Ch_n^{3/2}||\tilde{V}(y)(t_n) - V(y)(t_n)||.
\]  

(2.6.2.14)

We therefore obtain

\[
|\tilde{U}_n^h - y(t_n)| \leq Ch_n^{1/2}||\tilde{u}_n^h - y||_{0,n} + Ch_n^{3/2}||\tilde{V}(y)(t_n) - V(y)(t_n)||. \tag{2.6.2.15}
\]

We note that for \( J \leq M \) and the \( u_n^h(t_{n_l}) = U_{n_l}^h \), inequality (2.6.2.13) can be rear-
ranged in the form

\[ ||\bar{u}_n^h - y||_{0,n} \leq Ch^{1/2}_n \sum_{\ell=1}^{n-1} |\bar{U}_\ell^h - y(\ell)| + Ch^{m+1}||y^{(m+1)}||_{0,n} \]

\[ + C h^{3/2}_n |\bar{V}(y)(t_n) - V(y)(t_n)| \]

\[ \leq Ch^{1/2}_n \sum_{\ell=1}^{n-1} h^{1/2}_\ell ||\bar{u}_\ell^h - y||_{0,\ell} + C \sum_{\ell=1}^{n} h^{3/2}_\ell |\bar{V}(y)(\ell) - V(y)(\ell)| \]

\[ + C h^{m+1}||y^{(m+1)}||_{0,n}. \]  

(2.6.2.16)

Therefore,

\[ ||\bar{u}_n^h - y||_{0,n} \leq Ch^{m+1}||y^{(m+1)}||_{0,n} + C \sum_{l=1}^{n} h^{3/2}_l |\bar{V}(y)(t_l) - V(y)(t_l)|. \]

Here \( || \cdot ||_{0,n} \) is the \( L^2 \)-norm over \([t_{n-M}, t_n]\). Thus (2.6.2.8) and (2.6.2.10) hold true when \( j = 0 \). From (2.6.2.14) and Gronwall’s lemma we also know that (2.6.2.7) and (2.6.2.9) hold. Inequalities (2.6.2.10) for \( 1 \leq j \leq m + 1 \) are obtained by using the inequalities

\[ ||\bar{u}_n^h - y||_{j,n} \leq ||\bar{u}_n^h - \bar{u}_n^h||_{j,n} + ||\bar{u}_n^h - y||_{j,n} \]

and

\[ ||\bar{u}_n^h - \bar{u}_n^h||_{j,n} \leq C h^{-j} ||\bar{u}_n^h - \bar{u}_n^h||_{0,n}. \]

The following theorem is the analogue of Theorem 2.6.1.2.

**Theorem 2.6.2.2.** Assume that the assumptions of Theorem 2.6.2.1 hold and take \( \{t_n\}_{n=1}^{N} \) as the Gaussian points in \( I := [0, T] \). Then for sufficiently small \( h \) we have

\[ \max\{||\bar{U}_n^h - y(t_n)||: n = 0, \ldots, N\} \leq Ch^{2m+2-J}. \]

**Proof.** Choose \( \bar{v}_n^i = (\bar{V}_0, \bar{v}_1^i, \ldots, \bar{v}_N^i) \) such that \( \bar{v}_1^i(t_0) = \bar{V}_0 \) and \( \bar{v}_n^i(t_n) = \bar{v}_n^{i+1}(t_n), \) \( n = 1, \ldots, N - 1 \). Substitute that \( \bar{v}_n^i \) in (2.6.2.11) and sum up over \( j = 1, \ldots, n \). This
yields

\[
(\tilde{U}_n^h - y(t_n))\tilde{v}_n^h(t_n) = \int_0^{t_n} (\tilde{u}^h - y)(\tilde{v}^h)'dt - \int_0^{t_n} a(t)(\tilde{u}^h - y)\tilde{v}^h dt + \int_0^{t_n} [\tilde{V}(\tilde{u}^h) - V(y)]\tilde{v}^h dt.
\]  

(2.6.2.17)

Let \( \tilde{w} \) in \( H^{m+2-J}([0, t]; \Omega) \) be the solution of

\[
\begin{align*}
\tilde{w}' - [a(t) + \omega_{nn}k(0)t]\tilde{w} &= 0, \\
\tilde{w}(t_n) &= \tilde{U}_n^h - y(t_n),
\end{align*}
\]

for \( t \in [0, t_n] \). Let \( \tilde{w}^h \) be a continuous piecewise interpolating polynomial of degree \( m + 1 - J \) of \( \tilde{w} \) such that \( \tilde{w}^h(t_n) = \tilde{U}_n^h - y(t_n) \). From Sobolev interpolation theory [22], we deduce

\[
||\tilde{w} - \tilde{w}^h||_1 \leq Ch^{m+1-J}||\tilde{w}^{(m+2-J)}||_0.
\]

Now we use the following Lemma 2.6.2.1 to express the norm of \( \tilde{w}^{(m+2-J)} \) in terms of \( \tilde{U}_n^h - y(t_n) \).

**Lemma 2.6.2.1.** Fix \( s \) in \( [0, T] \). Suppose that \( g \) is such that \( g|_{I_n} \in H^{m+1-J}(I_n; \Omega) \) for \( n = 1, \ldots, N \), and that \( \tilde{w} \) in \( H^1([0, T]; \Omega) \) satisfies the equation

\[
\tilde{w}'(t) - [a(t) + \omega_{nn}k(0)t]\tilde{w}(t) = g(t),
\]

for \( t \in [0, t_n] \). Then there exists a constant \( C \) independent of \( g \) and \( s \in [0, t_n] \) such that

\[
||\tilde{w}||_{m+2-J} \leq C\{||\tilde{w}(s)|| + ||g||_{m+1-J}\}.
\]

**Proof.** The lemma is a special case of Lemma 2.6.1.2 without the memory term. Compare also with Delfour and Dubeau [40].

We then have

\[
||\tilde{w} - \tilde{w}^h||_1 \leq Ch^{m+1-J}|\tilde{U}_n^h - y(t_n)|.
\]  

(2.6.2.18)
Set $\tilde{v}^h = \tilde{w}^h$ in (2.6.2.17)

$$|\tilde{U}^h_n - y(t_n)|^2 \leq \int_0^{t_n} (\tilde{w}^h - y)[(\tilde{w} - \tilde{w}^h)' - a(t)(\tilde{w} - \tilde{w}^h)]dt \quad (2.6.2.19)$$

$$- \int_0^{t_n} \tilde{w} \omega_{nn} t(\tilde{w}^h - y)dt + \int_0^{t_n} [\tilde{V}(\tilde{w}^h) - V(y)]\tilde{w}^h dt.$$

Since

$$\tilde{V}(\tilde{u}^h) - V(y) = \tilde{V}(\tilde{u}^h) - \tilde{V}(y) + \tilde{V}(y) - V(y)$$

$$= \tilde{V}(y) - V(y) + \sum_{i=1}^{n-1} \omega_{ni} (\tilde{U}^h_i - y(t_i))$$

$$+ \omega_{nn} t(\tilde{u}^h(t) - y(t)),$$  

and from Lemma 2.6.2.1 we have

$$||\tilde{w}^h||_0 \leq ||\tilde{w} - \tilde{w}^h||_0 + ||\tilde{w}||_0$$

$$\leq Ch^{m+2-J} |\tilde{U}^h_n - y(t_n)| + C|\tilde{U}^h_n - y(t_n)|, \quad (2.6.2.21)$$

Combining $\{(2.6.2.18), (2.6.2.19), (2.6.2.20), (2.6.2.21), \text{ and Lemma 2.6.2.1}\}$ yields

$$|\tilde{U}^h_n - y(t_n)| \leq Ch^{m+1-J} ||\tilde{u}^h - y||_0 + Ch^{2m+2-J} ||y^{(m)}||_0$$

$$+ C \sum_{i=1}^{n-1} \omega_{ni} k(t_{n-1} - t_i)|\tilde{U}^h_i - y(t_i)|. \quad (2.6.2.22)$$

Hence, Gronwall’s lemma leads to

$$|\tilde{U}^h_n - y(t_n)| \leq Ch^{2m+2-J}.$$

**Remark 2.6.2.1.** We conclude that it is not substantially more difficult to analyze the mesh-dependent Galerkin method for nonstandard Volterra integro-differential equations containing memory terms of the form $V_N^\mu(y)(t) := \int_0^t k(t-s)G(y(t), y(s))ds$. We leave the details to the interested readers.
Chapter 3

The discontinuous Galerkin method for delay VIDEs

In this chapter we focus on three kinds of delay Volterra integro-differential equations. We show the regularities of those problems and thus construct and analyze the robust adaptive discontinuous Galerkin methods for them. The readers may wish to consult Brunner and Zhang [28], Hale [58], Hale and Verduyn Lunel [59], and Bellen and Zennaro [12] and the references therein for the background materials and related results about delay differential or integro-differential equations.

3.1 Primary discontinuities of several classes of delay Volterra integro-differential equations

3.1.1 Delay VIDEs with weakly singular kernels

Let us consider

\[
\begin{align*}
&y'(t) = f(t, y(t)) + \int_0^t (t - s)^{-\alpha} G(s, y(s), y(\theta(s))) ds, \quad t \in I := [0, T], \\
&y(t) = \phi(t), \quad t \in [\bar{a}, 0],
\end{align*}
\]  

(3.1.1.1)

where we assume \(0 < \alpha < 1\) and

(i) \(f, G, \phi\) are sufficiently smooth.
(ii) $\theta(t) := t - \tau(t)$, with $\tau$ sufficiently smooth, $t > \tau(t) \geq \tau_0 > 0$ ($t \in I$).

Moreover, $\theta$ is strictly increasing on $I$ and $\bar{a} = \inf_{t \geq 0} \theta(t) < 0$.

(iii) The points $\{\xi_{\mu}\}$ are defined by

$$
\theta(\xi_{\mu}) = \xi_{\mu} - \tau(\xi_{\mu}) =: \xi_{\mu-1}, \forall \mu \geq 1,
$$

(3.1.1.2)

where $\xi_0 := 0$. Obviously

$$
\xi_{\mu+1} - \xi_{\mu} \geq \tau_0 > 0, \forall \mu \geq 0.
$$

For simplicity we denote $\bar{G}(s) := G(s, y(s), y(\theta(s)))$. We shall use the following formula frequently.

$$
H(t) := \int_\xi^t (t-s)^{-\alpha} \bar{G}(s) ds
= \frac{1}{1-\alpha} \bar{G}(\xi)(t-\xi)^{1-\alpha} + \frac{1}{(1-\alpha)(2-\alpha)} \bar{G}'(\xi)(t-\xi)^{2-\alpha}
+ \cdots + \frac{1}{(1-\alpha)_{m+1}} \bar{G}^{(m)}(\xi)(t-\xi)^{m+1-\alpha}
+ \frac{1}{(1-\alpha)_{m+1}} \int_0^t \bar{G}^{(m+1)}(s)(t-s)^{m+1-\alpha} ds,
$$

(3.1.1.3)

with $(1-\alpha)_m := (1-\alpha)(2-\alpha) \cdots (m-\alpha)$. We remark that (3.1.1.3) can be obtained by using repeated integration by parts.

**Definition 3.1.1.1.** If the solution of (3.1.1.1) and its derivatives of order less than, or equal to $\mu$ are continuous at some points $\xi \in I$ but the derivatives of order $\mu + 1$ is not, then $\xi$ is called a *primary discontinuity* of problem (3.1.1.1)

Denote $J[\mu] := (\xi_{\mu-1}, \xi_{\mu+1}]$ ($\mu \geq 0$), where $\xi_{-1} := \bar{a}$. We shall describe the primary discontinuities in solutions for (3.1.1.1) as the following theorem.
Theorem 3.1.1.1. The primary discontinuities of problem (3.1.1.1) are the points \( \xi_\mu (\mu = 0, 1, \ldots) \) generated by (3.1.1.2). To be more precise, \( y \in C^{2\mu, 1-\alpha}(J[\mu]) \), but \( y^{(2\mu+1)} \) is not continuous at the point \( \xi_\mu \), provided the assumptions (i) and (ii) hold.

Remark 3.1.1.1. We use \( C^\beta(I) \) \((0 < \beta < 1)\) to denote the well-known Hölder space: \( V \) is in \( C^\beta(I) \) if, for any \( t_1, t_2 \in I \) \((t_1 \neq t_2)\), we have

\[
|V(t_1) - V(t_2)| \leq L \cdot |t_1 - t_2|^\beta.
\]

A function \( V \) is in \( C^{\mu, \beta}(I) \) \((\mu \in \mathbb{N}, \mu \geq 1)\) if \( V \in C^\mu(I) \) and \( V^{(\mu)} \in C^\beta(I) \). We set \( C^{0, \beta}(I) = C^\beta(I) \).

Proof. The proof is based on the method of steps.

(1) Consider the regularity of the solution for (3.1.1.1) at the point \( \xi_0 := 0 \). It is possible to satisfy the condition \( y(0) = \phi(0) \), but not, in general, also the condition \( y'(0+) = \phi'(0-) \). The continuity of the derivative of the solution can be guaranteed at the initial point \( 0 \) only for deliberately chosen \( \phi(t) \), and such a function \( \phi(t) \) must satisfy the condition \( \phi'(0-) = f(0, \phi(0)) \).

(2) Consider the regularity at the point \( \xi_1 \). We write the equation (3.1.1.1) as

\[
\begin{cases}
  y'(t) = f(t, y(t)) + H_1(t), & t \in I := [0, T], \\
  y(t) = \phi(t), & t \in [\bar{a}, 0],
\end{cases}
\]

where

\[
H_1(t) := \int_0^t (t - s)^{-\alpha}G(s)ds.
\]

From the formula (3.1.1.3), we obtain

\[
H_1(t) = \frac{1}{1-\alpha} \bar{G}(0)t^{1-\alpha} + \frac{1}{1-\alpha} \int_0^t \bar{G}'(s)(t-s)^{1-\alpha}ds.
\]

Thus

\[
H'_1(t) = \bar{G}(0)t^{-\alpha} + \int_0^t \bar{G}'(s)(t-s)^{-\alpha}ds.
\]
So $y''$ is continuous at the points $\xi_{\mu}$ ($\mu \geq 1$). But $y^{(3)}$ is discontinuous at $\xi_{1}$. Now we are going to prove that

$$\int_{0}^{t} \overline{G}'(s)(t-s)^{-\alpha}ds \in C^{1-\alpha}(J[1]). \quad (3.1.1.4)$$

For any $t_0$, $t_1 \in J[1]$ and without loss of generality we assume $t_0 \leq \xi_1 \leq t_1$. From

$$\left| \int_{0}^{t_1} \overline{G}'(s)(t_1-s)^{-\alpha}ds - \int_{0}^{t_0} \overline{G}'(s)(t_0-s)^{-\alpha}ds \right|$$

$$\leq \left| \int_{t_0}^{t_1} \overline{G}'(s)(t_1-s)^{-\alpha}ds \right| + \left| \int_{0}^{t_0} \left( \overline{G}'(s)(t_1-s)^{-\alpha} - \overline{G}'(s)(t_0-s)^{-\alpha} \right)ds \right|$$

$$\leq L_{\overline{G}'} \left\{ \frac{2(t_1 - t_0)^{1-\alpha}}{1-\alpha} + \frac{t_0^{1-\alpha}}{1-\alpha} - \frac{t_1^{1-\alpha}}{1-\alpha} \right\}$$

$$\leq \frac{3L_{\overline{G}'}(t_1 - t_0)^{1-\alpha}},$$

where $L_{\overline{G}'}$ is the upper bound of $|\overline{G}'(s)|$ in $J[1]$. So $y \in C^{3-\alpha}(J[1])$.

(3) Consider now the regularity at the point $\xi_{2}$. We write the equation (3.1.1.1) as

$$\begin{cases} y'(t) = f(t, y(t)) + \int_{0}^{\xi_1} (t-s)^{-\alpha} \overline{G}(s)ds + H_2(t), \ t \in I := [0, T], \\ y(t) = \phi(t), \ t \in [\bar{a}, 0], \end{cases}$$

where

$$H_2(t) := \int_{\xi_1}^{t} (t-s)^{-\alpha} \overline{G}(s)ds.$$

By using the formula (3.1.1.3) again, we obtain

$$H_2(t) = \frac{1}{1-\alpha} \overline{G}(\xi_1)(t - \xi_1)^{1-\alpha} + \frac{1}{(1-\alpha)(2-\alpha)} \overline{G}'(\xi_1)(t - \xi_1)^{2-\alpha}$$

$$+ \frac{1}{(1-\alpha)(2-\alpha)(3-\alpha)} \overline{G}''(\xi_1)(t - \xi_1)^{3-\alpha}$$

$$+ \frac{1}{(1-\alpha)(2-\alpha)(3-\alpha)} \int_{\xi_1}^{t} \overline{G}^{(3)}(t-s)^{3-\alpha}ds.$$
Thus we can calculate the derivatives of order up to three of $H_2(t)$ as:

\[
H'_2(t) = \frac{1}{1-\alpha} \mathcal{G}'(\xi_1)(t - \xi_1)^{-\alpha} + \frac{1}{1-\alpha} \mathcal{G}'(\xi_1)(t - \xi_1)^{1-\alpha}
\]

\[
+ \frac{1}{(1-\alpha)(2-\alpha)} \mathcal{G}''(\xi_1)(t - \xi_1)^{2-\alpha}
\]

\[
+ \frac{1}{(1-\alpha)(2-\alpha)} \int_{\xi_1}^{t} \mathcal{G}^{(3)}(s)(t-s)^{2-\alpha} ds.
\]

\[
H''_2(t) = -\alpha \mathcal{G}'(\xi_1)(t - \xi_1)^{-\alpha-1} + \mathcal{G}'(\xi_1)t(t - \xi_1)^{-\alpha} + \frac{1}{1-\alpha} \mathcal{G}''(\xi_1)(t - \xi_1)^{1-\alpha}
\]

\[
+ \frac{1}{1-\alpha} \int_{\xi_1}^{t} \mathcal{G}^{(3)}(s)(t-s)^{(1-\alpha)} ds.
\]

\[
H^{(3)}_2(t) = \alpha(\alpha+1) \mathcal{G}(\xi_1)(t - \xi_1)^{-\alpha-2} - \alpha \mathcal{G}'(\xi_1)(t - \xi_1)^{-\alpha-1}
\]

\[
+ \mathcal{G}''(\xi_1)(t - \xi_1)^{-\alpha} + \int_{\xi_1}^{t} \mathcal{G}^{(3)}(s)(t-s)^{\alpha} ds.
\]

Hence $y^{(4)}$ is continuous at the point $\xi_\mu$ ($\mu \geq 2$). But $y^{(5)}$ is discontinuous at $\xi_2$. Furthermore along the lines proving (3.1.1.4) we can verify that

\[
\int_{\xi_1}^{t} \mathcal{G}^{(3)}(s)(t-s)^{-\alpha} ds \in C^{1-\alpha}([2]).
\]

(4) We suppose $y \in C^{2\mu,1-\alpha}(J^{[\mu]})$ and $y \in C^{2\mu}(\xi_m)$ ($m \geq \mu$). Now we consider the regularity at $\xi_{\mu+1}$. We write the equation (3.1.1.1) as

\[
\begin{align*}
\begin{cases}
  y'(t) = f(t, y(t)) + \int_{0}^{\xi_\mu} (t-s)^{-\alpha} \mathcal{G}(s) ds + H_{\mu+1}(t), & t \in I := [0, T], \\
y(t) = \phi(t), & t \in [\bar{a}, \bar{b}],
\end{cases}
\end{align*}
\]

where

\[
H_{\mu+1}(t) := \int_{\xi_\mu}^{t} (t-s)^{-\alpha} \mathcal{G}(s) ds.
\]

We write $H_{\mu+1}(t)$ as, by using (3.1.1.3),

\[
H_{\mu+1}(t) = \frac{1}{1-\alpha} \mathcal{G}(\xi_\mu)(t - \xi_\mu)^{1-\alpha} + \frac{1}{1-\alpha} \mathcal{G}'(\xi_\mu)(t - \xi_\mu)^{2-\alpha}
\]

\[
+ \cdots + \frac{1}{(1-\alpha)(2-\alpha)} \mathcal{G}^{(2\mu)}(\xi_\mu)(t - \xi_\mu)^{2\mu+1-\alpha}
\]

\[
+ \frac{1}{(1-\alpha)(2\mu+1)} \int_{\xi_\mu}^{t} \mathcal{G}^{(2\mu+1)}(s)(t-s)^{2\mu+1-\alpha} ds. \quad (3.1.1.5)
\]
Thus we can calculate the derivatives of order up to $2\mu + 1$ of $H_{\mu+1}(t)$ as:

\[
H'_{\mu+1}(t) = \frac{1}{1-\alpha}G'(\xi_\mu)(t - \xi_\mu)^{1-\alpha} + \frac{1}{(1-\alpha)2\mu}G^{(2\mu)}(\xi_\mu)(t - \xi_\mu)^{2\mu-\alpha} + \cdot \cdot \cdot + \frac{1}{(1-\alpha)2\mu}\int_{\xi_\mu}^t G^{(2\mu+1)}(s)(t - s)^{2\mu-\alpha}ds.
\]

\[
H^{(2\mu+1)}_{\mu+1}(t) = (\alpha)_{2\mu}G(\xi_\mu)(t - \xi_\mu)^{-\alpha - 2\mu} + \cdot \cdot \cdot + G^{(2\mu)}(\xi_\mu)(t - \xi_\mu)^{-\alpha} + \frac{1}{(1-\alpha)2\mu}\int_{\xi_\mu}^t G^{(2\mu+1)}(s)(t - s)^{-\alpha}ds.
\]

Hence $y \in C^{2(\mu+1),1-\alpha}(J[\mu+1])$.

Let us consider now

\[
\begin{cases}
  y'(t) = f(t, y(t), y(\theta(t))) + \int_0^t (t - s)^{-\alpha}G(s, y(s), y(\theta(s)))ds, & t \in I := [0, T], \\
y(t) = \phi(t), & t \in [\bar{a}, 0],
\end{cases}
\]

with the assumptions (i), (ii), (iii) in (3.1.1.1).

**Theorem 3.1.1.2.** The primary discontinuities of problem (3.1.1.8) are the points $\xi_\mu$ ($\mu = 0, 1, \ldots$) generated by (3.1.1.2). To be more precise, $y \in C^{\mu,1-\alpha}(J[\mu])$, but $y^{(\mu+1)}$ is not continuous at the point $\xi_\mu$, provided the assumptions (i) and (ii) hold.

**Proof.** From [125] we know the primary discontinuities of problem

\[
\begin{cases}
  y'(t) = f(t, y(t), y(\theta(t))), & t \in I := [0, T], \\
y(t) = \phi(t), & t \in [\bar{a}, 0],
\end{cases}
\]

are the points $\xi_\mu$ ($\mu = 0, 1, \ldots$) generated by (3.1.1.2). To be more precise, $y^{(\mu)}$ is continuous at the point $\xi_\mu$ and $y^{(\mu+1)}(\xi_{\mu+1})$ is bounded, but $y^{(\mu+1)}$ is not continuous at $\xi_\mu$, provided the assumptions (i) and (ii) hold. Thus, recalling Theorem 3.1.1.1, we obtain Theorem 3.1.1.2.
**Theorem 3.1.1.3.** Consider

\[
\begin{cases}
y'(t) = f(t, y(t), y(\theta(t)), y'(\theta(t))) + \int_0^t (t-s)^{-\alpha}G(s, y(s), y(\theta(s)))ds, & t \in I := [0, T], \\
y(t) = \phi(t), & t \in [\bar{a}, 0].
\end{cases}
\]

with the assumptions (i), (ii), (iii) in (3.1.1.1). There is no smoothing to the solutions of (3.1.1.10); more precisely, \( y \in C^{1-\alpha}(J^\mu), \forall \mu \geq 1. \)

**Proof.** It is known that there is no smoothing to the solutions of neutral delay differential equations (see [125]), so there is no smoothing to the solutions of (3.1.1.10). Following the lines in the proof of Theorem 3.1.1.1, we can verify that

\[ y \in C^{1-\alpha}(J^\mu), \forall \mu \geq 1. \]

**Remark 3.1.1.2.** The delay integro-differential equations may include terms such as

\[ \int_0^t K(t-s)G(y(\theta(s)))ds, \quad (3.1.1.11) \]

and

\[ \int_0^{\theta(t)} K(t-s)G(y(s))ds. \quad (3.1.1.12) \]

A natural question arises: “What is the difference between the regularity of the delay integro-differential equation with the term (3.1.1.11) and that with the term (3.1.1.12)?”

The term (3.1.1.11) can be expressed as

\[ \int_{\theta(0)}^{\theta(t)} K(t-\theta^{-1}(s))(\theta^{-1}(s))'G(y(s))ds. \quad (3.1.1.13) \]

Therefore, if the kernel function \( K \) is sufficiently smooth, then the regularities of (3.1.1.11) and (3.1.1.12) are the same. But when \( K \) has weakly singular behavior, their regularities will be different. Discussion of Section 3.1.2 will explain this well.
3.1.2 Delay functional VIDEs of Hale’s type

Consider

\[
\begin{cases}
\frac{d}{dt} \left( y(t) - \int_0^{\theta(t)} K(t - s)G(y(s))ds \right) = f(t, y(t), y(\theta(t))), & t \in I := [0, T], \\
y(t) = \phi(t), & t \in [\bar{a}, 0],
\end{cases}
\] (3.1.2.1)

where we assume that

(a) \( K \) is sufficiently smooth, and

(b) (i), (ii), (iii) in (3.1.1.1) hold.

