STABILITY AND BIFURCATION IN SYSTEMS OF TRI-NEURONS WITH MULTIPLE TIME DELAYS

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STABILITY AND BIFURCATION IN SYSTEMS OF TRI-NEURONS WITH MULTIPLE TIME DELAYS

by

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Abstract

This Master thesis consists of four chapters, mainly considering the stability and bifurcation in the systems of delay differential equations representing the neural network models containing tri-neurons with time-delayed connections.

In Chapter 1, some background of neural networks and the motivation of this work are briefly addressed.

In Chapter 2, we mainly show the stability analysis. By constructing Liapunov functional, we obtain the global stability condition. Then we show the delay-independent and delay-dependent conditions for local stability respectively.

In Chapter 3, we discuss the bifurcations. By using the center manifold theory and normal form method, we propose the transcritical , pitchfork and Hopf bifurcation analysis.

In the last chapter, by using the global Hopf bifurcation result and high-dimensional Bendixson's criterion, we show that the local Hopf bifurcation can be extended globally after certain critical values of delay.

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Chapter 1

INTRODUCTION

A neural network, which is a system of neurons, could be a piece of hardware, a computer, an algorithm and so on [27]. In this thesis, we only consider the artificial neural network (ANN) which is designed to model the way in which the brain performs a particular task or function. Such a network is usually implemented by electronic components or simulated in software on a digital computer. Mathematically, ANN is usually described by a system of differential equations (continuous time) or difference equations (discrete time). For each single neuron, the simple structure results in a simple mathematical equation. However, when many simple neurons are connected to form a neural network, which results in a system of coupled differential equations, the whole network could have very rich dynamics and thus admit various applications [11, 16, 19, 22, 25, 30, 41, 42].

The first mathematical model of neural network was presented by McCulloch and Pitts in 1943 [36], in which the network is described by the system of difference equations

$$x_i(t+1) = s\left(\sum_{j=1}^{n} w_{ij}x_j(t) - \theta_i\right), \quad i = 1, 2, \ldots, n, \tag{1.1}$$

where x_i is the state variable associated with neuron i, w_{ij} represents the synaptic coupling strengths between neurons j and i, θ_i is a threshold and the transfer function $s(x)$ is the unit step function. McCulloch and Pitts showed that such a network can carry out any logical calculation and thus can be viewed as a kind of computer performing in parallel manner.

The theory and applications of neural networks have been greatly developed since Cohen and Grossberg's paper [10] and Hopfield's paper [28] were published in 1980s. In [10], the well-known Cohen-Grossberg neural network model was described by a system of ordinary differential equations

$$\dot{x}_i(t) = a_i(x_i(t)) \left(b_i(x_i(t)) - \sum_{j=1}^{n} t_{ij} s_j(x_j(t)) \right), \quad i = 1, 2, \ldots, n. \tag{1.2}$$

In [28], Hopfield proposed the network by the following system

$$C_i \dot{x}_i(t) = -\frac{x_i(t)}{R_i} + \sum_{j=1}^{n} w_{ij} s_j(x_j(t)) + J_i, \quad i = 1, 2, \ldots, n, \tag{1.3}$$

which was implemented by electric circuits to fulfill various tasks such as linear programming. Due to the promising potential for the tasks of classification, associative memory, parallel computations, and their ability to solve difficult optimization problems, (1.2) and (1.3) have attracted great attention from the scientific world. Various generalizations and modifications of (1.2) and (1.3) have been proposed and studied. For the Hopfield neural networks, see Bélair [2], Cao and Wu [7], Guan et al. [18], Hirsch [26], Lu [33], Matsuoka [35], and van den Driessche and Zou [43]. While, for the Cohen-Grossberg type neural networks, see Wang and Zou [45, 46, 47], and Ye et al. [56].

Since the finite speeds of the switching and transmission of signals in the network, time delays do exist in the neural network and thus should be incorporated. More details about introducing the time delay into the equations of neural network models can be found in Marcus and Westervelt [34], Myers [37] and Wu [52]. More details about how delay affects the dynamics can be found in [1, 3, 8, 9, 15, 21, 32, 38, 39, 44, 49, 54, 58].

Marcus and Westervelt [34] first introduced a single delay into (1.3) and considered the following system of delay differential equations

$$C_i\dot{x}_i(t) = -\frac{x_i(t)}{R_i} + \sum_{j=1}^{n} w_{ij}s_j(x_j(t-\tau)) + J_i, \ \ i = 1, 2, \ldots, n. \tag{1.4}$$

They discussed the model experimentally and numerically and presented that delay could destroy stability and cause sustained oscillations. Eq. (1.3) has also been studied by Wu [51], Wu and Zou [55]. Gopalsamy and He [17], van den Driessche and Zou [43] considered the generalized model with multiple delays

$$\dot{x}_i(t) = -b_i x_i(t) + \sum_{j=1}^{n} w_{ij}s_j(x_j(t-\tau_{ij})) + J_i, \quad i = 1, 2, \ldots, n.$$

For system (1.2), Ye et al. [56] introduced delays by considering the following system of delay differential equations

$$\dot{x}_i(t) = -a_i(x_i(t))\left(b_i(x_i(t)) - \sum_{k=0}^{K}\sum_{j=1}^{n} w_{ij}^{(k)} s_j(x_j(t-\tau_k))\right), \ \ i = 1, 2, \ldots, n.$$

In [1] Baldi and Atiya considered a cyclical ring of neurons with delay interaction. Later Campbell [4] generalized the model into a ring network where each element has two time delays and investigated both the stability of the equilibrium and the bifurcation when the stability is lost. In [40], Shayer and Campbell considered a network of a pair of neurons with time-delayed connections between the neurons and time delayed feedback from each neuron itself, and showed how time delays affect not only the stability of equilibrium but also the bifurcation when the stability is lost. In [53], Wu et al. studied the symmetric network of tri-neurons with one time delay. In [5], Campbell et al. proposed the symmetric network of tri-neurons with two different delays. In [6], Campbell et al. considered the cyclical ring of neurons with $n = 4$ when the delays in the communications between each pair of adjacent neurons are identical. In [57], Yuan and Campbell generalized a ring of identical elements with time delayed nearest neighbor coupling.

The global existence of the periodic solution to the mathematical models of population dynamics has attracted much attention due to its theoretical and practical significance. We know that periodic solutions can arise from the Hopf bifurcation in delay differential equations. However, these periodic solutions are generally local. Therefore, it is important to extent the non-constant periodic solutions from local Hopf bifurcation globally. In [13], Erbe et al. proposed the global Hopf bifurcation theorem with a purely topological argument. Later Krawcewicz [29] et al. first applied this global Hopf bifurcation theorem to a neural functional differential equation. Thereafter, a lot of researchers have investigated the global existence of periodic solutions for retarded functional differential equations, for instance: Li and Muldowney [31], Wei and Li [48], Wei and Yuan [50], and Wu [51], etc.

In this thesis, we consider a Hopfield-type network of tri-neurons coupled in any possible way with identified connection strength. An architecture of such network can be shown in Fig 1.1:

Mathematically, we have the following functional differential equations:

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + \alpha f(x_1(t-\tau_s)) + a_{12}\beta g(x_2(t-\tau_n)) + a_{13}\beta g(x_3(t-\tau_n)) \\ \dot{x}_2(t) &= -x_2(t) + a_{21}\beta g(x_1(t-\tau_n)) + \alpha f(x_2(t-\tau_s)) + a_{23}\beta g(x_3(t-\tau_n)), \quad (1.5) \\ \dot{x}_3(t) &= -x_3(t) + a_{31}\beta g(x_1(t-\tau_n)) + a_{32}\beta g(x_2(t-\tau_n)) + \alpha f(x_3(t-\tau_s)) \end{aligned}$$

where a_{ij}, $(i \neq j, i, j = 1, 2, 3)$ has the value 1 or 0, depending whether the neurons i and j are connected or not; α, $\beta \in \mathbb{R}$ denote the strength in self-connection and neighboring-

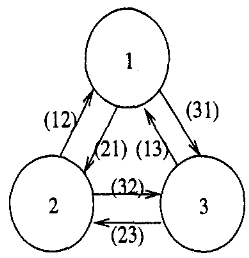

Figure 1.1: The architecture of a network of tri-neurons

connection respectively; τ_s, $\tau_n \geq 0$ denote the delay in self-connection and neighboring-connection respectively; Furthermore f, g are assumed adequately smooth, e.g. f, $g \in C^3$, and satisfy the following condition:

(C1) $f(0) = g(0) = 0$, $f'(0) = g'(0) = 1$, and $-\infty < \lim_{x \to \pm\infty} f(x),\ g(x) < \infty$.

(C2) $f'(x) > 0$, $g'(x) > 0$ for all $x \in \mathbb{R}$; $xf''(x) < 0$, $xg''(x) < 0$ for all $x \neq 0$.

The goal of this thesis is to investigate how time delays affect the dynamics of solutions by studying the stability and the bifurcation of the model, and the existence of periodic solutions. Accordingly, this thesis is organized as follows:

The next chapter is about the stability analysis. We show the global stability condition with Liapunov functional. Then we obtain the delay-independent and delay-dependent local stability condition. In Chapter 3, after using the center manifold theory and normal form method, we have the transcritical, pitchfork and Hopf bifurcation analysis. In Chapter 4, the local Hopf bifurcation implies the global Hopf bifurcation after certain critical values of delay.

Chapter 2

STABILITY ANALYSIS

2.1 Global Stability

We usually use the Liapunov second method to analyse the global stability. Following the method in [57], we have the following theorem:

Theorem 2.1.1 *If* $\max_{1\leq k\leq 3}\sum_{j=1,j\neq k}^{3}\frac{(a_{jk}+a_{kj})}{2}|\beta| < 1-|\alpha|$, *then the trivial solution in system (1.5) is globally asymptotically stable.*

Proof: For system (1.5), we construct a liapunov function

$$\begin{aligned} V(x)(t) &= \sum_{j=1}^{3} x_j^2(t) + |\alpha| \sum_{j=1}^{3} \int_{t-\tau_s}^{t} f^2(x_j(v))\,\mathrm{d}v \\ &\quad + |\beta| \int_{t-\tau_n}^{t} [(a_{21}+a_{31})g^2(x_1(v)) + (a_{12}+a_{32})g^2(x_2(v)) + (a_{13}+a_{23})g^2(x_3(v))]\,\mathrm{d}v. \end{aligned}$$

$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t}|_{(1.5)} = & \ 2\sum_{j=1}^{3}[x_j(t)\dot{x}_j(t)] + |\alpha|\sum_{j=1}^{3}[f^2(x_j(t)) - f^2(x_j(t-\tau_s))] \\
& +|\beta|[(a_{21}+a_{31})(g^2(x_1(t)) - g^2(x_1(t-\tau_n))) \\
& +(a_{12}+a_{32})(g^2(x_2(t)) - g^2(x_2(t-\tau_n))) + (a_{13}+a_{23})(g^2(x_3(t)) - g^2(x_3(t-\tau_n)))] \\
= & \ 2x_1(t)[-x_1(t) + \alpha f(x_1(t-\tau_s)) + a_{12}\beta g(x_2(t-\tau_n)) + a_{13}\beta g(x_3(t-\tau_n))] \\
& +2x_2(t)[-x_2(t) + a_{21}\beta g(x_1(t-\tau_n)) + \alpha f(x_2(t-\tau_s)) + a_{23}\beta g(x_3(t-\tau_n))] \\
& +2x_3(t)[-x_3(t) + a_{31}\beta g(x_1(t-\tau_n)) + a_{32}\beta g(x_2(t-\tau_n)) + \alpha f(x_3(t-\tau_s))] \\
& +|\alpha|\sum_{j=1}^{3}[f^2(x_j(t)) - f^2(x_j(t-\tau_s))] + |\beta|[(a_{21}+a_{31})(g^2(x_1(t)) - g^2(x_1(t-\tau_n))) \\
& +(a_{12}+a_{32})(g^2(x_2(t)) - g^2(x_2(t-\tau_n))) + (a_{13}+a_{23})(g^2(x_3(t)) - g^2(x_3(t-\tau_n)))] \\
\le & \ -2\sum_{j=1}^{3}x_j^2(t) + |\alpha|\sum_{j=1}^{3}[x_j^2(t) + f^2(x_j(t-\tau_s))] + |\beta|[a_{12}(x_1^2(t) + g^2(x_2(t-\tau_n))) \\
& +a_{13}(x_1^2(t) + g^2(x_3(t-\tau_n))) + a_{21}(x_2^2(t) + g^2(x_1(t-\tau_n))) + a_{23}(x_2^2(t) + g^2(x_3(t-\tau_n))) \\
& +a_{31}(x_3^2(t) + g^2(x_1(t-\tau_n))) + a_{32}(x_3^2(t) + g^2(x_2(t-\tau_n)))] \\
& +|\alpha|\sum_{j=1}^{3}[f^2(x_j(t)) - f^2(x_j(t-\tau_s))] + |\beta|[(a_{21}+a_{31})(g^2(x_1(t)) - g^2(x_1(t-\tau_n))) \\
& +(a_{12}+a_{32})(g^2(x_2(t)) - g^2(x_2(t-\tau_n))) + (a_{13}+a_{23})(g^2(x_3(t)) - g^2(x_3(t-\tau_n)))] \\
\le & \ -2\sum_{j=1}^{3}x_j^2(t) + |\beta|(a_{12}x_1^2(t) + a_{13}x_1^2(t) + a_{21}x_2^2(t) + a_{23}x_2^2(t) + a_{31}x_3^2(t) + a_{32}x_3^2(t)) \\
& +|\alpha|\sum_{j=1}^{3}x_j^2(t) + |\alpha|\sum_{j=1}^{3}f^2(x_j(t)) + |\beta|(a_{21}+a_{31})g^2(x_1(t)) \\
& +|\beta|(a_{12}+a_{32})g^2(x_2(t)) + |\beta|(a_{13}+a_{23})g^2(x_3(t))
\end{aligned}
$$

Rewrite

$$f(x_j(t)) = p_j(t)x_j(t), \quad g(x_j(t)) = q_j(t)x_j(t),$$

where

$$p_j(t) = \int_0^1 f'(vx_j(t))\mathrm{d}v, \quad q_j(t) = \int_0^1 g'(vx_j(t))\mathrm{d}v.$$

From the conditions (C1) and (C2), there exist $p^*, q^* \in (0,1]$ such that $p_j(t) \le p^*$, $q_j(t) \le q^*$, $(j = 1, 2, \dots, n)$. Thus we have

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &\le -2\sum_{j=1}^{3} x_j^2(t) + |\beta|(a_{12}x_1^2(t) + a_{13}x_1^2(t) + a_{21}x_2^2(t) + a_{23}x_2^2(t) + a_{31}x_3^2(t) + a_{32}x_3^2(t)) \\
&\quad +|\alpha|\sum_{j=1}^{3} x_j^2(t) + |\alpha|p^*\sum_{j=1}^{3} x_j^2(t) + |\beta|q^*(a_{21} + a_{31})x_1^2(t) \\
&\quad +|\beta|q^*(a_{12} + a_{32})x_2^2(t) + |\beta|q^*(a_{13} + a_{23})x_3^2(t) \\
&\le -(2-2|\alpha|-(a_{12}+a_{13}+a_{21}+a_{31})|\beta|)x_1^2(t) - (2-2|\alpha|-(a_{12}+a_{21}+a_{23}+a_{32})|\beta|)x_2^2(t) \\
&\quad -(2 - 2|\alpha| - (a_{13} + a_{23} + a_{31} + a_{32})|\beta|)x_3^2(t)
\end{aligned}$$

If the given condition $\max_{1\le k\le 3} \sum_{j=1, j\ne k}^{3} \frac{(a_{jk}+a_{kj})}{2}|\beta| < 1 - |\alpha|$ holds, then we have $\frac{\mathrm{d}V}{\mathrm{d}t}|_{(1.5)} < 0$.

According to the liapunov second method, the trivial solution is globally asymptotically stable. □

2.2 Absolutely Stable Condition for Local Stability

For a delay differential equation, the linearization of the system at its equilibrium point gives us an exponential polynomial equation (a transcendental characteristic equation). We know that the equilibrium point is stable if and only if all the eigenvalues of the exponential polynomial equation have negative real parts, and unstable if and only if at least one root has a positive real part. Thus, the bifurcation may take place when the real part of a certain eigenvalue changes from negative to zero or to positive.

There is a possibility that if the coefficients of the exponential polynomial satisfy certain conditions, the real parts of all eigenvalues remain negative for all values of the delay, then the corresponding delay differential system is said to be absolutely stable. A general result in Hale et al. [21] states that a delay system is absolutely stable if and only if the corresponding ODE system is asymptotically stable and the characteristic equation has no purely imaginary roots.

At the trivial equilibrium, the linearization of system (1.5) is

$$\begin{aligned}
\dot{u}_1(t) &= -u_1(t) + \alpha u_1(t-\tau_s) + a_{12}\beta u_2(t-\tau_n) + a_{13}\beta u_3(t-\tau_n) \\
\dot{u}_2(t) &= -u_2(t) + a_{21}\beta u_1(t-\tau_n) + \alpha u_2(t-\tau_s) + a_{23}\beta u_3(t-\tau_n), \\
\dot{u}_3(t) &= -u_3(t) + a_{31}\beta u_1(t-\tau_n) + a_{32}\beta u_2(t-\tau_n) + \alpha u_3(t-\tau_s)
\end{aligned} \tag{2.1}$$

and the corresponding characteristic equation is

$$\det \begin{bmatrix} \lambda+1-\alpha e^{-\lambda\tau_s} & -a_{12}\beta e^{-\lambda\tau_n} & -a_{13}\beta e^{-\lambda\tau_n} \\ -a_{21}\beta e^{-\lambda\tau_n} & \lambda+1-\alpha e^{-\lambda\tau_s} & -a_{23}\beta e^{-\lambda\tau_n} \\ -a_{31}\beta e^{-\lambda\tau_n} & -a_{32}\beta e^{-\lambda\tau_n} & \lambda+1-\alpha e^{-\lambda\tau_s} \end{bmatrix} = 0,$$

i.e.

$$\begin{aligned}
P(\lambda) \triangleq\ & (\lambda+1-\alpha e^{-\lambda\tau_s})^3 - (a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32})\beta^3 e^{-3\lambda\tau_n} \\
& -(a_{23}a_{32} + a_{13}a_{31} + a_{12}a_{21})(\lambda+1-\alpha e^{-\lambda\tau_s})\beta^2 e^{-2\lambda\tau_n} = 0.
\end{aligned}$$

To see the different connections among the three neurons more clearly, we separate Fig. 1.1 into several cases which are shown in Fig. 2.1 by the 'form' of characteristic equation $P_i(\lambda) = 0$:

Case 1: $a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 1$, which is studied in [5].

$$P_1(\lambda) \triangleq [\lambda+1-\alpha e^{-\lambda\tau_s} - 2\beta e^{-\lambda\tau_n}][\lambda+1-\alpha e^{-\lambda\tau_s} + \beta e^{-\lambda\tau_n}]^2 = 0.$$

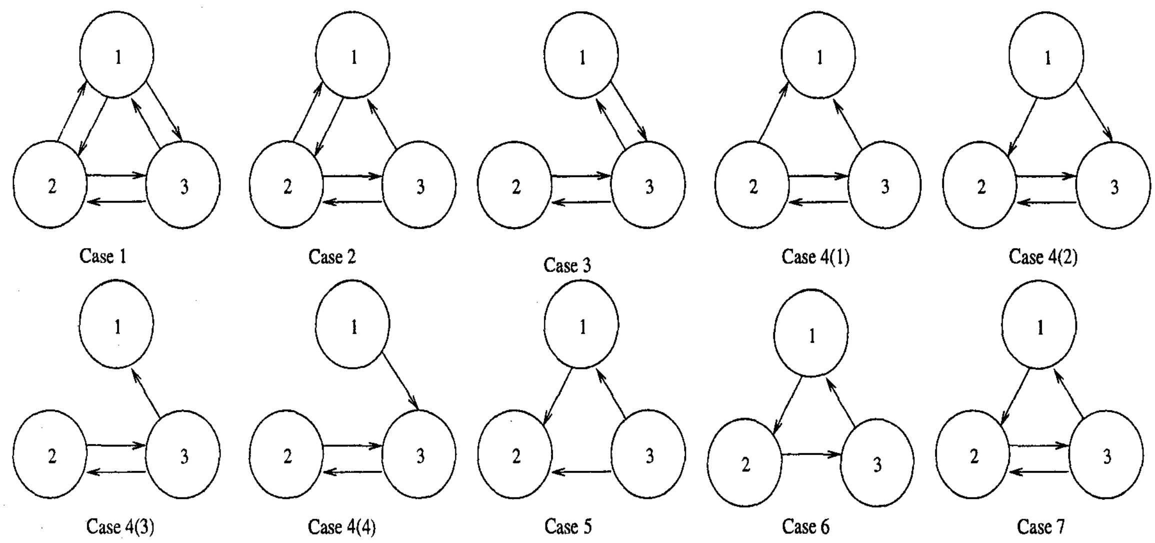

Figure 2.1: Tri-neurons with different connection about all cases

Case 2: $a_{12} = a_{13} = a_{21} = a_{23} = a_{32} = 1, a_{31} = 0$.

$$P_2(\lambda) \triangleq [\lambda+1-\alpha e^{-\lambda\tau_s}+\beta e^{-\lambda\tau_n}][\lambda+1-\alpha e^{-\lambda\tau_s}-\frac{1+\sqrt{5}}{2}\beta e^{-\lambda\tau_n}][\lambda+1-\alpha e^{-\lambda\tau_s}-\frac{1-\sqrt{5}}{2}\beta e^{-\lambda\tau_n}]=0.$$

Case 3: $a_{13} = a_{23} = a_{31} = a_{32} = 1, a_{12} = a_{21} = 0$, which is a chain.

$$P_3(\lambda) \triangleq [\lambda+1-\alpha e^{-\lambda\tau_s}][\lambda+1-\alpha e^{-\lambda\tau_s}+\sqrt{2}\beta e^{-\lambda\tau_n}][\lambda+1-\alpha e^{-\lambda\tau_s}-\sqrt{2}\beta e^{-\lambda\tau_n}] = 0. \quad (2.2)$$

Case 4 (1): $a_{12} = a_{13} = a_{23} = a_{32} = 1, a_{21} = a_{31} = 0$.

Case 4 (2): $a_{21} = a_{23} = a_{31} = a_{32} = 1, a_{12} = a_{13} = 0$.

Case 4 (3): $a_{13} = a_{23} = a_{32} = 1, a_{12} = a_{21} = a_{31} = 0$.

Case 4 (4): $a_{23} = a_{31} = a_{32} = 1, a_{12} = a_{13} = a_{21} = 0$.

$$P_4(\lambda) \triangleq [\lambda + 1 - \alpha e^{-\lambda\tau_s}][\lambda + 1 - \alpha e^{-\lambda\tau_s} - \beta e^{-\lambda\tau_n}][\lambda + 1 - \alpha e^{-\lambda\tau_s} + \beta e^{-\lambda\tau_n}] = 0.$$

Case 5: $a_{13} = a_{21} = a_{23} = 1, a_{12} = a_{31} = a_{32} = 0$.

$$P_5(\lambda) \triangleq [\lambda + 1 - \alpha e^{-\lambda\tau_s}]^3 = 0.$$

Case 6: $a_{13} = a_{21} = a_{32} = 1, a_{12} = a_{23} = a_{31} = 0$, which is studied in [4].

$$\begin{aligned} P_6(\lambda) \triangleq \quad & [\lambda + 1 - \alpha e^{-\lambda\tau_s} - \beta e^{-\lambda\tau_n}][\lambda^2 + (2 - 2\alpha e^{-\lambda\tau_s} + \beta e^{-\lambda\tau_n})\lambda \\ & + \alpha^2 e^{-2\lambda\tau_s} + \beta^2 e^{-2\lambda\tau_n} - \alpha\beta e^{-2\lambda(\tau_s+\tau_n)} - 2\alpha e^{-\lambda\tau_s} - \beta e^{-\lambda\tau_n} + 1] = 0. \end{aligned}$$

Case 7: $a_{13} = a_{21} = a_{23} = a_{32} = 1, a_{12} = a_{31} = 0$.

$$\begin{aligned} P_7(\lambda) \triangleq & \lambda^3 - (3\alpha e^{-\lambda\tau_s} - 3)\lambda^2 - (6\alpha e^{-\lambda\tau_s} - 3\alpha^2 e^{-2\lambda\tau_s} + \beta^2 e^{-2\lambda\tau_n} - 3)\lambda \\ & - \alpha^3 e^{-3\lambda\tau_s} + 3\alpha^2 e^{-2\lambda\tau_s} - 3\alpha e^{-\lambda\tau_s} - \beta^3 e^{-3\lambda\tau_n} - \beta^2 e^{-2\lambda\tau_n} + \alpha\beta^2 e^{-\lambda(\tau_s+2\tau_n)} + 1 = 0. \end{aligned}$$

2.2.1 Delay $\tau_s \neq \tau_n$

First of all, let us consider the characteristic equation

$$\Delta_1(\lambda) = \lambda + 1 - \alpha e^{-\lambda\tau_s} - \beta e^{-\lambda\tau_n} = 0. \tag{2.3}$$

Using the method in [40], we have the following lemma:

Lemma 2.2.1 *If the parameters α and β satisfy $|\alpha| + |\beta| < 1$, all the roots in Eq (2.3) have negative real parts.*

Proof: Let $\lambda = \mu + i\omega$, μ, $\omega \in \mathbb{R}$, and separate Δ_1 into real and imaginary parts to yield $\Delta_1(\lambda) = R_1(\mu, \omega) + iI_1(\mu, \omega)$, where

$$\begin{aligned} R_1(\mu, \omega) &= \mu + 1 - \alpha e^{-\mu\tau_s}\cos(\omega\tau_s) - \beta e^{-\mu\tau_n}\cos(\omega\tau_n), \\ I_1(\mu, \omega) &= \omega + \alpha e^{-\mu\tau_s}\sin(\omega\tau_s) + \beta e^{-\mu\tau_n}\sin(\omega\tau_n). \end{aligned}$$

Since

$$R_1(\mu,\omega) \geq \mu + 1 - |\alpha|e^{-\mu\tau_s} - |\beta|e^{-\mu\tau_n} \triangleq R_{01}(\mu),$$

$$R_{01}(0) = 1 - |\alpha| - |\beta| > 0,$$

and

$$R_1'(\mu) = 1 + |\alpha|\tau_s e^{-\mu\tau_s} + |\beta|\tau_n e^{-\mu\tau_n} > 0,$$

we have $R_{01}(\mu) > 0$ for $\mu \geq 0$ and $R_1(\mu,\omega) > 0$ for all $\mu \geq 0$, $\omega \in \mathbb{R}$.

Now let $\lambda = \mu + i\omega$ be an arbitrary root of the characteristic equation $\Delta_1(\lambda) = 0$. Then μ and ω must satisfy $R_1(\mu,\omega) = 0$ and $I_1(\mu,\omega) = 0$. But from the discussion above this implies $\mu < 0$. Thus all the roots in Eq (2.3) have negative real parts. □

Apply the above lemma to each cases, then we have the following theorem:

Theorem 2.2.2 *If the parameters α, β satisfy the condition for certain case in the following table, the trivial equilibrium in the corresponding case is stable:*

Case	*Condition*
1	$\|\alpha\| + 2\|\beta\| < 1$
2	$\|\alpha\| + \frac{1+\sqrt{5}}{2}\|\beta\| < 1$
3	$\|\alpha\| + \sqrt{2}\,\|\beta\| < 1$
4	$\|\alpha\| + \|\beta\| < 1$
5	$\|\alpha\| < 1$

Proof: We take **Case 3** as an example:

Consider the characteristic equation $P_3 = 0$ for **Case 3**. From Lemma 2.2.1, we know that if the parameters α, β satisfy

$$\begin{cases} |\alpha| < 1, \\ |\alpha| + \sqrt{2}|\beta| < 1, \\ |\alpha| + \sqrt{2}|\beta| < 1, \end{cases}$$

i.e.

$$|\alpha| + \sqrt{2}|\beta| < 1,$$

then all the roots in $P_3 = 0$ have negative real part, which implies that the trivial equilibrium is stable.

Similarly, we can get the other conditions for the other cases.□

2.2.2 Delay $\tau_s = \tau_n = \tau$

For the characteristic equation

$$\Delta_2(\lambda) = \lambda + 1 - (\alpha + \beta i)e^{-\lambda\tau} = 0, \tag{2.4}$$

we have:

Lemma 2.2.3 *If the parameters α and β satisfy $\alpha^2 + \beta^2 < 1$, all the roots in Eq. (2.4) must have negative real parts.*

Proof: Let $\lambda = \mu + i\omega$, μ, $\omega \in \mathbb{R}$, and separate Δ_2 into real and imaginary parts to yield $\Delta_2(\lambda) = R_2(\mu, \omega) + iI_2(\mu, \omega)$, where

$$R_2(\mu, \omega) = \mu + 1 - \alpha e^{-\mu\tau}\cos(\omega\tau) - \beta e^{-\mu\tau}\sin(\omega\tau).$$

Since

$$\begin{aligned} R_2(\mu,\omega) &= \mu+1-\sqrt{\alpha^2+\beta^2}e^{-\mu\tau}\sin(\omega\tau-\varphi), \ \ (\varphi=\arctan\frac{\beta}{\alpha}) \\ &\geq \mu+1-\sqrt{\alpha^2+\beta^2}e^{-\mu\tau} \triangleq R_{02}(\mu), \end{aligned}$$

$$R_{02}(0)=1-\sqrt{\alpha^2+\beta^2}>0,$$

and

$$R'_{02}(\mu)=1+\tau\sqrt{\alpha^2+\beta^2}e^{-\mu\tau}>0,$$

we have $R_{02}(\mu)>0$ for $\mu\geq 0$ and $R_2(\mu,\omega)>0$ for all $\mu\geq 0$, $\omega\in\mathbb{R}$.

The rest of the proof follows from that in Lemma 2.2.1. □

By applying Lemma 2.2.1 and the lemma above to each case, we have the following theorem:

Theorem 2.2.4 *If the parameters α, β satisfy the condition for certain case in the following table, then the trivial equilibrium in the corresponding case is stable:*

Case	*Condition*
1	$\lvert\alpha-\beta\rvert<1$, $\lvert\alpha+2\beta\rvert<1$
2	$\lvert\alpha+\frac{1-\sqrt{5}}{2}\beta\rvert<1$, $\lvert\alpha+\frac{1+\sqrt{5}}{2}\beta\rvert<1$
3	$\lvert\alpha-\sqrt{2}\beta\rvert<1$, $\lvert\alpha+\sqrt{2}\beta\rvert<1$
4	$\lvert\alpha-\beta\rvert<1$, $\lvert\alpha+\beta\rvert<1$
5	$\lvert\alpha\rvert<1$
6	$\lvert\alpha+\beta\rvert<1$, $\alpha^2-\alpha\beta+\beta^2<1$
7	$\lvert\alpha+(\frac{\kappa}{6}+\frac{2}{\kappa})\beta\rvert<1$, $\left[\alpha-\left(\frac{\kappa}{12}+\frac{1}{\kappa}\right)\beta\right]^2+\left[\frac{\sqrt{3}}{2}\left(\frac{\kappa}{6}-\frac{2}{\kappa}\right)\beta\right]^2<1$, *where* $\kappa=(108+12\sqrt{69})^{1/3}$

Proof: For **Case 6**, the corresponding characteristic equation P_6 is

$$[\lambda+1-(\alpha+\beta)e^{-\lambda\tau}][\lambda^2+(2-(2\alpha-\beta)e^{-\lambda\tau})\lambda+(\alpha^2+\beta^2-\alpha\beta)e^{-2\lambda\tau}-(2\alpha-\beta)e^{-\lambda\tau}+1]=0,$$

i.e.

$$[\lambda+1-(\alpha+\beta)e^{-\lambda\tau}][\lambda+1-(\alpha-\frac{\beta}{2}+\frac{\sqrt{3}}{2}\beta i)e^{-\lambda\tau}][\lambda+1-(\alpha-\frac{\beta}{2}-\frac{\sqrt{3}}{2}\beta i)e^{-\lambda\tau}]=0.$$

From Lemma 2.2.3, we know the stability condition we need is:

$$\begin{cases} |\alpha+\beta|<1, \\ (\alpha-\frac{\beta}{2})^2+(\frac{\sqrt{3}}{2}\beta)^2<1, \\ (\alpha-\frac{\beta}{2})^2+(-\frac{\sqrt{3}}{2}\beta)^2<1, \end{cases}$$

i.e.

$$|\alpha+\beta|<1 \text{ and } \alpha^2-\alpha\beta+\beta^2<1.$$

So we get the result for **Case 6**.

For **Case 7**, the corresponding characteristic equation is

$$\begin{aligned} P_7 = \lambda^3 - (3\alpha e^{-\lambda\tau}-3)\lambda^2 - (6\alpha e^{-\lambda\tau} - 3\alpha^2 e^{-2\lambda\tau} + \beta^2 e^{-2\lambda\tau} - 3)\lambda \\ -\alpha^3 e^{-3\lambda\tau} + 3\alpha^2 e^{-2\lambda\tau} - 3\alpha e^{-\lambda\tau} - \beta^3 e^{-3\lambda\tau} - \beta^2 e^{-2\lambda\tau} + \alpha\beta^2 e^{-3\lambda\tau} + 1 = 0, \end{aligned}$$

which can be factored as

$$P_7 = P_{71}P_{72}P_{73} = 0,$$

where

$$\begin{aligned} P_{71} &= \lambda+1-(\alpha+(\frac{\kappa}{6}+\frac{2}{\kappa})\beta)e^{-\lambda\tau}, \\ P_{72} &= \lambda+1-(\alpha-(\frac{\kappa}{12}+\frac{1}{\kappa})\beta-\frac{\sqrt{3}}{2}(\frac{\kappa}{6}-\frac{2}{\kappa})\beta i)e^{-\lambda\tau}, \\ P_{73} &= \lambda+1-(\alpha-(\frac{\kappa}{12}+\frac{1}{\kappa})\beta+\frac{\sqrt{3}}{2}(\frac{\kappa}{6}-\frac{2}{\kappa})\beta i)e^{-\lambda\tau}, \\ \kappa &= (108+12\sqrt{69})^{1/3}. \end{aligned}$$

Apply Lemma 2.2.3, we can obtain the stability condition for $P_{71} = 0$, $P_{72} = 0$ and $P_{73} = 0$ respectively, as:

$$\begin{cases} |\alpha + (\frac{\kappa}{6} + \frac{2}{\kappa})\beta| < 1, \\ \left[\alpha - \left(\frac{\kappa}{12} + \frac{1}{\kappa}\right)\beta\right]^2 + \left[-\frac{\sqrt{3}}{2}\left(\frac{\kappa}{6} - \frac{2}{\kappa}\right)\beta\right]^2 < 1, \\ \left[\alpha - \left(\frac{\kappa}{12} + \frac{1}{\kappa}\right)\beta\right]^2 + \left[\frac{\sqrt{3}}{2}\left(\frac{\kappa}{6} - \frac{2}{\kappa}\right)\beta\right]^2 < 1, \end{cases}$$

Therefore, with the common conditions, all the roots of $P_3 = 0$ have negative real parts.

Similarly, we can get the results for the other cases. □

2.3 Stable Condition for Local Stability

2.3.1 Delay $\tau_s = \tau_n = \tau$

If there exist $\nu > 0$, $M > 0$, such that $|x_t(\phi)| \leq Me^{-\nu t}|\phi|$ for all $t \geq 0$, $\phi \in C$, we say that the trivial solution of the system is exponentially asymptotically stable.

Now let us consider the system

$$\dot{x}(t) = -x(t) + \alpha x(t - \tau), \tag{2.5}$$

we have the following lemma:

Lemma 2.3.1 *The trivial equilibrium in system (2.5) is exponentially asymptotically stable if and only if the parameter α satisfy $\alpha^* < \alpha < 1$, where α^* is the negative root of* $\frac{1}{\tau}\arccos\frac{1}{\alpha} - \sqrt{\alpha^2 - 1} = 0$.

Proof: The corresponding characteristic equation for (2.5) is

$$\Delta_3(\lambda) = \lambda + 1 - \alpha e^{-\lambda\tau}. \tag{2.6}$$

In order to apply the Theorem 2.19 given by Stepan in [42], assume $\lambda = i\omega$ is a root of (2.6), i.e.

$$\Delta_3(i\omega) = R(\omega) + iS(\omega) = i\omega + 1 - \alpha\cos\omega\tau + \alpha i\sin\omega\tau = 0,$$

where

$$R(\omega) = 1 - \alpha\cos\omega\tau, \quad S(\omega) = \omega + \alpha\sin\omega\tau.$$

Therefore,

$$R(0) = 1 - \alpha > 0 \tag{2.7}$$

If the number s of the zeros of S is odd, then $S'(0) > 0$, which means

$$\alpha > -\frac{1}{\tau}. \tag{2.8}$$

Then

$$S(\omega) = \omega + \alpha\sin(\omega\tau) > \omega - \frac{1}{\tau}\sin(\omega\tau) = \frac{\omega\tau - \sin(\omega\tau)}{\tau} > 0$$

for $\omega > 0$, implying that S has only one zero root, hence $s = 1$. The stability condition is satisfied trivially:

$$\sum_{k=1}^{s-1}(-1)^k \mathrm{sgn} R(\sigma_k) = 0.$$

If the number s of the non-negative zeros of S is even, since $S(0) = 0$, $S(+\infty) = +\infty$, we have

$$S'(0) = 1 + \alpha\tau < 0,$$

i.e.

$$\alpha < -\frac{1}{\tau}. \tag{2.9}$$

And the stability condition with $m = 0$ in the Theorem 2.19 in [42] has the actual form

$$\sum_{k=1}^{s-1}(-1)^k \mathrm{sgn} R(\sigma_k) = -1, \tag{2.10}$$

where $\sigma_{s-1} \geq \cdots \geq \sigma_0 = 0$ denote the non-negative real zeros of S.

Consider $P(\alpha) = \frac{1}{\tau} \arccos \frac{1}{\alpha} - \sqrt{\alpha^2 - 1}$ with $\alpha < -1$. Since

$$P'(\alpha) = \frac{-1}{\alpha\tau\sqrt{\alpha^2 - 1}} - \frac{\alpha}{\sqrt{\alpha^2 - 1}} > 0$$

and $P(-1) = \frac{\pi}{\tau} > 0$, $P(-\infty) \to -\infty$, $P(\alpha)$ has only one zero, that is only one α^*. From the condition

$$\alpha > \alpha^*, \tag{2.11}$$

we have

$$P(\alpha) = S(\zeta_1) > 0,$$

where ζ_1 is the smallest positive roots of $R(\omega) = 0$, i.e. $\zeta_1 = \frac{1}{\tau} \arccos \frac{1}{\alpha}$.

Now let us consider the value of $\mathrm{sgn} R(\sigma_k)$.

Firstly, we know $S(0) = 0$, $S'(0) < 0$, so there exist $\varepsilon > 0$ such that $S(\varepsilon) < 0$. From $S(\zeta_1) > 0$, we have $\sigma_1 < \zeta_1$ and

$$R(\sigma_1) > 0, \tag{2.12}$$

since $R(x) > 0$ for any $x \in (0, \zeta_1)$.

Secondly, let us consider σ_{2k} $(k \in \mathbb{Z}^+)$. We know $S'(\sigma_{2k}) < 0$, i.e. $\alpha \cos(\sigma_{2k}\tau) < -\frac{1}{\tau}$. So we can obtain

$$R(\sigma_{2k}) = 1 - \alpha \cos(\sigma_{2k}\tau) > 1 + \frac{1}{\tau} > 0. \tag{2.13}$$

Finally, let us consider σ_{2k-1} $(k \in \mathbb{Z}^+)$. We will prove $\sigma_2 - \sigma_1 < \frac{2\pi}{\tau}$. Suppose $\sigma_2 - \sigma_1 \geq \frac{2\pi}{\tau}$, then $S'(\omega) = 1 + \alpha\tau\cos(\omega\tau)$ will have more than one zero between σ_2 and σ_1, since the period of $S'(\omega)$ is $\frac{2\pi}{\tau}$. It is a contradiction. So we know that $\sigma_1 + \frac{2\pi}{\tau}$ is not in (σ_1, σ_2). Since $S(\sigma_1 + \frac{2\pi}{\tau}) = \left(\sigma_1 + \frac{2\pi}{\tau}\right) + \alpha \sin\left[(\sigma_1 + \frac{2\pi}{\tau})\tau\right] = \frac{2\pi}{\tau} > 0$ and $S(x) < 0$ for any $x \in (\sigma_2, \sigma_3)$, we can obtain $\sigma_3 - \sigma_1 < \frac{2\pi}{\tau}$. Then we have $\sigma_3 < \frac{2\pi}{\tau} + \sigma_1 < \frac{2\pi}{\tau} + \zeta_1 = \zeta_3$, since $\zeta_{2k+1} = \frac{2k\pi}{\tau} + \zeta_1$. By mathematical induction, we have

$$\sigma_{2k-1} < \zeta_{2k-1}. \tag{2.14}$$

We know $R'(\zeta_{2k}) = \alpha\tau\sin(\omega\tau) > 0$, where ζ_i is the positive zeros of $R(\omega)$ and $\zeta_1 < \zeta_2 < \ldots$, i.e. $\alpha\sin(\zeta_{2k}\tau) > 0$, so we have $S(\zeta_{2k}) = \zeta_{2k} + \alpha\sin(\zeta_{2k}\tau) > 0$. Since $R(\zeta_{2k-1}) = 0$, $R'(\zeta_{2k-1}) < 0$, there exists $\varepsilon > 0$ such that $R(\zeta_{2k-1} + \varepsilon) < 0$. From $R(\sigma_{2k}) > 0$, we have $\zeta_{2k} < \sigma_{2k}$. Since $\zeta_{2k} < \sigma_{2k} < \sigma_{2k+1}$ and inequality (2.14), we can obtain

$$\zeta_{2k} < \sigma_{2k+1} < \zeta_{2k+1}.$$

So we have

$$R(\sigma_{2k-1}) > 0. \tag{2.15}$$

From (2.12), (2.13) and (2.15), obviously they satisfy the condition (2.10).

The conditions (2.7) and (2.11) are just the conditions in the theorem, while (2.8) and (2.9) show that the result in the theorem is independent of s. □

Remark 2.3.1 *The Theorem 2.19 given by Stepan in [42] is as following:*

Consider the n-dimensional linear autonomous RFDE

$$\dot{x}(t) = \int_{-\infty}^{0} [\mathrm{d}\eta(\theta)]x(t+\theta)$$

and suppose that there exists a scalar $\nu > 0$ such that

$$\int_{-\infty}^{0} e^{-\nu\theta}|\mathrm{d}\eta_{jk}(\theta)| < +\infty, \quad j,k = 1,\ldots,n.$$

The characteristic function assumes the form

$$D(\lambda) = \det(\lambda I - \int_{-\infty}^{0} e^{\lambda\theta}\mathrm{d}\eta(\theta)).$$

Let $\rho_1 \geq \ldots \rho_r \geq 0$ and $\sigma_1 \geq \ldots \geq \sigma_s = 0$ denotes the non-negative real zeros of R and S respectively, where

$$R(\omega) = \Re D(i\omega), \quad S(\omega) = \Im D(i\omega).$$

The trivial solution $x = 0$ of the RFDE is exponentially asymptotically stable if and only if

$$n = 2m,$$
$$S(\rho_k) \neq 0, \quad k = 1, \ldots, r,$$
$$\sum_{k=1}^{r}(-1)^k \mathrm{sgn} S(\rho_k) = (-1)^m m;$$

or

$$n = 2m + 1,$$
$$R(\sigma_k) \neq 0, \quad k = 1, \ldots, s-1,$$
$$R(0) > 0,$$
$$\sum_{k=1}^{s-1}(-1)^k \mathrm{sgn} S(\sigma_k) + \frac{1}{2}((-1)^s + (-1)^m) + (-1)^m m = 0,$$

where m is integer.

Remark 2.3.2 *If the parameters satisfy exponentially asymptotically stable condition (C_j) $(j = 1, 2, 3)$ for the equations $E_j = 0$ respectively, the solution of the equation $E_1 * E_2 * E_3 = 0$ is exponentially asymptotically stable under the common condition of each (C_j).*

Proof : *There exist $\nu_i > 0$, $i = 1, 2, 3$, $M_i > 0$, such that $|x_{it}(\phi)| \leq M_i e^{-\nu_i t}|\phi|$ for the equations $E_i = 0$ for all $t \geq 0$, $\phi \in C$, since the trivial solution of the equations is exponentially asymptotically stable.*

Let $\nu = \min(\nu_i)$, $x_t = x_{it}$, there exists $M = \max\{M_i\} > 0$, such that $|x_t(\phi)| \leq Me^{-\nu t}|\phi|$ for the equation $E_1 * E_2 * E_3 = 0$ for all $t \geq 0$, $\phi \in C$. $\square$

Consequently, we have the following theorem:

Theorem 2.3.2 *The trivial equilibrium in the following case is exponentially asymptotically stable if and only if the parameters α, β satisfy the corresponding condition in the following table, where α^* is defined in Lemma 2.3.1:*

Case	*Condition*
1	$\alpha^* < \alpha - \beta < 1$, $\alpha^* < \alpha + 2\beta < 1$
2	$\alpha^* < \alpha + \frac{1-\sqrt{5}}{2}\beta < 1$, $\alpha^* < \alpha + \frac{1+\sqrt{5}}{2}\beta < 1$
3	$\alpha^* < \alpha - \sqrt{2}\beta < 1$, $\alpha^* < \alpha + \sqrt{2}\beta < 1$
4	$\alpha^* < \alpha - \beta < 1$, $\alpha^* < \alpha + \beta < 1$
5	$\alpha^* < \alpha < 1$

Now we have obtain the delay-independent stable condition of local stability for **Case 1** to **Case 5**, next we will consider the following equation in order to obtain the result for **Case 6** and **Case 7**.