We describe the primary discontinuities of problem (3.1.2.1) in the following theorem.

**Theorem 3.1.2.1.** The primary discontinuities of problem (3.1.2.1) are the points \( \xi_\mu (\mu = 0, 1, \ldots) \) generated by (3.1.1.2). To be more precise, \( y^{(\mu)} \) is continuous at \( \xi_\mu \) but \( y^{(\mu + 1)} \) is, in general, not provided the assumptions (i) and (ii) hold.

**Proof.** We rewrite the left-hand side of (3.1.2.1) as

\[
\frac{d}{dt} \left( y(t) - \int_0^{\theta(t)} K(t - s)G(y(s))ds \right) = y'(t) - \theta'(t)K(t - \theta(t))G(y(\theta(t))) - \int_0^{\theta(t)} K(t - s)G(y(s))ds.
\] (3.1.2.2)

Thus the remaining lines of the proof are easily generated by using the method of steps (cf. Brunner and Zhang [28]).

For the equation:

\[
\begin{cases}
\frac{d}{dt} \left( y(t) - \int_0^{\theta(t)} (t - s)^{-\alpha}G(y(s))ds \right) = f(t, y(t), y(\theta(t))), & t \in I := [0, T], \\
y(t) = \phi(t), & t \in [\bar{a}, 0],
\end{cases}
\] (3.1.2.3)
where $1 < \alpha < 1$ and assuming (i), (ii), (iii) in (3.1.2.1) hold, we have Theorem 3.1.2.2.

**Theorem 3.1.2.2.** The primary discontinuities of problem (3.1.2.3) are the points $\xi_\mu$ ($\mu = 0, 1, \ldots$) generated by (3.1.1.2). To be more precise, $y^{(\mu)}$ is continuous at $\xi_\mu$ but $y^{(\mu+1)}$ is, in general, not provided the assumptions (i) and (ii) hold. If, in addition, we assume $y \in C^{1-\alpha}(J[0])$, then $y \in C^{\mu,1-\alpha}(J^{[\mu]})$, but $y^{(\mu+1)}$ is not continuous at $\xi_{\mu+1}$.

**Proof.** Since

$$
\frac{d}{dt} \left( y(t) - \int_0^{\theta(t)} (t-s)^{-\alpha} G(y(s))ds \right) = y'(t) - \theta'(t)(t-\theta(t))^{-\alpha} G(y(\theta(t))) + \alpha \int_0^{\theta(t)} (t-s)^{-1-\alpha} G(y(s))ds,
$$

and $t-\theta(t) = \tau(t) \geq \tau_0 > 0$ hold, assuming that $y$ is continuous at the point $\xi_0 := 0$, we can realize the assertion with the method of steps as in Brunner and Zhang [28]. If we assume, in addition, $y \in C^{1-\alpha}(J[0])$, then we can prove the second assertion still by using the method of steps.

We now consider

$$
\begin{cases}
\frac{d}{dt} \left( y(t) - \int_\theta^{\theta(t)} K(t-s)G(y(s))ds \right) = f(t, y(t), y(\theta(t))), & t \in I := [0, T], \\
y(t) = \phi(t), & t \in [\bar{a}, 0],
\end{cases}
$$

(3.1.2.5)

where we assume $K$ is sufficiently smooth and (i), (ii), (iii) in (3.1.1.1) hold. The primary discontinuities of problem (3.1.2.5) are described by Theorem 3.1.2.3.

**Theorem 3.1.2.3.** The primary discontinuities of problem (3.1.2.5) are the points $\xi_\mu$ ($\mu = 0, 1, \ldots$) generated by (3.1.1.2). More precisely, $y^{(\mu)}$ is continuous at $\xi_\mu$ but $y^{(\mu+1)}$ is not, in general, provided the assumptions (i) and (ii) hold.
Proof. We rewrite the left-hand side of (3.1.2.5)
\[
\frac{d}{dt} \left( y(t) - \int_{\theta(t)}^{t} K(t - s)G(y(s))ds \right)
\]
\[
= y'(t) + \theta'(t)K(t - \theta(t))G(y(\theta(t))) - K(0)G(y(t))
\]
\[
- \int_{\theta(t)}^{t} K_t(t - s)G(y(s))ds.
\]
(3.1.2.6)
Hence the proof can be completed by using the method of steps (cf. Brunner and Zhang [28]).

Consider now
\[
\left\{ \begin{array}{l}
\frac{d}{dt} \left( y(t) - \int_{\theta(t)}^{t} (t - s)^{-\alpha}G(y(s))ds \right) = f(t, y(t), y(\theta(t))), \quad t \in I := [0, T], \\
y(t) = \phi(t), \quad t \in [\bar{a}, 0],
\end{array} \right.
\]
(3.1.2.7)
where we assume $0 < \alpha < 1$ and (i), (ii), (iii) in (3.1.1.1). This case is different from
the one treated in Theorem 3.1.2.2, since we cannot use the techniques in the proof
of Theorem 3.1.2.1. We describe the primary discontinuities of problem (3.1.2.7) in
the following theorem.

**Theorem 3.1.2.4.** There is no smoothing to the solution of (3.1.2.7). To be more
precise. $y \in C^{1,1-\alpha}(J^{[\mu]})$, for all $\mu \geq 1$, where $C$ is independent on $\mu$, provided the
assumptions (i) and (ii) hold.

Proof. Consider first the regularity of the solution for (3.1.2.7) at the point $\xi_0 := 0$.
It is possible to choose $y(0) = \phi(0)$. The continuity of the derivative of the solution
can be guaranteed at the initial point $\xi_0 := 0$ only for $\phi(t)$ satisfying the condition
\[
\phi'(0-) = \frac{d}{dt} \left( \int_{\theta(t)}^{t} (t - s)^{-\alpha}G(\phi(s))ds \right) + f(0, \phi(0), \phi(\theta(0))).
\]
Consider now the regularity at the point $\xi_1$. We write the equation (3.1.2.7) as
\[
\left\{ \begin{array}{l}
y'(t) = f(t, y(t), y(\theta(t))) + \frac{d}{dt} (H_1(t)), \quad t \in I := [0, T], \\
y(t) = \phi(t), \quad t \in [\bar{a}, 0],
\end{array} \right.
\]
where

\[ H_1(t) := \int_{\theta(t)}^{t} (t - \theta(t))^{-\alpha} \overline{G}(s) ds, \]

where \( \overline{G}(s) := G(y(s)) \). From the formula (3.1.1.3), we obtain

\[ H_1(t) = \frac{1}{1 - \alpha} \overline{G}(\theta(t)) t^{1-\alpha} + \frac{1}{1 - \alpha} \int_{\theta(t)}^{t} \overline{G}'(s)(t - s)^{1-\alpha} ds. \]

Thus

\[ \frac{d}{dt} (H_1(t)) = \overline{G}(\theta(t))(t - \theta(t))^{-\alpha} (1 - \theta'(t)) + \int_{\theta(t)}^{t} \overline{G}'(s)(t - s)^{-\alpha} ds, \]

where \( \overline{G}'(s) = G_y(y(s)) y'(s) \). Similarly, we calculate that

\[ \frac{d^2 H_1(t)}{dt^2} = \left[ \overline{G}(\theta(t)) \cdot (t - \theta(t))^{-\alpha} \cdot (1 - \theta'(t)) \right]' + \overline{G}'(\theta(t))(t - \theta(t))^{-\alpha} (1 - \theta'(t)) + \int_{\theta(t)}^{t} \overline{G}''(s)(t - s)^{-\alpha} ds. \]

Since

\[ \overline{G}'(\theta(t)) = G_y(y(\theta(t))) y'(\theta(t)). \]

we see that \( y'' \) is discontinuous at the point \( \xi_1 \) and thus there is no smoothing to the solution of (3.1.2.7). To prove that \( y \in C^{1,1-\alpha}(J^{[\alpha]}) \), it is sufficient to verify that

\[ \left| \int_{\theta(t)}^{t} (t - s)^{-\alpha} \overline{G}'(s) ds \right| \leq C(t - \xi_1)^{1-\alpha}. \tag{3.1.2.8} \]

Since \( \overline{G}'(t) \) is continuous in \( [\bar{a}, T] \), we have

\[ \left| \int_{\theta(t)}^{t} (t - s)^{-\alpha} \overline{G}'(s) ds \right| \leq L_{\overline{G}} \int_{\theta(t)}^{t} (t - s)^{-\alpha} ds \leq C(t - \theta(t))^{1-\alpha} \leq C(t - \xi_1)^{1-\alpha}. \tag{3.1.2.9} \]

In the last step of (3.1.2.9), we use that \( \theta \) is strictly increasing in \( I \) and thus

\[ \xi_1 = \theta(\xi_2) > \theta(t) > \theta(\xi_1) = \xi_0, \quad \forall t \in (\xi_1, \xi_2). \]
Following the above process we can establish that

\[ y'(t) \leq C(t - \xi_\mu)^{1-\alpha}, \ \forall \mu \geq 1. \]

### 3.1.3 Delay VIDEs of neutral type with smooth kernels

Consider the following delay VIDEs of neutral type:

\[
\begin{aligned}
y'(t) &= f(t, y(t), y(\theta(t))) + \int_0^t K(t-s)G(s, y(s), y'(s), y'(\theta(s)))ds, \quad t \in I := [0, T], \\
y(t) &= \phi(t), \quad t \in [a, 0],
\end{aligned}
\]

(3.1.3.1)

where we assume that

(a) \( K \) is sufficiently smooth, and

(b) (i), (ii), (iii) in (3.1.1.1) hold.

We describe the primary discontinuities of problem (3.1.3.1) in the following theorem.

**Theorem 3.1.3.1.** The primary discontinuities of problem (3.1.3.1) are generated inductively by the recursion (3.1.1.2), where \( \xi_0 := 0 \). More precisely, \( y^{(\mu)} \) and lower-order derivatives are continuous at \( \xi_\mu \), but \( y^{(\mu+1)} \) is, in general, not under the assumptions (a) and (b) for (3.1.3.1).

**Proof.** (1) We consider the regularity of the solution for (3.1.3.1) at the point \( \xi_0 := 0 \).

It is possible to choose \( \phi \) to satisfy \( y(0) = \phi(0) \). However in general, \( \phi'(0-) \neq y'(0+) \).

Hence \( y \) is continuous at \( \xi_\mu (\mu \geq 0) \), but \( y' \) is not continuous at \( \xi_0 \).

(2) Consider the regularity at \( \xi_1 \). Obviously, \( y' \) is continuous at \( \xi_\mu (\mu \geq 1) \).

But since the expression for \( y'' \) at the point \( \xi_1 \) include \( y' \) at the point \( \xi_0 \), \( y'' \) is not continuous at the point \( \xi_1 \).
(3) Consider the regularity at \( \xi_2 \). Set \( \overline{G}(s) := G(s, y(s), y'(s), y'(\theta(s))) \). We obtain

\[
y''(t) = f'(t, y(t), y(\theta(t))) + K(0)\overline{G}(t) + \int_0^t K(t - s)\overline{G}(s)ds.
\]

Hence \( y'' \) is continuous at \( \xi_\mu \) \((\mu \geq 2)\). But \( y^{(3)} \) is not continuous at \( \xi_2 \), since \( y^{(3)} \) includes \( y^{(2)} \) which is not continuous at \( \xi_2 \).

(4) Suppose \( y^{(\mu)} \) is continuous at \( \xi_\mu \) \((\mu \geq \mu)\). Consider now the regularity at \( \xi_{\mu+1} \). We know that \( y^{(\mu+1)}(\xi_\mu) \) includes \( y^{(m)}(\xi_\mu) \) \((m \leq \mu)\), so \( y^{(\mu+1)} \) is continuous at \( \xi_{\mu+1} \). But \( y^{(\mu+2)}(t) \) is discontinuous at \( \xi_{\mu+1} \), since \( y^{(\mu+2)}(\xi_{\mu+1}) \) includes \( y^{(\mu+1)}(\xi_\mu) \), and \( y^{(\mu+1)} \) is not continuous at \( \xi_{\mu+1} \).

### 3.1.4 Delay VIDEs of neutral type with weakly singular kernels

Consider

\[
\begin{cases}
y'(t) = f(t, y(t), y(\theta(t))) + \int_0^t (t - s)^{-\alpha}G(s, y(s), y'(s), y'(\theta(s)))ds, & t \in I := [0, T], \\
y(t) = \phi(t), & t \in [\bar{a}, 0],
\end{cases}
\]

where we assume \( 0 < \alpha < 1 \) and (i), (ii), (iii) in (3.1.1.1).

We describe the primary discontinuities of problem (3.1.4.1) as the following theorem.

**Theorem 3.1.4.1.** The primary discontinuities of problem (3.1.4.1) are generated inductively by the recursion (3.1.1.2), where \( \xi_0 := 0 \). More precisely, under the assumption (i), (ii) and (iii) in (3.1.1.1), \( y \in C^{\mu,1-\alpha}(J^{[\mu]}) \), but \( y^{(\mu+1)} \) is not continuous at \( \xi_{\mu} \) in general.

**Proof.** (1) We consider the regularity of solution for (3.1.4.1) at the point \( \xi_0 := 0 \). It is possible to choose \( \phi \) to satisfy \( y(0) = \phi(0) \). However in general \( \phi'(0-) \neq y'(0+) \).
Hence $y$ is continuous at $\xi_\mu$ ($\mu \geq 0$), but $y'$ is not continuous at $\xi_0$.

(2) Consider the regularity at $\xi_1$. Obviously, $y'$ is continuous at $\xi_1$, hence $y'$ is continuous at $\xi_\mu$ ($\mu \geq 1$). We can prove

$$H_1(t) := \int_0^t (t-s)^{-\alpha} \mathcal{G}(s) ds \in C^{1-\alpha}(J^{[1]}),$$

by following the lines proving (3.1.1.4).

(3) Consider the regularity at $\xi_2$. We write the equation (3.1.4.1) as

$$\begin{cases} y'(t) = f(t, y(t)) + \int_0^{\xi_2} (t-s)^{-\alpha} \mathcal{G}(s) ds + H_2(t), & t \in I := [0, T], \\ y(t) = \phi(t), & t \in [\bar{a}, 0] \end{cases}$$

where $H_2(t) := \int_{\xi_2}^t (t-s)^{-\alpha} \mathcal{G}(s) ds$. By using the formula (3.1.1.3), we obtain

$$H_2(t) := \frac{1}{1-\alpha} \mathcal{G}(\xi_1)(t-\xi_1)^{1-\alpha} + \frac{1}{1-\alpha} \int_{\xi_1}^t \mathcal{G}'(s)(t-s)^{1-\alpha} ds$$

$$H_2'(t) = \bar{\mathcal{G}}(\xi_1)(t-\xi_1)^{-\alpha} + \int_{\xi_1}^t \mathcal{G}'(s)(t-s)^{-\alpha} ds.$$ 

Hence $y''$ is continuous at $\xi_2$. Thus $y''$ is continuous at $\xi_\mu$ ($\mu \geq 2$). Furthermore we can prove that $y'' \in C^{1-\alpha}(J^{[\mu]})$.

We consider the regularity at $\xi_{\mu+1}$. We rewrite the equation (3.1.4.1) as

$$\begin{cases} y'(t) = f(t, y(t)) + \int_0^{\xi_\mu} (\xi_\mu-s)^{-\alpha} \mathcal{G}(s) ds + H_{\mu+1}(t), & t \in I := [0, T], \\ y(t) = \phi(t), & t \in [\bar{a}, 0] \end{cases}$$

where $H_{\mu+1}(t) = \int_{\xi_\mu}^t (t-s)^{-\alpha} \mathcal{G}(s) ds$.

$$H_{\mu+1}(t) = \frac{1}{1-\alpha} \mathcal{G}(\xi_{\mu})(t-\xi_{\mu})^{1-\alpha} + \frac{1}{(1-\alpha)(2-\alpha)} \mathcal{G}(\xi_{\mu})(t-\xi_{\mu})^{2-\alpha}$$

$$+ \cdots + \frac{1}{(1-\alpha)_{\mu+1}} \mathcal{G}^{(\mu)}(\xi_{\mu})(t-\xi_{\mu})^{\mu+1-\alpha}$$

$$+ \frac{1}{(1-\alpha)_{\mu+1}} \int_{\xi_\mu}^t \mathcal{G}^{(\mu+1)}(s)(t-s)^{\mu+1-\alpha} ds. \quad (3.1.4.2)$$
Thus we can calculate the derivatives of order up to $\mu + 1$ of $H_{\mu+1}(t)$ as:

\[
H'_{\mu+1}(t) = G(\xi_{\mu})(t - \xi_{\mu})^{-\alpha} + \frac{1}{1 - \alpha} G'(\xi_{\mu})(t - \xi_{\mu})^{1-\alpha} + \cdots + \frac{1}{(1 - \alpha)^{\mu+1}} \int_{\xi_{\mu}}^{t} G^{(\mu+1)}(s)(t - s)^{-\alpha} ds.
\]

Hence $y^{(\mu+1)}(t)$ is continuous at $\xi_{\mu+1}$. Furthermore we know that

\[
\int_{\xi_{\mu}}^{t} G^{(\mu+1)}(s)(t - s)^{-\alpha} ds \in C^{1-\alpha}(J^{[\mu+1]}),
\]

thus $y \in C^{\mu+1,1-\alpha}(J^{[\mu+1]})$.

### 3.2 The discontinuous Galerkin method for delay VIDEs

#### 3.2.1 The discontinuous Galerkin method for functional VIDEs of Hale's type

In this section we analyze the discontinuous Galerkin method for

\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{d}{dt} \left( y(t) - \int_{0}^{\theta(t)} K(t - s)G(y(s))ds \right) = f(t, y(t), y(\theta(t))), \quad t \in I := [0, T], \\
y(t) = \phi(t), \quad t \in [\bar{a}, 0],
\end{array} \right.
\end{aligned}
\]

(3.2.1.1)

where we assume $y \in \Omega \subset \mathbb{R}$ and

(a) $K$ is sufficiently smooth.

(b) (i), (ii), (iii) in (3.1.1.1).
Without loss of the generality we set $T = \xi_{M+1}$ for some $M \geq 1$, and introduce $Z_M := \{\xi_\mu : \mu = 0, 1, \ldots, M\}$.

Since, as we have already seen in Section 3.1.2, the solution of (3.2.1.1) suffers from a loss of regularity at the primary discontinuity points $\{\xi_\mu\}$, the meshes $I_h$ underlying the DG space will have to include these points if the DG solution is to attain its optimal global (or local) order. Thus, we shall employ meshes of the form

$$I_h := \bigcup_{\mu=0}^{M} I_h^{[\mu]},$$

with the local mesh given by

$$I_h^{[\mu]} := \{t_n^{[\mu]} : \xi_\mu = t_0^{[\mu]} < t_1^{[\mu]} < \ldots < t_{N_\mu}^{[\mu]} = \xi_{\mu+1}\} \quad (\xi_\mu \in Z_M).$$

Such a mesh is called a constrained mesh (with respect to $\theta$) for $I$. We introduce the following notations

$$I^{[\mu]} := (\xi_\mu, \xi_{\mu+1}], \quad I_n^{[\mu]} := (t_n^{[\mu]} - t_{n-1}^{[\mu]}, \quad \bar{I}^{[\mu]} := [\xi_\mu, \xi_{\mu+1}], \quad \bar{I}_n^{[\mu]} := [t_{n-1}^{[\mu]}, t_n^{[\mu]}],$$

and

$$|I^{[\mu]}| := \xi_{\mu+1} - \xi_\mu, \quad h_n^{[\mu]} := t_n^{[\mu]} - t_{n-1}^{[\mu]}; \quad h^{[\mu]} := \max_{(n)} h_n^{[\mu]}.$$

Consider now the local graded meshes of the form

$$t_n^{[\mu]} := \left(\frac{n}{N_\mu}\right)^{r_\mu} \cdot |I^{[\mu]}| = \left(\frac{n}{N_\mu}\right)^{r_\mu} \cdot (\xi_{\mu+1} - \theta(\xi_{\mu+1})) = \left(\frac{n}{N_\mu}\right)^{r_\mu} \cdot \tau(\xi_{\mu+1}), \quad 0 \leq n \leq N_\mu - 1 \quad (N_\mu \geq 2),$$

where the grading exponent $r_\mu \in \mathbb{R}$ will always be assumed to satisfy $r_\mu \geq 1$. We know that

$$h_n^{[\mu]} \leq h^{[\mu]} \leq r_\mu \cdot \tau(\xi_{\mu+1}) N_\mu^{-1}, \quad 0 \leq n \leq N_\mu - 1 \quad (N_\mu \geq 2).$$
In this presentation, for simplicity, we take $r_\mu = 1 \ (\forall \xi_\mu \in Z_M)$ (i.e., each $I_{h}^{[\mu]}$ is a uniform mesh) and $N_\mu := (N)^{\frac{1}{\mu+1}}$ for all $\xi_\mu \in Z_M$.

We recall the problem (3.2.1.1) and denote

\[ y^{[\mu]} := y|_{I^{[\mu]}}. \]

Let

\[ z(t) := y(t) - \int_{0}^{\theta(t)} K(t-s)G(y(s))ds. \] (3.2.1.4)

It is easily seen from (3.2.1.4) that $z(t)$ and $y(t)$ possess the same regularity. We have

\[ z'(t) = f(t, y(t), y(\theta(t))) \]
\[ = f \left( t, z(t) + \int_{0}^{\theta(t)} K(t-s)G(y(s))ds, y(\theta(t)) \right). \] (3.2.1.5)

We write (3.2.1.4) and (3.2.1.5) locally on the interval $I^{[\mu]}$.

\[ y^{[\mu]} = z^{[\mu]} + \sum_{i=0}^{\mu-2} \int_{I^{[i]}} K(t-s)G(y^{[i]}(s))ds + \int_{\xi_{\mu-1}}^{\theta(t)} K(t-s)G(y^{[\mu-1]}(s))ds. \] (3.2.1.6)

\[ (z^{[\mu]})' = f \left( t, z^{[\mu]} + \sum_{i=0}^{\mu-2} \int_{I^{[i]}} K(t-s)G(y^{[i]}(s))ds \right. \]
\[ + \left. \int_{\xi_{\mu-1}}^{\theta(t)} K(t-s)G(y^{[\mu-1]}(s))ds, y^{[\mu-1]}(\theta(t)) \right). \] (3.2.1.7)

Here we set $y^{[0]} := \phi(0)$.

We abbreviate

\[ \bar{f}(t) := f \left( t, z(t) + \int_{0}^{\theta(t)} K(t-s)G(y(s))ds, y(\theta(t)) \right), \]
\[ \bar{f}^{[\mu]} := f \left( t, z^{[\mu]} + \sum_{i=0}^{\mu-2} \int_{I^{[i]}} K(t-s)G(y^{[i]}(s))ds \right. \]
\[ + \left. \int_{\xi_{\mu-1}}^{\theta(t)} K(t-s)G(y^{[\mu-1]}(s))ds, y^{[\mu-1]}(\theta(t)) \right). \]
Define:

\[ V[\Lambda](I_h) := \left\{ \varphi \in L^2(I) : \varphi|_{I^n_\mu} \in P^{(\mu+1)}(I^n_\mu), \ \mu \in \mathbb{Z}_M \right\}, \]

where \( \Lambda := \{\mu + 1\}_{\mu=0}^M. \)

Now we are ready to define the \( DG(\Lambda) \) scheme to (3.2.1.1): Find \( z_h \in V[\Lambda](I_h), \)

such that
\[
\sum_{\mu=0}^M \sum_{n=1}^{N_\mu} \left( \int_{I^n_\mu} \left( [z^{[\mu]}_h]' - f^{[\mu]}_h(t) \right) X(t) dt + [z^{[\mu]}_h]_{n-1} X^+_n \right) + (z^{[0]}_h)_0^+ X^+_0 = \phi(0) X^+_0 \tag{3.2.1.8}
\]

for all \( X \in V[\Lambda](I_h). \) Here,

\[
f^{[\mu]}_h(t) := \frac{d}{dt} f(t, z_h(t) + \int_0^{\theta(t)} K(t-s) G(y_h(s)) ds, y_h(\theta(t))),
\]

\[
 f^{[\mu]}_h(t) := f \left( t, z^{[\mu]}_h + \sum_{i=0}^{\mu-2} \int_{I^{[i]}} K(t-s) G(y^{[i]}_h(s)) ds + \int_{\xi_{\mu-1}}^{\theta(t)} K(t-s) G(y^{[\mu-1]}_h(s)) ds, y^{[\mu-1]}_h(\theta(t)) \right).
\]

This represents a time-stepping method: Find

\[
z^{[\mu]}_h \in V^{(\mu)}(I^{[\mu]}) := \left\{ \varphi \in L^2(I^{[\mu]}) : \varphi|_{I^n_\mu} \in P^{(\mu+1)}(I^n_\mu) \right\},
\]

such that

\[
\sum_{n=1}^{N_\mu} \int_{I^n_\mu} \left( [z^{[\mu]}_h]' - f^{[\mu]}_h(t) \right) X(t) dt + \sum_{n=2}^{N_\mu} [z^{[\mu]}_h]_{n-1} X^+_n + (z^{[\mu]}_h)_{n-1}^+ X^+_n = (z^{[\mu]}_h)_0^+ X^+_0, \ \forall X \in V^{(\mu)}(I^{[\mu]}), \tag{3.2.1.9}
\]

for \( \mu = 0, 1, \ldots, M. \) Here, we set \( (z^{[\mu]}_h)_0^- := (z^{[\mu-1]}_h)_0^- \) (for \( \mu = 1, 2, \ldots, M \)) and \( (\xi^{[\mu]}_0)^- = \phi(0) \) (for \( \mu = 0 \)).