Let us consider

$$\ddot{x}(t) + (a_1 + a_2 x(t-\tau))\dot{x}(t) + a_3 x(t-2\tau) + a_4 x(t-\tau) + a_5 = 0, \qquad (2.16)$$

we have the following lemma:

Lemma 2.3.3 *If the parameters a_1, a_2, a_3, a_4 and a_5 satisfy $a_3 + a_4 + a_5 > 0$ and one of four conditions below, then the trivial equilibrium in system (2.16) is exponentially asymp-*

totically stable:

$$
\begin{aligned}
&(1) \quad a_3 > 0, \ \ a_4 > 0, \ \ a_1 - |a_2| - 2a_3\tau - a_4\tau > 0, \\
&(2) \quad a_3 < 0, \ \ a_4 > 0, \ \ a_1 - |a_2| - 2a_3\tau\cos\zeta_0 - a_4\tau > 0, \\
&(3) \quad a_3 > 0, \ \ a_4 < 0, \ \ a_1 - |a_2| - 2a_3\tau - a_4\tau\cos\zeta_0 > 0, \\
&(4) \quad a_3 < 0, \ \ a_4 < 0, \ \ a_1 - |a_2| - 2a_3\tau\cos\zeta_0 - a_4\tau\cos\zeta_0 > 0,
\end{aligned}
$$

where

$$\zeta_0 = \tan\zeta_0, \zeta_0 \in [\pi, 2\pi].$$

Proof: The corresponding characteristic equation for system (2.16) is

$$\Delta_4(\lambda) = \lambda^2 + (a_1 + a_2 e^{-\lambda\tau})\lambda + a_3 e^{-2\lambda\tau} + a_4 e^{-\lambda\tau} + a_5 = 0.$$

Let us substitute $\lambda = i\omega$ into $\Delta_4(\lambda) = 0$ and separate

$$\Delta_4(i\omega) = (i\omega)^2 + (a_1 + a_2 e^{-i\omega\tau})(i\omega) + a_3 e^{-2i\omega\tau} + a_4 e^{-i\omega\tau} + a_5 = 0$$

into real and imaginary parts, then we have

$$
\begin{aligned}
R_4(\omega) &= -\omega^2 + a_2\omega\sin(\omega\tau) + a_3\cos(2\omega\tau) + a_4\cos(\omega\tau) + a_5, \\
S_4(\omega) &= a_1\omega + a_2\omega\cos(\omega\tau) - a_3\sin(2\omega\tau) - a_4\sin(\omega\tau).
\end{aligned}
$$

It is easy to check

$$R_4(0) = a_3 + a_4 + a_5 > 0,$$

and

$$\lim_{\omega\to\infty} R_4(\omega) = -\infty.$$

So the number r of the positive zeros ρ_k of R_4 is odd, i.e.

$$\sum_{k=1}^{k}(-1)^k \mathrm{sgn} S_4(\rho_k) = -1.$$

If $a_3 > 0$, $a_4 > 0$, we can get

$$S_4(\omega) \geq S_4^-(\omega) = (a_1 - |a_2| - 2a_3\tau - a_4\tau)\omega > 0.$$

To estimate the lower bound for $S_4(\omega)$ when a_3 or a_4 is negative, we need to get the maximum ξ such that $\sin x \geq \xi x$ for $x \geq 0$ except the point $x = 0$, which means $x \in [\pi, \frac{3}{2}\pi]$, i.e. to find the minimum of $h(x) = \frac{\sin x}{x}$. From $h'(x) = \frac{x\cos x - \sin x}{x^2} = 0$, we denote the root of $x\cos x - \sin x = 0$ as ξ_0, i.e. $\xi_0 = \tan \xi_0$. Since

$$h''(\xi_0) = \frac{2\xi_0 \sin\xi_0 - 2\xi_0^2 \cos\xi_0 - \xi_0^3 \cos\xi_0}{\xi_0^4} = -\frac{\sin\xi_0}{\xi_0} > 0,$$

therefore, the minimum value of $h(x)$ is $\cos\xi_0$ and $\frac{\sin x}{x} \geq \frac{\sin\xi_0}{\xi_0} = \cos\xi_0$, i.e. $\sin x \geq x\cos\xi_0$, if $x \geq 0$.

Similarly, we have

if $a_3 < 0$, $a_4 > 0$,

$$S_4(\omega) \geq S_4^-(\omega) = (a_1 - |a_2| - 2a_3\tau\cos\zeta_0 - a_4\tau)\omega > 0;$$

if $a_3 > 0$, $a_4 < 0$,

$$S_4(\omega) \geq S_4^-(\omega) = (a_1 - |a_2| - 2a_3\tau - a_4\tau\cos\zeta_0)\omega > 0;$$

if $a_3 < 0$, $a_4 < 0$,

$$S_4(\omega) \geq S_4^-(\omega) = (a_1 - |a_2| - 2a_3\tau\cos\zeta_0 - a_4\tau\cos\zeta_0)\omega > 0. \quad \square$$

Apply Lemma 2.3.3 to **Case 6** and **Case 7**, we have the following theorems:

Theorem 2.3.4 *The trivial equilibrium in* **Case 6** *is exponentially asymptotically stable if the parameters* α *and* β *satisfy* $\alpha^* < \alpha + \beta < 1$ *and one of two conditions below, and* α^* *is defined in Lemma 2.3.1:*

$$(1) \qquad -2\alpha + \beta > 0,$$

$$2 + 2\alpha - \beta - 2(\alpha^2 + \beta^2 - \alpha\beta)\tau + (2\alpha - \beta)\tau > 0,$$

$$(2) \qquad -2\alpha + \beta < 0, \;\; \alpha^2 + \beta^2 - \alpha\beta - 2\alpha + \beta + 1 > 0,$$

$$2 + 2\alpha - \beta - 2(\alpha^2 + \beta^2 - \alpha\beta)\tau + (2\alpha - \beta)\tau \cos\zeta_0 > 0,$$

where ζ_0 *is defined in Lemma 2.3.3.*

Theorem 2.3.5 *The trivial equilibrium in* **Case 7** *is exponentially asymptotically stable if the parameters* α *and* β *satisfy* $\alpha^* < \alpha + \left(\frac{\kappa}{6} + \frac{2}{\kappa}\right)\beta < 1$ *and one of two conditions below, where* α^* *is defined in Lemma 2.3.1:*

$$(1) \qquad \alpha - \left(\tfrac{\kappa}{12} + \tfrac{1}{\kappa}\right)\beta > 0,$$

$$2 - 2\left|\alpha - \left(\tfrac{\kappa}{12} + \tfrac{1}{\kappa}\right)\beta\right| - 2\left[\left(\alpha - \left(\tfrac{\kappa}{12} + \tfrac{1}{\kappa}\right)\beta\right)^2 + \tfrac{3}{4}\left(\tfrac{\kappa}{6} - \tfrac{2}{\kappa}\right)^2 + \alpha - \left(\tfrac{\kappa}{12} + \tfrac{1}{\kappa}\right)\beta\right]\tau > 0,$$

$$(2) \qquad \alpha - \left(\tfrac{\kappa}{12} + \tfrac{1}{\kappa}\right)\beta < 0,$$

$$2 - 2\left|\alpha - \left(\tfrac{\kappa}{12} + \tfrac{1}{\kappa}\right)\beta\right| - 2\left[\left(\alpha - \left(\tfrac{\kappa}{12} + \tfrac{1}{\kappa}\right)\beta\right)^2 + \tfrac{3}{4}\left(\tfrac{\kappa}{6} - \tfrac{2}{\kappa}\right)^2 + \alpha - \left(\tfrac{\kappa}{12} + \tfrac{1}{\kappa}\right)\beta\cos\zeta_0\right]\tau > 0,$$

where ζ_0 *is defined in Lemma 2.3.3,* $\kappa = (108 + 12\sqrt{69})^{1/3}$.

From the definition of ζ_0 which is defined in Lemma 2.3.3, we can get $\cos\zeta_0 \approx -0.217$. Let $\tau = 1$, we obtain the exponentially asymptotical stability region on $\alpha - \beta$ plane as Fig. (2.2), from Theorem 2.3.4 for **Case 6**.

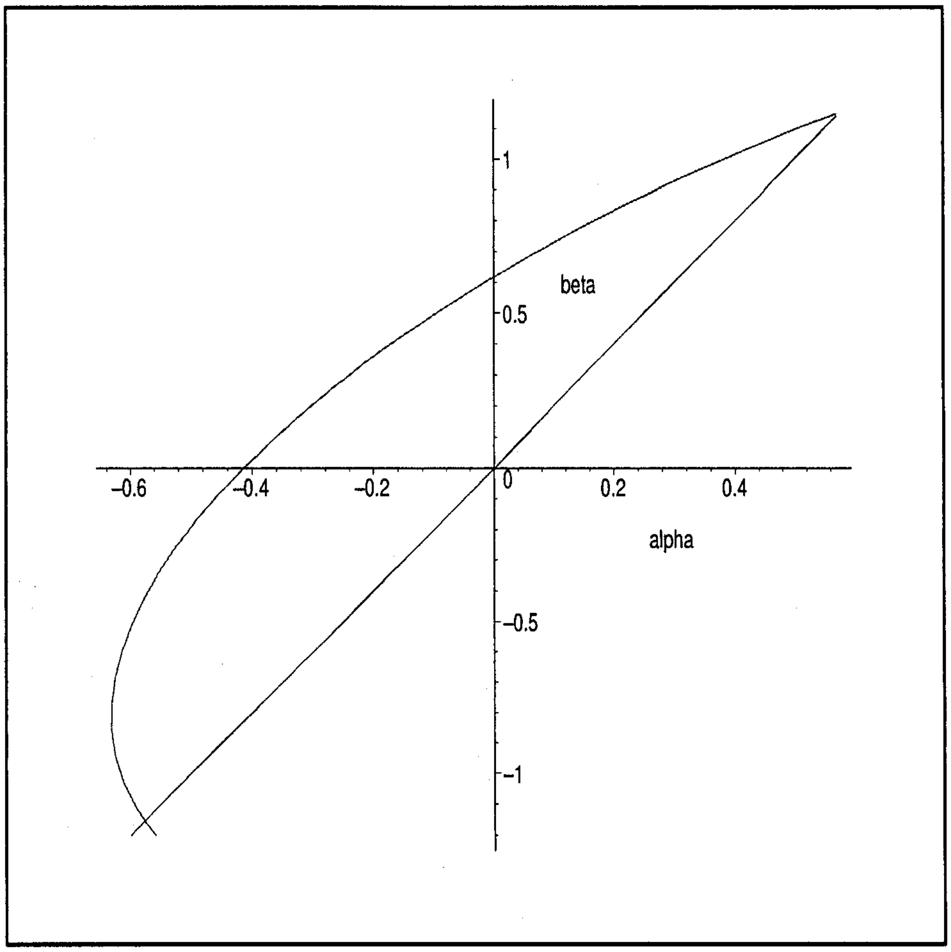

Figure 2.2: The stability region from Theorem 2.3.4 for **Case 6** when $\tau = 1$

2.3.2 Delay $\tau_s \neq \tau_n$

Lemma 2.3.6 *All roots in Eq (2.3) have negative real parts, if the parameters satisfy* $-\frac{1}{2} < \alpha\tau_s < 0$ *and one of the two conditions below:*

1) $\alpha < -1$, $|\beta| < -\alpha$;

2) $0 < |\beta| < |1+\alpha|$.

Proof: 1) Let $\lambda = \mu + i\omega$ in $\Delta_1 = \lambda + 1 - \alpha e^{-\lambda\tau_s} - \beta e^{-\lambda\tau_n} = 0$. Separating it into real and imaginary parts, we obtain

$$\mu = -1 + \alpha e^{-\mu\tau_s}\cos(\omega\tau_s) + \beta e^{-\mu\tau_n}\cos(\omega\tau_n) \tag{2.17}$$

and

$$\omega = -\alpha e^{-\mu\tau_s}\sin(\omega\tau_s) - \beta e^{-\mu\tau_n}\sin(\omega\tau_n). \tag{2.18}$$

From condition $0 \leq \tau_s < -\frac{1}{2\alpha}$, we have $0 \leq \omega\tau_s < 1$, i.e. $\frac{1}{2} < \cos(1) < \cos(\omega\tau_s) \leq 1$ and $0 \leq \sin(\omega\tau_s) < \sin(1) < 1$.

Eliminating the last term in (2.17) and (2.18) results in

$$M(\mu) \triangleq (\mu+1)^2 + \omega^2 - 2\alpha e^{-\mu\tau_s}[(\mu+1)\cos(\omega\tau_s) - \omega\sin(\omega\tau_s)] + \alpha^2 e^{-2\mu\tau_s} - \beta^2 e^{-2\mu\tau_n} = 0.$$

So

$$M(0) = 1 - 2\alpha\cos(\omega\tau_s) + \alpha^2 + \omega^2 + 2\alpha\omega\sin(\omega\tau_s) - \beta^2.$$

Since $\alpha < 0$, $\sin(\omega\tau_s) < \omega\tau_s$ and $\tau_s < -\frac{1}{2\alpha}$, then

$$\omega^2 + 2\alpha\omega\sin(\omega\tau_s) \geq \omega^2(1 + 2\alpha\tau_s) > 0.$$

Meanwhile, $\alpha < 0$, $\cos(\omega\tau_s) > 0$ and $\beta^2 < \alpha^2$, so we have $M(0) > 0$.

Taking the derivative of $M(\mu)$ with respect to μ, we obtain

$$\begin{aligned}\frac{\partial M}{\partial \mu} &= 2\{\tau_n\beta^2 e^{-2\mu\tau_n} - \alpha\omega\tau_s e^{-\mu\tau_s}\sin(\omega\tau_s) + (\mu+1)[1 + \alpha\tau_s e^{-\mu\tau_s}\cos(\omega\tau_s)] \\ &\quad -\alpha e^{-\mu\tau_s}[\cos(\omega\tau_s) + \alpha\tau_s e^{-\mu\tau_s}]\}.\end{aligned} \tag{2.19}$$

Since $\alpha < 0$, $\omega \geq 0$, $\tau_s \geq 0$, $\tau_n \geq 0$, $\mu \geq 0$, $\sin(\omega\tau_s) \geq 0$ and $\cos(\omega\tau_s) > 0$, we obtain the first two terms in (2.19) are nonnegative.

From $0 \leq \tau_s < -\frac{1}{2\alpha}$ and $\mu \geq 0$, i.e. $0 < e^{-\mu\tau_s} \leq 1$, and $\cos(\omega\tau_s) \leq 1$, we have

$$(\mu+1)[1+\alpha\tau_s e^{-\mu\tau_s}\cos(\omega\tau_s)] > (\mu+1)(1-\frac{1}{2}) \geq 0.$$

From $\alpha < 0$, $\frac{1}{2} < \cos(1) < \cos(\omega\tau_s)$, $\tau_s < -\frac{1}{2\alpha}$, and $0 < e^{-\mu\tau_s} \leq 1$, we have

$$-\alpha e^{-\mu\tau_s}[\cos(\omega\tau_s) + \alpha\tau_s e^{-\mu\tau_s}] > -\alpha e^{-\mu\tau_s}(\cos(1) - \frac{1}{2}) > 0.$$

Thus, $\frac{\partial M}{\partial \mu} > 0$ for $\mu \geq 0$, therefore $M(0) > 0$ if $\mu \geq 0$. Therefore if $M(\mu) = 0$, then $\mu < 0$, i.e. all roots of the characteristic equation have negative real part.

2)We will use Rouché's theorem to prove it.

Let $f_1(\lambda) = \lambda + 1 - \alpha e^{-\lambda\tau_s}$ and $f_2(\lambda) = -\beta e^{-\lambda\tau_n}$. Consider the contour C_R which consists of the semicircle $z = Re^{i\theta}$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and the line $z = iy$, $-R \leq y \leq R$.

On the semicircle, $|f_2(\lambda)| = |\beta|e^{-Re(\lambda)\tau_n} \leq |\beta|$ and $f_1(\lambda) = R + O(1)$, so $|f_1(\lambda)|$ can be made as large as we like (in particular larger than $|\beta|$) by taking R large enough. Thus $|f_1(\lambda)| > |f_2(\lambda)|$ on the semicircle.

On the line, $|f_2(\lambda)| = |-\beta e^{-\lambda\tau_n}| = |\beta| > 0$ and

$$\begin{aligned}
|f_1(\lambda)| &= |1 - \alpha\cos(y\tau_s) + i(y + \alpha\sin(y\tau_s))| \\
&= \sqrt{1 - 2\alpha\cos(y\tau_s) + \alpha^2 + y^2 + 2\alpha y\sin(y\tau_s)} \\
&\geq \sqrt{(1+\alpha)^2 + y^2(1+2\alpha\tau_s)} \\
&\geq |1+\alpha| > 0,
\end{aligned}$$

since $-\frac{1}{2} < \alpha\tau_s < 0$. So when $|\beta| < |1+\alpha|$, $|f_1(\lambda)| > |f_2(\lambda)|$ on C_R for R sufficiently large.

$R \to \infty$ shows that f_1 and f_2 have the same number of zeros in the right half of the complex plane. But we have seen that $f_1(\lambda)$ has no zeros with nonnegative real part if

$-\frac{1}{2} < \alpha\tau_s < 0$ (we have proved it from the result 1) since here $\beta = 0$). Hence using Rouché's Theorem, we know that all zeros of $\Delta_1(\lambda) = f_1(\lambda) + f_2(\lambda)$ have negative real parts under the given conditions. □

Remark 2.3.3 *Fix τ_s and show the stability condition in Lemma 2.3.6 as the shadow in Fig. (2.3).*

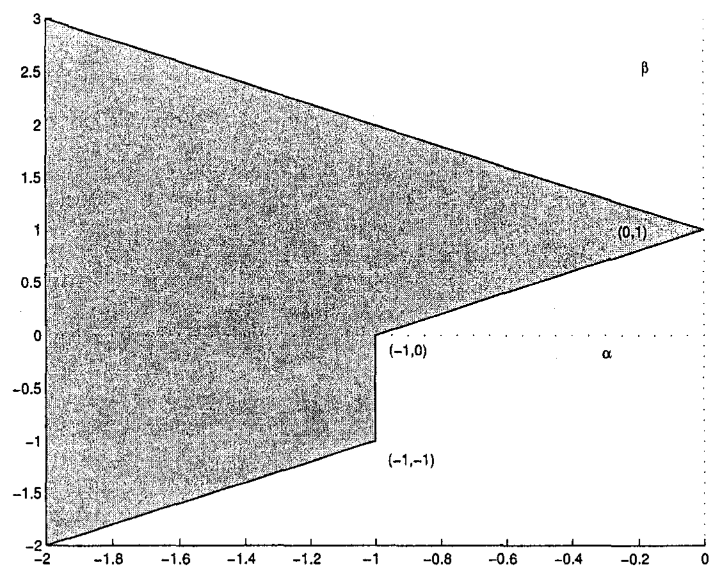

Figure 2.3: The stable condition for local stability when $\tau_s \neq \tau_n$, $\tau_s = 0.25$

Based on the lemma above, we have the following theorem:

Theorem 2.3.7 *If the parameters α, β satisfy the condition for certain case in the following table, the trivial equilibrium in the corresponding case is stable:*

Case	*Condition*
1	$-\frac{1}{2} < \alpha\tau_s < 0$, *1)* $0 < \|\beta\| < \frac{1}{2}\|1+\alpha\|$, *2)*$\alpha < -1$, $\|\beta\| < -\frac{1}{2}\alpha$
2	$-\frac{1}{2} < \alpha\tau_s < 0$, *1)* $0 < \|\beta\| < \frac{\sqrt{5}-1}{2}\|1+\alpha\|$, *2)*$\alpha < -1$, $\|\beta\| < \frac{1-\sqrt{5}}{2}\alpha$
3	$-\frac{1}{2} < \alpha\tau_s < 0$, *1)* $0 < \|\beta\| < \frac{\sqrt{2}}{2}\|1+\alpha\|$, *2)*$\alpha < -1$, $\|\beta\| < -\frac{\sqrt{2}}{2}\alpha$
4	$-\frac{1}{2} < \alpha\tau_s < 0$, *1)* $0 < \|\beta\| < \|1+\alpha\|$, *2)*$\alpha < -1$, $\|\beta\| < -\alpha$
5	$\alpha^* < \alpha < 1$, *where* α^* *is defined in Lemma 2.3.1*
6	$-\frac{1}{2} < \alpha\tau_s < 0$, $0 < \|\beta\| < \|1+\alpha\|$
7	$-\frac{1}{2} < \alpha\tau_s < 0$, $0 < \|\beta\| < \frac{\sqrt{5}-1}{2}\|1+\alpha\|$

Proof: We take **Case 3** as an example:

The stability condition for $P_3 = 0$ is

$$\begin{cases} -\frac{1}{2} < \alpha\tau_s < 0, & 1)\ 0 < |\beta| < |1+\alpha|, \quad 2)\ \alpha < -1, |\beta| < -\alpha, \\ -\frac{1}{2} < \alpha\tau_s < 0, & 1)\ 0 < |\beta| < \frac{\sqrt{2}}{2}|1+\alpha|, \quad 2)\ \alpha < -1, |\beta| < -\frac{\sqrt{2}}{2}\alpha, \\ -\frac{1}{2} < \alpha\tau_s < 0, & 1)\ 0 < |\beta| < \frac{\sqrt{2}}{2}|1+\alpha|, \quad 2)\ \alpha < -1, |\beta| < -\frac{\sqrt{2}}{2}\alpha, \end{cases}$$

i.e.

$$-\frac{1}{2} < \alpha\tau_s < 0,\ 1)\ 0 < |\beta| < \frac{\sqrt{2}}{2}|1+\alpha|, 2)\ \alpha < -1, |\beta| < -\frac{\sqrt{2}}{2}\alpha.$$

Similarly, we can get the same results for the other cases except for **Case 6** and **Case 7**.

For **Case 6**, the corresponding characteristic equation is

$$P_6 = (\lambda + 1 - \alpha e^{-\lambda\tau_s})^3 - (\beta e^{-\lambda\tau_n})^3 = 0.$$

Let $f_1(\lambda) = \lambda + 1 - \alpha e^{-\lambda\tau_s}$ and $f_2(\lambda) = -\beta e^{-\lambda\tau_n}$. By Lemma **??**, if the parameters satisfy $-\frac{1}{2} < \alpha\tau_s < 0$ and $0 < |\beta| < |1+\alpha|$, then $|f_1(\lambda)| > |f_2(\lambda)|$, i.e. $|f_1^3(\lambda)| > |f_2^3(\lambda)|$. All roots of $f_1(\lambda)$ have negative real parts when $-\frac{1}{2} < \alpha\tau_s < 0$. Therefore, the result is hold in the theorem obtained from Rouché Theorem.