Also, (3.2.1.9) can be interpreted as a local time stepping method: Find \( z^{[\mu]}_h \in P^{(\mu+1)}(I^{[\mu]}), \)

such that

\[
\int_{I^{[\mu]}_n} \left( [z^{[\mu]}_h]' - f^{[\mu]}_h(t) \right) X(t) dt + (z^{[\mu]}_h)_{n-1}^+ X^+_n = (z^{[\mu]}_h)_n^- X^+_n \tag{3.2.1.10}
\]
for all $X \in \mathcal{P}^{(\mu+1)}(I^\mu_n)$. Again, we set $(z^\mu_n)_0^+ := (z^{\mu-1}_n)_0^-$. Hence define

$$
y^\mu_h := z^\mu_h + \sum_{i=0}^{\mu-2} \int_{I^i} K(t-s)G(y^i_h(s))ds + \int_{\xi_{\mu-1}}^\theta(t) K(t-s)G(y^i_h(s))ds.$$

(3.2.1.11)

We set

$$e_y := y - y_h, \quad e_z := z - z_h, \quad e^\mu_y := e_y|_{J^\mu}, \quad e^\mu_z := e_z|_{J^\mu}.$$

Subtracting (3.2.1.11) from (3.2.1.6) we obtain

$$e^\mu_y := e^\mu_z + \sum_{i=0}^{\mu-2} \int_{I^i} K(t-s)[G(y^i(s)) - G(y^i_h(s))]ds + \int_{\xi_{\mu-1}}^\theta(t) K(t-s)[G(y^i(s)) - G(y^i_h(s))]ds.$$

Thus we have

$$||e^\mu_y||_{J_{\mu}} \leq ||e^\mu_z||_{J_{\mu}} + \sum_{i=0}^{\mu-2} K\mathcal{L}_G ||I^i|| ||e^i_y||_{J_{\mu}} + K\mathcal{L}_G ||I^{\mu-1}|| ||e^{\mu-1}_z||_{J_{\mu-1}},$$

where $\mathcal{K} := ||K||_{[a,T]}$, $L_f := ||f||_{\Omega}$, $L_G := ||G||_{\Omega}$. Hence, from Gronwall’s lemma, we have

$$||e^\mu_y||_{J_{\mu}} \leq ||e^\mu_z||_{J_{\mu}} \exp(\mathcal{K}L_G \xi_{\mu}).$$

(3.2.1.12)

Now we estimate the error $||e^\mu_z||_{J_{\mu}}$. Define the projection $Iz^\mu \in \mathcal{V}^{(\mu)}(I^\mu)$, $\xi_{\mu} \in Z_M$, by

$$(Iz^\mu)_n := (z^\mu)_n^-, \quad 1 \leq n \leq N_{\mu},$$

(3.2.1.13)

$$\int_{J^\mu_n} (Iz^\mu) X'(t)dt := \int_{J^\mu_n} z^\mu X'(t)dt$$

(3.2.1.14)

for all $X \in \mathcal{P}^{(\mu+1)}(I^\mu_n), \quad 1 \leq n \leq N_{\mu}$. The approximation properties of $I$ in (3.2.1.13) and (3.2.1.14) have been thoroughly investigated in [96]: On the generic subinterval $I^\mu_n$ there holds

$$||z^\mu - Iz^\mu||_{J^\mu_n} \leq C(h^\mu_n)^{\mu+1}||z^\mu||^{(\mu+1)}_{J^\mu_n},$$

(3.2.1.15)
where $C$ is independent of $h_n^{[\mu]}$.

We split the error $e^{[\mu]} = z^{[\mu]} - z_h^{[\mu]} := \rho^{[\mu]} + \eta^{[\mu]}$ into $\rho^{[\mu]} := z^{[\mu]} - \mathcal{I}z^{[\mu]}$ and $\eta^{[\mu]} := \mathcal{I}z^{[\mu]} - z_h^{[\mu]}$. It is easily seen that $\eta^{[\mu]}$ satisfies
\begin{equation}
\int_{t_n^{[\mu]}}^{t_{n+1}^{[\mu]}} (\eta^{[\mu]})'X(t)dt + (\eta^{[\mu]})_{n-1}^+X_n^+ = \int_{t_n^{[\mu]}}^{t_{n+1}^{[\mu]}} [\tilde{f}^{[\mu]}(t) - \tilde{f}_h^{[\mu]}(t)]X(t)dt + (\eta^{[\mu]})_{n-1}^-X_n^-,
\end{equation}
for all $X \in \mathcal{P}^{(\mu+1)}(I_n^{[\mu]})$. Equivalently,
\begin{equation}
-\int_{t_n^{[\mu]}}^{t_{n+1}^{[\mu]}} (\eta^{[\mu]})X'(t)dt + (\eta^{[\mu]})_{n}^-X_n^- = \int_{t_n^{[\mu]}}^{t_{n+1}^{[\mu]}} [\tilde{f}^{[\mu]}(t) - \tilde{f}_h^{[\mu]}(t)]X(t)dt + (\eta^{[\mu]})_{n-1}^-X_n^-.
\end{equation}
for all $X \in \mathcal{P}^{(\mu+1)}(I_n^{[\mu]})$.

Lemma 3.2.1.1. We have
\begin{equation}
[(\eta^{[\mu]})_{n}^-]^2 \leq 2L_f \int_{I_n^{[\mu]}} |\rho^{[\mu]}|^2 dt + (4L_f + L_GK) \int_{I_n^{[\mu]}} [\eta^{[\mu]}]^2 dt + L_f h_n^{[\mu]} \sum_{i=0}^{\mu-1} |I^{[i]}||e^{[i-1]}||t_{I^{[i]}}|| \leq \sum_{i=0}^{\mu-1} |I^{[i]}||e^{[i]}||t_{I^{[i]}}|| + [(\eta^{[\mu]})_{n-1}^-]^2.
\end{equation}

Proof. We take $X = \eta^{[\mu]}$ in (3.2.1.16) and obtain
\begin{align*}
\frac{1}{2}[(\eta^{[\mu]})_{n}^-]^2 + \frac{1}{2}[(\eta^{[\mu]})_{n-1}^+]^2 & \leq L_f \int_{I_n^{[\mu]}} \left( \left| e^{[\mu]}_z + \sum_{i=0}^{\mu-2} \int_{I^{[i]}} K(t-s)[G(y^{[i]}(s)) - G(y_h^{[i]}(s))]ds \right| \right. \\
& \left. + \int_{\xi_{n-1}}^{\theta(t)} K(t-s)[G(y_h^{[i]}(s)) - G(y_h^{[i-1]}(s))]ds \right) |\eta^{[\mu-1]}|dt + \left. \frac{1}{2}[(\eta^{[\mu]})_{n-1}^-]^2 \right. \\
& \left. + \frac{1}{2}[(\eta^{[\mu]})_{n-1}^+]^2 \right).
\end{align*}
This yields
\begin{align*}
[(\eta^{[\mu]})_{n}^-]^2 & \leq 2L_f \int_{I_n^{[\mu]}} |e^{[\mu]}_z||\eta^{[\mu]}|dt + 2L_f \int_{I_n^{[\mu]}} |e^{[\mu-1]}(\theta(t))| \cdot |\eta^{[\mu]}|dt \\
& \quad + 2L_f L_GK \int_{I_n^{[\mu]}} \left( \sum_{i=0}^{\mu-1} |I^{[i]}||e^{[i]}|t_{I^{[i]}}|| \right) |\eta^{[\mu]}|dt + [(\eta^{[\mu]})_{n-1}^-]^2.
\end{align*}
Since $|e^{[\mu]}_z||\eta^{[\mu]}| \leq |\rho^{[\mu]}|^2 + \frac{3}{2}|\eta^{[\mu]}|^2$, the lemma is proved.

**Lemma 3.2.1.2.** We have

\[
\int_{I_k}^n [(\eta^{[\mu]})']^2 (t - t_{n-1}^{[\mu]}) dt \\
\leq 6h_n^2 L_f^2 \int_{I_k}^n |\rho^{[\mu]}|^2 dt + 6h_n^2 L_f^2 \int_{I_k}^n |\eta^{[\mu]}|^2 dt + 3(h_n^2) L_f^2 ||e^{[\mu-1]}_y||^2_{\tilde{I}_{n-1}} \\
+ 3(h_n^2) L_f^2 L_G K^2 \mu \max_{i=0}^{\mu-1} |I_{[i]}|^2 \sum_{i=0}^{\mu-1} ||e^{[i]}_y||^2_{\tilde{I}_{[i]}}.
\]

**Proof.** To verify the lemma, we select $X = (\eta^{[\mu]})' (t - t_{n-1}^{[\mu]})$ in (3.2.1.17) and obtain

\[
\int_{I_k}^n [(\eta^{[\mu]})']^2 (t - t_{n-1}^{[\mu]}) dt = \int_{I_k}^n [\tilde{f}^{[\mu]}(t) - \tilde{f}_h^{[\mu]}(t)] (\eta^{[\mu]})' (t - t_{n-1}^{[\mu]}) dt \\
\leq \left( \int_{I_k}^n (t - t_{n-1}^{[\mu]}) [\tilde{f}^{[\mu]}(t) - \tilde{f}_h^{[\mu]}(t)]^2 dt \right)^{1/2} \cdot \left( \int_{I_k}^n (t - t_{n-1}^{[\mu]}) [\eta^{[\mu]}']^2 dt \right)^{1/2}.
\]

Hence we have

\[
\int_{I_k}^n [(\eta^{[\mu]})']^2 (t - t_{n-1}^{[\mu]}) dt \\
\leq h_n^2 \int_{I_k}^n [\tilde{f}^{[\mu]}(t) - \tilde{f}_h^{[\mu]}(t)]^2 dt \\
\leq h_n^2 \int_{I_k}^n \left[ L_f |e^{[\mu]}_z| + L_f |e^{[\mu-1]}_y| (\theta(t)) + L_f L_G K \sum_{i=0}^{\mu-1} |I_{[i]}| \cdot ||e^{[i]}_y||_{\tilde{I}_{[i]}} \right]^2 dt \\
\leq 3h_n^2 L_f^2 \int_{I_k}^n |e^{[\mu]}_z|^2 dt + 3(h_n^2) L_f^2 ||e^{[\mu-1]}_y||^2_{\tilde{I}_{n-1}} \\
+ 3(h_n^2) L_f^2 L_G K^2 \mu \max_{i=0}^{\mu-1} |I_{[i]}|^2 \sum_{i=0}^{\mu-1} ||e^{[i]}_y||^2_{\tilde{I}_{[i]}} \\
\leq 6h_n^2 L_f^2 \int_{I_k}^n |\rho^{[\mu]}|^2 dt + 6h_n^2 L_f^2 \int_{I_k}^n |\eta^{[\mu]}|^2 dt + 3(h_n^2) L_f^2 ||e^{[\mu-1]}_y||^2_{\tilde{I}_{n-1}} \\
+ 3(h_n^2) L_f^2 L_G K^2 \mu \max_{i=0}^{\mu-1} |I_{[i]}|^2 \sum_{i=0}^{\mu-1} ||e^{[i]}_y||^2_{\tilde{I}_{[i]}}, \tag{3.2.1.18}
\]

thus completing the proof.
Lemma 3.2.1.3. We have

\[ \left( \int_{I^\mu_n} \eta^{[\mu]}(t) \right)^2 \leq 2 (h_n^{[\mu]})^2 [(\eta^{[\mu]}^{-1})^2 + 2 \int_{I_n^\mu} [\eta^{[\mu]} - \eta^{[\mu]}]_n^2 dt + 4 (h_n^{[\mu]})^3 L^2 \int_{I_n^\mu} [\rho^{[\mu]}]_n^2 dt + 4 (h_n^{[\mu]})^3 L^2 \int_{I_n^\mu} [\eta^{[\mu]}]_n^2 dt + \frac{4}{3} (h_n^{[\mu]})^6 L^2 \sum_{i=0}^{\mu-1} \max I_i^{[\mu]} \sum_{i=0}^{\mu-1} \left\| e_i^{[\mu]} \right\|_2^2. \]

Proof. We choose \( X = t_{n+1}^{[\mu]} - t \) in (3.2.1.17) to obtain

\[ \int_{I_n^\mu} \eta^{[\mu]}(t) dt - h_n^{[\mu]} \eta^{[\mu]} \eta^{[\mu]} = \int_{I_n^\mu} [f^{[\mu]}(t) - f^{[\mu]}(t)](t_{n+1}^{[\mu]} - t) dt. \]

Hence, by the Cauchy-Schwarz inequality, we have

\[ \left( \int_{I_n^\mu} \eta^{[\mu]}(t) dt \right)^2 \leq 2 (h_n^{[\mu]})^2 [(\eta^{[\mu]}^{-1})^2 + 2 \int_{I_n^\mu} [f^{[\mu]}(t) - f^{[\mu]}(t)]^2 dt \int_{I_n^\mu} (t_{n+1}^{[\mu]} - t)^2 dt. \]

Recalling the second inequality of (3.2.1.18) we finish the proof.

To derive the error estimates we need also the following two lemmas from [96].

Lemma 3.2.1.4. There holds

\[ \int_{I_n^\mu} [\varphi(t)]^2 dt \leq \frac{1}{h_n^{[\mu]}} \int_{I_n^\mu} \varphi(t) dt + \frac{1}{2} \int_{I_n^\mu} (t_{n+1}^{[\mu]} - t)(t - t_{n+1}^{[\mu]}) [\varphi(t)]^2 dt \quad (3.2.1.19) \]

for all \( \varphi(t) \in \mathcal{P}^{\mu+1}(I_n^{[\mu]}). \)

Lemma 3.2.1.5. There holds

\[ \| \varphi \|_{I_n^{[\mu]}}^2 \leq C \log(\mu + 2) \int_{I_n^{[\mu]}} [\varphi(t)]^2 (t - t_{n+1}^{[\mu]}) dt + C [\varphi^{-1}]^2 \]

for all \( \varphi \in \mathcal{P}^{\mu+1}(I_n^{[\mu]}), \mu \geq 0. \) The constant \( C \) is independent of \( I_n^{[\mu]} \) and \( \mu \).

Now we are ready to prove the error estimates.
Corollary 3.2.1.1. Let $K := \|K\|_{[0,T]}$, $L_f := \|f\|_\Omega$ and $L_G := \|G\|_\Omega$. Then, for $h_n^{[\mu]} L_f$ small enough, we have

$$
\|e_z^{[\mu]}\|_{l^{[\mu]}} \leq C(L_f, L_G, K, T, \mu) \|z^{[\mu]} - I z^{[\mu]}\|_{l^{[\mu]}}. \tag{3.2.1.20}
$$

Proof. Combine Lemma 3.2.1.3 and Lemma 3.2.1.4 into

\[
\left(\int_{t_n^{[\mu]}} \eta^{[\mu]}(t) dt \right)^2 \leq c(h_n^{[\mu]} )^2 \left[ (\eta^{[\mu]} )_{n-1} \right]^2 + c(h_n^{[\mu]} )^3 L_f^2 \int_{t_n^{[\mu]}} [\rho^{[\mu]} ]^2 dt \\
+ c(h_n^{[\mu]} )^2 L_f^2 \left( \int_{t_n^{[\mu]}} \eta^{[\mu]}(t) dt \right)^2 + c(h_n^{[\mu]} )^4 L_f^2 \int_{t_n^{[\mu]}} \left[(\eta^{[\mu]} \right)^2 (t - t_n^{[\mu]}) dt \\
+ c(h_n^{[\mu]} )^6 L_f^2 \|e_y^{[\mu-1]}\|_{l^{[\mu-1]}}^2 + c(h_n^{[\mu]} )^6 L_f^2 L_G^2 \|K^{[\mu]}\|_{l^{[\mu]}} \mu \max_{i=0}^{\mu-1} |I[i]|^2 \sum_{i=0}^{\mu-1} ||e_y^{[i]}||_{l^{[i]}}^2 ,
\]

where $c$ is a generic constant independent of any parameter. Hence for $h_n^{[\mu]} L_f$ small enough, we have

\[
\left(\int_{t_n^{[\mu]}} \eta^{[\mu]}(t) dt \right)^2 \leq c(h_n^{[\mu]} )^2 \left[ (\eta^{[\mu]} )_{n-1} \right]^2 + c(h_n^{[\mu]} )^3 L_f^2 \int_{t_n^{[\mu]}} [\rho^{[\mu]} ]^2 dt \\
+ c(h_n^{[\mu]} )^4 L_f^2 \int_{t_n^{[\mu]}} \left[(\eta^{[\mu]} \right)^2 (t - t_n^{[\mu]}) dt + c(h_n^{[\mu]} )^6 L_f^2 \|e_y^{[\mu-1]}\|_{l^{[\mu-1]}}^2 \\
+ c(h_n^{[\mu]} )^6 L_f^2 L_G^2 \|K^{[\mu]}\|_{l^{[\mu]}} \mu \max_{i=0}^{\mu-1} |I[i]|^2 \sum_{i=0}^{\mu-1} ||e_y^{[i]}||_{l^{[i]}}^2 . \tag{3.2.1.21}
\]
We combine Lemma 3.2.1.1, Lemma 3.2.1.2 and Lemma 3.2.1.4 to obtain

\[
\int_{I_n^\mu} [(\eta^\mu(t))']^2 (t - t_{n-1}^\mu) dt + [(\eta^\mu)_n^-]^2
\]

\[
\leq cL_f \int_{I_n^\mu} [\rho^\mu]^2 dt + cL_f \int_{I_n^\mu} [\eta^\mu]^2 dt + [(\eta^\mu)_n^-]^2
\]

\[
+ cL_f h_n^\mu ||e_{y}^{\mu-1}||^2_{f_{y u-1}} + cL_f L_G K T h_n^\mu \sum_{i=0}^{\mu-1} ||e_{y}^i||^2_{f_{y i}}
\]

\[
\leq cL_f \int_{I_n^\mu} [\rho^\mu]^2 dt + \frac{cL_f}{h_n^\mu} \left( \int_{I_n^\mu} [\eta^\mu] dt \right)^2 + cL_f h_n^\mu \int_{I_n^\mu} [(\eta^\mu)]^2 (t - t_{n-1}^\mu) dt
\]

\[
+ [(\eta^\mu)_n^-]^2 + cL_f h_n^\mu ||e_{y}^{\mu-1}||^2_{f_{y u-1}} + cL_f L_G K T h_n^\mu \sum_{i=0}^{\mu-1} ||e_{y}^i||^2_{f_{y i}}.
\]

Using (3.2.1.21) we obtain

\[
\int_{I_n^\mu} [(\eta^\mu(t))']^2 (t - t_{n-1}^\mu) dt + [(\eta^\mu)_n^-]^2
\]

\[
\leq cL_f \int_{I_n^\mu} [\rho^\mu]^2 dt + cL_f h_n^\mu L_f \int_{I_n^\mu} [(\eta^\mu(t))']^2 (t - t_{n-1}^\mu) dt + cL_f h_n^\mu [(\eta^\mu)_n^-]^2
\]

\[
+ [(\eta^\mu)_n^-]^2 + cL_f h_n^\mu ||e_{y}^{\mu-1}||^2_{f_{y u-1}} + cL_f L_G K T h_n^\mu \sum_{i=0}^{\mu-1} ||e_{y}^i||^2_{f_{y i}}. \quad (3.2.1.22)
\]

Iterating (3.2.1.22) yields

\[
\int_{I_n^\mu} [(\eta^\mu(t))']^2 (t - t_{n-1}^\mu) dt + [(\eta^\mu)_n^-]^2
\]

\[
\leq cL_f \sum_{i=1}^{n} h_i^\mu ||\rho^\mu||^2_{f_{i}^\mu} + cL_f \sum_{i=1}^{n} \left( \int_{I_n^\mu} [(\eta^\mu)]^2 (t - t_{i-1}^\mu) dt + [(\eta^\mu)_i^-]^2 \right)
\]

\[
+ cL_f |I^\mu| \cdot ||e_{y}^{\mu-1}||^2_{f_{y u-1}} + cL_f L_G K T |I^\mu| \sum_{i=0}^{\mu-1} ||e_{y}^i||^2_{f_{y i}}.
\]

For all sufficiently small \(h_n^\mu L_f\), Gronwall's lemma can be applied and gives

\[
\int_{I_n^\mu} [(\eta^\mu(t))']^2 (t - t_{n-1}^\mu) dt + [(\eta^\mu)_n^-]^2
\]

\[
\leq \left( cL_f |I^\mu| \cdot ||\rho^\mu||^2_{f_{i}^\mu} + cL_f |I^\mu| \cdot ||e_{y}^{\mu-1}||^2_{f_{y u-1}} + cL_f L_G K T |I^\mu| \sum_{i=0}^{\mu-1} ||e_{y}^i||^2_{f_{y i}} \right)
\]

\[
\cdot \exp(cL_f |I^\mu|).
\]
Observing Lemma 3.2.1.5 we then find
\[ ||\eta^\mu||_{I_{0}[\mu]}^2 \leq \log(\max(\mu+1,2))cL_f|I^{[\mu]}| \left[ ||\rho^\mu||_{I_{0}[\mu]}^2 + ||e^\mu_{y}||_{I_{0}[\mu-1]}^2 + L_GK\sum_{i=0}^{\mu-1} ||e^\mu_{i}||_{I_{0}[\mu]}^2 \right]. \]

We obtain, by using the triangle inequality,
\[ ||e^\mu_{z}||_{I_{0}[\mu]}^2 \leq C(L_f, T, \mu)||\rho^\mu||_{I_{0}[\mu]}^2 + c\log(\max(\mu + 1,2))L_f|I^{[\mu]}| \cdot ||e^\mu_{y}||_{I_{0}[\mu-1]}^2 \]
\[ + c\log(\max(\mu + 1,2))L_fL_GK\sum_{i=0}^{\mu-1} ||e^\mu_{i}||_{I_{0}[\mu]}^2. \]

From (3.2.1.12) we know that
\[ ||e^\mu_{y}||_{I_{0}[\mu]}^2 \leq C(L_f, T, \mu)||\rho^\mu||_{I_{0}[\mu]}^2 \]
\[ + c\log(\max(\mu + 1,2))L_f|I^{[\mu]}| \exp(2K\log \xi_{\mu-1}) ||e^\mu_{y}||_{I_{0}[\mu-1]} \]
\[ + c\log(\max(\mu + 1,2))L_fL_GK\sum_{i=0}^{\mu-1} \exp(2K\log \xi_{i}) ||e^\mu_{i}||_{I_{0}[\mu]}^2. \]

We use Gronwall’s lemma again and obtain
\[ ||e^\mu_{z}||_{I_{0}[\mu]} \leq C(L_f, L_G, \mathcal{K}, T, \mu)||\rho^\mu||_{I_{0}[\mu]}^2. \]

Our main result is presented as the following theorem.

**Theorem 3.2.1.1.** Under the assumptions in Corollary 3.2.1.1, we have
\[ ||e_{y}||_{I} \leq C(L_f, L_G, \mathcal{K}, T, \mu, M, y) \cdot \frac{1}{N}. \] (3.2.1.23)

**Proof.** From Corollary 3.2.1.1 and (3.2.1.15) we have
\[ ||e^\mu_{z}||_{I_{0}[\mu]} \leq C(L_f, L_G, \mathcal{K}, T, \mu, z)[h^\mu_{n}]^{\mu+1}. \] (3.2.1.24)

Combining (3.2.1.24) and (3.2.1.12) we arrive at
\[ ||e^\mu_{y}||_{I_{0}[\mu]} \leq C(L_f, L_G, \mathcal{K}, T, \mu, y)[h^\mu_{n}]^{\mu+1}. \] (3.2.1.25)

We obtain
\[ ||e_{y}||_{I} \leq C(L_f, L_G, \mathcal{K}, T, \mu, M, y) \cdot \frac{1}{N}. \]
3.2.2 The DG method for delay VIDEs of neutral type with weakly singular kernels

In this section we establish the convergence results of the DG method for equations with weakly singular kernels (3.1.2.3), (3.1.2.7), and (3.1.4.1).