For **Case 7**, the corresponding characteristic equation is

$$P_7 = (\lambda + 1 - \alpha e^{-\lambda\tau_s})^3 - \beta^3 e^{-3\lambda\tau_n} + (\lambda + 1 - \alpha e^{-\lambda\tau_s})\beta^2 e^{-2\lambda\tau_n} = 0.$$

Let $f_{01}(\lambda) = (\lambda + 1 - \alpha e^{-\lambda\tau_s})^3 - \beta^3 e^{-3\lambda\tau_n}$ and $f_{02}(\lambda) = (\lambda + 1 - \alpha e^{-\lambda\tau_s})\beta^2 e^{-2\lambda\tau_n}$.

From the condition $0 < |\beta| < \frac{\sqrt{5}-1}{2}|1+\alpha|$, we have $0 < |\beta| < |1+\alpha|$, then we get $|f_1(\lambda)| > |f_2(\lambda)|$ from Lemma 2.3.6.

$$\begin{aligned} |f_{01}(\lambda)| - |f_{02}(\lambda)| &= |f_1^3(\lambda) - f_2^3(\lambda)| - |f_1(\lambda) f_2^2(\lambda)| \\ &> |f_1(\lambda)|^3 - |f_2(\lambda)|^3 - |f_1(\lambda)| \cdot |f_2(\lambda)|^2 \\ &> |f_2(\lambda)| \cdot (|f_1(\lambda)|^2 - |f_2(\lambda)|^2 - |f_1(\lambda)| \cdot |f_2(\lambda)|) \\ &= |f_2(\lambda)| \left(|f_1(\lambda)| + \frac{-1+\sqrt{5}}{2}|f_2(\lambda)| \right) \left(|f_1(\lambda)| - \frac{1+\sqrt{5}}{2}|f_2(\lambda)| \right). \end{aligned}$$

From $0 < \frac{\sqrt{5}+1}{2}|\beta| < |1+\alpha|$, i.e. $0 < |\beta| < \frac{\sqrt{5}-1}{2}|1+\alpha|$, by Lemma 2.3.6, we have $|f_1(\lambda)| - \frac{1+\sqrt{5}}{2}|f_2(\lambda)| > 0$. Therefore, $|f_{01}(\lambda)| - |f_{02}(\lambda)| > 0$.

According to Rouché Theorem, we obtain the final result. □

2.4 Unstability

For

$$\Delta_1(\lambda) = \lambda + 1 - \alpha e^{-\lambda\tau_s} - \beta e^{-\lambda\tau_n} = 0,$$

we have the following lemma:

Lemma 2.4.1 *If $\alpha + \beta > 1$, the characteristic equation $\Delta_1 = 0$ has a root with positive real part for all values of $\tau_s \geq 0$ and $\tau_n \geq 0$.*

Proof: We know

$$\Delta_1(0) = 1 - \alpha - \beta < 0$$

and

$$\lim_{\lambda \to +\infty} \Delta_1(\lambda) = +\infty$$

for all $\tau_s \geq 0$ and $\tau_n \geq 0$. Since $\Delta_1(\lambda)$ is a continuous function of λ, there exists $\lambda^* > 0$ s.t. $\Delta_1(\lambda^*) = 0$ for any $\tau_s \geq 0$, $\tau_n \geq 0$ and $\alpha + \beta > 1$. Thus, the characteristic equation $\Delta_1(\lambda) = 0$ has a positive real root for these parameters values. □

Since the characteristic equation of **Case 1** to **Case 5** can be transformed into the multiplication of three factors, apply Lemma 2.4.1 to each factor, we have the following theorem:

Theorem 2.4.2 *If the parameters α, β satisfy the condition for certain case in the following table, then the trivial equilibrium in the corresponding case is unstable:*

Case	*Condition*
1	$\alpha + 2\beta > 1, \quad \alpha - \beta > 1.$
2	$\alpha + \frac{1+\sqrt{5}}{2}\beta > 1, \quad \alpha + \frac{1-\sqrt{5}}{2}\beta > 1.$
3	$\alpha + \sqrt{2}\,\beta > 1, \quad \alpha - \sqrt{2}\,\beta > 1.$
4	$\alpha + \beta > 1, \quad \alpha - \beta > 1.$
5	$\alpha > 1.$

2.5 Curves of Characteristic Roots with Zero Real Part

In the following we will take the characteristic equation

$$\Delta_{31}\Delta_{32}\Delta_{33} \triangleq (\lambda+1-\alpha e^{-\lambda\tau_s})(\lambda+1-\alpha e^{-\lambda\tau_s}+\sqrt{2}\beta e^{-\lambda\tau_n})(\lambda+1-\alpha e^{-\lambda\tau_s}-\sqrt{2}\beta e^{-\lambda\tau_n})$$

in **Case 3** as an example to show the analysis. The approach in this section follows closely that of [5, 40, 57].

As the parameters vary, stability may be lost by the real root of the characteristic equation passing through zero or by the pair of complex conjugate roots passing through the imaginary axis. To determine the full region of stability of the trivial solution, we must describe the regions in parameter space where this occurs.

If the characteristic equation in **Case 3** has a simple zero root, we have $P_3(0) = 0$ and $P_3'(0) \neq 0$, where $P_3(0) = 0$ yields $(1-\alpha)(1-\alpha+\sqrt{2}\beta)(1-\alpha-\sqrt{2}\beta) = 0$. So we can get the boundaries $\beta = \beta_1 \triangleq \frac{\sqrt{2}}{2}(\alpha - 1)$ and $\beta = \beta_2 \triangleq \frac{\sqrt{2}}{2}(1-\alpha)$. Note that from theorem 2.4.2, $\alpha = 1$ cannot form part of the boundary of the stability region.

From the characteristic equation of each case, we have the following result about the zero root of the characteristic equation:

Lemma 2.5.1 *When β satisfies the condition for certain case in the following table and $\tau_n \neq \tau^*$, then for the corresponding case $\lambda = 0$ is a simple zero root of the corresponding characteristic equation, where*

$$\tau^* = \frac{1+\alpha\tau_s}{\alpha - 1}.$$

Case	*Condition*
1	$\beta = \frac{1-\alpha}{2} \triangleq \beta_{11}$
2	$\beta = \alpha - 1 \triangleq \beta_{21}$
2	$\beta = \frac{1-\sqrt{5}}{2}(\alpha - 1) \triangleq \beta_{22}$
2	$\beta = \frac{1+\sqrt{5}}{2}(\alpha - 1) \triangleq \beta_{23}$
3	$\beta = \frac{\sqrt{2}}{2}(1 - \alpha) \triangleq \beta_{31}$
3	$\beta = \frac{\sqrt{2}}{2}(\alpha - 1) \triangleq \beta_{32}$
4	$\beta = 1 - \alpha \triangleq \beta_{41}$
4	$\beta = \alpha - 1 \triangleq \beta_{21}$
6	$\beta = 1 - \alpha \triangleq \beta_{41}$

The characteristic equation in **Case 3** has a pair of pure imaginary roots $\pm i\omega$ when $P_3(\pm i\omega) = 0$. From $\Delta_{31}(i\omega) = 0$, we have:

$$1 - \alpha \sin(\omega\tau_s) = 0, \tag{2.20}$$

$$\omega + \alpha \sin(\omega\tau_s) = 0, \tag{2.21}$$

which yields the curve

$$\arccos\alpha + 2k\pi + \tau_s\sqrt{(\alpha^2 + 1)} = 0.$$

While from $\Delta_{32}(i\omega) = 0$, we have:

$$1 - \alpha \cos(\omega\tau_s) + \sqrt{2}\beta\cos(\omega\tau_n) = 0, \tag{2.22}$$

$$\omega + \alpha \sin(\omega\tau_s) - \sqrt{2}\beta\sin(\omega\tau_n) = 0. \tag{2.23}$$

Since the parameter space is four dimensional, it is difficult to visualize these regions. Thus we will focus on fixing the parameters α and τ_s, and describing curves in the β, τ_n plane where the characteristic equation has a zero root or a pair of pure imaginary roots.

This occurs along curves given by

$$\beta = \beta_1^{\pm} = \pm\frac{\sqrt{2}}{2}\sqrt{1 - 2\alpha\cos(\omega\tau_s) + \alpha^2 + \omega^2 + 2\alpha\omega\sin(\omega\tau_s)}, \tag{2.24}$$

$$\tau_{1k}^{+} = \begin{cases} \mathcal{T}_{2k}, & \alpha\cos(\omega\tau_s) - 1 > 0, \\ \mathcal{T}_{2k+1}, & \alpha\cos(\omega\tau_s) - 1 < 0, \end{cases} \tag{2.25}$$

$$\tau_{1k}^{-} = \begin{cases} \mathcal{T}_{2k+1}, & \alpha\cos(\omega\tau_s) - 1 > 0, \\ \mathcal{T}_{2k}, & \alpha\cos(\omega\tau_s) - 1 < 0, \end{cases} \tag{2.26}$$

where

$$\mathcal{T}_l(\omega) = \frac{1}{\omega}\left\{\arctan\left[\frac{\omega + \alpha\sin(\omega\tau_s)}{\alpha\cos(\omega\tau_s) - 1}\right] + l\pi\right\},$$

and $\arctan(x)$ is the principle branch of the inverse tangent function. Clearly, equations (2.25) and (2.26) represent an infinite family of curves.

Similarly, from $\Delta_{33}(i\omega) = 0$, we have:

$$\begin{aligned} 1 - \alpha\cos(\omega\tau_s) - \sqrt{2}\beta\cos(\omega\tau_n) &= 0, \quad (2.27) \\ \omega + \alpha\sin(\omega\tau_s) + \sqrt{2}\beta\sin(\omega\tau_n) &= 0. \quad (2.28) \end{aligned}$$

This occurs along curves given by

$$\beta = \beta_2^{\pm} = \pm\frac{\sqrt{2}}{2}\sqrt{1 - 2\alpha\cos(\omega\tau_s) + \alpha^2 + \omega^2 + 2\alpha\omega\sin(\omega\tau_s)}, \tag{2.29}$$

$$\tau_{2k}^{+} = \begin{cases} \mathcal{T}_{2k+1}, & \alpha\cos(\omega\tau_s) - 1 > 0; \\ \mathcal{T}_{2k}, & \alpha\cos(\omega\tau_s) - 1 < 0, \end{cases} \tag{2.30}$$

$$\tau_{2k}^{-} = \begin{cases} \mathcal{T}_{2k}, & \alpha\cos(\omega\tau_s) - 1 > 0; \\ \mathcal{T}_{2k+1}, & \alpha\cos(\omega\tau_s) - 1 < 0. \end{cases} \tag{2.31}$$

We define the following for the theorem later:

$$\begin{aligned}
F_{11} &= \max\{0, 1-|\alpha|\}, \\
F_{12} &= \min\{0, |\alpha|-1\}, \\
F_{21} &= \max\{-\frac{\sqrt{2}}{2}\beta, 1-|\alpha|\} \text{ as } \tau_n \text{ is varied on the curves } (\beta_1^-, \tau_{1n}^-), \\
F_{22} &= \min\{-\frac{\sqrt{2}}{2}\beta, |\alpha|-1\} \text{ as } \tau_n \text{ is varied on the curves } (\beta_1^+, \tau_{1n}^+), \\
F_{31} &= \max\{\frac{\sqrt{2}}{2}\beta, 1-|\alpha|\} \text{ as } \tau_n \text{ is varied on the curves } (\beta_2^-, \tau_{2n}^-), \\
F_{32} &= \min\{\frac{\sqrt{2}}{2}\beta, |\alpha|-1\} \text{ as } \tau_n \text{ is varied on the curves } (\beta_2^+, \tau_{2n}^+).
\end{aligned}$$

Now we describe the geometry of the curves defined above and how this geometry changes as α and τ_s vary.

Lemma 2.5.2

$$\begin{aligned}
\lim_{\omega\to 0^+} \beta_1^{\pm} &= \lim_{\omega\to 0^+} \beta_2^{\pm} = \pm\frac{\sqrt{2}}{2}|1-\alpha|, \\
\lim_{\omega\to 0^+} \tau_{1k}^{\pm} &= \lim_{\omega\to 0^+} \tau_{2k}^{\pm} = \infty, k > 0,
\end{aligned}$$

$$\lim_{\omega\to 0^+} \tau_{10}^{+} = \lim_{\omega\to 0^+} \tau_{20}^{-} = \begin{cases} \frac{1+\alpha\tau_s}{\alpha-1}, & \alpha > 1, \\ \infty, & \alpha \le 1, \end{cases} \tag{2.32}$$

$$\lim_{\omega\to 0^+} \tau_{10}^{-} = \lim_{\omega\to 0^+} \tau_{20}^{+} = \begin{cases} \frac{1+\alpha\tau_s}{\alpha-1}, & \alpha < 1, \\ -\infty, & \alpha = 1, \\ \infty, & \alpha > 1, \end{cases} \tag{2.33}$$

and

$$\lim_{\omega\to\infty} \beta_1^{\pm} = \lim_{\omega\to\infty} \beta_2^{\pm} = \pm\infty,$$
$$\lim_{\omega\to\infty} \tau_{10}^{\pm} = \lim_{\omega\to\infty} \tau_{20}^{\pm} = 0.$$

Proof: The proof follows from straightforward calculations. □

All points in parameter space where the characteristic equation has roots with zero real part have been determined. By varying one or more parameters in the parameter space, passing through such a point may cause a qualitative change in the type of solutions admitted by the DDE. Such bifurcation points are important, especially when they lie on the boundary of the stability region of the trivial solution, because they determine the observable behavior of the system.

Lemma 2.5.3 *If* $|\alpha| \leq 1$, *then*

1) *the curves* (β_1^+, τ_{1k}^+) *and* (β_2^+, τ_{2k}^+) *are bounded on the left by the line* $\beta = \frac{\sqrt{2}}{2}(1-|\alpha|)$;

2) *the curves* (β_1^-,τ_{1k}^-) *and* (β_2^-,τ_{2k}^-) *are bounded on the right by the line* $\beta=\frac{\sqrt{2}}{2}(|\alpha|-1)$.

Proof: From equations (2.22) and (2.23), which holds along $\beta = \beta_1^+$, we have

$$\beta_1^+ \geq -\beta_1^+ \cos(\omega\tau_{1k}^+) = \frac{\sqrt{2}}{2}(1 - \alpha\cos(\omega\tau_s)) \geq \frac{\sqrt{2}}{2}(1 - |\alpha|).$$

Since $\beta_1^+ = -\beta_1^-$, we have

$$\beta_1^- \leq -\beta_1^- \cos(\omega\tau_{1k}^-) = \frac{\sqrt{2}}{2}(1 - \alpha\cos(\omega\tau_s)) \leq \frac{\sqrt{2}}{2}(|\alpha| - 1).$$

The rest of the results can be shown in an analogous manner. □

Now we consider the boundary of the stability region when $\tau_s \neq \tau_n$. Generally, for

$$\Delta_1 = \lambda + 1 - \alpha e^{-\lambda\tau_s} - \beta e^{-\lambda\tau_n}.$$

If $\Delta_1(i\omega) = 0$, we have

$$\begin{aligned} 1 - \alpha\cos(\omega\tau_s) &= \beta\cos(\omega\tau_n), \\ \omega + \alpha\sin(\omega\tau_s) &= -\beta\sin(\omega\tau_n), \end{aligned}$$

which yields

$$\begin{aligned} \beta^{\pm} &= \pm\sqrt{1 + \alpha^2 + \omega^2 + 2\alpha\omega\sin(\omega\tau_s) - 2\alpha\cos(\omega\tau_s)}, \\ \tau_n^{+} &= \frac{1}{\omega}\arctan\frac{-\omega - \alpha\sin(\omega\tau_s)}{1 - \alpha\cos(\omega\tau_s)} + \begin{cases} \frac{2k\pi}{\omega} & 1 - \alpha\cos(\omega\tau_s) < 0, \\ \frac{(2k+1)\pi}{\omega} & 1 - \alpha\cos(\omega\tau_s) > 0, \end{cases} \\ \tau_n^{-} &= \frac{1}{\omega}\arctan\frac{-\omega - \alpha\sin(\omega\tau_s)}{1 - \alpha\cos(\omega\tau_s)} + \begin{cases} \frac{(2k+1)\pi}{\omega} & 1 - \alpha\cos(\omega\tau_s) < 0, \\ \frac{2k\pi}{\omega} & 1 - \alpha\cos(\omega\tau_s) > 0. \end{cases} \end{aligned}$$

So we know that on the $\beta - \tau_n$ plane the curves (β^+, τ_n^+) and (β^-, τ_n^-) establish the boundary of the stability region, i.e. the area around the equilibrium $(0,0,0)$ decided by these curves is the full stability region.

Let

$$\begin{aligned} F_1(\tau_n) &= \max\{\beta, 1 - |\alpha|\} \text{ as } \tau_n \text{ is varied on the curves } (\beta^-, \tau_n^-)\,, \\ F_2(\tau_n) &= \min\{\beta, |\alpha| - 1\} \text{ as } \tau_n \text{ is varied on the curves } (\beta^+, \tau_n^+)\,, \end{aligned}$$

then the parameter β satisfy $F_1(\tau_n) < \beta < F_2(\tau_n)$ if and only if the equilibrium point in the corresponding system is stable.

Then for **Case 3**,we have the following theorem:

Theorem 2.5.4 *The parameter β satisfies* $\max_{1\le i\le 3} F_{i1} < \beta < \min_{1\le i\le 3} F_{i2}$ *if and only if the equilibrium point in* **Case 3** *is stable.*

Similarly, we can obtain the results for other cases.

Now we consider the boundary of the stability region when $\tau_s = \tau_n$. For

$$\lambda + 1 - (\alpha + \beta)e^{-\lambda\tau} = 0,$$

let $\lambda = i\omega$ be the imaginary root, we have:

$$\begin{aligned} \beta^{\pm} &= \pm\sqrt{1+\omega^2} - \alpha, \\ \tau &= \frac{1}{\omega}\left(\arccos\frac{1}{\alpha+\beta} + 2k\pi\right). \end{aligned}$$

Let

$$\begin{aligned} F_{11}(\tau) &= \max\{\beta, 1\} \text{ as } \tau_n \text{ is varied on the curves } (\beta^-, \tau)\,, \\ F_{12}(\tau) &= \min\{\beta, -1\} \text{ as } \tau_n \text{ is varied on the curves } (\beta^+, \tau)\,, \end{aligned}$$

then the parameter β satisfies $F_{11}(\tau) < \beta < F_{12}(\tau)$ if and only if the equilibrium point in the corresponding system is stable.

Until now, we can obtain the boundary of the stability region when $\tau_s = \tau_n$ for **Case 1** to **Case 5**. Next we will prove the result for **Case 6** and **Case 7**.

For

$$\lambda + 1 - (\alpha + \beta i)e^{-\lambda\tau} = 0,$$

let $\lambda = i\omega$ be the imaginary root, we have

$$\begin{aligned} \beta^{\pm} &= \pm\sqrt{1+\alpha^2+\omega^2+2\alpha\omega\sin(\omega\tau_s) - 2\alpha\cos(\omega\tau_s)}, \\ \tau^{\pm} &= \frac{1}{\omega}(\arcsin\frac{1}{\sqrt{1+2\alpha^2+\omega^2+2\alpha\omega\sin(\omega\tau_s)-2\alpha\cos(\omega\tau_s)}} \\ &\quad - \arctan\frac{\alpha}{\pm\sqrt{1+\alpha^2+\omega^2+2\alpha\omega\sin(\omega\tau_s)-2\alpha\cos(\omega\tau_s)}} + k\pi). \end{aligned}$$

Let

$$\begin{aligned} F_1(\tau) &= \max\{\beta\} \text{ for every } \tau \text{ where } \beta, \tau \text{ in } (\beta^-, \tau^-) \text{ curves}, \\ F_2(\tau) &= \min\{\beta\} \text{ for every } \tau \text{ where } \beta, \tau \text{ in } (\beta^+, \tau^+) \text{ curves}, \end{aligned}$$

then the parameter β satisfies $F_1(\tau) < \beta < F_2(\tau)$ if and only if the equilibrium point is stable.

For **Case 6**, when we consider

$$\lambda + 1 - (\alpha - \frac{\beta}{2} + \frac{\sqrt{3}}{2}\beta i)e^{-\lambda\tau} = 0,$$

let $\lambda = i\omega$ be the imaginary root of the above equation, then we can get the expression for the parameters β and τ. After the definition of $F_{21}(\lambda)$ and $F_{22}(\lambda)$, we know that the parameters β satisfy $F_1(\tau) < \beta < F_2(\tau)$ if and only if the equilibrium point is stable.

Similarly, we can define $F_{31}(\lambda)$ and $F_{32}(\lambda)$ for $\lambda + 1 - (\alpha - \frac{\beta}{2} - \frac{\sqrt{3}}{2}\beta i)e^{-\lambda\tau} = 0$.

Apply this result to **Case 6**, since

$$P_6(\lambda) = [\lambda+1-(\alpha+\beta)e^{-\lambda\tau}][\lambda+1-(\alpha-\frac{\beta}{2}+\frac{\sqrt{3}}{2}\beta i)e^{-\lambda\tau}][\lambda+1-(\alpha-\frac{\beta}{2}-\frac{\sqrt{3}}{2}\beta i)e^{-\lambda\tau}] = 0,$$

we have the following theorem:

Theorem 2.5.5 *The parameter β satisfies $\max_{1\le i\le 3} F_{i1} < \beta < \min_{1\le i\le 3} F_{i2}$ if and only if the equilibrium point in* **Case 6** *is stable.*

Similarly, we can obtain the result for **Case 7**.

Now we have done enough to describe the full stability region of the trivial solution in the β, τ plane, as α and τ_s are varied. See Fig 2.4 as an example. Detailed analysis about the full stability region of **Case 1** can be seen in [5]. For similar analysis see [40].