In view of Theorem 3.1.2.4, the solution of (3.1.2.7) satisfies

\[ y \in C^{1,1-\alpha}(J^{[\mu]}) \quad (\forall \mu \geq 1). \]

Hence, \( DG(\Lambda) \) with \( \Lambda := \{ \mu + 1 \}_{\mu=0}^{\Lambda} \) for equation (3.1.2.7) can only yield

\[ \| y - y_h \|_{\tilde{I}^{[\mu]}} = \mathcal{O}(N_\mu^{-(2-\alpha)}), \]

on uniform meshes. If we use the \textit{graded meshes} with \textit{grading exponent}

\[ r_\mu := (\mu + 1)/(1 - \alpha) \]

(see (3.2.1.3)), then we may achieve the following theorem

\textbf{Theorem 3.2.2.1.} The error estimate of \( DG(\Lambda) \) method for equation (3.1.2.7) satisfies

\[ \| y - y_h \|_{\tilde{I}^{[\mu]}} = \mathcal{O}(N_\mu^{-(\mu+2)}). \quad (3.2.2.1) \]

\textit{Proof.} We can analyze the \( DG(\Lambda) \) for equation (3.1.2.7) by following that of Section 3.2.1 except the estimate (3.2.1.15). The paper [96] shows that on the generic subinterval \( I_n^{[\mu]} \) there holds

\[ \| z^{[\mu]} - \mathcal{I} z^{[\mu]} \|_{L^2(I_n^{[\mu]})} \leq C(\mu + 2) \| z^{[\mu]} - q \|_{L^2(I_n^{[\mu]})} + C \| \z^{[\mu]} - q' \|_{L^2(I_n^{[\mu]})}, \quad (3.2.2.2) \]

for any \( q \in \mathcal{P}^{(\mu+1)}(I_n^{[\mu]}) \). Here and throughout this section, positive constant \( C \) is independent of \( h_n^{[\mu]} \). Babuška and Suri [5] stated that

\[ \| z^{[\mu]} - \tilde{z} \|_{L^2(I_n^{[\mu]})} \leq CN_\mu^{-(\mu+2)}, \quad (3.2.2.3) \]
and

\[ \|(z[^\mu])' - \tilde{z}'\|_{L^2([|\mu|])} \leq CN_\mu^{-\mu+2}, \]  

(3.2.2.4)

where \( \tilde{z} \) is the FEM approximation to \( z[^\mu] \) on the graded meshes (3.2.1.3) with grading exponent

\[ r_\mu := (\mu + 1)/(1 - \alpha). \]

The original proof of (3.2.2.4) can be found in Gui and Babuška [54] (compare also Rice [93]).

Combining (3.2.2.2), (3.2.2.3), and (3.2.2.4) leads to

\[ \|(z[^\mu] - Iz[^\mu])\|_{[|\mu|]} \leq CN_\mu^{-\mu+2}. \]  

(3.2.2.5)

This estimate essentially helps us to complete the proof of (3.2.2.1).

The reads may compare Brunner [24, 23] which established these results in collocation methods on graded meshes for weakly singular Volterra integral and integro-differential equations.

Similarly, if we choose the grading exponent as

\[ r_\mu = (\mu + 1)/(1 - \alpha), \]

then we may obtain the following theorem

**Theorem 3.2.2.2.** We have the error estimate of DG(\( \Lambda \)) for equation (3.1.2.\( \alpha \))

\[ \|y - y_h\|_{[|\mu|]} = O(N_\mu^{-2\mu+1}). \]

**Proof.** The proof is similar to that of Theorem 3.2.2.1.

**Remark 3.2.2.1.** As for DG(\( \Lambda \)) for equation (3.1.4.1), we can establish the convergence results by combining the techniques in Section 3.2.1 and [80].
Chapter 4

Cascading multilevel discretization method for parabolic problems

4.1 Introduction

The two-grid method was first proposed by Xu [121, 120, 119] and later further studied by many others such as [4], [11], [38], [39], [81], [88], [116], [123], and [124]. So far the cascading multilevel discretization method has been investigated by [47], [71], [72], and [86].

The scheme of Marion and Xu [88] for the semi-linear parabolic equation (4.2.1.1) is based on two different finite element spaces one defined on a coarse grid with grid size $H$, and the other one on a fine grid with grid size $h \ll H$, respectively. Nonlinear and time-dependence are both treated in the coarse space, and only a fixed stationary equation needs to be solved on the fine space at each time. However, a question arises: “how to solve the two equations efficiently?” We realize that when the coarse grid size is small, solving the nonlinear time-dependent equation on the coarse grid is not trivial. In [86] and Section 4.2 of this thesis, we construct cascading multilevel algorithms (Algorithm A and Algorithm B) based on the scheme proposed by Marion and Xu [88]. In Algorithm A, only fixed stationary linear equations need
to be solved at each step and the result shows that the convergence rate is $O(h_J)$ in the energy norm $| \cdot |_1$, but it depends on the number of the grids. Algorithm B requires to solve both stationary linear equations and linear parabolic equations at each step, the total dimension of which equals to that of the corresponding level of the $P1$ conforming finite element space. The convergence rate of Algorithm B is also $O(h_J)$ in the energy norm $| \cdot |_1$ and is independent of the number of the grids. We also present Algorithm C, Algorithm D and Algorithm E for the parabolic equation with variable delays, parabolic equation with memory term and parabolic Fredholm equation.

4.2 Cascading multilevel discretization algorithms

In this section, we construct the cascading multilevel discretization algorithms and derive their convergence theorems.

4.2.1 Algorithm A

We consider the semilinear equation

$$u_t - \Delta u + f(u) = 0, \quad \text{in } \Omega \times \mathbb{R}^+,$$

with initial condition

$$u(x, 0) = \hat{u}(x) \quad \text{in } \Omega,$$

and boundary condition

$$u = 0, \quad \text{on } \partial \Omega,$$

where $\Omega \subset \mathbb{R}^d$ (with $d \leq 3$) is a bounded convex polygonal domain. The nonlinear term $f$ from $\mathbb{R}$ into $\mathbb{R}$ is assumed to be of class $C^4$ and its derivatives of order up
to four are bounded on $\mathbb{R}$. We start our analysis from the weak form of (4.2.1.1):

$$(u_t, v) + ((u, v)) + (f(u), v) = 0, \quad \forall v \in H^1_0(\Omega). \quad (4.2.1.2)$$

where $((u, v)) := (\nabla u, \nabla v)$, and $(\cdot, \cdot)$ is the $L_2$ inner product. We denote $|u|_1 := ((u, u))^{1/2}$, and $||u||_0 := (u, u)^{1/2}$. Let $T^h_j, \ j = 0, 1, \ldots, J$, be the nested quasi-uniform triangulations of $\Omega$, and let $V_j$ be the corresponding $P_1$ conforming finite element spaces. Thus

$$V_0 \subset V_1 \subset V_2 \cdots \subset V_J \subset H^1_0(\Omega).$$

We assume $h_j = h_{j-1}/2$ ($j = 1, \ldots, J$), without loss of generality. The corresponding $P_1$ conforming finite element approximation for (4.2.2.1) is: Find $u_j \in V_j$ such that

$$(u_{j,t}, v) + ((u_j, v)) + (f(u_j), v) = 0, \quad \forall v \in V_j, \quad (4.2.1.3)$$

with $u_j(0) = Q_j \hat{u}$, where the operator $Q_j$ is defined in (4.2.1.8) below. We present some results for the $P_1$ conforming finite element approximation to (4.2.1.1). First, we have the following estimates:

$$|v|_1 \leq ch_j^{-1}||v||_0, \quad ||v||_{L^\infty(\Omega)} \leq ch_j^{-d/p}||v||_{L^p(\Omega)}, \quad \text{for } 1 \leq p \leq \infty, \quad \forall v \in V_j, \quad (4.2.1.4)$$

which are the well-known inverse inequalities (cf. Ciarlet [35], Brenner and Scott [22], Xu [122]). We have the following estimates:

$$|(u - u_j)(t)|_1 \leq Ch_j^\sigma \log h_j^\sigma, \quad (4.2.1.5)$$

$$|||(u - u_j)^{(i)}(t)||_0 \leq Ch_j^{2|\sigma|} \log h_j^\sigma, \quad 0 \leq i \leq 2, \quad (4.2.1.6)$$

where $\sigma$ is some nonnegative constant. Here and throughout this chapter $C$ denotes the generic constant, which is independent of $h_j$ and $j$, but may depend on $t$. The proofs of (4.2.1.5) and (4.2.1.6) with $i = 0$ can be found in Johnson et al.
and in Crouzeix, Thomée, and Wahlbin [36]. Their techniques together with those in Thomée [113] can be extended in a straightforward manner to higher-order derivatives.

Now we consider the splitting of the space $V_j$:

$$V_j = V_{j-1} \oplus V_j^{j-1}, \quad \text{with } V_j^{j-1} := (I - Q_{j-1})V_j, \quad (4.2.1.7)$$

where $Q_{j-1} : L^2(\Omega) \rightarrow V_{j-1}$ is the $L^2$ orthogonal projection into $V_{j-1}$, as defined by

$$(Q_{j-1}v, v_j) = (v, v_j), \quad \forall v_j \in V_{j-1}. \quad (4.2.1.8)$$

Note also that $V_{j-1}$ and $V_j^{j-1}$ are orthogonal with respect to the scalar product $(\cdot, \cdot)$.

We introduce the operator $R_{j-1}^j : V_j \rightarrow V_j^{j-1}$ by setting

$$(R_{j-1}^j v_j, \chi) := ((v_j, \chi)), \quad \forall \chi \in V_j^{j-1}. \quad (4.2.1.9)$$

The cascading multilevel Algorithm A and Algorithm B associated with $(V_j, V_{j-1}, V_j^{j-1})$ consist of looking for an approximate solution

$$u^j := v^{j-1} + w^j, \quad \text{with } u^j \in V_j, \ v^{j-1} \in V_{j-1}, \ w^j \in V_j^{j-1} \quad (4.2.1.10)$$

for (4.2.2.1).

**Algorithm A**: Solve directly with respect to $v^0$

$$(v^0_t, \phi) + ((v^0, \phi)) + (f(v^0), \phi) = 0, \quad \forall \phi \in V_0, \quad (4.2.1.11)$$

$$v^0(0) = Q_0 \hat{u}. \quad (4.2.1.12)$$

For $j = 1, \ldots, J$, solve the linear system of equations for $w^j$:

$$(w^j, \chi) + (f(w^{j-1}), \chi) = 0, \quad \forall \chi \in V_j^{j-1}, \quad (4.2.1.13)$$

$$w^j = v^{j-1} + w^j. \quad (4.2.1.14)$$
Note that equation (4.2.1.12) defines $w^j$ uniquely in terms of $v^{j-1}$. We denote the corresponding mapping by

$$w^j = \Phi(v^{j-1}).$$  \hspace{1cm} (4.2.1.15)

In order to establish the convergence of Algorithm A, we shall prove or cite a number of lemmas and assertions.

The following lemmas (Lemma 4.2.1.1 and Lemma 4.2.1.2) and assertions ((4.2.1.18)-(4.2.1.25)) and their proofs can be found in Marion and Xu [88].

**Lemma 4.2.1.1.** Let $[v, \phi]_1 := (((I - R_j^{j-1})v, (I - R_j^{j-1})\phi))$. Then $v_{j-1}$ satisfies

$$(v_{j-1,t}, \phi) + [v_{j-1}, \phi]_1 = (u_{j,t}, R_j^{j-1}\phi) + (f(u_j), R_j^{j-1}\phi - \phi), \quad \forall \phi \in V_{j-1}. \hspace{1cm} (4.2.1.16)$$

**Lemma 4.2.1.2.** Let $||\phi||_1 := |(I - R_j^{j-1})\phi|_1$. Then there exist two constants $C_1$ and $C_2$, independent of the grid size, such that,

$$C_1|\phi|_1 \leq ||\phi||_1 \leq C_2|\phi|_1, \quad \forall \phi \in V_{j-1}. \hspace{1cm} (4.2.1.17)$$

For $t > 0$, we have

$$||w_j||_0 + ||w_{jt}||_0 \leq C h_j^{2}, \hspace{1cm} (4.2.1.18)$$

$$|w_j - \Phi(v_{j-1})|_1 \leq C h_j^{3}, \hspace{1cm} (4.2.1.19)$$

$$||w_j - \Phi(v_{j-1})||_0 \leq C h_j^{4}, \hspace{1cm} (4.2.1.20)$$

where $\Phi$ is as in (4.2.1.15). The following two assertions are essential in proving the convergence of Algorithm A and also Algorithm B for (4.2.1.1).

$$|w_j - w^j|_1 \leq C h_j^{3} + (1 + C_3 h_j)|e_j|_1, \hspace{1cm} (4.2.1.21)$$

$$||w_j - w^j||_0 \leq C h_j^{4} + (1 + C_4 h_j)|e_j|_1. \hspace{1cm} (4.2.1.22)$$
\( e_j := v_{j-1} - v^{j-1} \). The following estimates are also helpful in the proof of the results in the thesis:

\[
\| (f(u_j) - f(v^{j-1}), R_j^{j-1} \phi) \|_0 \leq Ch_{j-1} \| e_j \|_0 \| \phi \|_1 + Ch_j^3 \| \phi \|_1, \tag{4.2.1.23}
\]

\[
\| (f(u_j^j) - f(u_j), \phi) \|_0 \leq C\{ \| e_j \|_0 + h_{j-1} \| e_j \|_1 + h_j^4 \} \| \phi \|_0, \tag{4.2.1.24}
\]

\[
\| (u_j^j, R_j^{j-1} \phi) \|_1 \leq Ch_j^3 \| \phi \|_1. \tag{4.2.1.25}
\]

**Lemma 4.2.1.3.** We assume that

\[
\tilde{u}^j := \tilde{v}^{j-1} + \tilde{w}^j, \text{ with } \tilde{u}^j \in V_j, \tilde{v}^{j-1} \in V_{j-1}, \tilde{w}^j \in V_j^{j-1}, \tag{4.2.1.26}
\]

and

\[
(\tilde{v}^{j-1}, \phi) + ((\tilde{v}^{j-1}, \phi)) + (f(\tilde{v}^{j-1}), \phi) = 0, \quad \forall \phi \in V_{j-1}, \tag{4.2.1.27}
\]

\[
\tilde{v}^{j-1}(0) = Q_{j-1} \tilde{u}.
\]

\[
((\tilde{v}^{j-1} + \tilde{w}^j, \chi)) + (f(\tilde{v}^{j-1}), \chi) = 0, \quad \forall \chi \in V_j^{j-1}. \tag{4.2.1.28}
\]

In particular, let

\[
\tilde{u}^0 = u^0, \quad \tilde{v}^0 = v^0, \quad \tilde{w}^1 = w^1, \tag{4.2.1.29}
\]

and the initial value \( \tilde{u} \in L^2(\Omega) \) be given. Assume also that

\[
h_j^2 | \log h_j |^\sigma \leq h_{j-1}^2, \tag{4.2.1.30}
\]

where \( \sigma \) is as in (4.2.1.5) and (4.2.1.6). Then

\[
| u_j - \tilde{u}^j |_1 \leq C(K_1) h_j, \tag{4.2.1.31}
\]

where \( K_1 = 2(1 + C_2 h_{j-1}) \) and \( C_3 \) as in (4.2.1.21).
Proof. Firstly, we note that (4.2.1.30) is necessary for the proof of (4.2.1.18) (see Marion and Xu [88]). Let

\[
|u_j - \tilde{u}_j|_1 = |u_j - \tilde{u}^{j-1} - \tilde{w}_j|_1 = |u_j - u_{j-1} + w_j - \tilde{w}^j - w_j|_1 \\
\leq |w_j - \tilde{w}_j|_1 + |w_j|_1 + Ch_{j-1}. 
\] (4.2.1.32)

We estimate the term

\[
|w_j - \tilde{w}_j|_1 \leq Ch_{j-1}^3 + (1 + C_3h_{j-1})|v_{j-1} - \tilde{v}^{j-1}|_1 \\
= Ch_{j-1}^3 + (1 + C_3h_{j-1})|u_{j-1} - \tilde{u}^{j-1} + v_{j-1} - u_{j-1}|_1 \\
= Ch_{j-1}^3 + (1 + C_3h_{j-1})|u_{j-1} - \tilde{u}^{j-1} + v_{j-1} - u_{j} + u_{j} - u_{j-1}|_1 \\
\leq (1 + C_3h_{j-1})|u_{j-1} - \tilde{u}^{j-1}|_1 + |w_j|_1 + Ch_{j-1} + Ch_{j-1}^3. 
\] (4.2.1.33)

In view of (4.2.1.32) and (4.2.1.33), we have

\[
|u_j - \tilde{u}_j|_1 \leq (1 + C_3h_{j-1})|u_{j-1} - \tilde{u}^{j-1}|_1 + C|w_j|_1 + Ch_{j-1} + Ch_{j-1}^3. 
\]

Noting (4.2.1.18), the previous inequality arrives at

\[
|u_j - \tilde{u}_j|_1 \leq K_1|u_{j-1} - \tilde{u}^{j-1}|_1 + Ch_{j-1} \leq C(K_1)^j h_j, 
\] (4.2.1.34)

which is (4.2.1.31).

**Theorem 4.2.1.1.** The error estimate of Algorithm A for (4.2.1.1) is given by

\[
|u - u^J|_1 \leq C(2K_2)^J h_J, 
\] (4.2.1.35)

where \( K_2 = 1 + Ch_{j-1}^2 \).

Proof. From (4.2.1.27), we know that \( \tilde{v}^{j-1} = u_{j-1} \), thus (4.2.1.28) becomes,

\[
((u_{j-1} + \tilde{w}^j, \chi)) + (f(u_{j-1}), \chi) = 0, \quad \forall \chi \in V_{j}^{j-1}. 
\] (4.2.1.36)
And (4.2.1.12) is

\[(u^{-1} + w^j, \chi) + (f(u^{-1}), \chi) = 0, \quad \forall \chi \in V_j^{-1}.\]  \hspace{1cm} (4.2.1.37)

Combining (4.2.1.36) and (4.2.1.37), we obtain

\[(\tilde{w}^j - w^j, \chi) = -((u_{j-1} - u^{-1}, \chi)) - (f(u_{j-1}) - f(u^{-1}), \chi).\]  \hspace{1cm} (4.2.1.38)

Substituting \(\chi = \tilde{w}^j - w^j\) into (4.2.1.38), we get

\[|\tilde{w}^j - w^j|^2_1 = |(u_{j-1} - u^{-1}, \tilde{w}^j - w^j)) - (f(u_{j-1}) - f(u^{-1}), \tilde{w}^j - w^j)|^2_1 \]
\[\leq |u_{j-1} - u^{-1}|_1|\tilde{w}^j - w^j|^2_1 + C h_j^2 |u_{j-1} - u^{-1}|_1|\tilde{w}^j - w^j|_1.\]  \hspace{1cm} (4.2.1.39)

Hence,

\[|\tilde{w}^j - w^j|_1 \leq (1 + C h_j^2) |u_{j-1} - u^{-1}|_1.\]  \hspace{1cm} (4.2.1.40)

Now we estimate

\[|u_j - u^j|_1 = |u_j - \tilde{u}^j + \tilde{u}^j - u^j|_1 \]
\[= |u_j - \tilde{u}^j + u_{j-1} - u^{-1} + \tilde{w}^j - w^j|_1 \]
\[\leq |u_j - \tilde{u}^j|_1 + |u_{j-1} - u^{-1}|_1 + |\tilde{w}^j - w^j|_1 \]
\[\leq C(K_1)h_j + |u_{j-1} - u^{-1}|_1 + (1 + C h_j^2) |u_{j-1} - u^{-1}|_1 \]
\[\leq C(K_1)h_j + K_2 |u_{j-1} - u^{-1}|_1,\]  \hspace{1cm} (4.2.1.41)

where the second step follows from (4.2.1.13) and (4.2.1.26), and the fourth step is
based on (4.2.1.31) and (4.2.1.40). Therefore, (4.2.1.41) yields

\[
|u_j - u^J|_1 \leq C(K_1)^J h_J + K_2 |u_{j-1} - u^{J-1}|_1 \\
\leq C(K_1)^J h_J + CK_2 K_1^{J-1} h_{J-1} + K_2^2 |u_{j-2} - u^{J-2}| \\
\vdots \\
\leq C(K_1)^J h_J + CK_2 K_1^{J-1} h_{J-1} + \cdots + CK_2^J h_0 \\
\leq (2K_2)^J h_J.
\]

Consequently, we achieve

\[
|u - u^J|_1 = |u - u_J + u_J - u^J|_1 \leq |u - u_J|_1 + |u_J - u^J|_1 \leq C(2K_2)^J h_J,
\]

which is (4.2.1.35).

Remark 4.2.1.1. We know from Theorem 4.2.1.1 that when we fix the number of grids, the convergence rate of Algorithm A is $O(h_J)$.

### 4.2.2 Algorithm B

In this section, we construct Algorithm B, which solves both time-dependent linear and stationary linear equations at each level of the $P_1$ finite element space.

**Algorithm B:** Let

\[
u^0 = Q_0 \bar{u}.
\]

For $j = 2, \ldots, J$, solve

\[
(v_{j-1}^j, \phi) + ((v^{j-1} + w^j, \phi)) + (f(u^{j-1}), \phi) = 0, \quad \forall \phi \in V_{j-1},
\]

\[
((v^{j-1} + w^j, \chi)) + (f(u^{j-1}), \chi) = 0, \quad \forall \chi \in V_{j-1}^j,
\]

\[
u^j = v^{j-1} + w^j,
\]

for $v^{j-1}$ and $w^j$. 
Theorem 4.2.2.1. Under the assumptions of Theorem 2.1 we have

\[ |u - u_J|_1 \leq Ch_J, \]  

(4.2.2.5)

where the constant $C$ is independent of the grid size $h_J$ and the grid $J$.

Proof. We use the corresponding notation of Section 4.2.2.1 and we claim that

\[ \left( v^j_{t-1}, \phi \right) + \left[ v^j_{t-1}, \phi \right] = -(f(u^j_{t-1}), \phi) + (f(v^j_{t-1}), R^j_{t-1} \phi), \quad \forall \phi \in V_{j-1}. \]  

(4.2.2.6)

The proof of (4.2.2.6) is the analog of that of (4.2.1.16). Because of (4.2.2.6) and (4.2.1.16), $d_j := v_{j-1} - v^j_{t-1}$ satisfies

\[ \left( d_{j,t}, \phi \right) + \left[ d_j, \phi \right] = (u_{j,t}, R^j_{t-1}) + (f(u_j) - f(v^j_{t-1}), R^j_{t-1} \phi) + (f(v^j_{t-1}) - f(u_j), \phi). \]  

(4.2.2.7)

We estimate

\[ ||(f(u^j_{t-1}) - f(u_j), \phi)||_0 \leq C||u^j_{t-1} - u_j||_0||\phi||_0 \leq (||u^j_{t-1} - u_{j-1}||_0 + Ch_{j-1})||\phi||_0. \]  

(4.2.2.8)

Substituting $\phi = d_{j,t}$ into (4.2.2.7) and using (4.2.2.8), we see that

\[ ||d_{j,t}||_0^2 + [d, d_{j,t}] \leq (u_{j,t}, R^j_{t-1} d_{j,t}) + (f(u_j) - f(v^j_{t-1}), R^j_{t-1} d_{j,t}) \]
\[ + (||u^j_{t-1} - u_{j-1}||_0 + Ch_{j-1})||\phi||_0. \]  

(4.2.2.9)

The estimates (4.2.1.23), (4.2.1.25), (4.2.1.17) and Young’s inequality leads to

\[ \frac{d(||d_j||^2)}{dt} \leq Ch^2_{j-1} |d_j|_1^2 + Ch^2_{j-1} |u^j_{t-1} - u_{j-1}|_1^2 + Ch^2_{j-1}. \]

Hence,

\[ |d_j|_1^2 \leq Ch^2_{j-1} + Ch^2_{j-1} |u^j_{t-1} - u_{j-1}|_1^2 + Ch^2_{j-1} \int_0^t |u^j_{t-1} - u_{j-1}|_1^2 ds. \]
Thus we have

\[ |u_j - u^J|^2 \leq 2|d_j|^2 + 2|w_j - u^J|^2 \leq C|d_j|^2 + Ch_j^6 \]

\[ \leq Ch_j^2 + Ch_j^2 |u_{j-1} - u^J|^2 + Ch_j^4 \int_0^t |u_{j-1} - u^{j-1}|^2 ds. \]

Therefore,

\[ |u_j - u^J|_1 \leq Ch_{j-1} + Ch_{j-1} |u_{j-1} - u^J|_1. \quad (4.2.2.10) \]

Iterating (4.2.2.10) yields

\[ |u_j - u^J|_1 \leq Ch_{j-1} + Ch_{j-1} |u_{j-1} - u^{j-1}|_1 \]

\[ \leq Ch_{j-1} + Ch_{j-1} h_{j-2} + Ch_{j-1} h_{j-2} |u_{j-2} - u^{j-2}|_1 \]

\[ \vdots \]

\[ \leq Ch_{j-1} + Ch_{j-1} h_{j-2} + \cdots + Ch_{j-1} h_{j-2} \cdots h_1 \]

\[ \leq Ch_j [1 + h_1 (2^{-(j-3)} + 2^{-(j-3)} 2^{-(j-4)} + \cdots + 2^{-(j-3)} 2^{-(j-4)} \cdots 1)] \leq Ch_j (1 + Ch_1) \leq Ch_j. \]

**Remark 4.2.2.1.** In Algorithm A and Algorithm B, we need to solve the linear equations (4.2.1.12) and (4.2.2.3), respectively. For the further study of the linear systems in \( V^{j-1}_j \), the author refers to Marion and Xu [88].