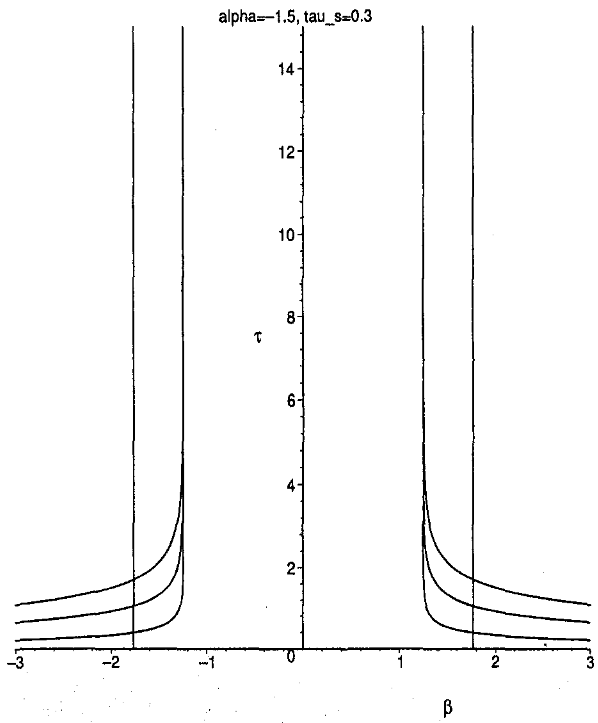

Figure 2.4: Region of local stability of the trivial solution for **Case 3**

Chapter 3

BIFURCATION ANALYSIS

3.1 Center Manifold Reduction and Normal Form

We consider a general functional differential delay equation:

$$\dot{x} = L(\mu)x_t + f(x_t, \mu),$$

with $x_t = x(t+\theta)$, $-h \le \theta \le 0$, $C = C([-h, 0], \mathbb{R})$, $L : C \to \mathbb{R}$ a linear operator, and $f \in C^r(C, \mathbb{R})$, $r \ge 1$. We assume that the linear part of the equation

$$\dot{x}(t) = L(\mu)x_t \tag{3.1}$$

has m eigenvalues with zero real parts and all the other eigenvalues have negative real parts. In such a situation, Hale [20] proved that there exists an m-dimensional invariant manifold in the state space C, which is called the center manifold, and that long term behavior of solutions to the nonlinear equation is well approximated by the flow on this manifold. Then we can decompose C as $C = P \oplus Q$. P is an m-dimensional subspace spanned by the solutions to (3.1) corresponding to the m zero real part eigenvalues; Q is the corresponding complementary space of P; P and Q are invariant under the flow associated with equation (3.1). Further, the flow on the center manifold is

$$x_t = \Phi z(t) + h(z(t), f),$$

where Φ is a basis for P, $h \in Q$, and x satisfies the ordinary differential equation

$$\dot{x} = Bx + \Psi(0)R(\Phi x + h(z(t), f), \mu),$$

where B is the $(m \times m)$ matrix of eigenvalues with null real part of (3.1), Ψ is the basis for the invariant subspace of the adjoint problem corresponding to P, which is normalized by $< \Psi, \Phi >= I$, where I is the $m \times m$ identity matrix. For $\phi \in C[-h, 0]$ and $\psi \in C[0, h]$, we introduce the bilinear operator associated with (3.1)

$$< \psi, \phi >= \psi(0)\phi(0) - \int_{-h}^{0} \int_{0}^{\theta} \psi(\xi - \theta) \mathrm{d}\eta(\theta) \phi(\xi) \mathrm{d}\xi,$$

where

$$\mathrm{d}\eta(\theta) = \begin{pmatrix} -\delta_0 + \alpha\delta_{\tau_s} & a_{12}\beta\delta_{\tau_n} & a_{13}\beta\delta_{\tau_n} \\ a_{21}\beta\delta_{\tau_n} & -\delta_0 + \alpha\delta_{\tau_s} & a_{23}\beta\delta_{\tau_n} \\ a_{31}\beta\delta_{\tau_n} & a_{32}\beta\delta_{\tau_n} & -\delta_0 + \alpha\delta_{\tau_s} \end{pmatrix} \mathrm{d}\theta,$$

and $\delta_\tau = \delta(\theta + \tau)$ is the Dirac distribution at the point $\theta = -\tau$, $h = \max(\tau_s, \tau_n)$.

When $\beta = \beta^*(\alpha)$, $\lambda = 0$ is a single zero root of the characteristic equation, the linearization system becomes

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + \alpha x_1(t - \tau_s) + a_{12}\beta^*(\alpha) x_2(t - \tau_n) + a_{13}\beta^*(\alpha) x_3(x - \tau_n) \\ \dot{x}_2(t) &= -x_2(t) + a_{21}\beta^*(\alpha) x_1(t - \tau_n) + \alpha x_2(t - \tau_s) + a_{23}\beta^*(\alpha) x_3(t - \tau_n). \quad (3.2) \\ \dot{x}_3(t) &= -x_3(t) + a_{31}\beta^*(\alpha) x_1(t - \tau_n) + a_{32}\beta^*(\alpha) x_2(t - \tau_n) + \alpha x_3(t - \tau_s) \end{aligned}$$

Suppose $\Phi \triangleq \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}_{3\times 1}$ is a solution of (3.2) corresponding to $\lambda = 0$. From

$$K =< \Phi^T, \Phi >= \Phi^T(0)\Phi(0) - \int_{-h}^{0}\int_{0}^{\theta} \Phi^T(\xi - \theta)\mathrm{d}\eta\Phi(\xi)\mathrm{d}\xi,$$

then the base in the complementary space is

$$\Psi(s) = K^{-1}\Phi^T(s) \triangleq (\psi_1, \psi_2, \psi_3)_{1\times 3}.$$

Until now the problem of describing the long term behavior of solutions to the delay differential equation has been reduced locally to the problem of describing the behavior of solutions to the 1-dimensional system of ordinary differential equation (3.1).

The following shows the rest of the details of the process:

By introducing a bifurcation parameter $\mu \in \mathbb{R}$ in β, i.e. $\beta = \beta^*(\alpha) + \mu$, then the system (1.5) becomes

$$\begin{aligned}
\dot{x}_1(t) &= -x_1(t)+\alpha f(x_1(t-\tau_s))+a_{12}(\beta^*(\alpha)+\mu)g(x_2(t-\tau_n))+a_{13}(\beta^*(\alpha)+\mu)g(x_3(x-\tau_n)) \\
\dot{x}_2(t) &= -x_2(t)+a_{21}(\beta^*(\alpha)+\mu)g(x_1(t-\tau_n))+\alpha f(x_2(t-\tau_s))+a_{23}(\beta^*(\alpha)+\mu)g(x_3(t-\tau_n)). \\
\dot{x}_3(t) &= -x_3(t)+a_{31}(\beta^*(\alpha)+\mu)g(x_1(t-\tau_n))+a_{32}(\beta^*(\alpha)+\mu)g(x_2(t-\tau_n))+\alpha f(x_3(t-\tau_s))
\end{aligned}$$

Rewriting the system as

$$\dot{x}(t) = L(\mu)x_t + F(x_t) + G(x_t),$$

for $\varphi_1, \varphi_2, \varphi_3 \in \mathbf{C}$, $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$,

$$L(\mu)\varphi = \begin{bmatrix} -\varphi_1(0) + \alpha\varphi_1(-\tau_s) + a_{12}(\beta^*(\alpha)+\mu)\varphi_2(-\tau_n) + a_{13}(\beta^*(\alpha)+\mu)\varphi_3(-\tau_n) \\ a_{21}(\beta^*(\alpha)+\mu)\varphi_1(-\tau_n) - \varphi_2(0) + \alpha\varphi_2(-\tau_s) + a_{23}(\beta^*(\alpha)+\mu)\varphi_3(-\tau_n) \\ a_{31}(\beta^*(\alpha)+\mu)\varphi_1(-\tau_n) + a_{32}(\beta^*(\alpha)+\mu)\varphi_2(-\tau_n) - \varphi_3(0) + \alpha\varphi_3(-\tau_s) \end{bmatrix},$$

$$F(\varphi) = \alpha[\frac{1}{2}f''(0)\varphi^2(-\tau_s) + \frac{1}{6}f'''(0)\varphi^3(-\tau_s) + \ldots],$$

$$G(\varphi) = \begin{bmatrix} (\beta^*(\alpha)+\mu)[\frac{1}{2}g''(0)[a_{12}\varphi_2^2(-\tau_n)+a_{13}\varphi_3^2(-\tau_n)]+\frac{1}{6}g'''(0)[a_{12}\varphi_2^3(-\tau_n)+a_{13}\varphi_3^3(-\tau_n)]] \\ (\beta^*(\alpha)+\mu)[\frac{1}{2}g''(0)[a_{21}\varphi_1^2(-\tau_n)+a_{23}\varphi_3^2(-\tau_n)]+\frac{1}{6}g'''(0)[a_{21}\varphi_1^3(-\tau_n)+a_{23}\varphi_3^3(-\tau_n)]] \\ (\beta^*(\alpha)+\mu)[\frac{1}{2}g''(0)[a_{31}\varphi_1^2(-\tau_n)+a_{32}\varphi_2^2(-\tau_n)]+\frac{1}{6}g'''(0)[a_{31}\varphi_1^3(-\tau_n)+a_{32}\varphi_2^3(-\tau_n)]] \end{bmatrix}.$$

Let $L(\mu) = L_0 + L_1(\mu) + o(|\mu|)$, where

$$L_0\varphi = \begin{bmatrix} -\varphi_1(0) + \alpha\varphi_1(-\tau_s) + a_{12}\beta^*(\alpha)\varphi_2(-\tau_n) + a_{13}\beta^*(\alpha)\varphi_3(-\tau_n) \\ -\varphi_2(0) + a_{21}\beta^*(\alpha)\varphi_1(-\tau_n) + \alpha\varphi_2(-\tau_s) + a_{23}\beta^*(\alpha)\varphi_3(-\tau_n) \\ -\varphi_3(0) + a_{31}\beta^*(\alpha)\varphi_1(-\tau_n) + a_{32}\beta^*(\alpha)\varphi_2(-\tau_n) + \alpha\varphi_3(-\tau_s) \end{bmatrix},$$

$$L_1(\mu)\varphi = \mu \begin{bmatrix} a_{12}\varphi_2(-\tau_n) + a_{13}\varphi_3(-\tau_n) \\ a_{21}\varphi_1(-\tau_n) + a_{23}\varphi_3(-\tau_n) \\ a_{31}\varphi_1(-\tau_n) + a_{32}\varphi_2(-\tau_n) \end{bmatrix}.$$

Since we only need the lowest order terms in Taylor series for the nonlinearity, let $h = 0$ here, then

$$\begin{aligned} R(\Phi z, \mu) = &\ [L(\mu) - L_0]\Phi z + F(\Phi z, \mu) + G(\Phi z, \mu) \\ = &\ L_1(\mu)\Phi z + F(\Phi z, \mu) + G(\Phi z, \mu) \end{aligned}$$

Combining the previous processes, we obtain the following results:

For **Case 1**, $\beta = \beta_{11}$:

It is easy to check $\Phi = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, so $\Psi = \frac{1}{3+3\alpha\tau_s+3(1-\alpha)\tau_n}(1,1,1)$, then the normal form up to the third order is:

$$\dot{u}(t) = \frac{1}{1+\alpha\tau_s+(1-\alpha)\tau_n}[2\mu u+\frac{1}{2}(\alpha f''(0)+(1-\alpha)g''(0))u^2+\frac{1}{6}(\alpha f'''(0)+(1-\alpha)g'''(0))u^3].$$

For **Case 2,** $\beta = \beta_{21}$:

We have $\Phi = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, so $\Psi = \frac{1}{2+2\alpha\tau_s+2(1-\alpha)\tau_n}(0,1,-1)$, then the normal form up to the third order is:

$$\dot{u}(t) = \frac{1}{1+\alpha\tau_s+(1-\alpha)\tau_n}[-\mu u+\frac{1}{6}(\alpha f'''(0)+(1-\alpha)g'''(0))u^3].$$

For **Case 2,** $\beta = \beta_{22}$:

We have $\Phi = \begin{pmatrix} 1 \\ 1 \\ \frac{\sqrt{5}-1}{2} \end{pmatrix}$, so $\Psi = \frac{2}{(7-\sqrt{5})(1+\alpha\tau_s+(1-\alpha)\tau_n)}(1,1,\frac{\sqrt{5}-1}{2})$, then the normal form up to the third order is:

$$\begin{aligned}\dot{u}(t) = \ & \tfrac{1}{(7-\sqrt{5})(1+\alpha\tau_s+(1-\alpha)\tau_n)}[(3\sqrt{5}+1)\mu u+(\sqrt{5}\alpha f''(0)+\tfrac{7-5\sqrt{5}}{2}(\alpha-1)g''(0))u^2 \\ & +\tfrac{1}{6}((11-3\sqrt{5})\alpha f'''(0)+(5\sqrt{5}-15)(\alpha-1)g'''(0))u^3].\end{aligned}$$

For **Case 2,** $\beta = \beta_{23}$:

We have $\Phi = \begin{pmatrix} 1 \\ 1 \\ -\frac{\sqrt{5}+1}{2} \end{pmatrix}$, so $\Psi = \frac{2}{(7+\sqrt{5})(1+\alpha\tau_s+(1-\alpha)\tau_n)}(1,1,-\frac{\sqrt{5}+1}{2})$, then the normal form up to the third order is:

$$\begin{aligned}\dot{u}(t) = \ & \tfrac{1}{(7+\sqrt{5})(1+\alpha\tau_s+(1-\alpha)\tau_n)}[(1-3\sqrt{5})\mu u+(-\sqrt{5}\alpha f''(0)+\tfrac{7+5\sqrt{5}}{2}(\alpha-1)g''(0))u^2 \\ & +\tfrac{1}{6}((11+3\sqrt{5})\alpha f'''(0)+(5\sqrt{5}+15)(1-\alpha)g'''(0))u^3].\end{aligned}$$

For **Case 3**, $\beta = \beta_{31}$:

We have $\Phi = \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix}$, so $\Psi = \frac{1}{4+4\alpha\tau_s+4(1-\alpha)\tau_n}(1, 1, \sqrt{2})$, then the normal form up to the third order is:

$$\begin{aligned}\dot{u}(t) = \; & \tfrac{1}{4+4\alpha\tau_s+4(1-\alpha)\tau_n}[4\sqrt{2}\mu u + (1+\sqrt{2})(\alpha f''(0) + (1-\alpha)g''(0))u^2 \\ & +(\alpha f'''(0) + (1-\alpha)g'''(0))u^3].\end{aligned}$$

For **Case 3**, $\beta = \beta_{32}$:

We have $\Phi = \begin{pmatrix} 1 \\ 1 \\ -\sqrt{2} \end{pmatrix}$, so $\Psi = \frac{1}{4+4\alpha\tau_s+4(1-\alpha)\tau_n}(1, 1, -\sqrt{2})$, then the normal form up to the third order is:

$$\begin{aligned}\dot{u}(t) = \; & \tfrac{1}{4+4\alpha\tau_s+4(1-\alpha)\tau_n}[-4\sqrt{2}\mu u + (1-\sqrt{2})(\alpha f''(0) + (1-\alpha)g''(0))u^2 \\ & +(\alpha f'''(0) + (1-\alpha)g'''(0))u^3].\end{aligned}$$

For **Case 4 (1)**, $\beta = \beta_{41}$:

We have $\Phi = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, so $\Psi = \frac{1}{6+6\alpha\tau_s+6(1-\alpha)\tau_n}(2, 1, 1)$, then the normal form up to the third order is:

$$\dot{u}(t)=\frac{1}{6+6\alpha\tau_s+6(1-\alpha)\tau_n}[6\mu u+(5\alpha f''(0)+3(1-\alpha)g''(0))u^2+(3\alpha f'''(0)+(1-\alpha)g'''(0))u^3].$$

For **Case 4 (1)**, $\beta = \beta_{21}$:

We have $\Phi = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, so $\Psi = \frac{1}{2+2\alpha\tau_s+2(1-\alpha)\tau_n}(0, 1, -1)$, then the normal form up to the third order is:

$$\dot{u}(t) = \frac{1}{1+\alpha\tau_s+(1-\alpha)\tau_n}[-\mu u + \frac{1}{6}(\alpha f'''(0) + (1-\alpha)g'''(0))u^3].$$

For **Case 4 (2)**, $\beta = \beta_{41}$:

We have $\Phi = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, so $\Psi = \frac{1}{2+2\alpha\tau_s+2(1-\alpha)\tau_n}(0,1,1)$, then the normal form up to the third order is:

$$\dot{u}(t) = \frac{1}{1+\alpha\tau_s+(1-\alpha)\tau_n}[\mu u+\frac{1}{2}(\alpha f''(0)+(1-\alpha)g''(0))u^2+\frac{1}{6}(\alpha f'''(0)+(1-\alpha)g'''(0))u^3].$$

For **Case 4 (2)**, $\beta = \beta_{21}$:

We have $\Phi = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, so $\Psi = \frac{1}{2+2\alpha\tau_s+2(1-\alpha)\tau_n}(0,1,-1)$, then the normal form up to the third order is:

$$\dot{u}(t) = \frac{1}{1+\alpha\tau_s+(1-\alpha)\tau_n}[-\mu u+\frac{1}{6}(\alpha f'''(0)+(1-\alpha)g'''(0))u^3].$$

For **Case 4 (3)**, $\beta = \beta_{41}$:

We have $\Phi = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, so $\Psi = \frac{1}{3+3\alpha\tau_s+3(1-\alpha)\tau_n}(1,1,1)$, then the normal form up to the third order is:

$$\dot{u}(t) = \frac{1}{1+\alpha\tau_s+(1-\alpha)\tau_n}[\mu u+\frac{1}{2}(\alpha f''(0)+(1-\alpha)g''(0))u^2+\frac{1}{6}(\alpha f'''(0)+(1-\alpha)g'''(0))u^3].$$

For **Case 4 (3)**, $\beta = \beta_{21}$:

We have $\Phi = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, so $\Psi = \frac{1}{3+3\alpha\tau_s+3(1-\alpha)\tau_n}(1,1,-1)$, then the normal form up to the third order is:

$$\dot{u}(t) = \frac{1}{1+\alpha\tau_s+(1-\alpha)\tau_n}[-\mu u+\frac{1}{6}(\alpha f''(0)+(\alpha-1)g''(0))u^2+\frac{1}{6}(\alpha f'''(0)+(1-\alpha)g'''(0))u^3].$$

For **Case 4 (4)**, $\beta = \beta_{41}$:

We have $\Phi = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, so $\Psi = \frac{1}{2+2\alpha\tau_s+2(1-\alpha)\tau_n}(0,1,1)$, then the normal form up to the third order is:

$$\dot{u}(t) = \frac{1}{1+\alpha\tau_s+(1-\alpha)\tau_n}[\mu u + \frac{1}{2}(\alpha f''(0)+(1-\alpha)g''(0))u^2 + \frac{1}{6}(\alpha f'''(0)+(1-\alpha)g'''(0))u^3].$$

For **Case 4 (4)**, $\beta = \beta_{21}$:

We have $\Phi = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, so $\Psi = \frac{1}{2+2\alpha\tau_s+2(1-\alpha)\tau_n}(0,1,-1)$, then the normal form up to the third order is:

$$\dot{u}(t) = \frac{1}{1+\alpha\tau_s+(1-\alpha)\tau_n}[-\mu u + \frac{1}{6}(\alpha f'''(0)+(1-\alpha)g'''(0))u^3].$$

For **Case 6**, $\beta = \beta_{41}$:

We have $\Phi = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, so $\Psi = \frac{1}{3+3\alpha\tau_s+3(1-\alpha)\tau_n}(1,1,1)$, then the normal form up to the third order is:

$$\dot{u}(t) = \frac{1}{1+\alpha\tau_s+(1-\alpha)\tau_n}[\mu u + \frac{1}{2}(\alpha f''(0)+(1-\alpha)g''(0))u^2 + \frac{1}{6}(\alpha f'''(0)+(1-\alpha)g'''(0))u^3].$$

Remark 3.1.1 *In general, there will be dependence on $f''(0)$, $g''(0)$ in the third order terms. However, the normal forms above are correct, since the expressions we need including the third order terms are for pitchfork bifurcation and it is assumed that $f''(0) = g''(0) = 0$.*

3.2 Bifurcation Analysis

To study the different bifurcations from the trivial equilibrium, we introduce another assumption, (C3)$f''(0) = g''(0) = 0$.

3.2.1 Transcritical Bifurcation

If the functions f and g satisfy (C1), but not (C3), the dynamical properties are determined by the normal form truncated at the second order for a certain case in the following, therefore, the system in the corresponding case undergoes a transcritical bifurcation:

Case	Normal form
Case 1, $\beta = \beta_{11}$	$\dot{u}(t) = \frac{1}{1+\alpha\tau_s+(1-\alpha)\tau_n}[2\mu u + \frac{1}{2}(\alpha f''(0) + (1-\alpha)g''(0))u^2]$
Case 2, $\beta = \beta_{22}$	$\dot{u}(t) = \frac{1}{(7-\sqrt{5})(1+\alpha\tau_s+(1-\alpha)\tau_n)}$ $[(3\sqrt{5}+1)\mu u + (\sqrt{5}\alpha f''(0) + \frac{7-5\sqrt{5}}{2}(\alpha-1)g''(0))u^2]$
Case 2, $\beta = \beta_{23}$	$\dot{u}(t) = \frac{1}{(7+\sqrt{5})(1+\alpha\tau_s+(1-\alpha)\tau_n)}$ $[(1-3\sqrt{5})\mu u - (\sqrt{5}\alpha f''(0) - \frac{7+5\sqrt{5}}{2}(\alpha-1)g''(0))u^2]$
Case 3, $\beta = \beta_{31}$	$\dot{u}(t) = \frac{1}{4+4\alpha\tau_s+4(1-\alpha)\tau_n}[4\sqrt{2}\mu u + (1+\sqrt{2})(\alpha f''(0) + (1-\alpha)g''(0))u^2]$
Case 3, $\beta = \beta_{32}$	$\dot{u}(t) = \frac{1}{4+4\alpha\tau_s+4(1-\alpha)\tau_n}[-4\sqrt{2}\mu u + (1-\sqrt{2})(\alpha f''(0) + (1-\alpha)g''(0))u^2]$
Case 4(1), $\beta = \beta_{41}$	$\dot{u}(t) = \frac{1}{6+6\alpha\tau_s+6(1-\alpha)\tau_n}[6\mu u + (5\alpha f''(0) + 3(1-\alpha)g''(0))u^2]$
Case 4(2), $\beta = \beta_{41}$	$\dot{u}(t) = \frac{1}{1+\alpha\tau_s+(1-\alpha)\tau_n}[\mu u + \frac{1}{2}(\alpha f''(0) + (1-\alpha)g''(0))u^2]$
Case 4(3), $\beta = \beta_{41}$	$\dot{u}(t) = \frac{1}{1+\alpha\tau_s+(1-\alpha)\tau_n}[\mu u + \frac{1}{2}(\alpha f''(0) + (1-\alpha)g''(0))u^2]$
Case 4(3), $\beta = \beta_{21}$	$\dot{u}(t) = \frac{1}{1+\alpha\tau_s+(1-\alpha)\tau_n}[-\mu u + \frac{1}{6}(\alpha f''(0) + (\alpha-1)g''(0))u^2]$
Case 4(4), $\beta = \beta_{41}$	$\dot{u}(t) = \frac{1}{1+\alpha\tau_s+(1-\alpha)\tau_n}[\mu u + \frac{1}{2}(\alpha f''(0) + (1-\alpha)g''(0))u^2]$

3.2.2 Pitchfork Bifurcation

If f, g satisfy both the assumptions (C1) and (C3), the second order terms in the normal form disappear, we have to consider the normal form up to the third order.

If the parameter α satisfies the condition for a certain case in the following, the pitchfork bifurcation takes place in the system for the corresponding case:

Case	Condition
1	$\beta = \beta_{11}$ and $A_{11} \triangleq \alpha f'''(0) + (1-\alpha)g'''(0) \neq 0$
2	$\beta = \beta_{21}$ and $A_{11} \neq 0$
2	$\beta = \beta_{22}$ and $A_{22} \triangleq (11 - 3\sqrt{5})\alpha f'''(0) + (5\sqrt{5} - 15)(\alpha - 1)g'''(0) \neq 0$
2	$\beta = \beta_{23}$ and $A_{23} \triangleq (11 + 3\sqrt{5})\alpha f'''(0) + (5\sqrt{5} + 15)(1 - \alpha)g'''(0) \neq 0$
3	$\beta = \beta_{31}$ and $A_{11} \neq 0$
3	$\beta = \beta_{32}$ and $A_{11} \neq 0$
4(1)	$\beta = \beta_{41}$ and $A_{41} \triangleq 3\alpha f'''(0) + (1-\alpha)g'''(0) \neq 0$
4(1)	$\beta = \beta_{21}$ and $A_{11} \neq 0$
4(2)(3)(4)	$\beta = \beta_{41}$ and $A_{11} \neq 0$
4(2)(3)(4)	$\beta = \beta_{21}$ and $A_{11} \neq 0$

Let $A_0 \triangleq 1 + \alpha\tau_s + (1-\alpha)\tau_n$, then in **Case 1**, when $\beta = \beta_{11}$, it is supercritical when $A_0 A_{11} < 0$ and subcritical when $A_0 A_{11} > 0$. Similarly, we can obtain the similar results for the other cases in the above table.

3.2.3 Hopf Bifurcation

To begin, we verify that the characteristic equation has a simple pair of pure imaginary roots which cross the imaginary axis with nonzero speed, which means that a Hopf bifurcation occurs [23]. Then, we follow with a center manifold analysis of the criticality of Hopf bifurcation and determine the stability of the bifurcating periodic orbits.

We take **Case 3** as an example to do the analysis. For **Case 3** denote the characteristic equation P_3 as

$$\Delta_{31}\Delta_{32}\Delta_{33} = [\lambda+1-\alpha e^{-\lambda\tau_s}][\lambda+1-\alpha e^{-\lambda\tau_s}+\sqrt{2}\beta e^{-\lambda\tau_n}][\lambda+1-\alpha e^{-\lambda\tau_s}-\sqrt{2}\beta e^{-\lambda\tau_n}] = 0.$$

We assume $\Delta_{31}(i\omega) \neq 0$, since the parameter space is about α, β, τ_s, τ_n, however there is no β and τ_n in Δ_{31}.

Consider $\Delta_{32}(i\omega) = 0$ when ω, α, β, τ_s, τ_n satisfy (2.22) (2.23). It is easy to see that under these conditions excluding some particular points, $\Delta_{31}(i\omega) \neq 0$, $\Delta_{33}(i\omega) \neq 0$. Thus we have

$$\frac{\partial P_3}{\partial \lambda}|_{\lambda=i\omega} = \frac{\partial \Delta_{32}(i\omega)}{\partial \lambda}\Delta_{31}(i\omega)\Delta_{33}(i\omega), \tag{3.3}$$

and to check these roots are simple, it is enough to check that

$$\frac{\partial \Delta_{32}}{\partial \lambda}|_{\lambda=i\omega} = (1+\alpha\tau_s e^{-\lambda\tau_s} - \sqrt{2}\beta\tau_n e^{-\lambda\tau_n})|_{\lambda=i\omega} \neq 0.$$

This leads to the conditions

$$\begin{aligned} k_{11} &= 1+\alpha\tau_s\cos(\omega\tau_s) - \sqrt{2}\beta\tau_n\cos(\omega\tau_n) \neq 0, \\ k_{12} &= \alpha\tau_s\sin(\omega\tau_s) - \sqrt{2}\beta\tau_n\sin(\omega\tau_n) \neq 0. \end{aligned} \tag{3.4}$$

Now we fix all the parameters except the delay τ_n and find the conditions to guarantee that $Re(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_n}|_{\lambda=i\omega}) \neq 0$, i.e. the transversality condition for Hopf bifurcation is satisfied. Suppose $\lambda = \lambda(\tau_n)$, then from the characteristic equation $P_3(\lambda, \tau_n) = 0$ we have

$$\frac{dP_3}{d\tau_n} = \frac{\partial P_3}{\partial \tau_n} + \frac{\partial P_3}{\partial \lambda}\frac{d\lambda}{d\tau_n} = 0,$$

which gives

$$\frac{d\lambda}{d\tau_n} = -\frac{\partial P_3}{\partial \tau_n} / \frac{\partial P_3}{\partial \lambda}.$$

From the discuss above, it is easy to see that

$$\frac{\partial P_3}{\partial \tau_n}|_{\lambda=i\omega} = \frac{\partial \Delta_{32}(i\omega)}{\partial \tau_n}\Delta_{31}(i\omega)\Delta_{33}(i\omega).$$

Putting this together with (3.3) gives

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_n}|_{\lambda=i\omega} = -\frac{\partial P_3}{\partial \tau_n} / \frac{\partial P_3}{\partial \lambda}|_{\lambda=i\omega} = -\frac{\partial \Delta_{32}(i\omega)}{\partial \tau_n} / \frac{\partial \Delta_{32}(i\omega)}{\partial \lambda}.$$

We have

$$\mathrm{Re}(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_n}|_{\lambda=i\omega}) = \frac{\sqrt{2}\beta\omega(\sin(\omega\tau_n) + \alpha\tau_s \sin(\omega(\tau_n - \tau_s)))}{k_{11}^2 + k_{12}^2},$$

where k_{11} and k_{12} are defined in (3.4). Then it is clear that the transversality condition is

$$\beta\omega(\sin(\omega\tau_n) + \alpha\tau_s \sin(\omega(\tau_n - \tau_s))) \neq 0. \tag{3.5}$$

Similarly, considering $\Delta_{33}(i\omega) = 0$, we have the following conditions

$$\begin{aligned} k_{21} &= 1 + \alpha\tau_s \cos(\omega\tau_s) + \sqrt{2}\beta\tau_n \cos(\omega\tau_n) \neq 0, \\ k_{22} &= \alpha\tau_s \sin(\omega\tau_s) + \sqrt{2}\beta\tau_n \sin(\omega\tau_n) \neq 0, \end{aligned} \tag{3.6}$$

and the transversality condition is the same as (3.5).

We summarize the above results in the following theorem:

Theorem 3.2.1 *For fix α, $\tau_s \geq 0$ and for β satisfying $\beta = \beta_1^{\pm}$ ($\beta = \beta_2^{\pm}$) as defined by (2.24)((2.29)), the system (1.5) for* **Case 3** *undergoes a Hopf bifurcation at $\tau_n = \tau_{1k}^{\pm}$ ($\tau_n = \tau_{2k}^{\pm}$) as defined by (2.25) and (2.26) ((2.30) and (2.31)) if conditions (3.4) ((3.6)) and (3.5) hold.*

To analyze Hopf bifurcation, generally [24] we can derive the explicit formula to determine the properties of the Hopf bifurcation at the critical value of the delay using the normal form and the center manifold theory, i.e. we can determine the direction, stability and period of these periodic solutions bifurcating from the steady state.

For $\phi \in C^1([-h,0],\mathbb{R}^3)$, define

$$A(\mu)\phi = \begin{cases} \frac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, & \theta \in [-h,0), \\ \int_{-h}^{0} \mathrm{d}\eta(\theta,\mu)\phi(\theta), & \theta = 0, \end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-h,0), \\ f(\mu,\phi), & \theta = 0. \end{cases}$$

Then system (1.5) is equivalent to

$$\dot{x}(t) = A(\mu)x_t + R(\mu)x_t,$$

where $x_t(\theta) = x(t+\theta)$ for $\theta \in [-h,0]$.

For $\psi \in C^1([0,h],(\mathbb{R}^3)^*)$, define the adjoint operator A^* as following:

$$A^*\psi(s) = \begin{cases} -\frac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & s \in (0,h], \\ \int_{-h}^{0} \mathrm{d}\eta^T(s,0)\psi(-s), & s = 0. \end{cases}$$

And the bilinear inner product

$$<\psi(s),\phi(\theta)>=\bar{\psi}(0)\phi(0) - \int_{-h}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta)\mathrm{d}\eta(\theta)\phi(\xi)\mathrm{d}\xi,$$

where $\eta(\theta) = \eta(\theta,0)$.

Then, as usual, we know

$$<\psi, A\phi>=<A^*\psi,\phi>,$$

where $(\phi, \psi) \in D(A) \times D(A^*)$, and the normalization condition $< q^*, q >= 1$.

We know that $\pm i\omega$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^*. We first need to compute the eigenvector of $A(0)$ and A^* associated with $\pm i\omega$.

Suppose that $q(\theta) = (1, q_1, q_2)^T e^{i\theta\omega}$ is the eigenvector of $A(0)$ corresponding to $i\omega$. Then $A(0)q(\theta) = i\omega q(\theta)$, i.e.

$$\begin{pmatrix} i\omega + 1 - \alpha e^{-i\omega\tau_s} & -a_{12}\beta e^{-i\omega\tau_n} & -a_{13}\beta e^{-i\omega\tau_n} \\ -a_{21}\beta e^{-i\omega\tau_n} & i\omega + 1 - \alpha e^{-i\omega\tau_s} & -a_{23}\beta e^{-i\omega\tau_n} \\ -a_{31}\beta e^{-i\omega\tau_n} & -a_{32}\beta e^{-i\omega\tau_n} & i\omega + 1 - \alpha e^{-i\omega\tau_s} \end{pmatrix} \begin{pmatrix} 1 \\ q_1 \\ q_2 \end{pmatrix} e^{i\omega\theta} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus, we can obtain $q(0)$ easily for each case.

Similarly, let $q^*(s) = D(1, q_1^*, q_2^*)e^{is\omega}$ be the eigenvector of $A^*(0)$ corresponding to $-i\omega$, then $A^*(0)q^*(s) = -i\omega q^*(s)$, i.e.

$$\begin{pmatrix} 1 & q_1^* & q_2^* \end{pmatrix} e^{i\omega s} \begin{pmatrix} i\omega + 1 - \alpha e^{-i\omega\tau_s} & -a_{12}\beta e^{-i\omega\tau_n} & -a_{13}\beta e^{-i\omega\tau_n} \\ -a_{21}\beta e^{-i\omega\tau_n} & i\omega + 1 - \alpha e^{-i\omega\tau_s} & -a_{23}\beta e^{-i\omega\tau_n} \\ -a_{31}\beta e^{-i\omega\tau_n} & -a_{32}\beta e^{-i\omega\tau_n} & i\omega + 1 - \alpha e^{-i\omega\tau_s} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

So for each case we have $q^*(0)$ obviously.

In the order to conform $< q^*(s), q(\theta) >= 1$, we need to assure the appropriate value of D. From the definition, we have

$$\begin{aligned} < q^*(s), q(\theta) > &= \bar{D}(1, \bar{q_1^*}, \bar{q_2^*})(1, q_1, q_2)^T - \int_{-h}^{0}\int_{\xi=0}^{\theta} \bar{D}(1, \bar{q_1^*}, \bar{q_2^*})e^{-i(\xi-\theta)\omega} \mathrm{d}\eta(\theta)(1, q_1, q_2)^T e^{i\xi\omega} \mathrm{d}\xi \\ &= \bar{D}\left\{1 + q_1\bar{q_1^*} + q_2\bar{q_2^*} - \int_{-h}^{0} (1, \bar{q_1^*}, \bar{q_2^*})\theta e^{i\theta\omega} \mathrm{d}\eta(\theta)(1, q_1, q_2)^T\right\}. \end{aligned}$$

Since

$$D_1 = (1, \bar{q_1^*}, \bar{q_2^*}) \left(\begin{pmatrix} -\alpha\tau_s e^{-i\omega\tau_s} & -a_{12}\beta\tau_n e^{-i\omega\tau_n} & -a_{13}\beta\tau_n e^{-i\omega\tau_n} \\ -a_{21}\beta\tau_n e^{-i\omega\tau_n} & -\alpha\tau_s e^{-i\omega\tau_s} & -a_{23}\beta\tau_n e^{-i\omega\tau_n} \\ -a_{31}\beta\tau_n e^{-i\omega\tau_n} & -a_{32}\beta\tau_n e^{-i\omega\tau_n} & -\alpha\tau_s e^{-i\omega\tau_s} \end{pmatrix} \begin{pmatrix} 1 \\ q_1 \\ q_2 \end{pmatrix} \right)$$

we have

$$< q^*(s), q(\theta) >= \bar{D}\{1 + q_1\bar{q_1^*} + q_2\bar{q_2^*} - D_1\}.$$

Thus, we can obtain D as

$$D = \frac{1}{1 + \bar{q_1}q_1^* + \bar{q_2}q_2^* - \bar{D_1}}.$$

The 'form' of the center manifold we need here in low dimension is

$$\dot{\mathbf{z}} = B\mathbf{z} + \Psi(0)R(\Phi\mathbf{z}),$$

where

$$B = \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} z \\ \bar{z} \end{pmatrix}, \quad \mathbf{\Psi(0)} = \begin{pmatrix} q^*(0) \\ \overline{q^*(0)} \end{pmatrix},$$

and

$$R(\Phi z) = F(\Phi z) + G(\Phi z),$$

$$F(\phi) = \alpha[\frac{1}{2}f''(0)\phi^2(-\tau_s) + \frac{1}{6}f'''(0)\phi^3(-\tau_s)],$$

$$G\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = \begin{bmatrix} \beta[\frac{1}{2}g''(0)(a_{12}\varphi_2^2(-\tau_n) + a_{13}\varphi_3^2(-\tau_n) + \frac{1}{6}g'''(0)(a_{12}\varphi_2^3(-\tau_n) + a_{13}\varphi_3^3(-\tau_n))] \\ \beta[\frac{1}{2}g''(0)(a_{21}\varphi_1^2(-\tau_n) + a_{23}\varphi_3^2(-\tau_n) + \frac{1}{6}g'''(0)(a_{21}\varphi_1^3(-\tau_n) + a_{23}\varphi_3^3(-\tau_n))] \\ \beta[\frac{1}{2}g''(0)(a_{31}\varphi_1^2(-\tau_n) + a_{32}\varphi_2^2(-\tau_n) + \frac{1}{6}g'''(0)(a_{31}\varphi_1^3(-\tau_n) + a_{32}\varphi_2^3(-\tau_n))] \end{bmatrix}.$$

We take **Case 3** $\Delta_{32}(i\omega) = 0$ as an example to do a detailed analysis:

It is easy to obtain

$$\Phi(\theta)\mathbf{z} = \begin{pmatrix} e^{i\omega\theta}z + e^{-i\omega\theta}\bar{z} \\ e^{i\omega\theta}z + e^{-i\omega\theta}\bar{z} \\ -\sqrt{2}e^{i\omega\theta}z - \sqrt{2}e^{-i\omega\theta}\bar{z} \end{pmatrix} \triangleq \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}.$$

Directly computing we have

$$\dot{z} = i\omega z + \frac{i-3\omega}{4(5\omega^2-2\omega+1)}[(2-\sqrt{2})\alpha\frac{f''(0)}{2}(e^{-i\omega\tau_s}z+e^{i\omega\tau_s}\bar{z})^2$$
$$-\sqrt{2}\beta(\frac{1}{2}g''(0)+\frac{1}{6}g'''(0))(e^{-i\omega\tau_n}z+e^{i\omega\tau_n}\bar{z})^2+(2-\sqrt{2})\alpha\frac{f'''(0)}{6}(e^{-i\omega\tau_s}z+e^{i\omega\tau_s}\bar{z})^3$$
$$+(4-\sqrt{2})\beta(\frac{1}{2}g''(0)+\frac{1}{6}g'''(0))(e^{-i\omega\tau_n}z+e^{i\omega\tau_n}\bar{z})^3].$$

Let $z = x + iy$, so we can rewrite the system above as

$$\begin{aligned}\dot{x} &= -\omega y - 3\omega\mathcal{E},\\ \dot{y} &= \omega x + \mathcal{E},\end{aligned}$$

where

$$\mathcal{E} = \frac{1}{4(5\omega^2-2\omega+1)}[(2-\sqrt{2})\alpha\frac{f''(0)}{2}(e^{-i\omega\tau_s}z+e^{i\omega\tau_s}\bar{z})^2$$
$$-\sqrt{2}\beta(\frac{1}{2}g''(0)+\frac{1}{6}g'''(0))(e^{-i\omega\tau_n}z+e^{i\omega\tau_n}\bar{z})^2+(2-\sqrt{2})\alpha\frac{f'''(0)}{6}(e^{-i\omega\tau_s}z+e^{i\omega\tau_s}\bar{z})^3$$
$$+(4-\sqrt{2})\beta(\frac{1}{2}g''(0)+\frac{1}{6}g'''(0))(e^{-i\omega\tau_n}z+e^{i\omega\tau_n}\bar{z})^3]$$
$$= a(e^{-i\omega\tau_s}z+e^{i\omega\tau_s}\bar{z})^2+b(e^{-i\omega\tau_n}z+e^{i\omega\tau_n}\bar{z})^2+c(e^{-i\omega\tau_s}z+e^{i\omega\tau_s}\bar{z})^3+d(e^{-i\omega\tau_n}z+e^{i\omega\tau_n}\bar{z})^3$$
$$= 4a(x\cos(\omega\tau_s)+y\sin(\omega\tau_s))^2+4b(x\cos(\omega\tau_n)+y\sin(\omega\tau_n))^2$$
$$+8c(x\cos(\omega\tau_s)+y\sin(\omega\tau_s))^3+8d(x\cos(\omega\tau_n)+y\sin(\omega\tau_n))^3,$$

and

$$\begin{aligned}a &= \frac{(2-\sqrt{2})\alpha}{8(5\omega^2-2\omega+1)}f''(0),\\ b &= \frac{-\sqrt{2}\beta}{4(5\omega^2-2\omega+1)}(\frac{1}{2}g''(0)+\frac{1}{6}g'''(0)),\\ c &= \frac{(2-\sqrt{2})\alpha}{24(5\omega^2-2\omega+1)}f'''(0),\\ d &= \frac{(4-\sqrt{2})\beta}{4(5\omega^2-2\omega+1)}(\frac{1}{2}g''(0)+\frac{1}{6}g'''(0)).\end{aligned}$$

According to [22], we can obtain the Lyapunov coefficient for the Hopf bifurcation:

$$
\begin{aligned}
N &= -24\omega(c\cos^3(\omega\tau_s) + d\cos^3(\omega\tau_n)) + 8c\sin^3(\omega\tau_s) + 8d\sin^3(\omega\tau_n) \\
&\quad -72\omega(c\cos(\omega\tau_s)\sin^2(\omega\tau_s) + d\cos(\omega\tau_s)\sin^2(\omega\tau_s)) \\
&\quad +8(c\cos^2(\omega\tau_s)\sin(\omega\tau_s) + d\cos^2(\omega\tau_n)\sin(\omega\tau_n)) \\
&\quad +(9\omega^2-1)(8a\sin(\omega\tau_s)\cos(\omega\tau_s) + 8b\sin(\omega\tau_s)\cos(\omega\tau_s))(4a+4b) \\
&\quad +3\omega(4a\cos^2(\omega\tau_s) + 4b\cos^2(\omega\tau_n))^2 - 3\omega(4a\sin^2(\omega\tau_s) + 4b\sin^2(\omega\tau_n))^2.
\end{aligned} \tag{3.7}
$$

So we have the following theorem:

Theorem 3.2.2 *The criticality of the bifurcation for* **Case 3** *is determined by the sign of Lyapunov coefficient (3.7). When $N > 0$, the bifurcation is subcritical and the Hopf bifurcation yields an unstable limit cycle; while $N < 0$, the bifurcation is supercritical and the Hopf bifurcation yields a stable limit cycle.*

Remark 3.2.1 *Using the same method, we can obtain similar results for other cases.*

We take **Case 2** as an example to do numerical continuation using DDE-BIFTOOL [12]. In Fig. 3.1, there are two Hopf bifurcation points when the y-axis $\frac{1}{3}(\max(x_1)+\max(x_2)+\max(x_3)) = 0$. As β is increasing, we know from Fig. 3.1 there are two periodic solutions, which are asynchronous ($x_1(t) \neq x_2(t) \neq x_3(t)$). Fig. 3.2(a) shows a standing wave (periodic solutions satisfy $u_{n-i}(t) = u_i(t - \frac{1}{2}p)$, $i(\text{mod } n)$, $t \in \mathbb{R}$) occurring in Fig.3.1 L1. Fig. 3.2(b) depicts a mirror-reflecting wave (periodic solutions satisfy $u_i(t) = u_{n-i}(t)$, $i(\text{mod } n)$, $t \in \mathbb{R}$) which occurs in Fig.3.1 L2.

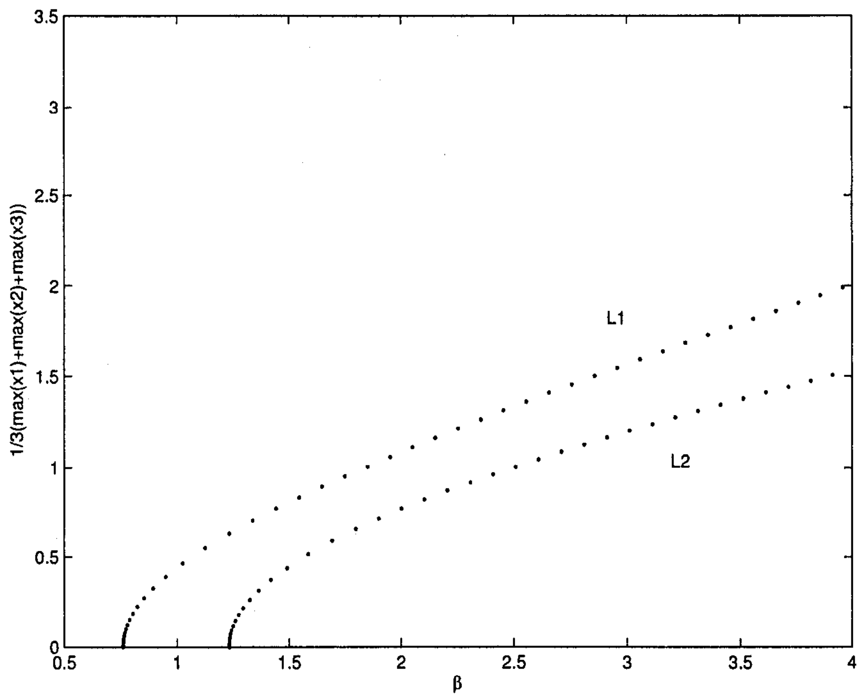

Figure 3.1: Numerical continuation of periodic solutions emanating from Hopf bifurcation with $\alpha = -1.5$, $\tau_s = 1$, $\tau_n = 1$ for **Case 2**.