### 4.3 Comparison and discussion

We summarize the presentation of Section 4.2 in Table 4.3, and add some discussions.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Convergence</th>
<th>Open question</th>
<th>Literature</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( O(h_j) ) (but dependent on ( J ))</td>
<td>CMGI</td>
<td>[88]</td>
</tr>
<tr>
<td>B</td>
<td>( O(h_j) ) (independent on ( J ))</td>
<td>CMGI</td>
<td></td>
</tr>
</tbody>
</table>
Algorithm A and Algorithm B are based on the $L_2$ decomposition techniques. Algorithm A needs only to solve a linear fixed stationary system of equations in $V_j^{-1}$ of each step. Algorithm B requires the solution of the linear parabolic equations in $V_{j-1}$ and the linear fixed stationary equations in $V_j^{-1}$ at each step. If we incorporate the classical iteration methods (called smoothers) into the linear stationary system or the linear system arising from the linear parabolic equations with discontinuous Galerkin (DG) time-stepping methods or other methods, then it is not difficult to formulate the cascading multigrid iteration method (CMGI), which is ongoing work. For the idea of cascading multigrid methods, Algorithm A avoids the solving of the linear parabolic equations. Hence, it is much easier to implement cascading multigrid iteration methods, but it can be used only when we fix the number of grids. Although Algorithm A and Algorithm B have the same convergence rate $O(h_J)$, Algorithm A is dependent of the number of grids $J$, while Algorithm B is not. So Algorithm B is more accurate than Algorithm A with respect to convergence: it has a smaller error constant.

4.4 Extensions to other parabolic problems

In this section we extend the analysis of the cascading multilevel discretization algorithms to parabolic partial differential equations with variable delays and with nonlinear memory terms.

4.4.1 Parabolic equation with delay argument

We consider

$$u_t(x,t) - \Delta u(x,t) + f(u(x,t),u(x,\theta(t))) = 0, \quad \forall x \in \Omega, \quad t \in I := [0,T], \quad (4.4.1.1)$$
with initial condition
\[ u(x, t) = \hat{u}(x, t), \quad \forall x \in \Omega, \forall t \leq 0, \]
and boundary condition
\[ u(x, t)|_{\partial \Omega} = 0, \quad \forall t \in I, \]
where \( \Omega \subset \mathbb{R}^d \) (with \( d \leq 3 \)) is a bounded convex polygonal domain. Consider the case of
\[ f(u, v) = f_1(u) + f_2(v). \]
Here \( f_1 \) and \( f_2 \) from \( \mathbb{R} \) into \( \mathbb{R} \) are assumed to be of class \( C^1 \) and their derivatives are bounded on \( \mathbb{R} \). The delay function \( \theta(t) \) will be subject to the following conditions (i)–(iii):
(i) \( \theta(t) = t - \tau(t), \quad \theta \in C^d(I) \) for some \( d \geq 0; \)
(ii) \( \tau(t) \geq \tau_0 > 0 \) for \( t \in I; \)
(iii) \( \theta \) is strictly increasing on \( I. \)

We define the points \( \{\xi_{\mu}\}, \mu = 0, 1, \ldots, M, \) by
\[ \theta(\xi_{\mu}) = \xi_{\mu-1}, \quad \mu \geq 1, \]
where \( \xi_0 := 0. \quad I^{[i]} := [\xi_{i-1}, \xi_i], \quad i = 1, \ldots, M. \) Furthermore we assume, without loss of generality, that \( T = \sum_{i=1}^{M} |I^{[i]}|. \) We note that we shall use the corresponding notations in Section 4.2.2.1. We shall analyze the P1 conforming finite element approximation to (4.4.1.1). The weak form of (4.4.1.1) is
\[ (u_t, v) + ((u, v)) + (f(u(x, t), u(x, \theta(t))), v) = 0, \quad \forall v \in H^1_0(\Omega). \]
If we set \( u^{[i]} := u(x, t), \ t \in I^{[i]} \), then (4.4.1.2) can be written as

\[
(u^{[i]}, v) + ((u^{[i]}), v) + (f_2(u^{[i]-1}(x, \theta(t))), v) = 0, \quad \forall v \in H^1_0(\Omega).
\]  

(4.4.1.3)

for all \( t \in I^{[i]} \). Correspondingly, the P1 conforming finite element approximation to (4.4.1.1) is

\[
(u_{J,t}, v) + ((u_J, v)) + (f(u_J(x, t), u_J(x, \theta(t))), v) = 0,
\]

(4.4.1.4)

for all \( v \in V_J \) with \( u_J := R_J \hat{u}(x, t) \ (t \leq 0) \), and

\[
(u^{[i]}_{J,t}, v) + ((u^{[i]}_J, v)) + (f_1(u^{[i]}_J(x, t)), v) + (f_2(u^{[i]-1}_J(x, \theta(t))), v) = 0,
\]

(4.4.1.5)

for all \( v \in V_J \) (\( t \in I^{[i]} \)), where \( u^{[i]}_J := u_J(x, t) \ (t \in I^{[i]} \)). We are now ready to present the convergence results of the P1 conforming finite element method for (4.4.1.1).

**Theorem 4.4.1.1.** Let \( u \) and \( u_J \) be the solution of (4.4.1.1) and (4.4.1.4), respectively. Then we have

\[
\|u - u_J\|_0 \leq C h^2_J \log h_J|\sigma, \\
\|u - u_J\|_1 \leq C h_J \log h_J|\sigma,
\]

where \( \sigma \) is as in (4.2.1.5) and (4.2.1.6).

**Proof.** For convenience, we define

\[
e(t) := u(x, t) - u_J(x, t), \quad e^{[i]} := u^{[i]}(x, t) - u^{[i]}_J(x, t), \quad \tilde{e}^{[i]}_J := \hat{u}^{[i]}_J - u^{[i]}_J,
\]

where \( \hat{u}^{[i]}_J \) is the solution of

\[
(u^{[i]}_{J,t}, v) + ((\tilde{u}^{[i]}_J, v)) + (f_1(\tilde{u}^{[i]}_J(x, t)), v) + (f_2(u^{[i]-1}(x, \theta(t))), v) = 0, \quad \forall v \in V_J, \quad (4.4.1.6)
\]
for $t \in I^i$. From (4.2.1.5) and (4.2.1.6), we know that

\[ ||u^i - \tilde{u}^i_j||_0 \leq Ch_j^2 \log h_j \sigma, \hspace{1em} \forall t \in I^i, \]

(4.4.1.7)

\[ ||u^i - \tilde{u}^i_j||_1 \leq Ch_j \log h_j \sigma, \hspace{1em} \forall t \in I^i. \]

(4.4.1.8)

Because of

\[ e^i_j(t) := u^i_j(x, t) - u^i_j(x, t) = (u^i - \tilde{u}^i_j) + (\tilde{u}^i_j - u^i_j), \]

(4.4.1.9)

we need only to estimate the term $\tilde{e}^i_j := \tilde{u}^i_j - u^i_j$. Subtracting (4.4.1.5) from (4.4.1.6), we obtain

\[ (\tilde{e}^i_j, v) + ((\tilde{e}^i_j, v)) + (f_1(\tilde{u}^i_j)(x, t)) - f_1(u^i_j)(x, t), v) \]

\[ + (f_2(u^{i-1})(x, \theta(t))) - f_2(u^{i-1})(x, \theta(t)), v) = 0, \hspace{1em} \forall v \in V_j. \]

(4.4.1.10)

Substituting $v := e^i_j$ into (4.4.1.10) leads to

\[ \frac{1}{2} \frac{d}{dt} \left( ||\tilde{e}^i_j||^2_0 \right) + ||\tilde{e}^i_j||^2_1 \leq C||e^i_j||^2_0 + C||e^{i-1}(\theta(t))||_0 ||e^i_j||_0, \]

and

\[ \frac{d}{dt} \left( ||\tilde{e}^i_j||^2_0 \right) \leq C||e^i_j||^2_0 + ||e^{i-1}(\theta(t))||^2_0. \]

Consequently, we have

\[ ||\tilde{e}^i_j||^2_0 \leq C||e^i_j(t_{i-1})||^2_0 + \int_{t_{i-1}}^t \exp(C \cdot (t - s))||e^{i-1}(\theta(s))||^2_0 ds. \]

Hence, the estimates

\[ ||\tilde{e}^i_j(t)||_0 \leq C||e^{i-1}(t)||_0, \]

(4.4.1.11)

\[ ||\tilde{e}^i_j||_1 \leq C||e^{i-1}(t)||_1, \]

(4.4.1.12)

hold. According to (4.4.1.9), (4.4.1.7) and (4.4.1.8), we get

\[ ||e^i(t)||_0 \leq ||u^i(t) - \tilde{u}^i_j(t)||_0 + ||\tilde{e}^i_j||_0 \leq Ch_j^2 \log h_j \sigma + C||e^{i-1}(t)||_0, \hspace{1em} \forall t \in I^i. \]
Similarly, we find

\[ |e^{[i]}(t)|_1 \leq Ch_j \log h_j |^\sigma + C|e^{[i-1]}(t)|_1, \quad \forall t \in I_{[i]}, \]

and this leads to

\[ ||e(t)||_0 \leq C h_j^2 \log h_j |^\sigma, \]
\[ |e(t)|_1 \leq C h_j \log h_j |^\sigma. \]

**Algorithm C:** Let

\[ u^0 = R_0 \hat{u}(x, t), \quad \forall t \leq 0. \quad \text{(4.4.1.13)} \]

For \( j = 2, \ldots, J, \) solve

\[ (u^j_t, v) + ((u^j, v)) + (f(u^{j-1}(x, t), u^{j-1}(x, \theta(t))), v) = 0, \quad \forall v \in V_j, \quad \text{(4.4.1.14)} \]

for \( u^j. \)

**Theorem 4.4.1.2.** Assume that

\[ |\log h_{j-1}|^\sigma \leq C. \quad \text{(4.4.1.15)} \]

Let \( u \) and \( u^J \) denote the solution of (4.4.1.1) and Algorithm C, respectively. Then we have

\[ |u - u^J|_1 \leq Ch_J, \]

where \( C \) is independent of \( h_J \) and the number of the grids.

**Proof.** Let \( \delta_j := u_j - u^j. \) Then subtracting (4.4.1.14) from (4.4.1.4), we get

\[ (\delta_j_t, v) + ((\delta_j, v)) + (f(u_j(x, t), u_j(x, \theta(t))) - f(u^{j-1}(x, t), u^{j-1}(x, \theta(t))), v) = 0. \quad \text{(4.4.1.16)} \]
Now we estimate

\[ \left\| (f(u_j(x, t), u_j(x, \theta(t))) - f(u^{j-1}(x, t), u^{j-1}(x, \theta(t))), v) \right\|_0 \quad (4.4.1.17) \]
\[ \leq \left\| (f(u_j(x, t), u_j(x, \theta(t))) - f(u^{j-1}(x, t), u^{j-1}(x, \theta(t))) \right\|_0 \cdot ||v||_0 \]
\[ \leq [C||u_j(x, t) - u_{j-1}(x, t) + \delta_{j-1}||_0 \]
\[ + C||u_j(x, \theta(t)) - u_{j-1}(x, \theta(t)) + \delta_{j-1}(x, \theta(t))||_0 \cdot ||v||_0 \]
\[ \leq [Ch^2_{j-1} \log h_{j-1}|^{2\sigma} + ||\delta_{j-1}(x, t)||_0 + ||\delta_{j-1}(x, \theta(t))||_0 \cdot ||v||_0. \]

Substituting \( v = \delta_{j,t} \) into \( (4.4.1.16) \) and combining \( (4.4.1.17) \), we obtain the estimate

\[ ||\delta_{j,t}||_0^2 + \frac{d(||\delta_j||^2)}{dt} \leq [Ch^2_{j-1} \log h_{j-1}|^{2\sigma} + ||\delta_{j-1}(x, t)||_0 + ||\delta_{j-1}(x, \theta(t))||_0 \cdot ||\delta_{j,t}||_0 \]
\[ \leq Ch^4_{j-1} \log h_{j-1}|^{4\sigma} + h^2_{j-1}||\delta_{j-1}(x, t)||_1^2 + h^2_{j-1}||\delta_{j-1}(x, \theta(t))||_1^2. \]

Hence, it holds that

\[ ||\delta_j||^2_1 \leq Ch^4_{j-1} \log h_{j-1}|^{4\sigma} + h^2_{j-1} \int_0^t (||\delta_{j-1}(x, s)||_1^2 + ||\delta_{j-1}(x, \theta(s))||_1^2)ds. \quad (4.4.1.18) \]

Iterating \( (4.4.1.18) \) leads to

\[ ||\delta_j||^2_1 \leq Ch^4_{j-1} \log h_{j-1}|^{4\sigma} + Ch^2_{j-1}h^4_{j-2} \log h_{j-2}|^{4\sigma} + \cdots \]
\[ + Ch^2_{j-1}h^2_{j-2} \cdots h^4_{1} \log h_1|^{4\sigma} \leq Ch^4_{j-1}. \quad (4.4.1.19) \]

In the last step of \( (4.4.1.19) \), we used the assumption \( (4.4.1.15) \). Recalling Theorem 4.4.1.1 and \( (4.4.1.19) \), we obtain

\[ |u - u^J|_1 \leq |u - u_J||_1 + |u_J - u^J||_1 \leq ch_J \log h_J|^{\sigma} + ch_J \leq ch_J, \]

which is our desired result.
4.4.2 Parabolic equation with memory term

We consider

\[ u_t - \Delta u + \int_0^t k(t-s)G(u(x,s))ds = 0, \quad \forall x \in \Omega, \; \forall t \in I := [0, T], \tag{4.4.2.1} \]

with initial condition

\[ u(x, 0) = \hat{u}(x), \quad \forall x \in \Omega, \]

and boundary condition

\[ u(x, t) = 0, \quad \forall x \in \partial \Omega, \; \forall t \in I, \]

where \( \Omega \subset \mathbb{R}^d \) (with \( d \leq 3 \)) is a bounded convex polygonal domain. The nonlinear term \( G \) from \( \mathbb{R} \) into \( \mathbb{R} \) is assumed to be of class \( C^1 \), with bounded derivatives on \( \mathbb{R} \). Furthermore we assume \( k \in C^1(I) \). The weak form of (4.4.2.1) is

\[ (u_t, v) + ((u, v)) + \int_0^t k(t-s)(G(u(x,s)), v)ds = 0, \quad \forall v \in H_0^1(\Omega), \]

where \( (G(u(x,s)), v) := \int_\Omega G(u(x,s))vdx \). Correspondingly, the \( P_1 \) conforming finite element approximation is given by

\[ (u_{j,t}, v) + ((u_j, v)) + \int_0^t k(t-s)(G(u_j(s)), v)ds = 0, \quad \forall v \in V_j, \tag{4.4.2.2} \]

with \( u_j(0) = R_j \hat{u} \). We know that

\[ |u - u_j|_1 \leq Ch_j, \tag{4.4.2.3} \]

which can be found in [34].

**Algorithm D:** Let

\[ u^0 = R_0 \hat{u}. \]
For \( j = 2, \ldots, J \), solve
\[
(u^j_t, v) + ((u^j, v)) + \int_0^t k(t-s) \left( G(u^{j-1}) + G'(u^{j-1})(u^j - u^{j-1}), v \right) ds = 0, \quad \forall v \in V_j,
\]
for \( u^j \).

**Theorem 4.4.2.1.** Assume that
\[
h_{j-1} \leq \frac{C}{(2\sqrt{2})^j}. \tag{4.4.2.5}
\]
Then
\[
|u - u^J|_1 \leq Ch_J,
\]
where \( C \) is independent on \( h_J \) and the number of the grids.

**Proof.** Let \( e_j := u_j - u^j \). Subtracting (4.4.2.4) from (4.4.2.2) yields
\[
(e_{j,t}, v) + ((e_j, v)) + \int_0^t k(t-s) \left( G(u_{j-1}) - G(u^{j-1}) + G'(u_{j-1})(u_j - u_{j-1}) - G'(u^{j-1})(u^j - u^{j-1}) + O((u_j - u_{j-1})^2), v \right) ds = 0, \quad \forall v \in V_j.
\]

Bringing \( v = e_{j,t} \) into (4.4.2.6), we arrive at
\[
\|e_{j,t}\|_0^2 + \frac{1}{2} \frac{d\|e_j\|_1^2}{dt} \leq C h_j^4 + C \left( \int_0^t |k(t-s)||e_{j-1}|_1 ds \right)^2.
\]

Hence, we achieve
\[
\frac{d\|e_j\|_1^2}{dt} \leq C h_j^4 + C h_{j-1}^2 \left( \int_0^t |k(t-s)||e_{j-1}|_1 ds \right)^2 \leq C(h_j^4 + 2 \int_0^t |e_{j-1}|_1^2 ds),
\]
by using \( |ab| \leq 2(a^2 + b^2) \).

Integrating (4.4.2.7) yields
\[
|e_j|_1^2 \leq C h_j^4 + 2t \int_0^t |e_{j-1}|_1^2 ds \leq C h_j^4 + C^2 h_{j-2}^4 + \cdots + C^{2^{j-2}} h_1^4 \leq C h_j^2 h_{j-1}^2.
\]
Therefore, by using assumption (4.4.2.5) we get

\[ |e_j|_1 \leq C h_j \quad (j = 1, \ldots, J). \quad (4.4.2.8) \]

Then (4.4.2.8) and (4.4.2.3) lead to

\[ |u - u^J|_1 \leq |u - u_J|_1 + |u_J - u^J|_1 \leq c h_J, \]

which completes the proof.

4.5 Application to parabolic Fredholm equation

4.5.1 Finite element method for parabolic Fredholm equation

Consider the parabolic Fredholm equation

\[ u_t - \Delta u = \int_{\Omega} f(u) dx, \quad \forall x \in \Omega, \quad t \in I := [0, T], \quad (4.5.1.1) \]

with initial condition

\[ u(x, 0) = \hat{u}(x), \quad \forall x \in \Omega, \]

and boundary condition

\[ u(x, t) = 0, \quad \forall x \in \partial \Omega, \quad t \in I. \]

We assume that \( f \) satisfies

\[ |f(u_1) - f(u_2)| \leq L_f \|u_1 - u_2\|_0, \quad \forall u_1, u_2 \in \Lambda \subset \mathbb{R}, \]

such that (4.5.1.1) possesses a unique solution \( u \in \Lambda \). We refer [32] for the general description of the parabolic Fredholm equation. We begin our analysis with the weak form of (4.5.1.1),

\[ (u_t, v) + ((u, v)) = \left( \int_{\Omega} f(u) dx, v \right), \quad \forall v \in H^1_0(\Omega). \quad (4.5.1.2) \]
The corresponding P1 conforming finite element approximation for (4.5.1.2) is: Find \( u_j \in V_j \) such that

\[
(u_{j,t}, v) + ((u_j, v)) = \left( \int_{\Omega} f(u_j) dx, v \right), \quad \forall v \in V_j,
\]

(4.5.1.3)

with \( u_j(0) = R_j \hat{u} \), where \( R_j : L^2(\Omega) \to V_j \) is defined by

\[
((u - R_j u, v)) = 0, \quad \forall v \in V_j.
\]

(4.5.1.4)

We know from Ciarlet [35] or Brenner and Scott [22] that

\[
||u - R_j u||_0 \leq C h_j^2,
\]

(4.5.1.5)

\[
||u - R_j u||_1 \leq C h_j,
\]

(4.5.1.6)

where the constant \( C \) is independent of \( h_j \).

**Theorem 4.5.1.1.** Let \( u \) and \( u_j \) be the solution of (4.5.1.2) and (4.5.1.3), respectively. Then we have

\[
||u - u_j||_0 \leq C h_j^2.
\]

(4.5.1.7)

\[
||u - u_j||_1 \leq C h_j,
\]

(4.5.1.8)

where the constant \( C \) is independent of \( h_j \).

**Proof.** Let

\[
e := u - u_j = u - R_j u + R_j u - u_j := \rho + \theta.
\]

Subtract (4.5.1.3) from (4.5.1.2),

\[
(e_t, v) + ((e, v)) = \left( \int_{\Omega} [f(u) - f(u_j)] dx, v \right), \quad \forall v \in V_j.
\]

(4.5.1.9)
Hence, we have the estimate

\[
(\theta_t, v) + ((\theta, v)) = \left( \int_{\Omega} [f(u) - f(R_j u)] dx, v \right) \\
+ \left( \int_{\Omega} [f(R_j u) - f(u_j)] dx, v \right) - (\rho_t, v).
\]

(4.5.1.10)

for all \( v \in V_j \), with \( \theta(0) = \hat{u}_j - R_j \hat{u} \) in \( \Omega \), where

\[
||\theta(0)||_0 \leq ||\hat{u}_j - \hat{u}||_0 + ||\hat{u} - R_j \hat{u}||_0 \leq Ch_j^2. \tag{4.5.1.11}
\]

Taking \( v = \theta \) in (4.5.1.10), we obtain

\[
\frac{1}{2} \frac{d(||\theta||_0^2)}{dt} + ||\theta||_1^2 \leq L_f|\Omega| \cdot ||\theta||_0^2 + L_f|\Omega| \cdot ||\rho||_0 ||\theta||_0 + ||\rho_t||_0 ||\theta||_0 \\
\leq (3L_f|\Omega| + 2)||\theta||_0 + 2L_f|\Omega| ||\rho||_0 + 2||\rho_t||_0.
\]

Consequently, it holds that

\[
\frac{d(||\theta||_0^2)}{dt} \leq 2(3L_f|\Omega| + 2)||\theta||_0^2 + 4L_f|\Omega| ||\rho||_0^2 + 4||\rho_t||_0^2. \tag{4.5.1.12}
\]

Integrate (4.5.1.12) with respect to \( t \):

\[
||\theta||_0^2 \leq ||\theta(0)||_0^2 + \int_0^t 2(3L_f|\Omega| + 2)||\theta||_0^2 ds + \int_0^t [4L_f|\Omega| ||\rho||_0^2 + 4||\rho_t||_0^2] ds.
\]

Gronwall's lemma leads to

\[
||\theta||_0^2 \leq C \left( ||\theta(0)||_0^2 + \int_0^t ||\rho||_0^2 + ||\rho_t||_0^2 ds \right).
\]

Therefore, we have

\[
||\theta||_0 \leq Ch_j^2, \tag{4.5.1.13}
\]

where \( C \) is independent of \( h_j \). Hence, we only need to derive (4.5.1.7).
To prove (4.5.1.7), choose \( v = \theta_t \) in (4.5.1.10) and arrive at

\[
\|\theta_t\|_0^2 + \frac{1}{2} \frac{d(\|\theta_t\|_0^2)}{dt} = (\int_\Omega [f(u) - f(R_j u)] dx, \theta_t) + (\int_\Omega [f(R_j u) - f(u_j)] dx, \theta_t) - (\rho_t, \theta_t) \\
\leq \|\int_\Omega [f(u) - f(R_j u)] dx\|_0 \cdot \|\theta_t\|_0 + \|\int_\Omega [f(R_j u) - f(u_j)] dx\|_0 \cdot \|\theta_t\|_0 + \|\rho_t\|_0 \cdot \|\theta_t\|_0 \\
\leq \frac{1}{2} \|\theta_t\|_0^2 + 2L_j^2 \Omega |\rho_t|^2 + 2L_j^2 \Omega \|\theta_t\|_0^2 + 2\|\rho_t\|_0^2.
\]

After eliminating the first term on the right-hand side and integrating with respect to \( t \), we obtain (in view of (4.5.1.13) and (4.5.1.11))

\[
|\theta|_1 \leq Ch_j,
\]

where \( C \) is independent of \( h_j \). Thus, we have verified Theorem 4.5.1.1.

4.5.2 Algorithm for parabolic Fredholm equation

Algorithm E: Let

\[
u^0 = R_0 \bar{u}.
\]

For \( j = 2, \ldots, J \), solve

\[
(u_j, v) + ((u_j, v)) = (\int_\Omega f(u^{j-1}) dx, v) ds, \quad \forall v \in V_j,
\]

for \( u^j \).