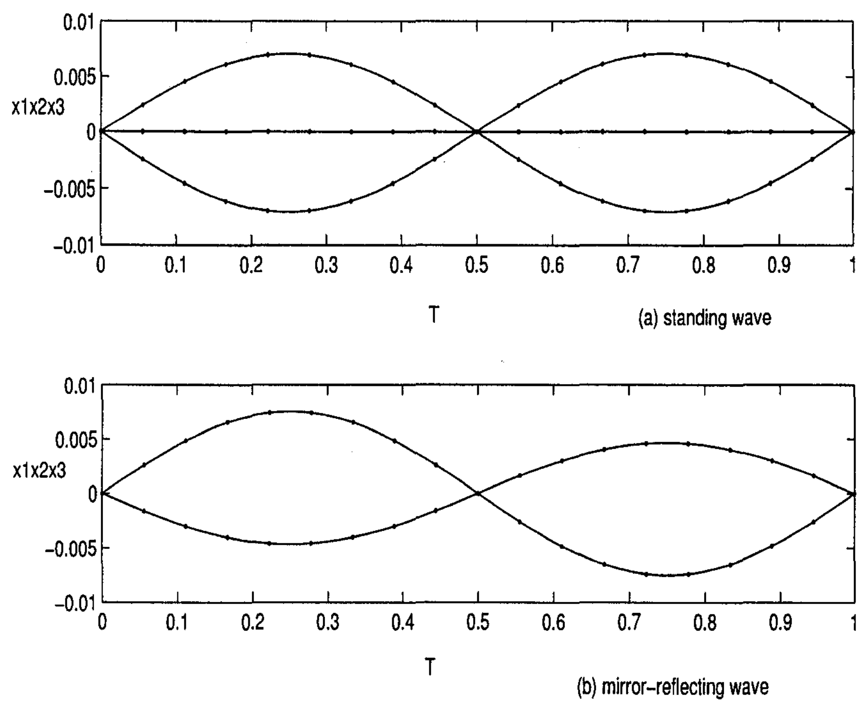

Figure 3.2: Hopf bifurcations of standing wave and mirror-reflecting wave with $\alpha = -1.5$, $\tau_s = 1$, $\tau_n = 1$ for **Case 2**.

Chapter 4

GLOBAL EXISTENCE OF PERIODIC SOLUTIONS WHEN $\tau_S = 0$

In delay differential equations we can obtain periodic solutions through Hopf bifurcation. However, most results are generally local. So it is important to discuss the extendibility of the non-constant periodic solution globally. By using a global Hopf bifurcation result due to Wu [51] and high-dimensional Bendixson's criterion due to Li and Muldowney [31], we can show that the local Hopf bifurcation implies the global Hopf bifurcation for a certain critical value of delay.

For system (1.5) if $\tau_s = 0$, $\tau_n = \tau$ we have

$$\begin{aligned}
\dot{x}_1(t) &= -x_1(t) + \alpha f(x_1(t)) + a_{12}\beta g(x_2(t-\tau)) + a_{13}\beta g(x_3(t-\tau)), \\
\dot{x}_2(t) &= -x_2(t) + a_{21}\beta g(x_1(t-\tau)) + \alpha f(x_2(t)) + a_{23}\beta g(x_3(t-\tau)), \qquad (4.1) \\
\dot{x}_3(t) &= -x_3(t) + a_{31}\beta g(x_1(t-\tau)) + a_{32}\beta g(x_2(t-\tau)) + \alpha f(x_3(t)),
\end{aligned}$$

and assume

(H_1) there exists $L > 0$ such that $|f(x)| \leq L$, $|g(x)| \leq L$ for $x \in \mathbb{R}$; The origin $(0,0,0)$ is the unique equilibrium.

The corresponding characteristic equation becomes

$$(\lambda+1-\alpha)^3-(a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32})\beta^3e^{-3\lambda\tau}-(a_{23}a_{32}+a_{13}a_{31}+a_{12}a_{21})(\lambda+1-\alpha)\beta^2e^{-2\lambda\tau}=0.$$

4.1 Stability Analysis

Let us consider the characteristic equation:

$$\Delta_5(\lambda) \triangleq \lambda + 1 - \alpha - \beta e^{-\lambda\tau} = 0. \tag{4.2}$$

We know that when $\tau = \tau_j$, $\lambda = i\omega$ is a root of (4.2) with

$$\omega = \sqrt{\beta^2 - (1-\alpha)^2},$$

where

$$\tau_j = \frac{1}{\sqrt{\beta^2 - (1-\alpha)^2}} \left[\arccos\left(\frac{1-\alpha}{\beta}\right) + j_0\pi \right], \tag{4.3}$$

$$j_0 = \begin{cases} 2j+1, & \text{when } \beta > 0 \quad (j = 0, 1, \ldots) \\ 2j. & \text{when } \beta < 0 \quad (j = 0, 1, \ldots) \end{cases}$$

For the system corresponding to (4.2), we have the following results:

Lemma 4.1.1

1) If the parameters α and β satisfy $\alpha + |\beta| < 1$, all the roots in Eq. (4.2) have negative real parts.

2) If $\alpha + \beta > 1$, then there exists a solution with a positive real part.

3) If the parameters α and β satisfy $|1-\alpha| < -\beta$, then the equilibrium $(0,0,0)$ is asymptotically stable when $\tau \in [0,\tau_0)$, and unstable when $\tau > \tau_0$, where τ_0 is defined in (4.3). Moreover, at $\tau = \tau_j$, $(j = 0, 1, \ldots)$, there exists a pair of purely imaginary roots in (4.2). Hopf bifurcation occurs near $(0,0,0)$.

The proof is similar to those in Lemma 2.2.1, (2.4.1) and (2.3.1). In order to guarantee a Hopf bifurcation, we need not only the existence of pure imaginary eigenvalues, i.e. $\Delta_5(i\omega) = 0$, but also the transversality condition $\frac{\partial \Delta_5(\lambda)}{\partial \lambda}|_{\lambda=i\omega} = 1 + \beta\tau e^{-i\omega\tau} \neq 0$, which is obvious.

Since the characteristic equation in **Case 2-Case 4** can be factored as the multiplication of three first-order quasi-polynomials and each factor has the similar construct to (4.2). The corresponding τ_j has the following form:

$$\begin{aligned}
\tau_{21j} &= \frac{1}{\sqrt{\beta^2-(1-\alpha)^2}}\left[\arccos\left(\frac{\alpha-1}{\beta}\right)+j_0\pi\right],\\
\tau_{22j} &= \frac{1}{\sqrt{\frac{3+\sqrt{5}}{2}\beta^2-(1-\alpha)^2}}\left[\arccos\left(\frac{(\sqrt{5}-1)(1-\alpha)}{2\beta}\right)+j_0\pi\right],\\
\tau_{23j} &= \frac{1}{\sqrt{\frac{3-\sqrt{5}}{2}\beta^2-(1-\alpha)^2}}\left[\arccos\left(\frac{(\sqrt{5}+1)(\alpha-1)}{2\beta}\right)+j_0\pi\right],\\
\tau_{31j} &= \frac{1}{\sqrt{2\beta^2-(1-\alpha)^2}}\left[\arccos\left(\frac{\sqrt{2}(1-\alpha)}{2\beta}\right)+j_0\pi\right],\\
\tau_{32j} &= \frac{1}{\sqrt{2\beta^2-(1-\alpha)^2}}\left[\arccos\left(\frac{\sqrt{2}(\alpha-1)}{2\beta}\right)+j_0\pi\right],\\
\tau_{41j} &= \tau_{21j},\\
\tau_{42j} &= \tau_j\,.
\end{aligned}$$

Rerange the critical value of time delay τ_{ikj} in **Case i** $(i = 2, 3, 4)$ as $\tau_{i0} < \tau_{i1} < \cdots <$

$\tau_{ij} < \cdots$, and the corresponding pure imaginary eigenvalue as $I\omega_i$ (I is the unit of pure imaginary root).

Theorem 4.1.2

1) If the parameters α, β satisfy the corresponding condition in the following table, then the equilibrium (0,0,0) is stable for any $\tau \geq 0$.

Case	*Condition*
2	$\alpha + \frac{1+\sqrt{5}}{2}\lvert\beta\rvert < 1$
3	$\alpha + \sqrt{2}\lvert\beta\rvert < 1$
4	$\alpha + \lvert\beta\rvert < 1$

2) If the parameters α, β satisfy the corresponding condition in the following table, then the equilibrium (0,0,0) is unstable for $\tau \geq 0$.

Case	*Condition*
2	$\alpha + \frac{1+\sqrt{5}}{2}\beta > 1$ *or* $\alpha - \beta > 1$
3	$\alpha + \sqrt{2}\beta > 1$ *or* $\alpha - \sqrt{2}\beta > 1$ *or* $\alpha > 1$
4	$\alpha + \beta > 1$ *or* $\alpha - \beta > 1$ *or* $\alpha > 1$

3) In addition, for **Case 2**, *if the parameters α, β satisfy*

(H_2) $\alpha - 1 < \beta < \frac{\sqrt{5}-1}{2}(\alpha - 1)$,

then the equilibrium (0,0,0) is asymptotically stable when $\tau \in [0, \tau_{20})$, and unstable when $\tau > \tau_{20}$. Moreover, at $\tau = \tau_{2j}$, $j = 0, 1, \ldots$, the corresponding system has a pair of pure imaginary roots, and undergoes Hopf bifurcation near $(0,0,0)$.

Therefore, we can determine the stability situation for each case in the parameter (α, β) plane. For instance, Fig (4.1) shows the stability region for **Case 2**: in **I**, the trivial solution is stable; in **II**, the trivial solution is conditional stable; in part **III**, the trivial solution is unstable.

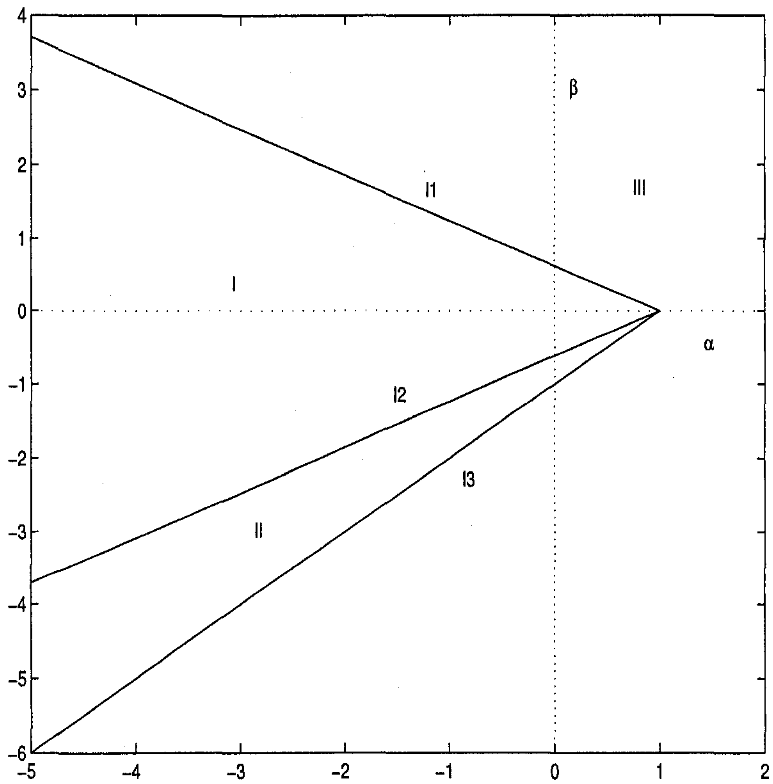

Figure 4.1: Stability region partition in the parameter α, β space for **Case 2**, where $l1$ is $\alpha + \frac{1+\sqrt{5}}{2}\beta = 1$, $l2$ is $\alpha - \frac{1+\sqrt{5}}{2}\beta = 1$, $l3$ is $\alpha - \beta = 1$.

4.2 Preliminary Results

To investigate the global existence of multiple periodic solutions for system (1.5) with $\tau_s = 0$, we need to combine the global Hopf bifurcation result and high-dimensional Bendixson's criterion.

Firstly we introduce the global Hopf bifurcation result of Wu [51], which we shall explain as the following:

Let X be a Banach space of bounded continuous mappings x: $\mathbb{R} \to \mathbb{R}^n$ with supreme norm. Consider the functional differential equation

$$x'(t) = F(x^t, \alpha, T), \tag{4.4}$$

where $F : X \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^n$ is completely continuous. Restrict F to a subspace of constant function x, we have a mapping $\hat{F} = F|_{\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+} : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^n$. Assume

(A1) $\hat{F} \in C^2$.

Let $\hat{x}_0 \in X$ be a constant mapping with value $x_0 \in \mathbb{R}^n$. The point $(\hat{x}_0, \alpha_0, T_0)$ is called a stationary solution of (4.4) if $\hat{F}(\hat{x}_0, \alpha_0, T_0) = 0$. Assume

(A2) $D_x\hat{F}(x, \alpha, T)|_{(\hat{x}_0,\alpha_0,T_0)}$ *is an isomorphism at each stationary solution* $(\hat{x}_0, \alpha_0, T_0)$.

Under the assumptions (A1) and (A2), by the implicit function theorem, for each stationary solution $(\hat{x}_0, \alpha_0, T_0)$, there exists $\epsilon_0 > 0$ and a C^1 mapping $y : B_{\epsilon_0}(\alpha_0, T_0) \to \mathbb{R}^n$ such that $\hat{F}(y(\alpha, T), \alpha, T) = 0$, for $(\alpha, T) \in B_{\epsilon_0}(\alpha_0, T_0) = (\alpha_0 - \epsilon_0, \alpha_0 + \epsilon_0) \times (T_0 - \epsilon_0, T_0 + \epsilon_0)$. Define the characteristic matrix at a stationary solution $(\hat{x}_0, \alpha_0, T_0)$ of (4.4), as

$$\Delta_{(\hat{x}_0,\alpha_0,T_0)}(\lambda) = \lambda I - D_\varphi F(\hat{x}_0, \alpha_0, T_0)(e^\lambda I),$$

where $D_\varphi F(\hat{x}_0, \alpha_0, T_0)$ is the derivative of $F(\varphi, \alpha, T)$ with respect to φ at $(\hat{x}_0, \alpha_0, T_0)$. The zeros of det $\Delta_{(\hat{x}_0,\alpha_0,T_0)}(\lambda) = 0$ are the characteristic roots. Note that (A2) is equivalent to assuming that $\lambda = 0$ is not a characteristic root of any stationary solution. Assume

(A3) $F(\varphi, \alpha, T)$ is differentiable with respect to φ. The characteristic matrix $\Delta_{(\hat{x}_0, \alpha_0, T_0)}(\lambda)$ is continuous in $(\alpha, T, \lambda) \in B_{\epsilon_0}(\alpha_0, T_0) \times \mathbb{C}$.

A stationary solution $(\hat{x}_0, \alpha_0, T_0)$ is said to be a center, if it has purely imaginary characteristic values of the form $im\frac{2\pi}{T_0}$ for certain positive integer m. A center $(\hat{x}_0, \alpha_0, T_0)$ is isolated if (i) it is the only center in a neighborhood of $(\hat{x}_0, \alpha_0, T_0)$, (ii) it has only finitely many purely imaginary characteristic values of the form $im\frac{2\pi}{T_0}$. Let $J(\hat{x}_0, \alpha_0, T_0)$ be the set of all such positive integers m at an isolated center $(\hat{x}_0, \alpha_0, T_0)$. Assume that

(A4) there exists $\epsilon, \delta \in (0, \epsilon_0)$ such that on $[\alpha_0 - \delta, \alpha_0 + \delta] \times \partial\Omega_{(\epsilon, T_0)}$, $\det \Delta_{(\hat{y}(\alpha, T), \alpha, T)}(u + im\frac{2\pi}{T}) = 0$ for some $m \in J(\hat{x}_0, \alpha_0, T_0)$ if and only if $\alpha = \alpha_0$, $u = 0$, $T = T_0$, where

$$\Omega_{(\epsilon, T_0)} = \{(u, T) : 0 < u < \epsilon, |T - T_0| < \epsilon\}.$$

Define

$$H_m^{\pm}(\hat{x}_0, \alpha_0, T_0)(u, T) = \det \Delta_{(\hat{y}(\alpha_0 \pm \delta, T), \alpha_0 \pm \delta, T)} \left(u + im\frac{2\pi}{T}\right). \tag{4.5}$$

Then (A4) implies that $H_m^{\pm}(\hat{x}_0, \alpha_0, T_0) \neq 0$ on $\partial\Omega_{(\epsilon, T_0)}$. Thus, the mth crossing number $\gamma_m(\hat{x}_0, \alpha_0, T_0)$ of $(\hat{x}_0, \alpha_0, T_0)$ can be defined, using topological degree of mappings $H_m^{\pm}$, as

$$\gamma_m(\hat{x}_0, \alpha_0, T_0) = \deg_B(H_m^{-}(\hat{x}_0, \alpha_0, T_0), \Omega_{(\epsilon, T_0)}) - \deg_B(H_m^{+}(\hat{x}_0, \alpha_0, T_0), \Omega_{(\epsilon, T_0)}). \tag{4.6}$$

It is shown in Wu [51] that $\gamma_m(\hat{x}_0, \alpha_0, T_0) \neq 0$ implies the existence of a local bifurcation of periodic solutions with periods near $\frac{T_0}{m}$.

In addition,

(A5) All centers of (4.4) are isolated and (A4) holds for each center $(\hat{x}_0, \alpha_0, T_0)$ and each

$m \in J(\hat{x}_0, \alpha_0, T_0)$.

(A6) For each bounded set $W \subseteq X \times \mathbb{R} \times \mathbb{R}_+$, *there exists constant* $L > 0$ *s.t.* $|F(\varphi, \alpha, T) - F(\psi, \alpha, T) \leq L \sup_{s \in \mathbb{R}} |\varphi(s) - \psi(s)|$, *for* $(\varphi, \alpha, T) \in W$.

The following is a global Hopf bifurcation result in Wu [51].

Theorem 4.2.1 *Assume that (A1)-(A6) hold. Let*

$$\Sigma(F) = \mathrm{Cl}\{(x, \alpha, T) : x \text{ is a } T-\text{periodic solutions of } (4.4)\} \subset X \times \mathbb{R} \times \mathbb{R}_+,$$

$$N(F) = \{(\hat{x}, \alpha, T) : F(\hat{x}, \alpha, T) = 0\}.$$

Let $C(\hat{x}_0, \alpha_0, T_0)$ *be the connected component in* $\Sigma(F)$ *of an isolated center* $(\hat{x}_0, \alpha_0, T_0)$. *Then either*

1) $C(\hat{x}_0, \alpha_0, T_0)$ *is unbounded, or*

2) $C(\hat{x}_0, \alpha_0, T_0)$ *is bound,* $C(\hat{x}_0, \alpha_0, T_0) \cap N(F)$ *is finite, and*

$$\sum_{(\hat{x}, \alpha, T) \in C(\hat{x}_0, \alpha_0, T_0) \cap N(F)} \gamma_m(\hat{x}, \alpha, T) = 0 \tag{4.7}$$

for all $m = 1, 2, \ldots$, *where* $\gamma_m(\hat{x}, \alpha, T)$ *is the* m*th crossing number of* $(\hat{x}, \alpha, T)$ *if* $m \in J(\hat{x}, \alpha, T)$, *or it is zero otherwise.*

By the theorem above, to show that $C(\hat{x}_0, \alpha_0, T_0)$ is unbounded, it suffices to show that the sum in (4.7) is nonzero, for a particular integer m.

Next, we review the high-dimensional bendixson's criterion of Li and Muldowney [31].

Consider a system of ordinary differential equations

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \ f \in C^1. \tag{4.8}$$

As shown in [31], to derive a high-dimensional Bendixson criterion, it is sufficient to show that the second compound equation

$$z'(t) = \frac{\partial f^{[2]}}{\partial x}(x(t, x_0))z(t), \tag{4.9}$$

with respect to a solution $x(t, x_0) \in D$ (where $D \in \mathbb{R}^n$ is an open connected set) to (4.8) is equi-uniformly asymptotically stable, namely, for each $x_0 \in D$, system (4.4) is uniformly asymptotically stable, and the exponential decay rate is uniform for x_0 in each compact subset of D. Here $\frac{\partial f^{[2]}}{\partial x}$ is the second additive compound matrix of the Jacobian matrix $\frac{\partial f}{\partial x}$. It is an $\begin{pmatrix} n \\ 2 \end{pmatrix} \times \begin{pmatrix} n \\ 2 \end{pmatrix}$ matrix, and thus (4.9) is a linear system of dimension $\begin{pmatrix} n \\ 2 \end{pmatrix}$.

For a 3×3 matrix

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix},$$

its second additive compound matrix $C^{[2]}$ is

$$C^{[2]} = \begin{pmatrix} c_{11} + c_{22} & c_{23} & -c_{13} \\ c_{32} & c_{11} + c_{33} & c_{12} \\ -c_{31} & c_{21} & c_{22} + c_{33} \end{pmatrix}. \tag{4.10}$$

If D is simply connected, the equi-uniform asymptotic stability of (4.9) precludes the existence of any invariant simple closed rectifiable curve in D, including periodic orbits. In particular, the following result is provided in [31].

Theorem 4.2.2 *Let $D \subset \mathbb{R}^n$ be a simply connected region. Assume that the family of linear systems*

$$z'(t) = \frac{\partial f^{[2]}}{\partial x}(x(t, x_0))z(t), \quad x_0 \in D$$

is equi-uniformly asymptotically stable. Then

1) D contains no simple closed invariant curves including periodic orbits, homoclinic orbits, heteroclinic cycles;

2) each semi-orbit in D converges to a single equilibrium.

In particular, if D is positively invariant and contains an unique equilibrium $\bar{x}$, then $\bar{x}$ is globally asymptotically stable in D.

The required uniform asymptotic stability of the family of linear systems (4.9) can be proved by constructing a suitable Lyapunov function. For instance, (4.9) is equi-uniformly asymptotically stable if there exists a positive definite function $V(z)$, such that $\frac{dV(z)}{dt}|_{(4.9)}$ is negative definite, where V and $\frac{dV(z)}{dt}|_{(4.9)}$ are both independent of x_0.

4.3 Nonexistence of Nonconstant Periodic Solution When $\tau = 0$

Consider the system (4.1) with $\tau = 0$,

$$\begin{aligned} \dot{x}_1 &= -x_1 + \alpha f(x_1) + a_{12}\beta g(x_2) + a_{13}\beta g(x_3), \\ \dot{x}_2 &= -x_2 + a_{21}\beta g(x_1) + \alpha f(x_2) + a_{23}\beta g(x_3), \\ \dot{x}_3 &= -x_3 + a_{31}\beta g(x_1) + a_{32}\beta g(x_2) + \alpha f(x_3), \end{aligned} \tag{4.11}$$

Under the following assumption:

(H_3) There exists $m_1, m_2 > 0$ s.t.

$$\mu(t) = \max\{\mu_1(t), \mu_2(t), \mu_3(t)\} < 0, \tag{4.12}$$

where

$$
\begin{aligned}
\mu_1(t) &= -2 + \alpha f'(x_1) + \alpha f'(x_2) + \frac{m_1}{m_2}|a_{23}\beta g'(x_3)| + m_1|a_{13}\beta g'(x_3)|, \\
\mu_2(t) &= -2 + \alpha f'(x_1) + \alpha f'(x_3) + \frac{m_2}{m_1}|a_{32}\beta g'(x_2)| + m_2|a_{12}\beta g'(x_2)|, \\
\mu_3(t) &= -2 + \alpha f'(x_2) + \alpha f'(x_3) + \frac{1}{m_1}|a_{31}\beta g'(x_1)| + \frac{1}{m_2}|a_{21}\beta g'(x_1)|.
\end{aligned}
$$

We have

Theorem 4.3.1 *If the Hypotheses (C1), (H_1) and (H_3) are satisfied, then the system (4.11) has no non-constant periodic solutions. Furthermore, the unique equilibrium (0,0,0) is globally asymptotically stable in* $\mathbb{R}^3$.