Theorem 4.5.2.1. Assume

\[
h_{j-1} \leq \frac{C}{(2\sqrt{2})^j}.
\]

Then

\[
|u - u^J|_1 \leq Ch_J,
\]

where \( C \) is independent of \( h_j \) and the number of grids.
Proof. Let $e_j := u_j - u^j$. Subtracting (4.5.2.1) from (4.5.1.3) yields

$$
(e_{j,t}, v) + ((e_j, v)) = \left( \int_{\Omega} [f(u_j) - f(u^{j-1})] dx, v \right) ds, \quad \forall v \in V_j,
$$

(4.5.2.3)

Substitute $v = e_{j,t}$ into (4.5.2.3) to obtain

$$
\begin{align*}
\|e_{j,t}\|^2_0 + \frac{1}{2} \frac{d(|e_j|^2)}{dt} & \leq \|e_{j,t}\|_0 \left\| \int_{\Omega} [f(u_j) - f(u_{j-1})] dx \right\|_0 \\
& + \|e_{j,t}\|_0 \left\| \int_{\Omega} [f(u_{j-1}) - f(u^{j-1})] dx \right\|_0 \\
& \leq \frac{1}{2}\|e_{j,t}\|^2_0 + L_j^2 |\Omega|^2 \|u_j - u_{j-1}\|^2_0 \\
& + L_j^2 |\Omega|^2 \|u_{j-1} - u^{j-1}\|^2_0.
\end{align*}
$$

(4.5.2.4)

Therefore, after eliminating the first term of right-hand side of (4.5.2.4) we find

$$
\frac{d(|e_j|^2)}{dt} \leq C h_{j-1}^4 + 2C |e_{j-1}|_1^2.
$$

(4.5.2.5)

Integrating (4.5.2.5) leads to

$$
|e_j|_1^2 \leq C h_{j-1}^4 + 2C \int_0^t |e_{j-1}|_1^2 ds \leq C h_{j-1}^4 + 2C h_{j-2}^4 + \cdots + 2^{j-2} C h_1^4 \leq C h_{j}^4 2^j h_{j-1}^2.
$$

In view of the assumption (4.5.2.2), we achieve

$$
|e_j|_1 \leq C h_j.
$$

(4.5.2.6)

and hence, combining (4.5.2.2) with (4.5.1.8),

$$
|u - u^j|_1 \leq |u - u_J|_1 + |u_J - u^j|_1 \leq C h_J,
$$

which completes the proof.
Chapter 5

The abstract cascading multigrid method in Besov spaces

It is important to mention that Bramble [19] contributed the general analysis of the V-cycle and the W-cycle in an abstract setting. Consider the FEM equation (5.2.1.2):

\[
A_k y_k = Q_k g,
\]

where \( A_k \) is symmetric positive definite. Let \( \lambda_{\text{max}} \) and \( \lambda_{\text{min}} \) denote the largest and smallest eigenvalue of \( A_k \), respectively. The condition number of \( A_k \) is defined by

\[
K(A_k) := \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}}.
\]

In view of the process of iteration, we know that the smaller the condition number \( K(A_k) \) the more effective the iteration method is. Therefore, we define the preconditioning operator (or: preconditioner)

\[
B_k : V_k \rightarrow V_k,
\]

such that the condition number \( K(B_k A_k) \) is as small as possible. Instead, we now solve the equation

\[
B_k A_k y_k = B_k Q_k g,
\]
by the iteration method. Indeed, this approach is more efficient. However the new problem arising is how to construct such a precondition operator $B_k$ with as little computational cost as possible. Bramble [19] and the references mentioned there showed that the multigrid processes, the V-cycle and the W-cycle, are the ideal methods. We shall prove that the cascading multigrid process is also a simple and robust method. Bornemann and Deuflhard [15] provided the numerical examples to compare the cascading multigrid algorithm with the V-cycle algorithm. We analyze this by using the general framework with the assumptions abstracted from the FEM discretization of a given problem. In fact, the abstract multigrid framework provided us with a good way to describe the method for more complicated problems. To see this, we shall apply the cascading multigrid algorithms to the heat equation with mild regularity in Besov spaces [6] and to the equation discretized by the interior penalty discontinuous Galerkin method (see [52] and references therein). In comparison with the analysis in Shi and Xu [107], a distinctive feature of our method is the use of block Jacobi and symmetric Gauss-Seidel iteration as smoothers. We also extend these methods to VIDEs.

5.1 Notations and definitions

5.1.1 Bilinear form and induced norm

A bilinear form $A(\cdot, \cdot)$ in a Hilbert space $V$ is called symmetric and elliptic, if it satisfies

1. (Symmetry) $A(u, v) = A(v, u), \ \forall u, v \in V$.

2. (Continuity) There exists a positive constant $C$, such that,

$$A(u, v) \leq C||u||_V||v||_V, \ \forall u, v \in V.$$
3. (Coercivity) There exists a positive constant $c$, such that,

$$A(u, u) \geq c\|u\|^2_V, \quad \forall u, v \in V.$$  

Here $\| \cdot \|_V$ is the norm in $V$.

We find that the elliptic bilinear forms can define norms in $V$. Let the elliptic bilinear form $A(\cdot, \cdot)$ define the norm $\| \| \cdot \|$ by

$$\| \| \cdot \| := (A(\cdot, \cdot))^{1/2},$$

and let the elliptic mesh-dependent bilinear form $A_k(\cdot, \cdot)$ induce the norm $\| \| \cdot \|_k$ via

$$\| \| \cdot \|_k := (A_k(\cdot, \cdot))^{1/2}.$$  

Define the time-dependent norm (see Section 5.2.2):

$$\| \| \cdot \|_\tau := (\tau^{-1}(\cdot, \cdot) + A(\cdot, \cdot))^{1/2},$$

and

$$\| \| \cdot \|_{k, \tau} := (\tau^{-1}(\cdot, \cdot) + A_k(\cdot, \cdot))^{1/2}.$$  

Here $(\cdot, \cdot)$ is defined by

$$(u, v) := \int_\Omega uv dx, \quad \forall u, v \in V,$$

and the temporal mesh size $\tau$ is a given positive number.

### 5.1.2 Function spaces and their norms

We introduce the multi-index notation. A multi-index is defined as

$$\alpha := (\alpha_1, \ldots, \alpha_n), \quad \alpha_i \in \mathbb{N}_0.$$
The length of $\alpha$ is given by

$$|\alpha| := \sum_{i=1}^{n} \alpha_i.$$ 

For $\phi \in C^\infty$, we let

$$D^\alpha \phi := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \phi.$$ 

Given a vector $x := (x_1, \ldots, x_n)$, we define $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.

Let $k$ be nonnegative integer and let

$$w \in L^1_{loc}(\Omega) := \{ w : w \in L^1(K), \forall \text{ compact } K \subset \text{interior } \Omega \}.$$ 

Suppose also that the weak derivatives $D^\alpha w$ exist for all $|\alpha| \leq k$. Define the Sobolev space norm

$$||w||_{H^k(\Omega)} := \left( \sum_{|\alpha| \leq k} ||D^\alpha w||_{L^2(\Omega)}^2 \right)^{1/2}.$$ 

We define the Sobolev spaces via

$$H^k(\Omega) := \{ w \in L^1_{loc}(\Omega) : ||w||_{H^k(\Omega)} < \infty \},$$

and

$$H^k_0(\Omega) := \{ w \in H^k(\Omega) : w|_{\partial \Omega} = 0 \}.$$ 

Let $s$ be a nonnegative real number and let $[s]$ denote the integer part of $s$.

Define the norm

$$||w||_{H^s(\Omega)} := \left( ||w||_{H^{|s|}(\Omega)}^2 + \sum_{|\alpha| = [s]} \int_{\Omega} \int_{\Omega} \frac{|w^\alpha(x) - w^\alpha(y)|^2}{|x - y|^{n+(s-[s])2}} dxdy \right)^{1/2}.$$ 

Then we define the fractional Sobolev spaces (Slobodeckij space) via

$$H^s(\Omega) := \{ w \in L^1_{loc}(\Omega) : ||w||_{H^s(\Omega)} < \infty \},$$

and

$$H^s_0(\Omega) := \{ w \in H^s(\Omega) : w|_{\partial \Omega} = 0 \}.$$
**Remark 5.1.2.1.** We direct the readers to the book [1] for more general definitions and properties of Sobolev spaces.

Kufner, John, and Fucik [79] and Triebel [114] discuss the method of defining fractional Sobolev spaces and Besov spaces by using the interpolation theory.

Let $X$ and $Y$ be two complex Banach spaces. Then we define, for $\theta \in (0, 1)$, the interpolation space

$$[X, Y]_{\theta, \infty} := \{ w \in Y : \sup_{t > 0} t^{-2\theta} K(t, w)^2 < \infty \},$$

and the $K$-functional by

$$K(t, w) := \inf_{w_0 \in X} (\|w_0\|^2_X + t^2 \|w - w_0\|^2_Y)^{1/2}.$$

For an appropriate bounded domain $\Omega \subset \mathbb{R}^n$, we define the Besov space via

$$B^s(\Omega) := (H^{s_0}(\Omega), H^{s_1}(\Omega))_{\theta, \infty},$$

where $\theta \in (0, 1)$, and $s_0$ and $s_1$ are nonnegative integers that satisfy $s_0 \neq s_1$, $s = (1 - \theta)s_0 + \theta s_1$.

**Remark 5.1.2.2.** If $s$ is a fractional and positive number, then $H^s(\Omega) := B^s(\Omega)$ is the fractional Sobolev space.

We now introduce the Besov space $B^s_{01}$ defined by the subspace interpolation of multilevel norms. The definition originates from [6].

Assume that

$$V_1 \subset V_2 \subset \cdots \subset V_k \subset \cdots$$

is a sequence of finite-dimensional subspaces of $H^1_0(\Omega)$ whose union is dense in $H^1_0(\Omega)$, and let $Q_k$ denote the $L^2(\Omega)$ orthogonal projection onto $V_k$ with $Q_0 := 0$ (see also}
(5.2.1.1) of Section 5.2). Let

$$B^{-s}(\Omega) := [L^2(\Omega), H^{-1}(\Omega)]_{s, \infty}.$$  

We introduce two norms, namely

$$\|w\|_{B^{-s}} := \sum_{k=1}^{\infty} (4^{k-1})^{-s/2} \| (Q_k - Q_{k-1})w \|_{L^2},$$

and

$$\|w\|_{B_{01}^{-s}}^2 := \|w\|_{B^{-s}}^2 + \|w\|_{B_1^{-s}}^2.$$  

The space $B_{01}^{-s}$ is then defined as the completion of $L^2(\Omega)$ with respect to the norm $\| \cdot \|_{B_{01}^{-s}}$.

5.2 The abstract cascading multigrid method

5.2.1 The cascading multigrid algorithm

Consider a finite-dimensional space $V$ equipped with an inner product $(\cdot, \cdot)$ and a bilinear form $A(\cdot, \cdot)$, with corresponding norms $\| \cdot \|$ and $\| \| \| \|$. We further assume that $A(\cdot, \cdot)$ is elliptic (i.e., continuous and coercive) and symmetric. We let the subspaces $V_k$ satisfy

$$V_0 \subset V_1 \subset \cdots \subset V_J \equiv V,$$

and we define the linear operator $A_k : V_k \to V_k$ by

$$(A_k u, v) := A(u, v), \text{ for all } u, v \in V_k.$$  

Obviously $A_k$ is symmetric and positive definite. Furthermore, we define two projectors $P_k : V_J \to V_k$ and $Q_k : V_J \to V_k$ by

$$A(P_k u, v) := A(u, v),$$
and

\[(Q_k u, v) := (u, v),\]  
(5.2.1.1)

for all \(u \in V_j\) and \(v \in V_k\). We also need to introduce a generic linear smoothing operator \(R_k : V_k \rightarrow V_k\) with the assumption that \(R_0 := A_0^{-1}\). For ease of analysis, we assume also that \(R_k\) is symmetric. To solve the equations for each level \(k := 0, 1, \ldots, J\):

\[A_k y_k = Q_k g,\]  
(5.2.1.2)

we shall now define the precondition operator \(B \equiv B_J\) by the following cascading multigrid algorithm.

**CMG Algorithm I:**

0) \(B_0 := A_0^{-1}\).

Define \(B_k\) implicitly in terms of \(B_{k-1}\), for \(k = 1, \ldots, J\):

1) For \(\ell = 1, \ldots, m_k\), we set

\[y^\ell_k := y^{\ell-1}_k + R_k(Q_k g - A_k y^{\ell-1}_k).\]

Here \(y^0_k := B_{k-1} Q_{k-1} g\).

2) \(B_k Q_k g := y^{m_k}_k\).

In order to analyze the CMG Algorithm I, we need to make the following four natural assumptions.

**Assumption 5.2.1.1.** Let \(\lambda_k\) denote the maximum eigenvalue of \(A_k\), i.e.,

\[\lambda_k := \sup_{v \in V_k} \frac{(A_k v, v)}{(v, v)}.\]  
(5.2.1.3)

For given \(\alpha \in [0, 1]\), there exists \(C_\alpha\) independent of \(k\) such that

\[(A_k^{1-\alpha}(I - P_{k-1})u, (I - P_{k-1})u) \leq C_\alpha \lambda_k^{-\alpha} A(u, u), \quad \text{for all } u \in V_k.\]
Assumption 5.2.1.2. Let $K_k := I - R_k A_k$, $R_{k,\omega} := \omega \lambda_k^{-1} I$ and $K_{k,\omega} := I - R_{k,\omega} A_k$. There exists $\omega \in (0, 1]$ such that

$$A(K_k v, K_k v) \leq A(K_{k,\omega/2} v, K_{k,\omega/2} v), \text{ for all } v \in V_k.$$

Hence it holds that

$$A(K_k^{\omega} v, K_k^{\omega} v) \leq A(K_{k,\omega/2}^{\omega} v, K_{k,\omega/2}^{\omega} v), \text{ for nonnegative number } m \text{ and for } v \in V_k.$$

Assumption 5.2.1.3. We suppose that, for the given constant $b > 0$,

$$c \frac{\lambda_J}{b^{J-k}} \leq \lambda_k \leq C \frac{\lambda_J}{b^{J-k}}.$$

Here and throughout the paper, $c$ and $C$ denote generic positive constants which are independent of $k$.

Assumption 5.2.1.4. For $\beta > 0$, let

$$m_k := \lfloor \beta^{J-k} m_J \rfloor,$$

where $\lfloor \cdot \rfloor$ means the greatest integer function.

Remark 5.2.1.1. We see that Assumption 5.2.1.1 is reasonable, if we refer to the elliptic boundary value problems with less than full regularity (cf. [15] [105]).

Remark 5.2.1.2. Bramble [19] tells us that block Jacobi and (symmetric) Gauss-Seidel are the examples of $R_k$ satisfying Assumption 5.2.1.2.

We need also the following lemma about a property of $K_{k,\omega}$, which is necessary for the analysis of the cascading multigrid method.

Lemma 5.2.1.1. The iterates of $K_{k,\omega}^{m_k}$ possess the following two properties:

$$A(K_{k,\omega}^{m_k} v, K_{k,\omega}^{m_k} v) \leq A(v, v), \quad (5.2.1.4)$$
and
\[ A(K_{k,\omega}^m v, K_{k,\omega}^m v) \leq C \frac{\lambda_k^{-\alpha}}{m_k^2} (A^{1-\alpha} u, v). \] (5.2.1.5)

for \( \omega \in (0, 1] \) and \( \alpha \in (0, 1] \).

**Proof.** See (pp. 164, Brenner and Scott [22]).

**Theorem 5.2.1.1.** Let \( B_J \) be defined by CMG Algorithm I and assume that Assumptions \{5.2.1.1, 5.2.1.2, 5.2.1.3, 5.2.1.4\} hold. Then we have

\[
A((I - B_J A_J)u, u) \leq \begin{cases} 
C \cdot \frac{1}{1 - (b/\beta)^{\alpha/2}} \cdot \frac{\lambda_j^{-\alpha/2}}{m_j^{\alpha/2}} A(u, u), & \text{for } \beta > b, \\
C \cdot J \cdot \frac{\lambda_j^{-\alpha/2}}{m_j^{\alpha/2}} A(u, u), & \text{for } \beta = b,
\end{cases}
\]

for all \( u \in V_J \).

**Proof.** We begin with the estimate:

\[
y_J - B_J A_J y_J = y_J - B_J Q_J g = y_J - y^{m_J}
\]

\[
= K_j^{m_J}(y_J - y^0) = K_j^{m_J}(y_J - B_{J-1} Q_{J-1}g)
\]

\[
= K_j^{m_J}(y_J - y_{J-1}) + K_j^{m_J}(y_{J-1} - B_{J-1} Q_{J-1}g)
\]

\[
\ldots
\]

\[
= K_j^{m_J}(y_J - y_{J-1}) + K_j^{m_J} K_{j-1}^{m_{J-1}} (y_{J-1} - y_{J-2})
\]

\[
+ \ldots + K_j^{m_J} K_{j-1}^{m_{J-1}} \ldots K_1^{m_1}(y_1 - y_0). \] (5.2.1.6)

In view of (5.2.1.2), we obtain

\[
y_k = A_k^{-1} Q_k Q_{J}^{-1} A_J y_J = A_k^{-1} Q_k A_J y_J, \quad \text{for } k = 0, \ldots, J - 1. \] (5.2.1.7)

Since Bramble [19] proved

\[ Q_k A_J = A_k P_k, \text{ on } V_k, \]
we have

\[ y_k = P_k y_J, \text{ for } k = 0, \ldots, J - 1. \]

Bringing these equations into (5.2.1.6) leads to

\[ I - B_J A_J = \sum_{k=1}^{J} \prod_{i=1}^{j-k+1} K_{j+1-i}^{m_{j+1-i}}(P_k - P_{k-1}). \]

Consequently by using the Cauchy-Schwartz inequality, we obtain

\[ A((I - B_J A_J)u, u) \]

\[ = \sum_{k=1}^{J} A \left( \prod_{i=1}^{j-k+1} K_{j+1-i}^{m_{j+1-i}}(P_k - P_{k-1})u, \prod_{i=1}^{j-k+1} K_{j+1-i}^{m_{j+1-i}}(P_k - P_{k-1})u \right) \]

\[ \leq \sum_{k=1}^{J} [A \left( \prod_{i=1}^{j-k+1} K_{j+1-i}^{m_{j+1-i}}(P_k - P_{k-1})u, \prod_{i=1}^{j-k+1} K_{j+1-i}^{m_{j+1-i}}(P_k - P_{k-1})u \right)]^{1/2} \cdot [A(u, u)]^{1/2}. \]

In view of \{Assumption 5.2.1.2, Lemma 5.2.1.1, Assumption 5.2.1.1\}, we estimate

\[ A \left( \prod_{i=1}^{j-k+1} K_{j+1-i}^{m_{j+1-i}}(P_k - P_{k-1})u, \prod_{i=1}^{j-k+1} K_{j+1-i}^{m_{j+1-i}}(P_k - P_{k-1})u \right) \]

\[ \leq A \left( K_{k,\omega/2}^{m_k}(P_k - P_{k-1})u, K_{k,\omega/2}^{m_k}(P_k - P_{k-1})u \right) \]

\[ \leq C \frac{\lambda_k^{-\alpha}}{m_k^\alpha} (A^{1-\alpha}(P_k - P_{k-1})u, (P_k - P_{k-1})u) \]

\[ \leq C \frac{\lambda_k^{-\alpha}}{m_k^\alpha} \lambda_k^{-\alpha} A(u, u). \]

The estimates (5.2.1.8) and (5.2.1.9) lead to

\[ A((I - B_J A_J)u, u) \leq C \left( \sum_{k=1}^{J} \frac{\lambda_k^{-\alpha}}{m_k^\alpha/2} \right) A(u, u). \]

Combining (5.2.1.10), Assumption 5.2.1.3 and Assumption 5.2.1.4, we obtain our desired result.

**Theorem 5.2.1.2.** Let \( N_k := \dim(V_k) \) and assume \( N_{k+1} \geq a N_k \). Then the computational cost of the CMG Algorithm I is given by

\[ \sum_{k=1}^{J} m_k N_k \leq \begin{cases} C \cdot \frac{1}{1 - \beta/a} \cdot m_J N_J, & \text{for } \beta < a, \\ C \cdot J \cdot m_J N_J, & \text{for } \beta = a. \end{cases} \]
Proof. $N_{k+1} \geq aN_k$ implies $N_k \leq a^{-(J-k)}N_J$. Recalling Assumption 5.2.1.4, we estimate
\[
\sum_{k=1}^{J} m_k N_k \leq C \sum_{k=1}^{J} \left( \frac{\beta}{a} \right)^{J-k} \cdot m_J N_J.
\]
Hence (5.2.1.11) is true.

Remark 5.2.1.3. The parameter $b$ in the expression of Theorem 5.2.1.1 solely depends on the original problem. It is well known that the finite element method for a partial differential equation of order $n$ leads to $b = 2^n$. The parameter $a$ in (5.2.1.11) (Theorem 5.2.1.2) depends only on the dimension $d$ of the domain, e.g., $a = 2^d$. When $2^d = a > b$, that is, $d > \lfloor \log b \rfloor$, the condition number $K(I - B_J A_J)$ is uniformly bounded by $O(\lambda_j^{-\alpha/2})$, and the amount of work is proportional to $O(N_J)$. The cascading algorithm is henceforth called optimal. The algorithm will be called near-optimal if it satisfies the following corollary.

Corollary 5.2.1.1 (Bornemann and Deuflhard (1996)). In case of $b = a = 2^d$, we choose $\beta = 2^d$ and the number of iterations on level $J$ as
\[
m_J = \lfloor m_* \cdot J^{d/\alpha} \rfloor.
\]
Then we have the estimates
\[
K(I - B_J A_J) \leq C\lambda_j^{-\alpha/2},
\]
and
\[
\sum_{k=1}^{J} m_k N_k \leq C m_* N_J (1 + \log N_J)^{1+\alpha/2}.
\]

5.2.2 The method for the heat equation

Shi and Xu [107] analyzed the cascading multigrid method for the heat equation with Richardson and Conjugate Gradient iteration as smoothers. We shall solve the
heat equation with mild regularity by the abstract cascading multigrid method with block Jacobi and symmetric Gauss-Seidel iteration as smoothers.

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a open polygonal domain with the largest corner angle $\omega$. We consider the heat equation:

$$u_t - \Delta u = f(x, t), \quad \text{on } \Omega \times [0, T], \quad (5.2.2.1)$$

with boundary condition

$$u(x, t) = 0, \quad \text{on } \partial \Omega \times [0, T],$$

and initial condition

$$u(x, 0) = u_0(x), \quad \text{on } \Omega,$$

where $f \in H^{-1+\alpha}(\Omega)$. The weak form of (5.2.2.1) is to find $u \in H^1_0(\Omega)$, with $u(x, 0) = u_0(x) \in H^{1+\alpha}(\Omega) \cap H^1_0(\Omega)$, such that

$$(u_t, v) + B(u, v) = (f, v), \quad \forall v \in H^1_0(\Omega), \quad t \in [0, T], \quad (5.2.2.2)$$

where the bilinear form $B$ is

$$B(u, v) = \int_{\Omega} \nabla u \cdot \nabla v dx, \quad \forall u, v \in H^1_0(\Omega),$$

and

$$(f, v) = \int_{\Omega} fv dx.$$

We use the backward Euler scheme for the time-stepping. Let

$$I_\tau := \{t_n : 0 =: t_0 < t_1 < \cdots < t_N := T\},$$

be the mesh on $I$ with

$$\tau_n := t_n - t_{n-1}, \quad \tau := \max_{(n)} \tau_n.$$
Moreover, we set

\[ w := u^n - u^{n-1}, \quad (g, v) := (f, v) - B(u^{n-1}, v). \]

Then (5.2.2.2) leads to the weak form with time-stepping: Find \( w \in H^1_0(\Omega) \) such that

\[ A_r(w, v) = (g, v), \quad \forall v \in H^1_0(\Omega), \quad (5.2.2.3) \]

where

\[ A_r(w, v) := \tau^{-1}(w, v) + B(w, v). \quad (5.2.2.4) \]

We conclude that (5.2.2.3) has a unique solution \( w \in H^{1+\alpha}(\Omega) \cap H^1_0(\Omega) \) which satisfies

\[ ||w||_{H^{1+\alpha}} \leq C||g||_{H^{-1+\alpha}}, \quad (5.2.2.5) \]

where \( 0 < \alpha < \alpha_0 = \frac{\pi}{\omega} \) (see Ciarlet [35] and Johnson [76]).

Let \( \mathcal{T}_k (k := 0, 1, \ldots, J) \) be a sequence of quasi-uniform triangular partitions of \( \Omega \) with mesh size \( h_k = h_02^{-k} \). Let \( V_k \) denote the P1 conforming finite element space on \( \mathcal{T}_k \). It is well known that

\[ V_0 \subset V_1 \subset \cdots \subset V_J \equiv V \subset H^1_0(\Omega). \]

We derive the discrete form of (5.2.2.3): Find \( w_k \in V_k \) such that

\[ A_r(w_k, v) = (g, v), \quad \forall v \in V_k. \quad (5.2.2.6) \]

Define

\[ (A_kw_k, v) := A_r(w_k, v), \quad \forall w_k, v \in V_k. \]

Then (5.2.2.6) can be expressed by

\[ A_kw_k = g_k, \quad (5.2.2.7) \]
where $g_k \in V_k$, $(g_k, v) := (g, v)$, $\forall v \in V_k$. It is well-known that
\[ ||w - w_k||_\tau \leq Ch_k^\alpha (1 + \tau^{-1} h_k)^{1/2}||g||_{H^{-1+\alpha}}, \quad (5.2.2.8) \]
where
\[ ||\cdot||_\tau := (\tau^{-1}(\cdot, \cdot) + B(\cdot, \cdot))^{1/2}. \]
We know from the definitions (5.2.1.3) and (5.2.2.4) that
\[ \lambda_k = \mathcal{O}(h_k^{-2}) + \tau^{-1}. \]
Since [10] observed that some commonly used iterative methods, like Richardson iteration, can already guarantee good convergence for $\tau \leq \lambda_J^{-1}$, we therefore only consider the case
\[ \lambda_J^{-1} \leq \tau \leq \frac{\lambda_J^{-1}}{\gamma_0}, \quad \text{for some } \gamma_0 \in (0, 1). \quad (5.2.2.9) \]
Now we shall verify that Assumptions \{(5.2.1.1), (5.2.1.2), (5.2.1.3), (5.2.1.4)\} hold true in this case. Let
\[ A := A_\tau. \]
From [19] and [113], we conclude that Assumption 5.2.1.1 is satisfied. Assumption 5.2.1.3 can be easily written as
\[ c \left( \frac{\lambda_J}{2^{2(j-k)}} + \frac{2^{2(j-k)} - 1}{2^{2(j-k)}} \tau^{-1} \right) \leq \lambda_k \leq C \left( \frac{\lambda_J}{2^{2(j-k)}} + \frac{2^{2(j-k)} - 1}{2^{2(j-k)}} \tau^{-1} \right). \quad (5.2.2.10) \]
Assumption 5.2.1.2 and \{(5.3.1.2), (5.3.1.3)\} concern only $R_k$. Therefore they remain true as discussed in Section 5.2.1. Thus, recalling (5.2.1.10), we obtain
\[ A((I - BJ)m_{\alpha/2}) \]
\[ \leq C \left( \sum_{k=1}^{J} \frac{\lambda_k^{-\alpha}}{m_{\alpha/2}} \right) A(w, w) \]
\[ \leq C \left( \frac{1}{1 + \gamma_0} \right)^{\alpha} m_{\alpha/2} \sum_{k=1}^{J} \left( \frac{2^\alpha}{m_{\alpha/2}} \right)^k A(w, w). \]
So (5.2.2.11) yields the following theorem. The last estimate in (5.2.2.11) reveals that the attainable (optimal or nearly optimal) order depends on the value of $\beta$; for $\beta = 4$ the estimate will contain the factor $J$, i.e., the number of the grid points. Compare also Bornemann and Deuflhard [15].

**Theorem 5.2.2.1.** Under Assumption 5.2.1.4, we have the following convergence estimate of CMG Algorithm I for (5.2.2.7):

$$A((I - B_J A_J)w, w) \leq \begin{cases} 
C(\alpha) \cdot \frac{1}{1 - \left(\frac{2}{\beta^{1/2}}\right)^2} \cdot \frac{h_j^q}{m_j^{\alpha/2}} A(w, w), & \text{for } \beta > 4, \\
C(\alpha) \cdot J \cdot \frac{h_j^q}{m_j^{\alpha/2}} A(w, w), & \text{for } \beta = 4,
\end{cases}$$

with $C(\alpha) := C(1/(1 + \gamma_0))^\alpha$. These estimates hold for all $w \in V_J$.

In view of

$$N_{k+1} \geq 2^d N_k,$$

we conclude that Theorem 5.2.1.2 remains true with $a = 2^d$. Clearly, when $d = 2$, CMG Algorithm I for (5.2.2.7) is near optimal; when $d = 3$, the algorithm is optimal.

For a detailed discussion on how to choose $\beta$ so that the cascading multigrid method has optimal accuracy and complexity, we refer to Shi and Xu [107] or Bornmann and Deuflhard [15].

**5.2.3 The cascading multigrid method in Besov spaces**

For $s \in (0, 1)$, define the interpolation space

$$[X, Y]_{s, \infty} := \{w \in Y : \sup_{t>0} t^{-2s} K(t, w)^2 < \infty\},$$

and the $K$-functional by

$$K(t, w) := \inf_{w_0 \in X} (||w_0||_X^2 + t^2 ||w - w_0||_Y^2)^{1/2}.$$
Recall the equation (5.2.2.3): Find $w \in H_0^1(\Omega)$ such that

$$A_{\tau}(w, v) = (g, v), \quad \forall v \in H_0^1(\Omega).$$

(5.2.3.1)

where

$$A_{\tau}(w, v) := \tau^{-1}(w, v) + B(w, v).$$

Recall (5.2.2.5): for $0 < \alpha < \alpha_0 = \frac{\pi}{\omega}$, we have the regularity

$$||w||_{H^{1+\alpha}} \leq C||g||_{H^{-1+\alpha}}, \quad \forall g \in H^{-1+\alpha_0}(\Omega).$$

For the critical case $\alpha = \alpha_0$, Bacuta, Bramble and Xu [6] proved the estimate

$$||w||_{B^{1+\alpha_0}(\Omega)} \leq C||g||_{B_{01}^{1+\alpha_0}(\Omega)}, \quad \forall g \in B_{01}^{1+\alpha_0}(\Omega),$$

(5.2.3.2)

where $B^{1+\alpha_0}(\Omega)$ is a standard Besov space and $B_{01}^{1+\alpha_0}(\Omega)$ was defined in [6]. Correspondingly, the convergence estimate of the $P1$ finite element method for (5.2.3.1) is

$$||w - w_k||_{k, q} \leq C h_k^{\alpha_0}(1 + \tau^{-1}h_k)^{1/2}||w||_{H^2(\Omega) \cap H_0^1(\Omega), H_0^1(\Omega); 1 - \alpha_0, \infty}$$

$$\leq C h_k^{\alpha_0}(1 + \tau^{-1}h_k)^{1/2}||g||_{B_{01}^{1+\alpha_0}(\Omega)}. $$

(5.2.3.3)

We begin our analysis with an analogue of equation (5.2.1.9). Triebel [114, pp. 59] proved the interpolation property

$$H^1(\Omega) = [H^1(\Omega), H^1(\Omega)]_{\theta, \infty}.$$

Hence, we have

$$A \left(K_{k, w/2}^{m_k}(P_k - P_{k-1})w, K_{k, w/2}^{m_k}(P_k - P_{k-1})w \right)$$

$$\leq C \sup_{t > 0} t^{-2(1 - \alpha_0)} \inf_{w_0 \in V_j} \left[ A \left(K_{k, w/2}^{m_k}(P_k - P_{k-1})w_0, K_{k, w/2}^{m_k}(P_k - P_{k-1})w_0 \right) \right.$$ 

$$+ t^2 A \left(K_{k, w/2}^{m_k}(P_k - P_{k-1})(w - w_0), K_{k, w/2}^{m_k}(P_k - P_{k-1})(w - w_0) \right)].$$

(5.2.3.4)
We use Lemma 5.2.1.1 and Assumption 5.2.1.1 with $\alpha = 0$ and $\alpha = 1$ to get the equations

$$A\left(K_{k,w/2}^m(P_k - P_{k-1})w_0, K_{k,w/2}^m(P_k - P_{k-1})w_0 \right)$$

$$\leq C \frac{\lambda_k^{-1}}{m_k} ((P_k - P_{k-1})w_0, (P_k - P_{k-1})w_0) \leq C \frac{\lambda_k^{-2}}{m_k} A(w_0, w_0), \quad (5.2.3.5)$$

and

$$A\left(K_{k,w/2}^m(P_k - P_{k-1})(w - w_0), K_{k,w/2}^m(P_k - P_{k-1})(w - w_0) \right) \leq C A(w - w_0, w - w_0). \quad (5.2.3.6)$$

Hence \{(5.2.3.4), (5.2.3.5), (5.2.3.6)\} lead to

$$A\left(K_{k,w/2}^m(P_k - P_{k-1})\lambda_k^{1/2} w, K_{k,w/2}^m(P_k - P_{k-1})\lambda_k^{1/2} w \right)$$

$$\leq C \frac{\lambda_k^{-2\alpha_0}}{m_k^{\alpha_0/2}} \sup_{t > 0} (m_k^{1/2} \lambda_k t)^{-2(1-\alpha_0)} \inf_{w_0 \in V_f} \left[ A(w_0, w_0) + (m_k^{1/2} \lambda_k t)^2 A(w - w_0, w - w_0) \right]$$

$$\leq C \frac{\lambda_k^{-2\alpha_0}}{m_k^{\alpha_0/2}} A(w, w). \quad (5.2.3.7)$$

Therefore we obtain

$$A((I - B_j A_j)w, w) \leq C \left( \sum_{k=1}^J \frac{\lambda_k^{\alpha_0}^{-\alpha_0}}{m_k^{\alpha_0/2}} \right) A(w, w).$$

This suggests the following theorem:

**Theorem 5.2.3.1.** Under Assumption 5.2.1.4, we have the convergence estimate of the CMG Algorithm I for (5.2.3.1), namely

$$A((I - B_j A_j)w, w) \leq \begin{cases} 
C \cdot \frac{1}{1 - \left(\frac{2}{\beta^{1/2}}\right)^{\alpha_0}} \cdot \frac{h_j^{\alpha_0}}{m_j^{\alpha_0/2}} A(w, w), & \text{for } \beta > 4, \\
C \cdot \frac{h_j^{\alpha_0}}{m_j^{\alpha_0/2}} A(w, w), & \text{for } \beta = 4,
\end{cases}$$

for all $w \in V_f$. 


5.2.4 The method for VIDEs

We consider
\[ u_t - \Delta u + \int_0^t k(t-s)Bu(x,s)ds = f(x,t), \quad x \in \Omega, \ t \in I := [0,T], \tag{5.2.4.1} \]
with initial condition
\[ u(x,0) = u_0(x), \quad x \in \Omega, \]
and boundary condition
\[ u(x,t) = 0, \quad x \in \partial \Omega, \ t \in I, \]
where \( \Omega \subset \mathbb{R}^d \) (\( d = 2, 3 \)) is a bounded convex polygonal domain. \( B \) is an elliptic differential operator of order up to two. We can easily follow the idea of Chapter 4 to analyze the cascading multigrid method for (5.2.4.1) with less than second-order \( B \). So it is interesting to analyze the case with dominant memory term (i.e., the operator \( B \) is second order). For ease of exposition, we suppose \( B := -\Delta \). In most cases of application, \( k \) is nonnegative. We want to use the trapezoidal rule (5.2.4.3) for the memory term, hence we assume \( k \in C^1(I) \). If \( k \) is weakly singular, we can use the left-rectangular rule (see [34]).

The weak form of (5.2.4.1) is to find \( u \in H_0^1(\Omega) \), with \( u(x,0) = u_0(x) \in H^{1+\alpha}(\Omega) \cap H_0^1(\Omega) \), such that
\[ (u_t, v) + B(u,v) + \int_0^t k(t-s)B(u(x,s),v)ds = (f,v), \quad \forall v \in H_0^1(\Omega), \ t \in I, \tag{5.2.4.2} \]
where the bilinear form \( B \) is as before:
\[ B(u,v) = \int_\Omega \nabla u \cdot \nabla v dx, \quad \forall u, v \in H_0^1(\Omega), \]
and also
\[ (f,v) = \int_\Omega f v dx. \]
We use the backward Euler scheme for time-stepping of (5.2.4.2) and the trapezoidal rule,
\[
\int_0^{t_n} p(s) ds \approx \frac{1}{2} \tau p(0) + \sum_{j=1}^{n-1} \tau p(t_j) + \frac{1}{2} \tau p(t_n), \tag{5.2.4.3}
\]
for discretizing the memory term.

Let \( k_j := k(t_n - t_j) \) (\( j = 0, \ldots, n \)). Then we get that: Find \( u^n - u^{n-1} =: w \in H^{1+\alpha}_0(\Omega) \), such that
\[
A_\tau(w, v) = (g, v), \quad \forall v \in H^1_0(\Omega), \tag{5.2.4.4}
\]
where
\[
A_\tau(w, v) := \tau^{-1}(w, v) + (1 + \frac{1}{2} \tau k_n) B(w, v),
\]
and
\[
(g, v) := (f, v) - (1 + \frac{1}{2} \tau k_n) B(u^{n-1}, v) - \frac{1}{2} \tau k_0 B(u_0, v) - \sum_{j=1}^{n-1} \tau k_j B(u^j, v).
\]
The same discussion as that in Section 5.2.2 leads to the \( P_1 \) conforming finite element approximation to (5.2.4.4): Find \( w_k \in V_k \subset H^1_0(\Omega) \), such that
\[
A_\tau(w_k, v) = (g, v), \quad \forall v \in V_k, \tag{5.2.4.5}
\]
and its error estimate is
\[
\|w - w_k\|_{\tau} \leq C h_k^\alpha (1 + \tau^{-1} k_0 h_k)^{1/2} \|g\|_{H^{-1+\alpha}}. \tag{5.2.4.6}
\]
We also define
\[
(A_k w_k, v) := A_\tau(w_k, v), \quad \forall w_k, \, v \in V_k. \tag{5.2.4.7}
\]
Then (5.2.4.5) can be written as
\[
A_k w_k = g_k, \tag{5.2.4.8}
\]
where
\( g_k \in V_k, (g_k, v) := (g, v), \forall v \in V_k.\)

We see that
\[
\lambda_k = C\left(1 + \frac{1}{2}T_k\right)h_k^{-2} + \tau^{-1}.
\]

Hence, we have
\[
c\left(\frac{(1 + \frac{1}{2}T_k)\lambda_J}{2^{2(J-k)}} + \frac{2^{2(J-k)} - 1}{2^{2(J-k)}}\tau^{-1}\right) \leq \lambda_k \leq C\left(\frac{(1 + \frac{1}{2}T_k)\lambda_J}{2^{2(J-k)}} + \frac{2^{2(J-k)} - 1}{2^{2(J-k)}}\tau^{-1}\right).
\]

(5.2.4.9)

Assumption 5.2.1.2 and \{(5.3.1.2), (5.3.1.3)\} concern only \(R_k\). They henceforth remain true as discussed in Section 5.2.1. Thus, recalling (5.2.1.10), we obtain
\[
A((I - B_JA_J)w, w) \leq C\left(\sum_{k=1}^{J} \frac{\lambda_k^{-\alpha}}{m_k^{\alpha/2}}\right)A(w, w)
\]

(5.2.4.10)

\[
\leq C \frac{h_j^\alpha}{m_j^{\alpha/2}} \sum_{k=1}^{J} \left(\frac{2^{\alpha}}{\beta^{\alpha/2}}\right) A(w, w).
\]

We used \(a + b \geq 2\sqrt{ab}\) in the calculation from (5.2.4.9) to (5.2.4.10).

So (5.2.4.10) yields the following theorem.

**Theorem 5.2.4.1.** Under Assumption 5.2.1.4, we have the following convergence estimate of the CMG Algorithm I for (5.2.4.8):

\[
A((I - B_JA_J)w, w) \leq \begin{cases}
C \cdot \frac{1}{1 - \left(\frac{2}{\beta^{1/2}}\right)} \cdot \frac{h_j^\alpha}{m_j^{\alpha/2}} A(w, w), & \text{for } \beta > 4, \\
C \cdot J \cdot \frac{h_j^\alpha}{m_j^{\alpha/2}} A(w, w), & \text{for } \beta = 4,
\end{cases}
\]

for all \(w \in V_J\).

**Remark 5.2.4.1.** We notice that compared with Section 5.2.2 and Section 5.3.2, the analysis of this section does not need the assumption (5.2.2.9).
5.3 Abstract cascading multigrid method with non-nested spaces and varying inner products and bilinear forms

In the previous sections we assumed that $V_0 \subset V_1 \subset \cdots \subset V_J = V$, and that $(\cdot, \cdot)$ and $A(\cdot, \cdot)$ are defined on all $V_k$. In this section, we allow the spaces $V_k$ to be not necessarily nested and define a symmetric elliptic bilinear form $A_k(\cdot, \cdot)$ and an inner product $(\cdot, \cdot)_k$ corresponding to $0 \leq k \leq J$.

5.3.1 The cascading multigrid algorithm

Consider the following setting:

(1) Let $V_0, \ldots, V_J$ be $J$ finite dimensional spaces with $V_J = V$, which are not necessarily nested.

(2) Define $J$ linear operators

$$I_k : V_{k-1} \rightarrow V_k, \quad k = 1, \ldots, J,$$

which connect the spaces.

(3) Let $(\cdot, \cdot)_k$ be an inner product on $V_k \times V_k$, with induced norm $\| \cdot \|_k$, and let $A_k(\cdot, \cdot)$ be symmetric and elliptic bilinear form on $V_k \times V_k$ with $A_J(\cdot, \cdot) = A(\cdot, \cdot)$ and induced norm $\| \| \cdot \|_k$.

(4) Define $A_k : V_k \rightarrow V_k$ by

$$(A_k u, v)_k := A_k(u, v), \quad \text{for all } v \in V_k.$$

(5) Define $P_{k-1} : V_k \rightarrow V_{k-1}$ and $Q_{k-1} : V_k \rightarrow V_{k-1}$ by

$$A_{k-1}(P_{k-1} u, v) := A_k(u, I_k v),$$
and
\[(Q_{k-1}u, v)_{k-1} := (u, I_k v)_k,\]
for all \(v \in V_{k-1}\).

(6) Let \(R_k\) be a linear symmetric operator \(R_k : V_k \to V_k\) with \(R_0 := A_0^{-1}\).

We shall solve the equation (5.2.1.2) on each level \(k := 0, 1, \ldots, J\) by the cascading multigrid algorithm.

Define the precondition operator \(B \equiv B_J\) by the

**CMG Algorithm II:**

0) \(B_0 := A_0^{-1}\);

Define \(B_k\) implicitly in terms of \(B_{k-1}\), for \(k = 1, \ldots, J\):

1) For \(\ell = 1, \ldots, m_k\), we set
\[y_k^\ell := y_{k-1}^\ell + R_k(Q_k g - A_k y_{k-1}^{\ell-1}).\]

Here \(y_k^0 := I_k B_{k-1} Q_{k-1} g\).

3) \(B_k Q_k g := y_k^{m_k}\).

Before we analyze the CMG Algorithm II, we add the following three natural assumptions.

**Assumption 5.3.1.1.** We suppose
\[A_k(I_k u, I_k u) \leq A_{k-1}(u, u), \text{ for all } u \in V_{k-1}.\]  

(5.3.1.1)

Using the Cauchy-Schwartz inequality, we can easily prove that (5.3.1.1) holds true if and only if
\[A_{k-1}(P_{k-1} u, P_{k-1} u) \leq A_k(u, u), \text{ for all } u \in V_k.\]
Assumption 5.3.1.2. Let $\lambda_k$ denote the maximum eigenvalue of $A_k$, i.e.,

$$\lambda_k := \sup_{v \in V_k} \frac{(A_k v, v)_k}{(v, v)_k}.$$

For given $\alpha \in (0, 1]$, there exists a constant $C_\alpha$, independent of $k$, such that

$$(A_k^{1-\alpha} (I - I_k P_{k-1}) u, (I - I_k P_{k-1}) u)_k \leq C_\alpha \lambda_k^{-\alpha} A_k (u, u), \quad \text{for all } u \in V_k$$

(compare with Assumption 5.2.1.1, $(\cdot, \cdot) \equiv (\cdot, \cdot)_k$).

Assumption 5.3.1.3. Let $K_k := I - R_k A_k$, $R_{k, \omega} := \omega \lambda_k^{-1} I$ and $K_{k, \omega} := I - R_{k, \omega} A_k$. There exists $\omega \in (0, 1]$ such that

$$A_k (K_k v, K_k v) \leq A_k (K_{k, \omega/2} v, K_{k, \omega/2} v), \quad \text{for all } v \in V_k.$$

Hence it holds that

$$A_k (K^m_k v, K^m_k v) \leq A_k (K^m_{k, \omega/2} v, K^m_{k, \omega/2} v), \quad \text{for nonnegative number } m \text{ and } v \in V_k$$

(compare with Assumption 5.2.1.2).

By using the same argument as in the proof of Lemma 5.2.1.1, we can prove the following lemma.

Lemma 5.3.1.1. $K^m_{k, \omega}$ satisfies the following two properties:

$$A_k (K^m_{k, \omega} v, K^m_{k, \omega} v) \leq A_k (v, v), \quad (5.3.1.2)$$

and

$$A_k (K^m_{k, \omega} v, K^m_{k, \omega} v) \leq C\frac{\lambda_k^{-\alpha}}{m^\alpha_k} (A_k^{1-\alpha} v, v)_k, \quad (5.3.1.3)$$

for $\omega \in (0, 1]$ and $\alpha \in (0, 1]$.

Now we are ready to present our main results.
Theorem 5.3.1.1. Let $B_J$ be defined by the CMG Algorithm II, and let Assumptions \{5.3.1.1, 5.3.1.2, 5.3.1.3, 5.2.1.3, 5.2.1.4\} hold. Then we have

$$A((I - B_JA_J)u, u) \leq \begin{cases} 
C \cdot \frac{1}{1 - (b/\beta)^{\alpha/2}} \cdot \frac{\lambda_{J}^{\alpha/2}}{m_{J}^{\alpha/2}} A(u, u), & \text{for } 3 > b, \\
C \cdot J \cdot \frac{\lambda_{J}^{\alpha/2}}{m_{J}^{\alpha/2}} A(u, u). & \text{for } \beta = b.
\end{cases}$$

for all $u \in V_J$.

Proof. We first estimate:

$$y_J - B_JA_Jy_J = y_J - B_JQ_Jg = y_J - y^{m_J} \quad (5.3.1.4)$$

$$= K_{mJ}^{mJ}(y_J - y^0) = K_{mJ}^{mJ}(y_J - I_JB_{J-1}Q_{J-1}g)$$

$$= K_{mJ}^{mJ}(y_J - I_Jy_{J-1}) + K_{mJ}^{mJ}(I_Jy_{J-1} - I_JB_{J-1}Q_{J-1}g)$$

$$\ldots \ldots$$

$$= K_{mJ}^{mJ}(y_J - I_Jy_{J-1}) + (K_{mJ}^{mJ}I_J)K_{mJ-1}^{mJ-1}(y_{J-1} - I_{J-1}y_{J-2})$$

$$+ \ldots + (K_{mJ}^{mJ}I_J)(K_{mJ-1}^{mJ-1}I_{J-1}) \ldots (K_{2}^{m2}I_2)K_{1}^{m1}(y_1 - I_1y_0).$$

From (5.2.1.2):

$$A_ky_k = Q_kg,$$

we derive

$$(A_ky_k, I_kv)_k = (Q_kg, I_kv)_k, \quad \forall v \in V_{k-1}.$$  

Hence,

$$A_k(y_k, I_kv) = (Q_kg, I_kv)_k, \quad \forall v \in V_{k-1}.$$  

In view of the definitions of $P_k$ and $Q_k$, we obtain

$$A_{k-1}(P_{k-1}y_k, v) = (Q_{k-1}g, v)_{k-1}, \quad \forall v \in V_{k-1},$$
which leads to
\[ A_{k-1}P_{k-1}y_k = Q_{k-1}g. \]

Therefore, we have
\[ y_{k-1} = P_{k-1}y_k, \quad \text{for} \quad k = 0, \ldots, J - 1. \]  \hspace{1cm} (5.3.1.5)

Combining (5.3.1.5) and (5.3.1.4) we find
\[ I - B_J A_J = K_{m}^{m_j}(I - I_J P_{J-1}) + \sum_{k=1}^{J-1} \prod_{i=1}^{J-k} (K_{j+1-i}^{m_{j+1-i}}i_{J+1-i}) \cdot K_{k}^{m_k} \cdot (I - I_k P_{k-1}) \cdot \prod_{i=1}^{J-k} P_{J-i}. \]

By the Cauchy-Schwarz inequality and Lemma 5.3.1.1,
\[
A((I - B_J A_J)u, u) \\
\leq \left[ A(K_{m}^{m_j}(I - I_J P_{J-1})u, K_{m}^{m_j}(I - I_J P_{J-1})u) \right]^{1/2} \cdot [A(u, u)]^{1/2} \\
+ \sum_{k=1}^{J-1} \left[ A_k \left( (K_{k}^{m_k} \cdot (I - I_k P_{k-1}) \cdot \prod_{i=1}^{J-k} P_{J-i}u, (K_{k}^{m_k} \cdot (I - I_k P_{k-1}) \cdot \prod_{i=1}^{J-k} P_{J-i})u \right) \right]^{1/2} \\
\cdot [A(u, u)]^{1/2}. \]  \hspace{1cm} (5.3.1.6)

From Assumption 5.3.1.3, Lemma 5.3.1.1, and Assumption 5.3.1.2, we have
\[
A(K_{m}^{m_j}(I - I_J P_{J-1})u, K_{m}^{m_j}(I - I_J P_{J-1})u) \\
\leq C \frac{\lambda_j^{1/2}}{m_j^{1/2}} (A^{1/2}(I - I_J P_{J-1})u, (I - I_J P_{J-1})u) \\
\leq C \frac{\lambda_j^{1/2}}{m_j^{1/2}} A(u, u), \]  \hspace{1cm} (5.3.1.7)

and in addition, using Assumption 5.3.1.1,
\[
A_k \left( (K_{k}^{m_k} \cdot (I - I_k P_{k-1}) \cdot \prod_{i=1}^{J-k} P_{J-i}u, (K_{k}^{m_k} \cdot (I - I_k P_{k-1}) \cdot \prod_{i=1}^{J-k} P_{J-i})u \right) \\
\leq C \frac{\lambda_k^{1/2}}{m_k^{1/2}} \left( A_k^{1/2}(I - I_k P_{k-1}) \prod_{i=1}^{J-k} P_{J-i}u, (I - I_k P_{k-1}) \prod_{i=1}^{J-k} P_{J-i})u \right)^{1/2} \\
\leq C \frac{\lambda_k^{1/2}}{m_k^{1/2}} A_k \left( \prod_{i=1}^{J-k} P_{J-i}u, \prod_{i=1}^{J-k} P_{J-i}u \right) \leq C \frac{\lambda_k^{1/2}}{m_k^{1/2}} A(u, u). \]  \hspace{1cm} (5.3.1.8)
Hence (5.3.1.6), (5.3.1.7) and (5.3.1.8) lead to

\[ A((I - B_j A_j)u, u) \leq C \left( \sum_{k=1}^{J} \frac{\lambda_k^{\alpha}}{m_k^{\alpha/2}} \right) A(u, u). \] (5.3.1.9)

Combining (5.3.1.9) and Assumption 5.2.1.3 and Assumption 5.2.1.4, we obtain our desired result.