Proof: Firstly we prove that solutions of (4.11) are bounded. Let

$$V(x_1, x_2, x_3) = \sum_{i=1}^{3} x_i^2.$$

Then the derivative of V along a solution of (4.11) is

$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t}|_{(4.11)} &= 2\sum_{i=1}^{3} x_i \dot{x}_i \\
&= 2\sum_{i=1}^{3} x_i(-x_i + \alpha f(x_i) + \sum_{j=1, j\neq i}^{3} a_{ij}\beta g(x_j)) \\
&= -2\sum_{i=1}^{3} x_i^2 + 2\alpha \sum_{i=1}^{3} x_i f(x_i) + 2\beta \sum_{i=1}^{3} \sum_{j=1, j\neq i}^{3} a_{ij} x_i g(x_j)
\end{aligned}
$$

From the assumption (C1) and (H_1), we have

$$\frac{\mathrm{d}V}{\mathrm{d}t}|_{(4.11)} \leq -2\sum_{i=1}^{3} x_i^2 + 2\alpha L \sum_{i=1}^{3} |x_i| + 2\beta L \sum_{i=1}^{3} \sum_{j=1, j\neq i}^{3} a_{ij}|x_i|.$$

There exists $M > 1$ s.t. $\sqrt{\sum_{i=1}^{3} x_i^2} \geq M$ which implies

$$\frac{\mathrm{d}V}{\mathrm{d}t}|_{(4.11)} \leq 0.$$

This shows that the solutions of system (4.11) are uniformly ultimately bounded.

Denote $x = (x_1, x_2, x_3)^T$ and

$$F(x) = \begin{pmatrix} -x_1 + \alpha f(x_1) + a_{12}\beta g(x_2) + a_{13}\beta g(x_3) \\ -x_2 + a_{21}\beta g(x_1) + \alpha f(x_2) + a_{23}\beta g(x_3) \\ -x_3 + a_{31}\beta g(x_1) + a_{32}\beta g(x_2) + \alpha f(x_3) \end{pmatrix},$$

then

$$\frac{\partial F}{\partial x} = \begin{pmatrix} -1 + \alpha f'(x_1) & a_{12}\beta g'(x_2) & a_{13}\beta g'(x_3) \\ a_{21}\beta g'(x_1) & -1 + \alpha f'(x_2) & a_{23}\beta g'(x_3) \\ a_{31}\beta g'(x_1) & a_{32}\beta g'(x_2) & -1 + \alpha f'(x_3) \end{pmatrix}.$$

$$\frac{\partial F^{[2]}}{\partial x} = \begin{pmatrix} -2 + \alpha f'(x_1) + \alpha f'(x_2) & a_{23}\beta g'(x_3) & -a_{13}\beta g'(x_3) \\ a_{32}\beta g'(x_2) & -2 + \alpha f'(x_1) + \alpha f'(x_3) & a_{12}\beta g'(x_2) \\ -a_{31}\beta g'(x_1) & a_{21}\beta g'(x_1) & -2 + \alpha f'(x_2) + \alpha f'(x_3) \end{pmatrix}.$$

The corresponding second compound system is

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \frac{\partial F^{[2]}}{\partial x} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix},$$

i.e.,

$$\begin{aligned} \dot{z}_1 &= (-2 + \alpha f'(x_1) + \alpha f'(x_2))z_1 + a_{23}\beta g'(x_3)z_2 - a_{13}\beta g'(x_3)z_3 \\ \dot{z}_2 &= a_{32}\beta g'(x_2)z_1 + (-2 + \alpha f'(x_1) + \alpha f'(x_3))z_2 + a_{12}\beta g'(x_2)z_3. \\ \dot{z}_3 &= -a_{31}\beta g'(x_1)z_1 + a_{21}\beta g'(x_1)z_2 + (-2 + \alpha f'(x_2) + \alpha f'(x_3))z_3 \end{aligned}$$

Let

$$W(z) = \max\{m_1|z_1|, m_2|z_2|, |z_3|\},$$

where $m_1, m_2 > 0$ are constant. By direct calculation, we have:

$$\frac{\mathrm{d}^+}{\mathrm{d}t} m_1|z_1| \le (-2+\alpha f'(x_1)+\alpha f'(x_2))\cdot m_1|z_1| + \frac{m_1}{m_2}|a_{23}\beta g'(x_3)|\cdot m_2|z_2| + m_1|a_{13}\beta g'(x_3)|\cdot|z_3|,$$
$$\frac{\mathrm{d}^+}{\mathrm{d}t} m_2|z_2| \le \frac{m_2}{m_1}|a_{32}\beta g'(x_2)|\cdot m_1|z_1| + (-2+\alpha f'(x_1)+\alpha f'(x_3))\cdot m_2|z_2| + m_2|a_{12}\beta g'(x_2)|\cdot|z_3|,$$
$$\frac{\mathrm{d}^+}{\mathrm{d}t}|z_3| \le \frac{1}{m_1}|a_{31}\beta g'(x_1)|\cdot m_1|z_1| + \frac{1}{m_2}|a_{21}\beta g'(x_1)|\cdot m_2|z_2| + (-2+\alpha f'(x_2)+\alpha f'(x_3))|z_3|,$$

where $\mathrm{d}^+/\mathrm{d}t$ denotes the right-hand derivative. From (H_3), it is obtained that

$$\frac{\mathrm{d}^+}{\mathrm{d}t} W(z(t)) \le \mu(t) W(z(t)).$$

From the boundedness of solution to (4.11), there exists a $\delta > 0$ s.t. $\mu(t) \le -\delta < 0$. So we have

$$W(z(t)) \le W(z(s))e^{-\delta(t-s)}, \quad t \ge s > 0.$$

This shows the equi-uniform asymptotic stability of the second compound system (4.13). So from Theorem 4.2.2 we get the conclusion of Theorem 4.3.1. □

Remark 4.3.1 *We understand that the assumption (H_3) is difficult to verify since no explicit expressions of solutions are available in general. While for some simple models, there are some efficient conditions for the differential equation independent of time delay. Since we focus on the delay differential equation in this thesis, we do not consider it further.*

4.4 Global Existence of Periodic Solutions

In this subsection, we will show that the local Hopf branches of (4.1) obtained in Theorem 4.1.2(2) can be extended for large values of the delay τ in **Case 2**.

Consider the Fuller space

$$\Sigma = \{(\hat{x}, \tau, T)\} : \hat{x} \text{ is a } T-\text{periodic solution of } (4.1)\} \subset X \times \mathbb{R} \times \mathbb{R}_+.$$

Obviously (4.1) does not depend explicitly on T. We will verify assumptions (A1)-(A6) in Theorem 4.2.1 for system (4.1) as following:

From assumption (C1) and (H_1), (A1) and (A6) are obviously satisfied. Since $(0,0,0)$ is the only equilibrium, thus all stationary solution are of the form $(\hat{0}, \tau, T)$. $\lambda = 0$ is not a characteristic root of the equilibrium $(0,0,0)$, thus (A2) is satisfied. The characteristic equation $q(\lambda)$ is continuous in $(\tau, T, \lambda) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{C}$, which verifies (A3).

A stationary solution $(\hat{0}, \tau, T)$ is a center if $(0,0,0)$ has purely imaginary eigenvalues of form $Im\frac{2\pi}{T}$. By Theorem 4.1.2(2) we know that this occurs if and only if $m = 1$, $\tau = \tau_{2k}$ and $T = \frac{2\pi}{\omega_2}$, and $\lambda(\tau_{2k}) = I\omega_2$. Therefore, the set of centers is given by

$$\left\{ \left(\hat{0}, \tau_{2k}, \frac{2\pi}{\omega_2}\right); k = 0, 1, 2, \ldots \right\},$$

and all centers are isolated. For fixed k, there exists $\epsilon > 0$, $\delta > 0$ and a smooth curve $\lambda : (\tau_{2k} - \delta, \tau_{2k} + \delta) \to \mathbb{C}$ s.t. $q(\lambda(\tau)) = 0$, $|\lambda(\tau) - I\omega_2| < \epsilon$ for all $\tau \in (\tau_{2k} - \delta, \tau_{2k} + \delta)$, and $\lambda(\tau_{2k}) = I\omega_2$.

Let

$$\Omega_\epsilon = \left\{ (\nu, T) : 0 < \nu < \epsilon, |T - \frac{2\pi}{\omega_2}| < \epsilon \right\}. \tag{4.13}$$

Clearly, if $|\tau - \tau_{2k}| < \delta$ and $(\nu, T) \in \partial\Omega_\epsilon$ satisfy $q(\nu + I\frac{2\pi}{T}) = 0$, then $\tau = \tau_{2k}$, $\nu = 0$ and $T = \frac{2\pi}{\omega_2}$. This verifies (A4) and (A5). Moreover, if

$$H_m^{\pm}\left(\hat{0}, \tau_{2k}, \frac{2\pi}{\omega_2}\right)(\nu, T) = \Delta_{(\hat{0}, \tau_{2k} \pm \delta, T)}\left(\nu + Im\frac{2\pi}{T}\right),$$

then, at $m = 1$,

$$\gamma_m\left(\hat{0}, \tau_{2k}, \frac{2\pi}{\omega_{20}}\right) = \deg_B\left(H_m^-\left(\hat{0}, \tau_{2k}, \frac{2\pi}{\omega_2}\right), \Omega_\epsilon\right) - \deg_B\left(H_B^+\left(\hat{0}, \tau_{2k}, \frac{2\pi}{\omega_2}\right), \Omega_\epsilon\right) = -1.$$

Thus the connected component $C(\hat{0}, \tau_{2k}, \frac{2\pi}{\omega_2})$ through $(\hat{0}, \tau_{2k}, \frac{2\pi}{\omega_2})$ in Σ is nonempty. Since the first crossing number of each center is always -1, by Theorem 4.2.1, we conclude that:

Lemma 4.4.1 *$C(\hat{0}, \tau_{2k}, \frac{2\pi}{\omega_i})$ is unbounded for each center $(\hat{0}, \tau_{2k}, \frac{2\pi}{\omega_2})$.*

Moreover, we can obtain that

Lemma 4.4.2 *Periodic solutions of (4.1) are uniformly bounded.*

Proof: Let $M = \max\{1, L(\alpha + \beta \sum_{i=1}^{3} \sum_{j=1, j\neq i}^{3} a_{ij})\}$ and $r(t) = \sqrt{\sum_{i=1}^{3} x_i^2(t)}$. Differentiating $r(t)$ along a solution of (4.1) we have

$$\begin{aligned}
\dot{r}(t) &= \frac{1}{r(t)} \sum_{i=1}^{3} x_i(t)\dot{x}_i(t) \\
&= \frac{1}{r(t)}[-\sum_{i=1}^{3} x_i^2(t) + \alpha \sum_{i=1}^{3} x_i(t) f(x_i(t)) + \beta \sum_{i=1}^{3} \sum_{j=1, j\neq i}^{3} a_{ij} x_i(t) g(x_j(t-\tau))] \\
&\leq \frac{1}{r(t)}[-\sum_{i=1}^{3} x_i^2(t) + \alpha L \sum_{i=1}^{3} |x_i(t)| + \beta L \sum_{i=1}^{3} \sum_{j=1, j\neq i}^{3} a_{ij} |x_i(t)|]
\end{aligned}$$

If there exists $t_0 > 0$ s.t. $r(t_0) = A \geq M$, then we have

$$\begin{aligned}
\dot{r}(t_0) &\leq \frac{1}{A}[-A^2 + A\alpha L + A\beta L \sum_{i=1}^{3} \sum_{j=1, j\neq i}^{3} a_{ij}] \\
&= -A + \alpha L + \beta L \sum_{i=1}^{3} \sum_{j=1, j\neq i}^{3} a_{ij} < 0.
\end{aligned}$$

It follows that if $x(t) = (x_1(t), x_2(t), x_3(t))^T$ is a periodic solution of system (4.1), then $r(t) < M$ for all t. This shows that the periodic solutions of (4.1) are uniformly bounded. $\square$

Lemma 4.4.3 *The periods in the periodic solution of (4.1) are uniformly bounded.*

Proof: Note that if $x(t)$ is a τ-periodic solution of system (4.1), then $x(t)$ is τ-periodic solution of the ordinary differential equation (4.11). Applying Theorem 4.2.2, we know that under Hypothesis (H_3), the system (4.11) has no non-constant periodic solutions. Therefore, system (4.1) has no non-constant τ-periodic solutions.

By the definition of τ_{2k}, we know that

$$\omega_2 \tau_{2k} > 2\pi, \quad k = 1, 2, \ldots$$

and hence

$$\frac{2\pi}{\omega_2} < \tau_{2k}, \quad k = 1, 2, \ldots$$

From Theorem 4.1.2(2), we know that $\tau_{20} > 0$. Hence for $\tau > \tau_{2k}$, there exists an integer m s.t. $\frac{\tau}{m} < \frac{2\pi}{\omega_2} < \tau$. Since system (4.1) has no non-constant τ-periodic solution, it has no $\frac{\tau}{n}$-periodic solution for any integer n. This implies that the period T of a periodic solution on the connected component $C(\hat{0}, \tau_{2k}, \frac{2\pi}{\omega_2})$ satisfies $\frac{\tau}{m} < T < \tau$. So we know that the period of the periodic solutions of the system (4.1) on $C(\hat{0}, \tau_{2k}, \frac{2\pi}{\omega_2})$ are uniformly bounded. □

Based on the above discussion, we present our main result as:

Theorem 4.4.4 *Suppose the Hypotheses (C1) and* $(H_1) - (H_3)$ *are satisfied. Then system (4.1) for* **Case 2** *has at least* $k + 1$ *non-constant periodic solutions when* $\tau > \tau_{2k}$, $k \geq 1$.

Proof: By Lemma 4.4.1, it is obvious that $C(\hat{0}, \tau_{2k}, \frac{2\pi}{\omega_2})$ is nonempty and unbounded. By Lemmas (4.4.2) and (4.4.3), the projection of $C(\hat{0}, \tau_{2k}, \frac{2\pi}{\omega_2})$ onto the x-space is bounded.

For **Case 2**, we prove $\omega_2 \tau_{20} > \frac{\pi}{2}$ with (H_2).

In fact, let $\lambda = i\omega$ be the imaginary root of $P_{21}(\lambda) = \lambda + 1 - \alpha + \beta e^{-\lambda\tau}$, $P_{22}(\lambda) = \lambda + 1 - \alpha - \frac{1+\sqrt{5}}{2}\beta e^{-\lambda\tau}$ and $P_{32}(\lambda) = \lambda + 1 - \alpha - \frac{1-\sqrt{5}}{2}\beta e^{-\lambda\tau}$ respectively, then separate it into real and imaginary parts we have:

$$\begin{cases} 1-\alpha = -\beta\cos(\omega_{21}\tau_{210}) \\ \omega_{21} = \quad \beta\sin(\omega_{21}\tau_{210}) \end{cases}, \quad \begin{cases} 1-\alpha = \dfrac{1+\sqrt{5}}{2}\beta\cos(\omega_{22}\tau_{220}) \\ \omega_{22} = -\dfrac{1+\sqrt{5}}{2}\beta\sin(\omega_{22}\tau_{220}) \end{cases}, \quad \begin{cases} 1-\alpha = \dfrac{1-\sqrt{5}}{2}\beta\cos(\omega_{23}\tau_{230}) \\ \omega_{23} = -\dfrac{1-\sqrt{5}}{2}\beta\sin(\omega_{23}\tau_{230}) \end{cases}.$$

Without loss of generality, let $\beta > 0$. From (H_2), we have $\alpha < 1$. So

$$\begin{cases} \cos(\omega_{21}\tau_{210}) < 0 \\ \sin(\omega_{21}\tau_{210}) > 0 \end{cases} \quad \begin{cases} \cos(\omega_{22}\tau_{220}) > 0 \\ \sin(\omega_{22}\tau_{220}) < 0 \end{cases} \quad \begin{cases} \cos(\omega_{23}\tau_{230}) < 0 \\ \sin(\omega_{23}\tau_{230}) > 0 \end{cases}$$

So we have $\omega_2\tau_{30} > \frac{\pi}{2}$, i.e. $\frac{2\pi}{\omega_2} < 4\tau_{30} \leq 4\tau_{3k}$, $k \geq 0$. Thus, the projection of $C(\hat{0}, \tau_{2k}, \frac{2\pi}{\omega_2})$ onto T-space is bounded. This implies that the projection of $C(\hat{0}, \tau_{2k}, \frac{2\pi}{\omega_2})$ onto the τ-space must be unbounded.

Applying Theorem 4.2.2, we know that system (4.1) has no non-constant periodic solutions when $\tau = 0$. Thus, the projection of $C(\hat{0}, \tau_{2k}, \frac{2\pi}{\omega_i})$ onto the τ-space must be an interval $[\tau, \infty)$ with $0 < \tau \leq \tau_{2k}$. This shows, for each $\tau > \tau_{2k} \geq \tau_{21}$ the system (4.1) has at least k periodic solutions. This completes the proof of Theorem 4.4.4. □

Therefore, under the Hypotheses (C1) (H_1) and (H_3), the unique equilibrium $(0, 0, 0)$ of system (4.1) with $\tau = 0$ is globally asymptotically stable in $\mathbb{R}^3$. However, under the Hypotheses (C1) and $(H_1) - (H_3)$, system (4.1) has at least k non-constant periodic solutions when $\tau > \tau_{2k}$ $(k \geq 1)$. This demonstrates how time delays affect the dynamics of system (4.1).

Remark 4.4.1 *Using the same procedure, we can get the similar result for the system with* $\tau_n = 0$ *but* $\tau_s \neq 0$.

4.5 Numerical Simulation

In order to demonstrate the global Hopf bifurcation results in Theorem 4.4.4, we use XppAuto to carry out the numerical simulations. In Fig. (4.2), we show that there exists a periodic solution when $f(x) = g(x) = \tanh(x)$, $\tau = 5$, $\alpha = -0.5$, $\beta = -1$, where τ is between the two Hopf bifurcation values $\tau_{20} = 4.55$ and $\tau_{21} = 14.90$. Fig. (4.3) shows that there exists a critical value of τ_c. When $\tau < \tau_c$ the equilibrium of the system is stable. When $\tau > \tau_c$ there exists at least a periodic solution.

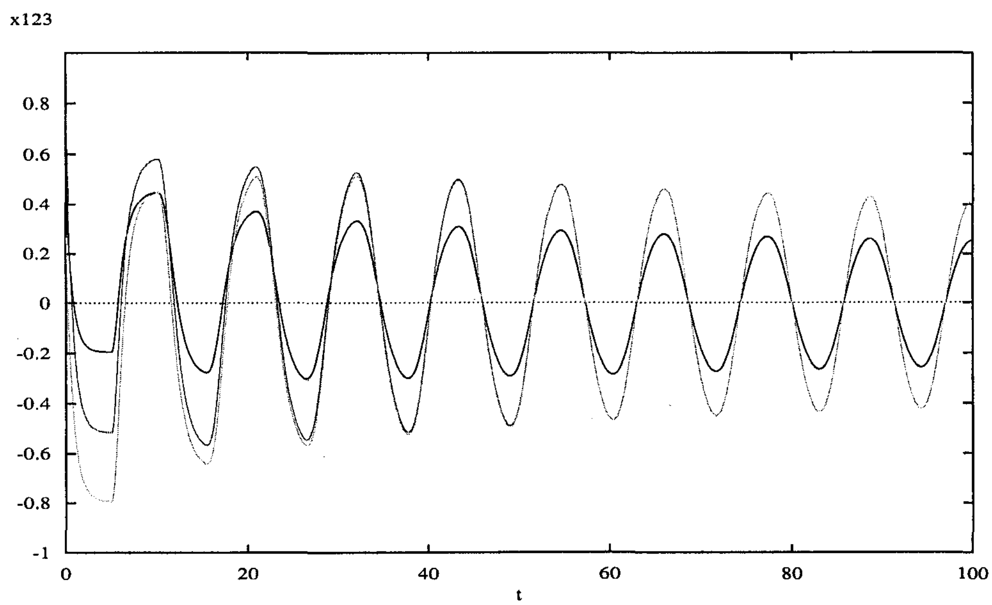

Figure 4.2: A periodic solution on $x_{1,2,3} - t$ spaces with $\tau = 5$, $\alpha = -0.5$, $\beta = -1$ and initial data $x_1 = 0.8$, $x_2 = 0.3$, $x_3 = 0.5$ in **Case 2**

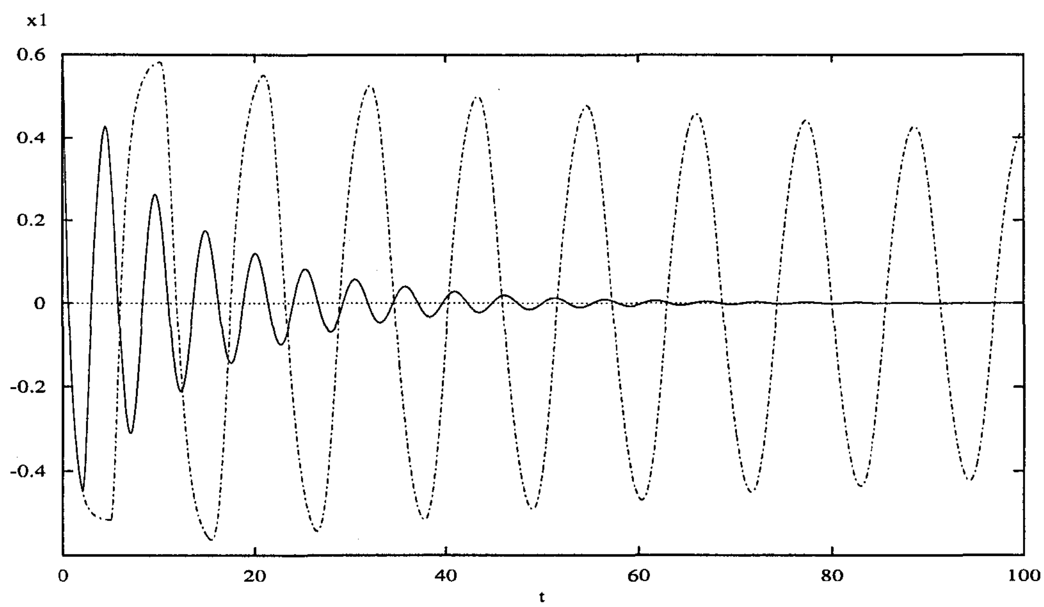

Figure 4.3: Numerical solution on $x_1 - t$ space for $\tau = 2, 5$, $\alpha = 0.5$, $\beta = -2$ in **Case 2**

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