The computational cost estimate theorem is the same as Theorem 5.2.1.2.

### 5.3.2 Application to interior penalty discontinuous Galerkin method

The V-cycle algorithm for the interior penalty discontinuous Galerkin method was presented in the paper by Gopalakrishnan and Kanschat [52], which was based on Arnold [3]. In this section, we analyze the CMG Algorithm II for the heat equation (5.2.2.1) with the interior penalty discontinuous Galerkin discretization. We will essentially use the notations in Section 5.3.1.

Let \( T_k (k := 0, 1, \ldots, J) \) be a quasi-uniform triangular partition of \( \Omega \) with the mesh size \( h_k = h_0 2^{-k} \). We define the multilevel spaces

\[ V_0 \subset V_1 \subset \cdots \subset V_J \equiv V, \]

by

\[ V_k := \{ v : v|_T \in \mathcal{P}^{(m)}(T), \forall T \in T_k \}, \]

where \( \mathcal{P}^{(m)}(T) \) denotes the polynomial with degree not exceeding \( m \) on \( T \).

To describe the interior penalty discontinuous Galerkin method, we need the spaces

\[ H_0^1(T_k) := \{ v \in L^2(\Omega) : v|_T \in H^1(T) \text{ and } v|_{\partial \Omega \cap T} = 0, \forall T \in T_k \}. \]
Let $\mathcal{E}_k$ denote the set of edges of the triangulation $\mathcal{T}_k$. If $e \in \mathcal{E}_k$ is an interior edge, we denote by $n_e$ one of the two unit normal vectors at $e$, and we define the jumps and averages of the normal derivatives (for $x \in e$) of $v \in H^1_0(\mathcal{T}_k)$ by

$$[v]_e(x) := \lim_{\delta \to 0^+} [v(x - \delta n_e) - v(x + \delta n_e)],$$

and

$$\langle \partial_n v \rangle_e(x) := \frac{1}{2} \lim_{\delta \to 0^+} [n_e \cdot \nabla v(x - \delta n_e) + n_e \cdot \nabla v(x + \delta n_e)].$$

If $e \subseteq \partial\Omega$, we fix $n_e$ to be the outward normal vector and let

$$[v]_e(x) := \lim_{\delta \to 0^+} v(x - \delta n_e) \quad \text{and} \quad \langle \partial_n v \rangle_e := \lim_{\delta \to 0^+} n_e \cdot \nabla v(x - \delta n_e).$$

Define $B_k(\cdot, \cdot)$ on $H^1_0(\mathcal{T}_k) \times H^1_0(\mathcal{T}_k)$ by

$$B_k(u, v) := \sum_{T \in \mathcal{T}_k} (\nabla u, \nabla v)_T + \sum_{e \in \mathcal{E}_k} \left( \frac{\sigma}{\ell_e} ([u], [v])_e - \langle \langle \partial_n u \rangle, [v] \rangle_e - \langle [u], \langle \partial_n v \rangle \rangle_e \right).$$

Here $\ell_e$ denotes the length of the edge $e$ and $\sigma$ is a positive parameter to be chosen later.

The weak form of (5.2.2.1) is defined by: Find $u \in H^1_0(\mathcal{T}_k)$, with $u(x, 0) = u_0(x) \in H^{1+\alpha}(\mathcal{T}_k) \cap H^1_0(\mathcal{T}_k)$, such that

$$(u_t, v) + B_k(u, v) = (f, v), \quad \forall v \in H^1_0(\mathcal{T}_k), \quad t \in [0, T].$$

As in Section 5.2.2, we use the backward Euler scheme for the time-stepping of (5.3.2.2). Then we derive the weak form with time-stepping: To find $w \in H^{1+\alpha}_0(\mathcal{T}_k)$ such that

$$A_{r,k}(w, v) := (g, v), \quad \forall v \in H^1_0(\mathcal{T}_k),$$

(5.3.2.3)
where
\[ A_{r,k}(w, v) := \tau^{-1}(w, v) + B_k(w, v), \]
and
\[ (g, v) := (f, v) - B_k(u^{n-1}, v). \]

The interior penalty discontinuous Galerkin approximation to (5.3.2.3):

Find \( w_k \in V_k \), such that
\[ A_{r,k}(w_k, v) = (g, v), \quad \forall v \in V_k. \]  \hspace{1cm} (5.3.2.4)

Define
\[ (A_k w_k, v) := A_k(w_k, v) := A_{r,k}(w_k, v), \quad \forall w_k, \ v \in V_k. \] \hspace{1cm} (5.3.2.5)

Then (5.3.2.4) can be expressed by
\[ A_k w_k = g_k, \] \hspace{1cm} (5.3.2.6)

where
\[ g_k \in V_k, \ (g_k, v) := (g, v), \quad \forall v \in V_k. \]

From \([3]\), we can easily obtain that
\[ |||w - w_k|||_{k, \tau} \leq Ch_k^\alpha (1 + \tau^{-1} h_k)^{1/2} ||g||_{H^{-1+\alpha}}. \] \hspace{1cm} (5.3.2.7)

From \([52]\), we know that
\[ c(\sigma + 1) h_k^{-2} + \tau^{-1} \leq \lambda_k \leq C(\sigma + 1) h_k^{-2} + \tau^{-1}. \]

Hence we have
\[ c \left( (\sigma + 1) \frac{\lambda_j}{2^2(J-k)} + \frac{2^2(J-k) - 1}{2^2(J-k)} \tau^{-1} \right) \leq \lambda_k \leq C \left( (\sigma + 1) \frac{\lambda_j}{2^2(J-k)} + \frac{2^2(J-k) - 1}{2^2(J-k)} \tau^{-1} \right). \] \hspace{1cm} (5.3.2.8)
In [52] it was also proved that Assumption 5.3.1.1 and Assumption 5.3.1.2 hold true. Assumption 5.3.1.3 and Lemma 5.3.1.1 are guaranteed, since they only concern the smoother operator $R_k$. We assume $\tau$ satisfies (5.2.2.9). So by recalling (5.3.1.9), we derive that

$$A((I - B_j A_j)w, w) \leq C \left( \sum_{k=1}^{J} \frac{\lambda_k^{-\alpha}}{m_k^{\alpha/2}} \right) A(w, w)$$

$$\leq C \left( \frac{1}{1 + \sigma + \gamma_0} \right)^{\alpha} \frac{h_j^{\alpha}}{m_j^{\alpha/2}} \sum_{k=1}^{J} \left( \frac{2^{\alpha}}{\beta^{\alpha/2}} \right)^{k} A(w, w).$$

The following theorem is therefore established.

**Theorem 5.3.2.1.** Under Assumption 5.2.1.4, we have the convergence estimate of CMG Algorithm II for (5.3.2.6):

$$A((I - B_j A_j)w, w) \leq \left\{ \begin{array}{ll}
C \cdot \frac{1}{1 - \left( \frac{2}{\beta^{1/2}} \right)^{\alpha}} \cdot \frac{h_j^{\alpha}}{m_j^{\alpha/2}} A(w, w), & \text{for } \beta > 4,
C \cdot \frac{h_j^{\alpha}}{m_j^{\alpha/2}} A(w, w), & \text{for } \beta = 4,
\end{array} \right.$$  

for all $w \in V_J$.  

The computational cost estimate is that of Theorem 5.2.1.2 with $a = 2^d$.  

**Remark 5.3.2.1.** The foregoing discussion reveals that the abstract setting of the cascading multigrid method provides a more feasible way to establish the convergence theorem of the method for the time-dependent problems with mild regularity, and for problems that are discretized by other new FEM methods. We believe it is also possible to extend this abstract framework to the mesh-free method described in, e.g., [14, 85].
5.3.3 Extension to VIDEs

In this section, we shall extend the CMG Algorithm II to (5.2.4.1) with the interior penalty discontinuous Galerkin approximation. We start from the interior penalty discontinuous Galerkin weak form of (5.2.4.1).

Find \( u \in H^1_0(T_k) \) with \( u(x, 0) = u_0(x) \in H^{1+\alpha}(T_k) \cap H^1_0(T_k) \), such that

\[
(u_t, v) + B_k(u, v) + \int_0^t B_k(u(x, s), v) ds = (f, v), \quad \forall v \in H^1_0(T_k), \quad t \in I, \tag{5.3.3.1}
\]

where \( B_k \) was defined by (5.3.2.1), i.e.,

\[
B_k(u, v) := \sum_{T \in \mathcal{T}_h} (\nabla u, \nabla v)_T + \sum_{e \in \mathcal{E}_h} \left( \frac{\sigma}{\ell_e} ([u], [v])_e - ([\partial_n u], [v])_e - ([u], [\partial_n v])_e \right).
\]

We use the backward Euler scheme to approximate (5.3.3.1), and the trapezoidal rule to discretize the memory term. Then we get the form: Find \( w \in H^1_0(T_k) \), such that

\[
A_{\tau,k}(w, v) = (g, v), \quad \forall v \in H^1_0(T_k), \tag{5.3.3.2}
\]

where

\[
A_{\tau,k}(w, v) := \tau^{-1}(w, v) + (1 + \frac{1}{2} \tau k_0) B_k(w, v),
\]

and

\[
(g, v) := (f, v) - (1 + \frac{1}{2} \tau k_0) B_k(u^{n-1}, v) - \frac{1}{2} \tau k_n B_k(u_0, v) - \sum_{j=1}^{n-1} \tau k_j B_k(u^j, v).
\]

The interior penalty discontinuous Galerkin approximation to (5.3.3.2) is formulated as: Find \( w_k \in V_k \) such that

\[
A_{\tau,k}(w_k, v) = (g, v), \quad \forall v \in V_k. \tag{5.3.3.3}
\]
Its error estimate:

\[ ||w - w_k||_{k, \tau} \leq Ch_k^\alpha (1 + \tau^{-1} h_k)^{1/2} ||g||_{H^{-1+\alpha}}. \]  \tag{5.3.3.4}

can be easily proved by using the techniques in [3].

Define

\[ (A_k w_k, v) := A_k(w_k, v) := A_{\tau,k}(w_k, v), \; \forall w_k, \; v \in V_k. \]  \tag{5.3.3.5}

Then (5.3.3.3) can be expressed by

\[ A_k w_k = g_k, \]  \tag{5.3.3.6}

where

\[ g_k \in V_k. \; (g_k, v) := (g, v), \; \forall v \in V_k. \]

From [52], we know that

\[ c(1 + \frac{1}{2} \tau k_0)(1 + \sigma) h_k^{-2} + \tau^{-1} \leq \lambda_k \leq C(1 + \frac{1}{2} \tau k_0)(1 + \sigma) h_k^{-2} + \tau^{-1}. \]

Therefore, we derive

\[ c \left( (1 + \frac{1}{2} \tau k_0)(1 + \sigma) \frac{\lambda_j}{2^2(J-k)} + \frac{2^2(J-k) - 1}{2^2(J-k)} \tau^{-1} \right) \leq \lambda_k \leq C \left( (1 + \frac{1}{2} \tau k_0)(1 + \sigma) \frac{\lambda_j}{2^2(J-k)} + \frac{2^2(J-k) - 1}{2^2(J-k)} \tau^{-1} \right). \]  \tag{5.3.3.7}

In [52] it was also verified that Assumption 5.3.1.2 holds true. Assumption 5.3.1.3 and Lemma 5.3.1.1 are guaranteed, since they only concern the smoother operator \( R_k \). We assume \( \tau \) satisfies (5.2.2.9). So by recalling (5.3.1.9), we obtain

\[ A((I - B_j A_j) w, w) \leq C \left( \sum_{k=1}^{J} \frac{\lambda_k^{-\alpha}}{m_k^{\alpha/2}} \right) A(w, w) \]

\[ \leq C \left( \frac{1}{1 + \sigma + 2\sqrt{2} (\sigma + 1)} \right)^{\alpha} \frac{h_j^{\alpha/2}}{m_j^{\alpha/2}} \sum_{k=1}^{J} \left( \frac{2^{\alpha/2}}{\beta^{\alpha/2}} \right)^{k} A(w, w). \]

The following theorem is henceforth established.
Theorem 5.3.3.1. Under Assumption 5.2.1.4, we have the convergence estimate of the CMG Algorithm II for (5.3.3.6)

\[
A((I - B_J A_J)w, w) \
\leq \begin{cases} 
C \frac{1}{1 - \left(\frac{2}{\beta^{1/2}}\right)^{\alpha}} \frac{h_j^\alpha}{m_j^{\alpha/2}} A(w, w), & \text{for } \beta > 4, \\
CJ \frac{h_j^\alpha}{m_j^{\alpha/2}} A(w, w), & \text{for } \beta = 4,
\end{cases}
\]

for all \( w \in V_J \).

5.4 History of cascading multigrid method

At present there exist three types of multigrid algorithms: the V-cycle algorithm, the W-cycle algorithm and the cascading multigrid algorithm. You may consult Hackbusch [55] and Bramble [19] for background material, and Brandt [21] and Trottenberg et al. [115] for references. The cascading algorithm is the new member of the family of multigrid methods. As a distinctive feature, the algorithm does not need nested correction at all and performs more iterations on coarser levels so as to obtain fewer iterations on finer levels. The first publication of this algorithm (Deuflhard [42] in 1994) contained rather convincing numerical results, but no theoretical justification. In 1996, Bornemann and Deuflhard [15] provided a theoretical analysis of this algorithm. In 1998 and 1999, Shi and Xu [104, 105] generalized the idea and applied it to nonconforming finite element method. Later many papers such as ([51], [106], [17], [103], [102], [107], [108], [112], [110], [126], [18], [87]) contributed to this area. In this thesis and [86], we provided an abstract cascacing multigrid, which is more general than [104, 105] and is applicable for the problems with mild regularity in Besov spaces and for the interior penalty discontinuous Galerkin method.
Chapter 6

Future works

In this chapter, we shall mention some future research topics growing out of this thesis.

6.1 Adaptive discontinuous Galerkin time-stepping for (partial) VIDEs with blow-up solutions

6.1.1 VIDEs with blow-up solutions

Consider the semilinear Volterra integro-differential equation:

\[
\begin{aligned}
    y'(t) + a(t)y(t) &= \mathcal{V}_c(y)(t), \quad \text{for } t > 0, \\
    y(0) &= y_0,
\end{aligned}
\]  

(6.1.1.1)

where \( \mathcal{V}_c(y)(t) := \int_0^t k(t-s)G(y(s))ds \). The solution of (6.1.1.1) will blow up in finite time under suitable assumptions on the functions \( a, k, \) and \( G \). For example, blow-up will occur in finite time under the following assumptions:

1. \( G(y) := y^p, \) for \( p > 1 \).
2. For \( \tau \geq 0, k \) satisfies

\[
k \in C^1, \quad k(\tau) \geq \text{const.} > 0, \quad \text{and } k'(\tau) \leq 0.
\]

3. \( y_0 \geq 0 \) and \( 0 < -a(t) < \text{const.} \) (for \( t > 0 \)).

(Compare Roberts and Olmstead [94] for the case of Volterra integral equations.)
and see Bellout [13] for PVIDEs in Section 6.1.2).

As shown in Stuart and Floater [111], the time-stepping method (one-point collocation method) with fixed temporal mesh is totally inadequate for dealing with ODEs with blow-up solutions. The same is true for the discontinuous Galerkin method on a fixed mesh, and one will resort to techniques for the computation of blow-up problems. There exist various methods for generation of adaptive meshes (e.g., defining the adaptive mesh according to the asymptotic profile of the solution near the blow-up time (if it is known)). Time-stepping based on a posteriori error estimates of DG is definitely one of the efficient approaches for mesh adaptivity.

Another approach for the blow-up problem (6.1.1.1) is the $p$- or $hp$-version of the adaptive DG time-stepping method (Brunner and Schötzau [27]). It will likely be very effective in dealing with blow-up equations, especially when the asymptotic behavior of the solution near the blow-up time is known.

### 6.1.2 PVIDEs with blow-up solutions

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary $\partial \Omega$ and

$$Q_t := \Omega \times (0, t), \quad \Gamma_t := \partial \Omega \times (0, t).$$

We consider the PVIDE:

$$u_t = \Delta u + \int_0^t k(t - s)G(u(x, s))ds, \quad \text{in } Q_t,$$

with initial condition

$$u(x, 0) = u_0(x) \geq 0, \quad \text{in } \Omega,$$

and boundary condition

$$u(x, t) = 0, \quad \text{on } \Gamma_t.$$
Bellout [13] proved that $u$ blows up in finite time, i.e.,

$$\exists T_b < \infty \text{ such that } \lim_{t \to T_b} \max_{x \in \Omega} u(x, t) = \infty,$$

if

$$G \in C^1, \ G(0) > 0; \ G'(\tau) > 0, \ G''(\tau) \geq 0 \ (\forall \tau \geq 0), \quad (6.1.2.2)$$

and

$$k \in C^1; \ k(\tau) \geq \text{const.} > 0, \ k'(\tau) \leq 0 \ (\forall \tau \geq 0). \quad (6.1.2.3)$$

Problem (6.1.2.1) can be viewed as the generalization of problem (6.1.1.1): because it now also involves a spatial variable, its numerical analysis and computation become much more complicated. Future efficient methods for the computation of (6.1.2.1) are based on moving meshes in which the spatial mesh is generated by appropriately chosen moving mesh PDEs (see, e.g., [67, 68, 69, 70], also Bandle and Brunner [7]).

Readers are referred to the book by Samarskii, Galaktionov, Kurdyumov, and Mikhailov [95], an the survey papers by Bandle and Brunner [8] and Souplet [109] for numerous references on theoretical and numerical blow-up.

### 6.2 The artificial boundary method for PVIDEs on unbounded spatial domains

#### 6.2.1 The artificial boundary method

Consider the following initial-boundary-value problem for the one-dimensional diffusion equation with memory term:

$$u_t + \int_0^t k(x, t - \tau)u(x, \tau)d\tau = \Delta u + f(x, t), \quad x \in \mathbb{R}^1, \ t \in I, \quad (6.2.1.1)$$

$$u|_{t=0} = u_0(x), \quad x \in \mathbb{R}^1, \quad (6.2.1.2)$$

$$u \to 0, \text{ as } |x| \to \infty. \quad (6.2.1.3)$$
where $I := [0, T]$, $\Delta u := \partial^2 u / \partial x^2$. Suppose that

(i) the functions $f$ and $u_0$ are continuous and have compact support:

$$\text{supp } \{f(x)\} \subset [0, 1], \quad \text{supp } \{u_0(x)\} \subset [0, 1];$$

(ii) the kernel $k$ satisfies $k(t, x) = k_0(t)$ when $x \notin (0, 1)$, with $k_0$ continuous or weakly singular.

In order to solve this problem numerically we introduce two artificial boundaries, as follows:

$$\Gamma_1 := \{x = 1 : 0 \leq t \leq T\}, \quad (6.2.1.4)$$
$$\Gamma_0 := \{x = 0 : 0 \leq t \leq T\}. \quad (6.2.1.5)$$

These artificial boundaries divide the given spatial-temporal domain into three sub-domains:

$$Q_1 := \{(x, t) : 1 < x < +\infty, \ 0 \leq t \leq T\},$$
$$Q_0 := \{(x, t) : -\infty < x < 0, \ 0 \leq t \leq T\},$$
$$Q_i := \{(x, t) : 0 \leq x \leq 1, \ 0 \leq t \leq T\}.$$

Consider first the restriction of the given initial-boundary-value problem (6.2.1.1)–(6.2.1.3) to the domain $Q_1$. Because of our assumptions (i) and (ii), $u = u(x, t)$ has to satisfy

$$u_t + \int_0^t k_0(t - \tau)u(x, \tau)d\tau = \Delta u, \quad 1 < x < \infty, \ 0 < t \leq T, \quad (6.2.1.6)$$
$$u|_{t=0} = 0, \quad 1 \leq x < \infty, \quad (6.2.1.7)$$
$$u \to 0, \quad \text{as } x \to \infty. \quad (6.2.1.8)$$
Using Laplace transform techniques one can show that (see Han, Brunner and Ma [61]) the exact (artificial) boundary conditions on \( \Gamma_1 \) and on \( \Gamma_0 \) are respectively given by

\[
\frac{\partial u(1, t)}{\partial x} = -\frac{1}{\sqrt{\pi}} \int_0^t \frac{H(t - \tau) \partial u(1, \tau)}{\sqrt{t - \tau}} \, d\tau
\]

and by

\[
\frac{\partial u(0, t)}{\partial x} = \frac{1}{\sqrt{\pi}} \int_0^t \frac{H(t - \tau) \partial u(0, \tau)}{\sqrt{t - \tau}} \, d\tau,
\]

with \( t \in [0, T] \) and with appropriate kernel \( H \).

By the artificial boundary conditions (6.2.1.9) and (6.2.1.10) the original initial-boundary-value problem (6.2.1.1)–(6.2.1.3) can thus be reduced to one defined on the bounded spatial-temporal computational domain \( Q_i \):

\[
\frac{\partial u}{\partial t} + \int_0^t k(x, t - \tau)u(x, \tau)\, d\tau = \Delta u + f(x, t), \quad (x, t) \in Q_i, \tag{6.2.1.11}
\]

\[
u|_{t=0} = u_0(x), \quad 0 \leq x \leq 1, \tag{6.2.1.12}
\]

\[
\frac{\partial u}{\partial x} \bigg|_{x=1} = -\frac{1}{\sqrt{\pi}} \int_0^t \frac{H(t - \tau) \partial u(1, \tau)}{\sqrt{t - \tau}} \, d\tau, \tag{6.2.1.13}
\]

\[
\frac{\partial u}{\partial x} \bigg|_{x=0} = \frac{1}{\sqrt{\pi}} \int_0^t \frac{H(t - \tau) \partial u(0, \tau)}{\sqrt{t - \tau}} \, d\tau. \tag{6.2.1.14}
\]

On \( Q_i \) the problem (6.2.1.11)–(6.2.1.14) is equivalent to (6.2.1.1)–(6.2.1.3).

Han and Huang [62, 63] proposed an artificial boundary method for the heat equation on unbounded domains. The method focuses on introducing an appropriate computational domain with an artificial boundary and adding the nonlocal boundary condition. Han, Brunner, and Ma [61] used this method to solve linear PVIDEs on unbounded spatial domains in \( \mathbb{R}^1 \). Work is currently being done on the extension to unbounded domains in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \).
6.2.2 The finite element method and the DG time-stepping for the reduced problems

Since the reduced problem (6.2.1.11)-(6.2.1.14) includes the artificial boundary conditions (6.2.1.9) and (6.2.1.10), it is important to verify the coercivity and continuity of the bilinear form $a(u, v)$ in the error estimate of the finite element approximation to the reduced problem. The reader may consult the following fundamental references:

1. Han and Wu [64]: artificial boundary method for Laplace’s equation and linear elastic equations on unbounded domains and error estimates for its finite element approximations.

2. Han and Bao [60]: the finite element approximation of elliptic problems on unbounded domains is formulated on a bounded domain using a nonlocal approximate artificial boundary condition and error estimates are based on the mesh size, the terms used in the approximate artificial boundary condition, and the location of the artificial boundary.

3. Han and Zheng [65]: mixed finite element methods and high-order local artificial boundary conditions for exterior problems of elliptic equations.

The analysis of the finite element method for the reduced problems coming from artificial boundary methods for parabolic PDEs and, especially, PVIDEs, is still at an early stage.

We see from (6.2.1.11)-(6.2.1.14) that the artificial boundary conditions are expressed in time integral form. Hence, it will be necessary to employ suitable quadrature for the boundary conditions while applying the discontinuous Galerkin time-stepping method to the reduced problem. The error analysis of the DG time-stepping for the reduced problem will be based on error estimates of its finite element method.
in space, the time-stepping error estimates, and the quadrature errors.

6.2.3 Blow-up problems on unbounded spatial domains

An obvious extension of (6.2.1.1)-(6.2.1.3) is the initial-boundary-value problem for the nonlinear diffusion equation with memory term:

\[ u_t = \Delta u + \int_0^t k(x, t - \tau)u^p(x, \tau)d\tau, \quad x \in \mathbb{R}^n, \quad t > 0, \quad p > 1, \quad (6.2.3.1) \]

\[ u|_{t=0} = u_0(x), \quad x \in \mathbb{R}^n, \quad (6.2.3.2) \]

\[ u \to 0, \quad \text{as} \quad |x| \to \infty. \quad (6.2.3.3) \]

The blow-up property for (6.2.3.1)-(6.2.3.3) with unbounded spatial domains is not yet known, in contrast to that for parabolic PDEs,

\[ u_t = \Delta u + u^p, \]

compare Fujita [48] (in \( \mathbb{R}^n \)), Bandle and Levine [9] (for sectorial domains), Bandle and Brunner [8], and their references. For the purpose of computation of the problem (6.2.3.1)-(6.2.3.3), we shall introduce the artificial boundaries and the corresponding artificial boundary conditions. We have be very careful on the choice of the location of the artificial boundary, otherwise it is possible that some blow-up points are not included into the computational domains. Is it true that the blow-up property of the blow-up of the reduced problem with nonlinear artificial boundary conditions is the same as that of the original problem (6.2.3.1)-(6.2.3.3)? If we have not established these corresponding analysis, how can we numerically detect the blow-up and determine the suitable location of the artificial boundaries?
Bibliography


